Reaction-diffusion equations, qualitative properties and population dynamics

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Résumé

Nous nous intéressons à certains problèmes issus des équations de réaction-diffusion et de leur application à la dynamique des populations.

La première partie traite des solutions stationnaires *stables* des équations de réaction-diffusion. Nous nous intéressons en particulier à l'influence de la géométrie du domaine sur l'existence de solutions stables non-constantes, appelées *patterns*. Nous établissons un critère de non-existence de patterns pour des domaines généraux.

Dans la deuxième partie, nous nous intéresserons à un modèle Hamilton-Jacobi pour la théorie de l'évolution darwinienne. Notre modèle présente un phénomène de *concentration*, c'est-à-dire que la population converge vers une masse de Dirac quand un paramètre d'échelle tend vers 0. Nous étudions le cas d'une population structurée en âge et en phénotype, soumise à une compétition entre individus. Dans un deuxième temps, nous ajoutons l'effet de mutations. Nous considérons également un modèle faisant intervenir un phénomène de *sauvetage évolutif*, dans lequel la population peut avoir une dynamique cyclique.

La troisième partie est consacrée à l'étude de systèmes d'équations de réaction-diffusion. Notre cadre contient le modèle d'épidémiologie SI, et étend certaines propriétés classiques à une classe plus large. Enfin, nous proposerons un modèle pour rendre compte de la dynamique des émeutes et de l'agitation sociale.

Abstract

We are interested in some problems arising in reaction-diffusion equations and their application to population dynamics.

The first part deals with *stable* stationary solutions of reaction-diffusion equations. More precisely, our aim is to understand the influence of the geometry of the domain on the existence of stable non-constant solutions, called *patterns*. We establish a criterion for the non-existence of patterns in general domains.

In the second part, we address a Hamilton-Jacobi model for Darwin's theory of evolution. This models features a *concentration* phenomenon, that is, the solution converges to a Dirac mass when a rescalling parameters goes to 0. We study the case of a population structured by age and phenotype, subject to competition between individuals. In a second step, we add the effect of mutations. We also consider a model which features a phenomenon of *evolutionary rescue*, in which the population can have cyclic dynamics.

The third part is devoted to the study of systems of reaction-diffusion equations. Our framework encompasses the epidemiological SI model, and extends some results to a broader class. Finally, we propose a model to account for the dynamics of riots and social unrest.

Remerciements

J'adresse mes premiers remerciements à mon directeur de thèse, Henri Berestycki, qui a su me proposer d'excellents sujets et être disponible chaque fois que j'avais besoin d'aide. Je le remercie sincèrement pour sa bienveillance. Ces années de thèses ont été passionnantes et inoubliables.

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Introduction

Chapitre A

Introduction Générale : équations de réaction-diffusion et dynamique des populations

An introduction in English is available on page 83.

Cette thèse porte sur certains problèmes issus des équations de réaction-diffusion et de la modélisation de la dynamique des populations. Dans cette introduction, nous proposons un bref historique et présentons le contexte général dans lequel s'inscrit ce travail.

En préambule, Il nous semble avant tout important de préciser brièvement ce que nous entendons par *modèle*, et en quoi consiste le rôle des mathématiques dans le processus de modélisation.

Un modèle est une expérience de pensée, un cadre représentatif idéalisé, reconnu approximatif et schématique, qui propose un point de vue (parmi d'autres possibles) sur un phénomène. Un modèle doit, d'une part, pouvoir rendre compte du phénomène étudié de manière précise. D'autre part, un modèle doit également satisfaire un critère de simplicité et d'esthétique. À la manière d'un bon croquis, un modèle doit obtenir un maximum de ressemblance en un minimum de traits.

Dans le processus de modélisation, le rôle des mathématiques réside principalement dans l'étude des propriétés intrinsèques des modèles. Les modèles sont étudiés en eux-même, à un niveau conceptuel. Les mathématiques n'étudient donc pas le réel directement, mais le langage avec lequel nous pensons le réel.

Contents

A.1 Bref historique

A.1.1 Premiers modèles discrets

Le premier modèle mathématique connu de dynamique des populations est dû à Leonardo Fibonacci, dans son *Liber Abaci* [147] de 1202. Il modélise la taille d'une population de lapins en prenant comme hypothèses :

— un couple de lapin adulte engendre, chaque mois, un nouveau couple de lapins.

— un lapin atteint la maturité au bout d'un mois.

En notant u_n le nombre de couples de lapin au n-ième mois, on en déduit la relation

$$u_{n+2} = u_{n+1} + u_n.$$

C'est la fameuse suite de Fibonacci. En partant d'un seul couple de lapins nouveaux-nés, c'est-à-dire $u_0 = u_1 = 1$, on en déduit

$$u_n \underset{n\to+\infty}{\sim} \frac{1}{\sqrt{5}}\Phi^n,$$

où $\Phi := \frac{1+\sqrt{5}}{2}$ est le nombre d'or. La taille de la population connait donc une croissance *exponentielle* (ou *géométrique*).

Un modèle comparable est proposé par Thomas Malthus dans son *Essay on the principle of population* [216] de 1798. Il suppose que la population croit à un taux constant $a > 0$ (qui peut être vu comme le taux de naissance moins le taux de mort). Cette hypothèse s'écrit

$$u_{n+1} = au_n.$$

On en déduit

$$u_n = a^n u_0,$$

donc la population a une croissance exponentielle, voir Figure A.4a

Cette observation est accompagnée d'une conclusion pessimiste : puisque, par ailleurs, la production de nourriture croît au plus *linéairement*, l'humanité sera vite confrontée à la famine… Voir Figure A.1.

A.1.2 Premiers modèles continus

En 1838, Pierre-François Verhulst reprend les travaux de Malthus dans sa *Notice sur la loi que la population poursuit dans son accroissement* [281]. Contrairement aux approches précédentes, il considère que le nombre d'individus $u(t)$ évolue continument en temps. En particulier, $u(t)$ prend des valeurs non-entière. Cela prend sens si l'on mesure la population avec une unité très grande (par exemple, le millier d'individus) : alors $u(t)$ représente, non plus le *nombre d'individus*, mais la *taille de la population.*

Verhulst traduit mathématiquement la loi de Malthus, sous la forme de l'équation différentielle

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = au(t).$$

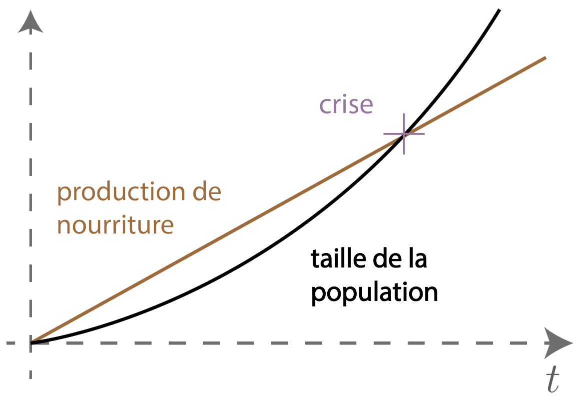
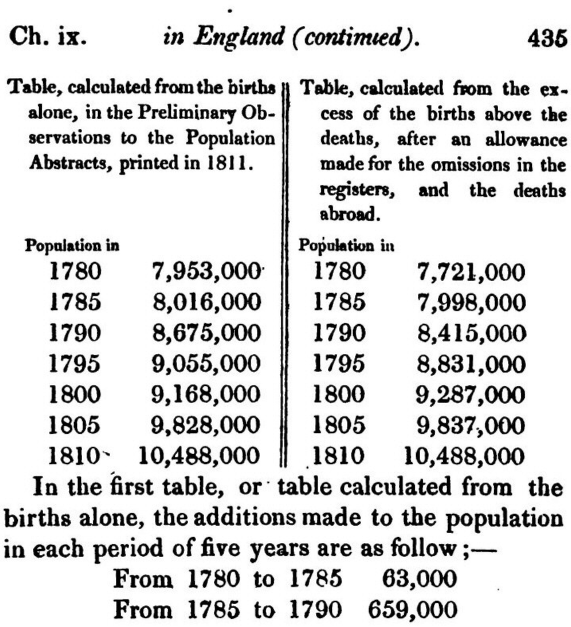


FIGURE A.1 – Schéma de la théorie malthusienne



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Table, calculated from the births alone, in the Preliminary Observations to the Population Abstracts, printed in 1811.		Table, calculated from the excess of the births above the deaths, after an allowance made for the omissions in the registers, and the deaths abroad.	
Population in		Population in	
1780	7,953,000	1780	7,721,000
1785	8,016,000	1785	7,998,000
1790	8,675,000	1790	8,415,000
1795	9,055,000	1795	8,831,000
1800	9,168,000	1800	9,287,000
1805	9,828,000	1805	9,837,000
1810	10,488,000	1810	10,488,000

In the first table, or table calculated from the births alone, the additions made to the population in each period of five years are as follow ;—

From 1780 to 1785 63,000
From 1785 to 1790 659,000

FIGURE A.2 – Table démographique prévisionnelle extraite des travaux de Malthus de 1798

Il en déduit

$$u(t) = u(0)e^{at},$$

et retrouve la conclusion de Malthus : la population a une croissance exponentielle, voir Figure A.4a.

Comme le modèle de Malthus n'est valable que dans un environnement où les ressources sont abondantes, Verhulst propose un modèle pour prendre en compte un phénomène de *saturation* induit par la limitation des ressources. Inspiré par les travaux de Quetelet de 1835 [254], Verhulst considère que le taux de croissance a doit décroitre à mesure que la taille de la population augmente. Le choix le plus simple étant de remplacer a par $a - bu(t)$, où $b > 0$ est une constante, Verhulst remplace l'équation de Malthus par *l'équation logistique* :

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = (a - bu(t))\,u(t). \tag{A.1}$$

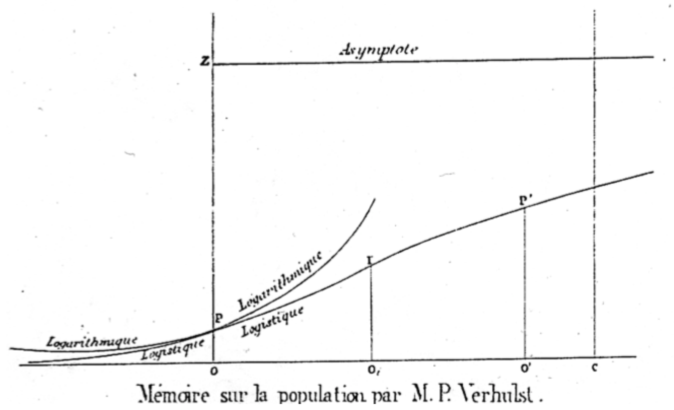


FIGURE A.3 – Représentation des fonctions "logarithmique" (*i.e.* exponentielle) et logistique dans le premier mémoire de Verhulst

Cette équation possède l'état d'équilibre constant

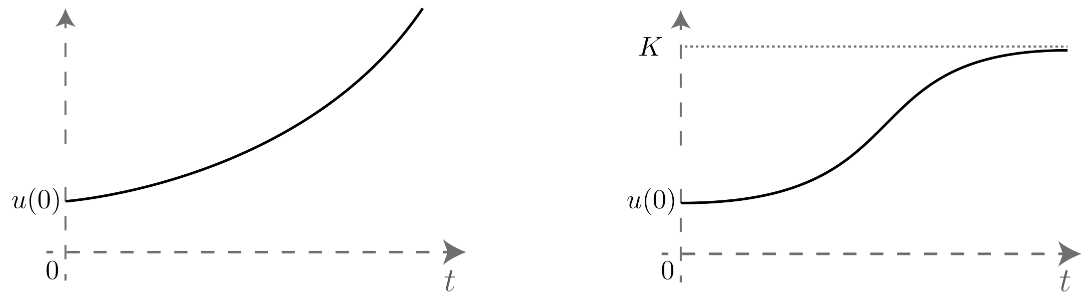
$$K = \frac{a}{b}.$$

La constante K est souvent appelée la *capacité de charge* du milieu (*carrying capacity*), et représente la taille de population maximale que le milieu peut accueillir à long terme. Les solutions de (A.1) s'écrivent

$$u(t) = \frac{Ku(0)}{(K - u(0))e^{-at} + u(0)}.$$

Ces solutions sont dites sigmoïdales, ou de manière informelle, *en forme de S*, voir Figure A.4b. Si $u(0) \leq K$, ces solutions sont croissantes et tendent vers K en temps long. Cela exprime le fait que la population se stabilise.

Pour un historique plus détaillé, nous laisserons le lecteur se référer à [24, 218].



(a) Dans le modèle de Malthus

(b) Dans le modèle de Verhulst

FIGURE A.4 – Évolution de la taille de la population $u(t)$

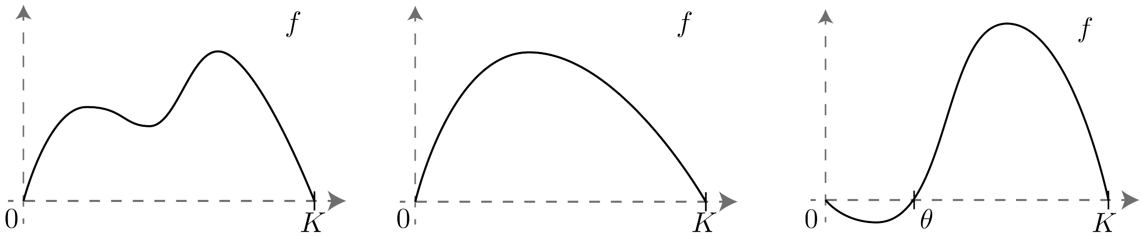
A.2 Équations de réaction-diffusion

A.2.1 Équations différentielles ordinaires

De manière plus générale, on considère que la taille de la population obéit à l'équation

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = f(u(t)),$$

pour une certaine fonction f, appelée la *non-linéarité*. On ne cherche dès lors plus à exprimer les solutions de manière explicite (car ce n'est plus possible), mais à dégager les propriétés qualitatives des solutions, en fonction de la forme de la non-linéarité. On considère en général les cas suivants :



(a) monostable **(b)** monostable KPP **(c)** bistable

FIGURE A.5 – Non-linéarités

— **Non-linéarité monostable.** Voir Figure A.5a. La fonction f est dite de type *monostable* si elle est de la forme

$$f(0) = 0 \quad ; \quad f \geq 0 \text{ sur } [0, K] \quad ; \quad f \leq 0 \text{ sur } [K, +\infty).$$

Cela exprime le fait que la population croit jusqu'à atteindre une taille maximale K, comme dans l'équation logistique (A.1).

— **Non-linéarité monostable-KPP.** On dit que f est de type *KPP* (pour Kolmogorov-Petrovski-Piskunov) si

$$x \mapsto \frac{f(x)}{x} \text{ est décroissante.}$$

Voir Figure A.5b. Cette hypothèse exprime le fait que le taux de croissance diminue à mesure que la taille de la population augmente. L'équation logistique de Verhulst (A.1) satisfait cette hypothèse.
De plus, on dit que f est *KPP au sens faible* si elle est monostable et telle que $f(x) \leq xf'(0)$.

— **Non-linéarité bistable.** Voir Figure A.5c. La fonction f est dite de type *bistable* si elle est de la forme

$$f(0) = 0 \quad ; \quad f \leq 0 \text{ sur } [0, \theta]$$
$$f \geq 0 \text{ sur } [\theta, K] \quad ; \quad f \leq 0 \text{ sur } [K, +\infty).$$

L'exemple le plus simple est

$$f(x) := x(K - x)(x - \theta).$$

Par rapport au cas monostable, on suppose ici que le taux de croissance de la population est négatif si sa taille est en dessous du seuil θ. Cette hypothèse est connue sous le nom de *l'effet Allee*. Du point de vue de la modélisation, elle est nécessaire quand on considère une population en faible effectif.
Sous cette hypothèse, deux scénarios sont possibles. Si $u(0) > \theta$, alors $u(t)$ converge en temps long vers K, comme dans le cas *monostable*. Si $u(0) < \theta$, alors $u(t)$ converge en temps long vers 0 (*i.e.* la population s'éteint).

A.2.2 Modèles avec espace

Les modèles précédents ne prennent pas en compte la répartition des individus dans l'espace. Considérons maintenant que la densité de population $u(t,x)$ dépend du temps $t \geq 0$ et de l'espace $x \in \mathbb{R}^n$, où $n \geq 1$ est la dimension.

Négligeons, pour un moment, la croissance de la population (*i.e.*, supposons $f = 0$) pour nous concentrer uniquement sur l'aspect spatial. Il nous faut modéliser le déplacement des individus dans la population que nous considérons.

Diffusion. Une première hypothèse, sans doute la plus simple, est de considérer que chaque individu se déplace selon un mouvement brownien (voir Figure A.6). Ce mouvement correspond à celui d'une particule dans un gaz. À l'échelle macroscopique (c'est-à-dire, quand on ne considère non pas chaque individu séparément, mais la densité d'un grand nombre d'individus) cette hypothèse se traduit par une *diffusion*. Cela s'exprime mathématiquement par l'opérateur Laplacien : la densité de population $u(t,x)$ vérifie

$$\partial_t u(t,x) - d\Delta_x u(t,x) = 0, \tag{A.2}$$

où

$$\Delta_x u(t,x) = \partial^2_{x_1} u(t,x) + \cdots + \partial^2_{x_n} u(t,x).$$

La constante $d \geq 0$ est le coefficient de diffusion, homogène à *longueur*^{2}.*temps*$^{-1}$, et rend compte de la mobilité des individus.

L'équation (A.2) est appelée *Équation de la Chaleur*. Toute solution $u(t,x)$ converge uniformément vers 0 en temps long. De plus, l'équation a un effet *régularisant*, c'est-à-dire que même si la donnée initiale $u(t=0,\cdot)$ est irrégulière, la fonction $u(t,\cdot)$ est lisse pour tout $t > 0$.

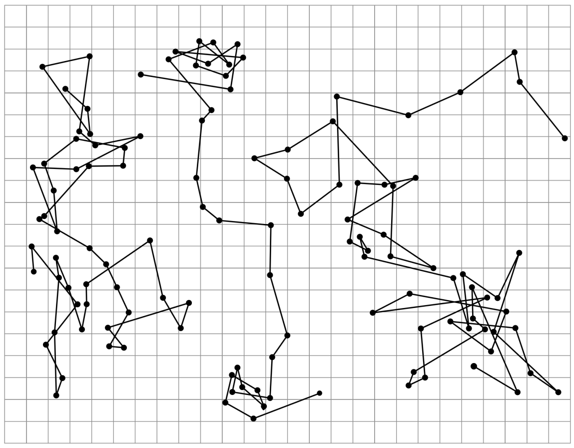


Figure A.6 – Mouvements browniens

Dérive (drift). Nous faisons l'hypothèse supplémentaire qu'une force, dirigée par un vecteur $\overrightarrow{b}$, induit une dérive de nos individus (pensons, par exemple, à une population de poissons soumis à une force de courant). La densité de population satisfait alors

$$\partial_t u(t,x) - d\Delta_x u(t,x) + \overrightarrow{b} \cdot \nabla_x u(t,x) = 0.$$

En l'absence de diffusion (*i.e.*, si $d = 0$), cette équation s'appelle une *équation de transport*.

Diffusion non-locale. Une autre hypothèse possible consiste à considérer qu'un individu à l'emplacement x *migre* vers un emplacement y avec une certaine probabilité $M(y-x)dy$. L'équation vérifiée par u s'écrit alors

$$\partial_t u(t,x) - \left(\int_{\mathbb{R}^n} M(x-y)u(t,y)dy - u(t,x)\right) = 0.$$

Il s'agit d'une équation de diffusion non-locale.

A.2.3 Conditions au bord

Supposons à présent que notre population évolue, non pas sur $\mathbb{R}^n$, mais sur un domaine quelconque $\Omega \subset \mathbb{R}^n$. Il nous faut modéliser le comportement d'un individu quand il touche le bord $\partial\Omega$. Nous considérons trois cas de figure.

— **Condition au bord de Dirichlet**. Elle correspond à l'hypothèse qu'un individu disparait (ou meurt) dès qu'il touche le bord. À l'échelle macroscopique, elle se traduit par

$$u = 0 \quad \text{sur } \partial\Omega.$$

— **Condition au bord de Neumann**. Elle correspond à l'hypothèse qu'un individu rebondit sur le bord. À l'échelle macroscopique, elle se traduit par

$$\partial_\nu u = 0 \quad \text{sur } \partial\Omega.$$

où ∂_ν est la dérivée normale sortante au domaine. Elle exprime le fait que le flux d'individus vers l'extérieur du domaine est nul.

— **Conditions au bord de Robin**. Elle correspond à l'hypothèse que les individus qui touchent le bord ont un taux de croissance $\gamma \in \mathbb{R}$. À l'échelle macroscopique, elle se traduit par

$$\partial_\nu u = \gamma u \quad \text{sur } \partial\Omega.$$

Le cas Neumann correspond à $\gamma = 0$, le cas Dirichlet correspond formellement à $\gamma = -\infty$.

A.2.4 Équations de réaction-diffusion

En considérant une population qui croit et diffuse, par exemple sur $\mathbb{R}^n$, nous obtenons une *équation de réaction-diffusion*,

$$\partial_t u(t,x) - d\Delta_x u(t,x) = f(u(t,x)), \quad x \in \mathbb{R}^n. \tag{A.3}$$

Cette équation possède de nombreuses propriétés, notamment en ce qui concerne la propagation spatiale de la solution. Les travaux pionniers dans ce domaine sont dus à Fisher [150], Kolmogorov, Petrovski, Piskunov [199], puis Fife, McLeod [148], Aronson, Weinberger [17], Bramson [64, 65].

Les équations de réaction-diffusion (A.3) admettent des solutions diverses. Premièrement, il y a les solutions *stationnaires*, qui ne dépendent pas du temps, et vérifient donc

$$-d\Delta u(x) = f(u(x)).$$

Une autre forme particulière de solution est *l'onde progressive* (voir Figure A.7), telle que $u(t,x) = U(ct - x \cdot e)$, où $c \geq 0$ est une vitesse, $e \in \mathbb{S}^n$ une direction, et $U : \mathbb{R} \to \mathbb{R}$ vérifie

$$cU' - dU'' = f(U).$$

Elle correspondent à un profil constant se déplaçant à vitesse c dans la direction e. Si f est de type *KPP*, on peut montrer qu'il existe une (unique) *onde progressive* de vitesse c si et seulement si $c \geq c_0 := 2\sqrt{f'(0)}$. Si f est de type *bistable*, il existe une onde progressive pour une unique vitesse $c_\star \in \mathbb{R}$, dont le signe est déterminé par celui de $\int_0^K f$.

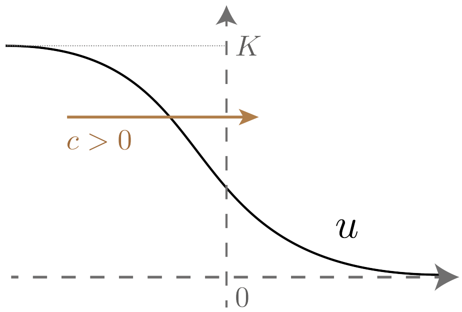


Figure A.7 – Onde progressive

Toujours si f est de type *KPP*, et si $u(t=0,x) \gneqq 0$ est à support compact, $u(t,x)$ converge vers K en temps long en se *propageant* dans l'espace à la vitesse c_0, c'est-à-dire

$$\forall c < c_0, \quad \lim_{t\to+\infty} \left(\inf_{|x|\leq ct} u(t,x) \right) = K,$$
$$\forall c > c_0, \quad \lim_{t\to+\infty} \left(\sup_{|x|\geq ct} u(t,x) \right) = 0.$$

On peut même montrer que la solution converge vers l'onde progressive de vitesse c_0 (avec un retard logarithmique), voir par exemple [173].

Les équations de réaction-diffusion sont utilisées dans des domaines d'application variés. De nombreux exemples issues de la biologie sont présentés dans l'ouvrage de Murray [241]. Par exemple, l'article de Skellam [266] modélise, entre autre, la propagation du rat-musqué en Europe centrale, voir Figure A.8. Dans un autre contexte, le célèbre article de Alan Turing [278] modélise la formation de *motifs* chez les organismes vivants (Figure A.9).

A.2.5 Objet de ce manuscrit

Nous nous intéressons dans ce manuscrit à certains problèmes issus des équations de réaction-diffusion et de leur utilisation dans la modélisation de la dynamique des populations. La première partie traite des solutions stationnaires *stables* des équations de réaction-diffusion. Dans la deuxième partie, nous nous intéresserons à la modélisation du phénomène de sélection naturelle dans la théorie de l'évolution darwinienne. Enfin, nous étudieront dans la troisième partie des systèmes d'équations de réaction-diffusion qui interviennent dans des modèles d'épidémiologie, et proposerons un modèle pour décrire la dynamique des émeutes et de l'agitation sociale.

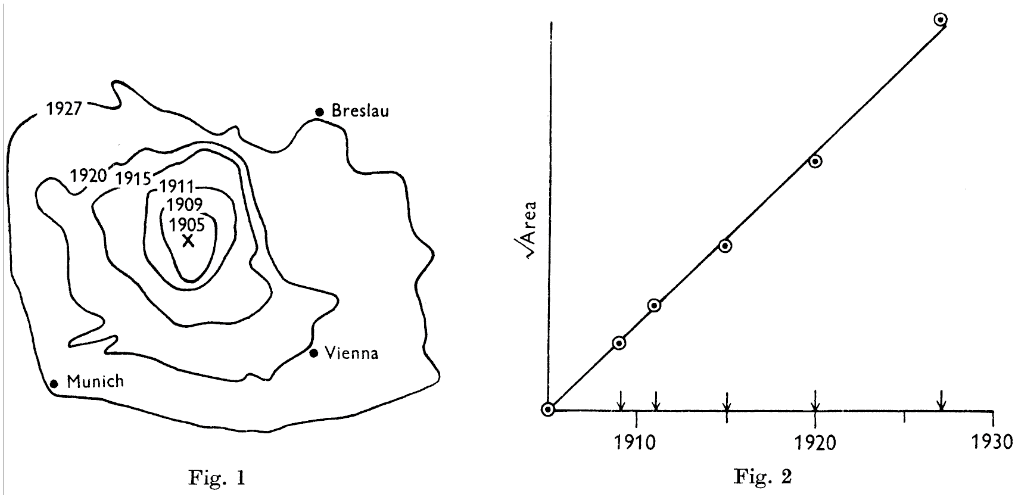


FIGURE A.8 – Illustrations issues de l'article de Skellam [266] décrivant la propagation à vitesse constante du rat-musqué en Europe centrale.

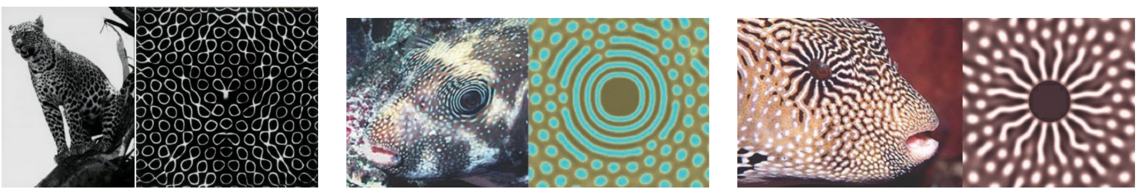


FIGURE A.9 – Trois exemples de motifs de Turing.
Partie gauche : photographie. Partie droite : simulation par ordinateur

Chapitre B

Introduction à la partie I : Propriétés qualitatives des solutions stables

Contents

B.1 Introduction

B.1.1 Présentation du problème

Cadre général

Dans la première partie de ce manuscrit, nous étudions certaines propriétés qualitatives des solutions stables d'équations elliptiques semi-linéaires avec condition de bord de Neumann. Nous considérons le problème suivant :

$$\begin{cases} -\Delta u(x) = f(u(x)) & \forall x \in \Omega, \\ \partial_\nu u(x) = 0 & \forall x \in \partial\Omega, \\ u \in C^2\left(\overline{\Omega}\right) \cap L^\infty(\Omega), \end{cases} \tag{B.1}$$

où $\Omega \subset \mathbb{R}^n$ est un domaine lisse, ∂_ν représente la dérivée selon la normale sortante au domaine, et f est une fonction réelle C^1.

Une solution est dite *stable* si la seconde variation de l'énergie en u est positive (possiblement dégénérée), c'est-à-dire, si u est un miniseur local (au sens faible) de l'énergie. Cette définition sera précisée par la suite. Du point de vue de la modélisation, les solutions stables sont les solutions qui ont un "sens physique" : elles représentent les seuls états potentiellement observables d'un système physique.

Notons que si $z \in \mathbb{R}$ est une racine stable de f, *i.e.* $f(z) = 0$ et $f'(z) \leq 0$, c'est une solution stable (triviale). Nous appellerons *pattern* toute solution *stable non-constante*. Nous nous intéressons en particulier au problème d'existence et de non-existence de *patterns*. Ce problème met en jeu de manière complexe la géométrie du domaine.

La question de l'existence de patterns intervient dans de nombreux domaines d'application, comme la formation de motifs en biologie [127, 241, 278, 287] ou en chimie [260], les transitions de phase [77], ou la propagation de populations biologiques [150, 266].

Notre point de départ est un célèbre théorème démontré par Casten, Holland [86], et indépendamment par Matano [217].

Theorem B.1 ([86, 217]) *Si le domaine Ω est borné et convexe, alors il n'existe pas de pattern à* (B.1)*.*

Nous insistons sur le fait que cette conclusion est valable quelque soit $f \in C^1$. Ce résultat a initié un foisonnement de développements mathématiques mettant en jeu des notions très profondes d'EDPs et de géométrie.

L'enjeu de notre travail est de tenter de mieux comprendre quels sont les critères géométriques qui assurent l'existence ou la non-existence de patterns. En effet, la littérature présente en quelque sorte un fossé entre, d'une part, les domaines convexes, et d'autre part, les domaines *haltères* (introduits dans la suite).

Nous donnerons dans cette section un critère quantitatif pour la non-existence de patterns dans un domaine quelconque, portant sur le signe de la valeur propre principale d'un certain opérateur linéaire. Ce critère est, à notre connaissance, le premier de cette nature. Notons également que la littérature traite, d'une part, des domaines convexes bornés, et d'autre part de l'espace entier $\mathbb{R}^n$; elle ne traite pas, ou peu, des domaines convexes non-bornés quelconques dans $\mathbb{R}^n$. Nous discuterons donc de l'extension de nos résultats aux domaines non-bornés. En outre, les techniques utilisées dans la littérature reposent de manière cruciale sur le caractère auto-adjoint de l'opérateur Laplacien. La méthode que nous donnons permet de traiter le cas d'opérateurs non-auto-adjoints. Nous donnons également des résultats de perturbations, une formulation asymptotique du théorème de Casten, Holland et Matano, ainsi qu'une estimation de *flatness* des patterns.

Nous introduisons ici le contenu de la partie I de ce manuscrit, c'est-à-dire les chapitres 1,2,3. Le chapitre 1 étudie l'extension du théorème de Casten, Holland, et Matano dans des domaines convexes non-bornés. Diverses extensions sont proposées au chapitre 2, notamment un critère quantitatif pour la non-existence de pattern. Par cette nouvelle approche, nous retrouvons et étendons les résultats du chapitre 1. Enfin, le chapitre 3 est consacré à l'introduction et l'analyse de la *valeur propre principale généralisée* dans des domaines non-bornés avec condition de Robin.

Definition de la stabilité

Une solution u de (B.1) peut être vue comme un point critique de l'énergie

$$\mathcal{E}(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u),$$

où F est une primitive de f. La forme quadratique associée à la seconde variation de E est

$$\mathcal{F}(\psi) = \mathcal{F}(u, f, \Omega)(\psi) := \int_\Omega |\nabla \psi|^2 - f'(u)\psi^2.$$

Posons

$$\lambda_1 = \lambda_1(u, f, \Omega) := \inf_{\substack{\psi \in H^1 \\ \|\psi\|_{L^2}=1}} \mathcal{F}(\psi).$$

Definition B.2 *Une solution u du problème* (B.1) *est dite stable si $\lambda_1 \geq 0$, et stable non-dégénérée si $\lambda_1 > 0$.*

Ainsi, une solution stable non-dégénérée est un minimum (non-dégénéré) de l'énergie. L'équation d'Euler-Lagrange associée à la fonctionnelle $\mathcal{F}$ est l'équation (B.1) linéarisée en u :

$$\begin{cases} -\Delta v - f'(u)v = 0 & \forall x \in \Omega, \\ \partial_\nu v(x) = 0 & \forall x \in \partial\Omega. \end{cases} \tag{B.2}$$

Si le domaine Ω est borné, alors λ_1 est une valeur propre de l'opérateur linéarisée, appelée *valeur propre principale*. Cette valeur propre possède de nombreuses propriétés fondamentales, et peut être définie pour d'autres opérateurs elliptiques (notamment des opérateurs non-auto-adjoints avec conditions de bord de Robin) et pour des domaines non-bornés. Ainsi, la Definition B.2 est assez flexible. Nous reviendrons sur ce point plus tard.

Remarque. Il existe d'autres manières de définir la stabilité d'une solution. La plus classique est sans doute la définition dynamique : une solution u de (B.1) est *stable d'un point de vue dynamique* si elle attire toute solution du problème parabolique

$$\begin{cases} \partial_t v(t,x) - \Delta v(t,x) = f(v(t,x)) & \forall (t,x) \in (0,+\infty)\times\Omega, \\ \partial_\nu v(t,x) = 0 & \forall (t,x) \in (0,+\infty)\times\partial\Omega, \\ v(0,x) = v_0(x) & \forall x \in \Omega, \end{cases} \tag{B.3}$$

avec une condition initiale v_0 suffisamment proche de u. En fait, si le domaine Ω est borné, nous avons la chaîne d'implications suivante :

$$\lambda_1 > 0 \quad \Rightarrow \quad \text{stabilité d'un point de vue dynamique} \quad \Rightarrow \quad \lambda_1 \geq 0.$$

Nous donnerons une discussion plus détaillée sur la validité de cette chaine d'implication dans les domaines non-bornés en section 1.7.2

B.1.2 Sujets connexes et état de l'art

Afin de présenter le contexte dans lequel s'inscrit notre travail, nous proposons un état de l'art ainsi qu'une introduction à quelques sujets connexes qui interviennent dans notre étude.

Le contre exemple de Matano et les domaines haltères

L'article pionnier de Matano [217] propose une étude générale du problème parabolique (B.3) sur un domaine borné. Entre autre, il propose un contre exemple au théorème B.1, c'est-à-dire qu'il construit un pattern à (B.1) pour un certain domaine Ω et une non-linéarité f.

La non-linéarité qu'il considère est de type *bistable*. Pour fixer les idées, posons

$$f(u) = u(1-u^2),$$

la non-linéarité d'Allen-Cahn. Le domaine considéré par Matano présente un goulot d'étranglement. Ces domaines sont dits de type *haltère* (*dumbbell domains*). Par exemple, considérons un domaine $\Omega_\varepsilon \subset \mathbb{R}^n$, $n \geq 2$ formé de l'union de deux convexes disjoints, connectés par un fin couloir cylindrique de rayon $0 < \varepsilon \ll 1$. Voir Figure B.1.

Matano démontre l'existence d'un pattern, si ε est suffisamment petit. Cela constitue un contre exemple au théorème B.1. L'antagonisme entre les domaines convexes et les domaines haltères illustre bien que le problème d'existence de patterns à (B.1) met en jeu, de manière complexe, les propriétés géométriques du domaine.

L'idée de Matano peut être déclinée pour démontrer l'existence de patterns dans de nombreuses situations. Notons que Ω_ε peut être vu comme une perturbation d'un ensemble non-connexe $\Omega_0 := D_1 \sqcup D_2$ avec D_1, D_2 deux convexes disjoints. La fonction $u_0 := \mathbb{1}_{D_1} - \mathbb{1}_{D_2}$ est un pattern sur Ω_0. Le contre exemple de Matano dans Ω_ε s'obtient en quelque sorte comme une perturbation de u_0. Cet argument est

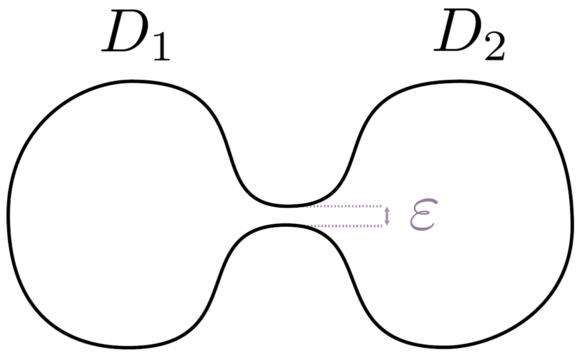


FIGURE B.1 – domaine haltère

rendu rigoureux par Hale et Vegas [172], qui construisent effectivement des patterns comme bifurcations de solutions triviales par perturbation du domaine.

Un cadre théorique plus général pour l'étude du problème parabolique (B.3) dans des domaines haltères est proposé par Arrieta, Carvalho, et Lozada-Cruz [19–21] (voir aussi Gadyl'shin [157]).

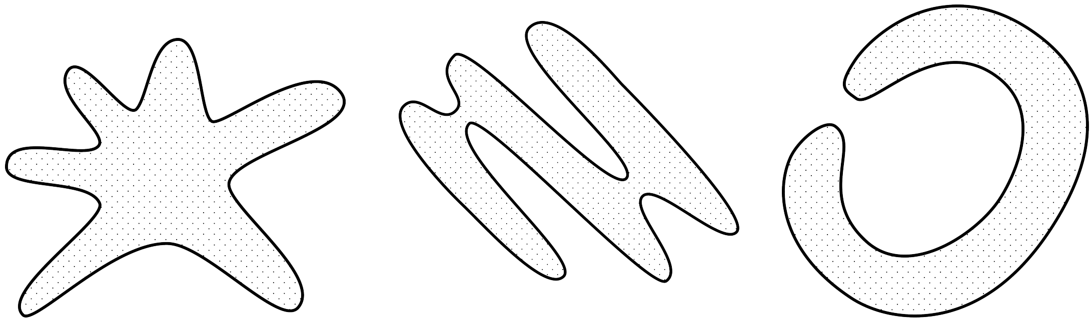
Lien avec l'invasion/blocage des populations

Considérons que la quantité u représente une densité de population. Dans un domaine haltère, le déplacement d'un individu d'un bout à l'autre du domaine est entravé par la présence du goulot d'étranglement. Au contraire, dans un domaine convexe, le déplacement des individus peut toujours s'effectuer en ligne droite. Cette observation suggère que l'existence de patterns est liée au fait que la géometrie du domaine entrave la diffusion des individus.

Cette heuristique peut être mise en lumière par certains résultats sur la propagation des ondes progressives. Ces ondes progressives sont des solutions particulières du problème parabolique (B.3) qui consistent en un profil se déplaçant à vitesse constante. Elle modélisent, entre autre, l'invasion d'un territoire par une population.

Berestycki, Hamel, et Matano [42] étudient l'influence d'un obstacle sur la propagation d'une onde progressive. Ils construisent une solution particulière $v(t, x)$ au problème parabolique (B.3), dans un domaine $\Omega = \mathbb{R}^n \backslash K$ où l'*obstacle* K est compact, et la non-linéarité f est de type bistable. Cette solution est dite *onde progressive généralisée*, car elle est définie pour tout $t \in \mathbb{R}$ et converge uniformément vers une onde progressive planaire quand $t \to -\infty$. La solution $v(t, x)$ converge en temps long vers une solution du problème stationnaire $u(x)$. Les auteurs montrent que, si l'obstacle est étoilé (Figure B.2a) ou axialement convexe (Figure B.2b), il y invasion, c'est-à-dire $u \equiv$ *constante*. En revanche, si l'obstacle présente un goulot d'étranglement (à la manière d'un domaine haltère, voir Figure B.2c), il y a blocage, c'est-à-dire $u \not\equiv$ *constante*. En tant qu'état limite d'un problème parabolique, u est, *de facto*, une solution stable de (B.1). On peut donc faire une analogie entre ces résultats et le problème d'existence de patterns.

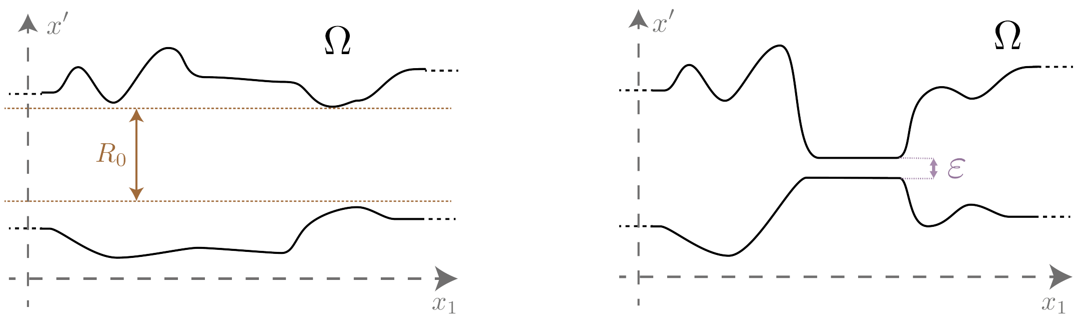
Un problème similaire est considéré par Berestycki, Bouhours et Chapuisat [40] dans des cylindres à section variable. Les auteurs démontrent qu'il y a invasion si le cylindre est suffisamment large et si toutes les sections sont étoilées (voir Figure B.3a). Au contraire, ils démontrent qu'il y a blocage si le cylindre présente un



(a) étoilé **(b)** axiallement convexe **(c)** haltère

FIGURE B.2 – Obstacles K

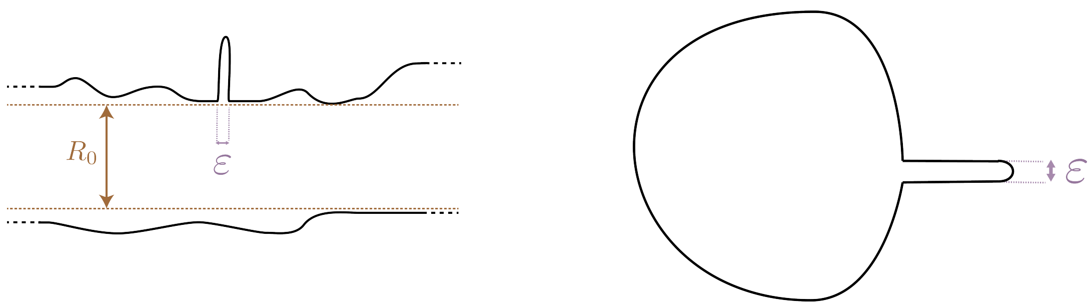
goulot d'étranglement (voir Figure B.3b). Des résultats analogues dans le cas d'une propagation non-locale sont obtenus par Brasseur, Coville, Hamel et Valdinoci [67], Brasseur, Coville [66].



(a) large, axiallement étoilé **(b)** avec un goulot d'étranglement

FIGURE B.3 – Cylindres

Ces exemples montrent une forte analogie entre le blocage d'onde progressive et l'existence de patterns. En général, le blocage implique l'existence de patterns. Cependant, la réciproque n'est pas vraie. En effet, il n'y a pas blocage si le domaine est étoilé et contient une boule suffisamment grande en son centre ; or ces domaines admettent parfois des patterns, voir Figures B.4a-B.4b.



(a) axiallement convexe **(b)** haltère étoilé

FIGURE B.4 – Domaines pour lesquels il existe des patterns mais il n'y pas blocage

Un modèle pour les transitions de phase

Un lien extrêmement fécond peut être établi entre les lignes de niveau des patterns et les surfaces minimales. Dans le but d'éclairer les développements qui vont suivre, nous commençons par donner une idée intuitive de comment un pattern de (B.1) peut modéliser une transition de phase dans la théorie de van der Waals [280], Cahn et Hilliard [77].

Considérons u comme la densité d'un fluide chimique à deux phases, à l'équilibre dans un domaine Ω. Nous supposons que chaque phase prise séparément est stable. Cette hypothèse suggère que u minimise une énergie avec un potentiel à deux puits. Pour fixer les idées, considérons le potentiel $F(u) = (1-u^2)^2$, de telle sorte à ce que les deux phases du fluide soient représentées par les valeurs ± 1. Dans le modèle le plus simple, l'énergie prend la forme suivante :

$$\int_\Omega F(u).$$

Ainsi, toute fonction u ne prenant que les valeurs ± 1 minimise l'énergie. D'un point de vue physique, ce modèle n'est pas satisfaisant, car l'interface entre les deux phases peut être complètement erratique. Il nous faut alors prendre en compte l'influence des forces de frottement microscopiques à l'intérieur du fluide. Cela se traduit par l'ajout d'un terme cinétique à l'énergie du système :

$$\mathcal{E}_\varepsilon(u) = \int_\Omega \frac{\varepsilon^2}{2}|\nabla u|^2 + F(u),$$

où $\varepsilon > 0$ est un paramètre d'intensité. La présence de ce terme de pénalisation empêche un minimiseur de sauter instantanément de $+1$ à -1, et force la transition de phase à se faire sur une longueur caractéristique ε. Tout minimiseur de $\mathcal{E}_\varepsilon$ est alors une solution stable de l'équation d'Euler-Lagrange associée :

$$\begin{cases} -\varepsilon^2 \Delta u_\varepsilon = f(u_\varepsilon) & \text{in } \Omega, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{B.4}$$

où f est la non-linéarité d'Allen-Cahn $f(u) := 2(u - u^3)$.

Pour $\varepsilon = 1$, nous retrouvons (B.1). Le cas $\varepsilon \ll 1$ s'interprète comme le cas d'un fluide très visqueux. Une autre interprétation est de voir u_ε comme une solution de $-\Delta u_\varepsilon = \varepsilon^{-2} f(u_\varepsilon)$, et ε^{-2} comme l'ordre de grandeur de la profondeur des puits de stabilité de ± 1. Une troisième interprétation possible est d'effectuer le changement d'échelle $x \longleftrightarrow \varepsilon x$, et de voir une solution de (B.4) comme une solution de $-\Delta u_\varepsilon = f(u_\varepsilon)$ dans le domaine agrandi $\varepsilon^{-1}\Omega$.

Lien avec les surfaces minimales

Dans une série de papiers pionniers, Modica, Mortola [237] et Modica [234] développent le cadre théorique de la Γ-convergence pour étudier le comportement d'une suite u_ε de minimiseurs de l'énergie $\mathcal{E}_\varepsilon$, quand $\varepsilon \to 0$. Modica démontre que, quitte à extraire une sous-suite, u_ε converge L^1_{loc} vers $\mathbb{1}_E - \mathbb{1}_{\Omega \setminus E}$, où E est un ensemble de périmètre minimal dans Ω.

Plus précisément, considérons la fonctionnelle de périmètre dans Ω

$$\mathcal{P}_\Omega(E) = \int_\Omega |\nabla \mathbb{1}_E| = \sup_{\substack{g \in C^1_0(\Omega) \\ |g| \le 1}} \left| \int_E \operatorname{div} g \right|,$$

où $C_0^1(\Omega)$ est l'ensemble des fonction C^1 à support compact dans Ω. Dire que l'ensemble E est de périmètre minimal dans Ω (ou plus brièvement, ∂E est une surface minimale) signifie

$$\mathcal{P}_A(E) \leq \mathcal{P}_A(F),$$

pour tout ouvert $A \subset \Omega$, et tout ouvert F qui coïncide avec E à l'extérieur de $\overline{A}$. Le résultat de Modica affirme que, si ε est petit, les minimiseurs de l'énergie ressemblent à une fonction constante par morceaux selon la partition $E \sqcup \Omega \backslash E$; la solution présente donc une transition de phase brutale, localisée autour d'une surface minimale ∂E.

Le résultat de Modica a ensuite été affiné par Caffarelli et Cordoba [74, 75], qui prouvent que les ensembles de sur-niveau $\{u_\varepsilon \geq \lambda\}$ convergent localement uniformément vers E (au sens de la distance de Hausdorff) pour tout $\lambda \in (0, 1)$ fixé. Cela confirme l'heuristique selon laquelle les lignes de niveau des patterns se comportent comme des surfaces minimales de Ω.

Kohn et Sternberg [198] établissent en quelque sorte une réciproque au résultat de Modica : étant donné un ensemble E de périmètre minimal dans Ω, il existe une suite u_ε de minimiseurs de $\mathcal{E}_\varepsilon$ qui converge L^1_{loc} vers $\mathbb{1}_E - \mathbb{1}_{\Omega \backslash E}$.

Ainsi, pourvu qu'il existe un tel ensemble E (non-trivial), Kohn et Sternberg démontrent qu'il existe un pattern au problème (B.4) pour $\varepsilon \ll 1$. Déterminer les ensembles E de périmètre minimal dans Ω est un problème assez ancien et en général difficile (lié au *problème de Plateau*, voir [273] pour un aperçu). Intuitivement, un tel ensemble E doit être tel que ∂E et $\partial \Omega$ s'intersectent orthogonalement en des points où Ω est concave, voir Figure B.5. Une surface minimale ∂E est formellement localisée au niveau d'un goulot d'étranglement du domaine.

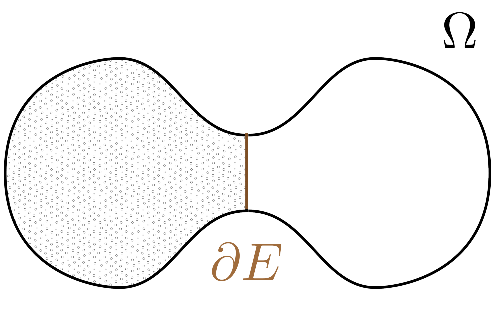


FIGURE B.5 – Domaine Ω qui possède un E minimisant le périmètre.

Un tel ensemble E existe donc typiquement dans les domaines haltères. Ainsi, le pattern de Kohn et Stenberg fait écho à celui de Matano. Au contraire, si le domaine Ω est convexe, il n'existe pas de tel ensemble E ; d'après le théorème B.1, il n'existe pas non plus de minimiseur non-constant de l'énergie. Nous retrouvons ici l'antagonisme entre les domaines convexes et les domaines haltères.

La théorie a été étendue à l'étude des minimiseurs de l'énergie sous contrainte de volume $\int_\Omega u_\varepsilon = m$ par de nombreux auteurs, comme Modica [235, 236], Owen [244], Sternberg [270], Kohn et Sternberg [198]. Dans ce cas, il peut exister des patterns (*i.e.*, des minimiseurs non-constants) dans les domaines convexes. On peut montrer cependant que les patterns ne présentent qu'une seule transition de phase entre ± 1 dans Ω. Le premier résultat dans cette direction est obtenu par Carr, Gurtin, et Slemrod [85], qui considèrent le problème sur un intervalle, et montrent que les miniseurs sont monotones (pour tout $\varepsilon > 0$). Ce résultat est étendu au cas des cylindres droits par Gurtin et Matano [169]. Enfin, pour un domaine borné convexe quelconque, Sternberg et Zumbrun [271, 272] montrent que l'interface de transition de phase est une surface connexe ; en d'autres termes, il ne peut y avoir qu'une seule transition de phase.

Nous mentionnons également l'extension de cette théorie aux minimiseurs à valeurs dans $\mathbb{R}^n$ (Fonseca, Tartar [154], Baldo [26], Caffarelli, Garofalo, Segala [76] Danielli, Garofalo [111]), ainsi que l'étude du problème dans $\Omega = \mathbb{R}^n$ (Lopes [210, 211]) et sur les variétés riemanniennes (Pacard et Ritoré [245]).

La conjecture de De Giorgi

Comme nous l'avons vu dans la section précédente, de manière heuristique, les lignes de niveau des solutions stables de (B.4) se comportent comme des surfaces minimales dans Ω. En procédant à un changement d'échelle $x \leftrightarrow \varepsilon x$, c'est-à-dire en zoomant autour d'un point x, le problème (B.4) à la limite $\varepsilon \to 0$ revient à considérer l'équation (B.1) dans tout $\mathbb{R}^n$. Ainsi, les lignes de niveau des solutions stables dans $\mathbb{R}^n$ devraient se comporter comme des surfaces (de dimension $n-1$) minimales de $\mathbb{R}^n$.

Simons [265] démontre qu'en dimension $n \leq 7$, toute surface minimale de $\mathbb{R}^n$ est un hyperplan. Un contre-exemple en dimension $n \geq 8$ a ensuite été proposé par Bombieri, De Giorgi, et Gusti [57]. En ce qui concerne les surface minimales qui, en outre, sont le graphe d'une fonction définie sur $\mathbb{R}^{n-1}$, on "gagne une dimension" : une telle surface est nécessairement un hyperplan si et seulement si $n \geq 8$ (ce problème est connu sous le nom de *problème de Bernstein*, voir [57, 72, 114, 184]). Cela mène De Giorgi à énoncer la fameuse conjecture suivante :

Conjecture: (De Giorgi) *Soit u une solution de $-\Delta u = u - u^3$ dans $\mathbb{R}^n$, telle que $|u| < 1$ et $\partial_{x_n} u > 0$. Tout ensemble de niveau de u est un hyperplan, au moins si $n \leq 8$.*

Le fait que toute ensemble de niveau u est un hyperplan revient à dire que u est plane, *i.e.* u ne varie que dans une seule direction de l'espace (qui, notons-le, n'est pas connue *a priori*). Il est alors facile de montrer que u est de la forme $\tanh \frac{x_1}{\sqrt{2}}$. Notons également que l'hypothèse $\partial_{x_n} u > 0$ implique *a priori* que toute ensemble de niveau de u est le graphe d'une fonction définie sur $\mathbb{R}^{n-1}$.

La conjecture de De Giorgi a donné lieu à une vaste littérature (voir [144,250,262, 286] pour un état de l'art). Il est démontré par Ghoussoub et Gui [160] en dimension $n \leq 2$, par Ambrosio et Cabré [10] en dimension $n = 3$, et par Savin [261, 262] en dimension $4 \leq n \leq 8$ sous l'hypothèse supplémentaire

$$\lim_{x_n \to \pm\infty} u(x', x_n) = \pm 1. \tag{B.5}$$

Par ailleurs, un contre exemple est donné par del Pino, Kowalczyk et Wei [115] en dimension $n \geq 9$ (ce contre exemple satisfait (B.5)). La conjecture est encore ouverte pour les dimensions $4 \leq n \leq 8$ sans l'hypothèse (B.5). Notons que les preuves sont, en général, valables pour une classe assez large de non-linéarités, et non uniquement pour la non-linéarité d'Allen-Cahn (voir [7, 72, 262]).

Si on considère le même problème en supposant que la limite dans (B.5) est uniforme en $x' \in \mathbb{R}^n$ (problème connu sous le nom de *conjecture de Gibbons*), alors la conclusion est vraie en toute dimension, comme démontré indépendamment par Farina [140], Berestycki, Hamel et Monneau [43], Barlow, Bass et Gui [39]. Il est également prouvé dans [39, 140] que la conclusion reste vraie si (B.5) est remplacée par l'hypothèse qu'au moins un ensemble de niveau est le graphe d'une fonction globalement Lipschitz définie sur $\mathbb{R}^{n-1}$.

Nous mentionnons également l'obtention de résultats partiels en dimension 4 et 5 par Ghoussoub et Gui [161], et la preuve de la conjecture en dimension 4 par Figalli et Serra [149] dans le cas du Laplacien fractionnaire.

Théorème de Liouville

La démonstration de la conjecture de De Giorgi en dimension $n \leq 3$ repose sur un *théorème de Liouville* établi par Berestycki, Caffarelli et Nirenberg [41]. Leur résultat peut être formulé de la manière suivante :

Theorem B.3 ([41]) *Soit $\varphi > 0$ et $\sigma \in C^2$ telle que $\varphi\sigma$ est bornée. Supposons que l'inégalité suivante est vérifiée*

$$\sigma \mathrm{div}\left(\varphi^2 \nabla \sigma\right) \geq 0 \quad \textit{dans } \mathbb{R}^n.$$

Si $n \leq 2$, alors σ est constante.

Pour $n = 1$ et $\varphi = 1$, on retrouve le fameux résultat de Liouville : toute fonction convexe et bornée dans $\mathbb{R}$ est constante.

Ce théorème a été le point de départ de nombreuses avancées sur la compréhension des solutions entières, et une vaste littérature lui est consacrée (voir les travaux de Moschini [240] ou Pinchover [249] pour un aperçu). En toute généralité, le théorème ne peut être étendu aux dimensions $n \geq 3$ d'après le contre-exemple de Barlow [38]. En revanche, les conditions "$\varphi\sigma$ borné" et "$n \leq 2$" peuvent être remplacées par la simple hypothèse

$$\int_{B_R} (\varphi\sigma)^2 = O(R^2), \quad \text{quand } R \to +\infty, \tag{B.6}$$

où B_R est la boule de rayon R. Cette condition est en quelque sorte optimale [158].

La stratégie communément adoptée pour démontrer la conjecture de De Giorgi (au moins en petite dimension) est de chercher à appliquer le théorème de Liouville avec φ la fonction propre principale de (B.2), et $\sigma_i := \frac{\partial_{x_i} u}{\varphi}$.

Patterns dans $\mathbb{R}^n$

L'hypothèse $\partial_{x_n} u > 0$ dans la conjecture de De Giorgi implique en particulier que la solution u considérée est stable. En effet, rappelons que u est stable si et seulement si

$$\lambda_1 := \inf_{\substack{\psi \in H^1 \\ \|\psi\|_{L^2}=1}} \left[\int_{\mathbb{R}^n} |\nabla\psi|^2 - f'(u)\psi^2\right] \geq 0. \tag{B.7}$$

Cette définition reste identique si les fonctions tests ψ sont choisies dans $C^1(\mathbb{R}^n)$ et à support compact. En posant $v := \partial_{x_n} u$, et en dérivant (B.1), nous obtenons

$-\Delta v - f'(u)v = 0$ dans $\mathbb{R}^n$. En multipliant cette équation par $\frac{\psi^2}{v}$ (pour une fonction test $\psi \in C^1(\mathbb{R}^n)$ à support compact), en intégrant sur $\mathbb{R}^n$, puis en intégrant par parties, nous obtenons

$$\begin{aligned}
0 &= \int_{\mathbb{R}^n} \nabla v \cdot \nabla \frac{\psi^2}{v} - f'(u)\psi^2 \\
&= 2\int_{\mathbb{R}^n} \frac{\psi}{v} \nabla v \cdot \nabla \psi - \frac{\psi^2}{v^2} |\nabla v|^2 - f'(u)\psi^2 \\
&\leq \int_{\mathbb{R}^n} |\nabla \psi|^2 - f'(u)\psi^2,
\end{aligned}$$

où nous utilisons l'inégalité de Young $2ab \leq a^2 + b^2$ dans la dernière étape. Ainsi, nous déduisons $\lambda_1 \geq 0$, *i.e.*, u est stable.

Il est donc naturel de considérer une variante de la conjecture de De Giorgi, en remplaçant l'hypothèse $\partial_{x_n} u > 0$ par l'hypothèse que u est stable.

Conjecture: (variante de De Giorgi) *Soit u une solution* stable *de $-\Delta u = u - u^3$ dans $\mathbb{R}^n$, telle que $|u| < 1$. Tout ensemble de niveau de u est un hyperplan, au moins si $n \leq 7$.*

Notons qu'avec la seule hypothèse de stabilité (plutôt que $\partial_{x_n} u > 0$), nous ne pouvons pas garantir que tout ensemble de niveau de u est le graphe d'une fonction définie sur $\mathbb{R}^{n-1}$. Cela explique pourquoi cette conjecture est énoncée pour $n \leq 7$ (contrairement à celle de De Giorgi, énoncée pour $n \leq 8$).

Il s'avère que la preuve de Ghoussoub et Gui est valable dans ce cadre plus faible : la conjecture est donc démontrée pour $n \leq 2$. Un contre exemple est exhibé par Pacard et Wei [246] en dimension paire supérieure ou égale à 8, prouvant ainsi que la condition $n \leq 7$ ne peut être relaxée. Les dimensions intermédiaires $3 \leq n \leq 7$ sont ouvertes.

Dancer [104] démontre cependant que si la solution bornée u est *stable non-dégénérée*, c'est-à-dire $\lambda_1 > 0$ dans (B.7), alors u est constante. Ce résultat est valable en toute dimension. Ainsi, il n'existe pas de pattern stable non-dégénéré dans $\mathbb{R}^n$: ce résultat peut être vu comme une extension du théorème B.1 à un domaine convexe non-borné (avec toutefois l'hypothèse d'une stabilité non-dégénérée).

Il peut donc exister des patterns dans $\mathbb{R}^n$, cependant leur stabilité est dégénérée, et ils sont nécéssairement plans si $n \leq 2$.

Rappelons que nous supposons *a priori* que les patterns sont des solutions *bornées*. Sans cette hypothèse, les résultats ne sont plus valables : par exemple, $u(x) := e^x$ est une solution stable non-dégénérée de $-u'' = -u$ dans $\mathbb{R}$).

Certains résultats plus précis sont également disponibles si on suppose que f est, par exemple, croissante, convexe, positive, *etc.* Voir l'ouvrage de Dupaigne [135] et les articles [105, 107, 136, 141, 145]. Voir aussi [76, 111] pour l'étude d'une classe générale d'équations sur $\mathbb{R}^n$.

Nous mentionnons également le résultat de Cabré et Capella [73] (affiné par Villegas [282]), qui montre qu'il existe des patterns radiaux dans $\mathbb{R}^n$ si et seulement si $n \geq 11$.

Résultats de symétrie et extension aux variétés

Matano [217] démontre que, si le domaine est un solide de révolution, toute solution stable est invariante selon l'axe de rotation. En outre, si la section est convexe, alors toute solution stable est constante. En particulier, cela implique la non-existence de patterns dans certains domaines non-convexes, comme les tores ou les anneaux.

Ce résultat suggère l'idée que les solutions stables héritent de certaines symétries du domaine. Nous avons rencontré la même idée dans la section précédente : une solution stable dans $\mathbb{R}^n$, $n \leq 2$, est nécessairement plane. On ne peut cependant pas s'attendre à des résultats analogues pour des symétries autres que celles par rotation ou translation : la raison est que seuls les opérateurs de dérivée cartésienne et angulaire commutent avec le Laplacien.

Il est naturel de considérer le problème sous d'autres géométries, et donc d'étudier les propriétés des solutions stables de (B.1) sur des variétés. Le premier article sur le sujet est dû à Jimbo [185]. Il établit une généralisation du théorème B.1 : si la variété M est compacte, de courbure de Ricci négative, et convexe (*i.e.*, la seconde forme fondamentale de ∂M est positive), alors il n'existe pas de pattern pour l'équation (B.1) dans M. Puis il construit un pattern sur une variété de type haltère avec la méthode de Matano [217].

Le résultat de symétrie angulaire de Matano est étendu aux surfaces de révolutions par Rubinstein et Wolansky [260], puis à des variétés sans bord plus générales par Gonçalves et Nascimento [166]. Ces derniers proposent également une construction très précise de patterns, qui dénote bien l'importance du signe de la courbure du Ricci. Nous mentionnons également les travaux de Bandle, Punzo, Tesei [28] et Tesei [253].

Enfin, des résultats analogues à ceux disponibles dans $\mathbb{R}^n$ autour de la conjecture de De Giorgi ont été obtenus pour des variétés riemanniennes sans bord non-compactes par Farina, Sire, Valdinoci [143] et Farina, Mari, Valdinoci [142].

Autres extensions

Il existes de nombreuses autres extensions au théorème B.1 et à ses variantes. Le cas des conditions de bord de Robin est étudié par Bandle, Mastrolia, Monticelli, Punzo [27], et celui d'une réaction au bord par Cònsul et Solà-Morales [100]. Nous mentionnons également Chanillo et Cabré [96] pour des propriétés qualitatives des solutions stables positives dans les domaines convexes avec condition de bord de Dirichlet.

Beaucoup d'auteurs se sont également intéressés à comprendre comment l'existence de patterns pouvait émerger de la non-homogénéité des coefficients, par exemple, si la diffusivité est variable [129, 130, 268, 291] ou si le terme de réaction est non-homogène [9, 69, 269].

Enfin, nous mentionnons que les résultats d'existence et de non-existence de patterns ont été obtenus pour des systèmes coopératifs [186, 194], puis étendus à des cas de systèmes activateur/inhibiteur [292].

B.2 Présentation des résultats

B.2.1 Critère pour la non-existence de patterns

Définitions

L'approche que nous proposons permet de traiter le cas d'opérateurs non-auto-adjoints homogènes, de la forme

$$-Lu := -\mathrm{div}\,(A \cdot \nabla u) - B \cdot \nabla u, \tag{B.8}$$

avec $B \in \mathbb{R}^n$ et A une matrice symétrique définie positive. Nous considérons donc le problème (B.1) en remplaçant le Laplacien $-\Delta$ par $-L$:

$$\begin{cases} -\,Lu(x) = f(u(x)) & \forall x \in \Omega, \\ \partial_{\nu_A} u(x) = 0 & \forall x \in \partial\Omega, \\ u \in C^2\left(\overline{\Omega}\right) \cap L^\infty(\Omega), \end{cases} \tag{B.9}$$

Ici, $\partial_{\nu_A} u := \nu \cdot A \cdot \nabla u$ est la dérivée co-normale de u pour L : cette condition de bord est la condition de Neumann naturelle associée à L. Comme précédemment, nous considérons $f \in C^1$ et Ω un domaine de classe C^2.

Nous adaptons la Définition B.2 à notre cadre :

Definition B.4 *Soit u une solution de* (B.9). *On considère la valeur propre principale $\lambda_1 := \lambda_1(u, f, \Omega)$ associée à l'opérateur linéarisé en u :*

$$\forall \psi \in C^2(\overline{\Omega}), \qquad \begin{cases} -\,L\psi - f'(u)\psi = 0 & \textit{sur } \Omega, \\ \partial_{\nu_A}\psi = 0 & \textit{sur } \partial\Omega. \end{cases}$$

La solution u est dite stable *si $\lambda_1 \geq 0$, et* stable non-dégénérée *si $\lambda_1 > 0$.*

Non-existence de patterns en domaine borné

Notre premier résultat est une extension du Théorème B.1 de Casten, Holland et Matano au cas d'un opérateur non-auto-adjoint.

Proposition B.5 *Si Ω est borné et convexe, il n'existe pas de pattern à* (B.9).

Comme nous allons le voir, ce résultat est en fait le cas particulier d'un théorème plus général. Pour cela, nous considérons λ_1^γ la valeur propre principale de l'opérateur linéarisé *modifié* :

$$\forall \psi \in C^2(\overline{\Omega}), \qquad \begin{cases} -\,L\psi - f'(u)\psi = 0 & \text{sur } \Omega, \\ \partial_{\nu_A}\psi + \gamma\psi = 0 & \text{sur } \partial\Omega, \end{cases} \tag{B.10}$$

où $\gamma(x)$ est la "courbure minimale" de $\partial\Omega$, c'est-à-dire la plus petite valeur propre de la seconde forme fondamentale de $\partial\Omega$ en x. En particulier, si $n = 2$, γ n'est rien d'autre que la courbure de $\partial\Omega$.

Notons que ce problème aux valeurs propres est un problème de Robin *indéfini*, en ce sens que nous ne faisons pas d'hypothèse de signe sur γ. Ce cas n'est pas tout à fait standard. La définition et les propriétés de la valeur propre principale λ_1^γ seront précisées plus tard. Nous donnons le théorème suivant :

Theorem B.6 *Supposons que Ω est borné, et soit u une solution de* (B.9). *Si $\lambda_1^\gamma \geq 0$, alors u est constante.*

Notons que les hypothèses "u stable" et "Ω convexe" de la Proposition B.5 sont ici combinées en la seule hypothèse "$\lambda_1^\gamma \geq 0$". Si Ω est convexe, alors $\gamma \gneqq 0$, et $\lambda_1 > \lambda_1^\gamma$. Ainsi, le théorème B.1 de Casten, Holland et Matano est un cas particulier du théorème précédent. La preuve du théorème est assez élémentaire (peut-être même davantage que la preuve classique du Théorème B.1), et consiste simplement à appliquer le principe du maximum à la fonction $W := \frac{|\nabla u|_A^2}{\varphi^2}$, où $|\cdot|_A$ est la norme induite par A et φ est la *fonction propre principale* associée à λ_1^γ.

Ce théorème donne ainsi un critère quantitatif pour la non-existence de patterns. Il ramène le problème au signe d'une valeur propre qui fait intervenir à la fois la non-linéarité et la géométrie du domaine. Nous laissons cependant la question de l'optimalité de ce critère pour des travaux futurs.

Résultats de perturbation

Le théorème B.6 permet de gagner de la flexibilité sur les hypothèses du théorème de Casten, Holland et Matano. On peut dès lors aisément démontrer de nombreux résultats de perturbation pour le théorème B.1. On peut, par exemple, montrer qu'il n'existe pas de patterns dans certains domaines non-convexes, ou bien montrer que certaines solutions instables sont nécessairement constantes. Ces questions se ramènent simplement à l'étude du signe de la valeur propre λ_1^γ. Nous donnons, comme complément, cette série de propriétés classiques.

Proposition B.7 *Soit $\Omega \subset \mathbb{R}^n$ un domaine C^2 borné, $-L$ comme dans* (B.8)*, $c : \Omega \to \mathbb{R}$ et $\gamma : \partial\Omega \to \mathbb{R}$ deux fonctions continues. On définit $\lambda_1^\gamma := \lambda_1^\gamma(-L-c,\Omega)$ comme la valeur propre principale de*

$$\begin{cases} -L\psi - c\psi = \lambda\psi & \textit{dans } \Omega, \\ \partial_{\nu_A}\psi + \gamma\psi = 0 & \textit{sur } \partial\Omega. \end{cases}$$

Alors :

1. *λ_1^γ est continue par rapport à des perturbations C^2 du domaine Ω.*
2. *$\lambda_1^\gamma(-L-c)$ est continue et strictement décroissante par rapport à c.*
3. *λ_1^γ est continue et strictement croissante par rapport à γ.*
4. *Si $-L$ est auto-adjoint, alors*

$$\lambda_1^\gamma := \inf_{\substack{\psi \in H^1(\Omega) \\ \|\psi\|_{L^2}=1}} \left[\int_\Omega |\nabla\psi|_A^2 - c\psi^2 + \int_{\partial\Omega} \gamma\psi^2 \right], \tag{B.11}$$

avec $|\cdot|_A^2$ la norme induite par A.
En particulier, les applications $c \mapsto \lambda_1^\gamma(-L-c)$ et $\gamma \mapsto \lambda_1^\gamma$ sont concaves.

B.2.2 Valeur propre principale généralisée aux domaines non-bornés

Avant de considérer le problème d'existence de patterns dans des domaines non-bornés, nous devons étendre la définition et les propriétés de la valeur propre principale d'un opérateur. Nous faisons donc un détour par cette question.

Définition

Nous considérons, en général, un domaine Ω de classe C^2 et un opérateur elliptique linéaire

$$-\mathcal{L}u(x) := -\mathrm{div}\,(A(x)\cdot\nabla u(x)) - B(x)\cdot\nabla u(x) - c(x)\psi(x), \quad \forall x\in\Omega,$$

où, $c:\Omega\to\mathbb{R}$, $B:\Omega\to\in\mathbb{R}^n$, et $A:\Omega\to\mathbb{R}^{n\times n}$ telle que $A(x)$ est une matrice symétrique définie positive (uniformément en $x\in\Omega$). Nous supposons que les coefficients sont bornés et continus.

Nous associons à l'opérateur $\mathcal{L}$ une condition de bord de Robin

$$\mathcal{B}^\gamma u(x) := \partial_{\nu_A}u(x) + \gamma(x)u(x) = 0, \quad \forall x\in\partial\Omega,$$

avec $\gamma(\cdot)$ une fonction bornée et continue définie sur $\partial\Omega$. La cas "Neumann" correspond à $\gamma=0$, et le cas "Dirichlet" correspond formellement à $\gamma=+\infty$.

Classiquement, nous appelons sous-solution (resp. sur-solution) toute fonction $u\in C^2(\overline{\Omega})$ telle que

$$\begin{cases} -\mathcal{L}u\le 0 \text{ (resp. } \ge 0) & \text{dans } \Omega,\\ \mathcal{B}^\gamma u\le 0 \text{ (resp. } \ge 0) & \text{sur } \partial\Omega.\end{cases}$$

Nous nous intéressons en particulier à établir un critère pour la validité du Principe du Maximum, défini comme suit :

Definition B.8 *On dit que $(\mathcal{L},\mathcal{B}^\gamma)$ satisfait le Principe du Maximum si toute sous-solution bornée supérieurement est négative.*

Il est bien connu que, si le domaine est borné, la condition

$$c(\cdot)\le 0 \text{ et } \gamma(\cdot)\ge 0$$

implique que $(\mathcal{L},\mathcal{B}^\gamma)$ satisfait le Principe du Maximum. Cependant, cette condition est très restrictive, et nous recherchons un critère plus flexible, voire optimal. Cela motive l'introduction de la *valeur propre principale.*

Le cas des domaines bornés. Supposons dans un premier temps que le domaine Ω est borné. Nous considérons le problème aux valeurs propre suivant :

$$\begin{cases} -\mathcal{L}\psi = \lambda\psi & \text{in } \Omega,\\ \partial_{\nu_A}\psi + \gamma\psi = 0 & \text{on } \partial\Omega.\end{cases} \tag{B.12}$$

Notons que la condition de Robin est *indéfinie*, c'est-à-dire que nous ne faisons aucune hypothèse de signe sur γ. La littérature considère d'avantage le cas $\gamma\ge 0$, surtout pour des raisons techniques ; or, la plupart des résultats restent valables dans le cas *indéfini.* Daners [109] montre notamment que le problème de Robin indéfini peut être réécrit (avec une transformation assez simple) en un problème de Robin ayant la même structure mais avec $\tilde{\gamma}\ge 0$.

En particulier, le théorème de Krein-Rutman donne l'existence d'une valeur propre λ_1 au problème (B.12), appelée la *valeur propre principale.* Cette valeur propre est réelle et minimise la partie réelle du spectre (pour cette raison, elle est parfois appelée *première valeur propre*). En outre, λ_1 est simple, et est la seule valeur propre associée à une fonction propre positive (appelée *fonction propre principale*). Nous laisserons le lecteur se référer aux ouvrages classiques pour plus de détails [162, 252].

Une propriété fondamentale est que la validité du Principe du Maximum pour l'opérateur $(\mathcal{L}, \mathcal{B}^\gamma)$ équivaut à la condition $\lambda_1 > 0$.

Le cas des domaines non-bornés. Le théorème de Krein-Rutman ne peut pas être appliqué si le domaine n'est pas borné. La notion de valeur propre principale peut néanmoins être étendue. En suivant l'approche de [47, 51], nous donnons la définition suivante :

$$\lambda_1^\gamma := \sup\left\{\lambda \in \mathbb{R} : (\mathcal{L} + \lambda, \mathcal{B}^\gamma) \text{ possède une sur-solution positive}\right\}.$$

Cette définition coïncide avec la définition classique dans le cas d'un domaine borné. Ainsi défini, λ_1^γ possède une fonction propre positive. Cependant, nous ne savons pas si λ_1^γ est simple. Nous allons voir que le signe de λ_1^γ peut constituer un critère pour la validité du Principe du Maximum.

Principe du Maximum

Opérateurs auto-adjoints. Nous considérons d'abord le cas d'un opérateur auto-adjoint, c'est-à-dire, nous supposons $B \equiv 0$ dans (B.8). Dans ce cas, nous pouvons exprimer λ_1^γ avec la formule variationnelle de Rayleigh-Ritz (B.11). Nous démontrons que le signe (strict) de λ_1^γ équivaut à la validité du Principe du Maximum.

Theorem B.9 *Supposons que $\mathcal{L}$ est auto-adjoint.*

1. *Si $\lambda_1 > 0$, $(\mathcal{L}, \mathcal{B}^\gamma)$ satisfait le Principe du Maximum.*
2. *Si $\lambda_1 < 0$, $(\mathcal{L}, \mathcal{B}^\gamma)$ ne satisfait pas le Principe du Maximum.*

Le théorème suivant établit la validité de ce qu'on pourrait appeler un *Principe du Maximum Critique* dans le cas $\lambda_1^\gamma \geq 0$, si le domaine satisfait une certaine condition de croissance à l'infini.

Theorem B.10 *Supposons que $\mathcal{L}$ est auto-adjoint et le domaine Ω satisfait :*

$$\Omega \cap \{|x| \leq R\} = O(R^2) \quad \text{when } R \to +\infty. \tag{B.13}$$

Soit φ une fonction propre associée à λ_1^γ. Si $\lambda_1^\gamma \geq 0$, alors toute sous-solution bornée supérieurement est soit négative, soit un multiple de φ.

Notons que l'hypothèse (B.13) fait écho à l'hypothèse (B.6) du théorème de Liouville. Une première conséquence de ce résultat est la simplicité de λ_1 si elle admet une fonction propre bornée.

