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Pengfei Huang

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$$\rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p + \nabla \cdot T + f$$

$$e^{i\pi} + 1 = 0$$

# THÈSE DE DOCTORAT

## Théorie de Hodge Non-Abélienne et des Spécialisations

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# Théorie de Hodge Non-Abélienne et des Spécialisations

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## Résumé

La première partie de cette thèse est la géométrie de la théorie de Hodge non-Abélienne, en particulier l'étude des propriétés géométriques des espaces de modules.

Le premier résultat principal de cette partie est la construction d'un système dynamique sur l'espace de modules des fibrés de Higgs, nous montrons que les points fixes de ce système dynamique sont exactement ceux fixés par l'action de  $\mathbb{C}^*$  sur l'espace de modules des fibrés de Higgs, c'est-à-dire tous les  $\mathbb{C}$ -VHS dans l'espace de modules. Dans le même temps, nous étudions sa première variation et son comportement asymptotique.

Le deuxième résultat principal de cette partie est la preuve d'une conjecture (forme faible) par Simpson sur la stratification de l'espace de modules des fibrés plats, nous prouvons que la strata d'opérateurs est la strata fermée unique de dimension minimale en étudiant l'espace de modules des chaînes holomorphes de type donné.

Le troisième résultat principal de cette partie est une généralisation de la construction de l'espace de twistor de Deligne–Hitchin dans le cas de surface de Riemann, nous construisons des sections holomorphes pour l'espace de twistor de Deligne–Hitchin généralisé, c'est-à-dire les sections de de Rham. Nous calculons les fibrés normaux de ces sections, et nous avons constaté que les sections de de Rham ont la propriété wight un, donc ceux sont des courbes rationnelles amples équilibrées. Dans le même temps, nous montrons le théorème de type Torelli pour l'espace de twistor. De plus, nous étudions les groupes d'automorphisme des espaces de modules de Hodge et de l'espace de twistor de Deligne–Hitchin généralisé.

La deuxième partie de cette thèse est l'étude de certaines spécialisations de la correspondance de Hodge non-Abélienne. Celui-ci comprend principalement deux chapitres, le premier est une preuve fondamentale d'une conjecture liée aux représentations de carquois proposée par Reineke en 2003, nous montrons pour les représentations de carquois de type  $A_n$ , il existe un système de poids tel que les représentations stables par rapport à ce système de poids sont précisément celles indécomposables. Pour la deuxième, nous construisons la correspondance de Kobayashi–Hitchin pour les fibrés de carquois sur les variétés Kähleriennes généralisées.

**Mot clés:** Théorie de Hodge non-Abélienne, Espace de modules, Système dynamique, Strata d'opérateurs, Espace de twistor, Section de de Rham, Théorème de Torelli, Groupe d'automorphisme, Représentation de carquois, Variété Kählerienne généralisée, Fibré de carquois



## Abstract

The first part of this thesis is the geometry of non-Abelian Hodge theory, especially the study of geometric properties of moduli spaces.

The first main result of this part is the construction of a dynamical system on the moduli space of Higgs bundles, we show that fixed points of this dynamical system are exactly those fixed by the  $\mathbb{C}^*$ -action on the moduli space of Higgs bundles, that is, all  $\mathbb{C}$ -VHS in the moduli space. At the same time, we study its first variation and asymptotic behaviour.

The second main result of this part is the proof of a conjecture (weak form) by Simpson on the stratification of the moduli space of flat bundles, we prove that the oper stratum is the unique closed stratum of minimal dimension by studying the moduli space of holomorphic chains of given type.

The third main result of this part is a generalization of construction of Deligne–Hitchin twistor space in Riemann surface case, we construct holomorphic sections for the generalized Deligne–Hitchin twistor space, namely the de Rham sections. We calculate the normal bundles of these sections, and we found that de Rham sections have wight one property, so they are balanced ample rational curves. We also show the Torelli-type theorem for this new twistor space. Moreover, we study the automorphism groups of the Hodge moduli spaces and the generalized Deligne–Hitchin twistor space.

The second part of this thesis is the study of some specializations of non-Abelian Hodge correspondence. This mainly includes two chapters, the first one is a fundamental proof of a conjecture related to quiver representations proposed by Reineke in 2003, we show for representations of quivers of  $A_n$ -type, there exists a weight system such that the stable representations with respect to this weight system are precisely these indecomposable ones. For the second one, we build the Kobayashi–Hitchin correspondence for quiver bundles over generalized Kähler manifolds.

**Key words:** Non-Abelian Hodge theory, Moduli space, Dynamical system, Oper stratum, Twistor space, De Rham section, Torelli theorem, Automorphism group, Quiver representation, Generalized Kähler manifold, Quiver bundle



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# Introduction en francais



## Introduction

Soit  $(X, \omega)$  une variété Kählerienne de dimension  $n$ , et  $E$  un fibré vectoriel complexe sur  $X$ . Pour  $k \in \mathbb{Z}_{\geq 0}$ , notons par  $\mathcal{A}^k(E) := \Gamma(X, E \otimes \wedge^k(T_{\mathbb{C}}^*X))$ , l'espace des  $k$ -formes lisses sur  $X$  avec des valeurs dans  $E$ , en particulier,  $\mathcal{A}^0(E) = \Gamma(X, E)$ , l'espace des sections lisses de  $E$ . Pour  $\lambda \in \mathbb{C}$ , une  $\lambda$ -connexion ( $C^\infty$ ) plat sur  $E$  est un opérateur  $\mathbb{C}$ -linéaire  $\mathbb{D}^\lambda : \Gamma(X, E) \rightarrow \mathcal{A}^1(E)$  satisfait la règle Leibniz  $\lambda$ -twisted suivante:

$$\mathbb{D}^\lambda(fs) = f\mathbb{D}^\lambda(s) + \lambda\partial(f) \otimes s + \bar{\partial}(f) \otimes s,$$

où  $f \in C^\infty(X, \mathbb{C})$ ,  $s \in \Gamma(X, E)$ , et  $\mathbb{D}^\lambda \circ \mathbb{D}^\lambda = 0$  sous l'extension naturelle  $\mathbb{D}^\lambda : \mathcal{A}^1(E) \rightarrow \mathcal{A}^2(E)$ . Dans ce cas, la paire  $(E, \mathbb{D}^\lambda)$  est appelé un fibré  $\lambda$ -plat.

En particulier, lorsque  $\lambda = 1$ , c'est la connexion plate habituelle, nous utilisons la notation  $\nabla$ , et dans ce cas, la paire  $(E, \nabla)$  est appelé un fibré plat; lorsque  $\lambda = 0$ , décomposer  $\mathbb{D}^0$  en sa partie  $(0, 1)$  et  $(1, 0)$ , et nous obtenons  $\mathbb{D}^0 = \bar{\partial}_E + \theta$ , la planéité de  $\mathbb{D}^0$  implique  $(\bar{\partial}_E)^2 = \bar{\partial}_E\theta = \theta \wedge \theta = 0$ , ceci définit un fibré de Higgs, qui est le triple  $(E, \bar{\partial}_E, \theta)$ .

Le fibré  $\lambda$ -plat est à l'origine présenté par Deligne comme la compréhension de l'idée de Hitchin sur la construction de l'espace twistor de l'espace de modules des fibrés de Higgs [Del89], qui généralise les fibrés plats et les fibrés de Higgs. Plus tard, cela est étudié et développé par Simpson comme la filtration de Hodge sur la cohomologie non-Abélienne [Sim95, Sim08]. Pour mettre l'accent sur les fibrés plats et les fibrés de Higgs, nous les décrivons ici séparément.

Donc sur  $(X, \omega)$ , nous définissons les trois objets géométriques suivants:

- (1) Fibré plat:  $(E, \nabla)$ ;
- (2) Fibré de Higgs:  $(E, \bar{\partial}_E, \theta)$ ;
- (3) Fibré  $\lambda$ -plat ( $\lambda \in \mathbb{C}$ ):  $(E, \mathbb{D}^\lambda)$ .

En géométrie algébrique, en particulier en théorie géométrique des invariants, il existe une notion très importante pour les fibrés vectoriels, c'est la stabilité. On dit qu'un fibré  $\lambda$ -plat  $(E, \mathbb{D}^\lambda)$  est stable (resp. semistable), si pour tout sous-faisceau  $\mathcal{F}$  cohérent sans torsion saturé non nul qui est invariant par  $\mathbb{D}^\lambda$ , c'est-à-dire  $\mathbb{D}^\lambda|_{\Gamma(X, \mathcal{F})} : \Gamma(X, \mathcal{F}) \rightarrow \mathcal{A}^1(\mathcal{F})$ , avec  $0 < \text{rk}(\mathcal{F}) < \text{rk}(E)$ , on a

$$\mu_\omega(\mathcal{F}) < \text{ (resp. } \leq \text{ ) } \mu_\omega(E),$$

où  $\mu_\omega(E) := \frac{\text{deg}(E)}{\text{rk}(E)}$  est appelée la pente, et  $\text{deg}(E) := \int_X c_1(\det(E)) \wedge [\omega]^{n-1}$  est la degré. Il est polystable, si il est somme directe de fibrés  $\lambda$ -plats stables de la même pente.

En particulier, lorsque  $\lambda = 1$ , pour les fibrés plats, on applique généralement les notations simple (ou équivalente, irréductible) et semisimple (ou équivalente, réductible) plutôt que stable et polystable<sup>1</sup>, Puisqu'elles correspondent à des représentations simples (ou équivalentes, irréductibles) et semisimples de  $\pi_1(X)$  par la correspondance de Riemann-Hilbert.

En géométrie différentielle, il existe également une notion très importante pour les fibrés vectoriels, c'est-à-dire les métriques d'Hermite-Einstein. Pour un fibré  $\lambda$ -plat donné  $(E, \mathbb{D}^\lambda)$ , avec une métrique Hermitienne  $h$ , alors  $h$  induit une décomposition unique de  $\mathbb{D}^\lambda$ :

$$\mathbb{D}^\lambda = \lambda\partial_h + \theta + \bar{\partial}_h + \lambda\theta_h^\dagger$$

tel que:

---

<sup>1</sup>C'est évidemment dans le cas du compact que les fibrés plats sont automatiquement semistables.

- $\nabla_h := \partial_h + \bar{\partial}_h$  est une connexion  $h$ -unitaire, c'est-à-dire  $dh(u, v) = h(\nabla_h(u), v) + h(u, \nabla_h(v))$ ;
- $\Phi_h := \theta_h + \theta_h^\dagger \in \mathcal{A}^1(\text{End}(E))$  est un opérateur auto-adjoint par rapport à  $h$ , c'est-à-dire  $h(\theta_h(u), v) = h(u, \theta_h^\dagger(v))$ .

Soit  $D_{h,\lambda}^c := \partial_h + \theta_h^\dagger - \bar{\lambda}(\bar{\partial}_h + \theta_h)$ , on définit  $G_{h,\lambda} := \frac{1}{(1+|\lambda|^2)^2} [\mathbb{D}^\lambda, D_{h,\lambda}^c]$ , qui est appelée la pseudo-courbure. La métrique  $h$  est appelée une métrique d'Hermite–Einstein si l'équation suivante est vraie:

$$\Lambda_\omega G_{h,\lambda} = c \text{Id}_E,$$

où  $c = \frac{-2\pi\sqrt{-1} \deg_\omega(E)}{(1+|\lambda|^2)\text{rk}(E)\text{Vol}(X)}$  est une constante déterminée par  $X$  et les invariants topologiques de  $E$ .

La théorie de Hodge non-Abélienne est un pont entre la géométrie algébrique et la géométrie différentielle, qui met en relation la stabilité des fibrés vectoriels et l'existence de métriques d'Hermite–Einstein. Ceci est principalement basé sur les travaux de Donaldson [Don87] et Corlette [Cor88] sur les applications harmoniques (fibrés plats), Hitchin [Hit87a] et Simpson [Sim87] sur les fibrés de Higgs, et plus tard le travail de Mochizuki sur fibrés  $\lambda$ -plats (principalement des cas non compacts), qui identifient pleinement la stabilité des fibrés vectoriels et l'existence de métriques d'Hermite–Einstein, et de plus, fournir une correspondance bijective entre les objets ci-dessus sur une variété Kählerienne compacte  $(X, \omega)$ . Ce pont offre beaucoup de possibilités sur l'application de méthodes géométriques différentielles pour étudier des problèmes géométriques algébriques (par exemple, étudier la géométrie des espaces de modules).

En particulier, si ces fibrés vectoriels ont des classes de Chern nulles (la première et la deuxième), alors ces métriques d'Hermite–Einstein seront des métriques de pluri-harmoniques (c'est-à-dire la métrique Hermitienne  $h$  telle que  $G_{h,\lambda} = 0$ , ce qui équivaut à dire quand  $\lambda \neq 0$ , la métrique Hermitienne  $h$  telle que  $(E, \bar{\partial}_h, \theta_h)$  devenir un fibré de Higgs; et quand  $\lambda = 0$ , la métrique Hermitienne  $h$  telle que  $(E, \mathbb{D}^1)$  devenir un fibré plat). L'existence de métriques de pluri-harmoniques peut être conclue dans les énoncés suivantes:

**Théorème 1.** *Soit  $(X, \omega)$  une variété Kählerienne compacte. Alors*

- (1) (Donaldson [Don87], Corlette [Cor88]) *Un fibré plat  $(E, \nabla)$  sur  $X$  admet une métrique de pluri-harmonique si et seulement si il est semisimple;*
- (2) (Hitchin [Hit87a], Simpson [Sim87]) *Un fibré de Higgs  $(E, \bar{\partial}_E, \theta)$  sur  $X$  admet une métrique de pluri-harmonique si et seulement si il est polystable de classes de Chern nulles;*
- (3) (Mochizuki [Moc06]) *Un fibré  $\lambda$ -plat ( $\lambda \neq 0$ )  $(E, \mathbb{D}^\lambda)$  sur  $X$  admet une métrique de pluri-harmonique si et seulement si il est polystable de classes de Chern nulles.*

*De plus, dans chaque cas, la métrique de pluri-harmonique est unique jusqu'aux multiplicités scalaires.*

Par conséquence, nous obtenons la correspondance entre ces objets, appelée la correspondance de Hodge non-Abélienne:

**Corollaire 1 (La Correspondance de Hodge Non-Abélienne).** *Soit  $(X, \omega)$  une variété Kählerienne compacte. Alors pour tout  $\lambda \in \mathbb{C}$ , nous avons la correspondance bijective suivante entre les catégories:*

- (1) *La catégorie des fibrés plats semisimples de rang  $r$ ;*
- (2) *La catégorie des fibrés de Higgs polystables de rang  $r$  et de classes de Chern nulles;*

(3) *La catégorie des fibrés  $\lambda$ -plats polystables de rang  $r$  et de classes de Chern nulles;*

*ils sont reliés par des métriques de pluri-harmoniques, ils sont donc équivalents à la catégorie des fibrés harmoniques de rang  $r$ .*

Il existe beaucoup de généralisations de ce correspondance, une généralisation naturelle considère les variétés non compactes comme des variétés de base, la correspondance qui en résulte est dû à Simpson [Sim90], Biquard [Biq97], Jost–Zuo [JZ97], Mochizuki [Moc06, Moc09] et autres. D'autres généralisations telles que considérer les groupes de Lie réels comme des groupes de structure, ou considérer les corps de caractéristique positive, les corps  $p$ -adiques comme des corps de base [BGPiR03, GPGiR09, OV07, Fal05, AGT16]. Nous ne prétendons pas donner plus de détails à ces sujets ici.

Notre objectif principal de cette thèse est d'appliquer la théorie de Hodge non-Abélienne ci-dessus pour étudier certains problèmes de géométrie algébrique et de géométrie différentielle.

## 0.1 La Géométrie de la Théorie de Hodge Non-Abélienne

Cette partie concerne principalement la géométrie de la correspondance de Hodge non-Abélienne, en particulier les propriétés géométriques des espaces de modules associés.

### 0.1.1 La Théorie de Hodge Non-Abélienne de la Version Compacte

Le deuxième chapitre de cette thèse, en particulier les deux premières sections (§1.1 et §1.2), est une introduction à la théorie de Hodge non-Abélienne, nous allons essayer d'expliquer cette théorie explicitement dans des pages limitées, en particulier les fibrés  $\lambda$ -plats qui joueront des rôles importants dans la première partie de la thèse. Nos nouveaux résultats de ce chapitre sont principalement dans la troisième section (§1.3), nous commencerons par une estimation de la norme des sections  $\mathbb{D}^\lambda$ -plats sur des surfaces de Riemann compactes, alors nous obtenons un théorème d'annulation, c'est-à-dire qu'il n'y a pas de sections  $\mathbb{D}^\lambda$ -plats sur les fibrés  $\lambda$ -plats stables de la première classe de Chern nulle sur des surfaces de Riemann compactes. Plus précisément, nous avons les énoncés suivantes:

**Théorème 2** (= Theorem 1.3.1, Corollary 1.3.2). *Soit  $(X, \omega)$  une surface de Riemann compacte et  $(E, \mathbb{D}^\lambda)$  un fibré  $\lambda$ -plat, avec une métrique Hermitienne  $h$ . Donc pour toute section localement  $\mathbb{D}^\lambda$ -plate  $s$  de  $E$ , nous avons l'inégalités suivantes:*

$$(1) \quad \Delta_\omega \log(|s|_h^2) \geq -\frac{2}{(1+|\lambda|^2)} |\Lambda_\omega G_{h,\lambda}|_h,$$

$$(2) \quad \Delta_\omega(|s|_h^2) \geq -\frac{2}{1+|\lambda|^2} |\Lambda_\omega G_{h,\lambda}|_h \cdot |s|_h^2.$$

où  $\Delta_\omega$  est l'opérateur Laplacien sur  $(X, \omega)$ .

*En particulier, si  $(E, \mathbb{D}^\lambda)$  est stable de la première classe de Chern nulle, alors il n'y a pas de section  $\mathbb{D}^\lambda$ -plate globale non nulle sur  $E$ .*

Ensuite, nous considérons quelques exemples, le premier est des fibrés  $\lambda$ -plats de rang 2 sur un disque unité ouvert obtenu à partir de “solutions fiduciales” à des équations de Hitchin découpées introduites dans [MSWW16], et nous calculons les solutions explicites des sections  $\mathbb{D}^\lambda$ -plates. Le deuxième exemple considère l’action de  $\mathbb{C}^*$ , nous calculons une solution spéciale de métrique de pluri-harmonique de tels fibrés  $\lambda$ -plats sur le disque unité ouvert, cela montre que pour un fibré  $\lambda$ -plat sur une variété non complète, s’il admet une métrique de pluri-harmonique, il n’est généralement pas unique.

### 0.1.2 Une Étude de la Géométrie des Espaces de Modules

Le troisième chapitre de cette thèse est une étude de la géométrie des espaces de modules, qui se compose de trois sections.

Dans la première section (§2.1), nous construisons un système dynamique sur l’espace de modules des fibrés de Higgs en appliquant la correspondance de Hodge non-Abélienne, et nous étudions ce système dynamique. Plus explicitement, par la construction par Simpson d’espaces de modules via la théorie géométrique des invariants [Sim94a, Sim94b, Sim95], pour une variété projective complexe lisse  $X$ , nous avons trois espaces de modules de les objets précédents et l’espace de modules de représentations de  $\pi_1(X)$ :

- (1) L’espace de modules de de Rham  $\mathcal{M}_{\text{dR}}(X, r)$ : l’espace de modules de fibrés plats de rang  $r$  sur  $X$ ;
- (2) L’espace de modules de Dolbeault  $\mathcal{M}_{\text{Dol}}(X, r)$ : l’espace de modules de fibrés de Higgs semistables de rang  $r$  de classes de Chern nulles sur  $X$ ;
- (3) L’espace de modules de Hodge  $\mathcal{M}_{\text{Hod}}(X, r)$ : l’espace de modules de fibrés  $\lambda$ -plats semistables de rang  $r$  de classes de Chern nulles sur  $X$ ;
- (4) L’espace de modules de Betti  $\mathcal{M}_{\text{B}}(X, r)$ : l’espace de modules de représentations  $\rho : \pi_1(X) \rightarrow \text{GL}(r, \mathbb{C})$ .

Tous ces espaces sont variétés quasi-projectives à la Simpson, désignées par  $M_{\text{dR}}(X, r)$ ,  $M_{\text{Dol}}(X, r)$ ,  $M_{\text{Hod}}(X, r)$  et  $M_{\text{B}}(X, r)$  les sous-ensembles qui paramètre les classes d’isomorphisme des objets stables, c’est-à-dire les loci lisses correspondants. Chaque espace est un sous-ensemble ouvert dense de l’espace de modules correspondant. En particulier, si on fixe un nombre complexe  $\lambda \in \mathbb{C}$ , soit  $\mathcal{M}_{\text{Hod}}^\lambda(X, r) := \mathcal{M}_{\text{Hod}}(X, r)|_\lambda$ , alors on a

- $\mathcal{M}_{\text{Hod}}^0(X, r) = \mathcal{M}_{\text{Dol}}(X, r)$ ,
- $\mathcal{M}_{\text{Hod}}^1(X, r) = \mathcal{M}_{\text{dR}}(X, r)$ ,
- $\mathcal{M}_{\text{Hod}}^\lambda(X, r) \cong_{\text{an}} \mathcal{M}_{\text{dR}}(X, \bar{\lambda})$ , où  $\lambda \neq 0$ , “ $\cong_{\text{an}}$ ” signifie l’isomorphisme analytique complexe.

On a évidemment ces propriétés pour les loci lisses  $M_{\text{dR}}(X, r)$ ,  $M_{\text{Dol}}(X, r)$ ,  $M_{\text{Hod}}(X, r)$  et  $M_{\text{B}}(X, r)$ . Soit  $\mathcal{M}_{\text{Hod}}(X, r, \mathcal{O}_X)$  (respectivement,  $M_{\text{Hod}}(X, r, \mathcal{O}_X)$ ) est l’espace de modules de fibrés  $\lambda$ -plats semistables (respectivement, stables) de rang  $r$  de classes de Chern nulles avec déterminants  $\mathcal{O}_X$  fixes sur  $X$ .

Notre construction du système dynamique commence par  $M_{\text{Dol}}(X, r)$ , pour tout  $(\lambda, t) \in \mathbb{C} \times \mathbb{C}^*$ , soit  $(E, \bar{\partial}_E, \theta)$  un fibré de Higgs stable avec une métrique de pluri-harmonique  $h$ , considérer le fibré  $\lambda$ -plat correspondant  $(E, \bar{\partial}_E + \lambda\theta_h^\dagger, \lambda\partial_{E,h} + \theta)$ , alors  $h$  est également une métrique de pluri-harmonique pour un tel fibré  $\lambda$ -plat; ensuite nous considérons le fibré  $t\lambda$ -plat  $(E, \bar{\partial}_E + \lambda\theta_h^\dagger, t(\lambda\partial_{E,h} +$

$\theta$ ) donné par l'action de  $\mathbb{C}^*$ , il existe des métriques de pluri-harmoniques notées  $h_t$  sur  $(E, \bar{\partial}_E + \lambda\theta_h^\dagger, t(\lambda\partial_{E,h} + \theta))$ , car l'action de  $\mathbb{C}^*$  ne change pas la stabilité et la trivialité des classes de Chern des fibrés  $\lambda$ -plats; enfin par la correspondance de Hodge non-Abélienne, cela détermine un fibré de Higgs noté  $(E, \bar{\partial}_{E,h_t}, \theta_{h_t})$ , qui porte  $h_t$  comme sa métrique de pluri-harmonique.

Par conséquent, on obtient un système dynamique lisse  $\psi_{(\lambda,t)}$  paramétré par  $(\lambda, t)$  sur  $M_{\text{Dol}}(X, r)$ :

$$\begin{aligned} \psi_{(\lambda,t)} : M_{\text{Dol}}(X, r) &\longrightarrow M_{\text{Dol}}(X, r), \\ (E, \bar{\partial}_E, \theta) &\longmapsto (E, \bar{\partial}_{E,h_t}, \theta_{h_t}). \end{aligned}$$

De plus,  $\psi_{(\lambda,t)}$  peut être défini sur  $\mathcal{M}_{\text{Dol}}(X, r)$  comme une application continue.

Par définition, les propriétés suivantes sont évidemment:

**Proposition 1** (= Proposition 2.1.1).

- (1)  $\psi_{(0,t)}$  est l'action de  $\mathbb{C}^*$  habituelle donnée par  $t$ , et  $\psi_{(\lambda,1)}$  est l'application d'identité;
- (2)  $\psi_{(\lambda,t)}$  satisfait la formule de type "cocycle" suivante:

$$\psi_{(\lambda t_1, t_2)} \circ \psi_{(\lambda, t_1)} = \psi_{(\lambda, t_1 t_2)};$$

- (3) Pour tout  $\lambda \in \mathbb{C}, t \in \mathbb{C}^*$ , les fibrés vectoriels stables dans l'espace de modules (c'est-à-dire les fibrés de Higgs avec champ de Higgs nulles) sont les points fixes du système dynamique  $\psi_{(\lambda,t)}$ .

Un problème important dans la théorie du système dynamique est l'étude des points fixes, dans cette section, en utilisant l'analyse sur des métriques de pluri-harmoniques, nous obtenons que le système dynamique a les mêmes points fixes avec l'action de  $\mathbb{C}^*$  sur l'espace de modules  $\mathcal{M}_{\text{Dol}}(X, r)$ :

**Théorème 3** (= Theorem 2.1.5). Pour la paire fixe  $(\lambda, t) \in \mathbb{C} \times \mathbb{C}^*$ , nous définissons l'ensemble de points fixes de l'action de  $(\lambda, t)$ :

$$F_{(\lambda,t)} := \{u \in \mathcal{M}_{\text{Dol}}(X, r) : \psi_{(\lambda,t)}(u) = u\},$$

alors l'ensemble de points fixes  $F := \bigcap_{(\lambda,t) \in \mathbb{C} \times \mathbb{C}^*} F_{(\lambda,t)}$  se compose de tous les points fixes de l'action de  $\mathbb{C}^*$  sur  $\mathcal{M}_{\text{Dol}}(X, r)$ , c'est-à-dire tout les  $\mathbb{C}$ -VHS dans  $\mathcal{M}_{\text{Dol}}(X, r)$ .

De plus, nous calculons l'ensemble stabilisateur de points spéciaux dans l'espace de modules pour le cas d'une surface de Riemann compacte:

$$\mathcal{C}_u = \{(\lambda, t) \in \mathbb{C} \times \mathbb{C}^* : \psi_{(\lambda,t)}(u) = u\}.$$

**Théorème 4** (= Theorem 2.1.7, Corollary 2.1.8). Soit  $X$  une surface de Riemann compacte, et  $u \in \mathcal{M}_{\text{Dol}}(X, r)$  un fibré de Higgs découplé avec champ de Higgs non nulle, alors l'ensemble stabilisateur est

$$\mathcal{C}_u = \mathbb{C} \times \{\mu_M, \mu_M^2, \dots, \mu_M^{M-1}, 1\},$$

où  $\mu_M = e^{\frac{2\pi i}{M}}$ ,  $M \leq r$  est une constante. De plus, pour tout  $(\lambda, t) \in \mathbb{C} \times \mathbb{C}^*$ , le système dynamique  $\psi_{(\lambda,t)}$  ne change pas la métrique pluri-harmonique lorsqu'il agit sur un fibré de Higgs découplé.

Parallèlement, nous étudions également la propriété locale de ce système dynamique en calculant la première variation de celui-ci en un point de l'espace de modules (cf. [Theorem 2.1.3](#)).

La deuxième section (§2.2) de ce chapitre est les stratifications des espaces de modules, qui est principalement dû à un résultat important de Simpson [[Sim10](#)]. C'est-à-dire que chaque fibré plat  $(E, \nabla)$  sur une surface de Riemann compacte  $X$  admet une filtration:

$$F^\bullet : E = F^0 \supset F^1 \supset \dots \supset F^k = 0$$

satisfait la transversalité de Griffiths  $\nabla(F^p) \subseteq F^{p-1} \otimes K_X$  ( $p = 1, \dots, k$ ), et tel que le fibré de Higgs gradué  $(\text{Gr}_F(E) := \bigoplus_{p=0}^{k-1} F^p/F^{p+1}, \text{Gr}_F(\nabla))$  est semistable. De plus, une telle filtration est unique si et seulement si le fibré de Higgs gradué est stable. Nous appelons une telle filtration une filtration de Simpson.

Pour tout  $(E, \bar{\partial}_E, D^\lambda)$  dans l'espace de modules  $\mathcal{M}_{\text{Hod}}(X, r)$  ( $\lambda \neq 0$ ), on considère l'action de  $\mathbb{C}^*$ , Simpson prouve que la limite  $\lim_{t \rightarrow 0} t \cdot (E, \bar{\partial}_E, D^\lambda)$  existe et est un point fixe de l'action de  $\mathbb{C}^*$  sur  $\mathcal{M}_{\text{Dol}}(X, r)$ , c'est-à-dire un  $\mathbb{C}$ -VHS, et satisfait à l'égalité suivante (dans l'espace de modules):

$$\lim_{t \rightarrow 0} t \cdot (E, \bar{\partial}_E, D^\lambda) = \lim_{t \rightarrow 0} t \cdot (E, \bar{\partial}_E, \lambda^{-1} D^\lambda) = (\text{Gr}_F(E), \text{Gr}_F(\lambda^{-1} D^\lambda)),$$

où  $(E, \bar{\partial}_E, \lambda^{-1} D^\lambda)$  est le fibré plat correspondant de  $(E, \bar{\partial}_E, D^\lambda)$ , et  $(\text{Gr}_F(E), \text{Gr}_F(\lambda^{-1} D^\lambda))$  est le fibré de Higgs gradué induit du fibré plat  $(E, \bar{\partial}_E, \lambda^{-1} D^\lambda)$ .

Cela montre qu'il existe des stratifications de type Bialynicki-Birula de  $\mathcal{M}_{\text{Hod}}^\lambda(X, r)$ ,  $\mathcal{M}_{\text{Dol}}(X, r)$  et  $\mathcal{M}_{\text{dR}}(X, r)$  en sous-ensembles fermés localement données par l'action de  $\mathbb{C}^*$ :

$$\mathcal{M}_{\text{Hod}}^\lambda(X, r) = \bigcup_{\alpha} G_{\alpha}^{\lambda}, \quad \mathcal{M}_{\text{Dol}}(X, r) = \bigcup_{\alpha} G_{\alpha}^0, \quad \mathcal{M}_{\text{dR}}(X, r) = \bigcup_{\alpha} G_{\alpha}^1,$$

où l'indice  $\alpha$  correspond à la décomposition de l'ensemble de points fixes de l'action de  $\mathbb{C}^*$  sur  $\mathcal{M}_{\text{Dol}}(X, r)$  en ses composantes connectées:  $P = \bigcup_{\alpha} P_{\alpha}$ . L'application  $G_{\alpha}^{\lambda} \rightarrow P_{\alpha}$  est obtenue en prenant la limite de l'action de  $\mathbb{C}^*$ , la fibre en chaque point est un sous-variété Lagrangienne de l'espace de modules correspondant.

Pour une étude plus approfondie de ces stratifications, dans l'article [[Sim10](#)], Simpson a proposé les trois conjectures importantes suivantes:

**Conjecture 1** (= [Conjecture 2.2.7](#), [Conjecture 2.2.9](#) and [Conjecture 2.2.11](#)).

- (1) (**La Conjecture de Feuilletage**) *Lors de la variation de l'indice  $\alpha$ , ces fibres Lagrangiennes de  $p_{\alpha}^1 : G_{\alpha}^1 \rightarrow P_{\alpha}$  s'emboîtent pour fournir une feuilletage lisse de l'espace de modules de de Rham  $\mathcal{M}_{\text{dR}}(X, r)$  avec chaque feuille fermée.*
- (2) (**La Conjecture d'Imbrication**) *Les stratifications de l'espace de modules de Dolbeault  $\mathcal{M}_{\text{Dol}}(X, r)$  et de l'espace de modules de de Rham  $\mathcal{M}_{\text{dR}}(X, r)$  sont tous deux imbriqués, et les dispositions pour les deux stratifications sont les mêmes. Ici, l'imbrication signifie qu'il existe une relation d'ordre partiel " $\leq$ " sur l'ensemble d'index telle que  $\overline{G_{\alpha}^i} = \bigcup_{\beta \leq \alpha} G_{\beta}^i$  pour  $i = 0, 1$ .*
- (3) (**La Conjecture de la Strate d'Opérateurs**) *Dans la stratification de l'espace de modules de de Rham  $\mathcal{M}_{\text{dR}}(X, r)$ , la strate d'opérateurs  $G_{\text{oper}}^1$  est la strate fermée unique et la strate unique de dimension minimale. Ici, un opérateur signifie un fibré plat qui admet une filtration spéciale.*

Dans cette section, nous décrivons la relation explicite entre les filtrations de Simpson et les filtrations de Harder–Narasimhan pour les fibrés vectoriels sous-jacents des fibrés plats de rang 3 (la description du résultat est très longue, veuillez vous référer à [Theorem 2.2.13](#)).

Dans le même temps, nous poursuivons notre étude du système dynamique  $\psi_{(\lambda,t)}$ , et introduisons cinq limites différentes associées à ce système dynamique:

- $\psi_{\underline{(0,0)}}(E, \bar{\partial}_E, \theta) := \lim_{t \rightarrow 0} \left( \psi_{(0,t)}(E, \bar{\partial}_E, \theta) \right) = \lim_{t \rightarrow 0} (E, \bar{\partial}_E, t\theta);$
- $\psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta) := \lim_{\lambda \rightarrow 0} \left( \psi_{(\lambda,0)}(E, \bar{\partial}_E, \theta) \right);$
- $\psi^{(0,0)}(E, \bar{\partial}_E, \theta) := \lim_{t \rightarrow 0} \left( \lim_{\lambda \rightarrow 0} \left( \psi_{(\lambda,t)}(E, \bar{\partial}_E, \theta) \right) \right);$
- $\psi^{\overline{(0,0)}}(E, \bar{\partial}_E, \theta) := \lim_{\lambda \rightarrow 0} \left( \lim_{t \rightarrow 0} \left( \psi_{(\lambda,t)}(E, \bar{\partial}_E, \theta) \right) \right);$
- $\psi_{(0,0)}(E, \bar{\partial}_E, \theta) := \lim_{(\lambda,t) \rightarrow (0,0)} \left( \psi_{(\lambda,t)}(E, \bar{\partial}_E, \theta) \right).$

Pour tout point de l'espace de modules de Dolbeault  $\mathcal{M}_{\text{Dol}}(X, r)$ , ces limites sont généralement différentes lorsqu'elles existent. Nous étudions ce problème et trouvons des points particuliers tels que ces limites existent et coïncident dans l'espace de modules.

**Théorème 5** (= [Theorem 2.2.16](#)). *Soit  $X$  une surface de Riemann compacte, si  $(E, \bar{\partial}_E, \theta) \in \mathcal{M}_{\text{Dol}}(X, r)$  est un  $\mathbb{C}$ -VHS, ou un fibré de Higgs découplé, alors les cinq limites existent et coïncident en  $\mathcal{M}_{\text{Dol}}(X, r)$ .*

Pour décrire le système dynamique  $\psi_{(\lambda,t)}$ , un point clé est de comprendre la relation explicite entre la métrique de pluri-harmonique  $h_t$  et le paramètre  $t$ . C'est très difficile à calculer, car il est difficile de résoudre les équations de Hitchin d'un fibré  $\lambda$ -plat sous l'action de  $\mathbb{C}^*$ . Nous donnons une description de  $h_t$  autour de  $t = 1$ :

**Théorème 6** (= [Theorem 2.2.22](#)). *Soit  $X$  une surface de Riemann compacte, fixer  $\lambda \in \mathbb{C}^*$  et supposer  $t \in \mathbb{R}^*$ , alors sur un petit voisinage de  $t = 1$ , la fonction  $f := \frac{h_t h^{-1} - \text{Id}}{(t-1)^2}$  à valeur  $\text{End}(E)$  est une fonction analytique réelle par rapport à  $t - 1$ .*

Dans la dernière section (§2.3) de ce chapitre, nous prouvons la conjecture de strate d'opérateurs d'une version faible, ce qui donne une réponse partielle à la conjecture de Simpson:

**Théorème 7** (= [Theorem 2.3.1, Dimension Minimale](#)). *La strate d'opérateurs  $G_{\text{oper}}^1$  est la strate fermée unique de dimension minimale  $r^2(g-1) + g + 1$  dans l'espace de modules de de Rham  $\mathcal{M}_{\text{dR}}(X, r)$ .*

La preuve de ce théorème repose sur la description de l'ensemble de points fixes  $P$ , en particulier ses composants connectés  $P_\alpha$ , via la théorie des chaînes holomorphes. Comme point fixe de l'action de  $\mathbb{C}^*$ , un  $\mathbb{C}$ -VHS peut être identifié avec une chaîne holomorphe d'un certain type. Nous terminons la preuve en calculant la dimension de l'espace de modules de chaînes holomorphes et la propriété d'irréductibilité de cet espace de modules.

### 0.1.3 La Construction de l'Espace de Twistor

La principale contribution de ce chapitre est dans le cas de la surface Riemann compacte, nous généralisons la construction de Deligne de l'espace de twistor en collant l'espaces de modules de Hodge  $\mathcal{M}_{\text{Hod}}(X, r)$  et  $\mathcal{M}_{\text{Hod}}(\bar{X}, r)$  sur  $X$  et son conjugué  $\bar{X}$  [Del89, Sim95], comme une interprétation de la construction par Hitchin de l'espace de twistor associé à l'espace de modules de Dolbeault  $M_{\text{Dol}}(X, r)$  avec la structure hyper-Kählerienne [HKLR87].

L'idée fondamentale de [HKLR87] sur la construction d'un espace de twistor associé à une variété hyper-Kählerienne  $M$  est un produit topologique de  $M$  et la ligne projective complexe  $\mathbb{P}^1$ , cet espace est noté  $\text{TW}(M) := M \times \mathbb{P}^1$ . La structure quaternionique  $(I, J, K = IJ)$  sur  $M$  induit une structure complexe  $\mathcal{I}$  sur  $\text{TW}(M)$ . Nous obtenons donc une variété complexe et toujours notée  $\text{TW}(M)$ , c'est l'espace de twistor de Hitchin, et nous appelons cette théorie la théorie de twistor de Hitchin.

L'idée de Deligne est en fait une interprétation de la théorie de twistor de Hitchin via la théorie de Hodge non-Abélienne. Pour l'espace de modules qui paramètre les classes d'isomorphisme des objets stables sur  $X$  ( $M_{\text{Dol}}(X, r)$ ,  $M_{\text{dR}}(X, r)$  et ainsi de suite), notée  $M^{\text{sm}}(X, r)$  la variété lisse sous-jacente. Alors  $M^{\text{sm}}(X, r)$  admet une structure hyper-Kählerienne induite par  $M_{\text{Dol}}(X, r)$  et  $M_{\text{dR}}(X, r)$  [Hit87a, Fuj91]. Cela donne un espace de twistor de Hitchin  $\text{TW}(M^{\text{sm}}(X, r))$  suivant [HKLR87]. L'idée principale de Deligne est de coller les espaces de modules de Hodge  $\mathcal{M}_{\text{Hod}}(X, r)$  et  $\mathcal{M}_{\text{Hod}}(\bar{X}, r)$   $X$  et  $\bar{X}$  via l'application de conjugaison complexe et la correspondance de Hodge non-Abélienne, cela donne un nouvel espace de twistor  $\text{TW}_{\text{DH}}(X, r)$  et une fibration  $\text{TW}_{\text{DH}}(X, r) \rightarrow \mathbb{P}^1$ . Nous appelons  $\text{TW}_{\text{DH}}(X, r)$  l'espace de twistor de Deligne–Hitchin, cela montre que le locus lisse  $\text{TW}_{\text{DH}}^{\text{sm}}(X, r)$  est isomorphe analytique complexe à l'espace de twistor de Hitchin  $\text{TW}(M^{\text{sm}}(X, r))$ .

Nous considérons maintenant le cas où  $X$  est une surface de Riemann compacte de genre  $g \geq 2$ , alors  $X$  peut être désigné comme la paire  $(\mathcal{X}, I)$  pour  $\mathcal{X}$  la surface fermée orientable connectée lisse sous-jacente, et  $I$  la structure complexe. Les classes d'isotropie des structures complexes sur  $\mathcal{X}$  peuvent être désignées par son espace Teichmüller  $\text{Teich}(\mathcal{X})$ , et le groupe étendu modulaire de Teichmüller  $\text{Mod}^{\diamond}(\mathcal{X}) := \text{Diff}(\mathcal{X})/\text{Diff}_0(\mathcal{X})$  agit sur cet espace, avec le quotient  $\mathcal{M}(\mathcal{X}) := \text{Teich}(\mathcal{X})/\text{Mod}^{\diamond}(\mathcal{X})$  appelé *l'espace de modules de structures complexes de Riemann sur  $\mathcal{X}$* , qui paramètre les structures de surface de Riemann sur  $\mathcal{X}$  sous biholomorphisme. Lorsque nous passons en revue la construction de Deligne, les deux espaces de modules de Hodge apparaissant dans le processus de collage sont sur  $X$  et sur son conjugué  $\bar{X}$ , ou de manière équivalente, sur  $(\mathcal{X}, I)$  et  $(\mathcal{X}, -I)$ . Donc pour une structure complexe choisie  $I' \in \mathcal{M}(\mathcal{X})$ , soit  $X' := (\mathcal{X}, I')$ , on peut coller les espaces de modules de Hodge  $\mathcal{M}_{\text{Hod}}(X, r)$  et  $\mathcal{M}_{\text{Hod}}(X', r)$  le long du chevauchement  $\mathcal{M}_{\text{Hod}}(X, r) \times_{\mathbb{C}} \mathbb{C}^* \cong \mathcal{M}_{\text{Hod}}(X', r) \times_{\mathbb{C}} \mathbb{C}^* \cong \mathcal{M}_{\text{B}}(\mathcal{X}, r) \times \mathbb{C}^*$  par Riemann–Hilbert correspondance qui couvre l'application  $\mathbb{C}^* \rightarrow \mathbb{C}^*$ ,  $\lambda \mapsto \lambda^{-1}$ . L'identification des fibrés  $\lambda$ -plats donnée par le collage de Deligne  $\mathbf{d}$  peut être explicitement écrite comme suit:

$$[E, \bar{\partial}_E, D^\lambda, \lambda] \longleftrightarrow [E, (\bar{\partial}_E + \lambda^{-1} D^\lambda)_{X'}^{0,1}, \lambda^{-1} (\bar{\partial}_E + \lambda^{-1} D^\lambda)_{X'}^{1,0}, \lambda^{-1}],$$

où  $(\bullet)_{X'}^{1,0}$  et  $(\bullet)_{X'}^{0,1}$  désignent respectivement les parties  $(1, 0)$  et  $(0, 1)$  correspondantes par rapport à la structure complexe  $I'$  sur  $X'$ . La variété analytique obtenue est appelée *l'espace de twistor de Deligne–Hitchin généralisé*, et est notée  $\text{TW}(X, X'; r)$ , ce qui donne en particulier,  $\text{TW}_{\text{DH}}(X, r) = \text{TW}(X, \bar{X}; r)$ .

Nos principaux résultats dans ce chapitre peuvent être conclus suivante:

**Theorem 1** (= Theorem 3.3.4, Theorem 3.3.7, et Theorem 3.3.11).

- (1) (**Propriété de Poids Un**) Pour un fibré  $\lambda_0$ -plat fixe  $(E, \bar{\partial}_E, D^{\lambda_0})$ , il détermine une section

holomorphe  $s_{\lambda_0}$  de l'espace de twistor de Deligne–Hitchin généralisé  $\mathrm{TW}(X, X'; r)$ , appelée la section de de Rham. Cette a une propriété de poids 1, c'est-à-dire que son fibré normal satisfait

$$N_{s_{\lambda_0}} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus \dim_{\mathbb{C}}(\mathcal{M}_{\mathrm{dR}}(X, r))}.$$

En particulier, les sections de de Rham sont les courbes rationnelles amples équilibrées de degré  $\dim_{\mathbb{C}}(\mathcal{M}_{\mathrm{dR}}(X, r))$ .

- (2) (**La Théorème de type Torelli**) Soit  $X, X' \in \mathcal{M}(\mathcal{X})$  et  $Y, Y' \in \mathcal{M}(\mathcal{Y})$  sont des surfaces de Riemann de genre  $g \geq 3$ . Si  $\mathrm{TW}(X, X'; r, \mathcal{O})$  est analytiquement isomorphe à  $\mathrm{TW}(Y, Y'; r, \mathcal{O})$ , alors soit  $X \cong Y, X' \cong Y'$ , soit  $X \cong Y', X' \cong Y$ . Oú  $\mathrm{TW}(X, X'; r, \mathcal{O})$  est l'espace de twistor de Deligne–Hitchin généralisé obtenu en collant  $\mathcal{M}_{\mathrm{Hod}}(X, r, \mathcal{O}_X)$  et  $\mathcal{M}_{\mathrm{Hod}}(X', r, \mathcal{O}_{X'})$ .
- (3) (**Groupes d'Automorphism**) Soit  $\mathrm{Aut}_0(\mathrm{TW}(X, X'; r))$  est la composante identité des groupes d'automorphisme holomorphe  $\mathrm{Aut}(\mathrm{TW}(X, X'; r))$  de  $\mathrm{TW}(X, X'; r)$ , alors chaque élément de  $\mathrm{Aut}_0(\mathrm{TW}(X, X'; r))$  mappe les fibres de  $\pi : \mathrm{TW}(X, X'; r) \rightarrow \mathbb{P}^1$  en fibres. De plus, ce groupe satisfait la suite exacte courte suivante

$$\mathrm{Id} \longrightarrow \mathfrak{K} \longrightarrow \mathrm{Aut}_0(\mathrm{TW}(X, X'; r)) \longrightarrow \mathbb{C}^* \longrightarrow \mathrm{Id},$$

où chaque élément de  $\mathfrak{K}$  préserve les fibres de  $\pi : \mathrm{TW}(X, X'; r) \rightarrow \mathbb{P}^1$ .

## 0.2 Les Spécialisations de la Correspondance de Hodge Non-Abélienne

Cette partie peut être considérée comme certaines applications de la théorie de Hodge non-Abélienne, plus précisément, certaines spécialisations de la correspondance de Hodge non-Abélienne. Nous construisons des correspondances spéciales de Hodge non-Abélienne.

### 0.2.1 Stabilité et Indécomposabilité des Représentations de Carquois

Le sixième chapitre de cette thèse est la théorie de Hodge non-Abélienne de la version de carquois, qui est basée sur une conjecture proposée par M. Reineke en 2003 [Rie03]:

**Conjecture 2** (= [Conjecture 5.1.1](#)). Soit  $Q$  un carquois de type Dynkin, alors il existe un système de poids  $\Theta$  sur  $Q$  tel que les représentations stables par rapport à  $\Theta$  sont précisément les représentations indécomposables.

L'importance de cette conjecture tente d'identifier des représentations stables et indécomposables de carquois de type Dynkin par rapport à un certain système de poids. En général, la stabilité est une condition plus forte que l'indécomposabilité, alors que cette conjecture peut les rendre équivalentes.

D. Juteau a trouvé des contre-exemples à cette conjecture pour les carquois de type  $D$  et  $E$ , une conjecture de Reineke modifiée est proposée suivante:

**Conjecture 3** (= [Conjecture 5.3.1](#), **La Conjecture de Reineke Modifiée**). Si  $Q$  est un carquois de type Dynkin, alors la catégorie abélienne  $\mathbf{Rep}_k(Q)$  est une catégorie stable maximale.

Un carquois  $Q$  est en fait un graphe orienté qui se compose de sommets finis et de flèches finies qui relient ces sommets. Plus explicitement, c'est un triple  $Q = (Q_0, Q_1, s, t)$ , où  $Q_0$  et  $Q_1$  sont des ensembles finis de sommets et de flèches, respectivement, et  $s, t : Q_1 \rightarrow Q_0$  sont des applications qui mappent une flèche  $a \in Q_1$  à son sommet de départ  $s(a)$  et à son sommet terminal  $t(a)$ , respectivement, qui peut être désigné comme suivant :

$$\bullet \xrightarrow{a} \bullet \\ s(a) \qquad t(a)$$

Soit  $k$  un corps algébriquement clos, une  $k$ -représentation du carquois  $Q$  consiste à placer un  $k$ -espace vectoriel de dimension finie à chaque sommet et une  $k$ -morphisme linéaire entre des  $k$ -espaces vectoriels à chaque flèche. C'est-à-dire que se compose d'une paire  $X = \{(X_i)_{i \in Q_0}, (X_a)_{a \in Q_1}\}$ , pour chaque  $X_i$  un  $k$ -espace vectoriel de dimension finie, et chaque  $X_a : X_{s(a)} \rightarrow X_{t(a)}$  une  $k$ -morphisme linéaire.

Une  $k$ -représentation  $X$  du carquois  $Q$  est indécomposable s'il n'est pas une somme directe de deux  $k$ -représentations non nulles de  $Q$ .

Notée par  $\mathbf{Rep}_k(Q)$  la catégorie de  $k$ -représentations du carquois  $Q$ . Pour un système de poids  $\Theta = (\theta_i)_{i \in Q_0} \in \mathbb{Z}^{|Q_0|}$ ,  $w(X) := \sum_{i \in Q_0} \theta_i \dim_k X_i$  et  $r(X) := \sum_{i \in Q_0} \dim_k X_i$  sont appelés respectivement la fonction de poids et la fonction de rangs sur la catégorie  $\mathbf{Rep}_k(Q)$ .

La notion de stabilité des  $k$ -représentations des carquois peut se définir naturellement, une  $k$ -représentation  $X$  du carquois  $Q$  est  $(w, r)$ -stable (reps.  $(w, r)$ -semistable), si pour tout sous-représentation proper non nul  $U$  de  $X$ , on a

$$\mu(U) < \mu(X) \text{ (resp. } \mu(U) \leq \mu(X)),$$

où  $\mu(X) := \frac{w(X)}{r(X)}$  est la pente de  $X$  par rapport à la fonction de poids  $w$  et à la fonction de rangs  $r$ . La notion de polystabilité peut être définie de manière similaire.

Notre principal résultat de ce chapitre est de fournir une preuve de la conjecture de Reineke modifiée pour les carquois de type  $A_n$  par construction combinatoire d'un système de poids spécial (nous l'appelons le système de poids intrinsèque). En particulier, la conjecture de Reineke est valable pour les carquois de type  $A_n$ .

**Théorème 8** (= [Theorem 5.3.2](#)). *Soit  $Q$  un carquois de type  $A_n$ , alors il existe un système de poids  $\Theta = (\theta_i)_{i \in Q_0} \in \mathbb{Z}^{|Q_0|}$  sur  $Q$  tel que les représentations stables par rapport à la fonction de poids  $w(X) = \sum_{i \in Q_0} \theta_i \dim X_i$  et la fonction de rangs  $r(X) = \sum_{i \in Q_0} \dim X_i$  sont précisément les représentations indécomposables. C'est-à-dire que la catégorie abélienne  $\mathbf{Rep}_k(Q)$  est une catégorie stable maximale.*

Dans le même temps, nous étudions également le système de poids intrinsèque via la théorie semi-invariante (pour les détails, reportez-vous à [Proposition 5.3.10](#)).

## 0.2.2 La Correspondance de Kobayashi–Hitchin pour les Fibrés de Carquois

Le dernier chapitre de cette thèse est une autre spécialisation de la correspondance de Hodge non-Abélienne, c'est-à-dire que la correspondance de Kobayashi–Hitchin pour les fibrés de carquois sur les variétés Kählerienne généralisées.

La correspondance de Kobayashi–Hitchin, est une correspondance entre la stabilité des fibrés vectoriels et l'existence de métriques d'Hermite–Einstein, qui est introduite au début de cette introduction, joue un rôle essentiel dans la théorie de Hodge non-Abélienne.

Dans ce chapitre, nous allons combiner la géométrie de Kählerienne généralisée et la théorie du carquois, en particulier la théorie des fibrés de carquois. Nous introduirons la notion de fibrés de carquois sur les variétés Kähleriennes généralisées, ainsi que la stabilité et les métriques d’Hermite–Einstein de ces fibrés. Une idée naturelle est de considérer la relation entre eux.

Notre principal résultat de ce chapitre est le suivant:

**Théorème 9** (= [Theorem 6.3.1](#)). *Soit  $Q = (Q_0, Q_1)$  un carquois,  $\mathcal{E} = (E, \phi)$  un  $Q$ -fibré de carquois  $I_{\pm}$ -holomorphe sur une variété Kählerienne généralisée compacte  $(X, I_+, I_-, g, b)$  de dimension  $n$  tel que  $g$  est une métrique de Gauduchon par rapport à  $I_+$  et  $I_-$ , alors  $\mathcal{E}$  est  $(\alpha, \sigma, \tau)$ -polystable si et seulement si  $\mathcal{E}$  admet une métrique de  $(\alpha, \sigma, \tau)$ -Hermite–Einstein.*



# Introduction in English



## Introduction

Let  $(X, \omega)$  be an  $n$ -dimensional Kähler manifold,  $E$  be a complex vector bundle over  $X$ . For  $k \in \mathbb{Z}_{\geq 0}$ , denoted by  $\mathcal{A}^k(E) := \Gamma(X, E \otimes \wedge^k(T_{\mathbb{C}}^*X))$  the space of smooth  $k$ -forms on  $X$  with values in  $E$ , in particular,  $\mathcal{A}^0(E) = \Gamma(X, E)$ , the space of smooth sections of  $E$ . For  $\lambda \in \mathbb{C}$ , a  $(C^\infty)$  flat  $\lambda$ -connection on  $E$  is a  $\mathbb{C}$ -linear operator  $\mathbb{D}^\lambda : \Gamma(X, E) \rightarrow \mathcal{A}^1(E)$  satisfies the following  $\lambda$ -twisted Leibniz rule:

$$\mathbb{D}^\lambda(fs) = f\mathbb{D}^\lambda(s) + \lambda\partial(f) \otimes s + \bar{\partial}(f) \otimes s,$$

where  $f \in C^\infty(X, \mathbb{C})$ ,  $s \in \Gamma(X, E)$ , and  $\mathbb{D}^\lambda \circ \mathbb{D}^\lambda = 0$  under the natural extension  $\mathbb{D}^\lambda : \mathcal{A}^1(E) \rightarrow \mathcal{A}^2(E)$ . In this case, the pair  $(E, \mathbb{D}^\lambda)$  is called a  $\lambda$ -flat bundle.

In particular, when  $\lambda = 1$ , this is the usual flat connection, we use the notation  $\nabla$ , and in this case, the pair  $(E, \nabla)$  is called a flat bundle; when  $\lambda = 0$ , split  $\mathbb{D}^0$  into its  $(0, 1)$ -part and  $(1, 0)$ -part, and we obtain  $\mathbb{D}^0 = \bar{\partial}_E + \theta$ , the flatness of  $\mathbb{D}^0$  implies  $(\bar{\partial}_E)^2 = \bar{\partial}_E\theta = \theta \wedge \theta = 0$ , this defines a Higgs bundle, that is the triple  $(E, \bar{\partial}_E, \theta)$ .

$\lambda$ -flat bundle was originally introduced by Deligne as the understanding of Hitchin's idea on the construction of twistor space of the moduli space of Higgs bundles [Del89], which generalizes flat bundles and Higgs bundles. Later this was studied and developed by Simpson as Hodge filtration on non-Abelian cohomology [Sim95, Sim08]. To emphasis flat bundles and Higgs bundles, here we state them separately.

Hence on  $(X, \omega)$ , we define the following three geometric objects:

- (1) Flat bundle:  $(E, \nabla)$ ;
- (2) Higgs bundle:  $(E, \bar{\partial}_E, \theta)$ ;
- (3)  $\lambda$ -flat bundle ( $\lambda \in \mathbb{C}$ ):  $(E, \mathbb{D}^\lambda)$ .

In algebraic geometry, especially in geometric invariant theory, there is a very important notion for vector bundles, that is the stability. For a given  $\lambda$ -flat bundle  $(E, \mathbb{D}^\lambda)$ , it is called stable (resp. semistable), if for any non-zero proper saturated torsion-free coherent subsheaf  $\mathcal{F}$  which is invariant under  $\mathbb{D}^\lambda$ , that is,  $\mathbb{D}^\lambda|_{\Gamma(X, \mathcal{F})} : \Gamma(X, \mathcal{F}) \rightarrow \mathcal{A}^1(\mathcal{F})$  and  $0 < \text{rk}(\mathcal{F}) < \text{rk}(E)$ , we have

$$\mu_\omega(\mathcal{F}) < \text{ (resp. } \leq \text{)} \mu_\omega(E),$$

where  $\mu_\omega(E) := \frac{\deg(E)}{\text{rk}(E)}$  is called the slope, and  $\deg(E) := \int_X c_1(\det(E)) \wedge [\omega]^{n-1}$  is the degree. It is polystable, if it is the direct sum of stable  $\lambda$ -flat bundles of the same slope.

In particular, when  $\lambda = 1$ , for flat bundles, one usually apply the notations simple (or equivalently, irreducible) and semisimple (or equivalently, reductive) rather than stable and polystable<sup>2</sup>, since they correspond to simple (or equivalently, irreducible) and semisimple representations of  $\pi_1(X)$  by Riemann–Hilbert correspondence.

In differential geometry, there is also a very important notion for vector bundles, that is, Hermitian–Einstein metrics. For a given  $\lambda$ -flat bundle  $(E, \mathbb{D}^\lambda)$ , together with a Hermitian metric  $h$ , then  $h$  induced a unique decomposition of  $\mathbb{D}^\lambda$ :

$$\mathbb{D}^\lambda = \lambda\partial_h + \theta + \bar{\partial}_h + \lambda\theta_h^\dagger$$

such that:

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<sup>2</sup>It's obviously in the compact case that flat bundles are automatically semistable.

- $\nabla_h := \partial_h + \bar{\partial}_h$  is a  $h$ -unitary connection, that is,  $dh(u, v) = h(\nabla_h(u), v) + h(u, \nabla_h(v))$ ;
- $\Phi_h := \theta_h + \theta_h^\dagger \in \mathcal{A}^1(\text{End}(E))$  is a self-adjoint operator with respect to  $h$ , that is,  $h(\theta_h(u), v) = h(u, \theta_h^\dagger(v))$ .

Let  $D_{h,\lambda}^c := \partial_h + \theta_h^\dagger - \bar{\lambda}(\bar{\partial}_h + \theta_h)$ , and define  $G_{h,\lambda} := \frac{1}{(1+|\lambda|^2)^2} [\mathbb{D}^\lambda, D_{h,\lambda}^c]$ , which is called the *pseudo-curvature*.  $h$  is called a Hermitian–Einstein metric if the following equation holds:

$$\Lambda_\omega G_{h,\lambda} = c \text{Id}_E,$$

where  $c = \frac{-2\pi\sqrt{-1} \deg_\omega(E)}{(1+|\lambda|^2)\text{rk}(E)\text{Vol}(X)}$  is a constant determined by  $X$  and topological invariants of  $E$ .

Non-Abelian Hodge theory, is a bridge between algebraic geometry and differential geometry, which relates the stability of vector bundles and the existence of Hermitian–Einstein metrics. This mainly based on the work of Donaldson [Don87] and Corlette [Cor88] on harmonic maps (flat bundles), Hitchin [Hit87a] and Simpson [Sim87] on Higgs bundles, and later Mochizuki’s work on  $\lambda$ -flat bundles (mainly non-compact case), which fully identifies the stability of vector bundles and the existence of Hermitian–Einstein metrics, and moreover, gives an one-to-one correspondence between above objects over compact Kähler manifold  $(X, \omega)$ . This bridge provides many possibilities on applying differential-geometric methods to study algebraic-geometric problems (for example, study the geometry of moduli spaces).

Specially, if these vector bundles have trivial Chern classes (first and second), then these Hermitian–Einstein metrics will being pluri-harmonic metrics (that is, the Hermitian metric  $h$  such that  $G_{h,\lambda} = 0$ , which is equivalent to say that when  $\lambda \neq 0$ , the Hermitian metric  $h$  such that  $(E, \bar{\partial}_h, \theta_h)$  becomes a Higgs bundle; and when  $\lambda = 0$ , the Hermitian metric  $h$  such that  $(E, \mathbb{D}^1 := \partial_h + \bar{\partial}_h + \theta_h + \theta_h^\dagger)$  becomes a flat bundle). The existence of pluri-harmonic metrics, can be concluded into the following statements:

**Theorem 2.** *Let  $(X, \omega)$  be a compact Kähler manifold. Then*

- (1) (Donaldson [Don87], Corlette [Cor88]) *A flat bundle  $(E, \nabla)$  over  $X$  admits a pluri-harmonic metric if and only if it is semisimple;*
- (2) (Hitchin [Hit87a], Simpson [Sim87]) *A Higgs bundle  $(E, \bar{\partial}_E, \theta)$  over  $X$  admits a pluri-harmonic metric if and only if it is polystable with trivial Chern classes;*
- (3) (Mochizuki [Moc06]) *A  $\lambda$ -flat bundle ( $\lambda \neq 0$ )  $(E, \mathbb{D}^\lambda)$  over  $X$  admits a pluri-harmonic metric if and only if it is polystable with trivial Chern classes.*

*Moreover, in each case, the pluri-harmonic metric is unique up to scalar multiplicities.*

As a consequence, we obtain the correspondence between these objects, called the non-Abelian Hodge correspondence:

**Corollary 1 (Non-Abelian Hodge Correspondence).** *Let  $(X, \omega)$  be a compact Kähler manifold. Then for any  $\lambda \in \mathbb{C}$ , we have the following one-to-one correspondence between categories:*

- (1) *The category of semisimple flat bundles of rank  $r$ ;*
- (2) *The category of polystable Higgs bundles of rank  $r$  with trivial Chern classes;*
- (3) *The category of polystable  $\lambda$ -flat bundles of rank  $r$  with trivial Chern classes;*

they are connected by pluri-harmonic metrics, hence they are equivalent with the category of harmonic bundles of rank  $r$ .

There are many generalizations of this result, a natural generalization is considering non-compact manifolds as base manifolds, the resulting correspondence is due to Simpson [Sim90], Biquard [Biq97], Jost–Zuo [JZ97], Mochizuki [Moc06, Moc09] and others. Other generalizations such as considering real Lie groups as structure groups, or considering fields of positive characteristic,  $p$ -adic fields as base fields [BGPiR03, GPGiR09, OV07, Fal05, AGT16]. We do not pretend to give more details here.

Our main aim of this thesis is applying above non-Abelian Hodge theory to study some problems in algebraic geometry and differential geometry.

### 0.3 The Geometry of Non-Abelian Hodge Theory

This part mainly concerns the geometry of non-Abelian Hodge correspondence, especially geometric properties of the related moduli spaces.

#### 0.3.1 Non-Abelian Hodge Theory of Compact Version

The second chapter of this thesis, especially the first two sections (§1.1 and §1.2), is an introduction of non-Abelian Hodge theory, we will try to explain this theory explicitly within limited pages, especially  $\lambda$ -flat bundles that will play important roles in the first part of the thesis. Our new results of this chapter are mainly in the third section (§1.3), we will begin with a norm estimate of  $\mathbb{D}^\lambda$ -flat sections over compact Riemann surfaces, then we obtain a vanishing theorem, that is, there is no  $\mathbb{D}^\lambda$ -flat sections on stable  $\lambda$ -flat bundles with trivial first Chern class over compact Riemann surfaces. More precisely, we have:

**Theorem 3** (= Theorem 1.3.1, Corollary 1.3.2). *Suppose  $(X, \omega)$  is a compact Riemann surface, and  $(E, \mathbb{D}^\lambda)$  is a  $\lambda$ -flat bundle, together with a Hermitian metric  $h$ . Then for any local non-zero  $\mathbb{D}^\lambda$ -flat section  $s$  of  $E$ , the following inequalities holds:*

(1)

$$\Delta_\omega \log(|s|_h^2) \geq -\frac{2}{(1+|\lambda|^2)} |\Lambda_\omega G_{h,\lambda}|_h,$$

(2)

$$\Delta_\omega(|s|_h^2) \geq -\frac{2}{1+|\lambda|^2} |\Lambda_\omega G_{h,\lambda}|_h \cdot |s|_h^2.$$

where  $\Delta_\omega$  is the usual Laplacian operator on  $(X, \omega)$ .

In particular, when  $(E, \mathbb{D}^\lambda)$  is stable with trivial first Chern class, then there is no non-trivial global  $\mathbb{D}^\lambda$ -flat section on  $E$ .

Then we consider some examples, the first one is rank 2  $\lambda$ -flat bundles over punctured unit disk obtained from “fiducial solutions” to decoupled Hitchin equations introduced in [MSWW16], and calculate the explicit solutions of  $\mathbb{D}^\lambda$ -flat sections. The second one is considering the  $\mathbb{C}^*$ -action, we calculate a special solution of pluri-harmonic metric on such  $\lambda$ -flat bundles over punctured unit disk. This shows for a  $\lambda$ -flat bundle over a non-complete manifold, if it admits a pluri-harmonic metric, then usually it is not unique.

### 0.3.2 A Study of the Geometry of Moduli Spaces

The third chapter of this thesis, is a study of the geometry of moduli spaces, which consists of three sections.

In the first section (§2.1), we construct a dynamical system on the moduli space of Higgs bundles by applying the non-Abelian Hodge correspondence, and we study this dynamical system. More explicitly, based on Simpson's construction of moduli spaces via geometric invariant theory [Sim94a, Sim94b, Sim95], for a smooth complex projective variety  $X$ , we have three moduli spaces of above three objects and the moduli space of representations of  $\pi_1(X)$ :

- (1) De Rham moduli space  $\mathcal{M}_{\text{dR}}(X, r)$ : the moduli space of flat bundles of rank  $r$  over  $X$ ;
- (2) Dolbeault moduli space  $\mathcal{M}_{\text{Dol}}(X, r)$ : the moduli space of semistable Higgs bundles of rank  $r$  with trivial Chern classes over  $X$ ;
- (3) Hodge moduli space  $\mathcal{M}_{\text{Hod}}(X, r)$ : the moduli space of semistable  $\lambda$ -flat bundles of rank  $r$  with trivial Chern classes over  $X$ ;
- (4) Betti moduli space  $\mathcal{M}_{\text{B}}(X, r)$ : the moduli space of representations  $\rho : \pi_1(X) \rightarrow \text{GL}(r, \mathbb{C})$ .

Simpson shows that all of these moduli spaces are quasi-projective varieties, denoted by  $M_{\text{dR}}(X, r)$ ,  $M_{\text{Dol}}(X, r)$ ,  $M_{\text{Hod}}(X, r)$  and  $M_{\text{B}}(X, r)$  the subset that parametrizes the isomorphism classes of stable (irreducible, simple) objects, that is, the corresponding smooth loci. And each space is a dense open subset of the corresponding moduli space. In particular, when fix a  $\lambda \in \mathbb{C}$ , let  $\mathcal{M}_{\text{Hod}}^\lambda(X, r) := \mathcal{M}_{\text{Hod}}(X, r)|_\lambda$ , then we have

- $\mathcal{M}_{\text{Hod}}^0(X, r) = \mathcal{M}_{\text{Dol}}(X, r)$ ,
- $\mathcal{M}_{\text{Hod}}^1(X, r) = \mathcal{M}_{\text{dR}}(X, r)$ ,
- $\mathcal{M}_{\text{Hod}}^\lambda(X, r) \cong_{\text{an}} \mathcal{M}_{\text{dR}}(X, r)$ , where  $\lambda \neq 0$ , “ $\cong_{\text{an}}$ ” means the complex analytic isomorphism.

Obviously these properties also hold for the smooth loci  $M_{\text{dR}}(X, r)$ ,  $M_{\text{Dol}}(X, r)$ ,  $M_{\text{Hod}}(X, r)$  and  $M_{\text{B}}(X, r)$ . Let  $\mathcal{M}_{\text{Hod}}(X, r, \mathcal{O}_X)$  (respectively,  $M_{\text{Hod}}(X, r, \mathcal{O}_X)$ ) be the moduli space of semistable (respectively, stable)  $\lambda$ -flat ( $\lambda \in \mathbb{C}$ ) bundles of rank  $r$  over  $X$  with vanishing Chern classes and fixed determinant  $\mathcal{O}_X$ .

Our construction of the dynamical system is beginning with  $M_{\text{Dol}}(X, r)$ , for any  $(\lambda, t) \in \mathbb{C} \times \mathbb{C}^*$ , choose a stable Higgs bundle  $(E, \bar{\partial}_E, \theta)$  with a pluri-harmonic metric  $h$ , consider the corresponding  $\lambda$ -flat bundle  $(E, \bar{\partial}_E + \lambda\theta_h^\dagger, \lambda\partial_{E,h} + \theta)$ , it's clear that  $h$  is also a pluri-harmonic metric for such  $\lambda$ -flat bundle; then we consider the  $t\lambda$ -flat bundle  $(E, \bar{\partial}_E + \lambda\theta_h^\dagger, t(\lambda\partial_{E,h} + \theta))$  given by  $\mathbb{C}^*$ -action, there exists a pluri-harmonic metric denoted as  $h_t$  on each  $(E, \bar{\partial}_E + \lambda\theta_h^\dagger, t(\lambda\partial_{E,h} + \theta))$ , since  $\mathbb{C}^*$ -action does not change the stability and triviality of Chern classes of  $\lambda$ -flat bundles; finally by non-Abelian Hodge correspondence, this determines a Higgs bundle denoted as  $(E, \bar{\partial}_{E,h_t}, \theta_{h_t})$ , which carries  $h_t$  as its pluri-harmonic metric.

Therefore, we obtain a  $(\lambda, t)$ -parametrized smooth dynamical system  $\psi_{(\lambda,t)}$  on  $M_{\text{Dol}}(X, r)$ :

$$\begin{aligned} \psi_{(\lambda,t)} : M_{\text{Dol}}(X, r) &\longrightarrow M_{\text{Dol}}(X, r), \\ (E, \bar{\partial}_E, \theta) &\longmapsto (E, \bar{\partial}_{E,h_t}, \theta_{h_t}). \end{aligned}$$

Moreover,  $\psi_{(\lambda,t)}$  can be defined on  $\mathcal{M}_{\text{Dol}}(X, r)$  as a continuous map.

The following properties are obvious from the definition:

**Proposition 1** (= Proposition 2.1.1).

- (1)  $\psi_{(0,t)}$  is the usual  $\mathbb{C}^*$ -action given by  $t$ , and  $\psi_{(\lambda,1)}$  is the identity map;
- (2)  $\psi_{(\lambda,t)}$  satisfies the following ‘‘cocycle’’-type formula:

$$\psi_{(\lambda t_1, t_2)} \circ \psi_{(\lambda, t_1)} = \psi_{(\lambda, t_1 t_2)};$$

- (3) For any  $\lambda \in \mathbb{C}, t \in \mathbb{C}^*$ , the stable vector bundles (that is, Higgs bundles with zero Higgs field) in the moduli space are fixed points of the dynamical system  $\psi_{(\lambda,t)}$ .

An important problem in dynamical system is studying the fixed points, in this section, by using the analysis on pluri-harmonic metrics, we obtain that the dynamical system shares the same fixed points with  $\mathbb{C}^*$ -action on the moduli space  $\mathcal{M}_{\text{Dol}}(X, r)$ :

**Theorem 4** (= Theorem 2.1.5). Fix  $(\lambda, t) \in \mathbb{C} \times \mathbb{C}^*$ , define the fixed point set of  $(\lambda, t)$ -action:

$$F_{(\lambda,t)} := \{u \in \mathcal{M}_{\text{Dol}}(X, r) : \psi_{(\lambda,t)}(u) = u\},$$

then the fixed point set  $F := \bigcap_{(\lambda,t) \in \mathbb{C} \times \mathbb{C}^*} F_{(\lambda,t)}$  consists of all the fixed points of  $\mathbb{C}^*$ -action on  $\mathcal{M}_{\text{Dol}}(X, r)$ , that is, all  $\mathbb{C}$ -VHS in  $\mathcal{M}_{\text{Dol}}(X, r)$ .

Moreover, we calculate the stabilizer set of special points in the moduli space for the case of compact Riemann surface:

$$\mathcal{C}_u = \{(\lambda, t) \in \mathbb{C} \times \mathbb{C}^* : \psi_{(\lambda,t)}(u) = u\}.$$

**Theorem 5** (= Theorem 2.1.7, Corollary 2.1.8). Suppose  $X$  is a compact Riemann surface, and let  $u \in \mathcal{M}_{\text{Dol}}(X, r)$  be a decoupled Higgs bundle with non-trivial Higgs field, then its stabilizer set is

$$\mathcal{C}_u = \mathbb{C} \times \{\mu_M, \mu_M^2, \dots, \mu_M^{M-1}, 1\},$$

where  $\mu_M = e^{\frac{2\pi i}{M}}$ ,  $M \leq r$  is a constant. Moreover, for any  $(\lambda, t) \in \mathbb{C} \times \mathbb{C}^*$ , the dynamical system  $\psi_{(\lambda,t)}$  does not change the pluri-harmonic metric when acts on a decoupled Higgs bundle.

Meanwhile, we also study the local property of this dynamical system by calculating the first variation of it at a point in the moduli space (cf. Theorem 2.1.3).

The second section (§2.2) of this chapter is the stratifications of moduli spaces, which is mainly based on an important result of Simpson [Sim10]. That is, each flat bundle  $(E, \nabla)$  over a compact Riemann surface  $X$  admits a filtration:

$$F^\bullet : E = F^0 \supset F^1 \supset \dots \supset F^k = 0$$

satisfies the Griffiths transversality  $\nabla(F^p) \subseteq F^{p-1} \otimes K_X$  ( $p = 1, \dots, k$ ), and such that the graded Higgs bundle  $(\text{Gr}_F(E) := \bigoplus_{p=0}^{k-1} F^p/F^{p+1}, \text{Gr}_F(\nabla))$  is semistable. Moreover, such filtration is unique if and only if the graded Higgs bundle is stable. We call such filtration a Simpson filtration.

For any point  $(E, \bar{\partial}_E, D^\lambda)$  in the moduli space  $\mathcal{M}_{\text{Hod}}(X, r)$  ( $\lambda \neq 0$ ), consider its  $\mathbb{C}^*$ -action, Simpson shows the limit  $\lim_{t \rightarrow 0} t \cdot (E, \bar{\partial}_E, D^\lambda)$  exists and is a fixed point of the  $\mathbb{C}^*$ -action on

$\mathcal{M}_{\text{Dol}}(X, r)$ , that is, a  $\mathbb{C}$ -VHS, and satisfies the following equality (in the moduli space):

$$\lim_{t \rightarrow 0} t \cdot (E, \bar{\partial}_E, D^\lambda) = \lim_{t \rightarrow 0} t \cdot (E, \bar{\partial}_E, \lambda^{-1} D^\lambda) = (\text{Gr}_F(E), \text{Gr}_F(\lambda^{-1} D^\lambda)),$$

where  $(E, \bar{\partial}_E, \lambda^{-1} D^\lambda)$  is the corresponding flat bundle of  $(E, \bar{\partial}_E, D^\lambda)$ , and  $(\text{Gr}_F(E), \text{Gr}_F(\lambda^{-1} D^\lambda))$  is the induced graded Higgs bundle of the flat bundle  $(E, \bar{\partial}_E, \lambda^{-1} D^\lambda)$ .

This shows, there are Bialynicki-Birula type stratifications of  $\mathcal{M}_{\text{Hod}}^\lambda(X, r)$ ,  $\mathcal{M}_{\text{Dol}}(X, r)$  and  $\mathcal{M}_{\text{dR}}(X, r)$  into locally closed subsets given by the  $\mathbb{C}^*$ -action:

$$\mathcal{M}_{\text{Hod}}^\lambda(X, r) = \bigcup_{\alpha} G_{\alpha}^{\lambda}, \quad \mathcal{M}_{\text{Dol}}(X, r) = \bigcup_{\alpha} G_{\alpha}^0, \quad \mathcal{M}_{\text{dR}}(X, r) = \bigcup_{\alpha} G_{\alpha}^1,$$

where index  $\alpha$  corresponds to the decomposition of the set of fixed points of  $\mathbb{C}^*$ -action on  $\mathcal{M}_{\text{Dol}}(X, r)$  into its connected components:  $P = \bigcup_{\alpha} P_{\alpha}$ . The map  $G_{\alpha}^{\lambda} \rightarrow P_{\alpha}$  is obtained by taking the limit of  $\mathbb{C}^*$ -action, the fiber at each point is a Lagrangian submanifold of the corresponding moduli space.

For further study of such stratifications, in [Sim10], Simpson proposed the following three important conjectures:

**Conjecture 1** (= [Conjecture 2.2.7](#), [Conjecture 2.2.9](#) and [Conjecture 2.2.11](#)).

- (1) (**Foliation Conjecture**) *When varying the index  $\alpha$ , these Lagrangian fibers of  $p_{\alpha}^1 : G_{\alpha}^1 \rightarrow P_{\alpha}$  fit together to provide a smooth foliation of the de Rham moduli space  $\mathcal{M}_{\text{dR}}(X, r)$  with each leaf closed.*
- (2) (**Nestedness Conjecture**) *The stratifications for the Dolbeault moduli space  $\mathcal{M}_{\text{Dol}}(X, r)$  and the de Rham moduli space  $\mathcal{M}_{\text{dR}}(X, r)$  are both nested, and the arrangements for both stratifications are the same. Here nestedness means there is a partial order relation “ $\leq$ ” on the index set such that  $\overline{G_{\alpha}^i} = \bigcup_{\beta \leq \alpha} G_{\beta}^i$  hold for  $i = 0, 1$ .*
- (3) (**Oper Stratum Conjecture**) *In the stratification of the de Rham moduli space  $\mathcal{M}_{\text{dR}}(X, r)$ , the oper stratum  $G_{\text{oper}}^1$  is the unique closed stratum and the unique stratum of minimal dimension. Here an oper means a flat bundle that admits a special filtration.*

In this section, we describe the explicit relation between Simpson filtrations and Harder-Narasimhan filtrations for underlying vector bundles of rank 3 flat bundles (the description of the result is very long, please refer to [Theorem 2.2.13](#)).

At the same time, we continue our study of the dynamical system  $\psi_{(\lambda, t)}$ , and introduce five different limits associated to this dynamical system:

- $\psi_{\underline{(0,0)}}(E, \bar{\partial}_E, \theta) := \lim_{t \rightarrow 0} \left( \psi_{(0,t)}(E, \bar{\partial}_E, \theta) \right) = \lim_{t \rightarrow 0} (E, \bar{\partial}_E, t\theta);$
- $\psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta) := \lim_{\lambda \rightarrow 0} \left( \psi_{(\lambda,0)}(E, \bar{\partial}_E, \theta) \right);$
- $\psi^{(0,0)}(E, \bar{\partial}_E, \theta) := \lim_{t \rightarrow 0} \left( \lim_{\lambda \rightarrow 0} \left( \psi_{(\lambda,t)}(E, \bar{\partial}_E, \theta) \right) \right);$
- $\psi^{\overline{(0,0)}}(E, \bar{\partial}_E, \theta) := \lim_{\lambda \rightarrow 0} \left( \lim_{t \rightarrow 0} \left( \psi_{(\lambda,t)}(E, \bar{\partial}_E, \theta) \right) \right);$
- $\psi_{(0,0)}(E, \bar{\partial}_E, \theta) := \lim_{(\lambda,t) \rightarrow (0,0)} \left( \psi_{(\lambda,t)}(E, \bar{\partial}_E, \theta) \right).$

For any point in the moduli space  $\mathcal{M}_{\text{Dol}}(X, r)$ , generally these limits are different when they exist. We study this problem, and find some special points such that these limits exist and coincide in the moduli space.

**Theorem 6** (= [Theorem 2.2.16](#)). *Let  $X$  be a compact Riemann surface, when  $(E, \bar{\partial}_E, \theta) \in \mathcal{M}_{\text{Dol}}(X, r)$  is a  $\mathbb{C}$ -VHS, or a decoupled Higgs bundle, then the five limits exist and coincide in  $\mathcal{M}_{\text{Dol}}(X, r)$ .*

To describe the dynamical system  $\psi_{(\lambda, t)}$ , a key point is to figure out the explicit relation between the pluri-harmonic metric  $h_t$  and the parameter  $t$ . This is very hard to compute, since it's hard to solve Hitchin equations of a  $\lambda$ -flat bundle under  $\mathbb{C}^*$ -action. We give a description of  $h_t$  near  $t = 1$ :

**Theorem 7** (= [Theorem 2.2.22](#)). *Let  $X$  be a compact Riemann surface, fix  $\lambda \in \mathbb{C}^*$  and assuming  $t \in \mathbb{R}^*$ , then over a small neighborhood of  $t = 1$ , the  $\text{End}(E)$ -valued function  $f := \frac{h_t h^{-1} - \text{Id}}{(t-1)^2}$  is real analytic with respect to  $t - 1$ .*

In the last section ([§2.3](#)) of this chapter, we prove a weak version of the oper stratum conjecture, which gives a partial answer to Simpson's conjecture:

**Theorem 8** (= [Theorem 2.3.1, Minimal Dimension](#)). *The oper stratum  $G_{\text{oper}}^1$  is the unique closed stratum of minimal dimension  $r^2(g-1) + g + 1$  in the de Rham moduli space  $\mathcal{M}_{\text{dR}}(X, r)$ .*

The proof of this theorem relies on the description of the fixed point set  $P$ , especially its connected components  $P_\alpha$ , via the theory of holomorphic chains. Since as a fixed point of  $\mathbb{C}^*$ -action, a  $\mathbb{C}$ -VHS can be identified with a holomorphic chain of certain type. We finish the proof via calculating the dimension of moduli space of chains and irreducibility of the moduli space.

### 0.3.3 Twistor Space Constriction

The main contribution of this chapter is in the case of compact Riemann surface, we generalize Deligne's construction of twistor space via gluing the Hodge moduli spaces  $\mathcal{M}_{\text{Hod}}(X, r)$  and  $\mathcal{M}_{\text{Hod}}(\bar{X}, r)$  over  $X$  and its conjugate  $\bar{X}$  [[Del89](#), [Sim95](#)], as an interpretation of Hitchin's construction of twistor space associated to the Dolbeault moduli space  $M_{\text{Dol}}(X, r)$  with hyper-Kähler structure [[HKLR87](#)].

The fundamental idea of [[HKLR87](#)] on constructing a twistor space associated to a hyper-Kähler manifold  $M$  is a topological product of  $M$  and the complex projective line  $\mathbb{P}^1$ , we denote this space as  $\text{TW}(M) := M \times \mathbb{P}^1$ . The quaternionic structure  $(I, J, K = IJ)$  on  $M$  induces a complex structure  $\mathcal{I}$  on  $\text{TW}(M)$ . So we obtain a complex manifold and still denoted as  $\text{TW}(M)$ , this is the Hitchin twistor space, and we call this theory the Hitchin twistor theory.

Deligne's idea is in fact an interpretation of Hitchin twistor theory via non-Abelian Hodge theory. For moduli space that parametrizes isomorphism classes of stable objects over  $X$  ( $M_{\text{Dol}}(X, r)$ ,  $M_{\text{dR}}(X, r)$  and so on), denoted by  $M^{\text{sm}}(X, r)$  the underlying smooth manifold. Then  $M^{\text{sm}}(X, r)$  carries a hyper-Kähler structure induced from  $M_{\text{Dol}}(X, r)$  and  $M_{\text{dR}}(X, r)$  [[Hit87a](#), [Fuj91](#)]. This gives a Hitchin twistor space  $\text{TW}(M^{\text{sm}}(X, r))$  following [[HKLR87](#)]. Deligne's main idea is gluing the Hodge moduli spaces  $\mathcal{M}_{\text{Hod}}(X, r)$  and  $\mathcal{M}_{\text{Hod}}(\bar{X}, r)$  over  $X$  and  $\bar{X}$  via complex conjugation map and non-Abelian Hodge correspondence, this gives a new twistor space  $\text{TW}_{\text{DH}}(X, r)$  together with a fibration  $\text{TW}_{\text{DH}}(X, r) \rightarrow \mathbb{P}^1$ . We call  $\text{TW}_{\text{DH}}(X, r)$  the Deligne–Hitchin twistor space, it's showed that the smooth locus  $\text{TW}_{\text{DH}}^{\text{sm}}(X, r)$  is complex analytically isomorphic to the Hitchin twistor space  $\text{TW}(M^{\text{sm}}(X, r))$ .

Now we consider the case when  $X$  is a compact Riemann surface of genus  $g \geq 2$ , since  $X$  can be denoted as the pair  $(\mathcal{X}, I)$  for  $\mathcal{X}$  the underlying smooth connected orientable closed surface, and  $I$  the complex structure. The isotropy classes of complex structures on  $\mathcal{X}$  can be denoted by its Teichmüller space  $\text{Teich}(\mathcal{X})$ , and the extended mapping class group  $\text{Mod}^\diamond(\mathcal{X}) := \text{Diff}(\mathcal{X})/\text{Diff}_0(\mathcal{X})$  acts on this space, with the quotient  $\mathcal{M}(\mathcal{X}) := \text{Teich}(\mathcal{X})/\text{Mod}^\diamond(\mathcal{X})$  called the *Riemann moduli space of complex structures on  $\mathcal{X}$* , which parametrizes the Riemann surface structures on  $\mathcal{X}$  under biholomorphism. When we review Deligne's construction, the two Hodge moduli spaces appearing in the gluing process are over  $X$  and its conjugate  $\bar{X}$ , or equivalently, over  $(\mathcal{X}, I)$  and  $(\mathcal{X}, -I)$ . So for a chosen complex structure  $I' \in \mathcal{M}(\mathcal{X})$  and let  $X' := (\mathcal{X}, I')$ , we can glue the Hodge moduli spaces  $\mathcal{M}_{\text{Hod}}(X, r)$  and  $\mathcal{M}_{\text{Hod}}(X', r)$  along the overlap  $\mathcal{M}_{\text{Hod}}(X, r) \times_{\mathbb{C}} \mathbb{C}^* \cong \mathcal{M}_{\text{Hod}}(X', r) \times_{\mathbb{C}} \mathbb{C}^* \cong \mathcal{M}_{\text{B}}(\mathcal{X}, r) \times \mathbb{C}^*$  via the Riemann–Hilbert correspondence that covers the map  $\mathbb{C}^* \rightarrow \mathbb{C}^*$ ,  $\lambda \mapsto \lambda^{-1}$ . The identification of  $\lambda$ -flat bundles given by the Deligne gluing  $\mathbf{d}$  can be explicitly written as

$$[E, \bar{\partial}_E, D^\lambda, \lambda] \longleftrightarrow [E, (\bar{\partial}_E + \lambda^{-1} D^\lambda)_{X'}^{0,1}, \lambda^{-1} (\bar{\partial}_E + \lambda^{-1} D^\lambda)_{X'}^{1,0}, \lambda^{-1}],$$

where  $(\bullet)_{X'}^{1,0}$  and  $(\bullet)_{X'}^{0,1}$  denote the corresponding (1,0)-part and (0,1)-part with respect to the complex structure  $I'$  on  $X'$ , respectively. The obtained analytic variety is called the *generalized Deligne–Hitchin twistor space*, and is denoted as  $\text{TW}(X, X'; r)$ , which makes in particular,  $\text{TW}_{\text{DH}}(X, r) = \text{TW}(X, \bar{X}; r)$ .

Our main results in this chapter can be concluded as the following:

**Theorem 9** (= [Theorem 3.3.4](#), [Theorem 3.3.7](#), and [Theorem 3.3.11](#)).

- (1) (**Weight One Property**) For a fixed  $\lambda_0$ -flat bundle  $(E, \bar{\partial}_E, D^{\lambda_0})$ , it determines a holomorphic section  $s_{\lambda_0}$  of the generalized Deligne–Hitchin twistor space  $\text{TW}(X, X'; r)$ , called a *de Rham section*. This section has weight one property, that is, its normal bundle satisfies

$$N_{s_{\lambda_0}} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus \dim_{\mathbb{C}}(\mathcal{M}_{\text{dR}}(X, r))}.$$

In particular, de Rham sections are balanced ample rational curves of degree  $\dim_{\mathbb{C}}(\mathcal{M}_{\text{dR}}(X, r))$ .

- (2) (**Torelli-type Theorem**) Let  $X, X' \in \mathcal{M}(\mathcal{X})$  and  $Y, Y' \in \mathcal{M}(\mathcal{Y})$  be Riemann surfaces with genus  $g \geq 3$ . If  $\text{TW}(X, X'; r, \mathcal{O})$  is analytically isomorphic to  $\text{TW}(Y, Y'; r, \mathcal{O})$ , then either  $X \cong Y, X' \cong Y'$ , or  $X \cong Y', X' \cong Y$ . Where  $\text{TW}(X, X'; r, \mathcal{O})$  is the generalized Deligne–Hitchin twistor space obtained by gluing  $\mathcal{M}_{\text{Hod}}(X, r, \mathcal{O}_X)$  and  $\mathcal{M}_{\text{Hod}}(X', r, \mathcal{O}_{X'})$ .
- (3) (**Automorphism Groups**) Let  $\text{Aut}_0(\text{TW}(X, X'; r))$  be the identity component of the holomorphic automorphism group  $\text{Aut}(\text{TW}(X, X'; r))$  of  $\text{TW}(X, X'; r)$ , then each element of  $\text{Aut}_0(\text{TW}(X, X'; r))$  maps fibers of  $\pi : \text{TW}(X, X'; r) \rightarrow \mathbb{P}^1$  to fibers. Moreover, this group satisfies the following short exact sequence

$$\text{Id} \longrightarrow \mathfrak{K} \longrightarrow \text{Aut}_0(\text{TW}(X, X'; r)) \longrightarrow \mathbb{C}^* \longrightarrow \text{Id},$$

where each element of  $\mathfrak{K}$  preserves the fibers of  $\pi : \text{TW}(X, X'; r) \rightarrow \mathbb{P}^1$ .

## 0.4 Some Specializations of Non-Abelian Hodge Correspondence

This part can be viewed as applications of non-Abelian Hodge theory, more precisely, some specializations of non-Abelian Hodge correspondence. We build some special non-Abelian Hodge correspondences.

### 0.4.1 Stability and Indecomposability of Representations of Quivers

The sixth chapter of this thesis is non-Abelian Hodge theory of quiver version, which is based on a conjecture proposed by M. Reineke in 2003 [Rie03]:

**Conjecture 2** (= [Conjecture 5.1.1](#)). *If  $Q$  is a quiver of Dynkin type, then there exists a weight system  $\Theta$  on  $Q$  such that the stable representations with respect to  $\Theta$  are precisely the indecomposable ones.*

The importance of this conjecture is trying to identify stable and indecomposable representations of quivers of Dynkin type with respect to certain weight system. In general, stability is a stronger condition than indecomposability, while this conjecture can make them being equivalent.