Corollary B.11 *Sous les mêmes hypothèses, supposons de plus que λ_1 admet une fonction propre bornée. Alors toute fonction propre associée λ_1^γ est un multiple de φ, i.e., λ_1 est simple.*

Nous donnons également le corollaire suivant.

Corollary B.12 *Sous les mêmes hypothèses, supposons $\lambda_1^\gamma = 0$, et soit φ une fonction propre associée. Alors $(\mathcal{L}, \mathcal{B}^\gamma)$ satisfait le Principe du Maximum si et seulement si φ n'est pas bornée.*

Nous établissons également que ces résultats s'étendent à certains opérateurs non-auto-adjoints pour lesquels, en quelque sorte, le drift induit est borné, voir Théorème 3.12.

Opérateurs non-auto-adjoints. Dans le cas non-auto-adjoint, nous considérons une autre quantité (dans l'esprit de [51]) :

$$\tilde{\lambda}_1^\gamma := \sup \{\lambda \in \mathbb{R} : (\mathcal{L} + \lambda, \mathcal{B}^\gamma) \text{ possède une sur-solution d'infimum positif}\}.$$

D'après la définition, nous avons $\tilde{\lambda}_1^\gamma \leq \lambda_1^\gamma$, mais nous ne savons pas s'il y a égalité. Le signe de $\tilde{\lambda}_1^\gamma$ est une condition suffisante pour la validité du Principe du Maximum.

Theorem B.13 *Supposons que Ω est uniformément $C^{2,\alpha}$ et satisfait une condition uniforme de boule intérieure. Si $\tilde{\lambda}_1^\gamma > 0$, alors $(\mathcal{L}, \mathcal{B}^\gamma)$ satisfait le Principe du Maximum.*

B.2.3 Non-existence de patterns en domaine non-borné

Nous discutons maintenant l'existence de patterns dans les domaines non-bornés. Nous examinons d'abord le cas d'un opérateur auto-adjoint. Dans un second temps, nous examinons le cas d'un opérateur non-auto-adjoint dont le drift est, en quelque sorte, borné. Enfin, nous examinons le cas général.

Cas auto-adjoint

Notre premier résultat concerne le cas *non-dégénéré* $\lambda_1^\gamma > 0$.

Theorem B.14 *Supponsons que L est auto-adjoint, soit $\Omega \subset \mathbb{R}^n$ un domaine possiblement non-borné et u une solution de* (B.9)*. Si $\lambda_1^\gamma > 0$, u est constante.*

Corollary B.15 *Si L est auto-adjoint, il n'existe pas de pattern stable non-dégénéré dans les domaines convexes (possiblement non-bornés).*

Dans le cas particulier $L = \Delta$ et $\Omega = \mathbb{R}^n$, nous retrouvons des résultats connus [104, 160].

Examinons maintenant le cas *possiblement dégénéré* $\lambda_1^\gamma \geq 0$.

Theorem B.16 *Supposons que L est auto-adjoint et que Ω satisfait :*

$$\Omega \cap \{|x| \leq R\} = O(R^2). \tag{B.14}$$

Soit u une solution de (B.9) *telle que $\lambda_1^\gamma \geq 0$.*

1. *Si Ω n'est pas un cylindre droit (i.e., Ω n'est pas de la forme $\mathbb{R} \times \omega$, $\omega \subset \mathbb{R}^{n-1}$), u est constante.*
2. *Si Ω est un cylindre droit, u est soit constante, soit une solution plane monotone connectant (z^-, z^+) deux racines stables de f tels que $\int_{z^-}^{z^+} f = 0$.*

En particulier, sous l'hypothèse que f n'admet pas de racines stables (z^-, z^+) telles que $\int_{z^-}^{z^+} f = 0$, ce théorème prouve la non-existence de patterns dans tout domaine convexe de $\mathbb{R}^2$, ainsi que dans certains domaines convexes de $\mathbb{R}^n$, $n > 2$ (comme un domaine cylindrique dont l'aire de la section croît au plus comme R). Dans le cas particulier $L = \Delta$ et $\Omega = \mathbb{R}^n$, alors $\lambda_1^\gamma = \lambda_1$, et nous retrouvons les résultats classiques sur la variante de la conjecture de De Giorgi (voir la discussion donnée en introduction). L'hypothèse (B.14) fait notamment écho à (B.6), et n'est pas seulement une restriction technique : le résultat ci-dessus est faux dans $\mathbb{R}^{2n}$ pour $2n \geq 8$, voir [246].

Cas non-auto-adjoint avec drift borné

Nous faisons l'hypothèse

$$\sup_{x\in\Omega} \left| B \cdot A^{-1} \cdot x \right| < +\infty.$$

Cette hypothèse exprime le fait que le drift (induit par le terme d'ordre 1) n'est pas dirigé vers l'infini. Sous cette hypothèse, nous montrons les mêmes résultats que dans le cas auto-adjoint.

Theorem B.17 *Soit u une solution de* (B.9)*.*
— *Si $\lambda_1^\gamma > 0$, u est constante.*
— *Supposons $\lambda_1^\gamma \geq 0$, Ω vérifie* (2.11) *et n'est pas un cylindre droit. Alors u est constante.*
— *Supposons $\lambda_1^\gamma \geq 0$, Ω vérifie* (2.11) *et est un cylindre droit. Alors u est soit constante, soit une solution plane monotone connectant (z^-, z^+) deux racines stables de f tels que $\int_{z^-}^{z^+} f = 0$.*

Cas général

Dans le cas général, nous établissons la non existence de solutions non-constantes telles que $\tilde{\lambda}_1^\gamma > 0$.

Theorem B.18 *Supposons que Ω est uniformément $C^{2,\alpha}$ et satisfait une condition de boule intérieure uniforme. Soit u une solution* (B.9)*. Si $\tilde{\lambda}_1^\gamma > 0$, u est constante.*

B.2.4 Autres extensions

Symétries asymptotiques

Nous donnons une autre extension du Théorème B.1 de Casten, Holland, et Matano, en proposant une formulation *asymptotique* du résultat. Nous considérons un domaine cylindrique qui est *asymptotiquement* convexe (voir Figure B.6), et nous montrons que toute solution stable *converge* vers une constante.

Commençons par le cas des solutions *stables non-dégénérées.*

Proposition B.19 *Soit $\Omega \subset \mathbb{R}^N$ un domaine cylindrique (à section variable) sur l'axe x_1, qui converge vers un cylindre droit $\Omega_\infty := \mathbb{R} \times \omega_\infty$ quand $x_1 \to +\infty$. Supposons que ω_∞ est convexe, et soit u une solution stable non-dégénérée de* (B.1)*. Alors $u(x_1, \cdot)$ converge C^2_{loc} vers une racine stable de f quand $x_1 \to +\infty$.*

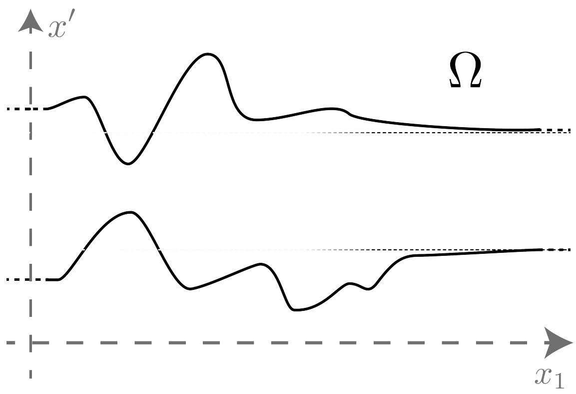


FIGURE B.6 – Cylindre asymptotiquement convexe

Notons que nous ne supposons pas que ω_∞ est borné.

Examinons maintenant le cas des solutions stables (possiblement dégénérées).

Proposition B.20 *Sous les mêmes hypothèses, mais avec une solution u seulement supposée stable. Supposons que les racines stables de f, notée (z_i), sont isolées, et que $\int_{z_i}^{z_j} f \neq 0$, pour $i \neq j$. Supposons de plus que Ω satisfait*

$$\Omega \cap \{|x| \leq R\} = O(R^2).$$

Alors $u(x_1, \cdot)$ converge C^2_{loc} vers une racine stable de f quand $x_1 \to +\infty$.

La section 2.3.3 contient d'autres résultats de symétrie, si le domaine limite Ω_∞ possède une invariance par translation ou rotation planaire. En particulier, nous avons le

Corollary B.21 *Soit $\Omega \subset \mathbb{R}^N$ un domaine cylindrique (à section variable) sur l'axe x_1, qui converge vers un cylindre droit $\Omega_\infty := \mathbb{R} \times \omega_\infty$, $w_\infty \subset \mathbb{R}^{n-1}$ borné, quand $x_1 \to +\infty$. Soit u une solution stable non-dégénérée de* (B.1). *Alors, quand $x_1 \to +\infty$, $u(x_1, \cdot)$ converge C^2_{loc} vers $u_\infty(\cdot)$, qui est une solution stable dans la section ω_∞ :*

$$\begin{cases} -\Delta_{x'} u_\infty = f(u_\infty) & \textit{dans } \omega_\infty, \\ \partial_\nu u_\infty = 0 & \textit{sur } \partial\omega_\infty. \end{cases}$$

***Flatness* des patterns**

En complément, nous proposons une estimation de la flatness des patterns en domaine borné quelconque. Cette estimation porte sur le gradient de u dans toutes les directions sauf une, et suggère que les patterns ont tendance à être plans. Cette même idée est exprimée dans la conjecture de De Giorgi et par les résultats de Gurtin et Matano [169].

Pour pouvoir énoncer notre résultat, nous introduisons la notion de *Spectral Gap*, qui exprime le fait que λ_1 est à une distance positive du reste du spectre de l'opérateur linéarisé (B.2). Soit u une solution de (B.2). Rappelons que

$$\lambda_1 := \inf_{\substack{\psi \in H^1 \\ \|\psi\|_{L^2}=1}} \mathcal{F}(\psi), \quad \text{avec} \quad \mathcal{F}(\psi) := \int_\Omega |\nabla\psi|^2 - f'(u)\psi^2,$$

et que λ_1 possède une fonction propre φ. Considérons l'hyperplan orthogonal à φ dans L^2 :

$$E := \mathrm{Vect}(\varphi)^\perp = \left\{\psi \in H^1 : \int_\Omega \psi\varphi = 0\right\},$$

et posons

$$\lambda_2 := \inf_{\substack{\psi\in E\\ \|\psi\|_{L^2}=1}} \mathcal{F}(\psi).$$

La propriété de *Spectral Gap* (voir par exemple [12, 97, 98]) assure $\lambda_2 - \lambda_1 > 0$. En particulier, si u est stable, alors $\lambda_2 > 0$. Notons que les domaines haltères sont des domaines pour lesquels $\lambda_2 - \lambda_1$ est très proche de 0.

Notre estimation de la flatness des patterns est la suivante.

Proposition B.22 *Soit Ω un domaine borné et u un pattern de* (B.1). *Il existe $(e_1, \dots, e_n)$ une base orthonormale de $\mathbb{R}^n$ telle que, pour tout $i \geq 2$,*

$$\|\partial_i u\|_{L^2}^2 \leq \frac{1}{\lambda_2} I_\gamma(\nabla u), \quad \forall i \in \{2, \cdots, n\},$$

où

$$I_\gamma(\nabla u) := -\int_{\partial\Omega} \gamma |\nabla u|^2$$

et $\gamma(x)$ est comme dans (B.10).

Si, en outre, u est une solution stable non-dégénérée de (B.1)*, alors*

$$\|\nabla u\|_{L^2}^2 \leq \left(\frac{1}{\lambda_1} + \frac{n-1}{\lambda_2}\right) I_\gamma(\nabla u).$$

Notons que $I_\gamma(\nabla u) \leq C\partial\Omega \sup_{\partial\Omega} \gamma^-$, où C est une borne sur $|\nabla u|$ (donnée, par exemple, par les estimations a priori classiques), et γ^- la partie négative de γ. Cela donne une estimation qui ne dépend pas de u. En particulier, si le domaine est convexe, alors $\gamma^- = 0$, et u est plane (mais, dans ce cas, le théorème B.1 implique déjà que u est constante).

Chapitre C

Introduction à la partie II : Concentration en masse de Dirac et modèles pour la sélection naturelle

C.1 Introduction

Dans cette section, nous abordons la modélisation de la théorie de l'évolution darwinienne [112]. L'approche mathématique de ce problème a connu récemment des progrès considérables, et est aujourd'hui un enjeu majeur en terme d'applications [165, 239, 263]. Ici, nous adoptons une méthode EDPs appelée *approche Hamilton-Jacobi*, introduite dans l'article de Diekmann, Jabin, Mischler, Perthame [125].

En guise d'introduction, nous considérons un exemple simple, dont l'étude est élémentaire. Cet exemple nous permettra d'introduire les éléments essentiels à notre étude, d'interpréter les résultats en terme de modélisation, et de donner un aperçu de la richesse de l'approche Hamilton-Jacobi.

Nous présenterons ensuite les résultats contenus dans la partie II ce manuscrit. Au chapitre 4, nous nous intéresserons au cas d'une population structurée en âge. Nous considérerons le même modèle avec l'ajout d'un phénomène de mutation au chapitre 5. Enfin, nous présenterons au chapitre 6 un modèle de *transmission génétique horizontale* qui met en jeu un phénomène de *sauvetage évolutif*, et induit une dynamique cyclique de la population.

C.1.1 L'approche Hamilton-Jacobi sur un exemple simple

Concentration vers une masse de Dirac

Commençons par donner une idée intuitive de l'approche adoptée, sur un exemple simple. Considérons une population, dont les individus présentent des différences phénotypiques. Nous supposons que le phénotype d'un individu est encodé dans une variable $y \in \mathbb{R}^n$; ainsi, dans toute cette section, y représente le *phénotype* (aussi appelé *trait*) d'un individu.

Notons $m(t, y)$ la densité d'individus de phénotype y au temps t, et $\rho(t) := \int_{\mathbb{R}^n} m(t, \cdot)$ la population totale. Supposons que la densité d'individus obéit à la dynamique suivante :

$$\begin{cases} \partial_t m(t,y) = (r(y) - \rho(t))m(t,y), \\ m(0,y) = m^0(y) > 0, \end{cases} \tag{C.1}$$

où $r(\cdot)$ est une fonction continue, bornée par deux constantes positives $\underline{r}$, $\overline{r}$. Par simplicité, nous supposons que r atteint son maximum $\overline{r}$ en un unique point $\bar{y}$.

Cette équation traduit deux hypothèses élémentaires de modélisation :

1. Les individus de phénotype y ont un taux croissance intrinsèque $r(y)$, appelée *fitness* [223]. On peut penser à r comme le taux de naissance moins le taux de mort intrinsèque. Ici, nous avons supposé qu'il existe un unique phénotype $\bar{y}$ de fitness optimale $\overline{r}$.
2. Le milieu a une capacité d'accueil limitée, ce qui induit une compétition entre les individus pour la survie. Ainsi, un taux de mort supplémentaire s'applique, lié à la taille de la population : par simplicité nous choisissons $-\rho(t)$.

Le modèle (C.1) peut être vu comme un système de Lotka-Volterra avec un nombre d'espèces indexé par $y \in \mathbb{R}^n$. L'intuition suggère que seuls les individus ayant la fitness optimale $\overline{r}$ puissent survivre en temps long. Mathématiquement, cela se traduirait par la convergence de $m(t,\cdot)$ vers une masse de Dirac $\rho_\infty \delta_{y=\bar{y}}$ quand $t \to +\infty$. Examinons cela plus en détails.

Le terme de compétition $-\rho(t)$ induit un *effet de saturation* sur la taille de la population. En effet, en intégrant l'équation selon $y \in \mathbb{R}^n$, nous obtenons

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) \leq (\overline{r} - \rho(t))\rho(t) \quad ; \quad \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) \geq (\underline{r} - \rho(t))\rho(t).$$

Ainsi, pourvu que $\rho(t=0) \in (\underline{r}, \overline{r})$ (ce que nous supposons, par simplicité), nous obtenons

$$\underline{r} \leq \rho(t) = \int_{\mathbb{R}^n} m(t,y)dy \leq \overline{r}. \tag{C.2}$$

L'équation (C.1) peut être intégrée explicitement :

$$m(t,y) = m^0(y) \exp\left(r(y)t - \int_0^t \rho\right),$$

et, après réécriture,

$$m(t,y) = m^0(y) \exp\Big((r(y) - \overline{r})t\Big) \exp\left(\overline{r}t - \int_0^t \rho\right).$$

Rappelons que, par simplicité, nous supposons que $r(\cdot)$ atteint son maximum $\overline{r}$ en un unique point $\bar{y} \in \mathbb{R}^n$. Écrit sous cette forme, et d'après (C.2), nous voyons que quand $t \to +\infty$, d'une part, $m(t,\cdot) \approx 0$ dans $\mathbb{R}^n \setminus \{\bar{y}\}$, et d'autre part, $\overline{r}t \approx \int_0^t \rho$. On peut effectivement montrer, au sens des distributions,

$$m(t,\cdot) \rightharpoonup \overline{r}\delta_{y=\bar{y}}, \quad \text{quand } t \to +\infty.$$

Cette convergence vers une masse de Dirac est appelée dans la littérature mathématique un phénomène de *concentration*. Du point de vu de la modélisation, elle signifie qu'en temps long, seul le phénotype optimal $\bar{y}$ survit : c'est un modèle pour la *sélection naturelle*.

Dynamique adaptative

Le résultat de concentration de la section précédente nous renseigne sur l'état de la population quand $t \to +\infty$. En revanche, il ne donne aucune information sur les états intermédiaires par lesquels passe la population. Nous nous intéressons à savoir *comment* la convergence précédente a lieu. Par exemple, si nous partons d'une population initiale qui ressemble à une masse de Dirac localisée en $y^0 \neq \bar{y}$, la population ressemble-t-elle, à tout instant, à une masse de Dirac ?

Pour répondre à cette question, nous utilisons une méthode, introduite dans ce contexte par Evans et Souganidis [139], qui vient de la théorie de l'homogénéisation. Nous introduisons dans l'équation un paramètre $\varepsilon \ll 1$, et effectuons le changement d'échelle $t' \longleftrightarrow \varepsilon^{-1}t$ (c'est-à-dire un dé-zoom en temps). Posons $m_\varepsilon(t,y) = m(\varepsilon^{-1}t,y)$, et $\rho_\varepsilon(t) = \rho(\varepsilon^{-1}t) = \int_{\mathbb{R}^n} m_\varepsilon(t,\cdot)$. Nous considérons donc l'équation suivante

$$\begin{cases} \varepsilon\partial_t m_\varepsilon(t,y) = (r(y) - \rho_\varepsilon(t))m_\varepsilon(t,y), \\ m_\varepsilon(0,y) = m_\varepsilon^0(y) > 0, \end{cases}$$

et supposons que la condition initiale ressemble à une masse de Dirac, c'est-à-dire à une Gaussienne de variance ε autour de y^0 :

$$m_\varepsilon^0(y) := \rho^0 \frac{1}{(\varepsilon\pi)^{n/2}} \exp\left(-\frac{|y-y^0|^2}{\varepsilon}\right), \quad \text{avec } \rho^0 \in (\underline{r}, \overline{r}).$$

D'après la forme de la condition initiale, il est commode d'effectuer un changement de variable, appelée *ansatz WKB* (ou *transformation de Hopf-Cole*) : posons $u_\varepsilon = \varepsilon\ln(m_\varepsilon)$, de manière à avoir $m_\varepsilon = \exp(\frac{u_\varepsilon}{\varepsilon})$. Nous avons

$$\partial_t u_\varepsilon(t,y) = r(y) - \rho_\varepsilon(t),$$

et nous déduisons

$$u_\varepsilon(t,y) = u_\varepsilon(0,y) + r(y)t - \int_0^t \rho_\varepsilon.$$

Étudions le comportement asymptotique de u_ε quand $\varepsilon \to 0$. Premièrement, remarquons que

$$u_\varepsilon(t=0,y) \underset{\varepsilon\to 0}{\longrightarrow} -|y-y^0|^2.$$

De plus, rappelons que d'après (C.2), nous avons $\underline{r} \leq \rho_\varepsilon(t) \leq \overline{r}$: la famille $(\rho_\varepsilon)_{\varepsilon>0}$ est bornée dans L^∞. L'espace L^∞ peut-être vu comme le dual de L^1. Par le théorème de Banach-Alaoglu, il existe une suite $\varepsilon_k \to 0$ telle que ρ_{ε_k} converge L^∞-faible-* vers un certain ρ. Nous avons, pour tout $t \geq 0$,

$$\int_0^t \rho_{\varepsilon_k} \underset{k\to+\infty}{\longrightarrow} \int_0^t \rho,$$

et nous déduisons

$$u_{\varepsilon_k} \underset{k\to+\infty}{\longrightarrow} u(t,y) := -|y-y^0|^2 + r(y)t - \int_0^t \rho.$$

Montrons maintenant que u ne dépend pas de l'extraction $\varepsilon_k \to 0$: cela prouvera, *a posteriori*, que u_ε converge vers u. D'après $\rho_\varepsilon(t) = \int_{\mathbb{R}^n} \exp\left(\frac{u_\varepsilon}{\varepsilon}\right)$, nous voyons bien que, pour tout $t \geq 0$,

$$\limsup_{k\to+\infty}\left(\sup_{y\in\mathbb{R}^n} u_{\varepsilon_k}(t,y)\right) > 0 \quad \Longrightarrow \quad \rho_{\varepsilon_k}(t) \underset{k\to+\infty}{\longrightarrow} +\infty,$$
$$\limsup_{k\to+\infty}\left(\sup_{y\in\mathbb{R}^n} u_{\varepsilon_k}(t,y)\right) < 0 \quad \Longrightarrow \quad \rho_{\varepsilon_k}(t) \underset{k\to+\infty}{\longrightarrow} 0.$$

Or, $\underline{r} \leq \rho_\varepsilon(t) \leq \overline{r}$, donc

$$\forall t \geq 0, \quad \sup_{y\in\mathbb{R}^n} u(t,y) = 0,$$

c'est-à-dire

$$\int_0^t \rho = \sup_{y\in\mathbb{R}^n}\left(-|y-y^0|^2 + r(y)t\right).$$

L'intégrale $\int_0^t \rho$ ne dépend donc pas de la sous suite ε_k. Nous avons ainsi prouvé la convergence de u_ε :

$$u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u(t,y) := -|y-y^0|^2 + r(y)t - \int_0^t \rho. \tag{C.3}$$

D'après $m_\varepsilon = \exp\left(\frac{u_\varepsilon}{\varepsilon}\right)$, et en notant

$$\mathcal{S} := \{(t,y) : u(t,y) = \sup u\} = \Big\{(t,y) : u(t,y) = 0\Big\},$$

nous déduisons que m_ε converge faiblement vers une mesure dont le support est inclus dans $\mathcal{S}$. Cette convergence établit que la population se *concentre* là où $u(t,\cdot)$ atteint son maximum (qui es égal à 0).

En supposant formellement que $u(t,\cdot)$ est concave, elle atteint son maximum en un unique point $\bar{y}(t)$, et nous avons

$$m_\varepsilon(t,y) \underset{\varepsilon\to 0}{\rightharpoonup} \rho(t)\delta_{y=\bar{y}(t)}.$$

En dérivant formellement l'égalité $\nabla_y u(t,\bar{y}(t)) = 0$, nous déduisons l'*Équation Canonique* :

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{y}(t) = -\left[\nabla_y^2 u(t,\bar{y}(t))\right]^{-1} \cdot \nabla_y r(\bar{y}(t)). \tag{C.4}$$

qui décrit la dynamique du point de concentration $\bar{y}(t)$.

Notons en particulier que $r(\bar{y}(t))$ est une fonction de Lyapunov :

$$\frac{\mathrm{d}}{\mathrm{d}t}r(\bar{y}(t)) = -\nabla_y r(\bar{y}(t)) \cdot \left[\nabla_y^2 u(t,\bar{y}(t))\right]^{-1} \cdot \nabla_y r(\bar{y}(t)) \geq 0.$$

Ainsi, $y(t)$ évolue selon la pente croissante de $r(\cdot)$, et converge vers $\bar{y}$.

Ajout de mutations

Ajoutant maintenant un dernier ingrédient à notre modèle : le phénomène de *mutation*. Nous supposons qu'un individu de phénotype y peut donner naissance à un individu de phénotype y' avec probabilité $M(y'-y)$, où M est par exemple un noyau Gaussien centré en 0. En notant respectivement $b(y)$ et $d(y)$ le taux de naissance et le taux de mort (de sorte à ce que $r(y) = b(y) - d(y)$), nous considérons l'équation

$$\begin{cases} \partial_t m(t,y) = \int_{\mathbb{R}^n} M(y'-y)b(y')m(t,y')dy' - (d(y)+\rho(t))m(t,y), \\ \rho(t) = \int_{\mathbb{R}^n} m(t,y)dy, \\ m(0,y) = m^0(y) > 0. \end{cases}$$

Ce modèle présente un phénomène de diffusion non-locale en y. Nous cherchons à obtenir des résultats analogues au cas sans mutation, c'est-à-dire, notamment, la concentration vers une masse de Dirac.

Comme précédemment, nous introduisons un paramètre ε et effectuons un changement d'échelle. Notons cependant que le terme de diffusion a pour effet de *lisser* la solution, et donc d'empêcher la concentration vers une masse de Dirac. Ainsi, le changement d'échelle en t doit s'accompagner d'un changement d'échelle en y : nous choisissons un changement d'échelle hyperbolique, et posons $m_\varepsilon(t,y) = m(\varepsilon^{-1}t, \varepsilon^{-1}y)$, $\rho_\varepsilon(t) = \rho(\varepsilon^{-1}t) = \int_{\mathbb{R}^n} m_\varepsilon(t,\cdot)$. L'équation devient

$$\varepsilon\partial_t m_\varepsilon(t,y) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} M\left(\frac{y'-y}{\varepsilon}\right) b(y')m_\varepsilon(t,y')dy' - (d(y)+\rho_\varepsilon(t))m_\varepsilon(t,y),$$

puis avec un changement de variable $z := \frac{y'-y}{\varepsilon}$,

$$\varepsilon\partial_t m_\varepsilon(t,y) = \int_{\mathbb{R}^n} M(z)b(y+\varepsilon z)m_\varepsilon(t,y+\varepsilon z)dz - (d(y)+\rho_\varepsilon(t))m_\varepsilon(t,y).$$

Procédons à l'*ansatz WKB*, c'est-à-dire posons $u_\varepsilon = \varepsilon \ln(m_\varepsilon)$. Nous obtenons

$$\partial_t u_\varepsilon(t,y) = \int_{\mathbb{R}^n} M(z)b(y+\varepsilon z)\exp\left(\frac{u_\varepsilon(t,y+\varepsilon z) - u_\varepsilon(t,y)}{\varepsilon}\right) - d(y) - \rho_\varepsilon(t).$$

En passant formellement à la limite $\varepsilon \to 0$, nous obtenons

$$\begin{cases} \partial_t u(t,y) = b(y)\int_{\mathbb{R}^n} M(z)\exp\left(\nabla_y u(t,y)\cdot z\right) - d(y) - \rho(t), \\ u(0,y) = -|y-y^0|^2. \end{cases} \tag{C.5}$$

C'est une équation de Hamilton-Jacobi.

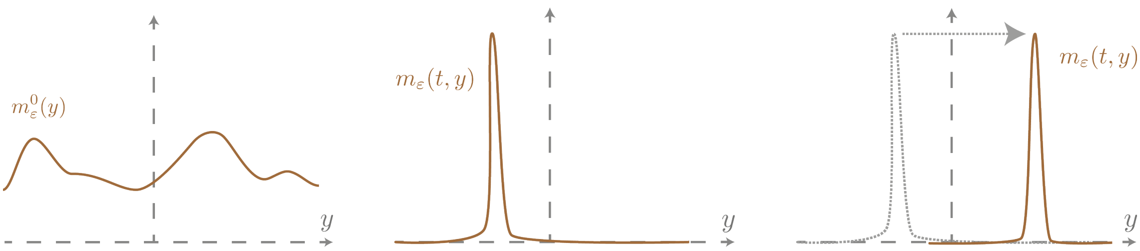
Cette équation est le point central de l'étude du modèle. Pour cette raison, la méthode que nous venons de présenter est souvent appelée l'*approche Hamilton-Jacobi*. Il est possible de démontrer des résultats analogues au cas sans mutations. Notamment, on peut montrer $\sup_{\mathbb{R}^n} u(t,\cdot) = 0$, et que la population se concentre là où $u(t,y) = 0$. On peut aussi, au moins formellement, obtenir l'Équation Canonique (C.4).

Cependant, l'ajout du terme de mutation peut présenter une difficulté majeure d'un point de vu mathématique. Premièrement, remarquons qu'une solution classique de (C.5) a peu de chance, *a priori*, d'être définie globalement en temps. La notion naturelle (et très pratique) dans ce cadre est celle des *solutions de viscosité*, introduite par Crandall et Lions dans les années 1980 (voir les ouvrages de référence [30,31,102,153]).

Le point délicat est de démontrer la convergence de u_ε vers u. Pour cela, la stratégie consiste à passer à la limite dans l'équation sur u_ε au sens des viscosités (qui est un sens assez faible), puis d'utiliser un résultat d'unicité sur l'équation limite, ce qui prouve, *a posteriori*, la convergence localement uniforme de u_ε vers u.

C.1.2 Interprétations

Nous voyons sur cet exemple simple la richesse de l'approche. Notre modèle permet de décrire la dynamique de la population sur deux échelles de temps. L'échelle de temps écologique t correspond au temps caractéristique que met une population monomorphe (*i.e.*, avec un seul $y \in \mathbb{R}^n$ fixé) à atteindre l'équilibre. À cette échelle, nous observons la concentration de la population autour d'un phénotype dominant $\bar{y}(t)$: c'est le phénomène de *sélection naturelle*. À l'échelle de temps évolutive $\varepsilon^{-1}t$, la population a, à tout instant, une structure monomorphe autour du phénotype $\bar{y}(t)$, qui lui-même évolue selon l'Équation Canonique (C.4) : c'est une description de la *dynamique adaptative*. Dit de manière informelle, au bout d'un temps assez court, la population ressemble à un Dirac (Figures C.1a-C.1b), puis, sur une échelle de temps plus longue, ce Dirac se déplace vers le phénotype optimal (Figure C.1c).



(a) temps initial **(b)** échelle écologique **(c)** échelle évolutive

FIGURE C.1 – État de la population $m_\varepsilon(t, y)$

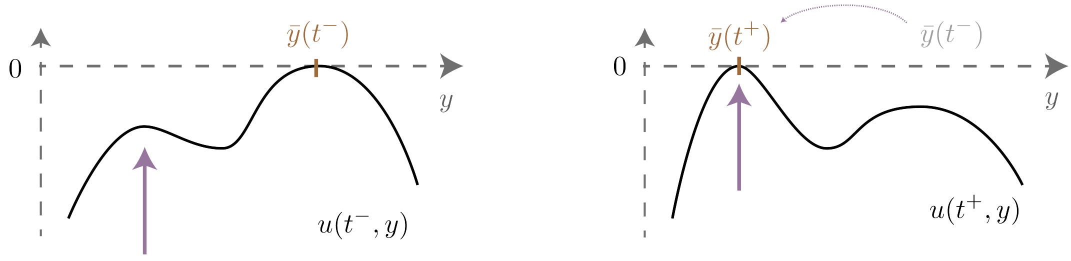
Notons cependant que la validité de l'Équation Canonique repose sur l'hypothèse formelle que $u(t, \cdot)$ est concave. Dans le cas sans mutation, nous voyons d'après (C.3) que, en temps long, la concavité de u dépend uniquement de la concavité de la fitness $r(\cdot)$. Supposer que la fitness est concave signifie qu'il n'y a qu'un seul phénotype optimal. Cette hypothèse est donc assez forte en terme de modélisation. Dans le cas avec mutations, nous voyons à travers l'équation (C.5) que la non-linéarité elle-même peut engendrer la perte de concavité de u.

En général, on ne s'attend pas à ce que la dynamique de $\bar{y}(t)$ suive (C.4) pour tout temps. Typiquement, la dynamique de $\bar{y}(t)$ peut présenter des sauts, correspondant à l'apparition d'un nouveau maximum global pour u, voir Figure C.2a-C.2b.

Par ailleurs, il peut exister un temps t_0 pour lequel $\bar{y}(t_0)$ est un maximum dégénéré (i.e $\nabla_y^2 u(t, \bar{y}(t)) = 0$). Ces point dégénérés peuvent donner lieu à des phénomènes de *branchage*, à l'issu desquels une population monomorphe devient polymorphe. Pour une analyse plus détaillée de l'Équation Canonique, nous laisserons le lecteur se référer à [18, 119, 220, 222].

C.1.3 Bref état de l'art

Kimura [192] a été le premier à modéliser les phénotypes par une variable continue $y \in \mathbb{R}^n$. L'approche Hamilton-Jacobi, telle que nous l'avons décrite, a été



(a) $t = t^-$ **(b)** $t = t^+$

FIGURE C.2 – Saut de $\overline{y}(t)$ lors de l'apparition d'un nouveau maximum

introduite par Diekmann, Jabin, Mischler et Perthame [125]. Elle a ensuite été développée [33, 36, 213] et adaptée dans différentes situations, par exemple : des modèles en espace [155, 183, 202], en particulier pour analyser les fronts d'invasion [8, 44, 60, 61, 63, 81, 277] , des modèles de populations sexuées [231, 258], ou des modèles avec plusieurs ressources [93–95, 230]. Nous citons également [79, 80], ainsi que [81, 87] qui présentent des résultats de concentration atypiques. Une introduction au sujet est proposée dans les thèses de Mirrahimi [227] et Taing [274].

Nous insistons sur le fait que le point de vu probabiliste est très fécond dans ce domaine, et tous les modèles que nous décrivons peuvent être déduits de modèles probabilistes individus-centrés [89–91, 119, 219]. La modélisation de la sélection darwinienne est aussi abordée dans le cadre de la théorie des jeux [174, 178] et des systèmes dynamiques [118, 124, 159].

C.2 Présentation des résultats

C.2.1 Population structurée en âge

En collaboration avec B. Perthame et C. Taing. Nous présentons ici le contenu du Chapitre 4.

Nous étudions le modèle d'une population structurée en *phénotype* et en *âge*. Soit $m_\varepsilon(t, x, y)$ une densité de population, fonction du temps $t \geq 0$, de l'âge $x \geq 0$, et du phénotype $y \in \mathbb{R}^n$, vérifiant

$$\begin{cases} \varepsilon \partial_t m_\varepsilon + \partial_x \left[A(x, y) m_\varepsilon \right] + \left(\rho_\varepsilon(t) + d(x, y) \right) m_\varepsilon = 0, \\ A(x = 0, y) m_\varepsilon \left(t, x = 0, y \right) = \displaystyle\int_{\mathbb{R}_+} b(x', y) m_\varepsilon(t, x', y) dx' dy, \\ \rho_\varepsilon(t) = \displaystyle\int_{\mathbb{R}_+} \int_{\mathbb{R}^n} m_\varepsilon(t, x, y) dx dy, \\ m_\varepsilon(t = 0, x, y) = m_\varepsilon^0(x, y) > 0. \end{cases} \tag{C.6}$$

Comme en introduction, $\varepsilon > 0$ est un paramètre de changement d'échelle ; la quantité $\rho_\varepsilon(t)$ représente la taille de la population au temps t ; le taux de mortalité contient un terme intrinsèque $d(x, y)$ et un terme de compétition $\rho(t)$. Le terme de bord signifie qu'un individu père donne naissance, au taux $b(x, y)$, à un individu d'âge $x = 0$. Notons que ce modèle ne contient pas de mutation. Le modèle avec mutations sera présenté à la section suivante.

Le terme de transport signifie que les individus "vieillissent" à vitesse $A(x,y)$. Pour plus de généralité, nous laissons ce paramètre dépendre de x, ce qui signifie que le transport peut ne pas se faire à vitesse constante. Ainsi, la variable x peut aussi bien représenter l'âge que toute autre quantité qui évolue au cours de la vie d'un individu et n'est pas transmis à la progéniture. On pourra penser, par exemple, à la taille, la maturation, la charge virale, *etc.*

Ce type de modèle pour les populations structurées en âge est souvent appelé *renewal equation* dans la littérature (d'après le terme de bord). Il est lié à l'équation de McKendrick-von Foerster (voir [247]). Il est étudié dans des contextes variés : populations structurées en taille [221, 233], modèles pour la division cellulaire [131, 224], croissance des tumeurs [3, 170], et beaucoup d'autres. Des modèles structurés en âge et en phénotype sont également considérés dans [82] et [83].

Approche formelle

Nous rappelons que l'*ansatz WKB* consiste au changement de variable :

$$m_\varepsilon(t,x,y) = e^{\frac{v_\varepsilon(t,x,y)}{\varepsilon}}.$$

Nous proposons une variante. Le principe est de procéder à un développement de Taylor $v_\varepsilon(t,x,y) = v_\varepsilon^1(t,y) + \varepsilon v_\varepsilon^2(t,x,y)$, de définir v_ε^1 comme un ansatz, puis d'estimer le terme d'erreur v_ε^2. Après réécriture, nous procédons au changement de variable suivant :

$$m_\varepsilon(t,x,y) = p_\varepsilon(t,x,y) e^{\frac{u_\varepsilon(t,y)}{\varepsilon}}. \tag{C.7}$$

Ici, $u_\varepsilon(t,y)$ sera notre ansatz, que nous allons définir *ad hoc* comme la solution d'une équation. Cette variante a pour avantage que le terme d'erreur p_ε que l'on doit estimer satisfait une équation linéaire, plutôt qu'une équation de Hamilton-Jacobi.

Nous devons maintenant trouver un bon candidat pour notre ansatz u_ε. En injectant (C.7) dans l'équation (C.6), nous obtenons

$$\begin{cases} \varepsilon \partial_t p_\varepsilon + \partial_x \left[A(x,y) p_\varepsilon \right] + d(x,y) p_\varepsilon + \left(\partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t) \right) p_\varepsilon = 0, \\ A(0,y) p_\varepsilon(t,0,y) = \int_{\mathbb{R}_+} b(x,y) p_\varepsilon(t,x,y) dx dy. \end{cases} \tag{C.8}$$

En posant formellement $\varepsilon \partial_t p_\varepsilon \approx 0$, nous considérons le problème aux valeurs propres : pour $y \in \mathbb{R}^n$ fixé, soit $(\Lambda(y), Q(x,y))$ l'unique solution de

$$\begin{cases} \partial_x \left[A(x,y) Q \right] + d(x,y) Q - \Lambda Q = 0, \\ A(0,y) Q(0,y) = \int_{\mathbb{R}_+} b(x,y) Q(x,y) dx, \\ Q(x,y) > 0, \quad \int_{\mathbb{R}_+} b(x,y) Q(x,y) dx = 1. \end{cases}$$

Intuitivement, $\Lambda(y)$ correspond à la *fitness effective*, et $Q(x,y)$ au profil d'âge à l'équilibre.

L'équation (C.8) suggère alors de définir u_ε comme solution de :

$$\begin{cases} \partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t) = -\Lambda(y), \\ u_\varepsilon(t=0,y) = u_\varepsilon^0(y), \end{cases}$$

pour une certaine condition initiale u_ε^0. Ainsi, p_ε verifie

$$\begin{cases} \varepsilon \partial_t p_\varepsilon + \partial_x \left[A(x,y) p_\varepsilon \right] - \Lambda(y) p_\varepsilon = 0, \\ A(0,y) p_\varepsilon(t,0,y) = \int_{\mathbb{R}_+} b(x,y) p_\varepsilon(t,x,y) dx dy, \end{cases}$$

et devrait donc se comporter comme un multiple de $Q(x,y)$ quand $\varepsilon \to 0$.

Principaux résultats

Sous des hypothèses générales sur les paramètres, et si la condition initiale est bien préparée, nous démontrons le théorème suivant, illustré aux Figures C.3a-C.3b-C.4-C.5.

Theorem C.1 *Quand $\varepsilon \to 0$:*

1. *$\rho_\varepsilon(t) = \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} m_\varepsilon(t,x,y)dxdy$ converge L^∞-faible-$\star$ vers un certain ρ.*
2. *p_ε converge vers un multiple de Q, pour une norme L^1 à poids.*
3. *u_ε converge localement uniformément vers une fonction u, solution de*

$$\begin{cases} \partial_t u(t,y) = -\Lambda(y) - \rho(t), & t > 0, y \in \mathbb{R}^n, \\ \sup\limits_{y \in \mathbb{R}^n} u(t,y) = 0, & \forall t > 0, \\ u(0,y) = u^0(y), & y \in \mathbb{R}^n. \end{cases}$$

4. *m_ε converge faiblement vers une mesure μ, dont le support est inclus dans $\mathcal{S} = \{(t,y) \in (0,\infty) \times \mathbb{R}^n | u(t,y) = 0\}$.*
5. *De plus, si on suppose que u^0 et $-\Lambda$ sont strictement concaves, alors*

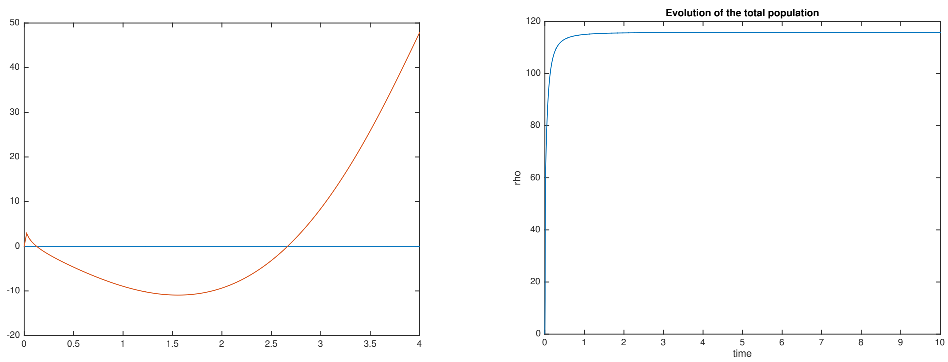
$$m_\varepsilon(t,x,y) \underset{\varepsilon \to 0}{\rightharpoonup} \rho(t) \frac{Q(x,y)}{\|Q(\cdot,y)\|_{L^1}} \delta_{y=\bar{y}(t)},$$

où $\bar{y}(t) \in \mathbb{R}^n$ satisfait l'Équation Canonique :

$$\dot{\bar{y}}(t) = \left(\nabla_y^2 u(t,\bar{y}(t))\right)^{-1} \cdot \nabla_y \Lambda\left(\bar{y}(t)\right).$$

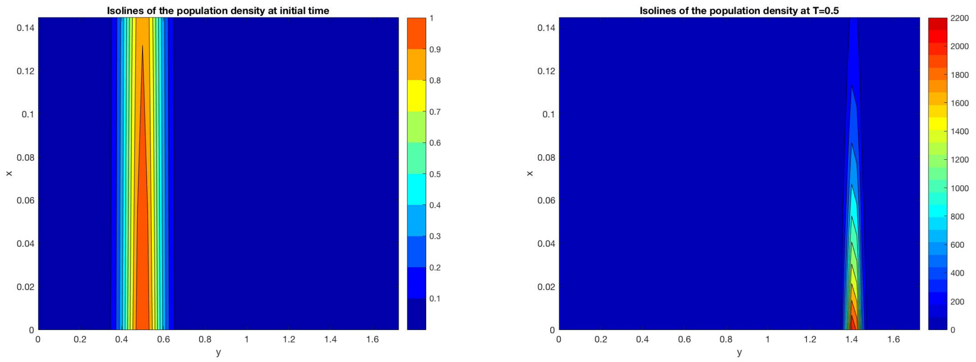
Dans ce cas, $\rho'(t) \geq 0$, et

$$u^0(\bar{y}(t)) = \int_0^t \rho(t')dt' - t\rho(t).$$



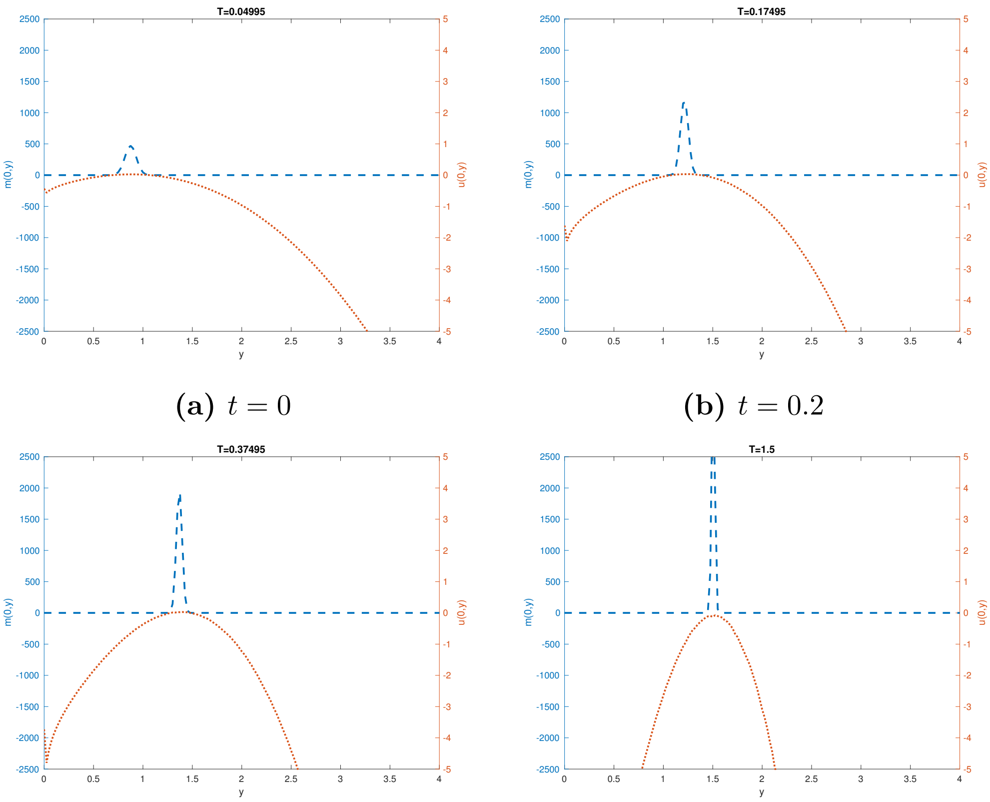
(a) Fitness effective $\Lambda(y)$ **(b)** Évolution de $\rho(t)$

FIGURE C.3



(a) $t = 0$ **(b)** $t = 0.5$

FIGURE C.4 – Isolignes de m_ε selon les variables (x, y).



(a) $t = 0$ **(b)** $t = 0.2$

(c) $t = 0.4$ **(d)** $t = 1.5$

FIGURE C.5 – Clichés de la distribution de la population en fonction de y. Ligne bleu pointillée= m_ε. Ligne rouge pointillée= u_ε.

C.2.2 Ajout d'un phénomène de mutation

En collaboration avec B. Perthame. Nous présentons le contenu du Chapitre 5.

Nous ajoutons un terme de mutation au modèle précédent. Nous considérons donc l'équation

$$\begin{cases} \varepsilon\partial_t m_\varepsilon + \partial_x \left[A(x,y)m_\varepsilon\right] + \left(\rho_\varepsilon(t) + d(x,y)\right) m_\varepsilon = 0, \\ A(x=0,y)m_\varepsilon\left(t,x=0,y\right) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} M(\frac{y'-y}{\varepsilon})b(x',y')m_\varepsilon(t,x',y')dx'dy', \\ \rho_\varepsilon(t) = \int_{\mathbb{R}_+}\int_{\mathbb{R}^n} m_\varepsilon(t,x,y)dxdy, \\ m_\varepsilon(t=0,x,y) = m_\varepsilon^0(x,y) > 0, \end{cases}$$

où $M(\cdot)$ est un noyau de probabilité, par exemple Gaussien. Le terme de mutation ajoute une difficulté majeure. En effet, la présence de diffusion (non-locale) rend difficile l'identification d'un problème aux valeurs propres limite. À la place, nous introduisons un problème *approché* (au prix d'une variable supplémentaire η), qui s'avère bien adapté à notre cadre ; il s'agit là de la principale trouvaille de notre approche.

Approche formelle

Variante de l'ansatz WKB. Comme décrit à la section précédente, notre stratégie est d'effectuer le changement de variable

$$m_\varepsilon(t,x,y) = p_\varepsilon(t,x,y)e^{\frac{u_\varepsilon(t,y)}{\varepsilon}},$$

où u_ε est un ansatz, et p_ε le terme d'erreur à estimer. En injectant cette forme dans l'équation, nous obtenons

$$\begin{cases} \varepsilon\partial_t p_\varepsilon + \partial_x \left[A(x,y)p_\varepsilon\right] + d(x,y)p_\varepsilon + \left(\partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t)\right) p_\varepsilon = 0, \\ A(0,y)p_\varepsilon\left(t,0,y\right) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} M(\frac{y'-y}{\varepsilon})e^{\frac{u_\varepsilon(t,y')-u_\varepsilon(t,y)}{\varepsilon}}b(x',y')p_\varepsilon(t,x',y')dx'dy'. \end{cases} \tag{C.9}$$

Après le changement de variable $z = \frac{y'-y}{\varepsilon}$, le terme de bord s'écrit

$$A(0,y)p_\varepsilon(t,0,y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}_+} M(z)e^{\frac{u_\varepsilon(t,y+\varepsilon z)-u_\varepsilon(t,y)}{\varepsilon}}b(x',y+\varepsilon z)p_\varepsilon(t,x',y+\varepsilon z)dx'dz.$$

Quand ε est petit, on peut formellement approximer

$$A(0,y)p(t,0,y) \approx \eta_\varepsilon(t,y)\int_{\mathbb{R}_+} b(x',y)p(t,x',y)dx', \tag{C.10}$$

où

$$\eta_\varepsilon(t,y) := \int_{\mathbb{R}^n} M(z)e^{\frac{u_\varepsilon(t,y+\varepsilon z)-u_\varepsilon(t,y)}{\varepsilon}}dz.$$

Alors, en posant formellement $\varepsilon\partial_t p_\varepsilon \approx 0$ dans la première ligne de (C.9), nous aboutissons au problème *approché* :

$$\begin{cases} \partial_x \left[A(x,y)p_\varepsilon(t,x,y)\right] + d(x,y)p_\varepsilon(t,x,y) + \left(\partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t)\right) p_\varepsilon(t,x,y) \approx 0, \\ A(0,y)p_\varepsilon\left(t,0,y\right) \approx \eta_\varepsilon(t,y)\int_{\mathbb{R}_+} b(x',y)p(t,x',y)dx'. \end{cases}$$

En considérant $\eta_\varepsilon(t,y)$ comme un paramètre, nous posons le problème aux valeurs propres suivant : pour $(y,\eta) \in \mathbb{R}^n \times (0,+\infty)$ fixé, soit $(\Lambda(y,\eta), Q(x,y,\eta))$ l'unique solution de

$$\begin{cases} \partial_x \left[A(x,y)Q\right] + d(x,y)Q - \Lambda(y,\eta)Q = 0, \quad \forall x > 0, \\ A(0,y)Q(0,y,\eta) = \eta \int_{\mathbb{R}_+} b(x,y)Q(x,y,\eta)dx, \\ Q(x,y,\eta) > 0, \quad \int_{\mathbb{R}_+} b(x,y)Q(x,y,\eta)dx = 1. \end{cases}$$

Définition d'un ansatz. Ces calculs formels suggèrent de définir notre ansatz u_ε comme solution de

$$\begin{cases} \partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t) = -\Lambda\left(y, \eta_\varepsilon(t,y)\right), \\ \eta_\varepsilon(t,y) = \int_{\mathbb{R}^n} M(z) e^{\frac{u_\varepsilon(t,y+\varepsilon z) - u_\varepsilon(t,y)}{\varepsilon}} dz, \\ u_\varepsilon(t=0,y) = u_\varepsilon^0(y), \end{cases} \tag{C.11}$$

pour certaines conditions initiales u_ε^0. Alors, en posant $Q_\varepsilon(t,x,y) := Q(x,y,\eta_\varepsilon(t,y))$ et $\Lambda_\varepsilon(t,y) := \Lambda(y,\eta_\varepsilon(t,y))$, nous avons

$$\begin{cases} \varepsilon\partial_t Q_\varepsilon + \partial_x \left[A(x,y)Q_\varepsilon\right] + d(x,y)Q_\varepsilon + \left(\partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t)\right) Q_\varepsilon = \varepsilon\partial_t Q_\varepsilon, \\ A(0,y)Q_\varepsilon\left(t,0,y\right) = \frac{1}{\varepsilon^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}_+} M(\frac{y'-y}{\varepsilon}) e^{\frac{u_\varepsilon(t,y') - u_\varepsilon(t,y)}{\varepsilon}} b(x',y')Q_\varepsilon(t,x',y')dx'dy'. \end{cases}$$

Notons que le terme de bord à $x = 0$ est obtenu grâce à la normalisation

$$\int_{\mathbb{R}_+} b(x,y)Q(x,y,\eta)dx = 1.$$

Excepté le membre de droite $\varepsilon\partial_t Q_\varepsilon$, nous voyons que Q_ε et p_ε sont solutions de la même equation (C.9), qui est linéaire et admet un principe de comparaison. Quitte à prouver de bonnes estimées sur $\varepsilon\partial_t Q_\varepsilon$, on peut borner p_ε supérieurement et inférieurement par des multiples de Q_ε (qui est uniformément $L^1 \cap L^\infty$). Cela justifie notre approche, en particulier l'approximation formelle (C.10).

Analyse de l'équation de Hamilton-Jacobi D'après ce qui précède, la clef de voûte de notre méthode réside dans le fait de prouver de bonnes estimations sur $\partial_t Q_\varepsilon$. Cela passe par une analyse poussée de notre ansatz u_ε, définie par l'équation (C.11). Notons que cette équation n'est pas autonome, à cause du terme $\rho_\varepsilon(t)$. Nous contournons cette difficulté en posant

$$U_\varepsilon(t,y) = u_\varepsilon(t,y) + \int_0^t \rho_\varepsilon(s)ds.$$

Ainsi,

$$\eta_\varepsilon(t,y) = \int_{\mathbb{R}^n} M(z) e^{\frac{U_\varepsilon(t,y+\varepsilon z) - U_\varepsilon(t,y)}{\varepsilon}} dz$$

et

$$\begin{cases} \partial_t U_\varepsilon(t,y) = -\Lambda\left(y, \int_{\mathbb{R}^n} M(z) e^{\frac{U_\varepsilon(t,y+\varepsilon z) - U_\varepsilon(t,y)}{\varepsilon}} dz\right) & \forall t \geq 0, \ \forall y \in \mathbb{R}^n, \\ U_\varepsilon(0,y) = u_\varepsilon^0(y) & \forall y \in \mathbb{R}^n, \end{cases}$$

Notons que U est alors défini de manière autonome.

À partir de cette équation, nous prouvons, *a priori*, que $\partial_t U_\varepsilon$ et $\nabla_y U_\varepsilon$ sont bornées, uniformément en $\varepsilon > 0$. Puis nous utilisons une sorte de convexité du Hamiltonien pour prouver que U_ε est semi concave, de quoi nous déduisons une borne $W^{2,1}_{loc}$ uniforme en $\varepsilon > 0$.

Quand $\varepsilon \to 0$, U_ε converge localement uniformément vers U, solution de viscosité de l'équation de Hamilton-Jacobi

$$\begin{cases} \partial_t U(t,y) = H(y, \nabla_y U) & \forall t \geq 0, \ \forall y \in \mathbb{R}^n, \\ U(0,y) = u^0(y) & \forall y \in \mathbb{R}^n. \end{cases} \tag{C.12}$$

avec un Hamiltonien

$$H(y,p) := -\Lambda\left(y, \int_{\mathbb{R}^n} M(z) e^{p\cdot z} dz\right).$$

On prouve, *a posteriori*, que $\sup u(t,\cdot) = 0$, et donc $\int_0^t \rho = \sup U(t,\cdot)$. Ainsi, ρ peut être vu comme une multiplicateur de Lagrange associé à la contrainte $\sup u(t,\cdot) = 0$.

Principaux résultats

Le théorème suivant rassemble nos principaux résultats.

Theorem C.2 *Sous des hypothèses générales sur les paramètres, et si la condition initiale est bien préparée, nous avons, quand $\varepsilon \to 0$:*

1. *La population m_ε s'éteint localement uniformément en dehors de l'ensemble*
$$\mathcal{S} := \left\{ t \geq 0, \ y \in \mathbb{R}^n : U(t,y) = \sup_{y' \in \mathbb{R}^n} U(t,y') \right\},$$
où $U \in W^{1,1}_{loc}$ est globalement Lipschitz, semi-convexe, et est une solution de viscosité de l'équation (C.12)*.*
2. *ρ_ε converge L^∞-faible-* vers $\rho \in L^\infty$ et*
$$\forall t > 0, \quad \int_0^t \rho = \sup_{y \in \mathbb{R}^n} U(t,y).$$
3. *En temps courts $t \in [0,T]$, nous avons $\mathcal{S} = \{(t, \bar{y}(t))\}$, où $\bar{y}(t)$ satisfait l'Équation Canonique :*
$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\bar{y}(t) = \left(\nabla_y^2 U(t,\bar{y}(t))\right)^{-1} \cdot \nabla_y \Lambda(\bar{y}(t),1) + \partial_\eta \Lambda(\bar{y}(t),1) \displaystyle\int_{\mathbb{R}^n} M(z) z dz, \\ \bar{y}(0) = \bar{y}^0. \end{cases}$$

La principale restriction sur les paramètres a est que nous supposons soit que $\frac{1}{A(\cdot,y)} \in L^1$, soit que $b(\cdot,y)$ a un support compact. Les autres hypothèses sont formulées directement sur le problème aux valeurs propres, et sont assez générales. Notons que, contrairement au cas sans mutation, nous ne savons pas montrer la convergence forte de p_ε.

C.2.3 Transmission Génétique Horizontale

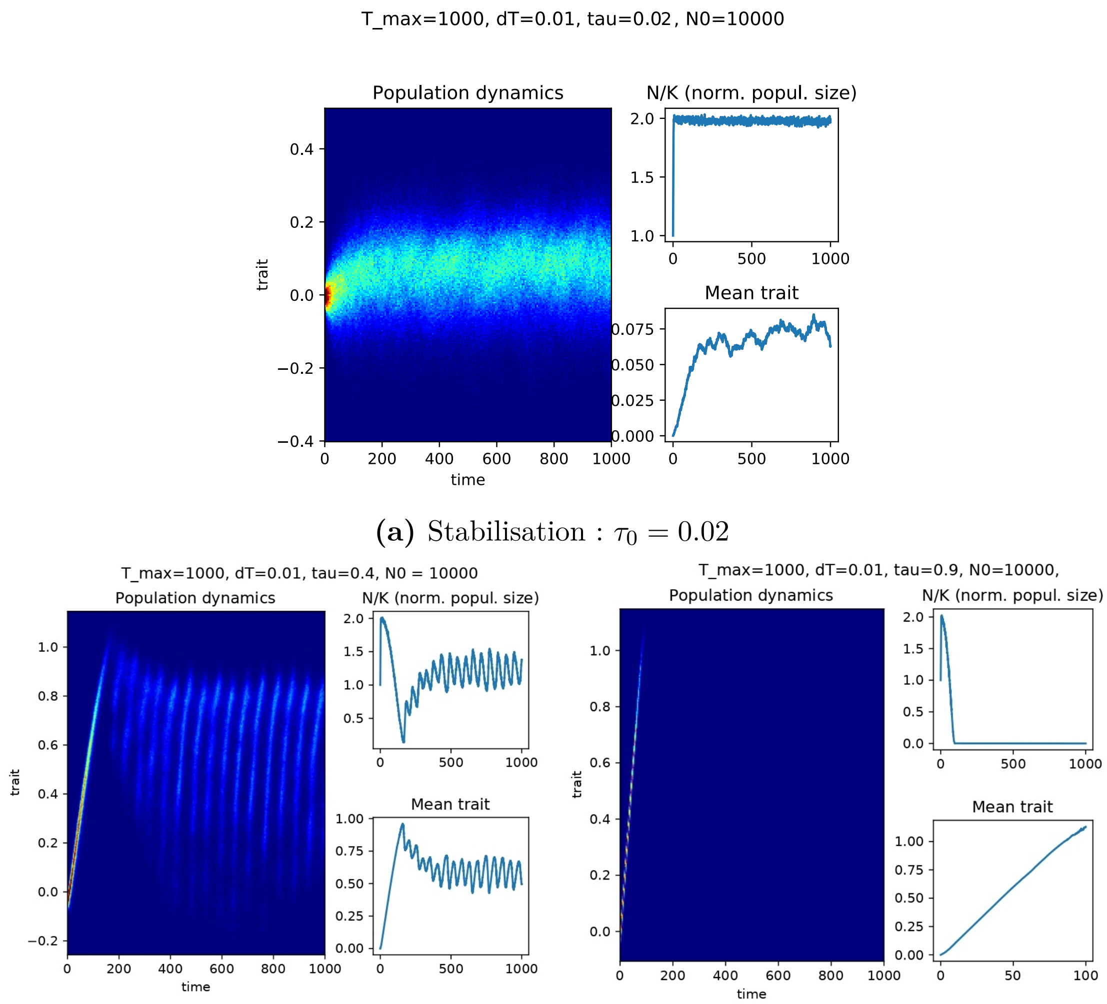
En collaboration avec V. Calvez, S. Figueroa Iglesias, H. Hivert, S. Méléard, et A. Melnykova. Nous présentons ici le contenu du Chapitre 6.

Un modèle pour le sauvetage évolutif

La *Transmission Génétique Horizontale* (TGH) est le transfert de matériel génétique entre deux organismes vivants, en opposition à la transmission verticale usuelle des parents à la progéniture. Ce phénomène joue un rôle important dans l'évolution de certaines bactéries, notamment concernant la résistance aux antibiotiques. Un phénomène de transmission horizontale intervient aussi dans le transfert de plasmides et de symbiotes [208, 215].

Un modèle stochastique est proposé dans [56, 156]), pour rendre compte de la dynamique d'une population sujette à la TGH. Les simulations numériques mettent en évidence une dynamique cyclique de la population. Intuitivement, tandis que la TGH conduit la population vers un phénotype délétère, il peut arriver que les rares individus non affecté par la TGH finisse par repeupler l'environnement, avant d'être à nouveau conduit vers un phénotype délétère.

Plus précisément, en fonction de l'intensité de la TGH, la dynamique peut suivre trois régimes : stabilisation, cycles, et extinction. Ces trois régimes sont illustrés Figure C.6.



(a) Stabilisation : $\tau_0 = 0.02$

(b) Cycles : $\tau_0 = 0.4$

(c) Extinction : $\tau_0 = 0.9$

FIGURE C.6 – Comportement de la population en fonction de l'intensité τ_0 de la TGH. L'axe horizontal correspond au temps, l'axe verticale au phénotype $y \in \mathbb{R}$.

La dynamique cyclique observée illustre un phénomène appelé le *sauvetage évolutif*. La compréhension de ce phénomène est un enjeu de recherche majeur, et s'avère difficile du point de vue mathématique [165, 239]. Comme nous l'avons vu dans la section précédente, l'approche Hamilton-Jacobi est efficace pour décrire les phénomènes de concentration, qui ont lieu dans des populations de grande taille. Or, le *sauvetage évolutif* repose sur des populations de petite taille, et il intéressant de comprendre en quelle mesure l'approche Hamilton-Jacobi est capable de capturer ce phénomène. Il s'agit de comprendre si les "passages à la limite", opérés pour passer du modèle stochastique à l'équation de Hamilton-Jacobi, ne nous font pas perdre trop d'information sur les populations de petite taille.

Nous proposons d'étudier cette question à travers l'exemple de la `TGH`. Nous présentons des simulations numériques et une analyse théorique formelle pour comparer le modèle stochastique avec le modèle Hamilton-Jacobi correspondant.

L'équation de Hamilton-Jacobi

À partir du modèle stochastique, nous aboutissons à l'équation de Hamilton-Jacobi suivante :

$$\partial_t u(t,y) = -(d(y)+\rho(t)) + b(y)\int_{\mathbb{R}} M(z)\exp\left(\nabla_y u(t,y)\cdot z\right)dz + \tau(y-\overline{y}(t)), \quad \text{(C.13)}$$

où

$$\bar{y}(t) = \text{argmax } u(t,\cdot).$$

Nous utilisons le même formalisme que dans l'équation (C.5). Nous supposons $n=1$ (donc y est une variable scalaire). Le terme $\tau(y-\overline{y}(t))$ tient compte de la `TGH`, et rend l'étude de l'équation non standard.

Pour fixer les idées, nous choisissons $d(y)=y^2$, $b(y)=1$, de manière à ce que la fitness $r(y)=1-y^2$ soit concave et optimale en 0. Le noyau de mutation est supposé être une Gaussienne de variance 1 centrée en 0. Le terme de `TGH` est supposé de la forme

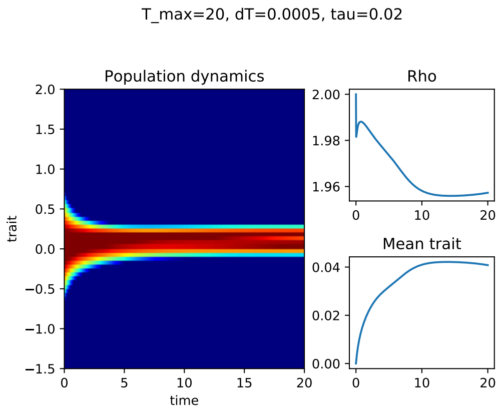
$$\tau(y) = \tau_0 \tanh\left(\frac{y}{\delta}\right),$$

où le paramètre $\tau_0 > 0$ correspond à l'intensité de la `TGH` (nous le considérons comme un paramètre de bifurcation), et δ règle la raideur de la transitions entre les valeurs $\pm\tau_0$. Ainsi, la `TGH` a pour effet de "pousser" les individus vers les $y>0$, et donc de mener la population vers des phénotypes délétères.

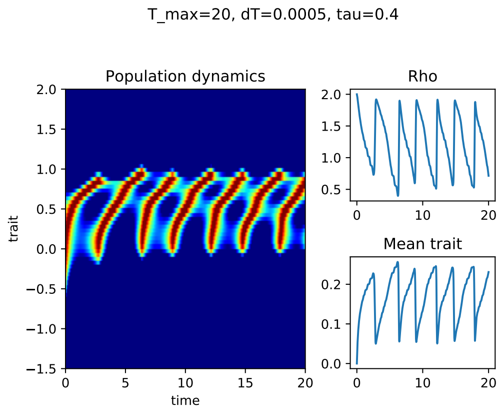
Résultats

Simulations numériques Simuler numériquement des équations comme (C.13) peut être délicat, notamment à cause du terme $\exp(\nabla_y u_\varepsilon \cdot z)$ et du terme de `TGH` qui est non-local. Nous proposons pour cela un schéma numérique *Asymptotic Preserving* [187, 195, 196].

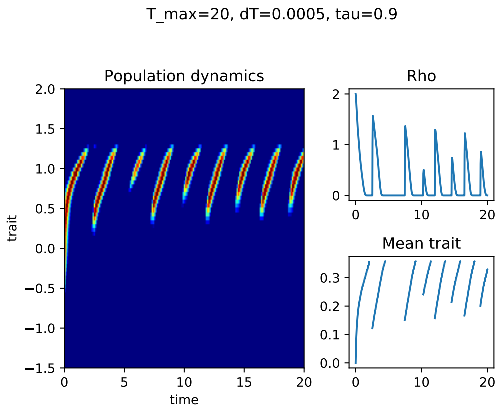
Les expériences que nous menons (Figure C.7) montrent que l'équation de Hamilton-Jacobi (C.13) est capable de reproduire qualitativement la dynamique observée dans le modèle stochastique.



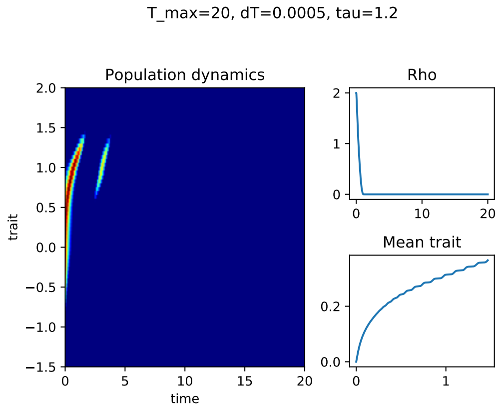
(a) Stabilisation : $\tau_0 = 0.02$



(b) Cycles : $\tau_0 = 0.4$



(c) Cycles avec épisodes d'extinction : $\tau_0 = 0.9$



(d) Extinction : $\tau_0 = 1.2$

FIGURE C.7 – Comportement de $u(t, y)$ en fonction de τ_0.

Analyse théorique formelle. Une analyse formelle de l'équation de Hamilton Jacobi permet une description précise de la dynamique. Il est notamment possible de prédire le régime de la dynamique (stabilisation, cycles, extinction) en fonction de la valeur de τ_0. Par exemple, la dynamique présente des cycles si et seulement si $\tau_0 > \tau_{cyc} := 4\delta^2$; passe par des épisodes d'extinction si et seulement si $\tau > \tau_{ext} := 2\delta$.

Ces valeurs seuils sont confirmées par les simulations numériques du modèle stochastique. L'approche Hamilton-Jacobi semble donc capable de capturer le phénomène de *sauvetage évolutif* de manière assez précise.

Lignées évolutives. Une *lignée évolutive* correspond à l'histoire phénotypique des ancêtres d'un individu. Nous présentons des expériences numériques en Figure C.8. D'une part, nous pouvons voir que toutes les lignées se concentrent" en une seule autour de $t \approx 400$. Cela signifie que tous les individus au temps final descendent du même ancêtre commun. Ce phénomène est appelé la *coalescence* [13, 14, 193]. D'autre part, nous voyons que les lignées restent proches du phénotype optimal $y = 0$ tout au long de la dynamique. Cela illustre bien que la population parvient à se maintenir grâce aux rares individus non affecté par la `TGH` : c'est le principe du *sauvetage évolutif.*

Nous pensons que les lignées correspondent, en moyenne, aux *caractéristiques* de l'équation de Hamilton-Jacobi (C.13).

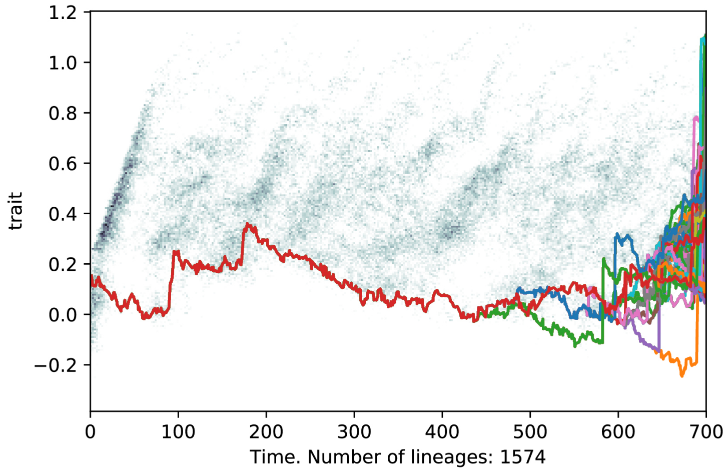


FIGURE C.8 – Lignées évolutives sur le modèle stochastique.

Chapitre D

Introduction à la partie III : Systèmes de réaction-diffusion et modélisation de l'agitation sociale

En collaboration avec H. Berestycki et L. Rossi.

D.1 Introduction

Présentation du problème

Dans cette partie, nous considérons le système de deux équations de réaction-diffusion suivant, pour $t \geq 0$, $x \in \mathbb{R}^n$,

$$\begin{cases} \partial_t u(t,x) = d_1 \Delta_x u(t,x) + \Phi(u(t,x), v(t,x)), \\ \partial_t v(t,x) = d_2 \Delta_x v(t,x) + \Psi(u(t,x), v(t,x)), \end{cases} \tag{D.1}$$

avec des conditions initiales $v(0,x) = v_0 > 0$ et $u(0,x) = u_0(x) \gneqq 0$, $d_1 > 0, d_2 \geq 0$. Notre objectif est d'étudier le comportement asymptotique de (u,v) et l'existence de fronts de transition.

Notre hypothèse principale est :

$$\Psi(0,v) = 0. \tag{D.2}$$

Cela signifie que, si $u = 0$, la dynamique de v consiste en une simple diffusion. Mis à part quelques hypothèses techniques supplémentaires, nous gardons une approche assez générale. En particulier, nous ne faisons pas d'hypothèse de monotonie.

Notons que l'hypothèse (D.2) implique que $(0,v)$ est un état d'équilibre, pour toute constante $v > 0$. Nous nous intéressons principalement aux solutions qui sont issues d'une petite perturbation d'un tel état d'équilibre. Ainsi, nous considérons une donnée initiale $u_0(\cdot) \gneqq 0$ à support compact ayant une petite norme L^∞. Par simplicité, nous supposons toujours que v_0 est constant.

La philosophie de notre est approche est de considérer v_0 comme un paramètre de bifurcation pour décrire la dynamique de (D.1). Comme nous allons le voir, de nombreuses propriétés qualitatives peuvent être déduites du signe de

$$K_0 := \partial_u \Phi(0, v_0).$$

Notons que cette quantité ne dépend pas de l'équation vérifiée par v (*i.e.* d_2 et Ψ).

Le célèbre modèle épidémiologique SI (pour *Susceptibles* et *Infectés*) est contenu dans nos hypothèses, avec l'identification $S(t,x) \equiv v(t,x)$ et $I(t,x) \equiv u(t,x)$. Dans sa version la plus simple (avec espace), le modèle SI s'écrit

$$\begin{cases} \partial_t I = \Delta I + \beta SI - \gamma I, \\ \partial_t S = -\beta SI, \end{cases} \tag{D.3}$$

avec $\beta, \gamma > 0$. Le modèle SI correspond donc au cas $\Phi(u,v) = \beta uv - \gamma u$, $\Psi(u,v) = -uv$. Nous remarquons que, dans ce cas, $v(t,x) \leq v_0$ (car $\Psi \leq 0$), et

$$\Phi(u(t,x), v(t,x)) \leq \partial_u \Phi(0, v_0) u(t,x).$$

Cette observation suggère d'accorder une attention particulière au cas, dit *inhibiteur*, où

$$\Phi(u,v) \leq \partial_u \Phi(0,v_0) u, \quad \forall u, v \geq 0. \tag{D.4}$$

Le terme *inhibiteur* vient de l'idée que, dès que le système quitte l'état d'équilibre $(0, v_0)$, le terme de réaction $\Phi(u,v)$ décroît, et la dynamique de u est *inhibée*. Notons en particulier que cette hypothèse implique que $\Phi(\cdot, v)$ est *KPP* (au sens faible). La condition (D.4) peut sembler très restrictive à première vue. Cependant, si par exemple $\Psi \leq 0$, on sait que $v(t,x) \leq v_0$. Si, de plus, $v \mapsto \Phi(u,v)$ est croissante, et $u \mapsto \frac{\Phi(u,v)}{u}$ décroissante (*i.e.* $\Phi(\cdot, v)$ est *KPP*), alors on peut restreindre $\Phi(u, \cdot)$ à $[0, v_0]$, et (D.4) est vérifiée.

Nous verrons que, sous les hypothèses (D.2)-(D.4), certains aspects de la dynamique de (D.1) sont gouvernés par le linéarisé de la première équation autour de $u = 0$, $v = v_0$:

$$\partial_t u = d_1 \Delta_x u + \partial_u \Phi(0, v_0) u. \tag{D.5}$$

Formellement, nous verrons que u se comporte comme la solution d'une équation *KPP* scalaire. En particulier, le signe de K_0 joue un rôle déterminent. D'une part, si $K_0 < 0$, l'état d'équilibre $(0, v_0)$ est stable, et donc absorbe la dynamique. D'autre part, si $K_0 > 0$, alors $(0, v_0)$ est répulsif et un phénomène de propagation a lieu, à la vitesse $c_0 := 2\sqrt{d_1 \partial_u \Phi(0, v_0)}$.

En général, sans l'hypothèse (D.4), nous verrons que u se comporte plutôt comme la solution d'une équation *monostable*.