D. Juteau has found counterexamples to this conjecture for quivers of  $D$ - and  $E$ -type, a modified Reineke's conjecture is proposed as following:

**Conjecture 3** (= [Conjecture 5.3.1](#), **Modified Reineke's Conjecture**). *If  $Q$  is a quiver of Dynkin type, then the abelian category  $\mathbf{Rep}_k(Q)$  is a maximal stable category.*

A quiver  $Q$  is in fact an oriented graph that consists of finite vertices and finite arrows that connect these vertices. More explicitly, it is a triple  $Q = (Q_0, Q_1, s, t)$ , where  $Q_0$  and  $Q_1$  are finite sets of vertices and arrows, respectively, and  $s, t : Q_1 \rightarrow Q_0$  are maps that map an arrow  $a \in Q_1$  to its starting vertex  $s(a)$  and terminal vertex  $t(a)$ , respectively, which can be denoted as the following:

$$\begin{array}{ccc} \bullet & \xrightarrow{a} & \bullet \\ s(a) & & t(a) \end{array}$$

Fix an algebraically closed field  $k$ , a  $k$ -representation of the quiver  $Q$  is by putting a finite dimensional  $k$ -vector space at each vertex and a  $k$ -linear map between  $k$ -vector spaces at each arrow. That is, a pair  $X = \{(X_i)_{i \in Q_0}, (X_a)_{a \in Q_1}\}$ , for each  $X_i$  a finite dimensional  $k$ -vector space, and each  $X_a : X_{s(a)} \rightarrow X_{t(a)}$  a  $k$ -linear map.

A  $k$ -representation  $X$  of the quiver  $Q$  is indecomposable if it is not a direct sum of two non-trivial  $k$ -representations of  $Q$ .

Denoted by  $\mathbf{Rep}_k(Q)$  the category of  $k$ -representations of the quiver  $Q$ . For a given weight system  $\Theta = (\theta_i)_{i \in Q_0} \in \mathbb{Z}^{|Q_0|}$ ,  $w(X) := \sum_{i \in Q_0} \theta_i \dim_k X_i$  and  $r(X) := \sum_{i \in Q_0} \dim_k X_i$  are called a weight function and a rank function on the category  $\mathbf{Rep}_k(Q)$ , respectively.

The notion of stability of  $k$ -representations of quivers can be defined naturally, a  $k$ -representation  $X$  of the quiver  $Q$  is  $(w, r)$ -stable (reps.  $(w, r)$ -semistable), if for any non-trivial proper subrepresentation  $U$  of  $X$ , we have

$$\mu(U) < \mu(X) \quad (\text{resp. } \mu(U) \leq \mu(X)),$$

where  $\mu(X) := \frac{w(X)}{r(X)}$  is the slope of  $X$  with respect to the weight function  $w$  and the rank function  $r$ . Polystability can be defined similarly.

Our main result of this chapter is providing a proof of the modified Reineke's conjecture for quivers of  $A_n$ -type by combinatorial construction of a special weight system (we call it intrinsic weight system). In particular, the original Reineke's conjecture holds for quivers of  $A_n$ -type.

**Theorem 10** (= [Theorem 5.3.2](#)). *If  $Q$  is a quiver of  $A_n$ -type, then there exists a weight system  $\Theta = (\theta_i)_{i \in Q_0} \in \mathbb{Z}^{|Q_0|}$  such that stable representations with respect to the weight function  $w(X) = \sum_{i \in Q_0} \theta_i \dim X_i$  and the rank function  $r(X) = \sum_{i \in Q_0} \dim X_i$  are precisely the indecomposable ones, that is, the category  $\mathbf{Rep}_k(Q)$  is a maximal stable category.*

Meanwhile, we also study the intrinsic weight system via semi-invariant theory (for the details, refer to [Proposition 5.3.10](#)).

## 0.4.2 Kobayashi–Hitchin Correspondence for Quiver Bundles

The last chapter of this thesis, is another specialization of non-Abelian Hodge correspondence, that is, the Kobayashi–Hitchin correspondence for quiver bundles over generalized Kähler manifolds.

The Kobayashi–Hitchin correspondence, is a correspondence between stability of vector bundles and existence of Hermitian–Einstein metrics, which is introduced at the beginning of this introduction, plays an essential role in non-Abelian Hodge theory.

In this chapter we will combine generalized Kähler geometry and quiver theory, especially the theory of quiver bundles. We will introduce the notion of quiver bundles over generalized Kähler manifolds, as well as stability and Hermitian–Einstein metrics for such bundles. A natural idea is to consider the relationship between them.

Our main result of this chapter can be characterized as the following:

**Theorem 11** (= [Theorem 6.3.1](#)). *Let  $Q = (Q_0, Q_1)$  be a quiver,  $\mathcal{E} = (E, \phi)$  be an  $I_{\pm}$ -holomorphic  $Q$ -bundle over an  $n$ -dimensional compact generalized Kähler manifold  $(X, I_+, I_-, g, b)$  such that  $g$  is Gauduchon with respect to both  $I_+$  and  $I_-$ , then  $\mathcal{E}$  is  $(\alpha, \sigma, \tau)$ -polystable if and only if  $\mathcal{E}$  admits an  $(\alpha, \sigma, \tau)$ -Hermitian–Einstein metric.*

## Part I

# The Geometry of Non-Abelian Hodge Correspondence



# Chapter 1

## Non-abelian Hodge Theory of Compact Version

The non-Abelian Hodge theory, which is mainly based on the work of Corlette [Cor88] and Donaldson [Don87] on harmonic maps (flat bundles), and the work of Hitchin [Hit87a] and Simpson [Sim87] on Higgs bundles, gives a correspondence between semisimple flat bundles over a compact Kähler manifold  $X$  and polystable Higgs bundles with vanishing Chern classes over the same manifold. This correspondence can be generalized to a statement concerning  $\lambda$ -flat bundles, a topic introduced by Deligne [Del89] and further developed by Simpson [Sim95, Sim08, Sim10]. More precisely, for arbitrary  $\lambda_1, \lambda_2 \in \mathbb{C}$ , Mochizuki [Moc06] established a correspondence between the categories of polystable  $\lambda_1$ -flat bundles with vanishing Chern classes and polystable  $\lambda_2$ -flat bundles with vanishing Chern classes. In particular, taking  $\lambda_1 = 1$  and  $\lambda_2 = 0$ , this correspondence recovers the original non-Abelian Hodge correspondence. Their results are obtained by showing the existence of certain special metrics on certain bundles with additional conditions and can be concluded as follows (and will be explained explicitly later):

**Theorem 1.0.1.** *Let  $(X, \omega)$  be a compact Kähler manifold. Then*

- (1) (Donaldson [Don85], Corlette [Cor88]) *A flat bundle  $(E, \nabla)$  over  $X$  admits a pluri-harmonic metric if and only if it is semisimple;*
- (2) (Hitchin [Hit87a], Simpson [Sim87]) *A Higgs bundle  $(E, \bar{\partial}_E, \theta)$  over  $X$  admits a pluri-harmonic metric if and only if it is polystable with vanishing Chern classes;*
- (3) (Mochizuki [Moc06]) *A  $\lambda$ -flat bundle  $(E, \mathbb{D}^\lambda)$  ( $\lambda \neq 0$ ) over  $X$  admits a pluri-harmonic metric if and only if it is polystable with vanishing Chern classes.*

*Moreover, in each case, the pluri-harmonic metric is unique up to scalar multiplicities.*

As a result, the non-Abelian Hodge correspondence reads:

**Corollary 1.0.2 (Non-Abelian Hodge Correspondence).** *Let  $(X, \omega)$  be a compact Kähler manifold. Then for each  $\lambda \in \mathbb{C}$ , there is an one to one correspondence between the following categories:*

- (1) *The category of semisimple flat bundles of rank  $r$ ;*
- (2) *The category of polystable Higgs bundles of rank  $r$  with vanishing Chern classes;*
- (3) *The category of polystable  $\lambda$ -flat bundles of rank  $r$  with vanishing Chern classes;*

The equivalences are through pluri-harmonic metrics, hence all the above categories are equivalent to the category of harmonic bundles of rank  $r$ .

Therefore, in our setting, especially in this thesis, non-Abelian Hodge theory means a correspondence between flat bundles, Higgs bundles,  $\lambda$ -flat bundles, and harmonic bundles.

All the work mentioned above arise from the study of a classical problem in the non-Higgs setting: a correspondence between the existence of certain special metrics on a vector bundle and the stability of that bundle, called the Kobayashi–Hitchin correspondence. This correspondence builds a bridge between algebraic-geometric side of stability of vector bundles and differential-geometric side of existence of pluri-harmonic metrics. The study of this kind of problem can be dated back to Narasimhan and Seshadri’s work on the stability of vector bundles [NS65]. Their theorem states that a vector bundle over a compact Riemann surface is stable if and only if it arises from an irreducible projectively unitary representation of the fundamental group of that Riemann surface. The Narasimhan–Seshadri theorem was latter reproved by Donaldson with a differential geometric method [Don83], which relates the stability of vector bundles and the existence of certain special metrics. This celebrating idea was later generalized to higher dimensional compact Kähler manifolds by Donaldson [Don85], Uhlenbeck and Yau [UY86].

There is a natural field arising from non-Abelian Hodge theory, the study of the corresponding moduli spaces of these objects. From Simpson’s work on the construction of the moduli spaces [Sim94a, Sim94b, Sim95], we have the following four moduli spaces:

- *Betti moduli space*  $\mathcal{M}_B(X, r)$ : the moduli space of rank  $r$  representations  $\pi_1(X) \rightarrow \mathrm{GL}(r, \mathbb{C})$ ;
- *de Rham moduli space*  $\mathcal{M}_{\mathrm{dR}}(X, r)$ : the moduli space of rank  $r$  flat bundles over  $X$ ;
- *Dolbeault moduli space*  $\mathcal{M}_{\mathrm{Dol}}(X, r)$ : the moduli space of semistable rank  $r$  Higgs bundles over  $X$  with vanishing Chern classes;
- *Hodge moduli space*  $\mathcal{M}_{\mathrm{Hod}}(X, r)$ : the moduli space of semistable rank  $r$   $\lambda$ -flat bundles over  $X$  with vanishing Chern classes.

The study of these moduli spaces arising from non-Abelian Hodge theory shows that they are also related. More precisely, we have

**Theorem 1.0.3 (NAHC of Moduli Spaces Version, [Sim94a, Sim94b, Sim95]).**

- (1) *The de Rham moduli space  $\mathcal{M}_{\mathrm{dR}}(X, r)$  and the Betti moduli space  $\mathcal{M}_B(X, r)$  are complex analytically isomorphic as complex analytic varieties:*

$$\mathcal{M}_{\mathrm{dR}}(X, r) \stackrel{\mathrm{an}}{\cong} \mathcal{M}_B(X, r).$$

- (2) *The Dolbeault moduli space  $\mathcal{M}_{\mathrm{Dol}}(X, r)$  and the de Rham moduli space  $\mathcal{M}_{\mathrm{dR}}(X, r)$  are homeomorphic as topological spaces:*

$$\mathcal{M}_{\mathrm{Dol}}(X, r) \stackrel{\mathrm{homeo}}{\cong} \mathcal{M}_{\mathrm{dR}}(X, r),$$

*and they are  $C^\infty$  isomorphic over the subset of isomorphism classes of stable objects (smooth locus).*

- (3) *The Hodge moduli space  $\mathcal{M}_{\mathrm{Hod}}(X, r)$  has a fibration  $p: \mathcal{M}_{\mathrm{Hod}}(X, r) \rightarrow \mathbb{C}$  such that the fibers over 0 and 1 are exactly  $p^{-1}(0) = \mathcal{M}_{\mathrm{Dol}}(X, r)$  and  $p^{-1}(1) = \mathcal{M}_{\mathrm{dR}}(X, r)$ , respectively.*

- (4) Let  $\mathcal{M}_{\text{Hod}}^\lambda(X, r) := p^{-1}(\lambda)$ , when  $\lambda \neq 0$ , then  $\mathcal{M}_{\text{Hod}}^\lambda(X, r)$  and  $\mathcal{M}_{\text{dR}}(X, r)$  are algebraic isomorphic as algebraic varieties:

$$\mathcal{M}_{\text{Hod}}^\lambda(X, r) \stackrel{\text{alg}}{\cong} \mathcal{M}_{\text{dR}}(X, r).$$

## 1.1 Corlette–Simpson Correspondence

In this section, we will give an introduction to Higgs bundles, flat bundles and their correspondence based on [Don87, Cor88, Hit87a, Sim87] and [Sim92], we will try to give more details to make it to be a self-contained part.

Let  $X$  be an  $n$ -dimensional complex projective variety with the polarization given by a fixed ample line bundle  $L = \mathcal{O}_X(1)$ , let  $\omega$  be a Kähler form that represents  $c_1(L)$ . For convenience, we give the following notations that would be used later:

- $C^\infty(X, \mathbb{C})$ : the space of smooth complex valued functions on  $X$ ;
- $X_{\mathbb{R}}$ : the underlying  $2n$ -dimensional real manifold;
- $\Lambda^k(X) := \Lambda^k(T^*X_{\mathbb{R}})$ : the  $k$ -th exterior algebra bundle of  $T^*X_{\mathbb{R}}$ ;
- $\mathcal{A}^k(X) := \Gamma(X, \Lambda^k(X))$ , the space of smooth sections of the bundle  $\Lambda^k(X)$ , i.e, the space of smooth real  $k$ -forms on  $X$ ;
- $T_{\mathbb{C}}^*X := T^*X_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ : the complexified cotangent bundle;
- $T_{\mathbb{C}}^*X = (T^*X)^{1,0} \oplus (T^*X)^{0,1}$ : the decomposition into its (1,0)-part and (0,1)-part;
- $\Omega_X^{p,q}$ : the exterior algebra bundle of  $(p, q)$ -type, that is,  $\Omega_X^{p,q} = \Lambda^p((T^*X)^{1,0}) \otimes \Lambda^q((T^*X)^{0,1})$ ;
- $\Lambda_{\mathbb{C}}^k(X) := \Lambda^k(T_{\mathbb{C}}^*X)$ : the  $k$ -th exterior algebra bundle of  $T_{\mathbb{C}}^*(X)$ . Easy to see that

$$\Lambda_{\mathbb{C}}^k(X) = \bigoplus_{p+q=k} \Omega_X^{p,q};$$

- $\mathcal{A}^{p,q}(X, E) := \Gamma(X, E \otimes \Omega_X^{p,q})$ , the space of smooth sections of the bundle  $E \otimes \Omega_X^{p,q}$ , where  $E$  is any complex vector bundle over  $X$ , sometimes we write it as  $\mathcal{A}^{p,q}(E)$ ;
- $\mathcal{A}^k(X, E) := \Gamma(X, E \otimes \Lambda_{\mathbb{C}}^k(X))$ , the space of smooth sections of the bundle  $E \otimes \Lambda_{\mathbb{C}}^k(X)$ , sometimes we write it as  $\mathcal{A}^k(E)$ , in particular,  $\mathcal{A}^0(X, E) = \Gamma(X, E)$ , the space of smooth sections of  $E$ . Easy to see,

$$\mathcal{A}^k(E) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(E);$$

- $\Omega_X^p$ : the sheaf of holomorphic  $p$ -forms on  $E$ , in particular,  $\Omega_X^1 = \Lambda^n((T^*X)^{1,0})$ .

### Higgs bundles

Let  $E$  be a complex vector bundle over  $X$ , and let  $\bar{\partial}_E : \mathcal{A}^0(E) \rightarrow \mathcal{A}^{0,1}(E)$  be  $\mathbb{C}$ -linear operator such that

- (1)  $\bar{\partial}_E(fs) = \bar{\partial}(f)s + f\bar{\partial}_E(s)$  for any  $f \in C^\infty(X, \mathbb{C})$  and  $s \in \mathcal{A}^0(E)$ ;
- (2)  $(\bar{\partial}_E)^2 = 0$ ;

where the second condition is defined under the natural extension  $\bar{\partial}_E : \mathcal{A}^{p,q}(E) \rightarrow \mathcal{A}^{p,q+1}(E)$  for any integers  $p, q \geq 0$ . Such a pair  $(E, \bar{\partial}_E)$  is called a *holomorphic vector bundle*, sometimes we use a single word  $\mathcal{E}$  to denote a holomorphic vector bundle. By Koszul–Malgrange theorem, this is equivalent to the usual definition of a holomorphic vector bundle by holomorphic transition functions.

**Definition 1.1.1.** A *Higgs bundle* over  $X$  is a holomorphic vector bundle  $(E, \bar{\partial}_E)$  together with a map  $\theta : E \rightarrow E \otimes \Omega_X^1$  which is holomorphic and integrable, i.e.  $\bar{\partial}_E(\theta) = 0$  and  $\theta \wedge \theta = 0$ . Such a  $\theta$  is called a *Higgs field* and the triple  $(E, \bar{\partial}_E, \theta)$  denotes the Higgs bundle.

**Definition 1.1.2.** A Higgs bundle  $(E, \bar{\partial}_E, \theta)$  over  $X$  is called *stable* (resp. *semistable*) if for any proper coherent subsheaf  $F$  of  $0 < \text{rk}(F) < \text{rk}(E)$  and  $\theta(F) \subseteq F \otimes \Omega_X^1$  such that  $E/F$  is torsion-free, we have

$$\mu(F) < \text{ (resp. } \leq) \mu(E),$$

where  $\mu(E) := \frac{\deg(E)}{\text{rk}(E)}$  denotes the *slope* of  $E$ , and  $\deg(E) := \int_X c_1(E) \cdot [\omega]^{n-1}$  is the *degree* of  $E$ . It is called *polystable* if it is the direct sum of stable Higgs bundles of the same slope  $\mu(E)$ .

Given a hermitian metric  $h$  on the Higgs bundle  $(E, \bar{\partial}_E, \theta)$ , then  $h$  and  $\bar{\partial}_E$  uniquely determines an  $(1,0)$ -type operator  $\partial_{E,h}$  such that  $\nabla_h := \partial_{E,h} + \bar{\partial}_E$  is a unitary connection. Here a *connection* on a vector bundle  $E$  is a  $\mathbb{C}$ -linear operator  $\nabla : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$  that satisfies the Leibniz rule  $\nabla(fs) = df \otimes s + f\nabla(s)$  for any  $f \in C^\infty(X, \mathbb{C})$  and  $s \in \mathcal{A}^0(E)$ . A connection  $\nabla$  is *unitary* if it preserves the metric  $h$ , i.e. if  $dh(u, v) = h(\nabla(u), v) + h(u, \nabla(v))$  for any  $u, v \in \mathcal{A}^0(E)$ . This unique unitary connection is the *Chern connection*, and its curvature  $F_h := \nabla_h \circ \nabla_h$  is called the *Chern curvature*. A connection  $\nabla$  on  $E$  is said to be *flat* if its curvature  $F_\nabla := \nabla \circ \nabla$  vanishes under the natural extension  $\nabla : \mathcal{A}^k(X, E) \rightarrow \mathcal{A}^{k+1}(X, E)$  for any integer  $k \geq 0$ .

With  $h$ , the Higgs field  $\theta$  determines an adjoint operator  $\theta_h^\dagger \in \mathcal{A}^{0,1}(\text{End}(E))$  by

$$h(\theta(u), v) = h(u, \theta_h^\dagger(v))$$

for any  $u, v \in \mathcal{A}^0(E)$ . The renewed connection  $\mathbb{D}^1 := \partial_{E,h} + \bar{\partial}_E + \theta + \theta_h^\dagger$  is called the *Hitchin–Simpson connection*, and its curvature  $F_{(\bar{\partial}_E, \theta, h)} := \mathbb{D}^1 \circ \mathbb{D}^1$  is called the *Hitchin–Simpson curvature*.

The holomorphic object Higgs field was introduced by Hitchin [Hit87a] for solving certain Yang–Mills equations over a Riemann surface that generalizes the classical correspondence arising from Narasimhan and Seshadri’s work relating holomorphic vector bundles and unitary connections over a Riemann surface [NS65], and later generalized to higher dimensional compact varieties by Donaldson [Don85], Uhlenbeck and Yau [UY86]. Hitchin’s work was generalized to higher dimensional case by Simpson [Sim87], where he built the Kobayashi–Hitchin correspondence that relating the algebraic notion of stability and the existence of certain good metrics.

**Definition 1.1.3.**  $h$  is called a *Hermitian–Einstein metric* (or a *harmonic metric*) on the Higgs bundle  $(E, \bar{\partial}_E, \theta)$  if the Hitchin–Simpson curvature  $F_{(\bar{\partial}_E, \theta, h)}$  satisfies the following equation:

$$\Lambda_\omega F_{(\bar{\partial}_E, \theta, h)} = c \cdot \text{Id}_E, \tag{1.1}$$

where  $c = \frac{-2\pi\sqrt{-1}\mu_L(E)}{\text{Vol}(X)}$  is a scalar constant. If in particular, the Hitchin–Simpson connection  $\mathbb{D}^1$  is flat, i.e. if  $F_{(\bar{\partial}_E, \theta, h)} = 0$ , then  $h$  is called a *pluri-harmonic metric*, and in this case,  $(E, \bar{\partial}_E, \theta, h)$  is called a *harmonic bundle*.

Let  $F_{(\bar{\partial}_E, \theta, h)}^\perp := F_{(\bar{\partial}_E, \theta, h)} - \frac{1}{r} \text{Tr}(F_{(\bar{\partial}_E, \theta, h)}) \text{Id}_E$  be the trace-free part of the Hitchin–Simpson curvature, then (1.1) is equivalent to

$$\Lambda_\omega F_{(\bar{\partial}_E, \theta, h)}^\perp = 0. \quad (1.2)$$

From the definition we can see that a Higgs bundle  $(E, \bar{\partial}_E, \theta)$  together with a pluri-harmonic metric  $h$  gives rise to a flat bundle structure, that is,  $(E, \mathbb{D}^1 := \partial_{E, h} + \bar{\partial}_E + \theta + \theta_h^\dagger)$  is a flat bundle. The following theorem of Kobayashi–Hitchin correspondence for Higgs bundles is due to Hitchin for rank 2 Higgs bundles over compact Riemann surface, and Simpson for higher rank Higgs bundles over general compact Kähler manifolds.

**Theorem 1.1.4** (Hitchin [Hit87a], Simpson [Sim87]). *Let  $(X, L, \omega)$  be as above, a Higgs bundle  $(E, \bar{\partial}_E, \theta)$  over  $X$  is  $\mu_L$ -polystable if and only if it admits a Hermitian–Einstein metric, moreover, such a metric is unique up to scalar multiplicity.*

Moreover, Simpson showed the following Bogomolov–Gieseker inequality for Higgs bundles:

**Proposition 1.1.5** (Simpson [Sim87]). *If  $h$  is a Hermitian–Einstein metric on  $(E, \bar{\partial}_E, \theta)$ , then*

$$\left(2c_2(E) - \frac{r-1}{r}c_1(E)^2\right) \cdot [\omega]^{n-2} = C \int_X |F_{(\bar{\partial}_E, \theta, h)}^\perp|_{h, \omega}^2 \geq 0, \quad (1.3)$$

where  $r = \text{rk}(E)$ . In particular, if moreover,  $c_1(E) \cdot [\omega]^{n-1} = ch_2(E) \cdot [\omega]^{n-2} = 0$ , then  $F_{(\bar{\partial}_E, \theta, h)} = 0$ , that is,  $h$  is a pluri-harmonic metric and  $(E, \bar{\partial}_E, \theta, h)$  is a harmonic bundle.

## Flat Bundles

Now let  $V$  be a complex vector bundle with a connection  $\nabla : \mathcal{A}^0(V) \rightarrow \mathcal{A}^1(V)$  that is flat, i.e.,  $\nabla^2 = 0$  under the extension  $\nabla : \mathcal{A}^1(V) \rightarrow \mathcal{A}^2(V)$ .

**Definition 1.1.6.** A flat bundle  $(V, \nabla)$  is called *irreducible* (or equivalently, *simple*) if it has no non-zero proper flat subbundle, and it is called *semisimple* if it is a direct sum of irreducible flat bundles.

**Remark 1.1.7.** In fact, we can introduce the stability for flat bundles, that is, a flat bundle  $(V, \nabla)$  is called *stable* if any non-zero proper flat subbundle has degree strictly less than it, and it is called *polystable* if it is the direct sum of stable flat bundles. It's easy to see that, over compact manifolds, a flat bundle is stable if and only if it is irreducible, and it is polystable if and only if it is semisimple.

Given a hermitian metric  $h$  on  $V$ , there is a unique decomposition of  $\nabla$ :

$$\nabla = \nabla_h + \Phi_h$$

such that  $\nabla_h$  is a unitary connection and  $\Phi_h \in \mathcal{A}^1(\text{End}(V))$  is a self-adjoint operator. This decomposition is easy to make, in fact, we can decompose  $\nabla$  into its (1,0) and (0,1) parts:  $\nabla = \nabla^{1,0} + \nabla^{0,1}$ , then define a unique (0,1)-type operator  $\delta''_{V, h}$  such that  $\nabla^{1,0} + \delta''_{V, h}$  preserves  $h$  and a unique (1,0)-type operator  $\delta'_{V, h}$  such that  $\nabla^{0,1} + \delta'_{V, h}$  preserves  $h$ , let

$$\begin{aligned} \partial_{V, h} &= \frac{1}{2}(\nabla^{1,0} + \delta'_{V, h}), & \bar{\partial}_{V, h} &= \frac{1}{2}(\nabla^{0,1} + \delta''_{V, h}), \\ \theta_{V, h} &= \frac{1}{2}(\nabla^{1,0} - \delta'_{V, h}), & \theta_{V, h}^\dagger &= \frac{1}{2}(\nabla^{0,1} - \delta''_{V, h}), \end{aligned}$$

then  $\nabla_h := \partial_{V,h} + \bar{\partial}_{V,h}$  and  $\Phi_h := \theta_{V,h} + \theta_{V,h}^\dagger$  is the desired decomposition. From the flatness of  $\nabla$  and the definition of  $\delta'_{V,h}$  and  $\delta''_{V,h}$ , we have

$$\begin{cases} (\nabla^{1,0})^2 = (\nabla^{0,1})^2 = \nabla^{1,0}\nabla^{0,1} + \nabla^{0,1}\nabla^{1,0} = 0, \\ (\delta'_{V,h})^2 = (\delta''_{V,h})^2 = \delta'_{V,h}\delta''_{V,h} + \delta''_{V,h}\delta'_{V,h} = 0. \end{cases} \quad (1.4)$$

Let  $D''_{V,h} = \bar{\partial}_{V,h} + \theta_{V,h}$ , it's easy to check that it satisfies the Leibniz rule:

$$D''_{V,h}(fs) = \bar{\partial}(f)s + fD''_{V,h}(s),$$

let  $G_h := D''_{V,h} \circ D''_{V,h}$  be the *pseudo-curvature*.

**Definition 1.1.8.**  $h$  is called a *harmonic metric* on the flat bundle  $(V, \nabla)$  if the pseudo-curvature  $G_h$  satisfies that  $\Lambda_\omega G_h = 0$ , if in particular,  $G_h = 0$ , i.e, if  $(\bar{\partial}_{V,h} + \theta_{V,h})^2 = 0$ , then  $h$  is called a *pluri-harmonic metric* and in this case,  $(V, \nabla, h)$  is called a *harmonic bundle*.

From the definition we can see that a flat bundle  $(V, \nabla)$  together with a pluri-harmonic metric  $h$  gives rise to a Higgs bundle structure, that is,  $(V, \partial_{V,h}, \theta_{V,h})$  is a Higgs bundle.

Let  $G$  be a complex reductive group, and  $K \subseteq G$  be a maximal compact subgroup. Since any hermitian metric  $h$  on a flat principal  $G$ -bundle  $P$  is in fact a  $K$ -reduction, that is, a principal  $K$ -subbundle  $P_K$  with  $P = P_K \times_K G$ . Any flat  $G$ -bundle is equivalent to a representation  $\rho : \pi_1(X) \rightarrow G$ , and it is irreducible (resp. semisimple, or equivalently, reductive) if and only if the associated monodromy representation is irreducible (resp. semisimple, or equivalently, reductive), here a representation  $\rho : \pi_1(X) \rightarrow G$  is called *semisimple* (or equivalently, *reductive*) if the Zariski closure of  $\rho(\pi_1(X))$  in  $G$  is a reductive group. A  $K$ -reduction is equivalent to a section of the fiber bundle  $P/K = P \times_G G/K$ , which can be thought as a  $\rho$ -equivariant map  $h_\rho : \tilde{X} \rightarrow G/K$ . Thus a metric on a flat  $G$ -bundle is equivalent to a  $\rho$ -equivariant map  $h_\rho : \tilde{X} \rightarrow G/K$ .

The fundamental theorem of flat bundles is due to Donaldson [Don87] for rank 2 case and Corlette [Cor88] for general case:

**Theorem 1.1.9** (Donaldson [Don87], Corlette [Cor88]). *Let  $X$  be a compact Riemannian manifold and let  $G$  be a complex reductive group, a representation  $\rho : \pi_1(X) \rightarrow G$  is reductive if and only if there exists a  $\rho$ -equivariant harmonic map  $h_\rho : \tilde{X} \rightarrow G/K$ .*

Now we give a brief introduction to harmonic maps here, references could be [ABC<sup>+</sup>96]. Let  $(M, g)$  and  $(N, h)$  be two Riemannian manifolds, let  $f : M \rightarrow N$  be a smooth map between them, then the differential  $df$  can be thought as a section of the bundle  $TM \otimes f^*TN$ , we define the *energy* of  $f$  to be

$$e(f) := \frac{1}{2} \int_M |df|_{g \otimes f^*h}^2 d\text{Vol}_M,$$

$f$  is called a *harmonic map* if it is a critical point of the energy functional, that is, if it satisfies the Euler–Lagrange equation:

$$\Delta(f) := *d_\nabla *df = 0, \quad (1.5)$$

where  $d_\nabla : \mathcal{A}^k(M, f^*TN) \rightarrow \mathcal{A}^{k+1}(M, f^*TN)$  is the exterior differential operator induced from the pull-back Levi-Civita connection  $f^*\nabla^N : \Gamma(M, f^*TN) \rightarrow \mathcal{A}^1(M, f^*TN) = \Gamma(M, T^*M \otimes f^*TN)$  on the pull-back bundle  $f^*TN$ , where  $\nabla^N : \Gamma(X, TN) \rightarrow \mathcal{A}^1(M, TN) = \Gamma(M, T^*M \otimes TN)$  is the Levi-Civita connection of  $(N, h)$ , that is,  $d_\nabla = f^*\nabla^N$  on  $\mathcal{A}^0(M, f^*TN) = \Gamma(X, T^*M \otimes f^*TN)$

and  $d_{\nabla}(\alpha \otimes s) := d\alpha \otimes + (-1)^{\deg(\alpha)} \alpha \otimes f^* \nabla^N(s)$  for any  $\alpha \in \mathcal{A}^k(M)$  and  $s \in \Gamma(M, f^*TN)$ . And  $*$  :  $\mathcal{A}^k(M, f^*TN) \rightarrow \mathcal{A}^{\dim(M)-k}(M, f^*TN)$  is the induced Hodge star operator defined by

$$\langle \alpha \wedge * \beta \rangle := \langle \alpha, \beta \rangle d\text{Vol}_M,$$

where  $\langle \bullet, \bullet \rangle$  is the induced inner product on  $\mathcal{A}^k(M, f^*TN)$ .

If  $(M, g, J)$  is a hermitian manifold of complex dimension  $n$ , that is,  $g$  is compatible with the complex structure  $J$ , let  $\omega(\bullet, \bullet) := g(J\bullet, \bullet)$  be the associated fundamental 2-form, then  $*df = \frac{\omega^{n-1}}{(n-1)!} \wedge Jdf =: \frac{\omega^{n-1}}{(n-1)!} \wedge d^c f$ , where  $d^c f := Jdf = df \circ J$ . Therefore, (1.5) is equivalent to

$$d_{\nabla}(\omega^{n-1} \wedge d^c f) = 0, \quad (1.6)$$

if moreover,  $(M, g, J)$  is Kähler, i.e,  $d\omega = 0$ , then  $f$  is harmonic if and only if

$$\omega^{n-1} \wedge d_{\nabla} d^c f = 0. \quad (1.7)$$

Hence in particular, if  $M$  is a Riemann surface, then  $f$  is harmonic if and only if  $d_{\nabla} d^c f = 0$ , this equation only depends on the complex structure on  $M$  (not depend on the hermitian structure). From this, we have the notion of pluri-harmonic map.

Let  $(M, g, J)$  be a complex manifold and  $(N, h)$  a Riemannian manifold, a map  $f : M \rightarrow N$  is called a *pluri-harmonic map* if the restriction to any 1-dimensional complex submanifold is harmonic. From above analysis,  $f$  is pluri-harmonic if and only if  $d_{\nabla} d^c f = 0$ .

There is no general relation between pluri-harmonic maps and harmonic maps, but when  $M$  is Kähler, then from above we know that every pluri-harmonic map is harmonic. The other direction needs more constraints on  $M$  and  $N$ , this was first proved by Siu [Siu80] when  $M$  is compact Kähler and  $N$  is Kähler with certain curvature conditions, and later generalized by Sampson [Sam86] to general Riemannian manifolds  $N$  with non-positive hermitian curvature introduced by him.

**Theorem 1.1.10** (Siu [Siu80], Sampson [Sam86]). *Let  $M$  be a compact Kähler manifold and let  $N$  be a Riemannian manifold with non-positive hermitian curvature, if  $f : M \rightarrow N$  is harmonic, then it is pluri-harmonic.*

**Remark 1.1.11.** Siu–Sampson theorem also holds for  $\rho$ -equivariant harmonic maps, that is, for any compact Kähler manifold  $M$  and any Riemannian manifold  $N$  of non-positive hermitian curvature, let  $\rho : \pi_1(M) \rightarrow \text{Iso}(N)$  be a reductive representation of  $\pi_1(M)$ , where  $\text{Iso}(N)$  is the group of isometries of  $N$ . Then by Donaldson–Corlette theorem, there exists a  $\rho$ -equivalent harmonic map  $h_{\rho} : \tilde{M} \rightarrow N$ , this map is pluri-harmonic. In particular, one takes  $N = G/K$  a symmetric space,  $\text{Iso}(N) = G$ .

**Lemma 1.1.12 (Kähler Identities, [Sim87, Sim92]).** *Let  $(X, \omega)$  be a compact Kähler manifold,*

(1) *If  $(E, \bar{\partial}_E, \theta)$  is a Higgs bundle over  $X$  with a hermitian metric  $h$ , then*

$$(D_h^c)^* = -\sqrt{-1}[\Lambda_{\omega}, \mathbb{D}^1], \quad (\mathbb{D}^1)^* = \sqrt{-1}[\Lambda_{\omega}, D_h^c],$$

*where  $D_h^c = D'' - D'_h$  for  $D'' = \bar{\partial}_E + \theta$  and  $D'_h = \partial_{E,h} + \theta_h^{\dagger}$ ,  $(\ )^*$  is the formal adjoint with respect to the metric  $h$ , in particular,*

$$(D'_h)^* = \sqrt{-1}[\Lambda_{\omega}, D''], \quad (D'')^* = -\sqrt{-1}[\Lambda_{\omega}, D'_h].$$

(2) If  $(V, \nabla)$  is a flat bundle over  $X$  with a hermitian metric  $h$ , then

$$(D_{V,h}^c)^* = -\sqrt{-1}[\Lambda_\omega, \nabla], \quad (\nabla)^* = \sqrt{-1}[\Lambda_\omega, D_{V,h}^c],$$

where  $D_{V,h}^c = D_{V,h}'' - D_{V,h}'$  for  $D_{V,h}'' = \bar{\partial}_{V,h} + \theta_{V,h}$  and  $D_{V,h}' = \partial_{V,h} + \theta_{V,h}^\dagger$ , in particular,

$$(\delta'_{V,h})^* = \sqrt{-1}[\Lambda_\omega, \nabla^{0,1}], \quad (\delta''_{V,h})^* = -\sqrt{-1}[\Lambda_\omega, \nabla^{1,0}].$$

Returning back to the case of a flat bundle  $(V, \nabla)$  with a hermitian metric  $h$  and the  $\rho$ -equivariant map  $h_\rho : \pi_1(X) \rightarrow \mathrm{GL}(r, \mathbb{C})/U(r)$ , the reason for  $h$  being a harmonic metric is due to the following fact:

**Lemma 1.1.13** ([Sim92]).  *$h$  is a harmonic metric if and only if  $h_\rho$  is a harmonic map.*

*Proof.* It suffices to show that  $\Lambda_\omega G_h = 0$  is the Euler–Lagrange equation for the energy functional  $e(h_\rho) = 1/2 \int_X |dh_\rho|^2 d\mathrm{Vol}_X$ . In fact,  $dh_\rho = \theta_{V,h} + \theta_{V,h}^\dagger$ , so the Euler–Lagrange equation for  $e(h_\rho)$  is  $(\delta'_{V,h})^*(\theta_{V,h}) + (\delta''_{V,h})^*(\theta_{V,h}^\dagger) = 0$  [Sim92]. By Kähler identities for flat bundles (cf. Lemma 1.1.12), this is equivalent to

$$\sqrt{-1}\Lambda_\omega(\nabla^{0,1}(\theta_{V,h}) - \nabla^{1,0}(\theta_{V,h}^\dagger)) = 0,$$

hence equivalent to

$$\sqrt{-1}\Lambda_\omega(\nabla^{1,0}\delta''_{V,h} + \delta''_{V,h}\nabla^{1,0} - \nabla^{0,1}\delta'_{V,h} - \delta'_{V,h}\nabla^{0,1}) = 0.$$

Since

$$\begin{aligned} G_h &= (\bar{\partial}_{V,h} + \theta_{V,h})^2 \\ &= \frac{1}{4}(\nabla^{0,1}\delta''_{V,h} + \delta''_{V,h}\nabla^{0,1} - \nabla^{1,0}\delta'_{V,h} - \delta'_{V,h}\nabla^{1,0}) + \frac{1}{4}(\nabla^{1,0}\delta''_{V,h} + \delta''_{V,h}\nabla^{1,0} - \nabla^{0,1}\delta'_{V,h} - \delta'_{V,h}\nabla^{0,1}), \end{aligned}$$

this gives  $\Lambda_\omega G_h = \frac{1}{4}\Lambda_\omega(\nabla^{1,0}\delta''_{V,h} + \delta''_{V,h}\nabla^{1,0} - \nabla^{0,1}\delta'_{V,h} - \delta'_{V,h}\nabla^{0,1})$ .  $\square$

Similarly, for  $(X, \omega)$  compact Kähler,  $h$  is a pluri-harmonic metric if and only if  $h_\rho$  is a pluri-harmonic map, the Siu–Sampson theorem for  $\rho$ -equivariant map (Remark 1.1.11) implies that  $h_\rho$  is harmonic if and only if it is pluri-harmonic [ABC<sup>+</sup>96, Cor88]. Here we give a proof of this in the viewpoint of flat bundles.

**Lemma 1.1.14** ([Sim92, Lemma 1.1]). *Let  $h$  be a harmonic metric on the flat bundle  $(V, h)$  over a compact Kähler manifold  $(X, \omega)$ , then  $h$  is pluri-harmonic.*

*Proof.* It suffices to show that  $\Lambda_\omega G_h$  implies  $G_h = 0$ .

From  $D_{V,h}^c = D_{V,h}'' - D_{V,h}'$  and  $\nabla = D_{V,h}'' + D_{V,h}'$  we have

$$G_h = (D_{V,h}'')^2 = \frac{1}{4}(\nabla D_{V,h}^c + D_{V,h}^c \nabla).$$

From conclusion (1.4) and the fact  $D_{V,h}^c = \delta''_{V,h} - \delta'_{V,h}$ , we have  $(D_{V,h}^c)^2 = 0$ . Since  $G_h$  is an  $\mathrm{End}(V)$ -valued 2-form, we have the Bianchi identities

$$\nabla G_h = 0 = D_{V,h}^c G_h = 0.$$

On the other hand,

$$\begin{aligned} D_{V,h}^c &= D''_{V,h} - D'_{V,h} = \bar{\partial}_{V,h} - \partial_{V,h} + \theta_{V,h} - \theta_{V,h}^\dagger \\ &= \nabla^{0,1} - \theta_{V,h}^\dagger - (\nabla^{1,0} - \theta_{V,h}) + \theta_{V,h} - \theta_{V,h}^\dagger \\ &= \nabla^{0,1} - \nabla^{1,0} + 2(\theta_{V,h} - \theta_{V,h}^\dagger), \end{aligned}$$

again by (1.4) we have

$$\nabla \cdot (\nabla^{0,1} - \nabla^{1,0}) + (\nabla^{0,1} - \nabla^{1,0}) \cdot \nabla = 0.$$

Therefore,

$$G_h = \frac{1}{2} \left( \nabla \cdot (\theta_{V,h} - \theta_{V,h}^\dagger) + (\theta_{V,h} - \theta_{V,h}^\dagger) \cdot \nabla \right) = \frac{1}{2} \nabla (\theta_{V,h} - \theta_{V,h}^\dagger).$$

Now from the Kähler identities for flat bundles (cf. Lemma 1.1.12),  $\Lambda_\omega G_h = 0$  and  $D_{V,h}^c G_h = 0$  imply  $\nabla^* G_h = 0$ . As a result, we have

$$\|G_h\|_{L^2} = \frac{1}{2} \int_X \langle \nabla (\theta_{V,h} - \theta_{V,h}^\dagger), G_h \rangle = \frac{1}{2} \int_X \langle \theta_{V,h} - \theta_{V,h}^\dagger, \nabla^* G_h \rangle = 0.$$

□

### Corlette–Simpson correspondence

So over compact Kähler manifolds, unlike the case of Higgs bundles, there is no difference for a metric on a flat bundle being harmonic or being pluri-harmonic. So the equation  $G_h = 0$  for a flat bundle  $(V, \nabla)$  being a harmonic bundle is over determined, the equation  $\Lambda_\omega G_h = 0$  is enough. From Theorem 1.1.4 and Proposition 1.1.5 we can see that the equation  $F_{(\bar{\partial}_E, \theta, h)} = 0$  for a Higgs bundle  $(E, \bar{\partial}_E, \theta)$  being a harmonic bundle is also over determined, the conditions  $\Lambda_\omega F_{(\bar{\partial}_E, \theta, h)} = c \cdot \text{Id}_E$  and  $ch_1(E) \cdot [\omega]^{n-1} = ch_2(E) \cdot [\omega]^{n-2} = 0$  are enough. Therefore, this produces a correspondence between flat bundles and Higgs bundles via harmonic bundles (Corlette–Simpson correspondence, see Corollary 1.1.16).

Reviewing above results, let  $(V, \nabla)$  be a flat bundle of rank  $r$  over  $X$ , and let  $\rho : \pi_1(X) \rightarrow \text{GL}(r, \mathbb{C})$  be the associated monodromy representation. Given a hermitian metric  $h$  on  $V$ , let  $h_\rho : \tilde{X} \rightarrow \text{GL}(r, \mathbb{C})/U(r)$  be the corresponding  $\rho$ -equivariant map, then we have the following diagram:

$$\begin{array}{ccc} h \text{ pluri-harmonic metric} & \longleftrightarrow & h_\rho \text{ pluri-harmonic map} \\ \downarrow \uparrow & & \downarrow \uparrow \\ X \text{ Kähler} & \begin{array}{c} \uparrow \\ X \text{ compact Kähler} \\ \downarrow \end{array} & X \text{ Kähler} & \begin{array}{c} \uparrow \\ X \text{ compact Kähler} \\ \downarrow \end{array} \\ h \text{ harmonic metric} & \longleftrightarrow & h_\rho \text{ harmonic map} \end{array}$$

Therefore, the Donaldson–Corlette theorem ( Theorem 1.1.9) can be rewrote as the existence of harmonic (pluri-harmonic) metrics on flat bundles, which is a type of Kobayashi–Hitchin correspondence for flat bundles.

**Theorem 1.1.15** (Donaldson [Don87], Corlette [Cor88]). *Let  $(X, L, \omega)$  be as above, then a flat bundle  $(V, \nabla)$  over  $X$  admits a pluri-harmonic metric if and only if it is semisimple, and moreover, such a pluri-harmonic metric is unique up to scalar multiplicities.*

If the hermitian metric  $h$  on the Higgs bundle  $(E, \bar{\partial}_E, \theta)$  is a pluri-harmonic metric, then it is also a pluri-harmonic metric for the associated flat bundle  $(E, \mathbb{D}^1 = \partial_{E,h} + \bar{\partial}_E + \theta + \theta_h^\dagger)$ . And conversely, if the hermitian metric  $h$  on the flat bundle  $(V, \nabla)$  is a pluri-harmonic metric, then it is also a pluri-harmonic metric for the associated Higgs bundle  $(V, \bar{\partial}_{V,h}, \theta_{V,h})$ . Thus we have the following correspondence of flat bundles and Higgs bundles:

**Corollary 1.1.16 (Corlette–Simpson Correspondence, [Sim92]).** *Let  $(X, L, \omega)$  be as above, then there is an equivalence between the category of semisimple flat bundles  $(V, \nabla)$  and that of  $\mu_L$ -polystable Higgs bundles  $(E, \bar{\partial}_E, \theta)$  with  $ch_1(E) \cdot [\omega]^{n-1} = ch_2(E) \cdot [\omega]^{n-2} = 0$ , both of them are equivalent to the category of harmonic bundles.*

**Remark 1.1.17.** (1) By extension arguments ([Sim92, Corollary 3.4, Theorem 2]), that is, any flat bundle over  $X$  is an extension of semisimple flat bundles, and any semistable Higgs bundle with vanishing Chern classes is an extension of stable Higgs bundles with vanishing Chern classes. Above equivalence can be extended to an equivalence of the following two categories ([Sim92, Corollary 3.10]):

- The category of flat bundles of rank  $r$  over  $X$ ;
- The category of semistable Higgs bundles of rank  $r$  over  $X$  with vanishing Chern classes.

(2) There is a special kind of flat bundles, and a special kind of Higgs bundles, called complex variations of Hodge structure ( $\mathbb{C}$ -VHS for simplicity, and polarized  $\mathbb{C}$ -VHS in the sense of Griffiths), and systems of Hodge bundles, respectively. They correspond to the fixed points of  $\mathbb{C}^*$ -action on the corresponding moduli spaces (cf. [Sim92, Lemma 4.1, Corollary 4.2]). In particular, we have the following one-to-one correspondence:

- Flat bundles which are  $\mathbb{C}$ -VHS;
- Higgs bundles having the structures of systems of Hodge bundles.

In [Sim94a, Sim94b], by the technique of geometric invariant theory, Simpson constructed and showed the existence of the following three coarse moduli spaces:

- (1) The de Rham moduli space  $\mathcal{M}_{\text{dR}}(X, r)$ , that is, the coarse moduli space of rank  $r$  vector bundles over  $X$  with integrable connections;
- (2) The Dolbeault moduli space  $\mathcal{M}_{\text{Dol}}(X, r)$ , that is, the coarse moduli space of rank  $r$  semistable Higgs bundles over  $X$  with vanishing Chern classes;
- (3) The Betti moduli space  $\mathcal{M}_{\text{B}}(X, r)$ , that is, the coarse moduli space of representations  $\rho : \pi_1(X) \rightarrow \text{GL}(r, \mathbb{C})$ .

Moreover, he showed:

**Theorem 1.1.18 (NAHC of Moduli Spaces Version, Part I, [Sim94a, Sim94b]).**

- (1) *The Riemann–Hilbert correspondence gives the complex analytic isomorphism of  $\mathcal{M}_{\text{dR}}(X, r)$  and  $\mathcal{M}_{\text{B}}(X, r)$  as complex analytic varieties:*

$$\mathcal{M}_{\text{dR}}(X, r) \stackrel{\text{an.}}{\cong} \mathcal{M}_{\text{B}}(X, r).$$

- (2) *The Corlette–Simpson correspondence gives the homeomorphism of  $\mathcal{M}_{\text{dR}}(X, r)$  and  $\mathcal{M}_{\text{Dol}}(X, r)$  as topological spaces:*

$$\mathcal{M}_{\text{Dol}}(X, r) \stackrel{\text{homeo.}}{\cong} \mathcal{M}_{\text{dR}}(X, r).$$

In fact, in [Sim94a], Simpson provides two methods on constructing  $\mathcal{M}_{\text{Dol}}(X, r)$ , one is thinking a Higgs bundles as a coherent sheaf on the contangent bundle  $T^*X$ , the other is thinking a Higgs bundle as a module over  $\Lambda^{\text{Dol}} := \text{Sym}^\bullet(TX)$  and then using the geometric invariant theory of  $\Lambda$ -modules, the constructions are beautiful, but also very technical, we will give a brief introduction to the construction in next chapter.

## 1.2 Flat $\lambda$ -Connections and Mochizuki Correspondence

The notion of flat  $\lambda$ -connection as an interpolation of usual flat connection and Higgs field was suggested by Deligne [Del89], illustrated by Simpson in [Sim95] and further studied in [Sim08, Sim10].

We have seen in §1.1 that the two different ways to define a harmonic bundle are in fact equivalent, in this section, we will give another equivalent way to define a harmonic bundle by the language of  $\lambda$ -flat bundles.

Let  $(X, L, \omega)$  be as in the first section, let  $\mathcal{E} = (E, \bar{\partial}_E)$  be a holomorphic vector bundle over  $X$ , with  $E$  the underlying complex vector bundle. The following preliminaries of  $\lambda$ -flat bundles mainly come from [Sim95] and [Moc06] (see also [HH19, Hua20]).

**Definition 1.2.1.** Assume  $\lambda \in \mathbb{C}$ .

- (1) A *holomorphic  $\lambda$ -connection* on  $\mathcal{E}$  is a  $\mathbb{C}$ -linear map  $D^\lambda : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^1$  that satisfies the following  $\lambda$ -twisted Leibniz rule:

$$D^\lambda(fs) = fD^\lambda s + \lambda s \otimes df,$$

where  $f$  and  $s$  are holomorphic sections of  $\mathcal{O}_X$  and  $E$ , respectively. If  $(D^\lambda + \bar{\partial}_E) \circ (D^\lambda + \bar{\partial}_E) = 0$  under the natural extension  $D^\lambda : \mathcal{E} \otimes \Omega_X^p \rightarrow \mathcal{E} \otimes \Omega_X^{p+1}$  for any integer  $p \geq 0$ , then we call  $D^\lambda$  a (holomorphic) *flat  $\lambda$ -connection* and  $\mathcal{E}$  a (holomorphic)  *$\lambda$ -flat bundle*.

- (2) A  *$C^\infty$   $\lambda$ -connection* on  $E$  is a  $\mathbb{C}$ -linear map  $\mathbb{D}^\lambda : \mathcal{A}^0(E) \rightarrow \mathcal{A}^1(E)$  that satisfies the following  $\lambda$ -twisted Leibniz rule:

$$\mathbb{D}^\lambda(fs) = f\mathbb{D}^\lambda s + \lambda s \otimes \partial f + s \otimes \bar{\partial} f,$$

where  $f \in C^\infty(X, \mathbb{C})$  and  $s \in \mathcal{A}^0(E)$ . If  $\mathbb{D}^\lambda \circ \mathbb{D}^\lambda = 0$  under the natural extension  $\mathbb{D}^\lambda : \mathcal{A}^p(E) \rightarrow \mathcal{A}^{p+1}(E)$  for any integer  $p \geq 0$ , then we call  $\mathbb{D}^\lambda$  a ( $C^\infty$ ) *flat  $\lambda$ -connection*, and  $E$  a ( $C^\infty$ )  *$\lambda$ -flat bundle*.

Clearly,  $\lambda = 1$  and  $0$  correspond to the usual flat connection and Higgs field, respectively. If we work on projective curve, then every  $\lambda$ -connection is automatically flat. Giving a holomorphic flat  $\lambda$ -connection  $D^\lambda$  on  $\mathcal{E}$  is equivalent to giving a  $C^\infty$  flat  $\lambda$ -connection  $\mathbb{D}^\lambda$  on  $E$ . From now on, we will denote a  $\lambda$ -flat bundle as  $(E, \bar{\partial}_E, D^\lambda, \lambda)$  (sometimes,  $(E, d''_E, D^\lambda, \lambda)$ , or  $(E, d''_E, d'_E, \lambda)$ ) in holomorphic category and  $(E, \mathbb{D}^\lambda, \lambda)$  in  $C^\infty$  category, or simply, just as  $(E, \bar{\partial}_E, D^\lambda)$  (sometimes,  $(E, d''_E, D^\lambda)$ , or  $(E, d''_E, d'_E)$ ) and  $(E, \mathbb{D}^\lambda)$  when there is no ambiguity.

Let  $(E, \mathbb{D}^\lambda)$  be a  $\lambda$ -flat bundle, and  $h$  be a hermitian metric on  $E$ , then  $h$  induces a unique decomposition of  $\mathbb{D}^\lambda$ :

$$\mathbb{D}^\lambda = \lambda \partial_h + \theta_h + \bar{\partial}_h + \lambda \theta_h^\dagger$$

such that  $\nabla_h := \partial_h + \bar{\partial}_h$  is a  $h$ -unitary connection, and  $\Phi_h := \theta_h + \theta_h^\dagger \in \mathcal{A}^1(\text{End}(E))$  is a self-adjoint operator.

In fact, when  $\lambda = 0$ , this decomposition is the decomposition into different types which defines the Higgs bundle structure, that is,  $\mathbb{D}^0 = \bar{\partial}_E + \theta$  for  $\bar{\partial}_E$  defines a holomorphic structure on  $E$  and  $\theta$

defines a Higgs field. When  $\lambda \neq 0$ , we decompose  $\mathbb{D}^\lambda$  into its (1,0)-part  $d'_E$  and (0,1)-part  $d''_E$  that defines a holomorphic structure on  $E$ . From  $h$  and  $d'_E$ , we have a (0,1)-operator  $\delta''_h$  determined by the condition

$$\lambda \partial h(u, v) = h(d'_E u, v) + h(u, \delta''_h v).$$

Similarly,  $h$  and  $d''_E$  provide a (1,0)-operator  $\delta'_h$  via the condition

$$\bar{\partial} h(u, v) = h(d''_E u, v) + h(u, \delta'_h v).$$

One easily checks that

$$\begin{cases} \delta'_h(fs) = f\delta'_h(s) + \partial(f) \otimes s \\ \delta''_h(fs) = f\delta''_h(s) + \bar{\lambda}\bar{\partial}(f) \otimes s \end{cases}$$

for any  $f \in C^\infty(X, \mathbb{C})$  and  $s \in \mathcal{A}^0(E)$ . We introduce the following four operators

$$\begin{aligned} \partial_h &:= \frac{1}{1+|\lambda|^2} \left( \bar{\lambda} d'_E + \delta'_h \right), & \bar{\partial}_h &:= \frac{1}{1+|\lambda|^2} \left( d''_E + \lambda \delta''_h \right), \\ \theta_h &:= \frac{1}{1+|\lambda|^2} \left( d'_E - \lambda \delta'_h \right), & \theta_h^\dagger &:= \frac{1}{1+|\lambda|^2} \left( \bar{\lambda} d''_E - \delta''_h \right). \end{aligned}$$

They satisfy that

$$\begin{aligned} d'_E &= \lambda \partial_h + \theta_h, & d''_E &= \bar{\partial}_h + \lambda \theta_h^\dagger, \\ \delta'_h &= \partial_h - \bar{\lambda} \theta_h, & \delta''_h &= \bar{\lambda} \bar{\partial}_h - \theta_h^\dagger, \end{aligned}$$

By direct calculation we can see that  $\partial_h$  and  $\bar{\partial}_h$  obey the usual Leibniz rule:

$$\begin{cases} \partial_h(fs) = f\partial_h(s) + \partial(f) \otimes s \\ \bar{\partial}_h(fs) = f\bar{\partial}_h(s) + \bar{\partial}(f) \otimes s \end{cases}$$

for any  $f \in C^\infty(X, \mathbb{C})$  and  $s \in \mathcal{A}^0(E)$ , so  $\nabla_h := \partial_h + \bar{\partial}_h$  is an 1-connection (in fact this is a  $h$ -unitary connection, we will show this later). By definition,  $\theta_h \in C^\infty(X, \text{End}(E) \otimes \Omega_X^{1,0})$  and  $\theta_h^\dagger \in C^\infty(X, \text{End}(E) \otimes \Omega_X^{0,1})$ , and they are both  $\mathcal{O}_X$ -linear. Moreover,  $\theta_h^\dagger$  is the adjoint of  $\theta_h$  in the sense that  $h(\theta_h(u), v) = h(u, \theta_h^\dagger(v))$ .

**Proposition 1.2.2.**  $\nabla_h$  is a  $h$ -unitary connection, that is,

$$dh(u, v) = h(\nabla_h(u), v) + h(u, \nabla_h(v))$$

for any  $u, v \in \mathcal{A}^0(E)$ .

*Proof.* It suffices to show  $\partial h(u, v) = h(\partial_h(u), v) + h(u, \bar{\partial}_h(v))$ . In fact, the left hand side equals

$$\begin{aligned} h(\partial_h(u), v) + h(u, \bar{\partial}_h(v)) &= (1+|\lambda|^2)^{-1} (h(d''_E(u), v) + \lambda h(\delta''_h(u), v) + \lambda h(u, d'_E(v)) + h(u, \delta'_h(v))) \\ &= (1+|\lambda|^2)^{-1} (\bar{\partial} h(u, v) + \lambda h(u, d'_E(v)) + \lambda h(\delta''_h(u), v)), \end{aligned}$$

while  $h(u, d'_E(v)) = \overline{h(d'_E(v), u)} = \overline{\lambda \partial h(v, u) - h(v, \delta''_h(u))} = \bar{\lambda} \bar{\partial} h(u, v) - h(\delta''_h(u), v)$ , combining all these gives the desired result.  $\square$

Note in particular, when  $\lambda = 0$ , then  $\mathbb{D}^0 = \bar{\partial}_E + \theta$ ,  $\partial_h = \delta'_h = \partial_{E,h}$  is the (1,0)-type operator determined by  $\bar{\partial}_E$  and  $h$ .  $\bar{\partial}_h = \bar{\partial}_E$ ,  $\theta_h = \theta$ , and  $\theta_h^\dagger = -\delta''_h = \theta_h^\dagger$  is the adjoint operator of  $\theta$  with respect to  $h$ .

Now we assume  $\lambda \neq 0$ , introduce the following two operators

$$D''_{h,\lambda} = \bar{\lambda}(\bar{\partial}_h + \theta_h), \quad D'_{h,\lambda} = \partial_h + \theta_h^\dagger,$$

then

$$\begin{cases} D^c_{h,\lambda} := D''_{h,\lambda} - D'_{h,\lambda} = \delta''_h - \delta'_h \\ \mathbb{D}^\lambda = (\bar{\lambda})^{-1} D''_{h,\lambda} + \lambda D'_{h,\lambda}. \end{cases} \quad (1.8)$$

The  $\text{End}(E)$ -valued 2-form  $G_{h,\lambda} := (D''_{h,\lambda})^2 = (\bar{\lambda})^2(\bar{\partial}_h + \theta_h)^2$  is called the *pseudo-curvature*.

**Definition 1.2.3.**  $h$  is called a *harmonic metric* on the  $\lambda$ -flat bundle  $(E, \mathbb{D}^\lambda)$  if the pseudo-curvature  $G_{h,\lambda}$  satisfies that  $\Lambda_\omega G_{h,\lambda} = 0$ . In particular,  $G_{h,\lambda} = 0$ , i.e. if  $(\bar{\partial}_h + \theta_h)^2 = 0$ , then  $h$  is called a *pluri-harmonic metric*, and in this case,  $(E, \mathbb{D}^\lambda, h)$  is called a *harmonic bundle*.

From the definition we can see that a  $\lambda$ -flat bundle  $(E, \mathbb{D}^\lambda)$  together with a pluri-harmonic metric  $h$  gives rise to a Higgs bundle structure, that is,  $(E, \bar{\partial}_h, \theta_h)$  is a Higgs bundle.

For  $\lambda$ -flat bundles over compact Kähler manifold, we also have Kähler identities:

**Lemma 1.2.4 (Kähler Identities, [Moc06]).** *Let  $(X, \omega)$  be a compact Kähler manifold and  $(E, \mathbb{D}^\lambda)$  be a  $\lambda$ -flat bundle with a hermitian metric  $h$ , then*

$$(D^c_{h,\lambda})^* = -\sqrt{-1}[\Lambda_\omega, \mathbb{D}^\lambda], \quad (\mathbb{D}^\lambda)^* = \sqrt{-1}[\Lambda_\omega, D^c_{h,\lambda}].$$

The following assertion shows that for any  $\lambda$ -flat bundle  $(E, \mathbb{D}^\lambda)$  over a compact Kähler manifold together with a hermitian metric  $h$ , the equation  $G_{h,\lambda} = 0$  for  $h$  to be pluri-harmonic is over determined, as we have seen in [Lemma 1.1.14](#) for flat bundles.

**Lemma 1.2.5.** *For  $\lambda \neq 0$ , let  $h$  be a harmonic metric on the  $\lambda$ -flat bundle  $(E, \mathbb{D}^\lambda)$  over a compact Kähler manifold  $(X, \omega)$ , then  $h$  is pluri-harmonic.*

*Proof.* It suffices to show that  $\Lambda_\omega G_{h,\lambda} = 0$  implies  $G_{h,\lambda} = 0$ . By definition, the flatness of  $\mathbb{D}^\lambda$  is equivalent to  $(d'_E)^2 = (d''_E)^2 = d'_E d''_E + d''_E d'_E = 0$ , which implies  $(\delta'_h)^2 = (\delta''_h)^2 = \delta'_h \delta''_h + \delta''_h \delta'_h = 0$ . Therefore,  $(D^c_{h,\lambda})^2 = 0$ .

From (1.8) we have

$$G_{h,\lambda} = (D''_{h,\lambda})^2 = \frac{\bar{\lambda}|\lambda|^2}{(1+|\lambda|^2)^2} \left( \mathbb{D}^\lambda D^c_{h,\lambda} + D^c_{h,\lambda} \mathbb{D}^\lambda \right). \quad (1.9)$$

Note that  $G_{h,\lambda}$  is an  $\text{End}(E)$ -valued 2-form, so it satisfies the Bianchi identities

$$\mathbb{D}^\lambda G_{h,\lambda} = 0, \quad D^c_{h,\lambda} G_{h,\lambda} = 0.$$

On the other hand,

$$\begin{aligned} D_{h,\lambda}^c &= D_{h,\lambda}'' - D_{h,\lambda}' = \bar{\lambda}\bar{\partial}_h - \partial_h + \bar{\lambda}\theta_h - \theta_h^\dagger \\ &= \bar{\lambda}(d_E'' - \lambda\theta_h^\dagger) - \lambda^{-1}(d_E' - \theta_h) + \bar{\lambda}\theta_h - \theta_h^\dagger \\ &= (\bar{\lambda}d_E'' - \lambda^{-1}d_E') + (1 + |\lambda|^2)(\lambda^{-1}\theta_h - \theta_h^\dagger), \end{aligned}$$

combing with (1.9) gives

$$G_{h,\lambda} = \frac{\bar{\lambda}|\lambda|^2}{1 + |\lambda|^2} \mathbb{D}^\lambda(\lambda^{-1}\theta_h - \theta_h^\dagger).$$

Now use the Kähler identities for  $\lambda$ -flat bundles,  $\Lambda_\omega G_{h,\lambda} = 0$  and  $D_{h,\lambda}^c G_{h,\lambda} = 0$ , we have

$$(\mathbb{D}^\lambda)^* G_{h,\lambda} = \sqrt{-1}[\Lambda_\omega, D_{h,\lambda}^c]G_{h,\lambda} = 0.$$

Therefore, we have the desired result

$$\begin{aligned} \|G_{h,\lambda}\|_{L^2} &= \int_X \langle G_{h,\lambda}, G_{h,\lambda} \rangle d\text{Vol}_X = \frac{\bar{\lambda}|\lambda|^2}{1 + |\lambda|^2} \int_X \langle \lambda^{-1}\theta_h - \theta_h^\dagger, (\mathbb{D}^\lambda)^* G_{h,\lambda} \rangle, \\ &= 0. \end{aligned}$$

□

**Remark 1.2.6.** Our proof here is similar to the proof of Lemma 1.1.14 given by Simpson [Sim92]. But we should note that, this property does not hold when  $\lambda = 0$ , as we have seen in the last section, the metric  $h$  for a Higgs bundle  $(E, \bar{\partial}_E, \theta)$  satisfies the Hermitian–Einstein equation  $\Lambda_\omega F_{(\bar{\partial}_E, \theta, h)} = c \cdot \text{Id}_E$  (or even  $\Lambda_\omega F_{(\bar{\partial}_E, \theta, h)} = 0$ ) is not enough to make  $h$  to be pluri-harmonic. This reflects the big difference between Higgs bundles and flat bundles, but much similarity between flat bundles and  $\lambda$ -flat bundles ( $\lambda \neq 0$ ), we can also see this fact from the viewpoint of moduli spaces later.

**Proposition 1.2.7** ([Moc06, HH19]). *For  $\lambda \neq 0$ , let  $(E, \mathbb{D}^\lambda)$  be a  $\lambda$ -flat bundle over a compact Kähler manifold  $(X, \omega)$  together with a hermitian metric  $h$ . Then the following statements are equivalent:*

- (1)  $h$  is harmonic, i.e,  $\Lambda_\omega G_{h,\lambda} = 0$ ;
- (2)  $h$  is pluri-harmonic, i.e,  $G_{h,\lambda} = (\bar{\lambda})^2(\bar{\partial}_h + \theta_h)^2 = 0$ ;
- (3)  $\theta_h^2 = 0 = \bar{\partial}_h(\theta)$ ;
- (4)  $(\theta_h^\dagger)^2 = 0 = \partial_h(\theta_h^\dagger)$ .

*Proof.* Denote by  $(\cdot)_h^\dagger$  the adjoint with respect to the metric  $h$ , then by direct calculation we have the following adjoint formulas (see also [Moc06]):

$$(\theta_h^\dagger)^2 = -(\theta_h^2)_h^\dagger, \quad \partial_h^2 = -(\bar{\partial}_h^2)_h^\dagger, \quad (\partial_h(\theta_h^\dagger))_h^\dagger = \bar{\partial}_h(\theta_h)$$

and the relation  $\bar{\lambda}\bar{\partial}_h^2 + \lambda(\theta_h^\dagger)^2 = 0$ . Therefore, when  $\lambda \neq 0$ ,  $(\bar{\partial}_h + \theta_h)^2 = 0$  if and only if  $\theta_h^2 = 0 = \bar{\partial}_h(\theta_h)$ . □

The problem on the existence of pluri-harmonic metrics on given  $\lambda$ -flat bundles was studied and solved by Mochizuki in [Moc06]. In this paper, by using Donaldson functional and analytic method, he built the quasi-projective version of Kobayashi–Hitchin correspondence for  $\lambda$ -flat bundles. Moreover, he obtained the equivalence between the category of  $\mu_L$ -polystable  $\lambda_1$ -flat bundles with vanishing Chern classes and that of  $\mu_L$ -polystable  $\lambda_2$ -flat bundles with vanishing Chern classes for arbitrary  $\lambda_1, \lambda_2 \in \mathbb{C}$ , which in particular, generalizes the Corlette–Simpson correspondence to non-compact case. But in this section, we just need his results for compact case, and we will call it *Mochizuki correspondence*.

**Definition 1.2.8.** A  $\lambda$ -flat bundle  $(E, \mathbb{D}^\lambda)$  over  $X$  is called  $\mu_L$ -stable (resp.  $\mu_L$ -semistable) if for any  $\lambda$ -flat subbundle  $(V, \mathbb{D}^\lambda|_V)$  of  $0 < \text{rk}(V) < \text{rk}(E)$ , we have the following inequality

$$\mu_L(V) < (\text{resp. } \leq) \mu_L(E).$$

It is  $\mu_L$ -polystable if it decomposes as a direct sum of  $\mu_L$ -stable  $\lambda$ -flat bundles with the same slope.

**Remark 1.2.9.** In our case, the base manifold  $X$  is compact, if  $\lambda \neq 0$ , then for any  $\lambda$ -flat bundle  $(E, \mathbb{D}^\lambda)$  over  $X$ ,  $\mathbb{D}^1 := \lambda^{-1}d'_E + d''_E$  is a usual connection, which is flat. This means that  $\text{deg}(E) = 0$ , and all Chern classes are trivial. Therefore, any  $\lambda$ -flat bundle over compact manifold is automatically  $\mu_L$ -semistable, and  $\mu_L$ -stable just means that it has no non-zero  $\lambda$ -flat subbundle.

A *morphism* between two  $\lambda$ -flat bundles  $(E, \mathbb{D}^\lambda_E)$  and  $(F, \mathbb{D}^\lambda_F)$  is a morphism  $f : E \rightarrow F$  of vector bundles such that the following diagram commutes:

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \mathbb{D}^\lambda_E \downarrow & & \downarrow \mathbb{D}^\lambda_F \\ E \otimes \Omega_X^1 & \xrightarrow{f \otimes \text{id}} & F \otimes \Omega_X^1 \end{array}$$

Denoted by  $\text{Hom}((E, \mathbb{D}^\lambda_E), (F, \mathbb{D}^\lambda_F))$  the set of all such morphisms, and in particular,  $\text{End}(E, \mathbb{D}^\lambda)$  is the set of all endomorphisms of  $(E, \mathbb{D}^\lambda)$ .

**Lemma 1.2.10.** *If  $(E, \mathbb{D}^\lambda)$  is a  $\mu_L$ -stable  $\lambda$ -flat bundle, then  $\text{End}(E, \mathbb{D}^\lambda) = \mathbb{C} \cdot \text{Id}$ .*

*Proof.* Let  $f \in \text{End}(E, \mathbb{D}^\lambda)$  be a non-zero endomorphism, we first prove it is an isomorphism. Notice that we have the following short exact sequence:

$$0 \longrightarrow \text{Ker}(f) \longrightarrow E \longrightarrow \text{Im}(f) \longrightarrow 0. \quad (1.10)$$

**Step 1.**  $f$  is injective:

By (1.10),  $(\text{Ker}(f), \mathbb{D}^\lambda|_{\text{Ker}(f)})$  is a  $\lambda$ -flat subbundle of  $(E, \mathbb{D}^\lambda)$  with  $0 \leq \text{rk}(\text{Ker}(f)) < \text{rk}(E)$ , and  $(\text{Im}(f), \mathbb{D}^\lambda|_{\text{Im}(f)})$  is a  $\lambda$ -flat subbundle of  $(E, \mathbb{D}^\lambda)$  with  $0 < \text{rk}(\text{Im}(f)) \leq \text{rk}(E)$ . The stability of  $(E, \mathbb{D}^\lambda)$  implies  $\text{Ker}(f) = 0$ .

**Step 2.**  $f$  is surjective:

Since  $\text{Ker}(f) = 0$ , from (1.10) we have  $\text{Im}(f) = E$ .

**Step 3.** There exists some constant  $\eta \in \mathbb{C}$  such that  $f = \eta \cdot \text{Id}$ :

Choose a point  $x \in X$ , then the induced map on fibers  $f_x : E_x \rightarrow E_x$  is an isomorphism, let  $\eta$  be an eigenvalue. Consider the endomorphism  $f - \eta \text{Id}$ , clearly it is not an isomorphism, hence  $f - \eta \text{Id} \equiv 0$ .  $\square$

The Kobayashi–Hitchin correspondence for  $\lambda$ -flat bundles is due to Mochizuki [Moc06]:

**Theorem 1.2.11 (Mochizuki Correspondence, compact version).** *Let  $(X, L, \omega)$  be as in §1.1 and assume  $\lambda \neq 0$ . Then a  $\lambda$ -flat bundle  $(E, \mathbb{D}^\lambda)$  over  $X$  admits a pluri-harmonic metric if and only if it is  $\mu_L$ -polystable. Moreover, such a metric is unique up to scalar multiplicities.*

In some contexts, people usually call Corlette–Simpson correspondence as non-Abelian Hodge correspondence, in this thesis, non-Abelian Hodge correspondence (“NAHC” for simplicity) is a more general version that relates  $\lambda$ -flat bundles.

**Corollary 1.2.12 (Non-Abelian Hodge Correspondence, [Moc06]).** *Let  $(X, L, \omega)$  be as in §1.1, then for arbitrary  $\lambda_1, \lambda_2 \in \mathbb{C}$ , there is an one to one correspondence between the following categories:*

- (1) *The category of  $\mu_L$ -polystable  $\lambda_1$ -flat bundles of rank  $r$  over  $X$  with vanishing first and second Chern characteristics;*
- (2) *The category of  $\mu_L$ -polystable  $\lambda_2$ -flat bundles of rank  $r$  over  $X$  with vanishing first and second Chern characteristics.*

*Moreover, this equivalence preserves tensor products, direct sums and duals.*

*Proof.* Denote by  $\mathcal{C}_\lambda^{\text{poly}}$  the category of  $\mu_L$ -polystable  $\lambda$ -flat bundles of rank  $r$  with vanishing first and second Chern characteristics (when  $\lambda \neq 0$ , the vanishing of Chern characteristics hold automatically!). For any  $(E, \mathbb{D}^{\lambda_1}) \in \mathcal{C}_{\lambda_1}^{\text{poly}}$  with the pluri-harmonic metric  $h$ , then  $h$  induces the decomposition  $\mathbb{D}^{\lambda_1} = \lambda_1 \partial_h + \theta_h + \bar{\partial}_h + \lambda_1 \theta_h^\dagger$ . Note that in the case  $\lambda_1 = 0$ , this decomposition also exists, that is,  $\mathbb{D}^0 = \bar{\partial}_E + \theta$ . Where  $\bar{\partial}_E$  is the  $(0,1)$ -part of  $\mathbb{D}^0$ , which defines a holomorphic structure on  $E$ .  $\theta$  is an  $\mathcal{O}_X$ -linear holomorphic 1-form, which defines a Higgs field.  $\partial_h = \partial_{E,h}$ ,  $\bar{\partial}_h = \bar{\partial}_E$ ,  $\theta_h = \theta$  and  $\theta_h^\dagger$  is the adjoint of  $\theta$  with respect to the metric  $h$ . This decomposition gives rise to a flat  $\lambda_2$ -connection on  $E$ :  $\mathbb{D}^{\lambda_2} := \lambda_2 \partial_h + \theta_h + \bar{\partial}_h + \lambda_2 \theta_h^\dagger$ , with the new holomorphic structure  $\bar{\partial}_h + \lambda_2 \theta_h^\dagger$ . Clearly,  $h$  is also the pluri-harmonic metric for the  $\lambda_2$ -flat bundle  $(E, \mathbb{D}^{\lambda_2})$ . Therefore, the equivalence functor is the following:

$$\begin{aligned} \Xi_{\lambda_1, \lambda_2} : \mathcal{C}_{\lambda_1}^{\text{poly}} &\longrightarrow \mathcal{C}_{\lambda_2}^{\text{poly}}, \\ (E, \mathbb{D}^{\lambda_1}) &\longmapsto (E, \mathbb{D}^{\lambda_2}). \end{aligned}$$

□

**Remark 1.2.13.** (1) When  $\lambda_1 = 0, \lambda_2 = 1$  or  $\lambda_1 = 1, \lambda_2 = 0$ , this is in fact the Corlette–Simpson correspondence:

$$(E, \mathbb{D}^0 = \bar{\partial}_E + \theta, h) \longmapsto (E, \mathbb{D}^1 := \partial_{E,h} + \theta + \bar{\partial}_E + \theta_h^\dagger, h)$$

and the inverse

$$(V, \mathbb{D}^1 = \nabla, h) \longmapsto (V, \mathbb{D}^0 := \bar{\partial}_{V,h} + \theta_{V,h}, h).$$

(2) When  $\{\lambda_1, \lambda_2\} = \{0, \lambda\} (\lambda \neq 0)$ , this correspondence is:

$$(E, \mathbb{D}^\lambda, h) \longmapsto (E, \mathbb{D}^0 := \bar{\partial}_h + \theta_h, h)$$

and the inverse

$$(E, \mathbb{D}^0 = \bar{\partial}_E + \theta, h) \longmapsto (E, \mathbb{D}^\lambda := \lambda \partial_{E,h} + \theta + \bar{\partial}_E + \lambda \theta_h^\dagger, h).$$

In particular, when  $\lambda = 1$ , this is exactly the Corlette–Simpson correspondence (Corollary 1.1.16).

(3) When  $\{\lambda_1, \lambda_2\} = \{1, \lambda\} (\lambda \neq 0, 1)$ , this correspondence is:

$$(E, \mathbb{D}^\lambda, h) \longmapsto (E, \mathbb{D}^1 := \partial_h + \theta_h + \bar{\partial}_h + \theta_h^\dagger, h)$$

and the inverse

$$(V, \mathbb{D}^1 = \nabla, h) \longmapsto (V, \mathbb{D}^\lambda := \lambda \partial_{V,h} + \theta_{V,h} + \bar{\partial}_{V,h} + \lambda \theta_{V,h}^\dagger, h).$$

**Remark 1.2.14.** (1) A natural direction of the study of non-Abelian Hodge correspondence is the generalization to non-compact case. In regular case (tame case), Simpson established the correspondence between tame harmonic bundles and parabolic Higgs bundles for open curves [Sim90]. The higher dimensional generalization was obtained by Biquard [Biq97] for smooth divisor case and by Mochizuki [Moc06, Moc09] for general case. In irregular case (wild case), Biquard and Boalch built the correspondence for curves [BB04]. Later Mochizuki generalized it to higher dimensional case [Moc11]. Recently, the authors in [GKPT19] built the correspondence to the context of projective varieties with Kawamata log terminal (brief as klt) singularities.

(2) Another natural generalization is considering the corresponding case for real Lie groups. In [BGPiR03, GPGiR09], the authors considered principal  $G$ -Higgs bundles for real Lie group  $G$ . They studied the Kobayashi–Hitchin correspondence for principal  $G$ -Higgs bundles, and therefore, they built the non-Abelian Hodge correspondence for such objects.

(3) The non-Abelian Hodge theory for varieties over a field of characteristic  $p$  was built by Ogus and Vologodsky in [OV07]. The  $p$ -adic version was suggested by Faltings in [Fal05] and finished by Abbes, Gros and Tsuji in [AGT16].

As a direct application of non-Abelian Hodge correspondence, we have the following correspondence unifying the usual Corlette–Simpson correspondence and Riemann–Hilbert correspondence.

**Corollary 1.2.15** (Corlette–Simpson correspondence version of Corollary 1.2.12). *Let  $(X, L, \omega)$  be as above. Then for any  $\lambda \in \mathbb{C}$ , there is an one-to-one correspondence between the following categories:*

- (1) *The category of  $\mu_L$ -polystable  $\lambda$ -flat bundles of rank  $r$  over  $X$  with vanishing first and second Chern characteristics;*
- (2) *The category of semisimple representations of the fundamental group  $\pi_1(X)$  into  $\mathrm{GL}(r, \mathbb{C})$ .*

*Moreover, this equivalence preserves tensor products, direct sums and duals.*

*Proof.* For the case of  $\lambda = 0$ , we have the usual Simpson correspondence. So we assume  $\lambda \neq 0$ . Let  $(E, \mathbb{D}^\lambda)$  be a  $\mu_L$ -polystable  $\lambda$ -flat bundle (with trivial characteristic numbers), then there is a pluri-harmonic metric  $h$  on  $E$ . Therefore, we get

$$\begin{aligned} 0 &= (\mathbb{D}^\lambda)^2 = (\lambda \partial_h + \bar{\partial}_h + \theta_h + \lambda \theta_h^\dagger)^2 \\ &= \lambda(R(h) + [\theta_h, \theta_h^\dagger] + \partial_h \theta_h + \bar{\partial}_h \theta_h^\dagger), \end{aligned}$$

where  $R(h) = (\nabla_h)^2$  is the curvature of the unitary connection  $\nabla_h$ . Hence

$$R(h) + [\theta_h, \theta_h^\dagger] = \partial_h \theta_h = \bar{\partial}_h \theta_h^\dagger = 0,$$

which implies  $(E, \bar{\partial}_h, \theta_h, h)$  is a harmonic Higgs bundle associated with a semisimple representation  $\rho : \pi_1(X) \rightarrow \mathrm{GL}(r, \mathbb{C})$  by Hitchin–Simpson correspondence.

Conversely, if we have a semisimple representation  $\rho : \pi_1(X) \rightarrow \mathrm{GL}(r, \mathbb{C})$ , then we have a Higgs bundle  $(E, \bar{\partial}_E, \theta, h)$  with the pluri-harmonic metric  $h$ . Therefore, it gives rise to a flat  $\lambda$ -connection  $\mathbb{D}^\lambda = d'_E + d''_E$  with

$$d'_E = \lambda \partial_{E,h} + \theta, \quad d''_E = \bar{\partial}_E + \lambda \theta_h^\dagger,$$

where  $\partial_{E,h}$  is an  $(1,0)$ -type operator such that  $\partial_{E,h} + \bar{\partial}_E$  is a  $h$ -unitary connection, and  $\theta_h^\dagger$  is the adjoint of  $\theta$  with respect to  $h$ . Clearly,  $h$  is also a pluri-harmonic metric for the  $\lambda$ -flat bundle  $(E, \mathbb{D}^\lambda)$ , hence it is polystable with vanishing Chern characteristics.

Since pluri-harmonic metrics preserve tensor products, direct sums and duals, the equivalence described as above also preserves them.  $\square$

### 1.3 Some Estimates and Examples

In this section, we will generalize some results for usual flat bundles to the case of  $\lambda$ -flat bundles ( $\lambda \neq 0$ ). Meanwhile, we will give some examples, this part is based on the paper [HH19].

**Theorem 1.3.1.** *Assume  $\lambda \neq 0$ . Let  $X$  be a Riemann surface with Kähler form  $\omega$ , and let  $(E, \mathbb{D}^\lambda)$  be a  $\lambda$ -flat bundle over  $X$  with a hermitian metric  $h$ . Then we have the following estimates:*

(1) *for any local non-zero  $\mathbb{D}^\lambda$ -flat section  $s$  of  $E$ , we have*

$$\Delta_\omega \log(|s|_h^2) \geq -\frac{2}{1+|\lambda|^2} |\Lambda_\omega G_{h,\lambda}|_h,$$

where  $\Delta_\omega$  denotes the usual Laplacian on  $(X, \omega)$ ;

(2) *for any local non-zero  $\mathbb{D}^\lambda$ -flat section  $s$  of  $E$ , we have*

$$\Delta_\omega(|s|_h^2) \geq -\frac{2}{1+|\lambda|^2} |\Lambda_\omega G_{h,\lambda}|_h \cdot |s|_h^2.$$

*Proof.* (1) Let  $s$  be a local non-zero  $\mathbb{D}^\lambda$ -flat section, namely we have

$$\begin{aligned} d'_E s &= (\lambda \partial_h + \theta_h) s = 0, \\ d''_E s &= (\bar{\partial}_h + \lambda \theta_h^\dagger) s = 0, \end{aligned}$$

then

$$\begin{aligned} \bar{\partial} h(s, s) &= h(d''_E s, s) + h(s, \delta'_h s) = h(s, \delta'_h s), \\ \lambda \partial h(s, s) &= h(d'_E s, s) + h(s, \delta''_h s) = h(s, \delta''_h s), \end{aligned}$$

which gives rise to

$$\lambda \partial \bar{\partial} h(s, s) = \lambda \partial h(s, \delta'_h s) = h(d'_E s, \delta'_h s) + h(s, \delta''_h \delta'_h s) = h(s, \delta''_h \delta'_h s).$$

Therefore, we obtain

$$\lambda \partial \bar{\partial} \log(|s|_h^2) = \frac{\lambda \partial \bar{\partial} |s|_h^2}{|s|_h^2} - \frac{\lambda \partial |s|_h^2 \wedge \bar{\partial} |s|_h^2}{|s|_h^4} = \frac{h(s, \delta''_h \delta'_h s)}{|s|_h^2} - \frac{h(s, \delta''_h s) \wedge h(s, \delta'_h s)}{|s|_h^4}.$$

We calculate

$$\begin{aligned}
h(s, \delta_h'' \delta_h'(s)) &= h(s, (\bar{\lambda} \bar{\partial}_h - \theta_h^\dagger)(\partial_h - \bar{\lambda} \theta_h)(s)) \\
&= h(s, \bar{\lambda} \bar{\partial}_h \partial_h(s)) - h(s, \theta_h^\dagger \partial_h(s)) + h(s, \bar{\lambda} \theta_h^\dagger \theta_h(s)) - h(s, \bar{\lambda}^2 \bar{\partial}_h \theta_h(s)) \\
&= (1 + |\lambda|^2)(-\lambda |\theta_h^\dagger(s)|_h^2 + \frac{1}{\bar{\lambda}} |\theta_h(s)|_h^2) + \frac{\lambda}{1 + |\lambda|^2} h(s, G_{h,\lambda} s),
\end{aligned}$$

where we have used

$$\begin{aligned}
h(s, \bar{\lambda} \bar{\partial}_h \partial_h(s)) &= -h(s, \bar{\lambda} \bar{\partial}_h \frac{\theta_h}{\lambda}(s)) = -h(s, \frac{\bar{\lambda}}{\lambda} (\bar{\partial}_h \theta_h)(s)) - h(s, \bar{\lambda} \theta_h \theta_h^\dagger(s)) \\
&= -\lambda |\theta_h^\dagger(s)|_h^2 + \frac{|\lambda|^2}{\bar{\lambda}(1 + |\lambda|^2)^2} h(s, G_{h,\lambda} s), \\
h(s, \theta_h^\dagger \partial_h(s)) &= -h(s, \theta_h^\dagger \frac{\theta_h}{\lambda}(s)) = -\frac{1}{\lambda} |\theta_h(s)|_h^2, \\
h(s, \bar{\lambda}^2 \bar{\partial}_h \theta_h(s)) &= h(s, \bar{\lambda}^2 (\bar{\partial}_h \theta_h)(s)) + h(s, (\bar{\lambda})^2 \theta_h \lambda \theta_h^\dagger(s)) \\
&= |\lambda|^2 \lambda |\theta_h^\dagger(s)|_h^2 - \frac{\lambda |\lambda|^2}{(1 + |\lambda|^2)^2} h(s, G_{h,\lambda} s).
\end{aligned}$$

Meanwhile, we also calculate

$$\begin{aligned}
h(s, \delta_h'(s)) &= h(s, (\partial_h - \bar{\lambda} \theta_h)(s)) = h(s, (-\lambda^{-1} - \bar{\lambda}) \theta_h(s)) = -\frac{(1 + |\lambda|^2)}{\bar{\lambda}} h(s, \theta_h(s)), \\
h(s, \delta_h''(s)) &= h(s, (\bar{\lambda} \bar{\partial}_h - \theta_h^\dagger)(s)) = -(1 + |\lambda|^2) h(s, \theta_h^\dagger(s)).
\end{aligned}$$

Combining all the above equalities yields

$$\begin{aligned}
-\Delta_\omega \log(|s|_h^2) &= 2\sqrt{-1} \Lambda_\omega \partial \bar{\partial} \log(|s|_h^2) \\
&= 2\sqrt{-1} \Lambda_\omega \left[ - (1 + |\lambda|^2) \frac{|\theta_h^\dagger(s)|_h^2}{|s|_h^2} + \frac{1 + |\lambda|^2}{|\lambda|^2} \frac{|\theta_h(s)|_h^2}{|s|_h^2} \right. \\
&\quad \left. - \frac{(1 + |\lambda|^2)^2}{|\lambda|^2} \frac{h(s, \theta_h(s)) \wedge h(s, \theta_h^\dagger(s))}{|s|_h^4} + \frac{1}{1 + |\lambda|^2} \frac{h(s, G_{h,\lambda} s)}{|s|_h^2} \right] \\
&\leq \frac{2}{1 + |\lambda|^2} |\Lambda_\omega G_{h,\lambda} s|_h,
\end{aligned}$$

where the last line is derived by Cauchy–Schwarz inequality.

(2) By (1), we immediately have

$$\begin{aligned}
-\Delta_\omega(|s|_h^2) &= 2\sqrt{-1} \Lambda_\omega \partial \bar{\partial}(|s|_h^2) \\
&= 2\sqrt{-1} \Lambda_\omega \left[ - (1 + |\lambda|^2) |\theta_h^\dagger(s)|_h^2 + \frac{1 + |\lambda|^2}{|\lambda|^2} |\theta_h(s)|_h^2 + \frac{1}{1 + |\lambda|^2} h(s, G_{h,\lambda} s) \right] \\
&\leq \frac{2}{1 + |\lambda|^2} |\Lambda_\omega G_{h,\lambda}|_h \cdot |s|_h^2.
\end{aligned}$$

□

**Corollary 1.3.2.** *Let  $X$  be a compact Riemann surface and  $(E, \mathbb{D}^\lambda)$  be a stable  $\lambda$ -flat bundle with vanishing first Chern class, then there is no non-trivial global  $\mathbb{D}^\lambda$ -flat section of  $E$ .*

*Proof.* When  $\lambda = 0$ , this claim follows from Theorem 3.1 in [Car14]. Now we assume  $\lambda \neq 0$ . Let  $h$  be the pluri-harmonic on this stable  $\lambda$ -flat bundle  $(E, \mathbb{D}^\lambda)$ , and  $s$  be a non-trivial global  $\mathbb{D}^\lambda$ -flat section, then the function  $|s|_h^2$  is sub-harmonic by Theorem 1.3.1, (2). If  $X$  is compact,  $|s|_h^2$  is a nonzero constant, hence the section  $s$  generates a trivial line subbundle of  $(E, \mathbb{D}^\lambda)$ , which contradicts with the stability of  $(E, \mathbb{D}^\lambda)$ .  $\square$

**Definition 1.3.3.** Let  $\mathbb{O} = \{z : 0 \leq |z| < 1\}$  be the unit disk, and  $\mathbb{O}^* = \{z : 0 < |z| < 1\}$  be the punctured unit disk, and  $(E, D^\lambda)$  be a  $\lambda$ -flat bundle over  $\mathbb{O}^*$ . The  $\lambda$ -connection  $D^\lambda$  is called *regular* if for some trivialization of  $E$  it can be written as

$$D^\lambda = \lambda\partial + \frac{N(z)}{z}dz,$$

where  $N(z)$  is a matrix of holomorphic functions on  $\mathbb{O}$ .

**Proposition 1.3.4.** *Assume  $\lambda \neq 0$ . Let  $(E, D^\lambda)$  be a bundle over  $\mathbb{O}^*$  with a regular  $\lambda$ -connection  $D^\lambda$ , and  $s(z)$  be a  $\lambda$ -flat section defined in some punctured neighbourhood of the origin. Then  $s(z)$  can be extended meromorphically to origin.*

*Proof.* Write  $D^\lambda = \lambda\partial + \frac{N(z)}{z}dz$  under some trivialization of  $E$ . Then for any  $\epsilon \in (0, 1)$ , one defines

$$A = \sup_{|z| \leq \epsilon} \|N(z)\|, \quad B = \sup_{|z| = \epsilon} \|s(z)\|,$$

where  $\|\bullet\|$  denotes the standard norm for matrix and vector. Pick up a point  $z_0$  with  $|z_0| = \epsilon$  in the sector where  $s(z)$  is defined, and define  $f(r) = \|s(rz_0)\|^2$  for  $0 < r \leq 1$ . Since  $s(z)$  is a  $\lambda$ -flat section, i.e.  $\frac{ds}{dz} = -\frac{Ns(z)}{\lambda z}$ , we have

$$\begin{aligned} \left| \frac{df}{dr} \right| &= \left| 2z_0 s(rz_0) \frac{ds}{dz} \Big|_{z=rz_0} \right| \leq 2\epsilon \|s(rz_0)\| \cdot \left\| \frac{Ns(rz_0)}{\lambda rz_0} \right\| \\ &\leq \frac{2\epsilon A}{|\lambda rz_0|} \|s(rz_0)\|^2 = \frac{2Af(r)}{|\lambda|r}. \end{aligned}$$

Assume  $f(r)$  is not zero identically and  $f(r_0) = 0$  for some  $r_0 \in (0, 1]$ , then we write  $f(r) = (r - r_0)^n g(r)$  for some positive integer  $n$  and some analytic function  $g(r)$  with  $g(r_0) \neq 0$ . However, the above inequality implies

$$\left| ng(r) - (r - r_0) \frac{dg(r)}{dr} \right| \leq \frac{2A}{|\lambda|r} |r - r_0| \cdot |g(r)|,$$

thus  $g(r_0) = 0$ . Therefore, we can assume  $f(r)$  is zero identically or positive over  $(0, 1]$ . When  $f(r)$  is positive, we have

$$\frac{d \log f(r)}{dr} \geq -\frac{2A}{|\lambda|r},$$

hence

$$f(r) \leq f(1)r^{-\frac{2A}{|\lambda|}},$$

which yields the estimate

$$\|s(z)\| \leq B \left( \frac{|z|}{\epsilon} \right)^{\frac{-A}{|\lambda|}}$$

for  $|z| \leq \epsilon$ . The conclusion follows.  $\square$

**Example 1.3.5.** Let  $E$  be a rank 2 Hermitian vector bundle over  $\mathbb{O}^*$  with the local unitary frame  $\{v_1, v_2\}$ . In [MSWW16], the authors introduced the so-called “fiducial solution” of Hitchin equations expressed in terms of the frame  $\{v_1, v_2\}$  as follows

$$\begin{cases} A = \frac{1}{8} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} dz & d\bar{z} \\ z & \bar{z} \end{pmatrix} \\ \phi = \begin{pmatrix} 0 & \sqrt{|z|} \\ \frac{z}{\sqrt{|z|}} & 0 \end{pmatrix} dz, \end{cases}$$

that solves the decoupled Hitchin equations

$$F_A = 0, \quad [\phi, \phi^\dagger] = 0, \quad \bar{\partial}_A \phi = 0,$$

where  $F_A$  denotes the curvature of  $A$ . Let  $\mu \in \mathbb{C}^*$  be a constant, then we have a  $\lambda$ -flat connection  $\mathbb{D}_\mu^\lambda = d'_E + d''_E$  with

$$d'_E = \lambda \partial_A + \phi, \quad d''_E = \bar{\partial}_A + \mu \phi^\dagger,$$

then a  $\mathbb{D}_\mu^\lambda$ -flat section  $s = \begin{pmatrix} f(z, \bar{z}) \\ g(z, \bar{z}) \end{pmatrix}$  should satisfy the following equations

$$\begin{cases} \lambda \frac{\partial f}{\partial z} + \frac{\lambda f}{8z} + \sqrt{|z|} g = 0, \\ \lambda \frac{\partial g}{\partial z} - \frac{\lambda g}{8z} + \frac{z}{\sqrt{|z|}} f = 0, \\ \frac{\partial f}{\partial \bar{z}} - \frac{1}{8} \frac{f}{\bar{z}} + \mu \frac{\bar{z}}{\sqrt{|z|}} g = 0, \\ \frac{\partial g}{\partial \bar{z}} + \frac{1}{8} \frac{g}{\bar{z}} + \mu \sqrt{|z|} f = 0. \end{cases} \quad (1.11)$$

Let  $z \rightarrow 0$  in equations (1.11), we have

$$f(z, \bar{z}) \longrightarrow z^{-\frac{1}{8}} \bar{z}^{\frac{1}{8}}, \quad g(z, \bar{z}) \longrightarrow z^{\frac{1}{8}} \bar{z}^{-\frac{1}{8}},$$

hence we can assume that

$$f(z, \bar{z}) = z^{-\frac{1}{8}} \bar{z}^{\frac{1}{8}} u(z, \bar{z}), \quad g(z, \bar{z}) = z^{\frac{1}{8}} \bar{z}^{-\frac{1}{8}} v(z, \bar{z})$$

with  $\lim_{z \rightarrow 0} u = \lim_{z \rightarrow 0} v = 1$ . Then  $u(z, \bar{z})$  and  $v(z, \bar{z})$  should satisfy the following equations

$$\begin{cases} \lambda \frac{\partial u}{\partial z} + \sqrt{z}v = 0, \\ \lambda \frac{\partial v}{\partial z} + \sqrt{z}u = 0, \\ \frac{\partial u}{\partial \bar{z}} + \mu\sqrt{\bar{z}}v = 0, \\ \frac{\partial v}{\partial \bar{z}} + \mu\sqrt{\bar{z}}u = 0, \end{cases}$$

which imply

$$\frac{\partial u}{\partial(\frac{z^{\frac{3}{2}}}{\lambda\mu})} = \frac{\partial u}{\partial(\bar{z}^{\frac{3}{2}})}, \quad \frac{\partial v}{\partial(\frac{z^{\frac{3}{2}}}{\lambda\mu})} = \frac{\partial v}{\partial(\bar{z}^{\frac{3}{2}})}.$$

Therefore, we can write

$$u(z, \bar{z}) = U\left(\frac{z^{\frac{3}{2}}}{\lambda\mu} + \bar{z}^{\frac{3}{2}}\right), \quad v(z, \bar{z}) = V\left(\frac{z^{\frac{3}{2}}}{\lambda\mu} + \bar{z}^{\frac{3}{2}}\right).$$

Introducing new variable  $X = \frac{z^{\frac{3}{2}}}{\lambda\mu} + \bar{z}^{\frac{3}{2}}$ , we have

$$\begin{cases} \frac{3}{2\mu} \frac{\partial U}{\partial X} + V = 0, \\ \frac{3}{2\mu} \frac{\partial V}{\partial X} + U = 0, \\ \frac{3}{2} \frac{\partial U}{\partial X} + \mu V = 0, \\ \frac{3}{2} \frac{\partial V}{\partial X} + \mu U = 0, \end{cases}$$

which can be solved easily

$$\begin{aligned} U(X) &= C_1 \exp\left(\frac{2\mu}{3}X\right) + C_2 \exp\left(-\frac{2\mu}{3}X\right), \\ V(X) &= -C_1 \exp\left(\frac{2\mu}{3}X\right) + C_2 \exp\left(-\frac{2\mu}{3}X\right), \end{aligned}$$

where  $C_1$  and  $C_2$  are two constants. Consequently, any local  $\mathbb{D}_\mu^\lambda$ -flat section  $s$  is the  $\mathbb{C}$ -linear combination of the following two sections

$$\begin{aligned} s_1 &= \begin{pmatrix} z^{-\frac{1}{8}} \bar{z}^{\frac{1}{8}} \exp\left(\frac{2}{3\lambda}z^{\frac{3}{2}} + \frac{2\mu}{3}\bar{z}^{\frac{3}{2}}\right) \\ -z^{\frac{1}{8}} \bar{z}^{-\frac{1}{8}} \exp\left(\frac{2}{3\lambda}z^{\frac{3}{2}} + \frac{2\mu}{3}\bar{z}^{\frac{3}{2}}\right) \end{pmatrix}, \\ s_2 &= \begin{pmatrix} z^{-\frac{1}{8}} \bar{z}^{\frac{1}{8}} \exp\left(-\frac{2}{3\lambda}z^{\frac{3}{2}} - \frac{2\mu}{3}\bar{z}^{\frac{3}{2}}\right) \\ z^{\frac{1}{8}} \bar{z}^{-\frac{1}{8}} \exp\left(-\frac{2}{3\lambda}z^{\frac{3}{2}} - \frac{2\mu}{3}\bar{z}^{\frac{3}{2}}\right) \end{pmatrix}. \end{aligned}$$

One easily checks that  $\Delta \log(|s|_h^2) = 0$ .

**Example 1.3.6.** (Continue after [Example 1.3.5](#)) Let  $\lambda' = t\lambda$ . We want to find pluri-harmonic metric  $h_t$  for the  $\lambda'$ -flat bundle  $(E, \mathbb{D}^{\lambda'} = td'_E + d''_E)$  with  $\mu = \lambda$ . Denote the matrix form of  $h_t$  in terms of the frame  $\{v_1, v_2\}$  by  $H_t$ .

Write  $t \cdot d'_E = \lambda' \partial_A + t\phi$ ,  $d''_E = \bar{\partial}_A + \lambda(1 - |t|^2)\phi^\dagger + \lambda'(t\phi)^\dagger$ , where  $\phi^\dagger$  denotes the adjoint of  $\phi$  with respect to the normal hermitian metric, that is,

$$\phi^\dagger = \begin{pmatrix} 0 & \frac{\bar{z}}{\sqrt{|z|}} \\ \sqrt{|z|} & 0 \end{pmatrix} d\bar{z}.$$

Let  $\phi' := t\phi$ , note that  $\bar{\partial}_A + \lambda(1 - |t|^2)\phi^\dagger$  is integrable and satisfies

$$(\bar{\partial}_A + \lambda(1 - |t|^2)\phi^\dagger)(\phi') = 0$$

since the decoupled Hitchin's equations in [Example 1.3.5](#) hold. So  $(E, \bar{\partial}_A + \lambda(1 - |t|^2)\phi^\dagger, \phi)$  is a Higgs bundle, we want it becomes the associated Higgs bundle  $(E, \bar{\partial}_{h_t}, \theta_{h_t})$  of the harmonic  $\lambda'$ -flat bundle  $(E, d''_E, d'_E, h_t)$ . This means:

$$\begin{cases} \partial_A = \partial_{h_t}, \\ \bar{\partial}_A + \lambda(1 - |t|^2)\phi^\dagger = \bar{\partial}_{h_t}, \\ \phi' = \theta_{h_t}, \\ (\phi')^\dagger = (\phi')^\dagger_{h_t}. \end{cases}$$

Moreover, to make each  $h_t$  pluri-harmonic,  $\partial_{h_t} + \bar{\partial}_{h_t}$  need to be unitary with respect to  $h_t$ ,  $\theta_{h_t}^\dagger$  should be the adjoint of  $\theta_{h_t}$  with respect to  $h_t$ .

In conclusion,  $H_t$  should satisfy the following equations

$$\begin{cases} \bar{H}_t^{-1} \cdot (\bar{\phi}')^T \bar{H}_t = (\phi')^\dagger, \\ \partial H_t = (A^{1,0})^T \cdot H_t + H_t \cdot \overline{(A^{0,1} + \lambda(1 - |t|^2)\phi^\dagger)}, \\ \bar{\partial} H_t = (A^{0,1} + \lambda(1 - |t|^2)\phi^\dagger)^T \cdot H_t + H_t \cdot \overline{A^{1,0}}, \end{cases}$$

where  $(\bullet)^T$  denotes the transpose of matrices. We can express  $H_t$  as

$$H_t = \begin{pmatrix} a(z, \bar{z}) & b(z, \bar{z}) \\ b(z, \bar{z}) & c(z, \bar{z}) \end{pmatrix},$$

then we have

$$\left\{ \begin{array}{l} a = c, \\ b \cdot \bar{z}^{\frac{1}{2}} = \bar{b} \cdot z^{\frac{1}{2}}, \\ \frac{\partial a}{\partial z} = \bar{\lambda}(1 - |t|^2)b\sqrt{|z|}, \\ \frac{\partial a}{\partial \bar{z}} = \lambda(1 - |t|^2)b\frac{\bar{z}}{\sqrt{|z|}}, \\ \frac{\partial b}{\partial z} = \frac{b}{4z} + \bar{\lambda}(1 - |t|^2)a\frac{z}{\sqrt{|z|}}, \\ \frac{\partial b}{\partial \bar{z}} = -\frac{b}{4\bar{z}} + \lambda(1 - |t|^2)a\sqrt{|z|}. \end{array} \right.$$

It can be resolved as follows

$$\begin{aligned} a(z, \bar{z}) &= f(\bar{z}) \exp\left(\frac{2}{3}\bar{\lambda}(1 - |t|^2)z^{\frac{3}{2}}\right) + g(\bar{z}) \exp\left(-\frac{2}{3}\bar{\lambda}(1 - |t|^2)z^{\frac{3}{2}}\right), \\ b(z, \bar{z}) &= \frac{z^{\frac{1}{2}}}{\sqrt{|z|}} \left( f(\bar{z}) \exp\left(\frac{2}{3}\bar{\lambda}(1 - |t|^2)z^{\frac{3}{2}}\right) - g(\bar{z}) \exp\left(-\frac{2}{3}\bar{\lambda}(1 - |t|^2)z^{\frac{3}{2}}\right) \right), \end{aligned}$$

where

$$\begin{aligned} f(\bar{z}) &= C_1 \exp\left(\frac{2}{3}\lambda(1 - |t|^2)\bar{z}^{\frac{3}{2}}\right) + C_2 \exp\left(-\frac{2}{3}\lambda(1 - |t|^2)\bar{z}^{\frac{3}{2}}\right), \\ g(\bar{z}) &= C_2 \exp\left(\frac{2}{3}\lambda(1 - |t|^2)\bar{z}^{\frac{3}{2}}\right) + C_3 \exp\left(-\frac{2}{3}\lambda(1 - |t|^2)\bar{z}^{\frac{3}{2}}\right), \end{aligned}$$

for constants  $C_1, C_2$  and  $C_3$ .

**Remark 1.3.7.** This example exhibits that for a  $\lambda$ -flat bundle over a non-complete manifold, if it admits a pluri-harmonic metric, then the pluri-harmonic metric is usually not unique.

## Chapter 2

# A Study of the Geometry of Moduli Spaces

Let  $X$  be a smooth complex projective variety and let  $G$  be a complex reductive Lie group. Picking a basepoint  $x \in X$ , following [Sim94a, Sim94b, Sim95], there are four *representation spaces* (they represent certain moduli functors):

- (1) *Betti representation space*  $\mathcal{R}_B(X, x, G) := \text{Hom}(\pi_1(X, x), G)$ , the space of representations of  $\pi_1(X, x)$  into  $G$ ;
- (2) *De Rham representation space*  $\mathcal{R}_{\text{dR}}(X, x, G)$ , the moduli scheme of  $(E, \nabla, \xi)$ , where  $E$  is a principle  $G$ -bundle with a flat connection  $\nabla$  and  $\xi : E_x \cong G$  is a frame for  $E$  at  $x$ ;
- (3) *Dolbeault representation space*  $\mathcal{R}_{\text{Dol}}(X, x, G)$ , the moduli scheme of  $(E, \theta, \xi)$ , where  $(E, \theta)$  is a semistable principle Higgs  $G$ -bundle with vanishing (rational) Chern classes and  $\xi : E_x \cong G$  is a frame for  $E$  at  $x$ ;
- (4) *Hodge representation space*  $\mathcal{R}_{\text{Hod}}(X, x, G)$ , the moduli scheme of  $(E, \nabla_\lambda, \xi, \lambda)$ , where  $\lambda \in \mathbb{C}$ ,  $(E, \nabla_\lambda)$  is a semistable principle  $\lambda$ -flat  $G$ -bundle with vanishing (rational) Chern classes and  $\xi : E_x \cong G$  is a frame for  $E$  at  $x$ .

The group  $G$  acts on these representation spaces such that the universal categorical quotients exist, and the quotients are independent of the choice of basepoint. They are the corresponding moduli spaces, and the points in each moduli space parametrize the closed orbits in the corresponding representation space under the action of  $G$ . All these moduli spaces are quasi-projective varieties and can be denoted as follows:

- (1)  $\mathcal{M}_B(X, G) := \mathcal{R}_B(X, x, G) // G$ ;
- (2)  $\mathcal{M}_{\text{dR}}(X, G) := \mathcal{R}_{\text{dR}}(X, x, G) // G$ ;
- (3)  $\mathcal{M}_{\text{Dol}}(X, G) := \mathcal{R}_{\text{Dol}}(X, x, G) // G$ ;
- (4)  $\mathcal{M}_{\text{Hod}}(X, G) := \mathcal{R}_{\text{Hod}}(X, x, G) // G$ .

There is a fibration  $\mathcal{M}_{\text{Hod}}(X, G) \rightarrow \mathbb{C}$  such that the fibers over 0 and 1 are  $\mathcal{M}_{\text{Dol}}(X, G)$  and  $\mathcal{M}_{\text{dR}}(X, G)$ , respectively. Let  $M_B(X, G)$ ,  $M_{\text{dR}}(X, G)$ ,  $M_{\text{Dol}}(X, G)$  and  $M_{\text{Hod}}(X, G)$  be the corresponding smooth locus, as Zariski dense open subsets and parametrize the isomorphism classes of stable objects. In particular, when  $G = \text{GL}(r, \mathbb{C})$ , we will just apply the notations  $\mathcal{R}_{B/\text{dR}/\text{Dol}/\text{Hod}}(X, x, r)$ ,  $\mathcal{M}_{B/\text{dR}/\text{Dol}/\text{Hod}}(X, r)$  and  $M_{B/\text{dR}/\text{Dol}/\text{Hod}}(X, r)$  for simplicity.

For any  $\lambda \in \mathbb{C}$ , let  $\mathcal{M}_{\text{Hod}}^\lambda(X, r) := \mathcal{M}_{\text{Hod}}(X, r)|_\lambda =: \mathcal{R}_{\text{Hod}}^\lambda(X, x, r) // \text{GL}(r, \mathbb{C})$  (resp.  $M_{\text{Hod}}^\lambda(X, r) := M_{\text{Hod}}(X, r)|_\lambda$ ). Let  $U(r) \subseteq \text{GL}(r, \mathbb{C})$  be the maximal compact Lie group (compact real form), let  $\mathcal{R}_{\text{Hod}}^{\lambda, U(r)}(X, x, r) \subseteq \mathcal{R}_{\text{Hod}}^\lambda(X, x, r)$  consisting of points  $(E, \nabla_\lambda, \xi)$  that admit a pluri-harmonic metric<sup>1</sup> compatible with the frame  $\xi$  at  $x$ , such condition fixes the pluri-harmonic metric uniquely [Sim08]. The group  $U(r)$  acts on  $\mathcal{R}_{\text{Hod}}^{\lambda, U(r)}(X, r)$  such that  $\mathcal{M}_{\text{Hod}}^\lambda(X, r) = \mathcal{R}_{\text{Hod}}^{\lambda, U(r)}(X, x, r) / U(r)$ .

**Proposition 2.0.1.** *Let  $X$  be a compact Riemann surface of genus  $g \geq 2$ , and assume  $r \geq 2$ . We define  $\mathring{M}_{\text{Hod}}^\lambda(X, r) = \mathcal{M}_{\text{Hod}}^\lambda(X, r) \setminus M_{\text{Hod}}^\lambda(X, r)$ . If both  $M_{\text{Hod}}^\lambda(X, r)$  and  $\mathring{M}_{\text{Hod}}^\lambda(X, r)$  are nonempty, then we have*

$$\text{codim}_{\mathbb{C}}(\mathring{M}_{\text{Hod}}^\lambda(X, r)) \geq 2.$$

*Proof.* For any partition  $\vec{r} = (r_1, \dots, r_k) \in \mathbb{Z}_+^{\oplus k}$  with  $\sum_{i=1}^k r_i = r$  and  $1 < k \leq r$ , let  $M_{\text{Hod}}^\lambda(X, \vec{r}) := M_{\text{Hod}}^\lambda(X, r_1) \times \dots \times M_{\text{Hod}}^\lambda(X, r_k)$ . We introduce the following map

$$\delta_{\vec{r}} : M_{\text{Hod}}^\lambda(X, \vec{r}) \longrightarrow \mathcal{M}_{\text{Hod}}^\lambda(X, r)$$

by  $((E_1, \bar{\partial}_{E_1}, D_1^\lambda), \dots, (E_k, \bar{\partial}_{E_k}, D_k^\lambda)) \mapsto (\bigoplus_{i=1}^k E_i, \bigoplus_{i=1}^k \bar{\partial}_{E_i}, \bigoplus_{i=1}^k D_i^\lambda)$ .

Since  $\delta_{\vec{r}}$  is injective, we have

$$\dim_{\mathbb{C}}(\mathring{M}_{\text{Hod}}^\lambda(X, r)) = \dim_{\mathbb{C}} \bigcup_{\{\vec{r}\}} \text{Im}(\delta_{\vec{r}}) = \max_{\{\vec{r}\}} \{\dim_{\mathbb{C}} M_{\text{Hod}}^\lambda(X, \vec{r})\}.$$

Hitchin and Simpson calculated the dimension of moduli space [Hit87a, Sim92, Sim94a]:

$$\dim_{\mathbb{C}}(M_{\text{Hod}}^\lambda(X, r_i)) = \dim_{\mathbb{C}}(M_{\text{Dol}}(X, r_i)) = 2r_i^2(g-1) + 2,$$

then one can easily show that

$$\max_{\{\vec{r}\}} \{\dim_{\mathbb{C}} M_{\text{Hod}}^\lambda(X, \vec{r})\} = 2(g-1)((r-1)^2 + 1) + 4,$$

which means that  $\text{codim}_{\mathbb{C}}(\mathring{M}_{\text{Hod}}^\lambda(X, r)) = 4(g-1)(r-1) - 2 \geq 2$ .  $\square$

The proof of the following theorem is after Simpson (Lemma 7.13 in [Sim94b], Lemma 8.1 in [Sim10]) essentially.

**Theorem 2.0.2.** *The natural quotient map  $q : \mathcal{R}_{\text{Hod}}^{\lambda, U(r)}(X, x, r) \rightarrow \mathcal{M}_{\text{Hod}}^\lambda(X, r)$  is proper.*

*Proof.* The cases of  $\lambda = 0, 1$  have been proved by Simpson (Corollary 7.12 and Corollary 7.15 in [Sim94b]). For the case of  $\lambda \neq 0, 1$ , we consider a sequence  $\{(E_i, d''_{E_i}, D_i^\lambda, \beta_i)\}$  lying inside the inverse image of a compact subset of  $\mathcal{M}_{\text{Hod}}^\lambda(X, r)$ , where  $\beta_i$  is a frame on  $E_{ix}$ , and let  $h_i$  be the unique pluri-harmonic metric on  $(E_i, d''_{E_i}, D_i^\lambda, \beta_i)$ . It suffices to show that the characteristic polynomials of the corresponding Higgs fields  $\{\theta_{h_i}\}$  are uniformly bounded in  $C^0$ -norm. By the map  $(E, d''_E, D^\lambda, \beta) \mapsto (E, d''_E, \lambda^{-1}D^\lambda, \beta)$  and the Riemann–Hilbert correspondence,  $\mathcal{M}_{\text{Hod}}^\lambda(X, r)$  is complex analytically isomorphic to  $\mathcal{M}_{\text{B}}(X, r)$ . Let  $\rho_i$  be the monodromy representation corresponding to  $(E_i, d''_{E_i}, D_i^\lambda, \beta_i)$ , then  $\{\rho_i\}$  lie over a compact subset of  $\mathcal{M}_{\text{B}}(X, r)$ , hence the norms

<sup>1</sup>For general complex reductive Lie group  $G$  and a maximal compact subgroup  $K \subseteq G$ , a metric on a principle  $\lambda$ -flat  $G$ -bundle  $(E, \nabla_\lambda)$  is a  $K$ -reduction, i.e., a principal  $K$ -bundle  $E_K$  with  $E_K \times_K G \cong E$ , such that  $\nabla_\lambda$  has the decomposition  $\nabla_\lambda = \bar{\partial}_K + \lambda \theta_K^\dagger + \lambda \partial_K + \theta$  with  $\bar{\partial}_K + \partial_K$  a connection on  $E_K$  and  $\theta_K + \theta_K^\dagger$  a self-adjoint section of  $\text{ad}(E) \otimes \omega_X^1$ , where  $\text{ad}(E) := E \times_G \mathfrak{g}$  is the adjoint bundle. This metric is pluri-harmonic if  $\theta_K^2 = 0 = \bar{\partial}_K(\theta_K)$ .

$\{|\rho_i(\gamma)|\} = \{\sqrt{\text{Tr}(\rho_i(\gamma)\rho_i^\dagger(\gamma))}\}$  are uniformly bounded for any generator  $\gamma$  of  $\pi_1(X, x)$ . The limit point of  $\{\rho_i(\gamma)\}$  is denoted by  $\rho^{(\infty)}(\gamma)$ . By virtue of non-Abelian Hodge correspondence, each  $\rho_i$  produces another irreducible monodromy representation  $\tilde{\rho}_i$  of  $\pi_1(X, x)$  given by the flat bundle  $(E, \bar{\partial}_{h_i} + \bar{\theta}_{h_i}, \partial_{h_i} + \theta_{h_i}, \beta_i)$ . Then the norms  $\{|\tilde{\rho}_i(\gamma)|\}$  are also uniformly bounded. Indeed, we consider a family of flat bundles  $(E, \bar{\partial}_{h_i} + t^{-1}\theta_{h_i}^\dagger, \partial_{h_i} + t\theta_{h_i})$  parameterized by  $t \in \mathbb{C}^*$ , and the associated family of monodromy representations is denoted by  $\rho_t^{(i)}$ . It is clear that the map  $t \mapsto |\rho_t^{(i)}(\gamma)|$  is continuous. We have the bound  $|\tilde{\rho}_i(\gamma)| \leq C$ . If  $|\tilde{\rho}_i(\gamma)|$  tends to infinity, then for any constant  $C_1 > C$ , there is a sequence  $\{t_i\}$  which lie in a curve segment joining  $\lambda^{-1}$  to 1 but not passing through 0 such that  $|\rho_{t_i}^{(i)}(\gamma)| = C_1$ . By Theorem 1 in [Sim91], the map  $\rho \mapsto |\rho(\gamma)|$  from  $\mathcal{M}_B(X, r)$  to  $\mathbb{R}$  is proper, thus we may assume  $\{\rho_{t_i}^{(i)}\}$  has a limit point  $\rho_\diamond$ , then  $|\rho_\diamond(\gamma)| = C_1$ . We can also assume the sequence  $\{t_i\}$  has the limit point  $t_\infty$ , then  $\rho_\diamond(\gamma) = \rho_{t_\infty}^{(\infty)}(\gamma)$  due to the separatedness of moduli space, whose norm has a bound  $C_2$ . If one takes up  $C_1 > C_2$ , we will get a contradiction, which lead to the uniform bound of  $\{|\tilde{\rho}_i(\gamma)|\}$ .

Consequently, by Corollary 6 in [Sim91], the  $L^2$ -norms  $\{|\theta_{h_i}|\}_{L^2}$  are uniformly bounded. Since the maximum norm of an eigenvalue of a holomorphic matrix is a subharmonic function, the eigenforms of  $\theta_{h_i}$  are uniformly bounded in  $C^0$ .

So far, we prove the claim on the characteristic polynomial of Higgs fields  $\{\theta_{h_i}\}$ . Therefore, Lemma 2.8 in [Sim92] or Proposition 7.9 in [Sim94b] implies that there is a harmonic bundle  $(E, \bar{\partial}, \theta, h, \beta)$ , a subsequence  $\{i'\}$ , and  $C^\infty$ -automorphisms  $g_{i'}$  such that  $g_{i'}^*(h_{i'}) = h$ , both  $g_{i'}^*(\bar{\partial}_{h_{i'}}) - \bar{\partial}$  and  $g_{i'}^*(\theta_{h_{i'}}) - \theta$  converge to zero strongly in the operator norm for operators from  $L_1^p$  to  $L^p$  for  $p > 1$ , and the frames  $g_{i'}^*(\beta_{i'})$  converge to  $\beta$ . Since the  $\lambda$ -flat bundle can be treated as certain  $\Lambda$ -module in the sense of Simpson [Sim08], Theorem 5.12 in [Sim94a] is valid for this case, hence there is a subsequence  $\{(E_{i'}, d_{E_{i'}}'', D_{i'}^\lambda, \beta_{i'})\}$  converges to a point  $(E, \bar{\partial} + \lambda\theta_h^\dagger, \lambda\partial_h + \theta, \beta)$  in  $\mathcal{R}_{\text{Hod}}^{\lambda, U(r)}(X, x, r)$ .  $\square$

**Corollary 2.0.3 (NAHC of Moduli Spaces Version, Part II).** *We have the following homeomorphism of  $\mathcal{M}_{\text{Hod}}^\lambda(X, r)$  and  $\mathcal{M}_{\text{Dol}}(X, r)$  as topological spaces:*

$$\mathcal{M}_{\text{Hod}}^\lambda(X, r) \stackrel{\text{homeo.}}{\cong} \mathcal{M}_{\text{Dol}}(X, r).$$

*Proof.* A key step has been completed in the proof of above theorem, the remaining arguments are totally parallel to Theorem 7.18 in [Sim94b], so we omit them here.  $\square$

## 2.1 A Dynamical System on the Dolbeault Moduli Space

Let  $X$  be a smooth complex projective variety. The Dolbeault moduli space  $M_{\text{Dol}}(X, r)$  that parametrizes the isomorphism classes of stable Higgs bundles of rank  $r$  over  $X$  with vanishing (rational) Chern classes, equipped with an algebraic action by  $\mathbb{C}^*$ . The fixed points are those Higgs bundles having the structure of systems of Hodge bundles<sup>2</sup>. In differential geometric interpretation, they correspond to the critical points of the Morse–Bott functional defined by the  $L^2$ -norm of the Higgs fields [Hit87a]. The  $\mathbb{C}^*$ -action and the fixed points provide a stratification for  $M_{\text{Dol}}(X, r)$  (also for  $\mathcal{M}_{\text{Dol}}(X, r)$ ), called the *Bialynicki-Birula stratification*. There is another stratification of  $M_{\text{Dol}}(X, r)$  (also for  $\mathcal{M}_{\text{Dol}}(X, r)$ ) called the *Shatz stratification*, which is obtained by the Harder–Narasimhan type of the underlying vector bundles. These two stratifications are

<sup>2</sup>In many contexts, these fixed points are called “complex variations of Hodge structure”, in this thesis, we will also apply this expression, or for simplicity, “C-VHS” when there is no ambiguity.

different in general, but they are coincide in rank two case (cf. [Hau98] for the case of Riemann surface).

In this part, we will use the  $\mathbb{C}^*$ -action on  $M_{\text{Dol}}(X, r)$  and Mochizuki correspondence (Theorem 1.2.11) on the existence of pluri-harmonic metrics on (poly-)stable  $\lambda$ -flat bundles with vanishing Chern classes to construct a two parameter dynamical system on  $M_{\text{Dol}}(X, r)$ . This dynamical system recovers the usual  $\mathbb{C}^*$ -action and the identity map by choosing special parameters, and it satisfies some ‘‘cocycle’’-type formula (Proposition 2.1.1).

To study this dynamical system, we calculate the first variation of the dynamical system acts on an integral curve passing through an original point with holomorphic infinitesimal deformation of that point (Theorem 2.1.3). Moreover, we study the fixed points of that dynamical system and show that all  $\mathbb{C}$ -VHS lying in the moduli space are exactly the fixed points (Theorem 2.1.5). Meanwhile, we calculate the stabilizer set of some special points in the moduli space for the case of compact Riemann surface (Theorem 2.1.7).

This section is mainly based on [HH19, Hua20].

### 2.1.1 $\mathbb{C}^*$ -Action on Moduli Spaces

The Dolbeault moduli space  $\mathcal{M}_{\text{Dol}}(X, r)$  admits a natural  $\mathbb{C}^*$ -action [Hit87a, Sim92]:

$$t \cdot [(E, \bar{\partial}_E, \theta)] := [(E, \bar{\partial}_E, t\theta)].$$

This action is well-defined because it does not change the stability and vanishing of Chern classes (so it also acts on the moduli space  $M_{\text{Dol}}(X, r)$  of stable points). In [Sim95], Simpson showed the existence of the coarse moduli space  $\mathcal{M}_{\text{Hod}}(X, r)$  of semistable  $\lambda$ -flat bundles of rank  $r$  over  $X$  with vanishing Chern classes. This moduli space admits a fibration  $\pi : \mathcal{M}_{\text{Hod}}(X, r) \rightarrow \mathbb{C}$ ,  $[E, \bar{\partial}_E, D^\lambda, \lambda] \mapsto \lambda$  such that  $\pi^{-1}(1) = \mathcal{M}_{\text{dR}}(X, r)$  and  $\pi^{-1}(0) = \mathcal{M}_{\text{Dol}}(X, r)$ <sup>3</sup>.

Moreover, the  $\mathbb{C}^*$ -action on  $\mathcal{M}_{\text{Dol}}(X, r)$  extends to an action of  $\mathbb{C}^*$  on  $\mathcal{M}_{\text{Hod}}(X, r)$ :

$$t \cdot [E, \bar{\partial}_E, D^\lambda, \lambda] := [E, \bar{\partial}_E, tD^\lambda, t\lambda].$$

This action is well-defined and sends a flat  $\lambda$ -connection to a flat  $t\lambda$ -connection, so the fixed points must lie inside the fiber at  $\lambda = 0$ , i.e, in  $\mathcal{M}_{\text{Dol}}(X, r)$ . By [Sim92, Lemma 4.1], the fixed points in  $\mathcal{M}_{\text{Dol}}(X, r)$  are  $\mathbb{C}$ -VHS, that is, of the form

$$\left[ E = \bigoplus_{p=1}^k E^p, \bar{\partial}_E = \begin{pmatrix} \bar{\partial}_{E^1} & & \\ & \ddots & \\ & & \bar{\partial}_{E^k} \end{pmatrix}, \theta : E^p \rightarrow E^{p-1} \otimes \Omega_X^1 \right].$$

Following notations in [Sim10], let  $P$  be the subset of  $\mathcal{M}_{\text{Dol}}(X, r)$  that consists of all the fixed points, and let  $P = \bigcup_\alpha P_\alpha$  be the decomposition into its connected components. This fixed point set will play an important role on the stratifications of moduli spaces, which will be described in §2.2.

### 2.1.2 Constructing the Dynamical System

Due to Mochizuki correspondence, we can construct a new action on  $M_{\text{Dol}}(X, r)$ , which generalizes the usual  $\mathbb{C}^*$ -action. Fixing some  $(\lambda, t) \in \mathbb{C} \times \mathbb{C}^*$ , for any stable Higgs bundle  $[E, \bar{\partial}_E, \theta] \in M_{\text{Dol}}(X, r)$

<sup>3</sup>It's obviously that these properties also hold for  $M_{\text{dR}/\text{Dol}/\text{Hod}}(X, r)$ .

with a pluri-harmonic metric  $h$ , we have a stable  $\lambda$ -flat bundle  $[E, d''_E = \bar{\partial}_E + \lambda\theta_h^\dagger, D^\lambda = \lambda\partial_{E,h} + \theta] \in M_{\text{Hod}}^\lambda(X, r)$ , and hence a stable  $\lambda'$ -flat bundle  $[E, d''_E, D^{\lambda'} = t(\lambda\partial_{E,h} + \theta)] \in M_{\text{Hod}}^{\lambda'}(X, r)$  for  $\lambda' = t\lambda$ . The latter one admits a pluri-harmonic metric  $h_t$ , which gives rise to a stable Higgs bundle  $[E, \bar{\partial}_{E,h_t}, \theta_{h_t}] \in M_{\text{Dol}}(X, r)$  by Mochizuki's theorem. We conclude the above process in the following:

$$\begin{aligned} M_{\text{Dol}}(X, r) &\longrightarrow M_{\text{Hod}}^\lambda(X, r) \longrightarrow M_{\text{Hod}}^{\lambda'}(X, r) \longrightarrow M_{\text{Dol}}(X, r) \\ [E, \bar{\partial}_E, \theta] &\longmapsto [E, \bar{\partial}_E + \lambda\theta_h^\dagger, \lambda\partial_{E,h} + \theta] \longmapsto [E, \bar{\partial}_E + \lambda\theta_h^\dagger, t(\lambda\partial_{E,h} + \theta)] \longmapsto [E, \bar{\partial}_{E,h_t}, \theta_{h_t}]. \end{aligned}$$

As a summary, Mochizuki correspondence provides a two-parameter dynamical system (i.e. a smooth self-map  $\psi_{(\lambda,t)}$ ) on  $M_{\text{Dol}}(X, r)$ :

$$\begin{aligned} \psi_{(\lambda,t)} : M_{\text{Dol}}(X, r) &\longrightarrow M_{\text{Dol}}(X, r) \\ (E, \bar{\partial}_E, \theta) &\longmapsto (E, \bar{\partial}_{E,h_t}, \theta_{h_t}), \end{aligned}$$

and we also call it the  $(\lambda, t)$ -action. From the viewpoint of twistor theory (cf. the next chapter), to describe the image  $u'_0$  of a given point  $u_0 = [E, \bar{\partial}_E, \theta]$  under  $\psi_{(\lambda,t)}$ , within the Deligne–Hitchin twistor space, we first walk along the preferred section passing through  $u_0$  until the point  $u_\lambda$  lying in the fiber at  $\lambda$ , then we walk along the de Rham section determined by  $u_\lambda$  until the point  $u_{t\lambda}$  lying in the fiber at  $t\lambda$ , finally we walk along the new preferred section determined by  $u_{t\lambda}$  and get to the point lying in the fiber at 0, this point is exactly  $u'_0$ . Clearly,  $\psi_{(\lambda,t)}$  can be also defined on  $\mathcal{M}_{\text{Dol}}(X, r)$  as a continuous map.

We can see that

**Proposition 2.1.1.** *Let  $\psi_{(\lambda,t)}$  be the dynamical system on  $M_{\text{Dol}}(X, r)$  constructed above, then*

- (1)  $\psi_{(0,t)}$  is the usual  $\mathbb{C}^*$ -action by  $t$  on  $M_{\text{Dol}}(X, r)$ , and  $\psi_{(\lambda,1)}$  is the identity morphism,
- (2)  $\psi_{(\lambda,t)}$  satisfies the following ‘‘cocycle’’-type formula:

$$\psi_{(\lambda t_1, t_2)} \circ \psi_{(\lambda, t_1)} = \psi_{(\lambda, t_1 t_2)},$$

- (3) A stable vector bundle (with zero Higgs field) in  $M_{\text{Dol}}(X, r)$  is a fixed point of  $\psi_{(\lambda,t)}$  for any  $\lambda \in \mathbb{C}, t \in \mathbb{C}^*$ .

*Proof.* These arguments are trivial facts and can be obtained by directly calculation from definition, so we omit the proof.  $\square$

### 2.1.3 The First Variation of the Dynamical System

Let  $u = [E, \bar{\partial}_E, \theta] \in M_{\text{Dol}}(X, r)$  with a pluri-harmonic metric  $h$ , then the tangent space of  $M_{\text{Dol}}(X, r)$  at  $u$  is given by the first hypercohomology  $\mathbb{H}^1(\mathbf{C}^\bullet)$  of the Higgs complex [Sim92]

$$\mathbf{C}^\bullet : \text{End}(E) \xrightarrow{\theta^\wedge} \text{End}(E) \otimes_{\mathcal{O}_X} \Omega_X^1 \xrightarrow{\theta^\wedge} \cdots .$$

By Kähler identities, there is an isomorphism

$$\begin{aligned} \mathbb{H}^1(\mathbf{C}^\bullet) &\simeq \mathcal{H}^1(E, \theta) := \{(\alpha, \beta) \in \Omega_X^{0,1}(\text{End}(E)) \oplus \Omega_X^{1,0}(\text{End}(E)) \\ &\quad : (\partial_{E,h} + \theta_h^\dagger)(\alpha + \beta) = (\bar{\partial}_E + \theta)(\alpha + \beta) = 0\}. \end{aligned}$$

**Definition 2.1.2.** ([HH17]) The pair  $(\alpha, \beta) \in \mathcal{H}^1(E, \theta)$  is called an *infinitesimal deformation* of the Higgs bundle  $[E, \bar{\partial}_E, \theta]$ . In particular, if  $\partial_{E,h}\alpha = \bar{\partial}_E\beta = 0$ ,  $(\alpha, \beta)$  is called a *holomorphic infinitesimal deformation*.

Now we assume  $X$  is a compact Riemann surface. Consider the family  $u(s) := [E, \bar{\partial}_{E_s}, \theta_s]$  lying in  $M_{\text{Dol}}(X, r)$  with parameter  $s$  such that  $u(0) = u$  and  $\frac{du(s)}{ds}|_{s=0} = (\alpha, \beta) \in \mathcal{H}^1(E, \theta)$ . The pluri-harmonic metric for the Higgs bundle  $[E, \bar{\partial}_{E_s}, \theta_s]$  is denoted by  $h(s)$  with  $h(0) = h$ , and fixing  $\lambda, t$ , the pluri-harmonic metric for the  $\lambda'$ -flat bundle  $[E, d''_{E_s} = \bar{\partial}_{E_s} + \lambda(\theta_s)^\dagger_{h(s)}, d'_{E_s} = t(\lambda\partial_{E_s, h(s)} + \theta_s)]$  is denoted by  $h_t(s)$  with  $h_t(0) = h_t$ , which yields operators  $\delta'_{E_s} := \delta'_{E, h_t(s)}, \delta''_{E_s} := \delta''_{E, h_t(s)}$ . There is an integral curve  $\gamma$  in  $M_{\text{Dol}}(X, r)$  passing through the point  $u$  with tangent vector  $(\alpha, \beta)$ , the  $(\lambda, t)$ -action maps this curve to another curve  $\gamma'$ , we can study its local property at the point  $\psi_{(\lambda, t)}(u)$  by calculating the variations of each order of the dynamical system  $\psi_{(\lambda, t)}$ , here we calculate the first order:

**Theorem 2.1.3.** *Assume the original point  $u$  and the parameters  $\lambda, t$  are chosen to satisfy  $h_t = h$ , and assume  $\frac{du(s)}{ds}|_{s=0} = (\alpha, \beta)$  is a holomorphic infinitesimal deformation, then*

$$\frac{d\psi_{(\lambda, t)}u(s)}{ds}\Big|_{s=0} = \left(\alpha + \frac{\lambda(1 - |t|^2)}{1 + |\lambda|^2}\beta_h^\dagger, \frac{t(1 + |\lambda|^2)}{1 + |\lambda|^2}\beta\right).$$

*Proof.* We write  $h_t(s) = h_t H_t(s)$ , and  $d'_E = d'_{E_0}, d''_E = d''_{E_0}, \delta'_E = \delta'_{E_0}, \delta''_E = \delta''_{E_0}$ , then choosing a local  $h_t$ -unitary frame  $\{e_i\}$  of  $E$ , we have

$$\begin{aligned} \lambda' \partial h_t(s)(e_i, e_j) &= \lambda' \partial h_t(H_t(s)e_i, e_j) = h_t(d'_E(H_t(s)e_i), e_j) + h_t(H_t(s)e_i, \delta''_E e_j) \\ &= h_t(H_t(s)d'_{E_s} e_i, e_j) + h_t(H_t(s)e_i, \delta''_{E_s} e_j), \\ \bar{\partial} h_t(s)(e_i, e_j) &= \bar{\partial} h_t(H_t(s)e_i, e_j) = h_t(d''_E(H_t(s)e_i), e_j) + h_t(H_t(s)e_i, \delta'_E e_j) \\ &= h_t(H_t(s)d''_{E_s} e_i, e_j) + h_t(H_t(s)e_i, \delta'_{E_s} e_j). \end{aligned}$$

Taking derivative with respect to  $s$  and evaluating at  $s = 0$  give rise to

$$\begin{aligned} h_t(d'_E \left(\frac{dH_t(s)}{ds}\Big|_{s=0}\right) e_i, e_j) &= h_t\left(\frac{d(d'_{E_s})}{ds}\Big|_{s=0} e_i, e_j\right) + h_t\left(e_i, \frac{d(\delta''_{E_s})}{ds}\Big|_{s=0} e_j\right), \\ h_t(d''_E \left(\frac{dH_t(s)}{ds}\Big|_{s=0}\right) e_i, e_j) &= h_t\left(\frac{d(d''_{E_s})}{ds}\Big|_{s=0} e_i, e_j\right) + h_t\left(e_i, \frac{d(\delta'_{E_s})}{ds}\Big|_{s=0} e_j\right), \end{aligned}$$

which implies that

$$\begin{aligned} \frac{d(\delta'_{E_s})}{ds}\Big|_{s=0} &= \delta'_E \left(\frac{dH_t(s)}{ds}\Big|_{s=0}\right) - \left(\frac{d(d'_{E_s})}{ds}\Big|_{s=0}\right)^\dagger_{h_t}, \\ \frac{d(\delta''_{E_s})}{ds}\Big|_{s=0} &= \delta''_E \left(\frac{dH_t(s)}{ds}\Big|_{s=0}\right) - \left(\frac{d(d''_{E_s})}{ds}\Big|_{s=0}\right)^\dagger_{h_t}. \end{aligned}$$

On the other hand, from the pluri-harmonicity of  $h_t(s)$ , namely

$$[d''_{E_s} + \lambda' \delta''_{E_s}, d'_{E_s} - \lambda' \delta'_{E_s}] = -\lambda' [d''_{E_s}, \delta'_{E_s}] + \lambda' [d'_{E_s}, \delta''_{E_s}] = 0,$$

it follows that

$$\begin{aligned} & (\delta'_E d''_E \frac{dH_t(s)}{ds} |_{s=0} - \delta''_E d'_E \frac{dH_t(s)}{ds} |_{s=0}) + (d''_E (\frac{d(d''_{E_s})}{ds} |_{s=0})^\dagger_{h_t} - d'_E (\frac{d(d'_{E_s})}{ds} |_{s=0})^\dagger_{h_t}) \\ & - (\delta'_E \frac{d(d''_{E_s})}{ds} |_{s=0} - \delta''_E \frac{d(d'_{E_s})}{ds} |_{s=0}) = 0. \end{aligned} \quad (2.1)$$

Due to Proposition 3.2 in [CW19], we have

$$\frac{d(d''_{E_s})}{ds} |_{s=0} = \alpha + \lambda \beta_h^\dagger, \quad \frac{d(d'_{E_s})}{ds} |_{s=0} = -\lambda' \alpha_h^\dagger + t \beta.$$

The condition  $h_t = h$  leads to

$$\delta'_E = \partial_{E,h} - \bar{\lambda} \theta, \quad \delta''_E = \bar{\lambda}' \bar{\partial} - \bar{t} \theta_h^\dagger,$$

then by (2.1), since  $(\alpha, \beta)$  is a holomorphic infinitesimal deformation, we have

$$\mathbb{D}_{h_t, \lambda'}^c \mathbb{D}^{\lambda'} \frac{dH_t(s)}{ds} |_{s=0} = 0,$$

which implies  $\mathbb{D}^{\lambda'} \frac{dH_t(s)}{ds} |_{s=0} = 0$  by applying the Kähler identities in Lemma 1.2.4. But  $\lambda'$ -flat bundle  $(E, \mathbb{D}^{\lambda'} = d'_E + d''_E)$  is stable,  $\frac{dH_t(s)}{ds} |_{s=0}$  has to be constant. Therefore, from the calculation of

$$\frac{d}{ds} |_{s=0} \left( \frac{1}{1 + |\lambda'|^2} (d''_{E_s} + \lambda' \delta''_{E_s, h_t(s)}), \frac{1}{1 + |\lambda'|^2} (d'_{E_s} - \lambda' \delta'_{E_s, h_t(s)}) \right),$$

the desired result immediately follows.  $\square$

### 2.1.4 Fixed Points of the Dynamical System

In this subsection, we study the fixed points of the dynamical system  $\psi_{(\lambda, t)}$ . We first introduce the following definition.

**Definition 2.1.4.** For a fixed pair  $(\lambda, t) \in \mathbb{C} \times \mathbb{C}^*$ , one defines the *fixed points set* of  $(\lambda, t)$ -action as  $F_{(\lambda, t)} = \{u \in \mathcal{M}_{\text{Dol}}(X, r) : \psi_{(\lambda, t)}(u) = u\}$ . For a fixed Higgs bundle  $u \in \mathcal{M}_{\text{Dol}}(X, r)$ , one defines the *stabilizer set* as  $\mathcal{C}_u = \{(\lambda, t) \in \mathbb{C} \times \mathbb{C}^* : \psi_{(\lambda, t)}(u) = u\}$ .

The following result shows that the dynamical system shares the same fixed points with the usual  $\mathbb{C}^*$ -action, that is, all  $\mathbb{C}$ -VHS in the moduli space are the fixed points.

**Theorem 2.1.5.** *Let  $F = \bigcap_{(\lambda, t) \in \mathbb{C} \times \mathbb{C}^*} F_{(\lambda, t)}$ . Then  $F$  consists of all the  $\mathbb{C}$ -VHS lying in  $\mathcal{M}_{\text{Dol}}(X, r)$ .*

*Proof.* Let  $u = (E, \bar{\partial}_E, \theta) \in \mathcal{M}_{\text{Dol}}(X, r)$  be a  $\mathbb{C}$ -VHS of the following form:

$$\left( E = \bigoplus_{i=1}^k E_i, \quad \bar{\partial}_E = \begin{pmatrix} \bar{\partial}_{E_1} & & \\ & \ddots & \\ & & \bar{\partial}_{E_k} \end{pmatrix}, \quad \theta = \begin{pmatrix} 0 & & & \\ \theta_1 & 0 & & \\ & \ddots & \ddots & \\ & & & \theta_{k-1} & 0 \end{pmatrix} \right),$$

where  $\theta_i : E_i \rightarrow E_{i+1} \otimes \Omega_X^1$ , we only need to show  $\mathcal{C}_u = \mathbb{C} \times \mathbb{C}^*$ . Let  $\eta_\lambda$  be the map that sends a harmonic Higgs bundle  $(E, \bar{\partial}_E, \theta, h)$  to its corresponding harmonic  $\lambda$ -flat bundle  $(E, \bar{\partial}_E +$

$\lambda\theta_h^\dagger, \lambda\partial_{E,h} + \theta$ ), and let  $\varphi_t$  be the natural  $\mathbb{C}^*$ -action by  $t$ . Then in fact the dynamical system  $\psi_{(\lambda,t)}$  can be expressed as

$$\psi_{(\lambda,t)} = \eta_{t\lambda}^{-1} \circ \varphi_t \circ \eta_\lambda. \quad (2.2)$$

To show  $\mathcal{C}_u = \mathbb{C} \times \mathbb{C}^*$ , it suffices to show  $\varphi_t \circ \eta_\lambda(u) = \eta_{t\lambda}(u)$  holds in  $\mathcal{M}_{\text{Dol}}(X, r)$ , which means there exists a  $C^\infty$ -automorphism  $g \in \text{Aut}(E)$  such that

$$g(\varphi_t \circ \eta_\lambda(u))g^{-1} = \eta_{t\lambda}(u). \quad (2.3)$$

Indeed, under the pluri-harmonic metric  $h$  on  $u$ ,  $\varphi_t \circ \eta_\lambda(u)$  and  $\eta_{t\lambda}$  can be written as follows:

$$\begin{aligned} \varphi_t \circ \eta_\lambda(u) &= \left( E, \begin{pmatrix} \bar{\partial}_{E_1} & \lambda(\theta_1)_h^\dagger & & \\ & \ddots & \ddots & \\ & & \bar{\partial}_{E_{k-1}} & \lambda(\theta_{k-1})_h^\dagger \\ & & & \bar{\partial}_{E_k} \end{pmatrix} \right), \begin{pmatrix} t\lambda\partial_{E_1,h} & & & \\ t\theta_1 & t\lambda\partial_{E_2,h} & & \\ & \ddots & \ddots & \\ & & t\theta_{k-1} & t\lambda\partial_{E_k,h} \end{pmatrix} \right); \\ \eta_{t\lambda}(u) &= \left( E, \begin{pmatrix} \bar{\partial}_{E_1} & t\lambda(\theta_1)_h^\dagger & & \\ & \ddots & \ddots & \\ & & \bar{\partial}_{E_{k-1}} & t\lambda(\theta_{k-1})_h^\dagger \\ & & & \bar{\partial}_{E_k} \end{pmatrix} \right), \begin{pmatrix} t\lambda\partial_{E_1,h} & & & \\ \theta_1 & t\lambda\partial_{E_2,h} & & \\ & \ddots & \ddots & \\ & & \theta_{k-1} & t\lambda\partial_{E_k,h} \end{pmatrix} \right). \end{aligned}$$

It's easy to see that (2.3) has a solution

$$g = \begin{pmatrix} \text{Id}_{E_1} & & & \\ & t^{-1}\text{Id}_{E_2} & & \\ & & \ddots & \\ & & & t^{-k+1}\text{Id}_{E_k} \end{pmatrix}.$$

Therefore,  $\psi_{(\lambda,t)}(u) = u$  for any  $(\lambda, t) \in \mathbb{C} \times \mathbb{C}^*$ .  $\square$

**Definition 2.1.6.** ([MSWW16, Moc16]) A Higgs bundle  $(E, \bar{\partial}_E, \theta)$  over  $X$  is called a *decoupled Higgs bundle* if there is a Hermitian metric  $h$  on  $E$  satisfying  $R(h) = (\partial_{E,h} + \bar{\partial}_E)^2 = 0$  and  $\theta \wedge \theta_h^\dagger = 0$ . In this case,  $h$  is called a *decoupling metric*.

**Theorem 2.1.7.** *Let  $X$  be a compact Riemann surface, and  $u \in \mathcal{M}_{\text{Dol}}(X, r)$  represents a decoupled Higgs bundle with non-trivial Higgs field, then  $\mathcal{C}_u = \mathbb{C} \times \{\mu_M, \mu_M^2, \dots, \mu_M^{M-1}, 1\}$ , where  $\mu_M = e^{\frac{2\pi i}{M}}$  for some positive integer  $M \leq r$ , is a root of unity.*

*Proof.* Case I:  $\lambda \neq 0, |t| = 1$ .

In this case, we first show  $\psi_{(\lambda,t)}$  is the usual  $S^1$ -action by  $t$  on  $\mathcal{M}_{\text{Dol}}(X, r)$ .

Let  $(E, \mathbb{D}^\lambda, h)$  be a stable  $\lambda$ -flat bundle with the pluri-harmonic metric  $h$ . The operators  $\delta'_{h_t}, \delta''_{h_t}, \partial_{h_t}, \bar{\partial}_{h_t}, \theta_{h_t}, \theta_{h_t}^\dagger$  can be defined via  $(\mathbb{D}^\lambda, h_t)$  and  $(\mathbb{D}^{\lambda'}, h_t)$ , respectively, where  $\mathbb{D}^{\lambda'} = t d'_E + d''_E$  for  $\mathbb{D}^\lambda = d'_E + d''_E$ . In order to distinguish them, we add subscripts  $\lambda, \lambda'$  for these operators. Then by definition, we have

$$\delta'_{h_t, \lambda'} = \delta'_{h_t, \lambda}, \quad \delta''_{h_t, \lambda'} = \bar{t} \delta''_{h_t, \lambda},$$

hence

$$\begin{aligned}\bar{\partial}_{h_t, \lambda'} &= \frac{1}{1 + |t\lambda|^2} \left( d''_E + |t|^2 \lambda \delta''_{h_t, \lambda} \right), & \partial_{h_t, \lambda'} &= \frac{1}{1 + |t\lambda|^2} \left( |t|^2 \bar{\lambda} d'_E + \delta'_{h_t, \lambda} \right), \\ \theta_{h_t, \lambda'}^\dagger &= \frac{\bar{t}}{1 + |t\lambda|^2} \left( \bar{\lambda} d''_E - \delta''_{h_t, \lambda} \right), & \theta_{h_t, \lambda'} &= \frac{t}{1 + |t\lambda|^2} \left( d'_E - \lambda \delta'_{h_t, \lambda} \right).\end{aligned}\tag{2.4}$$

When  $|t| = 1$ , we arrive at

$$\begin{aligned}\bar{\partial}_{h_t, \lambda'} &= \bar{\partial}_{h_t, \lambda}, & \partial_{h_t, \lambda'} &= \partial_{h_t, \lambda}, \\ \theta_{h_t, \lambda'}^\dagger &= \bar{t} \theta_{h_t, \lambda}, & \theta_{h_t, \lambda'} &= t \theta_{h_t, \lambda}\end{aligned}$$

It follows from  $\bar{\partial}_{h_t, \lambda'}^2 = \bar{\partial}_{h_t, \lambda'} \theta_{h_t, \lambda'} = \theta_{h_t, \lambda'} \wedge \theta_{h_t, \lambda'} = 0$  that  $\bar{\partial}_{h_t, \lambda}^2 = \bar{\partial}_{h_t, \lambda} \theta_{h_t, \lambda} = \theta_{h_t, \lambda} \wedge \theta_{h_t, \lambda} = 0$ , namely,  $h_t$  is also a pluri-harmonic metric on  $(E, \mathbb{D}^\lambda)$ . Then by the uniqueness of pluri-harmonic metric, we have  $h_t = c \cdot h$  for some constant  $c$  when  $|t| = 1$ . Consequently, the morphism  $\psi_{(\lambda, t)}$  sends a stable Higgs bundle  $(E, \bar{\partial}_E, \theta)$  to another one  $(E, \bar{\partial}_E, t\theta)$ , namely,  $\psi_{(\lambda, t)}$  is just the usual  $S^1$ -action by  $t$  on  $\mathcal{M}_{\text{Dol}}(X, r)$ .

For reader's convenience, we give a proof of the uniqueness of pluri-harmonic metrics here. One writes  $h_t = h \cdot s_t$ , where  $s_t$  is a self-adjoint endomorphism of  $E$  with respect to both  $h$  and  $h_t$ . Since

$$\begin{cases} \delta'_{h_t, \lambda} = \delta'_{h, \lambda} + s_t^{-1} \delta'_{h, \lambda} s_t, \\ \delta''_{h_t, \lambda} = \delta''_{h, \lambda} + s_t^{-1} \delta''_{h, \lambda} s_t, \end{cases}\tag{2.5}$$

we have

$$G_{h_t, \lambda} - G_{h, \lambda} = \frac{\bar{\lambda} |\lambda|^2}{(1 + |\lambda|^2)^2} \left( \mathbb{D}^\lambda (s_t^{-1}) D_{h, \lambda}^c s_t + s_t^{-1} \mathbb{D}^\lambda D_{h, \lambda}^c s_t \right).$$

Define the Laplacian operator  $\Delta_{h, \omega}^\lambda := -\sqrt{-1} \Lambda_\omega \mathbb{D}^\lambda D_{h, \lambda}^c$  on  $(E, \mathbb{D}^\lambda)$ , then it follows that

$$\begin{aligned}\int_X \Delta_{h, \omega}^\lambda \text{Tr}(s_t) [\omega]^n &= -\frac{(1 + |\lambda|^2)^2}{\bar{\lambda} |\lambda|^2} \int_X \text{Tr} \left( s_t \sqrt{-1} (\Lambda_\omega G_{h_t, \lambda} - \Lambda_\omega G_{h, \lambda}) - \sqrt{-1} \Lambda_\omega (\mathbb{D}^\lambda s_t \cdot s_t^{-1} D_{h, \lambda}^c s_t) \right) [\omega]^n \\ &= -\frac{(1 + |\lambda|^2)^2}{\bar{\lambda} |\lambda|^2} \int_X |(\mathbb{D}^\lambda s_t) s_t^{-\frac{1}{2}}|_{h, \omega}^2 [\omega]^n,\end{aligned}$$

which implies  $\mathbb{D}^\lambda s_t = 0$ . Therefore we obtain the decomposition of  $E$  into  $\mathbb{D}^\lambda$ -flat subbundles as the eigenbundles of  $s_t$ . However,  $(E, \mathbb{D}^\lambda)$  is a stable  $\lambda$ -flat bundle, by [Lemma 1.2.10](#),  $s_t$  has to be a scalar multiplication.

Now let  $(E, \bar{\partial}_E, \theta)$  be a decoupled Higgs bundle with non-trivial Higgs field  $\theta$ . If it is a fixed point of  $\psi_{(\lambda, t)}$  for  $|t| = 1$ , thus there is a  $C^\infty$ -automorphism  $g \in \text{Aut}(E)$  such that

$$\begin{cases} g \bar{\partial}_E g^{-1} = 0, \\ g \theta g^{-1} = t \theta.\end{cases}\tag{2.6}$$

Since  $(E, \bar{\partial}_E)$  is already a polystable bundle, there is a decomposition  $(E, \bar{\partial}_E) = \bigoplus_{i=1}^N (E_i, \bar{\partial}_{E_i})$  into stable bundles  $(E_i, \bar{\partial}_{E_i})$  with vanishing Chern classes. By the first equation of [\(2.6\)](#),  $g$  must be of

the following form

$$g = \begin{pmatrix} a_1 \text{Id}_{E_1} & & & \\ & \ddots & & \\ & & \ddots & \\ & & & a_N \text{Id}_{E_N} \end{pmatrix}$$

for nonzero constants  $a_1, \dots, a_N$ . Denote by  $pr_{E_i}$  the projection  $pr_{E_i} : E \otimes K_X \rightarrow E_i \otimes K_X$ , if there exists  $i$  such that  $pr_{E_i} \circ \theta|_{E_i} : E_i \rightarrow E \otimes K_X \rightarrow E_i \otimes K_X$  is nonzero, then the second equation of (2.6) admits a solution for such  $g$  if and only if  $t = 1$ . If each  $pr_{E_i} \circ \theta|_{E_i}$  vanishes, since  $\theta$  is nonzero and  $[\theta, \theta_h^\dagger] = 0$ , there exist  $i_1 \neq i_2 \neq \dots \neq i_M$  for  $1 \leq i_1, \dots, i_M \leq N$  such that  $pr_{E_{i_{j+1}}} \circ \theta|_{E_{i_j}} : E_{i_j} \rightarrow E_{i_{j+1}} \otimes K_X$  for  $1 \leq j \leq M-1$  and  $pr_{E_{i_1}} \circ \theta|_{E_{i_M}} : E_{i_M} \rightarrow E_{i_1} \otimes K_X$  are all nonzero. Therefore by the second equation of (2.6) we have

$$a_{i_1} = ta_{i_2}, a_{i_2} = ta_{i_3}, \dots, a_{i_{M-1}} = ta_{i_M}, a_{i_M} = ta_{i_1}, \quad (2.7)$$

thus  $t$  has to be  $M$ -roots of the unity. Moreover, all components  $a_1, \dots, a_N$  are solved by a series of equations as the form of (2.7).

Case II:  $\lambda = 0, t \in \mathbb{C}^*$ .

In this case,  $\psi(\lambda, t)$ -action is just the usual  $\mathbb{C}^*$ -action by  $t$  on  $\mathcal{M}_{\text{Dol}}(X, r)$ . The similar arguments as above shows that a decoupled Higgs bundle with non-trivial Higgs field cannot be a fixed point of  $\psi_{(\lambda, t)}$  unless  $t$  is certain  $M$ -roots of the unity for some  $M \leq r$ .

Case III:  $\lambda \neq 0, |t| \neq 1$ .

Let  $(E, \bar{\partial}_E, \theta) \in \mathcal{M}_{\text{Dol}}(X, r)$  be a decoupled Higgs bundle with non-trivial Higgs field  $\theta$  and decoupling metric  $h$ . We show that  $(E, \bar{\partial}_E, \theta)$  cannot be a fixed point of  $\psi_{(\lambda, t)}$ .

One writes

$$\begin{cases} td'_E = t\lambda \left( \partial_{E, h} + \frac{t-a}{t\lambda} \theta \right) + a\theta, \\ d''_E = \bar{\partial}_E + \lambda(1-t\bar{a})\theta_h^\dagger + t\lambda\bar{a}\theta_h^\dagger, \end{cases}$$

for some  $a \in \mathbb{C}$ , then  $(E, \bar{\partial}_E + \lambda(1-t\bar{a})\theta_h^\dagger, a\theta)$  is a Higgs bundle. Note that  $(\partial_{E, h} - \bar{\lambda}(1-t\bar{a})\theta) + (\bar{\partial}_E + \lambda(1-t\bar{a})\theta_h^\dagger)$  is a unitary connection with respect to  $h$ . If one takes

$$a = t \frac{1 + |\lambda|^2}{1 + |\lambda'|^2},$$

we find that  $h$  is the pluri-harmonic metric both for the  $\lambda'$ -flat bundle  $(E, \mathbb{D}^{\lambda'} = td'_E + d''_E)$  and the Higgs bundle  $(E, \bar{\partial}_E + \lambda(1-t\bar{a})\theta_h^\dagger, a\theta)$ . Therefore, we get

$$\psi_{(\lambda, t)}((E, \bar{\partial}_E, \theta)) = (E, \bar{\partial}_E + \lambda(1-t\bar{a})\theta_h^\dagger, a\theta).$$

If  $(E, \bar{\partial}_E, \theta)$  is a fixed point of  $\psi_{(\lambda, t)}$ , there is a  $C^\infty$ -automorphism  $g \in \text{Aut}(E)$  such that

$$\begin{cases} g\bar{\partial}_E g^{-1} - \lambda(1-t\bar{a})\theta_h^\dagger = 0, \\ g\theta g^{-1} - a\theta = 0, \end{cases} \quad (2.8)$$

note that the first equation of (2.8) can be rewrote as

$$\bar{\partial}_E(\log(g^{-1})) - \lambda(1 - t\bar{a})\theta_h^\dagger = 0, \quad (2.9)$$

acting  $\partial_{E,h}$  on both sides of (2.9), applying Kähler identity, and finally integrate over  $X$ , which implies  $\bar{\partial}_E(\log(g^{-1})) = 0$ .

Since  $\lambda \neq 0, |t| \neq 1$ , this gives  $\theta_h^\dagger = 0$  from (2.9), and hence  $\theta = 0$ , which is a contradiction.

Combining these three cases together, we complete the proof of the theorem.  $\square$

**Corollary 2.1.8.** (1) For any  $(\lambda, t) \in \mathbb{C} \times \mathbb{C}^*$ , the dynamical system does not change the pluri-harmonic metric when acts on a decoupled Higgs bundle.

(2) Assume  $\lambda$  and  $t$  are both real numbers with  $|t| < 1$  and  $t\lambda \neq 1$ , then for any decoupled Higgs bundle  $(E, \bar{\partial}_E, \theta)$  with the decoupling metric denoted as  $h$ , we have the limit

$$\lim_{n \rightarrow \infty} \psi_{(\lambda,t)}^n(E, \bar{\partial}_E, \theta) = \left( E, \bar{\partial}_E + \frac{\lambda(1+t)}{1-t\lambda^2} \theta_h^\dagger, 0 \right).$$

*Proof.* (1) In the proof of Theorem 2.1.7, for any decoupled Higgs bundle  $(E, \bar{\partial}_E, \theta)$  with the decoupling metric denoted as  $h$ , which is also the pluri-harmonic metric. Then for any  $(\lambda, t) \in \mathbb{C} \times \mathbb{C}^*$ , we have

$$\psi_{(\lambda,t)}(E, \bar{\partial}_E, \theta, h) = \left( E, \bar{\partial}_E + \frac{\lambda(1-|t|^2)}{1+|t\lambda|^2} \theta_h^\dagger, \frac{t(1+|\lambda|^2)}{1+|t\lambda|^2} \theta, h \right).$$

(2) This is directly from (1) and calculation.  $\square$

**Remark 2.1.9.** In Theorem 2.1.5 and Theorem 2.1.7 we consider two extreme cases, which can be treated as descriptions of  $\mathbb{C}$ -VHS and decoupled Higgs bundles from the viewpoint of dynamical system, since they can be viewed the limits of  $\mathbb{C}^*$ -action when  $t \rightarrow 0$  and  $t \rightarrow \infty$ , respectively.

## 2.2 Stratifications of Moduli Spaces

Throughout this section,  $X$  denotes a smooth complex projective curve (compact Riemann surface) of genus  $g \geq 2$ .

Following the description in Section 2.1.1, the natural  $\mathbb{C}^*$ -action on the Dolbeault moduli space  $\mathcal{M}_{\text{Dol}}(X, r)$  extends to an action of  $\mathbb{C}^*$  on the Hodge moduli space  $\mathcal{M}_{\text{Hod}}(X, r)$ . By a work of Simpson [Sim10], for each  $[E, \bar{\partial}_E, D^\lambda] \in \mathcal{M}_{\text{Hod}}(X, r)$ , the limit  $\lim_{t \rightarrow 0} t \cdot [E, \bar{\partial}_E, D^\lambda]$  exists and is a fixed point that lies in  $P$ . Therefore, this action gives a Bialynicki-Birula stratification of  $\mathcal{M}_{\text{Hod}}(X, r)$  into locally closed subsets (see Section 2.2.2). Restricting the stratification to the fiber over 0, it recovers the classical Bialynicki-Birula stratification of  $\mathcal{M}_{\text{Dol}}(X, r)$ . Restricting the stratification to the fiber over 1, we will have a stratification of  $\mathcal{M}_{\text{dR}}(X, r)$  into locally closed subsets, and moreover, the space of opers, appears as a special stratum. This stratification is new to us, provides a new direction on the study of  $\mathcal{M}_{\text{dR}}(X, r)$ .

On the other hand, in the same paper, Simpson showed each flat bundle over  $X$  admits a filtration satisfies Griffiths transversality condition and such that the induced graded Higgs bundle is semistable (see Theorem 2.2.3). And moreover, the limit of  $\mathbb{C}^*$ -action on that flat bundle, as a fixed point in  $\mathcal{M}_{\text{Dol}}(X, r)$ , is S-equivalent to the graded Higgs bundle, hence coincide in the moduli space  $\mathcal{M}_{\text{Dol}}(X, r)$ . This property provides a possibility on the description of certain flat bundles

(e.g, [Theorem 2.2.13](#)), and moreover, plays an important role on the study of  $\mathcal{M}_{\text{dR}}(X, r)$  (e.g, on the proof of [Conjecture 2.2.9](#) for rank 2).

This section can be treated as an application of non-Abelian Hodge theory to the study of the de Rham moduli space  $\mathcal{M}_{\text{dR}}(X, r)$ . In this section, we will describe the stratifications of moduli spaces given by  $\mathbb{C}^*$ -action. Meanwhile, some conjectures related to the study of  $\mathcal{M}_{\text{dR}}(X, r)$  will be introduced.

### 2.2.1 Simpson Filtrations on Flat Bundles

For the  $\mathbb{C}^*$ -action on the Dolbeault moduli space  $\mathcal{M}_{\text{Dol}}(X, r)$ . Since the Hitchin map

$$h : \mathcal{M}_{\text{Dol}}(X, r) \rightarrow \bigoplus_{i=1}^r H^0(X, K_X^i)$$

is proper and  $\mathbb{C}^*$ -equivariant, for any  $[E, \bar{\partial}_E, \theta] \in \mathcal{M}_{\text{Dol}}(X, r)$ , the limit  $\lim_{t \rightarrow 0} t \cdot [E, \bar{\partial}_E, \theta]$  exists and is a fixed point of this action.

There is no analogue of the Hitchin map for the Hodge moduli space  $\mathcal{M}_{\text{Hod}}(X, r)$ . However, for each  $[E, \bar{\partial}_E, D^\lambda, \lambda] \in \mathcal{M}_{\text{Hod}}(X, r)$ , the limit  $\lim_{t \rightarrow 0} t \cdot [E, \bar{\partial}_E, D^\lambda, \lambda]$  of the  $\mathbb{C}^*$ -action still exists and is a fixed point that lies in some  $P_\alpha$  ([\[Sim10\]](#)). In particular, the limit  $\lim_{t \rightarrow 0} t \cdot (E, \nabla)$  of a flat bundle  $(E, \nabla)$  exists and is a fixed point. Moreover, this limit can be described by the existence of a special filtration of this flat bundle. This filtration, is found by Simpson, we will call it a *Simpson filtration* throughout the whole thesis.

**Definition 2.2.1.** ([\[Sim10\]](#), see also [\[HH19, Hua20\]](#)) Let  $E$  be a vector bundle over  $X$  with a flat connection  $\nabla : E \rightarrow E \otimes_{\mathcal{O}_X} \Omega_X^1$ . A decreasing filtration  $F^\bullet$  of  $E$  by strict subbundles

$$E = F^0 \supset F^1 \supset \dots \supset F^k = 0$$

is called a *Simpson filtration* if it satisfies the following two conditions:

1. Griffiths transversality:  $\nabla : F^p \rightarrow F^{p-1} \otimes_{\mathcal{O}_X} \Omega_X^1$  for  $p = 1, \dots, k$ ;
2. graded-semistability: the associated graded Higgs bundle  $(\text{Gr}_F(E), \text{Gr}_F(\nabla))$ , where  $\text{Gr}_F(E) = \bigoplus_{p=0}^{k-1} E^p$  with  $E^p = F^p/F^{p+1}$  and  $\text{Gr}_F(\nabla) : E^p \rightarrow E^{p-1} \otimes_{\mathcal{O}_X} \Omega_X^1$  induced from  $\nabla$ , is a semistable Higgs bundle.

Such a triple  $(E, \nabla, F^\bullet)$  is called a *partial oper*.

**Remark 2.2.2.** When in particular the filtration  $F^\bullet$  is of full flag, i.e,  $k = \text{rk}(E)$ , and each induced map  $\text{Gr}_F(\nabla) : E^p \rightarrow E^{p-1} \otimes_{\mathcal{O}_X} \Omega_X^1$  is an isomorphism, then a partial oper is called an *oper*.

Simpson proved the following nice theorem [\[Sim10\]](#).

**Theorem 2.2.3.** *Let  $(E, \nabla)$  be a flat bundle over a smooth projective curve  $X$ .*

- (1) *There exist Simpson filtrations  $F^\bullet$  on  $(E, \nabla)$ , this means any flat bundle has partial oper structure.*
- (2) *Let  $F_1^\bullet$  and  $F_2^\bullet$  be two Simpson filtrations on  $(E, \nabla)$ , then the associated graded Higgs bundles  $(\text{Gr}_{F_1}(E), \text{Gr}_{F_1}(\nabla))$  and  $(\text{Gr}_{F_2}(E), \text{Gr}_{F_2}(\nabla))$  are  $S$ -equivalent.*
- (3)  *$(E, \nabla, F^\bullet)$  is graded-stable if and only if the Simpson filtration is unique (up to indices shifting).*

$$(4) \lim_{t \rightarrow 0} t \cdot (E, \nabla) = [\mathrm{Gr}_F(E), \mathrm{Gr}_F(\nabla)] \in \mathcal{M}_{\mathrm{Dol}}(X, r).$$

In [Sim10], Simpson gave a wonderful iterated process to show the existence of Simpson filtration for  $(E, \nabla)$ . We now sketch how it works.

Suppose  $(E, \nabla)$  admits a filtration

$$F^\bullet : 0 \subset F^{k-1} \subset \dots \subset F^0 = E$$

that satisfies the Griffiths transversality  $\nabla(F^p) \subset F^{p-1} \otimes \Omega_X^1$ , and such that the associated Higgs bundle  $(V, \theta) := (\mathrm{Gr}_F(E), \mathrm{Gr}_F(\nabla))$  is not semistable. To see the existence of such filtration, we can begin with the trivial filtration  $0 \subset F^0 = E$ , the graded Higgs bundle will be  $(\mathrm{Gr}_F(E), \mathrm{Gr}_F(\nabla)) = (E, 0)$ . Then applying the following iteration process, we can always have a such filtration. Take  $H \subset (V, \theta)$  to be the maximal destabilizing subsheaf, which is known being unique and a subbundle of  $V$ , and the quotient  $V/H$  is also a subbundle of  $E$ . As a sub-Higgs bundle of a system of Hodge bundles,  $H$  also has a structure of system of Hodge bundles, and is a sub-system of Hodge bundles of  $(V, \theta)$ , that is,  $H = \bigoplus H^p$  with each  $H^p = H \cap \mathrm{Gr}_F^p(E) \subset F^p(E)/F^{p+1}(E)$  being a strict subbundle.

The new filtration  $G^\bullet$  is defined as

$$G^p := \mathrm{Ker} \left( E \rightarrow \frac{E/F^p(E)}{H^{p-1}} \right).$$

It satisfies the Griffiths transversality since  $\theta(H^p) \subset H^{p-1} \otimes \Omega_X^1$ , and it fits into the exact sequence

$$0 \longrightarrow \mathrm{Gr}_{F^p}^p(E)/H^p \longrightarrow \mathrm{Gr}_G^p(E) \longrightarrow H^{p-1} \longrightarrow 0.$$

The idea for iterating is that, if the new resulting graded Higgs bundle  $(\mathrm{Gr}_G(E), \mathrm{Gr}_G(\nabla))$  is still not semistable, then we continue this process to obtain a new graded Higgs bundle.

Simpson showed this iterating process will always stop at a semistable graded Higgs bundle by introducing three invariants that are bounded, the iteration process will strictly decrease these invariants in lexicographic order.

For a system of Hodge bundles  $F = \bigoplus_p F^p$ , denoted by  $F^{[k]}$  the system of Hodge bundles with Hodge index shifted by  $k$ , that is,  $(F^{[k]})^p := F^{p-k}$ . Then above exact sequence gives

$$0 \longrightarrow \mathrm{Gr}_F(E)/H \longrightarrow \mathrm{Gr}_G(E) \longrightarrow H^{[1]} \longrightarrow 0. \tag{2.10}$$

Let  $A \subset F$  be the maximal destabilizing subbundle, let  $\xi(F) := \frac{\sum \mathrm{rk}(F^p)p}{\sum \mathrm{rk}(F^p)}$  be the center of gravity, and let  $\gamma(F) := \xi(F/A) - \xi(A)$ . Define the three invariants of  $F$  as

$$(\alpha(F), \beta(F), \gamma(F)),$$

where  $\alpha(F) := \mu(A)$  and  $\beta(F) := \mathrm{rk}(A)$ .

For a filtration  $F^\bullet$  appeared in the iterating process, the three invariants  $(\alpha, \beta, \gamma)$  for  $(\mathrm{Gr}_F(E), \mathrm{Gr}_F(\nabla))$  are shifted invariant, and moreover, they take only finitely many values when iterating.

The key point is to show that when iterating and obtain a graded Higgs bundle that is not semistable, then the triple of invariants for the new graded Higgs bundle is strictly less than the old graded Higgs bundle in lexicographic order.

In fact, let  $F^\bullet$  and  $G^\bullet$  be the old and new filtration, respectively, such that the associated graded Higgs bundle  $(\mathrm{Gr}_F(E), \mathrm{Gr}_F(\nabla))$  is not semistable. Let  $I \subset (\mathrm{Gr}_G(E), \mathrm{Gr}_G(\nabla))$  be the maximal

destabilizing subbundle, then (2.10) induces a map  $f : I \rightarrow H^{[1]}$  of two maximal destabilizing objects, if this map is non-zero, then  $\mu(I) \leq \mu(H^{[1]}) = \mu(H)$ , as both of them are semistable. If this map is zero, then  $I$  can be identified as a subbundle of  $\text{Gr}_F(E)/H$ , in this case,  $I$  is also a maximal destabilizing object of  $\text{Gr}_F(E)/H$  (otherwise, it will contradict to the fact that  $I$  is the maximal destabilizing object of  $\text{Gr}_G(E)$ ). From the construction of Harder–Narasimhan filtration we can see that in this case,  $\mu(H) > \mu(I)$ .

If the equality holds, namely  $\mu(I) = \mu(H)$ , then the map  $f : I \rightarrow H^{[1]}$  is non-zero, consider the exact sequence

$$0 \longrightarrow \text{Ker}(f) \longrightarrow I \longrightarrow \text{Im}(f) \longrightarrow 0.$$

If  $\text{Ker}(f)$  is non-zero, as  $I$  and  $H^{[1]}$  are both semistable with same slope, then  $\mu(\text{Ker}(f)) = \mu(I) = \mu(\text{Im}(f)) = \mu(H)$ . But  $\text{Ker}(f)$  can be identified as a subbundle of  $\text{Gr}_F(E)/H$ , which gives  $\mu(H) > \mu(\text{Ker}(f))$ , a contradiction. Therefore,  $f$  is injective and  $\text{rk}(I) \leq \text{rk}(H^{[1]}) = \text{rk}(H)$ .

If the second equality also holds, namely  $\mu(I) = \mu(H)$  and  $\text{rk}(I) = \text{rk}(H)$ , then  $I$  and  $H^{[1]}$  are isomorphic. Hence

$$\begin{aligned} \gamma(\text{Gr}_G(E)) &= \xi(\text{Gr}_G(E)/I) - \xi(I) \\ &= \xi(\text{Gr}_G(E)/H^{[1]}) - \xi(H^{[1]}) \\ &= \xi(\text{Gr}_F(E)/H) - \xi(H) - 1 \\ &= \gamma(\text{Gr}_F(E)) - 1. \end{aligned}$$

**Corollary 2.2.4** ([Sim10]). *Let  $[E, \bar{\partial}_E, D^\lambda] \in \mathcal{M}_{\text{Hod}}(X, r)$  be any  $\lambda$ -flat bundle ( $\lambda \neq 0$ ) over  $X$  in the moduli space, then*

$$\lim_{t \rightarrow 0} t \cdot [E, \bar{\partial}_E, D^\lambda] = \lim_{t \rightarrow 0} t \cdot [E, \bar{\partial}_E, \lambda^{-1} D^\lambda] \in \mathcal{M}_{\text{Dol}}(X, r),$$

where  $(E, \bar{\partial}_E, \lambda^{-1} D^\lambda)$  is the flat bundle (1-flat bundle) associated to  $(E, \bar{\partial}_E, D^\lambda)$ .

By definition, the filtration for a  $\text{GL}(r, \mathbb{C})$ -oper is a special Simpson filtration.

**Corollary 2.2.5.** *Each oper  $(E, \nabla, F^\bullet)$  over  $X$  is graded-stable, in particular,  $(E, \nabla)$  has  $F^\bullet$  as the only Simpson filtration (up to indices translation).*

*Proof.* Let  $(\bigoplus_{i=0}^{r-1} E^i, \theta)$  be the associated graded Higgs bundle, where each  $E^i$  is a line bundle and each  $\theta|_{E^i} : E^i \rightarrow E^{i-1} \otimes \Omega_X^1$  is an isomorphism. This means

$$\deg(E^i) = \deg(E^{i-1}) + 2g - 2 = \dots = \deg(E^0) + i(2g - 2),$$

note that each  $\theta$ -invariant non-zero proper subbundle of  $(\bigoplus_{i=0}^{r-1} E^i, \theta)$  has the form  $\bigoplus_{i=0}^k E^i$  ( $0 \leq k < r - 1$ ), with

$$\deg\left(\bigoplus_{i=0}^k E^i\right) = (k + 1) \deg(E^0) + k(k + 1)(g - 1),$$

so

$$\mu\left(\bigoplus_{i=1}^k E^i\right) = \deg(E^0) + k(g - 1) < \deg(E^0) + (r - 1)(g - 1) = \mu(E),$$

this means  $(E, \nabla, F^\bullet)$  is graded-stable. In particular, by (3) of Theorem 2.2.3,  $F^\bullet$  is the only Simpson filtration for  $(E, \nabla)$ .  $\square$

**Remark 2.2.6.** The non-uniqueness of Simpson filtrations on a flat bundle is easy to see. In fact, any irreducible, rank 2 flat bundle of degree 0 with the underlying vector bundle strictly semistable which is an extension of a degree 0 line bundle admits more than one Simpson filtration. One is the trivial filtration, and the other one has two terms with the first term the extension of line bundle. The two resulting graded Higgs bundles are automatically semistable and  $S$ -equivalent to each other, as a unique representative point in the Dolbeault moduli space, which parametrizes the limit point of the  $\mathbb{C}^*$ -action.

### 2.2.2 Stratifications of Moduli Spaces

Following [Sim10], we introduce the following set:

$$G_\alpha := \left\{ [E, \bar{\partial}_E, D^\lambda, \lambda] \in \mathcal{M}_{\text{Hod}}(X, r) \mid \lim_{t \rightarrow 0} t \cdot [E, \bar{\partial}_E, D^\lambda, \lambda] \in P_\alpha \right\},$$

then these  $G_\alpha$  gives a Bialynicki-Birula type stratification<sup>4</sup> of the Hodge moduli space  $\mathcal{M}_{\text{Hod}}(X, r)$

$$\mathcal{M}_{\text{Hod}}(X, r) = \bigcup_{\alpha} G_\alpha$$

into locally closed subsets. There is a natural projection  $p_\alpha : G_\alpha \rightarrow P_\alpha$  by taking the limit of the  $\mathbb{C}^*$ -action:

$$\begin{aligned} p_\alpha : G_\alpha &\longrightarrow P_\alpha, \\ [E, \bar{\partial}_E, D^\lambda, \lambda] &\longmapsto \lim_{t \rightarrow 0} t \cdot [E, \bar{\partial}_E, D^\lambda, \lambda]. \end{aligned}$$

Restricting the stratification to the fiber over any  $\lambda \in \mathbb{C}$ , we have the stratification of  $\mathcal{M}_{\text{Hod}}^\lambda(X, r)$

$$\mathcal{M}_{\text{Hod}}^\lambda(X, r) = \bigcup_{\alpha} G_\alpha^\lambda := \bigcup_{\alpha} \left( G_\alpha \cap \pi^{-1}(\lambda) \right)$$

into locally closed subsets. In particular, we have the stratifications of  $\mathcal{M}_{\text{Dol}}(X, r)$  and  $\mathcal{M}_{\text{dR}}(X, r)$

$$\mathcal{M}_{\text{Dol}}(X, r) = \bigcup_{\alpha} G_\alpha^0, \quad \mathcal{M}_{\text{dR}}(X, r) = \bigcup_{\alpha} G_\alpha^1$$

into locally closed subsets. The first one is in fact the Bialynicki-Birula stratification of  $\mathcal{M}_{\text{Dol}}(X, r)$  given by the  $\mathbb{C}^*$ -action. The second one is called the *oper stratification* of the de Rham moduli space  $\mathcal{M}_{\text{dR}}(X, r)$ , since the space of opers appears as a special stratum (Theorem 2.2.3 and Corollary 2.2.4). For any  $\lambda \in \mathbb{C}$ , the projection  $p_\alpha$  restricts on  $G_\alpha^\lambda$  gives  $p_\alpha^\lambda : G_\alpha^\lambda \rightarrow P_\alpha$ . Over each point  $u \in P_\alpha$  which is stable, denoted by  $G_\alpha^\lambda(u) := (p_\alpha^\lambda)^{-1}(u)$  the fiber of the projection  $p_\alpha^\lambda$  over  $u$ . In particular,  $G_\alpha^0(u) = (p_\alpha^0)^{-1}(u)$ , and  $G_\alpha^1(u) = (p_\alpha^1)^{-1}(u)$ . By Bialynicki-Birula theory, the fiber  $G_\alpha^0(u)$  is an affine space. Moreover, in [CW19], by applying the conformal limit techniques, the authors showed each  $G_\alpha^1(u)$  is also affine ([CW19, Corollary 1.5]).

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<sup>4</sup>Here a stratification of a topological space means this space can be expressed as the disjoint union of locally closed subsets, while in general, defining a stratification needs an extra condition, that is, these closed subsets should satisfy a nestedness property (see the description before Conjecture 2.2.9).

Denoted by NAH the natural map from Corlette–Simpson correspondence:

$$\begin{aligned} \text{NAH} : M_{\text{Dol}}(X, r) &\longrightarrow M_{\text{dR}}(X, r), \\ [(E, \bar{\partial}_E, \theta, h)] &\mapsto [E, \mathbb{D}^1 := \partial_{E,h} + \theta + \bar{\partial}_E + \theta_h^\dagger, h]. \end{aligned} \quad (2.11)$$

Moreover, the authors of [CW19] showed that for each stable point  $u \in P_\alpha$ , the image  $\text{NAH}(G_\alpha^0(u))$  of the fiber  $G_\alpha^0(u)$  under non-Abelian Hodge correspondence intersects with the fiber  $G_\alpha^1(u)$  transversely at  $\text{NAH}(u)$ .

The non-Abelian Hodge correspondence shares no light on the study of the de Rham moduli space  $\mathcal{M}_{\text{dR}}(X, r)$  from the study of the Dolbeault moduli space  $\mathcal{M}_{\text{Dol}}(X, r)$ , since  $\mathcal{M}_{\text{dR}}(X, r)$  and  $\mathcal{M}_{\text{Dol}}(X, r)$  share very few similarities as algebraic spaces. However, the strata  $G_\alpha^1$  play an important role on the understanding of  $\mathcal{M}_{\text{dR}}(X, r)$ . Especially the Lagrangian property for each fiber of  $G_\alpha^1$  over stable  $\mathbb{C}$ -VHS ( $\in P_\alpha$ ) induces a natural question on the relationship between these Lagrangian fibers:

**Conjecture 2.2.7 (Foliation Conjecture, [Sim10]).** *When varying  $\alpha$ , these Lagrangian fibers of  $p_\alpha^1 : G_\alpha^1 \rightarrow P_\alpha$  fit together to provide a smooth foliation of  $\mathcal{M}_{\text{dR}}(X, r)$  with each leaf closed.*

This conjecture is still open, one progress was recently made by the authors of [LSS13] for the case of moduli space of rank 2 parabolic connections on  $\mathbb{P}^1$  minus 4 points.

This closedness property for  $\mathcal{M}_{\text{Dol}}(X, r)$  is clearly not right, since any fiber contained in the compact nilpotent cone would be not closed. In fact, if it is closed, as a subset of a compact space, it is compact also, as it is affine, this could not happen.