Le modèle épidémiologique SI

Le modèle SI a été introduit par Kermack et McKendrick [191] et est absolument incontournable en modélisation des épidémies, tant du point de vue de la théorie [176, 177, 179, 226, 259, 264, 283] que des applications [11, 25, 242]. Une vaste littérature lui est consacrée, notamment en ce qui concerne l'étude de la propagation spatiale. Les travaux pionniers sont dus à Kendall [190], Mollison [238], Thieme [275], Aronson [16] pour la vitesse asymptotique de propagation, Atkinson, Reuter [22], Diekmann [120–123], Brown, Carr [68] pour les ondes de transition, et Radcliffe, Rass [255–257] pour l'étude du système n-dimensionnel. Les ondes progressives ont également été étudiées par Hosono, Ilyas [180], Zhao, Wang [298], et beaucoup d'autres [6,134,189,288–290,293,294,299,300]. La théorie a été étendue au cas d'équations différées en temps, d'abord par Thieme, Zhao [276], puis par d'autres auteurs [204, 284, 285]. Enfin, la propagation en milieux hétérogène périodique est étudiée, entre autres, par Ducrot, Giletti [133] et Ducasse [132].

Cependant, la plupart des approches mathématiques disponibles pour étudier le système (D.3) repose sur sa forme explicite, et peuvent difficilement être généralisées à une classe plus large de systèmes. En effet, la méthode consiste souvent à procéder à un changement de variable particulier qui réduit le système à une seule équation scalaire. Plus précisément, en intégrant la deuxième ligne de (D.3), on obtient

$$S(t,x) = S_0 e^{-\beta R(t,x)}, \quad \text{avec} \quad R(t,x) = \int_0^t I(\cdot, x).$$

Puis, en injectant cette formule dans la première ligne, et en intégrant sur $(0,t)$, on trouve

$$\partial_t R = \Delta_x R + f(R) + I_0, \quad \text{où} \quad f(z) := S_0\left(1 - e^{-\beta z}\right) - \gamma z.$$

Sous l'hypothèse $S_0 > \frac{\gamma}{\beta}$, f satisfait la condition *KPP* (faible), c'est-à-dire

$$\exists A > 0 : f(0) = f(A) = 0, \quad f > 0 \text{ dans } (0,A), \quad \forall z \in (0,A), f(z) \leq f'(0)z.$$

Ainsi, dans $\mathbb{R}^n \backslash \text{supp } I_0$, le système (D.3) est ramené à une seule équation *KPP*, pour laquelle de nombreux résultats classiques sont disponibles. Formellement, la dynamique de R est caractérisée par la linéarisation de cette équation en $R = 0$ (qui s'avère coïncider avec l'équation (D.5), avec nos notations).

Cette approche présente à la fois l'avantage et l'inconvénient de reposer sur des calculs explicites : la méthode est simple mais difficilement transposable. Le contenu de cette section peut être vus comme une nouvelle approche sur le modèle *SI*, qui étend certains résultats connus à une classe générale de systèmes. Nous mentionnons également que nos preuves sont parfois plus simples que celles disponibles dans la littérature sur le modèle *SI*.

D.2 Étude théorique

Nous présentons ici le contenu du Chapitre 7.

Hypothèses et notations

Hypothèses. Dans cette section, nous supposons $d_1 > 0$, $d_2 \geq 0$, et

$$\begin{cases} \Psi(0,v) = 0 & \text{(Hypothèse principale)} \\ \exists M > 0,\ \forall v \geq M,\ \Psi(\cdot, v) \leq 0 & \text{(Saturation en } v\text{)} \\ \Phi(0,\cdot) = 0 \quad \text{and} \quad \Psi(\cdot, 0) = 0, & \text{(Homogénéité)} \\ \|\Phi\|_{C^2}, \|\Psi\|_{C^2} < +\infty & \text{(Régularité)} \end{cases}$$

Les non-linéarités que nous considérons sont typiquement de la forme

$$\Phi(u,v) = vu(1-u) - \gamma u,$$
$$\Psi(u,v) = uvf(u,v),$$

où $\gamma > 0$ et f est régulière et bornée.

En ce qui concerne les conditions initiales, nous supposons :

$$v_0 \in (0, M),\ u_0(\cdot) \gneqq 0,\ u_0(\cdot) \in L^\infty(\mathbb{R}^n), \text{ et } u_0 \text{ est à support compact.}$$

Notations. Posons

$$K_0 := \partial_u \Phi(0, v_0), \quad \overline{K} := \sup_{\substack{u \geq 0 \\ v \in (0,M)}} \frac{\Phi(u,v)}{u},$$

et, si $K_0 > 0$,

$$c_0 := 2\sqrt{d_1 K_0}, \quad \overline{c} := 2\sqrt{d_1 \overline{K}}.$$

Notons que le cas *inhibiteur* (D.4) correspond à $K_0 = \overline{K}$ et $c_0 = \overline{c}$.

Le cas *inhibiteur*

Considérons tout d'abord le cas *inhibiteur*, c'est-à-dire faisons l'hypothèse (D.4). Nous rappelons que ce cas comprend le modèle SI.

Stabilité. Le premier résultat établit que $(0, v_0)$ est attracteur quand $K_0 < 0$, et est répulsif quand $K_0 > 0$.

Theorem D.1

— *Si $K_0 < 0$ et $d_2 > 0$,*

$$\lim_{t \to +\infty} \left(\sup_{x \in \mathbb{R}^n} |(u(t,x), v(t,x)) - (0, v_0)| \right) = 0.$$

— *Si $K_0 > 0$, il existe $\delta_0 > 0$ indépendant de $u_0(\cdot) \gneqq 0$ tel que*

$$\limsup_{t \to +\infty} \left(\sup_{x \in \mathbb{R}^n} |(u(t,x), v(t,x)) - (0, v_0)| \right) \geq \delta_0.$$

Ce théorème exprime le fait qu'il existe un phénomène de seuil sur v_0, à travers le signe de K_0. Notons qu'en particulier, le cas $K_0 > 0$ satisfait le *Hair-trigger effect.* Notons également que, dans le modèle SI, la condition $K_0 > 0$ correspond à $S_0 > \frac{\gamma}{\beta}$.

Vitesse de propagation. Le théorème précédent ne dit rien sur comment la solution se propage dans l'espace quand $K_0 > 0$. Nous donnons le résultat suivant.

Theorem D.2

$$\forall c < c_0, \quad \limsup_{t \to +\infty} \left(\sup_{|x| \geq ct} |(u(t,x), v(t,x)) - (0, v_0)| \right) > 0,$$

$$\forall c > c_0, \quad \lim_{t \to +\infty} \left(\sup_{|x| \geq ct} |(u(t,x), v(t,x)) - (0, v_0)| \right) = 0.$$

Ce résultat exprime, en quelque sorte, qu'un phénomène de propagation à lieu à la vitesse c_0. Cependant, cette formulation peut induire en erreur, car le théorème précédent implique seulement que la $\limsup_{t \to +\infty}$ de la vitesse de propagation est égale à c_0.

Ondes de transition. Nous nous intéressons maintenant à l'étude des ondes de transition, c'est-à-dire aux solutions de (D.1) de la forme $u(t,x) = U(x \cdot e + ct)$, $v(t,x) = V(x \cdot e + ct)$, pour $c \geq 0$ (la vitesse de propagation), $e \in \mathbb{S}^n$ et avec une condition de bord en $-\infty$. Plus précisément, en posant $\xi := x \cdot e + ct \in \mathbb{R}$, nous considérons le système,

$$\begin{cases} cU'(\xi) = d_1 U''(\xi) + \Phi(U(\xi), V(\xi)), \\ cV'(\xi) = d_2 V''(\xi) + \Psi(U(\xi), V(\xi)), \\ U,\ V \text{régulières, bornées, strictement positives} \end{cases} \tag{D.6}$$

avec la condition

$$\begin{cases} U(-\infty) = 0, \\ V(-\infty) = v_0. \end{cases} \tag{D.7}$$

Nous donnons un résultat d'existence et de non-existence pour les ondes de transition.

Theorem D.3

— *Si $K_0 < 0$, il n'existe pas d'onde de transition.*

— *Si $K_0 > 0$, il n'existe pas d'onde de transition pour les vitesses $c < c_0$, et il existe une onde transition pour toute vitesse $c > c_0$.*

Dans le cas particulier du modèle SI, la preuve pour la non-existence d'ondes de transition repose classiquement sur une méthode Tauberienne avec une transformée de Laplace [126, 293, 296, 297]. Ici, nous proposons une preuve plus simple. Nous insistons sur le fait que nous n'avons pas imposé de condition en $+\infty$. Ainsi, une onde de transition peut avoir des formes diverses (nous verrons des exemples par la suite).

Le cas général

Dans cette section, nous ne faisons plus l'hypothèse (D.4) et examinons le cas général.

Stabilité. Comme dans la section précédente, la stabilité de $(0, v_0)$ est déterminée par le signe de $K_0 > 0$.

Theorem D.4

— *Si $K_0 < 0$, il existe $\varepsilon_0 > 0$ tel que pour tout $u_0(\cdot) \leq \varepsilon_0$ à support compact :*

$$si\ d_2 > 0, \quad \lim_{t \to +\infty} \left(\sup_{x \in \mathbb{R}^n} |(u(t,x), v(t,x)) - (0, v_0)| \right) = 0.$$

— *Si $K_0 > 0$, il existe $\delta_0 > 0$ tel que*

$$\limsup_{t \to +\infty} \left(\sup_{x \in \mathbb{R}^n} |(u(t,x), v(t,x)) - (0, v_0)| \right) \geq \delta_0.$$

Vitesse de propagation. Déterminons maintenant une borne supérieure et inférieure sur la vitesse asymptotique de propagation.

Theorem D.5 *Il existe* $c_\star \in [c_0, \overline{c}]$ *tel que*

$$\forall c < c_\star, \quad \limsup_{t\to+\infty} \left(\sup_{|x|\geq ct} |(u(t,x), v(t,x)) - (0, v_0)| \right) > 0,$$
$$\forall c > c_\star, \quad \lim_{t\to+\infty} \left(\sup_{|x|\geq ct} |(u(t,x), v(t,x)) - (0, v_0)| \right) = 0.$$

Ondes de transition. Examinons maintenant les ondes de transition, c'est-à-dire les solutions de (D.6)-(D.7). Nous donnons d'abord un résultat de non-existence.

Theorem D.6 *Si* $K_0 > 0$*, il n'existe pas d'onde de transition pour* $c < c_0$*.*

Pour notre résultat d'existence, nous faisons l'hypothèse d'une *saturation* sur u :

$$\exists M' > 0,\ \forall u \geq M', \quad \Phi(u, \cdot) \leq 0. \tag{D.8}$$

Considérons également

$$g(c) := \frac{\sqrt{c^2 - c_0^2} - \sqrt{c^2 - \overline{c}^2}}{2c}, \quad \forall c > \overline{c}.$$

Notons que $g(\cdot)$ est décroissante, et que $g(+\infty) = 0$. Cette fonction mesure, en quelque sorte, le défaut d'*inhibition*.

Le résultat suivant traite du cas où $\Phi(\cdot, v)$ est *KPP* (dans un sens faible).

Theorem D.7 *Supposons* (D.8) *et*

$$\Phi(u, v) \leq \partial_u \Phi(0, v) u.$$

Si $K_0 > 0$*,* $c > \overline{c}$ *et*

$$d_1 \geq d_2 g(c), \tag{D.9}$$

il existe une onde de transition de vitesse c*.*

Notons que l'ensemble des c pour lesquels (D.9) est vérifié est de la forme $[\tilde{c}, +\infty)$, avec $\tilde{c} = g^{-1}\left(\frac{d_1}{d_2}\right)$. Formellement, la quantité $\tilde{c}$ peut être interprétée comme la vitesse à laquelle se propage le défaut d'inhibition. En particulier, si

$$d_1 \geq d_2 \sqrt{1 - \frac{K_0}{\overline{K}}},$$

le résultat précédent implique l'existence d'ondes de transition pour tout $c > \overline{c}$.

Considérons désormais le cas où $\Phi(\cdot, v)$ n'est pas *KPP*.

Theorem D.8 *Supposons* (D.8) *et* $K_0 > 0$*. Si* $c > \overline{c}$ *vérifie* (D.9) *et*

$$h(c) := c + \sqrt{c^2 - \overline{c}} - 2\sqrt{c^2 - c_0^2} > 0, \tag{D.10}$$

il existe une onde de transition de vitesse c*.*

L'hypothèse (D.10) porte, en quelque sorte, sur le défaut d'inhibition dû au fait que $\Phi(\cdot, v)$ n'est pas *KPP*. Quand $c \to +\infty$, cette condition devient $\overline{K} < 2K_0$.

D.3 Modélisation de l'Agitation Sociale

Nous présentons ici le contenu du Chapitre 8.

D.3.1 Introduction

La motivation principale de l'étude du système (D.1) vient de la modélisation des émeutes et de l'agitation sociale. Pour une littérature sur ce sujet, voir [23, 58, 59, 113, 137, 167, 168, 203, 295] et les références attenantes. Notre approche est dans l'esprit des travaux [46, 48, 52], dans lesquels un système d'équations de réaction-diffusion est considéré pour rendre compte de la dynamique des émeutes.

Nous appelons *Agitation Sociale*, notée `AS`, la quantité d'activité émeutière et de désordre civil. Nous pouvons penser à l'`AS` comme la somme de toutes les actions illégales, pondérées par leur importance respective. Notre but n'est pas de discuter les origines de l'`AS`. Nous proposons plutôt un modèle, construit à partir d'hypothèses simples, pour rendre compte des patterns récurrents observés sur le terrain.

Notre modèle fait également intervenir un niveau de *Tension Sociale*, notée `TS`, par lequel nous entendons une quantité représentant le ressentiment d'une population envers la société, qu'il soit pour des raisons politiques, économiques, ou sociales.

Hypothèses de modélisation

Notre approche repose sur l'hypothèse que l'`AS` et la `TS` obéissent à une dynamique couplée. Une hypothèse centrale est de *négliger la dynamique intrinsèque de la* `TS`, en l'absence d'`AS`. Cela nous permettra de nous concentrer plus précisément sur l'interaction entre l'`AS` et la `TS`. Commençons par lister les caractéristiques récurrentes observées dans la dynamique de l'`AS` (qui seront nos hypothèses de modélisation) et par définir notre vocabulaire.

Premièrement, les mouvements d'`AS` ont souvent lieu sous la forme d'*irruptions* épisodiques, communément appelées *émeutes*, *révolutions*, *etc.* Cependant, il est souvent admis que ces *irruptions* d'`AS` sont déclenchées par un unique *événement exogène* (ou *évènement déclencheur*). On peut penser à cet *événement exogène* comme la goutte d'eau qui fait déborder le vase.

Le fait qu'un seul *évènement exogène* puisse déclencher, ou non, une *irruption* d'`AS` dépend du niveau de `TS`. La `TS` joue, en quelque sorte, un rôle d'*activateur* : si la `TS` est suffisamment grande, un petit *évènement exogène* déclenche une *irruption* d'`AS` ; en revanche, si la `TS` est trop faible, ce même évènement sera suivit par un prompt *retour au calme*.

Ces observations suggèrent, du point de vue de la modélisation, qu'un mécanisme de *relaxation* a lieu sur l'`AS` dans une contexte de basse `TS`. Cette relaxation rend compte de facteurs variés, comme la fatigue, la répression policière, l'incarcération, *etc.*

Au contraire, une forte `TS` active une *croissance endogène* de l'`AS`. En d'autres termes, si la `TS` est au dessus d'une valeur seuil, alors un mécanisme d'auto-renforcement a lieu sur l'`AS`. Ce phénomène est analogue à la propagation d'une flamme : une *croissance endogène* est activée quand la température est suffisamment élevée. On peut interpréter ce facteur *endogène* comme la dimension grégaire des mouvements sociaux : plus le mouvement est large, et plus un individu est susceptible d'y prendre part.

Naturellement, ce mécanisme d'auto-renforcement doit être contrebalancé par un effet de *saturation*, qui rend compte du nombre limité d'individus, de ressources, de biens à endommager *etc.*

Une autre caractéristique importante et communément observée dans les *irruptions* d'`AS` est l'expansion géographique. Ce phénomène peut être vu, en quelque sorte, comme résultant du déplacement des émeutiers. Un exemple frappant est le cas des émeutes françaises de 2005 (voir [58, 267]), qui ont été déclenchées par la mort de deux jeunes hommes tentant d'échapper à la police à Clichy-sous-Bois, une banlieue pauvre de la région parisienne. Cet évènement a eu lieu dans un contexte de haute tension sociale, et a été l'étincelle d'un mouvement émeutier qui s'est répandu dans tout le pays et a duré plus de trois semaines.

Bien que les irruptions d'`AS` puissent prendre des formes très diverses, une première classification naïve serait de distinguer les *émeutes*, qui durent quelques semaines puis s'éteignent, et les *révolutions*, qui durent plus longtemps et peuvent résulter en des changements politiques et sociaux majeurs (on pourra penser à la Révolution Française ou au Printemps Arabe [201, 214]. Voir également [15]).

Une *émeute* peut être interprétée comme une *irruption* d'`AS` qui *dissipe* la `TS`. Une fois que la `TS` retombe en dessous d'une valeur seuil, l'`AS` décroit et finalement s'éteint. Ce cas est appelé *inhibiteur de tension*. Il correspond qualitativement à l'irruption d'une épidémie, qui se propage jusqu'à ce que le nombre d'individus susceptibles tombe en dessous d'une valeur seuil. Ce comportement est bien capturé par le modèle SI, avec $S =$`TS` et $I =$`AS`.

Une *révolution* peut être interprétée comme une *irruption* d'`AS` qui *accroît* la `TS`. Ce phénomène de rétroaction positive provoque une escalade vers un état durable de haute `AS`. Ce cas est appelé *excitateur de tension*. Du point de vue de la modélisation, il correspond qualitativement à un système coopératif.

À travers l'exemple de l'*émeute* et de la *révolution*, nous voyons que la dynamique de l'`AS` suggère différentes classes de modèles : d'une part, les modèles épidémiologiques, d'autre part, les systèmes monotones. Dans la littérature, ces deux classes de modèles sont habituellement considérées séparément. Notre objectif est de proposer un cadre unique qui couvre la possibilité d'une *émeute* et d'une *révolution*.

Construction du modèle

Dans le même esprit que [46, 48, 52] et d'après la section précédente, nous proposons un modèle mathématique pour rendre compte de la dynamique de l'`AS` et de la `TS`. L'`AS` sera représentée par $u(t,x)$, fonction du temps $t \geq 0$ et de la localisation $x \in \mathbb{R}^n$, et la `TS` sera représentée par $v(t,x)$. Notre modèle prend la forme générale d'un système d'équations de réaction-diffusion

$$\begin{cases} \partial_t u(t,x) = d_1 \Delta_x u(t,x) + \Phi(u(t,x), v(t,x)), \\ \partial_t v(t,x) = d_2 \Delta_x v(t,x) + \Psi(u(t,x), v(t,x)), \\ u(0,x) := u_0(x), \quad v(0,x) := v_0(x). \end{cases}$$

avec $d_1 > 0$, $d_2 \geq 0$ et Φ, Ψ que nous allons spécifier.

Premièrement, nous faisons une hypothèse d'homogénéité, c'est-à-dire $\Phi(0,v) = 0$. En d'autres termes, le niveau de base d'`AS` en l'absence d'évènements inhabituels est normalisé à 0. Nous modélisons l'*évènement exogène* comme une petite perturbation de l'état $u = 0$. Cette perturbation est incluse dans la condition initiale $u_0(x) \gneqq 0$, que l'on choisit à support compact et de petite norme L^∞. Les termes de diffusion $d_1\Delta_x u(t,x)$ et $d_2\Delta_x v(t,x)$ décrivent l'influence qu'un lieu a sur ses voisins géographiques.

Le terme Φ est choisi de la forme

$$\Phi(u,v) := r(v)G(u) - \omega u.$$

Le facteur *endogène* (le mécanisme d'auto-renforcement) est modélisé par la fonction $G(\cdot)$, que l'on choisit de type *KPP*, par exemple $G(z) = z(1-z)$. Le paramètre $\omega > 0$ est le taux naturel de *relaxation* de l'`AS` en l'absence de facteur *endogène*. Le facteur endogène est modulé par $r(v(t,x))$, qui modélise le rôle d'*activateur* joué par la `TS`. Nous choisissons $r(\cdot)$ postif et croissant. Ce terme peut-être vu comme un "bouton on/off". Par exemple, $r(\cdot)$ peut être linéaire $r(v) = v$, ou de forme sigmoïdale :

$$r(v) := \frac{1}{1 + e^{(v-\alpha)\beta}}.$$

Ici, $\alpha \geq 0$ est une valeur seuil et β règle la raideur de la transitions entre l'état relaxé et l'état excité. Le cas $\beta = +\infty$ correspond à $r(v) := \mathbb{1}_{v>\alpha}$.

En notant $v_\star$ la valeur seuil de v au dessus de laquelle une *irruption* de u survient après un *évènement exogène*, nous avons

$$v_\star := r^{-1}\left(\frac{\omega}{G'(0)}\right).$$

Le terme Ψ est particulièrement important puisqu'il modélise la rétroaction que u opère sur v. Comme nous avons fait l'hypothèse de *négliger la dynamique intrinsèque de la* `TS` en l'absence d'`AS`, nous supposons

$$\Psi(0,v) = 0.$$

Cela peut-être considéré comme notre hypothèse principale. Dans le même esprit, nous choisissons v_0 constant, de sorte à ce que $(u = 0, v = v_0)$ soit un état d'équilibre. De plus, nous faisons une hypothèse d'homogénéité, c'est-à-dire $\Phi(u,0) = 0$, ainsi qu'une hypothèse de saturation sur v, c'est-à-dire $\forall v \geq 1,\ \Psi(\cdot,v) \leq 0$. Naturellement, nous imposons $v_\star \in (0,1)$ pour couvrir la possibilité d'une *irruption* d'`AS` ou bien d'un *retour au calme*. Nous supposons donc

$$\frac{\omega}{G'(0)} \in (r(0), r(1)).$$

D'après ce qui précède, Ψ peut-être choisi de la forme

$$\Psi(u,v) := uvf(u,v)$$

pour une fonction f à déterminer. En particulier, nous considérons les deux exemples suivants. Chaque cas illustre un comportement qualitatif différent.

1. *Cas inhibiteur de tension : $f \leq 0$.* Dans ce cas, une *irruption* de u va *dissiper* v. On s'attend à ce que ce cas décrive une *émeute* et soit qualitativement analogue au modèle SI. Par exemple, on peut prendre

$$f(u,v) := -1.$$

2. *Cas excitateur de tension : $f \geq 0$.* Dans ce cas, une *irruption* de u va *accroître* v. On s'attend à ce que ce cas décrive une *révolution* et soit qualitativement analogue à un système coopératif. Par exemple, on peut prendre

$$f(u,v) := (1-v).$$

D.3.2 Analyse du modèle

Définitions et hypothèses

D'après la section précédente, nous considérons $u(t,x)$ (représentant l'`AS`) et $v(t,x)$ (représentant la `TS`) solutions de

$$\begin{cases} \partial_t u = d_1 \Delta_x u + r(v)G(u) - \omega u, \\ \partial_t v = d_2 \Delta_x v + uvf(u,v), \\ u(0,x) := u_0(x), \quad v(0,x) := v_0, \end{cases} \tag{D.11}$$

sous les hypothèses :

— $d_1 > 0, d_2 \geq 0$; $\omega > 0$.

— $u_0(\cdot)$ est à support compact et $0 \lneqq u_0(\cdot) < 1$; v_0 est constant et $0 < v_0 < 1$.

— $G(\cdot)$ est de type KPP, *i.e.*

$$u \mapsto \frac{G(u)}{u} \text{ décroit} \quad \text{et} \quad G(0) = G(1) = 0.$$

Par exemple, $G(u) = u(1-u)$.

— $r(\cdot)$ est régulière, positive et croissante.

Nous nous intéressons également aux ondes de transition, c'est-à-dire aux vitesses $c > 0$ et aux profils (U,V) vérifiant

$$\begin{cases} cU' = d_1 U'' + r(V)G(U) - \omega U, \\ cV' = d_2 V'' + UV(1-V)(V - {}^1\!/_2), \\ U, V \text{ régulières, bornées et strictement positives,} \end{cases}$$

avec la condition

$$\begin{cases} U(-\infty) = 0, \\ V(-\infty) = v_0. \end{cases}$$

Principaux résultats

Nous proposons au Chapitre 8 une étude détaillée sur deux cas particuliers. Le premier correspond au cas *inhibiteur de tension*, avec $f(u,v) = -1$, et décrit la dynamique d'une *émeute* ; le deuxième correspond au cas *excitateur de tension*, avec $f(u,v) = 1-v$, et décrit la dynamique d'une *révolution*.

Nous présentons dans cette introduction un cas *mixte*, qui inclut les deux cas précédents et permet de modéliser à la fois la dynamique des *émeutes* et des *révolutions*. Considérons le problème (D.11) avec

$$f(u,v) := (1-v)(v - {}^1\!/_2),$$

$$v_\star := r^{-1}\left(\frac{\omega}{G'(0)}\right) \in (0, 1/2)\,.$$

Ce modèle présente un double effet de seuil. Du point de vue de la modélisation, un *évènement exogène* peut provoquer trois scénarios différents :

— `TS` initiale faible : *retour au calme.*

— `TS` initiale intermédiaire : *irruption d'une émeute.*

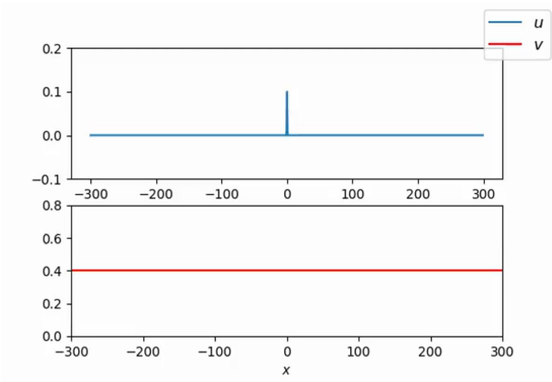
— `TS` initiale élevée : *irruption d'une révolution.*

Le théorème suivant résume nos principaux résultats. Il est illustré en Figure D.1-D.2

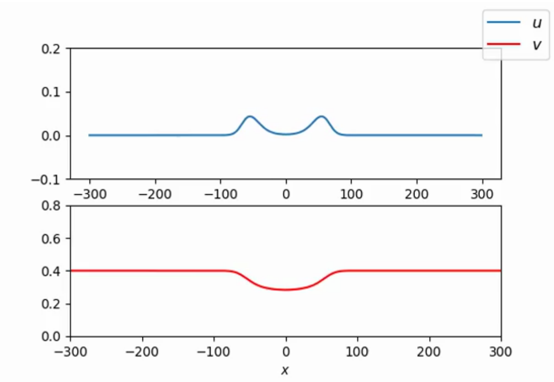
Theorem D.9

1. *Cas* $v_0 \in (0, v_\star)$. *Il n'existe pas d'onde de transition. Si* $d_2 > 0$, $(u(t,\cdot), v(t,\cdot))$ *converge uniformément vers* $(0, v_0)$.
2. *Cas* $v_0 \in (v_\star, 1/2)$. *Voir Figure D.1. Il n'existe pas d'onde de transition pour les vitesses* $c < c_0 := 2\sqrt{d_1\,(r(v_0)G'(0) - \omega)}$, *et il en existe pour les vitesses* $c > c_0$. *De plus, les ondes de transition* (U, V) *sont telles que* V *est décroissante,* $U(+\infty) = 0$ *et* $V(+\infty) = V_\infty$ *pour un certain* $V_\infty \le v_\star$.
3. *Cas* $v_0 \in (1/2; 1)$. *Voir Figure D.2. Il n'existe pas d'onde de transition pour les vitesses* $c < c_0$, *et il en existe pour les vitesses* $c > \overline{c} := 2\sqrt{d_1\,(r(1)G'(0) - \omega)}$ *satisfaisant* (D.9). *De plus, les ondes de transition* (U, V) *sont telles que* V *est croissante,* $V(+\infty) = 1$ *et* $U(+\infty) = u_\star$, *où* $u_\star$ *est l'unique solution positive de* $r(1)G(z) - \omega z = 0$.

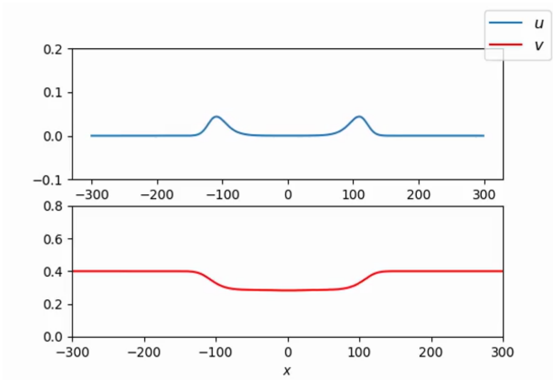
   La solution $(u(t,\cdot), v(t,\cdot))$ *converge localement uniformément vers* $(u_\star, 1)$, *et se propage avec une vitesse asymptotique* $c_\star \in [c_0, \overline{c}]$.



(a) $t = 0$

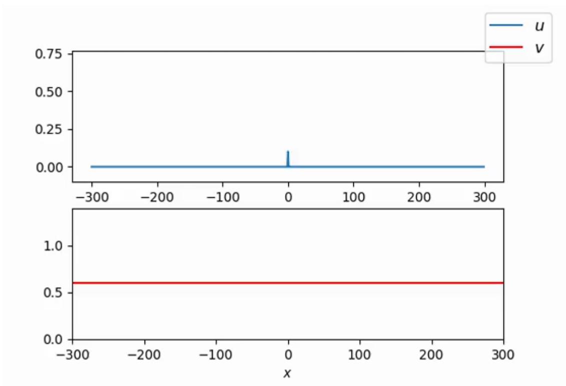


(b) $t = 150$

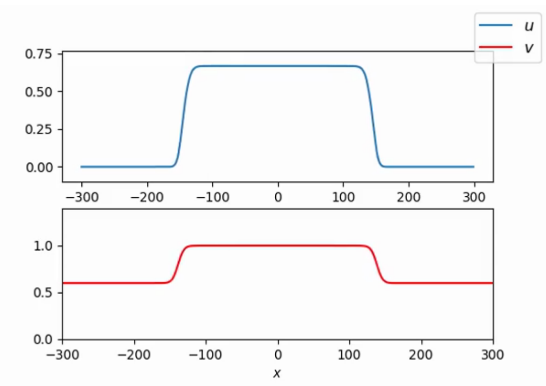


(c) $t = 270$

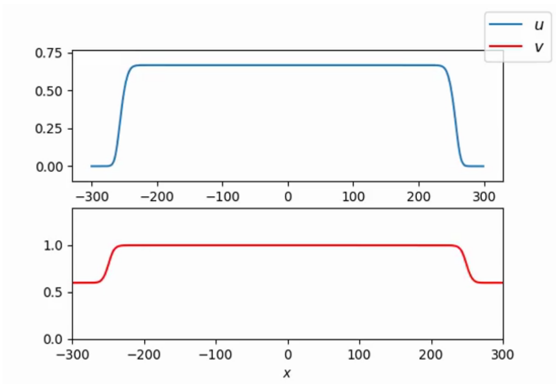
FIGURE D.1 – Émeute. $v_0 = 0.4$



(a) $t = 0$



(b) $t = 150$



(c) $t = 270$

FIGURE D.2 – Révolution. $v_0 = 0.6$

Introduction$'$ (English)

Contents

Chapter A′

General Introduction: Reaction-Diffusion equations and modeling of population dynamics

This thesis focuses on some problems arising from reaction-diffusion equations and modelization of population dynamics. In this introduction, we propose a brief history and present the general context of this work.

As a preamble, it seems important to us to briefly specify what we mean by *model*, and what is the role of mathematics in the modelling process.

A model is an experience of thought, an idealized and schematic representative framework, which proposes a point of view (among others possible) on a phenomenon. On the one hand, a model must be able to accurately reflect the phenomenon under study. On the other hand, a model must also satisfy a criterion of simplicity and aesthetics. Like a good sketch, a model must obtain maximum resemblance in a minimum number of strokes.

In the modelling process, the role of mathematics lies mainly in the study of the intrinsic properties of models. The models are studied in themselves, at a conceptual level. Mathematics therefore does not study reality directly, but the language with which we think about reality.

Contents

A′.1 A brief history

A′.1.1 First discrete models

The first known mathematical model of population dynamics is due to Leonardo Fibonacci, in his 1202 *Liber Abaci* [147]. It models the size of a rabbit population, under the assumptions:

— a couple of adult rabbits generate, each month, a new couple of rabbits.

— a rabbit reaches maturity after one month.

Denoting u_n the number of rabbit pairs at n-th month, we deduce the relationship

$$u_{n+2} = u_{n+1} + u_n.$$

It is the famous Fibonacci sequence. Starting from a single pair of newborn rabbits, *i.e.* $u_0 = u_1 = 1$, we deduce

$$u_n \underset{n \to +\infty}{\sim} \frac{1}{\sqrt{5}} \Phi^n,$$

where $\Phi := \frac{1+\sqrt{5}}{2}$ is the golden ratio. We see that the size of the population grows *exponentially.*

A comparable model is proposed by Thomas Malthus in his 1798 *Essay on the principle of population* [216]. He assumes that the population grows with a constant rate a >0 (which can be seen as the birth rate minus the death rate). This hypothesis is written as follows

$$u_{n+1} = a u_n.$$

From this, we deduce

$$u_n = a^n u_0,$$

and the population has exponential growth, see Figure A′.4a.

This observation is followed by a pessimistic conclusion: since, on the other hand, the production of food grows at most *linearily*, humanity will soon face famine. See Figure A′.1.

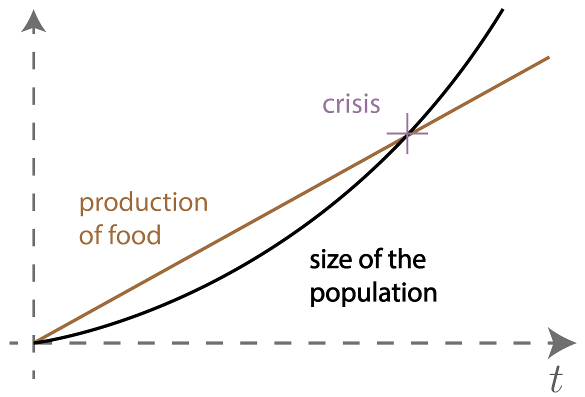


Figure A′.1 – Diagram of the Malthusian theory

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Table, calculated from the births alone, in the Preliminary Observations to the Population Abstracts, printed in 1811.		Table, calculated from the excess of the births above the deaths, after an allowance made for the omissions in the registers, and the deaths abroad.	
Population in		Population in	
1780	7,953,000	1780	7,721,000
1785	8,016,000	1785	7,998,000
1790	8,675,000	1790	8,415,000
1795	9,055,000	1795	8,831,000
1800	9,168,000	1800	9,287,000
1805	9,828,000	1805	9,837,000
1810	10,488,000	1810	10,488,000

In the first table, or table calculated from the births alone, the additions made to the population in each period of five years are as follow;—

From 1780 to 1785 63,000
From 1785 to 1790 659,000

Figure A′.2 – Forecast demographic table from Malthus' work of 1798

A′.1.2 First continuous models

In 1838, Pierre-François Verhulst revists Malthus' work in his *Notice sur la loi que la population poursuit dans son accroissement* [281]. Unlike previous approaches, he considers that the number of individuals $u(t)$ has a continuous dependance on time. In particular, $u(t)$ takes non-integer values. This makes sense if we measure the population with a very large unit (for example, the thousand of individuals): then $u(t)$ no longer represents the *number of individuals*, but the *size of the population.*

Verhulst mathematically translates Malthus' law into the form of the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = au(t).$$

He deduces

$$u(t) = u(0)e^{at},$$

and recovers Malthus' conclusion: the population has exponential growth, see Figure A′.4a.

Since Malthus' model is only valid in an environment where resources are abundant, Verhulst proposes a model to take into account a *saturation phenomenon* induced by resource limitations. Inspired by Quetelet's work of 1835 [254], Verhulst considers that the growth rate a should decrease as the population size increases. The simplest choice being to replace a by $a - bu(t)$, where $b > 0$ is a constant, Verhulst replaces Malthus' equation with *the logistic equation*:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = (a - bu(t))\,u(t). \tag{A′.1}$$

This equation admits the constant state of equilibrium

$$K = \frac{a}{b}.$$

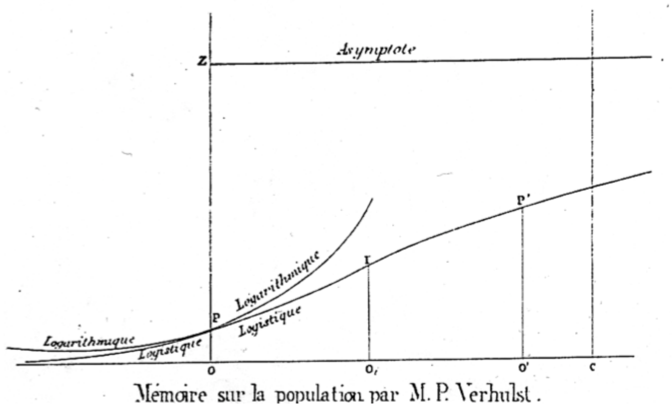


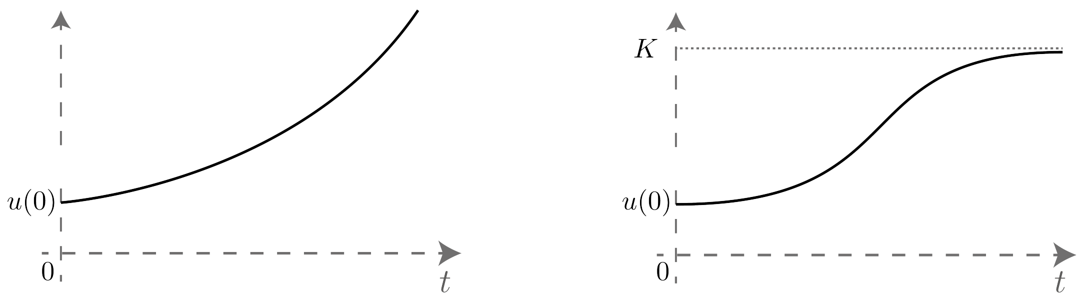
Figure A′.3 – Representation of the "logarithmic" (*i.e.* exponential) and logistic functions in Verhulst's first paper

The constant K is often referred to as the *carrying capacity* of the medium, and represents the maximum population size that the medium can host in the long term. The solutions of (A′.1) are written

$$u(t) = \frac{Ku(0)}{(K - u(0))e^{-at} + u(0)}.$$

These solutions are called sigmoidal, or informally, S-shaped, see Figure A′.4b. If $u(0) \leq K$, these solutions are increasing and converge to K in long time. This expresses the fact that the population stabilizes.

For a more detailed history, we let the reader refer to [24, 218].



(a) In Malthus' model **(b)** In Verhulst's model

Figure A′.4 – Evolution of the population size $u(t)$

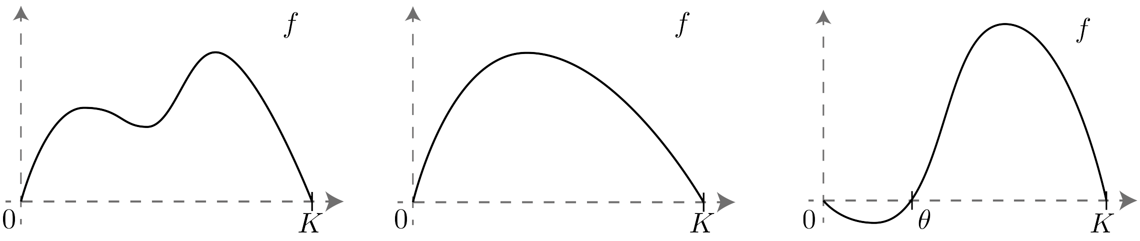
A′.2 Reaction-Diffusion Equations

A′.2.1 Ordinary Differential Equations

More generally, the population size is assumed to be governed by the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = f(u(t)),$$

for a certain function f, called the *nonlinearity*. The aim is no longer to express the solutions explicitly (because this is no longer possible), but to identify the qualitative



(a) monostable **(b)** monostable KPP **(c)** bistable

Figure A′.5 – Types of nonlinearity

properties of the solutions, depending on the form of the nonlinearity. The following cases are generally considered:

- **Monostable nonlinearity.** See Figure A′.5a. The function f is said to be of the *monostable* type if
$$f(0)=0 \quad ; \quad f \geq 0 \text{ on } [0,K] \qquad\qquad f \leq 0 \text{ on } [K,+\infty).$$
This expresses the fact that the population grows to a maximum size of K, as in the logistics equation (A′.1).
- **Monostable-KPP nonlinearity.** The function f is said to be of the *KPP* type (for Kolmogorov-Petrovski-Piskunov) if it is monostable and
$$x \mapsto \frac{f(x)}{x} \text{ is decreasing.}$$
See Figure A′.5b. This hypothesis expresses the fact that the growth rate decreases as the population size increases. Verhulst's logistic equation (A′.1) satisfies this hypothesis.
We say that f is *KPP in a weak sense* if it is monostable and such that $f(x) \leq xf'(0)$.
- **Bistable nonlinearity.** See Figure A′.5c. The function f is said to be of the *bistable* type if
$$f(0)=0 \quad ; \quad f \leq 0 \text{ on } [0,\theta]$$
$$f \geq 0 \text{ on } [\theta,K] \quad ; \quad f \leq 0 \text{ on } [K,+\infty).$$
The simplest example is
$$f(x) := x(K-x)(x-\theta).$$
Compared to the monostable case, it is assumed here that population's growth rate is negative if its size is below a threshold θ . This hypothesis is known as the Allee effect. From a modelling point of view, it is needed when considering small populations.
Under this assumption, two scenarios are possible. If $u(0) > \theta$, then $u(t)$ converges in long time to K, as in the *monostable* case. If $u(0) < \theta$, then $u(t)$ converges in long time to 0 (*i.e.* the population goes extinct).

A′.2.2 Models with space

Previous models do not take into account the distribution of individuals in space. Let us now consider that the population density $u(t,x)$ depends on time $t \geq 0$ and location $x \in \mathbb{R}^n$, where $n \geq 1$ is the dimension.

Let us neglect, for a moment, the population growth (*i.e.*, assume $f = 0$) to focus only on the spatial aspect. We need to model the movement of individuals.

Diffusion. A first hypothesis, probably the simplest, is to consider that each individual moves according to a Brownian motion (see Figure A′.6). This movement corresponds to that of a particle in a gas. At the macroscopic scale (*i.e.*, when we do not consider each individual separately, but the density of a large number of individuals) this hypothesis results in a *diffusion*. This is expressed mathematically by the Laplacian operator: the population density $u(t,x)$ satisfies

$$\partial_t u(t,x) - d\Delta_x u(t,x) = 0, \tag{A′.2}$$

where

$$\Delta_x u(t,x) = \partial^2_{x_1} u(t,x) + \cdots + \partial^2_{x_n} u(t,x).$$

The constant $d \geq 0$ is the diffusion coefficient, homogeneous to *length*^{2}.*time*$^{-1}$, and stands for the mobility of individuals.

Equation (A′.2) is called the *Heat Equation*. Any solution $u(t,x)$ converges uniformly to 0 in long time. In addition, the equation has a *regularizing* effect, that is, even if the initial data $u(t=0,\cdot)$ is irregular, the function $u(t,\cdot)$ is smooth for any $t > 0$.

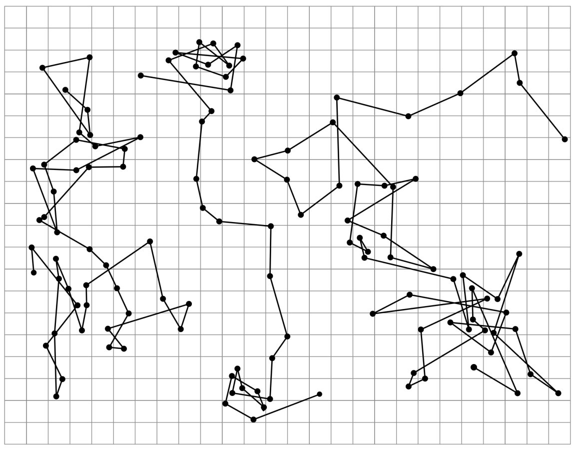


Figure A′.6 – Brownian motions

Drift. We make the additional assumption that a force, directed by a vector $\overrightarrow{b}$, induces a drift of our individuals (think, for example, of a population of fish subjected to a current force). The population density then satisfies

$$\partial_t u(t,x) - d\Delta_x u(t,x) + \overrightarrow{b} \cdot \nabla_x u(t,x) = 0.$$

If there is no diffusion (*i.e.*, if $d = 0$), this equation is called a *transport equation*.

Nonlocal diffusion. Another possible hypothesis is to consider that an individual at the location x *migrates* to a location y with a certain probability $M(y-x)dy$. The equation satisfied by u is then written

$$\partial_t u(t,x) - \left(\int_{\mathbb{R}^n} M(x-y)u(t,y)dy - u(t,x) \right) = 0.$$

This is a nonlocal diffusion equation.

A′.2.3 Boundary conditions

Now, let us suppose that our population evolves, not on $\mathbb{R}^n$, but on some domain $\Omega \subset \mathbb{R}^n$. We need to model individuals' behavior when they touch the border of the domain. We consider three scenarios.

— **Dirichlet boundary conditions**. It corresponds to the hypothesis that an individual vanishes (or dies) as soon as he touches the edge. At the macroscopic level, it results in
$$u = 0 \quad \text{on } \partial\Omega.$$

— **Neumann boundary conditions**. It corresponds to the hypothesis that an individual bounces off the edge. At the macroscopic level, it results in
$$\partial_\nu u = 0 \quad \text{on } \partial\Omega.$$
where ∂_ν is the normal outward derivative to the domain. It expresses that the flux of individuals outside the domain is zero.

— **Robin boundary conditions**. It corresponds to the assumption that individuals who touch the edge have a growth rate of $\gamma \in \mathbb{R}$. At the macroscopic level, it results in
$$\partial_\nu u = \gamma u \quad \text{on } \partial\Omega.$$
The Neumann case corresponds to $\gamma = 0$, the Dirichlet case formally corresponds to $\gamma = -\infty$.

A′.2.4 Reaction-Diffusion Equations

By considering a population that both grows and spreads, for example on $\mathbb{R}^n$, we obtain a *reaction-diffusion equation*,
$$\partial_t u(t,x) - d\Delta_x u(t,x) = f(u(t,x)), \quad x \in \mathbb{R}^n. \tag{A′.3}$$
This equation has many properties, especially with regard to the spatial propagation of the solutions. The pioneering work in this field is due to Fisher [150], Kolmogorov, Petrovski, Piskunov [199], then Fife, McLeod [148], Aronson, Weinberger [17], Bramson [64, 65].

The reaction-diffusion equations (A′.3) admit various solutions. First, there are the stationary solutions, which do not depend on time, and therefore satisfy
$$-d\Delta u(x) = f(u(x)).$$

Another particular form of solution is *the traveling wave* (see Figure A′.7), such that $u(t,x) = U(ct - x\cdot e)$, where $c \geq 0$ is a speed, $e \in \mathbb{S}^n$ a direction, and $U : \mathbb{R} \to \mathbb{R}$ satisfies
$$cU' - dU'' = f(U).$$
Traveling waves correspond to a constant profile moving at a speed of c in the e direction. If f is of the *KPP* type, we can show that there exists a (unique) *traveling wave* with speed c, if and only if $c \geq c_0 := 2\sqrt{f'(0)}$. If f is of the *bistable* type, there exists a traveling wave for a single speed $c_\star \in \mathbb{R}$, whose sign is determined by that of $\int_0^K f$.

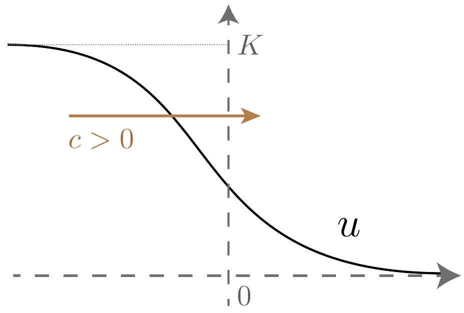


Figure A′.7 – Traveling wave

If f is still of the *KPP* type, and if $u(t = 0, x) \gneqq 0$ is compact, then $u(t,x)$ converges to K in long time and *propagates* through space with speed c_0, that is

$$\forall c < c_0, \quad \lim_{t\to+\infty} \left(\inf_{|x|\leq ct} u(t,x) \right) = K,$$
$$\forall c > c_0, \quad \lim_{t\to+\infty} \left(\sup_{|x|\geq ct} u(t,x) \right) = 0.$$

We can even show that the solution converges to the traveling wave of speed c_0 (with a logarithmic delay), see for example [173].

Reaction-Diffusion equations are used in various fields of application. Many examples from biology are presented in Murray's book [241]. For example, Skellam's article [266] models, among other things, the spread of muskrats in Central Europe, see Figure A′.8. In another context, Alan Turing's famous article [278] models the

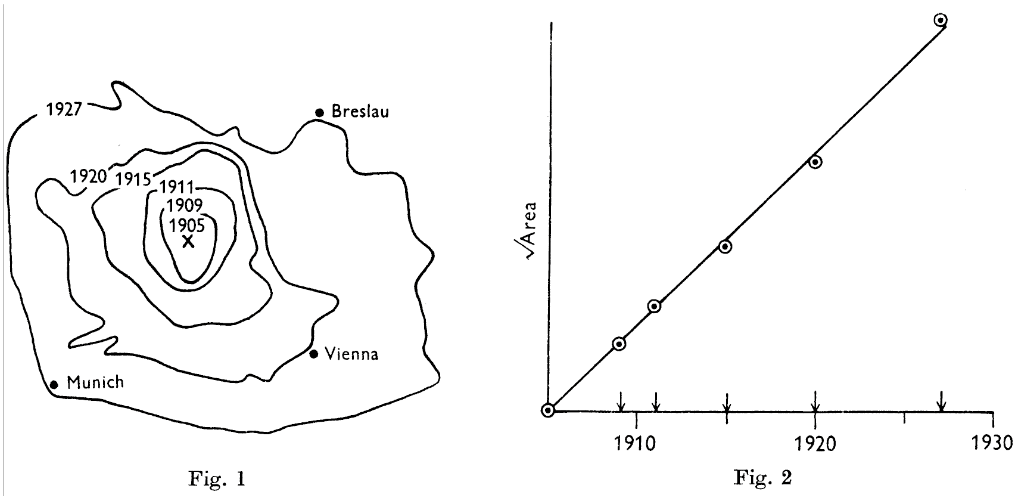


Figure A′.8 – Illustrations from Skellam's article [266] describing the propagation at constant speed of muskrats in Central Europe.

formation of *patterns* in living organisms (Figure A′.9).

A′.2.5 Purpose of this manuscript

In this manuscript, we are interested in some problems arising from reaction-diffusion equations and their use in modelling population dynamics. The first part

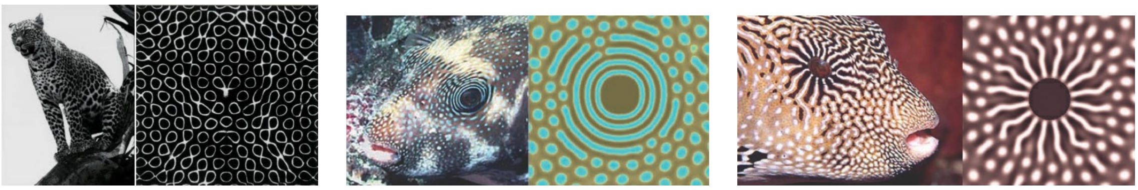


Figure A′.9 – Three examples of Turing's patterns. Left side: picture. Right side: computer simulation

deals with *stable* stationary solutions of reaction-diffusion equations. In the second part, we focus on the modelling of the natural selection phenomenon in the Darwinian theory of evolution. Finally, in the third part, we study systems of reaction-diffusion equations that arise from epidemiological models, and propose a model to describe the dynamics of riots and social unrest.

Chapter B′

Introduction to Part I: Qualitative properties of stable solutions

Contents

B′.1 Introduction

B′.1.1 Presentation of the problem

Framework

In the first part of this manuscript, we study some qualitative properties of stable solutions of semilinear elliptic equations with Neumann boundary conditions. We consider the following problem:

$$\begin{cases} -\Delta u(x) = f(u(x)) & \forall x \in \Omega, \\ \partial_\nu u(x) = 0 & \forall x \in \partial\Omega, \\ u \in C^2\left(\overline{\Omega}\right) \cap L^\infty(\Omega), \end{cases} \tag{B′.1}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth domain, ∂_ν is the outward normal derivative, and f is a C^1 nonlinearity.

A solution is said to be *stable* if the second variation of energy in u is positive (possibly degenerate), *i.e.*, if u is a local minimizer (in the weak sense) of energy. This definition will be further clarified later. From a modelling perspective, stable solutions are solutions that have a "physical meaning": they represent the only potentially observable states of a physical system.

Note that if $z \in \mathbb{R}$ is a stable root of f, *i.e.* $f(z) = 0$ and $f'(z) \leq 0$, it is a (trivial) stable solution. We call *pattern* any non-constant *stable* solution. We are particularly interested in the problem of the existence and non-existence of *patterns*. This problem involves the geometry of the domain in a complex way.

The question of the existence of patterns occurs in many fields of application, such as the formation of patterns in biology [127, 241, 278, 287] or chemistry [260], phase transitions [77], or the propagation of biological populations [150, 266].

Our starting point is a famous theorem proved by Casten, Holland [86], and independently by Matano [217].

Theorem B′.1 ([86, 217]) *If the domain Ω is bounded and convex, there exists no pattern to* (B′.1).

We insist on the fact that this conclusion is valid regardless of $f \in C^1$. This result initiated a profusion of mathematical developments involving deep notions of PDE and geometry.

The purpose of our work is to try to better understand which geometric criteria ensure the existence or non-existence of patterns. Indeed, the literature presents a gap between convex domains on the one hand, and *dumbbell* domains (introduced later) on the other hand.

In this section we give a quantitative criterion for the non-existence of patterns in general domains, related to the sign of the principal eigenvalue of a certain linear operator. To our knowledge, this criterion is the first of this nature. It should also be noted that the literature deals with bounded convex domains on the one hand, and with the entire space $\mathbb{R}^n$ on the other hand; it does not deal, or only slightly, with general unbound convex domains. We will therefore discuss the extension of our results to unbounded domains. In addition, the methods used in the literature crucially rely on the self-adjoint nature of the Laplacian operator. The method we give allows us to treat the case of non-self-adjoint operators. We also give perturbation results, an asymptotic formulation of the theorem of Casten, Holland and Matano, as well as a *flatness* estimate for patterns.

We introduce here the content of Part I, that is, the chapters 1,2,3. In Chapter 1, we study the extension of the Casten, Holland, and Matano theorem to unbounded convex domains. Various extensions are proposed in Chapter 2, including a quantitative criterion for the non-existence of patterns. In this Chapter, we recover and refine the results of Chapter 1. Finally, Chapter 3 is devoted to the introduction and analysis of the generalized *principal eigenvalue* in unbounded domains with Robin boundary conditions.

Definition of stability

A solution u of (B′.1) can be seen as a critical point of the energy

$$\mathcal{E}(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 + F(u),$$

where F is a primitive of f. The quadratic form associated with the second variation of E is

$$\mathcal{F}(\psi) = \mathcal{F}(u, f, \Omega)(\psi) := \int_\Omega \Omega|\nabla \psi|^2 - f'(u)\psi^2.$$

Let us define

$$\lambda_1 = \lambda_1(u, f, \Omega) := \inf_{\substack{\psi \in H^1 \\ \|\psi\|_{L^2}=1}} \mathcal{F}(\psi).$$

Definition B′.2 *A solution u of* (B′.1) *is said to be stable if $\lambda_1 \geq 0$, and stable non-degenerate if $\lambda_1 > 0$.*

Thus, a stable non-degenerate solution is a (non-degenerate) minimum of the energy. The Euler-Lagrange equation associated with the functional $\mathcal{F}$ is obtained as a linearization of (B′.1) around u:

$$\begin{cases} -\Delta v - f'(u)v = 0 & \forall x \in \Omega, \\ \partial_\nu v(x) = 0 & \forall x \in \partial\Omega. \end{cases} \tag{B′.2}$$

If the domain Ω is bounded, then λ_1 is an eigenvalue of the linearized operator, called the *principal eigenvalue*. This eigenvalue has many fundamental properties, and can be defined for other elliptic operators (including non-self-adjoint operators with Robin boundary conditions) and for unbounded domains. Thus, Definition B′.2 is quite flexible. We will come back to this point later.

Remark. There are other ways to define the stability of a solution. The most classical is undoubtedly the dynamical definition: a solution u of (B′.1) is *stable from a dynamical point of view* if it attracts any solution of the parabolic problem

$$\begin{cases} \partial_t v(t,x) - \Delta v(t,x) = f(v(t,x)) & \forall (t,x) \in (0,+\infty) \times \Omega, \\ \partial_\nu v(t,x) = 0 & \forall (t,x) \in (0,+\infty) \times \partial\Omega, \\ v(0,x) = v_0(x) & \forall x \in\in \Omega, \end{cases} \tag{B′.3}$$

with an initial condition v_0 sufficiently close to u. In fact, if the domain Ω is bounded, we have the following chain of implications:

$$\lambda_1 > 0 \quad \Rightarrow \quad \text{stability from a dynamical point of view} \quad \Rightarrow \quad \lambda_1 \geq 0.$$

We give a more detailed discussion on the validity of this chain of implications in unbounded domains in section 1.7.2

B′.1.2 Related topics and state of the art

In order to present the context of our work, we propose a state of the art and an introduction to some of the related topics that arise from our study.

The counter example of Matano and the dumbbell domains

Matano's pioneering article [217] proposes a general study of the parabolic problem (B′.3) on bounded domains. Among other things, he constructs a counterexample to Theorem B′.1, *i.e.* he constructs a pattern to (B′.1) for a certain domain Ω and a nonlinearity f.

The non-linearity he considers is of the *bistable* type. To fix ideas, let us consider

$$f(u) = u(1 - u^2),$$

called the Allen-Cahn nonlinearity. The domain considered by Matano features a bottleneck. These domains are called *dumbbell domains*. For example, consider a domain $\Omega_\varepsilon \subset \mathbb{R}^n$, $n \geq 2$ formed by the union of two disjointed convex sets, connected by a thin cylindrical corridor of radius $0 < \varepsilon \ll 1$. See Figure B′.1.

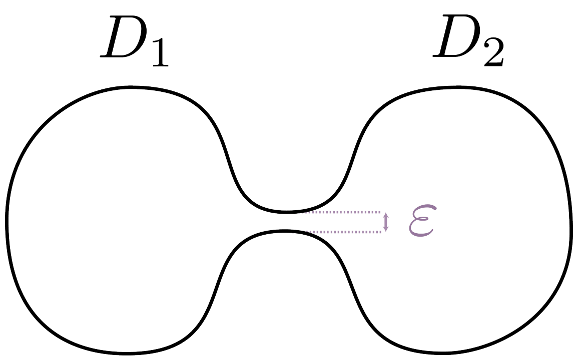


Figure B′.1 – domaine haltère

Matano proves the existence of a pattern, if ε is small enough. This is a counterexample to Theorem B′.1. The antagonism between convex and dumbbell domains illustrates that the problem of the existence of patterns to (B′.1) involves the geometry of the domain in a complex way.

Matano's idea can be adapted to construct patterns in many situations. Note that Ω_ε can be seen as a perturbation of a non-connected set $\Omega_0 := D_1 \sqcup D_2$ with D_1, D_2 two disjoint convex sets. The function $u_0 := \mathbb{1}_{D_1} - \mathbb{1}_{D_2}$ is a pattern on Ω_0. Matano's counter-example in Ω_ε is somehow obtained as a perturbation of u_0. This argument is made rigorous by Hale and Vegas [172], who actually build patterns as bifurcations of trivial solutions by perturbation of the domain.

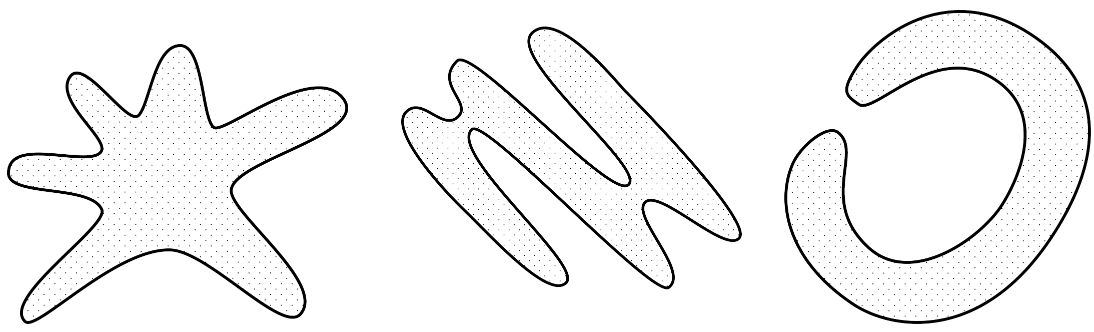
A general theoretical framework for the study of the parabolic problem (B′.3) in dumbbell domains is provided by Arrieta, Carvalho, and Lozada-Cruz [19–21] (see also Gadyl'shin [157]).

Link with invasion/blocking of populations

Let us consider that the quantity u represents a population density. In a dumbbell domain, the movement of an individual from one end of the domain to the other is hindered by the bottleneck. On the contrary, in a convex domain, the movement of individuals can always be done in a straight line. This observation suggests that the existence of patterns is linked to the fact that the geometry of the domain hinders the diffusion of individuals.

This heuristic can be highlighted by some results on the propagation of traveling waves. These traveling waves are particular solutions of the parabolic problem (B′.3) which consist of a profile moving at a constant speed. They model, among other things, the invasion of a territory by a population.

Berestycki, Hamel, and Matano [42] study the influence of an obstacle on the propagation of a traveling wave. They build a particular solution $v(t, x)$ to the parabolic problem (B′.3), in a domain $\Omega = \mathbb{R}^n \backslash K$ where the "obstacle" K is compact, and the nonlinearity f is of the bistable type. This solution is called a *generalized traveling wave*, because it is defined for any $t \in \mathbb{R}$ and converges uniformly towards a planar traveling wave when $t \to -\infty$. The solution $v(t, x)$ converges in long time towards some $u(x)$, which is a solution of the stationary problem. The authors show that, if the obstacle is star-shaped (Figure B′.2a) or axially convex (Figure B′.2b), there is invasion, *i.e.* $u \equiv constant$. On the other hand, if the obstacle contains a bottleneck (like a dumbbell domain, see Figure B′.2c), there is a blockage, *i.e.* $u \not\equiv constant$. As a limiting state of a parabolic problem, u is, *de facto*, a stable solution of (B′.1). We can therefore make an analogy between these results and the problem of the existence of patterns.

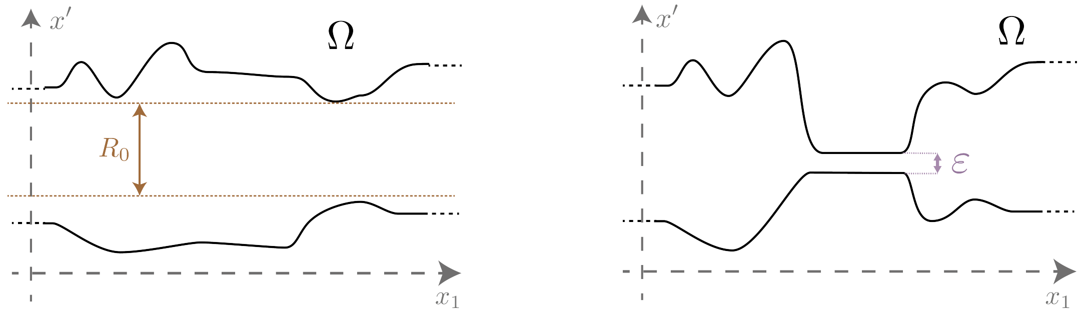


(a) star-shaped **(b)** axially convex **(c)** dumbbell

Figure B′.2 – Obstacles K

A similar problem is considered by Berestycki, Bouhours and Chapuisat [40] in cylinders with varying cross-section. The authors prove that there is invasion if the cylinder is large enough and all sections are star-shaped (see Figure B′.3a). On the contrary, they show that there is a blockage if the cylinder has a bottleneck (see

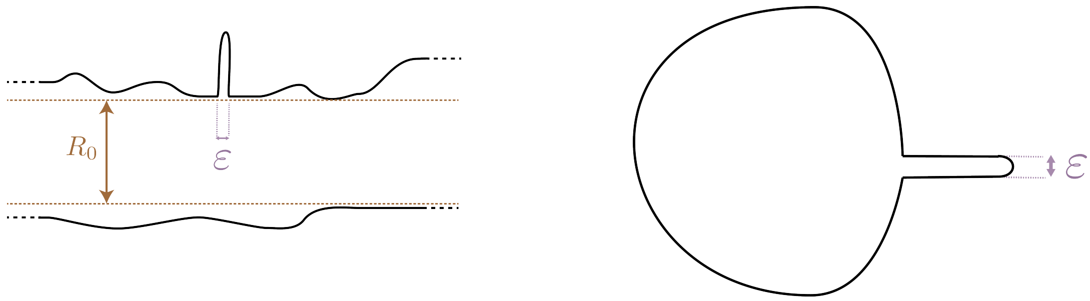
Figure B′.3b). Similar results in the case of non-local propagation are obtained by Brasseur, Coville, Hamel, Valdinoci [67], Brasseur, Coville [66].



(a) large, axially star-shaped **(b)** with a bottleneck

Figure B′.3 – Cylindres

These examples show a strong analogy between traveling wave blocking and the existence of patterns. In general, blocking implies the existence of patterns. However, the reverse is not true. Indeed, there is no blockage if the domain is star-shaped and contains a sufficiently large ball in its center; however, these domains sometimes admit patterns, see Figures B′.4a-B′.4b.



(a) axially convex **(b)** dumbbell star-shaped

Figure B′.4 – Domains for which there exists pattern but there is no blockage

A model for phase transitions

An extremely fecund link can be established between the level sets of patterns and minimal surfaces. In order to shed light on the developments that follow, we begin by giving an intuitive idea of how a pattern of (B′.1) can model a phase transition in the theory of van der Waals [280], Cahn and Hilliard [77].

Let us consider u as the density of a two-phase chemical fluid, at equilibrium in a domain Ω. We assume that each phase taken separately is stable. This hypothesis suggests that u minimizes an energy with a two-well potential. To fix ideas, consider the potential $F(u) = (1-u^2)^2$, so that the two phases of the fluid are represented by the values ± 1. In the simplest model, the energy takes the following form:

$$\int_\Omega F(u).$$

Thus, any function u taking only the values ± 1 minimizes energy. From a physical point of view, this model is not satisfactory, because the interface between the two phases can be completely erratic. We must then take into account the influence of microscopic friction forces inside the fluid. This results in adding a kinetic term to the energy of the system:

$$\mathcal{E}_\varepsilon(u) = \int_\Omega \frac{\varepsilon^2}{2}|\nabla u|^2 + F(u),$$

where $\varepsilon > 0$ is an intensity parameter. The presence of this penalisation term prevents a minimizer from instantly jumping from $+1$ to -1, and forces the phase transition to occur over a characteristic length of ε. Any minimizer of $\mathcal{E}_\varepsilon$ is then a stable solution of the associated Euler-Lagrange equation:

$$\begin{cases} -\varepsilon^2 \Delta u_\varepsilon = f(u_\varepsilon) & \text{in } \Omega, \\ \partial_\nu u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{B′.4}$$

with Allen-Cahn's nonlinearity $f(u) := 2(u - u^3)$.

For $\varepsilon = 1$, we recover (B′.1). The case $\varepsilon \ll 1$ is interpreted as the case of a very viscous fluid. Another interpretation is to see u_ε as a solution of $-\Delta u_\varepsilon = \varepsilon^{-2} f(u_\varepsilon)$, and ε^{-2} as the order of magnitude of the depth of the stability wells of ± 1. A third possible interpretation is to do the rescaling $x \longleftrightarrow \varepsilon x$, and to see a solution of (B′.4) as a solution of $-\Delta u_\varepsilon = f(u_\varepsilon)$ in the expanded domain $\varepsilon^{-1}\Omega$.

Link with minimal surfaces

In a series of pioneering papers, Modica, Mortola [237] and Modica [234] develop the theoretical framework of the Γ-convergence to study the behavior of a sequence u_ε of minimizers of the energy $\mathcal{E}_\varepsilon$, when $\varepsilon \to 0$. Modica shows that, up to extraction of a subsequence, u_ε converges L^1_{loc} to $\mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}$, where E is a set with minimum perimeter in Ω.

More precisely, let us consider the perimeter functional in Ω

$$\mathcal{P}_\Omega(E) = \int_\Omega |\nabla \mathbb{1}_E vert = \sup_{g\in C^1_0(\Omega)\ |g|\le 1} \left| \int_E \operatorname{div} g \right|,$$

where $C^1_0(\Omega)$ is the set of compactly supported C^1 functions in Ω. Saying that the set E is of minimum perimeter in Ω (or shortly, ∂E is a minimal surface) means

$$\mathcal{P}_A(E) \le \mathcal{P}_A(F),$$

for any open set $A \subset \Omega$, and any open set F that coincides with E outside $\overline{A}$. Modica's result states that, if ε is small, the energy minimizers look like a piecewise-constant function in the partition $E \sqcup \Omega\setminus E$; the solution therefore features a sharp phase transition, located around a minimal surface ∂E.

Modica's result was then refined by Caffarelli and Cordoba [74, 75], which prove that the superlevel sets $\{u_\varepsilon \ge \lambda\}$ converge locally uniformly to E (in the sense of Hausdorff distance) for any fixed $\lambda \in (0, 1)$. This confirms the heuristic that pattern level sets behave as minimal surfaces of Ω.

Kohn and Sternberg [198] somehow establish a converse statement to Modica's result: given a set E with minimum perimeter in Ω, there exists u_ε a minimizers of $\mathcal{E}_\varepsilon$ that converges L^1_{loc} to $\mathbb{1}_E - \mathbb{1}_{\Omega\setminus E}$.

Thus, provided there exists such a (non-trivial) set E, Kohn and Sternberg prove that there exists a pattern to problem (B′.4) for $\varepsilon \ll 1$. Determining the sets E with minimum perimeter sets in Ω is an old and often difficult problem (related to the *Plateau problem*, see [273] for an overview). Intuitively, such a set E must be so that ∂E and $\partial\Omega$ intersect orthogonally at points where Ω is concave, see Figure B′.5. A minimal surface ∂E is formally located at a bottleneck of the domain.

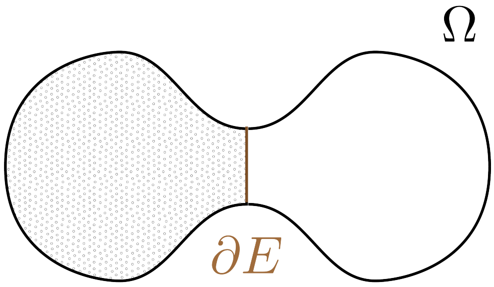


Figure B′.5 – Domain Ω that has a set E with minimum perimeter.

Therefore, such a set E typically exists in dumbbell domains. Thus, Kohn and Stenberg's pattern echoes Matano's. On the contrary, if the domain Ω is a convex domain, there exists no such set E; accordingly, from Theorem B′.1, there exists no non-constant energy minimizer as well. We find again the antagonism between convex and dumbbell domains.

The theory has been extended to the study of energy minimizers under a volume constraint $\int_\Omega u_\varepsilon = m$ by many authors, such as Modica [235, 236], Owen [244], Sternberg [270], Kohn and Sternberg [198]. In this case, there may exist patterns (*i.e.*, non-constant minimizers) in convex domains. However, we can show that patterns have only one phase transition between ± 1 in Ω. The first result in this direction is obtained by Carr, Gurtin, and Slemrod [85], who consider the problem on an interval, and show that minimizers are monotonic (for any $\varepsilon > 0$). This result is extended to the case of straight cylinders by Gurtin and Matano [169]. Finally, for any convex bounded domain, Sternberg and Zumbrun [271,272] show that the phase transition interface is a connected surface; in other words, there is at most one phase transition.

We also mention the extension of this theory to minimizers with values in $\mathbb{R}^n$ (Fonseca and Tartar [154], Baldo [26],Caffarelli, Garofalo, Segala [76] Danielli, Garofalo [111]), as well as the study of the problem in $\Omega = \mathbb{R}^n$ (Lopes [210, 211]) and on Riemannian manifolds (Pacard and Ritoré [245]).

De Giorgi's conjecture

As we saw in the previous section, heuristically, the level sets of the stable solutions of (B′.4) behave as minimal surfaces in Ω. By making a rescalling $x \leftrightarrow \varepsilon x$, i.e. zooming around a point x, equation (B′.4) at the limit $\varepsilon \to 0$ reduces to equation (B′.1) in $\mathbb{R}^n$. Thus, the level sets of the stable solutions in $\mathbb{R}^n$ should behave as minimal surfaces (of dimension $n-1$) of $\mathbb{R}^n$.

Simons [265] shows that, in dimension $n \leq 7$, any minimal surface of $\mathbb{R}^n$ is a hyperplane. A counter-example in dimension $n \geq 8$ was then proposed by Bombieri, De Giorgi, and Gusti [57]. Regarding minimal surfaces which, in addition, are the graph of a function defined on $\mathbb{R}^{n-1}$, we "gain one dimension": such a surface is necessarily a hyperplane if and only if $n \geq 8$ (this problem is known as *Bernstein's problem*, see [57, 72, 114, 184]). This leads De Giorgi to state the following famous conjecture.

Conjecture: (De Giorgi) *Let u be a solution of $-\Delta u = u - u - u^3$ in $\mathbb{R}^n$, such that $|u| < 1$ and $\partial_{x_n} u > 0$. The level sets of u are hyperplanes, at least if $n \leq 8$.*

The fact that any level sets of u is a hyperplane means that u is flat, *i.e.* u only depends on one direction in space (which is not known *a priori*). It is then easy to show that u has the form $\tanh \frac{x_1}{\sqrt{2}}$. It should also be noted that assumption $\partial_{x_n} u > 0$ implies that any level set of u is the graph of a function defined on $\mathbb{R}^{n-1}$.

De Giorgi's conjecture has given rise to a vast literature (see [144,250,262,286] for a state of the art). The conjecture is proved by Ghoussoub and Gui [160] in dimension $n \leq 2$, by Ambrosio and Cabré [10] in dimension $n = 3$, and by Savin [261, 262] in dimension $4 \leq n \leq 8$ under the additional assumption

$$\lim \lim_{x_n \to \pm\infty} u(x'', x_n) = \pm 1. \tag{B′.5}$$

In addition, a counter example is given by del Pino, Kowalczyk and Wei [115] in dimension $n \geq 9$ (this counterexample satisfies (B′.5)). The conjecture is still open for dimensions $4 \leq n \leq n \leq 8$ without the additional assumption (B′.5). Note that the proofs are, in general, valid for a rather large class of nonlinearities, and not only for Allen-Cahn's nonlinearity (see [7, 72, 262]).

If we consider the same problem assuming that the limit in (B′.5) is uniform in $x' \in \mathbb{R}^n$ (problem known as *Gibbons' conjecture*), then the conclusion holds in any dimension, as proved, independently, by Farina [140], Berestycki, Hamel and Monneau [43], Barlow, Bass and Gui [39]. It is also proved in [39, 140] that the conclusion remains true if (B′.5) is replaced by the assumption that at least one level set is the graph of a globally Lipschitz function defined on $\mathbb{R}^{n-1}$.

We also mention the partial results obtained in dimension 4 and 5 by Ghoussoub and Gui [161], and the proof of the conjecture in dimension 4 by Figalli and Serra [149] for the fractional Laplacian.

Liouville Theorem

The proof of De Giorgi's conjecture in dimensions $n \leq 3$ relies on a Liouville type result established by Berestycki, Caffarelli and Nirenberg [41]. Their result can be formulated as follows:

Theorem B′.3 (Liouville, [41]) *Let $\varphi > 0$ and $\sigma \in C^2$ such that $\varphi\sigma$ is bounded. Suppose that the following inequality holds*

$$\sigma \mathrm{div}\left(\varphi^2 \nabla \sigma\right) \geq 0 \quad \textit{in } \mathbb{R}^n.$$

If $n \leq 2$, then σ is constant.

For $n = 1$ and $\varphi = 1$, we recover the famous Liouville result: any convex and bounded function in $\mathbb{R}$ is constant.

This theorem has been the starting point for many advances in the study of entire solutions, and a vast literature is devoted to it (see the works of Moschini [240] and Pinchover [249] for an overview). In general, the theorem cannot be extended to the dimensions $n \geq 3$ according to Barlow's counterexample [38]. On the other hand, the conditions "$\varphi\sigma$ bounded" and "$n \leq 2$" can be replaced by the single assumption

$$\int_{B_R} (\varphi\sigma)^2 = O(R^2), \quad \text{when } R \to +\infty, \tag{B$'$.6}$$

where B_R is a ball of radius R. This condition is somehow optimal [158].