We give a more explicit explanation here, define the following indexed sets by the limit of  $\mathbb{C}^*$ -action:

$$D_\alpha^0 := \left\{ [E, \bar{\partial}_E, \theta] \in \mathcal{M}_{\text{Dol}}(X, r) \mid \lim_{t \rightarrow \infty} t \cdot [E, \bar{\partial}_E, \theta] \in P_\alpha \right\}.$$

Then by Hausel’s thesis [Hau98], these sets fit together into the nilpotent cone:

$$h^{-1}(0) = \bigcup_{\alpha} D_\alpha^0, \quad (2.12)$$

which is a deformation retraction of the whole moduli space  $\mathcal{M}_{\text{Dol}}(X, r)$ . Let  $u \in P_\alpha$  be a fixed point such that the whole fiber  $G_\alpha^0(u) = \{[E, \bar{\partial}_E, \theta] \mid \lim_{t \rightarrow 0} t \cdot [E, \bar{\partial}_E, \theta] = u\}$  is contained in the nilpotent cone, that is,  $G_\alpha^0(u) \subseteq h^{-1}(0)$ . Take any  $v \in G_\alpha^0(u)$  that is not a fixed point, as  $t \cdot v \in G_\alpha^0(u)$  for all  $t \in \mathbb{C}^*$  and  $G_\alpha^0(u)$  is closed, both  $\lim_{t \rightarrow 0} t \cdot v$  and  $\lim_{t \rightarrow \infty} t \cdot v$  lie in  $G_\alpha^0(u)$ . Note the first limit is the fixed point  $u$ , and the second limit is also a fixed point by (2.12). By definition,  $G_\alpha^0(u)$  can contain only one fixed point,  $u$ , this means the two limits of a non-fixed point should coincide, this couldn’t happen since the fixed point sets are ordered by the energy functional.

Following this idea, with a discussion with Simpson, he told me the following pure algebraic-geometric result:

**Lemma 2.2.8.** *Let  $\mathbb{G}_m$  the multiplicative group, and  $Y$  be any algebraic variety. Suppose  $\mathbb{G}_m$  acts on  $Y$  with open dense orbit isomorphic to  $\mathbb{G}_m$ , and such that the two endpoints are identified. Then there does not exist an ample linearized line bundle on  $Y$ .*

*Proof.* Let  $y$  be any point lies in the open dense orbit and let  $z$  be an extra point not in the orbit, denote by  $\mathcal{O}_y$  the orbit. Then by assumption,  $\mathcal{O}_y$  and  $\{z\}$  are the only two orbits of this action, and moreover,  $\lim_{t \rightarrow 0} t \cdot y = \lim_{t \rightarrow \infty} t \cdot y = z$ . Suppose  $Y$  has an ample  $\mathbb{G}_m$ -linearized line bundle  $L$ , then there exists an invariant section of  $L^{\otimes n}$  for some  $n \in \mathbb{Z}_+$ . Hence the linear action of  $\mathbb{G}_m$

on the fiber  $L_{\lim_{t \rightarrow 0} t \cdot y}$  has positive weight, while it acts on the fiber  $L_{\lim_{t \rightarrow \infty} t \cdot y}$  has negative weight, which is a contradiction.  $\square$

With this Lemma, we can see in another way why  $G_\alpha^0(u)$  could not have two endpoints of the  $\mathbb{C}^*$ -action on a non-fixed point identified. Since the Dolbeault moduli space  $\mathcal{M}_{\text{Dol}}(X, r)$  has an ample  $\mathbb{G}_m$ -linearized line bundle (see Simpson's construction of the moduli space [Sim94a, Sim94b]), so it could not have such  $G_\alpha^0(u)$  inside with the property. Therefore,  $G_\alpha^0(u)$  could not have a non-fixed point with two endpoints of the  $\mathbb{C}^*$ -action identified.

Let  $P_0$  be the irreducible component of  $P$  that consists of all those Higgs bundles of rank  $r$  with underlying vector bundles semistable and Higgs field zero, that is,  $P_0 = \mathcal{U}(X, r)$ , the moduli space of semistable vector bundles of rank  $r$  and degree 0, which is known being an irreducible variety. Then the corresponding stratum  $G_0^1$  in  $\mathcal{M}_{\text{dR}}(X, r)$  is the unique open stratum that consists of those flat bundles  $(E, \bar{\partial}_E, \partial + \varphi)$ , where  $(E, \bar{\partial}_E)$  is a polystable vector bundle,  $\varphi \in H^0(X, \text{End}(E) \otimes \Omega_X^1)$ , and  $\partial$  is the unique unitary flat connection, with trivial Simpson filtration. On the Dolbeault side, the corresponding stratum  $G_0^0$  is a dense open subset of  $\mathcal{M}_{\text{Dol}}(X, r)$  and can be identified with the cotangent bundle  $T^*\mathcal{U}(X, r)$ , we call these strata the *lowest strata*. And if we take  $u = [E, \bar{\partial}_E, 0] \in P_0$ , then  $G_0^0(u) = H^0(X, \text{End}(E) \otimes \Omega_X^1) \subseteq \mathcal{M}_{\text{Dol}}(X, r)$ , the space of Higgs fields on  $E$ , the authors in [PPN19] showed it is closed if and only if  $E$  is very stable, i.e, there is no non-zero nilpotent Higgs field on  $E$ .

A *uniformizing Higgs bundle* is a Higgs bundle of the form

$$\left( E = \bigoplus_{i=1}^r L_i, \bar{\partial}_E = \bigoplus_{i=1}^r \bar{\partial}_{L_i}, \bigoplus_{i=1}^{r-1} (\theta_i : L_i \xrightarrow{\cong} L_{i+1} \oplus K_X) \right),$$

it is clearly determined by  $L_1 \in \text{Pic}^{(r-1)(g-1)}(X)$ , so the space  $P_u$  of uniformizing Higgs bundles is in fact  $\text{Sym}^{r-1}(L \oplus L')$  with  $L' \cong L \oplus K_X^{-1}$ , which is parametrized by  $L \in \text{Pic}^{g-1}(X)$ , the space is known to be a connected component of the fixed point set  $P$ . It corresponds to the oper stratum  $G_{\text{oper}}^1$  in  $\mathcal{M}_{\text{dR}}(X, r)$  and the stratum  $G_{\text{Hit}}^0$  in  $\mathcal{M}_{\text{Dol}}(X, r)$ , called the *Hitchin stratum*, as a similarity of Hitchin component for  $\text{SL}(r, \mathbb{C})$  case. Moreover, the oper stratum  $G_{\text{oper}}^1$  is closed in  $\mathcal{M}_{\text{dR}}(X, r)$ , since the Hitchin stratum  $G_{\text{Hit}}^0$  is closed in  $\mathcal{M}_{\text{Dol}}(X, r)$  [Sim10].

In [Sim10], Simpson proposed another method to study the behaviour of the stratifications. Let  $M$  be a (quasi-)projective variety with a stratification of locally closed subsets  $M = \bigcup_{\alpha \in \Lambda} G_\alpha$ , we call this stratification *nested* if there is a partial order  $(\Lambda, \leq)$  such that

$$\overline{G_\alpha} = \bigcup_{\beta \leq \alpha} G_\beta,$$

this implies the partial order is defined as:

$$\beta \leq \alpha \iff G_\beta \subseteq \overline{G_\alpha}.$$

The partially ordered indexed set is called the *arrangement* of the strata.

**Conjecture 2.2.9 (Nestedness Conjecture).** *The stratifications for  $\mathcal{M}_{\text{Dol}}(X, r)$  and  $\mathcal{M}_{\text{dR}}(X, r)$  are both nested, and the arrangements for both stratifications are the same.*

Simpson himself studied it for rank 2 case by using the beautiful technique of deformation theory [Sim10] (for readers' convenience, we give a detailed proof here), but for the higher rank case, it is still an open problem.

**Theorem 2.2.10** ([Sim10, Theorem 7.6]). *Conjecture 2.2.9 is true when  $r = 2$ .*

*Proof.* When  $r = 2$ , any point  $u \in P$  has the form  $[E = E^1 \oplus E^2, \bar{\partial}_E = \begin{pmatrix} \bar{\partial}_{E^1} & 0 \\ 0 & \bar{\partial}_{E^2} \end{pmatrix}, \theta : E^1 \rightarrow E^2 \otimes K_X]$  or  $[E, \bar{\partial}_E, \theta = 0] \in \mathcal{U}(X, 2)$ . In fact, from [Hit87a], the irreducible components of  $P$  are indexed by  $P_e$  for  $0 \leq e \leq g - 1$ . When  $e = 0$ ,  $P_0 = \mathcal{U}(X, 2)$ , and when  $0 < e \leq g - 1$ ,

$$P_e = \left\{ [E = E^1 \oplus E^2, \bar{\partial}_E = \bar{\partial}_{E^1} \oplus \bar{\partial}_{E^2}, \theta : E^1 \rightarrow E^2 \otimes K_X] \mid \theta \neq 0, \deg(E^1) = e, \deg(E^2) = -e \right\}.$$

In fact,  $\theta \neq 0$  and the semistability condition imply that  $0 \leq e \leq g - 1$ ,

- when  $e = 0$ ,  $\deg(E^1) = \deg(E^2) = 0$ ,  $E = E^1 \oplus E^2$  is a strictly semistable vector bundle, which is a point in  $P_0$ ;
- when  $0 < e \leq g - 1$ ,  $\theta \in \Gamma((E^1)^* \otimes E^2 \otimes K_X)$  of non-negative degree  $2g - 2 - 2e$ , it determines an effective divisor  $D$  of degree  $2g - 2 - 2e$ . Therefore, each point in  $P_e$  is fully determined by  $E^1$  of degree  $e$  and  $E^2 = E^1 \otimes TX \otimes \mathcal{O}_X(D)$ , this means that  $P_e \cong \text{Pic}^e(X) \times \text{Sym}^{2g-2-2e}(X)$ , which is known to be a smooth irreducible variety.

So  $\mathcal{M}_{\text{dR}}(X, 2)$  is stratified by  $G_e^1$  for  $0 \leq e \leq g - 1$ :

- $G_0^1 = \{[V, \nabla] \mid V \in \mathcal{U}(X, 2), \nabla : V \rightarrow V \otimes K_X\}$ , which is a dense open subset of  $\mathcal{M}_{\text{dR}}(X, 2)$  and can be identified with  $T^*\mathcal{U}(X, 2)$ ;
- $G_e^1 = \{[0 \subsetneq F^1 \subsetneq V, \nabla \text{ irreducible} \mid [E = F^1 \oplus V/F^1, \theta := \nabla : F^1 \rightarrow V/F^1 \otimes K_X] \in P_e\}$ ;

similar for  $\mathcal{M}_{\text{Dol}}(X, 2)$  and  $G_e^0$ , it suffices to show  $\overline{G_\alpha^1} = \bigcup_{\beta \leq \alpha} G_\beta^1$ , i.e, for this case, is to show

$$\overline{G_e^1} = G_e^1 \cup G_{e+1}^1 \cup \cdots \cup G_{g-1}^1, \quad 1 \leq e \leq g - 1.$$

**Step 1:**  $G_e^1 \subseteq \overline{G_{e-1}^1}$  for  $2 \leq e \leq g - 1$ :

we will show this by deformation theory, that is, for any point  $(V, \nabla)$  in  $G_e^1$ , by using the deformation theory to find a sequence of points  $\{(V_t, \nabla_t)\}$  in  $G_{e-1}^1$  with limit  $\lim_{t \rightarrow 0}(V_t, \nabla_t) = (V, \nabla)$ . Each  $(V, \nabla) \in G_e^1$  has the Simpson filtration

$$0 \subsetneq F^1 \subsetneq V$$

such that  $\nabla : F^1 \rightarrow E \otimes K_X$ , where  $\deg(F^1) = e$ . So by Simpson's [Theorem 2.2.3](#) it has limiting point of  $\mathbb{C}^*$ -action as

$$\lim_{t \rightarrow 0}(V, t \cdot \nabla) = (E = F^1 \oplus V/F^1, \theta := \nabla : F^1 \rightarrow V/F^1 \otimes K_X),$$

which is known to be stable as a Higgs bundle.

Choose any  $p \in X$  and define  $L_p := F^1(-p)$ , which has degree  $e - 1$ , denoted by  $\varphi$  the inclusion map  $\varphi : L_p \rightarrow F^1 \hookrightarrow V$ , that fits into the short exact sequence

$$0 \longrightarrow L_p \longrightarrow F^1 \longrightarrow \mathbb{C}_p \longrightarrow 0, \quad (2.13)$$

where  $\mathbb{C}_p$  is the skyscraper sheaf supported at  $p$ . Our aim is to show that we can deform  $\varphi$  to be an injective map so that  $L_p$  is a line subbundle of  $V$ , thus the new obtained filtration and the associated graded Higgs bundle will located in the next stratum  $G_{e-1}^1$ .

The deformation complex of the quadruple  $(V, \nabla, L_p, \varphi)$  is

$$\mathbf{C}^\bullet : \text{End}(V) \oplus \text{End}(L_p) \xrightarrow{\Theta} (\text{End}(V) \otimes K_X) \oplus \text{Hom}(L_p, V),$$

where

$$\Theta = \begin{pmatrix} \nabla & 0 \\ \bullet \circ \varphi & \varphi \circ \bullet \end{pmatrix},$$

associated to the deformation complex, there is a long exact sequence

$$\cdots \longrightarrow \mathbb{H}^i(\text{End}(L_p) \rightarrow \text{Hom}(L_p, V)) \longrightarrow \mathbb{H}^i(\mathbf{C}^\bullet) \longrightarrow \mathbb{H}^i(\text{End}(V) \otimes \Omega_X^\bullet) \longrightarrow \cdots,$$

as a point in  $G_e^1$ ,  $(V, \nabla)$  is gr-stable, hence it is irreducible itself. The infinitesimal deformation of  $(V, \nabla)$  is given by the first hyper-cohomology of the complex

$$(\text{End}(V) \otimes \Omega_X^\bullet, \nabla) := \text{End}(V) \xrightarrow{\nabla} \text{End}(V) \otimes \Omega_X^1,$$

since  $(V, \nabla)$  is irreducible,  $\mathbb{H}^2(\text{End}(V) \otimes \Omega_X^\bullet) \cong \mathbb{C}$ , i.e, the obstruction vanishes. To show the deformation complex  $\mathbf{C}^\bullet$  is unobstructed, which is equivalent to show  $\mathbb{H}^2(\mathbf{C}^\bullet) \cong \mathbb{H}^2(\text{End}(V) \otimes \Omega_X^\bullet) \cong \mathbb{C}$ , it suffices to show

$$\mathbb{H}^1(\text{End}(V) \otimes \Omega_X^\bullet) \rightarrow \mathbb{H}^2(\text{End}(L_p) \rightarrow \text{Hom}(L_p, V)).$$

There is an induced filtration on the complex  $(\text{End}(V) \otimes \Omega_X^\bullet, \nabla)$  defined as

$$\begin{aligned} F^p(\text{End}(V)) &:= \{\alpha \in \text{End}(V) : \alpha(F^q(V)) \subseteq F^{p+q}(V) \text{ for all } q\}, \\ F^p(\text{End}(V) \otimes \Omega_X^1) &:= F^{p-1}(\text{End}(V)) \otimes \Omega_X^1, \end{aligned}$$

so these filtered terms are given by

$$\begin{aligned} F^1(\text{End}(V)) &= \{\alpha \in \text{End}(V) : \alpha(F^1) = 0, \alpha(V) \subseteq F^1\} \\ \subseteq F^0(\text{End}(V)) &= \{\alpha \in \text{End}(V) : \alpha(F^1) \subseteq F^1\} \\ \subseteq F^{-1}(\text{End}(V)) &= \text{End}(V) \end{aligned}$$

and

$$\begin{aligned} F^2(\text{End}(V) \otimes \Omega_X^1) &= F^1(\text{End}(V)) \otimes \Omega_X^1 \\ \subseteq F^1(\text{End}(V) \otimes \Omega_X^1) &= F^0(\text{End}(V)) \otimes \Omega_X^1 \\ \subseteq F^0(\text{End}(V) \otimes \Omega_X^1) &= F^{-1}(\text{End}(V)) \otimes \Omega_X^1, \end{aligned}$$

from the Griffiths transversality for  $(V, F^\bullet)$  we can easily check that

$$\nabla : F^p(\text{End}(V)) \longrightarrow F^{p-1}(\text{End}(V)) \otimes \Omega_X^1 = F^p(\text{End}(V) \otimes \Omega_X^1),$$

so we have a filtered complex  $(\text{End}(V) \otimes \Omega_X^1, F^\bullet, \nabla)$ , and associated to it, we have the spectral

sequence

$$\mathbb{H}^i(\mathrm{Gr}_F^p(\mathrm{End}(V) \otimes \Omega_X^\bullet), \theta) = E_1^{p,i-p} \Rightarrow \mathrm{Gr}_F^p \mathbb{H}^i(\mathrm{End}(V) \otimes \Omega_X^\bullet, \nabla),$$

where  $\theta$  is induced by  $\nabla$ . By our assumption  $(F^1 \oplus V/F^1, \theta)$  is a stable Higgs bundle, then we have  $E_1$ -degeneration (see Lemma 7.1 in [Sim10]), which gives the isomorphism

$$\begin{aligned} \mathrm{Gr}_F^p \mathbb{H}^1(\mathrm{End}(V) \otimes \Omega_X^\bullet, \nabla) &\cong \mathbb{H}^1(\mathrm{Gr}_F^p(\mathrm{End}(V) \otimes \Omega_X^\bullet), \theta) \\ &= \mathbb{H}^1\left(\mathrm{Gr}_F^p(\mathrm{End}(V)) \xrightarrow{\theta} \mathrm{Gr}_F^{p-1}(\mathrm{End}(V)) \otimes \Omega_X^1\right). \end{aligned}$$

It follows that

$$\begin{aligned} \mathrm{Gr}_F^2 \mathbb{H}^1(\mathrm{End}(V) \otimes \Omega_X^\bullet, \nabla) &= H^0(\mathrm{Hom}(V/F^1, F^1) \otimes \Omega_X^1), \\ \mathrm{Gr}_F^1 \mathbb{H}^1(\mathrm{End}(V) \otimes \Omega_X^\bullet, \nabla) &= \mathbb{H}^1(\mathrm{Hom}(V/F^1, F^1) \xrightarrow{\theta} (\mathrm{End}(F^1) \oplus \mathrm{End}(V/F^1)) \otimes \Omega_X^1), \end{aligned}$$

$$\begin{aligned} \mathrm{Gr}_F^0 \mathbb{H}^1(\mathrm{End}(V) \otimes \Omega_X^\bullet, \nabla) &= \mathbb{H}^1(\mathrm{End}(F^1) \oplus \mathrm{End}(V/F^1) \xrightarrow{\theta} \mathrm{Hom}(F^1, V/F^1) \otimes \Omega_X^1), \\ \mathrm{Gr}_F^{-1} \mathbb{H}^1(\mathrm{End}(V) \otimes \Omega_X^\bullet, \nabla) &= H^1(\mathrm{Hom}(F^1, V/F^1)). \end{aligned}$$

In particular, we have the surjective morphism  $\mathbb{H}^1 \rightarrow \mathrm{Gr}_F^{-1} \mathbb{H}^1$ , that is,

$$\mathbb{H}^1(\mathrm{End}(V) \otimes \Omega_X^\bullet) \rightarrow H^1(\mathrm{Hom}(F^1, V/F^1)). \quad (2.14)$$

Applying  $\mathrm{Hom}(-, V/F^1)$  to (2.13) we have the long exact sequences

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(\mathbb{C}_p, V/F^1) \longrightarrow \mathrm{Hom}(F^1, V/F^1) \longrightarrow \mathrm{Hom}(L_p, V/F^1) \longrightarrow \mathrm{Ext}^1(\mathbb{C}_p, V/F^1) \\ \longrightarrow \mathrm{Ext}^1(F^1, V/F^1) \longrightarrow \dots \end{aligned}$$

Note that  $\mathrm{Hom}(\mathbb{C}_p, V/F^1) = 0$  and  $\mathrm{Ext}^1(\mathbb{C}_p, V/F^1) = \mathbb{C}_p$ , so we have the short exact sequence

$$0 \longrightarrow \mathrm{Hom}(F^1, V/F^1) \longrightarrow \mathrm{Hom}(L_p, V/F^1) \longrightarrow \mathbb{C}_p \longrightarrow 0,$$

which induces the long exact sequence of cohomology as follows:

$$0 \longrightarrow \mathbb{C} \longrightarrow H^1(\mathrm{Hom}(F^1, V/F^1)) \longrightarrow H^1(\mathrm{Hom}(L_p, V/F^1)) \longrightarrow 0, \quad (2.15)$$

where we used the fact  $H^0(\mathrm{Hom}(L_p, V/F^1)) = 0$  since  $\mathrm{deg}(\mathrm{Hom}(L_p, V/F^1)) < 0$  and  $H^i(\mathbb{C}_p) = \mathbb{C}$  for  $i = 0$  and 0 for  $i > 0$ . Now applying  $\mathrm{Hom}(L_p, -)$  to (2.13) we obtain the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{End}(L_p) \longrightarrow \mathrm{Hom}(L_p, F^1) \longrightarrow \mathrm{Hom}(L_p, \mathbb{C}_p) \longrightarrow \mathrm{Ext}^1(L_p, L_p) \\ \longrightarrow \mathrm{Ext}^1(L_p, F^1) \longrightarrow \mathrm{Ext}^1(L_p, \mathbb{C}_p) \longrightarrow \dots, \end{aligned}$$

which induces the surjective morphism

$$H^1(\mathrm{End}(L_p)) \longrightarrow H^1(\mathrm{Hom}(L_p, F^1)) \longrightarrow 0 \quad (2.16)$$

since  $\mathrm{Hom}(L_P, \mathbb{C}_p) \cong \mathbb{C}_p$  and  $\mathrm{Ext}^1(L_p, \mathbb{C}_p) = 0$ . From (2.14) and (2.15) we have the surjective morphism

$$\mathbb{H}^1(\mathrm{End}(V) \otimes \Omega_X^\bullet) \rightarrow H^1(\mathrm{Hom}(L_p, V/F^1)) \quad (2.17)$$

and there is an element in the kernel of the first surjective morphism which maps to a non-zero element of  $H^1(\mathrm{Hom}(F^1, V/F^1))$ . On the other hand, applying  $\mathrm{Hom}(L_p, -)$  to the short exact sequence

$$0 \rightarrow F^1 \rightarrow V \rightarrow V/F^1 \rightarrow 0$$

gives the long exact sequence

$$H^1(\mathrm{Hom}(L_p, F^1)) \rightarrow H^1(\mathrm{Hom}(L_p, V)) \rightarrow H^1(\mathrm{Hom}(L_p, V/F^1)) \rightarrow 0, \quad (2.18)$$

combining (2.16)-(2.18), we obtain the surjective morphism

$$\mathbb{H}^1(\mathrm{End}(V) \otimes \Omega_X^\bullet, \nabla) \oplus H^1(\mathrm{End}(L_p)) \rightarrow H^1(\mathrm{Hom}(L_p, V)). \quad (2.19)$$

Therefore,  $\mathbb{H}^1(\mathrm{End}(V) \otimes \Omega_X^\bullet, \nabla) \rightarrow \mathbb{H}^2(\mathrm{End}(L_p) \rightarrow \mathrm{Hom}(L_p, V))$ , which means the quadruple  $(V, \nabla, L_p, \varphi)$  is unobstructed. And the map (2.19) has an element in its kernel that maps to a non-zero element of  $H^1(\mathrm{Hom}(F^1, V/F^1))$ , which means there is a deformation of  $(V, \nabla, L_p, \varphi)$  doesn't extend to a deformation of  $F^1 \subseteq V$ . Hence we have a family of quadruples  $(V_t, \nabla_t, (L_p)_t, \varphi_t)$  such that each  $\mathrm{deg}((L_p)_t) = e - 1$  and each  $\varphi_t : (L_p)_t \hookrightarrow V_t$  is a subbundle morphism, each point  $(0 \subsetneq (L_p)_t \subsetneq V_t, \nabla_t)$  is graded-stable, hence in the stratum  $G_{e-1}^1$  and with limit  $\lim_{t \rightarrow 0} (V_t, \nabla_t) = (V, \nabla)$ .

**Step 2:**  $\overline{G_e^1} \subseteq G_e^1 \cup G_{e+1}^1 \cup \cdots \cup G_{g-1}^1$  for  $1 \leq e \leq g - 1$ :

In rank 2 case, the Simpson filtration for a flat bundle coincides with the Harder–Narasimhan filtration of the underlying vector bundle. Since the Harder–Narasimhan type is upper semi-continuous, which means that if we have a family of flat bundles  $\{(F_t^1 \subsetneq V_t, \nabla_t)\}_t$  in  $G_e^1$  with limit point  $(F_0^1 \subsetneq V_0, \nabla_0)$ , then  $\mathrm{deg}(F_0^1) \geq \mathrm{deg}(F_t^1)$ . This is equivalent to say, each family of flat bundles in the stratum  $G_e^1$  has limit point either in  $G_e^1$  itself, or in the upper strata, that is, in  $G_e^1 \cup G_{e+1}^1 \cup \cdots \cup G_{g-1}^1$ .  $\square$

Based on this, Simpson proposed another conjecture in [Sim10]:

**Conjecture 2.2.11 (Oper Stratum Conjecture).** *The oper stratum  $G_{\mathrm{oper}}^1$  is the unique closed stratum and the unique stratum of minimal dimension.*

**Corollary 2.2.12** ([Sim10]). *Conjecture 2.2.11 is true when  $r = 2$ .*

*Proof.* As we have seen in the proof of Conjecture 2.2.9 for  $r = 2$  (Theorem 2.2.10),

$$\overline{G_e^1} = G_e^1 \cup G_{e+1}^1 \cup \cdots \cup G_{g-1}^1, \quad 0 \leq e \leq g - 1.$$

In particular, one takes  $e = g - 1$ , then  $\overline{G_{g-1}^1} = G_{g-1}^1$ , that is, the oper stratum is closed, and obviously it is the unique closed stratum of minimal dimension.  $\square$

In [GnR17], the authors considered the relation between the Białynicki-Birula stratification and the Shatz stratification of  $\mathcal{M}_{\mathrm{Dol}}(X, 3)$ , where the Shatz stratification is given by the Harder–Narasimhan type of underlying vector bundles of Higgs bundles. With the inspiration, we can consider the relation between the partial oper stratification given by the Simpson filtrations and

the Shatz stratification of  $\mathcal{M}_{\text{dR}}(X, 3)$ , where the Shatz stratification is given by the Harder–Narasimhan type of underlying vector bundles of flat bundles.

For a flat bundle  $(E, \nabla)$  of rank 3, suppose  $E$  is not a stable vector bundle. Then the Harder–Narasimhan type of  $E$  can be given as follows:

1. Type (1,2), that is, the Harder–Narasimhan filtration is given by  $0 \subsetneq H^1 \subsetneq E$  with  $\text{rk}(H^1) = 1$  and  $\text{deg}(H^1) = d_1$ . In this case,  $H_1 \subseteq E$  is the maximal destabilizing subsheaf, so  $d_1 > 0$ .
2. Type (2,1), that is, the Harder–Narasimhan filtration is given by  $0 \subsetneq H^1 \subsetneq E$  with  $\text{rk}(H^1) = 2$  and  $\text{deg}(H^1) = d_1$ . As in (1),  $H_1 \subseteq E$  is the maximal destabilizing subsheaf, so  $d_1 > 0$ .
3. Type (1,1,1), that is, the Harder–Narasimhan filtration is given by  $0 \subsetneq H^1 \subsetneq H^2 \subsetneq E$  with  $\text{rk}(H^1) = 1$ ,  $\text{deg}(H^1) = d_1$  and  $\text{rk}(H^2) = 2$ ,  $\text{deg}(H^2/H^1) = d_2$ . In this case,  $H_1 \subseteq E$  is the maximal destabilizing subsheaf, and  $H_2/H_1 \subseteq E/H_1$  is the maximal destabilizing subsheaf, hence  $d_1 > 0$  and  $d_1 + d_2 > 0$ .

**Theorem 2.2.13.** *Let  $(E, \nabla) \in \mathcal{M}_{\text{dR}}(X, 3)$  be an irreducible flat bundle of rank 3 that is graded-stable, and such that  $E$  is not a stable vector bundle. Then its Simpson filtration is determined by its Harder–Narasimhan filtration as follows:*

- (1) *If the Harder–Narasimhan type of  $E$  is (1,2) as above. Let  $I \subset E/H^1$  be the sub line bundle by saturating the subsheaf  $\theta(H^1) \otimes K_X^{-1} \subset E/H^1$ , where  $\theta : H^1 \rightarrow E/H^1 \otimes K_X$  is the non-zero map induced by  $\nabla$ . Then*

- (1.1)  $0 < d_1 < g - 1$  &  $d_1 - 2g + 2 \leq \text{deg}(I) < -d_1$ , *the Simpson filtration coincides with the Harder–Narasimhan filtration. Hence*

$$\lim_{t \rightarrow 0} t \cdot (E, \nabla) = [H^1 \oplus E/H^1, \theta].$$

- (1.2)  $0 < d_1 < g - 1$  &  $-d_1 < \text{deg}(I) \leq -\frac{d_1}{2}$  *or*  $g - 1 < d_1 \leq \frac{4g-4}{3}$  &  $d_1 - 2g + 2 \leq \text{deg}(I) \leq -\frac{d_1}{2}$ , *in either case, the Simpson filtration is given by*

$$0 \subsetneq H^1 \subsetneq F^1 \subsetneq E,$$

where  $F^1 = \text{Ker}(E \rightarrow \frac{E/H^1}{I}) \subset E$  is a rank 2 subbundle. Hence

$$\lim_{t \rightarrow 0} t \cdot (E, \nabla) = [H^1 \oplus I \oplus \frac{E/H^1}{I}, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_1 & 0 & 0 \\ 0 & \varphi_2 & 0 \end{pmatrix}],$$

where  $\varphi_1 : H^1 \rightarrow I \otimes K_X$  is induced by  $\theta$  and  $\varphi_2 : I \rightarrow \frac{E/H^1}{I} \otimes K_X$  is induced by  $\nabla : F^1 \rightarrow E \otimes K_X$ .

- (2) *If the Harder–Narasimhan type of  $E$  is (2,1) as above. Let  $\theta : H^1 \rightarrow E/H^1 \otimes K_X$  be the non-zero induced map and let  $N := \text{Ker}(H^1 \rightarrow E/H^1 \otimes K_X) \subset H^1$  be the sub line bundle. Then*

- (2.1)  $0 < d_1 < g - 1$  &  $2d_1 - 2g + 2 \leq \text{deg}(N) < 0$ , *the Simpson filtration coincides with the Harder–Narasimhan filtration. Hence*

$$\lim_{t \rightarrow 0} t \cdot (E, \nabla) = [H^1 \oplus E/H^1, \theta].$$

(2.2)  $0 < d_1 < g - 1$  &  $0 < \deg(N) \leq \frac{d_1}{2}$  or  $g - 1 < d_1 \leq \frac{4g-4}{3}$  &  $2d_1 - 2g + 2 \leq \deg(N) \leq \frac{d_1}{2}$ ,  
in either case, the Simpson filtration is given by

$$0 \subsetneq N \subsetneq G^1 \subsetneq E,$$

where  $G^1 := \text{Ker}(E \rightarrow E/H^1) \subset E$  is a rank 2 subbundle. Hence

$$\lim_{t \rightarrow 0} t \cdot (E, \nabla) = [N \oplus H^1/N \oplus E/H^1, \begin{pmatrix} 0 & 0 & 0 \\ \psi_1 & 0 & 0 \\ 0 & \psi_2 & 0 \end{pmatrix}],$$

where  $\psi_1 : N \rightarrow H^1/N \otimes K_X$  and  $\psi_2 : H^1/N \rightarrow E/H^1 \otimes K_X$  are induced by  $\nabla : N \rightarrow H^1 \otimes K_X$ .

(3) If the Harder–Narasimhan type of  $E$  is  $(1,1,1)$  as above. Let  $I \subset E/H^1$  be the sub line bundle as defined in (1) and  $N \subset H^2$  be the sub line bundle as defined in (2). Then

(3.1)  $\max\{-d_1, 2d_2 - d_1\} < \deg(I) \leq d_2$ , the Simpson filtration is given by

$$0 \subsetneq H^1 \subsetneq F^1 \subsetneq E,$$

where  $F^1 = \text{Ker}(E \rightarrow \frac{E/H^1}{I}) \subset E$  is a rank 2 subbundle. Hence

$$\lim_{t \rightarrow 0} t \cdot (E, \nabla) = [H^1 \oplus I \oplus \frac{E/H^1}{I}, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_1 & 0 & 0 \\ 0 & \varphi_2 & 0 \end{pmatrix}],$$

where  $\varphi_1 : H^1 \rightarrow I \otimes K_X$  is induced by  $\theta$  in (1) and  $\varphi_2 : I \rightarrow \frac{E/H^1}{I} \otimes K_X$  is induced by  $\nabla : F^1 \rightarrow E \otimes K_X$ . In particular, if  $\deg(I) = d_2$ , then  $F^1 = H^2$ , that is, the Simpson filtration coincides with the Harder–Narasimhan filtration.

(3.2)  $d_1 - 2g + 2 \leq \deg(I) < \min\{-d_1, 2d_2 - d_1\}$ ,

(3.2.1)  $d_2 < 0$ ,

then the Simpson filtration is given by

$$0 \subsetneq H^1 \subsetneq E.$$

Hence

$$\lim_{t \rightarrow 0} t \cdot (E, \nabla) = [H^1 \oplus E/H^1, \theta],$$

where  $\theta : H^1 \rightarrow E/H^1 \otimes K_X$  is induced from  $\nabla : H^1 \rightarrow E \otimes K_X$ .

(3.2.2)  $d_2 > 0$ ,

•  $2(d_1 + d_2) - 2g + 2 \leq \deg(N) < 0$ , the Simpson filtration is given by

$$0 \subsetneq H^2 \subsetneq E.$$

Hence

$$\lim_{t \rightarrow 0} t \cdot (E, \nabla) = [H^2 \oplus E/H^2, \theta],$$

where  $\theta : H^2 \rightarrow E/H^2 \otimes K_X$  is induced from  $\nabla : H^2 \rightarrow E \otimes K_X$ .

- $0 < \deg(N) \leq d_1$ , the Simpson filtration is given by

$$0 \subsetneq N \subsetneq H^2 \subsetneq E.$$

Hence

$$\lim_{t \rightarrow 0} t \cdot (E, \nabla) = [N \oplus H^2/N \oplus E/H^2, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_1 & 0 & 0 \\ 0 & \varphi_2 & 0 \end{pmatrix}],$$

where  $\psi_1 : N \rightarrow H^2/N \otimes K_X$  and  $\theta$  in (1) and  $\psi_2 : H^2/N \rightarrow E/H^2 \otimes K_X$  are induced by  $\nabla : N \rightarrow H^2 \otimes K_X$ . In particular, if  $\deg(N) = d_1$ , then  $N = H^1$ , that is, the Simpson filtration coincides with the Hardar–Narasimhan filtration.

*Proof.* (1) and (2) are dual to each other, here we just prove (1). Since  $H^1 \subset E$  is the maximal destabilizing subbundle, we have  $\mu(H^1) = d_1 > 0$ , and since  $I$  is the sub line bundle of the semistable bundle  $E/H^1$ , we have  $\mu(I) \leq \mu(E/H^1) = -\frac{d_1}{2}$ . On the other hand, the induced map  $\theta : H^1 \rightarrow I \otimes K_X$  is non-zero, which gives  $\deg(I) \geq d_1 - 2g + 2$ . These give the maximal bound of  $d_1$  and  $\deg(I)$  as follows:

$$0 < d_1 \leq \frac{4g-4}{3}, \quad d_1 - 2g + 2 \leq \deg(I) \leq -\frac{d_1}{2}.$$

Consider the induced graded Higgs bundle  $(H^1 \oplus E/H^1, \theta)$ , it is graded-stable if and only if  $\mu(H^1 \oplus I) < 0$ , that is,  $\deg(I) < -d_1$ . Therefore, if the conditions in (1.1) are satisfied, the Simpson filtration is  $0 \subsetneq H^1 \subsetneq E$ , with the associated gr-stable Higgs bundle  $(H^1 \oplus E/H^1, \theta)$ . When  $(H^1 \oplus E/H^1, \theta)$  is not semistable, then its maximal destabilizing subbundle is  $H^1 \oplus I$ , which should satisfy  $\mu(H^1 \oplus I) > 0$ . By Simpson's iteration process, the next filtration is

$$0 \subsetneq H^1 \subsetneq F^1 \subsetneq E,$$

where  $F^1 = \text{Ker}(E \rightarrow \frac{E/H^1}{I}) \subset E$  is a rank 2 subbundle. The associated graded Higgs bundle is

$$(H^1 \oplus I \oplus \frac{E/H^1}{I}, \begin{pmatrix} 0 & 0 & 0 \\ \varphi_1 & 0 & 0 \\ 0 & \varphi_2 & 0 \end{pmatrix}), \text{ with } \varphi_1 : H^1 \rightarrow I \otimes K_X \text{ is induced by } \theta \text{ and } \varphi_2 : I \rightarrow \frac{E/H^1}{I} \otimes K_X$$

is induced by  $\nabla : F^1 \rightarrow E \otimes K_X$ . It is gr-stable if and only if  $\mu(\frac{E/H^1}{I}) < 0$ , which can be divided into the two kinds of bounds for  $d_1$  and  $\deg(I)$  as in (1.2), in either case, the Simpson filtration is  $0 \subsetneq H^1 \subsetneq F^1 \subsetneq E$ . But if the associated graded Higgs bundle is not gr-semistable, we should have  $\mu(\frac{E/H^1}{I}) > 0$ , that is,  $\deg(I) > -d_1$ , in this case,  $d_1$  and  $\deg(I)$  have bounds  $0 < d_1 < g - 1, d_1 - 2g + 2 \leq \deg(I) < -d_1$ . By Simpson's iteration process, the next filtration is  $0 \subsetneq H^1 \subsetneq E$ , which comes back to the case (1.1), the associated graded Higgs bundle will be stable, and the iteration process stop here. Therefore, we finish the proof of (1).

(3) Since  $H^1 \subset E$  is the maximal destabilizing subsheaf and  $H^2/H^1 \subset E/H^1$  is the maximal destabilizing subsheaf. And the induced morphisms  $H^1 \rightarrow I \otimes K_X$  and  $H^2/N \rightarrow E/H^2 \otimes K_X$  are both non-zero, we have the maximal bounds of  $\deg(I)$  and  $\deg(N)$  as follows:

$$d_1 - 2g + 2 \leq \deg(I) \leq d_2, \quad 2(d_1 + d_2) - 2g + 2 \leq \deg(N) \leq d_1.$$

By the uniqueness of maximal destabilizing subsheaf,  $\deg(I) = d_2$  if and only if  $I = H^2/H^1$ , and  $\deg(N) = d_1$  if and only if  $N = H^1$ .

Look at the graded Higgs bundle  $(H^1 \oplus E/H^1, \theta)$ , where  $\theta : H^1 \rightarrow E/H^1 \otimes K_X$  is the induced map from  $\nabla : H^1 \rightarrow E \otimes K_X$ . It is stable if and only if  $\mu(H^2/H^1) < 0$  and  $\mu(H^1 \oplus I) < 0$ , that is,  $d_2 < 0$  and  $\deg(I) < -d_1$ , this is the case (3.2.1). When it is not stable, then the maximal destabilizing subsheaf should have positive slope, the possible maximal destabilizing subsheaf is  $H^1 \oplus I$  or  $H^2/H^1$ :

(a) If  $\mu(H^1 \oplus I) > \mu(H^2/H^1)$  and  $\mu(H^1 \oplus I) > 0$ , that is,  $\max\{2d_2 - d_1, -d_1\} < \deg(I) \leq d_2$ , then  $(H^1 \oplus E/H^1, \theta)$  has maximal destabilizing subsheaf  $H^1 \oplus I$ . By Simpson's iteration, the next filtration is  $0 \subsetneq H^1 \subsetneq F^1 \subsetneq E$  for  $F^1 = \text{Ker}(E \rightarrow \frac{E/H^1}{I}) \subset E$  a rank 2 subbundle. Easy to see that its associated graded Higgs bundle is stable.

(b) If  $\mu(H^2/H^1) > \mu(H^1 \oplus I)$  and  $\mu(H^2/H^1) > 0$ , that is,  $d_2 > 0$  and  $d_1 - 2g + 2 \leq \deg(I) < 2d_2 - d_1$ , then  $(H^1 \oplus E/H^1, \theta)$  has maximal destabilizing subsheaf  $H^2/H^1$ . By Simpson's iteration, the next filtration is  $0 \subsetneq H^2 \subsetneq E$ , but we should discuss the stability of the graded Higgs bundle  $(H^2 \oplus E/H^2, \theta')$ :

- it is stable if and only if  $\mu(N) < 0$ , that is, when  $d_2 > 0$ ,  $d_1 - 2g + 2 \leq \deg(I) < 2d_2 - d_1$  and  $2(d_1 + d_2) - 2g + 2 \leq \deg(N) < 0$ , the Simpson filtration is  $0 \subsetneq H^2 \subsetneq E$ ;
- if it is not semistable, then its maximal destabilizing subsheaf is  $N$  and should satisfy  $\mu(N) > 0$ , and by Simpson's iteration, the next filtration is  $0 \subsetneq N \subsetneq H^2 \subsetneq E$ . Its associated Higgs bundle is stable, so the iteration stops.

Combining all the above, we obtain the statement (3).  $\square$

For each  $\alpha$ , let  $(G_\alpha^1)^{\text{VHS}} \subset G_\alpha^1$  be the subset that consists of polarized  $\mathbb{C}$ -VHS which can be identified with those Higgs bundles having the structure of systems of Hodge bundles (i.e, the fixed point set  $P_\alpha \subset G_\alpha^0$ ) by non-Abelian Hodge correspondence, that is,  $(G_\alpha^1)^{\text{VHS}} = \text{NAH}(P_\alpha)$ . Simpson guessed in [Sim10] that points in  $G_\alpha^1 \setminus (G_\alpha^1)^{\text{VHS}}$  do not relate to the points in  $G_\alpha^0$  via the non-Abelian Hodge correspondence, this is the following conjecture:

**Conjecture 2.2.14.**

$$(G_\alpha^1)^{\text{VHS}} = G_\alpha^1 \cap \text{NAH}(G_\alpha^0).$$

For each polarized  $\mathbb{C}$ -VHS  $(E, \nabla)$  such that the corresponding monodromy representation is irreducible, then its Simpson filtration coincides with its Hodge filtration, this provides a method to construct its Hodge filtration.

### 2.2.3 Asymptotic Behaviour of the Dynamical System

After introducing Simpson filtrations for flat bundles, we now come back to study the asymptotic behaviour of the dynamical system  $\psi_{(\lambda, t)}$  on  $\mathcal{M}_{\text{Dol}}(X, r)$  by applying Simpson's [Theorem 2.2.3](#). We first introduce the following notations:

**Definition 2.2.15.** Given  $(E, \bar{\partial}_E, \theta) \in \mathcal{M}_{\text{Dol}}(X, r)$  which admits a pluri-harmonic metric  $h$ , we define following five limits

- $\psi_{\underline{(0,0)}}(E, \bar{\partial}_E, \theta) := \lim_{t \rightarrow 0} \left( \psi_{(0,t)}(E, \bar{\partial}_E, \theta) \right) = \lim_{t \rightarrow 0} (E, \bar{\partial}_E, t\theta);$
- $\psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta) := \lim_{\lambda \rightarrow 0} \left( \psi_{(\lambda,0)}(E, \bar{\partial}_E, \theta) \right);$

- $\psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta) := \lim_{t \rightarrow 0} \left( \lim_{\lambda \rightarrow 0} \left( \psi_{(\lambda,t)}(E, \bar{\partial}_E, \theta) \right) \right);$
- $\psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta) := \lim_{\lambda \rightarrow 0} \left( \lim_{t \rightarrow 0} \left( \psi_{(\lambda,t)}(E, \bar{\partial}_E, \theta) \right) \right);$
- $\psi_{(0,0)}(E, \bar{\partial}_E, \theta) := \lim_{(\lambda,t) \rightarrow (0,0)} \left( \psi_{(\lambda,t)}(E, \bar{\partial}_E, \theta) \right).$

where in the second limit, the map  $\psi_{(\lambda,0)}$  is defined by Simpson's [Theorem 2.2.3](#), that is,

$$\begin{aligned} \psi_{(\lambda,0)}(E, \bar{\partial}_E, \theta) &= \lim_{t \rightarrow 0} \left( E, \bar{\partial}_E + \lambda \theta_h^\dagger, t \partial_{E,h} + t \lambda^{-1} \theta \right) \\ &= \left( \text{Gr}_{F_\lambda}(E_\lambda), \text{Gr}_{F_\lambda}(\nabla_\lambda) \right), \end{aligned}$$

where  $h$  is a pluri-harmonic metric on  $(E, \bar{\partial}_E, \theta)$ ,  $(E_\lambda, \nabla_\lambda) = (E, \bar{\partial}_E + \lambda \theta_h^\dagger, \partial_{E,h} + \lambda^{-1} \theta)$  is the associated flat bundle, and  $F_\lambda^\bullet$  denotes the Simpson filtration on  $(E_\lambda, \nabla_\lambda)$ .

In general, it is not clear that above limits coincide if they exist. The following theorem confirms the coincidence of these limits for some special cases.

**Theorem 2.2.16.** *Let  $X$  be a compact Riemann surface.*

- (1) *If  $(E, \bar{\partial}_E, \theta) \in \mathcal{M}_{\text{Dol}}(X, r)$  is a  $\mathbb{C}$ -VHS, or a decoupled Higgs bundle, then the above limits exist and coincide in  $\mathcal{M}_{\text{Dol}}(X, r)$ .*
- (2) *Let  $(E, \bar{\partial}_E, \theta) \in \mathcal{M}_{\text{Dol}}(X, 3)$  and assume the maximal destabilizing subbundle of  $(E, \bar{\partial}_E)$  is preserved by  $\theta_h^\dagger$  for the pluri-harmonic metric  $h$  on  $(E, \bar{\partial}_E, \theta)$ . If the limit  $\psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta)$  exists, then it coincides with the limit  $\psi_{(0,0)}(E, \bar{\partial}_E, \theta)$ .*

*Proof.* (1) (i) Let  $(E, \bar{\partial}_E, \theta) \in \mathcal{M}_{\text{Dol}}(X, r)$  be a  $\mathbb{C}$ -VHS. Since it is a fixed point of  $(\lambda, t)$ -action for any  $(\lambda, t) \in \mathbb{C} \times \mathbb{C}^*$  by [Theorem 2.1.5](#), we have

$$\psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta) = \psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta) = \psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta) = \psi_{(0,0)}(E, \bar{\partial}_E, \theta) = (E, \bar{\partial}_E, \theta).$$

Hence we only need to show  $\psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta) = (E, \bar{\partial}_E, \theta)$ . For  $\lambda \neq 0$ , write

$$(E, \bar{\partial}_E, \theta) = \left( \bigoplus_{i=1}^k (E_i, \bar{\partial}_{E_i}), \bigoplus_{i=1}^{k-1} (\theta_i : E_i \rightarrow E_{i+1} \otimes K_X) \right),$$

then by virtue of the pluri-harmonic metric  $h$  on  $(E, \bar{\partial}_E, \theta)$ , we have a holomorphic flat connection

$$\nabla = \partial_{E,h} + \lambda^{-1} \theta = \begin{pmatrix} \partial_{E_1,h} & & & & \\ \lambda^{-1} \theta_1 & \partial_{E_2,h} & & & \\ & \ddots & \ddots & & \\ & & & \lambda^{-1} \theta_{k-1} & \partial_{E_k,h} \end{pmatrix}$$

with respect to the holomorphic structure

$$\bar{\partial}'_E = \bar{\partial}_E + \lambda \theta_h^\dagger = \begin{pmatrix} \bar{\partial}_{E_1} & \lambda(\theta_1)_h^\dagger & & & \\ & \ddots & \ddots & & \\ & & \bar{\partial}_{E_{k-1}} & \lambda(\theta_{k-1})_h^\dagger & \\ & & & \bar{\partial}_{E_k} & \end{pmatrix}.$$

There is a Simpson filtration  $F^\bullet$  on  $(E, \bar{\partial}'_E, \nabla)$  given by  $F^p := \bigoplus_{i=1}^{k-p} E_i$ ,  $0 \leq p \leq k-1$  since one easily checks that

$$\nabla F^p \subset F^{p-1} \otimes K_X, \quad \bar{\partial}'_E F^p = 0.$$

It follows that  $\psi_{(\lambda,0)}(E, \bar{\partial}'_E, \theta) = (E, \bar{\partial}_E, \lambda^{-1}\theta)$  from Simpson's [Theorem 2.2.3](#).

Therefore,  $\psi_{\overline{(0,0)}}(E, \bar{\partial}'_E, \theta) = \lim_{\lambda \rightarrow 0} (E, \bar{\partial}'_E, \lambda^{-1}\theta) = (E, \bar{\partial}_E, \theta)$ .

(ii) Let  $(E, \bar{\partial}_E, \theta) \in \mathcal{M}_{\text{Dol}}(X, r)$  be a decoupled Higgs bundle with a decoupling metric  $h$ . We can assume  $\theta$  is non-zero. By definition, it is obvious that

$$\psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta) = \psi_{(0,0)}(E, \bar{\partial}_E, \theta) = (E, \bar{\partial}_E).$$

A key observation is that if two Higgs bundles  $(\bar{\partial}_E + a\theta_h^\dagger, b\theta)$  and  $(\bar{\partial}_E + a'\theta_h^\dagger, b'\theta)$  for  $a, a', b, b' \in \mathbb{C}^*$  are equivalent, we must have  $a = a'$  and  $|\frac{b'}{b}| = 1$ . Indeed, suppose there is a  $C^\infty$ -automorphism  $g \in \text{Aut}(E)$  such that

$$\begin{cases} g \circ \bar{\partial}_E \circ g^{-1} + ag\theta_h^\dagger g^{-1} = \bar{\partial}_E + a'\theta_h^\dagger, \\ bg\theta g^{-1} = b'\theta, \end{cases} \quad (2.20)$$

let  $\bar{\partial}'_E := \bar{\partial}_E + a\theta_h^\dagger$ , which defines a new holomorphic on  $E$ . Then the first equation of (2.20) can be rewrote as

$$g \circ \bar{\partial}'_E \circ g^{-1} = \bar{\partial}'_E + (a - a')\theta_h^\dagger,$$

same argument with (2.8) and (2.9) implies  $a = a'$ . Choose a point  $x \in X$  and a neighborhood  $U \subseteq X$  of  $x$  such that  $\theta|_U \neq 0$ , and one expresses  $\theta = \Theta dz$  for matrix-valued function  $\Theta$  over  $U$ . Due to the condition  $[\theta, \theta_h^\dagger] = 0$ , the  $r \times r$  matrix  $\Theta|_{x'}$  cannot be nilpotent for some point  $x' \in U$ , hence it is diagonalizable. Then from the second equation it follows that  $|\frac{b'}{b}| = 1$ . As a consequence, we can calculate the limits (in the sense of equivalent classes) from [Corollary 2.1.8](#)

$$\begin{aligned} & \lim_{(\lambda,t) \rightarrow (0,0)} \left( E, \bar{\partial}_E + \frac{\lambda(1-|t|^2)}{1+|t\lambda|^2} \theta_h^\dagger, \frac{t(1+|\lambda|^2)}{1+|t\lambda|^2} \theta \right) \\ &= \lim_{\lambda \rightarrow 0} \lim_{t \rightarrow 0} \left( E, \bar{\partial}_E + \frac{\lambda(1-|t|^2)}{1+|t\lambda|^2} \theta_h^\dagger, \frac{t(1+|\lambda|^2)}{1+|t\lambda|^2} \theta \right) = \lim_{\lambda \rightarrow 0} (E, \bar{\partial}_E + \lambda \theta_h^\dagger) \\ &= \lim_{t \rightarrow 0} \lim_{\lambda \rightarrow 0} \left( E, \bar{\partial}_E + \frac{\lambda(1-|t|^2)}{1+|t\lambda|^2} \theta_h^\dagger, \frac{t(1+|\lambda|^2)}{1+|t\lambda|^2} \theta \right) = \lim_{t \rightarrow 0} (E, \bar{\partial}_E, t\theta) \\ &= (E, \bar{\partial}_E), \end{aligned}$$

which leads to  $\psi_{\overline{(0,0)}}(E, \bar{\partial}'_E, \theta) = \psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta) = \psi_{(0,0)}(E, \bar{\partial}_E, \theta) = (E, \bar{\partial}_E)$ .

(2) (i) Consider the family of flat bundles  $\{(E_\lambda, \nabla_\lambda) := (E, \bar{\partial}_E + \lambda\theta_h^\dagger, \partial_{E,h} + \lambda^{-1}\theta)\}_{\lambda \in \mathbb{C}}$ . We first assume the underlying family of flat bundles  $\{(E, \bar{\partial}_E + \lambda\theta_h^\dagger)\}$  are not semistable on some small open punctured neighborhood of  $\lambda = 0$ , then by the openness of semistability, the one at  $\lambda = 0$  is also not semistable. That is,  $(E, \bar{\partial}_E)$  is not semistable.

Let  $L \subseteq (E, \bar{\partial}_E)$  be the maximal destabilizing subbundle, then obviously  $\deg(L) > 0$ . And by assumption, for each  $\lambda \in \mathbb{C}^*$ ,  $L$  is also the maximal destabilizing subbundle of  $(E, \bar{\partial}_E + \lambda\theta_h^\dagger)$ . Let  $L^\perp$  be the orthogonal complement of  $L$  in  $E$  with respect to  $h$ , that is, we have  $C^\infty$ -decomposition  $E \cong L \oplus E/L$  such that  $E/L$  can be identified with  $L^\perp$ .

$L$  is preserved by  $\bar{\partial}_E$  and  $\theta_h^\dagger$ , so with respect to above decomposition, we can write  $\bar{\partial}_E$  and  $\theta$  as the following:

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial}_1 & \alpha \\ 0 & \bar{\partial}_2 \end{pmatrix}, \quad \theta = \begin{pmatrix} \theta_1 & 0 \\ \beta & \theta_2 \end{pmatrix},$$

where  $\beta$  must be non-zero, otherwise  $(L, \bar{\partial}_E|_L, \theta|_L)$  is a Higgs subbundle of  $(E, \bar{\partial}_E, \theta)$  with positive degree, this is impossible. In rank 2 case, the Simpson filtration on a flat bundle coincides with the Harder–Narasimhan filtration of the underlying vector bundle, so we have

$$\psi_{(0,0)}(E, \bar{\partial}_E, \theta) = \lim_{\lambda \rightarrow 0} \left( E, \begin{pmatrix} \bar{\partial}_1 + \lambda\bar{\theta}_1 & 0 \\ 0 & \bar{\partial}_2 + \lambda\bar{\theta}_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \bar{\alpha} + \lambda^{-1}\beta & 0 \end{pmatrix} \right).$$

Taking the  $C^\infty$ -automorphism  $g = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \in \text{Aut}(E)$ , by directly calculation, we have

$$\begin{aligned} g \circ \begin{pmatrix} \bar{\partial}_1 + \lambda\bar{\theta}_1 & 0 \\ 0 & \bar{\partial}_2 + \lambda\bar{\theta}_2 \end{pmatrix} \circ g^{-1} &= \begin{pmatrix} \bar{\partial}_1 + \lambda\bar{\theta}_1 & 0 \\ 0 & \bar{\partial}_2 + \lambda\bar{\theta}_2 \end{pmatrix} \\ g \circ \begin{pmatrix} 0 & 0 \\ \bar{\alpha} + \lambda^{-1}\beta & 0 \end{pmatrix} \circ g^{-1} &= \begin{pmatrix} 0 & 0 \\ \lambda\bar{\alpha} + \beta & 0 \end{pmatrix} \end{aligned}$$

For the following two Higgs bundles

$$\left( E, \begin{pmatrix} \bar{\partial}_1 + \lambda\bar{\theta}_1 & 0 \\ 0 & \bar{\partial}_2 + \lambda\bar{\theta}_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \lambda\bar{\alpha} + \beta & 0 \end{pmatrix} \right), \quad \left( E, \begin{pmatrix} \bar{\partial}_1 + \lambda'\bar{\theta}_1 & 0 \\ 0 & \bar{\partial}_2 + \lambda'\bar{\theta}_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \lambda'\bar{\alpha} + \beta & 0 \end{pmatrix} \right),$$

they represent a same point in the moduli space  $\mathcal{M}_{\text{Dol}}(X, r)$  if and only if  $\lambda = \lambda'$ , hence

$$\psi_{(0,0)}(E, \bar{\partial}_E, \theta) = \left( E, \begin{pmatrix} \bar{\partial}_1 & 0 \\ 0 & \bar{\partial}_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} \right) = \lim_{t \rightarrow 0} (E, \bar{\partial}_E, t\theta),$$

As a conclusion,  $\psi_{(0,0)}(E, \bar{\partial}_E, \theta) = \psi_{(0,0)}(E, \bar{\partial}_E, \theta)$ .

(ii) If the family of flat bundles  $\{(E, \bar{\partial}_E + \lambda\theta_h^\dagger)\}$  are semistable on some small open punctured neighborhood of  $\lambda = 0$ , then by assumption,  $(E, \bar{\partial}_E)$  is also semistable, otherwise the maximal destabilizing subbundle of  $(E, \bar{\partial}_E)$  is a subbundle of  $(E, \bar{\partial}_E + \lambda\theta_h^\dagger)$ , with positive degree. In this

case, it's easy to see that

$$\psi_{\underline{(0,0)}}(E, \bar{\partial}_E, \theta) = (E, \bar{\partial}_E) = \psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta).$$

□

For a fixed  $t \in \mathbb{C}^*$ , then we have a family of one-parameter dynamical system  $\{\varphi_\lambda := \psi_{(\lambda,t)}\}_{\lambda \in \mathbb{C}}$ , clearly,  $\varphi_0$  is the usual  $\mathbb{C}^*$ -action given by  $t$ . The following theorem says, the family  $\varphi_\lambda$  is continuous at  $\lambda = 0$ , that is,  $\lim_{\lambda \rightarrow 0} \varphi_\lambda = \varphi_0$ .

**Theorem 2.2.17.** *For any  $(E, \bar{\partial}_E, \theta) \in \mathcal{M}_{\text{Dol}}(X, r)$ , the third limit  $\psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta)$  in Definition 2.2.15 exists, and coincides with the first limit  $\psi_{\underline{(0,0)}}(E, \bar{\partial}_E, \theta)$ , that is, a  $\mathbb{C}$ -VHS.*

*Proof.* Let  $h$  be the pluri-harmonic metric on the Higgs bundle  $(E, \bar{\partial}_E, \theta) \in \mathcal{M}_{\text{Dol}}(X, r)$ ,  $h_t$  be the pluri-harmonic metric on  $\psi_{(\lambda,t)}(E, \bar{\partial}_E, \theta)$ .

Writing  $h_t = h \cdot s$  with  $s = e^\chi$  for  $\chi \in \text{End}(E)$ , by virtue of formulas (2.4) and (2.5), the direct calculation shows that the image of  $(\lambda, t)$ -action on  $(E, \bar{\partial}_E, \theta)$  is given by  $\psi_{(\lambda,t)}(E, \bar{\partial}_E, \theta) = (E, \bar{\partial}'_E, \theta')$ , where

$$\begin{aligned} \bar{\partial}'_E(\lambda, t) &= \bar{\partial}_E + \frac{\lambda(1 - |t|^2)}{1 + |t\lambda|^2} \theta_h^\dagger + \frac{\lambda|t|^2}{1 + |t\lambda|^2} s^{-1}(\bar{\lambda}\bar{\partial}_E - \theta_h^\dagger)s, \\ \theta'(\lambda, t) &= \frac{t(1 + |\lambda|^2)}{1 + |t\lambda|^2} \theta - \frac{\lambda t}{1 + |t\lambda|^2} s^{-1}(\partial_{E,h} - \bar{\lambda}\theta)s. \end{aligned}$$

The condition  $\bar{\partial}'_E \theta' = 0$  gives rise to a equation satisfied by  $s$ . Fix  $t \in \mathbb{C}^*$ , we can always choose a small open punctured neighborhood  $U$  of  $\lambda = 0$ , such that  $(E, \bar{\partial}'_E(\lambda, t), \theta'(\lambda, t))$  are not equivalent to each other (that is, they represent different points in the moduli space).

Therefore, we have

$$\lim_{\lambda \rightarrow 0} \left( \psi_{(\lambda,t)}(E, \bar{\partial}_E, \theta) \right) = \psi_{(0,t)}(E, \bar{\partial}_E, \theta) = (E, \bar{\partial}_E, t\theta).$$

□

**Corollary 2.2.18.** (1) *For a Higgs bundle  $(E, \bar{\partial}_E, \theta) \in \mathcal{M}_{\text{Dol}}(X, r)$ , if the forth limit  $\psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta)$ , or the last limit  $\psi_{(0,0)}(E, \bar{\partial}_E, \theta)$  in Definition 2.2.15 exists, then the limit must be a  $\mathbb{C}$ -VHS.*

(2) *Define  $F^\circ := \bigcap_{(\lambda,t) \in \mathbb{C}^* \times \mathbb{C}^*} F_{(\lambda,t)}$ , then  $F^\circ = F$ .*

*Proof.* (1) Write  $\psi_{\overline{(0,0)}}(E, \bar{\partial}_E, \theta) = (E, \bar{\partial}'_E, \theta')$ . On one hand, we have

$$\begin{aligned} \lim_{\tilde{t} \rightarrow 0} \lim_{\lambda \rightarrow 0} \lim_{t \rightarrow 0} \psi_{(\lambda,t,\tilde{t})}(E, \bar{\partial}'_E, \theta') &= \lim_{\tilde{t} \rightarrow 0} \lim_{\lambda \rightarrow 0} \lim_{t \rightarrow 0} \psi_{(\lambda,t,\tilde{t})} \circ \psi_{(\lambda,t)}(E, \bar{\partial}_E, \theta) \\ &= \lim_{\tilde{t} \rightarrow 0} \lim_{\lambda \rightarrow 0} \lim_{t \rightarrow 0} \psi_{(\lambda,\tilde{t})}(E, \bar{\partial}_E, \theta) \\ &= \lim_{\tilde{t} \rightarrow 0} (E, \bar{\partial}'_E, \theta') = (E, \bar{\partial}'_E, \theta'). \end{aligned}$$

On the other hand, we have

$$\lim_{\tilde{t} \rightarrow 0} \lim_{\lambda \rightarrow 0} \lim_{t \rightarrow 0} \psi_{(\lambda,t,\tilde{t})}(E, \bar{\partial}'_E, \theta') = \lim_{\tilde{t} \rightarrow 0} \psi_{(0,\tilde{t})}(E, \bar{\partial}'_E, \theta') = \lim_{\tilde{t} \rightarrow 0} (E, \bar{\partial}'_E, \tilde{t}\theta'),$$

compare these two limits, we find that  $(E, \bar{\partial}'_E, \theta')$  must be a  $\mathbb{C}$ -VHS.

(2) Let  $(E, \bar{\partial}_E, \theta) \in F^\circ$ , then

$$\begin{aligned} \lim_{t \rightarrow 0} \lim_{\lambda \rightarrow 0} \psi_{(\lambda, t)}(E, \bar{\partial}_E, \theta) &= \lim_{t \rightarrow 0} (E, \bar{\partial}_E, \theta) = (E, \bar{\partial}_E, \theta) \\ &= \lim_{t \rightarrow 0} \psi_{(0, t)}(E, \bar{\partial}_E, \theta) = \lim_{t \rightarrow 0} (E, \bar{\partial}_E, t\theta), \end{aligned}$$

which means that  $(E, \bar{\partial}_E, \theta)$  is a  $\mathbb{C}$ -VHS, hence by [Theorem 2.1.5](#), we have  $F^\circ = F$ .  $\square$

**Remark 2.2.19.** (1) When the base field is of positive characteristic, in [[LSZ19](#), [LSYZ19](#)], the authors construct the so called Higgs–de Rham flow via  $\psi_{(1,0)}$ . More precisely, this flow begins with a graded Higgs bundle  $(E_0, \bar{\partial}_{E_0}, \theta_0)$ , then along the non-Abelian Hodge correspondence of positive characteristic version (that is, the Ogas–Vologodsky correspondence) to obtain a flat bundle and taking the graded Higgs bundle via Simpson filtration, this process is in fact  $\{\psi_{(1,0)}^n\}_{n \in \mathbb{N}}$ . As a result, they build the correspondence between Fontaine modules and periodic Higgs–de Rham flows (that is, there exists some  $n$  such that  $\psi_{(1,0)}^n(E_0, \bar{\partial}_{E_0}, \theta_0) \cong (E_0, \bar{\partial}_{E_0}, \theta_0)$ ) over projective schemes.

(2) For a harmonic Higgs bundle  $(E, \bar{\partial}_E, \theta, h) \in \mathcal{M}_{\text{Dol}}(X, r)$ , it determines a family of flat connections

$$D_{\lambda, t} := \bar{\partial}_E + \lambda t \theta_h^\dagger + \partial_{E, h} + \lambda^{-1} t \theta,$$

where  $\lambda \in \mathbb{C}^*$ ,  $t \in \mathbb{R}^+$ , and  $h_t$  is the pluri-harmonic metric on  $(E, \bar{\partial}_E, t\theta)$ . Fix  $\hbar := \lambda t^{-1}$ , then we have a family of flat connections parametrized by  $t$ :

$$D_{\hbar, t} = \bar{\partial}_E + \hbar t^2 \theta_h^\dagger + \partial_{E, h} + \hbar^{-1} \theta,$$

the limit  $\lim_{t \rightarrow 0} D_{\hbar, t}$ , when exists, called the *conformal limit* of  $(E, \bar{\partial}_E, \theta)$ . In [[Gai14](#)], Gaiotto proposed the following conjecture:

**Conjecture 2.2.20 (Gaiotto’s Conformal Limit Conjecture, [[Gai14](#)]).** *When  $u := (E, \bar{\partial}_E, \theta) \in G_{\text{Hit}}^0$ , then its conformal limit  $\lim_{t \rightarrow 0} D_{\hbar, t}(u)$  exists and located in  $G_{\text{oper}}^1$ . And this gives an analytic isomorphism between fibers  $G_{\text{Hit}}^0(u) \cong G_{\text{oper}}^1(u)$ .*

This conjecture was verified by the authors in [[DFK<sup>+</sup>16](#)] recently, and with some generalization by the authors in [[CW19](#)], that is, this conjecture holds true for any  $u := (E, \bar{\partial}_E, \theta) \in G_\alpha^0$  such that the limit  $\lim_{t \rightarrow 0} t \cdot$  is stable.

Now we give the following property to see how conformal limits work for our case.

**Proposition 2.2.21.** *Let  $X$  be a compact Riemann surface, and  $(t, \lambda) \in \mathbb{R}^* \times \mathbb{R}$ , then for a Higgs bundle  $(E, \bar{\partial}_E, \theta) \in \mathcal{M}_{\text{Dol}}(X, r)$  such that the limit  $\lim_{t \rightarrow 0} t \cdot (E, \bar{\partial}_E, \theta)$  is stable, then*

$$\psi_{(0,0)}(E, \bar{\partial}_E, \lambda\theta) = \psi_{(0,0)}(E, \bar{\partial}_E, \theta) = \lim_{t \rightarrow 0} (E, \bar{\partial}_E, t\theta).$$

*Proof.* Define a new dynamical system on the moduli space  $\mathcal{M}_{\text{Dol}}(X, r)$ :

$$\phi_{(t, \lambda, \eta)} : \mathcal{M}_{\text{Dol}}(X, r) \longrightarrow \mathcal{M}_{\text{Dol}}(X, r)$$

parametrized by  $(t, \lambda, \eta) \in \mathbb{C}^* \times \mathbb{C} \times \mathbb{C}^*$  as follows:

$$\begin{aligned} (E, \bar{\partial}_E, \theta) &\xrightarrow{\mathbb{C}^*\text{-action}} (E, \bar{\partial}_E, t\theta) \xrightarrow{\text{NAHC}} (E, \bar{\partial}_E + \lambda \bar{t} \theta_{h_t}^\dagger, \lambda \partial_{E, h_t} + t\theta) \\ &\xrightarrow{\mathbb{C}^*\text{-action}} (E, \bar{\partial}_E + \lambda \bar{t} \theta_{h_t}^\dagger, \eta \lambda \partial_{E, h_t} + \eta t \theta) \xrightarrow{\text{NAHC}} (E, \bar{\partial}_{E, h_\eta}, \theta_{h_\eta}). \end{aligned}$$

Then for  $(t, \lambda) \in \mathbb{R}^* \times \mathbb{R}$ , we have

$$\lim_{\lambda \rightarrow 0} \lim_{t \rightarrow \lambda} \left( \phi_{(t, \lambda, \eta)}(E, \bar{\partial}_E, \theta) \right) = \lim_{\lambda \rightarrow 0} \left( \text{Gr}_{F_\lambda}(E_\lambda), \text{Gr}_{F_\lambda}(\nabla_\lambda) \right),$$

where  $(E_\lambda, \nabla_\lambda) = (E, \bar{\partial}_E + |\lambda|^2 \theta_{h_\lambda}^\dagger, \partial_{E, h_\lambda} + \theta)$  is the flat bundle, and  $F_\lambda^\bullet$  is a Simpson filtration on  $(E_\lambda, \nabla_\lambda)$ .

From above results related to conformal limits, the limit  $\lim_{\lambda \rightarrow 0} (E, \bar{\partial}_E + |\lambda|^2 \theta_{h_\lambda}^\dagger, \partial_{E, h_\lambda} + \theta)$  exists and as an oper, we denote it as  $(E, \bar{\partial}'_E, \nabla')$ , therefore,

$$\lim_{\lambda \rightarrow 0} \lim_{t \rightarrow \lambda} \left( \phi_{(t, \lambda, \eta)}(E, \bar{\partial}_E, \theta) \right) = \lim_{t \rightarrow 0} (E, \bar{\partial}'_E, t \nabla').$$

On the other hand,

$$\lim_{\lambda \rightarrow 0} \lim_{t \rightarrow \lambda} \left( \phi_{(t, \lambda, \eta)}(E, \bar{\partial}_E, \theta) \right) = \lim_{\lambda \rightarrow 0} \lim_{t \rightarrow 0} \left( \phi_{(t, \lambda, \eta)}(E, \bar{\partial}_E, \theta) \right) = \lim_{t \rightarrow 0} (E, \bar{\partial}_E, t\theta).$$

Note that  $\lim_{t \rightarrow 0} (E, \bar{\partial}'_E, t \nabla') = \lim_{\lambda \rightarrow 0} (\psi_{(\lambda, 0)}(E, \bar{\partial}_E, \lambda \theta)) = \psi_{\overline{(0, 0)}}(E, \bar{\partial}_E, \lambda \theta)$ .

In conclusion, for a Higgs bundle  $(E, \bar{\partial}_E, \theta) \in \mathcal{M}_{\text{Dol}}(X, r)$  such that the limit  $\lim_{t \rightarrow 0} (E, \bar{\partial}_E, t\theta)$  is stable, we have

$$\psi_{\overline{(0, 0)}}(E, \bar{\partial}_E, \lambda \theta) = \psi_{\underline{(0, 0)}}(E, \bar{\partial}_E, \theta) = \lim_{t \rightarrow 0} (E, \bar{\partial}_E, t\theta).$$

□

Let  $h$  be the pluri-harmonic metric on  $(E, \bar{\partial}_E, \theta) \in M_{\text{Dol}}(X, r)$  and  $h_t$  be the pluri-harmonic metric on  $\psi_{\lambda, t}(E, \bar{\partial}_E, \theta)$ . To characterize the self-map  $\psi_{(\lambda, t)}$ , a crucial step is to understand how  $h_t$  depends on  $t$ . The following theorem gives a description of  $h_t$  around  $t = 1$ .

**Theorem 2.2.22.** *Let  $X$  be a compact Riemann surface. Fixing  $\lambda \in \mathbb{C}^*$  and assuming  $t \in \mathbb{R}^*$ , then over a small neighborhood of  $t = 1$ , the  $\text{End}(E)$ -valued function  $f := \frac{h_t h^{-1} - \text{Id}}{(t-1)^2}$  is real analytic with respect to  $t - 1$ .*

*Proof.* Fixing  $\lambda \in \mathbb{C}^*$ , denoted by  $A_\lambda(t, \chi) = 0$  the equation appeared in [Theorem 2.2.17](#) for which  $s$  should be satisfied to preserve the condition  $\bar{\partial}'_E \theta' = 0$ . Obviously this equation has a solution  $(t, s) = (1, 0)$ . The linearization of  $A_\lambda$  reads

$$\begin{aligned} L_{A_\lambda}(t, \dot{\chi}) &= \frac{t\lambda}{1 + |t\lambda|^2} \bar{\partial}_E \partial_{E, h} \dot{\chi} + \frac{t|\lambda|^2(1 - |t|^2)}{1 + |t\lambda|^2} [\theta, \bar{\partial}_E \dot{\chi}] + \frac{t\lambda^2(1 - |t|^2)}{(1 + |t\lambda|^2)^2} [\theta_h^\dagger, \partial_{E, h} \dot{\chi}] \\ &\quad + \frac{t\lambda|t|^2(1 + |\lambda|^2)}{(1 + |t\lambda|^2)^2} [\theta, [\theta_h^\dagger, \dot{\chi}]] - \frac{t\lambda|\lambda|^2(1 - |t|^2)}{(1 + |t\lambda|^2)^2} [\theta_h^\dagger, [\theta, \dot{\chi}]], \end{aligned}$$

hence

$$L_{A_\lambda}|_{(1, 0)}(\dot{\chi}) = \frac{\lambda}{1 + |\lambda|^2} (\bar{\partial}_E \partial_{E, h} \dot{\chi} + [\theta, [\theta_h^\dagger, \dot{\chi}]]).$$

If  $L_{A_\lambda}|_{(1,0)}(\dot{\chi}) = 0$ , we must have  $\dot{\chi} = 0$  since  $(E, \bar{\partial}_E, \theta)$  is a stable Higgs bundle, namely the linear 2-order differential operator  $L_{A_\lambda}|_{(1,0)}$  has a trivial kernel. Moreover, it can be deformed to an elliptic operator  $\frac{\lambda}{1+|\lambda|^2}\bar{\partial}_E\partial_{E,h}$ , thus has index zero, which means  $L_{A_\lambda}|_{(1,0)}$  is also surjective. Assuming  $t \in \mathbb{R}^*$ , due to implicit function theorem, there is a real analytic  $\text{End}(E)$ -valued function  $\chi$  in  $(t-1)$  over a small neighborhood  $0 \leq |t-1| \leq \varepsilon$  such that the equation  $A_\lambda(t, \chi) = 0$  holds. Then substituting the Taylor series  $\chi = \sum_{i=1}^{\infty} \chi_i(t-1)^i$  and Taylor series in  $(t-1)$  of coefficient functions into the above equation, we immediately find that  $\chi_1 = 0$ , that is,  $\chi \sim O((t-1)^2)$ .  $\square$

## 2.3 Oper Stratum Conjecture

In this section, let  $X$  be a smooth projective curve of genus  $g \geq 2$ , our main aim is to show the following result, which is a confirmation of weak oper stratum conjecture:

**Theorem 2.3.1 (Minimal Dimension).** *The oper stratum  $G_{\text{oper}}^1$  is the unique closed stratum of minimal dimension  $r^2(g-1) + g + 1$  in  $\mathcal{M}_{\text{dR}}(X, r)$ , the Hitchin stratum  $G_{\text{Hit}}^0$  is the unique stratum of minimal dimension  $r^2(g-1) + g + 1$  in  $\mathcal{M}_{\text{Dol}}(X, r)$ .*

For each index  $\alpha$ , the projections  $p_\alpha^0 : G_\alpha^0 \rightarrow P_\alpha$  and  $p_\alpha^1 : G_\alpha^1 \rightarrow P_\alpha$  are known being Zariski locally trivial fiber bundles, with fibers affine spaces of dimension half of the dimension of the corresponding moduli space [HT03, CW19]. Therefore,

$$\dim(G_\alpha^i) = r^2(g-1) + 1 + \dim(P_\alpha),$$

it suffices to show  $\dim(P_\alpha) \geq \dim(P_u) = g$ , with equality holds only when  $P_\alpha = P_u$ .

This section is arranged as the following. Since a connected component  $P_\alpha$  of the fixed point set  $P$  of  $\mathbb{C}^*$ -action on  $\mathcal{M}_{\text{Dol}}(X, r)$  can be represented by the moduli space of holomorphic chains of certain type, in Section 2.3.1, we will give a brief introduction to the theory of holomorphic chains based on [ACGPS06, BGPG04, Hei16]. For the cases of rank 3 and rank 4, the geometric structure of the corresponding moduli spaces of holomorphic chains are known explicitly [ACGPS06, BGPG04, Got94], so we will give a proof to Theorem 2.3.1 by explicit descriptions of moduli spaces in Section 2.3.2 and Section 2.3.3. For the case of general rank, in [ACGPS06], the authors calculated the dimension of the related moduli space by deformation theory, but we still do not the geometric structure of the moduli space, so we will provide a general proof to Theorem 2.3.1 in Section 2.3.4.

### 2.3.1 Holomorphic Chains

In this section, we will give a brief introduction to the theory of holomorphic chains and its relation with systems of Hodge bundles. For more details, see references [ACGPS06, BGPG04, Hei16].

**Definition 2.3.2.** (1) A *holomorphic chain*<sup>5</sup> of length  $l$  is a tuple  $\mathcal{C}_\bullet = (\mathcal{E}_i, i = 1, \dots, l; \varphi_i, i = 1, \dots, l-1)$  that consists of holomorphic vector bundles  $\mathcal{E}_i$  over  $X$  and holomorphic morphisms  $\varphi_i : \mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$ , we write it as

$$\mathcal{C}_\bullet : \mathcal{E}_1 \xrightarrow{\varphi_1} \mathcal{E}_2 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{l-1}} \mathcal{E}_l.$$

The collection  $(\vec{r}, \vec{d}) := (\text{rk}(\mathcal{E}_1), \dots, \text{rk}(\mathcal{E}_l); \deg(\mathcal{E}_1), \dots, \deg(\mathcal{E}_l))$  is called the *type* of the chain. If  $\varphi_i \neq 0$  for  $1 \leq i \leq l-1$ , then the chain is said to be *indecomposable*.

<sup>5</sup>Later on, we will also apply the notion ‘‘chain’’ when there is no ambiguity.

(2) Given a collection of real numbers  $\vec{\alpha} = (\alpha_1, \dots, \alpha_{l-1}) \in \mathbb{R}^{l-1}$ , called the *stability parameter*. The  $\vec{\alpha}$ -slope of a chain  $\mathcal{C}_\bullet$  of type  $(\vec{r}, \vec{d})$  is defined as

$$\mu_{\vec{\alpha}}(\mathcal{C}_\bullet) := \frac{\sum_{i=1}^l d_i + \sum_{i=1}^{l-1} \alpha_i r_i}{\sum_{i=1}^l r_i},$$

we may sometimes denote it as  $\mu_{\vec{\alpha}}(\vec{r}, \vec{d})$ . It is called  $\vec{\alpha}$ -stable (resp.  $\vec{\alpha}$ -semistable) if for any proper subchain  $\mathcal{C}'_\bullet \subseteq \mathcal{C}_\bullet$ , we have

$$\mu_{\vec{\alpha}}(\mathcal{C}'_\bullet) < (\text{resp. } \leq) \mu_{\vec{\alpha}}(\mathcal{C}_\bullet).$$

It is  $\vec{\alpha}$ -polystable if it is a direct sum of  $\vec{\alpha}$ -stable chains of same  $\vec{\alpha}$ -slope, here a subchain means a collection  $(\mathcal{E}'_i, i = 1, \dots, l)$  of coherent subsheaves  $\mathcal{E}'_i \subseteq \mathcal{E}_i, i = 1, \dots, l$  such that  $\varphi_i(\mathcal{E}'_i) \subseteq \mathcal{E}'_{i+1}, i = 1, \dots, l-1$ .

**Remark 2.3.3.** In some contexts, for example [BGP04, GP13], the stability parameters are defined to have the  $l$ -th term  $\alpha_l$ . In fact, if we let  $\vec{\alpha} = (\alpha_1, \dots, \alpha_l)$  and  $\vec{\alpha}_l := (\alpha_l, \dots, \alpha_l)$ , then

$$\mu_{\vec{\alpha} - \vec{\alpha}_l}(\mathcal{C}_\bullet) = \mu_{\vec{\alpha}}(\mathcal{C}_\bullet) - \alpha_l,$$

which does not affect the stability. Hence we can always assume  $\alpha_l = 0$ , and define the stability parameter as  $\vec{\alpha} = (\alpha_1, \dots, \alpha_{l-1}) \in \mathbb{R}^{l-1}$ .

If we release the notion of chains being the collection of coherent sheaves, then the category of chains is known to be an Abelian category [ACGPS06]. So each chain admits a unique Harder–Narasimhan filtration, and each  $\vec{\alpha}$ -semistable chain admits a Jordan–Hölder filtration such that the associated graded term is unique up to isomorphism, which gives the natural notion of  $S$ -equivalent classes of chains. Therefore, the construction of moduli space of semistable vector bundles over projective curves by GIT method can be applied to the construction of moduli space of chains of fixed type and stability parameters.

**Definition 2.3.4.** Define a partial order for  $\vec{r}: \vec{r}' < \vec{r}$  if  $r_i < r_i$  for all  $i$ . The stability parameter  $\vec{\alpha}$  is *critical* for a chain  $\mathcal{C}_\bullet$  of type  $(\vec{r}, \vec{d})$  if there exist  $(\vec{r}', \vec{d}')$  with  $\vec{r}' < \vec{r}$  and  $\vec{\beta} := (\beta_1, \dots, \beta_l) \in \mathbb{R}^l$  such that  $\mu_{\vec{\alpha}}(\vec{r}', \vec{d}') = \mu_{\vec{\alpha}}(\vec{r}, \vec{d})$  and  $\mu_{\vec{\beta}}(\vec{r}', \vec{d}') \neq \mu_{\vec{\beta}}(\vec{r}, \vec{d})$ .

If we denote by  $\mathcal{M}_{\text{Dol}}(X, r, d)$  the moduli space of semistable Higgs bundles over  $X$  of rank  $r$  and degree  $d$  (when  $d = 0$ , we write it as  $\mathcal{M}_{\text{Dol}}(X, r)$  for simplicity), then the fixed points of the  $\mathbb{C}^*$ -action are those systems of Hodge bundles<sup>6</sup>. In fact, we can study the connected pieces of fixed point set by studying the moduli space of chains of certain type and certain stability parameter:

**Proposition 2.3.5.** *There is an equivalence between holomorphic chains and systems of Hodge bundles.*

*Proof.* Given a chain  $\mathcal{C}_\bullet: \mathcal{E}_1 \xrightarrow{\varphi_1} \mathcal{E}_2 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{l-1}} \mathcal{E}_l$  of type  $(\vec{r}, \vec{d})$ , then we have a system of Hodge bundles

$$(E, \theta) := \left( \bigoplus_{i=1}^l E_i, \bigoplus_{i=1}^{l-1} \theta_i \right),$$

<sup>6</sup>We have applied the notion “ $\mathbb{C}$ -VHS” to denote the systems of Hodge bundles with degree 0, and for other degree, we will still apply the notion “system of Hodge bundles”.

where  $E_i = \mathcal{E}_i \otimes K_X^{-(l-i)}$ ,  $i = 1, \dots, l$  and  $\theta_i := \varphi_i \otimes \text{Id}_{K_X^{-(l-i)}} : E_i \rightarrow E_{i+1} \otimes K_X$ ,  $i = 1, \dots, l-1$ . The type of this system of Hodge bundles is  $(r_1, r_2, \dots, r_l; d_1 + r_1(l-1)(2g-2), d_2 + r_2(l-2)(2g-2), \dots, d_l)$ .

On the other hand, if we have a system of Hodge bundles  $(E = \bigoplus_{i=1}^l E_i, \theta = \bigoplus_{i=1}^{l-1} (\theta_i : E_i \rightarrow E_{i+1} \oplus K_X))$  of type  $(r_1, \dots, r_l; d_1, \dots, d_l)$ , then we can construct a holomorphic chain

$$\mathcal{C}_\bullet : \mathcal{E}_1 \xrightarrow{\varphi_1} \mathcal{E}_2 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_{l-1}} \mathcal{E}_l,$$

where  $\mathcal{E}_i := E_i \otimes K_X^{-(l-i)}$  and  $\varphi_i := \theta_i \otimes \text{Id}_{K_X^{-(l-i)}} : \mathcal{E}_i \rightarrow \mathcal{E}_{i+1}$ ,  $i = 1, \dots, l-1$ . The type of this chain is  $(\vec{r}, \vec{d}) = (r_1, r_2, \dots, r_l; d_1 - r_1(l-1)(2g-2), d_2 - r_2(l-2)(2g-2), \dots, d_l)$ .  $\square$

Therefore, the study of (semistable) systems of Hodge bundles of type  $(r_1, \dots, r_l; d_1, \dots, d_l)$  is equivalent to study the holomorphic chains of type  $(\vec{r}, \vec{d}) = (r_1, r_2, \dots, r_l; d_1 - r_1(l-1)(2g-2), d_2 - r_2(l-2)(2g-2), \dots, d_2)$  and with stability parameter

$$\vec{\alpha}_{\text{Higgs}} := ((l-1)(2g-2), (l-2)(2g-2), \dots, (2g-2)),$$

we call these chains the corresponding holomorphic chains, we will use the notation  $\vec{\alpha} > \vec{\alpha}_{\text{Higgs}}$  to denote  $\alpha_i - \alpha_{i+1} > 2g-2$  for  $i = 1, \dots, l-1$  (where we set  $\alpha_l = 0$ ). So we can give the following definitions:

**Definition 2.3.6.** A system of Hodge bundles of type  $(\vec{r}, \vec{d}) = (r_1, \dots, r_l; d_1, \dots, d_l)$  is

- (1) *indecomposable* if the corresponding holomorphic chain is indecomposable;
- (2) *stable* (resp. *semistable*, *polystable*) if the corresponding holomorphic chain is  $\vec{\alpha}_{\text{Higgs}}$ -stable (resp.  $\vec{\alpha}_{\text{Higgs}}$ -semistable,  $\vec{\alpha}_{\text{Higgs}}$ -polystable);
- (3) *coming from a  $\mathbb{C}$ -VHS* if it is polystable of degree 0, i.e,  $\sum_{i=1}^l d_i = 0$ .

The authors in [GPH13] gave the necessary conditions for the existence of  $\vec{\alpha}$ -semistable chains:

**Proposition 2.3.7** ([GPH13, Proposition 4]). *Let  $\mathcal{C}_\bullet$  be a  $\vec{\alpha}$ -semistable chain of length  $l$ , where  $\vec{\alpha} = (\alpha_1, \dots, \alpha_l) \in \mathbb{R}^l$  is a stability parameter satisfying  $\alpha_1 > \dots > \alpha_l = 0$ , and let  $\mu = \mu_{\vec{\alpha}}(\mathcal{C}_\bullet)$ . Then*

- (1) *for all  $j \in 2, \dots, l$ , we have*

$$\frac{\sum_{i=j}^l (d_i + \alpha_i r_i)}{\sum_{i=j}^l r_i} \leq \mu;$$

- (2) *for all  $j$  such that  $r_j = r_{j+1}$ , we have*

$$d_j \leq d_{j+1};$$

- (3) *for all  $1 \leq k < j \leq l$  such that  $r_k < \min\{r_{k+1}, \dots, r_j\}$ , we have*

$$\frac{\sum_{i \notin [k, j]} (d_i + \alpha_i r_i) + (j - k + 1)d_k + (\sum_{i=k}^j \alpha_i) r_k}{\sum_{i \notin [k, j]} r_i + (j - k + 1)r_k} \leq \mu;$$

(3) for all  $1 \leq k < j \leq l$  such that  $r_k > \max\{r_{k+1}, \dots, r_j\}$ , we have

$$\frac{\sum_{i=k}^{j-1} (d_i - d_j + \alpha_i (r_i - r_j))}{\sum_{i=k}^{j-1} (r_i - r_j)} \leq \mu.$$

Equivalently,  $\vec{\alpha}$ -semistable chains exist only when non of the following subchains has greater  $\vec{\alpha}$ -slope and non of the following quotient chains has smaller  $\vec{\alpha}$ -slope [GPH13]:

1. for any  $j \in 2, \dots, l$ , then  $\mathcal{C}_\bullet^j : 0 \longrightarrow \dots \longrightarrow 0 \longrightarrow \mathcal{E}_j \longrightarrow \dots \longrightarrow \mathcal{E}_l$  would be a subchain;
2. for some  $j$  such that with  $r_j = r_{j+1}$ , if  $d_j > d_{j+1}$ , then  $\varphi_j : \mathcal{E}_j \rightarrow \mathcal{E}_{j+1}$  could not be injective, so

$$\mathcal{E}'_\bullet : \dots \longrightarrow 0 \longrightarrow \text{Ker}(\varphi_j) \longrightarrow 0 \longrightarrow \dots$$

would be a subchain, and

$$\mathcal{E}''_\bullet : \dots \longrightarrow 0 \longrightarrow \text{Coker}(\varphi_j) \longrightarrow 0 \longrightarrow \dots$$

would be a quotient subchain, this could not happen by  $\vec{\alpha}$ -semistability;

3. for any  $1 \leq k < j \leq l$  such that  $r_k < \min\{r_{k+1}, \dots, r_j\}$ , then

$$\mathcal{E}'_\bullet : \mathcal{E}_1 \longrightarrow \dots \longrightarrow \mathcal{E}_{k-1} \longrightarrow \mathcal{E}_k \longrightarrow \dots \longrightarrow \mathcal{E}_k \longrightarrow \mathcal{E}_{j+1} \longrightarrow \dots \longrightarrow \mathcal{E}_l$$

would be a subchain;

4. for any  $1 \leq k < j \leq l$  such that  $r_k > \max\{r_{k+1}, \dots, r_j\}$ , then

$$\mathcal{E}''_\bullet : \mathcal{E}_1 \longrightarrow \dots \longrightarrow \mathcal{E}_{k-1} \longrightarrow \mathcal{E}_j \longrightarrow \dots \longrightarrow \mathcal{E}_j \longrightarrow \mathcal{E}_{j+1} \longrightarrow \dots \longrightarrow \mathcal{E}_l$$

would be a quotient chain.

Following [GPHS14, GPH13], let  $\text{Chain}(\vec{r}, \vec{d})$  be the stack of chains of fixed type  $(\vec{r}, \vec{d})$ , which is known to be an algebraic stack, locally of finite type [GPHS14]. For stability parameter  $\vec{\alpha}$  of real numbers, let  $\text{Chain}_{\vec{\alpha}}^{ss}(\vec{r}, \vec{d}) \subseteq \text{Chain}(\vec{r}, \vec{d})$  be the open substack of  $\vec{\alpha}$ -semistable chains. Let  $(\vec{r}, \vec{d})$  be the type that satisfies the condition (2) of Proposition 2.3.7, define the convex stability region in  $\mathbb{R}^{l-1}$  [BGPGH18]:

$$\text{Stab}(\vec{r}, \vec{d}) := \left\{ \alpha \in \mathbb{R}^{l-1} \mid \vec{\alpha} > \vec{\alpha}_{\text{Higgs}} \text{ and } \vec{\alpha} \text{ satisfies (1), (3), (4) of Proposition 2.3.7} \right\}.$$

**Theorem 2.3.8** ([BGPGH18, Theorem 3.2, Theorem 4.1]).

- (1) For type  $(\vec{r}, \vec{d}) \in \mathbb{Z}_+^l \times \mathbb{Z}^l$  and stability parameter  $\vec{\alpha} > \vec{\alpha}_{\text{Higgs}}$ , then the stack  $\text{Chain}_{\vec{\alpha}}^{ss}(\vec{r}, \vec{d})$  is non-empty and irreducible if and only if  $(\vec{r}, \vec{d})$  satisfies condition (2) of Proposition 2.3.7 and  $\vec{\alpha} \in \text{Stab}(\vec{r}, \vec{d})$ ;
- (2) For type  $(\vec{r}, \vec{d}) \in \mathbb{Z}_+^l \times \mathbb{Z}^l$ , then the coarse moduli space  $\text{Chain}_{\vec{\alpha}_{\text{Higgs}}}^{ss}(\vec{r}, \vec{d}) \in \mathbb{Z}_+^l \times \mathbb{Z}^l$  of the stack  $\text{Chain}_{\vec{\alpha}_{\text{Higgs}}}^{ss}(\vec{r}, \vec{d}) \in \mathbb{Z}_+^l \times \mathbb{Z}^l$  is non-empty and irreducible if and only if  $(\vec{r}, \vec{d})$  satisfies condition (2) of Proposition 2.3.7 and  $\vec{\alpha}_{\text{Higgs}} \in \overline{\text{Stab}(\vec{r}, \vec{d})}$ .

Let  $\mathcal{M}_{\text{VHS}}(X, r) \subseteq \mathcal{M}_{\text{Dol}}(X, r)$  be the subvariety that consists of  $\mathbb{C}$ -VHS's of fixed rank  $r$ , this is exactly the fixed point set  $P$  of  $\mathbb{C}^*$ -action. Denoted by  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \subseteq \mathcal{M}_{\text{VHS}}(X, r)$  the subvariety that consists of indecomposable  $\mathbb{C}$ -VHS's of type  $(\vec{r}, \vec{d}) := (r_1, \dots, r_l; d_1, \dots, d_l) \in \mathbb{Z}_{>0}^l \times \mathbb{Z}^l$  such that

$$\begin{cases} |\vec{r}| := \sum_{i=1}^l r_i = r, \\ |\vec{d}| := \sum_{i=1}^l d_i = 0. \end{cases}$$

Let  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d}) \subseteq \mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  be the subvariety consists of stable objects.

From [Proposition 2.3.7](#), we have the following properties:

**Proposition 2.3.9.** *If  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  is non-empty and let  $(E, \theta) := (\bigoplus_{i=1}^l E_i, \bigoplus_{i=1}^{l-1} (\theta_i : E_i \rightarrow E_{i+1})) \in \mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$ , then*

- (1) for all  $1 < j \leq l$ , we have  $\sum_{i=j}^l d_i < 0$ ;
- (2) for all  $j$  such that  $r_j = r_{j+1}$ , we have  $d_j - d_{j+1} \leq (2g - 2)r_j$ ;
- (3) for all  $1 \leq k < j \leq l$  such that  $r_k < \min\{r_{k+1}, \dots, r_j\}$ , we have

$$-\sum_{i=k+1}^j d_i + (j - k)(d_k - (j - k + 1)(g - 1)r_k) \leq 0;$$

- (4) for all  $1 \leq k < j \leq l$  such that  $r_k > \max\{r_{k+1}, \dots, r_j\}$ , we have

$$\sum_{i=k}^{j-1} d_i - (j - k)(d_j + (j - k + 1)(g - 1)r_j) \leq 0;$$

- (5) for all  $1 \leq j \leq l - 1$ , we have

$$d_j \leq [2(l - j) - 1]r_j(g - 1).$$

*Proof.* The first four inequalities directly follows from [Proposition 2.3.7](#). Now we prove the last one. Let  $P_{l-1}$  be the first term in the Harder-Narasimhan filtration of  $E_{l-1}$ . Consider the induced morphism  $\theta_{l-1} : E_{l-1} \otimes T_X \rightarrow E_l$ , under which, the image of  $P_{l-1} \otimes T_X$  is denoted by  $P'_{l-1}$ . Then the Higgs-semistability of system of Hodge bundles implies that

$$\deg(P_{l-1}) + \deg(P'_{l-1}) \leq 0.$$

Since  $P_{l-1} \otimes T_X$  is a semistable vector bundle and the induced map  $P_{l-1} \otimes T_X \rightarrow P'_{l-1}$  is a quotient map, we have

$$\mu(P_{l-1} \otimes T_X) = \mu(P_{l-1}) + \mu(T_X) \leq \mu(P'_{l-1}) \leq -\frac{\deg(P_{l-1})}{\text{rk}(P'_{l-1})},$$

from which it follows that

$$\mu(E_{l-1}) \leq \mu(P_{l-1}) \leq \frac{\text{rk}(P'_{l-1})}{\text{rk}(P_{l-1}) + \text{rk}(P'_{l-1})} \mu(K_X) \leq g - 1.$$

By recursion, we have

$$\mu(E_j) \leq \mu(P_j) \leq \mu(K_X) + \mu(P_{j+1}) \leq [2(l - j) - 1](g - 1)$$

for  $1 \leq j \leq l - 2$ . Hence, we derive the final inequalities.  $\square$

Conversely, from (2) of [Theorem 2.3.8](#) and [Proposition 2.3.9](#) we immediately have:

**Corollary 2.3.10.** *For type  $(\vec{r}, \vec{d})$  that satisfies (1)-(4) of [Proposition 2.3.9](#),  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  is non-empty and irreducible.*

*Proof.* By (2) of [Theorem 2.3.8](#), for type  $(\vec{r}, \vec{d})$  that satisfies (1)-(4) of [Proposition 2.3.9](#), there is a semistable Higgs bundle as the desired form. However, the condition (1) of [Proposition 2.3.9](#) guarantees that such Higgs bundle is polystable and with each  $\theta_i$  is non-zero.  $\square$

This means the connected components  $P_\alpha$  of the fixed point set  $P$  are of the form  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  with type  $(\vec{r}, \vec{d})$  satisfies (1)-(4) of [Proposition 2.3.9](#). Therefore, to show [Theorem 2.3.1](#), it suffices to show

$$\dim_{\mathbb{C}} \left( \mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \right) \geq g$$

whenever  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  is non-empty, and the equality holds if and only if the type  $(\vec{r}, \vec{d})$  is chosen such that  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) = P_u$ , i.e, the connected component of uniformizing Higgs bundles.

### 2.3.2 Proof for Rank Three Case

**Lemma 2.3.11.** *When  $r = 3$ , let  $(\vec{r}, \vec{d})$  be the type satisfies (1)-(4) of [Proposition 2.3.9](#), then  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  is not empty, irreducible with dimension*

$$g \leq \dim_{\mathbb{C}} \left( \mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \right) \leq 9g - 8.$$

*In particular, the equality on the left hand side holds true only when*

$$(\vec{r}, \vec{d}) = (1, 1, 1; 2g - 2, 0, -2g + 2),$$

*in this case,  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  is the connected component of uniformizing Higgs bundles. And the equality on the right hand side holds true only when  $(\vec{r}, \vec{d}) = (3; 0)$ , in this case,  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  is the moduli space of semistable vector bundles of rank 3 and degree 0 over  $X$ .*

*Proof.* Firstly, we can explicitly describe  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  as follows:

Case I: When  $(\vec{r}, \vec{d}) = (3; 0)$ , in this case,  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) = \mathcal{U}(X, 3, 0)$ , the moduli space of semistable vector bundles of rank 3 and degree 0 over  $X$ , which is known being an irreducible variety of dimension  $9g - 8$ .

Case II: When  $(\vec{r}, \vec{d}) = (1, 1, 1; d_1, d_2, -d_1 - d_2)$ , then from [Proposition 2.3.9](#) the degree type is known to be satisfying

$$\begin{cases} d_1 > 0, \\ d_1 + d_2 > 0, \\ d_1 - d_2 \leq 2g - 2, \\ d_1 + 2d_2 \leq 2g - 2. \end{cases} \quad (2.21)$$

We have the isomorphism [\[Got94\]](#)

$$\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \cong \text{Jac}^{d_1}(X) \times \text{Sym}^{d_2 - d_1 + 2g - 2}(X) \times \text{Sym}^{-d_1 - 2d_2 + 2g - 2}(X),$$

which maps  $(E_1, E_2, E_3; \varphi_1, \varphi_2)$  onto  $(E_1, \text{div}(\varphi_1), \text{div}(\varphi_2))$ , where  $\text{Jac}^{d_1}(X)$  is the moduli space of line bundles of degree  $d_1$  over  $X$ ,  $\text{Sym}^p(X)$  denotes the space of effective divisors of degree  $p$  in  $X$  which is isomorphic to  $X^p/S_p$  for the symmetry group  $S_p$ , and  $\text{div}(\theta_i)$  stands for the divisor determined by the morphism  $\theta_i$ . Conversely, for a given line bundle  $L_0 \in \text{Jac}^{d_1}(X)$ , and effective divisors  $D_1 \in \text{Sym}^{d_2 - d_1 + 2g - 2}(X)$ ,  $D_2 \in \text{Sym}^{-d_1 - 2d_2 + 2g - 2}(X)$ , we have unique line bundles  $L_i = \mathcal{O}_X(D_i)$  with non-zero sections  $\varphi_i$  up to multiplication by non-zero scalars,  $i = 1, 2$ . Then we get an indecomposable system of Hodge bundles

$$(L_0, L_0 \otimes L_1 \otimes K_X^{-1}, L_0 \otimes L_1 \otimes L_2 \otimes K_X^{-2}; \varphi_1, \varphi_2),$$

which is polystable under the condition [\(2.21\)](#).

Case III: When  $(\vec{r}, \vec{d}) = (1, 2; d_1, -d_1)$ , then from [Proposition 2.3.9](#) the degree type is known to be satisfying

$$0 < d_1 \leq g - 1. \quad (2.22)$$

When  $0 < d_1 < g - 1$ , then  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  is non-empty and birationally equivalent to a  $\mathbb{P}^N$ -fibration over  $\text{Jac}^{-2d_1 + 2g - 2}(X) \times \text{Jac}^{d_1 - 2g + 2}(X)$  for  $N = -3d_1 + 5g - 6$  [\[BGPG04, Got94\]](#). When  $d_1 = g - 1$ , then  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  is empty and  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \cong \text{Jac}^0(X) \times \text{Jac}^{-g+1}(X)$  [\[BGPG04\]](#), which is known to be irreducible.

Case IV: When  $(\vec{r}, \vec{d}) = (2, 1; d_1, -d_1)$ , then from [Proposition 2.3.9](#) the degree type is known to be satisfying

$$0 < d_1 \leq g - 1. \quad (2.23)$$

When  $0 < d_1 < g - 1$ , then  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  is non-empty and birationally equivalent to a  $\mathbb{P}^N$ -fibration over  $\text{Jac}^{2d_1 + 2(2g - 2)}(X) \times \text{Jac}^{-d_1}(X)$  for  $N = -3d_1 + 5g - 6$  [\[BGPG04, Got94\]](#). When  $d_1 = g - 1$ , then  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  is empty and  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \cong \text{Jac}^{-2g+2}(X) \times \text{Jac}^{-g+1}(X)$  [\[BGPG04\]](#), which is known to be irreducible.

Therefore, we have the dimension formula

$$\dim_{\mathbb{C}} \left( \mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \right) = \begin{cases} 9g - 8, & (\vec{r}, \vec{d}) = (3; 0); \\ 5g - 4 - 2d_1 - d_2, & (\vec{r}, \vec{d}) = (1, 1, 1; d_1, d_2, -d_1 - d_2); \\ 7g - 6 - 3d_1, & (\vec{r}, \vec{d}) = (1, 2; d_1, -d_1) \text{ or } (2, 1; d_1, -d_1) \\ & \text{with } 0 < d_1 < g - 1, \\ 2g, & (\vec{r}, \vec{d}) = (1, 2; g - 1, -g + 1) \text{ or } (2, 1; g - 1, -g + 1). \end{cases}$$

On the other hand, by (2.21)-(2.23), we have the inequalities

$$0 < 2d_1 + d_2 \leq \begin{cases} 4g - 4, & (\vec{r}, \vec{d}) = (1, 1, 1; d_1, d_2, -d_1 - d_2); \\ g - 1, & (\vec{r}, \vec{d}) = (1, 2, d_1, d_2 = -d_1) \text{ or } (2, 1; d_1, d_2 = -d_1), \end{cases}$$

which leads to the inequalities on  $\dim_{\mathbb{C}} \mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  as follows:

$$\begin{cases} \dim_{\mathbb{C}} \left( \mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \right) = 9g - 8, & (\vec{r}, \vec{d}) = (3; 0); \\ g \leq \dim_{\mathbb{C}} \left( \mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \right) < 5g - 4, & (\vec{r}, \vec{d}) = (1, 1, 1; d_1, d_2, -d_1 - d_2); \\ 4g - 3 \leq \dim_{\mathbb{C}} \left( \mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \right) < 7g - 6, & (\vec{r}, \vec{d}) = (1, 2; d_1, -d_1) \text{ or } (2, 1; d_1, -d_1). \end{cases}$$

The desired results immediately follows.  $\square$

Therefore, we confirm the weak oper stratum conjecture for rank 3 case:

**Theorem 2.3.12.** *When  $r = 3$ , the oper stratum  $G_{\text{oper}}^1$  is the unique closed stratum of minimal dimension  $10g - 8$  in  $\mathcal{M}_{\text{dR}}(X, 3)$ , and the Hitchin stratum  $G_{\text{Hit}}^0$  is the unique stratum of minimal dimension  $10g - 8$  in  $\mathcal{M}_{\text{Dol}}(X, 3)$ .*

### 2.3.3 Proof for Rank Four Case

**Lemma 2.3.13.** *When  $r = 4$ , let  $(\vec{r}, \vec{d})$  be the type satisfies (1)-(4) of Proposition 2.3.9, then  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  is not empty, irreducible with dimension*

$$g \leq \dim_{\mathbb{C}} \left( \mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \right) \leq 16g - 15.$$

*In particular, the equality on the left hand side holds true only when*

$$(\vec{r}, \vec{d}) = (1, 1, 1, 1; 3g - 3, g - 1, g + 1, -3g + 3),$$

*in this case,  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  is the connected component of uniformizing Higgs bundles. And the equality on the right hand side holds true only when  $(\vec{r}, \vec{d}) = (4; 0)$ , in this case,  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  is the moduli space of semistable vector bundles of rank 4 and degree 0 over  $X$ .*

*Proof.* As in rank 3 case, we have an explicit description of  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  based on [ACGPS06, BPG04]:

Case I: When  $(\vec{r}, \vec{d}) = (4; 0)$ , in this case,  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) = \mathcal{U}(X, 4, 0)$ , the moduli space of semistable vector bundles of rank 4 and degree 0 over  $X$ , which is known being an irreducible variety of dimension  $16g - 15$ .

Case II: When  $(\vec{r}, \vec{d}) = (1, 1, 1, 1; d_1, d_2, d_3, -d_1 - d_2 - d_3)$ , then from [Proposition 2.3.9](#) the degree type is known to be satisfying

$$\left\{ \begin{array}{l} d_1 > 0, \\ d_1 + d_2 > 0, \\ d_1 + d_2 + d_3 > 0, \\ d_1 - d_2 \leq 2g - 2, \\ d_2 - d_3 \leq 2g - 2, \\ d_1 + d_2 + 2d_3 \leq 2g - 2. \end{array} \right. \quad (2.24)$$

We have the isomorphism

$$\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \cong \text{Jac}^{d_1}(X) \times \text{Sym}^{d_2 - d_1 + 2g - 2}(X) \times \text{Sym}^{d_3 - d_2 + 2g - 2}(X) \times \text{Sym}^{-d_1 - d_2 - 2d_3 + 2g - 2}(X)$$

which maps  $(E_1, E_2, E_3, E_4; \varphi, \varphi, \varphi)$  to  $(E_1, \text{div}(\varphi), \text{div}(\varphi), \text{div}(\varphi))$ . In fact, for any line bundle  $L_0 \in \text{Jac}^{d_1}(X)$ , and effective divisors  $D_1 \in \text{Sym}^{d_2 - d_1 + 2g - 2}(X)$ ,  $D_2 \in \text{Sym}^{d_3 - d_2 + 2g - 2}(X)$ ,  $D_3 \in \text{Sym}^{-d_1 - d_2 - 2d_3 + 2g - 2}(X)$ , we have line bundles  $L_i = \mathcal{O}(D_i)$  with non-zero sections  $\varphi_i$  up to multiplication by non-zero scalars,  $i = 1, 2, 3$ . Then we get an indecomposable system of Hodge bundles

$$(L_0, L_0 \otimes L_1 \otimes K_X^{-1}, L_0 \otimes L_1 \otimes L_2 \otimes K_X^{-2}, L_0 \otimes L_1 \otimes L_2 \otimes L_3 \otimes K_X^{-3}; \varphi_1, \varphi_2, \varphi_3),$$

which is polystable under the conditions [\(2.24\)](#).

Case III: When  $(\vec{r}, \vec{d}) = (1, 3; d_1, -d_1)$ , then from [Proposition 2.3.9](#) the degree type is known to be satisfying

$$0 < d_1 \leq g - 1. \quad (2.25)$$

When  $0 < d_1 < g - 1$ , then  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  is non-empty and birationally equivalent to a  $\mathbb{P}^N$ -fibration over  $\mathcal{U}^s(X, 2, -2d_1 + 2g - 2) \times \text{Jac}^{d_1 - 2g + 2}(X)$  for  $N = -4d_1 + 8g - 9$  [[BGPG04](#)], where  $\mathcal{U}^s(X, r, d)$  denotes the moduli space of stable vector bundles of rank  $r$  and degree  $d$  over  $X$ . When  $d_1 = g - 1$ , then  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  is empty and  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \cong \mathcal{U}(X, 2, 0) \times \text{Jac}^{-g+1}(X)$  [[BGPG04](#)].

Case IV: When  $(\vec{r}, \vec{d}) = (3, 1; d_1, -d_1)$ , then from [Proposition 2.3.9](#) the degree type is known to be satisfying

$$0 < d_1 \leq g - 1. \quad (2.26)$$

When  $0 < d_1 < g - 1$ , then  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  is non-empty and birationally equivalent to a  $\mathbb{P}^N$ -fibration over  $\mathcal{U}^s(X, 2, 2d_1 - 6g + 6) \times \text{Jac}^{-d_1}(X)$  [[BGPG04](#)] for  $N = -4d_1 + 8g - 9$ . When  $d_1 = g - 1$ , then  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  is empty and  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \cong \mathcal{U}(X, 2, -4g + 4) \times \text{Jac}^{-g+1}(X)$

[BGPG04].

Case V: When  $(\vec{r}, \vec{d}) = (2, 2; d_1, -d_1)$ , then from [Proposition 2.3.9](#) the degree type is known to be satisfying

$$0 < d_1 \leq 2g - 2. \quad (2.27)$$

$\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  is birationally equivalent to a  $\mathbb{P}^N$ -fibration over  $\mathcal{U}^s(X, 2, d_1 - 4g + 4) \times \text{Sym}^{-2d_1 + 4g - 4}(X)$  for  $N = -2d_1 + 4g - 4$  [BGPG04].

Case VI: When  $(\vec{r}, \vec{d}) = (1, 1, 2; d_1, d_2, -d_1 - d_2)$ , then from [Proposition 2.3.9](#) the degree type is known to be satisfying

$$\begin{cases} d_1 > 0, \\ d_1 + d_2 > 0, \\ d_1 - d_2 \leq 2g - 2, \\ d_1 + 2d_2 \leq 2g - 2. \end{cases} \quad (2.28)$$

When the last inequality is strict, then  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  is non-empty and birationally equivalent to a  $\mathbb{P}^N$ -fibration over  $\text{Jac}^{d_2 - 2g + 2}(X) \times \text{Jac}^{-d_1 - 2d_2 - 2g + 2}(X) \times \text{Sym}^{d_2 - d_1 + 2g - 2}(X)$  for  $N = -d_1 - 3d_2 + 5g - 6$  [ACGPS06]. When the last equality holds, then  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  is empty and  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \cong \text{Jac}^0(X) \times \text{Jac}^{d_2}(X) \times \text{Sym}^{d_2 - d_1 + 2g - 2}(X)$ .

Case VII: When  $(\vec{r}, \vec{d}) = (2, 1, 1; d_1, d_2, -d_1 - d_2)$ , then from [Proposition 2.3.9](#) the degree type is known to be satisfying

$$\begin{cases} 0 < d_1 \leq 2g - 2, \\ d_1 + d_2 > 0, \\ d_1 + 2d_2 \leq 2g - 2, \\ d_1 - d_2 \leq 2g - 2. \end{cases} \quad (2.29)$$

When the last inequality is strict, then  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  is non-empty and birationally equivalent to a smooth irreducible variety of dimension  $9g - 8 - 2d_1$  [ACGPS06]. When the last equality holds, then  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  is empty and  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \cong \text{Jac}^{-4g + 4}(X) \times \text{Jac}^{d_1 - 4g + 4}(X) \times \text{Sym}^{-3d_1 + 6g - 6}(X)$ .

Case VIII: When  $(\vec{r}, \vec{d}) = (1, 2, 1; d_1, d_2, -d_1 - d_2)$ , then from [Proposition 2.3.9](#) the degree type is known to be satisfying

$$\begin{cases} d_1 > 0, \\ d_1 + d_2 > 0, \\ d_1 - d_2 \leq 2g - 2, \\ d_1 + 2d_2 \leq 2g - 2. \end{cases} \quad (2.30)$$

When  $\vec{d} \neq (2g - 2, 0, -2g + 2)$ , then  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  is non-empty and birationally equivalent

lent to a  $\mathbb{P}^N$ -fibration over  $\text{Jac}^{d_1-4g+4}(X) \times \text{Jac}^{d_2-d_1}(X) \times \text{Sym}^{-2d_1-d_2+4g-4}(X)$  for  $N = -2d_1 - d_2 + 4g - 5$  [ACGPS06]. When  $\vec{d} = (2g - 2, 0, -2g + 2)$ , then  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  is empty and  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \cong \text{Jac}^{-2g+2}(X) \times \text{Jac}^{-g+1}(X)$ .

Therefore, the dimension  $\dim_{\mathbb{C}}(\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}))$  of each case is given by

$$\left\{ \begin{array}{ll} 16g - 15, & (\vec{r}, \vec{d}) = (4; 0); \\ 7g - 6 - 2d_1 - d_2 - d_3, & (\vec{r}, \vec{d}) = (1, 1, 1, 1; d_1, d_2, d_3, -d_1 - d_2 - d_3); \\ 13g - 12 - 4d_1, & (\vec{r}, \vec{d}) = (1, 3; d_1, -d_1) \text{ or } (3, 1; d_1, -d_1), d_1 \neq g - 1; \\ 5g - 3, & (\vec{r}, \vec{d}) = (1, 3; g - 1, -g + 1) \text{ or } (3, 1; g - 1, -g + 1); \\ 12g - 11 - 4d_1, & (\vec{r}, \vec{d}) = (2, 2; d_1, -d_1); \\ 9g - 8 - 2d_1 - 2d_2, & (\vec{r}, \vec{d}) = (1, 1, 2; d_1, d_2, -d_1 - d_2), d_1 + 2d_2 \neq 2g - 2; \\ 5g - 3 - \frac{3}{2}d_1, & (\vec{r}, \vec{d}) = (1, 1, 2; d_1, g - 1 - \frac{d_1}{2}, -g + 1 - \frac{d_1}{2}); \\ 9g - 8 - 2d_1, & (\vec{r}, \vec{d}) = (2, 1, 1; d_1, d_2, -d_1 - d_2), d_1 - d_2 \neq 2g - 2; \\ 8g - 6 - 3d_1, & (\vec{r}, \vec{d}) = (2, 1, 1; d_1, -2g + 2 + d_1, 2g - 2 - 2d_1); \\ 10g - 9 - 4d_1 - 2d_2, & (\vec{r}, \vec{d}) = (1, 2, 1; d_1, d_2, -d_1 - d_2), (d_1, d_2, -d_1 - d_2) \neq (2g - 2, 0, -2g + 2); \\ 2g, & (\vec{r}, \vec{d}) = (1, 2, 1; 2g - 2, 0, -2g + 2). \end{array} \right.$$

On the other hand, by (2.24)-(2.30), we have the inequalities

$$0 < 2d_1 + d_2 + d_3 \leq \left\{ \begin{array}{ll} 6g - 6, & (\vec{r}, \vec{d}) = (1, 1, 1, 1; d_1, d_2, d_3, -d_1 - d_2 - d_3); \\ g - 1, & (\vec{r}, \vec{d}) = (1, 3; d_1, d_2 = -d_1) \text{ or } (3, 1; d_1, d_2 = -d_1), d_3 = 0; \\ 2g - 2, & (\vec{r}, \vec{d}) = (2, 2; d_1, d_2 = -d_1), d_3 = 0; \\ 2g - 2, & (\vec{r}, \vec{d}) = (2, 1, 1; d_1, d_2, -d_1 - d_2); \end{array} \right.$$

$$0 < d_1 + d_2 \leq 2g - 2, (\vec{r}, \vec{d}) = (1, 1, 2; d_1, d_2, -d_1 - d_2);$$

$$0 < 2d_1 + d_2 \leq 4g - 4, (\vec{r}, \vec{d}) = (1, 2, 1; d_1, d_2, -d_1 - d_2).$$

Hence, we arrive at (we take  $\mathcal{M}$  to represent  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  for simplicity)

$$\left\{ \begin{array}{ll} \dim_{\mathbb{C}} \mathcal{M} = 16g - 15, & (\vec{r}, \vec{d}) = (4; 0); \\ g \leq \dim_{\mathbb{C}} \mathcal{M} < 7g - 6, & (\vec{r}, \vec{d}) = (1, 1, 1, 1; d_1, d_2, d_3, -d_1 - d_2 - d_3); \\ 5g - 3 \leq \dim_{\mathbb{C}} \mathcal{M} < 13g - 12, & (\vec{r}, \vec{d}) = (1, 3; d_1, -d_1) \text{ or } (3, 1; d_1, -d_1); \\ 5g - 4 \leq \dim_{\mathbb{C}} \mathcal{M} < 12g - 11, & (\vec{r}, \vec{d}) = (2, 2; d_1, -d_1); \\ 2g \leq \dim_{\mathbb{C}} \mathcal{M} \leq 9g - 8, & (\vec{r}, \vec{d}) = (1, 1, 2; d_1, d_2, -d_1 - d_2) \text{ or } (2, 1, 1; d_1, d_2, -d_1 - d_2); \\ 2g - 1 \leq \dim_{\mathbb{C}} \mathcal{M} < 10g - 9, & (\vec{r}, \vec{d}) = (1, 2, 1; d_1, d_2, -d_1 - d_2). \end{array} \right.$$

The desired results immediately follows.  $\square$

Therefore, we confirm the oper stratum conjecture for rank 4 case:

**Theorem 2.3.14.** *When  $r = 4$ , the oper stratum  $G_{\text{oper}}^1$  is the unique closed stratum of minimal dimension  $17g - 15$  in  $\mathcal{M}_{\text{dR}}(X, 4)$ , and the Hitchin stratum  $G_{\text{Hit}}^0$  is the unique stratum of minimal dimension  $17g - 15$  in  $\mathcal{M}_{\text{Dol}}(X, 4)$ .*

### 2.3.4 Higher Rank Cases

In general cases, the dimension of  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  was calculated by authors in [ACGPS06] by deformation theory:

**Proposition 2.3.15** ([ACGPS06, Theorem 3.8 (iv)]). *If  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  is not empty for  $(\vec{r}, \vec{d}) = (r_1, \dots, r_l; d_1, \dots, d_l)$ , then its dimension is given by*

$$\dim_{\mathbb{C}} \left( \mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \right) = (g-1) \sum_{i=1}^l r_i(r_i + r_{i+1}) + \sum_{i=1}^l r_i(d_{i+1} - d_{i-1}) + 1,$$

where one assigns  $r_{l+1} = d_0 = d_{l+1} = 0$ .

**Lemma 2.3.16.** *Let  $(\vec{r}, \vec{d})$  be the type satisfies (1)-(4) of Proposition 2.3.9, then  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  is not empty, irreducible with dimension*

$$g \leq \dim_{\mathbb{C}} \left( \mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \right) \leq r^2(g-1) + 1.$$

In particular, the equality on the left hand side holds true only when

$$(\vec{r}, \vec{d}) = (1, \dots, 1; (r-1)(g-1), (r-3)(g-1), \dots, -(r-1)(g-1)),$$

in this case,  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  is the connected component of uniformizing Higgs bundles. And the equality on the right hand side holds true only when  $(\vec{r}, \vec{d}) = (r; 0)$ , in this case,  $\mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  is the moduli space of vector bundles of rank  $r$  and degree 0 over  $X$ .

*Proof.* Since stability is an open condition, one can consider the problem at the level of stacks. Let  $\mathfrak{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$  be the corresponding moduli stack, and let  $\mathfrak{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  be the substack consisting of stable objects. It follows from  $\mathfrak{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  being a  $\mathbb{G}_m$ -gerbe over its coarse moduli space  $\mathcal{M}_{\text{ind-VHS}}^s(\vec{r}, \vec{d})$  [GPHS14] that

$$\dim_{\mathbb{C}} \left( \mathfrak{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \right) = \dim_{\mathbb{C}} \left( \mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \right) - 1.$$

Let  $\text{Bun}^{r_i, d_i}(X)$  be the moduli stack of vector bundles of rank  $r_i$  and degree  $d_i$  over  $X$ . Since there is  $r_i$  such that  $\text{Bun}^{r_i, d_i}(X) \subset \mathfrak{M}_{\text{ind-VHS}}(\vec{r}, \vec{d})$ , if  $r_i > 1$ , we have

$$\dim_{\mathbb{C}} \left( \mathfrak{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \right) \geq \dim_{\mathbb{C}} \left( \text{Bun}^{r_i, d_i}(X) \right) = r_i^2(g-1) > g-1.$$

Therefore, it suffices to show  $g \leq \dim_{\mathbb{C}} \left( \mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \right)$  when  $\vec{r} = (1, \dots, 1)$ . By Proposition 2.3.15, we have

$$\begin{aligned} \dim_{\mathbb{C}} \left( \mathcal{M}_{\text{ind-VHS}}(\vec{r}, \vec{d}) \right) &= (g-1)(2r-1) + \sum_{i=1}^{r-1} (d_{i+1} - d_i) + 1 \\ &\geq (g-1)(2r-1) - (r-1)(2g-2) + 1 \\ &= g, \end{aligned}$$

where we have used [Proposition 2.3.9](#) for the second inequality. In particular, the equality holds if and only if  $\vec{d} = ((r-1)(g-1), (r-3)(g-1), \dots, (-r+1)(g-1))$ .

For the inequality on the right hand side in the lemma and the the condition for reaching the equality, we will see this from the proof of [Theorem 3.3.7](#), which says, the moduli space  $\mathcal{U}(X, r, 0)$  of semistable vector bundles, as a connected component of the fixed points set  $P$ , is the only connected component in the nilpotent cone that reaches the maximal dimension  $\frac{1}{2} \dim_{\mathbb{C}} (\mathcal{M}_{\text{Dol}}(X, r)) = r^2(g-1) + 1$ .  $\square$

Therefore, we confirm the weak oper stratum conjecture for higher rank cases:

**Theorem 2.3.17.** *The oper stratum  $G_{\text{oper}}^1$  is the unique closed stratum of minimal dimension  $r^2(g-1) + g + 1$  in  $\mathcal{M}_{\text{dR}}(X, r)$ , and the Hitchin stratum  $G_{\text{Hit}}^0$  is the unique stratum of minimal dimension  $r^2(g-1) + g + 1$  in  $\mathcal{M}_{\text{Dol}}(X, r)$ .*

## Chapter 3

# Twistor Structure Construction

In this chapter we will introduce twistor theory, then apply the non-Abelian Hodge correspondence to study this theory, and obtain some interesting results.

In the first section, we will introduce the basic twistor theory based on [HKLR87]. The fundamental idea on constructing a twistor space associated to a hyper-Kähler manifold  $M^1$  is topological a product of this manifold and the complex projective line  $\mathbb{P}^1$ , the quaternionic structure  $(I, J, K = IJ)$  on  $M$  induces a complex structure  $\mathcal{I}$  on  $M \times \mathbb{P}^1$ , and hence gives a complex manifold denoted as  $\text{TW}(M)$ , we call it the *Hitchin twistor space* associated to  $M$ .

In the second section, we will introduce Deligne's idea on interpreting this twistor theory via non-Abelian Hodge correspondence. As moduli space parametrizing isomorphism classes of stable objects in non-Abelian Hodge theory, the underlying smooth manifold  $M^{\text{sm}}(X, r)$  carries a natural hyper-Kähler structure (and therefore, quaternionic), and hence gives a Hitchin twistor space,  $\text{TW}(M^{\text{sm}}(X, r))$ . Deligne's idea is, this twistor space can be interpreted by gluing the Hodge moduli spaces  $\mathcal{M}_{\text{Hod}}(X, r)$  and  $\mathcal{M}_{\text{Hod}}(\bar{X}, r)$  via complex conjugation and non-Abelian Hodge correspondence to obtain a new space  $\text{TW}_{\text{DH}}(X, r)$  together with a fibration (holomorphic projection)  $\text{TW}_{\text{DH}}(X, r) \rightarrow \mathbb{P}^1$ . This resulting space is called the *Deligne–Hitchin twistor space*, and the subset of smooth point locus (such that the projection is smooth)  $\text{TW}_{\text{DH}}^{\text{sm}}(X, r)$  is analytically isomorphic to  $\text{TW}(M^{\text{sm}}(X, r))$ .

The most difference between these two methods is that, for the first one (Hitchin et al.), one begins with a manifold with quaternionic structure (that is,  $M^{\text{sm}}(X, r)$ ), then taking the product with the complex projective line  $\mathbb{P}^1$ , and finally, the quaternionic structure of  $M^{\text{sm}}(X, r)$  and the natural complex structure of  $\mathbb{P}^1$  gives a twistor structure for the product space. For the second one (Deligne), one glues the Hodge moduli spaces over  $X$  and  $\bar{X}$  via complex conjugation and non-Abelian Hodge correspondence. The preferred sections gives the trivialization (in fact  $C^\infty$  isomorphism)  $\text{TW}_{\text{DH}}^{\text{sm}}(X, r) \cong M^{\text{sm}}(X, r) \times \mathbb{P}^1$  and the weight one property for preferred sections implies the quaternionic structure.

In the last section, we focus on Riemann surface  $X$ . Denoted  $X$  as  $(\mathcal{X}, I)$ , for  $\mathcal{X}$  a smooth connected orientable closed surface of genus at least 2, and  $I$  the complex structure that induces the Riemann surface structure. Then the space of isotropy classes of complex structures on  $\mathcal{X}$  can be identified with its Teichmüller space. The extended mapping class group acts on the Teichmüller space, and therefore, induces an action of the outer automorphism group  $\text{Out}(\pi_1(\mathcal{X}))$  on the Teichmüller space. When we trace back to Deligne's construction, the two Hodge moduli spaces appearing in the gluing process are over  $X$  and its conjugate  $\bar{X}$ , or equivalently, over  $(\mathcal{X}, I)$  and  $(\mathcal{X}, -I)$ . The map from a complex structure  $I$  to  $-I$  stands for a non-trivial element in

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<sup>1</sup>In fact, in [HKLR87], a twistor structure can be constructed for any complex manifold with a quaternionic structure.

$\text{Out}(\pi_1(\mathcal{X}))$ ). Hence we can generalize this to any non-trivial element  $\gamma$ . The resulting two Hodge moduli spaces can be also glued together to give a new twistor space, where we call it  $\gamma$ -twistor space, and denoted as  $\text{TW}_\gamma(X, r)$ . We study this twistor space, and obtain some interesting results. This chapter is mainly based on [HH19, Hua20].

### 3.1 General Twistor Construction

In [HKLR87], the authors provide a construction of twistor space for any hyper-Kähler manifold (In general, for differential manifolds with quaternionic structures). Let  $M$  be a hyper-Kähler manifold with three complex structures  $(I, J, K = IJ)$ . The stereographic projection of the complex projective line  $\mathbb{P}^1 \rightarrow S^2$

$$\lambda = u + iv \mapsto \left( x = \frac{1 - |\lambda|^2}{1 + |\lambda|^2}, y = \frac{2u}{1 + |\lambda|^2}, z = \frac{2v}{1 + |\lambda|^2} \right)$$

defines a family of complex structures  $I_\lambda := xI + yJ + zK$  on  $M$ .

The *twistor space*  $\text{TW}(M)$  of  $M$  is a  $C^\infty$  trivialization  $\text{TW}(M) \cong M \times \mathbb{P}^1$ .  $I_\lambda$  determines an almost complex structure  $\mathcal{I}$  on  $\text{TW}(M)$ , let  $a = m \times \lambda \in \text{TW}(M)$ , then

$$\mathcal{I} : T_a \text{TW}(M) = T_m M \oplus T_\lambda \mathbb{P}^1 \rightarrow T_m M \oplus T_\lambda \mathbb{P}^1$$

is given by  $(I_\lambda, I_0)$ , where  $I_0 : T_\lambda \mathbb{P}^1 \rightarrow T_\lambda \mathbb{P}^1$  is the usual complex structure on  $\mathbb{P}^1$  given by  $I_0(v) = iv$ . Clearly, this  $\mathcal{I}$  is an almost complex structure, and moreover, it is integrable [Sal82, HKLR87], thus the twistor space  $\text{TW}(M)$  is a complex manifold of dimension  $\dim_{\mathbb{C}}(\text{TW}(M)) = 1 + \dim_{\mathbb{C}}(M)$ .

The twistor space  $\text{TW}(M)$  has the following properties:

- (1) the projection  $\pi : \text{TW}(M) \rightarrow \mathbb{P}^1$  is holomorphic;
- (2) there is an antilinear involution  $\sigma : \text{TW}(M) \rightarrow \text{TW}(M)$ ,  $(m, \lambda) \mapsto (m, -\bar{\lambda}^{-1})$ , which covers the antipodal involution  $\sigma_{\mathbb{P}^1} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ ,  $\lambda \mapsto -\bar{\lambda}^{-1}$ , so it gives a real structure on  $\text{TW}(M)$ ;
- (3) for any  $m \in M$ , the section  $\{m\} \times \mathbb{P}^1 \subset \text{TW}(M)$  is holomorphic and  $\sigma$ -invariant, we call it a *preferred section*, in some contexts [HKLR87, Sal82], it is called a *twistor line*;
- (4) weight 1 property: the normal bundle along any preferred section is isomorphic to  $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus \dim_{\mathbb{C}}(M)}$ .

**Proposition 3.1.1.** *Preferred sections are  $\sigma$ -invariant. Moreover, locally, preferred sections are the only  $\sigma$ -invariant holomorphic sections.*

If we denoted by  $\text{Pre}$  the set of preferred sections in the Douady moduli space  $\text{Sec}$  of holomorphic sections. Then “locally” means there exists an open neighborhood  $U$  of  $\text{Pre}$  in  $\text{Sec}$  such that preferred sections are the only  $\sigma$ -invariant sections in  $U$  [KV98]. To my knowledge, whether Proposition 3.1.1 holds globally is unknown:

**Question 3.1.2.** Are the preferred sections the only  $\sigma$ -invariant sections?

Moreover, let  $p : \text{TW}(M) \rightarrow M$  be the natural projection, then the twistor space  $\text{TW}(M)$  admits a natural hermitian metric  $g := p^*g_M + \pi^*g_{\text{FS}}$ , where  $g_M$  is the hyper-Kähler metric on  $M$  and  $g_{\text{FS}}$  is the Fubini–Study metric on  $\mathbb{P}^1$ . This metric makes the twistor space  $\text{TW}(M)$  into a *balanced manifold* [KV98] (i.e, the associated fundamental form  $\omega(\bullet, \bullet) := g(\mathcal{I}\bullet, \bullet)$  satisfies the weak closedness condition  $d(\omega^{\dim_{\mathbb{C}} M}) = 0$ ).

Now we apply above ideas to the moduli spaces appeared in non-Abelian Hodge theory.

Let  $X$  be a smooth complex projective variety, as introduced in last chapter, denoted by  $M_{\mathrm{dR}}(X, r)$  and  $M_{\mathrm{Dol}}(X, r)$  the open subsets of smooth points of the moduli spaces  $\mathcal{M}_{\mathrm{dR}}(X, r)$  and  $\mathcal{M}_{\mathrm{Dol}}(X, r)$ , respectively. From non-Abelian Hodge theory, we have  $C^\infty$  isomorphism  $M_{\mathrm{dR}}(X, r) \cong M_{\mathrm{Dol}}(X, r)$  coming from the homeomorphism between the underlying topological spaces [Sim94a, Sim94b, Sim95]. Let  $M^{\mathrm{sm}}(X, r)$  be the underlying smooth differentiable manifold. Then by the work of Hitchin ([Hit87a] for 1 dimensional base manifold) and Fujiki ([Fuj91] for higher dimensional Kähler manifolds as the base),  $M^{\mathrm{sm}}(X, r)$  carries a hyper-Kähler structure that consists of three complex structures  $I, J, K := IJ = -JI$  for  $I$  comes from the complex structure of the base  $X$  such that  $(M^{\mathrm{sm}}(X, r), I)$  is analytically isomorphic to  $M_{\mathrm{Dol}}(X, r)$ , and  $J$  comes from the complex structure of the Lie group  $G = \mathrm{GL}(r, \mathbb{C})$  such that  $(M^{\mathrm{sm}}(X, r), J)$  is analytically isomorphic to  $M_{\mathrm{dR}}(X, r)$ . Moreover,  $IJ =: K$  gives the third complex structure, which makes  $(M^{\mathrm{sm}}(X, r), I, J, K)$  has the structure of a hyper-Kähler manifold. Hence Hitchin's idea for constructing the twistor space of hyper-Kähler manifolds can be applied to  $M^{\mathrm{sm}}(X, r)$ . Therefore, we obtain a twistor space for  $M^{\mathrm{sm}}(X, r)$ , denoted as  $\mathrm{TW}(M^{\mathrm{sm}}(X, r))$ , and called it the *Hitchin twistor space*.

### 3.2 Deligne's Interpretation

Deligne's original aim was to understand the Hitchin twistor space  $\mathrm{TW}(M^{\mathrm{sm}}(X, r))$  via  $\lambda$ -connections. The idea is gluing the Hodge moduli space  $\mathcal{M}_{\mathrm{Hod}}(X, r)$  over  $X$  and the Hodge moduli space  $\mathcal{M}_{\mathrm{Hod}}(\bar{X}, r)$  over its conjugate chart  $\bar{X}$  to obtain a twistor space  $\mathrm{TW}_{\mathrm{DH}}(X, r)$  such that the smooth locus  $\mathrm{TW}_{\mathrm{DH}}^{\mathrm{sm}}(X, r)$  is analytically isomorphic to  $\mathrm{TW}(M^{\mathrm{sm}}(X, r))$ . This was described and further studied by Simpson, he interpreted  $\mathcal{M}_{\mathrm{Hod}}(X, r)$  as the Hodge filtration on the non-Abelian de Rham cohomology  $\mathcal{M}_{\mathrm{dR}}(X, r)$ . He showed the Griffiths transversality and the regularity of the Gauss–Manin connection for this filtration [Sim95]. Here we introduce their ideas based on Simpson's papers [Sim95, Sim08].

Let  $X$  be a complex projective variety with a fixed base point  $x$ . The complex conjugation  $\varphi : X \rightarrow \bar{X}, x \mapsto \bar{x}$  induces an isomorphism

$$\varphi_* : \pi_1(X, x) \rightarrow \pi_1(\bar{X}, \bar{x}) \quad (3.1)$$

of fundamental groups. With same notations used before, let  $\mathcal{M}_{\mathrm{Hod}}(X, r)$  be the moduli space of semistable  $\lambda$ -flat bundles of rank  $r$  with vanishing Chern classes, let  $\mathcal{M}_{\mathrm{B}}(X, r)$  be the moduli space of representations  $\rho : \pi_1(X) \rightarrow \mathrm{GL}(r, \mathbb{C})$ . The Riemann-Hilbert correspondence gives the complex analytic isomorphism [Sim94a]:

$$\mathrm{RH}_X : \mathcal{M}_{\mathrm{dR}}(X, r) \xrightarrow{\cong} \mathcal{M}_{\mathrm{B}}(X, r). \quad (3.2)$$

The  $\mathbb{C}^*$ -action on  $\mathcal{M}_{\mathrm{Hod}}(X, r)$  gives the algebraic isomorphism:

$$\begin{aligned} r_X : \mathcal{M}_{\mathrm{Hod}}(X, r) \times_{\mathbb{C}} \mathbb{C}^* &\xrightarrow{\cong} \mathcal{M}_{\mathrm{dR}}(X, r) \times \mathbb{C}^* \\ [E, \bar{\partial}_E, D^\lambda, \lambda] &\longmapsto ([E, \bar{\partial}_E, \lambda^{-1} D^\lambda], \lambda). \end{aligned} \quad (3.3)$$

The gluing isomorphism  $\mathbf{d}_\varphi : \mathcal{M}_{\mathrm{Hod}}(X, r) \times_{\mathbb{C}} \mathbb{C}^* \rightarrow \mathcal{M}_{\mathrm{Hod}}(\bar{X}, r) \times_{\mathbb{C}} \mathbb{C}^*$  is given as follows:

First by isomorphisms (3.1)-(3.3), any point  $[E, \bar{\partial}_E, D^\lambda, \lambda]$  in  $\mathcal{M}_{\mathrm{Hod}}(X, r) \times_{\mathbb{C}} \mathbb{C}^*$  determines a representation  $\rho(\lambda^{-1} D^\lambda) \circ \varphi_*^{-1} : \pi_1(\bar{X}, \bar{x}) \rightarrow \mathrm{GL}(r, \mathbb{C})$ , where  $\rho(\lambda^{-1} D^\lambda) : \pi_1(X, x) \rightarrow \mathrm{GL}(r, \mathbb{C})$

is the monodromy corresponds to the flat connection  $\lambda^{-1}D^\lambda$ . Then by the conjugate version of isomorphisms (3.2),  $\rho(\lambda^{-1}D^\lambda) \circ \varphi_*^{-1}$  corresponds to a flat bundle over  $\bar{X}$ . Finally, take  $\lambda^{-1} \in \mathbb{C}^*$  and fix the obtained flat bundle at this point, by the conjugate version of (3.3) this gives a  $\lambda^{-1}$ -flat bundle over  $\bar{X}$ . The obtained  $\lambda^{-1}$ -flat bundle over  $\bar{X}$  is  $[E, \lambda^{-1}D^\lambda, \lambda^{-1}\bar{\partial}_E, \lambda^{-1}]$ . Therefore, the gluing isomorphism is

$$\begin{aligned} \mathbf{d}_\varphi : \mathcal{M}_{\text{Hod}}(X, r) \times_{\mathbb{C}} \mathbb{C}^* &\cong \mathcal{M}_{\text{Hod}}(\bar{X}, r) \times_{\mathbb{C}} \mathbb{C}^* \\ [E, \bar{\partial}_E, D^\lambda, \lambda] &\leftrightarrow [E, \lambda^{-1}D^\lambda, \lambda^{-1}\bar{\partial}_E, \lambda^{-1}], \end{aligned}$$

which covers the map  $\mathbb{C}^* \rightarrow \mathbb{C}^*, \lambda \mapsto \lambda^{-1}$ . Therefore, we obtained a space  $\text{TW}_{\text{DH}}(X, r)$ , called the *Deligne–Hitchin twistor space*, together with a fibration  $\text{TW}_{\text{DH}}(X, r) \rightarrow \mathbb{P}^1$  which extends the projection  $\pi : \mathcal{M}_{\text{Hod}}(X, r) \rightarrow \mathbb{C}^*$ . The fibers of this fibration are  $\mathcal{M}_{\text{Dol}}(X, r)$  at  $\lambda = 0$ ,  $\mathcal{M}_{\text{Dol}}(\bar{X}, r)$  at  $\lambda = \infty$ , and analytically isomorphic to  $\mathcal{M}_{\text{dR}}(X, r)$  at  $\lambda \neq 0, \infty$ .

Let  $[E, \bar{\partial}_E, \theta, h] \in \mathcal{M}_{\text{Dol}}(X, r)$  be a harmonic Higgs bundle with the pluri-harmonic metric  $h$ , then it determines a holomorphic section  $p : \mathbb{P}^1 \rightarrow \text{TW}_{\text{DH}}(X, r)$ :

$$\lambda \longmapsto [E, \bar{\partial}_E + \lambda\theta_h^\dagger, \lambda\partial_{E,h} + \theta, \lambda],$$

where  $\theta_h^\dagger$  and  $\partial_{E,h}$  are the unique operators determined by  $(\theta, h)$  and  $(\bar{\partial}_E, h)$ , respectively. This section is called a *preferred section*.

We can also define an antilinear involution  $\sigma : \text{TW}_{\text{DH}}(X, r) \rightarrow \text{TW}_{\text{DH}}(X, r)$  that covers the antipodal involution  $\sigma_{\mathbb{P}^1} : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ . This map is defined by gluing the antiholomorphic isomorphisms  $\sigma_{\text{Hod}, X} : \mathcal{M}_{\text{Hod}}(X, r) \rightarrow \mathcal{M}_{\text{Hod}}(\bar{X}, r)$  and  $\sigma_{\text{Hod}, \bar{X}} : \mathcal{M}_{\text{Hod}}(\bar{X}, r) \rightarrow \mathcal{M}_{\text{Hod}}(X, r)$ , where

$$\begin{aligned} \sigma_{\text{Hod}, X} : \mathcal{M}_{\text{Hod}}(X, r) &\rightarrow \mathcal{M}_{\text{Hod}}(\bar{X}, r) \\ [E, \bar{\partial}_E, D^\lambda, \lambda] &\mapsto [\bar{E}^*, (\bar{\partial}_E)^*, (\bar{D}^\lambda)^*, -\bar{\lambda}]. \end{aligned}$$

When  $(E, \bar{\partial}_E, D^\lambda)$  is a harmonic  $\lambda$ -flat bundle, by taking a pluri-harmonic metric  $h$ , then we have the isomorphism

$$[\bar{E}^*, (\bar{\partial}_E)^*, (\bar{D}^\lambda)^*, -\bar{\lambda}] \cong [E, \delta'_h, -\delta''_h, -\bar{\lambda}],$$

where the operators  $\delta'_h$  and  $\delta''_h$  can be found in in Chapter 1, Section 1.2.

Therefore, over these points admit pluri-harmonic metrics (that is, polystable points, in particular, stable points), we can write the involution  $\sigma$  as follows:

$$\begin{aligned} \sigma : \text{TW}_{\text{DH}}(X, r) &\rightarrow \text{TW}_{\text{DH}}(X, r) \\ [E, \bar{\partial}_E, D^\lambda, \lambda] &\mapsto [E, \bar{\lambda}^{-1}\delta''_h, -\bar{\lambda}^{-1}\delta'_h, -\bar{\lambda}^{-1}]. \end{aligned}$$

In fact,  $\sigma$  is the product of the following 3 involutions [Sim08]:

- (1) an antiholomorphic involution

$$\begin{aligned} C : \text{TW}_{\text{DH}}(X, r) &\rightarrow \text{TW}_{\text{DH}}(X, r) \\ [E, \bar{\partial}_E, D^\lambda, \lambda] &\mapsto [E, \bar{\lambda}^{-1}\bar{D}^\lambda, \bar{\lambda}^{-1}\bar{\partial}_E, \bar{\lambda}^{-1}] \end{aligned}$$

obtained by gluing complex conjugations of  $\lambda$ -flat bundles;

(2) a holomorphic involution

$$\begin{aligned} D : \mathrm{TW}_{\mathrm{DH}}(X, r) &\rightarrow \mathrm{TW}_{\mathrm{DH}}(X, r) \\ [E, \bar{\partial}_E, D^\lambda, \lambda] &\mapsto [E^*, (\bar{\partial}_E)^*, (D^\lambda)^*, \lambda] \end{aligned}$$

obtained by gluing duals of  $\lambda$ -flat bundles;

(3) a holomorphic involution

$$\begin{aligned} N : \mathrm{TW}_{\mathrm{DH}}(X, r) &\rightarrow \mathrm{TW}_{\mathrm{DH}}(X, r) \\ [E, \bar{\partial}_E, D^\lambda, \lambda] &\mapsto [E, \bar{\partial}_E, -D^\lambda, -\lambda] \end{aligned}$$

obtained by  $-1 \in \mathbb{C}^*$  acts on  $\mathrm{TW}_{\mathrm{DH}}(X, r)$ .

By definition, any point in the twistor space determines a unique preferred section. Therefore, the set of preferred sections gives a homeomorphism

$$\mathrm{TW}_{\mathrm{DH}}(X, r) \cong \mathcal{M}_{\mathrm{Dol}}(X, r) \times \mathbb{P}^1,$$

which is a  $C^\infty$  isomorphism over smooth points [Sim95]:

$$\mathrm{TW}_{\mathrm{DH}}^{\mathrm{sm}}(X, r) \cong M_{\mathrm{Dol}}(X, r) \times \mathbb{P}^1.$$

**Proposition 3.2.1.** *Preferred sections are  $\sigma$ -invariant. Moreover, locally, preferred sections are the only  $\sigma$ -invariant holomorphic sections.*

Preferred sections are invariant under the antilinear involution  $\sigma$  is easy to see. Here “locally” means that for any given preferred section  $p : \mathbb{P}^1 \rightarrow \mathrm{TW}_{\mathrm{DH}}(X, r)$ , there exists an open neighbourhood  $U \subseteq \mathrm{TW}_{\mathrm{DH}}(X, r)$  of  $p$  such that preferred sections are the only  $\sigma$ -invariant sections in  $U$ . This is true because the normal bundle along any preferred section is isomorphic to  $(\mathcal{O}_{\mathbb{P}^1}(1))^{\oplus \dim_{\mathbb{C}}(\mathcal{M}_{\mathrm{dR}}(X, r))}$ .

In [Sim95], Simpson asked the following question:

**Question 3.2.2.** Does Proposition 3.2.1 holds globally? That is, are the preferred sections the only  $\sigma$ -invariant sections?

In [Sim95, Sim08], Simpson showed this question is true for twistor space of rank 1 bundles. Recently, in [BHR19], the authors constructed holomorphic  $\sigma$ -invariant but not preferred sections  $\mathbb{P}^1 \rightarrow \mathrm{TW}_{\mathrm{DH}}(X, 2)$  for twistor space  $\mathrm{TW}_{\mathrm{DH}}(X, 2)$  of rank 2 bundles over compact Riemann surface of  $g \geq 2$ .

Now we give a brief conclusion on above two different kinds of constructing the twistor spaces. In Hitchin's constructing, we first have a hyper-Kähler manifold (the moduli spaces are hyper-Kähler)  $M^{\mathrm{sm}}(X, r)$ , then the twistor space has a naturally structure of  $M^{\mathrm{sm}}(X, r) \times \mathbb{P}^1$ . The complex structure is induced from a family of complex structures (given by the quaternionic structure) on  $M^{\mathrm{sm}}(X, r)$  and the natural complex structure on  $\mathbb{P}^1$ . While from Deligne's interpretation, the twistor space is obtained by gluing the Hodge moduli spaces  $\mathcal{M}_{\mathrm{Hod}}(X, r)$  and  $\mathcal{M}_{\mathrm{Hod}}(\bar{X}, r)$  over  $X$  and the conjugate  $\bar{X}$ , respectively. This gluing is obtained via the algebraic isomorphism  $r_X : \mathcal{M}_{\mathrm{Hod}}(X, r) \times_{\mathbb{C}} \mathbb{C}^* \cong \mathcal{M}_{\mathrm{dR}}(X, r) \times \mathbb{C}^*$  and the conjugate algebraic isomorphism  $r_{\bar{X}} : \mathcal{M}_{\mathrm{Hod}}(\bar{X}, r) \times_{\mathbb{C}} \mathbb{C}^* \cong \mathcal{M}_{\mathrm{dR}}(\bar{X}, r) \times \mathbb{C}^*$ , the analytic isomorphisms  $\mathrm{RH}_X : \mathcal{M}_{\mathrm{dR}}(X, r) \cong \mathcal{M}_{\mathrm{B}}(X, r)$  and  $\mathrm{RH}_{\bar{X}} : \mathcal{M}_{\mathrm{dR}}(\bar{X}, r) \cong \mathcal{M}_{\mathrm{B}}(\bar{X}, r)$  given by the Riemann–Hilbert correspondence, and the identification  $\bar{\varphi} :$

$\mathcal{M}_B(X, r) = \mathcal{M}_B(\bar{X}, r)$  induced from the conjugate map  $\varphi : X \rightarrow \bar{X}$ , that is, maps a representation  $\rho : \pi_1(X, x) \rightarrow \mathrm{GL}(r, \mathbb{C})$  to  $\rho \circ \varphi_*^{-1} : \pi_1(\bar{X}, \bar{x}) \rightarrow \mathrm{GL}(r, \mathbb{C})$ . That can be outlined by the following diagram:

$$\begin{array}{ccc}
 \mathcal{M}_{\mathrm{Hod}}(X, r) \times_{\mathbb{C}} \mathbb{C}^* & \xrightarrow[\cong_{\mathrm{an}}]{\mathrm{d}\varphi} & \mathcal{M}_{\mathrm{Hod}}(\bar{X}, r) \times_{\mathbb{C}} \mathbb{C}^* \\
 \cong_{\mathrm{alg}} \downarrow r_X & & \cong_{\mathrm{alg}} \uparrow r_{\bar{X}}^{-1} \\
 \mathcal{M}_{\mathrm{dR}}(X, r) \times \mathbb{C}^* & & \mathcal{M}_{\mathrm{dR}}(\bar{X}, r) \times \mathbb{C}^* \\
 \cong_{\mathrm{an}} \downarrow \mathrm{RH}_X \times \mathrm{id}_{\mathbb{C}^*} & & \cong_{\mathrm{an}} \uparrow \mathrm{RH}_{\bar{X}}^{-1} \times \mathrm{id}_{\mathbb{C}^*} \\
 \mathcal{M}_B(X, r) \times \mathbb{C}^* & \xrightarrow[\cong]{\tilde{\varphi} \times c} & \mathcal{M}_B(\bar{X}, r) \times \mathbb{C}^*
 \end{array}$$

where  $c : \mathbb{C}^* \rightarrow \mathbb{C}^*, \lambda \mapsto \lambda^{-1}$ .

Preferred sections give the twistor space the product structure, and the weight 1 property implies the quaternionic structure. This viewpoint shows that a quaternionic structure on  $M^{\mathrm{sm}}(X, r)$  is equivalent to the weight 1 property of a preferred section [Sim08].

Moreover, the twistor spaces arising from these two different methods are analytically isomorphic:

**Theorem 3.2.3** ([Sim95, Theorem 4.2]). *The twistor space  $\mathrm{TW}_{\mathrm{DH}}^{\mathrm{sm}}(X, r)$  is analytically isomorphic to  $\mathrm{TW}(M^{\mathrm{sm}}(X, r))$ .*

**Remark 3.2.4.** Now we come back to see Question 3.1.2, from the answer to Question 3.2.2 given by the authors in [BHR19], in general, Question 3.1.2 does not have a positive answer, that is, preferred sections in the Hitchin twistor space  $\mathrm{TW}(M)$  are not the only  $\sigma$ -invariant sections. In Simpson's proof of Theorem 3.2.3, the complex analytic isomorphism identifies preferred sections in  $\mathrm{TW}(M^{\mathrm{sm}}(X, r))$  and in  $\mathrm{TW}_{\mathrm{DH}}^{\mathrm{sm}}(X, r)$ , hence at least Question 3.2.2 is not right for  $\mathrm{TW}_{\mathrm{DH}}^{\mathrm{sm}}(X, 2)$ .

Therefore, we can propose the following question:

**Question 3.2.5.** When  $M$  is a compact hyper-Kähler manifold, are preferred sections in the Hitchin twistor space  $\mathrm{TW}(M)$  the only  $\sigma$ -invariant sections?

### 3.3 A Generalization

In [HH19] (see also [Hua20]), we concentrated on the case with Riemann surfaces as the base, we generalized Deligne's construction to obtain the so called *generalized Deligne–Hitchin twistor space*.

#### 3.3.1 Construction

Let  $X$  be a compact Riemann surface of genus  $g \geq 2$  and let  $\mathcal{X}$  be the underlying smooth surface. Take  $G = \mathrm{GL}(r, \mathbb{C})$ . Associated to the surface  $\mathcal{X}$ , there are some geometric objects:

- (1) The fundamental group  $\pi_1(\mathcal{X})$ :

$$\pi_1(\mathcal{X}) = \langle \alpha_1, \beta_1, \dots, \alpha_g, \beta_g : \prod_{i=1}^g \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1} = 1 \rangle.$$

- (2) The Betti moduli space  $\mathcal{M}_B(\mathcal{X}, r)$  (introduced in last chapter):

$$\mathcal{M}_B(\mathcal{X}, r) = \mathrm{Rep}^{\mathrm{red}}(\mathcal{X}, G)/G,$$

where  $\text{Rep}^{\text{red}}(\mathcal{X}, G) := \text{Hom}^{\text{red}}(\pi_1(\mathcal{X}), G)$ , the space of reductive representations of  $\pi_1(\mathcal{X})$  into  $G$ . And  $G$  acts on  $\text{Rep}^{\text{red}}(\mathcal{X}, G)$  by conjugation.  $\text{Rep}^{\text{red}}(\mathcal{X}, G)$  is an affine variety by the embedding

$$\text{Rep}^{\text{red}}(\mathcal{X}, G) \hookrightarrow G^{2g}, \quad \rho \longmapsto (\rho(\alpha_1), \dots, \rho(\beta_g)),$$

and  $\mathcal{M}_B(\mathcal{X}, G)$  is a quasi-projective variety.  $M_B(X, r) := \text{Rep}^{\text{irr}}(\mathcal{X}, G)/G$ , the moduli space of irreducible representations, as a Zariski dense open subset of  $\mathcal{M}_B(X, r)$ , which is a smooth quasi-projective variety.

(3) The Teichmüller space  $\text{Teich}(\mathcal{X})$ :

$$\text{Teich}(\mathcal{X}) := \{\text{isotopy classes of complex structures on } \mathcal{X}\},$$

which is identified with the equivalent classes of marked hyperbolic structures on  $\mathcal{X}$ . It's known that  $\text{Teich}(\mathcal{X})$  is homeomorphic to  $\mathbb{R}^{6g-6}$ .

Fix  $\lambda \in \mathbb{C}$ , let  $\mathcal{M}_{\text{Hod}}^\lambda(X, r)$  be the moduli space of semistable  $\lambda$ -flat bundles over  $X$  of rank  $r$  and with vanishing Chern classes,  $M_{\text{Hod}}^\lambda(X, r)$  the smooth locus of stable objects. Then  $\mathcal{M}_{\text{Hod}}(X, r) = \bigcup_{\lambda \in \mathbb{C}} \mathcal{M}_{\text{Hod}}^\lambda(X, r)$ . When  $\lambda \neq 0$ , we have the analytic isomorphism (in fact, algebraic isomorphism)

$$\mu_\lambda : \mathcal{M}_{\text{Hod}}^\lambda(X, r) \xrightarrow{\cong} \mathcal{M}_{\text{dR}}(X, r)$$

given by the map  $(E, \bar{\partial}_E, D^\lambda) \mapsto (E, \bar{\partial}_E, \nabla = \lambda^{-1}D^\lambda)$ , with its inverse  $\mu_\lambda^{-1} : (E, \bar{\partial}_E, \nabla) \mapsto (E, \bar{\partial}_E, D^\lambda = \lambda\nabla)$  (rescaling the twistor parameter  $\lambda$ ). Moreover, by Mochizuki correspondence between  $\mathcal{M}_{\text{Hod}}^\lambda(X, r)$  and  $\mathcal{M}_{\text{Dol}}(X, r)$  and the homeomorphism  $\mathcal{M}_{\text{Dol}}(X, r)$  and  $\mathcal{M}_{\text{dR}}(X, r)$ , we can define another homeomorphism

$$\nu_\lambda : \mathcal{M}_{\text{Hod}}^\lambda(X, r) \xrightarrow{\cong} \mathcal{M}_{\text{dR}}(X, r),$$

which is a  $C^\infty$  isomorphism over smooth locus:  $M_{\text{Hod}}^\lambda(X, r) \cong M_{\text{dR}}(X, r)$ .

The extended mapping class group

$$\text{Mod}^\diamond(\mathcal{X}) := \pi_0(\text{Diff}(\mathcal{X})) = \text{Diff}(\mathcal{X})/\text{Diff}_0(\mathcal{X})$$

acts on the Teichmüller space  $\text{Teich}(\mathcal{X})$  by precomposition. The quotient  $\mathcal{M}(\mathcal{X}) := \text{Teich}(\mathcal{X})/\text{Mod}^\diamond(\mathcal{X})$  is called the *Riemann moduli space of complex structures on  $\mathcal{X}$* , which parametrizes the Riemann surface structures on  $\mathcal{X}$  under biholomorphism.

A generalization of Deligne gluing is the following:

We can write  $X = (\mathcal{X}, I)$  with the complex structure  $I \in \mathcal{M}(\mathcal{X})$  that determines the Riemann surface structure, choose a new complex structure  $I' \in \mathcal{M}(\mathcal{X})$  and let  $X' = (\mathcal{X}, I')$  be the new Riemann surface. Then by the analytic isomorphism  $\mu_\lambda$  and the Riemann–Hilbert correspondence, we can glue the Hodge moduli spaces  $\mathcal{M}_{\text{Hod}}(X, r)$  and  $\mathcal{M}_{\text{Hod}}(X', r)$  along the overlap  $\mathcal{M}_{\text{Hod}}(X, r) \times_{\mathbb{C}} \mathbb{C}^* \cong \mathcal{M}_{\text{Hod}}(X', r) \times_{\mathbb{C}} \mathbb{C}^* \cong \mathcal{M}_B(\mathcal{X}, r) \times \mathbb{C}^*$  that covers the map  $\mathbb{C}^* \rightarrow \mathbb{C}^*, \lambda \mapsto \lambda^{-1}$ .

More precisely, choose  $I' \in \mathcal{M}(\mathcal{X})$ . For  $[E, \bar{\partial}_E, D^\lambda, \lambda] \in \mathcal{M}_{\text{Hod}}(X, r) \times_{\mathbb{C}} \mathbb{C}^*$ , then it corresponds to the flat bundle  $[E, \bar{\partial}_E, \lambda^{-1}D^\lambda, \lambda]$  via the isomorphism  $\mu_\lambda$ , and hence corresponds to a point  $(\rho(\lambda^{-1}D^\lambda), \lambda) \in \mathcal{M}_B(\mathcal{X}, r) \times \mathbb{C}^*$  via the Riemann–Hilbert correspondence, where  $\rho(\lambda^{-1}D^\lambda) : \pi_1(\mathcal{X}) \rightarrow G$  is the associated monodromy representation. Again by the Riemann–Hilbert correspondence, it corresponds to some  $[E', \bar{\partial}_{E'}, \nabla, \lambda^{-1}] \in \mathcal{M}_{\text{dR}}(X', r) \times \mathbb{C}^*$  at the point

$\lambda^{-1}$ , consequently, we obtain some  $[E', \bar{\partial}_{E'}, D'^{\lambda^{-1}} = \lambda^{-1}\nabla, \lambda^{-1}] \in \mathcal{M}_{\text{Hod}}(X', r) \times_{\mathbb{C}} \mathbb{C}^*$  via  $\mu_{\lambda^{-1}}^{-1}$ . From the construction, it's not hard to see that

$$[E', \bar{\partial}_{E'}, D'^{\lambda^{-1}}, \lambda^{-1}] = [E, (\bar{\partial}_E + \lambda^{-1}D^\lambda)_{X'}^{0,1}, \lambda^{-1}(\bar{\partial}_E + \lambda^{-1}D^\lambda)_{X'}^{1,0}, \lambda^{-1}],$$

where  $(\bullet)_{X'}^{1,0}$  and  $(\bullet)_{X'}^{0,1}$  denote the corresponding (1,0)-part and (0,1)-part with respect to the complex structure  $I'$  on  $X'$ , respectively. That is, for an 1-form  $\alpha$ ,

$$(\alpha)_{X'}^{1,0} = \frac{\sqrt{-1} + I'}{2\sqrt{-1}}\alpha, \quad (\alpha)_{X'}^{0,1} = \frac{\sqrt{-1} - I'}{2\sqrt{-1}}\alpha$$

since  $I'(\alpha) = \sqrt{-1}(\alpha)_{X'}^{1,0} - \sqrt{-1}(\alpha)_{X'}^{0,1}$  and  $\sqrt{-1}\alpha = \sqrt{-1}(\alpha)_{X'}^{1,0} + \sqrt{-1}(\alpha)_{X'}^{0,1}$ .

Therefore, we define an analytic isomorphism again called the *Deligne isomorphism*

$$\mathbf{d} : \mathcal{M}_{\text{Hod}}(X, r) \times_{\mathbb{C}} \mathbb{C}^* \rightarrow \mathcal{M}_{\text{Hod}}(X', r) \times_{\mathbb{C}} \mathbb{C}^*$$

that covers the map  $\mathbb{C}^* \rightarrow \mathbb{C}^*$ ,  $\lambda \mapsto \lambda^{-1}$ . Now we can use this isomorphism  $\mathbf{d}$  to glue together two analytic spaces  $\mathcal{M}_{\text{Hod}}(X, r)$  and  $\mathcal{M}_{\text{Hod}}(X', r)$  along their open sets. The resulting space is denoted by  $\text{TW}(X, X'; r)$ , called the *generalized Deligne–Hitchin twistor space*. In particular, it forms a fibration  $\pi : \text{TW}(X, X'; r) \rightarrow \mathbb{P}^1$  with the fibers  $\pi^{-1}(\lambda)$  analytically isomorphic to  $\mathcal{M}_{\text{B}}(\mathcal{X}, r)$  for  $\lambda \in \mathbb{P}^1 \setminus \{0, \infty\}$ , and the fibers  $\pi^{-1}(0) = \mathcal{M}_{\text{Dol}}(X, r)$ ,  $\pi^{-1}(\infty) = \mathcal{M}_{\text{Dol}}(X', r)$ .

In conclusion, our construction is along the following diagram:

$$\begin{array}{ccc} \mathcal{M}_{\text{Hod}}(X, r) \times_{\mathbb{C}} \mathbb{C}^* & \xrightarrow[\cong_{\text{an}}]{\mathbf{d}} & \mathcal{M}_{\text{Hod}}(X', r) \times_{\mathbb{C}^*} \mathbb{C}^* \\ \cong_{\text{alg}} \downarrow r_X & & \cong_{\text{alg}} \uparrow r_{X'}^{-1} \\ \mathcal{M}_{\text{dR}}(X, r) \times \mathbb{C}^* & & \mathcal{M}_{\text{dR}}(X', r) \times \mathbb{C}^* \\ \cong_{\text{an}} \downarrow \text{RH}_X \times \text{id}_{\mathbb{C}^*} & & \cong_{\text{an}} \uparrow \text{RH}_{X'}^{-1} \times \text{id}_{\mathbb{C}^*} \\ \mathcal{M}_{\text{B}}(\mathcal{X}, r) \times \mathbb{C}^* & \xrightarrow[\cong]{\text{id} \times c} & \mathcal{M}_{\text{B}}(\mathcal{X}, r) \times \mathbb{C}^* \end{array}$$

where  $c : \mathbb{C}^* \rightarrow \mathbb{C}^*$ ,  $\lambda \mapsto \lambda^{-1}$ .

On the other hand, if we replace the isomorphism  $\mu_\lambda$  by the isomorphism  $\nu_\lambda$  in the above construction, then we can obtain another twistor space  $\mathcal{TW}(X, X'; r)$  which is diffeomorphic to  $\text{TW}(X, X'; r)$ .

### 3.3.2 Another Description

Let  $\mathbf{N} := S^2 \setminus \{(1, 0, 0)\}$ ,  $\mathbf{S} := S^2 \setminus \{(-1, 0, 0)\}$ . Consider the projections

$$p_{\mathbf{N}} : \mathbf{N} \longrightarrow \mathbb{C}, \quad (x_1, x_2, x_3) \mapsto \frac{x_2 + \sqrt{-1}x_3}{1 - x_1}$$

with the inverse

$$p_{\mathbf{N}}^{-1} : \mathbb{C} \longrightarrow \mathbf{N}, \quad \lambda \mapsto \left( \frac{|\lambda|^2 - 1}{|\lambda|^2 + 1}, \frac{2\text{Re}\lambda}{|\lambda|^2 + 1}, \frac{2\text{Im}\lambda}{|\lambda|^2 + 1} \right),$$

and

$$p_{\mathbf{S}} : \mathbf{S} \longrightarrow \mathbb{C}, \quad (x_1, x_2, x_3) \mapsto \frac{x_2 - \sqrt{-1}x_3}{1 + x_1}$$

with the inverse

$$p_{\mathbf{S}}^{-1} : \mathbb{C} \longrightarrow \mathbf{S}, \quad \lambda \mapsto \left( \frac{1 - |\lambda|^2}{|\lambda|^2 + 1}, \frac{2\operatorname{Re}\lambda}{|\lambda|^2 + 1}, \frac{2\operatorname{Im}\lambda}{|\lambda|^2 + 1} \right).$$

As described as previous section, let  $M^{\operatorname{sm}}(X, r)$  and  $M^{\operatorname{sm}}(X', r)$  be the underlying smooth differentiable manifolds of the moduli space  $M_{\operatorname{Dol}}(X, r)$  and  $M_{\operatorname{Dol}}(X', r)$ , respectively. Then they are diffeomorphic and they carry the hyperkähler structures  $(\mathbb{I}, \mathbb{J}, \mathbb{K} := \mathbb{I}\mathbb{J} = -\mathbb{J}\mathbb{I})$  and  $(\mathbb{I}', \mathbb{J}, \mathbb{K}' := \mathbb{I}'\mathbb{J} = -\mathbb{J}\mathbb{I}')$ , respectively. For the product  $M^{\operatorname{sm}}(X, r) \times \mathbf{S}$ , at  $\vec{x} \in \mathbf{S}$ , we assign the fiber  $M^{\operatorname{sm}}(X, r)$  with the complex structure

$$\mathbb{I}_{\vec{x}} = x_1\mathbb{I} + x_2\mathbb{J} + x_3\mathbb{K},$$

and for the product  $M^{\operatorname{sm}}(X', r) \times \mathbf{N}$ , at  $\vec{x}' \in \mathbf{N}$ , we assign the fiber  $M^{\operatorname{sm}}(X', r)$  with the complex structure

$$\mathbb{J}_{\vec{x}'} = x'_1\mathbb{I}' + x'_2\mathbb{J} + x'_3\mathbb{K}',$$

respectively. By Simpson's result, when  $\vec{x} = (x_1, x_2, x_3)$ ,  $\vec{x}' = (-x_1, x_2, x_3)$ , there is a diffeomorphism  $f_{\vec{x}, \vec{x}'} : M^{\operatorname{sm}}(X, r) \rightarrow M^{\operatorname{sm}}(X', r)$  such that it is a biholomorphism

$$(M^{\operatorname{sm}}(X, r), \mathbb{I}_{\vec{x}}) \cong (M^{\operatorname{sm}}(X', r), \mathbb{J}_{\vec{x}'}).$$

Moreover, by virtue of these diffeomorphisms, we can glue  $M^{\operatorname{sm}}(X, r) \times \mathbf{S}$  and  $M^{\operatorname{sm}}(X', r) \times \mathbf{N}$  along the overlap  $\mathbf{S} \cap \mathbf{N}$  to make the topological product  $M^{\operatorname{sm}}(X, r) \times S^2$  into a holomorphic fibration over  $\mathbb{P}^1$  that is exactly biholomorphically equivalent to the smooth locus of the generalized Deligne–Hitchin twistor space  $\operatorname{TW}^{\operatorname{sm}}(X, X'; r) := M_{\operatorname{Hod}}(X, r) \cup_{\mathbf{d}} M_{\operatorname{Hod}}(X', r)$ , where the complex structures on fibers have been defined as above and the complex structure on base is the standard one of  $\mathbb{P}^1$ .

### 3.3.3 $\mathbb{C}^*$ -Action, De Rham Sections and Preferred Sections

The Deligne–Hitchin twistor space introduced in the last section is an important example of  $\gamma$ -twistor space, we can obtain it by choosing  $\gamma$  being the complex conjugation [Sim95, Sim08]. Here we briefly recall it.

**Example 3.3.1** (Deligne–Hitchin twistor space, [Sim95, Sim08]). Let  $X' = \bar{X} := (\mathcal{X}, -I)$  be the complex conjugate of  $X$ , which is generally not isomorphic to  $X = (\mathcal{X}, I)$  unless there is a real structure on  $X$ . Then the Deligne isomorphism  $\mathbf{d} : \mathcal{M}_{\operatorname{Hod}}(X, r) \times_{\mathbb{C}} \mathbb{C}^* \rightarrow \mathcal{M}_{\operatorname{Hod}}(\bar{X}, r) \times_{\mathbb{C}} \mathbb{C}^*$  is explicitly given by

$$((E, \bar{\partial}_E, D^\lambda), \lambda) \mapsto ((E, \lambda^{-1}D^\lambda, \lambda^{-1}\bar{\partial}_E), \lambda^{-1}).$$

The Deligne–Hitchin twistor space  $\operatorname{TW}_{\operatorname{DH}}(X, r)$  is obtained by gluing  $\mathcal{M}_{\operatorname{Hod}}(X, r)$  and  $\mathcal{M}_{\operatorname{Hod}}(\bar{X}, r)$  with the isomorphism  $\mathbf{d}$ . There is an antiholomorphic involution  $\sigma$  on  $\operatorname{TW}_{\operatorname{DH}}(X, r)$  covering the antipodal involution  $\lambda \mapsto -\bar{\lambda}^{-1}$  of  $\mathbb{P}^1$ . It is defined by gluing together antilinear isomorphisms  $i : \mathcal{M}_{\operatorname{Hod}}(X, r) \rightarrow \mathcal{M}_{\operatorname{Hod}}(\bar{X}, r)$  and  $i^{-1} : \mathcal{M}_{\operatorname{Hod}}(\bar{X}, r) \rightarrow \mathcal{M}_{\operatorname{Hod}}(X, r)$ , where  $i$  is given by

$$((E, \bar{\partial}_E, D^\lambda), \lambda) \mapsto ((\bar{E}^*, (\bar{\partial}_E)^*, -(\bar{D}^\lambda)^*), -\bar{\lambda}).$$

When  $(E, \bar{\partial}_E, D^\lambda)$  admits a pluri-harmonic metric  $h$ , then we have an isomorphism

$$(\bar{E}^*, (\bar{\partial}_E)^*, -(\bar{D}^\lambda)^*, -\bar{\lambda}) \cong ((E, \delta'_h, -\delta''_h), -\bar{\lambda}).$$

**Theorem 3.3.2.** *Let  $\{(E, \bar{\partial}''_{E_i}, D_i^\lambda)\}$  be a sequence lying in  $M_{\text{Hod}}^\lambda(X, r)$ , and assume it has a limit point  $(E, \bar{\partial}''_\infty, D_\infty^\lambda)$ . Denoted by  $(E, \bar{\partial}_i, \theta_i)$  and  $(E, \bar{\partial}_\infty, \theta_\infty)$  the Higgs bundles corresponding to  $(E, \bar{\partial}''_{E_i}, D_i^\lambda)$  and  $(E, \bar{\partial}''_\infty, D_\infty^\lambda)$ , respectively. Then the sequence  $\{(E, \bar{\partial}_i, \theta_i)\}$  converges to the point  $(E, \bar{\partial}_\infty, \theta_\infty)$  in the space  $M_{\text{Dol}}(X, r)$ .*

*Proof.* Actually, this property is an ingredient of the proof of [Corollary 2.0.3](#), and can be proved by Theorem 5.12 in [[Sim94a](#)] and the separateness of moduli space as what Simpson has done in [[Sim94b](#)]. Now we revisit it by twistor theory. By assumption, we can assume the sequence  $\{(E, \bar{\partial}''_{E_i}, D_i^\lambda)\}$  lie in a sufficiently small open subset of  $M_{\text{Hod}}^\lambda(X, r)$ . Every point  $(E, \bar{\partial}''_{E_i}, D_i^\lambda)$  gives rise to a preferred section of Deligne–Hitchin twistor space. From the proof of [Theorem 2.0.2](#), the sequence  $\{(E, \bar{\partial}_i, \theta_i)\}$  has a limit point denoted by  $(\tilde{E}, \tilde{\partial}_\infty, \tilde{\theta}_\infty)$ . It is known that each preferred section has the weight 1 property, that is, the normal bundle of a preferred section is isomorphic to the bundle  $(\mathcal{O}(1))^{\oplus \dim_{\mathbb{C}}(M_{\text{dR}}(X, r))}$  over  $\mathbb{P}^1$ . Hence we get a holomorphic section joining  $(E, \bar{\partial}''_\infty, D_\infty^\lambda, \lambda)$  to  $(\tilde{E}, \tilde{\partial}_\infty, \tilde{\theta}_\infty, 0)$ , this section must be  $\sigma$ -invariant since it lies in the closure of a  $\sigma$ -invariant subset. However, as we have seen in the last section (cf. [Proposition 3.2.1](#)), locally every  $\sigma$ -invariant holomorphic section is preferred, thus  $(\tilde{E}, \tilde{\partial}_\infty, \tilde{\theta}_\infty) = (E, \bar{\partial}_\infty, \theta_\infty)$ .  $\square$

The  $\mathbb{C}^*$ -action on  $M_{\text{Hod}}(X, r)$  (see [Section 2.1.1, Chapter 2](#)) can be extended to the entire generalized Deligne–Hitchin twistor space  $\text{TW}(X, X'; r)$ . Indeed, one defines the  $\mathbb{C}^*$ -action on  $M_{\text{Hod}}(X', r)$  by

$$t \cdot [E, \bar{\partial}_E, D^\lambda, \lambda] := [E, \bar{\partial}_E, t^{-1}D^\lambda, t^{-1}\lambda],$$

then the following diagram commutes

$$\begin{array}{ccc} M_{\text{Hod}}(X, r) & \xrightarrow{t} & M_{\text{Hod}}(X, r) \\ \mathbf{d} \downarrow & & \mathbf{d} \downarrow \\ M_{\text{Hod}}(X', r) & \xrightarrow{t} & M_{\text{Hod}}(X', r). \end{array}$$

Fixing a twistor parameter  $\lambda_0 \in \mathbb{C}^*$ , associated to a given  $\lambda_0$ -flat bundle  $(E, \bar{\partial}_E, D^{\lambda_0}) \in M_{\text{Hod}}^{\lambda_0}(X, r)$ , we have a holomorphic section

$$s_{\lambda_0} : \lambda \longmapsto (E, \bar{\partial}_E, \lambda \lambda_0^{-1} D^{\lambda_0})$$

of  $M_{\text{Hod}}(X, r) \times_{\mathbb{C}} \mathbb{C}^* \rightarrow \mathbb{C}^*$ . By Simpson's [Theorem 2.2.3](#), the limit  $\lim_{\lambda \rightarrow 0} s_{\lambda_0}(\lambda)$  exists and is a fixed point of the  $\mathbb{C}^*$ -action, hence  $s_{\lambda_0}(\lambda)$  can be holomorphically extended to  $\lambda = 0$ . By Deligne isomorphism, we have

$$\mathbf{d}(s_{\lambda_0}(\lambda)) = (E, (\lambda_0^{-1} D^{\lambda_0} + \bar{\partial}_E)_{X'}^{0,1}, \lambda^{-1} (\lambda_0^{-1} D^{\lambda_0} + \bar{\partial}_E)_{X'}^{1,0})$$

as a section of  $M_{\text{Hod}}(X', r) \times_{\mathbb{C}} \mathbb{C}^* \rightarrow \mathbb{C}^*$ . Again by Simpson's [Theorem 2.2.3](#),  $\mathbf{d}(s_{\lambda_0}(\lambda))$  can be holomorphically extended to  $\lambda = \infty$ , and the limit  $\lim_{\lambda \rightarrow \infty} \mathbf{d}(s_{\lambda_0}(\lambda))$  is also a fixed point of  $\mathbb{C}^*$ -action. The extended section is also denoted by  $s_{\lambda_0}$ , and is called the *de Rham section* associated to  $(E, \bar{\partial}_E, D^{\lambda_0})$ .

Given a section  $s : \mathbb{P}^1 \rightarrow \text{TW}(X, X'; r)$  of the generalized Deligne–Hitchin twistor space  $\text{TW}(X, X'; r)$ , its normal bundle is given by the short exact sequence

$$0 \longrightarrow T_{\mathbb{P}^1} \longrightarrow s^*T_{\text{TW}(X, X'; r)} \longrightarrow N_s \longrightarrow 0.$$

For the complex  $(\text{End}(E) \otimes \Omega_X^\bullet, \lambda_0^{-1}D^{\lambda_0})$  associated to the  $\lambda_0$ -flat bundle  $(E, \bar{\partial}_E, D^{\lambda_0}, \lambda_0)$ , there is an isomorphism

$$\mathbb{H}^1(\text{End}(E) \otimes \Omega_X^\bullet, \lambda_0^{-1}D^{\lambda_0}) \cong \mathcal{H}^1(E, D^{\lambda_0}) := \left\{ \begin{array}{l} (\alpha, \beta) \in \Omega_X^{0,1}(\text{End}(E)) \oplus \Omega_X^{1,0}(\text{End}(E)) \\ : D^{\lambda_0}\alpha + \lambda_0\bar{\partial}_E\beta = \bar{\lambda}_0\delta'_{h,\lambda_0}\alpha - \delta''_{h,\lambda_0}\beta = 0 \end{array} \right\}$$

due to Kähler identities of flat  $\lambda_0$ -connection (Lemma 1.2.4), where  $\delta'_{h,\lambda_0}$  and  $\delta''_{h,\lambda_0}$  are defined by the pluri-harmonic metric  $h$  on  $(E, \bar{\partial}_E, D^{\lambda_0})$ . For any  $\lambda \in \mathbb{P}^1$ , since stability is an open condition, we have the isomorphism  $N_s|_\lambda \cong \mathcal{H}^1(E, D^{\lambda_0})$ .

By Grothendieck's theorem,

$$N_s \cong \bigoplus_{i=1}^{\kappa} \mathcal{O}_{\mathbb{P}^1}(n_i),$$

where  $\kappa = \dim_{\mathbb{C}}(\mathcal{H}^1(E, D^{\lambda_0})) = \dim_{\mathbb{C}}(M_B(\mathcal{X}, G))$ . The degree  $\deg(N_s) = \sum_{i=1}^{\kappa} n_i$  is called the *weight* of the section  $s$ .

In [Ver14], Verbitsky introduced the notion of ample rational curves, a rational curve  $s$  in an algebraic variety is called *ample* if its normal bundle is positive, that is, if  $N_s \cong \bigoplus_i \mathcal{O}(i), i > 0$ . Since preferred sections of Deligne–Hitchin twistor space have weight 1 property, they are ample rational curves. Similarly, we introduce the following notations for  $\gamma$ -twistor space:

**Definition 3.3.3** ([Ver14]). A section  $s : \mathbb{P}^1 \rightarrow \text{TW}(X, X'; r)$  is called

- (1) a *balanced rational curve* if  $n_1 = \dots = n_\kappa$ ;
- (2) a *positive rational curve* if its weight  $\deg(N_s) = \sum_{i=1}^{\kappa} n_i$  is a positive integer;
- (3) an *ample rational curve* if each  $n_i$  is a positive integer.

By Kodaira's deformation theory [Kod62], the infinitesimal deformations of the section  $s$  are given by elements in the cohomology  $H^0(\mathbb{P}^1, N_s)$ , and the corresponding obstruction is described by the cohomology  $H^1(\mathbb{P}^1, N_s)$ . Hence if  $s$  is a positive rational curve, then any infinitesimal deformation of  $s$  is unobstructed.

For de Rham sections of the Deligne–Hitchin twistor space  $\text{TW}(X, X'; r)$ , we have the following property:

**Theorem 3.3.4.** *Let  $s_{\lambda_0} : \mathbb{P}^1 \rightarrow \text{TW}(X, X'; r)$  be a de Rham section of the generalized Deligne–Hitchin twistor space  $\text{TW}(X, X'; r)$ , then*

- (1) *The weight 1 property holds, namely*

$$N_{s_{\lambda_0}} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus \kappa}.$$

*Therefore, de Rham sections of the generalized Deligne–Hitchin twistor space  $\text{TW}(X, X'; r)$  are balanced ample rational curves;*

- (2) *The infinitesimal deformations of  $s_{\lambda_0}$  are unobstructed*

*Proof.* Let  $s_{\lambda_0}$  be a de Rham section of the generalized Deligne–Hitchin twistor space  $\mathrm{TW}(X, X'; r)$ , take  $(\alpha, \beta) \in \mathcal{H}^1(E, D^{\lambda_0})$ , under the Deligne isomorphism  $\mathbf{d}_\gamma$ , we have

$$\begin{aligned} & \mathbf{d}(E, \bar{\partial}_E + \alpha, \lambda(\lambda_0^{-1}D^{\lambda_0} + \beta)) \\ &= \left( E, (\lambda_0^{-1}D^{\lambda_0} + \bar{\partial}_E + \alpha + \beta)_{X'}^{0,1}, \lambda^{-1}(\lambda_0^{-1}D^{\lambda_0} + \bar{\partial}_E + \alpha + \beta)_{X'}^{1,0} \right), \end{aligned}$$

which means that the transition function of the bundle  $N_{s_{\lambda_0}}$  over  $\mathbb{P}^1$  is determined by

$$\alpha + \lambda\beta \mapsto \lambda^{-1} \left( \frac{\sqrt{-1} + I'}{2\sqrt{-1}}(\alpha + \beta) + \lambda \frac{\sqrt{-1} - I'}{2\sqrt{-1}}(\alpha + \beta) \right),$$

which means that  $N_{s_{\lambda_0}} \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus \kappa}$ .