The common strategy to prove De Giorgi's conjecture (at least in small dimensions) is to apply the Liouville Theorem with φ the principal eigenfunction of (B$'$.2), and $\sigma_i := \frac{\partial_{x_i} u}{\varphi}$.

Patterns in $\mathbb{R}^n$

The assumption $\partial_{x_n} u > 0$ in De Giorgi's conjecture implies in particular that the solution considered is stable. Indeed, let us recall that u is stable if and only if

$$\lambda_1 := \inf_{\substack{\psi \in H^1 \\ |\psi\|_{L^2}=1}} \left[\int_{\mathbb{R}^n} |\nabla\psi|^2 - f'(u)\psi^2 \right] \geq 0. \tag{B$'$.7}$$

The definition of λ_1 remains the same if the test functions ψ are chosen in $C^1(\mathbb{R}^n)$ and with compact support. Setting $v := \partial_{x_n} u$, and differentiating (B$'$.1), we get $-\Delta v - f'(u)v = 0$ in $\mathbb{R}^n$. Multiplying this equation by $\frac{\psi^2}{v}$ (for a test function $\psi \in C^1(\mathbb{R}^n)$ with compact support), integrating on $\mathbb{R}^n$, then integrating by parts, we obtain

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} \nabla v \cdot \nabla \frac{\psi^2}{v} - f'(u)\psi^2 \\ &= 2\int_{\mathbb{R}^n} \frac{\psi}{v} \nabla v \cdot \nabla\psi - \frac{\psi^2}{v^2} |\nabla v|^2 - f'(u)\psi^2 \\ &\leq \int_{\mathbb{R}^n} |\nabla\psi|^2 - f'(u)\psi^2, \end{aligned}$$

where we use Young's inequality $2ab \leq a^2 + b^2$ in the last step. So, we deduce $\lambda_1 \geq 0$, *i.e.*, u is stable.

It is therefore natural to consider a variant of De Giorgi's conjecture, replacing the assumption $\partial_{x_n} u > 0$ with the assumption that u is stable.

Conjecture: (De Giorgi's variant) *Let u be a* stable *solution of $-\Delta u = u - u^3$ in $\mathbb{R}^n$, such that $|u| < 1$. The level sets of u is a hyperplanes, at least if $n \leq 7$.*

Note that with the stability assumption (instead of $\partial_{x_n} u > 0$), we cannot guarantee that every level set of u is the graph of a function defined on $\mathbb{R}^{n-1}$. This explains why this conjecture is stated for $n \leq 7$ (unlike De Giorgi's conjecture, stated for $n \leq 8$).

It turns out that Ghoussoub and Gui's proof is valid in this weaker context: the conjecture is therefore proved for $n \leq 2$. A counterexample is provided by Pacard and Wei [246] in even dimensions greater than or equal to 8, thus proving that the condition $n \leq 7$ is optimal. The intermediate dimensions $3 \leq n \leq n \leq 7$ are open.

However, Dancer [104] shows that if the bounded solution u is *stable non-degenerate*, *i.e.* $\lambda_1 > 0$ in (B′.7), then u is constant. This result holds in any dimension. Thus, exists no stable non-degenerate pattern in $\mathbb{R}^n$: this result can be seen as an extension of the theorem B′.1 to $\mathbb{R}^n$ (with the assumption of non-degenerate stability, however).

So there may exist patterns in $\mathbb{R}^n$, but their stability is degenerate, and they are necessarily flat if $n \leq 2$.

We recall that we assume *a priori* that patterns are bounded solutions. Without this assumption, the results are no longer valid: for example, $u(x) := e^x$ is a stable non-degenerate solution of $-u'' = -u$ in $\mathbb{R}$.

Some more precise results are also available if we assume that f is, for example, increasing, convex, positive, *etc.* See Dupaigne's book [135] and the articles [105, 107, 136, 141, 145]. See also [76, 111] for a study of a general class of equations in $\mathbb{R}^n$.

We also mention the result of Cabré and Capella [73] (refined by Villegas [282]) which establishes that radial patterns exist in $\mathbb{R}^n$ if and only if $n \geq 11$.

Symmetry results and extension to manifolds

Matano [217] shows that, if the domain is a solid of revolution, any stable solution is invariant with respect to the axis of rotation. In addition, if the section is convex, then any stable solution is constant. In particular, this implies the non-existence of patterns in certain non-convex domains, such as torus or rings.

This result suggests the idea that stable solutions inherit some symmetries of the domain. We met the same idea in the previous section: a stable solution in $\mathbb{R}^n$, $n \leq 2$, is necessarily flat. However, similar results cannot be expected for symmetries other than those by rotation or translation: the reason is that only Cartesian and angular derivative operators commute with the Laplacian.

It is natural to consider the problem under other geometries, and therefore to study the properties of stable solutions of (B′.1) on manifolds. The first article on the subject is due to Jimbo [185]. It establishes a generalization of Theorem B′.1: if the manifold M is compact, of negative Ricci curvature, and convex (*i.e.*, the second fundamental form of ∂M is nonnegative), then there exists no pattern to (B′.1) in M. Then, he constructs a pattern on a *dumbbell* type manifold, inspired by Matano's method [217].

Matano's angular symmetry result is extended to surfaces of revolution by Rubinstein and Wolansky [260], then to more general manifolds without boundary by Gonçalves and Nascimento [166]. The latter also offers a very precise pattern construction, which clearly highlights the importance of the sign of the Ricci curvature. We also mention the work of Bandle, Punzo, Tesei [28] and Tesei [253].

Finally, results similar to those available in $\mathbb{R}^n$ around the De Giorgi conjecture have been obtained for non-compact Riemannian manifold without boundary by Farina, Sire, Valdinoci [143] and Farina, Mari, Valdinoci [142].

Other extensions

There are many other extensions of Theorem B′.1 and its variants. The case of Robin boundary conditions is studied by Bandle, Mastrolia, Monticelli, Punzo [27], and the case of a boundary reaction by Cònsul and Solà-Morales [100]. We also mention the article of Chanillo and Cabré [96] for qualitative properties of positive stable solutions in convex domains with Dirichlet boundary conditions.

Many authors have also been interested in understanding how the existence of patterns could emerge from the non-homogeneity of coefficients, for example, if diffusivity is variable [129, 130, 268, 291] or if the reaction term is non-homogeneous [9, 69, 269].

Finally, we mention that the results of existence and non-existence of patterns were obtained for cooperative systems [186, 194], then extended to cases of activator/inhibitor systems [292].

B′.2 Presentation of the results

B′.2.1 Criterion for the non-existence of patterns

Definitions

Our approach makes it possible to deal with non-self-adjoint operators, of the form

$$-Lu := -\mathrm{div}\,(A \cdot \nabla u) - B \cdot \nabla u, \tag{B′.8}$$

with $B \in \mathbb{R}^n$ and A a symmetric positive-definite matrix. We thus consider equation (B′.1) by replacing the Laplacian $-\Delta$ by $-L$:

$$\begin{cases} -Lu(x) = f(u(x)) & \forall x \in \Omega, \\ \partial_{\nu_A} u(x) = 0 & \forall x \in \partial\Omega, \\ u \in C^2\left(\overline{\Omega}\right) \cap L^\infty(\Omega), \end{cases} \tag{B′.9}$$

Here, $\partial_{\nu_A} u := \nu \cdot A \cdot \nabla u$ is the co-normal derivative of u for L: this boundary condition is the natural Neumann condition associated with L. As before, we consider $f \in C^1$ and Ω a C^2 domain. In this section, we will focus on the case of a bounded domain Ω.

We adapt Definition B′.2 to our framework:

Definition B′.4 *Let u be a solution of* (B′.9). *We consider the principal eigenvalue $\lambda_1 := \lambda_1(u, f, \Omega)$ associated with the linearized operator in u:*

$$\forall \psi \in C^2(\overline{\Omega}), \quad \begin{cases} -L\psi - f'(u)\psi = 0 & \text{in } \Omega, \\ \partial_{\nu_A} \psi = 0 & \text{on } \partial\Omega. \end{cases}$$

The solution u is said to be stable *if $\lambda_1 \geq 0$, and* stable non-degenerate *if $\lambda_1 > 0$.*

Non-existence of patterns in bounded domains

Our first result is an extension of the Theorem B′.1 of Casten, Holland and Matano in the case of a non-self-adjoint operator.

Proposition B′.5 *If Ω is bounded and convex, there exists no pattern to* (B′.9)*.*

As we are about to see, this result is in fact a particular case of a more general theorem. We consider λ_1^γ the principal eigenvalue of a *modified* linearized operator:

$$\forall \psi \in C^2(\overline{\Omega}), \quad \begin{cases} -L\psi - f'(u)\psi = 0 & \text{in } \Omega, \\ \partial_{\nu_A}\psi + \gamma\psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{B′.10}$$

where $\gamma(x)$ is the "minimum curvature" of $\partial\Omega$, *i.e.* the smallest eigenvalue of the second fundamental form of $\partial\Omega$ at x. In particular, if $n = 2$, γ is nothing but the curvature of $\partial\Omega$.

Note that this eigenvalue problem is a Robin *indefinite* problem, in the sense that we do not make a sign assumption on γ. This case is not quite standard. The definition and properties of the principal eigenvalue λ_1^γ will be specified later. We give the following theorem:

Theorem B′.6 *Assume that Ω is bounded, and let u be a solution of* (B′.9)*. If $\lambda_1^\gamma \geq 0$, then u is constant.*

Note that the assumptions "u stable" and "Ω convex" of Proposition B′.5 are here combined into the single assumption "$\lambda_1^\gamma \geq 0$". In particular, if Ω is convex, then $\gamma \gneqq 0$, and $\lambda_1 > \lambda_1^\gamma$. Thus, Casten, Holland and Matano's Theorem B′.1 is a special case of the above theorem. The proof of the theorem is quite elementary (perhaps even more than the classical proof of theorem B′.1), and consists in applying the Maximum Principle to $W := \frac{|\nabla u|_A^2}{\varphi^2}$, where $|\cdot|_A$ is the norm induced by A and φ is the *principal eigenfunction* associated with λ_1^γ.

This theorem thus gives a quantitative criterion for the non-existence of patterns. It reduces the problem to the sign of an eigenvalue that involves both the nonlinearity and the geometry of the domain. However, we leave the question of the optimality of this criterion for future work.

Perturbation results

Theorem B′.6 allows to gain flexibility on the hypotheses of the Casten, Holland and Matano's theorem. It is then easy to prove many perturbation results for Theorem B′.1. We can, for example, show that there exists no pattern in some non-convex domains, or show that some unstable solutions are necessarily constant. These questions simply boil down to the study of the sign of the eigenvalue λ_1^γ. We give, as a complement, this series of classical properties.

Proposition B′.7 *Let $\Omega \subset \mathbb{R}^n$ be a C^2 bounded domain, $-L$ as in* (B′.8)*, $c : \Omega \to \mathbb{R}$ and $\gamma : \partial\Omega \to \mathbb{R}$ two continuous functions. We define $\lambda_1^\gamma := \lambda_1^\gamma(-L - c, \Omega)$ as the principal eigenvalue of*

$$\begin{cases} -L\psi - c\psi = \lambda\psi & \textit{in } \Omega, \\ \partial_{\nu_A}\psi + \gamma\psi = 0 & \textit{on } \partial\Omega. \end{cases}$$

Then:

1. *λ_1^γ is continuous with respect to C^2 perturbations of the domain Ω.*
2. *$\lambda_1^\gamma(-L-c)$ is continuous and decreasing with respect to c.*
3. *λ_1^γ is continuous and increasing with respect to γ.*
4. *If $-L$ is self-adjoint, then*

$$\lambda_1^\gamma := \inf_{\substack{\psi \in H^1(\Omega) \\ \|\psi\|_{L^2}=1}} \left[\int_\Omega |\nabla \psi|_A^2 - c\psi^2 + \int_{\partial\Omega} \gamma \psi^2 \right], \tag{B′.11}$$

with $|\cdot|_A^2$ the norm induced by A.
In particular, the mappings $c \mapsto \lambda_1^\gamma(-L-c)$ and $\gamma \mapsto \lambda_1^\gamma$ are concave.

B′.2.2 Generalized principal eigenvalue in unbounded domains

Before considering the problem of the existence of patterns in unbounded domains, we have to extend the definition and properties of the principal eigenvalue. So we make a digression to this question. More details are given in Chapter 3.

Definition

We generally consider a C^2 domain Ω domain and a linear elliptic operator

$$-\mathcal{L}u(x) := -\mathrm{div}\,(A(x)\cdot\nabla u(x)) - B(x)\cdot\nabla u(x) - c(x)\psi(x), \quad \forall x \in \Omega,$$

where, $c : \Omega \to \mathbb{R}$, $B : \Omega \to \in \mathbb{R}^n$, and $A : \Omega \to \mathbb{R}^{n\times n}$ such that $A(x)$ is positive-definite (uniformly in $x \in \Omega$). We assume that the coefficients are bounded and continuous.

We associate the operator $\mathcal{L}$ with Robin boundary conditions

$$\mathcal{B}^\gamma u(x) := \partial_{\nu_A} u(x) + \gamma(x)u(x) = 0, \quad \forall x \in \partial\Omega,$$

with $\gamma(\cdot)$ a bounded and continuous function defined on $\partial\Omega$. The Neumann case corresponds to $\gamma = 0$, and the Dirichlet case formally corresponds to $\gamma = +\infty$.

Classically, $u \in C^2(\overline{\Omega})$ is said to be a sub-solution (or super-solution) if

$$\begin{cases} -\mathcal{L}u \le 0 \text{ (resp. } \ge 0) & \text{in } \Omega, \\ \mathcal{B}^\gamma u \le 0 \text{ (resp. } \ge 0) & \text{on } \partial\Omega. \end{cases}$$

In particular, we are interested in establishing a criterion for the validity of the Maximum Principle, defined as follows:

Definition B′.8 *It is said that $(\mathcal{L}, \mathcal{B}^\gamma)$ satisfies the Maximum Principle if any sub-solution with finite supremum is nonpositive.*

It is well known that, if the domain is bounded, the condition

$$c(\cdot) \le 0 \text{ and } \gamma(\cdot) \ge 0$$

implies that $(\mathcal{L}, \mathcal{B}^\gamma)$ satisfies the Maximum Principle. However, this condition is very restrictive, and we are looking for a more flexible, even optimal criterion. This is a motive for the introduction of the *principal eigenvalue*.

The case of a bounded domain. Let us first assume that the domain Ω is bounded. We consider the following eigenproblem:

$$\begin{cases} -\mathcal{L}\psi = \lambda\psi & \text{in } \Omega, \\ \partial_{\nu_A}\psi + \gamma\psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{B′.12}$$

Note that the Robin boundary conditions are *indefinite*, that is, we do not make any assumption on the sign of γ. The literature considers more the case $\gamma \geq 0$, mostly for technical reasons; however, most of the classical results remain valid in the *indefinite* case. Namely, Daners [109] shows that the indefinite Robin problem can be rewritten (with a rather simple transformation) into a Robin problem with the same structure but with $\tilde{\gamma} \geq 0$.

In particular, the Krein-Rutman theory gives the existence of an eigenvalue λ_1 to (B′.12), called the *principal eigenvalue*. This eigenvalue is real and minimizes the real part of the spectrum (for this reason, it is sometimes called *first eigenvalue*). In addition, λ_1 is simple, and is the only eigenvalue associated with a positive eigenfunction (called *principal eigenfunction*). We let the reader refer to [162, 252] for more details.

A fundamental property is that the validity of the Maximum Principle for the operator $(\mathcal{L}, \mathcal{B}^\gamma)$ is equivalent to the condition $\lambda_1 > 0$.

Extension to unbounded domains. Krein-Rutman's theory cannot be applied if the domain is unbounded. However, the notion of the principal eigenvalue can be extended. Following the approach of [47, 51], we give the following definition:

$$\lambda_1^\gamma := \sup \left\{ \lambda \in \mathbb{R} : (\mathcal{L} + \lambda, \mathcal{B}^\gamma) \text{ admits a positive super-solution} \right\}.$$

This definition coincides with the classical definition in the case of a bounded domain. With this definition, λ_1^γ admits a positive eigenfunction. However, we do not know if λ_1^γ is simple. We will see that the sign of λ_1^γ can be a criterion for the validity of the Maximum Principle.

The Maximum Principle

Self-adjoint operators. We first consider the case of a self-adjoint operator, that is, we assume $B \equiv 0$ in (B′.8). In this case, we can express λ_1^γ through the Rayleigh-Ritz variationnal formula (B′.11). We show that the (strict) sign of λ_1^γ is equivalent to the validity of the Maximum Principle.

Theorem B′.9 *Suppose $\mathcal{L}$ is self-adjoint.*

1. *If $\lambda_1 > 0$, $(\mathcal{L}, \mathcal{B}^\gamma)$ satisfies the Maximum Principle.*
2. *If $\lambda_1 < 0$, $(\mathcal{L}, \mathcal{B}^\gamma)$ does not satisfy the Maximum Principle.*

As we will see, there is no general answer for the degenerate case $\lambda_1^\gamma = 0$. The following theorem establishes the validity of what could be called a *Critical Maximum Principle* in the case $\lambda_1^\gamma \geq 0$, if the domain satisfies a certain growth condition at infinity.

Theorem B′.10 *Suppose that $\mathcal{L}$ is self-adjoint and that the domain Ω satisfies:*

$$\Omega \cap \{|x| \leq R\} = O(R^2) \quad \textit{when } R \to +\infty. \tag{B′.13}$$

Let φ be an eigenfunction associated with λ_1^γ. If $\lambda_1^\gamma \geq 0$, then a sub-solution with finite supremum is either nonpositive or a multiple of φ.

Note that condition (B′.13) echoes condition (B′.6) from the Liouville Theorem. A first consequence of this result is the simplicity of λ_1 if it admits a bounded eigenfunction.

Corollary B′.11 *Under the same conditions, further assume that λ_1 admits a bounded eigenfunction. Then, any eigenfunction associated with λ_1^γ is a multiple of φ, i.e., λ_1 is simple.*

We also give the following corollary.

Corollary B′.12 *Under the same conditions, further assume $\lambda_1^\gamma = 0$, and let φ be an associated eigenfunction. Then, $(\mathcal{L}, \mathcal{B}^\gamma)$ satisfies the Maximum Principle if and only if φ is not bounded.*

We also establish that those results extend to some particular non-self-adjoint operators for which the induced drift is somehow bounded. See Theorem 3.12

Non-self-adjoint operators. In the non-self-adjoint case, we consider another quantity (in the spirit of [51]):

$$\tilde{\lambda}_1^\gamma := \sup\left\{\lambda \in \mathbb{R} : (\mathcal{L} + \lambda, \mathcal{B}^\gamma) \text{ admits a super-solution with positive infimum}\right\}.$$

From their definitions, we have $\lambda_1^\gamma \leq \lambda_1^\gamma$, but we do not know if equality holds. The sign of $\tilde{\lambda}_1^\gamma$ is a sufficient condition for the validity of the Maximum Principle.

Theorem B′.13 *Assume that Ω is uniformly $C^{2,\alpha}$ and satisfies a uniform inner ball condition. If $\tilde{\lambda}_1^\gamma > 0$, then $(\mathcal{L}, \mathcal{B}^\gamma)$ satisfies the Maximum Principle.*

B′.2.3 Non-existence of patterns in unbounded domains

We now discuss the existence of patterns in unbounded domains. We first consider the case of a self-adjoint operator. In a second step, we examine the case of a non-auto-adjoint operator whose drift is, in a way, limited. Finally, we examine the general case.

Self-adjoint case

The first result deals with the *non-degenerate* case $\lambda_1^\gamma > 0$.

Theorem B′.14 *Assume L is self-adjoint, let $\Omega \subset \mathbb{R}^n$ be a possibly unbounded domain and u be a solution of* (B′.9)*. If $\lambda_1^\gamma > 0$, u is constant.*

In particular:

Corollary B′.15 *If L is self-adjoint, there exists is no stable non-degenerate pattern in convex (possibly unbounded) domains.*

In the particular case $L = \Delta$, $\Omega = \mathbb{R}^n$, we recover some known results [104, 160].

Let us now deal with the *possibly degenerate* case $\lambda_1^\gamma \geq 0$.

Theorem B′.16 *Assume L is self-adjoint and Ω satisfies:*

$$\Omega \cap \{|x| \leq R\} = O(R^2). \tag{B′.14}$$

Let u be a solution of (B′.9) *such that $\lambda_1^\gamma \geq 0$.*

1. *If Ω is not a straight cylinder (i.e., Ω is not of the form $\mathbb{R} \times \omega$, $\omega \subset \mathbb{R}^{n-1}$), u is constant.*
2. *If Ω is a straight cylinder, u is either constant or a monotonic flat solution connecting two stable roots (z^-, z^+) of f such that $\int_{z^-}^{z^+} f = 0$.*

In particular, under the assumption that f does not admit stable roots (z^-, z^+) such that $\int_{z^-}^{z^+} f = 0$, this theorem proves the non-existence of patterns in any convex domain of $\mathbb{R}^2$, as well as in some convex domains of $\mathbb{R}^n$, $n > 2$ (like a cylindrical domain whose section's volume grows at most like R). In the particular case $L = \Delta$ and $\Omega = \mathbb{R}^n$, then $\lambda_1^\gamma = \lambda_1$, and we recover the classical results on the variant of De Giorgi's conjecture (see the discussion given in the introduction). The condition (B′.14) echoes (B′.6), and is more than a technical restriction: the above result does not hold in $\mathbb{R}^{2n}$ for $2n \geq 8$, see [246].

Non-self-adjoint case with a bounded drift

We make the following assumption:

$$\sup_{x \in \Omega} \left| B \cdot A^{-1} \cdot x \right| < +\infty. \tag{B′.15}$$

This condition somehow expresses that the drift (induced by the first order term) is neither from nor towards infinity. Under this assumption, we have the same results as in the self-adjoint case, namely,

Theorem B′.17 *Assume* (B′.15) *and let u be a solution of* (B′.9)*.*

— *If $\lambda_1^\gamma > 0$, then u is constant.*

— *Assume $\lambda_1^\gamma \geq 0$, Ω satisfies* (B′.14) *and is not a straight cylinder. Then u is constant.*

— *$\lambda_1^\gamma \geq 0$, Ω satisfies* (B′.14) *and is a straight cylinder. Then u is either constant, or is a flat monotonic solution connecting two stable roots (z^-, z^+) of f such that $\int_{z^-}^{z^+} f = 0$.*

General case

In the general case, we establish the non-existence of non-constant solutions such that $\tilde{\lambda}_1^\gamma > 0$.

Theorem B′.18 *Assume Ω is a (unbounded) uniformly smooth domain with a uniform interior ball condition, and let u be a solution of* (B′.9)*. If $\tilde{\lambda}_1^\gamma > 0$, then u is constant.*

B′.2.4 Other extensions

Asymptotic symmetries

We give another extension of Casten, Holland, and Matano's Theorem B′.1, by proposing an *asymptotic* formulation of the result. We consider a cylindrical domain that is *asymptotically* convex (see Figure B′.6), and we show that any stable solution *converges* to a constant.

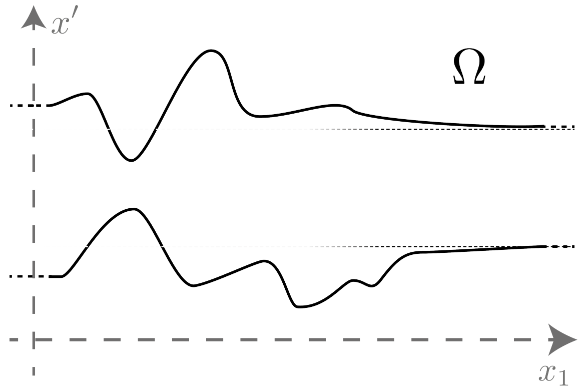


Figure B′.6 – Asymptotically convex cylinder

Let us start with the case of stable non-degenerate solutions. The result is stated informally, a more precise version will be given in section 2.3.

Proposition B′.19 *Let $\Omega \subset \mathbb{R}^N$ be a cylindrical domain (with varying section) on the x_1 axis, which converges to a straight cylinder $\Omega_\infty := \mathbb{R} \times \omega_\infty$ when $x_1 \to +\infty$. Suppose that ω_∞ is convex, and let u be a stable non-degenerate solution of* (B′.1)*. Then, $u(x_1, \cdot)$ converges C^2_{loc} to a stable root of f when $x_1 \to +\infty$.*

Note that we do not assume that ω_∞ is bounded.

Let us now deal with the case of stable (possibly degenerate) solutions.

Proposition B′.20 *Under the same assumptions, but with a solution u only assumed to be stable. Suppose that the stable roots of f, noted (z_i), are isolated, and that $\int_{z_i}^{z_j} f \neq 0$, for $i \neq j$. Let us further assume that Ω satisfies*

$$\Omega \cap \{|x| \leq R\} = O(R^2).$$

Then, $u(x_1, \cdot)$ converges C^2_{loc} to a stable root of f when $x_1 \to +\infty$.

Section 2.3.3 contains other symmetry results, if the limiting domain Ω_∞ has an invariance by translation or planar rotation. In particular, we have the

Corollary B′.21 *Let $\Omega \subset \mathbb{R}^N$ be a cylindrical domain (with a variable section) on the x_1 axis, which converges to a straight cylinder $\Omega_\infty := \mathbb{R} \times \omega_\infty$, $w_\infty \subset \mathbb{R}^{n-1}$, when $x_1 \to +\infty$. Let u be a stable non-degenerate solution of* (B′.1)*. Then, when $x_1 \to +\infty$, $u(x_1, \cdot)$ converges C^2_{loc} to $u_\infty(\cdot)$, which is a stable solution in the section ω_∞:*

$$\begin{cases} -\Delta_{x'} u_\infty = f(u_\infty) & \text{in } \omega_\infty, \\ \partial_\nu u_\infty = 0 & \text{on } \partial\omega_\infty. \end{cases}$$

Flatness of patterns

As a complement, we propose an estimate on the flatness of patterns in general bounded domain. This estimate focuses on the gradient of u in all but one direction, and suggests that patterns tend to be flat. This same idea is expressed in De Giorgi's conjecture and in the results of Gurtin and Matano [169].

To be able to state our result, we introduce the notion of *Spectral Gap*, which expresses the fact that λ_1 is at positive distance from the remaining spectrum of the linearized operator. Let u be a solution of (B′.2). We recall that

$$\lambda_1 := \inf_{\substack{\psi\in H^1\\ \|\psi\|_{L^2}=1}} \mathcal{F}(\psi), \quad \text{with} \quad \mathcal{F}(\psi) := \int_\Omega |\nabla\psi|^2 - f'(u)\psi^2,$$

and that λ_1 admits an eigenfunction φ. Let us consider the orthogonal hyperplane to φ in L^2:

$$E := \mathrm{Vect}(\varphi)^\perp = \left\{\psi \in H^1 : \int_\Omega \psi\varphi = 0\right\},$$

and set

$$\lambda_2 := \inf_{\substack{\psi\in E\\ \|\psi\|_{L^2}=1}} \mathcal{F}(\psi).$$

The property of *Spectral Gap* (see for example [12, 97, 98]) ensures $\lambda_2 - \lambda_1 > 0$. In particular, if u is stable, then $\lambda_2 > 0$. Note that dumbbell domains are domains for which $\lambda_2 - \lambda_1$ is typically very small.

Our estimate on the flatness of patterns is as follows.

Proposition B′.22 *Let Ω be a bounded domain and u a pattern of* (B′.1)*. There exists an orthonormal basis $(e_1, \ldots, e_n)$ such that, for every $i \geq 2$,*

$$\|\partial_i u\|_{L^2}^2 \leq \frac{1}{\lambda_2} I_\gamma(\nabla u), \quad \forall i \in \{2, \cdots, n\},$$

where

$$I_\gamma(\nabla u) := -\int_{\partial\Omega} \gamma|\nabla u|^2$$

and $\gamma(x)$ is as in (B′.10)*.*

If, in addition, u is a stable non-degenerate, then

$$\|\nabla u\|_{L^2}^2 \leq \left(\frac{1}{\lambda_1} + \frac{n-1}{\lambda_2}\right) I_\gamma(\nabla u).$$

Note that $I_\gamma(\nabla u) \leq C\partial\Omega\Omega \sup_{\partial\Omega}\gamma^-$, where C is a bound on $|\nabla u|$ (given, for example, by standard a priori estimates), and γ^- is the negative part of γ. This gives an estimate independent of u. In particular, if the domain is convex, then $\gamma^- = 0$, and u is flat (but, in this case, Theorem B′.1 already implies that u is constant).

Chapter C′

Introduction to part II: Concentration into Dirac mass and models for natural selection

C′.1 Introduction

In this section, we discuss the modelling of Darwinian theory of evolution [112]. The mathematical approach to this problem has recently made considerable progress, and has become a major issue in terms of applications [165, 239, 263]. Here, we follow a PDE method called the *Hamilton-Jacobi approach*, introduced by Diekmann, Jabin, Mischler, Perthame [125].

As an introduction, we consider a simple example, the study of which is elementary. This example allows us to introduce the essential material of our study, interpret the results in terms of modelling, and give an overview of the richness of the Hamilton-Jacobi approach.

We then present the results contained in Part II. In Chapter 4, we focus on the case of an age-structured population. We consider the same model with the addition of a mutation phenomenon in Chapter 5. Finally, we present in Chapter 6 a model for *horizontal gene transfer* that involves an evolutionary rescue phenomenon, which induces a cyclic population dynamic.

C′.1.1 The Hamilton-Jacobi approach on a s simple example

Concentration into a Dirac mass

Let us start by giving an intuitive idea of the approach adopted, using a simple example. Let us consider a population, whose individuals have phenotypic differences. We assume that an individual's phenotype is encoded in a variable $y \in \mathbb{R}^n$; thus, throughout this section, y represents an individual's *phenotype* (also called *trait*).

Note $m(t, y)$ the density of individuals with phenotype y at time t, and $\rho(t) := \int_{\mathbb{R}^n} m(t, \cdot)$ the total population. Suppose that the density of individuals obeys the following dynamics:

$$\begin{cases} \partial_t m(t, y) = (r(y) - \rho(t))m(t, y), \\ m(0, y) = m^0(y) > 0, \end{cases} \tag{C′.1}$$

where $r(\cdot)$ is a continuous function, bounded by two positive constants $\underline{r}$, $\overline{r}$. For simplicity, we assume that r reaches its maximum $\overline{r}$ in a single point $\bar{y}$.

This equation reflects two basic modelling assumptions:

1. Individuals with phenotype y have an intrinsic growth rate $r(y)$, called *fitness* [223]. We can think of r as the birth rate minus the intrinsic death rate. Here, we assumed that there is a single phenotype $\bar{y}$ of optimal fitness $\overline{r}$.
2. The environment has a limited carrying capacity, which leads to competition between individuals for survival. Thus, an additional death rate applies, related to the size of the population: for simplicity we choose $-\rho(t)$.

The model (C′.1) can be seen as a Lotka-Volterra system with a number of species indexed by $y \in \mathbb{R}^n$. Intuition suggests that only individuals with the optimal fitness $\overline{r}$ can eventually survive. Mathematically, this would result in the convergence of $m(t,\cdot)$ to a Dirac mass $\rho_\infty \delta_{y=\bar{y}}$ when $t \to +\infty$. Let us look at this in more details.

The competition term $-\rho(t)$ induces a *saturation effect* on the size of the population. Indeed, by integrating the equation over $y \in \mathbb{R}^n$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) \leq (\overline{r} - \rho(t))\rho(t) \quad ; \quad \frac{\mathrm{d}}{\mathrm{d}t}\rho(t) \geq (\underline{r} - \rho(t))\rho(t).$$

So, as long as $\rho(t=0) \in (\underline{r}, \overline{r})$ (which we assume, for simplicity), we get

$$\underline{r} \leq \rho(t) = \int_{\mathbb{R}^n} m(t,y)dy \leq \overline{r}. \tag{C′.2}$$

This a priori bound on ρ is what we call a *saturation effect* on population size.

Equation (C′.1) can be explicitly solved:

$$m(t,y) = m^0(y) \exp\left(r(y)t - \int_0^t \rho\right),$$

and, after rewriting,

$$m(t,y) = m^0(y) \exp\Big((r(y) - \overline{r})t\Big) \exp\left(\overline{r}t - \int_0^t \rho\right).$$

Let us recall that, for simplicity, we assume that $r(\cdot)$ reaches its maximum $\overline{r}$ on a single $\bar{y} \in \mathbb{R}^n$. Written in this form, and according to (C′.2), we see that, when $t \to +\infty$, on the one hand, $m(t,\cdot) \approx 0$ in $\mathbb{R}^n \setminus \{\bar{y}\}$, and on the other hand, $\overline{r}t \approx \int_0^t \rho$. We can indeed show, in the sense of distributions,

$$m(t,\cdot) \rightharpoonup \overline{r}\delta_{y=\bar{y}}, \quad \text{when } t \to +\infty.$$

This convergence towards a Dirac mass is called a phenomenon of *concentration* in the mathematical literature. From a modelling point of view, it means that only the optimal phenotype $\bar{y}$ eventually survives: it is a model for *natural selection*.

Adaptive dynamics

The concentration result of the previous section gives us information on the state of the population when $t \to +\infty$. However, it does not provide any information on the intermediate states through which the population goes. We are interested in understanding how the previous convergence occurs. For example, if we start from an initial population that looks like a Dirac mass located in $y^0 \neq \bar{y}$, does the population look like a mass of Dirac at any time?

To answer this question, we use a method, introduced in this context by Evans and Souganidis [139], which comes from the theory of homogenization. We introduce a parameter $\varepsilon \ll 1$ into the equation, and do the rescaling $t' \longleftrightarrow \varepsilon^{-1} t$ (*i.e.* a time unzoom). Let us assume $m_\varepsilon(t,y) = m(\varepsilon^{-1}t, y)$, and $\rho_\varepsilon(t) = \rho(\varepsilon^{-1}t) = \int_{\mathbb{R}^n} m_\varepsilon(t,\cdot)$. We therefore consider the following equation

$$\begin{cases} \varepsilon \partial_t m_\varepsilon(t,y) = (r(y) - \rho_\varepsilon(t)) m_\varepsilon(t,y), \\ m_\varepsilon(0,y) = m_\varepsilon^0(y) > 0, \end{cases}$$

and assume that the initial condition resembles a Dirac mass, *i.e.* is a Gaussian of variance ε around y^0:

$$m_\varepsilon^0(y) := \rho^0 \frac{1}{(\varepsilon\pi)^{n/2}} \exp\left(-\frac{|y-y^0|^2}{\varepsilon}\right), \quad \text{with } \rho^0 \in (\underline{r}, \overline{r}).$$

According to the form of the initial condition, it is convenient to make a variable change, called *ansatz WKB* (or *Hopf-Cole transformation*): let us define $u_\varepsilon = \varepsilon \ln(m_\varepsilon)$, so that we have $m_\varepsilon = \exp(\frac{u_\varepsilon}{\varepsilon})$. We have

$$\partial_t u_\varepsilon(t,y) = r(y) - \rho_\varepsilon(t),$$

and we deduce

$$u_\varepsilon(t,y) = u_\varepsilon(0,y) + r(y)t - \int_0^t \rho_\varepsilon.$$

Let us study the asymptotics of u_ε when $\varepsilon \to 0$. First, it should be noted that

$$u_\varepsilon(t=0,y) \underset{\varepsilon\to 0}{\longrightarrow} -|y-y^0|^2.$$

In addition, let us recall that, according to (C′.2), we have $\underline{r} \leq \rho_\varepsilon(t) \leq \overline{r}$: the family $(\rho_\varepsilon)_{\varepsilon>0}$ is bounded to L^∞. The L^∞ space can be seen as the dual of L^1. By the Banach-Alaoglu theorem, there is a sequence $\varepsilon_k \to 0$ such that ρ_{ε_k} converges L^∞-weak-* towards some ρ. We have, for every $t \geq 0$,

$$\int_0^t \rho_{\varepsilon_k} \underset{k\to+\infty}{\longrightarrow} \int_0^t \rho,$$

and we deduce

$$u_{\varepsilon_k} \underset{k\to+\infty}{\longrightarrow} u(t,y) := -|y-y^0|^2 + r(y)t - \int_0^t \rho.$$

Now let us show that u does not depend on the extraction $\varepsilon_k \to 0$: it will prove, *a posteriori*, that u_ε converges towards u. From $\rho_\varepsilon(t) = \int_{\mathbb{R}^n} \exp\left(\frac{u_\varepsilon}{\varepsilon}\right)$, we see that, for every $t \geq 0$,

$$\limsup_{k\to+\infty}\left(\sup_{y\in\mathbb{R}^n} u_{\varepsilon_k}(t,y)\right) > 0 \quad \Longrightarrow \quad \rho_{\varepsilon_k}(t) \underset{k\to+\infty}{\longrightarrow} +\infty,$$

$$\limsup_{k\to+\infty}\left(\sup_{y\in\mathbb{R}^n} u_{\varepsilon_k}(t,y)\right) < 0 \quad \Longrightarrow \quad \rho_{\varepsilon_k}(t) \underset{k\to+\infty}{\longrightarrow} 0.$$

Now, $\underline{r} \leq \rho_\varepsilon(t) \leq \overline{r}$, so

$$\forall t \geq 0, \quad \sup_{y\in\mathbb{R}^n} u(t,y) = 0,$$

that is to say

$$\int_0^t \rho = \sup_{y\in\mathbb{R}^n} \left(-|y-y^0|^2 + r(y)t\right).$$

The integral $\int_0^t \rho$ therefore does not depend on the sub-sequence ε_k. We have thus proven the convergence of u_ε:

$$u_\varepsilon \underset{\varepsilon\to 0}{\longrightarrow} u(t,y) := -|y-y^0|^2 + r(y)t - \int_0^t \rho. \tag{C′.3}$$

From $m_\varepsilon = \exp\left(\frac{u_\varepsilon}{\varepsilon}\right)$, and setting

$$\mathcal{S} := \{(t,y) : u(t,y) = \sup u\} = \Big\{(t,y) : u(t,y) = 0\Big\},$$

we deduce that m_ε converges weakly towards a measure whose support is included in $\mathcal{S}$. This convergence establishes that the population is concentrated where $u(t,\cdot)$ reaches its maximum (which itself is equal to 0).

Assuming formally that $u(t,\cdot)$ is concave, it reaches its maximum at a single point $\bar{y}(t)$, and we have

$$m_\varepsilon(t,y) \underset{\varepsilon\to 0}{\rightharpoonup} \rho(t)\delta_{y=\bar{y}(t)}.$$

By formally differentiating $\nabla_y u(t,\bar{y}(t)) = 0$, we infer the *Canonical Equation*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{y}(t) = -\left[\nabla_y^2 u(t,\bar{y}(t))\right]^{-1} \cdot \nabla_y r(\bar{y}(t)). \tag{C′.4}$$

which describes the dynamics of the concentration point $\bar{y}(t)$.

In particular, $r(\bar{y}(t))$ is a Lyapunov function:

$$\frac{\mathrm{d}}{\mathrm{d}t} r(\bar{y}(t)) = -\nabla_y r(\bar{y}(t)) \cdot \left[\nabla_y^2 u(t,\bar{y}(t))\right]^{-1} \cdot \nabla_y r(\bar{y}(t)) \geq 0.$$

Thus, $y(t)$ evolves according to the increasing slope of $r(\cdot)$, and converges to $\bar{y}$.

Addition of mutations

Let us now add one last ingredient to our model: the phenomenon of *mutation*. We assume that an individual of phenotype y can give birth to an individual of phenotype y' with probability $M(y'-y)$, where M is for example a Gaussian kernel centered in 0. By noting respectively $b(y)$ and $d(y)$ the birth rate and the death rate (so that $r(y) = b(y) - d(y)$), we consider the equation

$$\begin{cases} \partial_t m(t,y) = \displaystyle\int_{\mathbb{R}^n} M(y'-y) b(y') m(t,y') dy' - (d(y) + \rho(t)) m(t,y), \\ \rho(t) = \displaystyle\int_{\mathbb{R}^n} m(t,y) dy, \\ m(0,y) = m^0(y) > 0. \end{cases}$$

This model features a non-local diffusion phenomenon in y. Our aim is to obtain similar results to the case without mutation, in particular, the concentration towards a Dirac mass.

As before, we introduce a parameter ε and make a rescaling. However, it should be noted that the diffusion term has the effect of *smoothing* the solution, and thus prevents concentration towards a Dirac mass. The rescaling in t must then go along with a rescaling in y: we choose a hyperbolic rescaling, and set $m_\varepsilon(t,y) = m(\varepsilon^{-1}t, \varepsilon^{-1}y)$, $\rho_\varepsilon(t) = \rho(\varepsilon^{-1}t) = \int_{\mathbb{R}^n} m_\varepsilon(t,\cdot)$. The equation becomes

$$\varepsilon\partial_t m_\varepsilon(t,y) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} M\left(\frac{y'-y}{\varepsilon}\right) b(y')m_\varepsilon(t,y')dy' - (d(y)+\rho_\varepsilon(t))m_\varepsilon(t,y),$$

and, with a change of variable $z := \frac{y'-y}{\varepsilon}$,

$$\varepsilon\partial_t m_\varepsilon(t,y) = \int_{\mathbb{R}^n} M(z)b(y+\varepsilon z)m_\varepsilon(t,y+\varepsilon z)dz - (d(y)+\rho_\varepsilon(t))m_\varepsilon(t,y).$$

Let us perform the WKB *ansatz*, that is, set $u_\varepsilon = \varepsilon\ln(m_\varepsilon)$. We obtain

$$\partial_t u_\varepsilon(t,y) = \int_{\mathbb{R}^n} M(z)b(y+\varepsilon z)\exp\left(\frac{u_\varepsilon(t,y+\varepsilon z)-u_\varepsilon(t,y)}{\varepsilon}\right) - d(y) - \rho_\varepsilon(t).$$

By formally passing to the limit $\varepsilon\to 0$, we obtain

$$\begin{cases} \partial_t u(t,y) = b(y)\int_{\mathbb{R}^n} M(z)\exp\left(\nabla_y u(t,y)\cdot z\right) - d(y) - \rho(t), \\ u(0,y) = -|y-y^0|^2. \end{cases} \tag{C′.5}$$

It is a Hamilton-Jacobi equation.

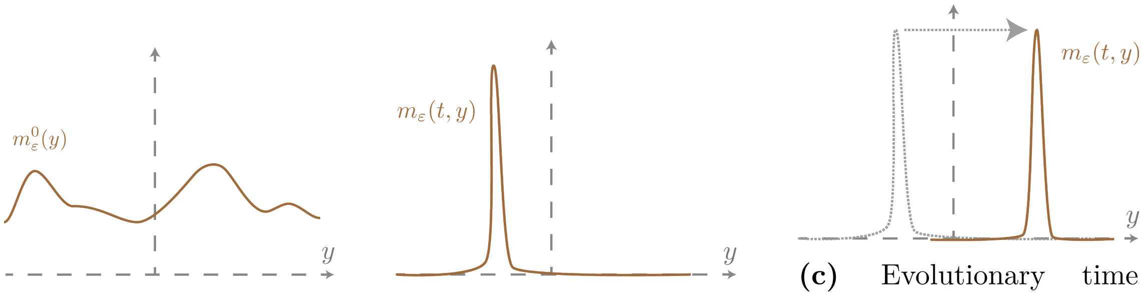
This equation is the central point of the analysis of the model. For this reason, the method we presented is often called the *Hamilton-Jacobi approach.* Similar results can be proved in the case without mutations. In particular, we can show $\sup_{\mathbb{R}^n} u(t,\cdot) = 0$, and that the population is concentrated where $u(t,y)=0$. At least formally, we can also recover the Canonical Equation (C′.4).

However, adding the mutation term can result in significant mathematical difficulties. First, it should be noted that a classical solution of (C′.5) is *a priori* unlikely to be defined globally in time. The natural (and very convenient) concept in this context is that of viscosity solutions, introduced by Crandall and Lions in the 1980s (see the reference books [30, 31, 102, 153]).

The tricky point is to prove the convergence of u_ε towards some u. To do this, the strategy consists in passing to the limit in the equation on u_ε in the sense of viscosities (which is a rather weak sense), then using a uniqueness result on the limit equation, which proves, *a posteriori*, the locally uniform convergence of u_ε to u.

C′.1.2 Interpretation

We see in this simple example the richness of the approach. Our model describes population dynamics on two time scales. The ecological time scale t corresponds to the characteristic time it takes for a monomorphic population (*i.e.*, with only one $y\in\mathbb{R}^n$ set) to reach equilibrium. At this scale, we observe the concentration of the population around a dominant phenotype $\bar{y}(t)$: this is the phenomenon of *natural selection*. On the evolutionary time scale $\varepsilon^{-1}t$, the population has, at any given time, a monomorphic structure around the phenotype $\bar{y}(t)$, which itself evolves according to the Canonical Equation (C′.4): this is a description of *adaptive dynamics*. Loosely speaking, after a relatively short time, the population looks like a Dirac mass (Figures C′.1a-C′.1b), then, on a longer time scale, this Dirac mass moves towards the optimal phenotype (Figure C′.1c).

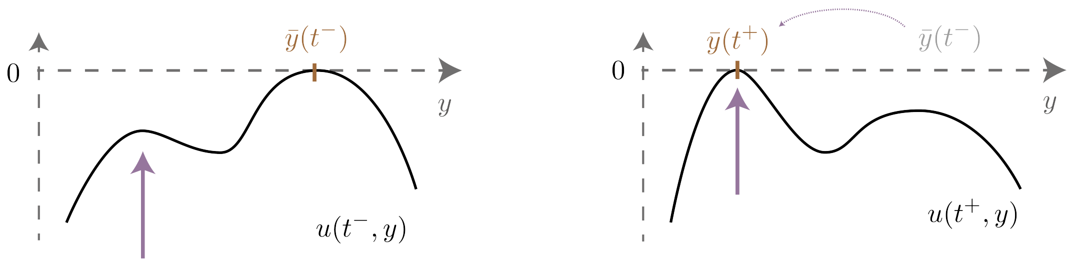


(a) Initial time **(b)** Ecological time scale **(c)** Evolutionary time scale

Figure C′.1 – Evolution of the population $m_\varepsilon(t,y)$

It should be noted, however, that the validity of the Canonical Equation is based on the formal assumption that $u(t,\cdot)$ is concave. In the case without mutation, we see from (C′.3) that, in long time, the concavity of u depends only on that of the fitness $r(\cdot)$. Assuming that fitness is concave means that there is only one optimal phenotype. This hypothesis is quite strong in terms of modelling. In the case with mutations, we see through equation (C′.5) that the nonlinearity itself can generate a loss of concavity of u.

In general, the dynamics of $\bar{y}(t)$ are not expected to follow (C′.4) for all times. Typically, the dynamics of $\bar{y}(t)$ can feature jumps, corresponding to the arising of a new global maximum for u, see Figure C′.2a-C′.2b.



(a) $t = t^-$ **(b)** $t = t^+$

Figure C′.2 – Jump of $\bar{y}(t)$ when a new maximum arises

In addition, there may be a time t_0 for which $\bar{y}(t_0)$ is a degenerate maximum (*i.e.* $\nabla_y^2 u(t,\bar{y}(t)) = 0$). These degenerate points can give rise to *branching* phenomena, after which a monomorphic population becomes polymorphic. For a more detailed analysis of the Canonical Equation, we let the reader refer to [18, 119, 220, 222].

C′.1.3 Brief state of the art

Kimura [192] was the first to model phenotypes by a continuous variable $y \in \mathbb{R}^n$. The Hamilton-Jacobi approach, as described above, was introduced by Diekmann, Jabin, Mischler and Perthame [125]. It was then developed [33,36,213] and adapted in different situations, for example: space models [155, 183, 202], in particular to analyze invasion fronts [8,44,60,63,63,81,277], models for gendered populations [231, 258], or models with several resources [93–95, 230]. We also mention [79, 80], as well as [81, 87] which present atypical concentration results. An introduction to the subject is provided in Mirrahimi's [227] and Taing's [274] thesis.

We insist on the fact that the probabilistic point of view is very fecund in this field, and all the models we describe can be deduced from individual-based stochastic models [89–91, 119, 219]. The modelling of Darwinian selection is also addressed in the context of game theory [174, 178] and dynamical systems [118, 124, 159].

C′.2 Presentation of the results

C′.2.1 Age-structured populations

In collaboration with B. Perthame et C. Taing. We present here the content of Chapter 4.

We study the model of a population structured in *phenotype* and *age*. Let $m_\varepsilon(t, x, y)$ be a population density, depending on time $t \geq 0$, age $x \geq 0$, and phenotype $y \in \mathbb{R}^n$, satisfying

$$\begin{cases} \varepsilon\partial_t m_\varepsilon + \partial_x \left[A(x, y)m_\varepsilon\right] + (\rho_\varepsilon(t) + d(x, y))\, m_\varepsilon = 0, \\ A(x = 0, y)m_\varepsilon\,(t, x = 0, y) = \displaystyle\int_{\mathbb{R}_+} b(x', y)m_\varepsilon(t, x', y)dx'dy, \\ \rho_\varepsilon(t) = \displaystyle\int_{\mathbb{R}_+}\int_{\mathbb{R}^n} m_\varepsilon(t, x, y)dxdy, \\ m_\varepsilon(t = 0, x, y) = m^0_\varepsilon(x, y) > 0. \end{cases} \tag{C′.6}$$

As in the introduction, $\varepsilon > 0$ is a rescaling parameter; the quantity $\rho_\varepsilon(t)$ represents the size of the population at time t; the mortality rate contains an intrinsic term $d(x, y)$ and a competition term $\rho(t)$. The boundary condition means that a father individual gives birth, at rate $b(x, y)$, to an individual with age $x = 0$. It should be noted that this model does not feature mutations. The model with mutations will be presented in the next section.

The transport term means that individuals "age" at a speed $A(x, y)$. For more generality, we let this parameter depend on x, which means that the transport may not occur at a constant speed. This way, the variable x can represent age as well as any other quantity that increases over an individual's lifetime and is not transmitted to offspring. We could think, for example, of size, maturation, viral load, etc.

This type of model for age-structured populations is often referred to in the literature as the *renewal equation*. It is related to the McKendrick-von Foerster equation (see [247]). It is studied in various contexts: size structured populations [221, 233], models for cell division [131, 224], tumor growth [3, 170], and many others. Age and phenotype structured models are also considered in [82] and [83].

Formal approach

We recall that the *WKB ansatz* consists in the change of variable:

$$m_\varepsilon(t,x,y) = e^{\frac{v_\varepsilon(t,x,y)}{\varepsilon}}.$$

We propose a variant. The principle is to do a Taylor expansion $v_\varepsilon(t,x,y) = v_\varepsilon^1(t,y) + \varepsilon v_\varepsilon^2(t,x,y)$, define v_ε^1 as an ansatz, then estimate the error term v_ε^2. After rewriting, we proceed to the following change of variable:

$$m_\varepsilon(t,x,y) = p_\varepsilon(t,x,y)e^{\frac{u_\varepsilon(t,y)}{\varepsilon}}. \tag{C′.7}$$

Here, $u_\varepsilon(t,y)$ will be our ansatz, which we will define *ad hoc* as the solution to an equation. The advantage of this variant is that the error term p_ε that must be estimated satisfies a linear equation, rather than a Hamilton-Jacobi equation.

We now need to find a good candidate for our ansatz u_ε. By injecting (C′.7) into equation (C′.6), we obtain

$$\begin{cases} \varepsilon\partial_t p_\varepsilon + \partial_x\left[A(x,y)p_\varepsilon\right] + d(x,y)p_\varepsilon + \left(\partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t)\right)p_\varepsilon = 0, \\ A(0,y)p_\varepsilon\left(t,0,y\right) = \int_{\mathbb{R}_+} b(x,y)p_\varepsilon(t,x,y)dxdy. \end{cases} \tag{C′.8}$$

By formally setting $\varepsilon\partial_t p_\varepsilon \approx 0$, we consider the eigenproblem: for fixed $y \in \mathbb{R}^n$, let $(\Lambda(y), Q(x,y))$ be the unique solution of

$$\begin{cases} \partial_x\left[A(x,y)Q\right] + d(x,y)Q - \Lambda Q = 0, \\ A(0,y)Q(0,y) = \int_{\mathbb{R}_+} b(x,y)Q(x,y)dx, \\ Q(x,y) > 0, \quad \int_{\mathbb{R}_+} b(x,y)Q(x,y)dx = 1. \end{cases}$$

Intuitively, $\Lambda(y)$ corresponds to the effective fitness, and $Q(x,y)$ to the age profile at equilibrium.

Equation (C′.8) then suggests to define u_ε as the solution of:

$$\begin{cases} \partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t) = -\Lambda\left(y\right), \\ u_\varepsilon(t=0,y) = u_\varepsilon^0(y), \end{cases}$$

for some initial conditions u_ε^0. So, p_ε satisfies

$$\begin{cases} \varepsilon\partial_t p_\varepsilon + \partial_x\left[A(x,y)p_\varepsilon\right] - \Lambda(y)p_\varepsilon = 0, \\ A(0,y)p_\varepsilon\left(t,0,y\right) = \int_{\mathbb{R}_+} b(x,y)p_\varepsilon(t,x,y)dxdy, \end{cases}$$

and should therefore behave as a multiple of $Q(x,y)$ when $\varepsilon \to 0$.

Main results

Under general assumptions on the parameters, and if the initial condition is well prepared, we prove the following theorem, illustrated in Figures C′.3a-C′.3b-C′.4-C′.5.

Theorem C′.1 *When $\varepsilon \to 0$:*

1. $\rho_\varepsilon(t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}_+} m_\varepsilon(t,x,y)dxdy$ *converges L^∞-weak-$\star$ to some ρ.*

2. p_ε *converges to a multiple of* Q*, for a weighted* L^1 *norm.*

3. u_ε *converges locally uniformly to some* u*, solution of*

$$\begin{cases} \partial_t u(t,y) = -\Lambda(y) - \rho(t), & t > 0, y \in \mathbb{R}^n, \\ \sup\limits_{y \in \mathbb{R}^n} u(t,y) = 0, & \forall t > 0, \\ u(0,y) = u^0(y), & y \in \mathbb{R}^n. \end{cases}$$

4. m_ε *weakly converges towards a measure* μ*, whose support is included in* $\mathcal{S} = \{(t,y) \in (0,\infty) \times \mathbb{R}^n | u(t,y) = 0\}$*.*

5. Moreover, if we assume that u^0 *and* $-\Lambda$ *are strictly concave, then*

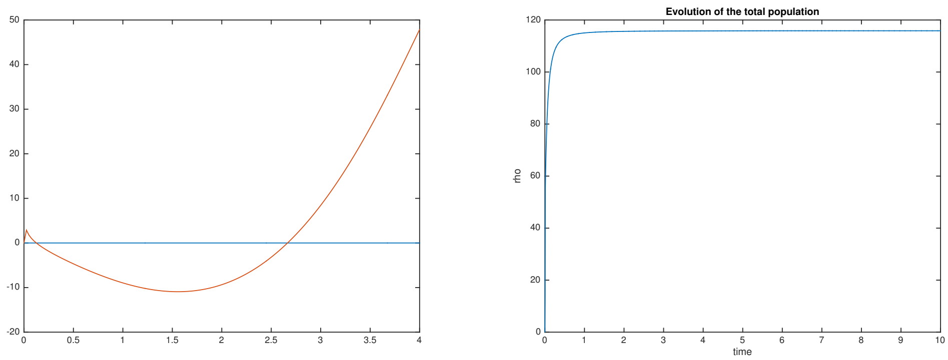
$$m_\varepsilon(t,x,y) \underset{\varepsilon \to 0}{\rightharpoonup} \rho(t) \frac{Q(x,y)}{\|Q(\cdot,y)\|_{L^1}} \delta_{y=\bar{y}(t)},$$

where $\bar{y}(t) \in \mathbb{R}^n$ *satisfies the Canonical Equation:*

$$\dot{\bar{y}}(t) = \left(\nabla_y^2 u(t, \bar{y}(t)) \right)^{-1} \cdot \nabla_y \Lambda \left(\bar{y}(t) \right).$$

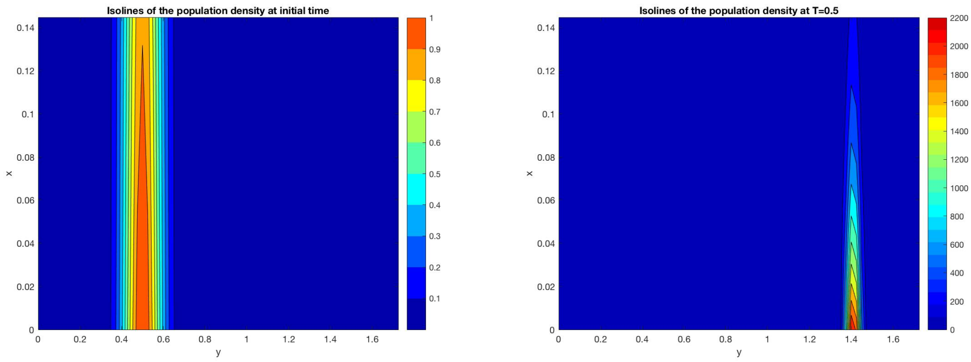
In this case, $\rho'(t) \geq 0$*, and*

$$u^0(\bar{y}(t)) = \int_0^t \rho(t')dt' - t\rho(t).$$



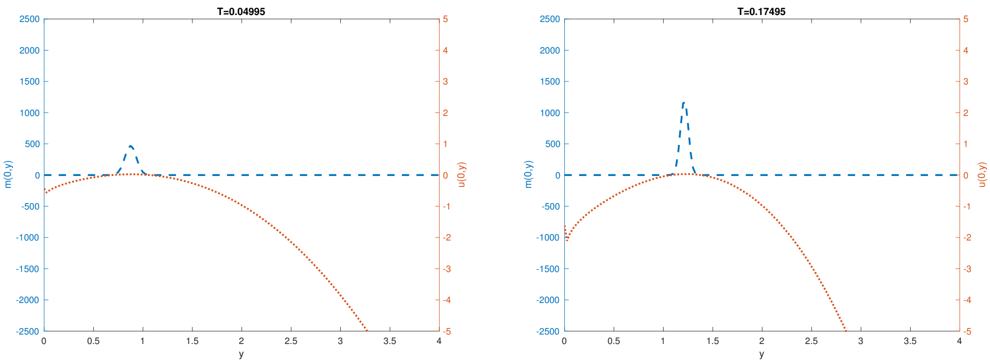
(a) Effective fitness $\Lambda(y)$ **(b)** Evolution of $\rho(t)$

Figure C′.3

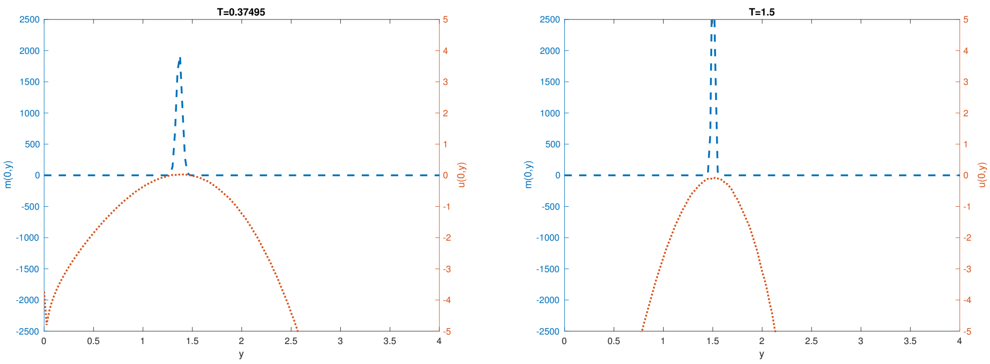


(a) $t = 0$ **(b)** $t = 0.5$

Figure C′.4 – Level lines of m_ε with respect to (x, y).



(a) $t = 0$ **(b)** $t = 0.2$



(c) $t = 0.4$ **(d)** $t = 1.5$

Figure C′.5 – Snapshots of the population distribution. Blue dotted line= m_ε. Red dotted line= u_ε.

C′.2.2 Adding a mutation phenomenon

In collaboration with B. Perthame. We introduce the content of Chapter 5.

We add a mutation term to the previous model. We consider the equation

$$\begin{cases} \varepsilon\partial_t m_\varepsilon + \partial_x\left[A(x,y)m_\varepsilon\right] + \left(\rho_\varepsilon(t) + d(x,y)\right)m_\varepsilon = 0, \\ A(x=0,y)m_\varepsilon\left(t,x=0,y\right) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} M(\frac{y'-y}{\varepsilon})b(x',y')m_\varepsilon(t,x',y')dx'dy', \\ \rho_\varepsilon(t) = \int_{\mathbb{R}_+}\int_{\mathbb{R}^n} m_\varepsilon(t,x,y)dxdy, \\ m_\varepsilon(t=0,x,y) = m_\varepsilon^0(x,y) > 0, \end{cases}$$

where $M(\cdot)$ is a probability kernel, for example Gaussian. The mutation term adds a significant difficulty. Indeed, the presence of (non-local) diffusion makes it difficult to identify a formal limiting problem. Instead, we introduce an *approximate* problem (at the expanse of an additional variable η), which is well suited to our framework; this is the main finding of our approach.

Formal approach

A variant of the WKB ansatz. As described in the previous section, our strategy is to make the change of variable

$$m_\varepsilon(t,x,y) = p_\varepsilon(t,x,y)e^{\frac{u_\varepsilon(t,y)}{\varepsilon}},$$

where u_ε is an ansatz, and p_ε the error term to estimate. By injecting this form into the equation, we obtain

$$\begin{cases} \varepsilon\partial_t p_\varepsilon + \partial_x\left[A(x,y)p_\varepsilon\right] + d(x,y)p_\varepsilon + \left(\partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t)\right)p_\varepsilon = 0, \\ A(0,y)p_\varepsilon\left(t,0,y\right) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} M(\frac{y'-y}{\varepsilon})e^{\frac{u_\varepsilon(t,y')-u_\varepsilon(t,y)}{\varepsilon}}b(x',y')p_\varepsilon(t,x',y')dx'dy'. \end{cases} \tag{C′.9}$$

After a change of variable $z = \frac{y'-y}{\varepsilon}$, the boundary term is written

$$A(0,y)p_\varepsilon(t,0,y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}_+} M(z)e^{\frac{u_\varepsilon(t,y+\varepsilon z)-u_\varepsilon(t,y)}{\varepsilon}}b(x',y+\varepsilon z)p_\varepsilon(t,x',y+\varepsilon z)dx'dz.$$

When ε is small, we can formally approximate

$$A(0,y)p(t,0,y) \approx \eta_\varepsilon(t,y)\int_{\mathbb{R}_+} b(x',y)p(t,x',y)dx', \tag{C′.10}$$

where

$$\eta_\varepsilon(t,y) := \int_{\mathbb{R}^n} M(z)e^{\frac{u_\varepsilon(t,y+\varepsilon z)-u_\varepsilon(t,y)}{\varepsilon}}dz.$$

So, by formally putting $\varepsilon\partial_t p_\varepsilon \approx 0$ in the first line of (C′.9), we end up with the *approximate* problem:

$$\begin{cases} \partial_x\left[A(x,y)p_\varepsilon(t,x,y)\right] + d(x,y)p_\varepsilon(t,x,y) + \left(\partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t)\right)p_\varepsilon(t,x,y) \approx 0, \\ A(0,y)p_\varepsilon\left(t,0,y\right) \approx \eta_\varepsilon(t,y)\int_{\mathbb{R}_+} b(x',y)p(t,x',y)dx'. \end{cases}$$

Considering $\eta_\varepsilon(t,y)$ as a parameter, we consider the following eigenproblem: for fixed $(y,\eta) \in \mathbb{R}^n \times (0,+\infty)$, let $(\Lambda(y,\eta), Q(x,y,\eta))$ be the unique solution of

$$\begin{cases} \partial_x\left[A(x,y)Q\right] + d(x,y)Q - \Lambda(y,\eta)Q = 0, \quad \forall x > 0, \\ A(0,y)Q(0,y,\eta) = \eta\int_{\mathbb{R}_+} b(x,y)Q(x,y,\eta)dx, \\ Q(x,y,\eta) > 0, \quad \int_{\mathbb{R}_+} b(x,y)Q(x,y,\eta)dx = 1. \end{cases}$$

Defining the ansatz. These formal calculations suggest to define our ansatz u_ε as a solution of

$$\begin{cases} \partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t) = -\Lambda\left(y, \eta_\varepsilon(t,y)\right), \\ \eta_\varepsilon(t,y) = \int_{\mathbb{R}^n} M(z) e^{\frac{u_\varepsilon(t,y+\varepsilon z) - u_\varepsilon(t,y)}{\varepsilon}} dz, \\ u_\varepsilon(t=0,y) = u_\varepsilon^0(y), \end{cases} \tag{C′.11}$$

for some initial conditions u_ε^0. So, setting $Q_\varepsilon(t,x,y) := Q(x,y,\eta_\varepsilon(t,y))$ and $\Lambda_\varepsilon(t,y) := \Lambda(y,\eta_\varepsilon(t,y))$, we have

$$\begin{cases} \varepsilon\partial_t Q_\varepsilon + \partial_x\left[A(x,y)Q_\varepsilon\right] + d(x,y)Q_\varepsilon + \left(\partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t)\right) Q_\varepsilon = \varepsilon\partial_t Q_\varepsilon, \\ A(0,y)Q_\varepsilon\left(t,0,y\right) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} M(\frac{y'-y}{\varepsilon}) e^{\frac{u_\varepsilon(t,y') - u_\varepsilon(t,y)}{\varepsilon}} b(x',y')Q_\varepsilon(t,x',y')dx'dy'. \end{cases}$$

Note that the boundary term is obtained from the normalization

$$\int_{\mathbb{R}_+} b(x,y)Q(x,y,\eta)dx = 1.$$

The above equation suggests that p_ε should behaves as Q_ε when $\varepsilon \to 0$. Except for the right-hand member $\varepsilon\partial_t Q_\varepsilon$, we see that Q_ε and p_ε are solutions of the same equation (C′.9), which is linear and admits a comparison principle. If we prove good estimates on $\varepsilon\partial_t Q_t Q_\varepsilon$, we infer that p_ε is bounded from above and below by multiples of Q_ε (which is uniformly $L^1 \cap L^\infty$). This justifies our approach, in particular the formal approximation (C′.10).

Analysis of the Hamilton-Jacobi equation From the above, the cornerstone of our method is to prove good estimates on $\partial_t Q_\varepsilon$. This requires an in-depth analysis of our ansatz u_ε, defined by the equation (C′.11). Note that this equation is not autonomous, because of the term $\rho_\varepsilon(t)$. We bypass this difficulty by setting

$$U_\varepsilon(t,y) = u_\varepsilon(t,y) + \int_0^t \rho_\varepsilon(s)ds.$$

So,

$$\eta_\varepsilon(t,y) = \int_{\mathbb{R}^n} M(z) e^{\frac{U_\varepsilon(t,y+\varepsilon z) - U_\varepsilon(t,y)}{\varepsilon}} dz$$

and

$$\begin{cases} \partial_t U_\varepsilon(t,y) = -\Lambda\left(y, \int_{\mathbb{R}^n} M(z) e^{\frac{U_\varepsilon(t,y+\varepsilon z) - U_\varepsilon(t,y)}{\varepsilon}} dz\right) & \forall t \geq 0,\ \forall y \in \mathbb{R}^n, \\ U_\varepsilon(0,y) = u_\varepsilon^0(y) & \forall y \in \mathbb{R}^n. \end{cases}$$

Note that U is then defined in an autonomous way.

From this equation, we prove, *a priori*, that $\partial_t U_\varepsilon$ and $\nabla_y U_\varepsilon$ are bounded, uniformly in $\varepsilon > 0$. Then, we use the fact that the Hamiltonian is somehow convex to prove that U_ε is semi-concave, from which we deduce a ε-uniform $W^{2,1}_{loc}$ estimate.

When $\varepsilon \to 0$, U_ε converges locally uniformly towards U, which is a viscosity solution of the Hamilton-Jacobi equation

$$\begin{cases} \partial_t U(t,y) = H(y, \nabla_y U) & \forall t \geq 0,\ \forall y \in \mathbb{R}^n, \\ U(0,y) = u^0(y) & \forall y \in \mathbb{R}^n, \end{cases} \tag{C′.12}$$

with a Hamiltonien

$$H(y,p) := -\Lambda\left(y, \int_{\mathbb{R}^n} M(z)e^{p\cdot z}dz\right).$$

We prove, *a posteriori*, that $\sup u(t,\cdot) = 0$, and therefore $\int_0^t \rho = \sup U(t,\cdot)$. Thus, ρ can be seen as a Lagrange multiplier associated with the constraint $\sup u(t,\cdot) = 0$.

Main results

The following theorem sums up our main results.

Theorem C′.2 *Under general assumptions on the parameters, and if the initial condition is well prepared, we have, when $\varepsilon \to 0$:*

1. *The population m_ε vanishes locally uniformly outside the set*

$$\mathcal{S} := \left\{ t \geq 0,\ y \in \mathbb{R}^n : U(t,y) = \sup_{y' \in \mathbb{R}^n} U(t,y') \right\},$$

   where $U \in W^{1,1}_{loc}$ is globally Lipschitz, semiconvex, and is a viscosity solution of (C′.12).

2. *ρ_ε converges L^∞-weak-* to $\rho \in L^\infty$ and*

$$\forall t > 0, \quad \int_0^t \rho = \sup_{y \in \mathbb{R}^n} U(t,y).$$

3. *For a short time interval $t \in [0,T]$, we have $\mathcal{S} = \{(t, \bar{y}(t))\}$, where $\bar{y}(t)$ satisfies the Canonical Equation:*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\bar{y}(t) = \left(\nabla_y^2 U(t,\bar{y}(t))\right)^{-1} \cdot \nabla_y \Lambda(\bar{y}(t),1) + \partial_\eta \Lambda(\bar{y}(t),1) \displaystyle\int_{\mathbb{R}^n} M(z)zdz, \\ \bar{y}(0) = \bar{y}^0. \end{cases}$$

The main restriction on the parameters is that we assume either that $\frac{1}{A(\cdot,y)} \in L^1$, or that $b(\cdot,y)$ has a compact support. The other assumptions are formulated directly on the limiting eigenproblem, and are quite general. Note that, unlike the case without mutations, we do not prove the strong convergence of p_ε.

C′.2.3 Horizontal Gene Transfer

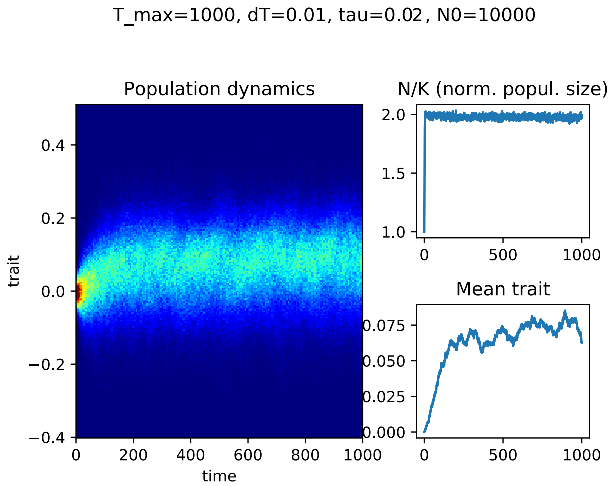
In collaboration with V. Calvez, S. Figueroa Iglesias, H. Hivert, S. Méléard, et A. Melnykova. We introduce here the content of Chapter 6.

A model for evolutionary rescue

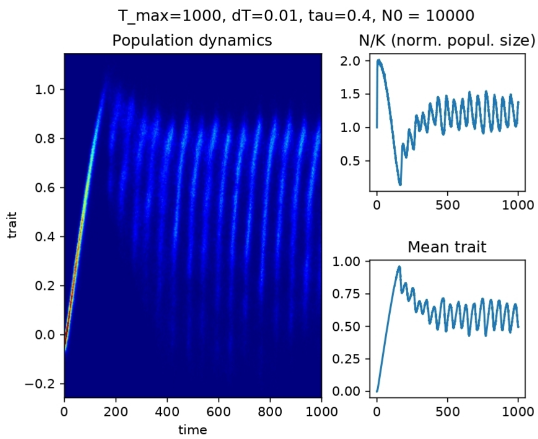
The Horizontal Gene Transfer (HT) is the transfer of genetic material between two living organisms, as opposed to the usual vertical transmission from parents to offspring. This phenomenon plays an important role in the evolution of certain bacteria, particularly with regard to antibiotic resistance. A phenomenon of horizontal transmission also occurs in the transfer of plasmids and symbionts [208, 215].

A stochastic model is proposed in [56, 156], to account for the dynamics of a population subject to HT. Numerical simulations reveal cyclic dynamics. Intuitively, while HT leads the population to a deleterious phenotype, it can happen that the few individuals not affected by HT end up repopulating the environment, before being led again to deleterious phenotypes.

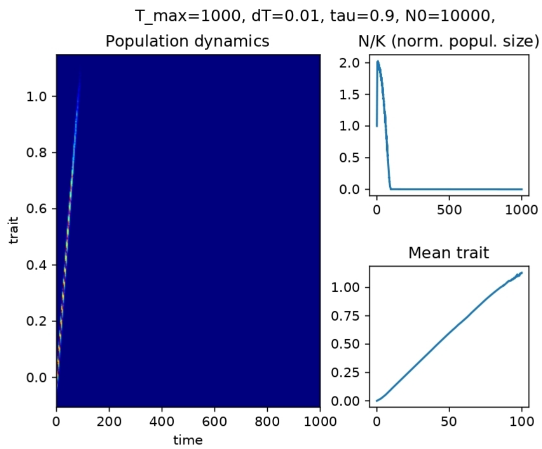
More precisely, depending on the intensity of the HT, the dynamics can follow three regimes: stabilization, cycles, and extinction. These three regimes are illustrated in Figure C′.6.



(a) Stabilization: $\tau_0 = 0.02$



(b) Cycles: $\tau_0 = 0.4$



(c) Extinction: $\tau_0 = 0.9$

Figure C′.6 – Evolution of the population, depending on τ_0 the intensity of HT. The horizontal axis corresponds to time, the vertical axis to the phenotype $y \in \mathbb{R}$.

The observed cyclic dynamics illustrate a phenomenon called *evolutionary rescue*. Understanding this phenomenon is a major research challenge, and proves to be difficult from a mathematical point of view [165, 239]. As we saw in the previous section, the Hamilton-Jacobi approach is effective in describing concentration phenomena, which occur in large populations. Evolutionary rescue relies on small populations, and it is interesting to understand to what extent the Hamilton-Jacobi approach is able to capture this phenomenon. The question is to understand whether the "limiting procedures" performed from the stochastic model to the Hamilton-Jacobi equation do not lose too much information on small populations.

We propose to study this question through the example of the HT. We present numerical simulations and formal theoretical analysis to compare the stochastic model with the corresponding Hamilton-Jacobi model.

The Hamilton-Jacobi equation

From the stochastic model, we end up with the following Hamilton-Jacobi equation:

$$\partial_t u(t,y) = -(d(y)+\rho(t)) + b(y)\int_{\mathbb{R}} M(z)\exp\left(\nabla_y u(t,y)\cdot z\right)dz + \tau(y-\overline{y}(t)), \quad (\text{C}'.13)$$

where

$$\bar{y}(t) = \text{argmax } u(t,\cdot).$$

We use the same formalism as in equation (C′.5). We assume $n=1$ (so y is a scalar variable). As in the previous sections, $d(y)$ represents the death rate associated with the phenotype $y \in \mathbb{R}^n$, $b(y)$ the birth rate, $M(\cdot)$ the kernel of mutations. The function $\rho(t)$ can be seen as the Lagrange multiplier for the constraint $\sup_{\mathbb{R}^n} u(t,\cdot) = 0$. The term $\tau(y-\overline{y}(t))$ stands for the effect of *HT*, and makes the study of the equation non-standard.

To fix ideas, we choose $d(y) = y^2$, $b(y) = 1$, so that the fitness $r(y) = 1 - y^2$ is concave and optimal at 0. The mutation kernel is assumed to be a Gaussian of variance 1 centered in 0. The *HT* term is assumed to be of the form

$$\tau(y) = \tau_0 \tanh\left(\frac{y}{\delta}\right),$$

where the parameter $\tau_0 > 0$ corresponds to the intensity of *HT* (we consider it as a bifurcation parameter), and δ sets the stiffness of the transition between the values $\pm\tau_0$. Thus, the effect of *HT* is to "push" individuals towards $y > 0$, and thus to lead the population towards deleterious phenotypes.