(2) Since  $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(i)) = 0$  whenever  $i \geq 0$ , the second claim follows from the standard Kodaira deformation theory [Kod62]  $\square$

**Remark 3.3.5.** By [HKLR87], a preferred section  $p : \mathbb{P}^1 \rightarrow \mathrm{TW}_{\mathrm{DH}}(X, r)$  of the Deligne–Hitchin twistor space  $\mathrm{TW}_{\mathrm{DH}}(X, r)$  is fully characterized by the following three properties:

- (1) It is holomorphic;
- (2) It has weight 1 property, or equivalently, it is a balanced (ample) rational of weight  $\kappa$ ;
- (3) It is  $\sigma$ -invariant.

It's obviously that the last condition could not be deleted, since de Rham sections of the Deligne–Hitchin twistor space  $\mathrm{TW}_{\mathrm{DH}}(X, r)$  have the weight 1 property. As we have remarked in the previous content, recently in [BHR19], the authors constructed holomorphic and  $\sigma$ -invariant, but not preferred sections for  $\mathrm{TW}_{\mathrm{DH}}(X, 2)$  (for  $X$  a compact Riemann surface). Hence the second condition is also necessary, but we do not know whether these three conditions fully characterize the preferred sections.

The fixed points of the  $\mathbb{C}^*$ -action on the generalized Deligne–Hitchin twistor space  $\mathrm{TW}(X, X'; r)$  lie in the fibers  $\pi^{-1}(0)$  and  $\pi^{-1}(\infty)$ , namely the  $\mathbb{C}$ -VHS in the corresponding Dolbeault moduli space  $\mathcal{M}_{\mathrm{Dol}}(X, r)$  and  $\mathcal{M}_{\mathrm{Dol}}(X', r)$ , respectively.

Look back on the stratifications of moduli spaces into locally closed strata introduced in Section 2.2.2:

$$\mathcal{M}_{\mathrm{Hod}}(X, r) = \bigcup_{\alpha} G_{\alpha}, \quad \mathcal{M}_{\mathrm{Hod}}^{\lambda}(X, r) = \bigcup_{\alpha} G_{\alpha}^{\lambda}, \quad \mathcal{M}_{\mathrm{Dol}}(X, r) = \bigcup_{\alpha} G_{\alpha}^0, \quad \mathcal{M}_{\mathrm{dR}}(X, r) = \bigcup_{\alpha} G_{\alpha}^1,$$

we can define these stratifications in the level of twistor space. More precisely, define

- $\tilde{G}_{\alpha} := \{\Theta \in \mathrm{TW}(X, X'; r) : \lim_{t \rightarrow 0} t \cdot \Theta = (u, 0), u \in P_{\alpha}\}$ ;
- $\tilde{G}_{\alpha}^{\lambda} := \tilde{G}_{\alpha} \cap \tilde{\pi}^{-1}(\lambda) = \{[E, d''_E, D^{\lambda}] \in M_{\mathrm{Hod}}^{\lambda}(X, r) : \lim_{t \rightarrow 0} t \cdot [E, d''_E, D^{\lambda}] = u, u \in P_{\alpha}\}$ ;
- $\tilde{G}_{\alpha}^{\lambda}(u) := \{[E, d''_E, D^{\lambda}] \in M_{\mathrm{Hod}}^{\lambda}(X, r) : \lim_{t \rightarrow 0} t \cdot [E, d''_E, D^{\lambda}] = u\}$ ;

where  $\alpha$  is an index,  $\lambda \in \mathbb{C}$ , and  $u \in P_\alpha$  stands for a  $\mathbb{C}$ -VHS. Then it's easy to see that

$$\tilde{G}_\alpha^\lambda = G_\alpha^\lambda, \quad \tilde{G}_\alpha^\lambda(u) = G_\alpha^\lambda(u).$$

So from now on, we will use the notations  $G_\alpha^\lambda, G_\alpha^\lambda(u), \dots$ . For any  $\lambda_1, \lambda_2 \in \mathbb{C}^*$ , and any  $u \in P_\alpha$ , it's clear that  $G_\alpha^{\lambda_1}(u) \cong G_\alpha^{\lambda_2}(u)$  as analytic varieties.

For  $\lambda \neq 0$  and  $u \in P_\alpha$  which stands for a stable  $\mathbb{C}$ -VHS, then by Mochizuki correspondence,  $G_\alpha^\lambda(u)$  gives rise to a smooth submanifold  $Y_\alpha^\lambda(u)$  of  $\mathcal{M}_{\text{Dol}}(X, r)$ , and in particular,  $Y_\alpha^1(u)$  is the image of  $G_\alpha^1(u)$  under non-Abelian Hodge correspondence. When  $\lambda = 0$ , set  $Y_\alpha^0(u) := G_\alpha^0(u)$ . By [Theorem 2.2.16](#),  $Y_\alpha^\lambda(u)$  contains  $u$ , we give the followings notions of smooth submanifolds of  $\mathcal{M}_{\text{Dol}}(X, r)$ :

- $Y_\alpha^\lambda(u)$  is called a  $\lambda$ -Hodge fiber over  $u$ , in particular,
  - $Y_\alpha^1(u)$  is called a *de Rham fiber* over  $u$ ;
  - $Y_\alpha^0(u)$  is called a *Higgs fiber* over  $u$ .

Some interesting properties are studied and obtained by Simpson in [[Sim95](#)], and Collier and Wentworth in [[CW19](#)]:

- $G_\alpha^\lambda(u)$  is locally closed in  $\mathcal{M}_{\text{Hod}}^\lambda(X, r)$  and is isomorphic to an affine spaces for any  $\lambda \in \mathbb{C}$ ;
- $G_\alpha^\lambda(u)$  is a Lagrangian submanifold with respect to the corresponding holomorphic symplectic forms on  $\mathcal{M}_{\text{Hod}}^\lambda(X, r)$  for any  $\lambda \in \mathbb{C}$ ;
- $G_\alpha^{\lambda_1}(u)$  is biholomorphic to  $G_\alpha^{\lambda_2}(u)$  for any  $\lambda_1, \lambda_2 \in \mathbb{C}$ ;
- $Y_\alpha^0(u)$  is transverse to  $Y_\alpha^1(u)$  at  $u$ .

Moreover, in [[Sim95](#)], Simpson conjectured that the Lagrangian fiber  $G_\alpha^1(u)$  is closed in  $\mathcal{M}_{\text{dR}}(X, r)$  and produces a foliation of the moduli space when varying  $\alpha$  (cf. [Conjecture 2.2.7](#)). This conjecture of rank two in the parabolic context was proved by the authors of [[LSS13](#)]. However, as pointed out by Simpson,  $G_\alpha^0(u)$  is not closed in  $\mathcal{M}_{\text{Dol}}(X, r)$  in general. For example, if we take  $u = (E, 0)$  for any stable vector bundle  $E$ , then  $G_\alpha^0(u) = H^0(X, \text{End}(E) \otimes \Omega_X^1)$ , the authors of [[PPN19](#)] showed that  $G_\alpha^0(u)$  is closed in  $\mathcal{M}_{\text{Dol}}(X, r)$  if and only if  $E$  is very stable, i.e. there is no non-zero nilpotent Higgs field on  $E$ .

We have the following *half-dimensional property*:

**Proposition 3.3.6.** *Let  $u \in M_{\text{Dol}}(X, r)$  be a stable  $\mathbb{C}$ -VHS. Then the tangent space  $T_v Y_u^\lambda$  at  $v \in Y_u^\lambda$  is half-dimensional of  $M_{\text{Dol}}(X, r)$ .*

*Proof.* Write  $u = (\mathcal{E}, \theta) = (\bigoplus_{i=1}^k E_i, \bigoplus_{i=1}^{k-1} \theta_i : E_i \rightarrow E_{i+1} \otimes K_X)$ , and let  $s = (E, D^\lambda) \in M_{\text{Hod}}^\lambda(X, r)$  be the preimage of  $v$  under Mochizuki correspondence. Then there is a Simpson filtration  $\{F^\bullet\} : 0 \subset F^{k-1} \subset \dots \subset F^0 = E$  on the flat bundle  $(E, \lambda^{-1}D^\lambda)$  such that  $(\text{Gr}_F(E), \text{Gr}_F(\lambda^{-1}D^\lambda)) = u$ . The Simpson filtration  $\{F^\bullet\}$  produces a filtration on  $\text{End}(E)$  by

$$F^p(\text{End}(E)) := \{\varphi \in \text{End}(E) : \varphi : F^q \rightarrow F^{p+q} \text{ for any } q\},$$

hence the graded objects are given by  $\mathfrak{E}^p := \text{Gr}_F^p(\text{End}(E)) = \bigoplus_{i=1}^k \text{Hom}(E_i, E_{i+p})$ . Then there is a filtration on  $\text{End}(E) \otimes \Omega_X^1$  by

$$F^p(\text{End}(E) \otimes \Omega_X^1) = F^{p-1}(\text{End}(E)) \otimes \Omega_X^1,$$

which induces a filtration on the hypercohomology  $\mathbb{H}^i(\Omega_X^\bullet(\text{End}(E)), \lambda^{-1}\widetilde{D}^\lambda)$ . By Lemma 7.1 in [Sim95], we have

$$\begin{aligned} T_s G_\alpha^\lambda(u) &\cong T_{s'} Y_\alpha^\lambda(u) \simeq F^1(\mathbb{H}^1(\Omega_X^\bullet(\text{End}(E)), \lambda^{-1}D^\lambda)) \\ &\cong \bigoplus_{p=1}^{k-1} \mathbb{H}^1\left(\text{Gr}_F^p(\text{End}(E)) \xrightarrow{\text{Gr}_F(\lambda^{-1}D^\lambda)} \text{Gr}_F^{p-1}(\text{End}(E)) \otimes K_X\right) \\ &\cong \left\{ (\alpha, \beta) \in \Omega_X^{0,1}\left(\bigoplus_{p=1}^{k-1} \mathfrak{e}^p\right) \oplus \Omega_X^{1,0}\left(\bigoplus_{p=0}^{k-1} \mathfrak{e}^p\right) : \bar{\partial}_\mathcal{E}\beta + [\theta, \alpha] = \partial_{\mathcal{E},h}\alpha + [\theta_h^\dagger, \beta] = 0 \right\}, \end{aligned}$$

where  $h$  is the pluri-harmonic metric on  $(\mathcal{E}, \theta)$ . The dimension of the last space can be calculated by Riemann-Roch as done in Lemma 3.6 of [CW19].  $\square$

Let  $\mathcal{M}_{\text{Hod}}(X, r, \mathcal{O}_X)$  (respectively,  $M_{\text{Hod}}(X, r, \mathcal{O}_X)$ ) be the moduli space of semistable (respectively, stable)  $\lambda$ -flat ( $\lambda \in \mathbb{C}$ ) bundles of rank  $r$  over  $X$  with vanishing Chern classes and fixed determinant  $\mathcal{O}_X$ , and let  $M_{\text{Hod}}^s(X, r, \mathcal{O}_X)$  be the Zariski dense open subset of  $M_{\text{Hod}}(X, r, \mathcal{O}_X)$  that consists of  $\lambda$ -flat bundles such that the underlying vector bundles are stable, its fiber over a fixed  $\lambda \in \mathbb{C}$  is denoted as  $M_{\text{Hod}}^{\lambda,s}(X, r, \mathcal{O}_X)$ . Let  $\text{TW}(X, X'; r, \mathcal{O})$  (respectively,  $\text{TW}^s(X, X'; r, \mathcal{O})$ ) be the generalized Deligne-Hitchin twistor space for a chosen  $X' \in \mathcal{M}(\mathcal{X})$ , which is obtained by gluing  $\mathcal{M}_{\text{Hod}}(X, r, \mathcal{O}_X) \times_{\mathbb{C}} \mathbb{C}^*$  and  $\mathcal{M}_{\text{Hod}}(X', r, \mathcal{O}_{X'}) \times_{\mathbb{C}} \mathbb{C}^*$  (respectively, gluing  $M_{\text{Hod}}(X, r, \mathcal{O}_X) \times_{\mathbb{C}} \mathbb{C}^*$  and  $M_{\text{Hod}}(X', r, \mathcal{O}_{X'}) \times_{\mathbb{C}} \mathbb{C}^*$ ) together via Deligne isomorphism **d**.

The following theorem is viewed as the Torelli-type theorem for the generalized Deligne-Hitchin twistor space, where the approach of the proof follows from [BGHL09]. The key point is the bound on the dimension of irreducible components of the moduli space of  $\mathbb{C}$ -VHS from above half-dimensional property.

**Theorem 3.3.7.** *Let  $X, X' \in \mathcal{M}(\mathcal{X})$  and  $Y, Y' \in \mathcal{M}(\mathcal{Y})$  be Riemann surfaces with genus  $g \geq 3$ . If  $\text{TW}(X, X'; r, \mathcal{O})$  is analytically isomorphic to  $\text{TW}(Y, Y'; r, \mathcal{O})$ , then either  $X \cong Y, X' \cong Y'$ , or  $X \cong Y', X' \cong Y$ .*

*Proof.* Let  $(\text{TW}(X, X'; r, \mathcal{O}))^{\mathbb{C}^*} \subseteq \text{TW}(X, X'; r, \mathcal{O})$  be the fixed point locus under the  $\mathbb{C}^*$ -action. Then by above half-dimensional property, for an irreducible component  $Z$  of  $(\text{TW}(X, X'; r, \mathcal{O}))^{\mathbb{C}^*}$ , we have

$$\dim_{\mathbb{C}}(Z) \leq \frac{1}{2} M_{\text{dR}}(X, r).$$

In particular, since the dimension of each irreducible component of the nilpotent cone is a Lagrangian variety, hence has half of the dimension of the moduli space, which is due to Laumon [Lau87] and Faltings [Fal93]. Then by a result of Simpson (Lemma 11.9 in [Sim94b]), above equality holds if and only if either  $Z = \mathcal{U}^s(X, r)$ , the moduli space of stable bundles of rank  $r$  over  $X$  with vanishing first Chern classes, or  $Z = \mathcal{U}^s(X', r)$ .

We have defined the  $\mathbb{C}^*$ -action on  $\text{TW}(X, X'; r, \mathcal{O})$ , which gives rise to a non-trivial holomorphic vector field on  $\text{TW}(X, X'; r, \mathcal{O})$ . By Corollary 3.4 in [BGHL09], the restriction of the tangent bundle  $T\text{TW}(X, X'; r, \mathcal{O})$  to  $\mathcal{U}^s(X, r)$  or  $\mathcal{U}^s(X', r)$  has no non-zero holomorphic sections. Let  $\iota(\mathcal{U}^s(X, r)) \subseteq \text{TW}(Y, Y'; r, \mathcal{O})$  be the image under the composition  $\mathcal{U}^s(X, r) \hookrightarrow \text{TW}(X, X'; r, \mathcal{O}) \cong \text{TW}(Y, Y'; r, \mathcal{O})$ , hence  $\iota(\mathcal{U}^s(X, r))$  is contained in  $(\text{TW}(Y, Y'; r, \mathcal{O}))^{\mathbb{C}^*}$ , which implies either  $\iota(\mathcal{U}^s(X, r)) = \mathcal{U}^s(Y, r)$  or  $\iota(\mathcal{U}^s(X, r)) = \mathcal{U}^s(Y', r)$ .

Then the conclusion follows from Theorem E in [KP95].  $\square$

### 3.3.4 Automorphism Groups

In this subsection, we study the automorphism groups of generalized Deligne–Hitchin twistor spaces.

First we calculate the automorphism groups of Hodge moduli spaces, which generalizes the result of Biswas and Heller [BH17] to the case of higher rank, we always assume  $g \geq 4$  and  $r \geq 2$  when there is no specification.

**Proposition 3.3.8.** *The Picard group  $\text{Pic}(M_{\text{Hod}}(X, r, \mathcal{O}_X))$  is isomorphic to  $\mathbb{Z}$ .*

*Proof.* Let  $\mathring{M}_{\text{Hod}}(X, r, \mathcal{O}_X) = M_{\text{Hod}}(X, r, \mathcal{O}_X) \setminus M_{\text{Hod}}^s(X, r, \mathcal{O}_X)$ . It is known that the codimension of  $\mathring{M}_{\text{Hod}}(X, r, \mathcal{O}_X)$  in  $M_{\text{Hod}}(X, r, \mathcal{O}_X)$  is at least 3 [BM09, BGL11], which implies that the pullback morphism provides an isomorphism between Picard groups  $\text{Pic}(M_{\text{Hod}}(X, r, \mathcal{O}_X))$  and  $\text{Pic}(M_{\text{Hod}}^s(X, r, \mathcal{O}_X))$ .

Recall that for an algebraic fibration  $f : \chi \rightarrow S$  of varieties with geometric connected fibers, there is an exact sequence

$$\text{Id} \longrightarrow \text{Pic}(S) \longrightarrow \text{Pic}(\chi) \longrightarrow \text{Pic}(\chi/S)(S) \longrightarrow \text{Br}(S),$$

where  $\text{Pic}(\chi/S)(S)$  is the group of sections of the relative Picard scheme  $\text{Pic}(\chi/S)$  and  $\text{Br}(S)$  denotes the Brauer group of the base  $S$ . For our case, the fibers of  $\pi : M_{\text{Hod}}(X, r, \mathcal{O}_X) \rightarrow \mathbb{C}$  are irreducible varieties,  $\text{Pic}(\mathbb{C})$  and  $\text{Br}(\mathbb{C})$  are both trivial. Hence, there are isomorphisms

$$\text{Pic}(M_{\text{Hod}}(X, r, \mathcal{O}_X)) \cong \text{Pic}(M_{\text{Hod}}^s(X, r, \mathcal{O}_X)) \cong \text{Pic}(M_{\text{Hod}}^s(X, r, \mathcal{O}_X)/\mathbb{C})(\mathbb{C}).$$

Denote the restriction  $\pi|_{M_{\text{Hod}}^s(X, r, \mathcal{O}_X)} : M_{\text{Hod}}^s(X, r, \mathcal{O}_X) \rightarrow \mathbb{C}$  by  $\pi_s$ . Then the fiber  $\pi_s^{-1}(0)$  is canonically isomorphic to the cotangent bundle  $T^*\mathcal{U}^s(X, r, \mathcal{O}_X)$  of  $\mathcal{U}^s(X, r, \mathcal{O}_X)$ , and the fiber  $\pi_s^{-1}(\lambda)$  for any  $\lambda \in \mathbb{C}^*$  is a torsor associated to  $T^*\mathcal{U}^s(X, r, \mathcal{O}_X)$ . Again by the above exact sequence, we have

$$\text{Pic}(\pi_s^{-1}(\lambda)) \cong \text{Pic}(\mathcal{U}^s(X, r, \mathcal{O}_X))$$

for any  $\lambda \in \mathbb{C}$ . From a Drezet–Narasimhan’s famous result [DN89], which states that

$$\text{Pic}(\mathcal{U}^s(X, r, \mathcal{O}_X)) \cong \mathbb{Z},$$

it immediately follows that  $\text{Pic}(M_{\text{Hod}}(X, r, \mathcal{O}_X)) \cong \mathbb{Z}$ . □

Let  $\text{Aut}\left(M_{\text{Hod}}(X, r, \mathcal{O}_X), M_{\text{Hod}}^s(X, r, \mathcal{O}_X)\right)$  be the subgroup of  $\text{Aut}(M_{\text{Hod}}(X, r, \mathcal{O}_X))$  consisting of the automorphisms which preserve both  $M_{\text{Hod}}^s(X, r, \mathcal{O}_X)$  and  $\mathring{M}_{\text{Hod}}(X, r, \mathcal{O}_X)$ .

**Lemma 3.3.9.**

$$\text{Aut}\left(M_{\text{Hod}}(X, r, \mathcal{O}_X), M_{\text{Hod}}^s(X, r, \mathcal{O}_X)\right) \cong \text{Aut}(M_{\text{Hod}}^s(X, r, \mathcal{O}_X)).$$

*Proof.* Since  $M_{\text{Hod}}^s(X, r, \mathcal{O}_X)$  is a dense open subset of  $M_{\text{Hod}}(X, r, \mathcal{O}_X)$ . We only need to show that the restriction morphism

$$\text{Aut}\left(M_{\text{Hod}}(X, r, \mathcal{O}_X), M_{\text{Hod}}^s(X, r, \mathcal{O}_X)\right) \rightarrow \text{Aut}(M_{\text{Hod}}^s(X, r, \mathcal{O}_X))$$

via  $\sigma \mapsto \sigma|_Y$  is surjective, namely any automorphism of  $M_{\text{Hod}}^s(X, r, \mathcal{O}_X)$  can be lifted to an automorphism of  $M_{\text{Hod}}(X, r, \mathcal{O}_X)$ . Let  $\sigma$  be an elements of  $\text{Aut}(M_{\text{Hod}}^s(X, r, \mathcal{O}_X))$ , which can be viewed as a pseudo-automorphism of  $M_{\text{Hod}}(X, r, \mathcal{O}_X)$ . Then  $\sigma$  induces an automorphism of  $\text{Pic}(M_{\text{Hod}}(X, r, \mathcal{O}_X))$ , by the above [Proposition 3.3.8](#), it must be the identity. Choose  $\Lambda$  to be a very ample line bundle on  $M_{\text{Hod}}(X, r, \mathcal{O}_X)$ , since

$$H^0(M_{\text{Hod}}(X, r, \mathcal{O}_X), \Lambda) \cong H^0(M_{\text{Hod}}^s(X, r, \mathcal{O}_X), \Lambda|_{M_{\text{Hod}}^s(X, r, \mathcal{O}_X)})$$

due to Hartog's theorem,  $\sigma$  induces an isomorphism of  $H^0(M_{\text{Hod}}(X, r, \mathcal{O}_X), \Lambda)$ , hence an automorphism of  $\mathbb{P}(H^0(M_{\text{Hod}}(X, r, \mathcal{O}_X), \Lambda))$ , which gives rise to an automorphism of  $M_{\text{Hod}}(X, r, \mathcal{O}_X)$ .  $\square$

**Theorem 3.3.10.** *The group  $\text{Aut}(M_{\text{Hod}}(X, r, \mathcal{O}_X), M_{\text{Hod}}^s(X, r, \mathcal{O}_X))$  satisfies the following short exact sequence*

$$\text{Id} \longrightarrow \mathfrak{V} \longrightarrow \text{Aut}(M_{\text{Hod}}(X, r, \mathcal{O}_X), M_{\text{Hod}}^s(X, r, \mathcal{O}_X)) \longrightarrow \text{Aut}(\mathcal{U}^s(X, r, \mathcal{O}_X)) \times \mathbb{C}^* \longrightarrow \text{Id}.$$

Let  $\text{ad}(E)$  be the subbundle of  $\text{End}(E)$  consisting of trace zero endomorphisms, and  $\mathcal{H}$  be a tautological bundle over  $\mathcal{U}^s(X, r, L)$  such that the fiber is  $H^0(X, \text{ad}(E) \otimes K_X)$  over a point  $E \in \mathcal{U}^s(X, r, \mathcal{O}_X)$ , then the abelian group  $\mathfrak{V}$  is isomorphic to  $\text{Mor}(\mathbb{C}, H^0(\mathcal{U}^s(X, r, \mathcal{O}_X), \mathcal{H}))$  consisting of the algebraic morphisms from  $\mathbb{C}$  to the space  $H^0(\mathcal{U}^s(X, r, \mathcal{O}_X), \mathcal{H})$  of algebraic sections of  $\mathcal{H}$ .

*Proof.* Let  $\sigma$  be an automorphism of  $M_{\text{Hod}}(X, r, \mathcal{O}_X)$ . Denote the following composition

$$\pi^{-1}(\lambda) \xrightarrow{\sigma} M_{\text{Hod}}(X, r, \mathcal{O}_X) \xrightarrow{\pi} \mathbb{C}$$

by  $\tilde{\sigma}_\lambda : \pi^{-1}(\lambda) \rightarrow \mathbb{C}$ . If  $\lambda \in \mathbb{C}^*$ , it has been shown that there are no nonconstant algebraic functions on  $M_{\text{Hod}}^\lambda(X, r, \mathcal{O}_X)$  [[Sim91](#), [Sin20](#)]. Hence,  $\tilde{\sigma}_\lambda$  is a constant map, in other words,  $\sigma$  maps the fiber  $\pi^{-1}(\lambda)$  to another fiber  $\pi^{-1}(\lambda')$  for some  $\lambda' \in \mathbb{C}$ . By Riemann–Hilbert correspondence,  $M_{\text{dR}}(X, r, \mathcal{O}_X)$  is biholomorphic to the affine variety  $M_{\text{B}}(X, \text{SL}(2, \mathbb{C}))$ , hence it does not contain any compact submanifold of positive dimension, however, by properness of Hitchin map [[Hit87a](#), [Hit87b](#)],  $M_{\text{Dol}}(X, r, \mathcal{O}_X)$  contains many compact submanifolds of positive dimensions, namely,  $\pi^{-1}(\lambda)$  is not biholomorphic to  $\pi^{-1}(0)$  for any  $\lambda \in \mathbb{C}^*$ . It follows that  $\lambda'$  actually belongs to  $\mathbb{C}^*$ , and  $\sigma$  preserves the fiber  $\pi^{-1}(0)$ . Therefore,  $\sigma$  induces an algebraic automorphism  $\tau_\sigma : \mathbb{C} \rightarrow \mathbb{C}$  satisfying

- $\pi \circ \sigma = \tau_\sigma \circ \pi$ ,
- $\tau_\sigma(0) = 0$ ,

namely,  $\tau$  is expressed as  $\tau_\sigma(z) = \varepsilon_\sigma \cdot z$  for some  $\varepsilon_\sigma \in \mathbb{C}^*$ .

Now we assume  $\sigma \in \text{Aut}(M_{\text{Hod}}^{\lambda, s}(X, r, \mathcal{O}_X))$ . We claim that there is a map  $\eta : \mathbb{C} \rightarrow \text{Aut}(\mathcal{N}(X, r, \mathcal{O}_X))$  such that the following diagram commutes

$$\begin{array}{ccc} \pi_s^{-1}(\lambda) & \xrightarrow{\sigma} & \pi_s^{-1}(\varepsilon_\sigma \lambda) \\ f_\lambda \downarrow & & f_{\varepsilon_\sigma \lambda} \downarrow \\ \mathcal{U}^s(X, r, \mathcal{O}_X) & \xrightarrow{\eta(\lambda)} & \mathcal{U}^s(X, r, \mathcal{O}_X), \end{array}$$

where  $f_\lambda : \pi_s^{-1}(\lambda) \rightarrow \mathcal{U}^s(X, r, \mathcal{O}_X)$  is just forgetful map. Indeed, we consider the following composition

$$\xi_\lambda : \pi_s^{-1}(\lambda) \xrightarrow{f_\lambda} \mathcal{U}^s(X, r, \mathcal{O}_X) \xrightarrow{\alpha} \text{Alb}(\mathcal{U}^s(X, r, \mathcal{O}_X)),$$

where the second map  $\alpha$  is the Albanese map to Albanese variety  $\text{Alb}(\mathcal{U}^s(X, r, \mathcal{O}_X))$  associated to  $\mathcal{U}^s(X, r, \mathcal{O}_X)$ . For a point  $e \in \mathcal{U}^s(X, r, \mathcal{O}_X)$ , since there is no algebraic map from the affine space  $f_\lambda^{-1}(e)$  to the abelian variety  $\text{Alb}(\mathcal{U}^s(X, r, \mathcal{O}_X))$ , the composition  $\xi_{\varepsilon\sigma\lambda} \circ \sigma \circ f_\lambda^{-1}$  maps  $e$  to a point  $\kappa$  lying in  $\text{Alb}(\mathcal{U}^s(X, r, \mathcal{O}_X))$ . If there is no point  $e'$  such that  $\sigma(f_\lambda^{-1}(e)) \subseteq f_{\varepsilon\sigma\lambda}^{-1}(e')$ , then  $\dim_{\mathbb{C}}(\alpha^{-1}(\kappa)) = \dim_{\mathbb{C}}(\mathcal{U}^s(X, r, \mathcal{O}_X))$ , which is impossible.

Therefore, the composition

$$\eta(\lambda) := f_{\varepsilon\sigma\lambda} \circ \phi \circ f_\lambda^{-1} : \mathcal{U}^s(X, r, \mathcal{O}_X) \rightarrow \mathcal{U}^s(X, r, \mathcal{O}_X)$$

is well-defined.

The automorphism group  $\text{Aut}(\mathcal{U}^s(X, r, \mathcal{O}_X))$  has been studied well. More concretely, there are two approaches of producing automorphisms of  $\text{Aut}(\mathcal{U}^s(X, r, \mathcal{O}_X))$  [KP95]:

- for  $\ell \in \text{Aut}(X)$ , and  $L \in \text{Pic}^0(X)$  with  $L^r \cong \mathcal{O}_X$ , send  $E \in \mathcal{U}^s(X, r, \mathcal{O}_X)$  to  $L \otimes \ell^*E$ ;
- for  $\ell \in \text{Aut}(X)$ , and  $L \in \text{Pic}^0(X)$  with  $L^r \cong \mathcal{O}_X$ , send  $E \in \mathcal{U}^s(X, r, \mathcal{O}_X)$  to  $L \otimes \ell^*E^\vee$ .

In other words,  $\text{Aut}(\mathcal{U}^s(X, r, \mathcal{O}_X))$  satisfies the following short exact sequence

$$\text{Id} \longrightarrow \mathcal{G} \longrightarrow \text{Aut}(\mathcal{U}^s(X, r, \mathcal{O}_X)) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \text{Id},$$

where the subgroup  $\mathcal{G}$  fits in the short exact sequence

$$\text{Id} \longrightarrow \text{Pic}^0(X)_r \longrightarrow \mathcal{G} \longrightarrow \text{Aut}(X) \longrightarrow \text{Id}$$

for  $\text{Pic}^0(X)_r$  consisting of the  $r$ -torsion points in  $\text{Pic}^0(X)$ . From the above characterization, we find that  $\text{Aut}(\mathcal{U}^s(X, r, \mathcal{O}_X))$  is a finite group. Consequently, the map  $\eta : \mathbb{C} \rightarrow \text{Aut}(\mathcal{U}^s(X, r, \mathcal{O}_X))$  is a constant map, and then the image is denoted by  $\eta_\sigma$ .

So far, we obtain a group homomorphism

$$\begin{aligned} \Theta : \text{Aut}(M_{\text{Hod}}^s(X, r, \mathcal{O}_X)) &\longrightarrow \text{Aut}(\mathcal{U}^s(X, r, \mathcal{O}_X)) \times \mathbb{C}^* \\ \sigma &\longmapsto (\eta_\sigma, \varepsilon_\sigma). \end{aligned}$$

To show this homomorphism is surjective, we only need to note that there is a natural  $\mathbb{C}^*$ -action on  $M_{\text{Hod}}^s(X, r, \mathcal{O}_X)$ , and an element of  $\text{Aut}(\mathcal{U}^s(X, r, \mathcal{O}_X))$  can be extended to an automorphism of  $M_{\text{Hod}}^s(X, r, \mathcal{O}_X)$  since there are natural ways to define the tensor product, pullback and dual on  $\lambda$ -connections.

Finally, we consider the kernel of the homomorphism  $\Theta$ . Assume  $\sigma \in \text{Ker}(\Theta)$ , then  $\sigma(\pi_s^{-1}(\lambda)) = \pi_s^{-1}(\lambda)$  and  $\sigma$  preserves the fibers of  $\pi_s^{-1}(\lambda) \rightarrow \mathcal{U}^s(X, r, \mathcal{O}_X)$  for any  $\lambda \in \mathbb{C}$ . Then we have a map

$$\rho_\lambda^\sigma : \pi_s^{-1}(\lambda) \rightarrow \mathcal{A}$$

defined by the composition

$$(E, D^\lambda) \mapsto (E, \sigma(D^\lambda) - D^\lambda) \mapsto (E, h_E(\sigma(D^\lambda) - D^\lambda)),$$

where  $\mathcal{A} = \bigoplus_{i=1}^r H^0(X, K_X^{\otimes i})$  is the Hitchin space, and  $h_E : H^0(X, \text{ad}(E) \otimes K_X) \rightarrow \mathcal{A}$  is the Hitchin map given by  $\vartheta \mapsto (\text{Tr}(\vartheta^2), \dots, \text{Tr}(\vartheta^r))$  [Hit87b]. When  $\lambda \in \mathbb{C}^*$ , since  $\text{codim}_{\mathbb{C}}(\pi^{-1}(\lambda) \setminus \pi_s^{-1}(\lambda)) \geq 3$ , there are also no nonconstant algebraic functions on  $\pi_s^{-1}(\lambda)$ . Hence  $\rho_\lambda^\sigma$  is a constant map, and then the image is denoted by  $\tilde{\rho}_\lambda^\sigma$ . Thus each  $\sigma \in \text{Ker}(\Theta)$  gives rise to a morphism

$$\begin{aligned} \rho : \mathbb{C}^* &\longrightarrow \mathcal{A} \\ \lambda &\longmapsto \tilde{\rho}_\lambda^\sigma, \end{aligned}$$

which can be extended to the entire  $\mathbb{C}$ . Moreover, since  $\dim_{\mathbb{C}} H^0(X, \text{ad}(E) \otimes K_X) = \dim_{\mathbb{C}} \mathcal{A} = (r^2 - 1)(g - 1)$  [Hit87b], the map  $(E, D^\lambda) \mapsto (E, \sigma(D^\lambda) - D^\lambda)$  is independent of  $D^\lambda$ , hence yields a section  $\tilde{\sigma}_\lambda$  of  $\mathcal{H}$ . Consequently, we obtain a morphism

$$\begin{aligned} \varrho_\sigma : \mathbb{C}^* &\longrightarrow H^0(\mathcal{U}^s(X, r, \mathcal{O}_X), \mathcal{H}) \\ \lambda &\longmapsto \tilde{\sigma}_\lambda, \end{aligned}$$

which provides an isomorphism  $\text{Ker}(\Theta) \simeq \text{Mor}(\mathbb{C}, H^0(\mathcal{U}^s(X, r, \mathcal{O}_X), \mathcal{H}))$  via  $\sigma \mapsto \varrho_\sigma$ .

Hence we complete the proof.  $\square$

In [BH17], the authors proved that the automorphism of the Deligne–Hitchin twistor space which is homotopic to the identity automorphism maps fibers of  $\text{TW}_{\text{DH}}(X, r) \rightarrow \mathbb{P}^1$  to fibers. In the following, we will generalize this property to the generalized Deligne–Hitchin twistor space  $\text{TW}(X, X'; r)$ .

**Theorem 3.3.11.** *Let  $\text{Aut}_0(\text{TW}(X, X'; r))$  be the identity component of the holomorphic automorphism group  $\text{Aut}(\text{TW}(X, X'; r))$  of  $\text{TW}(X, X'; r)$ , then it satisfies the following short exact sequence*

$$\text{Id} \longrightarrow \mathfrak{K} \longrightarrow \text{Aut}_0(\text{TW}(X, X'; r)) \longrightarrow \mathbb{C}^* \longrightarrow \text{Id},$$

where each element of  $\mathfrak{K}$  preserves the fiber of the fibration  $\pi : \text{TW}(X, X'; r) \rightarrow \mathbb{P}^1$ .

*Proof.* Let  $\sigma \in \text{Aut}_0(\text{TW}(X, X'; r))$ . Firstly, we show that  $\sigma$  maps fibers of  $\pi : \text{TW}(X, X'; r) \rightarrow \mathbb{P}^1$  to fibers. Fixing  $\lambda_0 \in \mathbb{P}^1$ , if  $\sigma$  does not map the fiber  $\pi^{-1}(\lambda_0)$  to a fiber, then there exists  $\lambda_1 \in \mathbb{C}^*$  and a small analytic open subset  $U \subset \mathbb{P}^1 \setminus \{0, \infty\}$  of  $\lambda_1$  such  $U \subset \pi(\sigma(\pi^{-1}(\lambda_0)))$ . Let  $u_1 = (E, \bar{\partial}_E, D^{\lambda_1}) \in \sigma(\pi^{-1}(\lambda_0)) \cap \pi^{-1}(\lambda_1)$ . By the isomorphism

$$\text{TW}(X, X'; r) \Big|_{\mathbb{P}^1 \setminus \{0, \infty\}} \cong \mathcal{M}_{\text{dR}}(X, r) \times \mathbb{C}^*,$$

there is a projection

$$p_v : T_{u_1} \sigma(\pi^{-1}(\lambda_0)) \rightarrow T_{u'_1} \mathcal{M}_{\text{dR}}(X, r)$$

where  $u'_1 = (E, \bar{\partial}_E, \lambda_1^{-1} D^{\lambda_1})$ . Consider the de Rham section  $s_{u_1}$  associated to  $u_1$ . Then there exists  $(\alpha, \beta) \in T_{u_1} \sigma(\pi^{-1}(\lambda_0))$  with nonzero  $\alpha \in \Omega_X^{0,1}(\text{End}(E))$  and  $\beta \in \Omega_X^{1,0}(\text{End}(E))$  such that  $p_v(\alpha, \beta)$  lies in  $N_{s_{u_1}}|_{\lambda_2} \simeq T_{u'_1} \mathcal{M}_{\text{dR}}(X, r)$  with some  $\lambda_2 \neq \lambda_1 \in U$ .

By Theorem 3.3.4, we can take three points  $(\alpha, \beta)$  lying in  $T_{u_1} \sigma(\pi^{-1}(\lambda_0))$  to determine a section  $s'_{u_1}$  of the normal bundle  $N_{s_{u_1}}$  such that it yields a section  $\widetilde{s'_{u_1}}$  of  $\pi : \text{TW}(X, X'; r) \rightarrow \mathbb{P}^1$  which intersects the fiber  $\pi^{-1}(\lambda_0)$  three times. This contradicts the fact that the element of  $\text{Aut}_0(\text{TW}(X, X'; r))$  maps a section to another section.

As a consequence, the holomorphic automorphism  $\sigma \in \text{Aut}_0(\text{TW}(X, X'; r))$  induces a holomorphic automorphism  $\tau_\sigma \in \text{Aut}(\mathbb{P}^1)$ , which has a form

$$\tau_\sigma(z) = \frac{az + b}{cz + d}$$

with  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}(2, \mathbb{C})$ . Since  $\sigma$  must map the fiber  $\pi^{-1}(0)$  to itself or to the fiber  $\pi^{-1}(\infty)$ , we have

$$\tau_\sigma(z) = \begin{cases} \varepsilon_\sigma \cdot z, & \tau_\sigma(0) = 0, \tau_\sigma(\infty) = \infty; \\ \frac{z}{\varepsilon_\sigma}, & \tau_\sigma(0) = \infty, \tau_\sigma(\infty) = 0, \end{cases}$$

for some  $\varepsilon_\sigma \in \mathbb{C}^*$ . The theorem follows. □



## Part II

# Some Specializations of Non-Abelian Hodge Correspondence



## Chapter 4

# The Geometry of Parabolic Non-Abelian Hodge Correspondence

The notion of parabolic structure was introduced by Metha and Seshadri [MS80] when they studied the unitary representations of the fundamental group of a punctured Riemann surface, as a generalization of Narasimhan and Seshadri's work on unitary representations of the fundamental group of compact Riemann surfaces [NS65]. In their interpretation, “parabolic structure” on a vector bundle over compact Riemann surface means at each punctured point, the bundle admits a flag of non-trivial subspaces of the fiber at that point, and each subspace associated with a “weight” of real number in  $[0, 1)$ . Under their setting and by introducing the parabolic stability, they established a correspondence between the category of (semi)stable parabolic bundles of parabolic degree 0 and the category of irreducible (semisimple) unitary representations of the fundamental group of the punctured surface. They also constructed the moduli space of semistable parabolic vector bundles of fixed rank and wight system with parabolic degree 0, and they show it is a normal projective variety. Later on, the definition of parabolic vector bundles (sheaves) for higher dimensional projective varieties was given by Maruyama and Yokogawa [MY92], where they constructed the coarse moduli space of stable parabolic sheaves, and showed it was a quasi-projective variety. The moduli space of semistable parabolic sheaves was considered by Yokogawa in [Yok93].

The notion of Higgs field for parabolic vector bundles over Riemann surface was introduced by Simpson [Sim90], in this paper, Simpson gives a different description of parabolic structure for vector bundles over Riemann surface. In his setting, the parabolic structure is given by a left continuous and decreasing sequence of sheaves indexed by real numbers, and the parabolic weights are those indices such that the sequence jump. This definition is equivalent to that of Metha and Seshadri, his method was applied by Maruyama and Yokogawa for constructing the moduli space, especially for higher dimensional generalization [MY92, Yok93]. In [Sim90], Simpson produced an analytic method by introducing the notion of tame harmonic metric which was arising from the study of Kobayashi–Hitchin correspondence. As a result, he built the one-to-one correspondence between stable tame harmonic bundles, stable parabolic Higgs bundles of parabolic degree 0 and irreducible representations of the fundamental groups, as well as stable parabolic local systems of parabolic degree 0. In [Yok93], Yokogawa also constructed the moduli space of semistable parabolic Higgs bundles, and showed it was a projective varieties. In addition, he also introduced the Hitchin morphism for parabolic Higgs bundles, and showed it was projective, as a generalization of classical result.

The higher dimensional version of Kobayashi–Hitchin correspondence for parabolic Higgs bundles and the non-compact version of Corlette–Simpson correspondence are due to Mochizuki in his

two important papers [Moc06, Moc09]. As we have already seen in the first part of this thesis, his results applying to compact case give us many lights on the understanding twistor spaces.

In this chapter, firstly we will give a basic introduction to parabolic Higgs bundle based on many well-known papers and books, including many definitions and notations that will be useful for the following chapters. In the first section, we will give the basic definition of parabolic vector bundles and parabolic Higgs bundles over compact Riemann surface, the main references are [MS80, Ses82, Sim90] and [GPGM07]. In the second part of this chapter, we will introduce the higher dimensional generalization, most of these can be found in [Moc06] and [IS07, IS08]. In [MY92] and [Yok93], the authors considered the parabolic structure with all components of the divisor combined together, but in [Moc06] and [IS07, IS08], the definition were modified to consider different filtration to give the parabolic structure for different component of the divisor, we will use this general consideration.

## 4.1 Non-Abelian Hodge Theory of Parabolic Version

### 4.1.1 General Definitions

Not like in the compact case, the parabolic structures for vector bundles and Higgs bundles are more complicated and abstract, there are several different definitions of these structures, but they coincide in dimension 1. Our main references for this section are [IS07, Moc06], see also [Hua18].

Let  $X$  be a connected smooth complex projective variety of  $\dim_{\mathbb{C}}(X) = n$ , let  $D = \bigcup_{i=1}^k D_i \subseteq X$  be a reduced simple normal crossing divisor with each component  $D_i$  smooth and irreducible.

Parabolic structures are usually defined in two different but equivalent ways.

#### Usual definition of parabolic structures.

**Definition 4.1.1** (Mochizuki, Iyer–Simpson). A *parabolic sheaf* on  $(X, D)$  is a pair  $\mathcal{E}_* = \{\mathcal{E}, \{\mathcal{E}_\alpha\}_{\alpha \in \mathbb{R}^k}\}$  that consists of a torsion-free coherent  $\mathcal{O}_X$ -module  $\mathcal{E}$  together with a collection of torsion-free coherent  $\mathcal{O}_X$ -modules  $\mathcal{E}_\alpha$  indexed by multi-indices  $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$  that satisfies following conditions:

- (1)  $\mathcal{E} = \mathcal{E}_0$  for  $\mathbf{0} = (0, \dots, 0)$ ;
- (2) **increasing:**  $\mathcal{E}_\alpha \subseteq \mathcal{E}_\beta$  whenever  $\alpha \leq \beta$  (i.e,  $\alpha_i \leq \beta_i$  for all  $i$ );
- (3) **normalization:**  $\mathcal{E}_{\alpha+\delta^i} = \mathcal{E}_\alpha(D_i) := \mathcal{E}_\alpha \otimes \mathcal{O}_X(D_i)$ , where  $\delta^i = (0, \dots, 1, \dots, 0) \in \mathbb{R}^k$  is a multi-index with the only 1 in the  $i$ -th position;
- (4) **semi-continuity:** for any given  $\alpha \in \mathbb{R}^k$ , there exists a constant  $c > 0$  such that for any multi-index  $\varepsilon$  with each  $0 \leq \varepsilon_i < c$ , we have  $\mathcal{E}_{\alpha+\varepsilon} = \mathcal{E}_\alpha$ .

$\mathcal{E}$  is usually called the *underlying sheaf* of  $\mathcal{E}_*$ . In particular, when all sheaves  $\mathcal{E}_\alpha$  are moreover locally free, then we call  $\mathcal{E}_*$  a *parabolic vector bundle*.

**Remark 4.1.2.** (1) Let  $i : X - D \hookrightarrow X$  be the natural inclusion, then  $i_*i^*(\mathcal{E})$  (direct image of the restricted sheaf  $\mathcal{E}$  on  $X - D$ ) is a quasi-coherent sheaf over  $X$ , and satisfies

$$i_*i^*(\mathcal{E}) = \bigcup_{\alpha \in \mathbb{R}^k} \mathcal{E}_\alpha.$$

We use  $\tilde{\mathcal{E}}$  to denote this sheaf, clearly we have  $\tilde{\mathcal{E}}|_{X-D} = \mathcal{E}|_{X-D}$ .

- (2) For any multi-index  $\mathbf{c} \in \mathbb{R}^k$ , then from the normalization condition, we can see that the parabolic structure is fully determined by the  $\mathbf{c}$ -truncation  ${}_{\mathbf{c}}\mathcal{E} := \{\mathcal{E}_\alpha\}_{\mathbf{c}-\delta < \alpha \leq \mathbf{c}}$ , where  $\delta = \sum_{i=1}^k \delta^i$ ;
- (3) By the semi-continuity condition, there exist jumping indices  $\alpha$ , i.e, these multi-indices  $\alpha$  such that  $\mathcal{E}_{\alpha-\varepsilon} \neq \mathcal{E}_\alpha$  for any  $\varepsilon > \mathbf{0}$ . Moreover, in a fixed interval  $(\mathbf{c} - \delta, \mathbf{c}]$ , the number of jumping indices is finite, since the divisor  $D$  has a finite number of irreducible components.

### Parabolic structures given by filtrations.

Fix a parabolic sheaf  $\mathcal{E}_*$  on  $(X, D)$  as defined in [Definition 4.1.1](#), and fix a multi-index  $\mathbf{c} \in \mathbb{R}^k$ . For each  $1 \leq i \leq k$ ,  $\mathcal{E}_{\mathbf{c}-\delta^i} = \mathcal{E}_{\mathbf{c}}(-D_i)$ , we have the exact sequence

$$0 \longrightarrow \mathcal{E}_{\mathbf{c}-\delta^i} \longrightarrow \mathcal{E}_{\mathbf{c}} \longrightarrow \mathcal{E}_{\mathbf{c}}|_{D_i} \longrightarrow 0$$

for  $\mathcal{E}_{\mathbf{c}}|_{D_i} = \frac{\mathcal{E}_{\mathbf{c}}}{\mathcal{E}_{\mathbf{c}-\delta^i}}$ , we can define a filtration  $\{\mathcal{F}_a^i\}_{c_i-1 < a \leq c_i}$  of the restricted sheaf  $\mathcal{E}_{\mathbf{c}}|_{D_i}$  by subsheaves as follows

$$\mathcal{F}_a^i := \left( \bigcup_{\substack{\alpha \leq \mathbf{c} \\ \alpha_i \leq a}} \mathcal{E}_\alpha \right) / \mathcal{E}_{\mathbf{c}-\delta^i} \subseteq \mathcal{E}_{\mathbf{c}} / \mathcal{E}_{\mathbf{c}-\delta^i} = \mathcal{E}_{\mathbf{c}}|_{D_i},$$

so  $\mathcal{F}_{c_i-1}^i = 0$  and  $\mathcal{F}_{c_i}^i = \mathcal{E}_{\mathbf{c}}|_{D_i}$ . Let  $\text{Gr}_a^{\mathcal{F}^i}(\mathcal{E}_{\mathbf{c}}) := \mathcal{F}_a^i / \mathcal{F}_{<a}^i$ , then (2) of [Remark 4.1.2](#) implies the set of jumping indices

$$\text{wt}^i(\mathcal{E}_{\mathbf{c}}) := \{a \in (c_i - 1, c_i] \mid \text{Gr}_a^{\mathcal{F}^i}(\mathcal{E}_{\mathbf{c}}) \neq 0\}$$

is finite. We call it the set of *parabolic weights of  $\mathcal{E}_{\mathbf{c}}$  along the divisor  $D_i$* .

For each  $1 \leq i \leq k$ , we write the associated parabolic weights as followings:

$$c_i - 1 = a_0^i < a_1^i < \cdots < a_{i_l-1}^i < a_{i_l}^i \leq c_i, \quad (4.1)$$

then the filtration is

$$0 = \mathcal{F}_{a_0^i}^i \subsetneq \mathcal{F}_{a_1^i}^i \subsetneq \cdots \subsetneq \mathcal{F}_{a_{i_l}^i}^i = \mathcal{E}_{\mathbf{c}}|_{D_i}. \quad (4.2)$$

Therefore, for any fixed multi-index  $\mathbf{c}$ , from a parabolic sheaf  $\mathcal{E}_*$ , we obtain the following pair

$$\left\{ \mathcal{E}_{\mathbf{c}}, \left\{ \mathcal{F}_{\alpha_i}^i \right\}_{\substack{1 \leq i \leq k \\ \alpha_i \in \{a_0^i, \dots, a_{i_l}^i\}}} \right\},$$

that satisfies (4.1)-(4.2), call it a *parabolic structure given by filtrations*.

On the other hand, given a parabolic structure given by filtrations  $\left\{ \mathcal{E}_{\mathbf{c}}, \left\{ \mathcal{F}_{\alpha_i}^i \right\}_{\substack{1 \leq i \leq k \\ \alpha_i \in \{a_0^i, \dots, a_{i_l}^i\}}} \right\}$  that satisfies (4.1)-(4.2). Then we can recover the parabolic sheaf. In fact, we just need to recover the  $\mathbf{c}$ -truncation  ${}_{\mathbf{c}}\mathcal{E}$ .

For a multi-index  $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k$ , we write  $\mathcal{E}_\alpha$  as  $\mathcal{E}_{(\alpha_1, \dots, \alpha_k)}$  to emphasize to explicit components.

Define semi-continuous sequence of subsheaves  $\tilde{\mathcal{F}}_{\alpha_i}^i$  of  $\mathcal{E}_{\mathbf{c}}|_{D_i}$  indexed by  $c_i - 1 \leq \alpha_i \leq c_i$  by the following rule:

$$\tilde{\mathcal{F}}_{\alpha_i}^i = \mathcal{F}_{a_j^i}^i, \quad a_j^i \leq \alpha_i < a_{j+1}^i, \quad 0 \leq j \leq i_l,$$

where one assigns  $a_{i_l+1}^i = a_{i_l}^i = c_i$ . Now let  $\mathcal{G}_{(c_1, \dots, c_{i-1}, \alpha_i, c_{i+1}, \dots, c_k)}$  be the subsheaf of  $\mathcal{E}_{(c_1, \dots, c_k)}$  that fits into the following exact sequence

$$0 \longrightarrow \mathcal{G}_{(c_1, \dots, \alpha_i, \dots, c_k)} \longrightarrow \mathcal{E}_{(c_1, \dots, c_k)} \longrightarrow \frac{\mathcal{E}_{\mathbf{c}}|_{D_i}}{\widetilde{\mathcal{F}}_{\alpha_i}^i} \longrightarrow 0,$$

set

$$\mathcal{E}_{(\alpha_1, \dots, \alpha_k)} := \bigcap_{i=1}^k \mathcal{G}_{(c_1, \dots, \alpha_i, \dots, c_k)} \subseteq \mathcal{E}_{\mathbf{c}}.$$

Then  $\{\mathcal{E}_{(\alpha_1, \dots, \alpha_k)}\}_{\substack{1 \leq i \leq k \\ c_i - 1 < \alpha_i \leq c_i}}$  is a collection of increasing and semi-continuous subsheaves (only finite terms) of  $\mathcal{E}_{\mathbf{c}}$  with  $\mathcal{E}_{(a_{1_l}^1, \dots, a_{k_l}^k)} = \mathcal{E}_{\mathbf{c}}$  and satisfies the property

$$\mathcal{E}_{\mathbf{c} - \delta^i} = \mathcal{E}_{\mathbf{c}}(-D_i), \quad i = 1, \dots, k.$$

This determines a  $\mathbf{c}$ -truncation  ${}_{\mathbf{c}}\mathcal{E} = \{\mathcal{E}_{\alpha}\}_{\mathbf{c} - 1 < \alpha \leq \mathbf{c}}$ , hence a parabolic structure.

Therefore, these two different methods of defining a parabolic structure are equivalent. We will alternatively use each one according to our convenience.

We give some basic examples of parabolic sheaves based on [IS07, IS08].

**Example 4.1.3.** (1) A *parabolic line bundle* on  $(X, D)$  is a parabolic sheaf  $\mathcal{L}$  such that all  $\mathcal{L}_{\alpha}$  are line bundles;

(2) Fix a multi-index  $\beta \in \mathbb{R}^k$ , define a parabolic line bundle  $\left(\mathcal{O}_X\left(\sum_{i=1}^k \beta_i D_i\right)\right)_*$  by setting

$$\left(\mathcal{O}_X\left(\sum_{i=1}^k \beta_i D_i\right)\right)_{\alpha} := \mathcal{O}_X\left(\sum_{i=1}^k [\alpha_i + \beta_i] D_i\right),$$

where  $[a]$  denotes the biggest integer no more than  $a$ ;

(3) Given a torsion-free coherent sheaf  $\mathcal{E}$  on  $X$ , there is a natural parabolic sheaf on  $(X, D)$  called the *trivial parabolic sheaf associated to  $\mathcal{E}$* :

$$\mathcal{E}_{\alpha} := \mathcal{E}\left(\sum_{i=1}^k [\alpha_i] D_i\right).$$

We call this trivial structure because jump happens only when each  $\alpha_i$  is an integer, and the underlying sheaf  $\mathcal{E}_0$  is  $\mathcal{E}$  itself.

(4) ([IS07, Lemma 2.1]) Any parabolic line bundle on  $(X, D)$  is isomorphic to  $\mathcal{L} \otimes \left(\mathcal{O}_X\left(\sum_{i=1}^k \beta_i D_i\right)\right)_*$  for some line bundle  $\mathcal{L}$  on  $X$  and multi-index  $\beta \in \mathbb{R}^k$ .

### Direct sum

Given two parabolic sheaf  $\mathcal{E}_*$  and  $\mathcal{F}_*$  over  $(X, D)$ , we define the *direct sum* parabolic sheaf  $(\mathcal{E} \oplus \mathcal{F})_*$  by setting

$$(\mathcal{E} \oplus \mathcal{F})_{\alpha} := \mathcal{E}_{\alpha} \oplus \mathcal{F}_{\alpha}.$$

Then the underlying sheaf is  $(\mathcal{E} \oplus \mathcal{F})_{\mathbf{0}} = \mathcal{E}_{\mathbf{0}} \oplus \mathcal{F}_{\mathbf{0}} = \mathcal{E} \oplus \mathcal{F}$ . Afterwards, we will use the notation  $\mathcal{E}_* \oplus \mathcal{F}_*$  to denote the direct sum of parabolic sheaves.

Denote by  $\text{wt}(\mathcal{E}_*) := \{\text{wt}^i(\mathcal{E}_{\mathbf{0}})\}_{i=1}^k \subseteq (-\mathbf{1}, \mathbf{0}]$  the set of parabolic weights of  $\mathcal{E}_*$  with values between  $-1$  and  $0$ .

Then we have

$$\text{wt}(\mathcal{E}_* \oplus \mathcal{F}_*) = \text{wt}(\mathcal{E}_*) \cup \text{wt}(\mathcal{F}_*).$$

### Tensor product

Given two parabolic sheaf  $\mathcal{E}_*$  and  $\mathcal{F}_*$  over  $(X, D)$ . We define the *tensor product* parabolic sheaf  $(\mathcal{E} \otimes \mathcal{F})_*$  by setting for any  $\alpha \in \mathbb{R}^k$ ,  $(\mathcal{E} \otimes \mathcal{F})_{\alpha}$  is the subsheaf of the quasi-coherent sheaf  $\widetilde{\mathcal{E} \otimes \mathcal{F}} := i_* i^*(\mathcal{E} \otimes \mathcal{F})$  generated by all  $\mathcal{E}_{\beta} \otimes \mathcal{F}_{\gamma}$  for  $\beta + \gamma \leq \alpha$ . Therefore,

$$(\mathcal{E} \otimes \mathcal{F})_{\alpha} = \sum_{0 < \beta \leq 1} \mathcal{E}_{\beta} \otimes \mathcal{F}_{\alpha - \beta} = \sum_{0 < \beta \leq 1} \mathcal{F}_{\beta} \otimes \mathcal{E}_{\alpha - \beta} = (\mathcal{F} \otimes \mathcal{E})_{\alpha}.$$

Then the underlying sheaf is  $(\mathcal{E} \otimes \mathcal{F})_{\mathbf{0}} = \sum_{0 < \beta \leq 1} \mathcal{E}_{\beta} \otimes \mathcal{F}_{-\beta} \supseteq \mathcal{E}_{\mathbf{1}} \otimes \mathcal{F}_{-\mathbf{1}} = \mathcal{E} \otimes \mathcal{F}$ , which contains the tensor product of the underlying sheaves  $\mathcal{E} \otimes \mathcal{F}$  as a subsheaf.

Afterwards, we will use the notation  $\mathcal{E}_* \otimes \mathcal{F}_*$  to denote the tensor product of parabolic sheaves.

Clearly, the parabolic weights of  $\mathcal{E}_* \otimes \mathcal{F}_*$  satisfies

$$\begin{aligned} \text{wt}(\mathcal{E}_* \otimes \mathcal{F}_*) = & \{ \alpha + \beta \mid \alpha \in \text{wt}(\mathcal{E}_*), \beta \in \text{wt}(\mathcal{F}_*), \alpha + \beta > -\mathbf{1} \} \\ & \cup \{ \alpha + \beta + \mathbf{1} \mid \alpha \in \text{wt}(\mathcal{E}_*), \beta \in \text{wt}(\mathcal{F}_*), \alpha + \beta \leq -\mathbf{1} \}. \end{aligned}$$

### Dual

Given a parabolic sheaf  $\mathcal{E}_*$  over  $(X, D)$ , for any multi-index  $\alpha \in \mathbb{R}^k$ , define  $\mathcal{E}_{\alpha-}$  being the left limit of the sheaf  $\mathcal{E}_{\alpha+\varepsilon}$  ( $\varepsilon < 0$ ) as  $\varepsilon \rightarrow 0$ .

As in [Bis03], we define the *dual* parabolic sheaf of  $\mathcal{E}_*$ , denoted as  $(\mathcal{E}^*)_*$  by setting

$$(\mathcal{E}^*)_{\alpha} := (\mathcal{E}_{-\alpha+1-})^*.$$

Then when the parabolic structure of  $\mathcal{E}_*$  is non-trivial, i.e, when  $\mathcal{E}_*$  has non-zero parabolic weights, the underlying sheaf  $(\mathcal{E}^*)_{\mathbf{0}}$  is strictly a subsheaf of  $\mathcal{E}^*$ .

In this case, the parabolic weights satisfy

$$\text{wt}((\mathcal{E}^*)_*) = \{ -1 - \alpha \mid \alpha \in \text{wt}(\mathcal{E}_*) \setminus (\text{wt}(\mathcal{E}_*) \cap \{\mathbf{0}\}) \} \cup (\text{wt}(\mathcal{E}_*) \cap \{\mathbf{0}\}).$$

### Parabolic morphisms

Let  $\mathcal{E}_*$  and  $\mathcal{F}_*$  be two parabolic sheaves over  $(X, D)$ .

**Definition 4.1.4.** A *parabolic morphism*  $f : \mathcal{E}_* \rightarrow \mathcal{F}_*$  is a collection of morphisms of sheaves  $\{f_{\alpha} : \mathcal{E}_{\alpha} \rightarrow \mathcal{F}_{\alpha}\}_{\alpha \in \mathbb{R}^k}$  that compatible with the inclusion of sheaves, that is, the following diagram

$$\begin{array}{ccc} \mathcal{E}_{\alpha} & \xrightarrow{f_{\alpha}} & \mathcal{F}_{\alpha} \\ \downarrow & & \downarrow \\ \mathcal{E}_{\beta} & \xrightarrow{f_{\beta}} & \mathcal{F}_{\beta} \end{array}$$

commutes for any multi-indices  $\alpha \leq \beta$ . Denoted by  $\text{Hom}_X(\mathcal{E}_*, \mathcal{F}_*)$  the set of all parabolic morphisms  $f : \mathcal{E}_* \rightarrow \mathcal{F}_*$ , and by  $\mathcal{H}om_X(\mathcal{E}_*, \mathcal{F}_*)$  the sheaf of parabolic morphisms.

A short sequence of parabolic sheaves  $0 \rightarrow \mathcal{E}_* \xrightarrow{f} \mathcal{F}_* \xrightarrow{g} \mathcal{W}_* \rightarrow 0$  is *exact* if the induced sequence  $0 \rightarrow \mathcal{E}_\alpha \xrightarrow{f_\alpha} \mathcal{F}_\alpha \xrightarrow{g_\alpha} \mathcal{W}_\alpha \rightarrow 0$  is exact for all multi-index  $\alpha$ .

The following facts are from [Bis03, Yok95]:

- $((\mathcal{E}^*)^*)^* = \mathcal{E}^*$ ;
- $(\mathcal{E}_* \otimes \mathcal{F}_*) \otimes \mathcal{W}_* = \mathcal{E}_* \otimes (\mathcal{F}_* \otimes \mathcal{W}_*)$ ;
- $\mathcal{E}_* \otimes (\mathcal{F}_* \oplus \mathcal{W}_*) = (\mathcal{E}_* \otimes \mathcal{F}_*) \otimes (\mathcal{E}_* \oplus \mathcal{W}_*)$ ;
- The functor by tensoring with a parabolic sheaf  $\bullet \otimes \mathcal{E}_*$  is right exact, it is exact if and only if the parabolic sheaf is locally free.

Let  $\mathcal{P}ar_{X,D}$  be the category of parabolic sheaves over  $(X, D)$ , then it is an abelian category with enough injective objects [Yok95].

### Parabolic subsheaves and parabolic quotient sheaves

For a parabolic sheaf  $\mathcal{E}_*$  over  $(X, D)$ , we take the  $\mathbf{0}$ -truncation  $\mathbf{0}\mathcal{E}$ , that is, the multi-index  $\alpha \in (-\mathbf{1}, \mathbf{0}]$ .

**Definition 4.1.5.** A parabolic sheaf  $\mathcal{F}_*$  over  $(X, D)$  is a *parabolic subsheaf* of  $\mathcal{E}_*$  if:

- (1)  $\mathcal{F}_\alpha \subseteq \mathcal{E}_\alpha$  for all  $\alpha$ ;
- (2) if  $\mathcal{F}_\beta \subseteq \mathcal{E}_\alpha$  for some  $\alpha < \beta$ , then  $\mathcal{F}_\beta = \mathcal{F}_\alpha$ .

**Definition 4.1.6.** A parabolic sheaf  $\mathcal{F}_*$  over  $(X, D)$  is a *parabolic quotient sheaf* of  $\mathcal{E}_*$  if there is a surjective parabolic morphism  $f : \mathcal{E}_* \rightarrow \mathcal{F}_*$  such that if  $f(\mathcal{E}_\beta) \subseteq \mathcal{F}_\alpha$  for some  $\alpha < \beta$ , then  $\mathcal{F}_\beta = \mathcal{F}_\alpha$ .

**Remark 4.1.7.** (1) Let  $\mathcal{E}_*$  be a parabolic sheaf over  $(X, D)$ , with  $\mathcal{E}$  the underlying sheaf. Then for any saturated subsheaf  $\mathcal{F} \subseteq \mathcal{E}$ , there is an induced parabolic structure on  $\mathcal{F}$  by setting

$$\mathcal{F}_\alpha := \mathcal{F} \cap \mathcal{E}_\alpha,$$

this makes  $\mathcal{F}_*$  being a parabolic subsheaf of  $\mathcal{E}_*$ , called the *induced parabolic subsheaf*. Not hard to see that this parabolic structure on  $\mathcal{F}$  (i.e, such that  $\mathcal{F}$  is the underlying sheaf) is the maximal one among all parabolic structures on  $\mathcal{F}$ . This is very important when we introduce the parabolic stability, and the stability condition only need to defined over subsheaves with induced parabolic structures.

(2) Let  $\mathcal{W}$  be a torsion free coherent sheaf which is a quotient sheaf of  $\mathcal{E}$ , that is, there is a surjective morphism of sheaves  $\varphi : \mathcal{E} \rightarrow \mathcal{W}$ . Then there is an induced parabolic structure on  $\mathcal{W}$  by setting

$$\mathcal{W}_\alpha := \varphi(\mathcal{E}_\alpha),$$

where  $\varphi(\mathcal{E}_\alpha)$  is the image sheaf under the composition  $\mathcal{E}_\alpha \hookrightarrow \mathcal{E} \xrightarrow{\varphi} \mathcal{W}$ . This makes  $\mathcal{W}_*$  being a parabolic sheaf, called the *induced parabolic quotient sheaf*. This parabolic structure on  $\mathcal{W}$  is the minimal one among all parabolic structures on  $\mathcal{W}$ . This is also important when we treat stability related problems.

### Locally abelian conditions

When the base variety is of higher dimension (i.e,  $\dim_{\mathbb{C}}(X) > 1$ ), then different irreducible components of the divisor set may intersect. When the parabolic structure is given by filtrations, then the fiber at a point which is located in a common intersection of divisor components should admit the same filtration with respect to different divisor components. So we should put some compatible condition on the filtrations on the intersection of divisor components, this is the “compatible filtrations” introduced by Mochizuki in [Moc06, Moc07], the compatibility condition is also considered by Li in [Li00] in a stronger sense, this notion is later reformulated by Iyer and Simpson as “locally abelian parabolic bundles” in [IS07, IS08]. Throughout this chapter, we will apply this notion.

**Definition 4.1.8.** A parabolic sheaf  $\mathcal{E}_*$  over  $(X, D)$  is called a *locally abelian parabolic bundle* for any point  $x \in X$ , there is a Zariski open neighborhood of that point such that in the neighborhood  $\mathcal{E}_*$  is isomorphic to a direct sum of parabolic line bundles.

Clearly, locally abelian condition is trivial when  $X$  is a surface. The following property, describes why locally abelian condition is important for parabolic sheaves over higher dimensional varieties.

**Theorem 4.1.9** ([Moc07, IS08, Bor07]). *Let  $\mathcal{E}_*$  be a locally abelian parabolic bundle over  $(X, D)$ , and let  $\mathcal{E} := \mathcal{E}_0$  be the underlying sheaf. Then  $\mathcal{E}_\alpha$  comes from previous construction of parabolic structure by unique filtrations  $\mathcal{F}_a^i$  of  $\mathcal{E}_c|_{D_i}$ , and we have the following properties:*

- (1) all sheaves  $\mathcal{E}_\alpha$  are locally free;
- (2) for each point  $P$  inside any non-empty intersection  $D_{i_1} \cap \cdots \cap D_{i_j}$ , the filtrations  $\mathcal{F}_{\bullet}^{i_1}, \dots, \mathcal{F}_{\bullet}^{i_j}$  of  $\mathcal{E}_P$  admit a common splitting, hence

$$\mathrm{Gr}_{l_1}^{\mathcal{F}^{i_1}} \cdots \mathrm{Gr}_{l_j}^{\mathcal{F}^{i_j}}(\mathcal{E}_P)$$

is independent of the order in which it taken;

- (3) the functions

$$\begin{aligned} f_{i_1 \cdots i_j} : D_{i_1} \cap \cdots \cap D_{i_j} &\longrightarrow \mathbb{Z}, \\ P &\longmapsto \mathrm{rk}\left(\mathrm{Gr}_{l_1}^{\mathcal{F}^{i_1}} \cdots \mathrm{Gr}_{l_j}^{\mathcal{F}^{i_j}}(\mathcal{E}_P)\right) \end{aligned}$$

are locally constant functions over  $D_{i_1} \cap \cdots \cap D_{i_j}$ .

*Conversely, suppose the parabolic structure is given by collection of sheaves  $\mathcal{E}_\alpha$  which come from filtrations, if it satisfies above properties (1)-(3), then the parabolic structure is locally abelian.*

Clearly, direct sum, tensor product, and dual preserve the locally abelian conditions. Denoted by  $\mathrm{Par}_{X,D}^{\mathrm{LA}} \subseteq \mathrm{Par}_{X,D}$  the full sub-category of locally abelian parabolic bundles over  $(X, D)$ . Then it is also an abelian category with enough injective objects.

#### 4.1.2 Special Case: Dimension 1

Historically, parabolic structure was originally introduced by Metha and Seshadri [MS80] when they studied the unitary representations of the fundamental group of a punctured Riemann surface, as

a generalization of Narasimhan and Seshadri's study of unitary representations of the fundamental group of compact Riemann surfaces [NS65]. More precisely, let  $X$  be a compact Riemann surface of genus  $g$ , and let  $D = x_1 + \cdots + x_k$  be an effectively divisor given by  $k$  points with the condition that  $2g - 2 + k > 0$ , Metha and Seshadri showed in [MS80] that there is a correspondence between irreducible unitary representations of  $\pi_1(X - D)$  and stable parabolic vector bundles on  $(X, D)$  of parabolic degree 0. Moreover, they also constructed the moduli space of such objects. Their definition of parabolic structure is given by filtrations, and later Simpson gave an equivalently definition by filtered sheaves when he introduced the Higgs field for parabolic vector bundles.

Now let  $X$  be a compact Riemann surface of genus  $g$ , and let  $D = x_1 + \cdots + x_k$  be an effectively divisor consists of  $k$  points with the condition that  $2g - 2 + k > 0$ .

**Definition 4.1.10.** A *parabolic sheaf*  $\mathcal{E}_*$  over  $(X, D)$  is a torsion free coherent sheaf  $\mathcal{E}$  over  $X$  such that at each  $x \in D$ , there is a filtration of  $\mathcal{E}_x$ :

$$0 \subsetneq \mathcal{E}_{x,1} \subsetneq \cdots \subsetneq \mathcal{E}_{x,l_x} = \mathcal{E}_x,$$

this filtration is called a *flag* of  $\mathcal{E}_x$ , together with a system of increasing real numbers, called *weights*:

$$-1 < a_{x,1} < \cdots < a_{x,l_x} \leq 0.$$

We call  $\mathcal{E}$  the *underlying sheaf* of  $\mathcal{E}_*$ .

**Remark 4.1.11.** (1) In fact in Metha–Seshadri's setting, the parabolic weights are real numbers between 1 and 0 in decreasing order, here the weights we choose are in accordance with our previous setting, that is, between -1 and 0 in increasing order. And we also adjust the following definition of parabolic sheaf given by filtered sheaves.

(2) Since any torsion free coherent sheaf over a Riemann surface is automatically locally free, it's reasonable to call a parabolic sheaf a *parabolic bundle*.

In [Sim90], Simpson supplied a new definition of parabolic structures by considering an increasing and left continuous family of sheaves indexed by real numbers, this is exactly what we applied in the first section to define a parabolic sheaf over higher dimensional varieties. More precisely, a parabolic structure on a torsion free coherent sheaf  $\mathcal{E}$  is the following:

at each  $x \in D$ , there is a  $\mathbb{R}$ -indexed family of torsion free coherent sheaves  $\mathcal{E}_\alpha^x$  such that

- (1)  $\mathcal{E}_0^x = \mathcal{E}$ ;
- (2) increasing:  $\mathcal{E}_\alpha^x \subseteq \mathcal{E}_\beta^x$  whenever  $\alpha \leq \beta$ ;
- (3) normalization: for any  $\alpha \in \mathbb{R}$ ,  $\mathcal{E}_{\alpha+1}^x = \mathcal{E}_\alpha^x(x)$ ;
- (4) right continuous: for any  $\alpha \in \mathbb{R}$ , there exists some  $\varepsilon > 0$  such that  $\mathcal{E}_{\alpha+\varepsilon}^x = \mathcal{E}_\alpha^x$ ;

the parabolic weights are exactly the following set of jumping indices:

$$a_x := \{\alpha \in (-1, 0] \mid \text{Gr}_\alpha^x := \mathcal{E}_\alpha^x / \mathcal{E}_{\alpha-\varepsilon}^x \neq 0\}.$$

Write the set of jumping indices increasingly as

$$a_x = \{a_{x,1}, \cdots, a_{x,l_x}\},$$

then this two definitions are equivalent by fitting into the following short exact sequence:

$$0 \longrightarrow \mathcal{E}_{a_{x,i}}^x \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_x/\mathcal{E}_{x,i} \otimes \mathcal{O}_{X,x} \longrightarrow 0.$$

Now let  $\mathcal{E}_* = \left\{ \mathcal{E}, \{ \mathcal{E}_{x,i} \}_{\substack{x \in D \\ 1 \leq i \leq l_x}} \right\}$  be a parabolic sheaf given by filtrations with increasing weight system  $\{-1 < a_{x,1} < \cdots < a_{x,l_x} \leq 0\}_{x \in D}$ .

**Definition 4.1.12.** The *parabolic degree* of  $\mathcal{E}_*$  is defined as

$$\deg^{\text{par}}(\mathcal{E}_*) = \deg(\mathcal{E}) - \sum_{x \in D} \sum_{i=1}^{l_x} a_{x,i} \text{rk}_x(\mathcal{E}_{x,i}/\mathcal{E}_{x,i-1}), \quad (4.3)$$

where one assigns  $\mathcal{E}_{x,0} = 0$  for each  $x \in D$ . The *parabolic slope* of  $\mathcal{E}_*$  is defined as

$$\mu^{\text{par}}(\mathcal{E}_*) := \frac{\deg^{\text{par}}(\mathcal{E}_*)}{\text{rk}(\mathcal{E})}. \quad (4.4)$$

**Remark 4.1.13.** (1) In [MS80] and some other papers, the parabolic weights they choose to define a parabolic structure over Riemann surfaces are real numbers decreasingly from 1 to 0, and “−” in (4.3) is “+”. In fact, this is exactly the same as ours by taking  $-a_{x,i}$  as weights.

(2) Let  $\mathcal{F} \subseteq \mathcal{E}$  be a coherent subsheaf, then there is a naturally induced parabolic structure on  $\mathcal{F}$  by setting  $\mathcal{F}_{x,i} = \mathcal{F}_x \cap \mathcal{E}_{x,i}$  for each  $x \in D$  and  $1 \leq i \leq l_x(\mathcal{E})$ . Note that  $\{\mathcal{F}_{x,i}\}_{1 \leq i \leq l_x(\mathcal{E})}$  may have two or more terms the same, so the filtration of  $\mathcal{F}_x$  is given by distinct ones and the resulting filtration is denoted as

$$0 \subsetneq \mathcal{F}_{x,1} \subsetneq \cdots \subsetneq \mathcal{F}_{x,l_x(\mathcal{F})} = \mathcal{F}_x,$$

for  $1 \leq j \leq l_x(\mathcal{F})$ , the induced parabolic weight is given by  $a_{x,j}^{\mathcal{F}} := \alpha_{x,i}^{\mathcal{E}}$ , where  $i$  is the smallest number such that  $\mathcal{F}_{x,j} \subseteq \mathcal{E}_{x,i}$ . Such a resulting parabolic sheaf  $\mathcal{F}_*$  is called an *induced parabolic subsheaf* of  $\mathcal{E}_*$ .

(3) Let  $\mathcal{W}$  be a torsion free quotient sheaf of  $\mathcal{E}$  with the surjective morphism  $f : \mathcal{E} \rightarrow \mathcal{W}$ , then there is a naturally induced parabolic structure on  $\mathcal{W}$  by setting  $\mathcal{W}_{x,i} = \mathcal{W}_x \cap f(\mathcal{E}_{x,i})$  for each  $x \in D$  and  $1 \leq i \leq l_x(\mathcal{E})$ . The distinct terms of  $\{\mathcal{W}_{x,i}\}_{1 \leq i \leq l_x(\mathcal{E})}$  give a filtration of  $\mathcal{W}_x$  as follows:

$$0 \subsetneq \mathcal{W}_{x,1} \subsetneq \cdots \subsetneq \mathcal{W}_{x,l_x(\mathcal{W})} = \mathcal{W}_x,$$

for  $1 \leq j \leq l_x(\mathcal{W})$ , the induced parabolic weight is given by  $\alpha_{x,j}^{\mathcal{W}} := \alpha_{x,i}^{\mathcal{E}}$ , where  $i$  is the smallest number such that  $\mathcal{Q}_{x,j} = f|_{\mathcal{E}_x}(\mathcal{E}_{x,i})$ . Such a resulting parabolic sheaf  $\mathcal{W}_*$  is called an *induced parabolic quotient sheaf* of  $\mathcal{E}_*$ .

**Proposition 4.1.14.** Let  $\mathcal{E}_*$  be a parabolic sheaf given by filtrations as above. Let  $\mathcal{F} \subseteq \mathcal{E}$  be a torsion free coherent subsheaf that fits into the following short exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{W} \longrightarrow 0,$$

then with the naturally induced parabolic structures on  $\mathcal{F}$  and  $\mathcal{W}$ , we have

$$\deg^{\text{par}}(\mathcal{E}_*) = \deg^{\text{par}}(\mathcal{F}_*) + \deg^{\text{par}}(\mathcal{W}_*).$$

*Proof.* This is because for each  $x \in D$ , the weight system satisfies

$$\{a_{x,1}^{\mathcal{E}}, \dots, a_{x,l_x(\mathcal{E})}^{\mathcal{E}}\} = \{a_{x,1}^{\mathcal{F}}, \dots, a_{x,l_x(\mathcal{F})}^{\mathcal{F}}\} \cup \{a_{x,1}^{\mathcal{W}}, \dots, a_{x,l_x(\mathcal{W})}^{\mathcal{W}}\}.$$

□

Let  $\Omega_X^1$  be the sheaf of holomorphic differential 1-forms on  $X$ , that is,  $\Omega_X^1 = K_X$ , the canonical line bundle of  $X$ .

**Definition 4.1.15.** A *parabolic Higgs sheaf* over  $(X, D)$  is a pair  $(\mathcal{E}_*, \theta)$ , where  $\mathcal{E}_*$  is a parabolic sheaf, and  $\theta : \mathcal{E} \rightarrow \mathcal{E} \otimes K_X(\log D)$  a morphism that preserves the parabolic structure, that is,  $\theta(\mathcal{E}_{x,i}) \subseteq \mathcal{E}_{x,i} \otimes K_X(\log D)|_x$  for each  $x \in D$  and  $1 \leq i \leq l_x$ . More strictly, if  $\theta(\mathcal{E}_{x,i}) \subseteq \mathcal{E}_{x,i-1} \otimes K_X(\log D)$  for each  $x \in D$  and  $1 \leq i \leq l_x$ , then  $(\mathcal{E}_*, \theta)$  is called a *strongly parabolic Higgs sheaf*.

**Definition 4.1.16.** A parabolic Higgs sheaf  $(\mathcal{E}_*, \theta)$  is called *stable* (resp. *semistable*) if for any saturated coherent subsheaf  $\mathcal{F} \subsetneq \mathcal{E}$  of  $0 < \text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E})$  with the naturally induced parabolic structure, and such that  $\theta(\mathcal{F}) \subseteq \mathcal{F} \otimes K_X(\log D)$  we have

$$\mu^{\text{par}}(\mathcal{F}_*) < \text{(resp. } \leq) \mu^{\text{par}}(\mathcal{E}_*).$$

Equivalently,  $(\mathcal{E}_*, \theta)$  is *stable* (resp. *semistable*) if for any torsion free quotient Higgs sheaf  $(\mathcal{W}, \theta')$  of  $(\mathcal{E}, \theta)$  of  $0 < \text{rk}(\mathcal{W}) < \text{rk}(\mathcal{E})$  with the naturally induced parabolic structure, we have

$$\mu^{\text{par}}(\mathcal{E}_*) < \text{(resp. } \leq) \mu^{\text{par}}(\mathcal{W}_*).$$

In the following, we give some examples of stable parabolic Higgs sheaves over Riemann surfaces.

**Example 4.1.17** (A basic example of stable parabolic Higgs sheaf, [BGG97]). Let  $X$  be a compact Riemann surface of genus  $g$ , and let  $D = x_1 + \dots + x_k$  be an effectively divisor consists of  $k$  points with the condition that  $2g - 2 + k > 0$ . Fix a square root  $\mathcal{L} = K_X^{\frac{1}{2}}$  for  $K_X$ , and let  $\eta = \mathcal{O}_X(D)$  be the line bundle give by the divisor  $D$ . Define  $\mathcal{E} := (\mathcal{L} \otimes \eta)^* \oplus \mathcal{L}$  be the rank 2 vector bundle. For each  $x \in D$ , consider the trivial filtration

$$0 \subsetneq \mathcal{E}_x$$

with the parabolic weight  $a_x = -\frac{1}{2}$  to  $\mathcal{E}_x$ . This gives a parabolic sheaf  $\mathcal{E}_*$  over  $(X, D)$ . Define the Higgs field

$$\theta = \begin{pmatrix} 0 & 1 \\ q_2 & 0 \end{pmatrix} \in H^0(X, \text{End}(\mathcal{E}) \otimes K_X \otimes \eta),$$

where  $1 \in H^0(X, \mathcal{L}^* \otimes (\mathcal{L} \otimes \eta)^* \otimes K_X \otimes \eta) = H^0(X, \mathcal{O})$  is the constant section and  $q_2 \in H^0(X, (\mathcal{L} \otimes \eta) \otimes \mathcal{L} \otimes K_X \otimes \eta) = H^0(X, K_X^2 \otimes \eta^2)$  is a quadratic differential. This defines a stable parabolic Higgs sheaf  $(\mathcal{E}_*, \theta)$  over  $(X, D)$  of degree 0. Indeed,

$$\text{deg}^{\text{par}}(\mathcal{E}_*) = \text{deg}(\mathcal{E}) - \sum_{x \in D} a_x \text{rk}_x(\mathcal{E}_x) = 0.$$

When  $q_2 \neq 0$ , then  $\mathcal{E}$  has no proper  $\theta$ -invariant subsheaf, so  $(\mathcal{E}_*, \theta)$  is automatically stable.

When  $q_2 = 0$ , then the only proper  $\theta$ -invariant parabolic subsheaf of  $\mathcal{E}_*$  is  $\mathcal{F} = (\mathcal{L} \otimes \eta)^*$  with the naturally induced parabolic structure, its degree is

$$\deg^{\text{par}}(\mathcal{F}) = \deg(F) + \sum_{x \in D} a_x \text{rk}_x(\mathcal{F}_x) = -(g-1+n) + \frac{1}{2} \times n < 0.$$

**Example 4.1.18** (Stable rank 2 parabolic Higgs bundle over  $\mathbb{P}^1$  with 3 parabolic points). Let  $X = \mathbb{P}^1$  and let  $D = x_1 + x_2 + x_3$  consists of three distinct points. Let  $(\mathcal{E}_*, \theta)$  be a parabolic Higgs bundle of rank 2, where the Higgs field  $\theta : \mathcal{E} \rightarrow \mathcal{E} \otimes K_X(D)$  satisfies the strong parabolic condition  $\theta(\mathcal{E}_{x_i,j}) \subseteq \mathcal{E}_{x_i,j-1} \otimes K_X(D)$ . Here  $K_X(D) = K_X \otimes \mathcal{O}_{\mathbb{P}^1}(D) = \mathcal{O}_{\mathbb{P}^1}(1)$ .

First we assume that at each parabolic point  $x_i$ , the filtration of  $\mathcal{E}_{x_i}$  is given by full flag  $0 \subsetneq \mathcal{E}_{x_i,1} \subsetneq \mathcal{E}_{x_i,2} = \mathcal{E}_{x_i}$ , and the weight system associated to the filtration is

$$-1 < a_i < -1 - a_i < 0,$$

so  $-1 < a_i < -\frac{1}{2}$ .

Since  $\mathcal{E}$  can be written as  $\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(n)$  for some integers  $m$  and  $n$ . In this case, assume  $\mathcal{E}_*$  is of parabolic degree 0, i.e.,

$$\deg^{\text{par}}(\mathcal{E}_*) = \deg(\mathcal{E}) - \sum_{i=1}^3 (a_i - 1 - a_i) = 0,$$

thus the underlying holomorphic bundle  $\mathcal{E}$  is of degree -3. We can write it as  $\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(-3-m)$ .

**Lemma 4.1.19.** *If  $(\mathcal{E}_*, \theta)$  is a stable rank 2 parabolic Higgs bundle with the parabolic structure given as above, then the underlying bundle has the form  $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ .*

*Proof.* Write the Higgs field as

$$\theta = \begin{pmatrix} 0 & \varphi \\ \omega & 0 \end{pmatrix},$$

where  $\varphi : \mathcal{O}_{\mathbb{P}^1}(-3-m) \rightarrow \mathcal{O}_{\mathbb{P}^1}(m+1)$  and  $\omega : \mathcal{O}_{\mathbb{P}^1}(m) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-m-2)$ .

If  $m \geq 0$ , then  $\omega$  must be non-zero, otherwise  $\mathcal{O}_{\mathbb{P}^1}(m)$  is  $\theta$ -invariant subbundle, with the naturally induced parabolic structure,  $(\mathcal{O}_{\mathbb{P}^1}(m))_*$  has nonnegative parabolic degree since

$$\deg^{\text{par}}(\mathcal{O}_{\mathbb{P}^1}(m)) \geq \deg(\mathcal{O}_{\mathbb{P}^1}(m)) \geq 0.$$

Hence  $m \leq -1$ , a contradiction.

If  $m \leq -3$ , then  $\varphi$  must be non-zero, otherwise  $\mathcal{O}_{\mathbb{P}^1}(-3-m)$  is  $\theta$ -invariant subbundle, with the naturally induced parabolic structure,  $(\mathcal{O}_{\mathbb{P}^1}(-3-m))_*$  has nonnegative parabolic degree since

$$\deg^{\text{par}}((\mathcal{O}_{\mathbb{P}^1}(-3-m))_*) \geq \deg(\mathcal{O}_{\mathbb{P}^1}(-3-m)) \geq 0.$$

Hence  $m \geq -2$ , a contradiction.

Therefore,  $m = -1$  or  $m = -2$ , in either case,  $\mathcal{E} \cong \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ . □

(1). If  $\omega$  and  $\varphi$  both non-zero, then there is no  $\theta$ -invariant subbundle. Hence  $(\mathcal{E}, \theta)$  is automatically parabolic stable.

(2). If  $\omega = 0$  and  $\varphi \neq 0$ , then the Higgs field is as

$$\theta = \begin{pmatrix} 0 & \varphi \\ 0 & 0 \end{pmatrix},$$

then the only  $\theta$ -invariant subbundle is  $L = \mathcal{O}_{\mathbb{P}^1}(-1)$ . We need to consider the parabolic structure on  $\mathcal{E}$ , this is a choice of a flag  $\mathcal{E}_{p_i,2} \subset \mathcal{E}_{p_i}$  for each point  $p_i$ . We first claim that  $\mathcal{E}_{p_i,2}$  cannot be contained by  $\mathcal{O}_{\mathbb{P}^1}(-1)_{p_i}$  for all  $p_i$ . If not, let  $\mathcal{E}_{p_i,2} = \mathcal{O}_{\mathbb{P}^1}(-1)_{p_i}$  for  $i = 1, 2, 3$ , then the stability condition gives

$$\deg^{\text{par}}(L) = -1 + \sum_{i=1}^3 (1 - \alpha_i) = 2 - (\alpha_1 + \alpha_2 + \alpha_3) < 0,$$

this is impossible by the choice of each  $\alpha_i < 1 - \alpha_i < 1$ . Similarly, we can also find that  $E_{p_i,2}$  cannot be contained by  $\mathcal{O}_{\mathbb{P}^1}(-1)_{p_i}$  for exactly two  $p_i$ .

(3). If  $\omega \neq 0$  and  $\varphi = 0$ , then the Higgs field is as

$$\theta := \begin{pmatrix} 0 & 0 \\ \omega & 0 \end{pmatrix},$$

then the only  $\theta$ -invariant subbundle is  $L = \mathcal{O}_{\mathbb{P}^1}(-2)$ .

To consider the stability condition, we just need to consider the two  $\theta$ -invariant line subbundle:  $\mathcal{O}_{\mathbb{P}^1}(-1)$  and  $\mathcal{O}_{\mathbb{P}^1}(-2)$ , since for other subbundle  $L = \mathcal{O}_{\mathbb{P}^1}(a)$  with  $a \leq -3$ , it has parabolic degree

$$\deg^{\text{par}}(L) = a + \sum_{i=1}^3 (1 - \alpha_i) < 0,$$

which automatically satisfies the stability condition.

**Example 4.1.20** (Stable rank 2 parabolic Higgs bundle over  $\mathbb{P}^1$  with 4 parabolic points).

In this example,  $D = p_1 + p_2 + p_3 + p_4$  is the divisor of four parabolic points. Let  $(\mathcal{E}_*, \theta)$  be a parabolic Higgs bundle, where the Higgs field  $\theta : \mathcal{E} \rightarrow \mathcal{E} \otimes K(D)$  satisfies the strong parabolic condition  $\theta(\mathcal{E}_{p_i,j}) \rightarrow \mathcal{E}_{p_i,j+1} \otimes K(D)$ . Here  $K(D) = K \otimes \mathcal{O}_{\mathbb{P}^1}(D) = \mathcal{O}_{\mathbb{P}^1}(2)$ .

We assume the parabolic structure is a full flag at each parabolic point and the weight system  $\alpha$  associated to the filtration  $\mathcal{E}_{p_i} = \mathcal{E}_{p_i,1} \supset \mathcal{E}_{p_i,2} \supset 0$  at each parabolic point is given by the following table

point	$p_1$	$p_2$	$p_3$	$p_4$
weight system	$0 = 0 < \alpha_1 < 1$	$0 = 0 < \alpha_2 < 1$	$0 = 0 < \alpha_3 < 1$	$0 < \alpha_4 < \alpha_5 < 1$

where  $\sum_{i=1}^2 \alpha_i = 1$ . Like the analysis in the last example, the underlying holomorphic bundle for the stable parabolic Higgs bundle  $(\mathcal{E}_*, \theta)$  with the above weight system is  $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ . Define a parabolic structure on  $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  by having the subspace aligning with  $\mathcal{O}_{\mathbb{P}^1}(-1)$  at each parabolic point  $p_i$ , taking the Higgs field

$$\theta := \begin{pmatrix} 0 & 0 \\ \omega & 0 \end{pmatrix},$$

then the only  $\theta$ -invariant subbundle is  $L = \mathcal{O}_{\mathbb{P}^1}(-1)$  which has parabolic degree of

$$\deg^{\text{par}}(L) = -1 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_5 = -\alpha_4 < 0 = \deg^{\text{par}}(E),$$

so this defined  $(\mathcal{E}_*, \theta)$  is stable parabolic Higgs bundle.

On the other hand, if we define the Higgs field

$$\theta := \begin{pmatrix} 0 & \varphi \\ 0 & 0 \end{pmatrix}$$

and define a parabolic structure on  $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$  by having the subspace aligning with  $\mathcal{O}_{\mathbb{P}^1}$  at each  $p_i$ , then the  $\theta$ -invariant subbundle  $\mathcal{O}_{\mathbb{P}^1}$  has strictly positive parabolic degree and so defined  $(\mathcal{E}_*, \theta)$  is not stable.

### 4.1.3 Non-Abelian Hodge Theory of Parabolic Version

Now we come back to the general case, let  $X$  be a connected smooth complex projective variety of  $\dim_{\mathbb{C}}(X) = n$ , let  $D = \bigcup_{i=1}^k D_i \subseteq X$  be a reduced simple normal crossing divisor with each component  $D_i$  smooth and irreducible. Let  $\Omega_X^1(\log D)$  be the sheaf of logarithmic differential 1-forms on  $X$ , that is, the sections of  $\Omega_X^1(\log D)$  are differential 1-forms that have a pole of order at most one at each component of  $D$ .

#### Regular parabolic $\lambda$ -flat bundles

Let  $\mathcal{E}_*$  be a parabolic sheaf over  $(X, D)$  indexed by  $\alpha \in \mathbb{R}^k$ , with  $\mathcal{E} := \mathcal{E}_0$  the underlying sheaf (i.e., torsion free coherent  $\mathcal{O}_X$ -module) and  $\tilde{\mathcal{E}} = \bigcup_{\alpha \in \mathbb{R}^k} \mathcal{E}_\alpha$  be the quasi-coherent sheaf.

**Definition 4.1.21.** Let  $\lambda \in \mathbb{C}$ , a *regular parabolic  $\lambda$ -connection* on  $\mathcal{E}_*$  is a  $\mathbb{C}$ -linear map

$$\mathbb{D}^\lambda : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}} \otimes \Omega_X^1(\log D)$$

that satisfies the  $\lambda$ -twisted Leibniz rule  $\mathbb{D}^\lambda(fs) = f\mathbb{D}^\lambda(s) + \lambda s \otimes df$  and preserves the parabolic structure, that is,  $\mathbb{D}^\lambda(\mathcal{E}_\alpha) \subseteq \mathcal{E}_\alpha \otimes \Omega_X^1(\log D)$  for each multi-index  $\alpha \in \mathbb{R}^k$ .

When  $\mathbb{D}^\lambda \circ \mathbb{D}^\lambda = 0$  under the extension  $\mathbb{D}^\lambda : \tilde{\mathcal{E}} \otimes \Omega_X^p(\log D) \rightarrow \tilde{\mathcal{E}} \otimes \Omega_X^{p+1}(\log D)$  for any integer  $p \geq 0$ , then we call  $\mathbb{D}^\lambda$  a *regular parabolic flat  $\lambda$ -connection*, and  $(\mathcal{E}_*, \mathbb{D}^\lambda)$  a *regular parabolic  $\lambda$ -flat sheaf*. More strictly, if for each multi-index  $\alpha \in \mathbb{R}^k$ , there exists some  $\varepsilon > 0$  such that  $\mathbb{D}^\lambda(\mathcal{E}_\alpha) \subseteq \mathcal{E}_{\alpha-\varepsilon} \otimes \Omega_X^1(\log D)$ , then we call  $\mathbb{D}^\lambda$  a *strongly regular parabolic flat  $\lambda$ -connection*, and  $(\mathcal{E}_*, \mathbb{D}^\lambda)$  a *strongly regular parabolic  $\lambda$ -flat sheaf*.

A (strongly) regular parabolic  $\lambda$ -flat sheaf  $(\mathcal{E}_*, \mathbb{D}^\lambda)$  is called a *locally abelian (strongly) regular parabolic  $\lambda$ -flat sheaf* if the underlying parabolic sheaf  $\mathcal{E}_*$  is a locally abelian parabolic bundle. In this case, we will briefly apply the notation *(strongly) regular parabolic  $\lambda$ -flat bundle* when there is no ambiguity.

From now on, we always assume  $\mathbb{D}^\lambda$  being flat. And sometimes we will add subscripts  $\mathcal{E}, \mathcal{F}, \dots$  to  $\mathbb{D}^\lambda$  in order to distinguish different objects.

In particular:

- when  $\lambda = 1$ , we will call  $\mathbb{D}^1$  a *regular parabolic flat connection*, and we usually apply the notation  $\nabla$ . In this case, the pair  $(\mathcal{E}_*, \nabla)$  is called a *regular parabolic flat sheaf*;
- when  $\lambda = 0$ , we will call  $\mathbb{D}^0$  a *regular parabolic Higgs field*, and we usually apply the notation  $\theta$ . In this case, the pair  $(\mathcal{E}_*, \theta)$  is called a *regular parabolic Higgs sheaf*.

Now let  $(\mathcal{E}_*, \mathbb{D}_{\mathcal{E}}^\lambda)$  and  $(\mathcal{F}, \mathbb{D}_{\mathcal{F}}^\lambda)$  be two regular parabolic  $\lambda$ -flat sheaves over  $(X, D)$ .

**Definition 4.1.22.** A *parabolic morphism*  $f : (\mathcal{E}_*, \mathbb{D}_{\mathcal{E}}^\lambda) \rightarrow (\mathcal{F}_*, \mathbb{D}_{\mathcal{F}}^\lambda)$  is a parabolic morphism  $f : \mathcal{E}_* \rightarrow \mathcal{F}_*$  of the underlying parabolic sheaves that compatible with the parabolic structures, that is, the following diagram

$$\begin{array}{ccc} \mathcal{E}_\alpha & \xrightarrow{f_\alpha} & \mathcal{F}_\alpha \\ \mathbb{D}_{\mathcal{E}}^\lambda \downarrow & & \mathbb{D}_{\mathcal{F}}^\lambda \downarrow \\ \mathcal{E}_\alpha \otimes \Omega_X^1(\log D) & \xrightarrow{f_\alpha \otimes \text{Id}} & \mathcal{F}_\alpha \otimes \Omega_X^1(\log D) \end{array}$$

commutes for any multi-index  $\alpha \in \mathbb{R}^k$ . Denotes by  $\text{Hom}((\mathcal{E}, \mathbb{D}_{\mathcal{E}}^\lambda), (\mathcal{F}, \mathbb{D}_{\mathcal{F}}^\lambda))$  and  $\mathcal{H}om((\mathcal{E}, \mathbb{D}_{\mathcal{E}}^\lambda), (\mathcal{F}, \mathbb{D}_{\mathcal{F}}^\lambda))$  the set and sheaf of parabolic morphisms of regular parabolic  $\lambda$ -flat sheaves, respectively.

### Tame pluri-harmonic metrics and adapted metrics

Tameness condition for pluri-harmonic metrics on Higgs bundle was introduced by Simpson in [Sim90], in which he studied the parabolic Higgs bundle over open projective curve. Later studied by Mochizuki in higher dimensional cases [Moc06, Moc07, Moc09]. Let's briefly recall it here.

Let  $(X, D)$  be as above, let  $(\mathcal{E}, \mathbb{D}^\lambda)$  be a usual  $\lambda$ -flat bundle over  $X - D$ . Let  $h$  be a pluri-harmonic metric on  $(\mathcal{E}, \mathbb{D}^\lambda)$ , then it induces a harmonic Higgs bundle  $(E, \bar{\partial}_h, \theta_h, h)$ , note that  $E$  is the underlying smooth vector bundle, and  $\mathcal{E} = (E, \bar{\partial}_h + \lambda \theta_h^\dagger)$  is the holomorphic bundle.

For any  $P \in X$ , let  $D_{j_1}, \dots, D_{j_m}$  are components of  $D$  that contain  $P$ , let  $\{U_P; z_1, \dots, z_n\}$  be a holomorphic coordinate around  $P$  such that  $U_P \cap D_{j_i} = \{z_i = 0\}$  for  $i = 1, \dots, m$ , this is called an *admissible coordinate*. Then the Higgs field  $\theta_h$  can be expressed as

$$\theta_h = \sum_{i=1}^m \theta_i(z) \frac{dz_i}{z_i} + \sum_{i=m+1}^n \theta_i(z) dz_i.$$

For each  $i = 1, \dots, n$  and a formal variable  $t$ , write the polynomial  $\det(t\text{Id}_{\mathcal{E}} - \theta_i(z))$  as

$$\det(t\text{Id}_{\mathcal{E}} - \theta_i(z)) = \sum_{l=1}^n a_l^i(z) t^l,$$

then the coefficients  $a_l^i(z)$  are holomorphic functions on  $U_P - \bigcup_{i=1}^m D_{j_i}$ .

**Definition 4.1.23.** A harmonic bundle  $(E, \bar{\partial}_E, \theta, h)$  over  $X - D$  is called a *tame harmonic bundle* over  $X - D$  if each  $a_j^i(z)$  can be extended to be a holomorphic function on  $U_P$  for any  $P \in X$ . In this case, we call  $h$  a *tame* pluri-harmonic metric. A harmonic  $\lambda$ -bundle  $(\mathcal{E}, \mathbb{D}^\lambda, h)$  over  $X - D$  is *tame* if the associated harmonic Higgs bundle  $(E, \bar{\partial}_h, \theta_h, h)$  is tame.

It's known that  $(E, \bar{\partial}_h, \theta_h, h)$  is tame if and only if there exists an extension  $(\bar{\mathcal{E}}, \bar{\theta})$  (cf. [Moc07, Lemma 22.1]), that is,  $\bar{\mathcal{E}}$  is a holomorphic vector bundle over  $X$  and  $\bar{\mathbb{D}}^\lambda : \bar{\mathcal{E}} \rightarrow \bar{\mathcal{E}} \otimes \Omega_X^1(\log D)$  is a Higgs field such that  $(\bar{\mathcal{E}}, \bar{\theta})|_{X-D} = (E, \bar{\partial}_h, \theta_h)$ <sup>1</sup>.

A naturally extension is given by the following, which is due to Mochizuki (cf. [Moc06, Moc09, Moc07]).

Now let  $\mathcal{E}$  be a holomorphic bundle over  $X - D$  with a hermitian metric  $h$ . Then  $h$  defines a parabolic structure on  $\mathcal{E}$  as follows.

<sup>1</sup>When  $\lambda = 1$ , this is just the Deligne extension for flat bundles (without any given metric) over quasi-projective varieties, and when  $\lambda = 0$ , the tameness condition is necessary to preserve the extension exists.

For each  $i = 1, \dots, k$ , let  $\sigma_i : \mathcal{O}_X \rightarrow \mathcal{O}_X(D_i)$  be the canonical section of  $\mathcal{O}_X(D_i)$  and let  $h_i$  be hermitian metric on  $\mathcal{O}_X(D_i)$ . The sheaf  $\mathcal{E}_\alpha(h)$  indexed by  $\alpha \in \mathbb{R}^k$  is, for any open subset  $U \subseteq X$ , let  $D_{i_1}, \dots, D_{i_m}$  be the irreducible components of  $D$  that intersect with  $U$ ,

$$\Gamma(U, \mathcal{E}_\alpha(h)) := \left\{ s \in \Gamma(U - D, \mathcal{E}) \mid |s|_h \leq C \cdot \prod_{j=1}^m |\sigma_{i_j}|_{h_{i_j}}^{-\alpha_{i_j} - \varepsilon} \text{ for some } C > 0, \forall \varepsilon > 0 \right\}.$$

Then  $\mathcal{E}_*(h) = \{\mathcal{E}_\alpha(h)\}_{\alpha \in \mathbb{R}^k}$  is a parabolic sheaf on  $(X, D)$ , called the *associated parabolic sheaf*.

Let  $\tilde{\mathcal{E}}(h) = \bigcup_{\alpha \in \mathbb{R}^k} \mathcal{E}_\alpha(h)$ .

**Definition 4.1.24.** For a parabolic sheaf  $\mathcal{E}_*$  over  $(X, D)$ , let  $\mathcal{E}' := \tilde{\mathcal{E}}|_{X-D} = \mathcal{E}_\alpha|_{X-D}$ , a hermitian metric  $h$  on  $\mathcal{E}'$  is called *adapted* to the parabolic structure of  $\mathcal{E}_*$  if  $\mathcal{E}'_\alpha(h) \cong \mathcal{E}_\alpha$  for any multi-index  $\alpha \in \mathbb{R}^k$ .

In general,  $\mathcal{E}_*(h)$  is not coherent [Moc06], but when  $(\mathcal{E}, \mathbb{D}^\lambda, h)$  is tame harmonic, then  $\mathcal{E}_*(h)$  is a locally abelian parabolic bundle due to Simpson for dimension 1 case [Sim90, Theorem 2], and Mochizuki for higher dimensional case [Moc09, Proposition 2.53]. This means that the associated parabolic sheaf of a tame harmonic bundle appears as an example of locally abelian parabolic bundle. We call this special associated sheaf as *prolongation of tame harmonic bundle* following Mochizuki.

**Proposition 4.1.25** (Simpson, Mochizuki, [Sim90, Moc09, Moc07]). *If  $(\mathcal{E}, \mathbb{D}^\lambda, h)$  is a tame harmonic bundle, then  $(\mathcal{E}_*(h), \mathbb{D}^\lambda)$  is a regular parabolic  $\lambda$ -flat bundle. In particular,  $(\mathcal{E}_\alpha(h), \mathbb{D}^\lambda)$  is an extension of  $(\mathcal{E}, \mathbb{D}^\lambda, h)$  for any  $\alpha \in \mathbb{R}^k$ .*

## Parabolic Chern characters

A very important idea for parabolic sheaves is that locally abelian parabolic bundles with rational parabolic weights can be viewed as usual vector bundles over certain Deligne–Mumford stack [Bis97, IS07, Bor07]. Hence the parabolic Chern characters of the parabolic bundle is defined to be the Chern characters of that vector bundle over the Deligne–Mumford stack.

The explicit formula for parabolic Chern characters are given by Iyer and Simpson in [IS08]. Taher calculated the first, second and third parabolic Chern characters based on Iyer–Simpson formula [Tah10], which coincide with the first and second parabolic Chern characters defined by Mochizuki in [Moc06]. Here we briefly introduce this based on their papers.

Let  $\mathcal{E}_*$  be a locally abelian parabolic bundle over  $(X, D)$ , for a multi-index  $\mathbf{c} \in \mathbb{R}^k$  as the base point, then let  ${}_{\mathbf{c}}\mathcal{E}$  be the  $\mathbf{c}$ -truncation.

**Definition 4.1.26 (Iyer–Simpson formula).** The *parabolic Chern character* of  $\mathcal{E}_*$  is defined as

$$\text{ch}^{\text{par}}(\mathcal{E}_*) = \text{ch}^{\text{par}}({}_{\mathbf{c}}\mathcal{E}) := \frac{\int_{c_{n-1}}^{c_n} \cdots \int_{c_1-1}^{c_1} e^{-\sum_{i=1}^k \alpha_i D_i} \text{ch}(\mathcal{E}_\alpha) d\alpha_1 \cdots d\alpha_n}{\int_{c_{n-1}}^{c_n} \cdots \int_{c_1-1}^{c_1} e^{-\sum_{i=1}^k \alpha_i D_i} d\alpha_1 \cdots d\alpha_n}, \quad (4.5)$$

where  $\text{ch}(\mathcal{E}_\alpha)$  is the usual Chern character of the locally free sheaf  $\mathcal{E}_\alpha$ .

**Remark 4.1.27.** (1) The numerator of (4.5) is independent of the choice of  $\mathbf{c}$ -truncation. Indeed, choose  $\beta = \alpha + \sum_{i=1}^k t_i \delta^i$  for any  $(t_1, \dots, t_k) \in \mathbb{R}^k$ , then  $\mathcal{E}_\beta = \mathcal{E}_\alpha(\sum_{i=1}^k t_i D_i)$ , and

$$\text{ch}(\mathcal{E}_\beta) = \text{ch}(\mathcal{E}_\alpha) \cdot e^{\sum_{i=1}^k t_i D_i}.$$

Hence achieves

$$\int_{c_n+t_n-1}^{c_n+t_n} \cdots \int_{c_1+t_1-1}^{c_1+t_1} e^{-\sum_{i=1}^k \beta_i D_i} \text{ch}(\mathcal{E}_\beta) d\beta_1 \cdots d\beta_n = \int_{c_n-1}^{c_n} \cdots \int_{c_1-1}^{c_1} e^{-\sum_{i=1}^k \alpha_i D_i} \text{ch}(\mathcal{E}_\alpha) d\alpha_1 \cdots d\alpha_n.$$

(2) If we choose another base point  $\mathbf{d} = \mathbf{c} + \sum_{i=1}^k t_i \delta^i$ , then

$$\text{ch}^{\text{par}}(\mathbf{d}\mathcal{E}) = \text{ch}^{\text{par}}(\mathbf{c}\mathcal{E}) \cdot e^{\sum_{i=1}^k t_i D_i}.$$

(3) The first parabolic Chern character  $\text{ch}_1^{\text{par}}(\mathcal{E}_*)$  can be obtained explicitly as follows:

$$\text{ch}_1^{\text{par}}(\mathcal{E}_*) = c_1(\mathcal{E}_\mathbf{c}) - \sum_{i=1}^k \left( \sum_{a \in \text{wt}^i(\mathcal{E}_\mathbf{c})} a \cdot \text{rk}_{D_i}(\text{Gr}_a^{\mathcal{F}^1}(\mathcal{E}_\mathbf{c})) \right) \cdot |D_i| \in H^2(X, \mathbb{R}), \quad (4.6)$$

where  $\text{rk}_{D_i}(\text{Gr}_a^{\mathcal{F}^1}(\mathcal{E}_\mathbf{c}))$  denotes the rank of the sheaf  $\text{Gr}_a^{\mathcal{F}^1}(\mathcal{E}_\mathbf{c})$  (as  $\mathcal{O}_{D_i}$ -module).

(4) The second parabolic Chern character is not easy to calculate, for our convenience, here we apply Mochizuki's result [Moc06]:

$$\begin{aligned} \text{ch}_2^{\text{par}}(\mathcal{E}_*) &= \text{ch}_2(\mathcal{E}_\mathbf{c}) + \sum_{i=1}^k \left( \sum_{a \in \text{wt}^i(\mathcal{E}_\mathbf{c})} a \cdot \iota_{i*} \left( c_1(\text{Gr}_a^{\mathcal{F}^i}(\mathcal{E}_\mathbf{c})) \right) \right) \\ &+ \frac{1}{2} \sum_{i=1}^k \left( \sum_{a \in \text{wt}^i(\mathcal{E}_\mathbf{c})} a^2 \cdot \text{rk}_{D_i}(\text{Gr}_a^{\mathcal{F}^i}(\mathcal{E}_\mathbf{c})) \cdot |D_i|^2 \right) \\ &+ \frac{1}{2} \sum_{1 \leq i \neq j \leq k} \sum_{\substack{a_i \in \text{wt}^i(\mathcal{E}_\mathbf{c}) \\ a_j \in \text{wt}^j(\mathcal{E}_\mathbf{c})}} \left( \sum_{D' \in \text{Irr}(D_i \cap D_j)} a_i a_j \cdot \text{rk}_{D'}(\text{Gr}_{a_i, a_j}^{\mathcal{F}^{i,j}}(\mathcal{E}_\mathbf{c})) \cdot |D'| \right) \in H^4(X, \mathbb{R}), \end{aligned} \quad (4.7)$$

where

- $\text{ch}_2(\mathcal{E}_\mathbf{c})$  is the second Chern character of the sheaf  $\mathcal{E}_\mathbf{c}$ ;
- $\iota : D_i \hookrightarrow X$  is the natural embedding, and  $\iota_{i*} : H^2(D_i) \rightarrow H^4(X)$  is the associated Gysin map;
- $\text{Irr}(D_i \cap D_j)$  is the set of irreducible components of  $D_i \cap D_j$ ;
- on the restricted sheaf  $\mathcal{E}_\mathbf{c}|_{D_i \cap D_j}$ , there are two increasing filtrations, namely  $\{\mathcal{F}_{a_i}^i\}_{c_i-1 \leq a_i \leq c_i}$  and  $\{\mathcal{F}_{a_j}^j\}_{c_j-1 \leq a_j \leq c_j}$  with  $\mathcal{F}_{c_i-1}^i = \mathcal{F}_{c_j-1}^j = 0$  and  $\mathcal{F}_{c_i}^i = \mathcal{F}_{c_j}^j = \mathcal{E}_\mathbf{c}|_{D_i \cap D_j}$ . This induces an increasing multi-indexed filtration  $\mathcal{F}_{a_i, a_j}^{i,j} := \mathcal{F}_{a_i}^i \cap \mathcal{F}_{a_j}^j$  with  $\mathcal{F}_{c_i-1, c_j-1}^{i,j} = 0$  and  $\mathcal{F}_{c_i, c_j}^{i,j} = \mathcal{E}_\mathbf{c}|_{D_i \cap D_j}$ . The graded term is given by

$$\text{Gr}_{a_i, a_j}^{\mathcal{F}^{i,j}} := \frac{\mathcal{F}_{a_i, a_j}^{i,j}}{\sum_{(b_i, b_j) \preceq (a_i, a_j)} \mathcal{F}_{b_i, b_j}^{i,j}},$$

where  $(b_i, b_j) \preceq (a_i, a_j)$  means  $b_i \leq a_i$  &  $b_j < a_j$  or  $b_i < a_i$  &  $b_j \leq a_j$ ;

- under the locally abelian condition, the filtrations  $\mathcal{F}_\bullet^i$  and  $\mathcal{F}_\bullet^j$  coincide on  $\mathcal{E}_\mathbf{c}|_{D_i \cap D_j}$ , hence

$$\text{Gr}_{a_i, a_j}^{\mathcal{F}^{i,j}} = \text{Gr}_{a_j}^{\mathcal{F}^j} \text{Gr}_{a_i}^{\mathcal{F}^i}.$$

The following properties state that for a torsion free coherent sheaf with trivial parabolic structure, the first Chern class of the parabolic sheaf is exactly the usual first Chern class of the sheaf itself (when choose  $\mathbf{0}$ -truncation).

**Proposition 4.1.28.** *Let  $\mathbf{c} = \mathbf{0}$ , then*

- (1)  $c_1^{\text{par}}((\mathcal{O}_X(\sum_{i=1}^k \beta_i D_i))_*) = \sum_{i=1}^k \beta_i |D_i| \in H^2(X, \mathbb{R})$ ;
- (2) *Let  $\mathcal{E}_*$  be the parabolic sheaf obtained from a torsion free coherent sheaf  $\mathcal{E}$  with trivial parabolic structure, then*

$$c_1^{\text{par}}(\mathcal{E}_*) = c_1(\mathcal{E}) \in H^2(X, \mathbb{R}).$$

*Proof.* Direct calculation from the formula (4.6), trivial.  $\square$

### Stability

After introduce the first parabolic Chern character, it's natural to introduce the stability for locally abelian parabolic bundles.

Now let  $L := \mathcal{O}_X(1)$  be an ample line bundle over the smooth complex projective variety  $X$  of  $\dim_{\mathbb{C}}(X) = n$ . Let  $\mathcal{E}_*$  be a parabolic sheaf over  $(X, D)$  with a regular parabolic flat  $\lambda$ -connection  $\mathbb{D}^\lambda$ . As above, choose a multi-index  $\mathbf{c} \in \mathbb{R}^k$  as a base point, and consider the  $\mathbf{c}$ -truncation  ${}_{\mathbf{c}}\mathcal{E}_*$  to define the parabolic Chern characters.

**Definition 4.1.29.** The *parabolic degree* of  $\mathcal{E}_*$  with respect to  $L$  is defined as

$$\text{deg}_L^{\text{par}}(\mathcal{E}_*) := \int_X \text{ch}_1^{\text{par}}(\mathcal{E}_*) \cdot c_1(L)^{n-1}.$$

The *parabolic slope* of  $\mathcal{E}_*$  is given by

$$\mu_L^{\text{par}}(\mathcal{E}_*) := \frac{\text{deg}_L^{\text{par}}(\mathcal{E}_*)}{\text{rk}(\mathcal{E}_*)},$$

where  $\text{rk}(\mathcal{E}_*) = \text{rk}(\mathcal{E}_\alpha)$ , the rank of locally free sheaf  $\mathcal{E}_\alpha$ , which is independent of the choice of multi-index  $\alpha \in (\mathbf{c} - \delta, \mathbf{c}]$ .

**Remark 4.1.30.** When  $n = 1$  and take  $\mathbf{0}$ -truncation, then the parabolic degree in fact coincides with the one we defined in (4.3) for parabolic sheaves over Riemann surfaces.

**Definition 4.1.31.** A regular parabolic  $\lambda$ -flat sheaf  $(\mathcal{E}_*, \mathbb{D}^\lambda)$  is called  $\mu_L^{\text{par}}$ -*stable* (resp.  $\mu_L^{\text{par}}$ -*semistable*) if for any saturated subsheaf  $\mathcal{F} \subseteq \mathcal{E}_{\mathbf{c}}$  of  $0 < \text{rk}(\mathcal{F}) < \text{rk}(\mathcal{E}_{\mathbf{c}})$  such that  $\mathbb{D}^\lambda(\mathcal{F}) \subseteq \mathcal{F} \otimes \Omega_X^1(\log D)$ , we have

$$\mu_L^{\text{par}}(\mathcal{F}_*) < \text{ (resp. } \leq) \mu_L^{\text{par}}(\mathcal{E}_*),$$

where  $\mathcal{F}_* \subseteq \mathcal{E}_*$  is the induced parabolic subsheaf (cf. (1) of Remark 4.1.7). It is called  $\mu_L^{\text{par}}$ -*polystable* if is the direct sum of  $\mu_L^{\text{par}}$ -stable parabolic sheaves of the same parabolic slope.

**Remark 4.1.32.** A regular parabolic  $\lambda$ -flat parabolic sheaf  $(\mathcal{E}_*, \mathbb{D}^\lambda)$  is  $\mu_L^{\text{par}}$ -stable (resp.  $\mu_L^{\text{par}}$ -semistable) if and only if for any quotient  $\lambda$ -flat sheaf  $(\mathcal{W}, \mathbb{D}_{\mathcal{W}}^\lambda)$  of  $(\mathcal{E}_{\mathbf{c}}, \mathbb{D}^\lambda)$  with  $0 < \text{rk}(\mathcal{W}) < \text{rk}(\mathcal{E}_{\mathbf{c}})$ , we have

$$\mu_L^{\text{par}}(\mathcal{E}_*) < \text{ (resp. } \leq) \mu_L^{\text{par}}(\mathcal{W}_*),$$

where  $\mathcal{W}_*$  is the induced parabolic quotient sheaf (cf. (2) of Remark 4.1.7).

### Non-Abelian Hodge theory of parabolic version

Parabolic sheaves are generalizations of usual sheaves to compact smooth varieties with divisor components (i.e, open varieties). As in compact version, an important project worth studying is the existence of certain special metrics on these objects, that is, the Kobayashi–Hitchin correspondence for regular parabolic  $\lambda$ -flat bundles.

When  $\lambda = 0$ , Simpson introduced the Higgs field for parabolic sheaves over compact Riemann surfaces with divisors, and built this correspondence [Sim90]; later this was generalized by Biquard to higher dimensional compact Kähler manifolds with smooth divisors [Biq97]; the more general case by simple normal crossing divisors are obtained by Mochizuki [Moc06].

When  $\lambda \neq 0$ , the correspondence of tame version is due to Mochizuki [Moc09].

**Theorem 4.1.33 (Kobayashi–Hitchin correspondence, [Moc09]).** *Let  $(X, L = \mathcal{O}_X(1))$  be a polarized variety with a reduced simple normal crossing divisor  $D$ . Let  $(\mathcal{E}_*, \mathbb{D}^\lambda)$  be a regular parabolic  $\lambda$ -flat bundle over  $(X, D)$ , set  $\mathcal{E}' = \tilde{\mathcal{E}}|_{X-D}$  for  $\tilde{\mathcal{E}} = \bigcup_{\alpha \in \mathbb{R}^k} \mathcal{E}_\alpha$ . Then the following statements are equivalent:*

- (1)  $(\mathcal{E}_*, \mathbb{D}^\lambda)$  is  $\mu_L^{\text{par}}$ -polystable with  $\deg_L^{\text{par}}(\mathcal{E}_*) = \int_X \text{ch}_2^{\text{par}}(\mathcal{E}_*) \cdot c_1(L)^{n-2} = 0$ ;
- (2)  $(\mathcal{E}', \mathbb{D}^\lambda)$  admits a tame pluri-harmonic metric  $h$  which is adapted to the parabolic structure.

Moreover, such a metric is unique up to scalar multiplications.

Now let  $\text{Par}_{\text{poly}}^\lambda$  be the category of  $\mu_L^{\text{par}}$ -polystable regular parabolic  $\lambda$ -flat bundles over  $(X, D)$  with  $\deg_L^{\text{par}}(\mathcal{E}_*) = \int_X \text{ch}_2^{\text{par}}(\mathcal{E}_*) \cdot c_1(L)^{n-2} = 0$ , as a consequence, we have the following correspondence:

**Corollary 4.1.34 (Non-Abelian Hodge correspondence, [Moc09]).** *For any  $\lambda_1, \lambda_2 \in \mathbb{C}$ , there is an equivalence of categories*

$$\text{Par}_{\text{poly}}^{\lambda_1} \cong \text{Par}_{\text{poly}}^{\lambda_2}$$

$$\mathbb{D}^{\lambda_1} = \bar{\partial}_h + \lambda_1 \theta_h^\dagger + \lambda_1 \partial_h + \theta_h \leftrightarrow \mathbb{D}^{\lambda_2} = \bar{\partial}_h + \lambda_2 \theta_h^\dagger + \lambda_2 \partial_h + \theta_h,$$

where  $h$  is the tame pluri-harmonic metric that adapted to the parabolic structure. Moreover, such equivalence preserves tensor products, direct sums and duals.

**Remark 4.1.35.** In particular, choose  $\lambda_1 = 1$  and  $\lambda_2 = 0$ , then this gives a correspondence between:

- $\mu_L^{\text{par}}$ -polystable regular parabolic flat bundles with trivial parabolic degree and second parabolic Chern character

and

- $\mu_L^{\text{par}}$ -polystable regular parabolic Higgs bundles with trivial parabolic degree and second parabolic Chern character.

This is exactly the parabolic version of Corlette–Simpson correspondence.

As a conclusion, we have the following equivalence of categories:

**Corollary 4.1.36.** *The following categories are equivalent:*

- (1) the category of tame harmonic bundles  $(E, \bar{\partial}_E, \theta, h)$  over  $X - D$ ;
- (2)  $\text{Par}_{\text{poly}}^\lambda$ ;
- (3) the category of semisimple flat bundles  $(\mathcal{E}, \nabla)$  over  $X - D$ .

*Proof.* The equivalence between (1) and (2) is from [Theorem 4.1.33](#). Indeed, given a tame harmonic bundle  $(E, \bar{\partial}_E, \theta, h)$  over  $X - D$ , then  $(\mathcal{E}, \mathbb{D}^\lambda, h)$  is a harmonic  $\lambda$ -flat bundle over  $X - D$ , where  $\mathcal{E} = (E, \bar{\partial}_h + \lambda\theta_h^\dagger)$ . Then the associated parabolic  $\lambda$ -flat sheaf  $(\mathcal{E}, \mathbb{D}^\lambda)$  is regular parabolic  $\lambda$ -flat bundle (cf. [Proposition 4.1.25](#)), and moreover,  $\mu_L^{\text{par}}$ -polystable with trivial parabolic degree and second parabolic Chern character (cf. [[Moc09](#), Proposition 2.55]).

The equivalence between (2) and (3) is due to a result by Jost and Zuo in [[JZ97](#)], which states that any semisimple flat bundle  $(\mathcal{E}, \nabla)$  over  $X - D$  admits a pluri-harmonic metric  $h$  which is tame<sup>2</sup>. Given a flat bundle  $(\mathcal{E}, \nabla)$  over  $X - D$ , then there is a Deligne extension  $(\bar{\mathcal{E}}, \bar{\nabla})$ , that is,  $\bar{\nabla} : \bar{\mathcal{E}} \rightarrow \bar{\mathcal{E}} \otimes \Omega_X^1(\log D)$  is a regular flat connection such that  $(\bar{\mathcal{E}}, \bar{\nabla})|_{X-D} = (\mathcal{E}, \nabla)$ . This extension naturally defines a regular parabolic flat bundle  $(\bar{\mathcal{E}}_*, \bar{\nabla})$  (of trivial parabolic structure) with trivial parabolic degree and second parabolic Chern character, and the semistability of  $(\mathcal{E}, \nabla)$  is exactly the  $\mu_L^{\text{par}}$ -polystability of  $(\bar{\mathcal{E}}_*, \bar{\nabla})$ . On the other hand, by [Theorem 4.1.33](#), the restriction of any element in  $\text{Par}_{\text{poly}}^1$  to  $X - D$  admits a tame pluri-harmonic metric, hence semisimple by Jost–Zuo [[JZ97](#)].  $\square$

## 4.2 Parabolic Higgs Bundles as Higgs Bundles over Deligne–Mumford Stacks

### 4.2.1 Higgs Bundles over Root Stacks

In [[IS07](#)], Iyer and Simpson showed that there is an equivalence between the category of locally abelian parabolic vector bundles over a smooth projective variety with normal crossing divisor and vector bundles on the associated Deligne–Mumford stack. This is similar to that of Borne’s result [[Bor07](#)], where he used the Maruyama and Yokogawa’s definition of parabolic bundles that all the components of divisors are combined together. In this part, we will generalize their result to Higgs case.

To construct the Deligne–Mumford stack associated to a projective variety with normal crossing divisor, we need some definitions and results on root stacks. For reader’s convenience, here we introduce this theory based on [[Cad07](#)] and [[BMW13](#)].

#### Root stacks

Let  $X$  be a scheme, let  $\mathcal{L}$  be an invertible sheaf over  $X$ , and  $s \in H^0(X, \mathcal{L})$  be a global section. Let  $n$  be a fixed positive integer.

**Definition 4.2.1** ([[Cad07](#)]). The category  $X_{(\mathcal{L}, s, n)}$  fibered in groupoids is defined as the following:

- (1) the objects over a scheme  $Y$  are quadruples

$$(f, \mathcal{M}, \varphi, t),$$

where  $f : Y \rightarrow X$  is a morphism of schemes,  $\mathcal{M}$  is an invertible sheaf over  $Y$ ,  $t \in H^0(Y, \mathcal{M})$  is a section of  $\mathcal{M}$  and  $\varphi : \mathcal{M}^{\otimes n} \rightarrow f^*\mathcal{L}$  is an isomorphism of invertible sheaves such that  $\varphi(t^{\otimes n}) = f^*(s)$ ;

---

<sup>2</sup>This metric is called the *Corlette–Jost–Zuo metric* by Mochizuki, and this result is later improved by Mochizuki in [[Moc07](#)], which states that any semisimple flat bundle over  $X - D$  admits a pluri-harmonic metric which is tame and purely imaginary, here purely imaginary means the eigenvalues of the residue data of the extended Higgs field along each irreducible component of  $D$  are of purely imaginary. And in both cases, the pluri-harmonic metric is unique up to flat automorphisms of the bundle (cf. [[Moc07](#), Lemma 25.28]).

- (2) the morphism from an object  $(f, \mathcal{M}, \varphi, t)$  (over a scheme  $Y$ ) and  $(f', \mathcal{M}', \varphi', t')$  (over a scheme  $Y'$ ) is a pair  $(g, \sigma)$ , where  $g : Y \rightarrow Y'$  is a morphism of schemes with  $f' \circ g = f$  and  $\sigma : \mathcal{M} \rightarrow g^* \mathcal{M}'$  is an isomorphism of sheaves such that  $\sigma(t) = g^*(t')$  and the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}^{\otimes n} & \xrightarrow{\sigma^{\otimes n}} & g^* \mathcal{M}'^{\otimes n} \\ \phi \downarrow & & \downarrow g^* \varphi' \\ f^* \mathcal{L} & \xrightarrow{=} & g^* f'^* \mathcal{L} \end{array}$$

If  $(h, \tau)$  is a morphism from the object  $(f', \mathcal{M}', \varphi', t')$  (over a scheme  $Y'$ ) to another object  $(f'', \mathcal{M}'', \varphi'', t'')$  (over a scheme  $Y''$ ), then we define the composition of  $(g, \sigma)$  and  $(h, \tau)$  as follows:

$$(h, \tau) \circ (g, \sigma) := (h \circ g, (g^* \tau) \circ \sigma) : (f, \mathcal{M}, \varphi, t) \rightarrow (f'', \mathcal{M}'', \varphi'', t'')$$

since the following diagrams commute:

$$\begin{array}{ccccc} \mathcal{M}^{\otimes n} & \xrightarrow{\sigma^{\otimes n}} & g^* \mathcal{M}'^{\otimes n} & \xrightarrow{g^* \tau^{\otimes n}} & g^* h^* \mathcal{M}''^{\otimes n} \\ \varphi \downarrow & & \downarrow g^* \varphi'^{\otimes n} & & \downarrow g^* h^* \tau \\ f^* \mathcal{L} & \xrightarrow{=} & g^* f'^* \mathcal{L} & \xrightarrow{=} & g^* h^* f''^* \mathcal{L} \end{array}$$

For an integer  $k$ , this root stack with one denominator can be generalized to multi-denominators  $\underline{n} = (n_1, \dots, n_k)$ . Fix a multi-denominators  $\underline{n} = (n_1, \dots, n_k)$ , let  $\underline{\mathcal{L}} = (\mathcal{L}_1, \dots, \mathcal{L}_k)$  be a multi-invertible sheaves and let  $\underline{s} = (s_1, \dots, s_k)$  be a multi-sections with each  $s_i \in H^0(X, \mathcal{L}_i)$ . Then this form a root stack denoted as  $X_{(\underline{\mathcal{L}}, \underline{s}, \underline{n})}$ , called the  $\underline{n}$ -th root stack of  $(\underline{\mathcal{L}}, \underline{s})$ . In fact, the objects of  $X_{(\underline{\mathcal{L}}, \underline{s}, \underline{n})}$  (over a scheme  $Y$ ) are quadruples

$$\{(f, \mathcal{M}_i, \varphi_i, t_i)\}_{i=1}^k,$$

where  $f : Y \rightarrow X$  is a morphism of schemes,  $\mathcal{M}_1, \dots, \mathcal{M}_k$  are invertible sheaves on  $Y$ , each  $t_i \in H^0(Y, \mathcal{M}_i)$  is a section of  $\mathcal{M}_i$  and each  $\varphi_i : \mathcal{M}_i^{\otimes n_i} \rightarrow f^* \mathcal{L}_i$  is an isomorphism of invertible sheaves such that  $\varphi_i(t_i^{\otimes n_i}) = f^*(s_i)$ .

Let  $\{(f, \mathcal{M}_i, \varphi_i, t_i)\}_{i=1}^k$  (over a scheme  $Y$ ) and  $\{(f', \mathcal{M}'_i, \varphi'_i, t'_i)\}_{i=1}^k$  (over a scheme  $Y'$ ) be two objects of the  $\underline{n}$ -th root stack  $X_{(\underline{\mathcal{L}}, \underline{s}, \underline{n})}$ . A morphism between them is a pair  $\{(g, \sigma_i)\}_{i=1}^k$ , where  $g : Y \rightarrow Y'$  is a morphism with  $f' \circ g = f$ , and for each  $i$ ,  $\sigma_i : \mathcal{M}_i \rightarrow g^* \mathcal{M}'_i$  is an isomorphism of sheaves such that  $\sigma_i(t_i) = g^*(t'_i)$  and such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{M}_i^{\otimes n_i} & \xrightarrow{\sigma_i^{\otimes n_i}} & g^* \mathcal{M}'_i^{\otimes n_i} \\ \varphi_i \downarrow & & \downarrow g^* \varphi'_i \\ f^* \mathcal{L}_i & \xrightarrow{=} & g^* f'^* \mathcal{L}_i \end{array}$$

Compositions of morphisms are similar as the one denominator case that we described above.

Now let  $X$  be a connected smooth complex projective variety of  $\dim_{\mathbb{C}}(X) = n$ , let  $D = \bigcup_{i=1}^k D_i \subseteq X$  be a reduced simple normal crossing divisor with each component  $D_i$  smooth and irreducible. Let  $\underline{n} = (n_1, \dots, n_k)$  be a multi-denominators. Let  $\mathcal{L}_i := \mathcal{O}_X(D_i)$  be the line bundle associated to  $D_i$ , and  $s_{D_i}$  be the tautological section of  $\mathcal{L}_i$  that vanishes along  $D_i$ . Then we have

the  $n$ -th root stack of  $(\underline{\mathcal{L}}, \underline{s})$ , we denote it as  $Z := X[\frac{D_1}{n_1}, \dots, \frac{D_k}{n_k}]$  in accordance with [IS07, IS08]. Then  $Z$  is a Deligne–Mumford stack (cf. [Cad07], Theorem 2.3.3), let  $p : Z \rightarrow X$  be the natural projection, then  $p$  is finite and  $X$  is a coarse moduli space for  $Z$  under  $p$  [Bor07, Cad07].

### Sheaf on Root Stack

Now we introduce the sheaf on algebraic stacks, which can be found in [LMB00, G01, BMW13, Ols06].

Let  $Z$  be an algebraic stack (here we are mainly interest on the case that it is the above Deligne–Mumford stack  $X[\frac{D_1}{n_1}, \dots, \frac{D_k}{n_k}]$ ), denoted by  $\text{Sch}/Z$  the category of  $Z$ -schemes such that each object is a pair  $(U, u)$ , where  $U$  is a scheme and  $u : U \rightarrow Z$  is a morphism over  $S$ . A morphism of two objects  $(U_1, u_1)$  and  $(U_2, u_2)$  is a morphism  $f : U_1 \rightarrow U_2$  of  $Z$ -schemes such that the following diagram commutes:

$$\begin{array}{ccc} U_1 & \xrightarrow{f} & U_2 \\ & \searrow u_1 & \swarrow u_2 \\ & & Z \end{array} \quad (4.8)$$

The lisse-étale site, denoted as  $Z_{\text{lisse-ét}}$ , is a full subcategory of  $\text{Sch}/Z$  such that for each object  $(U, u)$  of  $Z_{\text{lisse-ét}}$ ,  $u : U \rightarrow Z$  is a smooth morphism and each covering is étale, here we say a covering of an object  $(U, u)$  is a collection of maps  $\{(U_i, u_i) \rightarrow (U, u)\}_{i \in I}$  such that the underlying collection of scheme-morphisms  $\{f_i : U_i \rightarrow U\}_{i \in I}$  is a covering, that is,  $U = \bigcup_{i \in I} f_i(U_i)$ . If each  $f_i$  is an étale morphism, then we say the covering is an étale covering.

**Definition 4.2.2.** The **structure sheaf**  $\mathcal{O}_Z$  on the stack  $Z$  is defined to the sheaf on the lisse-étale site  $Z_{\text{lisse-ét}}$  which to any object  $(U, u)$  one associates the sheaf  $\mathcal{O}_U$ . A *sheaf* on  $Z$  is a sheaf of  $\mathcal{O}_Z$ -modules.

Now we give an explicit description of this. A (coherent, torsion-free, locally free) sheaf  $\mathcal{F}$  on the stack  $Z$  consists of the data  $(\{\mathcal{F}_{(U,u)}\}, \{\rho_f\})$ , where:

(1) For each object  $(U, u)$  of  $Z_{\text{lisse-ét}}$ ,  $\mathcal{F}_{(U,u)}$  is a (coherent, torsion-free, locally free) étale sheaf on  $U$ .

(2)  $\rho_f : \mathcal{F}_{(U_1, u_1)} \rightarrow f^* \mathcal{F}_{(U_2, u_2)}$  is a morphism of sheaves for each morphism  $f : (U_1, u_1) \rightarrow (U_2, u_2)$  of objects of  $Z_{\text{lisse-ét}}$  and in particular,  $\rho_f$  is an isomorphism if  $f$  is étale.

This data satisfy that for any composition  $(U_1, u_1) \xrightarrow{f} (U_2, u_2) \xrightarrow{g} (U_3, u_3)$  of objects in  $Z_{\text{lisse-ét}}$ , the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_{(U_1, u_1)} & \xrightarrow{\rho_f} & f^* \mathcal{F}_{(U_2, u_2)} \\ \rho_{g \circ f} \downarrow & & \downarrow f^* \rho_g \\ (g \circ f)^* \mathcal{F}_{(U_3, u_3)} & \xrightarrow{\cong} & f^* \circ g^* \mathcal{F}_{(U_3, u_3)} \end{array}$$

that is, it satisfies the cocycle condition  $\rho_{g \circ f} = f^* \rho_g \circ \rho_f$ .

To introduce the Higgs field on the sheaf over the stack  $Z$ , we first introduce the sheaf of differentials on  $Z$  as described in [G01]. We use the étale morphism to define  $\Omega_Z$ , let  $u : U \rightarrow Z$  be an étale morphism, then we set

$$\Omega_{Z, (U, u)} := \Omega_U,$$

if  $v : V \rightarrow Z$  is another étale morphism such that the diagram (4.8) commutes, then  $f : U \rightarrow V$  has to be étale and we have a canonical isomorphism  $\rho_f : \Omega_U \rightarrow f^*\Omega_V$ , which satisfies the cocycle condition as above.

This can be extended to any morphism  $Y \rightarrow Z$  where  $Y$  is a scheme by taking an étale atlas  $U = \coprod U_i \rightarrow Z$ , define

$$\Omega_{Z,(X \times_Z U, u \circ p_2)} := p_2^* \Omega_U$$

by the morphism  $X \times_Z U \xrightarrow{p_2} U \xrightarrow{u} Z$ , then  $\Omega_{Z,(X \times_Z U, u \circ p_2)}$  descends to give a sheaf  $\Omega_{Z,X}$  on  $X$  and this does not depend on the choice of étale atlas  $U$ , and the induced isomorphism satisfies the cocycle condition.

**Definition 4.2.3.** A *Higgs bundles* on the stack  $Z$  consists of a locally free coherent sheaf  $\mathcal{F}$  on  $Z$  together with a homomorphism  $\theta : \mathcal{F} \rightarrow \mathcal{F} \otimes \Omega_Z^1$  such that  $\theta \wedge \theta = 0$ , here the homomorphism means to each étale morphism  $u : U \rightarrow Z$  one assigns a homomorphism  $\theta_{(U,u)} : \mathcal{F}_{(U,u)} \rightarrow \mathcal{F}_{(U,u)} \otimes \Omega_{Z,(U,u)}^1 = \mathcal{F}_{(U,u)} \otimes \Omega_U^1$  such that for each commutative diagram (4.8), the following diagram commutes:

$$\begin{array}{ccc} \mathcal{F}_{(U,u)} & \xrightarrow{\theta_{(U,u)}} & \mathcal{F}_{(U,u)} \otimes \Omega_U^1 \\ \rho_f^{\mathcal{F}} \downarrow & & \downarrow \rho_f^{\mathcal{F}} \otimes \rho_f^{\Omega_Z^1} \\ f^* \mathcal{F}_{(V,v)} & \xrightarrow{f^* \theta_{(V,v)}} & f^* \mathcal{F}_{(V,v)} \otimes f^* \Omega_V^1 \end{array}$$

here we use the notation  $\rho_f^{\mathcal{F}}$  to denote the isomorphism is associated to the sheaf  $\mathcal{F}$ .

Now we introduce the stability of Higgs bundles over Deligne–Mumford stacks. In [Nir08], Nironi constructed the moduli space of semistable sheaves on projective Deligne–Mumford stacks by introducing the modified Hilbert polynomials with an extra term in the classical Hilbert polynomial. This extra term is a locally free sheaf  $\mathcal{E}$  on  $Z$  such that for every geometric point  $x \rightarrow Z$ , the representation of the stabilizer group  $\text{Stab}(x)$  contains every irreducible representation, this is called a *generating sheaf*. There is a standard normal crossing divisor  $D' = \sum_{i=1}^k D'_i$  on  $Z$  with  $n_i \cdot D'_i = p^*(D_i)$  for each  $i$  [Sim92]. In our case, the generating sheaf is the locally free sheaf

$$\mathcal{E} = \bigoplus_{i=1}^k \bigoplus_{j=1}^{n_i} \mathcal{O}_Z(jD'_j),$$

the modified Hilbert polynomial for a coherent sheaf  $\mathcal{F}$  pure of dimension  $d$  is defined as

$$P_{\mathcal{E}}(\mathcal{F}(m)) := \chi(\mathcal{F} \otimes \mathcal{E}^* \otimes p^* \mathcal{O}_X(m)) = \sum_{i=0}^d a_{\mathcal{E},i}(\mathcal{F}) \frac{m^i}{i!},$$

where  $\mathcal{O}_X(1)$  a very ample invertible sheaf on  $X$ ,  $\mathcal{E}$  is the generating sheaf, the pair  $(\mathcal{O}_X(1), \mathcal{E})$  is called a *polarization* of the stack  $Z$  ([Nir08]). The reduced Hilbert polynomial and the slope are given as

$$p_{\mathcal{E}}(\mathcal{F}) := \frac{P_{\mathcal{E}}(\mathcal{F})}{a_{\mathcal{E},d}(\mathcal{F})}$$

and

$$\mu_{\mathcal{E}}(\mathcal{F}) = \frac{a_{\mathcal{E},d-1}(\mathcal{F})}{a_{\mathcal{E},d}(\mathcal{F})},$$

respectively.

**Definition 4.2.4.** A Higgs bundle  $(F, \theta)$  on  $Z$  is called *stable* (resp. *semistable*) if for any saturated coherent subsheaf  $\mathcal{G}$  of  $F$  that is  $\theta$ -invariant, we have

$$p_{\mathcal{E}}(\mathcal{G}) < \text{ (resp. } \leq) p_{\mathcal{E}}(F).$$

It is called *slope stable* (resp. *slope semistable*) if

$$\mu_{\mathcal{E}}(\mathcal{G}) < \text{ (resp. } \leq) \mu_{\mathcal{E}}(F).$$

### 4.2.2 Correspondence

Let  $X$  be a projective variety and  $D = \sum_{i=1}^k D_i$  be a normal crossing divisor. Suppose all parabolic bundles over  $(X, D)$  have rational weights, and such that the weights have  $n_1, \dots, n_k$  as their denominators. Then there is a Deligne–Mumford stack denoted as  $Z = X[\frac{D_1}{n_1}, \dots, \frac{D_k}{n_k}]$  such that the natural map  $p : Z \rightarrow X$  is finite and  $X$  is the coarse moduli space.

**Theorem 4.2.5.** *There is an equivalence between the category of locally abelian parabolic Higgs bundles on  $(X, D)$  whose parabolic weights have denominators  $n_1, \dots, n_k$  and the category of Higgs bundles on  $Z$ . This equivalence preserves tensor products, direct sums and duals. Moreover, with the chosen generating sheaf, this equivalence preserves stability.*

*Proof.* The first statement is a generalization of Lemma 2.3 in [IS07] to Higgs case. This conclusion is similar as Theorem 4.7 in [BMW13], but they use Maruyam–Yokogawa’s definition of parabolic bundles, with all divisor components are combined together.

Let  $(\mathcal{F}, \theta)$  be a Higgs bundle on  $Z$ , then  $\mathcal{E} = p_*\mathcal{F}$  is a coherent sheaf on  $X$ , which is locally free [IS07]. The associated parabolic bundle  $\mathcal{E}_*$  on  $(X, D)$  can be defined by

$$\mathcal{E}_\alpha := p_*(\mathcal{F} \otimes \mathcal{O}_Z(\sum_i \alpha'_i D'_i)),$$

where each  $\alpha'_i$  is the biggest rational number no bigger than  $\alpha_i$  with its denominator dividing  $n_i$ . We define  $\Phi := p_*\theta$ . We need to show this  $\Phi$  is a parabolic Higgs field. Indeed,

$$p_*(\mathcal{F} \otimes \Omega_Z^1) \subset p_*(\mathcal{F} \otimes \Omega_Z^1(\log D')) = p_*(\mathcal{F} \otimes p^*\Omega_X^1(\log D)) = p_*\mathcal{F} \otimes \Omega_X^1(\log D),$$

the last equality holds by projection formula since  $p$  is finite. Therefore,  $\Phi$  defines a homomorphism  $\Phi : \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega_X^1(\log D)$ , and  $\Phi \wedge \Phi = 0$  since  $\theta$  has this property. Moreover, for each multi-index  $\alpha$ ,  $\Phi(\mathcal{E}_\alpha) \subset \mathcal{E}_\alpha \otimes \Omega_X^1(\log D)$ . What we left to show is such constructed parabolic Higgs bundle  $(\mathcal{E}_*, \Phi)$  is locally abelian, the proof is same as the proof of Lemma 2.3 in [IS07].  $\square$

In [IS07], Iyer and Simpson showed that the pullback of a locally abelian parabolic bundle is also a locally abelian bundle. This is also right for locally abelian parabolic Higgs bundles.

**Corollary 4.2.6.** *Let  $f : (X', D') \rightarrow (X, D)$  be a morphism of smooth varieties with normal crossing divisors such that  $f^{-1}(D) \subset D'$ . Then the pullback  $f^*(\mathcal{E}_*, \Phi) := (f^*\mathcal{E}_*, f^*\Phi)$  of a locally abelian parabolic Higgs bundle  $(\mathcal{E}_*, \Phi)$  on  $(X, D)$  is also a locally abelian parabolic Higgs bundle on  $(X', D')$ .*

*Proof.* This is a direct result of Lemma 2.6 in [IS07] and the above correspondence.  $\square$



## Chapter 5

# Stability and Indecomposability of Representations of Quivers

A quiver is in fact a finite oriented graph that consists of finite vertices and arrows connect these vertices. A representation of a quiver is obtained by viewing each vertex as a finite dimensional vector space and each arrow as a linear map between two vector spaces. The theory of quiver representations is an important area that relates with many fields like geometric invariant theory, cohomology theory, and Lie theory. One of the important problems in quiver representation theory is studying the classification of finite dimensional representations of a given quiver up to isomorphisms, this problem is very hard for general quivers, complete classifications are known for some special quivers, for example, quivers of Dynkin type.

On the other hand, by introducing weight systems, we naturally have the notion of stability of representations, a moduli related problem comes from this. We can construct the moduli space via Mumford's geometric invariant theory, and study the geometric properties of moduli spaces.

A non-Abelian Hodge theory related problem is studying the correspondence between stability and indecomposability of representations of special quivers. In general, stability implies indecomposability, but the other direction is usually not true.

In this chapter, we will introduce a study of this type problem based on the paper [HH20b]. More precisely, this is based on a conjecture proposed by Markus Reineke, that predicts for representations of quivers of Dynkin type, the correspondence is essentially true.

Our main result is the following:

**Theorem 5.0.1** (= [Theorem 5.3.2](#)). *If  $Q$  is a quiver of type  $A_n$ , then there exists a weight system  $\Theta = (\theta_i)_{i \in Q_0} \in \mathbb{Z}^{|Q_0|}$  such that the stable representations with respect to the weight function  $w(X) = \sum_{i \in Q_0} \theta_i \dim X_i$  and rank function  $r(X) = \sum_{i \in Q_0} \dim X_i$  are precisely the indecomposables, namely  $\mathbf{Rep}_k(Q)$  is a maximal stable category.*

### 5.1 Motivation

In his remarkable paper [Rie03] published on Invent. Math. in 2003, M. Reineke proposed the following stability-indecomposability correspondence conjecture:

**Conjecture 5.1.1** (Reineke, [Rie03]). *If  $Q$  is a quiver of Dynkin type, there exists a weight system  $\Theta$  on  $Q$  such that the stable representations are precisely the indecomposable ones.*

If this conjecture is true, it will have some valuable applications. For example, it can be used to study the stratification of representation varieties of Dynkin quiver [Rie03], and to study identities

between products of quantum dilogarithm series associated with Dynkin quivers [Kel11]. In this chapter, we will give an elementary proof of Reineke's conjecture for the quivers of  $A_n$ -type by combinatorial construction of a special weight system<sup>1</sup>.

On the other hand, Juteau [Jut17] has found some counterexamples for Reineke's conjecture in the quivers of  $D$ - and  $E$ -type by computer program (unpublished). Therefore, a modified version of original Reineke's conjecture is proposed as follows<sup>2</sup>.

**Conjecture 5.1.2** (= Conjecture 5.3.1). *Let  $Q$  be a Dynkin quiver, then the abelian category  $\text{Rep}_k(Q)$  is a maximal stable category.*

A natural extension is to consider the Reineke-type conjecture for certain triangulated categories with Bridgeland stability conditions<sup>3</sup>.

## 5.2 Quivers and Their Representations

In this section, we collect some basic materials of quiver theory (for more details, see [Jr.16]). Throughout the paper,  $k$  is assumed to be a fixed algebraically closed field.

**Definition 5.2.1.** (1) A *quiver*  $Q = (Q_0, Q_1, s, t)$  is a 4-tuple, where

- $Q_0$  and  $Q_1$  are finite sets of *vertices* and *arrows* respectively,
- $s, t : Q_1 \rightarrow Q_0$  map each arrow  $a \in Q_1$  to its *starting vertex*  $s(a)$  and *terminal vertex*  $t(a)$ , this is denoted as

$$s(a) \xrightarrow{a} t(a).$$

(2) Let  $Q = (Q_0, Q_1)$  be a quiver,  $Q' = (Q'_0, Q'_1)$  is called a *subquiver* if

- $Q'_0 \subset Q_0$  and  $Q'_1 \subset Q_1$ ,
- $s(a), t(a) \in Q'_0$  for all  $a \in Q'_1$ .

In particular, a subquiver  $Q'$  is called a *full subquiver* if furthermore,  $a \in Q'_1$  for all  $a \in Q_1$  satisfying  $s(a) \in Q'_0$ . For example, for the following quiver

$$\bullet \xrightarrow{a_1} \bullet \xrightarrow{a_2} \bullet \xrightarrow{a_3} \bullet,$$

- $\bullet \xrightarrow{a_3} \bullet$  is a subquiver, but not a full subquiver, while  $\bullet \xrightarrow{a_2} \bullet \xrightarrow{a_3} \bullet$  is a full subquiver.

(3) Let  $Q = (Q_0, Q_1)$  and  $Q' = (Q'_0, Q'_1)$  be two quivers, the *intersection* and *union* of these two quivers are defined as

$$Q \cap Q' := (Q_0 \cap Q'_0, Q_1 \cap Q'_1), \quad Q \cup Q' := (Q_0 \cup Q'_0, Q_1 \cup Q'_1),$$

<sup>1</sup>From the communications with Prof. Reineke, Prof. Hille and Prof. Juteau, we know that Hille and Juteau have a proof for this conjecture in  $A_n$ -case, however it is unpublished. Meanwhile, from the report of the referee, we know that in his paper [Kel11], Keller suggests to resolve this conjecture by the method of [IT09]. Moreover, recently we know from the paper [Kin20] that the authors of [AI19] have an independently proof of Reineke conjecture for quivers of  $A_n$ -type. The the paper [Kin20] also considers this problem from another viewpoint, and from the Remark 1.5 of it, the authors have the method to show counterexamples to the whole Reineke conjecture only exist for quivers of  $D_n$ -type ( $n \geq 9$ ),  $E_7$  and  $E_8$ -type, this means they have a complete solution to Reineke conjecture.

<sup>2</sup>We do not know whether such modified version is true for the Dynkin quivers (even for tame quivers), but so far as we know, Juteau and Hille [Jut17, Hil17] are attempting to prove it.

<sup>3</sup>Very recently, we note that the authors of [KOT19] prove that for the derived category  $\mathcal{D}^b(Q)$  of a Dynkin quiver  $Q$ , there exists a Bridgeland stability condition  $\sigma$  such that for an object  $E \in \mathcal{D}^b(Q)$  the following are equivalent: i)  $E$  is indecomposable, ii)  $E$  is exceptional, iii)  $E$  is  $\sigma$ -stable.

respectively.

- (4) A *path*  $\gamma$  of a quiver  $Q$  is a sequence  $a_1 \cdots a_n$  ( $n \geq 1$ ) of arrows that satisfies  $s(a_{i+1}) = t(a_i)$  for  $1 \leq i \leq n-1$ , and the starting vertex of  $a_1$  and terminal vertex of  $a_n$  are called *starting vertex* and *terminal vertex* of  $\gamma$ , respectively. We denote this path as

$$\gamma : \bullet \xrightarrow{a_1} \bullet \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} \bullet.$$

The *length* of a path  $\gamma$  is defined by the number of arrows in  $\gamma$ , we denote it as  $|\gamma|$ . For each vertex  $i \in Q_0$ , we denote  $e_i$  the *trivial path* that starts and terminates at  $i$ , i.e.,  $s(i) = t(i)$ . Clearly, each trivial path has length 0.

**Remark 5.2.2.** Clearly any subquiver of a quiver, intersections and unions of quivers are all quivers.

**Definition 5.2.3.** (1) Let  $Q = (Q_0, Q_1)$  be a quiver, a *representation* of  $Q$  over  $k$  (later we call it a  $k$ -representation, or just a representation for simplicity) is a pair  $X = \{(X_i)_{i \in Q_0}, (X_a)_{a \in Q_1}\}$ , where

- $(X_i)_{i \in Q_0}$  is a family of finite dimensional  $k$ -vector spaces associated to all vertices  $i \in Q_0$ ;
- $(X_a)_{a \in Q_1}$  a family of  $k$ -linear maps associated to all arrows  $a \in Q_1$ , i.e.,

$$X_a : X_{s(a)} \longrightarrow X_{t(a)}.$$

Define  $d = (d_i)_{i \in Q_0} \in \mathbb{Z}^{|Q_0|}$  for  $d_i = \dim_k X_i$ , and call it the *dimension vector* of the representation  $X$ .

- (2) Let  $X, Y$  be two  $k$ -representations of  $Q$ , a *morphism*  $u : X \rightarrow Y$  is a collection of linear maps  $u_i : X_i \rightarrow Y_i$  for all  $i \in Q_0$  such that for each arrow  $a \in Q_1$ , the following diagram commutes:

$$\begin{array}{ccc} X_{s(a)} & \xrightarrow{X_a} & X_{t(a)} \\ u_{s(a)} \downarrow & & \downarrow u_{t(a)} \\ Y_{s(a)} & \xrightarrow{Y_a} & Y_{t(a)}. \end{array}$$

We say the morphism  $u$  is an *isomorphism* if moreover each  $u_i$  is an isomorphism, and denote it as  $X \cong Y$ . We denote by  $\mathbf{Rep}_k(Q)$  the category of  $k$ -representations of  $Q$ .

- (3) Let  $u : X \rightarrow Y$  and  $v : Y \rightarrow Z$  be two morphisms of  $k$ -representations of  $Q$ , we define their *composition*  $v \circ u : X \rightarrow Z$  by taking  $(v \circ u)_i = v_i \circ u_i : X_i \xrightarrow{u_i} Y_i \xrightarrow{v_i} Z_i$  for each  $i \in Q_0$ . More precisely, if we take  $X = \{(X_i)_{i \in Q_0}, (X_a)_{a \in Q_1}\}$ ,  $Y = \{(Y_i)_{i \in Q_0}, (Y_a)_{a \in Q_1}\}$  and  $Z = \{(Z_i)_{i \in Q_0}, (Z_a)_{a \in Q_1}\}$ , then the following diagram commutes for each arrow  $a \in Q_1$ :

$$\begin{array}{ccc} X_{s(a)} & \xrightarrow{X_a} & X_{t(a)} \\ u_{s(a)} \downarrow & & \downarrow u_{t(a)} \\ Y_{s(a)} & \xrightarrow{Y_a} & Y_{t(a)} \\ v_{s(a)} \downarrow & & \downarrow v_{t(a)} \\ Z_{s(a)} & \xrightarrow{Z_a} & Z_{t(a)} \end{array}$$

(4) The *direct sum*  $W = X \oplus Y$  of two  $k$ -representations  $X$  and  $Y$  of  $Q$  is defined by the pair

$$W = \{(W_i)_{i \in Q_0}, (W_a)_{a \in Q_1}\} = \{(X_i \oplus Y_i)_{i \in Q_0}, (X_a \oplus Y_a)_{a \in Q_1}\},$$

where each linear map is given by

$$W_a = X_a \oplus Y_a : X_{s(a)} \oplus Y_{s(a)} \rightarrow X_{t(a)} \oplus Y_{t(a)}.$$

A  $k$ -representation  $W$  of  $Q$  is said to be *decomposable* if there exist non-zero  $k$ -representations  $X$  and  $Y$  such that  $W \cong X \oplus Y$ , otherwise it is said to be *indecomposable*.

- (5) Let  $X$  and  $Y$  be two  $k$ -representations of  $Q$ .  $X$  is said to be a *subrepresentation*  $Y$  if  $X_i \subseteq Y_i$  for all  $i \in Q_0$  and  $X_a = Y_a|_{X_{s(a)}} : X_{s(a)} \rightarrow X_{t(a)}$  for all  $a \in Q_1$ .
- (6) A  $k$ -representation of  $Q$  is called *simple* if it has no proper non-zero subrepresentations, and called *semisimple* if it is the direct sum of simple representations.
- (7) We say a  $k$ -representation  $X$  of  $Q$  is *thin* if  $\dim_k(X_i) \leq 1$  for all  $i \in Q_0$ , that is, if each linear space  $X_i$  is either 0 or  $k$ .

Giving a quiver  $Q$ , an important aim of quiver representation theory is to classify all  $k$ -representations of  $Q$  and all morphisms of  $k$ -representations up to isomorphism. An important theorem of Krull and Schmidt makes this classification problem easier, it states that every  $k$ -representation of a given quiver can be uniquely decomposed into the direct sum of indecomposable  $k$ -representations up to the ordering of indices.

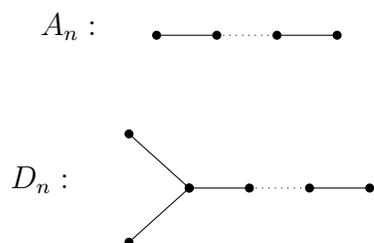
**Theorem 5.2.4** (Krull–Schmidt Classification Theorem). *Let  $X$  be a representation of a given quiver  $Q$ , then there is an isomorphism*

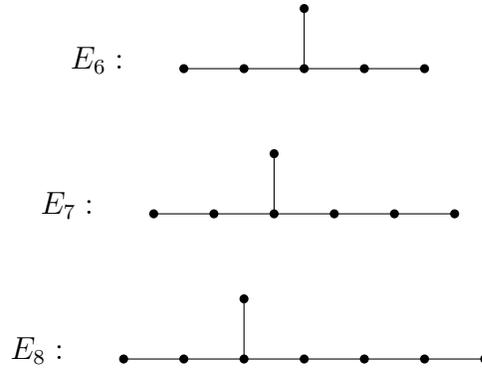
$$X \cong X_1 \oplus \cdots \oplus X_s,$$

where each  $X_i$  is indecomposable representation of  $Q$  and the decomposition is unique up to ordering, more precisely, if  $X \cong X_1 \oplus \cdots \oplus X_s$  and  $X \cong Y_1 \oplus \cdots \oplus Y_r$ , then  $s = r$  and  $X_i = Y_{f(i)}$  for each  $i \in \{1, \dots, s\}$ , where  $f : \{1, \dots, s\} \rightarrow \{1, \dots, r\}$  is a bijection.

A quiver  $Q$  is said to be of *finite type* if it has finitely many isomorphism classes of indecomposable representations. Gabriel’s classification theorem states that for a connected quiver  $Q$  without oriented cycles, the following are equivalent

- $Q$  is of finite type,
- the underlying graph of  $Q$  is a simply laced Dynkin diagram, namely one of the followings



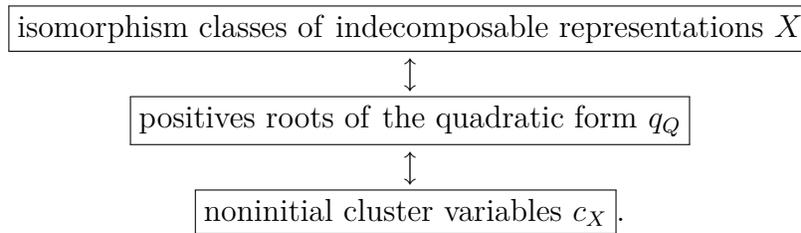


- the quadratic form

$$q_Q(\alpha) = \sum_{i \in Q_0} \alpha_i^2 - \sum_{a \in Q_1} \alpha_{s(a)} \alpha_{t(a)}$$

is positive definite, where  $\alpha \in \mathbb{Z}^{|Q_0|}$ .

Moreover one has the following bijective correspondence:



For a quiver  $Q$ , we express the underlying graph  $\Gamma_Q$  as a binary set  $\Gamma_Q =: \{(Q_0, Q'_1)\}$  of vertices and edges, where  $Q'_1$  is obtained by taking all arrows in  $Q_1$  as edges. For any edge  $l \in Q'_1$ , it corresponds to a unique arrow  $a \in Q_1$ , we denote  $s(l) = s(a)$  and  $t(l) = t(a)$  the starting vertex and terminal vertex of  $l$ , respectively.

**Definition 5.2.5.** The *support* of  $X$  is a subset of  $\Gamma_Q$  consisting of all vertices  $i$  with the assigning linear space  $X_i \neq 0$  and all edges connecting these vertices, that is:

$$\text{supp}(X) := \{(\tilde{Q}_0, \tilde{Q}_1) \mid i \in \tilde{Q}_0 \text{ if } X_i \neq 0; l \in \tilde{Q}_1 \text{ if } s(l), t(l) \in \tilde{Q}_0\} \subset \Gamma_Q.$$

The *support quiver*  $\text{supp}_X(Q)$  is given by recovering the arrows of all edges in  $\text{supp}(X)$ , that is, given by the vertices  $i$  with  $X_i \neq 0$  and the arrows  $a$  with  $X_a \neq 0$ .

The following facts are very obvious.

**Fact.** Let  $Q$  be a quiver, then

- (1) If  $X$  is an indecomposable  $k$ -representation of  $Q$ , then its support quiver  $\text{supp}_X(Q)$  is connected.
- (2) If two  $k$ -representations  $X$  and  $Y$  are isomorphic, then  $\text{supp}_X(Q) = \text{supp}_Y(Q)$ .
- (3) Let  $X$  be a thin representation of  $Q$ , then  $X$  is indecomposable if and only if the support quiver  $\text{supp}_X(Q)$  is connected.

Now we introduce the stability conditions for representations of quivers, our definition here is more abstract than that in many contexts of quiver representation theory, this is for our convenience of introducing the modified Reineke conjecture in the next section.

**Definition 5.2.6** ([DW11]). (1) Let  $\mathcal{A}$  be a category, and  $w, r$  be two functions on  $\mathcal{A}$ , called *weight function* and *rank function* respectively, such that  $r(X) \neq 0$  for any nonzero object  $X$  in  $\mathcal{A}$ . A object  $X \in \mathcal{A}$  is called  $(w, r)$ -*stable* (resp.  $(w, r)$ -*semistable*) if for any nonzero subobject  $U$  of  $X$ , we have

$$\mu(U) < \mu(X) \text{ (resp. } \mu(U) \leq \mu(X)),$$

where the slope function  $\mu(X)$  is defined by  $\mu(X) = \frac{w(X)}{r(X)}$ .

(2) Let  $w, r$  be weight function and rank function on a category  $\mathcal{A}$  respectively, all  $(w, r)$ -stable (resp.  $(w, r)$ -semistable) objects of  $\mathcal{A}$  form a full subcategory of  $\mathcal{A}$ , called  $(w, r)$ -stable (resp.  $(w, r)$ -semistable) subcategory. Two pairs  $(w, r)$  and  $(w', r')$  is called *stable-equivalent* (resp. *semistable-equivalent*) if they induce the same stable (resp. semistable) subcategories, we denote this as

$$(w, r) \sim_s (w', r') \text{ (resp. } (w, r) \sim_{ss} (w', r')).$$

(3) Let  $\mathcal{A}$  be an abelian category, if there exist an **additive** weight function  $w$  and a rank function  $r$  on  $\mathcal{A}$  such that the  $(w, r)$ -stable subcategory consists of all indecomposable objects, then we call  $\mathcal{A}$  a *maximal stable category*.

### 5.3 Reineke's Conjecture for Quivers of $A_n$ -Type

After introducing the abstract stability conditions for representations of quiver in last section, we can modify Reineke conjecture into the following form:

**Conjecture 5.3.1 (Modified Reineke's conjecture).** *Let  $Q$  be a Dynkin quiver, then the abelian category  $\mathbf{Rep}_k(Q)$  is a maximal stable category.*

In this section, we will confirm above conjecture for quivers of type  $A_n$ . Namely, we will prove the following main theorem:

**Theorem 5.3.2.** *If  $Q$  is a quiver of type  $A_n$ , then there exists a weight system  $\Theta = (\theta_i)_{i \in Q_0} \in \mathbb{Z}^{|Q_0|}$  such that the stable representations with respect to the weight function  $w(X) = \sum_{i \in Q_0} \theta_i \dim X_i$  and rank function  $r(X) = \sum_{i \in Q_0} \dim X_i$  are precisely the indecomposables, namely  $\mathbf{Rep}_k(Q)$  is a maximal stable category.*

#### 5.3.1 Intrinsic Weight System

Let  $Q$  be a quiver of type  $A_n$ , we put it horizontally and fix a reference direction from left to right so that assign numbers  $1, \dots, n$  to the vertices of  $Q$  along the reference direction, then we can classify all vertices into the following four types according to the directions of arrows attached to them:

- a vertex  $i \in Q_0$  is called of type I if it is only as the starting vertex of arrows linking to it;
- a vertex  $i \in Q_0$  is called of type II if it is only as the terminal vertex of arrows linking to it;
- a vertex  $i \in Q_0$  is called of type III if there is a path goes through  $i$  with the reference direction such that  $i$  is a vertex but neither a starting nor a terminal one of that path;

• a vertex  $i \in Q_0$  is called of type IV if there is a path goes through  $i$  with the direction opposite to the reference direction such that  $i$  is a vertex but neither a starting nor a terminal one of that path.

Now we define a weight system  $\Theta = \{\theta_i\}_{i \in Q_0}$  according to the type of each vertex as follows:

$$\theta_i = \begin{cases} l_i + r_i + 2l_i r_i, & i \text{ is a vertex of type I;} \\ -l_i - r_i - 2l_i r_i, & i \text{ is a vertex of type II;} \\ r_i - l_i, & i \text{ is a vertex of type III;} \\ l_i - r_i, & i \text{ is a vertex of type IV} \end{cases} \quad (5.1)$$

where  $l_i$  and  $r_i$  stands for the number of vertices on the left and right of the vertex  $i$ , respectively. Such weight system is called the *intrinsic weight system*, and the corresponding weight function  $w_\Theta(X) = \sum_{i \in Q_0} \theta_i \dim X_i$  for a representation  $X$  of  $Q$  is called the *intrinsic weight function*.

**Example 5.3.3.** The following quiver of  $A_n$ -type

$$\bullet_1 \longrightarrow \bullet_2 \longrightarrow \bullet_3 \longrightarrow \bullet_4 \longleftarrow \bullet_5 \longleftarrow \bullet_6 \longrightarrow \bullet_7, \quad (5.2)$$

has four types of vertices:

$$\begin{cases} I : & 1, 6, \\ II : & 4, 7, \\ III : & 2, 3, \\ IV : & 5, \end{cases}$$

Then the intrinsic weight system  $\Theta$  is given by  $\theta_1 = 6$ ,  $\theta_2 = 4$ ,  $\theta_3 = 2$ ,  $\theta_4 = -24$ ,  $\theta_5 = 2$ ,  $\theta_6 = 16$ ,  $\theta_7 = -6$ .

**Lemma 5.3.4.** *Let  $Q = \{Q_0, Q_1\}$  be a quiver of  $A_n$ -type with the intrinsic weight system  $\Theta = \{\theta_i\}_{i \in Q_0}$ . Then*

1. *along any path, the weights at the vertices contained in the path decrease.*
2. *the sum  $\sum_{i \in Q_0} \theta_i$  of weights at all vertices is exactly zero.*

*Proof.* (1) We just need to show along each arrow  $a \in Q_1$ , the weight decreases, i.e.,  $\theta_{s(a)} > \theta_{t(a)}$  for each arrow  $a \in Q_1$ . We can draw the quiver  $Q$  as follows:

$$Q : \quad \cdots \bullet_{s(a)} \xrightarrow{a} \bullet_{t(a)} \cdots,$$

if we consider the two vertices and arrows closed to  $\bullet_{s(a)} \xrightarrow{a} \bullet_{t(a)}$ , then we have the following four cases:

(i)  $Q : \quad \cdots \longrightarrow \bullet_{s(a)} \xrightarrow{a} \bullet_{t(a)} \longleftarrow \cdots$ , in this case, we have

$$\theta_{s(a)} = r_{s(a)} - l_{s(a)}, \quad \theta_{t(a)} = -l_{t(a)} - r_{t(a)},$$

(ii)  $Q : \quad \cdots \longleftarrow \bullet_{s(a)} \xrightarrow{a} \bullet_{t(a)} \longrightarrow \cdots$ , in this case, we have

$$\theta_{s(a)} = l_{s(a)} + r_{s(a)} + 2l_{s(a)} \cdot r_{s(a)}, \quad \theta_{t(a)} = r_{t(a)} - l_{t(a)},$$

(iii)  $Q : \cdots \longrightarrow \bullet_{s(a)} \xrightarrow{a} \bullet_{t(a)} \longrightarrow \cdots$ , in this case, we have

$$\theta_{s(a)} = r_{s(a)} - l_{s(a)}, \quad \theta_{t(a)} = r_{t(a)} - l_{t(a)},$$

(iv)  $Q : \cdots \longleftarrow \bullet_{s(a)} \xrightarrow{a} \bullet_{t(a)} \longleftarrow \cdots$ , in this case, we have

$$\theta_{s(a)} = l_{s(a)} + r_{s(a)} + 2l_{s(a)} \cdot r_{s(a)}, \quad \theta_{t(a)} = -l_{t(a)} - r_{t(a)} - 2l_{t(a)} \cdot r_{t(a)},$$

Since  $r_{s(a)} > r_{t(a)} \geq 0$  and  $0 \leq l_{s(a)} < l_{t(a)}$ , in each case, we have  $\theta_{s(a)} > \theta_{t(a)}$ .

(2) For any subquiver  $Q' = \{Q'_0, Q'_1\}$  of  $Q$ , we give each vertex  $i \in Q'_0$  a new weight:

$$\theta_i^{Q'} = \#\{a \mid s(a) = i, a \in Q'_1\} - \#\{a \mid t(a) = i, a \in Q'_1\} \in \{\pm 2, 0\},$$

the difference of numbers of arrows in  $Q'_1$  starting at  $i$  and numbers of arrows in  $Q'_1$  terminating at  $i$ , this construction immediately gives

$$\sum_{i \in Q'_0} \theta_i^{Q'} = 0.$$

Then for each vertex  $i \in Q_0$ , its weight  $\theta_i$  is the sum of all such new weights  $\theta_i^{Q'}$  for the connected subquiver  $Q'$  contains  $i$ :

$$\theta_i = \sum_{\substack{Q' \subset Q \text{ connected subquiver} \\ i \in Q'_0}} \theta_i^{Q'}.$$

If  $i$  is of type I, then near the vertex  $i$ , the quiver locally looks like  $\longleftarrow \bullet_i \longrightarrow$ , all subquivers contain  $i$  can be divided into three classes:

- (i)  $\cdots \longleftarrow \bullet_i$ ,
- (ii)  $\bullet_i \longrightarrow \cdots$ ,
- (iii)  $\cdots \longleftarrow \bullet_i \longrightarrow \cdots$ .

The first class has  $l_i$  subquivers, for each subquiver  $Q'$ , we have  $\theta_i^{Q'} = 1$ ; the second class has  $r_i$  subquivers, for each subquiver  $Q'$ , we have  $\theta_i^{Q'} = 1$ ; the third class has  $l_i r_i$  subquivers, for each subquiver  $Q'$ , we have  $\theta_i^{Q'} = 2$ . Consequently,

$$\sum_{\substack{Q' \subset Q \text{ connected subquiver} \\ i \in Q'_0}} \theta_i^{Q'} = l_i + r_i + 2l_i r_i = \theta_i.$$

If  $i$  is of type III, then near the vertex  $i$ , the quiver locally looks like  $\longrightarrow \bullet_i \longrightarrow$ , all subquivers contain  $i$  can be divided into three classes:

- (i)  $\cdots \longrightarrow \bullet_i$ ,
- (ii)  $\bullet_i \longrightarrow \cdots$ ,
- (iii)  $\cdots \longrightarrow \bullet_i \longrightarrow \cdots$ .

The first class has  $l_i$  subquivers, for each subquiver  $Q'$ , we have  $\theta_i^{Q'} = -1$ ; the second class has  $r_i$  subquivers, for each subquiver  $Q'$ , we have  $\theta_i^{Q'} = 1$ ; the third class has  $l_i r_i$  subquivers, for each

subquiver  $Q'$ , we have  $\theta_i^{Q'} = 0$ . Hence,

$$\sum_{\substack{Q' \subset Q \text{ connected subquiver} \\ i \in Q'_0}} \theta_i^{Q'} = r_i - l_i = \theta_i.$$

The cases of type II and IV are similar.

Therefore, the sum of all weights is calculated as

$$\sum_{i \in Q_0} \theta_i = \sum_{i \in Q_0} \left( \sum_{\substack{Q' \subset Q \text{ connected subquiver} \\ i \in Q'_0}} \theta_i^{Q'} \right) = \sum_{\substack{Q' \subset Q \\ \text{connected subquiver}}} \left( \sum_{i \in Q'_0} \theta_i^{Q'} \right) = 0.$$

We complete the proof. □

Denoted by  $I_{p,q}$  the following indecomposable thin representation (the orientation for the graph is just took as an example)

$$0 \longrightarrow \cdots \longrightarrow k_p \xrightarrow{1} \cdots \xrightarrow{1} k_q \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0,$$

where  $1 \leq p \leq q \leq n$ . Then the indecomposable representations of  $Q$  are classified by  $I_{p,q}$ 's, more precisely, we have

**Proposition 5.3.5** ([KMS06, LM98]). *Let  $X$  be a representation of a quiver  $Q$  of type  $A_n$ . Then  $X$  is indecomposable if and only if  $X$  is a thin representation whose support quiver is connected, that is,  $X$  is isomorphic to some  $I_{p,q}$ .*

**Lemma 5.3.6.** *Let  $Q$  be a quiver of  $A_n$ -type, and  $X$  is the indecomposable representation of type  $I_{1,n}$ , then  $X$  is stable with respect to the intrinsic weight function and rank function.*

*Proof.* To show the stability, we only need to prove the intrinsic weight function  $w_\Theta(X')$  on any proper subrepresentation  $X'$  of  $I_{1,n}$  is negative.

Obviously, the support quiver  $Q' = \text{supp}_{X'}(Q)$  is a proper full subquiver of  $Q$ . We first assume  $Q'$  is connected, then  $Q$  must look like as follows:

$$Q : \quad \cdots \bullet \longrightarrow Q' \longleftarrow \bullet \cdots ,$$

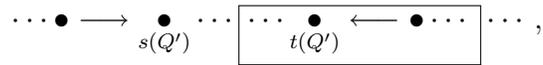
and denoted by  $s(Q')$  and  $t(Q')$  the two vertices at the boundary of  $Q'$ , one draws  $Q$  as follows:

$$Q : \quad \cdots \bullet \longrightarrow \underset{s(Q')}{\bullet} \cdots \underset{t(Q')}{\bullet} \longleftarrow \bullet \cdots .$$

Let  $l_{Q'}$  and  $r_{Q'}$  be the number of vertices on the left and right hand side of the whole  $Q'$ , respectively.

To compute the weight function  $w_\Theta(X')$ , we first separate  $Q'$  from  $Q$ , and view  $Q'$  as an independent quiver. Then  $Q'$  carries a weight system  $\Theta'$ , called the independent weight system, given by the manner described previously. Denoted by  $\theta_{\text{ind}}(Q')$  the sum of the weights belong to the independent weight system associated to the independent quiver  $Q'$ , then it identical equals to zero. By Lemma 5.3.4, the actual weight function  $w_\Theta(X')$  is the sum of  $\theta_{\text{ind}}(Q')$  and  $\theta_{\text{add}}(Q')$ , where  $\theta_{\text{add}}(Q')$  is the sum of the added new weights at the vertices in  $Q'_0$  caused by the connected subquivers containing not only vertices in  $Q'_0$  but also in  $Q_0 \setminus Q'_0$ . Such connected quivers are divided into three cases:

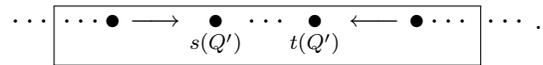
(i) the considered connected subquiver ( inside the box) contains vertices in  $Q'$  and some vertices only on the right of  $Q'$ , like the following:



(ii) the considered connected subquiver (inside the box) contains vertices in  $Q'$  and some vertices only on the left of  $Q'$ , like the following:



(iii) the considered connected subquiver ( inside the box) contains the whole  $Q'$  and some vertices both on the right and on the left of  $Q'$ , like the following:



The first case includes  $r_{Q'} \cdot |Q'_0|$  choices. A key observation is that each choice contributes a term  $-1$  to the sum of weights at the vertices of  $Q'$ . Indeed, let  $\tilde{Q}$  be a such connected subquiver which produce new weight for the vertices in  $\tilde{Q}_0$  as in the proof of [Lemma 5.3.4](#)

$$\theta_i^{\tilde{Q}} = \#\{a \mid s(a) = i, a \in \tilde{Q}_1\} - \#\{a \mid t(a) = i, a \in \tilde{Q}_1\}.$$

Then once we compute the sum  $\sum_{i \in \tilde{Q}_0 \cap Q'_0} \theta_i^{\tilde{Q}}$ , the inner arrows of  $\tilde{Q} \cap Q'$  do no work, only the arrow closest attaching to  $\tilde{Q} \cap Q'$  has effect by providing one term -1 in the sum. Similarly, the second case admits  $l_{Q'} \cdot |Q'_0|$  choices, and each case contributes a term  $-1$  to the sum; the third case contains  $l_{Q'} \cdot r_{Q'}$  choices, and each case contributes a term  $-2$  to the sum. Finally we reach

$$\theta_{\text{add}}(Q') = -r_{Q'} \cdot |Q'_0| - l_{Q'} \cdot |Q'_0| - 2r_{Q'} \cdot l_{Q'},$$

hence the weight function  $w_{\Theta}(X')$  is given by

$$\theta(Q') = \theta_{\text{ind}}(Q') + \theta_{\text{add}}(Q') = -r_{Q'} \cdot |Q'_0| - l_{Q'} \cdot |Q'_0| - 2r_{Q'} \cdot l_{Q'} < 0.$$

If  $Q'$  is not connected, we denote its connected components as  $Q^1, Q^2, \dots, Q^s$  which correspond to the direct summand  $X^i$  of the representation  $X'$ , then

$$w_{\Theta}(X') = \sum_{i=1}^s w_{\Theta}(X^i).$$

For each summand we have shown it is negative. □

### 5.3.2 Proof of the Main Theorem

We complete our proof of the main theorem by the following lemma.

**Lemma 5.3.7.** *Let  $Q$  be a quiver of  $A_n$ -type, then every indecomposable representation is stable with respect to the intrinsic weight function and rank function.*

*Sketch of Proof.* Here we just give a sketch of proof here, for more details, see paper [HH20b].

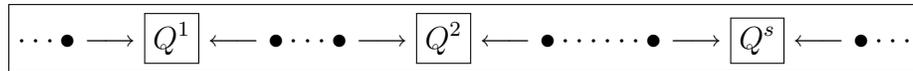
Let  $X$  be an indecomposable representation of  $Q$  and  $X' \subset X$  be any proper subrepresentation. Now  $X$  must be of type  $I_{p,q}$  with support quiver  $Q^X := \text{supp}_X(Q)$  connected, and the support quiver  $Q^{X'} := \text{supp}_{X'}(Q)$  of  $X'$  is a proper full subquiver of  $Q^X$ . Therefore our aim is to prove the following inequality holds for any proper full subquiver  $Q^{X'}$  of  $Q^X$ :

$$\frac{w_\Theta(X')}{|Q_0^{X'}|} < \frac{w_\Theta(X)}{|Q_0^X|}. \tag{5.3}$$

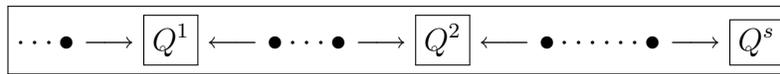
Let  $Q^{X'}$  has  $s$  connected components  $Q^1, \dots, Q^s$ , clearly each  $Q^i$  is a proper full subquiver of  $Q^X$ . To calculate the total weights  $w_\Theta(X)$  and  $w_\Theta(X')$ , similar as the proof of Lemma 5.3.6, we first separate  $Q^X$  from the whole quiver  $Q$  to get the sums  $\theta_{\text{ind}}(Q^X)$  ( $= 0$ ) and  $\theta_{\text{ind}}(Q^{X'})$  ( $< 0$ ) coming from the independent weight system on  $Q^X$ . Secondly, we calculate the sums  $\theta_{\text{add}}(Q^X)$  and  $\theta_{\text{add}}(Q^{X'})$  when the rest parts of  $Q$  are considered.

According to the relation of  $Q^i$  and  $Q^{X'}$ , we can divide our consideration into three different big cases, and when we take the rest part of  $Q$  into account, each big case can be divided into four different small cases.