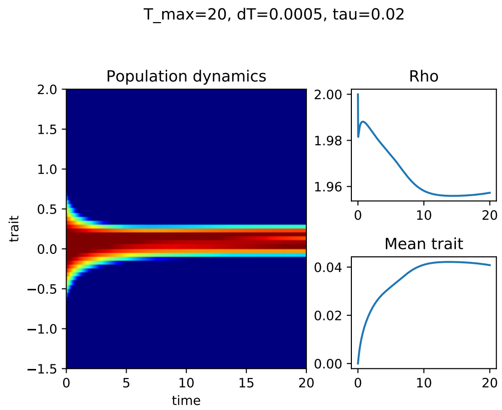
Results

Numerical simulations To numerically simulate equations such as (C′.13) is not straightforward, especially because of the term $\exp(\nabla_y u_\varepsilon \cdot z)$ and the *HT* term which is non-local. For this purpose, we propose an *Asymptotic Preserving* numerical scheme [187, 195, 196].

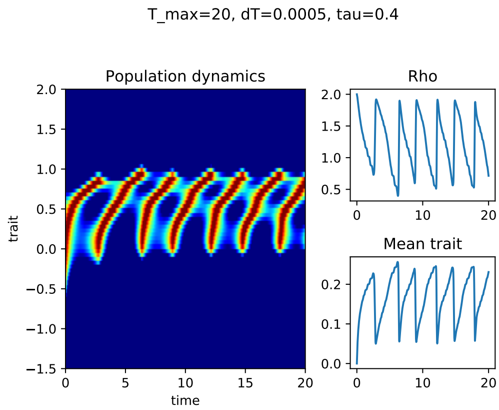
The experiments we conduct (Figure C′.7) show that the Hamilton-Jacobi equation (C′.13) is able to qualitatively reproduce the dynamics observed in the stochastic model.

Formal theoretical analysis. A formal analysis of the Hamilton Jacobi equation provides an accurate description of the dynamics. In particular, it is possible to predict the dynamic regime (stabilization, cycles, extinction) depending on the value of τ_0. For example, the dynamics presents cycles if and only if $\tau_0 > \tau_{cyc} := 4\delta^2$; goes through episodes of extinction if and only if $\tau > \tau_{ext} := 2\delta$.

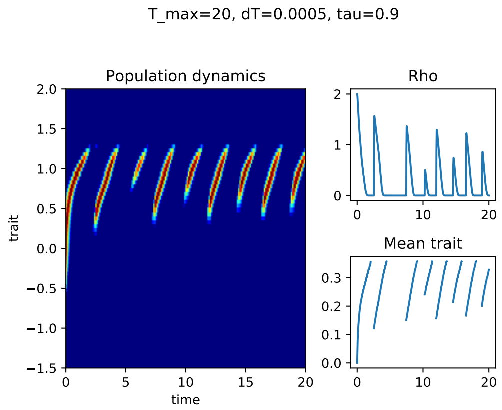
These threshold values are confirmed by numerical simulations on the stochastic model. The Hamilton-Jacobi approach therefore seems able to capture the phenomenon of *evolutionary rescue* accurately.



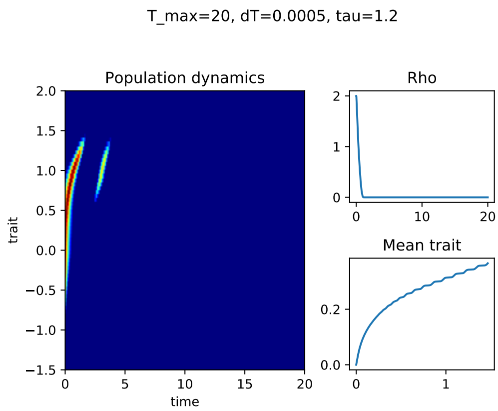
(a) Stabilization: $\tau_0 = 0.02$



(b) Cycles: $\tau_0 = 0.4$



(c) Cycles with episodes of extinction: $\tau_0 = 0.9$



(d) Extinction: $\tau_0 = 1.2$

Figure C$'$.7 – Behavior of $u(t, y)$ with various τ_0.

Lineages. A *lineage* corresponds to the phenotypic history of an individual's ancestors. We present numerical experiments in Figure C$'$.8. On the one hand, we can see that the lineages are all gathered in one around $t \approx 400$. This means that all individuals from final time descend from the same common ancestor. This phenomenon is called *coalescence* [13, 14, 193]. On the other hand, we see that the lineages remain close to the optimal phenotype $y = 0$ throughout the dynamics. This illustrates well that the population manages to sustain thanks to the rare individuals not affected by HT: it is the principle of evolutionary rescue.

We think that the lineages correspond, in mean, to the *characteristics* of the Hamilton-Jacobi (C$'$.13) equation.

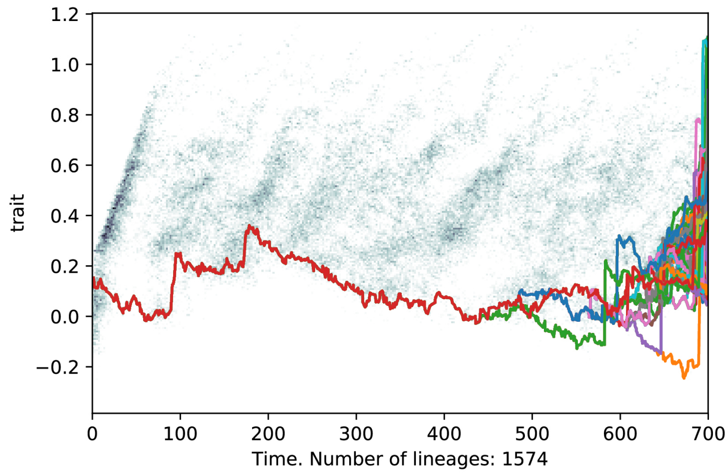


Figure C′.8 – Lineages on the stochastic model.

Chapter D′

Introduction to Part III: Reaction-diffusion systems and modelling of social unrest

In collaboration with H. Berestycki et L. Rossi.

D′.1 Introduction

Presentation of the problem

We consider the following system of two reaction-diffusion equations, for $t \geq 0$, $x \in \mathbb{R}^n$,

$$\begin{cases} \partial_t u(t,x) = d_1 \Delta_x u(t,x) + \Phi(u(t,x), v(t,x)), \\ \partial_t v(t,x) = d_2 \Delta_x v(t,x) + \Psi(u(t,x), v(t,x)), \end{cases} \tag{D′.1}$$

with initial conditions $v(0,x) = v_0 > 0$ and $u(0,x) = u_0(x) \gneqq 0$, $d_1 > 0, d_2 \geq 0$. Our aim is to study the asymptotic behavior of (u,v) and the existence of transition fronts.

Our main assumption is:

$$\Psi(0,v) = 0. \tag{D′.2}$$

This means that, if $u = 0$, the dynamics of v consists of pure diffusion. Apart from a few additional technical assumptions, we keep a rather general approach. In particular, we do not make any monotonicity assumptions.

Note that assumption (D′.2) implies that $(0,v)$ is a state of equilibrium, for any constant $v > 0$. We are mainly interested in solutions that emerges from a small perturbation of such a state of equilibrium. Thus, we consider an initial data $u_0(\cdot) \geqq 0$ with a compact support having a small L^∞ norm. For simplicity, we always assume that v_0 is constant.

The philosophy of our approach is to consider v_0 as a bifurcation parameter to describe the dynamics of (D′.1). As we will see, many qualitative properties can be deduced from the sign of

$$K_0 := \partial_u \Phi(0, v_0).$$

Note that this quantity does not depend on the equation satisfied by v (*i.e.*, d_2 and Ψ).

The famous epidemiological model SI (for *Susceptible* and *Infected*) is contained in our assumptions, with the identification $S(t,x) \equiv v(t,x)$ and $I(t,x) \equiv u(t,x)$. In its simplest version (with space), the SI model is written

$$\begin{cases} \partial_t I = \Delta I + \beta SI - \gamma I, \\ \partial_t S = -\beta SI, \end{cases} \tag{D′.3}$$

with $\beta, \gamma > 0$. The SI model therefore corresponds to the case $\Phi(u,v) = \beta uv - \gamma u$, $\Psi(u,v) = -uv$. We notice that, in this case, $v(t,x) \leq v_0$ (because $\Psi \leq 0$), and

$$\Phi(u(t,x), v(t,x)) \leq \partial_u \Phi(0, v_0) u(t,x).$$

This observation suggests that particular attention should be devoted to the case, called *inhibiting*, where

$$\Phi(u,v) \leq \partial_u \Phi(0,v_0) u, \quad \forall u, v \geq 0. \tag{D′.4}$$

The term *inhibiting* comes from the idea that, as soon as the system leaves the equilibrium state $(0, v_0)$, the reaction term $\Phi(u,v)$ decreases, and the dynamic of u is *inhibited*. Note in particular that this hypothesis implies that $\Phi(\cdot, v)$ is *KPP* (in a weak sense). The condition (D′.4) may seem very restrictive at first sight. However, if for example $\Psi \leq 0$, we know that $v(t,x) \leq v_0$. If, in addition, $v \mapsto \Phi(u,v)$ is increasing, and $u \mapsto \frac{\Phi(u,v)}{u}$ decreasing (*i.e.* $\Phi(\cdot, v)$ is *KPP*), then we can restrict $\Phi(u, \cdot)$ to $[0, v_0]$, and (D′.4) is fulfilled.

We will see that, under the assumptions (D′.2)-(D′.4), some aspects of the dynamics of (D′.1) are governed by the linearization of the first equation around $u = 0$, $v = v_0$:

$$\partial_t u = d_1 \Delta_x u + \partial_u \Phi(0, v_0) u. \tag{D′.5}$$

Formally, we will see that u behaves as the solution of a scalar *KPP* equation. In particular, the sign of K_0 plays a decisive role. On the one hand, if $K_0 < 0$, the equilibrium state $(0, v_0)$ is stable, and therefore absorbs the dynamics. On the other hand, if $K_0 > 0$, then $(0, v_0)$ is repulsive and a propagation phenomenon occurs, at speed $c_0 := 2\sqrt{d_1 \partial_u \Phi(0, v_0)}$.

In general, without the assumption (D′.4), we will see that u behaves rather like the solution of a *monostable* equation.

The epidemiological model SI

The SI model was introduced by Kermack and McKendrick [191] and is absolutely essential in epidemic modelling, both from the point of view of theory [176, 177, 179, 226, 259, 264, 283] and applications [11, 25, 242]. A vast literature is devoted to it, particularly regarding the study of spatial propagation. The pioneering works are due to Kendall [190], Mollison [238], Thieme [275], Aronson [16] for the asymptotic speed of propagation, Atkinson, Reuter [22], Diekmann [120–123], Brown, Carr [68] for transition waves, and Radcliffe, Rass [255–257] for studying the n-dimensional system. Traveling waves have also been studied by Hosono, Ilyas [180], Zhao, Wang [298], and many others [6,134,189,288–290,293,294,299,300]. The theory has been extended to the case of time-delayed equations, first by Thieme, Zhao [276], then by other authors [204,284,285]. Finally, propagation in periodic heterogeneous environments is studied, among others, by Ducrot, Giletti [133] and Ducasse [132].

However, most of the mathematical approaches available to study the system (D′.3) are based on its explicit form, and can hardly be generalized to a broader class of systems. Indeed, the method often consists in a particular change of variable that reduces the system to a single scalar equation. More precisely, by integrating the second line of (D′.3), we obtain

$$S(t,x) = S_0 e^{-\beta R(t,x)}, \quad \text{with} \quad R(t,x) = \int_0^t I(\cdot, x).$$

Then, by injecting this formula into the first line, and integrating on $(0,t)$, we find

$$\partial_t R = \Delta_x R + f(R) + I_0, \quad \text{où} \quad f(z) := S_0 \left(1 - e^{-\beta z}\right) - \gamma z.$$

Under the assumption $S_0 > \frac{\gamma}{\beta}$, f satisfies the (weak) *KPP* assumption, that is

$$\exists A > 0 : f(0) = f(A) = 0, \quad f > 0 \text{ dans } (0,A), \quad \forall z \in (0,A), f(z) \leq f'(0)z.$$

Thus, in $\mathbb{R}^n \backslash \text{supp } I_0$, the system (D′.3) is reduced to a single *KPP* equation, for which many classical results are available. Formally, the dynamics of R is governed by the linearization of this equation near $R = 0$ (which coincides with (D′.5), with our notations).

This approach has both the advantage and disadvantage of relying on explicit calculations: the method is simple but difficult to transpose. The results we present in this section can be seen as a new approach on the *SI* model, which extends some known results to a general class of systems. We also note that, even for the *SI* model itself, our proofs are sometimes simpler than the ones available in the literature.

D′.2 Theoritical analysis

We present here the content of Chapter 7.

Assumptions and notations

Assumptions. Throughout this section, we assume $d_1 > 0$, $d_2 \geq 0$, and

$$\begin{cases} \Psi(0,v) = 0 & \text{(Main assumption)} \\ \exists M > 0,\ \forall v \geq M,\ \Psi(\cdot, v) \leq 0 & \text{(Saturation on } v\text{)} \\ \Phi(0,\cdot) = 0 \quad \text{and} \quad \Psi(\cdot, 0) = 0, & \text{(Homogeneity)} \\ \|\Phi\|_{C^2}, \|\Psi\|_{C^2} < +\infty & \text{(Regularity)} \end{cases}$$

The nonlinearities we consider are typically of the form

$$\begin{aligned} \Phi(u,v) &= vu(1-u) - \gamma u, \\ \Psi(u,v) &= uvf(u,v), \end{aligned}$$

where $\gamma > 0$ and f is smooth and bounded.

As for the initial conditions, we assume:

$$v_0 \in (0, M),\ u_0(\cdot) \gneqq 0,\ u_0(\cdot) \in L^\infty(\mathbb{R}^n), \text{ and } u_0 \text{ has a compact support.}$$

Notations. Let us set

$$K_0 := \partial_u \Phi(0, v_0), \quad \overline{K} := \sup_{\substack{u \geq 0 \\ v \in (0,M)}} \frac{\Phi(u,v)}{u},$$

and, if $K_0 > 0$,

$$c_0 := 2\sqrt{d_1 K_0}, \quad \overline{c} := 2\sqrt{d_1 \overline{K}}.$$

Note that the *inhibiting* case (D′.4) corresponds to $K_0 = \overline{K}$ and $c_0 = \overline{c}$.

The *inhibiting* case

We first focus on the *inhibiting case*: further assume (D′.4). We recall that this case includes the *SI* model.

Stability. The first result establishes that $(0, v_0)$ is attractive when $K_0 < 0$, and repulsive when $K_0 > 0$.

Theorem D′.1

— *If* $K_0 < 0$ *and* $d_2 > 0$,

$$\lim_{t \to +\infty} \left(\sup_{x \in \mathbb{R}^n} |(u(t,x), v(t,x)) - (0, v_0)| \right) = 0.$$

— *If* $K_0 > 0$*, there exists* $\delta_0 > 0$ *independent of* $u_0(\cdot) \gneqq 0$ *such that*

$$\limsup_{t \to +\infty} \left(\sup_{x \in \mathbb{R}^n} |(u(t,x), v(t,x)) - (0, v_0)| \right) \geq \delta_0.$$

This theorem expresses a threshold phenomenon on v_0, through the sign of K_0. Note that, in particular, the case $K_0 > 0$ satisfies the *Hair-trigger effect.* It should also be noted that, in the *SI* model, the condition $K_0 > 0$ corresponds to $S_0 > \frac{\gamma}{\beta}$.

Propagation speed. The previous theorem says nothing about how the solution propagates through space when $K_0 > 0$. We give the following result.

Theorem D′.2

$$\forall c < c_0, \quad \limsup_{t \to +\infty} \left(\sup_{|x| \geq ct} |(u(t,x), v(t,x)) - (0, v_0)| \right) > 0,$$
$$\forall c > c_0, \quad \lim_{t \to +\infty} \left(\sup_{|x| \geq ct} |(u(t,x), v(t,x)) - (0, v_0)| \right) = 0.$$

This result expresses, in a way, that a propagation phenomenon occurs at the speed c_0. However, this formulation can be misleading, because the previous theorem only implies that the $\limsup_{t \to +\infty}$ of the propagation speed is equal to c_0.

Transition waves. We are now interested in the study of transition waves, i.e. solutions of (D′.1) of the form $u(t,x) = U(x \cdot e + ct)$, $v(t,x) = V(x \cdot e + ct)$, for $c \geq 0$ (the speed), $e \in \mathbb{S}^n$ and with a boundary condition at $-\infty$. More precisely, setting $\xi := x \cdot e + ct \in \mathbb{R}$, we consider the system,

$$\begin{cases} cU'(\xi) = d_1 U''(\xi) + \Phi(U(\xi), V(\xi)), \\ cV'(\xi) = d_2 V''(\xi) + \Psi(U(\xi), V(\xi)), \\ U,\ V \text{smooth, bounded, positive} \end{cases} \tag{D′.6}$$

along with

$$\begin{cases} U(-\infty) = 0, \\ V(-\infty) = v_0. \end{cases} \tag{D′.7}$$

We give a result of existence and non-existence for the transition waves.

Theorem D′.3

— *If $K_0 < 0$, there exists no transition wave.*

— *If $K_0 > 0$, there exists no transition wave with speed $c < c_0$, and there exists a transition wave for any speed $c > c_0$.*

In the particular case of the SI model, the proof for the non-existence of transition waves is classically based on a Tauberian method with a Laplace transform [126, 293, 296, 297]. Here, we propose a simpler proof. We insist on the fact that we do not imposed any conditions in $+\infty$. Thus, a transition wave can have various shapes (we will see examples later on).

The general case

Now, we drop assumption (D′.4) and deal with the general case.

Stability. As in the previous section, the stability of $(0, v_0)$ is determined by the sign of $K_0 > 0$.

Theorem D′.4

— *If $K_0 < 0$, there exists $\varepsilon_0 > 0$ such that for any $u_0(\cdot) \leq \varepsilon_0$ with compact support:*

$$\text{if } d_2 > 0, \quad \lim_{t \to +\infty} \left(\sup_{x \in \mathbb{R}^n} |(u(t,x), v(t,x)) - (0, v_0)| \right) = 0.$$

— *If $K_0 > 0$, there exists $\delta_0 > 0$ such that*

$$\limsup_{t \to +\infty} \left(\sup_{x \in \mathbb{R}^n} |(u(t,x), v(t,x)) - (0, v_0)| \right) \geq \delta_0.$$

Propagation speed. Let us now determine an upper and lower bound on the asymptotic speed of propagation.

Theorem D′.5 *There exists $c_\star \in [c_0, \overline{c}]$ such that*

$$\forall c < c_\star, \quad \limsup_{t \to +\infty} \left(\sup_{|x| \geq ct} |(u(t,x), v(t,x)) - (0, v_0)| \right) > 0,$$

$$\forall c > c_\star, \quad \lim_{t \to +\infty} \left(\sup_{|x| \geq ct} |(u(t,x), v(t,x)) - (0, v_0)| \right) = 0.$$

Transition waves. Now, let us deal with transition waves, i.e. solutions of (D′.6)-(D′.7). First of all, we give a result of non-existence.

Theorem D′.6 *If $K_0 > 0$, there exists no transition wave for $c < c_0$.*

For our existence result, we assume a *saturation* phenomenon on u:

$$\exists M' > 0,\ \forall u \geq M', \quad \Phi(u, \cdot) \leq 0. \tag{D′.8}$$

Let us also consider

$$g(c) := \frac{\sqrt{c^2 - c_0^2} - \sqrt{c^2 - \overline{c}^2}}{2c}, \quad \forall c > \overline{c}.$$

Note that $g(\cdot)$ is decreasing, and that $g(+\infty) = 0$. This function measures, in a way, the *lack of inhibition.*

The following result deals with the case where $\Phi(\cdot, v)$ is *KPP* (in a weak sense).

Theorem D′.7 *Suppose* (D′.8) *and*

$$\Phi(u, v) \leq \partial_u \Phi(0, v) u.$$

If $K_0 > 0$, $c > \overline{c}$ and

$$d_1 \geq d_2 g(c), \tag{D′.9}$$

there exists a transition wave with speed c.

Note that the set of c which satisfies (D′.9) is of the form $[\tilde{c}, +\infty)$, with $\tilde{c} = g^{-1}\left(\frac{d_1}{d_2}\right)$. Formally, the quantity $\tilde{c}$ can be interpreted as the speed at which the lack of inhibition spreads. In particular, if

$$d_1 \geq d_2 \sqrt{1 - \frac{K_0}{\overline{K}}},$$

the previous result implies the existence of transition waves for any $c > \overline{c}$.

Let us now consider the case where $\Phi(\cdot, v)$ is not *KPP.*

Theorem D′.8 *Suppose* (D′.8) *and $K_0 > 0$. If $c > \overline{c}$ satisfies* (D′.9) *and*

$$h(c) := c + \sqrt{c^2 - \overline{c}} - 2\sqrt{c^2 - c_0^2} > 0, \tag{D′.10}$$

there exists a transition wave with speed c.

Assumption (D′.10) deals, in a way, with the lack of inhibition due to the fact that $\Phi(\cdot, v)$ is not *KPP*. When $c \to +\infty$, this condition becomes $\overline{K} < 2K_0$.

D′.3 Modelling Social Unrest

We present here the content of Chapter 8.

D′.3.1 Introduction

The main motivation for studying the system (D′.1) comes from the modelling of riots and social unrest. For a literature on this subject, see [23, 58, 59, 113, 137, 167, 168, 203, 295] and references therein. Our approach is in the spirit of [46, 48, 52], in which a system of reaction-diffusion equations is considered to account for the dynamics of the riots.

We call *Social Unrest*, noted `SU`, the amount of riot activity and civil disorder. We can think of it as the sum of all illegal actions, weighted by their respective importance. Our goal is not to discuss the origins of `SU`. Instead; we propose a model, built on simple assumptions, to account for recurrent patterns observed in the field.

Our model also involves a level of *Social Tension*, noted `ST`, by which we mean a quantity representing a population's resentment towards society, whether for political, economic, or social reasons.

Modelling assumptions

Our approach relies on the assumption that `SU` and `ST` follows coupled dynamics. A central hypothesis is to neglect the intrinsic dynamics of the `SU`, in the absence of `ST`. This allows us to focus more precisely on the interaction between `SU` and `ST`. Let us start by listing the recurring characteristics observed in the dynamics of `SU` (which will be our modeling assumptions) and defining our vocabulary.

First of all, movements of `SU` often occur as episodic *bursts*, commonly called *riots*, *revolutions*, *etc*. However, it is commonly admitted that these *bursts* of `SU` are triggered by a single *exogenous event* (or *trigger event*). We can think of this exogenous event as the drop that breaks the camel's back.

Whether or not a single exogenous event can trigger a *burst* of `SU` depends on the level of `ST`. `ST` plays, in a way, the role of an activator: if `ST` is large enough, a small *exogenous event* triggers a *burst* of `ST`; on the other hand, if the `ST` is low enough, the same event will be followed by a prompt *resumption of calm*.

These observations suggest, from a modelling perspective, that a *relaxation* mechanism occurs on `SU` in a context of low `ST`. This relaxation accounts for various factors, such as fatigue, police repression, incarceration, etc.

On the contrary, a high `ST` activates an endogenous growth of `SU`. In other words, if `ST` is above a threshold value, then a self-reinforcing mechanism takes place on `ST`. This phenomenon is similar to a flame propagation: an *endogenous growth* is activated when the temperature is high enough. This factor can be interpreted as the gregarious dimension of social movements: the broader the movement, the more prone an individual is to join it.

Naturally, this self-reinforcement mechanism has to be counterbalanced by a *saturation* effect, which accounts for the limited number of individuals, resources, goods to be damaged *etc*.

Another important feature commonly observed during *bursts* of `SU` is the geographical spread. This phenomenon can be seen, in a way, as the result of the displacement of rioters. A striking example is the case of the 2005 French riots (see [58, 267]), which were triggered by the death of two young men trying to escape the police in Clichy-sous-Bois, a poor suburb of Paris. This event took place in a context of high social tension, and was the spark of a riot movement that spread throughout the country and lasted more than three weeks.

Although irruptions of `SU` can take many different forms, a first naive classification would be to distinguish between *riots*, which last a few weeks and then vanishes, and *revolutions*, which last longer and can result in major political and social changes (one could think of the French Revolution or the Arab Spring [201,214]. See also [15]).

A riot can be interpreted as a *burst* of `SU` which *decreases* `ST`. Once `ST` falls below a threshold value, `SU` decreases and eventually vanishes. This case is called *tension inhibiting*. It corresponds qualitatively to the outbreak of an epidemic, which spreads until the number of susceptible individuals falls below a threshold value. This behavior is well captured by the SI model, with $S =$`SU` and $I =$`ST`.

A *revolution* can be interpreted as a *burst* of `SU` that *increases* `ST`. This dynamics of positive feedback causes an escalation to a durable state of high `SU`. This case is called *tension enhancing*. From a modelling point of view, it corresponds qualitatively to a cooperative system.

Through the example of riots and revolutions, we see that the dynamics of `SU` suggests different classes of models: epidemiological models on the one hand, and monotone systems on the other hand. In the literature, these two classes of models are usually considered separately. Our aim is to propose a single framework that covers both possibilities of a *riot* and a *revolution*.

Construction of the model

In the same spirit as [46, 48, 52] and from the previous section, we propose a mathematical model to account for the dynamics of `SU` and `ST`. Throughout the section, `SU` is represented by $u(t, x)$, depending on time $t \geq 0$ and location $x \in \mathbb{R}^n$, and `ST` is represented by $v(t, x)$. Our model takes the general form of a system of reaction-diffusion equations

$$\begin{cases} \partial_t u(t, x) = d_1 \Delta_x u(t, x) + \Phi(u(t, x), v(t, x)), \\ \partial_t v(t, x) = d_2 \Delta_x v(t, x) + \Psi(u(t, x), v(t, x)), \\ u(0, x) := u_0(x), \quad v(0, x) := v_0(x). \end{cases}$$

with $d_1 > 0$, $d_2 \geq 0$ and Φ, Ψ that we will specify.

First, we make a homogeneity assumption, namely, $\Phi(0, v) = 0$. In other words, the base level of `SU` in the absence of unusual events is normalized to 0. We model the *exogenous event* as a small perturbation of the state $u = 0$. This perturbation is included in the initial condition $u_0(x) \gneqq 0$, which is chosen with compact support and with small L^∞ norm. The diffusion terms $d_1 \Delta_x u(t, x)$ and $d_2 \Delta_x v(t, x)$ describe the influence that a location has on its geographical neighbors.

The term Φ is chosen of the form

$$\Phi(u, v) := r(v)G(u) - \omega u.$$

The *endogenous growth* (or self-reinforcing mechanism) is modelled by the function $G(\cdot)$, which is chosen to be *KPP*, for example $G(z) = z(1 - z)$. The parameter $\omega > 0$ is the natural rate of *relaxation* of `SU` in the absence of *endogenous growth*. The endogenous factor is modulated by $r(v(t, x))$, which models the role of *activator* played by ST. We choose $r(\cdot)$ to be postive and increasing. This term can be seen as an "on/off switch". For example, $r(\cdot)$ can be linear $r(v) = v$, or sigmoidal:

$$r(v) := \frac{1}{1 + e^{(v-\alpha)\beta}}.$$

Here, $\alpha \geq 0$ is a threshold value and β sets the stiffness of the transition between the relaxed and excited states. The case $\beta = +\infty$ corresponds to $r(v) := \mathbb{1}_{v>\alpha}$.

Denoting $v_\star$ the threshold value of v above which a *burst* of u occurs after an *exogenous event*, we have

$$v_\star := r^{-1}\left(\frac{\omega}{G'(0)}\right).$$

The term Ψ is of particular importance since it models the feedback of u on v. As we assumed to neglect the intrinsic dynamics of ST in the absence of SU, we assume

$$\Psi(0, v) = 0.$$

This can be considered as our main assumption. In the same spirit, we choose v_0 to be constant, so that $(u = 0, v = v_0)$ is a state of equilibrium. In addition, we make a homogeneity assumption, namely $\Phi(u, 0) = 0$, as well as a saturation assumption on v, namely $\forall v \geq 1$, $\Psi(\cdot, v) \leq 0$. Naturally, we assume $v_\star \in (0, 1)$, to cover both possibilities of a *burst* of SU or a *resumption of calm*. We therefore assume that

$$\frac{\omega}{G'(0)} \in (r(0), r(1)).$$

Based on the above, Ψ can be chosen of the form

$$\Psi(u, v) := uvf(u, v)$$

for a function f to be determined. In particular, we consider the two following examples. Each case illustrates a different qualitative behavior.

1. *Tension inhibiting case:* $f \leq 0$. In this case, a *burst* of u will *decrease* v. This case is expected to describe a *riot* and to be qualitatively similar to the SI model. For example, we can take
$$f(u, v) := -1.$$
2. *Tension enhancing case:* $f \geq 0$. In this case, a *burst* of u will *increase* v. This case is expected to describe a *revolution* and to be qualitatively similar to a cooperative system. For example, we can take
$$f(u, v) := (1 - v).$$

D′.3.2 Analysis on the model

Definitions and assumptions

Based on the previous section, we consider $u(t, x)$ (representing SU) and $v(t, x)$ (representing ST) solutions of

$$\begin{cases} \partial_t u = d_1 \Delta_x u + r(v)G(u) - \omega u, \\ \partial_t v = d_2 \Delta_x v + uvf(u, v), \\ u(0, x) := u_0(x), \quad v(0, x) := v_0, \end{cases} \tag{D′.11}$$

under the assumptions:

— $d_1 > 0, d_2 \geq 0$; $\omega > 0$.
— $u_0(\cdot)$ is compactly supported and $0 \lneq u_0(\cdot) < 1$; v_0 is constant and $0 < v_0 < 1$.
— $G(\cdot)$ is of the KPP type, *i.e.*

$$u \mapsto \frac{G(u)}{u} \text{decreases} \quad \text{and} \quad G(0) = G(1) = 0.$$

For example, $G(u) = u(1-u)$.
— $r(\cdot)$ is smooth, positive and nondecreasing.
We are also interested in transition waves, i.e. speeds $c > 0$ and profiles (U, V) satisfying

$$\begin{cases} cU' = d_1 U'' + r(V)G(U) - \omega U, \\ cV' = d_2 V'' + UV(1-V)(V - 1/2), \\ U, V \text{ smooth, bounded and positive,} \end{cases}$$

with the condition

$$\begin{cases} U(-\infty) = 0, \\ V(-\infty) = v_0. \end{cases}$$

Main results

We propose in Chapter 8 a detailed study on two particular cases. The first corresponds to the *tension inhibiting* case, with $f(u,v) = -1$, and describes the dynamics of a *riot*; the second corresponds to the case *tension enhancing* case, with $f(u,v) = 1 - v$, and describes the dynamics of a *revolution*.

In this introduction, we present a mixed case, which somehow combines the two previous cases and allows us to model both the dynamics of riots and revolutions. Let us consider equation (D$'$.11) with

$$f(u,v) := (1-v)(v - 1/2),$$

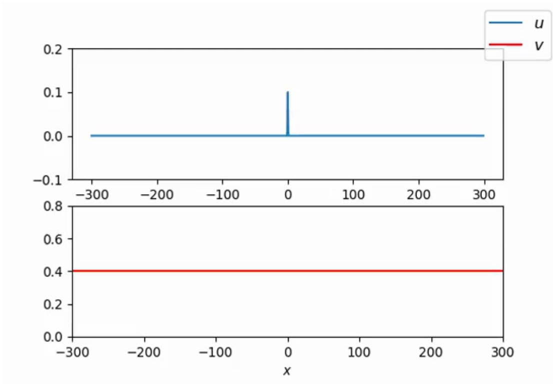
$$v_\star := r^{-1}\left(\frac{\omega}{G'(0)}\right) \in (0, 1/2).$$

This model features a double threshold phenomenon. From a modelling point of view, an exogenous event can result in three different scenarios:
— low initial `ST`: *resumption of calm.*
— intermediate initial `ST`: *burst of a riot.*
— high initial `ST`: *burst of a revolution.*
The following theorem summarizes our main results. It is illustrated in Figure D$'$.1 -D$'$.2

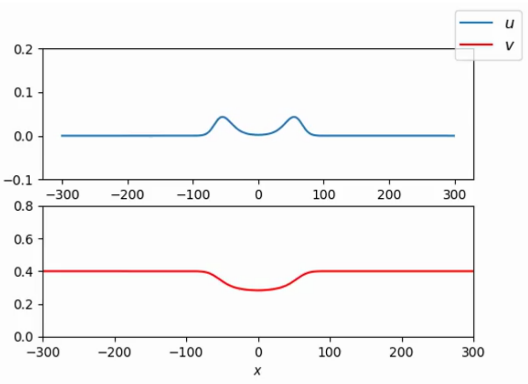
Theorem D$'$.9

1. <u>*Case $v_0 \in (0, v_\star)$*</u>. *There exists no transition wave. If $d_2 > 0$, $(u(t,\cdot), v(t,\cdot))$ converges uniformly towards* $(0, v_0)$.
2. <u>*Case $v_0 \in (v_\star, 1/2)$*</u>. *See Figure D$'$.1. There exists no transition wave for speeds $c < c_0 := 2\sqrt{d_1\left(r(v_0)G'(0) - \omega\right)}$, and there exist some for any speed $c > c_0$. In addition, the transition waves (U,V) are such that V is decreasing, $U(+\infty) = 0$ and $V(+\infty) = V_\infty$ for a certain $V_\infty \leq v_\star$.*

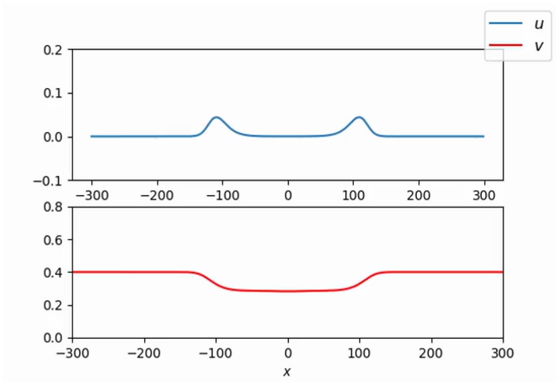
3. <u>*Case* $v_0 \in (1/2; 1)$</u>. *See Figure D′.2. There exists no transition wave for speeds* $c < c_0$*, and there exist some for any speed* $c > \bar{c} := 2\sqrt{d_1\left(r(1)G'(0) - \omega\right)}$ *satisfying* (D′.9)*. In addition, the transition waves* (U, V) *are such that* V *is increasing,* $V(+\infty) = 1$ *and* $U(+\infty) = u_\star$*, where* $u_\star$ *is the only positive solution to* $r(1)G(z) - \omega z = 0$.
Moreover, the solution $(u(t,\cdot), v(t,\cdot))$ *converges locally uniformly to* $(u_\star, 1)$*, and propagates with an asymptotic speed* $c_\star \in [c_0, \bar{c}]$.



(a) $t = 0$

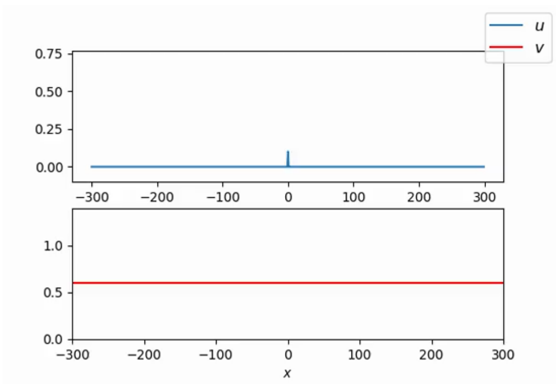


(b) $t = 150$

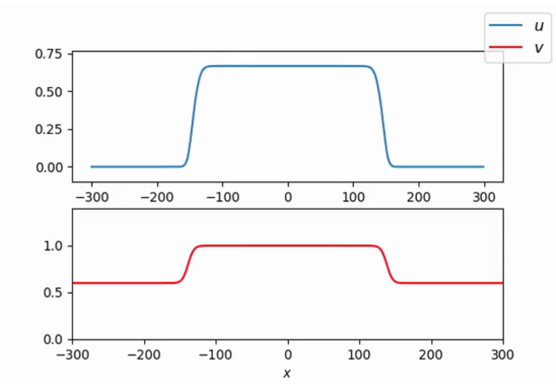


(c) $t = 270$

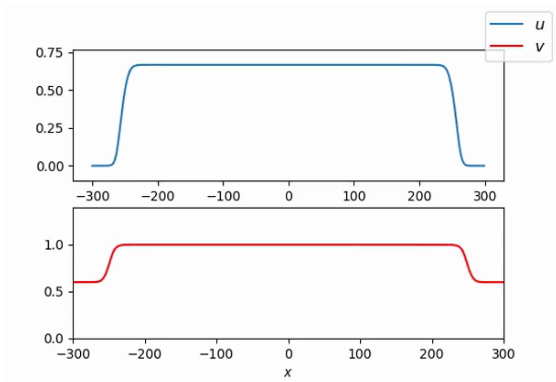
Figure D′.1 – Riot. $v_0 = 0.4$



(a) $t = 0$



(b) $t = 150$



(c) $t = 270$

Figure D′.2 – Revolution. $v_0 = 0.6$

Part I

Qualitative Properties of Stable Solutions

Chapter 1

Stable solutions of semilinear elliptic equations in unbounded domains

This paper establishes some properties for stable solutions of a semilinear elliptic equation with homogeneous Neumann boundary conditions in unbounded domains. A seminal result of Casten, Holland [86] and Matano [217] states that, in convex bounded domains, such solutions must be constant. This paper investigates if this property extends to *unbounded* convex domains. We give a positive answer for stable non-degenerate solutions, and for stable solutions if the domain Ω further satisfies $\Omega \cap \{|x| \leq R\} = O(R^2)$, when $R \to +\infty$. If the domain is a straight cylinder, a natural additional assumption is needed. We also derive some symmetry properties. Our results can be seen as an extension to more general domains of some results on the De Giorgi's conjecture.

Contents

1.1 Introduction

1.1.1 General Framework

Consider the following semilinear elliptic equation, with homogeneous Neumann boundary conditions

$$\begin{cases} -\Delta u(x) = f(u(x)) & \forall x \in \Omega, \\ \partial_\nu u(x) = 0 & \forall x \in \partial\Omega, \\ u \in C^{2,1}\left(\overline{\Omega}\right) \cap L^\infty(\Omega), \end{cases} \tag{1.1}$$

where ∂_ν denotes the outer normal derivative, f is a C^1 function and $\Omega \subset \mathbb{R}^n$ is a uniformly $C^{2,1}$ domain. In two seminal papers, Casten, Holland [86] and Matano [217] established the following: if the domain Ω is bounded and convex, then

$$\text{any stable solution of (1.1) in } \Omega \text{ is constant.} \tag{1.2}$$

In this work, we investigate if property (1.2) extends to unbounded convex domains. We also state some symmetry results, namely that stable solutions of (1.1) inherit symmetries from the domain's invariances by translations or planar rotations.

Note that we only consider *bounded* solutions and that our results do not hold for unbounded solutions. For example $u(x) := e^x$ is a nonconstant solution of $-u'' = -u$ in $\mathbb{R}$ which is stable non-degenerate. Note also that we need no other assumptions on f but smoothness. We choose to stick to this general context even if additional assumptions could lead to some stronger results. For example, Casten and Holland proved in [86] that if f is convex or concave, then (1.2) holds in any bounded domain, possibly not convex.

We point out that the question we address, in the case $\Omega = \mathbb{R}^n$, is strongly related to the De Giorgi's conjecture, to which an extensive literature is devoted (see [7, 10, 160, 251, 262] and references therein). This conjecture claims that any solution of the Allen-Cahn equation in $\mathbb{R}^n, n \leq 8$ which is monotonic in one variable must be planar. As monotonicity implies stability (see e.g Corollary 4.3 of [7]), the question of classifying stable solutions in $\mathbb{R}^n$ is crucial in this context and has been considered by many authors. To that extent, our results in the particular case of the whole space are already contained in several papers, see for example [104, 136, 160]. See also [73] for radial solutions. Moreover, the method we use is inspired by this literature, in particular, Theorem 1.7 from [41], of which a refined version is presented in Lemma 1.14. This work can thus be seen as an extension to more general domains of some results on the De Giorgi's conjecture.

Outline. The paper is organized as follows. We introduce the results in section 1.2 and give a brief discussion on the available counterexamples. In section 1.3, we recall the classical proof of the Casten, Holland and Matano result and prove Theorem 1.2.

In section 1.4 we state some general properties of the generalized principal eigenvalue. The proofs of Proposition 1.4 and Theorem 1.6 are done in section 1.5 and the proof of Theorem 1.3 in section 1.6. In the appendix, the reader can find more details on the generalized principal eigenvalue and on the notion of stability.

1.1.2 Definition of stability

We define stability through a linearization at equilibrium, as follows.

Definition 1.1 *For u a solution of* (1.1)*, define*

$$\lambda_1(u,\Omega) := \inf_{\substack{\psi\in\mathcal{H}^1\\ \|\psi\|_{L^2}=1}} \mathcal{F}_{(u,\Omega)}(\psi), \quad \textit{where } \mathcal{F}_{(u,\Omega)}(\psi) := \int_\Omega |\nabla\psi|^2 - f'(u)\psi.$$

The solution u is said to be stable if $\lambda_1(u,\Omega) \geq 0$ and stable non-degenerate if $\lambda_1(u,\Omega) > 0$.

In the sequel, we often omit to mention the dependance on (u,Ω) and simply write λ_1 or $\mathcal{F}$. It is important to note that, if the domain is bounded, λ_1 coincides with the classical principal eigenvalue of the linearized operator with Neumann boundary conditions. It formally corresponds to the lowest eigenvalue of the second variation of the energy associated with (1.1), thus $\lambda_1 > 0$ implies that the solution is a local minimum of the energy. The properties of λ_1 in unbounded domains have been studied in a general context (see [51] and references therein). In the appendix, we give a brief discussion on the link between this definition of stability and the usual dynamical point of view.

1.2 The results

1.2.1 Convex domains

The first result states that, considering stable *non-degenerate* solutions, property (1.2) fully extends to unbounded convex domain.

Theorem 1.2 *Let Ω be a convex domain. If u is a stable non-degenerate solution of* (1.1)*, then u is constant.*

Now, let us focus on the classification of stable possibly *degenerate* solutions. We need a further assumption on the size of the domain Ω at infinity, namely

$$|\Omega \cap \{|x| \leq R\}| = O\left(R^2\right), \quad \text{when } R \to +\infty. \tag{1.3}$$

This condition comes from Lemma 1.14, introduced below. Note that under this assumption, we allow for instance convex domains that are subdomains of $\mathbb{R}^2$, or of the form $\mathbb{R}^i \times \omega$ with ω bounded and $i \in \{1,2\}$, or of the form

$$\Omega = \left\{(x_1,x') \in \mathbb{R}\times\mathbb{R}^{n-1} : x' \in \omega(x_1)\right\},$$

where for all $x_1 \in \mathbb{R}$, $\omega(x_1) \subset \mathbb{R}^{n-1}$ with $|\omega(x_1)| = O(|x_1|)$ when $|x_1| \to +\infty$ ($\omega(x_1)$ can be empty for some ranges of values of x_1).

The case of a convex straight cylinder, namely $\Omega = \mathbb{R} \times \omega$ with $\omega \subset \mathbb{R}^n$ convex, turns out to be a specific case in our results. Note that it corresponds to domains for which convexity is degenerate in one direction. Indeed, some nonconstant stable solutions (consisting of planar waves) may exist in such domains. For example, the Allen-Cahn equation in $\mathbb{R}$

$$-u'' = u(1-u)(u - {}^1\!/_2)$$

admits the explicit solution $u : x \mapsto \frac{\tanh(x)+1}{2}$, which is stable (degenerate). Note that, in this example, the nonlinearity is balanced, that is, $\int_0^1 f = 0$. This leads up to the following assumption:

$$\forall (z_1, z_2) \in \mathcal{Z},\ z_1 \neq z_2 \quad \int_{z_1}^{z_2} f \neq 0, \tag{1.4}$$

where

$$\mathcal{Z} := \{z \in \mathbb{R} \mid f(z) = 0 \text{ and } f'(z) \leq 0\}$$

is the set of the stable zeros of f. Formally, the sign of $\int_{z_1}^{z_2} f$ corresponds to that of the speed c of a possible traveling wave connecting z_1 and z_2. Thus, assumption (1.4) prevents the existence of a stationary wave (with speed $c = 0$) connecting two stable states. Note that, if the domain is not a straight cylinder, planar waves do not exist anyway and this assumption is not needed.

Theorem 1.3 *Let $\Omega \subset \mathbb{R}^n$ be a convex domain which satisfies* (1.3) *and u be a stable solution of* (1.1)*.*

1. *If Ω is not a straight cylinder, then u is constant.*
2. *If Ω is a straight cylinder, then u is either constant, or a planar monotonic stationary wave connecting two stable roots $(z^-, z^+) \in \mathcal{Z}^2$,such that $\int_{z^-}^{z^+} f = 0$. As a consequence, if we further assume* (1.4) *then u is constant.*

Note that, in the case of a straight cylinder, the stability of any non-constant solution is expected to be degenerate, since a continuum of solutions is given by translations of the above-mentioned solution.

1.2.2 Symmetry properties

The following result deals with straight cylinders, possibly not convex. It is essentially classical, but we propose a proof for completeness.

Proposition 1.4 *Let $\Omega = \mathbb{R}^n \times \omega$ with $\omega \subset \mathbb{R}^m$ bounded and let u be a stable solution of* (1.1)*. For $x \in \Omega$, we generically denote $x = (x_1, \dots, x_n, x'_1, \dots, x'_m)$.*

1. *If u is stable non-degenerate, then u does not depend on $(x_1, \dots, x_n)$.*
2. *If u is stable degenerate and $n = 1$, then u is monotonic with respect to x_1.*
3. *If u is stable degenerate and $n = 2$, the dependance of u with respect to (x_1, x_2) is only through a single scalar variable $x_0 \in \mathbb{R}$. Moreover, u is monotonic with respect to x_0.*

We are now interested in cylinders which are invariant with respect to a planar rotation.

Definition 1.5 *A domain $\Omega \subset \mathbb{R}^{n+2}$ is said to be θ-invariant if $\Omega = \Omega' \times [0, 2\pi)$, where $\Omega' \subset \mathbb{R}^n \times \mathbb{R}^+$ in some cylindrical coordinates $(x, r, \theta) \in \mathbb{R}^n \times \mathbb{R}^+ \times [0, 2\pi)$.*

When considering a θ-invariant domain Ω, we further assume that the "radial section" is uniformly bounded:

$$\sup_{(x,r)\in\Omega'} r < +\infty. \tag{1.5}$$

In particular, it guarantees that if u is a solution of (1.1) in Ω, then $\partial_\theta u$ is bounded.

Theorem 1.6 *Let Ω be a θ-invariant domain which satisfies* (1.5) *and u be a stable solution of* (1.1). *Assume* (1.3) *or that u is stable non-degenerate. Then $\partial_\theta u = 0$.*

Corollary 1.7 *Under the same conditions, if we further assume that Ω' is convex, then Theorem 1.2 and Theorem 1.3 apply.*

As a consequence, property (1.2) holds in some nonconvex domains, such as cylinders whose section is a torus or an annulus.

The conclusions of Theorem 1.6 and Corollary 1.7 are classical when the domain is bounded [217]. Analogous results on manifold are also available [28, 142, 143, 185, 253, 260].

Proposition 1.4 and Theorem 1.6 deal with domains which are invariant with respect to a translation or a planar rotation. More generally, one can consider domains which are invariant with respect to a vector field $X(\cdot) \in C^1(\overline{\Omega}, \mathbb{R}^n)$ and ask whether stable solutions inherit the same symmetry, that is to say $v := \partial_X u = \nabla u \cdot X$ is zero. Based on the following observation, it is reasonable to think that the answer is negative in general. To fix ideas, let $\Omega \subset \mathbb{R}^2$ and let u be a stable solution of (1.1). Our method requires v to satisfy the linearized equation

$$-\Delta v - f'(u)v = 0 \quad \text{in } \Omega.$$

This essentially means that ∂_X and Δ commute, which implies $\partial_1 X_1 = \partial_2 X_2 = 0$ and $\partial_1 X_2 = -\partial_2 X_1$. Thus, the line integrals of X are parallel lines or concentric circles, i.e the domain is invariant with respect to a translation or a planar rotation, which is already covered by Proposition 1.4 and Theorem 1.6.

1.2.3 Counterexamples

The existence of a counterexample to, say, Theorem 1.3 can be investigated when relaxing the assumptions either on the convexity of the domain, or assumption (1.3).

For the latter case, we know from [246] that non-constant stable solutions exists in $\mathbb{R}^2 n$, $2n \geq 8$. the dimensions $3 \leq n \leq 7$ are open

Regarding counterexamples in bounded nonconvex domains, Matano constructs in [217] a *dumbbell domain* (consisting in two balls connected by a narrow passage, see the Introduction for more details) that admits a nonconstant stable solution, for a general class of bistable nonlinearities. Other constructions of non-constant stable solutions can be found in [40, 42, 172, 198].

Theorem 1.3 along with a blow-up argument leads to the following remarkable corollary. We further assume

$$\text{the zeros of } f \text{ are simple and isolated,}$$
$$\exists (M, m) \in \mathbb{R}^2 \text{ s.t } \begin{cases} f > 0 & \text{on } (-\infty, m), \\ f < 0 & \text{on } (M, +\infty). \end{cases} \tag{1.6}$$

Corollary 1.8 *Let $\Omega \subset \mathbb{R}^2$ be a smooth domain. We assume without loss that $0 \in \Omega$. For $\mu > 0$ we define the dilated domain $\Omega_\mu := \{\mu x,\ x \in \Omega\}$.*

If f satisfies (1.4) and (1.6), then there exists $\mu_* > 0$ such that (1.2) holds in Ω_μ for all $\mu \geq \mu^*$.

Proof A complete proof of Corollary 1.8 can be adapted from the proof of Theorem 5 in [104]. We only give a sketch of the proof. By contradiction, assume there exists a sequence u_n of nonconstant stable solutions of (1.1) in Ω_{μ_n}, for $\mu_n \to +\infty$. From assumption (1.6), we infer a uniform L^∞ bound for u_n, then a uniform $\mathcal{C}^{2,\alpha}$ bound by elliptic estimates. Thus we can extract a subsequence that converges to some u, which is a solution of (1.1) in the whole space $\mathbb{R}^2$. Using Corollary 1.13 (below), we can prove that u is stable. Now, from assumption (1.6) and the fact that u_n is not constant, we can show that u is not constant, which contradicts Theorem 1.3.■

This result may be put in perspective with the aforementioned Matano's counterexample. Corollary 1.8 somehow states that such a counterexample could not be achieved in a domain with no "narrow passage", regardless of its nonconvexity, at least for $n = 2$. See the introduction for further discussions.

1.3 Preliminaries

1.3.1 The classical case of bounded convex domains

To give a grasp of the method and the difficulties arising when the domain is unbounded, we recall the proof of (1.2) for bounded convex domains. Let Ω be a convex bounded domain, u a stable solution of (1.1) and set $v_i := \partial_{x_i} u$, for all $i \in \{1, \dots, n\}$.

Step 1. One the one hand, differentiating (1.1) with respect to x_i, we find that $v_i := \partial_{x_i} u$ satisfies the linearized equation

$$-\Delta v_i - f'(u) v_i = 0 \quad \text{in } \Omega. \tag{1.7}$$

From an integration by part we have

$$\mathcal{F}(v_i) = \int_{\partial\Omega} v_i \partial_\nu v_i = \frac{1}{2} \int_{\partial\Omega} \partial_\nu [v_i^2],$$

with $\mathcal{F} = \mathcal{F}_{(u,\Omega)}$ from Definition 1.1. On the other hand, as u is stable, we have $\mathcal{F}(\cdot) \geq 0$ and

$$0 \leq \mathcal{F}(v_i) \leq \sum_{k=1}^{n} \mathcal{F}(v_k) = \frac{1}{2} \int_{\partial\Omega} \partial_\nu \left[|\nabla u|^2 \right].$$

Step 2. When the domain is convex, the above integral turns out to be nonpositive, as stated in the following key lemma. This is where the *convexity* of the domain comes into play. It can be found in [86, 217], but a simple proof is presented at the end of the section for completeness. Note that the lemma still holds when the domain is unbounded.

Lemma 1.9 (**[86, 217]**) *Let Ω be a smooth convex domain. If u is a $\mathcal{C}^2$ function such that*

$$\partial_\nu u = 0 \quad on\ \partial\Omega, \tag{1.8}$$

then

$$\partial_\nu |\nabla u|^2 \leq 0 \quad on\ \partial\Omega.$$

We then conclude that for all $i \in \{1, \ldots, n\}$, we have $\mathcal{F}(v_i) = 0$ thus v_i minimizes $\mathcal{F}$. Note that, at this step, if we assume that u is not constant, i.e $v_i \not\equiv 0$ for some i, then we deduce $\lambda_1 = 0$, i.e u is *stable degenerate.*

Note also that, if Ω is unbounded, the computations are not licit and need to be adapted. This is done in section 1.3.2.

Step 3. Owing to the above conclusion, we deduce as a classical fact that for all $i \in \{1, \ldots, n\}$, v_i is a multiple of the principal eigenvalue associated to (1.7), which is denoted φ and is positive in $\overline{\Omega}$.

Note that if Ω is unbounded, this step also needs to be adapted. This is done in section 1.4.

Step 4. From $\partial_\nu u = 0$ on the closed surface $\partial\Omega$, we deduce that v_i vanishes on some point of the boundary. But as v_i is colinear to φ, we conclude $v_i \equiv 0$, which completes the proof.

Note that if Ω is a straight cylinder, the above conclusion fails and v_i may be a nonzero multiple of φ.

Before proving Lemma 1.9, we need the following definition.

Definition 1.10 *Let $\Omega \subset \mathbb{R}^n$. A "representation of the boundary" is a pair (ρ, U) where ρ is a C^2 function defined on U a neighborhood of $\partial\Omega$ such that*

$$\rho(x) \begin{cases} < 0 & \textit{if } x \in \Omega \cap U \\ = 0 & \textit{if } x \in \partial\Omega \\ > 0 & \textit{if } x \in U \backslash \overline{\Omega} \end{cases} \qquad \textit{and} \qquad \nabla\rho(x) = \nu(x) \quad \forall x \in \partial\Omega,$$

where $\nu(x)$ is the outer normal unit vector of $\partial\Omega$ at x.

It is classical that such a representation of the boundary always exists for $C^{2,1}$ domains, see e.g section 6.2 of [162].

Proof (of Lemma 1.9) Let us consider (ρ, U) a representation of the boundary for Ω. Equation (1.8) becomes

$$\nabla u \cdot \nabla \rho = 0 \quad \text{on } \partial\Omega.$$

As ∇u is tangential to $\partial\Omega$, we can differentiate the above equality with respect to the vector field ∇u. It gives, on $\partial\Omega$,

$$0 = \nabla\left(\nabla u \cdot \nabla\rho\right) \cdot \nabla u = \nabla u \cdot \nabla^2 u \cdot \nabla\rho + \nabla u \cdot \nabla^2 \rho \cdot \nabla u.$$

From this, we infer

$$\partial_\nu |\nabla u|^2 = \nabla\left(|\nabla u|^2\right) \cdot \nabla\rho = 2\nabla u \cdot \nabla^2 u \cdot \nabla\rho = -2\nabla u \cdot \nabla^2\rho \cdot \nabla u,$$

Since Ω is convex, for all $x_0 \in \partial\Omega$, we have that $\nabla^2\rho(x_0)$ is a nonnegative quadratic form in the tangent space of $\partial\Omega$ at x_0. As ∇u is tangential to $\partial\Omega$, we deduce from the above equation that $\partial_\nu|\nabla u|^2$ is nonpositive. ∎

In [86], the authors give the following remarkable geometrical interpretation of the above lemma. Consider a bounded convex domain $\Omega \subset \mathbb{R}^2$. As u satisfies Neumann boundary conditions, its level set cross the border $\partial\Omega$ orthogonally. Since the domain is convex, these level sets go apart one from each other as we move outward $\partial\Omega$. As $|\nabla u|$ corresponds to the inverse of the distance of two level sets, it implies that $|\nabla u|$ decreases as we move outward Ω, hence the result.

1.3.2 Non-degenerate stable solutions - proof of Theorem 1.2

The following proof is adapted from the first two steps of section 1.3.1. The method is inspired from [160]. As Ω is unbounded, the computations which lead to "$\mathcal{F}(v_i) = 0$" are not licit. We shall instead perform the computations on a truncated function $\chi_R v_i$. For $R > 0$, we set

$$\chi_R(x) := \chi\left(\frac{|x|}{R}\right), \quad \forall x \in \mathbb{R}^n, \tag{1.9}$$

for χ a smooth nonnegative function such that

$$\chi(z) = \begin{cases} 1 & \text{if } 0 \le z \le 1, \\ 0 & \text{if } z \ge 2, \end{cases} \qquad |\chi'| \le 2.$$

Lemma 1.11 *Let $v \in C^2\left(\overline{\Omega}\right)$ be bounded and satisfy*

$$v(\Delta v + f'(u)v) \ge 0 \quad \textit{in } \Omega, \tag{1.10}$$

and

$$\int_{\partial\Omega} \chi_R^2 v \partial_\nu v \le 0, \quad \forall R \gg 1. \tag{1.11}$$

If u is stable non-degenerate, then $v \equiv 0$.

Proof (of Lemma 1.11) By contradiction, assume $v \not\equiv 0$. By a standard elliptic argument, v cannot be identically zero on any subset $\Omega' \subset \Omega$. For $R > 0$, multiplying (1.10) by χ_R^2, integrating on Ω, using the divergence theorem and (1.11) we find

$$\begin{aligned} \mathcal{F}\left(\frac{\chi_R v}{\|\chi_R v\|_{L^2}}\right) &= \frac{\int_{\partial\Omega} \chi_R^2 v \partial_\nu v + \int_\Omega |\nabla \chi_R|^2 v^2}{\int_\Omega \chi_R^2 v^2} \\ &\le \frac{\int_\Omega |\nabla \chi_R|^2 v^2}{\int_\Omega \chi_R^2 v^2} \le \frac{\frac{4}{R^2}\int_{\Omega_{2R}\setminus\Omega_R} v^2}{\int_{\Omega_R} v^2} \le 4\ \alpha_R, \end{aligned}$$

denoting,

$$\Omega_R := \Omega \cap \{|x| \le R\}, \quad \mathcal{C}(R) := \int_{\Omega_R} v^2, \quad \alpha_R := \frac{\mathcal{C}(2R) - \mathcal{C}(R)}{R^2 \mathcal{C}(R)}.$$

Now, let us show

$$\liminf_{R\to+\infty} \alpha_R \le 0. \tag{1.12}$$

If (1.12) holds, then $\lambda_1 \leq 0$, which contradicts the fact that u is stable nondegenerate and thereby completes the proof. By contradiction, let us assume $\alpha_R \geq \delta > 0$. We have $\mathcal{C}(2R) \geq \delta R^2 \mathcal{C}(R)$. Iterating, we find, for R large enough

$$\mathcal{C}(2^j R) \geq K \left(\delta R^2\right)^j \quad \text{for all integer } j \geq 1,$$

where positive constants are generically denoted K. In addition, v is bounded, hence $\mathcal{C}(R) \leq KR^n$. We have

$$K \left(2^j R\right)^n \geq \left(\delta R^2\right)^j .$$

Fixing R large enough, we reach a contradiction as j goes to $+\infty$. Thereby, we have proved (1.12) and the proof is complete. ∎

We denote $v_i := \partial_{x_i} u$ for $i \in \{1, \dots, n\}$. As Ω is convex, Lemma 1.9 implies

$$\sum_{i=1}^{n} \int_{\partial\Omega} \chi_R^2 v_i \partial_\nu v_i = \frac{1}{2} \int_{\partial\Omega} \chi_R^2 \partial_\nu |\nabla u|^2 \leq 0. \tag{1.13}$$

From a differentiation of (1.1), we find that all the v_i satisfy (1.7). Moreover, since u is bounded, classical global Schauder estimates (see e.g Theorem 6.30 in [162]) ensure that all the v_i are bounded. Then, Lemma 1.11 implies in particular $\int_{\partial\Omega} \chi_R^2 v_i \partial_\nu v_i \geq 0$ for all i, i.e all the terms of the sum in (1.13) are nonnegative. As the sum is nonpositive, all the terms must be zero. From Lemma 1.11, we find $v_i \equiv 0$ for all i, i.e u is constant, which completes the proof.

1.4 Properties of λ_1

1.4.1 Existence of a positive eigenfunction

If the domain is bounded, it is classical that there exists a positive eigenfunction associated to λ_1. This property extends to unbounded domains, as stated in the following proposition.

Proposition 1.12 *Let $\Omega \subset \mathbb{R}^n$ and u be a solution of* (1.1). *There exists $\varphi \in W^{2,p}_{\mathrm{loc}}(\Omega)$, $\forall p \geq 1$, which is positive on $\overline{\Omega}$ and satisfies*

$$\begin{cases} -\Delta\varphi - f'(u)\varphi = \lambda_1 \varphi & \text{in } \Omega, \\ \partial_\nu \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.14}$$

φ is referred as a principal eigenfunction of the linearized operator.

This statement is adapted from the results of [51]. However, presenting a full proof would be too technical and slightly off topic. See the appendix and Chapter 3 for more details.

As a corollary, we can prove that $\lambda_1 \geq 0$ is equivalent to the existence of a positive supersolution. This result is essentially classical in the theory of linear elliptic equations. The proof is postponed to the appendix.

Corollary 1.13 *Let $\Omega \subset \mathbb{R}^n$ and u be a solution of* (1.1). *The two following statements are equivalent:*

1. *u is stable*
2. *There exists $\phi \in W^{2,p}_{\text{loc}}(\Omega)$, $\forall p \geq 1$, which is positive on $\overline{\Omega}$ and satisfies*

$$\begin{cases} -\Delta\phi - f'(u)\phi \geq 0 & \text{in } \Omega, \\ \partial_\nu \phi \geq 0 & \text{on } \partial\Omega. \end{cases} \tag{1.15}$$

In the sequel, φ could be replaced by any ϕ satisfying (1.15) for our purposes.

1.4.2 Liouville result, or the simplicity of λ_1

When the domain is bounded, it is classical that the only minimizers of $\mathcal{F}_{(u,\Omega)}$ are multiples of φ. The following lemma claims that, if the minimizer is a bounded function, the above conclusion extends to unbounded domains which satisfy (1.3). It is a refinement of Theorem 1.7 from [41].

Lemma 1.14 *Let $\Omega \subset \mathbb{R}^n$ satisfy* (1.3) *and let u be a stable solution of* (1.1)*. If v is smooth, bounded and satisfies* (1.10)-(1.11)*, then $v \equiv C\varphi$ for some constant C, where φ is defined in Proposition 1.12.*

Proof (of Lemma 1.14) We follow the method of [41]. Let us set $\sigma = \frac{v}{\varphi}$ and show that σ is constant. From (1.10), we deduce

$$\sigma\varphi\left(\varphi\Delta\sigma + 2\nabla\varphi \cdot \nabla\sigma + \sigma\left(\Delta\varphi + f'(u)\varphi\right)\right) \geq 0.$$

From (1.14) and $\lambda_1 \geq 0$, we obtain

$$\sigma\nabla \cdot \left(\varphi^2\nabla\sigma\right) \geq 0.$$

Multiplying by χ_R^2 (defined in (1.9)), integrating on Ω and using the divergence theorem, we find

$$\begin{aligned} 0 &\leq \int_{\partial\Omega} \chi_R^2\sigma\varphi^2\partial_\nu\sigma - \int_\Omega \varphi^2\nabla\left(\chi_R^2\sigma\right)\cdot\nabla\sigma \\ &= \int_{\partial\Omega} \chi_R^2\sigma\varphi^2\partial_\nu\sigma - \int_\Omega \varphi^2\chi_R^2|\nabla\sigma|^2 - 2\int_\Omega \varphi^2\chi_R\sigma\nabla\chi_R\cdot\nabla\sigma. \end{aligned}$$

As $\partial_\nu\varphi = 0$ on $\partial\Omega$, the boundary term reads as $\int_{\partial\Omega}\chi_R^2 v\partial_\nu v$, which is nonpositive from (1.11). Using the Cauchy-Schwarz inequality, we deduce

$$\int_\Omega \chi_R^2\varphi^2|\nabla\sigma|^2 \leq 2\sqrt{\int_{\Omega_{2R}\setminus\Omega_R}\chi_R^2\varphi^2|\nabla\sigma|^2}\sqrt{\int_\Omega v^2|\nabla\chi_R|^2}, \tag{1.16}$$

where $\Omega_R = \Omega \cap \{|x| \leq R\}$.

Now, assumption (1.3) implies

$$\int_\Omega v^2|\nabla\chi_R|^2 \text{ is bounded, uniformly in } R \geq 1. \tag{1.17}$$

From (1.16), we have that $\int_\Omega \chi_R^2\varphi^2|\nabla\sigma|^2$ is uniformly bounded. Using (1.16) again, we infer that it converges to 0 as $R \to \infty$. At the limit, we find $\int_\Omega \varphi^2|\nabla\sigma|^2 \leq 0$. Hence $\nabla\sigma = 0$, which ends the proof. ■

The cornerstone of the proof is that $\sigma\nabla\cdot(\varphi^2\nabla\sigma)\geq 0$ implies $\nabla\sigma=0$, where $\sigma:=\frac{v}{\varphi}$. The litterature refers to this property as the *Liouville property*. Originally introduced in [41], it has been extensively discussed (see [10, 39, 158, 160, 240]) and used to derive numerous results (e.g [43, 73, 104, 136]), in particular to prove the De Giorgi's conjecture in low dimension (see [7, 10, 72, 160]). Lemma 1.14 is a refinement of this property for domains with a boundary, instead of $\Omega=\mathbb{R}^n$. This is why we need a boundary condition (1.11).

This is the only step where (1.3) is needed, it is thus a natural question to ask if this assumption can be relaxed. In the proof, (1.3) is used to derive (1.17), thus the choice of χ_R seems crucial. However, in [160], the authors consider the optimal χ_R by taking a solution of the minimization problem

$$\inf_{\chi\in H^1(\mathbb{R}^2)}\left[\int_{R\leq|x|\leq R'}|\nabla\chi(x)|^2\mathrm{d}x,\ \xi(x)=\begin{cases}1 & \text{if } |x|\leq R\\ 0 & \text{if } |x|\geq R'\end{cases}\right]. \tag{1.18}$$

That, in fact, does not allow to substantially relax condition (1.3). In [38], Barlow uses a probabilistic approach to establish that the aforementioned Liouville property (and consequently Lemma 1.14) does not hold in $\Omega=\mathbb{R}^n$, $n\geq 3$. It is thus reasonable to think that condition (1.3) cannot be relaxed, yet this is an open question. We also cite [158], in which (1.3) is proved to be sharp, however we point out that, there, the condition $v\in L^\infty$ is not satisfied.

Note also that, in this work, we only apply Lemma 1.14 to functions v which are derivatives of u, which is a stronger condition than (1.10). In this context, not much is known about whether (1.3) could be relaxed. Indeed, up to the author's knowledge, the only available counterexamples are for $\Omega=\mathbb{R}^n$, $n\geq 9$, as a consequence of [251] in which the authors construct counterexamples to the De Giorgi's conjecture for $n\geq 9$.

However, we can sometimes relax (1.3) under further assumptions, either on Ω, f, or v. From a remark in [136], if $f\geq 0$, we can relax (1.3) to $\left|\Omega\cap\{|x|\leq R\}\right|=O\left(R^4\right)$. Note also that, if $\Omega=\mathbb{R}^n$, then (1.3) can be replaced by $v\in H^1(\Omega)$, or $v=o(|x|^{1-\frac{n}{2}})$, see [41, 73]. In addition, we can show that Lemma 1.14 holds for a large class of domains satisfying

$$|\Omega\cap\{|x|\leq R\}|=O\left(R^2\ln(R)\right),\quad \text{when } R\to+\infty.$$

More precisely, let Ω be of the form

$$\Omega:=\left\{(x,x')\in\mathbb{R}^2\times\mathbb{R}^n : x'\in\omega(x)\right\},$$

where $\forall x\in\mathbb{R}^n$, $\omega(x)\subset\mathbb{R}^n$ is bounded and

$$|\omega(x)|=O\left(\ln\left(|x|\right)\right)\quad \text{when } |x|\to+\infty.$$

Then, to show (1.17) we use the cut-off

$$\chi_R(x)=\begin{cases}1 & \text{if } |x|\leq R,\\ \dfrac{\ln R^2-\ln|x|}{\ln R^2-\ln R} & \text{if } R\leq|x|\leq R^2,\\ 0 & \text{if } |x|\geq R^2.\end{cases}$$

This cut-off was first introduced in [160] as a solution of (1.18) for $n=2$.

1.5 Proof of the symmetry properties

1.5.1 Proof of Proposition 1.4

We set $v_i := \partial_{x_i} u$ for $i \in \{1, \dots, n\}$, which is bounded and satisfies (1.10). Since Ω is straight in the directions x_i, we have $\partial_\nu v_i = 0$ on $\partial\Omega$, therefore v_i satisfies (1.11). Thus, the first assertion follows from Lemma 1.11. Next, the second assertion follows from Lemma 1.14 and the fact that $\varphi > 0$.

To show the second assertion (we assume $n = 2$), we follow an idea from [41,160] and apply Lemma 1.14 to $\partial_\xi u = (\partial_{x_1} u, \partial_{x_2} u) \cdot \xi$, for ξ being any unit vector of $\mathbb{R}^2$. We infer that there exists a constant C_ξ such that $\partial_\xi u = C_\xi \varphi$. Since C_ξ depends continuously on ξ, it must vanish for some ξ_0 when ξ moves on the sphere from direction x_1 to $-x_1$. Using a change of coordinates, we may assume $\xi_0 = x_2$. Hence $\partial_{x_2} u \equiv 0$ and u depends on one coordinate only. The monotonicty is deduced from the second assertion.

1.5.2 Proof of Theorem 1.6 and Corollary 1.7

We set $v := \partial_\theta u$. Note that in a Cartesian system of coordinates

$$z = (x_1, \dots, x_n, y_1, y_2) \in \mathbb{R}^{n+2},$$

we have $v(z) = y_2 \partial_{y_1} u(z) - y_1 \partial_{y_2} u(z)$. A direct computation shows that v satisfies (2.4). We also know that v is bounded from (1.5) and classical elliptic estimates. We use the following lemma.

Lemma 1.15 *If Ω is θ-invariant and u is a smooth function satisfying*

$$\partial_\nu u = 0 \quad \text{on } \partial\Omega, \tag{1.19}$$

then

$$\partial_\nu \partial_\theta u = 0 \quad \text{on } \partial\Omega.$$

Proof (of Lemma 1.15) Let us consider (ρ, U) a representation of the boundary for Ω (see Definition 1.10). Since Ω is θ-invariant, we can choose ρ to be θ-invariant, i.e $\partial_\theta \rho = 0$. It implies that ∂_θ and ∂_ν commute. We then differentiate (1.19) with respect to θ to prove the claim. ∎

As a consequence, v is bounded and satisfies (1.10)-(1.11). If u is stable non-degenerate, we conclude with Lemma 1.11. Assume (1.3). From Lemma 1.14, we deduce $v \equiv C\varphi$ for some constant C. Thus, v is of constant sign. For $(x, x', r, \theta) \in \Omega$ we have

$$\int_0^{2\pi} v(x, x', r, \theta) d\theta = 0.$$

Therefore $v \equiv 0$, which completes the proof of Theorem 1.6.

To prove Corollary 1.7, it suffices to note that, since $\partial_\theta u = 0$, $w := \partial_r u$ satisfies

$$w(\Delta w + f'(u) w) = \frac{1}{r^2} w^2 \quad \text{a.e in } \Omega,$$

thus w satisfies (1.10), hence the conclusion.

1.6 Proof of Theorem 1.3

As a consequence of Lemma 1.14 and Lemma 1.9, we have the following result.

Lemma 1.16 *For any $\xi \in \mathbb{R}^n$, $\nabla u \cdot e \equiv C_\xi \varphi$ for some constant C_ξ.*

PROOF We assume without loss of generality that $|\xi| = 1$, and that ξ coincide with e_1 for $(e_1, \dots, e_n)$ an orthonormal basis of $\mathbb{R}^n$. We set $v_i := \partial_{x_i} u$ for $i \in \{1, \cdots, n\}$. Differentiating (1.1), we find that v_i satisfies (1.10). Moreover, since u is bounded, classical global Schauder's estimates guarantee that all the v_i are bounded.

Now, we show that all the v_i satisfy (1.11). On the one hand, as Ω is convex, Lemma 1.9 implies

$$\sum_{i=1}^{n} \int_{\partial\Omega} \chi_R v_i \partial_\nu v_i = \frac{1}{2} \int_{\partial\Omega} \chi_R \partial_\nu |\nabla u|^2 \leq 0,$$

where χ_R is a cut-off function, as in Lemma 1.11. On the other hand, since $\partial_\nu \varphi = 0$ on $\partial\Omega$, Lemma 1.14 implies, in particular, that $\int_{\partial\Omega} \chi_R^2 v_i \partial_\nu v_i \geq 0$ for all i, i.e all the terms of the above sum are nonnegative. As the sum is nonpositive, all the terms must be zero.

Then, we apply Lemma 1.14 to conclude. ∎

Let us go back to the proof of Theorem 1.3. The mapping

$$\begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R} \\ \xi & \mapsto & C_\xi \end{array}$$

is linear and continuous, and thus vanishes on an hyperplane H. Hence, u depends on only one direction e.

If Ω is not straight in the direction e, there exists a point $x_0 \in \partial\Omega$ on which the outer normal derivative is not colinear with e. From $\partial_\nu u = 0$ on $\partial\Omega$, we deduce $\partial_e u(x_0) = 0$. From $\varphi > 0$ on $\overline{\Omega}$, we deduce $\partial_e u \equiv 0$, thus u is constant.

If Ω is straight in the direction e, then $\partial_e u$ may be a nonzero multiple of φ. To fix ideas, we assume without loss that e corresponds to the x_1 direction. Since $v_1 \equiv C_1 \varphi$, it is of constant sign, hence u is monotic. Since u is bounded, it has a limit z^+ when $x_1 \to +\infty$. Setting $u_n(x_1) = u(x_1 + n)$ and using classical elliptic estimates, we can extract a subsequence that converges in $C^{2,\alpha}$ to a stable solution u_∞ of (1.1) (note that Ω is invariant under translation in the x_1 direction). From $u_\infty \equiv z^+$, we deduce that z^+ must be a stable root of f. Identically, u has a constant limit $z^- \in \mathcal{Z}$ as $x_1 \to -\infty$.

If $z^+ = z^-$, then u is constant. Let us assume $z^- \neq z^+$, and fix $M > 0$. Multiplying $-u'' = f(u)$ by u' and integrating on $x_1 \in [-M, M]$ gives

$$\frac{1}{2}\left(u'(-M)^2 - u'(M)^2\right) = \int_{u(-M)}^{u(M)} f.$$

As $u'(\pm\infty) = 0$ (indeed, u' is integrable and u'' is bounded), when M goes to $+\infty$ we obtain $\int_{z^-}^{z^+} f = 0$. The proof of Theorem 1.3 is thereby complete.

Remark 1.17 *Note that if a stationary traveling wave exists, it is always a stable (degenerate) solution of* (1.1)*. This can be shown using Corollary 1.13 with $\varphi := |\partial_{x_1} u|$.*

1.7 Appendix

1.7.1 Generalized principal eigenvalue

The reader can find a more detailed description of the (generalized) principal eigenvalue in Chapter 3.

Given an elliptic operator L and a smooth bounded domain Ω along with some proper boundary conditions, the classical Krein-Rutman theory provides a minimal eigenvalue, refered as the *principal eigenvalue*. This notion has been extended to non-smooth and unbounded domains, under Dirichlet boundary conditions, see [47, 51]. Considering smooth unbounded domains, the approach of [51] can be adapted to Neumann boundary conditions, by substituting the functional space $W^{2,p}_{\text{loc}}(\Omega)$ with $\tilde{W}^{2,p}_{\text{loc}}(\Omega) := \left\{\psi \in W^{2,p}_{\text{loc}}(\Omega) : \partial_\nu \psi = 0 \text{ on } \partial\Omega\right\}$. Indeed, in [50] the authors define

$$\lambda_{1,N} := \sup\left\{\lambda \in \mathbb{R} : \exists \psi \in \tilde{W}^{2,n}_{\text{loc}}(\Omega),\ \psi > 0,\ (\Delta + f'(u) + \lambda)\psi \leq 0 \text{ a.e in } \Omega\right\}$$

and prove the existence of an associated positive eigenfunction, namely

Proposition 1.18 (Theorem 3.1 and Proposition 1 in [50]) *There exists $\varphi \in W^{2,p}_{\text{loc}}$, $\forall p > 1$, which is positive on $\overline{\Omega}$ and satisfies*

$$\begin{cases} -\Delta\varphi - f'(u)\varphi = \lambda_{1,N}\varphi & a.e \text{ in } \Omega, \\ \partial_\nu \varphi = 0 & a.e \text{ on } \partial\Omega. \end{cases}$$

In fact, the quantities $\lambda_{1,N}$ and λ_1 (from Definition 1.1) are equal, as stated in the following lemma. Note however that the definition of λ_1 relies on the fact that the operator $-\Delta - f'(u)$ is self-adjoint, whereas $\lambda_{1,N}$ can be defined for more general operators.

Lemma 1.19 *In the definition of λ_1, it is equivalent to take the infimum on compactly supported smooth test functions, namely*

$$\lambda_1(u, \Omega) = \inf_{\substack{\psi \in C^1_c(\overline{\Omega}) \\ \|\psi\|_{L^2} = 1}} \mathcal{F}_{(u,\Omega)}(\psi),$$

where $C^1_c(\overline{\Omega})$ is the space of continuously differentiable functions with compact support in $\overline{\Omega}$. As a consequence, $\lambda_1 = \lambda_{1,N}$.

Proof Recalling that $f'(u)$ is bounded, the first statement is deduced from the dominated convergence theorem and classical density results. The remaining can be adapted from the proof of Proposition 2.2 (iv) in [51] (which itself relies on [5, 47]). ∎

Combining Proposition 1.18 and Lemma 1.19, the proof of Proposition 1.12 follows. We now prove Corollary 1.13.

Proof (of Corollary 1.13) If u is stable, the existence of ϕ is a direct consequence of Proposition 1.12.

Conversely, assume that such a ϕ exists. Let ψ be a test function. Owing to Lemma 1.19, ψ can be chosen in $C^1_c(\overline{\Omega})$. Multiplying (1.14) by $\frac{\psi^2}{\phi}$, integrating and using the divergence theorem, we find

$$\begin{aligned}
0 &\leq \int_\Omega \nabla\phi \cdot \nabla\frac{\psi^2}{\phi} - f'(u)\psi^2 \\
&= 2\int_\Omega \frac{\psi}{\phi}\nabla\phi\cdot\nabla\psi - \frac{\psi^2}{\phi^2}\left|\nabla\phi\right|^2 - f'(u)\psi^2 \\
&\leq \int_\Omega \left|\nabla\psi\right|^2 - f'(u)\psi^2 = \mathcal{F}_{(u,\Omega)}(\psi),
\end{aligned}$$

using Young's inequality in the last step. ■

1.7.2 On the different definitions of stability

When considering stability from a dynamical point of vue, one can come up with the two following definitions.

Definition 1.20 *A solution u of* (1.1) *is said to be dynamically stable if given any $\varepsilon > 0$, there exists $\delta_0 > 0$ such that for any $v_0(x)$ with $\|v_0 - u\|_{L^\infty} \leq \delta_0$, we have*

$$\|v(t,\cdot) - u(\cdot)\|_{L^\infty} \leq \varepsilon, \quad \forall t > 0,$$

where $v(t,x)$ is the solution of the evolution problem

$$\begin{cases}
\partial_t v(t,x) - \Delta v(t,x) = f(v(t,x)) & \forall x\in\Omega,\ \forall t>0,\\
\partial_\nu v(t,x) = 0 & \forall x\in\partial\Omega,\ \forall t>0,\\
v(t=0,x) = v_0(x) & \forall x\in\Omega.
\end{cases} \tag{1.20}$$

Definition 1.21 *A solution u of* (1.1) *is said to be asymptotically stable if there exists $\delta_0 > 0$ such that for any $v_0(x)$ with $\|v_0 - u\|_{L^\infty} \leq \delta_0$, we have*

$$\|v(t,\cdot) - u(\cdot)\|_{L^\infty} \to 0, \quad \text{when } t\to+\infty,$$

where $v(t,x)$ is the solution of (1.20).

The following proposition clarifies the hierarchy of the different definitions of stability.

Proposition 1.22 *Let u be a solution of* (1.1) *and λ_1 from Definition 1.1. The following implications hold*

$$u \text{ asymptotically stable} \Rightarrow\ u \text{ dynamically stable} \Rightarrow \lambda_1 \geq 0.$$

Proof The first implication is trivial. Let us show the second implication by contradiction: assume $\lambda_1 < 0$ and that u is dynamically stable. From the dominated convergence theorem, there exists a bounded domain $\tilde{\Omega}\subset\Omega$ such that

$$\tilde{\lambda}_1 := \inf_{\substack{\psi\in\mathcal{H}^1(\tilde{\Omega})\\ \|\psi\|_{L^2(\tilde{\Omega})}=1}} \int_{\tilde{\Omega}} |\psi|^2 - f'(u)\psi^2 < 0.$$

Note that $\tilde{\Omega}$ could be replace by a larger subdomain of Ω, therefore we can assume without loss that $\partial\Omega$ and $\partial\tilde{\Omega}$ do not intersect tangentially. From technical but classical arguments (see, e.g. Theorem 3.1 in [50]) we know that there exists a positive function $\tilde{\varphi} \in W^{2,p}(\tilde{\Omega})$, $\forall p > 1$ such that

$$\begin{cases} -\Delta\tilde{\varphi} - f'(u)\tilde{\varphi} = \tilde{\lambda}_1\tilde{\varphi} & \text{in } \tilde{\Omega}, \\ \partial_\nu\tilde{\varphi} = 0 & \text{on } \partial\tilde{\Omega} \cap \partial\Omega, \\ \tilde{\varphi} = 0 & \text{on } \partial\tilde{\Omega}\backslash\partial\Omega. \end{cases}$$

We choose the normalization $\|\tilde{\varphi}\|_{L^\infty} = \delta$, with δ small enough such that $0 < \delta < \delta_0$ (where δ_0 is given in Definition 1.20) and

$$\eta_\delta := \sup_{\substack{\tilde{u}\in[\inf u, \sup u] \\ |h|\leq\delta}} \left| f'(\tilde{u}) - \frac{f(\tilde{u}+h) - f(\tilde{u})}{h} \right| < -\lambda_1. \tag{1.21}$$

We set $v_0 := u + \tilde{\varphi}$. On the one hand, as $\|v_0 - u\|_{L^\infty} \leq \delta$, the stability assumption implies $\|h(t,\cdot)\|_{L^\infty} \leq \delta$ for all time $t \geq 0$, where $h(t,x) := v(t,x) - u(x)$. On the other hand, $h(t,\cdot) > 0$ and satisfies

$$\begin{cases} \partial_t h(t,x) - \Delta h(t,x) \geq (f'(u(x)) - \eta_\delta)\, h(t,x) & \text{in } \tilde{\Omega}, \\ \partial_\nu h = 0 & \text{on } \partial\tilde{\Omega} \cap \partial\Omega, \\ h > 0 & \text{on } \partial\tilde{\Omega}\backslash\partial\Omega. \end{cases}$$

Using the parabolic comparison principle, we infer $h(t,\cdot) \geq \tilde{h}(t,\cdot)$ for all $t \geq 0$, where $\tilde{h}(t,x) := e^{-(\lambda_1+\eta_\delta)t}\tilde{\varphi}(x)$. From $\lambda_1 + \eta_\delta < 0$, we deduce that $\|h(t,\cdot)\|_{L^\infty}$ goes to infinity when t becomes large: contradiction. ∎

Remark 1.23 *Note that, in the proof, the perturbation $\tilde{\varphi}$ has a compact support and an arbitrarily small L^∞ norm. Thus, if $\lambda_1 < 0$ then* (1.20) *drives $u + h$ away from u for* any *h which is positive or negative on a sufficiently large subdomain $\tilde{\Omega} \subset \Omega$.*

One can ask whether

$$\lambda_1 > 0 \Rightarrow u \text{ asymptotically stable.} \tag{1.22}$$

Note that property (1.22) is classical in bounded domains (see Proposition 1.4.1 in [135]), but it is not clear whether it extends to unbounded domains.

Question: *Does* (1.22) *hold in any unbounded domain Ω?*

A consequence of our results is that (1.22) holds in unbounded *convex* domains.

Proposition 1.24 *Let $\Omega \subset \mathbb{R}^n$ be a $C^{2,1}$ convex domain (not necessarily bounded) and u be a solution of* (1.1). *Then*

$$\lambda_1 > 0 \Rightarrow \ u \textit{ asymptotically stable} \Rightarrow \ u \textit{ dynamically stable} \Rightarrow \lambda_1 \geq 0.$$

Proof We only have to show the first implication. Assume $\lambda_1 > 0$. We deduce from Theorem 1.2 that u is constant. Thus $\lambda_1 = -f'(u)$ and φ is constant. We choose δ_0 small enough such that $\eta_{\delta_0} \in (0, \frac{\lambda_1}{2})$, where η_δ is defined in (1.21). Let v_0 be as in Definition 1.21 (we use the same notations). We set $T := \sup\{t > 0 : \|h(t, \cdot)\|_{L^\infty} \leq \delta_0\}$. By continuity and the choice of v_0, we know that $T > 0$. We have

$$\begin{cases} \partial_t h(t,x) - \Delta h(t,x) \leq (f'(u) + \eta_\delta)\, h(t,x) & \forall t \in (0,T),\ x \in \Omega, \\ \partial_\nu h(t,x) = 0 & \forall t \in (0,T),\ x \in \partial\Omega. \end{cases}$$

From $f'(u)+\eta_\delta \leq -\frac{\lambda_1}{2}$ and the parabolic comparison principle, we obtain $\|h(t,\cdot)\|_{L^\infty} \leq \delta_0 e^{-\frac{\lambda_1}{2}t}$, for all $t \in (0,T)$. We deduce $T = +\infty$ and $\|h(t,\cdot)\|_{L^\infty} \to 0$ when $t \to +\infty$, thus u is asymptotically stable. ∎

Chapter 2

Variations on the Casten, Holland, and Matano Theorem

In this Chapter, we prove further qualitative properties for stable solutions. First, give a quantitative criterion for the non-existence of patterns in bounded domains. This method allows simple proofs, implies numerous perturbation results, and allows to treat the case of non-selfadjoint operators. We also study the extension of our results to unbounded domains and refine the results of Chapter 1.

In a second step, we prove asymptotic properties for stable solutions when the domain satisfies a geometrical property at infinity. In particular, we prove that a stable solution converge to a constant when the domain is convex at infinity. Finally, we prove an estimate on the flatness of stable solution in bounded nonconvex domains. As a complement, we discuss the isolation of stable solutions in unbounded domains.

Contents

2.1 Introduction

2.1.1 Framework

Consider the following semilinear elliptic equation, with homogeneous Neumann boundary conditions

$$\begin{cases} -\Delta u(x) = f(u(x)) & \forall x \in \Omega, \\ \partial_\nu u(x) = 0 & \forall x \in \partial\Omega, \\ u \in C^2\left(\overline{\Omega}\right) \cap L^\infty(\Omega), \end{cases} \tag{2.1}$$

where ∂_ν denotes the outer normal derivative, f is a C^1 function and $\Omega \subset \mathbb{R}^n$ is a $C^{2,\alpha}$ domain. This chapter deals with qualitative properties of stable solutions of (2.1). Stability is defined in Definition 2.2.

We call *pattern* all stable non-constant solution of (2.1). In two seminal papers, Casten, Holland, and Matano prove the following.

Theorem 2.1 ([86, 217]) *If Ω is bounded and convex, there exists no pattern to* (2.1).

We point out that Theorem 2.1 holds for any nonlinearity f.

In this chapter, we propose to extend Theorem 2.1 in several directions. We show that the assumptions of Theorem 2.1, that is, Ω convex and u stable, can be combined into a single assumption, namely, the nonnegativity of the linearized operator

$$\begin{cases} -\Delta\psi - f'(u)\psi = 0 & \text{in } \Omega, \\ \partial_{\nu_A}\psi + \gamma\psi = 0 & \text{on } \partial\Omega, \end{cases}$$

where, in dimension $n = 2$, $\gamma(\cdot)$ is the curvature of $\partial\Omega$. It gives a quantitative criterion for the non-existence of patterns in general domains. This new approach provides simple proofs, implies numerous perturbation results, and allows to extend Theorem 2.1 to both nonconvex domains and unstable solutions. In addition, it treats the case of a non-selfadjoint operators. We also propose an extension to unbounded domains.

In a second part, we establish an asymptotic version of Theorem 2.1. That is, considering an unbounded domain which is convex *at infinity* (say, when $x_1 \to +\infty$), we show that all stable solutions *converge* to a constant. We also derive some asymptotic symmetry properties for stable solutions when the domain is, *at infinitiy*, either straight or stable by a planar rotation.

Finally, we derive an estimate on the flatness of stable solutions in bounded nonconvex domains. By *flat*, we mean a solution which gradient is small in all but one direction. This result can be put in perspective with the results of [169] or the De Giorgi conjecture.

Let us give a precise definition of stability.

Definition 2.2 *Let $\Omega \subset \mathbb{R}^n$ be a smooth domain and u be a solution of* (2.1).

— *Consider the linearized equation of* (2.1) *around* u

$$\begin{cases} -\Delta\psi - f'(u)\psi = 0 & \text{in } \Omega, \\ \partial_\nu \psi = 0 & \text{on } \partial\Omega, \end{cases} \qquad \psi \in C^2(\overline{\Omega}). \tag{2.2}$$

We denote λ_1 *the principal eigenvalue associated with* (2.2)*, see Chapter 3.*

— *The solution* u *is said to be stable if* $\lambda_1(u,\Omega) \geq 0$ *and stable non-degenerate if* $\lambda_1(u,\Omega) > 0$.

It is important to note that λ_1 formally corresponds to the lowest eigenvalue of the second variation of the energy associated with (2.1), thus $\lambda_1 > 0$ implies that the solution is a local minimum of the energy. We recall the Rayleigh formula (for self-adjoint operators)

$$\lambda_1 := \inf_{\substack{\psi \in H^1(\Omega) \\ \|\psi\|_{L^2}=1}} \mathcal{F}(\psi), \quad \text{with } \mathcal{F}(\psi) = \int_\Omega |\nabla\psi|^2 - f'(u)\psi^2. \tag{2.3}$$

Another important property of λ_1 is that it is associated with a positive eigenfunction.

Proposition 2.3 *There exists* $\varphi \in C^2$*, which is positive on* $\overline{\Omega}$ *and satisfies*

$$\begin{cases} -\Delta\varphi - f'(u)\varphi = \lambda_1\varphi & \text{in } \Omega, \\ \partial_\nu \varphi = 0 & \text{on } \partial\Omega. \end{cases}$$

The definition and the properties of λ_1 are discussed in more details in Chapter 3.

Outline. First, we recall the classical proof of Theorem 2.1 in section 2.1.2. A criterion for the non-existence of patterns in general bounded domains is proposed in section 2.2. We also discuss the extension of the results to unbounded domains. An asymptotic formulation of the Casten, Holland, and Matano Theorem is established in section 2.3. In section 2.4 we establish a flatness estimate for stable solution. As a complement, we give in section 2.5 a discussion on the isolation of stable solutions.

2.1.2 The classical proof of Casten, Holland, and Matano

We first recall the proof of (2.1) for bounded convex domains. Let Ω be a convex bounded domain, u a stable solution of (2.1) and set $v_i := \partial_{x_i} u$, for all $i \in \{1, \dots, n\}$.

Step 1. On the one hand, differentiating (2.1) with respect to x_i, we find that $v_i := \partial_{x_i} u$ satisfies the linearized equation

$$-\Delta v_i - f'(u)v_i = 0 \quad \text{in } \Omega. \tag{2.4}$$

From an integration by part we have

$$\mathcal{F}(v_i) = \int_{\partial\Omega} v_i \partial_\nu v_i = \frac{1}{2}\int_{\partial\Omega} \partial_\nu [v_i^2],$$

with $\mathcal{F} = \mathcal{F}_{(u,\Omega)}$ from Definition 2.2. On the other hand, as u is stable, we have $\mathcal{F}(\cdot) \geq 0$ and

$$0 \leq \mathcal{F}(v_i) \leq \sum_{k=1}^{n} \mathcal{F}(v_k) = \frac{1}{2}\int_{\partial\Omega} \partial_\nu \left[|\nabla u|^2\right].$$

Step 2. When the domain is convex, the above integral turns out to be non-positive, as stated in the following key lemma. This is where the *convexity* of the domain comes into play. It can be found in [86, 217], but a simple proof is presented at the end of the section for completeness. The lemma was first stated for *bounded convex* domain, but the same proof allows to state a more general result.

Definition 2.4 *For a smooth domain $\Omega \subset \mathbb{R}^n$, we denote by $\gamma : \partial\Omega \to \mathbb{R}$ the minimal curvature of $\partial\Omega$ (i.e the least eigenvalue of the second fundamental form). If $n = 2$, $\gamma(\cdot)$ is nothing but the curvature of $\partial\Omega$.*

Lemma 2.5 ([86, 217]) *Let Ω be a smooth domain and $\gamma(\cdot)$ from Definition 2.4. Let u be a C^2 function such that*

$$\partial_\nu u = 0 \quad on\ \partial\Omega. \tag{2.5}$$

Then

1. *$\partial_\nu |\nabla u|^2 \leq -\gamma |\nabla u|^2$ on $\partial\Omega$.*
2. *If $n = 2$, then $\partial_\nu |\nabla u|^2 = -\gamma |\nabla u|^2$ on $\partial\Omega$.*
3. *In particular, if Ω is convex then $\partial_\nu |\nabla u|^2 \leq 0$ on $\partial\Omega$.*

We then conclude that for all $i \in \{1, \dots, n\}$, we have $\mathcal{F}(v_i) = 0$ thus v_i minimizes $\mathcal{F}$. Note that, at this step, if we assume that u is not constant, i.e $v_i \not\equiv 0$ for some i, then we deduce $\lambda_1 = 0$, i.e u is *stable degenerate*.

Step 3. Owing to the above conclusion, we deduce as a classical fact that for all $i \in \{1, \dots, n\}$, v_i is a multiple of the principal eigenvalue associated to (2.4), which is denoted φ and is positive in $\overline{\Omega}$.

Step 4. From $\partial_\nu u = 0$ on the closed surface $\partial\Omega$, we deduce that v_i vanishes on some point of the boundary. But as v_i is colinear to φ, we conclude $v_i \equiv 0$, which completes the proof.

Before proving Lemma 2.5, we need the following definition.

Definition 2.6 *Let $\Omega \subset \mathbb{R}^n$. A "representation of the boundary" is a pair (ρ, U) where ρ is a C^2 function defined on U a neighborhood of $\partial\Omega$ such that*

$$\rho(x) \begin{cases} < 0 & if\ x \in \Omega \cap U \\ = 0 & if\ x \in \partial\Omega \\ > 0 & if\ x \in U \backslash \overline{\Omega} \end{cases} \qquad and \qquad \nabla\rho(x) = \nu(x) \quad \forall x \in \partial\Omega,$$

where $\nu(x)$ is the outer normal unit vector of $\partial\Omega$ at x.

It is classical that such a representation of the boundary always exists for $C^{2,1}$ domains, see e.g section 6.2 of [162].

PROOF (OF LEMMA 2.5) Let us consider (ρ, U) a representation of the boundary for Ω. Equation (2.5) becomes

$$\nabla u \cdot \nabla \rho = 0 \quad \text{on } \partial\Omega.$$

As ∇u is tangential to $\partial\Omega$, we can differentiate the above equality with respect to the vector field ∇u. It gives, on $\partial\Omega$,

$$0 = \nabla\left(\nabla u \cdot \nabla \rho\right) \cdot \nabla u = \nabla u \cdot \nabla^2 u \cdot \nabla \rho + \nabla u \cdot \nabla^2 \rho \cdot \nabla u.$$

From this, we infer

$$\partial_\nu |\nabla u|^2 = \nabla\left(|\nabla u|^2\right) \cdot \nabla \rho = 2\nabla u \cdot \nabla^2 u \cdot \nabla \rho = -2\nabla u \cdot \nabla^2 \rho \cdot \nabla u,$$

Now, note that $\underline{\gamma}$ is defined as the lowest eigenvalue of $\nabla^2\rho$ restricted to the tangent space of $\partial\Omega$, i.e for all $x \in \partial\Omega$

$$\gamma(x) = \inf\left\{X \cdot \nabla^2\rho(x) \cdot X : X \in \mathbb{R}^n, \|X\| = 1, X \perp \nu(x)\right\}.$$

Since ∇u is tangential to $\partial\Omega$, this concludes the proof of the two first statements, while the third comes from the fact that a convex domain has a nonnegative curvature everywhere. ■

In [86], the authors give the following remarkable geometrical interpretation of the third assertion of Lemma 2.5. Consider a bounded convex domain $\Omega \subset \mathbb{R}^2$. As u satisfies Neumann boundary conditions, its level set cross the border $\partial\Omega$ orthogonally. Since the domain is convex, these level sets go apart one from each other as we move outward $\partial\Omega$. As $|\nabla u|$ corresponds to the inverse of the distance of two level sets, it implies that $|\nabla u|$ decreases as we move outward Ω, hence the result.

2.2 Extension to non-self adjoint operators

2.2.1 Quantitative criterion for the non-existence of patterns

We consider problem (2.1) replacing the Laplacian $-\Delta$ with a more general elliptic operator. Let $-L$ be an homogeneous linear elliptic operator, i.e $-L$ is of the form, for all $u \in C^2$,

$$-Lu := -\text{div}\left(A \cdot \nabla u\right) - B \cdot \nabla u, \tag{2.6}$$

where $B \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ is a positive definite matrix. Consider the following semilinear elliptic equation

$$\begin{cases} -Lu(x) = f(u(x)) & \forall x \in \Omega, \\ \partial_{\nu_A} u(x) = 0 & \forall x \in \partial\Omega, \\ u \in C^{2,1}\left(\overline{\Omega}\right) \cap L^\infty(\Omega). \end{cases} \tag{2.7}$$

Here, $\partial_{\nu_A} u := \nu \cdot A \cdot \nabla u$, where ν denotes the outer normal direction of $\partial\Omega$. The nonlinearity f is, as before, of class C^1. We will focus in this section on the case where the domain Ω is smooth and *bounded*.

The approach of Casten-Holland and Matano introduced in section 2.1.2 crucially relies on the self-adjoint structure of the operator $-\Delta$ through the Rayleigh formula (2.3). Thus, we have to propose a completely different approach.

As in Definition 2.2, we define stability by the nonnegativity of the linearized operator.

Definition 2.7 *For u be a solution of* (2.7)*, we define the principal eigenvalue λ_1 associated with the linearized problem, for $\psi \in C^2(\Omega)$,*

$$\begin{cases} -L\psi - f'(u)\psi = 0 & \text{in } \Omega, \\ \partial_{\nu_A}\psi = 0 & \text{on } \partial\Omega, \end{cases}$$

The solution u is said to be stable if $\lambda_1 \geq 0$ and stable non-degenerate if $\lambda_1 > 0$.

Our first result is that the property of Casten Holland and Matano extends to non-selfadjoint operators.

Proposition 2.8 *Assume Ω is convex and bounded, and let u be a stable solution of* (2.7)*. Then u is constant.*

As we are about to see, this result turns out to be a specific case of a more general theorem. First, we need to consider a *modified* linearized equation.

Definition 2.9 *For u a solution of* (2.1)*, we consider the following problem, for $\psi \in C^2(\Omega)$,*

$$\begin{cases} -L\psi - f'(u)\psi = 0 & \text{in } \Omega, \\ \partial_{\nu_A}\psi + \gamma\psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.8}$$

with $\gamma(\cdot)$ from Definition 2.4. We define λ_1^γ as the principal eigenvalue of (2.8)

This definition is motivated by Lemma 2.5. Note that, although no sign is assumed on γ, the principal eigenvalue is well defined and satisfies all the classical properties, see Chapter 3.

Theorem 2.10 *Assume Ω is bounded, and let u be a solution of* (2.7)*. If $\lambda_1^\gamma \geq 0$, then u is constant.*

Note that, if Ω is convex, then $\gamma \gneqq 0$, and $\lambda_1 > \lambda_1^\gamma$ (see Proposition 2.12). Hence, the Casten-Holland and Matano property (Proposition 2.8) is a particular case of the above theorem. The assumptions "u stable" and "Ω convex" are somehow combined in the single assumption "$\lambda_1^\gamma \geq 0$".

The proof relies on the following observation, which is deduced from classical computations.

Lemma 2.11 *For u a solution of* (2.7)*, let φ be the principal eigenfunction associated with λ_1^γ. We denote $|\cdot|_A$ the norm induced by A (that is, $|X|_A = X \cdot A \cdot X$, for all $X \in \mathbb{R}^n$), and we set $W := \frac{|\nabla u|_A^2}{\varphi^2}$. We have*

$$\begin{cases} -\dfrac{1}{\varphi^2}\nabla \cdot \left(\varphi^2 A \cdot \nabla W\right) - B \cdot \nabla W = -2\lambda_1^\gamma W - 2\displaystyle\sum_{i=0}^n \left| \nabla \left(\frac{\partial_i u}{\varphi} \right) \right|_A^2 & \text{in } \Omega, \\ \partial_\nu W \leq 0 & \text{on } \partial\Omega. \end{cases} \tag{2.9}$$

Proof (of Theorem 2.10) Assume $\lambda_1^\gamma \geq 0$. From (2.9) and the strong Maximum Principle, we deduce that W is a constant. In particular, we obtain , $\forall i \in \{1, \dots, n\}$, $\left|\nabla\left(\frac{\partial_i u}{\varphi^\gamma}\right)\right|_A^2 \equiv 0$, i.e., $\partial_i u \equiv C_i \varphi^\gamma$ for some constant $C_i \in \mathbb{R}$. Then, from $\partial_{\nu_A} u = 0$ on $\partial\Omega$, we deduce that $\partial_i u$ vanishes on some point of the boundary. But $\varphi^\gamma > 0$ on $\overline{\Omega}$, thus $v_i \equiv 0$. ∎

2.2.2 Perturbation results

Theorem 2.10 allows more flexibility on the assumptions of the Casten, Holland and Matano property (Theorem 2.1). Numerous perturbation results can be easily deduced. For example, we can show, on the one hand, that there exists no patterns in some nonconvex domains, or on the other hand, that some unstable solutions must be constant. These questions reduce to the study of the sign of the principle eigenvalue λ_1^γ. We give, as a complement, the following classical properties.

Proposition 2.12 *Let $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, $-L$ as in* (2.6) *and $c : \Omega \to \mathbb{R}$ a continuous function. We define $\lambda_1^\gamma := \lambda_1^\gamma(-L - c, \Omega)$ as the principal eigenvalue of*

$$\begin{cases} -L\psi - c\psi = \lambda\psi & \text{in } \Omega, \\ \partial_{\nu_A}\psi + \gamma\psi = 0 & \text{on } \partial\Omega. \end{cases}$$

1. *λ_1^γ is continuous under smooth (say C^2) perturbations of the domain Ω.*
2. *Let c_1, $c_2 : \Omega \to \mathbb{R}$ be two continuous functions. We have*

$$c_1 \lneqq c_2 \quad \Rightarrow \quad \lambda_1^\gamma(-L - c_2) < \lambda_1^\gamma(-L - c_1),$$

and

$$\inf_\Omega [c_1 - c_2] \leq \lambda_1^\gamma(-L - c_2) - \lambda_1^\gamma(-L - c_1) \leq \sup_\Omega [c_1 - c_2].$$

3. *Let γ_1, $\gamma_2 : \partial\Omega \to \mathbb{R}$ be two continuous functions. Assume $\gamma_1 \lneqq \gamma_2$. Then*

$$\lambda_1^{\gamma_1} < \lambda_1^{\gamma_2}.$$

4. *If $-L$ is self adjoint (i.e $B = 0$ in* (2.6)*), then*

$$\lambda_1^\gamma := \inf_{\substack{\psi \in H^1(\Omega) \\ \|\psi\|_{L^2} = 1}} \left[\int_\Omega |\nabla\psi|_A^2 - c\psi^2 + \int_{\partial\Omega} \gamma\psi^2\right] \tag{2.10}$$

with $|\nabla\psi|_A^2 := \nabla\psi \cdot A \cdot \nabla\psi$. In particular, the mappings $c \mapsto \lambda_1^\gamma(-L, \Omega)$ and $\gamma \mapsto \lambda_1^\gamma(-L, \Omega)$ are concave.

The first statement can be found the litterature, see [106, 108, 171] and references therein. The second and third points are directly deduced from the definition and standard calculations. For the fourth statement, the Rayleigh formula is classical, and the second point comes from the fact that λ_1 is expressed as the infimum of linear function of c and γ.

2.2.3 Extension to unbounded domains

When the domain is unbounded, the principal eigenvalue does not exist in the classical sense, and the stability of a solution has to be defined. We thus need to generalize the definition of λ_1^γ (and λ_1). We let the reader refer to Chapter 3 for a more detailed discussion on this subject. Here, we only introduce the tools we need to derive our results. For readability, we first state our results and postpon the proofs at the end of the section.

Self-adjoint case

Let us first deal with the self-adjoint case, i.e., assume $B = 0$ in (2.6). We extend the definition of λ_1^γ to unbounded domains through the Rayleigh-Ritz formula (2.10). In the case $\lambda_1^\gamma > 0$, we have the following extension of Theorem 2.10.

Theorem 2.13 *Assume L is self-adjoint. Let Ω be possibly unbounded and u be a solution of* (2.7). *If $\lambda_1^\gamma > 0$, then u is constant.*

Proof Define, for all $\psi \in H^1(\Omega)$,

$$\mathcal{F}^\gamma(\psi) := \int_\Omega |\nabla\psi|_A^2 - f'(u)\psi^2 + \oint_{\partial\Omega} \gamma\psi^2,$$

the functional associated with λ_1^γ through (2.10). Denoting $v_i := \partial_{x_i} u$, from Lemma 2.5, we have on the boundary $\partial\Omega$,

$$\sum_{i=1}^n v_i \partial_{\nu_A} v_i - \gamma v_i^2 = \frac{1}{2} D_{\nu_A} |\nabla u|^2 \leq -\gamma |\nabla u|^2 \leq 0.$$

Then, we apply the same method as for Theorem 1.2. ∎

In the particular case $\Omega = \mathbb{R}^n$, then $\lambda_1^\gamma = \lambda_1$ and we recover some classical results [104, 160]. In the possibly degenerate case $\lambda_1^\gamma \geq 0$, we have the following result.

Theorem 2.14 *Assume L is self-adjoint and Ω satisfies the growth condition:*

$$\Omega \cap \{|x| \leq R\} = O(R^2). \tag{2.11}$$

Let u be a solution of (2.7) *such that $\lambda_1^\gamma \geq 0$.*

1. *Assume Ω is not a straight cylinder, i.e. Ω is not of the form $\mathbb{R} \times \Omega'$ with $\Omega' \subset \mathbb{R}^{n-1}$. Then u is constant.*
2. *Assume Ω is a straight cylinder. Then u is either constant, or a planar monotonic stationary wave connecting two stable roots (z^-, z^+) of f such that $\int_{z^-}^{z^+} f = 0$.*

Proof As in the proof of Theorem 2.13, the proof is deduced from the one of Theorem 1.3. ∎

Note that in the case of straight cylinder, we have $\gamma = 0$, thus $\lambda_1 = \lambda^\gamma$, and this result reduces to Theorem 1.3

Let us point out that the condition (2.11) is more than a technical restriction. In particular, the above result does not hold in $\mathbb{R}^{2n}$ for $2n \geq 8$, see [246]. This is related to the De Giorgi's conjecture, see the Introduction or Chapter 1 for more details.

Non-self-adjoint case

Let us now deal with case of a non-self-adjoint operator. Since the Rayleigh-Ritz formula is not avalaible anymore, we need to propose a new definition for λ_1^γ. We set

$$\lambda_1^\gamma := \sup\left\{\lambda \in \mathbb{R} : \exists \psi \in C^{2,\alpha},\ \begin{array}{ll} -L\psi - f'(u)\psi \geq \lambda\psi & \text{in } \Omega, \\ \partial_{\nu_A}\psi + \gamma\psi \geq 0 & \text{on } \partial\Omega. \end{array}\right\}.$$

This quantity is, in many aspect, a good generalization of the principal eigenvalue, see Chapter 3. In particular, it satisfies the Rayleigh-Ritz formula (2.10) if the operator is self-adjoint.

Case of a bounded drift. We make the following assumption:

$$\sup_{x\in\Omega}\left|B \cdot A^{-1} \cdot x\right| < +\infty.$$

This condition somehow expresses that the drift (induced by the first order term) is neither from nor towards infinity. Under this assumption, we have the same results as in the self-adjoint case, namely,

Theorem 2.15 *Let u be a solution of* (2.7).
- *If $\lambda_1^\gamma > 0$, then u is constant.*
- *Assume $\lambda_1^\gamma \geq 0$, Ω satisfies* (2.11) *and is not a straight cylinder. Then u is constant.*
- *$\lambda_1^\gamma \geq 0$, Ω satisfies* (2.11) *and is a straight cylinder. Then u is either constant, or a planar monotonic solution connecting two stable roots (z^-, z^+) of f such that $\int_{z^-}^{z^+} f = 0$.*

Proof Set

$$\zeta(x) = e^{B\cdot A^{-1}\cdot x}.$$

The function $\zeta(x)$ is bounded from above and below by two positive constants, and, for any $u \in C^2(\overline{\Omega})$,

$$\nabla \cdot (\zeta A \cdot \nabla u) = [\nabla \cdot (A \cdot \nabla u) + B \cdot \nabla u]\,\zeta.$$

From this, the proofs of Theorem 2.13 and Theorem 2.14 can easily be adapted. ■

General Case. In the general case, we do not know if positivity of λ_1^γ guarantees the validity of the Maximum Principle. Instead, one can define

$$\tilde{\lambda}_1^\gamma := \sup\left\{\lambda \in \mathbb{R} : \exists \psi \in C^{2,\alpha},\ \inf_\Omega \psi > 0,\ \begin{array}{ll} -L\psi - f'(u)\psi \geq \lambda\psi & \text{in } \Omega \\ \partial_{\nu_A}\psi + \gamma\psi \geq 0 & \text{on } \partial\Omega \end{array}\right\}.$$

In general, we have $\lambda_1 \geq \tilde{\lambda}_1$, but we do not know if equality holds. We have the following result.

Theorem 2.16 *Assume Ω is a (unbounded) uniformly smooth domain with a uniform interior ball condition, and let u be a solution of* (2.7). *If $\tilde{\lambda}_1^\gamma > 0$, then u is constant.*

Proof Assume $\tilde{\lambda}_1^\gamma > 0$. Let $\lambda \in (0, \tilde{\lambda}_1^\gamma)$ and $\psi \in C^2(\Omega)$ such that $\inf_\Omega \psi > 0$ and

$$\begin{cases} -L\psi - f'(u)\psi \geq 0 & \text{in } \Omega, \\ \partial_{\nu_A}\psi + \gamma\psi \geq 0 & \text{on } \partial\Omega. \end{cases}$$

We set $V := \frac{|\nabla u|_A^2}{\psi}$. We have that $\sup V < +\infty$ and, from standard computations,

$$\begin{cases} -LV - f'(u)V \leq 0 & \text{in } \Omega, \\ \partial_\nu V + \gamma V \leq 0 & \text{on } \partial\Omega. \end{cases}$$

From the Maximum Principle proved in Theorem 3.13, we deduce $V = 0$. ■

2.3 Asymptotic results

2.3.1 Statement

We propose in this chapter to establish an asymptotic formulation the Casten, Holland, and Matano property. That is, considering a domain which is *asymptotically* convex (say, when $x_1 \to +\infty$, see Figure 2.1), we show that all stable solutions *converge* to a constant. We also derive some symmetry properties for stable solutions when the domain is *asymptotically* straight in one direction or invariant with respect to a planar rotation. The method relies on recent results from Chapter 1, in which we study the Casten-Holland and Matano property in unbounded domains.

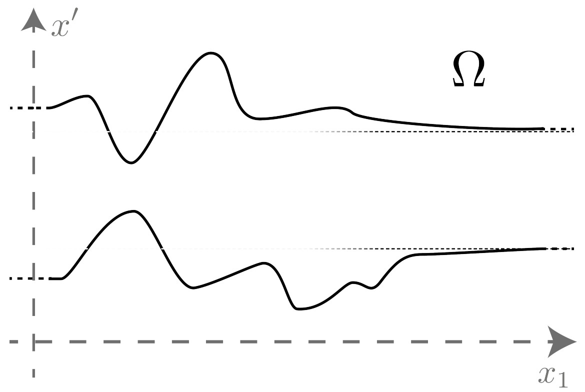


Figure 2.1 – Asymptotically convex cylinder

First, we give a definition for the convergence of a domain.

Definition 2.17 *Let $\Omega \subset \mathbb{R}^N$ be a uniformly $C^{2,\alpha}$ domain. For any real sequence $y \in \mathbb{R}$, we define the translated domains (on the x_1-axis)*

$$\Omega[y] := \{(x_1, \ldots, x_{N+1}) : (x_1 + y, \ldots, x_N) \in \Omega\}.$$

We say that Ω converges to $\Omega_\infty \subset \mathbb{R}^N$ if $\Omega[y] \underset{y\to+\infty}{\longrightarrow} \Omega_\infty$ in the sense that its boundary converges to that of Ω_∞ in the $C^{2,\alpha}_{\text{loc}}$ topology.

We now state the main results of this chapter. The first one deals with *stable non-degenerate* solutions of (2.1), when the domain is asymptotically convex.

Theorem 2.18 *Assume $\Omega \subset \mathbb{R}^N$ is uniformly $C^{2,\alpha}$ and converges to a convex domain $\Omega_\infty \subset \mathbb{R}^N$. Let u be a stable non-degenerate solution of* (2.1)*. Then u converges to a stable zero of f, C^2_{loc}-uniformly when $x_1 \to -\infty$.*

Note that Ω_∞ must be of the form $\mathbb{R} \times \omega_\infty$, for a convex domain $\omega_\infty \subset \mathbb{R}^{N-1}$.

Let us deal with the case of stable (possibly degenerate) solutions. As in Chapter 1, we need the following additional assumption:

$$\forall (z_1, z_2) \in \mathcal{Z},\ z_1 \neq z_2 \quad \int_{z_1}^{z_2} f \neq 0, \tag{2.12}$$

where

$$\mathcal{Z} := \{z \in \mathbb{R} \mid f(z) = 0 \text{ and } f'(z) \leq 0\}$$

is the set of the stable zeros of f. This assumption essentially prevents the existence of stationary planar waves. We also need the following technical assumption on the growth of the domain at infinity:

$$\limsup_{R \to +\infty} \frac{|\Omega_\infty \cap \{|x| \leq R|\}}{R^2} < +\infty. \tag{2.13}$$

We have evidence to believe that this assumption cannot be substentially relaxed. See section 1.4.1.4.2 for more details.

Theorem 2.19 *Assume $\Omega \subset \mathbb{R}^N$ is uniformly $C^{2,\alpha}$ and converges to a convex domain $\Omega_\infty \subset \mathbb{R}^N$ which satisfies* (2.13)*. We further assume that f has only isolated zeros and satisfies* (2.12)*. If u be a stable solution of* (2.1)*, it converges to a stable zero of f, C^2_{loc}-uniformly when $x_1 \to -\infty$.*

2.3.2 Proofs

Assume $\Omega \subset \mathbb{R}^N$ is uniformly $C^{2,\alpha}$ and converges to a convex domain $\Omega_\infty \subset \mathbb{R}^N$. Let u be a stable non-degenerate solution of (2.1) For technical reason, we need to extend u in all $\mathbb{R}^N$.

Lemma 2.20 *We can extend u to a uniformly $C^{2,\alpha}$ function (still denoted u) which is defined in $\mathbb{R}^N$, coincide with u on $\overline{\Omega}$, and is identically 0 in $\mathbb{R}^N \backslash U$, U being a neighborhood of $\overline{\Omega}$.*

Proof As u satisfies Neumann boundary conditions, we can extend it by reflexion on an open set $U \supset \overline{\Omega}$. Note that, since the domain is uniformly $C^{2,\alpha}$, we can choose U such that

$$\inf_{(x,y) \in \Omega \times \partial U} |x - y| > 0.$$

The extended function, denoted $\tilde{u}$, satsfies an elliptic equation on U for which classical global $C^{2,\alpha}$ estimate hold (see e.g. Theorem 6.30 in [162]). This procedure is classical but technical, see for instance Appendix A in [50]. From this, we infer global $C^{2,\alpha}$ estimates for $\tilde{u}$ in $\overline{\Omega}$.

Finally, choosing any open set $\tilde{U}$ such that $\Omega \subset \tilde{U} \subset U$, we can define $\tilde{\tilde{\mathrm{u}}} \in C^{2,\alpha}(\mathbb{R}^N \backslash \Omega)$ which coincide with $\tilde{u}$ in $\tilde{U}$ and is identically 0 in $\mathbb{R}^N \backslash U$. ∎

We consider the ω-limit set

$$\Sigma_u := \bigcap_{y\in\mathbb{R}} \overline{\{u[y'] : y' \geq y\}} \subset L^\infty_{\mathrm{loc}}(\mathbb{R}^N), \tag{2.14}$$

where

$$\forall y \in \mathbb{R}, \quad u[y] : (x_1, \ldots, x_N) \mapsto u(x_1 + y, \ldots, x_N).$$

The topological closure should be understood in the L^∞_{loc} sense. The key point is the following observation, which states that a solution is always "more stable" at infinity.

Lemma 2.21 *Let $u_\infty \in \Sigma_u$. It is a solution of* (2.1) *in* Ω_∞. *Moreover,* $\lambda_1(u, \Omega) \leq \lambda_1(u_\infty, \Omega_\infty)$.

Proof Let $u_\infty \in \Sigma_u$. There exists a sequence $y_n \to +\infty$ such that $u[y_n] \to u_\infty$. As a consequence of the uniform $C^{2,\alpha}$ estimates on $\overline{\Omega}$ (see the proof of Lemma 2.20), we deduce that the convergence $u[y_n] \to u_\infty$ occurs in C^2_{loc}. Thus, u_∞ is a solution of (2.1) in Ω_∞.

Consider $\varphi \in C^{2,\alpha}$ given by Proposition 3.3, define $a_n := \varphi(y_n)$ and $\varphi_n := \frac{1}{a_n}\varphi[z_n]$. Note that $\lambda_1(u[y_n], \Omega[y_n]) = \lambda_1(u, \Omega)$. We have

$$\begin{cases} -\Delta\varphi_n - f'(u[y_n])\varphi_n = \lambda_1(u, \Omega)\varphi_n & \text{a.e in } \Omega[y_n], \\ \partial_\nu \varphi_n = 0 & \text{a.e on } \partial\Omega[y_n]. \end{cases}$$

We can extend by reflexion φ_n on a neighborhood of $\overline{\Omega[y_n]}$ on which it satisfies an elliptic equation, thus can apply the Harnack inequality in $\overline{\Omega_\infty}$ to infer that φ_n is locally bounded, uniformly in n (for more details, see the proof of Proposition 1 p.30 in [50]). Thanks to classical elliptic estimates, we can extract a subsequence (still denoted n) such that φ_n converges to some $\varphi_\infty \in C^{2,\alpha}(\Omega_\infty)$ in the C^2_{loc} sense. Then, almost everywhere in Ω_∞, $\varphi_\infty > 0$ and $(\Delta + f'(u_\infty) + \lambda_1(u, \Omega))\varphi_\infty \leq 0$. From Corollary 1.13, we deduce $\lambda_1(u_\infty, \Omega_\infty) \geq \lambda_1(u, \Omega)$. ■

We also need the following general result. It can be compared to Theorem 2.9 in [217], dealing with evolution problems.

Lemma 2.22 Σ_u *is a connected set in the* $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ *topology.*

Proof By contradiction, assume there exists $Y \subset \Sigma_u$ both open and closed, $Y \neq \emptyset$ and $Y \neq \Sigma_u$. Note that Σ_u is a compact subset of $L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, thus Y is compact. As, in addition, $\Sigma_u \backslash Y[z]$ is a closed set, there exists a an open set $V \subset L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ and a closed set $F \subset L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ such that $Y \subsetneq V \subsetneq F$ and $Y = \Sigma_u \cap F$.

Since $Y \neq \emptyset$ and $Y \neq \Sigma_u$, there exists a real sequence $(y_n)_{n\geq 0}$ such that $y_n \to +\infty$ and, for all integer $n \geq 0$,

$$u[y_{2n}] \in V, \qquad u[y_{2n+1}] \notin F.$$

By continuity of the mapping $y \mapsto u[z]$, we deduce that, for all $n \geq 0$, there exists $\tilde{y}_n \in [y_{2n}, y_{2n+1}]$ such that

$$u[\tilde{y}_n] \in F \backslash V.$$

As the sequence $u[\tilde{y}_n]$ is uniformly $C^{2,\alpha}$, it converges up to an extraction to some $u_\infty \in \Sigma_u$. But we also have $u_\infty \in F \backslash V$, thus $u_\infty \in Y$ and $u_\infty \notin Y$: contradiction. ■

We are now ready to prove the main result of the paper.

PROOF (OF THEOREM 2.18 AND THEOREM 2.19) Let $u_\infty \in \Sigma_u$. From Lemma 2.21, u_∞ is a stable solution of (2.1) in Ω_∞, which is convex. From Theorem 1.2 and Theorem 1.3, we deduce that u_∞ is constant. We have

$$\Sigma_u \subset \mathcal{Z} := \{z \in \mathcal{Z} : f'(z) \leq 0\} .$$

Assume f has only isolated zeros. Then Σ_u is a discrete set, and is also connected (Lemma 2.22). Hence, it is a singleton, which achieves the proof.

Assume instead that u is stable non-degenerate, then we have

$$\Sigma_u \subset \mathcal{Z}^\star := \{z \in \mathcal{Z} : f'(z) < 0\} ,$$

which is a discrete set, and we conclude as above. ■

2.3.3 Other asymptotic symmetries

This section is devoted to state some asymptotic properties of stable solutions of (2.1) when the domain is asymptotically straight in one direction or invariant with respect to a planar rotation. We define what we mean for a domain to satisfy a geometrical property *asymptotically*. Note that we allow the domain not to converge to a limiting domain (in the sense of Definition 2.17).

Definition 2.23 *Let $\Omega \subset \mathbb{R}^N$ be a uniformly $C^{2,\alpha}$ domain. For any real sequence $y_n \to +\infty$, $\Omega[y_n]$ converges (up to an extraction) to some $\Omega_\infty \subset \mathbb{R}^N$ (in the sense of Definition 2.17). We define the set of all possible limiting domains*

$$\Gamma_\Omega := \left\{\Omega_\infty \subset \mathbb{R}^N : \exists y_n \to +\infty,\ \Omega[y_n] \text{ converges to } \Omega_\infty\right\}.$$

We say that such a domain Ω satisfies a geometrical property asymptotically if every $\Omega_\infty \in \Gamma_\Omega$ satisfies this property.

Proposition 2.24 *Let $\Omega \subset$ be a uniformly $C^{2,\alpha}$ domain which is asymptotically straight in a direction $e \in \mathbb{S}^N$. Let u be a stable solution of* (2.1). *If u is stable non-degenerate, then $\partial_e u$ converges to 0, C^1_{loc}-uniformly when $x_1 \to +\infty$.*

PROOF Let $y_n \to +\infty$. Up to an extraction (still denoted y_n), $u[z_n]$ converges to some $u_\infty \in \Sigma_u$ in $C^{2,\alpha}_{\mathrm{loc}}$ and $\Omega[y_n]$ converges to some $\Omega_\infty \in \Gamma_\Omega$. From Lemma 2.21, we deduce that u_∞ is a stable non-degenerate solution of (2.1) in Ω_∞. As Ω_∞ is straight in the direction e, using Proposition 1.4 we deduce that $\partial_e u_\infty \equiv 0$, hence the result. ■

In the particular case where the direction e coincides with that of x_1, we have the following.

Proposition 2.25 *Let $\Omega \subset$ be a uniformly $C^{2,\alpha}$ domain which converges to a domain $\mathbb{R} \times \omega_\infty$, where $\omega_\infty \subset \mathbb{R}^{N-1}$ is a bounded domain. We generically denote $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^n$. Let u be a stable non-degenerate solution of* (2.1). *Then $u(x_1, x')$ converges to some $u_\infty(x')$, C^1_{loc}-uniformly when $x_1 \to +\infty$, where u_∞ is a stable solution of*

$$\begin{cases} -\Delta_{x'} U = f(U) & \text{in } \omega_\infty, \\ \partial_\nu U = 0 & \text{on } \partial\omega_\infty. \end{cases}$$

Proof We apply the same method as in the proof of Theorem 2.18, using Proposition 1.4 and the fact that stable non-degenrate solutions are isolated among solutions (see Lemma 2.30 for more details). ■

We now turn to the case where the limiting domains Ω_∞ are invariant with respect to a planar rotation.

Definition 2.26 *We say that a domain $\Omega_\infty \subset \mathbb{R}^{N+2}$, $N \geq 0$ is θ-invariant if $\Omega_\infty = \Omega'_\infty \times [0, 2\pi)$, where $\Omega'_\infty \subset \mathbb{R}^N \times \mathbb{R}^+$ in some cylindrical coordinates $(x, r, \theta) \in \mathbb{R}^N \times \mathbb{R}^+ \times [0, 2\pi)$.*

We also assume the radial section to be uniformly bounded, namely, in the cylindrical system of coordinates:

$$\sup_{(x,r,\theta)\in\Omega} r < +\infty. \tag{2.15}$$

Proposition 2.27 *Let $\Omega \subset$ be a uniformly $C^{2,\alpha}$ domain which satisfies* (2.15) *and is asymptotically θ-invarient. Let u be a stable solution of* (2.1)*. Assume either that u is stable non-degenerate or that Ω satisfies* (2.13)*. Then $\partial_\theta u \to 0$, C^1_{loc}-uniformly when $x_1 \to +\infty$.*

Proof We proceed as in the proof of Proposition 2.24 and we use Theorem 1.6. ■

2.4 Flatness estimate

In this section, we are interested in estimating the flatness of stable solutions of (2.1) in bounded domains. A solution is said to be flat if its gradient is small in all but one direction.

We need the following classical result.

Lemma 2.28 (Spectral Gap) *Defining*

$$\lambda_2 := \inf \left\{ \mathcal{F}(\psi) : \psi \in H^1(\Omega),\ \|\psi\|_{L^2} = 1,\ \int \psi\varphi = 0 \right\},$$

with $\mathcal{F}$ defined in (2.3)*, we have*

$$\lambda_2 - \lambda_1 > 0.$$

This property is known as the *Spectral Gap*. Numerous estimate on $\lambda_2 - \lambda_1$ are available in the litterature, see [12, 97, 98]. For Dirichlet boundary conditions, see [175] and references therein. We now state the main result of this section.

Proposition 2.29 *Let us consider u a stable solution of* (2.1)*. There exists $(e_1, \ldots, e_n)$ an orthonormal basis of $\mathbb{R}^n$ such that, for $v_i = \frac{\partial u}{\partial e_i}$, we have*

$$\|v_i\|^2_{L^2} \leq \frac{1}{\lambda_2} I_\gamma(\nabla u), \quad \forall i \in \{2, \cdots, n\},$$

where

$$I_\gamma(\nabla u) := -\int_{\partial\Omega} \gamma |\nabla u|^2$$

and γ from Definition 2.4.
If we further assume that u is stable non-denerate, then

$$\|\nabla u\|^2_{L^2} \leq \left(\frac{1}{\lambda_1} + \frac{n-1}{\lambda_2} \right) I_\gamma(\nabla u).$$

Since u is bounded, standard elliptic estimates implies $|\nabla u| \leq C$ for some $C > 0$ which only depends on f and Ω. Then,

$$I_\gamma(\nabla u) \leq C|\partial\Omega| \sup_{\partial\Omega} \gamma^-$$

(where $^-$ denoted the negative part), which gives a bound independant of u. In particular, if Ω is convex, then $\gamma \geq 0$ and our estimates reduces to $v_2, \ldots, v_n \equiv 0$, i.e., u is flat (however, in this case, the Casten, Holland, and Matano property already imply $\nabla u = 0$).

Proof Denoting $\langle \cdot, \cdot \rangle$ the usual $L^2(\Omega)$ scalar product and φ the principal eigenfunction (given by Proposition 2.3), we have

$$L^2(\Omega) = \mathrm{Vect}(\varphi) \oplus \mathrm{Vect}(\varphi)^\perp.$$

Hence, for any vector $\xi \in \mathbb{R}^n$, we can write $\nabla u \cdot \xi = a_\xi \varphi + \psi_\xi$ with $a_\xi := \langle \nabla u \cdot \xi, \varphi \rangle$, $\psi_\xi \in L^2$ and $\int \varphi \psi_\xi = 0$. The continuous linear form

$$\begin{array}{ccc} \mathbb{R}^n & \to & \mathbb{R} \\ \xi & \mapsto & a_\xi \end{array}$$

vanishes on an hyperplane H. We consider $(e_2, \ldots, e_n)$ an orthonormal basis of H, that we supplement with a unit vector $e_1 \in H^\perp$ (so that $(e_1, e_2, \ldots, e_n)$ is an orthonormal basis of $\mathbb{R}^n$). We set $v_i = \nabla u \cdot e_i$, and a_i, ψ_i the corresponding elements of the L^2 decomposition. By construction, we have $a_2 = \cdots = a_n = 0$.

Let us fix $i \in \{1, \ldots, n\}$. On the one hand, we have

$$\mathcal{F}(v_i) = a_i^2 \mathcal{F}(\varphi) + \mathcal{F}(\psi_i) \geq a_i^2 \lambda_1 \|\varphi\|_{L^2}^2 + \lambda_2 \|\psi_i\|_{L^2}^2 \geq \lambda_2 \|\psi_i\|_{L^2}^2.$$

On the other hand, from an integration by part we have

$$\mathcal{F}(v_i) = \frac{1}{2} \int_{\partial\Omega} \partial_\nu \left[v_i^2 \right],$$

and since $\mathcal{F}(\cdot) \geq 0$,

$$\mathcal{F}(v_i) \leq \sum_{k=1}^n \mathcal{F}(v_k) = \frac{1}{2} \int_{\partial\Omega} \partial_\nu \left[|\nabla u|^2 \right] \leq - \int_{\partial\Omega} \gamma(x) |\nabla u|^2 dx,$$

where the last inequality comes from Lemma 2.5. We end up with

$$\|\psi_i\|_{L^2}^2 \leq -\frac{1}{\lambda_2} \int_{\partial\Omega} \gamma(x) |\nabla u|^2 dx,$$

and we conclude with $\|v_i\|_{L^2}^2 = \|\psi_i\|_{L^2}^2$ for $i \in \{2, \ldots, n\}$.

If u is stable non-degenerate, we use the same method to show $\|v_i\|_{L^2}^2 \leq \frac{1}{\lambda_1} I_\gamma(\nabla u)$. ■

As a complement, we propose an approach to estimate the L^2 norm of v_1 when u is stable, possibly degenerate. As in the previous proof, denoting $\langle \cdot, \cdot \rangle$ the usual $L^2(\Omega)$ scalar product and φ the principal eigenfunction (given by Proposition 2.3), we write $v_1 = a_1 \varphi + \psi_1$ with $a_1 := \langle v_1, \varphi \rangle$ and $\psi \in L^2$ such that $\int_\Omega \varphi \psi_1 = 0$. We choose the normalization $\|\varphi\|_{L^2} = 1$. First, from the above proof, we have $\|\psi_1\|_{L^2}^2 \leq \frac{1}{\lambda_2} I_\gamma(\nabla u)$.

Now, let us find a bound on a_1. From $\partial_\nu u = 0$ on $\partial\Omega$, we know that there exists $x_1 \in \partial\Omega$ such that $v_1(x_1) = 0$. For all $x \in \Omega$, $|v_1(x)| < Cd_1(x)$, where $C = \|\nabla u\|_\infty$ and $d_1(x)$ is the geodesic distance between x and x_1 in Ω. Then, we set $V := \left\{x \in \overline{\Omega} : Cd_1(x) \leq \frac{1}{2}a_1\varphi(x)\right\}$. We have

$$\|\psi_1\|_{L^2}^2 = \|v_1 - a_1\varphi\|_{L^2}^2 \geq \int_V (v_i - a_i\varphi)^2 \geq \frac{1}{4}a_1^2 \left(\inf_\Omega \varphi\right)^2 |V|.$$

We denote by r_1 the radius of the interior ball condition at x_1, i.e. Ω contains a ball of radius r_1 which is tangent with $\partial\Omega$ on x_1. Recalling that $\inf_\Omega \varphi > 0$, we set

$$\underline{r} := \min\left(r_1, \frac{1}{4C}a_1 \inf_\Omega \varphi\right) > 0.$$

Then V contains a ball of radius $\underline{r}$ and $|V| \geq k_n \underline{r}^n$, where k_n is the volume of the unit ball. We end up with

$$a_1 \leq \max\left(\frac{2\|\psi_1\|_{L^2}}{\sqrt{k_n} r_1^{n/2} \inf_\Omega \varphi}, \frac{4^{\frac{n+1}{n+2}} \|\psi_1\|_{L^2}^{\frac{2}{n+2}}}{C^{\frac{2}{n+2}} \inf_\Omega \varphi}\right).$$

2.5 Appendix: isolation of stable solutions

In this section, we give a brief discussion on the isolation of stable solutions of (2.1) in the set of all solutions. Note that this question is crucial in the proof of Theorem 2.18, since the key point is to show that Σ_u is a discrete set. When considering the L^∞ topology, we have the following.

Lemma 2.30 *Let $\Omega \subset \mathbb{R}^N$ be a smooth domain (possibly unbounded). We denote S the set of solutions of* (2.1) *in Ω. Let $u \in S$ be stable non-degenerate or unstable non-degenerate (i.e. $\lambda_1 < 0$). Then, u is isolated in S for the $L^\infty(\Omega)$ topology.*

This result is essentially classical, at least for bounded domains. We give a proof at the end of the section.

Note that, in general, this result fails for L^∞_{loc} topology. For example, consider the Allen-Cahn equation in $\mathbb{R}$

$$-u'' = u(1-u)(u - 1/2).$$

This equation admits an explicit solution $u : x \mapsto \frac{\tanh(x)+1}{2}$ which is stable (degenerate). On the one hand, the family of the translated solutions $u_a(\cdot) = u(\cdot - a)$ converges to 0 when $a \to +\infty$ in the L^∞_{loc} topology. On the other hand, 0 is a stable non-degenerate solution (because $f'(0) < 0$).

One could argue that the above counterexample relies on the fact that the nonlinearity is balanced, that is, $\int_0^1 f = 0$ with $f(u) := u(1-u)(u - 1/2)$). However, one can build similar counter examples for unbalanced nonlinearities by considering a ground states, which have been proved to exist in most cases, see e.g [45].

In an attempt to extend the method of section 2.3, we address the following question.

Question: *Is Σ_u from* (2.14) *always a singleton when u is stable non-degenerate?*

We think the answer is negative. However we are not able to exhibit a counterexample.

Proof (of Lemma 2.30) By contradiction, assume there exists $u_k \in S$, $u_{k+1} \not\equiv u_k$, a sequence that converges to u which is stable non-degenerate (the case unstable non-degenerate is similar). We set $v_k := u_{k+1} - u_k$. For all k, u_k is a solution of (2.1) in Ω, thus

$$\begin{cases} -\Delta v_k - c_k(x) v_k = 0 & \text{in } \Omega, \\ \partial_\nu v_k = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.16}$$

where

$$c_k(x) := \frac{f(u_{k+1}(x)) - f(u_k(x))}{u_{k+1}(x) - u_k(x)}.$$

As f is C^1 and u is bounded, $c_k(x)$ converges uniformly to $f'(u(x))$ when $k \to +\infty$.

Formally, we have

$$\mathcal{F}_{(u,\Omega)}\left(\frac{v_k}{\|v_k\|_{L^2}}\right) \leq \|f'(u) - c_k\|_\infty \xrightarrow[k\to 0]{} 0,$$

which contradicts the fact that u is stable non-degenerate. But the former calculation is not licit when Ω is unbounded. To make it rigorous, we use the cut-off function χ_R defined in (1.9).

Multiplying (2.16) by χ_R^2, integrating on Ω, using the divergence theorem and the boundary condition in (2.16) we find

$$\begin{aligned} \mathcal{F}_{(u,\Omega)}\left(\frac{\chi_R v_k}{\|\chi_R v_k\|_{L^2}}\right) &= \frac{\int_\Omega \chi_R^2 v_k^2 (c_k - f'(u))}{\int_\Omega \chi_R^2 v_k^2} + \frac{\int_\Omega |\nabla \chi_R|^2 v_k^2}{\int_\Omega \chi_R^2 v_k^2} \\ &\leq \|c_k - f'(u)\|_{L^\infty(\Omega_R)} + 4\, \alpha_{k,R}, \end{aligned}$$

where

$$\mathcal{C}_k(R) := \int_{\Omega_R} v_k^2, \quad \alpha_{k,R} := \frac{\mathcal{C}_k(2R) - \mathcal{C}_k(R)}{R^2 \mathcal{C}_k(R)}.$$

We claim

$$\forall k \geq 0, \quad \liminf_{R\to+\infty} \alpha_{k,R} \leq 0. \tag{2.17}$$

As $\|c_k - f'(u)\|_{L^\infty(\Omega_R)}$ goes to 0 when $k \to +\infty$, uniformly in R, the claim (2.17) implies that $\mathcal{F}_{(u,\Omega)}$ can be made arbitrarily small, which contradicts the fact that u is stable non-degenerate and achieves the proof.

We now prove (2.17). By contradiction, fix $k \geq 0$ and assume $\alpha_{k,R} \geq \delta > 0$ for all $R \geq 1$. We have $\mathcal{C}_k(2R) \geq \delta R^2 \mathcal{C}_k(R)$. Iterating, we find, for R large enough

$$\mathcal{C}_k(2^j R) \geq K\left(\delta R^2\right)^j, \quad \text{for all integer } j \geq 1,$$

where positive constants are generically denoted K. Besides v is bounded, hence $\mathcal{C}_k(R) \leq K R^n$. Hence

$$K\left(2^j R\right)^n \geq \left(\delta R^2\right)^j.$$

Fixing R large enough, we reach a contradiction as j goes to $+\infty$. ∎

Chapter 3

The generalized Robin principal eigenvalue

In this section, we introduce the principal eigenvalue of an elliptic operator with indefinite Robin boundary condition. The word indefinite refers to the fact that no sign assumption is made on the zero order coeficient of the boundary condition. Then, we extend this definition to unbounded domains and prove the existence of a generalized eigenfunction.

In the self adjoint case, we show that the positivity of the principal eigenvalue is equivalent to the validity of the Maximum Principle. If the domain satisfies a certain growth condition at infinity, we show that the nonnegativity of the principal eigenvalue implies what we call the *Critical* Maximum Principle.

In the non-self-adjoint case, we give a necessary and a sufficient condition for the validity of the Maximum Principle.

Contents

3.1 Definition of the principal eigenvalue

3.1.1 Framework

Let $\Omega \subset \mathbb{R}^n$ be a smooth domain (possibly unbounded) and $-\mathcal{L}$ be a linear elliptic operator, i.e $-\mathcal{L}$ is of the form, for all $u \in C^2(\overline{\Omega})$,

$$-\mathcal{L}u(x) := -\text{div}\left(A(x)\cdot\nabla u(x)\right) - B(x)\cdot\nabla u(x) - c(x)\psi(x), \quad \forall x \in \Omega, \tag{3.1}$$

where, $c : \Omega \to \mathbb{R}$, $B : \Omega \to \in \mathbb{R}^n$, and $A : \Omega \to \mathbb{R}^{n\times n}$ such that $A(x)$ is a positive symmetric matrix, uniformly in $x \in \Omega$. For simplicity, we will always assume that the coefficients are smooth and bounded.

We associate the operator $-\mathcal{L}$ with some homogeneous Robin boundary conditions

$$\mathcal{B}^\gamma u(x) = 0, \quad \forall x \in \partial\Omega,$$

where

$$\mathcal{B}^\gamma u(x) := \partial_{\nu_A} u(x) + \gamma(x)u(x),$$

with $\gamma(\cdot)$ a smooth and bounded function defined on $\partial\Omega$, $\partial_{\nu_A} u := \nu \cdot A \cdot \nabla u$, and $\nu(x)$ the outer normal direction of $\partial\Omega$ at x. The Neumann boundary conditions corresponds to $\mathcal{B}^\gamma$ with $\gamma = 0$, and the Dirichlet boundary conditions formally corresponds to $\gamma = +\infty$.

Definition 3.1 *We say that $u \in C^2(\overline{\Omega})$ is a subsolution (resp. supersolution) of $(\mathcal{L}, \mathcal{B})$ in Ω if and only if*

$$\begin{cases} -\mathcal{L}u \leq 0 \ \textit{(resp.} \geq 0\textit{)} & \textit{in } \Omega, \\ \mathcal{B}^\gamma u \leq 0 \ \textit{(resp.} \geq 0\textit{)} & \textit{on } \partial\Omega. \end{cases}$$

We are concern with the validity of the Maximum Principle, defined as follows.

Definition 3.2 *We say that $(\mathcal{L}, \mathcal{B}^\gamma)$ satisfies the Maximum Principle in Ω if every subsolution with finite supremum is nonpositive.*

It is well know that, if $c \leq 0$ and $\gamma \geq 0$, then $(\mathcal{L}, \mathcal{B}^\gamma)$ satisfies the Maximum Principle in Ω. However, those conditions are very restrictive and we need a more flexible criterion for the validity of the Maximum Principle. It motivates the introduction of the *principal eigenvalue*.

3.1.2 The principal eigenvalue on bounded domains

Let us assume in this section that Ω is bounded. We consider the following eigenvalue problem with indefinite Robin boundary conditions

$$\begin{cases} -\mathcal{L}\psi = \lambda\psi & \text{in } \Omega, \\ \partial_{\nu_A}\psi + \gamma\psi = 0 & \text{on } \partial\Omega, \\ \lambda \in \mathbb{R} \quad ; \quad \psi \in C^2(\overline{\Omega}). \end{cases} \tag{3.2}$$

The term *indefinite* comes from the fact that no sign is assumed on γ. This problem has already been considered by many authors, see, for instance [4, 99, 109, 110, 197, 200, 279] and references therein. See also [70] for a survey on the Robin problem.

Nevertheless, the litterature devotes much more attention to the case $\gamma \geq 0$. The reason is mostly technical. However, it is noted in [109] that (3.2) can be re-written in a similar form but with a new boundary coefficient $\tilde{\gamma}$ which is nonnegative, while preserving the structure of the operator $(\mathcal{L}, \mathcal{B}^\gamma)$. This shows that most of the known results in the case $\gamma \geq 0$ extend to the general case.

In particular, if Ω is bounded, the classical Krein-Rutman theory provides the existence of an eigenvalue λ_1 of problem (3.2) (refered as the *principal eigenvalue*) associated with a positive eigenfunction φ_1 (refered as the *principal eigenfunction*). This eigenvalue is real, simple, and has the lowest real part among all eigenvalues. In addition, it is the only eigenvalue associated with a positive eigenfunction.

The key property of λ_1 is the following: if Ω is bounded, then

$$(\mathcal{L}, \mathcal{B}^\gamma) \text{ satisfies the Maximum Principle in } \Omega \text{ if and only if } \lambda_1 > 0.$$

This result is classical, see, e.g. [162, 252]. Another important (and closely related) result is that, if Ω is bounded, then

$$(\mathcal{L}, \mathcal{B}^\gamma) \text{ admits a positive supersolution in } \Omega \text{ if and only if } \lambda_1 \geq 0. \tag{3.3}$$

3.1.3 Extension to unbounded domains

The Krein-Rutman theory cannot be applied if Ω is unbounded because the resolvent of $-\mathcal{L}$ is not compact, and we have to find another way to define λ_1. Property (3.3) motivates the following definition

$$\lambda_1 := \sup \left\{ \lambda \in \mathbb{R} : (\mathcal{L} + \lambda, \mathcal{B}^\gamma) \text{ admits a positive supersolution} \right\}. \tag{3.4}$$

This definition is inspired by [51]. As long as γ is bounded, λ_1 is well defined and finite. The following sections are devoted to the study of λ_1.

3.1.4 Existence of a positive eigenfunction

A remarkable property of λ_1 is that, even when the domain is unbounded, it is associated with a positive eigenfunction.

Proposition 3.3 *Let $\Omega \subset \mathbb{R}^n$ be a smooth (possibly unbounded) domain and $-\mathcal{L}$ an elliptic operator as in* (3.1). *There exists $\varphi_1 \in C^2(\overline{\Omega})$ which is positive on $\overline{\Omega}$ and satisfies*

$$\begin{cases} -L\varphi_1 = \lambda_1 \varphi_1 & \text{in } \Omega, \\ \partial_{\nu_A}\varphi_1 + \gamma\varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

We refer to φ_1 as a principal eigenfunction of (3.2).

Proof The proof follows closely [51] (see the proofs of Theorem 3.1 and Proposition 1). For any $R > 0$, let $D_R \subset \mathbb{R}^n$ be a smooth connected open set such that $B_R \subset D_R \subset B_{2R}$, with B_R the ball of radius R. We also choose D_R to be increasing with R, and to be such that $\partial D_R \cap \partial\Omega$ is a C^2 $(n-2)$-dimensional manifold. Set $\Omega_R := \Omega \cap D_R$ and consider the eigenvalue problem with mixed boundary conditions

$$\begin{aligned} &-\mathcal{L}\psi = \lambda\psi && \text{a.e in } \Omega_R, \\ &\partial_{\nu_A}\psi + \gamma\psi = 0 && \text{a.e on } \partial\Omega \cap D_R, \\ &\psi = 0 && \text{a.e on } \Omega \cap \partial D_R. \end{aligned} \tag{3.5}$$

From the results of Liberman [205], we know that all classical results (Schauder estimate, Maximum Principle, solvability, etc.) hold from the mixed boundary value problem above. As Ω_R is bounded, the Krein-Rutman theorem provides a pair of principal eigenelements $(\lambda_1^R, \varphi_1^R)$. From Hopf's lemma, we have $\varphi > 0$ on $\overline{\Omega}$. We choose the normalization $\varphi_1(0) = 1$. Note that we impose Dirichlet boundary conditions on $\Omega \cap \partial D_R$ to ensure a decreasing monotonicity of $R \mapsto \lambda_1^R$ (which would not be true with other boundary conditions, see for example [163, 164] and references therein). Hence, λ_1^R converge to some $\underline{\lambda}_1$ when $R \to +\infty$.

Now, fix a compact $0 \in K \subset \overline{\Omega}$ and assume R is large enough so that $K \subset \overline{\Omega}_R \backslash D_R$. From Theorem 3.3 in [206] and Theorem 4.3 in [207], we derive a Harnack estimate, that is,

$$\sup_K \varphi_1^R \leq C \inf_K \varphi_1^R$$

with a constant C independent of R. From $\varphi(0) = 1$, we deduce that φ_1^R is bounded in K, uniformly in R. From classical Schauder estimates, we deduce that φ_1^R is $C^{2,\alpha}(K)$, uniformly in R. Up to extraction of a subsequence, φ_1^R converges to some φ_1 in $C^2(K)$. From a diagonal argument, we are provided with $\varphi_1 \in C^2(\overline{\Omega})$ which satisfies

$$\begin{cases} -L\varphi_1 = \underline{\lambda}_1\varphi_1 & \text{in } \Omega, \\ \partial_{\nu_A}\varphi_1 + \gamma\varphi_1 = 0 & \text{on } \partial\Omega, \end{cases}$$

and $\varphi_1 > 0$ on $\overline{\Omega}$. Consequently $\lambda_1 = \underline{\lambda}_1$, which achieves the proof. ■

As a direct consequence,

Corollary 3.4 $(\mathcal{L}, \mathcal{B}^\gamma)$ *admits a positive supersolution in* Ω *if and only if* $\lambda_1 \geq 0$.

3.2 The self-adjoint case

We first deal with the case of self-adjoint operators, that is, we assume throughout the section that $B \equiv 0$ in (3.1).

3.2.1 Rayleigh-Ritz variational formula

In the self-adjoint case, the principal eigenvalue can be expressed through the so-called Rayleigh-Ritz variational formula. This result is classical in bounded domains.

Proposition 3.5 *Assume* Ω *is smooth (possibly unbounded) and that* $-L$ *is a self-adjoint elliptic operator. For* λ_1 *defined in* (3.4)*, we have*

$$\lambda_1 := \inf_{\substack{\psi \in H^1(\Omega) \\ \|\psi\|_{L^2} = 1}} \mathcal{F}(\psi), \tag{3.6}$$

where

$$\mathcal{F}(\psi) := \mathcal{F}(\Omega, -L, \gamma)(\psi) = \int_\Omega |\nabla\psi|_A^2 - c\psi^2 + \int_{\partial\Omega} \gamma\psi^2,$$
$$|\nabla\psi|_A^2 := \nabla\psi \cdot A \cdot \nabla\psi.$$

To prove this result, first note that from the dominated convergence theorem and classical density results, it is equivalent to take the infimum on compactly supported smooth test functions in (3.6), namely

$$\lambda_1 = \inf_{\substack{\psi \in C_c^1(\overline{\Omega}) \\ \|\psi\|_{L^2}=1}} \mathcal{F}(\psi),$$

where $C_c^1(\overline{\Omega})$ is the space of continuously differentiable functions with compact support in $\overline{\Omega}$. The remaining can be adapted from the proof of Proposition 2.2 (iv) in [51] (which itself relies on [5, 47]).

3.2.2 The Maximum Principle

We have the following fundamental result.

Theorem 3.6 *Let Ω be possibly unbounded and assume that $\mathcal{L}$ is self adjoint.*

1. *If $\lambda_1 > 0$ then $(\mathcal{L}, \mathcal{B}^\gamma)$ satisfies the Maximum Principle in Ω.*
2. *If $\lambda_1 < 0$ then $(\mathcal{L}, \mathcal{B}^\gamma)$ does not satisfy the Maximum Principle in Ω.*

We mention that the above result is not known in the case of Dirichlet boundary conditions. We will see in the next section that, in contrast with the situation where the domain is bounded, no general answer holds for the degenerate case $\lambda_1 = 0$.

Proof We prove the first statement. Let v be a subsolution of $(\mathcal{L}, \mathcal{B}^\gamma)$ with finite supremum in Ω. We set $v_+ := \sup(v, 0)$ the positive part of v. For $R > 0$, we define a cut-off function

$$\chi_R(x) := \chi\left(\frac{|x|}{R}\right), \quad \forall x \in \mathbb{R}^n, \tag{3.9}$$

with χ a smooth nonnegative function such that

$$\chi(z) = \begin{cases} 1 & \text{if } 0 \le z \le 1, \\ 0 & \text{if } z \ge 2, \end{cases} \qquad |\chi'| \le 2.$$

We have

$$-\chi_R^2 v_+ \nabla \cdot (A \cdot \nabla v) - c\chi_R^2 v_+^2 \le 0 \quad \text{in } \Omega.$$

and Integrating over Ω and using the divergence theorem, we find

$$\int_{\partial\Omega} -\chi_R^2 v_+ \partial_{\nu_A} v + \int_\Omega \nabla\left(\chi_R^2 v_+\right) \cdot A \cdot \nabla v - c\chi_R^2 v_+^2 \le 0.$$

Note that $\mathbb{1}_{v\ge 0}\nabla v = \nabla v_+$. From the boundary conditions and

$$\nabla\left(\chi_R^2 v_+\right) \cdot A \cdot \nabla v = |\nabla \chi_R v_+|_A^2 - |\nabla\chi_R|_A^2 v_+^2,$$

we find

$$\mathcal{F}(\chi_R v_+) \leq \int_\Omega |\nabla \chi_R|_A^2 v_+^2.$$

Assume $v_+ \not\equiv 0$ and let us show $\lambda_1 \leq 0$. We have

$$\mathcal{F}\left(\frac{\chi_R v_+}{\|\chi_R v_+\|_{L^2}}\right) \leq \frac{\int_\Omega |\nabla \chi_R|_A^2 v_+^2}{\int_\Omega \chi_R^2 v_+^2} \leq \frac{\frac{K}{R^2}\int_{\Omega_{2R}\setminus\Omega_R} v_+^2}{\int_{\Omega_R} v_+^2} \leq K\ \alpha_R,$$

for some constant $K > 0$ (independant of R), and

$$\Omega_R := \Omega \cap \{|x| \leq R\}, \quad \mathcal{C}(R) := \int_{\Omega_R} v_+^2, \quad \alpha_R := \frac{\mathcal{C}(2R) - \mathcal{C}(R)}{R^2 \mathcal{C}(R)}.$$

Now, let us show

$$\liminf_{R\to+\infty} \alpha_R \leq 0. \tag{3.10}$$

If (3.10) holds, then $\lambda_1 \leq 0$, which proves *1.* By contradiction, assume $\alpha_R \geq \delta > 0$. We have $\mathcal{C}(2R) \geq \delta R^2 \mathcal{C}(R)$. Iterating, we find, for R large enough

$$\mathcal{C}(2^j R) \geq K\left(\delta R^2\right)^j \quad \text{for all integer } j \geq 1,$$

where positive constants are generically denoted K. In addition, v is bounded, hence $\mathcal{C}(R) \leq K R^n$. We have

$$K\left(2^j R\right)^n \geq \left(\delta R^2\right)^j.$$

Fixing R large enough, we reach a contradiction as j goes to $+\infty$. Thereby, we have proved (3.10) and the proof of the first statement is achieved.