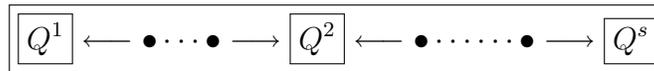
Case I: all  $Q^i$  are in the interior of  $Q^X$ , illustrated as follows:



Case II: there is a full subquiver of  $Q^X$  (without loss of generality, assumed to be  $Q^s$ ) that shares one boundary vertex with  $Q^X$ , illustrated as follows:



Case III: there are two full subquivers of  $Q^X$  (assumed to be  $Q^1$  and  $Q^s$ ) each of which has one boundary vertex coincides with that of  $Q^X$ , illustrated as follows:



Denoted by Cases in each case above, then each case can be divided into the following four different sub-cases:

(a) the two arrows near  $Q^X$  both point into  $Q^X$ :



(b) the left arrow near  $Q^X$  is out from  $Q^X$  and the right one points into  $Q^X$ :



(c) the two arrows near  $Q^X$  are both out from  $Q^X$ :



(d) the left arrow near  $Q^X$  points into  $Q^X$  and the right one is out from  $Q^X$ :

$$\cdots \bullet \longrightarrow \boxed{\text{Cases}} \longrightarrow \bullet \cdots .$$

Our method is analyzing each case and show the stability. □

**Definition 5.3.8.** Let  $\text{Ind}(Q)$  be the (finite) set of the isomorphism classes indecomposable representations over  $k$  of a Dynkin quiver  $Q$ . For a nonempty subset  $U \subseteq \text{Ind}(Q)$ , one introduces a subset  $S_{\mathbb{Z}}(Q, U)$  of  $\mathbb{Z}^n$  with  $n = |Q_0|$  as

$$S_{\mathbb{Z}}(Q, U) = \left\{ \Theta = (\theta_1, \dots, \theta_n) \in \mathbb{Z}^n \left| \begin{array}{l} \text{each element of } U \text{ is stable with respect to the} \\ \text{corresponding weight function } w_{\Theta} \text{ and rank function } r \end{array} \right. \right\}.$$

Obviously,  $S_{\mathbb{Z}}(Q, U) \subseteq S_{\mathbb{Z}}(Q, V)$  if  $V \subseteq U$ .  $S_{\mathbb{Z}}(Q, U)$  is determined by finitely many linear inequalities  $f_1(\Theta) > 0, \dots, f_m(\Theta) > 0$ , then we define a subset  $S_{\mathbb{R}}(Q, U)$  of  $\mathbb{R}^n$  as

$$S_{\mathbb{R}}(Q, U) = \{ \Theta = (\theta_1, \dots, \theta_n) \in \mathbb{R}^n : f_1(\Theta) > 0, \dots, f_m(\Theta) > 0 \},$$

and define a convex polyhedral cone  $C(Q, U)$  of  $\mathbb{R}^n$  as the closure

$$C(Q, U) = \overline{S_{\mathbb{R}}(Q, U)}.$$

The faces of maximal dimension in a cone  $C(Q, U)$  are called the *walls* in  $\mathbb{R}^n$ .

**Corollary 5.3.9** ([Rie03]). *Let  $Q$  be  $q$  quiver of  $A_n$ -type, then the cardinality of  $S_{\mathbb{Z}}(Q, \text{Ind}(Q))$  is infinite. Moreover, for any element in  $S_{\mathbb{Z}}(Q, \text{Ind}(Q))$ , with respect to the corresponding weight function and rank function*

- (1) any semistable representation is polystable;
- (2) the Hader–Narasimhan strata are precisely  $\text{GL}(Q, d)$ -orbits in  $\mathbf{Rep}(Q, d)$ .

### 5.3.3 Revisit Intrinsic Weight System via Semi-Invariant Theory

**Proposition 5.3.10.** *For each indecomposable representation  $I_{p,q}$  of the quiver  $Q$  of  $A_n$ -type, one defines weight systems  $\Theta(I_{p,q})$  and  $\Theta'(I_{p,q})$  as follows*

$$\Theta(I_{p,q})_i = \begin{cases} 1, & p < i < q \text{ and } i \text{ is a vertex of type I,} \\ -1, & p < i < q \text{ and } i \text{ is a vertex of type II,} \\ 0, & p < i < q \text{ and } i \text{ is a vertex of type III or IV,} \\ 1, & i = p \text{ and } i \text{ is a vertex of type I or III; } i = p = q, \\ 0, & i = p < q \text{ and } i \text{ is a vertex of type II or IV,} \\ 0, & i = p - 1 \text{ and } i \text{ is a vertex of type I or III,} \\ -1, & i = p - 1 \text{ and } i \text{ is a vertex of type II or IV,} \\ 0, & i = q + 1 \text{ and } i \text{ is a vertex of type I or IV,} \\ -1, & i = q + 1 \text{ and } i \text{ is a vertex of type II or III,} \\ 0, & i < p - 1; i > q + 1, \end{cases}$$

$$\Theta'(I_{p,q})_i = \begin{cases} 1, & p < i < q \text{ and } i \text{ is a vertex of type I,} \\ -1, & p < i < q \text{ and } i \text{ is a vertex of type II,} \\ 0, & p < i < q \text{ and } i \text{ is a vertex of type III or IV,} \\ -1, & i = p \text{ and } i \text{ is a vertex of type II or IV; } i = p = q, \\ 0, & i = p < q \text{ and } i \text{ is a vertex of type I or III,} \\ 1, & i = p - 1 \text{ and } i \text{ is a vertex of type I or III,} \\ 0, & i = p - 1 \text{ and } i \text{ is a vertex of type II or IV,} \\ 1, & i = q + 1 \text{ and } i \text{ is a vertex of type I or IV,} \\ 0, & i = q + 1 \text{ and } i \text{ is a vertex of type II or III,} \\ 0, & i < p - 1; i > q + 1, \end{cases}$$

then the intrinsic weight system  $\Theta$  can be written as

$$\Theta = \sum_{I_{p,q}} c(I_{p,q})\Theta(I_{p,q}) \quad (\text{or } \Theta = \sum_{I_{p,q}} c(I_{p,q})\Theta'(I_{p,q})),$$

where the sum runs through all indecomposable representations of  $Q$ , and the coefficients  $c(I_{p,q})$ 's are **non-negative integers**, moreover the sum can be taken over the indecomposable representations  $I_{p,q}$  satisfying if  $p \neq 1$  is a vertex of type I or IV then  $q = n$  or  $q \neq n$  is of type II or IV; if  $p$  is a vertex of type II or III then  $q \neq n$  is of type I or III; if  $p = 1$  then  $q \neq n$  is a vertex of type I or III (or satisfying if  $p \neq 1$  is a vertex of type II or III then  $q = n$  or  $q \neq n$  is one of type I or III; if  $p$  is a vertex of type I or IV then  $q \neq n$  is one of type II or IV; if  $p = 1$  then  $q \neq n$  is one of type II or IV).

*Proof.* We prove this proposition by virtue of the semi-invariant theory. Let us first recall it briefly. For a representation  $X$  of a general quiver  $Q$  with the dimension vector  $d$ , every weight system  $W = (W_i) \in \mathbb{Z}^{|Q_0|}$  on  $Q$  defines a character  $\chi_\Theta$  of reductive algebraic group  $\text{GL}(Q, d) = \prod_{i \in Q_0} \text{GL}(d_i)$  acting on  $X$  as a homomorphism

$$\begin{aligned} \chi_W : \text{GL}(Q, d) &\longrightarrow k^\times, \\ g = (g_i : g_i \in \text{GL}(d_i)) &\longmapsto \prod_{i \in Q_0} \det(g_i)^{W_i}. \end{aligned}$$

Conversely, every character of  $\text{GL}(d)$  must look like the above form. Let

$$\mathbf{Rep}(Q, d) = \bigoplus_{a \in Q_1} \text{Hom}(k^{d_{s(a)}}, k^{d_{t(a)}})$$

be the affine variety of representations of  $Q$  with dimension vector  $d$ , a polynomial function  $f$  in  $k[\mathbf{Rep}(Q, d)]$  is called a  $W$ -semi-invariant if  $g \cdot f = \chi_W(g)f$  for any  $g \in \text{GL}(X)$ . Denoted by  $\text{SI}_W(Q, d)$  the vector space of  $W$ -semi-invariants, then the direct sum

$$\text{SI}(Q, d) = \bigoplus_{W \in \mathbb{Z}^{|Q_0|}} \text{SI}_W(Q, d)$$

carries a ring structure, called the *ring of semi-invariants*. Moreover,  $\text{SI}(Q, d) = k[\mathbf{Rep}(Q, d)]^{\text{SL}(Q, d)}$

for  $\mathrm{SL}(Q, d) = \prod_{i \in Q_0} \mathrm{SL}(d_i)$  is the ring of polynomials in  $k[\mathbf{Rep}(Q, d)]$  which is stable under the action of  $\mathrm{SL}(d)$ . Let  $X, Y$  be two representations of a quiver  $Q$  of  $A_n$ -type with dimension vectors  $d_X, d_Y$  respectively, the Euler inner product is given by

$$\begin{aligned} \langle d_X, d_Y \rangle &= \dim_k \mathrm{Hom}_Q(X, Y) - \dim_k \mathrm{Ext}_Q(X, Y) \\ &= \sum_{i \in Q_0} (d_X)_i (d_Y)_i - \sum_{a \in Q_1} (d_X)_{s(a)} (d_Y)_{t(a)} \\ &= \sum_{i, i+1 \in Q_0} ((d_X)_i (d_Y)_i - (\widehat{d_X})_i (\widehat{d_Y})_{i+1}), \end{aligned}$$

where  $i+1$  stands for the next vertex of  $i$  along the reference direction, and

$$\begin{aligned} (\widehat{d_X})_i &= \begin{cases} (d_X)_i, & i \text{ is a vertex of type I or III,} \\ (d_X)_{i+1}, & i \text{ is a vertex of type II or IV;} \end{cases} \\ (\widehat{d_Y})_i &= \begin{cases} (d_Y)_i, & i \text{ is a vertex of type II or III,} \\ (d_Y)_{i-1}, & i \text{ is a vertex of type I or IV.} \end{cases} \end{aligned}$$

Define a map  $f_X^Y : \bigoplus_{i \in Q_0} \mathrm{Hom}(X_i, Y_i) \rightarrow \bigoplus_{a \in Q_1} \mathrm{Hom}(X_{s(a)}, Y_{t(a)})$  by

$$(f_i)_{i \in Q_0} \mapsto (f_{t(a)} X_a - Y_a f_{s(a)})_{a \in Q_1}.$$

If  $\langle d_X, d_Y \rangle = 0$ , the matrix of  $f_X^Y$  is a square matrix, then one can define a semi-invariant  $c(X, Y) = \det f_X^Y$  of the action  $\mathrm{GL}(Q, d_X) \times \mathrm{GL}(Q, d_Y)$  on  $\mathbf{Rep}(Q, d_X) \times \mathbf{Rep}(Q, d_Y)$ . For a fixed representation  $X$  (or  $Y$ ), the restriction  $c(X, Y)$  to  $\{X\} \times \mathbf{Rep}(Q, d_Y)$  (or  $\mathbf{Rep}(Q, d_X) \times \{Y\}$ ) defines a semi-invariant  $c_X(Y)$  (or  $c^Y(X)$ ) in  $\mathrm{SI}(Q, d_Y)$  with respect to the weight system  $W_X = \{(W_X)_i\}_{i \in Q_0}$ , where

$$(W_X)_i = \langle d_X, d_i \rangle = \begin{cases} (d_X)_i, & i \text{ is a vertex of type I;} \\ (d_X)_i - (d_X)_{i-1} - (d_X)_{i+1}, & i \text{ is a vertex of type II;} \\ (d_X)_i - (d_X)_{i-1}, & i \text{ is a vertex of type III;} \\ (d_X)_i - (d_X)_{i+1}, & i \text{ is a vertex of type IV,} \end{cases}$$

for  $(d_i)_j = \delta_{ij}$  (or  $\mathrm{SI}(Q, d_X)$ ) with respect to the weight system  $W^Y = \{(W^Y)_i\}_{i \in Q_0}$ , where

$$(W^Y)_i = -\langle d_i, d_Y \rangle = \begin{cases} -(d_Y)_i + (d_Y)_{i-1} + (d_Y)_{i+1}, & i \text{ is a vertex of type I;} \\ -(d_Y)_i, & i \text{ is a vertex of type II;} \\ -(d_Y)_i + (d_Y)_{i+1}, & i \text{ is a vertex of type III;} \\ -(d_Y)_i + (d_Y)_{i-1}, & i \text{ is a vertex of type IV} \end{cases}.$$

Derksen and Weyman's remarkable theorem [DW00] asserts that the semi-invariants of type  $c_X(Y)$  (or  $c^Y(X)$ ) span all the weight spaces in the rings  $\mathrm{SI}(Q, d_Y)$  (or  $\mathrm{SI}(Q, d_X)$ ). One can easily check

$$W_{I_{p,q}} = \Theta(I_{p,q}), \quad W^{I_{p,q}} = \Theta'(I_{p,q}).$$

on the other hand,  $I_{1,n}$  is stable with respect to the intrinsic weight function, thus for each subrepresentations  $R \subseteq I_{p,q}$  we have  $w(R) < 0$ , then by King's result, there exists an  $m > 0$  and  $f \in \mathrm{SI}(Q, \vec{n})_{m\Theta}$  such that  $f(I_{1,n}) \neq 0$ , where  $\vec{n} = (1, \dots, 1)$  denotes for the dimension vector of

$I_{1,n}$ . Derksen–Weyman theorem implies that the set  $\Sigma(Q, d) = \{W : \text{SI}(Q, d)_W \neq 0\}$  is saturated, therefore  $\text{SI}(Q, \vec{n})_{\Theta} \neq 0$ . The ring  $\text{SI}(Q, \vec{n})$  is generated by all  $c^{I_{p,q}}$ 's with  $\langle d_{I_{p,q}}, \vec{n} \rangle = 0$  (or  $c_{I_{p,q}}$ 's with  $\langle \vec{n}, d_{I_{p,q}} \rangle = 0$ ).

All these facts combined together lead to the final conclusion.  $\square$

**Remark 5.3.11.** We write  $\Theta = \Theta^+ - \Theta^-$ , where  $\Theta^+ = \{\Theta_i^+\}$  with  $\Theta_i^+ = \max\{\theta_i, 0\}$  and  $\Theta^- = \{\Theta_i^-\}$  with  $\Theta_i^- = \max\{-\theta_i, 0\}$ . For a dimension vector  $d$ , if  $\sum_{i \in Q_0} d_i \Theta_i \neq 0$ , then there is only trivial  $\Theta$ -semi-invariant. Therefore, we assume  $\sum_{i \in Q_0} d_i \Theta_i = 0$ , i.e.,  $\sum_{i \in Q_0} d_i \Theta_i^+ = \sum_{i \in Q_0} d_i \Theta_i^- = l$ , then for a representation  $X \in \mathbf{Rep}(Q, d)$ , one can define an  $l \times l$  matrix

$$A : \bigoplus_{i \in Q_0} X_i^{\Theta_i^+} \longrightarrow \bigoplus_{i \in Q_0} X_i^{\Theta_i^-},$$

where each block  $A_{ij} \in \text{Hom}(X_i, X_j)$  has a form

$$A_{ij} = \begin{cases} X(p_{i,j}), & \text{if there exists a path } p_{i,j} \text{ from } i \text{ to } j, \\ 0, & \text{otherwise,} \end{cases}$$

with  $X(p_{i,j})$  denoting the composition of the morphisms  $V_a$  for the arrows  $a$ 's consisting of the path  $p_{i,j}$ . Then  $\det A$  is a semi-invariant in  $\text{SI}(Q, d)_{\Theta}$ , and such semi-invariants generate the space  $\text{SI}(Q, d)_{\Theta}$ .



## Chapter 6

# Kobayashi–Hitchin Correspondence for Quiver Bundles over Generalized Manifolds

The non-Abelian Hodge theory begins from the study of the existence of Hermitian–Einstein metrics on holomorphic bundles that satisfy some special stability conditions. This is exactly what we call the Kobayashi–Hitchin correspondence.

This theory has a vast development since 1980s, many people work on this field from different points of view, and many interesting results are obtained. Usually, the study of this kind of problem can be generalized in the following directions:

- (1) Consider more general base manifolds: for example, general Hermitian manifolds with Gauduchon metric, or non-compact Kähler manifolds satisfying some analytic conditions [LY87, Moc20];
- (2) Consider more general bundles: for example, generalized vector bundles, or quiver bundles [ACGP03b, HMS16];
- (3) Consider more general stability conditions: for example, relax to semistability or to  $\alpha$ -stability parametrized by some weight system  $\alpha$  [LZ15, NZ18];
- (4) Consider singularities for Hermitian–Einstein connections and parabolic structure on vector bundles [Sim90, Moc06, Moc09, Biq97, Li00].

In this chapter, our considerations focus on generalized Kähler manifold as the base manifold and quiver bundle as the gauge theoretic system. This is mainly based on the paper [HH20a].

Generalized Kähler manifold was first discovered by Gates, Hull, and Roček as the target space of  $N = (2, 2)$  sigma model [GHR84], and then reformulated under the context of Hitchin’s generalized complex geometry [Hit03, Gua11] by Gualtieri [Gua14]. There are abundant candidates for generalized Kähler manifold, for example, any degenerate del Pezzo surface and all Hirzebruch surface admits non-trivial generalized Kähler structures [Hit07]. On the other hand, quiver bundle coming from quiver gauge theory consists of a set of vector bundles and a set of morphisms between these bundles [ACGP03b, ACGP03a].

In this chapter, we first introduce the notion of holomorphic quiver bundles over a generalized Kähler manifold, and introduce suitable stability and good metric for them. We should be faced with some new features in our setting: such stability depends on several real parameters reflecting

the generalized Kähler structure on base manifold and quiver structure on gauge theoretic system, and such metric satisfies a series of mutually coupled equations. Then we prove certain set-theoretic Kobayashi–Hitchin correspondence, namely we have the following main theorem which generalizes the results in [ACGP03b, HMS16].

**Theorem 6.0.1** (= Theorem 6.3.1). *Let  $Q = (Q_0, Q_1)$  be a quiver, and  $\mathcal{E} = (E, \phi)$  be an  $I_{\pm}$ -holomorphic  $Q$ -bundle over an  $n$ -dimensional compact generalized Kähler manifold  $(X, I_+, I_-, g, b)$  such that  $g$  is Gauduchon with respect to both  $I_+$  and  $I_-$ , then  $\mathcal{E}$  is  $(\alpha, \sigma, \tau)$ -polystable if and only if  $\mathcal{E}$  admits an  $(\alpha, \sigma, \tau)$ -Hermitian–Einstein metric.*

More related questions (generalizations) are proposed. On one hand, since a type of interesting generalized Kähler manifolds, so-called generalized Calabi–Yau manifolds appear in compactification of Type II string theory, must be non-compact, we can generalize such correspondence to the non-compact case. On the other hand, the parameters appeared in the definition of stability form a parameter space of stability conditions which is partitioned into chambers, studying the wall-crossing phenomenon on this space is also an interesting topic, maybe the Kobayashi–Hitchin correspondence can play some role.

## 6.1 Generalized Kähler Manifolds and Quiver Bundles

### 6.1.1 Generalized Kähler Manifolds

Let  $X$  be a manifold, we introduce the following definitions from generalized complex geometry based on [Hit03, Gua11, Gua14].

**Definition 6.1.1.** A *generalised complex structure* on  $X$  is an endomorphism  $J \in \text{End}(TX \oplus T^*X)$  such that

- (1)  $J^2 = -\text{Id}$ ;
- (2)  $\langle Ju, v \rangle = -\langle u, Jv \rangle$ , where the inner product  $\langle \cdot, \cdot \rangle$  on  $TX \otimes T^*X$  is the natural inner product defined by

$$\langle X + \xi, Y + \eta \rangle := \frac{1}{2}(i_X \eta + i_Y \xi).$$

- (3) Let  $V^{1,0}$  be the subbundle of the complexified bundle  $(TX \oplus T^*X) \otimes \mathbb{C}$  defined by the  $\sqrt{-1}$ -eigenspaces of  $J$ , then  $V^{1,0}$  is Courant involutive, that is, it is closed under the Courant bracket on  $TX \oplus T^*X$  defined as follows:

$$[[X + \xi, Y + \eta]] := [X, Y] + L_X \eta - L_Y \xi - \frac{1}{2}d(i_X \eta - i_Y \xi)$$

A generalized Kähler manifold refers to the geometric object defined by the following two equivalent approaches.

**Definition 6.1.2** ([Gua11]). A manifold  $X$  is called a *generalized Kähler manifold* if it carries two generalized complex structures  $J_1, J_2 \in \text{End}(TX \oplus T^*X)$  satisfying

- (1)  $J_1 J_2 = J_2 J_1$ ,
- (2) the symmetric pairing  $G(A, B) := \langle J_1(A), J_2(B) \rangle$  is positive-definite for any non-zero smooth sections  $A, B \in \Gamma(TX \oplus T^*X)$ , where  $\langle \cdot, \cdot \rangle$  denotes the inner product on  $TX \oplus T^*X$  defined as above.

$G$  is called a *generalized Kähler metric*.

**Definition 6.1.3** ([Gua14]). A manifold  $X$  is called a *generalized Kähler manifold* if it carries the data  $(I_+, I_-, g, b)$ , where

- $I_\pm$  are two complex structures on  $X$ ,
- $g$  is a Riemannian metric on  $X$ ,
- $b$  is a two-form on  $X$ ,
- $I_\pm$  are parallel with respect to the connections  $\nabla^\pm = \nabla \pm \frac{1}{2}g^{-1}\mathbb{H}$ , respectively, where  $\nabla$  is the Levi-Civita connection of  $g$  and  $\mathbb{H} = db$ .

The generalized Calabi–Yau manifold is an important kind of generalized Kähler manifold.

**Definition 6.1.4** ([Hit03]). A *generalized Calabi–Yau manifold* is a generalized Kähler manifold  $(X, J_1, J_2)$  such that both nowhere vanishing pure spinors  $\psi_1, \psi_2$  corresponding to  $J_1, J_2$ , respectively satisfy the following conditions

- $d\psi_1 = d\psi_2 = 0$ ,
- $(\psi_1, \bar{\psi}_1) = (\psi_2, \bar{\psi}_2)$ ,

where  $(\bullet, \bullet)$  is the Mukai pairing.

**Remark 6.1.5.** More generally, one defines the twisted generalized Kähler manifold as the manifold  $X$  with 4-tuple  $(I_+, I_-, g, \mathbb{H})$ , where  $I_\pm, g$  is the same as above, and  $\mathbb{H}$  is a closed 3-form such that  $I_\pm$  are parallel with respect to the connections  $\nabla^\pm = \nabla \pm \frac{1}{2}g^{-1}\mathbb{H}$ , respectively. Similarly, one can also introduce the twisted generalized Calabi–Yau manifold by replacing the first condition on pure spinors by  $d_{\mathbb{H}}\psi_1 = d_{\mathbb{H}}\psi_2 = 0$  for  $d_{\mathbb{H}} = d + \mathbb{H}\wedge$ . When  $\mathbb{H}$  is an exact 3-form, they reduce to the generalized Kähler manifold and generalized Calabi–Yau manifold defined as above.

**Definition 6.1.6** ([HMS16]). Let  $(X, I_+, I_-, g, b)$  be a generalized Kähler manifold, and  $E$  be a complex vector bundle over  $X$ .  $E$  is called  *$I_\pm$ -holomorphic* if there are two operators  $\bar{\partial}_\pm : C^\infty(E) \rightarrow C^\infty(E \otimes T_{I_\pm}^{0,1}X)$  such that they define a holomorphic structure on  $E$  with respect to  $I_\pm$  respectively.

Given an  $I_\pm$ -holomorphic vector bundle  $(E, \bar{\partial}_+, \bar{\partial}_-)$ , denote by  $\iota$  the natural isomorphisms between  $\bar{L}_\pm$  and  $T_{I_\pm}^{0,1}X$ , one defines  $\bar{D}_{\pm, s}(v) := \bar{\partial}_{\pm, \iota(s)}(v)$  for  $s \in C^\infty(\bar{L}_\pm)$  and  $v \in C^\infty(E)$ , where  $L_+ = L_1 \cap L_2$ ,  $L_- = L_1 \cap \bar{L}_2$  with  $L_1, L_2$  be  $\sqrt{-1}$ -eigensubbundles of  $(TX \oplus T^*X) \otimes \mathbb{C}$  with respect to the generalized complex structure  $J_{1,2}$  determined by

$$J_{1,2} = \frac{1}{2} \begin{pmatrix} I_+ \pm I_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(I_+^* \pm I_-^*) \end{pmatrix}$$

for Kähler forms  $\omega_\pm = g(I_\pm \cdot, \cdot)$ . Then  $\bar{D} = \bar{D}_+ + \bar{D}_- : C^\infty(E) \rightarrow C^\infty(E \otimes \bar{L}_1)$  defines a generalized holomorphic bundle with respect to  $J_1$  if and only if  $\bar{\partial}_+ \bar{\partial}_- + \bar{\partial}_- \bar{\partial}_+ = 0$  [HMS16].

Moreover, we make the following assumptions on the  $n$ -dimensional generalized Kähler manifold  $(X, I_+, I_-, g, b)$  in this paper:

- $g$  is Gauduchon, i.e.,  $dd_\pm^c \omega_\pm^{n-1} = 0$  and  $d\text{Vol}_g = \frac{1}{n!} \omega_\pm^n$ , where  $d_\pm^c = I_\pm \circ d \circ I_\pm$ ;
- $X$  is compact.

The first assumption is not too restrictive. It can be satisfied for generalized Kähler 4-manifolds automatically, and for real compact Lie groups. On the second assumption, we have the following no-go type theorem.

**Proposition 6.1.7.** (1) *A compact twisted generalized Kähler surface has even first Betti number if  $\mathbb{H}$  is exact, and has odd first Betti number if  $\mathbb{H}$  is not exact.*

(2) *A compact twisted generalized Calabi–Yau manifold must be a usual Calabi–Yau manifold.*

*Proof.* (Sketch) The first result has been proved by the authors of [ACGP03b]. We only prove the second claim. The structure of generalized Calabi–Yau reduces the structure group  $O(2n, 2n)$  of  $TX \oplus T^*X$  to  $SU_+(n) \times SU_-(n)$ , then there are two globally defined  $SU_\pm(n)$ -invariant spinors  $\xi_\pm$ . The constraints on pure spinors can be rewritten in terms of  $\xi_\pm$  [Wit06]

$$\begin{aligned} (\nabla_M \pm \frac{1}{4} M \lrcorner \mathbb{H}) \cdot \xi_\pm &= 0, \\ (df \pm \frac{1}{2} \mathbb{H}) \cdot \xi_\pm &= 0 \end{aligned}$$

for  $\forall M \in C^\infty(TX)$ , exact three-form  $\mathbb{H} = db$  and smooth function  $f = \log \frac{1}{(\psi_1, \psi_1)}$ , where  $\nabla$  denotes the spin connection with respect to  $g$ , and  $\cdot$  stands for the Clifford multiplication. We only need to show if  $X$  is compact then  $\mathbb{H}$  vanishes. Indeed, the following equations are derived from the above conditions

$$\begin{aligned} R_{\mu\nu}^{(g)} - \frac{1}{4} \mathbb{H}_{\mu\alpha\beta} \mathbb{H}_{\nu\gamma\delta} g^{\alpha\delta} g^{\beta\delta} + 2\nabla_\mu \nabla_\nu f &= 0, \\ g^{\mu\alpha} \nabla_\mu (e^{-2f} \mathbb{H}_{\alpha\beta\gamma}) &= 0, \\ R^{(g)} + 4g^{\mu\nu} \nabla_\mu \nabla_\nu f - 4g^{\mu\nu} \nabla_\mu \nabla_\nu f - \frac{1}{12} g^{\mu\alpha} g^{\nu\beta} g^{\lambda\gamma} \mathbb{H}_{\mu\nu\lambda} \mathbb{H}_{\alpha\beta\gamma} &= 0. \end{aligned}$$

After taking trace we get  $g^{\mu\nu} \nabla_\mu \nabla_\nu e^{-2f} - \frac{1}{6} e^{-2f} g^{\mu\alpha} g^{\nu\beta} g^{\lambda\gamma} \mathbb{H}_{\mu\nu\lambda} \mathbb{H}_{\alpha\beta\gamma} = 0$ , then integrating over  $X$  implies the vanishing of  $\mathbb{H}$  if  $X$  is compact.  $\square$

Now let  $(E, \bar{\partial}_+, \bar{\partial}_-)$  be an  $I_\pm$ -holomorphic bundle over a generalized Kähler manifold  $X$ , fix a Hermitian metric  $H$  on  $E$ , then there is a unique Chern-connection compatible with the complex structures  $I_\pm$  respectively, given by  $D_H^\pm := \partial_H^\pm + \bar{\partial}_\pm$ , whose curvature form is denoted by  $\mathbb{F}_H^\pm$ . Then we define the degrees associated to the two Chern connections as follows:

$$\deg_\pm(E) := \frac{\sqrt{-1}}{2\pi} \int_X \text{Tr}(\mathbb{F}_H^\pm) \wedge \omega_\pm^{n-1},$$

which are independent of the choice of Hermitian metric  $H$  on  $E$ , since for any two Hermitian metrics  $H$  and  $H'$  on  $E$ , we have  $\text{Tr}(\mathbb{F}_H^\pm) = \text{Tr}(\mathbb{F}_{H'}^\pm) + \partial_\pm \bar{\partial}_\pm (\log \det((H')^{-1}H))$ .

## 6.1.2 Quiver Bundles and Stability

We give the following definitions of quiver bundles.

**Definition 6.1.8.** (1) A *quiver*  $Q = (Q_0, Q_1, h, t : Q_1 \rightarrow Q_0)$  is a 4-tuple, where

- $Q_0$  and  $Q_1$  are finite sets of vertices and arrows, respectively,

- $h, t : Q_1 \rightarrow Q_0$  map each arrow  $a \in Q_1$  to its head  $h(a)$  and tail  $t(a)$ , respectively.
- (2) A  $Q$ -sheaf on a complex manifold  $X$  is a pair  $\mathcal{E} = (E, \phi)$ , where  $E = \{E_i\}_{i \in Q_0}$  is a collection of sheaves of  $\mathcal{O}_X$ -modules and  $\phi = \{\phi_a\}_{a \in Q_1}$  a collection of morphisms  $\phi_a : E_{t(a)} \rightarrow E_{h(a)}$ . In particular, if each  $E_i$  is locally free,  $\mathcal{E}$  is called a  $Q$ -bundle. A  $Q$ -subsheaf of  $\mathcal{E} = (E, \phi)$  is a  $Q$ -sheaf  $\mathcal{E}' = (E', \phi')$  such that  $E'_i$  is a subsheaf of  $E_i$  for each vertex  $i$  and  $\phi'_a = \phi_a|_{E_{h(a)}}$  for each arrow  $a$ .
- (3) A morphism  $f : \mathcal{E} \rightarrow \mathcal{F}$  between two  $Q$ -sheaves  $\mathcal{E} = (E, \phi)$  and  $\mathcal{F} = (F, \varphi)$  is a collection of morphisms  $f_i : E_i \rightarrow F_i$  such that for each arrow  $a \in Q_1$ , the following diagram commutes:

$$\begin{array}{ccc} E_{t(a)} & \xrightarrow{\phi_a} & E_{h(a)} \\ f_{t(a)} \downarrow & & \downarrow f_{h(a)} \\ F_{t(a)} & \xrightarrow{\varphi_a} & F_{h(a)}. \end{array}$$

- (4) A Hermitian metric on a  $Q$ -bundle  $\mathcal{E} = (E, \phi)$  is a collection  $H = \{H_i\}_{i \in Q_0}$  of Hermitian metrics  $H_i$  on  $E_i$ . For each arrow  $a \in Q_1$ , by virtue of the Hermitian metrics at tail and head, the morphism  $\phi_a$  has a smooth adjoint  $\phi_a^{*H} : E_{h(a)} \rightarrow E_{t(a)}$  with respect to the Hermitian metrics at tail and head, that is,  $H_{h(a)}(\phi_a(u), v) = H_{t(a)}(u, \phi_a^{*H}(v))$  for any sections  $u, v$  of  $E_{h(a)}, E_{t(a)}$ .
- (5) A  $Q$ -bundle  $\mathcal{E} = (E, \phi)$  on a generalized Kähler manifold  $(X, I_+, I_-, g, b)$  is called  $I_\pm$ -holomorphic if
- each  $E_i, i \in Q_0$ , is an  $I_\pm$ -holomorphic bundle, i.e.,  $E_i$  carries two holomorphic structures  $\bar{\partial}_{+i}, \bar{\partial}_{-i}$  with respect to  $I_\pm$ , respectively,
  - each  $\phi_a, a \in Q_1$ , is  $I_\pm$ -holomorphic, namely

$$\begin{cases} \phi_a \circ \bar{\partial}_{+t(a)} = \bar{\partial}_{+h(a)} \circ \phi_a, \\ \phi_a \circ \bar{\partial}_{-t(a)} = \bar{\partial}_{-h(a)} \circ \phi_a. \end{cases}$$

- (6) A morphism  $f : \mathcal{E} \rightarrow \mathcal{F}$  between two  $I_\pm$ -holomorphic  $Q$ -bundles  $\mathcal{E} = (E, \phi)$  and  $\mathcal{F} = (F, \varphi)$  is a collection of  $I_\pm$ -holomorphic morphisms  $f_i : (E_i, \bar{\partial}_{+i}, \bar{\partial}_{-i}) \rightarrow (F_i, \bar{\partial}'_{+i}, \bar{\partial}'_{-i})$ , such that for each arrow  $a \in Q_1$ , the following diagram commutes:

$$\begin{array}{ccc} (E_{t(a)}, \bar{\partial}_{+t(a)}, \bar{\partial}_{-t(a)}) & \xrightarrow{\phi_a} & (E_{h(a)}, \bar{\partial}_{+h(a)}, \bar{\partial}_{-h(a)}) \\ f_{t(a)} \downarrow & & \downarrow f_{h(a)} \\ (F_{t(a)}, \bar{\partial}'_{+t(a)}, \bar{\partial}'_{-t(a)}) & \xrightarrow{\varphi_a} & (F_{h(a)}, \bar{\partial}'_{+h(a)}, \bar{\partial}'_{-h(a)}). \end{array}$$

- (7) An  $I_\pm$ -holomorphic  $Q$ -bundle  $\mathcal{E} = (E, \phi)$  is said to be *simple* if any endomorphism  $f : \mathcal{E} \rightarrow \mathcal{E}$  must have the form  $f = \{c_i \text{Id}_{E_i}\}_{i \in Q_0}$  for constants  $c_i \in \mathbb{C}$ .

Similarly, we can naturally introduce the stability for  $Q$ -bundles.

**Definition 6.1.9** ([ACGP03b, HMS16]). Let  $\mathcal{E} = (E, \phi)$  be an  $I_\pm$ -holomorphic  $Q$ -bundle.

- (1) A *coherent  $Q$ -subsheaf*  $\mathcal{F}$  of  $\mathcal{E}$  is a 4-tuple  $\mathcal{F} = (\mathcal{F}_+, \mathcal{F}_-, \mathbb{S}_+, \mathbb{S}_-)$ , where

- $\mathcal{F}_\pm = (F_\pm, \varphi)$  are  $Q$ -subsheaves of the  $Q$ -sheaves  $\mathcal{E}_\pm$ , where  $\mathcal{E}_\pm = (E_\pm = \{E_{\pm i} = (E_i, \bar{\partial}_{\pm i})\}_{i \in Q_0}, \phi = \{\phi_a\}_{a \in Q_1})$ , respectively,
- $\mathbb{S}_\pm = \{S_{\pm i}\}_{i \in Q_0}$  are collections of analytic subsets of  $(X, I_\pm)$ , respectively, such that for each  $i \in Q_0$ 
  - $S_i = S_{+i} \cup S_{-i}$  has codimension at least 2,
  - $F_{\pm i}|_{X \setminus S_{\pm i}}$  are locally free and  $F_{+i}|_{X \setminus S_i} = F_{-i}|_{X \setminus S_i} := F_i$  as smooth vector bundles.

(2) For any coherent subsheaf  $\mathcal{F}$  of  $\mathcal{E}$ , we define  $(\alpha, \sigma, \tau)$ -degree and  $(\alpha, \sigma, \tau)$ -slope as follows:

$$\begin{aligned} \deg_{\alpha, \sigma, \tau}(\mathcal{F}) &:= \sum_{i \in Q_0} \alpha_i \sigma_i \deg_+(F_{+i}) + \sum_{i \in Q_0} (1 - \alpha_i) \sigma_i \deg_-(F_{-i}) - \sum_{i \in Q_0} \tau_i \text{rk}(F_i), \\ \mu_{\alpha, \sigma, \tau}(\mathcal{F}) &:= \frac{\sum_{i \in Q_0} \alpha_i \sigma_i \deg_+(F_{+i})}{\sum_{i \in Q_0} \sigma_i \text{rk}(F_i)} + \frac{\sum_{i \in Q_0} (1 - \alpha_i) \sigma_i \deg_-(F_{-i})}{\sum_{i \in Q_0} \sigma_i \text{rk}(F_i)} - \frac{\sum_{i \in Q_0} \tau_i \text{rk}(F_i)}{\sum_{i \in Q_0} \sigma_i \text{rk}(F_i)}, \end{aligned}$$

where  $\alpha_i \in (0, 1)$ ,  $\sigma_i \in \mathbb{R}^+$ ,  $\tau_i \in \mathbb{R}$ , and  $\text{rk}(F_i) = \text{rk}(F_{+i}) = \text{rk}(F_{-i})$  denotes the rank of the corresponding sheaves.

(3) A  $Q$ -bundle  $\mathcal{E}$  is called  $(\alpha, \sigma, \tau)$ -stable (resp.  $(\alpha, \sigma, \tau)$ -semistable) if for any proper coherent  $Q$ -subsheaf  $\mathcal{F}$ , we have

$$\mu_{\alpha, \sigma, \tau}(\mathcal{F}) < \text{ (resp. } \leq) \mu_{\alpha, \sigma, \tau}(\mathcal{E}),$$

and  $\mathcal{E}$  is called  $(\alpha, \sigma, \tau)$ -polystable if it is the direct sum of  $(\alpha, \sigma, \tau)$ -stable  $Q$ -subsheaves of the same  $(\alpha, \sigma, \tau)$ -slope with  $\mathcal{E}$ .

Due to the classical extension theorem [BH99], we have the following extension theorem for the coherent  $Q$ -subsheaves.

**Proposition 6.1.10.** *For each  $i \in Q_0$ , there are unique holomorphic bundles  $\hat{F}_{\pm i}$  over  $(X, I_\pm)$  extending the bundles  $F_{\pm i}|_{X \setminus S_{\pm i}}$ , respectively, hence there is a unique  $I_\pm$ -bundle  $(\hat{F}_i, \hat{\partial}_+, \hat{\partial}_-)$  over  $(X, I_\pm)$  extending the  $I_\pm$ -bundle  $(F_i, \bar{\partial}_+, \bar{\partial}_-)$  over  $X \setminus S_i$ .*

As in the classical case, stability implies simpleness in our case.

**Proposition 6.1.11.** *Let  $f : \mathcal{E} \rightarrow \mathcal{F}$  be a morphism between two  $I_\pm$ -holomorphic  $Q$ -bundles  $\mathcal{E} = (E, \phi)$  and  $\mathcal{F} = (F, \varphi)$ .*

- (1) *If  $\mathcal{E}$  and  $\mathcal{F}$  are  $(\alpha, \sigma, \tau)$ -semistable, then  $\mu_{\alpha, \sigma, \tau}(\mathcal{E}) \leq \mu_{\alpha, \sigma, \tau}(\mathcal{F})$ .*
- (2) *If  $\mathcal{E}$  and  $\mathcal{F}$  are stable of the same  $(\alpha, \sigma, \tau)$ -slope, then  $f$  is an isomorphism.*
- (3) *If  $\mathcal{E}$  is  $(\alpha, \sigma, \tau)$ -stable, then it is simple.*

*Proof.* For any morphism  $f : \mathcal{E} \rightarrow \mathcal{F}$ , it is a collection of morphisms  $\{f_i : E_i \rightarrow F_i\}_{i \in Q_0}$ , so we can define the kernel, image, cokernel of  $f$  as collection of kernels  $\{\text{Ker}(f_i)\}_{i \in Q_0}$ , images  $\{\text{Im}(f_i)\}_{i \in Q_0}$  and cokernels  $\{\text{Coker}(f_i)\}_{i \in Q_0}$ . Therefore, we have the short exact sequence of  $I_\pm$ -holomorphic  $Q$ -bundles

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} \longrightarrow \mathcal{W} \longrightarrow 0,$$

as in the natural case.

Rewrite

$$\deg_{\alpha, \sigma, \tau}(\mathcal{E}) = \deg_{\sigma, \tau, +}(\mathcal{E}) + \deg_{\sigma, \tau, -}(\mathcal{E})$$

and

$$\mu_{\alpha,\sigma,\tau}(\mathcal{E}) = \alpha\mu_{\sigma,\tau,+}(\mathcal{E}) + (1 - \alpha)\mu_{\sigma,\tau,-}(\mathcal{E}),$$

where

$$\begin{aligned} \deg_{\sigma,\tau,+}(\mathcal{E}) &= \sum_{i \in Q_0} \alpha_i \left( \sigma_i \deg_+(E_{i+}) - \tau_i \text{rk}(E_i) \right), \\ \deg_{\sigma,\tau,-}(\mathcal{E}) &= \sum_{i \in Q_0} (1 - \alpha_i) \left( \sigma_i \deg_-(E_{i-}) - \tau_i \text{rk}(E_i) \right) \end{aligned}$$

and

$$\begin{aligned} \mu_{\sigma,\tau,+}(\mathcal{E}) &= \frac{\sum_{i \in Q_0} \alpha_i \left( \sigma_i \deg_+(E_{i+}) - \tau_i \text{rk}(E_i) \right)}{\sum_{i \in Q_0} \sigma_i \text{rk}(E_i)} = \frac{\deg_{\sigma,\tau,+}(\mathcal{E})}{\sum_{i \in Q_0} \sigma_i \text{rk}(E_i)}, \\ \mu_{\sigma,\tau,-}(\mathcal{E}) &= \frac{\sum_{i \in Q_0} (1 - \alpha_i) \left( \sigma_i \deg_-(E_{i-}) - \tau_i \text{rk}(E_i) \right)}{\sum_{i \in Q_0} \sigma_i \text{rk}(E_i)} = \frac{\deg_{\sigma,\tau,-}(\mathcal{E})}{\sum_{i \in Q_0} \sigma_i \text{rk}(E_i)}. \end{aligned}$$

We can find that  $\mu_{\alpha,\sigma,\tau}(\mathcal{F}) \leq \mu_{\alpha,\sigma,\tau}(\mathcal{E})$  if and only if  $\mu_{\alpha,\sigma,\tau}(\mathcal{E}) \leq \mu_{\alpha,\sigma,\tau}(\mathcal{W})$ . Then (1) – (3) hold, as in the usual case.  $\square$

## 6.2 Hermitian–Einstein Metrics

**Definition 6.2.1.** A Hermitian metric  $H$  on an  $I_{\pm}$ -holomorphic  $Q$ -bundle  $\mathcal{E} = (E, \phi)$  is called an  $(\alpha, \sigma, \tau)$ -Hermitian–Einstein metric if for each vertex  $i \in Q_0$  it satisfies the following equations

$$\begin{aligned} & \sqrt{-1}(\alpha_i \sigma_i \mathbb{F}_{H_i}^+ \wedge \omega_+^{n-1} + (1 - \alpha_i) \sigma_i \mathbb{F}_{H_i}^- \wedge \omega_-^{n-1}) \\ & + (n - 1)! \left( \sum_{a \in h^{-1}(i)} \phi_a \circ \phi_a^{*H} - \sum_{a \in t^{-1}(i)} \phi_a^{*H} \circ \phi_a \right) \text{Vol}_g \\ & = (n - 1)! \lambda (\tau_i + \gamma \sigma_i) \text{Id}_{E_i} \text{Vol}_g \end{aligned}$$

with constants  $\lambda = \frac{2\pi}{(n-1)! \int_X d\text{Vol}_g}$  and  $\gamma$ .

**Remark 6.2.2.** Taking trace and the sum over all vertices and then doing integral over  $X$  on both sides, we see that  $\gamma$  is exactly the slope  $\mu_{(\alpha,\sigma,\tau)}(\mathcal{E})$ .

We employ the following notations:

- $S(E_i, H_i)$  is the space of smooth  $H_i$ -Hermitian endomorphisms of  $E_i$ ,  $S^+(E_i, H_i) \subset S(E_i, H_i)$  is the open subset of positive-definite ones;
- $S(\mathcal{E}, H) = \prod_{i \in Q_0} S(E_i, H_i)$ ,  $S^+(\mathcal{E}, H) = \prod_{i \in Q_0} S^+(E_i, H_i)$ . The metric  $H$  induces a metric on  $S(\mathcal{E}, H)$ , also denoted by  $H$ , namely  $\langle f, g \rangle_H = \sum_{i \in Q_0} \langle f_i, g_i \rangle_{H_i}$  for  $f = (f_i)_{i \in Q_0}$ ,  $g = (g_i)_{i \in Q_0} \in S(\mathcal{E}, H)$ .
- $L_k^p(S)$  denotes the corresponding Sobolev space.

- The pointwise or global norms and inner products  $|\bullet|$ ,  $\langle \bullet, \bullet \rangle$ ,  $\|\bullet\|$ ,  $\langle\langle \bullet, \bullet \rangle\rangle_{L^2}$  are defined with respect to the metrics  $H_i$  or induced metric induced metrics on  $E_{h(a)} \otimes (E_{t(a)})^*$  from the metrics  $H_{h(a)}$  and  $H_{t(a)}$  unambiguously depending on the contexts.

**Proposition 6.2.3.** *Let  $H$  be an  $(\alpha, \sigma, \tau)$ -Hermitian-Einstein metric on an  $I_{\pm}$ -holomorphic  $Q$ -bundle  $\mathcal{E} = (E, \phi)$  over  $X$ , then we define*

$$\begin{aligned} C_1(E_i) &= \int_X \text{Tr}(\sqrt{-1}(\alpha_i \Lambda_+ \mathbb{F}_{H_i}^+ + (1 - \alpha_i) \Lambda_- \mathbb{F}_{H_i}^-)) d\text{Vol}_g, \\ C_2(E_i) &= \int_X \left( \text{Tr}((\alpha_i \mathbb{F}_{H_i}^+)^2) \wedge \frac{\omega_+^{n-2}}{(n-2)!} + \text{Tr}(((1 - \alpha_i) \mathbb{F}_{H_i}^-)^2) \wedge \frac{\omega_-^{n-2}}{(n-2)!} \right), \\ C'_2(E_i) &= \langle\langle \alpha_i \Lambda_+ \mathbb{F}_{H_i}^+, (1 - \alpha_i) \Lambda_- \mathbb{F}_{H_i}^- \rangle\rangle_{L^2}, \end{aligned}$$

where  $\Lambda_{\pm}$  is the adjoint of the operator of the wedge by  $\omega_{\pm}$  with respect to the metric  $g$ . When  $\alpha_i = \alpha_j = \alpha$  for  $\forall i, j \in Q_0$ , the following inequality holds

$$\sum_{i \in Q_0} \sigma_i (C_2(E_i) - C'_2(E_i)) + 2\lambda \sum_{i \in Q_0} (\tau_i + \mu_{(\alpha, \sigma, \tau)}(\mathcal{E}) \sigma_i) C_1(E_i) \geq 0.$$

*Proof.* By assumption we have

$$\begin{aligned} 0 &= \sum_{i \in Q_0} \frac{1}{\sigma_i} \left\| \sqrt{-1}(\alpha \sigma_i \Lambda_+ \mathbb{F}_{H_i}^+ + (1 - \alpha) \sigma_i \Lambda_- \mathbb{F}_{H_i}^-) + \sum_{a \in h^{-1}(i)} \phi_a \circ \phi_a^{*H} - \sum_{a \in t^{-1}(i)} \phi_a^{*H} \circ \phi_a \right. \\ &\quad \left. - \lambda(\tau_i + \mu_{\alpha, \sigma, \tau}(\mathcal{E}) \sigma_i) \text{Id}_{E_i} \right\|_{L^2}^2 \\ &= \sum_{i \in Q_0} \left( \alpha^2 \sigma_i \|\Lambda_+ \mathbb{F}_{H_i}^+\|_{L^2}^2 + (1 - \alpha)^2 \sigma_i \|\Lambda_- \mathbb{F}_{H_i}^-\|_{L^2}^2 + 2\alpha(1 - \alpha) \sigma_i \langle\langle \Lambda_+ \mathbb{F}_{H_i}^+, \Lambda_- \mathbb{F}_{H_i}^- \rangle\rangle_{L^2} \right. \\ &\quad \left. + \frac{1}{\sigma_i} \left\| \sum_{a \in h^{-1}(i)} \phi_a \circ \phi_a^{*H} - \sum_{a \in t^{-1}(i)} \phi_a^{*H} \circ \phi_a - \lambda(\tau_i + \mu_{\alpha, \sigma, \tau}(\mathcal{E}) \sigma_i) \text{Id}_{E_i} \right\|_{L^2}^2 \right) \\ &\quad - 2\lambda \sum_{i \in Q_0} (\tau_i + \mu_{(\alpha, \sigma, \tau)}(\mathcal{E}) \sigma_i) \int_X \text{Tr}(\sqrt{-1}(\alpha \Lambda_+ \mathbb{F}_{H_i}^+ + (1 - \alpha) \Lambda_- \mathbb{F}_{H_i}^-)) d\text{Vol}_g \\ &\quad + 2 \sum_{a \in Q_1} \text{Re} \langle\langle \phi_a, [\sqrt{-1}(\alpha \Lambda_+ \mathbb{F}_{H_i}^+ + (1 - \alpha) \Lambda_- \mathbb{F}_{H_i}^-), \phi_a] \rangle\rangle_{L^2}, \end{aligned}$$

where  $[A, \phi]_a = A_{h(a)} \circ \phi_a - \phi_a \circ A_{t(a)}$  for  $A \in \text{End}(\mathcal{E})$ . Then we find the desired inequality by virtue of the following identities

$$\|\Lambda_{\pm} \mathbb{F}_{H_i}^{\pm}\|_{L^2}^2 = \|\mathbb{F}_{H_i}^{\pm}\|_{L^2}^2 - \int_X \text{Tr}((\mathbb{F}_{H_i}^{\pm})^2) \wedge \frac{\omega_{\pm}^{n-2}}{(n-2)!},$$

$$\begin{aligned}
\langle\langle \phi_a, [\sqrt{-1}\Lambda_{\pm}\mathbb{F}_H^{\pm}, \phi]_a \rangle\rangle_{L^2} &= \langle\langle \phi_a, 2\sqrt{-1}\Lambda_{\pm}\bar{\partial}_{\pm}\partial_H^{\pm}\phi_a - (\partial_H^{\pm})^{*H,g}\partial_H^{\pm}\phi_a \rangle\rangle_{L^2} \\
&\quad + \langle\langle \phi_a, \frac{\sqrt{-1}}{(n-1)!} \star_g (\bar{\partial}_{\pm}(\omega_{\pm}^{n-1}) \wedge \partial_H^{\pm}\phi_a) \rangle\rangle_{L^2} \\
&= \|\partial_H^{\pm}\phi_a\|_{L^2}^2 - \frac{\sqrt{-1}}{(n-1)!} \int_X \bar{\partial}_{\pm}|\phi_a|_H^2 \wedge \partial_{\pm}(\omega_{\pm}^{n-1}) \\
&= \|\partial_H^{\pm}\phi_a\|_{L^2}^2,
\end{aligned}$$

where  $\star_g$  denotes the Hodge star with respect to  $g$ , the connections acting on  $\phi_a$  are the induced connections on  $E_{h(a)} \otimes (E_{t(a)})^*$ , and the  $I_{\pm}$ -holomorphicity of  $\phi_a$ 's plays a crucial role in the second identity.  $\square$

We end this section with some examples.

**Example 6.2.4.** 1. We first consider  $X = \mathbb{P}^1$  with the standard Kähler structure  $(I, \omega)$ , it can be retreated as a generalized Kähler manifold by taking  $I = I_+ = I_-, \omega = \omega_+ = \omega_-$ . Let  $Q = (Q_0, Q_1, h, t)$  be a quiver with  $Q_0 = \{i, j\}$ ,  $Q_1 = \{a\}$  and  $t(a) = i, h(a) = j$ , then we consider the  $I_{\pm}$ -holomorphic  $Q$ -bundle  $\mathcal{E} = (E, \phi)$  over  $\mathbb{P}^1$  given by  $E_i = \mathcal{O}(m_i), E_j = \mathcal{O}(m_j)$  for  $m_j \geq m_i$ , and  $0 \neq \phi_a \in H^0(\mathbb{P}^1, \mathcal{O}(m_j - m_i))$ . Obviously,  $\deg_{\pm} \mathcal{O}(m) = m$ , hence for the stability parameters  $\alpha_i, \alpha_j; \sigma_i, \sigma_j$  and  $\tau_i, \tau_j$ ,  $\mathcal{E}$  is  $(\alpha, \sigma, \tau)$ -stable if and only if the following inequality holds

$$\sigma_i \sigma_j (m_j - m_i) < \sigma_i \tau_j - \sigma_j \tau_i.$$

In particular, the parameters  $\sigma, \tau$  are subject to the condition

$$\sigma_i \tau_j - \sigma_j \tau_i > 0,$$

which gives the constraints on these parameters as follows:

- if  $\tau_i = 0$ , then  $\tau_j > 0$ ;
- if  $\tau_j = 0$ , then  $\tau_i < 0$ ;
- if  $\tau_i, \tau_j \neq 0$ , then  $\frac{\sigma_i}{\sigma_j} > \frac{\tau_i}{\tau_j}$ .

2. Now we consider the example of Hopf surfaces, which can be found in [Gua14] (Example 1.21) and [HMS16] (Section 4 for details). Let  $X$  be a standard Hopf surface, namely  $X = \mathbb{C}^2 \setminus \{(0, 0)\} / (2(z_1, z_2) \sim (z_1, z_2))$ , then  $X$  is diffeomorphic to  $S^3 \times S^1$ . Denote by  $I_+$  the induced complex structure from  $\mathbb{C}^2$ , the Hermitian metric is given by

$$g = \frac{1}{4\pi|z|^2} (dz_1 d\bar{z}_1 + dz_2 d\bar{z}_2),$$

for  $|z|^2 = z_1 \bar{z}_1 + z_2 \bar{z}_2$ , and the associated 2-form  $\omega_+ = gI_+$  is

$$\omega_+ = \frac{\sqrt{-1}}{4\pi|z|^2} (dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2).$$

One can specify another complex structure  $I_-$  by providing a generator

$$\Omega = \frac{1}{|z|^4} (\bar{z}_1 dz_1 + z_2 d\bar{z}_2) \wedge (\bar{z}_1 dz_2 - z_2 d\bar{z}_1)$$

for  $\Omega^{2,0}((X, I_+))$ . It is easy to check that  $(g, I_-)$  is also Hermitian, and the associated 2-form is given by

$$\omega_- = \frac{\sqrt{-1}}{4\pi|z|^4} ((\bar{z}_1 dz_1 + z_2 d\bar{z}_2) \wedge (z_1 d\bar{z}_1 + \bar{z}_2 dz_2) + (\bar{z}_1 dz_2 - z_2 d\bar{z}_1) \wedge (z_1 d\bar{z}_2 - \bar{z}_2 dz_1)).$$

Then  $(I_+, I_-, g, \mathbb{H})$  defines a twisted generalized Kähler structure on  $X$ , where  $\mathbb{H} = d_+^c \omega_+ = -d_-^c \omega_-$  [Gua14, HMS16]. Actually, the torsion of twisted generalized Kähler structures on  $X$  cannot be exact [Gua14]. There is a natural projection  $\text{pr} : X \rightarrow \mathbb{P}^1$  onto  $\mathbb{P}^1$  via  $(z_1, z_2) \mapsto [z_1 : z_2]$ , and this projection is holomorphic with respect to  $I_+$ . We set  $\mathcal{O}_+(m) := \text{pr}^* \mathcal{O}_{\mathbb{P}^1}(m)$  for all  $m \in \mathbb{Z}$ , where  $\mathcal{O}_{\mathbb{P}^1}(m)$  denotes the holomorphic line bundle on  $\mathbb{P}^1$  of degree  $m$ . Consider the inverse map  $\varrho : X \rightarrow X$ ,  $(z_1, z_2) \mapsto (z_1, z_2)^{-1} := \frac{1}{|z|^2}(\bar{z}_1, -z_2)$ , which is a biholomorphic map from  $(X, I_-)$  to  $(X, I_+)$ , and we introduce  $\mathcal{O}_-(m) := \varrho^* \mathcal{O}_+(m)$  for all  $m \in \mathbb{Z}$ . For simplicity, we denote  $\mathcal{O}_\pm(0)$  by  $\mathcal{O}_\pm$ . By Proposition 4.5 of [HMS16],  $\mathcal{O}_+(m)$  can be made into an  $I_\pm$ -holomorphic line bundle  $L_+(m) := (\mathbb{O}, \bar{\partial}_{m,+}, \bar{\partial}_{m,-})$  on  $(X, g, I_+, I_-, \mathbb{H})$  such that  $(\mathbb{O}, \bar{\partial}_{m,+}) \simeq \mathcal{O}_+(m)$  and  $(\mathbb{O}, \bar{\partial}_{m,-}) \simeq \mathcal{O}_-(-m)$ , where  $\mathbb{O}$  denotes the topologically trivial line bundle  $X \times \mathbb{C}$  on  $X$ . Similarly, the  $I_\pm$ -holomorphic line bundle associated to  $\mathcal{O}_-(m)$  is denoted by  $L_-(m) := (\mathbb{O}, \bar{\partial}'_{m,+}, \bar{\partial}'_{m,-})$  with isomorphisms  $(\mathbb{O}, \bar{\partial}'_{m,+}) \simeq \mathcal{O}_+(-m)$  and  $(\mathbb{O}, \bar{\partial}'_{m,-}) \simeq \mathcal{O}_-(m)$ . Moreover, one can show that [HMS16]

$$\begin{aligned} \deg_+ L_+(m) &= m, & \deg_- L_+(m) &= -m, \\ \deg_+ L_-(m) &= -m, & \deg_- L_-(m) &= m. \end{aligned}$$

Next we take the quiver  $Q$  be the same as in (1), and an  $I_\pm$ -holomorphic  $Q$ -bundle  $\mathcal{E} = (E, \phi)$  which is given by  $E_i = L_+(m_i), E_j = L_+(m_j)$  and  $\phi_a$ , where  $\phi_a$  must vanish if  $m_i \neq m_j$ . Assume  $m_i = m_j = m$  and  $\phi_a$  is non-zero, then  $\mathcal{E}$  is  $(\alpha, \sigma, \tau)$ -stable if and only if

$$2m\sigma_i\sigma_j(\alpha_j - \alpha_i) < \sigma_i\tau_j - \sigma_j\tau_i.$$

Finally, as the Example 4.11 in [HMS16], let  $V$  be a fixed smooth complex vector bundle of rank 2, we choose  $I_\pm$ -holomorphic structures  $\bar{\partial}_\pm^V$  on  $V$  as follows:

- $\bar{\partial}_+^V$  is  $I_+$ -holomorphic structure such that  $V_+ := (V, \bar{\partial}_+^V)$  is not isomorphic to a sum of two line bundles and is given by the non-trivial extension

$$0 \rightarrow \mathcal{O}_+ \xrightarrow{\chi_+} V_+ \rightarrow \mathcal{O}_+(-m_+) \rightarrow 0$$

for  $m_+ \in \mathbb{Z}^{>0}$ ,

- $\bar{\partial}_-^V$  is  $I_-$ -holomorphic structure such that  $V_- := (V, \bar{\partial}_-^V)$  is given by the non-trivial extension

$$0 \rightarrow \mathcal{O}_- \xrightarrow{\chi_-} V_- \rightarrow \mathcal{O}_-(m_-) \rightarrow 0$$

for  $m_- \in \mathbb{Z}^{\geq 2}$ .

We assume the images of  $\mathcal{O}_\pm$  in  $V_\pm$  coincide as smooth line subbundles of  $V$ . Then  $\tilde{L} := (\mathbb{O}, \bar{\partial}_{0,+}, \bar{\partial}'_{0,-})$  is the only  $I_\pm$ -holomorphic line subbundle of  $(V, \bar{\partial}_+^V, \bar{\partial}_-^V)$  [HMS16]. The  $I_\pm$ -holomorphic  $Q$ -bundle  $\mathcal{E}' = (E', \phi')$  is given by  $E'_i = \tilde{L}, E'_j = (V, \bar{\partial}_+^V, \bar{\partial}_-^V)$  and  $\phi'_a$  is determined by the inclusions  $\chi_\pm$ . To find the constraints on stability parameters, note that  $\mathcal{E}'$  has 3 proper

$Q$ -subbundles:

- (i)  $\mathcal{F} = (F, \phi)$ , where  $F_i = \tilde{L}$ ,  $F_j = \tilde{L}$  and  $\phi_a$  is induced by  $\phi'_a$ , which is identity;
- (ii)  $\mathcal{F} = (F, \phi)$ , where  $F_i = 0$ ,  $F_j = \tilde{L}$  and  $\phi_a = 0$ ;
- (iii)  $\mathcal{F} = (F, \phi)$ , where  $F_i = 0$ ,  $F_j = (V, \bar{\partial}_+^V, \bar{\partial}_-^V)$  and  $\phi_a = 0$ ,

therefore,  $\mathcal{E}'$  is  $(\alpha, \sigma, \tau)$ -stable if and only if the following inequalities are satisfied

$$\begin{aligned} (\sigma_i + \sigma_j)\sigma_j(\alpha_j m_+ - (1 - \alpha_j)m_-) &< \sigma_j\tau_i - \sigma_i\tau_j, \\ \sigma_j^2(\alpha_j m_+ - (1 - \alpha_j)m_-) &< -\sigma_j\tau_i + \sigma_i\tau_j, \\ \sigma_i\sigma_j(\alpha_j m_+ - (1 - \alpha_j)m_-) &> 2(\sigma_j\tau_i - \sigma_i\tau_j). \end{aligned}$$

### 6.3 Kobayashi–Hitchin Correspondence

After introducing the stability for  $Q$ -bundles and Hermitian–Einstein metrics, our main issue is to show the following Kobayashi–Hitchin correspondence:

**Theorem 6.3.1.** *Let  $Q = (Q_0, Q_1)$  be a quiver, and  $\mathcal{E} = (E, \phi)$  be an  $I_{\pm}$ -holomorphic  $Q$ -bundle over an  $n$ -dimensional compact generalized Kähler manifold  $(X, I_+, I_-, g, b)$  such that  $g$  is Gauduchon with respect to both  $I_+$  and  $I_-$ , then  $\mathcal{E}$  is  $(\alpha, \sigma, \tau)$ -polystable if and only if  $\mathcal{E}$  admits an  $(\alpha, \sigma, \tau)$ -Hermitian–Einstein metric.*

One direction is quite easy:

**Lemma 6.3.2.** *If there exists an  $(\alpha, \sigma, \tau)$ -Hermitian–Einstein metric on an  $I_{\pm}$ -holomorphic  $Q$ -bundle  $\mathcal{E} = (E, \phi)$  over an  $n$ -dimensional generalized Kähler manifold  $(X, I_+, I_-, g, b)$ , then  $\mathcal{E}$  is  $(\alpha, \sigma, \tau)$ -polystable.*

*Proof.* Let  $\mathcal{E}'$  be a proper coherent  $Q$ -subsheaf of  $\mathcal{E}$ . At each vertex  $i \in Q_0$ , one defines the orthogonal projections  $p_{\pm i} : E_{\pm i} \rightarrow E'_{\pm i}$ , which are defined outside  $S_{\pm i}$ , respectively, via the metric  $H_i$ , then we have

$$\deg_{\pm}(E'_{\pm i}) = \frac{(n-1)!}{2\pi} \int_{X \setminus S_{\pm i}} [\text{Tr}(\sqrt{-1}p_{\pm i} \circ \Lambda_{\pm} \mathbb{F}_{H_i}^{\pm}) - |\xi_i^{\pm}|_{H_i}^2] d\text{Vol}_g,$$

where  $\xi_i^{\pm} = \bar{\partial}_{\pm i} p_{\pm i}$  denote the second fundamental forms which are of class  $L^2$ . Hence, by assumption that  $H$  is a Hermitian–Einstein metric on  $\mathcal{E}$ , the degree is calculated as

$$\begin{aligned} \deg_{\alpha, \sigma, \tau}(\mathcal{E}') &= \mu_{\alpha, \sigma, \tau}(\mathcal{E}) \sum_{i \in Q_0} \sigma_i \text{rk}(E'_i) \\ &\quad - \frac{(n-1)!}{2\pi} \sum_{i \in Q_0} \int_{X \setminus S} (\alpha_i \sigma_i |\xi_i^+|_{H_i, g}^2 + (1 - \alpha_i) \sigma_i |\xi_i^-|_{H_i, g}^2) d\text{Vol}_g \\ &\quad - \frac{(n-1)!}{2\pi} \sum_{a \in Q_1} \int_{X \setminus S} |\phi_a^{\perp}|_H^2, \end{aligned}$$

where  $S = \bigcup_{i \in Q_0} S_i$ ,  $\phi_a^{\perp}$  is the composition  $(E'_{t(a)})^{\perp} \xrightarrow{\phi_a} E_{h(a)} \xrightarrow{P_{h(a)}} E'_{h(a)}$  for the orthogonal complement  $(E'_{t(a)})^{\perp}$  of  $E'_{t(a)}$  in  $E_{t(a)}$  defined outside  $S_{h(a)} \cup S_{t(a)}$ , and  $|\phi_a^{\perp}|_H^2$  is defined via the induced

metric  $H$  on  $E_{h(a)} \otimes (E_{t(a)})^*$ . It follows that  $\mathcal{E}$  is semistable. Assume  $\mathcal{E} = (E, \phi)$  is indecomposable, i.e.  $\mathcal{E}$  cannot be written as a direct sum of two  $Q$ -bundles, then either  $\xi_i \neq 0$  for some  $i \in Q_0$  or  $\phi_a^\perp \neq 0$  for some  $a \in Q_1$ , therefore  $\mu_{\alpha, \sigma, \tau}(\mathcal{E}') < \mu_{\alpha, \sigma, \tau}(\mathcal{E})$ , thus  $\mathcal{E}$  is stable.

Finally, we find that  $\mathcal{E}$  is  $(\alpha, \sigma, \tau)$ -polystable.  $\square$

The existence of Hermitian–Einstein metrics on  $Q$ -bundles is hard to show. We use the continuity method to show this direction, thus to show that if an  $I_\pm$ -holomorphic  $Q$ -bundle  $\mathcal{E} = (E, \phi)$  is  $(\alpha, \sigma, \tau)$ -stable, then there exist an  $(\alpha, \sigma, \tau)$ -Hermitian–Einstein metric  $H$  on it. The approach of proof we employed in [HH20a] mainly follows from [HMS16, LT95].

We fix a Hermitian metric  $H$  on an  $I_\pm$ -holomorphic  $Q$ -bundle  $\mathcal{E} = (E, \phi)$ . If  $\tilde{H} = Hf = \{H_i f_i\}_{i \in Q_0}$  is an  $(\alpha, \sigma, \tau)$ -Hermitian–Einstein metric for  $f \in S^+(H, \mathcal{E})$ , then at each vertex  $i$  we have

$$\begin{aligned} L_{(\alpha, \sigma, \tau)_i}(f) &:= K_{(\alpha, \sigma, \tau)}(H_i) + \sqrt{-1}(\alpha_i \sigma_i \Lambda_+ \bar{\partial}_+ (f_i^{-1} \partial_{H_i}^+ f_i) + (1 - \alpha_i) \sigma_i \Lambda_- \bar{\partial}_- (f_i^{-1} \partial_{H_i}^- f_i)) \\ &\quad + \sum_{a \in h^{-1}(i)} \phi_a \circ (\phi_a)^{* \tilde{H}} - \sum_{a \in t^{-1}(i)} (\phi_a)^{* \tilde{H}} \circ \phi_a \\ &= K_{(\alpha, \sigma, \tau)}(\tilde{H}_i) + \sum_{a \in h^{-1}(i)} \phi_a \circ (\phi_a)^{* \tilde{H}} - \sum_{a \in t^{-1}(i)} (\phi_a)^{* \tilde{H}} \circ \phi_a \\ &= 0, \end{aligned}$$

where

$$\begin{aligned} K_{(\alpha, \sigma, \tau)}(H_i) &= \sqrt{-1}(\alpha_i \sigma_i \Lambda_+ \mathbb{F}_{H_i}^+ + (1 - \alpha_i) \sigma_i \Lambda_- \mathbb{F}_{H_i}^-) - \lambda(\tau_i + \mu_{\alpha, \sigma, \tau}(\mathcal{E}) \sigma_i) \text{Id}_{E_i} \\ (\phi_a)^{* \tilde{H}} &= f_{t(a)}^{-1} \circ (\phi_a)^{* H} \circ f_{h(a)}. \end{aligned}$$

The perturbed equation is given by

$$L_{(\alpha, \sigma, \tau)_i}^\varepsilon(f) := L_{(\alpha, \sigma, \tau)_i}(f) + \varepsilon \log f_i = 0$$

for  $\varepsilon \in (0, 1]$ . Consider the set

$$J = \left\{ \varepsilon \in (0, 1] : \text{there exists } f^{(\varepsilon)} \in S^+(\mathcal{E}, H) \text{ such that } L_{(\alpha, \sigma, \tau)_i}^\varepsilon(f^{(\varepsilon)}) = 0 \text{ holds for each vertex } i \in Q_0. \right\}.$$

We will not plan to give a detailed proof here, since our method is an ingredient of [LT95]. Our method can be sketched as follows:

- First step: show  $J$  is non-empty by finding a solution to the simultaneous equations  $\{L_{(\alpha, \sigma, \tau)_i}^1(f) = 0\}_{i \in Q_0}$  has a solution, so  $1 \in J$ ;
- Second step: show  $J \subseteq (0, 1]$  is open by Implicit Function Theorem for Banach spaces;
- Third step: show  $J \subseteq (0, 1]$  is closed by the stability of  $Q$ -bundle (in fact, we only need the simpleness property) and elliptic regularity theory;
- Last step: show the limit  $\lim_{\varepsilon \rightarrow 0} f^{(\varepsilon)}$  exists and  $\tilde{H} = H \cdot \lim_{\varepsilon \rightarrow 0} f^{(\varepsilon)}$  is in fact the Hermitian–Einstein metric.

# Bibliography

- [ABC<sup>+</sup>96] J. Amorós, M. Burger, K. Corlette, D. Kotschick, and D. Toledo. *Fundamental groups of compact Kähler manifolds*, volume 44 of *Mathematical Surveys and Monographs*. American Mathematical Society, 1996.
- [ACGP03a] L. Álvarez-Cónsul and O. García-Prada. Dimensional reduction and quiver bundles. *J. Reine Angew. Math.*, 556:1–46, 2003.
- [ACGP03b] L. Álvarez-Cónsul and O. García-Prada. Hitchin–Kobayashi correspondence, quivers, and vortices. *Comm. Math. Phys.*, 238:1–33, 2003.
- [ACGPS06] L. Álvarez-Cónsul, O. García-Prada, and A. Schmitt. On the geometry of moduli spaces of holomorphic chains over compact Riemann surfaces. *Int. Math. Res. Pap.*, pages 1–82, 2006.
- [AGT16] A. Abbes, M. Gros, and T. Tsuji. *The  $p$ -adic Simpson correspondence*, volume 193 of *Ann. of Math. Studies*. Princeton University Press, 2016.
- [AI19] P. Apruzzese and K. Igusa. Stability conditions for affine type A. *Alg. and Repr. Theory*, 2019.
- [BB04] O. Biquard and P. Boalch. Wild non-abelian Hodge theory on curves. *Compos. Math.*, 140:179–204, 2004.
- [BGG97] I. Biswas, P. Gaiety, and S. Govindarajan. Parabolic Higgs bundles and Teichmüller spaces for punctured surfaces. *Trans. Amer. Math. Soc.*, 349:1551–1560, 1997.
- [BGHL09] I. Biswas, T. Gómez, N. Hoffman, and M. Logares. Torelli theorem for the Deligne–Hitchin moduli space. *Comm. Math. Phys.*, 290:357–369, 2009.
- [BGL11] I. Biswas, P. Gothen, and M. Logares. On moduli spaces of Hitchin pairs. *Math. Proc. Camb. Phil. Soc.*, 151:441–457, 2011.
- [BGPG04] S. Bradlow, O. García-Prada, and P. Gothen. Moduli spaces of holomorphic triples over compact Riemann surfaces. *Math. Ann.*, 328:299–351, 2004.
- [BGPGH18] S. Bradlow, O. García-Prada, P. Gothen, and J. Heinloth. Irreducibility of moduli of semistable chains and applications to  $U(p, q)$ -Higgs bundles. In *Geometry and Physics: A Festschrift in Honor of Nigel Hitchin*, volume 2, pages 455–470, 2018.
- [BGPiR03] S. Bradlow, O. García-Prada, and I. Mundet i Riera. Relative Hitchin–Kobayashi correspondence for principal pairs. *Quat. J. Math.*, 54:171–208, 2003.
- [BH99] N. Buchdahl and A. Harris. Holomorphic connections and extensions of complex vector bundles. *Math. Nachr.*, 201:29–39, 1999.

- [BH17] I. Biswas and S. Heller. On the automorphisms of a rank one Deligne–Hitchin moduli space. *SIGMA*, 13(072), 2017.
- [BHR19] I. Biswas, S. Heller, and M. Röser. Real holomorphic sections of the Deligne–Hitchin twistor space. *Comm. Math. Phys.*, 290:1099–1133, 2019.
- [Biq97] O. Biquard. Fibrés de Higgs et connexions intégrables: le cas logarithmique (diviseur lisse). *Ann. Sci. École Norm. Sup.*, 30:41–96, 1997.
- [Bis97] I. Biswas. Parabolic bundles as orbifold bundles. *Duke. Math. J.*, 88(2):305–325, 1997.
- [Bis03] I. Biswas. On the principal bundles with parabolic structure. *J. Math. Kyoto Univ.*, 43-2:305–332, 2003.
- [BM09] I. Biswas and V. Muñoz. Torelli theorem for moduli spaces of  $SL(r, \mathbb{C})$ -connections on a compact Riemann surface. *Comm. Contemp. Math.*, 11:1–26, 2009.
- [BMW13] I. Biswas, S. Majumder, and W. Wong. Parabolic Higgs bundles and  $\Gamma$ -Higgs bundles. *J. Aust. Math. Soc.*, 95:315–328, 2013.
- [Bor07] N. Borne. Fibrés paraboliques et champ des racines. *Int. Math. Res. Notices*, 16, 2007.
- [Cad07] C. Cadman. Using stacks to impose tangency conditions on curves. *Amer. J. Math.*, 129:405–427, 2007.
- [Car14] S. Cardona. On vanishing theorems for Higgs bundles. *Differ. Geom. Appl.*, 35:95–102, 2014.
- [Cor88] K. Corlette. Flat  $G$ -bundles with canonical metrics. *Jour. Diff. Geom.*, 28:361–382, 1988.
- [CW19] B. Collier and R. Wentworth. Conformal limits and the Bialynicki-Birula stratification of the space of  $\lambda$ -connections. *Adv. Math.*, 350:1193–1225, 2019.
- [Del89] P. Deligne. Various letters to C. Simpson. 1989.
- [DFK<sup>+</sup>16] O. Dumitrescu, L. Fredrickson, G. Kydonakis, R. Mazzeo, M. Mulase, and A. Neitzke. Opers versus nonabelian Hodge. *arXiv: 1607.02172*, 2016.
- [DN89] J. Drezet and M. S. Narasimhan. Groupe de Picard des variétés de modules de fibrés semi-stables sur les courbes algébriques. *Invent. Math.*, 97:53–94, 1989.
- [Don83] S. Donaldson. A new proof of a theorem of Narasimhan and Seshadri. *Jour. Diff. Geom.*, 18:269–277, 1983.
- [Don85] S. Donaldson. Anti self-dual Yang–Mills connections over compact algebraic surfaces and stable vector bundles. *Proc. London Maths. Soc.*, 50:1–26, 1985.
- [Don87] S. Donaldson. Twisted harmonic maps and the self-duality equations. *Proc. London Maths. Soc.*, 55:127–131, 1987.
- [DW00] H. Derksen and J. Weyman. Semi-invariants of quivers and saturation for Littlewood–Richardson coefficients. *J. Amer. Math. Soc.*, 13:467–479, 2000.

- [DW11] H. Derksen and J. Weyman. The combinatorics of quiver representations. *Ann. Inst. Fourier*, 61:1066–1131, 2011.
- [Fal93] G. Faltings. Stable  $G$ -bundles and projective connections. *Jour. Alge. Geom.*, 2:507–568, 1993.
- [Fal05] G. Faltings. A  $p$ -adic Simpson correspondence. *Adv. Math.*, 198:847–862, 2005.
- [Fuj91] A. Fujiki. HyperKähler structures on the moduli space of flat bundles. In *Prospects in complex geometry*, volume 1468 of *Lecture Notes in Mathematics*, pages 1–83. Springer, 1991.
- [G01] T. Gómez. Algebraic stacks. *Proc. Indian Acad. Sci. Math. Sci.*, 111:1–31, 2001.
- [Gai14] D. Gaiotto. Opers and TBA. *arXiv: 1403.6137*, 2014.
- [GHR84] S. Gates, C. Hull, and M. Roček. Twisted multiplets and new supersymmetric non-linear  $\sigma$ -models. *Nuclear Phys. B.*, 248:157–186, 1984.
- [GKPT19] D. Greb, S. Kebekus, T. Peternell, and B. Taji. Nonabelian Hodge theory for klt spaces and descent theorems for vector bundles. *Compo. Math.*, 155:289–323, 2019.
- [GnR17] P. Gothen and R. Zúñiga Rojas. Stratifications on the moduli space of Higgs bundles. *Portugal. Math.*, 74:127–148, 2017.
- [Got94] P. Gothen. The Betti numbers of the moduli space of rank 3 Higgs bundles on Riemann surface. *Internat. J. Math.*, 5:861–875, 1994.
- [GPGiR09] O. García-Prada, P. Gothen, and I. Mundet i Riera. The Hitchin–Kobayashi correspondence, Higgs pairs and surface group representations. *arXiv: 0909.4487*, 2009.
- [GPGM07] O. García-Prada, P. Gothen, and V. Muñoz. Betti numbers of the moduli space of rank 3 parabolic Higgs bundles. *Mem. Amer. Math. Soc.*, 187(879), 2007.
- [GPH13] O. García-Prada and J. Heinloth. The  $y$ -genus of the moduli space of  $\mathrm{PGL}_n$ -Higgs bundles on a curve (for degree coprime to  $n$ ). *Duke. Math. J.*, 162:2731–2749, 2013.
- [GPHS14] O. García-Prada, J. Heinloth, and A. Schmitt. On the motives of moduli of chains and Higgs bundles. *J. Eur. Math. Soc.*, 16:2617–2668, 2014.
- [Gua11] M. Gualtieri. Generalized complex geometry. *Ann. of Math.*, 174:75–123, 2011.
- [Gua14] M. Gualtieri. Generalized Kähler geometry. *Comm. Math. Phys.*, 331:297–331, 2014.
- [Hau98] T. Hausel. *Geometry of the moduli space of Higgs bundles*. PhD thesis, University of Cambridge, arXiv: math/0107040, 1998.
- [Hei16] J. Heinloth. The intersection form on moduli spaces of twisted  $\mathrm{PGL}_n$ -Higgs bundles vanishes. *Math. Ann.*, 365:1499–1526, 2016.
- [HH17] Z. Hu and P. Huang. Degenerate, strong and stable Yang–Mills–Higgs pairs. *Jour. Geom. Phys.*, 120:73–88, 2017.
- [HH19] Z. Hu and P. Huang. Flat  $\lambda$ -connections, Mochizuki correspondence and twistor spaces. *arXiv: 1905.10765*, 2019.

- [HH20a] Z. Hu and P. Huang. The Hitchin–Kobayashi correspondence for quiver bundles over generalized Kähler manifolds. *Jour. Geom. Anal.*, 2020.
- [HH20b] P. Huang and Z. Hu. Stability and indecomposability of the representations of quivers of  $A_n$ -type. *Comm. in Alg.*, 48:2905–2919, 2020.
- [Hil17] L. Hille. Private communications. April 2017.
- [Hit87a] N. Hitchin. The self-duality equations on a Riemann surface. *Proc. London Maths. Soc.*, 55:59–126, 1987.
- [Hit87b] N. Hitchin. Stable bundles and integrable systems. *Duke. Math. J.*, 54:91–114, 1987.
- [Hit03] N. Hitchin. Generalized Calabi–Yau manifolds. *Quat. J. Math.*, 54:281–308, 2003.
- [Hit07] N. Hitchin. Bihermitian structures on Del Pezzo surfaces. *Jour. Symp. Geom.*, 5:1–8, 2007.
- [HKLR87] N. Hitchin, A. Karlhede, U. Lindström, and M. Roček. Hyper-Kähler metrics and supersymmetry. *Comm. Math. Phys.*, 108:535–589, 1987.
- [HMS16] S. Hu, R. Moraru, and R. Seyyedali. A Kobayashi–Hitchin correspondence for  $I_{\pm}$ -holomorphic bundles. *Adv. Math.*, 287:519–566, 2016.
- [HT03] T. Hausel and M. Thaddeus. Mirror symmetry, Langlands duality, and the Hitchin system. *Invent. Math.*, 153:197–229, 2003.
- [Hua18] P. Huang. Moduli space of parabolic Higgs bundles. Poster presented in RIMS, 2018.
- [Hua20] P. Huang. Non-Abelian Hodge theory and related topics. *SIGMA*, 16(029):34 pages, 2020.
- [IS07] J. Iyer and C. Simpson. A relation between parabolic Chern characters of the de Rham bundles. *Math. Ann.*, 338(2):347–383, 2007.
- [IS08] J. Iyer and C. Simpson. The Chern character of a parabolic bundle, and a parabolic corollary of Reznikov’s theorem. In *Geometry and Dynamics of Groups and Spaces*, volume 265 of *Progress in Mathematics*, pages 439–485. Birkhäuser, Basel, 2008.
- [IT09] C. Ingalls and H. Thomas. Noncrossing partions and representations of quivers. *Compo. Math.*, 145:1553–1562, 2009.
- [Jr.16] A. Kirillov Jr. *Quiver Representations and Quiver Varieties*, volume Graduate Studies in Mathematics. American Mathematical Society, 2016.
- [Jut17] D. Juteau. Private communications. April 2017.
- [JZ97] J. Jost and K. Zuo. Harmonic maps of infinite energy and rigidity results for representations of fundamental groups of quasiprojective varieties. *Jour. Diff. Geom.*, 47:469–503, 1997.
- [Kel11] B. Keller. On cluster theory and quantum dilogarithm identities. In *Representations of Algebras and Related Topics*. European Mathematical Society, 2011.
- [Kin20] R. Kinser. The cone of weights of total stability for type  $\mathbb{A}$  quivers. *arXiv: 2002.12396*, 2020.

- [KMS06] A. Knutson, E. Miller, and M. Shimozono. Four positive formulae for type A quiver polynomials. *Invent. Math.*, 166:229–325, 2006.
- [Kod62] K. Kodaira. A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds. *Ann. of Math.*, 84:146–162, 1962.
- [KOT19] K. Kikuta, G. Ouchi, and A. Takahashi. Serre dimensions and stability conditions. *arXiv: 1907.10981*, 2019.
- [KP95] A. Kovidakis and T. Pantev. The automorphism group of the moduli space of semistable vector bundles. *Math. Ann.*, 302:225–268, 1995.
- [KV98] D. Kaledin and M. Verbitsky. Non-Hermitian Yang–Mills connections. *Selecta Math.*, 4:279–320, 1998.
- [Lau87] G. Laumon. Un analogue global du cône nilpotent. *Duke. Math. J.*, 54:91–114, 1987.
- [Li00] J. Li. Hermitian–Einstein metrics and Chern number inequalities on parabolic stable bundles over Kähler manifolds. *Comm. Anal. Geom.*, 3:445–475, 2000.
- [LM98] V. Lakshmibai and P. Magyar. Degeneracy schemes, quiver schemes, and Schubert varieties. *Int. Math. Res. Notices*, 12:627–640, 1998.
- [LMB00] G. Laumon and L. Moret-Bailly. *Champs algébriques*. Springer, 2000.
- [LSS13] F. Loray, M-H. Saito, and C. Simpson. Foliations on the moduli space of rank two connections on the projective line minus four points. *Séminaires et Congrès*, 27:115–168, 2013.
- [LSYZ19] G. Lan, M. Sheng, Y. Yang, and K. Zuo. Uniformization of  $p$ -adic curves via Higgs–de Rham flows. *J. Reine Angew. Math.*, 747:63–108, 2019.
- [LSZ19] G. Lan, M. Sheng, and K. Zuo. Semistable Higgs bundles and representations of algebraic fundamental groups. *Jour. Eur. Math. Soc.*, 21(10):3053–3112, 2019.
- [LT95] M. Lübke and A. Teleman. *The Kobayashi–Hitchin Correspondence*. World Scientific Publishing, 1995.
- [LY87] J. Li and S.-T. Yau. Hermitian Yang–Mills connections on non-Kähler manifolds. In *Mathematical aspects of string theory*. 1987.
- [LZ15] J. Li and X. Zhang. Existence of approximate Hermitian–Einstein structures on semi-stable Higgs bundles. *Cal. Var. PDE*, 52:783–795, 2015.
- [Moc06] T. Mochizuki. *Kobayashi–Hitchin correspondence for tame harmonic bundles and an application*, volume 309 of *Astérisque*. Société Mathématique de France, 2006.
- [Moc07] T. Mochizuki. Asymptotic behaviour of tame harmonic bundles and an application to pure twistor  $D$ -modules. *Mem. Amer. Math. Soc.*, 185(869-870), 2007.
- [Moc09] T. Mochizuki. Kobayashi–Hitchin correspondence for tame harmonic bundles, II. *Geom. Topol.*, 13(1):359–455, 2009.
- [Moc11] T. Mochizuki. *Wild harmonic bundles and wild pure twistor  $D$ -modules*. *Astérisque*. Société Mathématique de France, 2011.

- [Moc16] T. Mochizuki. Asymptotic behaviour of certain families harmonic bundles on Riemann surfaces. *J. Topol.*, 9(4):1021–1073, 2016.
- [Moc20] T. Mochizuki. Kobayashi–Hitchin correspondence for analytically stable bundles. *Trans. Amer. Math. Soc.*, 373:551–596, 2020.
- [MS80] V. Mehta and C. S. Seshadri. Moduli of vector bundles on curves with parabolic structures. *Math. Ann.*, 248(3):205–239, 1980.
- [MSWW16] R. Mazzeo, J. Swoboda, H. Weiss, and F. Witt. Ends of the moduli space of Higgs bundles. *Duke. Math. J.*, 165:2227–2271, 2016.
- [MY92] M. Maruyama and K. Yokogawa. Moduli of parabolic stable sheaves. *Math. Ann.*, 293(1):77–99, 1992.
- [Nir08] F. Nironi. Moduli spaces of semistable sheaves on projective Deligne–Mumford stacks. *arXiv: 0811.1949*, 2008.
- [NS65] M. S. Narasimhan and C. S. Seshadri. Stable and unitary vector bundles on a compact Riemann surface. *Ann. of Math.*, 82(2):540–567, 1965.
- [NZ18] Y. Nie and X. Zhang. Semistable Higgs bundles over compact Gauduchon manifolds. *Jour. Geom. Anal.*, 28:627–642, 2018.
- [Ols06] M. Olsson. *Algebraic spaces and stacks*. American Mathematical Society, 2006.
- [OV07] A. Ogus and V. Vologodsky. Nonabelian Hodge theory in characteristic  $p$ . *Inst. Hautes Études Sci. Publ. Math.*, 106:1–138, 2007.
- [PPN19] C. Pauly and A. Peón-Nieto. Very stable bundles and properness of the Hitchin map. *Geom. Dedicata*, 198:143–148, 2019.
- [Rie03] M. Rieneke. The Harder–Narasimhan system in quantum groups and cohomology of quiver moduli. *Invent. Math.*, 152:349–368, 2003.
- [Sal82] S. Salamon. Quaternionic Kähler manifolds. *Invent. Math.*, 67:143–171, 1982.
- [Sam86] J. Sampson. Applications of harmonic maps to Kähler geometry. In Y.-T. Siu, editor, *Contemp. Math.*, volume 49, pages 125–134, 1986.
- [Ses82] C. S. Seshadri. *Fibrés vectoriels sur les courbes algébriques*, volume 96 of *Astérisque*. Société Mathématique de France, 1982.
- [Sim87] C. Simpson. Constructing of variations of Hodge structure using Yang–Mills theory and applications to uniformization. *J. Amer. Math. Soc.*, 1(4):867–918, 1987.
- [Sim90] C. Simpson. Harmonic bundles on noncompact curves. *J. Amer. Math. Soc.*, 3(3):713–770, 1990.
- [Sim91] C. Simpson. A lower bound for the size of monodromy of systems of ordinary differential equations. In A. Fujiki et al., editor, *ICM-90 Satellite Conference Proceedings*, pages 198–230, 1991.
- [Sim92] C. Simpson. Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.*, 75:5–95, 1992.

- [Sim94a] C. Simpson. Moduli of representations of the fundamental group of a smooth projective variety, I. *Inst. Hautes Études Sci. Publ. Math.*, 79:47–129, 1994.
- [Sim94b] C. Simpson. Moduli of representations of the fundamental group of a smooth projective variety, II. *Inst. Hautes Études Sci. Publ. Math.*, 80:5–79, 1994.
- [Sim95] C. Simpson. The Hodge filtration on nonabelian cohomology. In *Proc. Sympos. Pure Math.*, volume 62, pages 217–281, 1995.
- [Sim08] C. Simpson. A weight two phenomenon for the moduli of rank one local systems on open varieties. In *Proc. Sympos. Pure Math.*, volume 78, pages 175–214, 2008.
- [Sim10] C. Simpson. Iterated destabilizing modifications for vector bundles with connection. In *Contemp. Math.*, volume 522, pages 183–206, 2010.
- [Sin20] A. Singh. On the moduli space of  $\lambda$ -connections. *arXiv: 2002.00358*, 2020.
- [Siu80] Y.-T. Siu. The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds. *Ann. of Math.*, 112:73–111, 1980.
- [Tah10] C. Taher. Calculating the parabolic Chern character of a locally abelian parabolic bundles. *manu. math.*, 132:169–198, 2010.
- [UY86] K. Uhlenbeck and S.-T. Yau. On existence of Hermitian–Yang–Mills connections in stable vector bundles. *Comm. Pure Appl. Math.*, 398:257–293, 1986.
- [Ver14] M. Verbitsky. Holography principle for twistor spaces. *Pure Appl. Math. Q.*, 10:325–354, 2014.
- [Wit06] F. Witt. Generalized  $G_2$ -manifolds. *Comm. Math. Phys.*, 265:275–303, 2006.
- [Yok93] K. Yokogawa. Compactification of moduli of parabolic sheaves moduli of parabolic Higgs sheaves. *J. Math. Kyoto Univ.*, 33(2):451–504, 1993.
- [Yok95] K. Yokogawa. Infinitesimal deformations of parabolic Higgs sheaves. *Internat. J. Math.*, 6(1):125–148, 1995.