Let us now prove the second statement. As in the proof of Proposition 3.3, we can consider the principal eigenelements $(\lambda_1^R, \varphi_1^R)$ on the truncated domain (3.5). Since λ_1^R converges to λ_1, if R is large enough, then $\lambda_1 < 0$ and φ_1^R is a bounded positive supersolution of $(\mathcal{L}, \mathcal{B}^\gamma)$ in Ω.

Note however that φ_1^R is not smooth in Ω. It is not a real obstruction, indeed, we chose to stick to smooth functions in our framework for simplicity, but all the arguments can be transposed to Sobolev spaces. Nevertheless, there is an elegant way to bypass this difficulty that was pointed to us by L.Rossi. Consider φ_1^R as an intial data of the associated parabolic equation, namely, we define $v(t,x)$ as the solution of

$$\begin{cases} \partial_t v - \mathcal{L}v = 0 & \text{in } (0,+\infty)\times\Omega, \\ \partial_{\nu_A} v + \gamma v = 0 & \text{on } (0,+\infty)\times\partial\Omega, \\ v(t=0,\cdot) = \varphi_1^R(\cdot) & \text{in } \Omega. \end{cases}$$

From the regularizing properties of parabolic equations, we kow that $v(t,\cdot)$ is smooth whenever $t > 0$. We also have $v(t,\cdot) > 0$, and $v(t,\cdot)$ bounded, at least for short times. In addition, since φ_1^R is a (generalized) subsolution, we know from classical result that $t \mapsto v(t,\cdot)$ is increasing, that is $-\mathcal{L}v \leq 0$, and v is subsolution. ■

3.2.3 The Critical Maximum Principle

We define what could be called a *Critical Maximum Principle.*

Definition 3.7 *We say that* $(\mathcal{L}, \mathcal{B})$ *satisfies the Critical Maximum Principle in* Ω *if, for a given principal eigenfunction* φ_1*, every subsolution with finite supremum is either nonpositive, or a multiple of* φ_1*.*

When the domain Ω is bounded, the Strong Maximum Principle implies that $(\mathcal{L}, \mathcal{B})$ satisfies the Critical Maximum Principle in Ω if and only if $\lambda_1 \geq 0$.

We propose the following extension to unbounded domains satisfying a certain growth condition at infinity.

Theorem 3.8 *Assume* $\mathcal{L}$ *is self-adjoint and* Ω *satisfies*

$$\Omega \cap \{|x| \leq R\} = O(R^2) \quad \textit{when } R \to +\infty. \tag{3.11}$$

Then, $(\mathcal{L}, \mathcal{B})$ *satisfies the Critical Maximum Principle in* Ω *if and only if* $\lambda_1 \geq 0$*.*

A first consequence of this theorem is the simplicity of λ_1 if a principal eigenfunction is bounded.

Corollary 3.9 *Under the same assumptions, further assume that there exists* φ *a* bounded *eigenfunction associated with* λ_1*. Then any eigenfunction of* λ_1 *is a multiple of* φ*, i.e.,* λ_1 *is simple.*

We also have the following

Corollary 3.10 *Under the same assumption, assume* $\lambda_1 = 0$ *and let* φ *be an associated eigenfunction. Then* $(\mathcal{L}, \mathcal{B}^\gamma)$ *satisfies the Maximum principle if and only if* φ *is not bounded.*

We see from this corollary that no general answer holds for the validity of the Maximum Principle when $\lambda_1 = 0$.

Proof (of Theorem 3.8) It can be seen from the proof of the second statement of Theorem 3.6 that, if $\lambda_1 < 0$, then $(\mathcal{L}, \mathcal{B})$ does not satisfy the Critical Maximum Principle in Ω.

Let us prove the converse statement. Let v be a bounded subsolution of $(\mathcal{L}, \mathcal{B}^\gamma)$ in Ω. We set $v_+ := \sup(v, 0)$ the positive part of v. Let us set $\sigma = \frac{v}{\varphi_1}$, $\sigma_+ = \frac{v_+}{\varphi_1}$ and show that σ_+ is constant. Note that, since $\varphi_1 > 0$ on $\overline{\Omega}$, this is sufficient to conclude.

Our method is inspired from [41]. From standard calculations, we deduce

$$\sigma_+ \varphi_1 \Big[\varphi_1 \nabla \cdot (A \cdot \nabla \sigma) + 2 \nabla \varphi_1 \cdot \nabla \sigma + \sigma \mathcal{L} \varphi_1 \Big] \geq 0.$$

From $\lambda_1 \geq 0$, we obtain

$$\sigma_+ \nabla \cdot (\varphi_1^2 \, A \cdot \nabla \sigma) \geq 0 \quad \text{on } \Omega.$$

Note also that $\partial_{\nu_A} \sigma \leq 0$ on $\partial\Omega$. For $R > 0$, we consider the cut-off function χ_R defined in (3.9), Multiplying the above equation by χ_R^2, integrating on Ω and using the divergence theorem, we find

$$\begin{aligned} 0 &\leq \int_{\partial\Omega} \chi_R^2 \sigma_+ \varphi_1^2 \partial_{\nu_A} \sigma - \int_\Omega \varphi_1^2 \nabla \left(\chi_R^2 \sigma_+ \right) \cdot A \cdot \nabla \sigma \\ &\leq - \int_\Omega \varphi_1^2 \chi_R^2 |\nabla \sigma_+|_A^2 - 2 \int_\Omega \varphi_1^2 \chi_R \sigma_+ \nabla \chi_R \cdot A \cdot \nabla \sigma_+. \end{aligned}$$

Using the Cauchy-Schwarz inequality, we deduce

$$\int_\Omega \chi_R^2 \varphi_1^2 |\nabla \sigma_+|_A^2 \leq 2\sqrt{\int_{\Omega_{2R}\setminus\Omega_R} \chi_R^2\varphi_1^2|\nabla\sigma_+|_A^2}\sqrt{\int_\Omega v_+^2|\nabla\chi_R|_A^2}, \tag{3.12}$$

where $\Omega_R = \Omega \cap \{|x| \leq R\}$.

Now, assumption (3.11) implies

$$\int_\Omega v_+^2|\nabla\chi_R|_A^2 \text{ is bounded, uniformly in } R \geq 1. \tag{3.13}$$

From (3.12), we have that $\int_\Omega \chi_R^2\varphi_1^2|_A\nabla\sigma_+|^2$ is uniformly bounded. Using (3.12) again, we infer that it converges to 0 as $R \to \infty$. At the limit, we find $\int_\Omega \varphi_1^2|\nabla\sigma_+|_A^2 \leq 0$. Hence $\nabla\sigma_+ = 0$, which ends the proof. ∎

The cornerstone of the proof is that $\sigma\nabla\cdot(\varphi_1^2 A\cdot\nabla\sigma) \geq 0$ implies $\nabla\sigma = 0$, where $\sigma := \frac{v}{\varphi_1}$. The litterature refers to this property as the *Liouville property*. Originally introduced in [41], it has been extensively discussed (see [10, 39, 158, 160, 240]) and used to derive numerous results (e.g [43, 73, 104, 136]), in particular to prove the De Giorgi's conjecture in low dimension (see [7, 10, 72, 160]).

A natural question to ask if assumption (3.11) can be relaxed. Let us first mention that assumption (3.11) is not needed if $\inf_\Omega \varphi_1 > 0$. This result is know in $\mathbb{R}^n$ (thus without any boundary conditions), but can be easily adapted to our case. We let the reader refer to [240] and references therein.

In the proof, (3.11) is used to derive (3.13), thus the choice of χ_R seems crucial. However, in [160], the authors consider the optimal χ_R by taking a solution of the minimization problem

$$\inf_{\chi\in H^1(\mathbb{R}^2)}\left[\int_{R\leq|x|\leq R'}|\nabla\chi(x)|^2\mathrm{d}x,\ \xi(x) = \begin{cases}1 & \text{if } |x|\leq R\\ 0 & \text{if } |x|\geq R'\end{cases}\right]. \tag{3.14}$$

That, in fact, does not allow to substantially relax condition (3.11). In [38], Barlow uses a probabilistic approach to establish that the aforementioned Liouville property (and consequently Theorem 3.8) does not hold in $\Omega = \mathbb{R}^n$, $n \geq 3$. It suggests that condition (3.11) cannot be substentially relaxed; yet we do not know any precise sharpness statement (in [158], a sharpness result is proved on the Liouville Property. But, there, the condition $v \in L^\infty$ is not satisfied).

However, we can sometimes relax (3.11) under further assumptions, either on Ω or admissible subsolutions v. First, we can adapt the arguments of [41, 73] to show that assumption (3.11) can be replaced by $v \in H^1(\Omega)$, or $v = o(|x|^{1-\frac{n}{2}})$. In addition, we can show that Theorem 3.8 holds for a large class of domains satisfying

$$|\Omega\cap\{|x|\leq R\}| = O\left(R^2\ln(R)\right), \quad \text{when } R\to+\infty.$$

More precisely, let Ω be of the form

$$\Omega := \left\{(x,x')\in\mathbb{R}^2\times\mathbb{R}^n : x'\in\omega(x)\right\},$$

where $\forall x\in\mathbb{R}^n$, $\omega(x)\subset\mathbb{R}^n$ is bounded. Then, condition (3.11) can be replaced with

$$|\omega(x)| = O\left(\ln\left(|x|\right)\right) \quad \text{when } |x|\to+\infty.$$

To show (3.13) we use the cut-off

$$\chi_R(x) = \begin{cases} 1 & \text{if } |x| \leq R, \\ \dfrac{\ln R^2 - \ln |x|}{\ln R^2 - \ln R} & \text{if } R \leq |x| \leq R^2, \\ 0 & \text{if } |x| \geq R^2. \end{cases}$$

This cut-off comes from the minimizer of (3.14) for $n = 2$.

3.3 Non-self-adjoint operators

When L is not self-adjoint, we know that $\lambda_1 \geq 0$ is a necessary condition for $(\mathcal{L}, \mathcal{B}^\gamma)$ to satisfy the Maximum Principle (from Theorem 3.6).

Proposition 3.11 *If $\lambda_1 < 0$, $(\mathcal{L}, \mathcal{B}^\gamma)$ does not satisfy the Maximum Principle in Ω.*

However, we do not know if $\lambda_1 > 0$ is a sufficient condition. We first deal with a case which reduces to the self-adjoint case, then we deal with the general case.

3.3.1 Case of a bounded drift

We make the following assumption:

$$\exists \zeta : \overline{\Omega} \to \mathbb{R} \in C^1,\ \underline{\zeta}, \overline{\zeta} > 0, \quad \underline{\zeta} \leq \zeta \leq \overline{\zeta} < +\infty, \quad A \cdot \nabla \zeta = B\zeta. \tag{3.15}$$

This assumption is automatically satisfied if the operator is self-adjoint (simply take $\zeta \equiv 1$). If A, B are constants, take $\zeta(x) = e^{A^{-1} \cdot B \cdot x}$ so that (3.15) reduces to

$$\left| A^{-1} \cdot B \cdot x \right| < +\infty.$$

This condition somehow expresses that the drift (induced by the first order term) is neither from nor towards infinity. Under this assumption, we have the same results as in the self-adjoint case, namely,

Theorem 3.12 *Assume* (3.15).

1. *If $\lambda_1 > 0$, $(\mathcal{L}, \mathcal{B}^\gamma)$ satisfies the Maximum Principle in Ω.*
2. *If $\lambda_1 \geq 0$ and Ω further satisfies* (3.11)*, $(\mathcal{L}, \mathcal{B}^\gamma)$ satisfies the Critical Maximum Principle.*

Proof Under assumption (3.15), we have, for any $u \in C^2(\overline{\Omega})$,

$$\nabla \cdot (\zeta A \cdot \nabla u) = [\nabla \cdot (A \cdot \nabla u) + B \cdot \nabla u]\, \zeta.$$

Then, the proofs of Theorem 3.6 and Theorem 3.8 can be easily adapted. ∎

Note that the Critical Maximum Principle has no chance to hold, in general, without assumption (3.15), as can be seen from the equation

$$v'' - v' = 0 \quad \text{on } \mathbb{R},$$

which is such that $\lambda_1 = 0$ and admits both 1 and e^x as solutions.

3.3.2 General case

In the spirit of [51], we define

$$\tilde{\lambda}_1 := \sup\left\{\lambda \in \mathbb{R} : \exists \psi \in C^2(\overline{\Omega}),\ \inf_{\Omega} \psi > 0,\ \psi \text{ is a supersolution of } (\mathcal{L} + \lambda, \mathcal{B}^\gamma)\right\}.$$

Compared to the definition of λ_1, here we impose $\inf_\Omega \psi > 0$ instead of $\psi > 0$. In general, we have $\tilde{\lambda}_1 \leq \lambda_1$, and we do not know if equality holds (see *Conjecture 1* in [51]). The positivity of $\tilde{\lambda}_1$ is a sufficient condition for the Maximum Principle to hold.

Theorem 3.13 *Assume $\Omega \subset \mathbb{R}^n$ is a uniformly $C^{2,\alpha}$ domain with a uniform interior ball condition.*

1. *If $\tilde{\lambda}_1 > 0$ then $(\mathcal{L}, \mathcal{B}^\gamma)$ satisfies the Maximum Principle in Ω.*
2. *If $\lambda_1 < 0$ then $(\mathcal{L}, \mathcal{B}^\gamma)$ does not satisfy the Maximum Principle in Ω.*

Proof The second assertion is already proved in Theorem 3.6 (we do not use in the proof that $\mathcal{L}$ is self-adjoint). Let us focus on the first assertion: assume $\tilde{\lambda}_1 > 0$, let v be a subsolution of $(\mathcal{L}, \mathcal{B}^\gamma)$ with finite supremum, and ψ be a supersolution $(\mathcal{L} + \lambda, \mathcal{B}^\gamma)$ with positive infimum, for some $\lambda \in (0, \tilde{\lambda}_1)$. Let us show $v \leq 0$. Up to renormalization, we can assume without loss that $\sup_\Omega v \leq 1$ and $\inf_\Omega \psi \geq 3$.

We start with a technical lemma, inspired by the proof of Proposition 5.2 from [51].

Lemma 3.14 *There exist two functions $\underline{u}, \overline{u} \in C^2(\overline{\Omega})$ such that*

$$v^+ \leq \underline{u} \leq 2 \leq \overline{u} \leq \psi$$

where $v^+(x) = \max[v(x), 0]$, and

$$\begin{cases} -\mathcal{L}\overline{u} = |c(x)|\overline{\theta}(\overline{u}) + \lambda \overline{u} & \Omega, \\ \mathcal{B}^\gamma \overline{u} = |\gamma(x)|\overline{\theta}(\overline{u}) & \partial\Omega, \end{cases} \qquad \begin{cases} -\mathcal{L}\underline{u} = |c(x)|\underline{\theta}(\underline{u}) & \Omega, \\ \mathcal{B}^\gamma \underline{u} = |\gamma(x)|\underline{\theta}(\underline{u}) & \partial\Omega, \end{cases}$$

where

$$\overline{\theta}(\cdot) \textit{ is smooth, nonnegative, nonincreasing, } \overline{\theta} = \begin{cases} 1 & \textit{on } (-\infty, 2] \\ 0 & \textit{on } [3, +\infty) \end{cases}$$

$$\underline{\theta}(\cdot) \textit{ is smooth, nonpositive, nonincreasing, } \underline{\theta} = \begin{cases} 0 & \textit{on } (-\infty, 1] \\ -1 & \textit{on } [2, +\infty) \end{cases}$$

Proof (of Lemma 3.14) Let us focus only on the construction of $\overline{u}$, since the construction of $\underline{u}$ can be done the same way. We denote

$$f(x, s) := |c(x)|\overline{\theta}(s) + \lambda s \quad ; \quad g(x, s) := |\gamma(x)|\overline{\theta}(s).$$

From $\inf_\Omega \psi \geq 3$, we have

$$\begin{cases} -\mathcal{L}\psi \geq f(x, \psi) & \Omega, \\ \mathcal{B}^\gamma \psi \geq g(x, \psi) & \partial\Omega, \end{cases}$$

and, denoting $\sigma \equiv 2$,

$$\begin{cases} -\mathcal{L}\sigma \leq f(x,\sigma) & \Omega, \\ \mathcal{B}^\gamma \sigma \leq g(x,\sigma) & \partial\Omega. \end{cases}$$

We construct $\overline{u}$ with Perron's iterative method on a truncated domain. For $R > 0$, we consider an increasing sequence of bounded Lipschitz subdomains $\Omega_R \subset \Omega$, such that $\bigcup_{R>0} \Omega_R = \Omega$. We denote $\Sigma_R = \partial\Omega_R \cap \partial\Omega$, and choose Ω_R such that the intersection of Σ_R and $\partial\Omega_R \backslash \Sigma_R$ is a C^2 $(n-2)$-dimensional manifold.

We define $u_0 = \psi$ and, for all $n \geq 0$, u_{n+1} as the unique solution of

$$\begin{cases} -\mathcal{L}u_{n+1} + Cu_{n+1} = f(x,u_n) + Cu_n & \Omega_R, \\ \mathcal{B}^\gamma u_{n+1} + \Gamma u_{n+1} = g(x,u_n) + \Gamma u_n & \Sigma_R, \\ u_{n+1} = \psi & \partial\Omega_R \backslash \Sigma_R \end{cases}$$

with $C := \sup_\Omega c$ and $\Gamma := -\inf_{\partial\Omega} \gamma$. From the results of Liberman [205], we know that all classical results (Schauder estimate, Maximum Principle, solvability, etc.) hold from the mixed boundary value problem above. First, the sequence u_n is well defined. Then, from the Maximum Principle, we can show by induction that

$$\sigma \leq u_{n+1} \leq u_n \leq \psi.$$

Besides, from Schauder Estimates, we know that u_n converges in $C^2(\overline{\Omega}_R)$ to a function u^R solution of

$$\begin{cases} -\mathcal{L}u^R = f(x,u^R) & \Omega_R, \\ \mathcal{B}^\gamma u^R = g(x,u^R) & \Sigma_R, \\ u^R = \psi & \partial\Omega_R \backslash \Sigma_R \end{cases}$$

Now, we need to prove estimates on u^R, uniformly in R, to pass to the limit $R \to +\infty$. Let us fix $R_0 > 0$, and assume $R > R_0$. From Theorem 3.3 in [206] and Theorem 4.3 in [207], we derive a Harnack estimate, that is,

$$\sup_{\overline{\Omega}_{R_0}} u^R \leq C \inf_{\overline{\Omega}_{R_0}} u^R \leq C \inf_{\overline{\Omega}_{R_0}} \psi,$$

with a constant C independent of R. Now, from classical Schauder estimates, u^R is uniformly $C^{2,\alpha}$ in $\overline{\Omega}_{R_0}$. Thus, u^R converges (up to extraction) to some $\overline{u}$ in $C^2_{loc}(\overline{\Omega})$ when $R \to +\infty$, which satisfies the required conditions. ∎

Let us now go back to the proof of Theorem 3.13. We set

$$t_0 := \inf\left\{t \geq 0 : \underline{u} \leq t\overline{u}\right\}.$$

By contradiction, assume $t_0 > 0$. Denoting $w := \underline{u} - t_0\overline{u}$, we have $w \leq 0$ and there exists a sequence $x_n \in \Omega$ such that $w(x_n) \to 0$. Without loss of generality, we assume either $d(x_n, \partial\Omega) \to 0$ or $\liminf d(x_n, \partial\Omega) > 0$. Set $\overline{u}_n(\cdot) := \overline{u}(\cdot + x_n)$, $\underline{u}_n(\cdot) := \underline{u}(\cdot + x_n)$, $w_n(\cdot) := w(\cdot + x_n)$, defined in the closure of $\Omega_n := \Omega - x_n$. Let B_1 denote the unit ball with center 0 and $V_n := \overline{\Omega}_n \cap B_1$.

As in the proof of Lemma 3.14, $\overline{u}$ satisfies the Harnack inequality, that is,

$$\sup_{V_n} \overline{u}_n \leq C \inf_{V_n} \overline{u}_n.$$

with a constant $C > 0$ independent of n. For n large enough, we have $w_n(0) \geq -1$, therefore, $\sup_{V_n} \overline{u}_n$ is bounded uniformly in n. From classical Schauder estimates, we deduce that $\overline{u}_n$, $\underline{u}_n$, and w_n are uniformly bounded in $C^{2,\alpha}(V_n)$.

Let $r > 0$ be the radius from the uniform inner ball condition and y_n such that $0 \in B_n := \{|x - y_n| \leq r\} \subset \Omega_n$. We also choose y_n such that, if $\liminf d(0, \Omega_n) > 0$, then $\limsup \sup |y_n| < r$. Assuming without loss that r is small enough, we have $\overline{B}_n \subset V_n$.

From the uniform $C^{2,\alpha}$ estimates, we deduce that w_n converges (up to a subsequence) to some w_∞ in $C^2(\overline{B}_\infty)$, where B_∞ is a ball of radius r and $0 \in \overline{B}_\infty$. We further have $w_\infty \leq 0$, $w_\infty(0) = 0$ and

$$-\mathcal{L} w_\infty \leq -t_0 \lambda \overline{u}_\infty \quad \text{in } B_\infty. \tag{3.16}$$

If $\liminf d(0, \Omega_n) > 0$, then $0 \in B_\infty$, and we deduce from the Strong Maximum Principle that $w_\infty \equiv 0$. If $d(0, \Omega_n) \to 0$, then $0 \in \partial B_\infty$. But from the boundary condition, we have

$$\partial_{\nu_A} w_\infty(0) \leq -\gamma(0) w_\infty(0) = 0.$$

From Hopf's lemma, we also deduce $w_\infty \equiv 0$.

From (3.16) and $\inf_{B_\infty} \overline{u} > 0$, we have $\lambda = 0$: contradiction. Thus $t_0 = 0$, $\underline{u} = v^+ = 0$, and $v \leq 0$. ∎

Part II

Dynamics of Concentration and Modeling of Natural Selection

Chapter 4

Dynamics of concentration of a population structured by age and phenotype I: case without mutations

In collaboration with Benoît Perthame and Cécile Taing.
This chapter is published as an article in Acta Applicandae Mathematicae [243].

We study a mathematical model describing the growth process of a population structured by age and a phenotypical trait, subject to aging and competition between individuals and rare mutations. Our goals are to describe the asymptotic behavior of the solution to a renewal type equation, and then to derive properties that illustrate the adaptive dynamics of such a population.

Our approach uses the eigenelements of a formal limiting operator, which depend on the structuring variables of the model and define an effective fitness. This method reduces the convergence proof to entropy estimates rather than estimates on the constrained Hamilton-Jacobi equation. Numerical tests illustrate the theory and show the selection of the fittest trait according to effective fitness.

We begin with a simplified model by discarding the effect of mutations, which allows us to introduce the main ideas and state the full result. We study the case with mutations in more details in Chapter 5. In this chapter, we construct a global solution of a Hamilton-Jacobi equation which will be useful in Chapter 5.

Contents

4.1 Introduction

4.1.1 The model

The mathematical description of competition between populations and selection phenomena leads to the use of nonlocal equations that are structured by a quantitative trait. A mathematical way to express the selection of the fittest trait is to prove that the population density concentrates as a Dirac mass (or a sum of Dirac masses) located on this trait. This result has been obtained for various models with parabolic ([33, 36, 213]) and integrodifferential equations ([35, 117, 212]). More generally, convergence to positive measures in selection-mutation models has been studied by many authors, see [2,71,82] for example. The question that we pose in the present paper is the long time behavior of the population density when the growth rate depends both on phenotypical fitness and age. This question brings up to consider the aging parameter and to use renewal type equations. Accordingly, the aim of this paper is to study the asymptotic behavior of the solutions, as $\varepsilon \to 0$, to the following model, with $x \geq 0$ and $y \in \mathbb{R}^n$:

$$\begin{cases} \varepsilon \partial_t m_\varepsilon(t,x,y) + \partial_x \left[A(x,y) m_\varepsilon(t,x,y)\right] + \left(\rho_\varepsilon(t) + d(x,y)\right) m_\varepsilon(t,x,y) = 0, \\ A(x=0,y) m_\varepsilon\left(t, x=0, y\right) = \dfrac{1}{\varepsilon^n} \displaystyle\int_{\mathbb{R}^n} \int_{\mathbb{R}_+} M\left(\frac{y'-y}{\varepsilon}\right) b(x',y') m_\varepsilon(t,x',y') dx' dy', \\ \rho_\varepsilon(t) = \displaystyle\int_{\mathbb{R}_+} \int_{\mathbb{R}^n} m_\varepsilon(t,x,y) dx dy, \\ m_\varepsilon(t=0,x,y) = m_\varepsilon^0(x,y) > 0. \end{cases} \tag{4.1}$$

We choose $m_\varepsilon(t,x,y)$ to represent the population density of individuals which, at time t, have age x and trait y. The function $A(x,y)$ is the speed of aging for individuals with age x and trait y. We denote with $\rho_\varepsilon(t)$ the total size of the population at time t. Here the mortality effect features the nonlocal term $\rho_\varepsilon(t)$, which represents competition, and an intrinsic death rate $d(x,y) > 0$. The condition at the boundary $x = 0$ describes the birth of newborns that happens with rate $b(x,y) > 0$ and with the probability kernel of mutation M. The terminology of "renewal equation" comes from this boundary condition. It is related to the McKendrick-von Foerster equation which is only structured in age (see [247] for a study of the linear equation).

This model has been extended with other structuring variables as size for example (see [221, 233]) and then with more variables (representing DNA content, maturation, etc.) to illustrate biological phenomena, among many others, like cell division (see [131,224]), proliferative and quiescent states of tumour cells (see [3,170]). Space structured problems have also been extensively studied (see [183,228,229,248]). The variable x can represent different biological quantities that evolve throughout the individual lifespan and that are not inherited at birth. These can be as diverse as, for example, the size of individuals, a physiological age, a parasite load, and many others. Therefore we assume that the progression speed A depends both on x and the trait y to keep the model (4.1) quite general. In the present paper, we refer to x as the age for simplicity. Studies in these contexts can be found in [82] about the existence of steady states for a selection-mutation model structured by physiological age and maturation age, which is modeled as a phenotypical trait.

The parameter $\varepsilon > 0$ is used for a time rescaling since we consider selection-mutation phenomena that occur in a longer time scale than in an individual life cycle. It is also introduced to consider rare mutations. This rescaling is a classical way to give a continuous formulation of the adaptive evolution of a phenotypically structured population, in particular to analyze the dynamics of "$\bar{y}_\varepsilon(t)$", the fittest trait at time t, which is solution to a form of a canonical equation from the framework of adaptive dynamics (see [88, 124, 125, 213]).

Here we observe two different time scales for our model. The first one is the individual life cycle time scale, i.e., the time for the population to reach the dynamical equilibrium for a fixed y. The second one is the evolutionary time scale, corresponding to the evolution of the population distribution with respect to the variable y. The mathematical expression of these two time scales is the property of variable separation

$$m_\varepsilon(t,x,y) \simeq \bar{\rho}(t)Q(x,y)\delta_{y=\bar{y}(t)},$$

when ε is close to 0, where $Q(x,y)$ is a normalized equilibrium distribution over age for a fixed y, $\bar{\rho}(t)$ the total population density and $\bar{y}(t)$ the fittest trait at the limit $\varepsilon \to 0$. In order to observe the asymptotic behavior of the solution to (4.1), the key point is to prove convergence results when ε vanishes, that is when the two time scales become completely separated. In other words, as ε vanishes, we observe the ecological equilibrium, and we focus on the evolutionary dynamics of the population density to identify $\bar{y}(t)$.

As a first step, we ignore mutations, i.e. we take $M(z) = \delta_0(z)$. Equation (4.1) becomes, for $t, x \geq 0$ and $y \in \mathbb{R}^n$,

$$\begin{cases} \varepsilon\partial_t m_\varepsilon(t,x,y) + \partial_x\left[A(x,y)m_\varepsilon(t,x,y)\right] + \left(\rho_\varepsilon(t) + d(x,y)\right)m_\varepsilon(t,x,y) = 0, \\ A(x=0,y)m_\varepsilon\left(t,x=0,y\right) = \displaystyle\int_{\mathbb{R}_+} b(x',y)m_\varepsilon(t,x',y)dx', \\ \rho_\varepsilon(t) = \displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} m_\varepsilon(t,x,y)dxdy, \\ m_\varepsilon(t=0,x,y) = m_\varepsilon^0(x,y) > 0. \end{cases} \tag{4.2}$$

The analysis of this simplified model allows us to introduce the main ideas of our work. In order to study the asymptotic behavior of the solution to (4.2), we consider the associated eigenproblem, that is to find, for each $y \in \mathbb{R}^n$, the solution

$(\Lambda(y), Q(x,y))$ to

$$\begin{cases} \partial_x \left[A(x,y)Q(x,y)\right] + d(x,y)Q(x,y) = \Lambda(y)Q(x,y), \\ A(x=0,y)Q(x=0,y) = \displaystyle\int_{\mathbb{R}_+} b(x',y)Q(x',y)dx', \\ Q(x,y) > 0, \quad \displaystyle\int_{\mathbb{R}^+} b(x',y)Q(x',y)dx' = 1. \end{cases} \tag{4.3}$$

We also define Φ, solution of the dual problem

$$\begin{cases} A(x,y)\partial_x\Phi(x,y) + \left[\Lambda(y) - d(x,y)\right]\Phi(x,y) = -b(x,y)\Phi(0,y), \\ \displaystyle\int_{\mathbb{R}_+} Q(x,y)\Phi(x,y)dx = 1. \end{cases} \tag{4.4}$$

The purpose of this paper is to introduce an alternative to the usual WKB method (see [36,125]) to prove the concentration phenomenon in the y variable for the model (4.2). Indeed we propose a new approach that consists in firstly introducing the exponential concentration singularity and secondly in estimating the corresponding age profile. The main idea is to define a function $u_\varepsilon(t,y)$ independent of x, and an "age profile" $p_\varepsilon(t,x,y)$, such that we can write $m_\varepsilon(t,x,y) = p_\varepsilon(t,x,y)\mathrm{e}^{\frac{u_\varepsilon(t,y)}{\varepsilon}}$. Then, we prove that u_ε converges uniformly to a function u, which zeros correspond to the potential concentration points of the population density when ε vanishes. Moreover, following earlier works, we prove that $p_\varepsilon(t,x,y)$ converges to the first eigenvector of the stationary problem introduced in (4.3) using the general relative entropy (GRE) principle (see [225] for an introduction).

This convergence result does not apply for the model (4.1) with mutations. Because of several technical obstructions, we cannot prove the full result (see Chapter 5 for a more detailed study of the case with mutation). However, we are able to derive some estimates resulting from the study of the formal limiting problem. Then we derive an approximation problem with a Hamilton-Jacobi equation satisfied by a sequence u_ε that we build, and we prove its convergence to the solution to the constrained Hamilton-Jacobi equation coming from the formal limiting problem. This constrained Hamilton-Jacobi formally determines the locations of the concentration points.

Recently, the asymptotic behavior of an age-structured equation with spatial jumps has been studied in [83] when the death rate vanishes and with a slowly decaying birth rate b; then the eigenproblem (4.3) does not have a solution. Also in [128], a concentration result has been proved for a model representing the evolutionary epidemiology of spore-producing plant pathogens in a host population, with infection age and pathogen strain structures.

More generally, the Hamilton-Jacobi approach to prove the concentration of the population density goes back to [125] and has been extensively used in works on a similar issue (see [93] for example). It also has been used in the context of front propagation theory for reaction-diffusion equations (see [32,37,139]). For example in the case of the simple Fisher-KPP equation, the dynamics of the front are described by the level set of a solution of a Hamilton-Jacobi equation. In this framework, it is naturally appropriate to use the theory of viscosity solutions to derive the convergence of the sequence u_ε (see [30,31,153] for an introduction to this notion). In this paper, we also prove a uniqueness result in the viscosity sense that is not standard since the Hamiltonian under investigation has exponential growth.

The paper is organized as follows. We first state the general assumptions in section 4.1.2. Section 4.2 is devoted to the formulation and the proof of the convergence results in the case without mutations. In section 4.3, we discuss the case with mutations and tackle the formal limit of the stationary problem. Finally, we present some numerics in section 4.2.4.

4.1.2 Assumptions

Since the analysis requires several technical assumptions on the coefficients and the initial data, we present them first.

Regularity of the coefficients. We assume that $x \mapsto b(x,y) > 0$ and $x \mapsto d(x,y) > 0$ are uniformly continuous, that $x \mapsto A(x,y)$ is $\mathcal{C}^1$ and such that, for all $y \in \mathbb{R}^n$,

$$\lim_{x \to +\infty} d(x,y) = +\infty, \tag{4.5}$$

$$0 < \underline{r} \leq b(x,y) - d(x,y) \leq \overline{r}, \tag{4.6}$$

$$0 < A_0 \leq A(x,y) \leq A_\infty, \quad \text{for two positive constants } A_0 \text{ and } A_\infty. \tag{4.7}$$

This set of assumptions is an example. It serves mostly to guarantee some properties of the spectral problem which are stated in Theorem 4.2. Only the conclusions of Theorem 4.2 are used in the present approach to the concentration phenomena.

Conditions on the initial data. We suppose that the total density is initially bounded

$$0 < \underline{\rho}^0 \leq \rho_\varepsilon^0 \leq \overline{\rho}^0, \tag{4.8}$$

with $\underline{\rho}_0$ and $\overline{\rho}_0$ two constants. Besides we assume the population to be well prepared for concentration, that is, we can write

$$m_\varepsilon^0(x,y) = p_\varepsilon^0(x,y) e^{\frac{u_\varepsilon^0(y)}{\varepsilon}},$$

where u_ε^0 is uniformly Lipschitz continuous and

$$\begin{cases} \exists k_0 > 0, \forall \varepsilon > 0, \forall (y,y') \in \mathbb{R}^{2n}, |u_\varepsilon^0(y) - u_\varepsilon^0(y')| \leq k_0 |y - y'|, \\ u_\varepsilon^0(y) \to u^0(y) \leq 0 \text{ uniformly in } y, \\ \exists!\ \bar{y}^0 \in \mathbb{R}^n, \max_{y \in \mathbb{R}^n} u^0(y) = u^0(\bar{y}^0) = 0, \\ e^{\frac{u_\varepsilon^0}{\varepsilon}} \underset{\varepsilon \to 0}{\longrightarrow} \delta_{\bar{y}^0}. \end{cases} \tag{4.9}$$

Finally, we assume that, for all $y \in \mathbb{R}^d$, there exist $\underline{\gamma}(y)$, $\overline{\gamma}(y)$ and $\gamma^0(y)$ positive such that, for all $\varepsilon > 0, x \in \mathbb{R}_+$,

$$\underline{\gamma}(y) Q(x,y) \leq p_\varepsilon^0(x,y) \leq \overline{\gamma}(y) Q(x,y), \tag{4.10}$$

$$\int_{\mathbb{R}_+} \left| p_\varepsilon^0(x,y) - \gamma^0(y) Q(x,y) \right| \Phi(x,y) dx \underset{\varepsilon \to 0}{\longrightarrow} 0, \quad \text{uniformly in } y, \tag{4.11}$$

where Q, Φ are eigenelements associated with the eigenproblem (4.3)-(4.4) which properties are analyzed in section 4.2.1.

Some notations: We define, for $x \in \mathbb{R}_+$, $y \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, the functions

$$f(x,y,\lambda) = \frac{b(x,y)}{A(x,y)} \exp\left(-\int_0^x \frac{d(x',y)-\lambda}{A(x',y)}dx'\right), \quad F(y,\lambda) = \int_{\mathbb{R}_+} f(x,y,\lambda)dx. \tag{4.12}$$

4.2 Case without mutations

We present our new approach to understand how solutions of (4.2) behave when ε vanishes. To prove that a concentration in the y variable may occur, we first consider the principal eigenvalue $\Lambda(y)$ of (4.3), and define u_ε as the solution of the equation

$$\begin{cases} \partial_t u_\varepsilon(t,y) = -\Lambda(y) - \rho_\varepsilon(t), \quad t>0, y \in \mathbb{R}^n, \\ u_\varepsilon(0,y) = u_\varepsilon^0, \quad y \in \mathbb{R}^n. \end{cases} \tag{4.13}$$

Then, we define p_ε such that

$$m_\varepsilon(t,x,y) = p_\varepsilon(t,x,y)e^{\frac{u_\varepsilon(t,y)}{\varepsilon}}, \tag{4.14}$$

and we prove that p_ε converges when $\varepsilon \to 0$ respectively to the eigenvector Q associated to Λ in some way that we will specify, using an entropy method. Thereafter we prove that u_ε converges locally uniformly as ε goes to 0. This section is devoted to the proof of the following theorem, which states the concentration of the population density on the fittest traits.

Theorem 4.1 *Assume* (4.5)–(4.11)*. Let m_ε be the solution of* (4.2)*, u_ε the solution of* (4.13)*, p_ε defined by the factorization* (4.14) *and* (Λ, Q) *defined in* (4.3)*. Then, the following assertions hold true:*

1. *$\rho_\varepsilon(t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}_+} m_\varepsilon(t,x,y)dxdy$ converges to a function ρ when ε vanishes in $L^\infty(0,\infty)$ weak-$\star$.*
2. *p_ε converges to a multiple of the normalized eigenvector Q for a weighted L^1 norm.*
3. *u_ε converges locally uniformly when ε vanishes to a continuous function u solution of*
$$\begin{cases} \partial_t u(t,y) = -\Lambda(y) - \rho(t), \quad t>0, y \in \mathbb{R}^n, \\ \sup\limits_{y\in\mathbb{R}^n} u(t,y) = 0, \quad \forall t>0, \\ u(0,y) = u^0(y), \quad y \in \mathbb{R}^n. \end{cases}$$
4. *Hence, m_ε converges weakly as ε vanishes to a measure μ which support is included in $\{(t,y) \in (0,\infty)\times\mathbb{R}^n | u(t,y) = 0\}$.*
5. *Furthermore, assuming u^0 and $-\Lambda$ to be strictly concave*
$$m_\varepsilon(t,x,y) \underset{\varepsilon\to 0}{\rightharpoonup} \rho(t)\frac{Q(x,y)}{\|Q(\cdot,y)\|_{L^1}}\delta_{y=\bar{y}(t)},$$
where $\bar{y}(t) \in \mathbb{R}^n$ satisfies a canonical differential equation.

4.2.1 The eigenproblem

We first study the eigenproblem (4.3) and the associated dual problem (4.4). The operator in (4.3), which is time independent, is obtained by formally taking $\varepsilon = 0$ in system (4.2) and by removing the formal limiting term $\rho(t)$. We point out that this approach relies on the observation that $\rho_\varepsilon(t)$ operates linearly on m_ε; therefore its effect on the eigenvalue Λ is no more than a shift. The following theorem states existence and uniqueness for these eigenelements as well as some properties.

Theorem 4.2 *We assume (4.5)–*(4.7)*. For a given $y \in \mathbb{R}^n$, there exists a unique triplet*
$(\Lambda(y), Q(x,y), \Phi(x,y))$ solution of (4.3)-(4.4). Moreover, the function $x \mapsto Q(x,y)$ is bounded and belongs to $L^1(0,\infty)$, the function $y \mapsto \Lambda(y)$ is $\mathcal{C}^1$ and we have

$$\partial_\lambda F > 0, \quad F(y,\Lambda(y)) = 1, \tag{4.15}$$

$$\nabla_y \Lambda(y) = -\frac{\nabla_y F(y,\Lambda(y))}{\partial_\lambda F(y,\Lambda(y))}, \quad \underline{r} \leq -\Lambda(y) \leq \overline{r}, \tag{4.16}$$

where F is defined in (4.12)*.*

The complete proof, which only uses classical arguments, is postponed to Appendix 4.4.2. We give here a formal idea of the method. The eigenfunction Q satisfies a linear differential equation that allows us to derive

$$Q(x,y) = \frac{1}{A(x,y)} \exp\left(-\int_0^x \frac{d(x',y) - \Lambda(y)}{A(x',y)} dx' \right). \tag{4.17}$$

From this formulation, we deduce that the eigenvalue $\Lambda(y)$ satisfies $F(y,\Lambda(y)) = 1$, for all $y \in \mathbb{R}^d$, where F is defined in (4.12). Since $\partial_\lambda F > 0$, the above equality determines a unique Λ, and therefore a unique Q. Similarly, we derive an explicit formula for Φ.

Note that Q represents the age distribution at equilibrium for a fixed y; thus it seems natural that it exponentially decreases. The eigenvalue Λ defines what we call the *effective fitness*. It drives the adaptive dynamics of the population, as discussed in what follows.

4.2.2 Concentration

Saturation of the population density

The nonlocal term ρ_ε in (4.2), which is also called competition term, can be interpreted as the pressure exerted by the total population on the survival of individuals with trait y. It leads the total population to be bounded. This saturation property also holds for the general model with mutations and is stated in its general form in Proposition 4.11.

Proposition 4.3 *We assume* (4.5)-(4.8) *and* (4.10)*, then,*

$$\forall t \geq 0, \quad \rho_m \leq \rho_\varepsilon(t) \leq \rho_M,$$

where $\rho_m := \min(\underline{r}, \underline{\rho}^0)$ and $\rho_M := \max(\bar{r}, \bar{\rho}^0)$. Hence, after extraction of a subsequence, ρ_ε converges weakly-$\star$ to a function ρ in $L^\infty(0,+\infty)$.

The proof of this result, using classical arguments, is postponed to Appendix 4.4.1 and is given as a particular case of Proposition 4.11.

Thereafter, in order to remove the restriction to a subsequence, we need a uniqueness statement to prove the assertion (i) of Theorem 4.1. It is done in Section 4.2.2.

We now introduce u_ε solution to (4.13), and we define $p_\varepsilon(t,x,y)$ by the factorization (4.14) that we recall

$$m_\varepsilon(t,x,y) = p_\varepsilon(t,x,y)e^{\frac{u_\varepsilon(t,y)}{\varepsilon}}.$$

We first prove the convergence of p_ε. This convergence result is needed to prove the convergence of u_ε and then the uniqueness of ρ and u.

Convergence of p_ε

We state the following theorem on the convergence of p_ε, which details the statement (ii) of Theorem 4.1.

Theorem 4.4 *We assume* (4.5)–(4.11). *With the constants defined in* (4.10)–(4.11) *and* (Q,Φ) *defined in Theorem 4.2,*

1. *we have* $\underline{\gamma}(y)Q(x,y) \le p_\varepsilon(t,x,y) \le \overline{\gamma}(y)Q(x,y)$ *for all* $t \ge 0$,
2. *moreover, the profile* p_ε *converges to the eigenfunction* Q *for a weighted* L^1 *norm. Namely, for* γ^0 *defined in assumption* (4.11) *we have, uniformly in* (t,y),

$$\int_{\mathbb{R}^+} \left| \frac{p_\varepsilon}{Q}(t,x,y) - \gamma^0(y) \right| Q(x,y)\Phi(x,y)dx \to 0 \quad \textit{when } \varepsilon \to 0,$$

The main ingredients of the proof are as follows: in a first step we prove that $\frac{p_\varepsilon}{Q}$ is bounded. Then we use an entropy method to prove that the convergence occurs in a weighted L^1 space. Our approach follows closely [225, 247].

Proof (Proof of Theorem 4.4) *First step: bounds on* $\frac{p_\epsilon}{Q}$. From (4.2) and (4.13)-(4.14), we infer that p_ε satisfies

$$\begin{cases} \varepsilon\partial_t p_\varepsilon(t,x,y) + \partial_x \left[A(x,y)p_\varepsilon(t,x,y)\right] + \left[d(x,y) - \Lambda(y)\right] p_\varepsilon(t,x,y) = 0, \\ A(x=0,y)p_\varepsilon(t,x=0,y) = \int_{\mathbb{R}^+} b(x',y)p_\varepsilon(t,x',y)dx'. \end{cases}$$

Moreover Q satisfies the same linear equation. Assumption (4.10) and the comparison principle for transport equations prove the first statement of Theorem 4.4.

Second step: Entropy inequality. In the sequel, we consider

$$v_\varepsilon(t,x,y) := \frac{p_\varepsilon(t,x,y)}{Q(x,y)} - \gamma^0(y).$$

We also define, for any function $f(t,x,y)$, the average

$$\langle f \rangle(t,y) := \int_{\mathbb{R}_+} f(t,x,y)b(x,y)Q(x,y)dx,$$

and we notice that a direct computation gives

$$\begin{cases} \varepsilon \partial_t v_\varepsilon(t,x,y) + A(x,y)\partial_x v_\varepsilon(t,x,y) = 0, \\ v_\varepsilon(t,x=0,y) = \langle v_\varepsilon \rangle (t,y). \end{cases}$$

Thus we have, in distribution sense

$$\varepsilon \partial_t |v_\varepsilon(t,x,y)| + A(x,y)\partial_x |v_\varepsilon(t,x,y)| = 0.$$

We now introduce the generalized relative entropy

$$E_\varepsilon(t,y) = \int_{\mathbb{R}_+} |v_\varepsilon(t,x,y)| Q(x,y)\Phi(x,y)dx$$

and compute

$$\begin{aligned} \varepsilon \partial_t E_\varepsilon(t,y) &= \int_{\mathbb{R}_+} \varepsilon |\partial_t v_\varepsilon(t,x,y)| Q(x,y)\Phi(x,y)dx \\ &= -\int_{\mathbb{R}_+} A(x,y)|\partial_x v_\varepsilon(t,x,y)| Q(x,y)\Phi(x,y)dx \\ &= -\left[|v_\varepsilon| AQ\Phi\right]_{x=0}^{\infty} + \int_{\mathbb{R}_+} |v_\varepsilon| \partial_x \left(AQ\Phi\right) dx. \end{aligned}$$

The function $|v_\varepsilon| AQ\Phi$ converges to 0 when x goes to infinity,. Indeed, v_ε is bounded from the assertion (i) of Theorem 4.4, A is bounded and, since an explicit computation of $Q\Phi$ gives

$$Q(x,y)\Phi(x,y) = \frac{\Phi(0,y)}{A(x,y)} \left(1 - \int_0^x \frac{b(x',y)}{A(x',y)} \exp\left(\int_0^{x'} \frac{\Lambda(y) - d(x'',y)}{A(x'',y)} dx''\right) dx'\right),$$

from the equality $F(y,\Lambda(y)) = 1$ in (4.15), we deduce that $Q\Phi$ goes to 0 as $x \to \infty$. Then,

$$\varepsilon \partial_t E_\varepsilon(t,y) = \Phi(0,y)\left|\langle v_\varepsilon \rangle\right|(t,y) - \Phi(0,y)\int_{\mathbb{R}_+} bQ|v_\varepsilon| dx.$$

Hence, using the Cauchy-Schwarz inequality,

$$\varepsilon \partial_t E_\varepsilon(t,y) = -\Phi(0,y)\left(\langle |v_\varepsilon| \rangle - |\langle v_\varepsilon \rangle|\right) \leq 0.$$

Therefore $0 \leq E_\varepsilon(t,y) \leq E_\varepsilon(0,y)$, and we conclude for (ii) using (4.11). ∎

Remark 4.5 *As v_ε is bounded, the convergence stated in (iii) occurs in all weighted L^p norms. Namely, for all $p \geq 1$*

$$\int_{\mathbb{R}_+} \left|\frac{p_\varepsilon}{Q}(t,x,y) - \gamma^0(y)\right|^p Q\Phi dx \longrightarrow 0, \quad \textit{when } \varepsilon \to 0.$$

Convergence of u_ε

Integrating (4.13), we obtain the explicit formula

$$u_\varepsilon(t,y) = u_\varepsilon^0(y) - t\Lambda(y) - \int_0^t \rho_\varepsilon(s)ds. \tag{4.18}$$

Hence, by (4.9) and Proposition 4.3, after extraction of a subsequence, u_ε converges locally uniformly to a function u which is given by

$$u(t,y) = u^0(y) - t\Lambda(y) - \int_0^t \rho(s)ds. \tag{4.19}$$

Next, we claim that

$$\sup_{y\in\mathbb{R}^n} u(t,y) = 0, \quad \forall t \geq 0. \tag{4.20}$$

Indeed, we recall $m_\varepsilon(t,x,y) = p_\varepsilon(t,x,y)e^{\frac{u_\varepsilon(t,y)}{\epsilon}}$ and $p_\varepsilon(t,x,y)$ converges in virtue of Theorem 4.4. If there existed a point y_0 for some t such that $u(t,y_0) > 0$, $\rho_\varepsilon(t)$ would diverge, which is a contradiction with Proposition 4.3. In a similar way, $\sup_y u(t,\cdot) < 0$ would imply $\rho_\varepsilon(t) \to 0$, which also contradicts Proposition 4.3. Hence (4.20) must hold.

Thus, up to extraction of a subsequence, m_ε weakly converges to a measure which support is included in the set $\{(t,y) \in [0,+\infty) \times \mathbb{R}^n | u(t,y) = 0\}$. Outside of this set, we know that the population density vanishes locally uniformly as $\varepsilon \to 0$.

Finally we prove the convergence of the whole sequence u_ε. From (4.19) and (4.20) we obtain

$$\int_0^t \rho(s)ds = \sup_{y\in\mathbb{R}^n} [u^0(y) - t\Lambda(y)], \quad \forall t \geq 0.$$

The uniqueness of the limit function ρ is therefore ensured, which implies that the full sequence ρ_ε converges to ρ. Then, the convergence of the full family u_ε follows from (4.18). Hence the statements (i),(iii) and (iv) of Theorem 4.1.

4.2.3 Properties of concentration points

Since we can explicitly integrate (4.13) to obtain (4.19), we can identify the points where the population concentrates, which are the points where u vanishes.

Proposition 4.6 *Let $t \in (0,\infty)$ and $\bar{y}(t) \in \mathbb{R}^n$ such that $u(t,\bar{y}(t)) = 0$, where u is given in* (4.19). *As $\bar{y}(t)$ is a maximum point of $u(t,\cdot)$, it satisfies*

$$\nabla_y u^0(\bar{y}(t)) = t\nabla_y \Lambda\left(\bar{y}(t)\right), \tag{4.21}$$

and we have

$$u^0(\bar{y}(t)) = \int_0^t \rho(t')dt' - t\rho(t). \tag{4.22}$$

Proof From equation (4.19) we derive

$$u^0(\bar{y}(t)) = t\Lambda(\bar{y}(t)) + \int_0^t \rho(t')dt'. \tag{4.23}$$

Besides, $\bar{y}(t)$ is a maximum point of $u(t,\cdot)$, therefore $\nabla_y u(t,\bar{y}(t)) = 0$ which proves (4.21). Moreover $\partial_t u(t,\bar{y}(t)) = 0$, and using (4.13) we obtain

$$\Lambda(\bar{y}(t)) = -\rho(t). \tag{4.24}$$

Thus, combining (4.23) and (4.24), we obtain equation (4.22) ∎

At this stage, the concentration of the population density on a single trait $\bar{y}(t)$ cannot be concluded yet because the above relation defines a hypersurface. There are two frameworks in which one can prove that the population is monomorphic, that is, the population converges in measure toward a Dirac mass located on a unique point $\bar{y}(t)$ at each time $t \geq 0$. The first framework assumes that y is one dimensional, and $y \mapsto \Lambda(y)$ is strictly monotonic. The second assumes, for $y \in \mathbb{R}^d$, that $u_\varepsilon^0(\cdot)$ and $-\Lambda(\cdot)$ are strictly concave uniformly in ε. The interested reader can refer to [36] and [213] for a complete analysis of these two cases.

In the framework of strict uniform concavity, we obtain the additional result of uniform regularity on u_ε and u, which enables to rigorously derive a form of canonical equation in the language of adaptive dynamics. This canonical equation gives the dynamics of the selected trait, that is, the evolution of the concentration point in an evolutionary time scale.

Theorem 4.7 *Assume that u^0 and $-\Lambda$ are strictly concave in a neighborhood of $\bar{y}^0$ defined in* (4.9). *Then $u(t,\cdot)$, given in* (4.19), *is locally strictly concave and there exists $T > 0$ such that for all $t \in (0,T)$, $u(t,\cdot)$ reaches its maximum* 0 *on a unique point $\bar{y}(t)$. Moreover $t \mapsto \bar{y}(t) \in \mathcal{C}^1(0,T)$ and its dynamics is described by the equation*

$$\dot{\bar{y}}(t) = \left(\nabla_y^2 u(t,\bar{y}(t))\right)^{-1} \cdot \nabla_y \Lambda\left(\bar{y}(t)\right), \quad \bar{y}(0) = \bar{y}^0. \tag{4.25}$$

Proof We are interested in the solutions $\bar{y}(t) \in \mathbb{R}^n$ of

$$\nabla_y u(t,\bar{y}(t)) = 0. \tag{4.26}$$

Note that u is strictly concave, because u^0 and $-\Lambda$ are. Therefore, such a $\bar{y}(t)$ must satisfy $u(t,\bar{y}(t)) = \max_y u(t,y) = 0$.

From (4.9) we know that at initial time there exists a unique solution $\bar{y}^0$ of (4.26). Besides, as u is strictly concave, $\nabla_y^2 u$ is invertible. Hence, thanks to the implicit functions theorem, there exists $T > 0$ such that for all $t \in (0,T)$, there exists a unique $\bar{y}(t) \in \mathbb{R}^n$ satisfying (4.26). Moreover, $t \mapsto \bar{y}(t)$ is a $\mathcal{C}^1$ function, and then differentiating (4.26) with respect to t, we obtain, using (4.19),

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\nabla_y u(t,\bar{y}(t))\right] = -\nabla_y \Lambda(\bar{y}(t)) + \left(\nabla_y^2 u(t,\bar{y}(t)\right) \cdot \dot{\bar{y}}(t),$$

and (4.25) follows. ∎

Remark 4.8 *Note that we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\Lambda\left(\bar{y}(t)\right)\right] = \left(\nabla_y \Lambda(\bar{y}(t))\right) \cdot \left(\nabla_y^2 u(t,\bar{y}(t))\right)^{-1} \cdot \left(\nabla_y \Lambda(\bar{y}(t))\right). \tag{4.27}$$

Then, we deduce that $\frac{\mathrm{d}}{\mathrm{d}t}\left[\Lambda\left(\bar{y}(t)\right)\right] \leq 0$. Therefore, if at initial time $\bar{y}^0$ belongs to a potential well of Λ, then $\bar{y}(t)$ remains bounded. Thus Theorem 4.7 holds globally in time and $\bar{y}(t)$ converges to a local minimum of Λ when t goes to infinity.

From Theorem 4.7 we infer the statement (v) of Theorem 4.1. We also give the following additional results. The first one is derived directly from (4.16), the second one from (4.24) and (4.27).

Corollary 4.9 *Under the same hypothesis as in Theorem 4.7, the critical points for evolutionary dynamics satisfy $\nabla_y F(y^*, \Lambda(y^*)) = 0$.*

Corollary 4.10 *Under the same hypothesis as in Theorem 4.7, we have $t \mapsto \rho(t) \in \mathcal{C}^1(0,T)$ and $\dot{\rho}(t) \geq 0$ for all $t \in (0,T)$.*

4.2.4 Numerical simulations

In order to complete the theory, we present numerical results. We perform a simulation of equation (4.2) with $\varepsilon = 5{\cdot}10^{-3}$. The numerical results allow to visualize u_ε and then the concentration dynamics of the population density. We choose the variable pair (x, y) to be in the set $[0, 1] \times [0, 4]$ which we discretize with the steps $\Delta x = \frac{1}{M}$ and $\Delta y = \frac{1}{N}$ with $M = 90, N = 40$. The time step Δt is chosen to be $5 \cdot 10^{-5}$ according to the CFL condition. We denote by $m_{i,j}^k$ the numerical solution at grid point $x_i = i\Delta x$, $y_j = j\Delta y$ and time $t_k = k\Delta t$. Equation (4.2) is solved by an implicit-explicit finite-difference method with the following scheme: for $i = 1, \dots, N$ and $j = 1, \dots, M$,

$$m_{i,j}^{k+1} = m_{i,j}^k - \frac{\Delta t}{\varepsilon}\frac{A(x_i, y_j)m_{i,j}^k - A(x_{i-1}, y_j)m_{i-1,j}^k}{\Delta x} - \frac{\Delta t}{\varepsilon}\left(\rho^k m_{i,j}^k - d(x_i, y_j)m_{i,j}^{k+1}\right), \tag{4.28}$$

and the boundary term is discretized as

$$A(0, y_j)m_{0,j}^{k+1} = \sum_{i=1}^{M} b(x_i, y_j)m_{i,j}^k,$$

which is necessary for computing when $i = 0$ in (4.28).

The numerics is performed using Matlab with parameters as follows. We choose the initial number of individuals to be 1000 and the final time $T = 1.5$. We choose the following functions A, b and d as follows.

$$A(x, y) = 1, \qquad b(x, y) = 10 \cdot \frac{y}{1 + x^2}, \qquad d(x, y) = y^3 \cdot (2 + x/3),$$

and the initial data

$$m^0(x, y) = p^0(x, y)e^{\frac{u^0(y)}{\varepsilon}},$$

with

$$p^0(x, y) = \exp(-0.8x), \qquad u^0(y) = -\frac{(y - 0.5)^2}{2}.$$

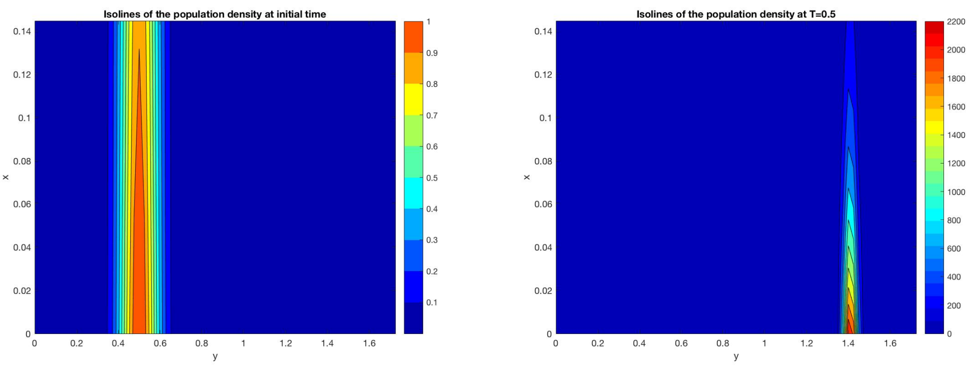


Figure 4.1 – Isolines in (x, y) of the population distribution

We choose to create a trade-off between the birth and death rates with regards to the y variable, by assuming that $y \mapsto b(x, y)$ and $y \mapsto d(x, y)$ are increasing, which means that higher natality also induces higher mortality. This assumption allows to determine an Evolutionary Stable Distribution or ESD from the language of adaptive dynamics, which gives the repartition of the fittest traits (see [78, 124, 182]). We do not know this ESD from the beginning; however, it is important to select, according to assumptions (4.5)-(4.6), a death rate with a stronger increase for large x than the growth rate with regards to the trait variable in order to avoid that the dominant traits go to infinity.

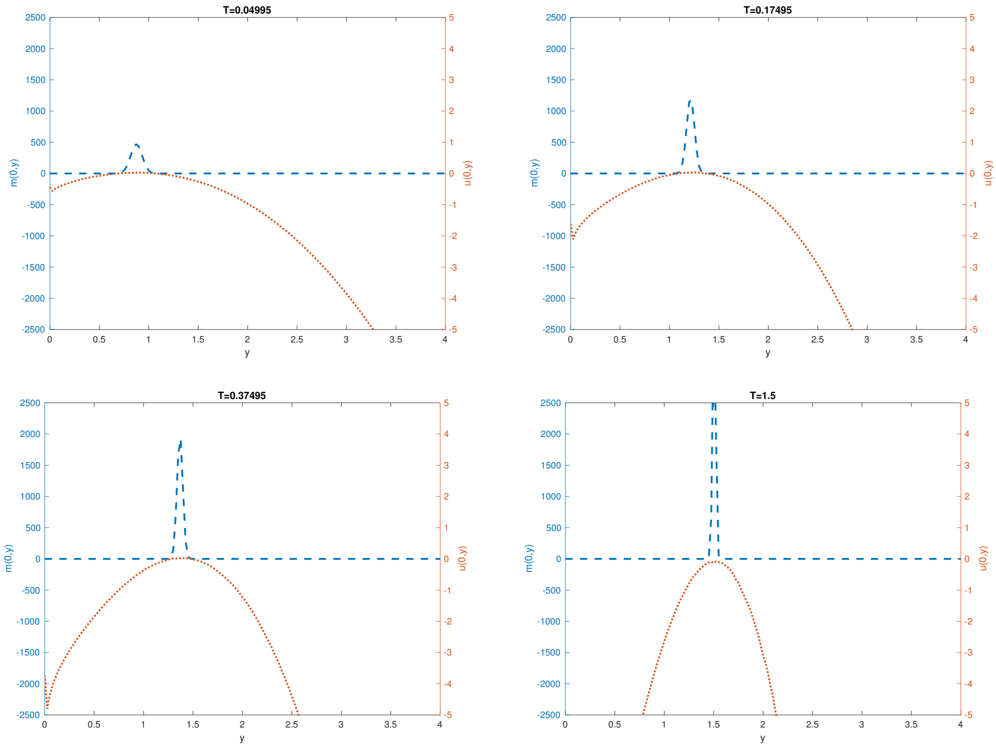


Figure 4.2 – Concentration dynamics: snapshots of the population distribution in y at four different times with respect to the trait variable. Blue dashed line= m_ε, red dotted line = u_ε.

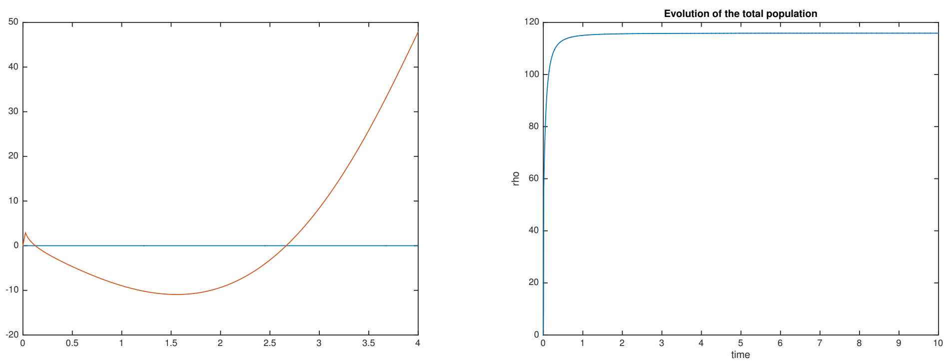


Figure 4.3 – Left: Principal eigenvalue $\Lambda(y)$. Right: Evolution of ρ over time

Figure 4.1 shows the population distribution with regards to y (abscissa) and x (ordinates) at two different times. The population has moved and concentrated to a location which is different from its initial one. One can observe this continuous evolution of the population distribution in Figure 4.2 where we show the distribution of individuals with age $x = 0$ at different times and identify an ESD.

The ESD can also be identified thanks to the principal eigenvalue. We show in Figure 4.3 the eigenvalue $\Lambda(y)$ solved by the Newton method using (4.15). From equation (4.25) one can notice that the equilibrium points have to satisfy $\nabla_y \Lambda(y) = 0$ and moreover that the dynamics of the concentration is directed towards the minimum points of $\Lambda(y)$, as predicted by our analysis.

4.3 Case with mutations

We turn to the model (4.1) including mutations. We use the same approach as in the previous section, that is, we write $m_\varepsilon(t,x,y) = p_\varepsilon(t,x,y)e^{\frac{u_\varepsilon(t,y)}{\varepsilon}}$ and insert this form in (4.1). We obtain

$$\begin{cases} \varepsilon\partial_t p_\varepsilon(t,x,y) + \partial_x\left[A(x,y)p_\varepsilon(t,x,y)\right] + d(x,y)p_\varepsilon(t,x,y) \\ \qquad\qquad\qquad = -(\rho_\varepsilon(t) + \partial_t u_\varepsilon(t,y))p_\varepsilon(t,x,y), \\ A(x=0,y)\, p_\varepsilon\,(t,x=0,y) \\ \qquad\qquad = \dfrac{1}{\varepsilon^n}\displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} M(\frac{y'-y}{\varepsilon})b(x',y')p_\varepsilon(t,x',y')e^{\frac{u_\varepsilon(t,y')-u_\varepsilon(t,y)}{\varepsilon}}dx'dy', \\ \rho_\varepsilon(t) = \displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} m_\varepsilon(t,x,y)dxdy, \\ p_\varepsilon(t=0,x,y) = p^0_\varepsilon(x,y) > 0. \end{cases} \tag{4.29}$$

With the change of variable $z = \frac{y'-y}{\varepsilon}$, the renewal term is written as

$$\begin{aligned} &A(x=0)\, p_\varepsilon(t,x=0,y) \\ &\qquad = \int_{\mathbb{R}^n}\int_{\mathbb{R}_+} M(z)e^{\frac{u_\varepsilon(t,y+\varepsilon z)-u_\varepsilon(t,y)}{\varepsilon}}b(x',y+\varepsilon z)p_\varepsilon(t,x',y+\varepsilon z)dx'dz. \end{aligned} \tag{4.31}$$

By taking formally the limit $\varepsilon \to 0$, we obtain

$$A(x=0)p(t,x=0,y) = \int_{\mathbb{R}^n} M(z)e^{\nabla_y u(t,y)\cdot z}dz \int_{\mathbb{R}_+} b(x',y)p(t,x',y)dx'.$$

Denoting

$$\eta(t,y) := \int_{\mathbb{R}^n} M(z)e^{\nabla_y u(t,y)\cdot z}dz,$$

the formal limit of (4.29) is written as

$$\begin{cases} \partial_x\left[A(x,y)p(t,x,y)\right] + d(x,y)p(t,x,y) = -(\rho(t) + \partial_t u(t,y))p(t,x,y), \\ A(x=0)p(t,x=0,y) = \eta(t,y)\displaystyle\int_{\mathbb{R}_+} b(x',y)p(t,x',y)dx', \\ \rho(t) = \displaystyle\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} m(t,x,y)dxdy, \\ p(t=0,x,y) = p^0(x,y) > 0, \quad u(t=0,y) = u^0(y). \end{cases}$$

With this form, one can consider the following eigenproblem: for fixed $(y,\eta) \in \mathbb{R}^n \times (0,+\infty)$, find $(\Lambda(y,\eta), Q(x,y,\eta))$, solution of

$$\begin{cases} \partial_x\left[A(x,y)Q(x,y,\eta)\right] + d(x,y)Q(x,y,\eta) = \Lambda(y,\eta)Q(x,y,\eta), \\ A(x=0,y)Q(x=0,y,\eta) = \eta\displaystyle\int_{\mathbb{R}_+} b(x',y)Q(x',y,\eta)dx', \\ Q(x,y,\eta) > 0, \quad \displaystyle\int_{\mathbb{R}_+} b(x,y)Q(x,y,\eta)dx = 1. \end{cases} \tag{4.32}$$

Using this eigenproblem, we will first compute the formal limit u of the sequence u_ε, and prove that it satisfies the following Hamilton-Jacobi equation

$$\begin{cases} \partial_t u(t,y) = -\Lambda\left(y, \int_{\mathbb{R}^n} M(z) e^{\nabla_y u(t,y)\cdot z} dz\right) - \rho(t), \quad t \geq 0, y \in \mathbb{R}^n, \\ u(0,y) = u^0(y), \quad y \in \mathbb{R}^n. \end{cases} \tag{4.33}$$

In this way, we formally recover the limit profile p using (4.32) with $\eta = \eta(t,y)$. Back to the question of adaptive dynamics, $\Lambda(y, \eta(t,y))$ defines the effective fitness of the population with trait y.

In what follows, we study this limit problem and construct a solution u. Actually the convergence of p_ε towards the solution Q of the eigenproblem (4.32) is an unsolved question. Indeed because of the particular form of the boundary condition (4.31), we do not know how to study the asymptotics of p_ε as $\varepsilon \to 0$. However, we construct a sequence u_ε from an approximation problem of (4.33) that is well defined, and we prove it converges to the solution of (4.33) in the viscosity sense.

To begin with, we state the saturation of the population density, and the existence and uniqueness of the eigenelements of (4.32).

4.3.1 Saturation and stationary problem

As in the case without mutations in the previous section, it still holds that the total population is bounded.

Proposition 4.11 *We assume* (4.5)–(4.8) *and* (4.10)*. Then there exist two constants $\rho_m, \rho_M > 0$ such that*

$$\forall t \geq 0, \; 0 < \rho_m \leq \rho_\varepsilon(t) \leq \rho_M.$$

where $\rho_m := \min(\underline{r}, \underline{\rho}^0)$ and $\rho_M := \max(\bar{r}, \bar{\rho}^0)$. Hence, after extracting a subsequence, ρ_ε converges to a function ρ in weak-$L^\infty(0,+\infty)$.*

We now establish the existence and uniqueness of the eigenelements in (4.32). Thus we introduce the associated dual problem: find $\Phi(x,y,\eta)$ solution of

$$\begin{cases} A(x,y)\partial_x \Phi(x,y,\eta) + \left[\Lambda(y,\eta) - d(x,y)\right]\Phi(x,y,\eta) = -\eta b(x,y)\Phi(0,y,\eta), \\ \int_{\mathbb{R}^+} Q(x,y,\eta)\Phi(x,y,\eta) dx = 1. \end{cases} \tag{4.34}$$

We also recall the definition (4.12) for the function F. The proof of the following theorem is given in Appendix 4.4.2.

Theorem 4.12 *We assume* (4.5)–(4.7)*. Given $y \in \mathbb{R}^n$ and $\eta \in \mathbb{R}_+$, there exists a unique triplet $(\Lambda(y,\eta), Q(x,y,\eta), \Phi(x,y,\eta))$ solution of* (4.32) *and* (4.34)*. The map $x \mapsto Q(x,y,\eta)$ is bounded and integrable, $y \mapsto \Lambda(y,\eta)$ is $\mathcal{C}^1$ and we have*

$$\partial_\lambda F > 0, \quad F(y, \Lambda(y,\eta)) = \frac{1}{\eta}, \tag{4.35}$$

$$\nabla_y \Lambda(y,\eta) = -\frac{\nabla_y F(y,\Lambda(y,\eta))}{\partial_\lambda F(y,\Lambda(y,\eta))}, \quad \partial_\eta \Lambda(y,\eta) = -\frac{1}{\eta^2 \partial_\lambda F(y,\Lambda(y,\eta))} < 0. \tag{4.36}$$

In the sequel we consider the effective Hamiltonian (fitness)

$$H(y,p) := -\Lambda\Big(y, \eta(p)\Big), \quad \eta(p) := \int_{\mathbb{R}^n} M(z)e^{p\cdot z}dz > 0. \tag{4.37}$$

Before constructing a solution to the associated Hamilton-Jacobi equation in the next section, we state the following result, which is proved in Appendix 4.4.3.

Proposition 4.13 *The mapping $p \mapsto H(y,p)$ is convex, for all $y \in \mathbb{R}^n$.*

4.3.2 The Hamilton-Jacobi equation

Here we consider the Hamilton-Jacobi equation (4.33) that we may write from (4.37) as

$$\begin{cases} \partial_t u(t,y) = H(y, \nabla_y u) - \rho(t), \\ u(0,y) = u^0(y), \quad y \in \mathbb{R}^n. \end{cases}$$

Our goal is to build a solution to this equation. Therefore, we introduce u_ε solution of an approximate problem motivated by the form in (4.29), which reads

$$\begin{cases} \partial_t u_\varepsilon(t,y) = -\Lambda\left(y, \int_{\mathbb{R}^n} M(z)e^{\frac{u_\varepsilon(t,y+\varepsilon z)-u_\varepsilon(t,y)}{\varepsilon}}dz\right) - \rho_\varepsilon(t), \\ u_\varepsilon(0,y) = u_\varepsilon^0(y), \quad y \in \mathbb{R}^n. \end{cases} \tag{4.38}$$

To simplify the Hamiltonian in equation (4.38), we set

$$U_\varepsilon(t,y) := u_\varepsilon(t,y) + \int_0^t \rho_\varepsilon(t')dt,$$

which satisfies

$$\partial_t U_\varepsilon(t,y) = -\Lambda\left(y, \int_{\mathbb{R}^n} M(z)e^{\frac{U_\varepsilon(t,y+\varepsilon z)-U_\varepsilon(t,y)}{\varepsilon}}dz\right). \tag{4.39}$$

For clarity, we set

$$\eta_\varepsilon(t,y) = \int_{\mathbb{R}^n} M(z)e^{\frac{U_\varepsilon(t,y+\varepsilon z)-U_\varepsilon(t,y)}{\varepsilon}}dz.$$

We state the following theorem, which is the main result of this section. The set of assumptions $(\mathcal{H})$ is presented below.

Theorem 4.14 *Assuming $(\mathcal{H})$ there exists a unique solution U_ε to (4.39). Furthermore, U_ε converges locally uniformly to a function U which is a viscosity solution of the equation*

$$\partial_t U(t,y) = H(y, \nabla_y U) = -\Lambda\left(y, \int_{\mathbb{R}^n} M(z)e^{\nabla_y U\cdot z}dz\right). \tag{4.40}$$

In other words, we prove a stability result in the language of the viscosity solutions theory (see [31]) in a situation where the Hamiltonian depends on $\nabla_y U$ with an exponential growth, which is the main difficulty here. The plan of the proof is as follows. Firstly, we consider the truncated equation associated with (4.39), for which classical results give existence and uniqueness of a global solution. Then we provide a uniform a priori estimate on the time derivative of the solution. It allows us to remove the truncation and to infer a global solution U_ε of (4.39). This proves the first step.

Secondly, we consider the semi-relaxed limits $\overline{U} := \limsup U_\varepsilon$ and $\underline{U} := \liminf U_\varepsilon$, and prove that they are respectively subsolution and supersolution of (4.40) in the viscosity sense. Then, an assumption of coercivity of $\eta \mapsto \Lambda(y,\eta)$ in (4.42), allows us to state that $\underline{U}$ is a Lipschitz function. Finally, using an uncommon uniqueness result on the Hamiltonian H, we prove that $\overline{U} = \underline{U}$, and conclude that U_ε converges locally uniformly to a viscosity solution of (4.40).

Assumptions ($\mathcal{H}$). We assume (4.9). In addition, for any compact interval I, we assume there exist two constants L_0, $L_1 > 0$, (depending on I) such that

$$\forall y \in \mathbb{R}^n, \forall \eta \in I, \quad \begin{cases} |\Lambda(y,\eta)| \le L_0, \\ |\partial_\eta \Lambda(y,\eta)| \le L_1. \end{cases} \tag{4.41}$$

We also assume

$$|\Lambda(y,\eta)| \to +\infty \text{ when } \eta \to +\infty \text{ or } \eta \to 0, \text{ uniformly in } y \in \mathbb{R}^n. \tag{4.42}$$

Finally, the following assumption is required for our uniqueness result, stated in Theorem 4.18. For all compact set $K_p \subset \mathbb{R}^n$, we assume there exist $C > 0, \gamma_1 \in [0,4), \gamma_2 \in [0,1)$ such that

$$\forall y \in \mathbb{R}^n, \forall p \in K_p, \quad \begin{cases} |\nabla_y H(y,p)| \le C\left(1 + |y|^{\gamma_1}\right), \\ |\nabla_p H(y,p)| \le C\left(1 + |y|^{\gamma_2}\right). \end{cases} \tag{4.43}$$

4.3.3 Global existence and a priori estimate

This section is devoted to the proof of the following Theorem, which is the first step towards Theorem 4.14.

Theorem 4.15 *Assume* (4.41)*. Then, for all $\varepsilon > 0$, there exists a unique global solution U_ε to the equation* (4.39)*, such that $|\partial_t U_\varepsilon(t,y)| \le L$ for a constant $L > 0$, uniformly in $\varepsilon > 0,\ t > 0,\ y \in \mathbb{R}^n$.*

The truncated problem

We first consider a truncated problem associated to (4.39). For a fixed $R > 0$, we define the function $\phi_R : \mathbb{R} \to \mathbb{R}$ which is smooth, increasing and satisfies the following conditions:

— $\phi_R(r) = r$ for $r \in [-\frac{R}{2}, \frac{R}{2}]$,
— $\phi_R(r) = R$ for $r \ge 2R$,
— $\phi_R(r) = -R$ for $r \le -2R$,
— $\phi_R' \ge 0$ is uniformly bounded.

Let $\varepsilon > 0$ be fixed. We consider the Cauchy problem

$$\begin{cases} \partial_t U_\varepsilon^R(t,y) = \phi_R\left(-\Lambda\left(y, \int_{\mathbb{R}^n} M(z) e^{\frac{U_\varepsilon^R(t,y+\varepsilon z) - U_\varepsilon^R(t,y)}{\varepsilon}} dz\right)\right), \\ U_\varepsilon^R(0,\cdot) = u_\varepsilon^0. \end{cases} \tag{4.44}$$

We state the following result

Lemma 4.16 *Assuming* (4.41)*, there exists a unique solution of* (4.44)*, defined globally in time.*

The proof is based on the Cauchy-Lipschitz Theorem and uses only classical arguments. It is left to the reader.

Estimate on the time derivative

The particular form of (4.44) allows us to infer uniform a priori estimates on $\partial_t U_\varepsilon^R$. It is stated in the following result.

Proposition 4.17 *For all $R > 0, \varepsilon > 0$, we have*

$$\|\partial_t U_\varepsilon^R\|_\infty \leq \|\partial_t u_\varepsilon^0\|_\infty := \|\Lambda(y, \eta_\varepsilon(0, y))\|_\infty .$$

As a consequence, there exists a positive constant L, independent of R and ε such that

$$\forall \varepsilon > 0, \forall R > 0, \forall t \geq 0, \forall y \in \mathbb{R}^n, \quad |\partial_t U_\varepsilon^R(t, y)| \leq L.$$

The complete proof is postponed to Appendix 4.4.4. However we give the formal idea here. As R is fixed, we simply write U_ε instead of U_ε^R. We set $V_\varepsilon(t, y) := \partial_t U_\varepsilon(t, y)$. Differentiating (4.40) with respect to t, we obtain

$$\partial_t V_\varepsilon(t, y) = \int_{\mathbb{R}^n} K_\varepsilon(t, y, z) \left(\frac{V_\varepsilon(t, y + \varepsilon z) - V_\varepsilon(t, y)}{\varepsilon} \right) dz,$$

where $K_\varepsilon(t, y, z) := -\partial_\eta \Lambda\left(y, \eta_\varepsilon(t, y)\right) M(z) e^{\frac{U_\varepsilon(t, y+\varepsilon z) - U_\varepsilon(t, y)}{\varepsilon}}$. Note that, thanks to (4.36), K_ε is positive. Then, if for some $t > 0$, $V_\varepsilon(t, \cdot)$ reaches its maximum at $\bar{y} \in \mathbb{R}^n$, we obtain the inequality

$$\partial_t V_\varepsilon(t, \bar{y}) = \int_{\mathbb{R}^n} K_\varepsilon(t, \bar{y}, z) \left(\frac{V_\varepsilon(t, \bar{y} + \varepsilon z) - V_\varepsilon(t, \bar{y})}{\varepsilon} \right) dz \leq 0.$$

Formally, it shows that the maximum value of V_ε is decreasing with time, that is,

$$\sup_y V_\varepsilon(t, y) \leq \sup_y V_\varepsilon(0, y) = \sup_y \partial_t u_\varepsilon^0.$$

With the same method we show $\inf_y \partial_t U_\varepsilon \geq \inf_y \partial_t u_\varepsilon^0$, which completes the first step of the proof. Then, using (4.41) and that u_ε^0 is a Lipschitz function from (4.9), we deduce an estimate on $\partial_t U_\varepsilon$, uniform in $R > 0$ and $\varepsilon > 0$.

Removing the truncation

From Proposition 4.17, $\partial_t U_\varepsilon^R(t, y) = \phi_R\left(-\Lambda\left(y, \eta_\varepsilon(t, y)\right)\right)$ is bounded uniformly in R. As $\phi_R \equiv \mathrm{Id}$ on $[-\frac{R}{2}, \frac{R}{2}]$, then, for R large enough, U_ε^R is also solution to the non-truncated problem (4.39). Conversely, a solution to (4.39) with a bounded time derivative is a solution to the truncated problem (4.44) for R large enough. Thus $U_\varepsilon := U_\varepsilon^R$ is the unique solution of (4.39) with $\|\partial_t U_\varepsilon\|_\infty \leq L$, for R large enough. The proof of Theorem 4.15 is thereby complete.

4.3.4 The semi-relaxed limits

We assume (4.41). Thanks to Theorem 4.15, there exists a constant $C > 0$ such that

$$|U_\varepsilon(t, y)| \leq |u_\varepsilon^0(y)| + Lt \leq C + Lt + k_0|y|, \quad \forall t > 0, \ \forall y \in \mathbb{R}^n, \tag{4.45}$$

uniformly in $\varepsilon > 0$. This allows us to consider the following semi-relaxed limits (see [34, 181])

$$\overline{U}(t,y) = \limsup_{\substack{x\to y\\ s\to t\\ \varepsilon\to 0}} U_\varepsilon(s,x), \quad \underline{U}(t,y) = \liminf_{\substack{x\to y\\ s\to t\\ \varepsilon\to 0}} U_\varepsilon(s,x). \tag{4.46}$$

Note that accordingly $\underline{U}$ and $\overline{U}$ satisfy the inequality (4.45). More precisely, from the uniform estimate on the time derivative stated in Theorem 4.15 we have

$$|\overline{U}(t,y) - u^0(y)| \leq Lt, \quad |\underline{U}(t,y) - u^0(y)| \leq Lt. \tag{4.47}$$

In this section, we prove

Theorem 4.18 *Assuming* (4.41)–(4.43)*, we have* $\overline{U} = \underline{U}$*.*

This result implies that U_ε converges locally uniformly to a solution U of equation (4.40), which completes the proof of Theorem 4.14.

Subsolution and supersolution

The following proposition is adapted from classical stability results for viscosity solutions of Hamilton-Jacobi equations (see [31]). Note that it slightly differs from the usual framework because of the nonlocal term $\eta_\varepsilon(t,y)$.

Proposition 4.19 *The semi-continuous functions* $\overline{U}$ *and* $\underline{U}$ *defined in* (4.46) *are respectively subsolution and supersolution of* (4.40) *in the viscosity sense in* $(0,\infty)\times\mathbb{R}^n$*. Also, for all* $T>0$*, the viscosity inequalities stand for* $t\in(0,T]$*.*

Proof (Proof of Proposition 4.19.) In order to prove that $\overline{U}$ is a viscosity subsolution of (4.40), since $\overline{U}$ is upper semi-continuous, let us consider a test function φ and a point (t_0,y_0) such that $\overline{U}-\varphi$ reaches a global maximum at (t_0,y_0). From classical results, there exists $(t_\varepsilon,y_\varepsilon)$ such that

$$\begin{cases} (t_\varepsilon,y_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} (t_0,y_0), \\ \max\limits_{t,y} U_\varepsilon - \varphi = (U_\varepsilon-\varphi)(t_\varepsilon,y_\varepsilon). \end{cases}$$

For all $z\in\mathbb{R}^n$, $\varphi(t_\varepsilon,y_\varepsilon+\varepsilon z) - U_\varepsilon(t_\varepsilon,y_\varepsilon+\varepsilon z) \geq \varphi(t_\varepsilon,y_\varepsilon) - U_\varepsilon(t_\varepsilon,y_\varepsilon)$, thus we have

$$\frac{\varphi(t_\varepsilon,y_\varepsilon+\varepsilon z)-\varphi(t_\varepsilon,y_\varepsilon)}{\varepsilon} \geq \frac{U_\varepsilon(t_\varepsilon,y_\varepsilon+\varepsilon z)-U_\varepsilon(t_\varepsilon,y_\varepsilon)}{\varepsilon}.$$

Since $\partial_\eta\Lambda<0$ from (4.36), equation (4.39) gives

$$\begin{aligned} \partial_t\varphi(t_\varepsilon,y_\varepsilon) &= -\Lambda\left(y_\varepsilon, \int_{\mathbb{R}^n} M(z) e^{\frac{U_\varepsilon(t_\varepsilon,y_\varepsilon+\varepsilon z)-U_\varepsilon(t_\varepsilon,y_\varepsilon)}{\varepsilon}} dz\right) \\ &\leq -\Lambda\left(y_\varepsilon, \int_{\mathbb{R}^n} M(z) e^{\frac{\varphi(t_\varepsilon,y_\varepsilon+\varepsilon z)-\varphi(t_\varepsilon,y_\varepsilon)}{\varepsilon}} dz\right). \end{aligned}$$

As ε goes to 0,

$$\partial_t\varphi(t_0,y_0) \leq -\Lambda\left(y_0, \int_{\mathbb{R}^n} M(z) e^{\nabla_y\varphi(t_0,y_0)\cdot z}\right) = H(y_0,\nabla_y\varphi(t_0,y_0)),$$

then $\overline{U}$ is a viscosity subsolution of (4.40). With the same method, we prove that $\underline{U}$ is a viscosity supersolution. It completes the first part of the proof. The second part of the statement is a well-known result, and proof can be found in [31]. ∎

A posteriori Lipschitz estimate on $\underline{U}$

The announced Lipschitz continuity of $\underline{U}$ is stated in the following result.

Proposition 4.20 *Assume* (4.41)–(4.42). *Then the lower semi-continuous function $\underline{U}$ defined in* (4.46) *is a L-Lipschitz function with $L > 0$ defined below.*

We first prove these two preliminary lemmas. We point out that (4.42) plays a crucial role in the proof.

Lemma 4.21 *Assume* (4.41)–(4.42). *Then there exist positive constants $\underline{\eta}$,$\bar{\eta}$, L_1 such that, uniformly in $\varepsilon > 0$, $\forall (t, y) \in (0, +\infty) \times \mathbb{R}^n$,*

$$\underline{\eta} \leq \eta_\varepsilon(t, y) \leq \bar{\eta}, \tag{4.49}$$

$$|\partial_\eta \Lambda\,(y, \eta_\varepsilon(t, y))| \leq L_1. \tag{4.51}$$

Proof From Theorem 4.15, we know $\partial_t U_\varepsilon(t, y) = -\Lambda(y, \eta_\varepsilon(t, y))$ is bounded for $(t, y) \in (0, +\infty) \times \mathbb{R}^n$, uniformly in $\varepsilon > 0$. From (4.42), we deduce that $\eta_\varepsilon(t, y)$ is bounded, which proves (4.49). Then we derive (4.51) directly from assumption (4.41). ∎

In what follows, we use the notation $\nabla U = (\partial_t U, \nabla_y U)$.

Lemma 4.22 *In the viscosity sense, $\nabla \underline{U}$ is bounded, that is, there exists a constant $L \geq k_0$ such that if ψ is a smooth function and $\underline{U} - \psi$ reaches its minimum at $(t_0, y_0) \in (0, +\infty) \times \mathbb{R}^n$, then*

$$|\partial_t \psi(t_0, y_0)| \leq L,$$

$$|\nabla_y \psi(t_0, y_0)|_\infty \leq L.$$

Proof Let ψ be a smooth function such that $\underline{U} - \psi$ reaches its minimum at (t_0, y_0). Similarly to the proof of Proposition 4.19, up to extraction of a subsequence, there exists a sequence of minimum points $(t_\varepsilon, y_\varepsilon)$ of $U_\varepsilon - \psi$ which converges to (t_0, y_0). As $\underline{U}$ is a supersolution, we obtain

$$\begin{aligned} -\Lambda\left(y_\varepsilon, \int_{\mathbb{R}^n} M(z) e^{\frac{\psi(t_\varepsilon, y_\varepsilon + \varepsilon z) - \psi(t_\varepsilon, y_\varepsilon)}{\varepsilon}} dz\right) & \\ \leq \partial_t \psi(t_\varepsilon, y_\varepsilon) = \partial_t U_\varepsilon(t_\varepsilon, y_\varepsilon) &= -\Lambda\,(y_\varepsilon, \eta_\varepsilon(t_\varepsilon, y_\varepsilon))\,. \end{aligned} \tag{4.53}$$

From the estimate on $\partial_t U_\varepsilon$ given by Theorem 4.15, we have, when ε goes to 0,

$$|\partial_t \psi(t_0, y_0)| \leq L.$$

Thus, from $\partial_\eta \Lambda < 0$, (4.49) and (4.53), we derive, as ε goes to 0,

$$\int_{\mathbb{R}^n} M(z) e^{\nabla_y \psi(t_0, y_0) \cdot z} dz \leq \overline{\eta}.$$

Since $M(z) > 0$, we deduce

$$|\nabla_y \psi(t_0, y_0)|_\infty \leq L',$$

for some constant L'. Setting $L := \max(L, L', k_0)$ achieves the proof. ∎

PROOF (PROOF OF PROPOSITION 4.20.) We want to prove that

$$\forall (t,t') \in (0,\infty)^2, (y,y') \in (\mathbb{R}^n)^2, \quad \underline{U}(t',y') - \underline{U}(t,y) \le L(|t-t'| + |y-y'|).$$

By contradiction, we assume that there exists $K > L$ such that, for some $(t_0, t'_0) \in (0,\infty)^2$ and $(y_0, y'_0) \in (\mathbb{R}^n)^2$,

$$\underline{U}(t'_0, y'_0) - \underline{U}(t_0, y_0) - K(|t_0 - t'_0| + |y_0 - y'_0|) > 0. \tag{4.54}$$

Let us define the test function $\psi(t,y) := \underline{U}(t'_0, y'_0) - K(|t-t'_0| + |y-y'_0|)$. As $k_0 < K$, from (4.45) we derive that $\psi(t,y) - \underline{U}(t,y) \to -\infty$ when $|y| \to \infty$. Because this function is upper semicontinuous, it reaches its maximum at a point $(\bar{t}, \bar{y}) \in [0,\infty) \times \mathbb{R}^n$. In order to apply Lemma 4.22 at $(\bar{t}, \bar{x})$, we have to prove that $\bar{t} > 0$ and that ψ is smooth in a neighborhood of $\bar{x}$. We prove the first assertion by contradiction. We assume $\bar{t} = 0$. From (4.47) and the Lipschitz continuity of u^0, we have

$$\underline{U}(t'_0, y'_0) - \underline{U}(\bar{t}, \bar{y}) \le \underline{U}(t'_0, y'_0) - u^0(y'_0) + u^0(y'_0) - u^0(\bar{y}) \le L(|t'_0 - \bar{t}| + |\bar{y} - y'_0|),$$

which contradicts (4.54). Thus $\bar{t} > 0$. Besides, using (4.54) we deduce $\bar{x} \neq x_0$, therefore ψ is smooth in a neighborhood of $\bar{x}$. Thus we can apply Lemma 4.22 and obtain $|\nabla \psi(\bar{t}, \bar{y})|_\infty = K \le L$, which is a contradiction. ∎

4.3.5 Uniqueness result

We prove Theorem 4.18. This implies that U_ε converges locally uniformly to a function U solution of (4.40) in the viscosity sense. Therefore, it completes the proof of Theorem 4.14.

We prove that a Lipschitz continuous supersolution remains above a subsolution provided it is the case at initial time. Namely, we prove $\underline{U} \equiv \overline{U}$, with the notations introduced in (4.46). We point out that this uniqueness result is not standard since our assumption (4.43) allows the Hamiltonian to have superlinear growth. The fact that $\underline{U}$ is Lipschitz continuous, as stated in Proposition 4.20, is used as a key ingredient.

PROOF (PROOF OF THEOREM 4.18.) From the definition of $\overline{U}$ and $\underline{U}$ given in (4.46), we know that $\underline{U} \le \overline{U}$. We prove the reverse inequality. We fix $T > 0$. By contradiction, we assume

$$\sigma := \sup_{\substack{y \in \mathbb{R}^n \\ t \in [0,T]}} (\overline{U}(t,y) - \underline{U}(t,y)) > 0.$$

From (4.45) and (4.46), there exists a constant $C > 0$ such that

$$\forall t > 0, \forall y \in \mathbb{R}^n, \quad |\underline{U}(t,y)| + |\overline{U}(t,y)| \le C + 2k_0|y|, \tag{4.55}$$

The same estimate also holds for $\underline{U}$. We use the classical method of doubling the variables in the framework of viscosity solutions (see [101, 102]). Let us fix $\alpha > 0$, $\delta \in [0,1]$ and set for all $t \in [0,T], t' \in [0,T], y \in \mathbb{R}^n, y' \in \mathbb{R}^n$, we define

$$V_\delta(t,y,t',y') := \Big[\overline{U}(t,y) - \alpha t - \delta |y|^2\Big] - \Big[\underline{U}(t',y') + \alpha t' + \delta |y'|^2\Big] - \frac{|y-y'|^2}{\delta^2} - \frac{|t-t'|^2}{\delta^2}.$$

Thanks to (4.55), V_δ reaches its maximum M_δ at a point $(t_\delta, y_\delta, t'_\delta, y'_\delta)$. In what follows we use the following lemma.

Lemma 4.23 *When δ vanishes, the estimates hold*

1. $|t_\delta - t'_\delta|, |y_\delta - y'_\delta| = O(\delta^\theta)$, *for* $\theta \in (0,2)$,
2. $|y_\delta|, |y'_\delta| = O(\frac{1}{\sqrt{\delta}})$,
3. $\liminf_{\delta\to 0} t_\delta, \liminf_{\delta\to 0} t'_\delta > 0$.

The proof of Lemma 4.23 is essentially technical. Note that the Lipschitz continuity of $\underline{U}$ is a key ingredient, since usual estimates cannot give any better result than $|y_\delta - y'_\delta| = O(\delta)$.

Proof (Proof of Lemma 4.23.) First, we prove that $|y_\delta|, |y'_\delta| = O(\frac{1}{\delta})$. For simplicity, all constants that do not depend on δ are denoted by K. We have

$$\begin{aligned}\forall (t,y,t',y') \in ([0,T]\times\mathbb{R}^n)^2, V_\delta(t,y,t',y') &\leq K + k_0(|y|+|y'|) - \delta(|y|^2+|y'|^2)\\ &\leq K + Kz - \delta z^2,\end{aligned}$$

where $z = \max(|y|,|y'|)$. This means that V_δ can be bounded from above by a second order polynomial function of z. Consequently, the points (y_δ, y'_δ) where V_δ reaches its maximum are bounded by z_0, maximum solution to the equation

$$V_\delta(0,0,0,0) - 1 = K + Kz - \delta z^2,$$

which writes under the form

$$z_0 = \frac{K+\sqrt{K+\delta K}}{\delta} = O(\frac{1}{\delta}).$$

Thus we infer

$$|y_\delta|,\ |y'_\delta| = O(\frac{1}{\delta}). \tag{4.56}$$

Now we prove the assertion 1 of Lemma 4.23. As $M_\delta \geq V_\delta(t_\delta, y_\delta, t_\delta, y_\delta)$, we have

$$\begin{aligned}\alpha(t'_\delta - t_\delta) + \delta(|y'_\delta|^2 - |y_\delta|^2) + \frac{|y_\delta - y'_\delta|^2}{\delta^2} + \frac{|t_\delta - t'_\delta|^2}{\delta^2}&\\ \leq \overline{U}(t_\delta,y_\delta) - \underline{U}(t'_\delta,y'_\delta) - \overline{U}(t_\delta,y_\delta) + \underline{U}(t_\delta,y_\delta)&\\ \leq \underline{U}(t_\delta,y_\delta) - \underline{U}(t'_\delta,y'_\delta) \leq L\left(|t_\delta - t'_\delta| + |y_\delta - y'_\delta|\right),&\end{aligned} \tag{4.58}$$

from the Lipschitz continuity of $\underline{U}$ stated in Proposition 4.20. Besides, from (4.56) we obtain

$$\delta\left(|y_\delta|^2 - |y'_\delta|^2\right) \leq \delta(|y_\delta| + |y'_\delta|)(|y_\delta| - |y'_\delta|) \leq K|y_\delta - y'_\delta|. \tag{4.59}$$

Consequently, using (4.59) in (4.58), we have

$$\frac{|y_\delta - y'_\delta|^2}{\delta^2}, \frac{|t_\delta - t'_\delta|^2}{\delta^2} \leq K\left(|t_\delta - t'_\delta| + |y_\delta - y'_\delta|\right). \tag{4.60}$$

As t_δ and t'_δ are bounded and using (4.56), we deduce

$$\frac{|y_\delta - y'_\delta|^2}{\delta^2}, \frac{|t_\delta - t'_\delta|^2}{\delta^2} = O\left(\frac{1}{\delta}\right),$$

and then,

$$|y_\delta - y'_\delta|, |t_\delta - t'_\delta| = O\left(\sqrt{\delta}\right).$$

Using this estimate in (4.60), we obtain a new estimate on $|y_\delta - y'_\delta|$ and $|t_\delta - t'_\delta|$,

$$|y_\delta - y'_\delta|, |t_\delta - t'_\delta| = O\left(\delta^{\frac{5}{4}}\right).$$

Then by the bootstrap argument, we prove for all $\theta \in (0,2)$ the estimates

$$|y_\delta - y'_\delta|, |t_\delta - t'_\delta| = O\left(\delta^{\theta}\right). \tag{4.61}$$

Hence the assertion 1 of Lemma 4.23.

Next, we prove the assertion 2. From $M_\delta \geq V_\delta(0,0,0,0)$, (4.47) and Proposition 4.20 we infer

$$\begin{aligned}\alpha(t_\delta + t'_\delta) + \delta\left(|y_\delta|^2 + |y'_\delta|^2\right) \leq\ & \overline{U}(t_\delta, y_\delta) - \underline{U}(t'_\delta, y'_\delta)\\ =\ & \left[\overline{U}(t_\delta, y_\delta) - u^0(y_\delta)\right] + \left[u^0(y_\delta) - \underline{U}(t_\delta, y_\delta)\right]\\ & + \left[\underline{U}(t_\delta, y_\delta) - \underline{U}(t'_\delta, y'_\delta)\right]\\ \leq\ & 2Lt_\delta + L(|t_\delta - t'_\delta| + |y_\delta - y'_\delta|).\end{aligned} \tag{4.62}$$

We deduce $\delta(|y_\delta|^2 + |y'_\delta|^2) = O(1)$, hence the assertion 2.

Finally, we prove the last assertion. By contradiction, we assume, up to extraction of a subsequence, that $t'_\delta \to 0$ as δ goes to 0. From (4.61), we deduce that t_δ converges to 0 as δ vanishes and then, from (4.62), we obtain $\delta(|y_\delta|^2 + |y'_\delta|^2) = o(1)$. We set $M := \max_{(t,y)\in[0,T]\times\mathbb{R}^n} V_\delta(t,y,t,y)$ and choose δ and α small enough to ensure $M \geq \frac{\sigma}{2}$. We write

$$\begin{aligned}\frac{\sigma}{2} \leq M \leq M_\delta \leq\ & \overline{U}(t_\delta, y_\delta) - \underline{U}(t'_\delta, y'_\delta)\\ & = [\overline{U}(t_\delta, y_\delta) - u^0(y_\delta)] + [u^0(y_\delta) - u^0(y'_\delta)] + [u^0(y'_\delta) - \underline{U}(t'_\delta, y'_\delta)]\\ & \leq L(t_\delta + t'_\delta) + k_0|y_\delta - y'_\delta|,\end{aligned}$$

where we used (4.47) for the last inequality. As δ goes to 0, we deduce from the previous inequality that $\sigma \leq 0$, contradiction. Thus $t'_\delta > 0$ uniformly in δ when δ goes to 0. Moreover we have $t_\delta - t'_\delta = o(1)$, hence the result. ∎

Now we go back to the proof of Theorem 4.18. We use that $\overline{U}$, and $\underline{U}$ are subsolution and supersolution in the viscosity sense. We define the test function

$$\varphi_{\alpha,\delta}(t,y) := \alpha t + \delta|y|^2 + \left[\underline{U}(t'_\delta, y'_\delta) + \alpha t'_\delta + \delta|y'_\delta|^2\right] + \frac{|y - y'_\delta|^2}{\delta^2} + \frac{|t - t'_\delta|^2}{\delta^2},$$

which is smooth and is such that $\overline{U} - \varphi_{\alpha,\delta}$ reaches its global maximum at the point (t_δ, y_δ). Since $\overline{U}$ is a subsolution of (4.40) ands $t_\delta \in (0,T]$, the viscosity inequality holds

$$\partial_t\varphi_{\alpha,\delta}(t_\delta, y_\delta) = \alpha + \frac{2}{\delta^2}(t_\delta - t'_\delta) \leq H\left(y_\delta, 2\delta y_\delta + \frac{2}{\delta^2}(y_\delta - y'_\delta)\right).$$

In the same way, since $\underline{U}$ is a supersolution, we derive

$$-\alpha + \frac{2}{\delta^2}(t_\delta - t'_\delta) \geq H\left(y'_\delta, -2\delta y'_\delta + \frac{2}{\delta^2}(y_\delta - y'_\delta)\right).$$

Subtracting this last inequality from the previous one and using Lemma 4.23, we obtain

$$
\begin{aligned}
2\alpha &\leq H\left(y_\delta, 2\delta y_\delta + \frac{2}{\delta^2}(y_\delta - y'_\delta)\right) - H\left(y'_\delta, -2\delta y'_\delta + \frac{2}{\delta^2}(y_\delta - y'_\delta)\right) \\
&\leq \left[H\left(y_\delta, 2\delta y_\delta + \frac{2}{\delta^2}(y_\delta - y'_\delta)\right) - H\left(y_\delta, -2\delta y'_\delta + \frac{2}{\delta^2}(y_\delta - y'_\delta)\right)\right] \\
&\quad + \left[H\left(y_\delta, -2\delta y'_\delta + \frac{2}{\delta^2}(y_\delta - y'_\delta)\right) - H\left(y'_\delta, -2\delta y'_\delta + \frac{2}{\delta^2}(y_\delta - y'_\delta)\right)\right] \\
&\leq 2\delta C(1 + |y_\delta|^{\gamma_2})|y_\delta + y'_\delta| + C(1 + |y_\delta|^{\gamma_1} + |y'_\delta|^{\gamma_1})|y_\delta - y'_\delta| \\
&= O(\delta^{1+\theta-\frac{\gamma_2}{2}}) + O(\delta^{\theta-\frac{\gamma_1}{2}}),
\end{aligned}
$$

fol all $\theta \in (0, 2)$. From assumption (4.43) we have $2\alpha = o(1)$, and as δ goes to 0, we find $\alpha \leq 0$, which is a contradiction. Therefore $\sigma = 0$ and we have $\overline{U} = \underline{U}$. The proof of Theorem 4.18 is thereby complete. ∎

4.4 Appendix

4.4.1 Saturation of the population denstity

We prove Proposition 4.3 and Proposition 4.11. Integrating (4.1) and using (4.6), we obtain

$$
\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\rho_\varepsilon(t) = -\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} \Big(\partial_x[A(x,y)m_\varepsilon(t,x,y)]dxdy + (d(x,y) + \rho_\varepsilon(t))m_\varepsilon(t,x,y)\Big)dxdy. \tag{4.63}
$$

First we prove that $A(x, y)m_\varepsilon(t, x, y)$ converges to 0 when x goes to infinity. Note that from (4.5) and the explicit formula for Q given in (4.17), we have

$$
\forall y \in \mathbb{R}^n, \quad \lim_{x\to\infty} Q(x,y) = \lim_{x\to\infty} \frac{1}{A(x,y)} \exp\left(-\int_0^x \frac{d(x',y) - \Lambda(y)}{A(x',y)}dx'\right) = 0.
$$

Since p^0_ε is bounded from (4.10), we deduce that m^0_ε converges to 0 when x goes to infinity. Besides, as A is bounded and m_ε satisfies (4.1) which is a transport equation, then a classical result implies that m_ε converges to 0 when x goes to infinity.

Then, integrating by parts in (4.63), we obtain

$$
\begin{aligned}
\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\rho_\varepsilon(t) &= \iint\limits_{\mathbb{R}^n\times\mathbb{R}_+} \left[\left(\frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} M(\frac{y'-y}{\varepsilon})dy\right) b(x,y') - d(x,y')\right] m_\varepsilon(t,x,y')dxdy' - \rho^2_\varepsilon(t) \\
&\leq \overline{r}\rho_\varepsilon(t) - \rho^2_\varepsilon(t).
\end{aligned}
$$

Therefore, using (4.8), we conclude

$$
0 \leq \rho_\varepsilon(t) \leq \max\left(\overline{r}, \rho^0_\varepsilon\right).
$$

The other inequality can be proved in the same way.

4.4.2 The eigenproblem - proof of Theorem 4.12 and Theorem 4.2

We only prove Theorem 4.12, as Theorem 4.2 is a particular case with $\eta = 1$. Equation (4.32) holds iff

$$Q(x,y,\eta) = Q(0,y,\eta)\text{exp}\left(-\int_0^x \frac{\partial_x A(x',y) + d(x',y) - \Lambda(y,\eta)}{A(x',y)}dx'\right),$$

and thanks to the condition at $x = 0$,

$$Q(x,y,\eta) = \eta\frac{1}{A(x,y)}\text{exp}\left(-\int_0^x \frac{d(x',y) - \Lambda(y,\eta)}{A(x',y)}dx'\right). \tag{4.64}$$

Multiplying by $b(x,y)$ and integrating with regard to the x variable, we obtain

$$\frac{1}{\eta} = F(y, \Lambda(y,\eta)). \tag{4.65}$$

A direct calculation gives $\partial_\lambda F > 0$, thus (4.65) ensures uniqueness for Λ and then for Q.

Moreover, as $F(y,+\infty) = +\infty$ and $F(y,-\infty) = 0$, there exists such a $\Lambda(y,\eta)$. Besides, defining Q as in (4.64) implies that Q is in $L^1 \cap L^\infty$, thanks to (4.5), thus it proves existence. Finally, using the implicit function theorem in (4.65) we deduce that $\Lambda(y,\eta)$ is $\mathcal{C}^1$ and (4.36) holds true.

For the dual equation (4.34), a simple calculation shows that the solution Φ must be given by

$$\Phi(x,y,\eta) = \Phi(0,y,\eta)e^{-\int_0^x \frac{\Lambda(y,\eta)-d(x',y)}{A(x',y)}dx'}\left(1 - \eta\int_0^x \frac{b(x',y)}{A(x',y)}e^{\int_0^{x'} \frac{\Lambda(y,\eta)-d(x'',y)}{A(x'',y)}dx''}\right),$$

where $\Phi(0,y,\eta) > 0$ is determined by the normalization $\int_{\mathbb{R}_+} Q(x,y,\eta)\Phi(x,y,\eta)dx = 1$.

Finally, we prove in the case without mutations

$$\forall y \in \mathbb{R}^n, \quad \underline{r} \le -\Lambda(y) \le \overline{r}.$$

Integrating (4.3) with respect to x, we have

$$-\Lambda(y) = \frac{\int_{\mathbb{R}_+}(b(x,y) - d(x,y))Q(x,y)dx}{\int_{\mathbb{R}_+} Q(x,y)dx}.$$

Thus, using (4.6), we obtain the announced result.

4.4.3 Convexity of the Hamiltonian - Proof of Proposition 4.13

We first state the following lemma. We recall that the definitions of $F(y,\lambda), \Lambda(y,\eta)$ and $\eta(p)$ are given in (4.12), (4.35) and (4.37).

Lemma 4.24 *We have*

$$\eta(p)\Big[\partial_\lambda F\Big(y,\Lambda(y,\eta(p))\Big)\Big]^2 \le \partial_\lambda^2 F\Big(y,\Lambda(y,\eta(p))\Big), \tag{4.66}$$

and

$$\Big[\partial_{p_i}\eta(p)\Big]^2 \le \eta(p)\partial_{p_i}^2\eta(p). \tag{4.67}$$

Proof (Proof of Lemma 4.24) We define and compute using (4.12)

$$g(x,y) := \int_0^x \frac{1}{A(x',y)}\mathrm{d}x', \quad \partial_\lambda F(y,\lambda) = \int_0^\infty g(x,y) f(x,y,\lambda)dx,$$

With these notations we may write

$$\partial_\lambda^2 F(y,\lambda) = \int_0^\infty g(x,y)^2 f(x,y,\lambda)dx.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\Big[\partial_\lambda F\Big(y,\Lambda(y,\eta(p))\Big)\Big]^2 \le \partial_\lambda^2 F\Big(y,\Lambda(y,\eta(p))\Big)\cdot F\Big(y,\Lambda(y,\eta(p))\Big),$$

and then thanks to (4.35) the first inequality follows. The second inequality is a simple consequence of the Cauchy-Schwarz inequality on $\eta(p) = \int_{\mathbb{R}^n} M(z)e^{p\cdot z}dz$. ■

We go back to the proof of Proposition 4.13. By differentiating twice (4.35) with respect to p_i, we obtain

$$\partial_\lambda F\Big(y,\Lambda(y,\eta(p))\Big) D_{p_i}\Lambda(y,\eta(p)) = -\frac{\partial_{p_i}\eta(p)}{\eta(p)^2}, \tag{4.68}$$

$$\partial_\lambda F\cdot D^2_{p_i}\Lambda(y,\eta(p)) + \partial_\lambda^2 F\cdot\Big[D_{p_i}\Lambda(y,\eta(p))\Big]^2 = -\frac{\partial_{p_i^2}\eta(p)}{\eta(p)^2} + 2\frac{\partial_{p_i}\eta(p)}{\eta(p)^3}.$$

Then using (4.66), (4.67) and (4.68), we derive

$$\begin{aligned}\partial_\lambda F\cdot D^2_{p_i}\Lambda(y,p) &= -\partial_\lambda^2 F\left[\frac{\partial_{p_i}\eta(p)}{\eta(p)^2\partial_\lambda F}\right]^2 - \frac{\partial_{p_i^2}\eta(p)}{\eta(p)^2} + 2\frac{\Big[\partial_{p_i}\eta(p)\Big]^2}{\eta(p)^3}\\ &\le -\frac{\Big[\partial_{p_i}\eta(p)\Big]^2}{\eta(p)^3} - \frac{\partial_{p_i^2}\eta(p)}{\eta(p)^2} + 2\frac{\Big[\partial_{p_i}\eta(p)\Big]^2}{\eta(p)^3}\\ &= -\frac{1}{\eta(p)^3}\left(\eta(p)\partial^2_{p_i}\eta(p) - \Big[\partial_{p_i}\eta(p)\Big]^2\right) \le 0,\end{aligned}$$

hence the announced convexity result on $p\mapsto H(y,p)$.

4.4.4 A priori bound on $\partial_t U_\varepsilon$ - Proof of Proposition 4.17

Our goal is to prove

$$\partial_t U^R_\varepsilon(t,y) \le \sup_{y\in\mathbb{R}^d}\partial_t U^{R,0}_\varepsilon := \sup_{y\in\mathbb{R}^n}\partial_t u^0_\varepsilon(0,y), \quad \forall R>0, \forall y\in\mathbb{R}^n, \forall t>0. \tag{4.69}$$

The reverse inequality can be obtained similarly. Note that from (4.41) we have that

$$\partial_t U^{0,R}_\varepsilon = -\Lambda\left(y, \int_{\mathbb{R}^n} M(z)e^{\frac{u^0_\varepsilon(y+\varepsilon z)-u^0_\varepsilon(y)}{\varepsilon}}dz\right) \text{ is bounded uniformly in } \varepsilon,$$

thus (4.69) allows us to conclude that $\partial_t U^R_\varepsilon$ is bounded uniformly in R and ε.

We prove (4.69) by contradiction. We assume that there exists $(T, y_0) \in (0, +\infty) \times \mathbb{R}^n$ such that

$$\partial_t U_\varepsilon^R(T, y_0) - \sup \partial_t U_\varepsilon^{R,0} > 0. \tag{4.70}$$

For conciseness, we define $V_\varepsilon^R(t, y) := \partial_t U_\varepsilon^R(t, y)$. For $\beta > 0$, $\alpha > 0$ small and for $t \in [0, T], y \in \mathbb{R}^n$, we also introduce

$$\varphi_{\alpha,\beta}(t, y) := V_\varepsilon^R(t, y) - \alpha t - \beta|y - y_0|.$$

We choose α small enough to ensure $\varphi_{\alpha,\beta}(T, y_0) > \varphi_{\alpha,\beta}(0, y_0) = \partial_t U_\varepsilon^{R,0}(y_0)$, which is possible thanks to assumption (4.70). From the definition of ϕ_R, we have $|V_\varepsilon^R(t, y)| \leq R$, therefore $\varphi_{\alpha,\beta}$ decreases to $-\infty$ as $|y| \to \infty$ and reaches its maximum on $[0, T] \times \mathbb{R}^n$ at a point $(\bar{t}, \bar{y})$. We have

$$\varphi_{\alpha,\beta}(\bar{t}, \bar{y} + \varepsilon z) \leq \varphi_{\alpha,\beta}(\bar{t}, \bar{y}), \quad \forall z \in \mathbb{R}^n,$$

and thus

$$\frac{V_\varepsilon^R(\bar{t}, \bar{y} + \varepsilon z) - V_\varepsilon^R(\bar{t}, \bar{y})}{\varepsilon} \leq \beta \frac{|\bar{y} + \varepsilon z| - |\bar{y}|}{\varepsilon} \leq \beta|z|, \quad \forall z \in \mathbb{R}^n. \tag{4.71}$$

Moreover, as u_ε^0 is k_0-Lipschitz continuous from (4.9), then we obtain for all $t > 0$, $(y, y') \in \mathbb{R}^{2n}$,

$$\begin{aligned}
&|U_\varepsilon^R(t, y) - U_\varepsilon^R(t, y')| \\
&\quad \leq |U_\varepsilon^R(t, y) - U_\varepsilon^{0,R}(y)| + |U_\varepsilon^{0,R}(y) - U_\varepsilon^{0,R}(y')| + |U_\varepsilon^{0,R}(y') - U_\varepsilon^R(t, y')| \\
&\quad \leq 2RT + k_0|y - y'|.
\end{aligned} \tag{4.73}$$

Next, we set

$$\eta_\varepsilon^R(t, y) := \int_{\mathbb{R}^n} M(z) e^{\frac{U_\varepsilon^R(t,y+\varepsilon z) - U_\varepsilon^R(t,y)}{\varepsilon}} dz,$$

$$\eta_\varepsilon^\pm := \int_{\mathbb{R}^n} M(z) e^{\pm\left(\frac{2RT}{\varepsilon} + k_0|z|\right)} dz,$$

and notice that $0 < \eta_\varepsilon^- \leq \eta_\varepsilon^R(t, y) \leq \eta_\varepsilon^+$.

We have chosen α such that $\varphi_{\alpha,\beta}(0, y_0) < \varphi_{\alpha,\beta}(T, y_0)$, then we know that $\bar{t} > 0$. Hence $\partial_t \varphi_{\alpha,\beta}(\bar{t}, \bar{y}) \geq 0$, that is $\partial_t V_\varepsilon^R(\bar{t}, \bar{y}) \geq \alpha$ (if $\bar{t} = T$ then $\partial_t V_\varepsilon^R(\bar{t}, \bar{y})$ stands for the left-derivative). Differentiating (4.44), we have

$$\partial_t V_\varepsilon^R(t, y) = \phi_R' \left(-\Lambda \left(y, \eta_\varepsilon^R\right)\right) \left(-\partial_\eta \Lambda \left(y, \eta_\varepsilon^R\right)\right) \Gamma_\varepsilon^R(t, y), \tag{4.76}$$

where $\Gamma_\varepsilon^R(t, y) := \int_{\mathbb{R}^n} M(z) e^{\frac{U_\varepsilon^R(t,y+\varepsilon z) - U_\varepsilon^R(t,y)}{\varepsilon}} \left(\frac{V_\varepsilon^R(t,y+\varepsilon z) - V_\varepsilon^R(t,y)}{\varepsilon}\right) dz$.

Writing (4.76) at $(\bar{t}, \bar{y})$, using (4.36) and (4.71)-(4.73), we have

$$\begin{aligned}
\alpha \leq \partial_t V_\varepsilon^R(\bar{t}, \bar{y}) &= \phi_R' \left(-\Lambda \left(y, \eta_\varepsilon^R(\bar{t}, \bar{y})\right)\right) \left(-\partial_\eta \Lambda \left(y, \eta_\varepsilon^R(\bar{t}, \bar{y})\right)\right) \Gamma_\varepsilon^R(\bar{t}, \bar{y}) \\
&\leq \beta \sup_{r \in \mathbb{R}} \phi_R'(r) \sup_{\substack{\eta \in (\eta_\varepsilon^-, \eta_\varepsilon^+) \\ y \in \mathbb{R}^n}} \left[-\partial_\eta \Lambda(y, \eta)\right] \left(\int_{\mathbb{R}^n} M(z) e^{\frac{U_\varepsilon^R(\bar{t},\bar{y}+\varepsilon z) - U_\varepsilon^R(\bar{t},\bar{y})}{\varepsilon}} |z| dz\right) \\
&\leq \beta \sup_{r \in \mathbb{R}} \phi_R'(r) \sup_{\substack{\eta \in (\eta_\varepsilon^-, \eta_\varepsilon^+) \\ y \in \mathbb{R}^n}} \left[-\partial_\eta \Lambda(y, \eta)\right] \left(\int_{\mathbb{R}^n} M(z) e^{\frac{2RT}{\varepsilon} + k_0|z|} |z| dz\right).
\end{aligned}$$

Hence $\alpha \leq \bar{C}\beta$, where $\bar{C}$ is a constant that does not depend on β. Then as β goes to 0, we obtain $\alpha \leq 0$, which is absurd. The proof is thereby achieved.

Chapter 5

Dynamics of concentration of a population structured by age and phenotype II: adding mutations

In collaboration with Benoît Perthame.

We study the model introduced in Chapter 4 with the addition of rare mutations, which adds a significant difficulty. Our goals are to describe the asymptotic behavior of the solution and the adaptive dynamics of such a population. We propose a variant of the standard WKB method, which sheds some light on the role of an approximate limiting eigenproblem. The proofs are performed on a more abstract level than in Chapter 4, which allows to give a better insight into the underlying structure and to relax some assumptions.

Contents

5.1 Introduction

5.1.1 Main results

We study a mathematical model describing the growth process of a population structured by age and a phenotypical trait, subject to aging, competition between individuals and rare mutations. Our goal is to describe the asymptotic behavior of the solution to a renewal type equation, in particular the selection of the fittest traits and the adaptative dynamics of such traits. Namely, for $\varepsilon > 0$, we choose $m_\varepsilon(t,x,y)$ to represent the population density of individuals which, at time $t \geq 0$, have age $x \geq 0$ and a quantitative phenotipical trait $y \in \mathbb{R}^n$, solution of

$$\begin{cases} \varepsilon\partial_t m_\varepsilon + \partial_x\left[A(x,y)m_\varepsilon\right] + \left(\rho_\varepsilon(t) + d(x,y)\right)m_\varepsilon = 0, \\ A(0,y)m_\varepsilon\left(t,0,y\right) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} M(\frac{y'-y}{\varepsilon})b(x',y')m_\varepsilon(t,x',y')dx'dy', \\ \rho_\varepsilon(t) = \int_{\mathbb{R}_+}\int_{\mathbb{R}^n} m_\varepsilon(t,x,y)dxdy, \\ m_\varepsilon(t=0,x,y) = m_\varepsilon^0(x,y) > 0. \end{cases} \tag{5.1}$$

The main result of this paper is that the population density $m_\varepsilon(t,x,y)$ concentrates when $\varepsilon \to 0$. In our approach, we make an ansatz by defining an *effective* fitness Λ and by constructing a function $U(t,y)$ solution of a Hamilton-Jacobi equation. The following theorem summarizes our results.

Theorem 5.1 (Concentration) *Under the assumptions of section 5.1.5, considering $\Lambda(y,\eta)$ defined in* (5.7) *and U defined in* (5.9)*, we have, when $\varepsilon \to 0$*

1. *The total population ρ_ε converges weakly to some positive $\rho \in L^\infty$ and*

$$\forall t > 0, \qquad \int_0^t \rho = \sup_{y\in\mathbb{R}^n} U(t,y).$$

2. *The population m_ε vanishes locally uniformly outside the set*

$$\mathcal{S} := \left\{t \geq 0,\ y \in \mathbb{R}^n : U(t,y) = \sup_{y'\in\mathbb{R}^n} U(t,y')\right\}, \tag{5.2}$$

where $U \in W^{1,1}_{loc}$ is globally Lipschitz, semiconvex, and is a viscosity solution of the Hamilton Jacobi equation (5.10).

3. *Under further assumptions on the initial conditions, and for small times $t \in [0,T]$, we have $\mathcal{S} = \{(t,\bar{y}(t))\}$, where $\bar{y}(t)$ follows the Canonical Equation:*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\bar{y}(t) = \left(\nabla_y^2 U(t,\bar{y}(t))\right)^{-1}\cdot\nabla_y\Lambda(\bar{y}(t),1) + \partial_\eta\Lambda(\bar{y}(t),1)\displaystyle\int_{\mathbb{R}^n} M(z)zdz, \\ \bar{y}(0) = \bar{y}^0. \end{cases}$$

The main restriction on the parameters is that we assume either $\frac{1}{A(\cdot,y)} \in L^1$ or that $b(\cdot, y)$ has a compact support, see (5.13)-(5.14) below. The other assumptions are formulated directly on the limiting eigenproblem and are quite general.

This work is the continuation of the study begun in [243] (i.e., Chapter 4), where a simplified version of the model is studied, discarding the effect of mutations. In this previous work, a variant of the classical WKB ansatz is introduced, which sheds some light on the role of the eigenelements of a limiting problem, and makes possible the estimates to be performed on a linear equation rather than on a Hamilton-Jacobi equation. In the present work, the mutation term in the second line of (5.1) adds a significant difficulty, since it becomes intricate to identify a proper limiting eigenproblem. Instead, we introduce an *approximate* limiting eigenproblem, at the expanse of an extra variable η, which turns out to be well suited for our purposes. The formal idea of the method is explained with more details in section 5.1.3. However, the results of Chapter 4 in the case without mutations are stronger since the asymptotic distribution in x is exactly identified. Because of technical obstructions, we are only able to prove a two-sided estimate in the present framework.

Similar results as Theorem 5.1 have been obtained for various models with parabolic [33, 36, 213] and integrodifferential equations [35, 117, 212]. More generally, convergence to positive measures in selection-mutation models has been studied by many authors, see [2, 71, 82] for example. Recently, the asymptotic behavior of an age-structured equation with spatial jumps has been determined in [83] when the death rate vanishes and with a slowly decaying birth rate b. Also in [116], a concentration result has been proved for a model representing the evolutionary epidemiology of spore-producing plant pathogens in a host population, with infection age and pathogen strain structures.

5.1.2 The model

Let us give some interpretations of the model (5.1). The function $A(x, y)$ is the speed of aging of individuals with age x and trait y. The total size of the population at time t is denoted with $\rho_\varepsilon(t)$. Here the mortality effect features the nonlocal term $\rho_\varepsilon(t)$, which represents competition, and an intrinsic death rate $d(x, y) > 0$. The condition at the boundary $x = 0$ describes the birth of newborns that happens with rate $b(x, y) > 0$ and with the probability kernel of mutation M.

The terminology of "renewal equation" comes from this boundary condition. It is related to the McKendrick-von Foerster equation which is only structured in age (see [247] for a study of the linear equation). This model has been extended with other structuring variables, as size [221,233], DNA content, maturation, etc., in the context of cell divisions [131,224], or proliferative and quiescent states of tumour cells [3,170]. Space structured problems have also been extensively studied [183, 228, 229, 248].

To keep the model (5.1) quite general, note that the progression speed A is allowed to depend on x. Thus, although the variable x is referred to as *age*, it can represent other biological quantities that evolve throughout the individual lifespan such as, for instance, the size of individuals, a physiological age, a parasite load, etc.

The rescaling parameter $\varepsilon > 0$ come from a hyperbolic rescalling of time and space. Accordingly, the dynamics are considered in two different time scales. The first one is the individual lifetime scale εt, i.e., the characteristic time for the population to reach the dynamical equilibrium for a fixed y. The second one is the evolutionary time scale t, corresponding to the evolution of the population distribution with respect to the variable y. Formally, at the limit when $\varepsilon \to 0$, the time scales are completely separated. This rescaling is a classical way to give a continuous formulation of the adaptive evolution of a phenotypically structured population (see [88, 124, 125, 213]). Note that the mutation kernel is supposed to be *thin-tailed*, i.e., it decreases faster than any exponential. A fat-tailed kernel needs a different rescaling, see [62].

5.1.3 Formal Approach and Method

To prove concentration of the population density, the usual approach relies on the WKB ansatz ([36, 125]), which consists in the change of variable

$$m_\varepsilon(t,x,y) = e^{\frac{v_\varepsilon(t,x,y)}{\varepsilon}}. \tag{5.3}$$

This form is motivated by the heuristical fact that a Dirac mass is nothing but a narrow Gaussian. Indeed, in a weak sense

$$\frac{1}{(\pi\varepsilon)^{\frac{n}{2}}} e^{-\frac{\|y-\bar{y}\|^2}{\varepsilon}} \underset{\epsilon\to 0}{\rightharpoonup} \delta_{\bar{y}=y}.$$

This approach has been extensively used in works on a similar issue (e.g. [32, 37, 93, 139] for example).

With the classical change of variable (5.3), at the limit $\varepsilon \to 0$, the function $v_\varepsilon(t,x,y)$ satisfies a constrained Hamilton-Jacobi equation, on which estimates can be difficult to prove. Here, we propose a variant of the method. The principle is to proceed to a Taylor expansion $v_\varepsilon(t,x,y) = v_\varepsilon^1(t,y) + \varepsilon v_\varepsilon^2(t,x,y)$, to make an ansatz for v_ε^1, and then to prove some estimates on v_ε^2. With a slight rewriting, we proceed to the change of variable

$$m_\varepsilon(t,x,y) = p_\varepsilon(t,x,y) e^{\frac{u_\varepsilon(t,y)}{\varepsilon}} \tag{5.4}$$

where $u_\varepsilon(t,y)$ will be our ansatz, defined *ad hoc* through a Hamilton Jacobi equation. Then, our goal is to prove some estimates on $p_\varepsilon(t,x,y)$. Note that p_ε satisfies a linear equation rather than a constrained Hamilton Jacobi equation: it makes thus possible the use of standard comparison principles.

We are now left with the task of finding a good candidate for $u_\varepsilon(t,y)$ and formally identifying $p_\varepsilon(t,x,y)$. Injecting (5.4) in (5.1) we find

$$\begin{cases} \varepsilon\partial_t p_\varepsilon + \partial_x\left[A(x,y)p_\varepsilon\right] + d(x,y)p_\varepsilon + \left(\partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t)\right)p_\varepsilon = 0, \\ A(0,y)p_\varepsilon\left(t,0,y\right) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} M\left(\frac{y'-y}{\varepsilon}\right) e^{\frac{u_\varepsilon(t,y')-u_\varepsilon(t,y)}{\varepsilon}} b(x',y')p_\varepsilon(t,x',y')dx'dy'. \end{cases} \tag{5.5}$$

With the change of variable $z = \frac{y'-y}{\varepsilon}$, the renewal term becomes

$$A(0,y)p_\varepsilon(t,0,y) = \int_{\mathbb{R}^n}\int_{\mathbb{R}_+} M(z) e^{\frac{u_\varepsilon(t,y+\varepsilon z)-u_\varepsilon(t,y)}{\varepsilon}} b(x', y+\varepsilon z)p_\varepsilon(t,x',y+\varepsilon z)dx'dz.$$

When ε is small, we can formally approximate

$$A(0,y)p(t,0,y) \approx \eta_\varepsilon(t,y)\int_{\mathbb{R}_+} b(x',y)p(t,x',y)dx', \tag{5.6}$$

where

$$\eta_\varepsilon(t,y) := \int_{\mathbb{R}^n} M(z)e^{\frac{u_\varepsilon(t,y+\varepsilon z)-u_\varepsilon(t,y)}{\varepsilon}}dz.$$

Then, formally putting $\varepsilon\partial_t p_\varepsilon \approx 0$ in the first line of (5.5), we end up with the following approximate problem

$$\begin{cases} \partial_x\left[A(x,y)p_\varepsilon\right] + d(x,y)p_\varepsilon + \left(\partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t)\right)p_\varepsilon = 0, \\ A(0,y)p_\varepsilon\left(t,0,y\right) = \eta_\varepsilon(t,y)\int_{\mathbb{R}_+} b(x',y)p(t,x',y)dx'. \end{cases}$$

Considering $\eta_\varepsilon(t,y)$ as a parameter, we introduce the following eigenproblem: for fixed $(y,\eta)\in\mathbb{R}^n\times(0,+\infty)$, find $(\Lambda(y,\eta),Q(x,y,\eta))$, solution of

$$\begin{cases} \partial_x\left[A(x,y)Q(\right] + d(x,y)Q - \Lambda(y,\eta)Q = 0, \quad \forall x>0, \\ A(0,y)Q(0,y,\eta) = \eta, \\ Q>0, \quad \int_{\mathbb{R}_+} b(x,y)Q(x,y,\eta)dx = 1. \end{cases} \tag{5.7}$$

Formally, Λ corresponds to the "effective fitness", and Q to the age profile at equilibirum.

This formal approach suggests to define u_ε as a solution of

$$\begin{cases} \partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t) = -\Lambda\left(y,\eta_\varepsilon(t,y)\right), \\ \eta_\varepsilon(t,y) = \int_{\mathbb{R}^n} M(z)e^{\frac{u_\varepsilon(t,y+\varepsilon z)-u_\varepsilon(t,y)}{\varepsilon}}dz, \\ u_\varepsilon(t=0,y) = u_\varepsilon^0(y), \end{cases} \tag{5.8}$$

for some initial conditions u_ε^0. Then, setting $Q_\varepsilon(t,x,y) := Q(x,y,\eta_\varepsilon(t,y))$ and $\Lambda_\varepsilon(t,y) := \Lambda(y,\eta_\varepsilon(t,y))$, we have

$$\begin{cases} \varepsilon\partial_t Q_\varepsilon + \partial_x\left[A(x,y)Q_\varepsilon\right] + d(x,y)Q_\varepsilon + \left(\partial_t u_\varepsilon(t,y) + \rho_\varepsilon(t)\right)Q_\varepsilon = \varepsilon\partial_t Q_\varepsilon, \\ A(0,y)Q_\varepsilon\left(t,0,y\right) = \frac{1}{\varepsilon^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}_+} M(\frac{y'-y}{\varepsilon})e^{\frac{u_\varepsilon(t,y')-u_\varepsilon(t,y)}{\varepsilon}}b(x',y')Q_\varepsilon(t,x',y')dx'dy'. \end{cases}$$

Note that the boundary term at $x=0$ is obtained by the definition of $\eta_\varepsilon(t,y)$ and the normalization $\int_{\mathbb{R}_+} b(x,y)Q(x,y,\eta)dx = 1$. Except for the term $\varepsilon\partial_t Q_\varepsilon$, we see that Q_ε and p_ε satisfy the same equation (5.5), which is linear and admits a comparison principle. After proving a good estimate on $\varepsilon\partial_t Q_\varepsilon$, we deduce that p_ε is bounded from above and below by multiples of Q_ε, which is sufficient to conclude. It justifies our approach, especially the approximation (5.6).

According to the above, the cornerstone of our method relies on finding good estimates on $\partial_t Q_\varepsilon$, or more precisely, on

$$\frac{\partial_t Q_\varepsilon}{Q_\varepsilon} = \partial_t\eta_\varepsilon(t,y)\frac{\partial_\eta Q}{Q}(x,y,\eta_\varepsilon(t,y)).$$

The term $\frac{\partial_\eta Q}{Q}$ can be computed explicitely, and is bounded if and only if $\frac{1}{A(\cdot,y)} \in L^1$. Let us explain formally how a bound on $\partial_t \eta_\varepsilon$ can be derived. Equation (5.8) is non-autonomous because of the term $\rho_\varepsilon(t)$. To avoid this difficulty, it is convenient to define

$$U_\varepsilon(t,y) = u_\varepsilon(t,y) + \int_0^t \rho_\varepsilon(s)ds.$$

We stress out that $u_\varepsilon(t,\cdot)$ and $U_\varepsilon(t,\cdot)$ only differ from a constant shift (in fact, we will see that, when ε vanishes, ρ_ε in (5.8) is nothing but a Lagragian multiplyer for the constrain: $\sup_{\mathbb{R}^n} u_\varepsilon(t,\cdot) = 0$, $\forall t \geq 0$). In particular,

$$\eta_\varepsilon(t,y) = \int_{\mathbb{R}^n} M(z) e^{\frac{U_\varepsilon(t,y+\varepsilon z) - U_\varepsilon(t,y)}{\varepsilon}} dz$$

and

$$\begin{cases} \partial_t U_\varepsilon(t,y) = -\Lambda\left(y, \int_{\mathbb{R}^n} M(z) e^{\frac{U_\varepsilon(t,y+\varepsilon z) - U_\varepsilon(t,y)}{\varepsilon}} dz\right) & \forall t \geq 0,\ \forall y \in \mathbb{R}^n, \\ U_\varepsilon(0,y) = u_\varepsilon^0(y) & \forall y \in \mathbb{R}^n, \end{cases} \tag{5.9}$$

thus U_ε satisfies a (nonlocal) time-autonomous equation.

When $\varepsilon \to 0$, U_ε formally converges to a function U solution of the Hamilton-Jacobi equation

$$\begin{cases} \partial_t U(t,y) = H(y, \nabla_y U) & \forall t \geq 0,\ \forall y \in \mathbb{R}^n, \\ U(0,y) = u^0(y) & \forall y \in \mathbb{R}^n. \end{cases} \tag{5.10}$$

with a Hamiltonian

$$H(y,p) := -\Lambda\left(y, \int_{\mathbb{R}^n} M(z) e^{p\cdot z} dz\right).$$

From this equation, it is classical to prove a priori bounds on $\partial_t U$, from which we deduce bounds on

$$\eta(t,y) = \int_{\mathbb{R}^n} M(z) e^{\nabla_y U(t,y)\cdot z} dz$$

and on $\nabla_y U$.

Besides, we prove that $p \mapsto H(y,p)$ is convex, which implies that U is somehow semi-convex, namely $\partial_t^2 U$ is bounded from below. Since $\partial_t U$ is bounded, we deduce a $\partial_t^2 U \in L^1_{loc}$. Then, using that $\partial_t^2 U = -\partial_t \eta \partial_\eta \Lambda$, we deduce $\partial_t \eta \in L^1_{loc}$, which was our goal.

Once we proved a uniform estimate on p_ε, Theorem 5.1 can be understood and formally justified as follows. On the one hand, the saturation term "$\rho_\varepsilon(t)$" in (5.1) implies the total population ρ_ε to be bounded, uniformly in $\varepsilon > 0$. On the other hand, from (5.4), the asymptotics of $m_\varepsilon(t,\cdot,\cdot)$ when ε vanishes are driven by the points y where $u(t,\cdot)$ is maximal, i.e where $U(t,\cdot)$ is maximal. In other words, when $\varepsilon \to 0$, m_ε vanishes outside the set $\mathcal{S}$ defined in (5.2). Then, we study the evolutionary dynamics through the dynamics of the point where $U(t,\cdot)$ reaches its supremum.

From the modelization point of view, it is a mathematical formulation of Natural Selection and Evolution. On an ecological time scale, only the phenotype $\bar{y}(t)$ which maximizes the ecological fitness $U(t,\cdot)$ can survive. On an evolutionary time scale we observe the dynamics of $\bar{y}(t)$.

5.1.4 Outline of the paper

Section 5.2 is devoted to the preliminary material, that is, the definition of the eigenelements Q, Λ and U_ε, a priori estimates, and the asymptotics of U_ε when ε vanishes. The results are stated in Section 5.2 and the proofs are postponed to section 5.4. In section 5.3, we proove our main result. We estimate the corresponding age profile p_ε in Section 5.3.1, before proving concentration in Section 5.3.2. Finally, we tackle the question of the adaptive dynamics in Section 5.3.3.

5.1.5 Assumptions

Most of our assumptions are formulated directly on the solution $(\Lambda(y,\eta), Q(x,y,\eta))$ of the limiting eigenproblem (5.7) and, therefore, may seem abstract to the reader. However, we think that, besides being more general, this formulation gives a better insight into the nature of our assumptions and into the spirit of our approach.

Examples

Before stating the general assumptions, we first give for the reader's convenience a more concrete set of assumptions on the parameters A, b, d which are sufficient to fulfill the general assumptions. Note that, in addition, we need the initial conditions to be "well prepared", which is not detailed here.

First, to avoid any difficulty when $|y| \to +\infty$, we can assume for example that A, b and d have a compact dependence on y. Namely, if ψ is a globally smooth diffeomorphism from $\mathbb{R}^n$ into the unit ball, we assume $A(x,y) := A_\star(x,\psi(y))$, $b(x,y) := b_\star(x,\psi(y))$, $d(x,y) := d_\star(x,\psi(y))$, where $A_\star$, $b_\star$ and $d_\star$ are defined on the closed unit ball. This way, the y space $\mathbb{R}^n$ is compactified.

The parameters can then be chosen to fullfill the following conditions, for all $x \geq 0$ and uniformly in $y \in \overline{B}_1$,

$$\begin{cases} A_\star(\cdot,\cdot),\ b_\star(x,\cdot),\ d_\star(x,\cdot) \text{ are } C^1, \\ b_\star(x,y) \geq 0, \quad d_\star(x,y) \geq 0, \quad A_\star(x,y) > 0, \\ b_\star(x,y) \leq K' e^{Kx}, \quad \text{for some } K, K' > 0 \\ M(\cdot) \text{ is a Gaussian probability kernel,} \\ \text{either } \dfrac{1}{A_\star(\cdot,y)} \text{ is integrable or } b_\star(\cdot,y) \text{ has a compact support,} \\ \underline{\eta} b_\star(x,y) - d_\star(x,y) \geq \underline{r}, \quad \text{for some } \underline{r} > 0, \end{cases}$$

where $\underline{\eta}$ is determined from the initial conditions, see (5.21).

In particular, the assumption in the last line can be substentially weakened. Indeed, from an integration of the first line in (5.7), this assumption implies

$$\Lambda(y,\eta) = \frac{\int_{\mathbb{R}_+} (d(x,y) - \eta b(x,y))\, Q(x,y,\eta) dx}{\int_{\mathbb{R}_+} Q(x,y,\eta) dx},$$

which ensures

$$\Lambda(y,\eta) \leq -\underline{r} < 0, \quad \forall \eta \geq \underline{\eta}.$$

In fact, this last inquality on Λ is sufficient in the sequel, see (5.21).

Assumptions on the parameters

We assume

$$b \geq 0, \quad d \geq 0, \quad A > 0 \text{ are continuous functions,}$$

$$A(\cdot,\cdot),\ b(x,\cdot),\ d(x,\cdot) \text{ are } C^1,$$

and the no degeneracy condition

$$d(\cdot,y)\ A(\cdot,y) \text{ are bounded in some small open set of } \mathbb{R}_+, \text{ uniformly in } y. \tag{5.11}$$

Regarding the mutation kernel, we assume that

$$M(\cdot) \geq 0 \text{ is a probability kernel and vanishes faster than any exponential,}$$

$$M(\cdot) \geq \beta \quad \text{in a neighborhood of 0.} \tag{5.12}$$

For instance, $M(\cdot)$ can be a Gaussian distribution or have a compact support. The second assumption, meaning that M is not degenerate, is not strictly needed, but allows to avoid some technicalities. Note that the case without mutations corresponds to $M = \delta_0$ and has been already treated in Chapter 4.

In addition, we need one of the two following conditions: assume either

$$\exists K > 0 \text{ such that } \forall y \in \mathbb{R}^n, \quad \int_0^{+\infty} \frac{1}{A(x',y)} dx' \leq K \tag{5.13}$$

or

$$\exists \overline{x} > 0 \text{ such that } \forall x \geq \overline{x},\ \forall y \in \mathbb{R}^n, \quad b(x,y) = 0. \tag{5.14}$$

This restriction is required in section 5.3.1 to prove an estimate on p_ε. Condition (5.13) implies that the transport towards $x = +\infty$ occurs in finite time (it can be seen on the characteristics). On the contrary, assumption (5.14) somehow compactifies the x-space $\mathbb{R}_+$ into $[0, \overline{x}]$. Our approach would also work if the support of $A(\cdot, y)$ is compact, but we omit this case for simplicity.

Assumption (5.13) turns out to be more natural in our context (from the mathematical point of view). It is a necessary and sufficient condition for the ratio $\frac{\partial_\eta Q}{Q}$ to be bounded. Under this assumption, we prove a stronger result in Lemma 5.12.

We also need the eigenvalue Λ of (5.7) to be well defined and differentiable (w.r.t η). As we will see in Proposition 5.2, setting, for $\lambda \in \mathbb{R}$,

$$F(y,\lambda) := \int_{\mathbb{R}_+} \frac{b(x,y)}{A(x,y)} \exp\left(\int_0^x \frac{\lambda - d(x',y)}{A(x',y)} dx' \right) dx, \tag{5.15}$$

we have that $\Lambda(y,\eta)$ is defined through $F(y, \Lambda(y,\eta)) = \frac{1}{\eta}$. We assume there exists $\overline{\Lambda} < 0$ such that

$$F(y, \overline{\Lambda}),\ \partial_\lambda F(y, \overline{\Lambda}) < +\infty \quad \text{for all } y \in \mathbb{R}^n.$$

This assumption is not very restrictive since it is automatically satisfied if $b(x,y) \leq K' e^{Kx}$, for some $K \leq -\overline{\Lambda}$.

Assumptions on the initial conditions.

We need the population $m_\varepsilon(t,x,y)$ to be "well prepared" for concentration. We assume that for all $\varepsilon > 0$ we can write $m_\varepsilon^0(x,y) = p_\varepsilon^0(x,y)e^{\frac{u_\varepsilon^0(y)}{\varepsilon}}$, according to (5.4), with u_ε^0 such that

$$u_\varepsilon^0 \text{ smoothly converges to a function } u^0 \text{ when } \varepsilon \text{ vanishes,}$$

$$\exists k^0 > 0 \text{ such that } \forall \varepsilon > 0,\ \forall y \in \mathbb{R}^n, \quad |\nabla_y u_\varepsilon^0(y)| \le k^0, \tag{5.16}$$

$$\exists C > 0 \text{ such that } \forall \varepsilon > 0,\ \forall y \in \mathbb{R}^n, \quad \partial_{y_i}^2 u_\varepsilon^0(y) \ge -C, \tag{5.17}$$

$$\underline{J}^0 \le \int_{\mathbb{R}^n} e^{\frac{u_\varepsilon^0(y)}{\varepsilon}} dy \le \overline{J}^0, \text{ for some } \underline{J}^0, \overline{J}^0 > 0, \tag{5.18}$$

and p_ε^0 such that, for some $\underline{\gamma}^0, \overline{\gamma}^0 > 0$:

$$\underline{\gamma}^0 \le \frac{p_\varepsilon^0(x,y)}{Q(x,y,\eta_\varepsilon^0(y))} \le \overline{\gamma}^0, \tag{5.19}$$

where Q is defined through (5.7) and

$$\eta_\varepsilon^0(y) := \int_{z\in\mathbb{R}^n} M(z) e^{\frac{u_\varepsilon^0(y+\varepsilon z) - u_\varepsilon^0(y)}{\varepsilon}} dz \tag{5.20}$$

accordingly with (5.30). An additionnal condition is also required on η_ε^0, see (5.21). Note that (5.18) and (5.19) ensures that $\rho_\varepsilon^0 := \iint_{\mathbb{R}_+\times\mathbb{R}^n} m_\varepsilon^0$ is bounded uniformly in $\varepsilon > 0$. Note also that assumption (5.18) implies $\sup_{\mathbb{R}^n} u = 0$.

Assumptions on the distribution of phenotypes

The following assumptions only deal with the parameters' dependence on $y \in \mathbb{R}^n$. In other words, if all the parameters have a compact dependance on y, then all the following assumptions are automatically satisfied.

First, we need a condition on the initial data, namely that $\eta_\varepsilon^0(y)$ (from (5.20)) is bounded and $\Lambda_\varepsilon^0(y) := \Lambda(y, \eta_\varepsilon^0(y))$ is bounded and negative. More precisely, we assume that there exists two negative constants $\underline{\Lambda} \le \overline{\Lambda} < 0$ and two positive constants $0 < \underline{\eta} \le \overline{\eta}$, such that for all $y \in \mathbb{R}^n$,

$$\frac{1}{\overline{\eta}} \le F(y, \underline{\Lambda}) \le F(y, \Lambda_\varepsilon^0(y)) := \frac{1}{\eta_\varepsilon^0(y)} \le F(y, \overline{\Lambda}) \le \frac{1}{\underline{\eta}}. \tag{5.21}$$

This assumption implies $\underline{\eta} \le \eta_\varepsilon^0(y) \le \overline{\eta}$ and $\underline{\Lambda} \le \Lambda_\varepsilon^0(y) \le \overline{\Lambda}$ (since $\lambda \mapsto F(y,\lambda)$ is increasing). We will see in Corollary 5.4 that those two inequalities hold for all times, namely $\underline{\eta} \le \eta_\varepsilon(t,y) \le \overline{\eta}$ and $\underline{\Lambda} \le \Lambda(y, \eta_\varepsilon(t,y)) \le \overline{\Lambda}$. Note that with the notations

$$\frac{1}{\underline{\eta}(y)} := F(y, \overline{\Lambda}), \quad \frac{1}{\overline{\eta}(y)} := F(y, \underline{\Lambda}), \tag{5.22}$$

assumption (5.21) can be written

$$\underline{\eta} \le \underline{\eta}(y) \le \eta_\varepsilon^0(y) \le \overline{\eta}(y) \le \overline{\eta}.$$

We stress out that this assumption implies $-\Lambda(y, \eta_\varepsilon(t,y)) > 0$. It means that every phenotype has a positive fitness, and is thus able to survive in absence of other phenotypes. This assumption is somehow restrictive, but it is reasonable, and allows to avoid some technicalities.

Finally, the next assumptions deal with the derivatives of Λ. We need $\partial_\eta \Lambda$, $\nabla_y \Lambda$ and $\nabla_y \partial_\eta \Lambda$ to be bounded, and Λ to be semiconvex. According to (5.29), we assume that there exists two constants $l, L > 0$ such that for all $y \in \mathbb{R}^n$, $\lambda \in [\underline{\Lambda}, \overline{\Lambda}]$,

$$l \leq \partial_\lambda F(y, \lambda) \leq L, \tag{5.23}$$

$$|\nabla_y F(y, \lambda)|, |\nabla_y \partial_\lambda F(y, \lambda)| \leq L, \tag{5.24}$$

$$\forall i \in \{1, \dots, n\}, \quad \partial^2_{y_i} F(y, \lambda) \geq -L. \tag{5.25}$$

5.2 Definition of the ansatz and a priori estimates

In this section, we first give a rigorous definition of (Λ, Q), which only uses standard arguments.

Then, we construct the ansatz U_ε, formally introduced in section 5.1.3, and state some a priori estimates. We use those results to derive estimates on η_ε and Λ which will be useful in the sequel. Note that the properties of U_ε can be studied separately from the rest of the paper. The proofs, which are technical, are postponed to section 5.4.

Finally, we study the asymptotics of U_ε when $\varepsilon \to 0$.

5.2.1 Formal Limiting Eigenproblem

We study the formal limiting problem (5.7) and state the existence of the eigenelements (Λ, Q), along with some properties. The proof only uses elementary arguments and is postponned to section 5.4.1.

Proposition 5.2 *Under the assumptions of section 5.1.5, for fixed $y \in \mathbb{R}^n$ and $\eta \in (\underline{\eta}(y), \overline{\eta}(y))$ (from* (5.22)*), there exists a unique couple $(\Lambda(y, \eta), Q(x, y, \eta))$ which satisfies* (5.7)*.*

Moreover, with F defined in (5.15)*, $\underline{\Lambda}$ and $\overline{\Lambda}$ defined in* (5.21)*, we have that $\Lambda(y, \eta)$ is continuously differentiable and $\forall y \in \mathbb{R}^n$, $\forall \eta \in (\underline{\eta}(y), \overline{\eta}(y))$,*

$$F(y, \Lambda(y, \eta)) = \frac{1}{\eta}, \tag{5.26}$$

$$\underline{\Lambda} \leq \Lambda(y, \eta) \leq \overline{\Lambda} < 0, \tag{5.27}$$

$$Q(x, y, \eta) = \eta \frac{1}{A(x, y)} \exp\left(\int_0^x \frac{\Lambda(y, \eta) - d(x', y)}{A(x', y)} dx' \right). \tag{5.28}$$

Proof See section 5.4.1. ■

We can express the derivatives of Λ with the ones of F from (5.26) (and the standard implicit function theorem). We have,

$$\partial_\eta \Lambda(y, \eta) = \frac{-1}{\eta^2 \partial_\lambda F(y, \Lambda(y, \eta))}, \quad \nabla_y \Lambda(y, \eta) = \frac{-\nabla_y F(y, \Lambda(y, \eta))}{\partial_\lambda F(y, \Lambda(y, \eta))}. \tag{5.29}$$

In particular, note that $\partial_\eta \Lambda < 0$, which turns out to be fundamental in the following section.

5.2.2 Construction of U_ε and a priori estimates

We give a rigourous definition of U_ε, formally introduced in (5.9).

Proposition 5.3 *Under the assumptions of section 5.1.5, for all $\varepsilon > 0$ there exists $U_\varepsilon(t, y)$ a solution of* (5.9). *In addition,*

$$-\overline{\Lambda} \leq \partial_t U_\varepsilon(t, y) \leq -\underline{\Lambda},$$

for all $\varepsilon > 0$, $t \geq 0$, $y \in \mathbb{R}^n$, where $\underline{\Lambda}, \overline{\Lambda} < 0$ are defined in (5.21).

Proof See section 5.4.2. ■

In fact, U_ε is the unique solution of (5.9) with a uniformly bounded time derivative.

As a direct consequence of Proposition 5.3, we deduce the following usefull corollary.

Corollary 5.4 *With $\underline{\Lambda}$, $\overline{\Lambda}$, $\underline{\eta}$, $\overline{\eta}$ defined in* (5.21), *and setting*

$$\eta_\varepsilon(t, y) := \int_{\mathbb{R}^n} M(z) e^{\frac{U_\varepsilon(t, y+\varepsilon z) - U_\varepsilon(t, y)}{\varepsilon}} dz, \tag{5.30}$$

$$\Lambda_\varepsilon(t, y) := \Lambda(y, \eta_\varepsilon(t, y)),$$

we have

$$\underline{\Lambda} \leq \Lambda_\varepsilon(t, y) \leq \overline{\Lambda} < 0,$$

and

$$\underline{\eta} \leq \eta_\varepsilon(t, y) \leq \overline{\eta}. \tag{5.31}$$

Proof Simply use Proposition 5.3, $\Lambda_\varepsilon = -\partial_t U_\varepsilon$ and assumption (5.21). ■

5.2.3 Further estimates

Proposition 5.5 *Under the assumptions of section 5.1.5, we have*

$$|\nabla_y U_\varepsilon(t, y)| \leq k^0 + \frac{L}{l\underline{\eta}^2} t,$$

for all $\varepsilon > 0$, $t \geq 0$, $y \in \mathbb{R}^n$, where k^0 is defined in (5.16), *$L, l > 0$ in* (5.23) *and $\underline{\eta}$ in* (5.31).

Proof See section 5.4.3. ■

Note that $\frac{L}{l\underline{\eta}^2}$ comes from a bound on $|\nabla_y \Lambda(y, \eta_\varepsilon)|$. We will see that, at the limit when $\varepsilon \to 0$, we can prove Lipschitz continuity globally in time.

We also need the following further a priori estimate.

Proposition 5.6 *The function U_ε is semiconvex, that is, $\partial_t^2 U_\varepsilon$, $\partial_{y_1}^2 U_\varepsilon$, ..., $\partial_{y_n}^2 U_\varepsilon$ are bounded from below, uniformly in $\varepsilon > 0$, $y \in \mathbb{R}^n$, locally uniformly in $t \geq 0$.*

Proof The idea is to use that the Hamiltonian is somehow convex. See section 5.4.4 for a detailed proof. ■

Corollary 5.7 *We have $U_\varepsilon \in W^{2,1}_{loc}$, uniformly in $\varepsilon > 0$. In addition, $\forall t \geq 0$,*

$$\int_0^t |\partial_t^2 U_\varepsilon(\cdot, y)|, \int_0^t \partial_{y_i}^2 |U_\varepsilon(\cdot, y)| \text{ are bounded uniformly in } \varepsilon > 0,\ y \in \mathbb{R}^n.$$

Proof See section 5.4.4. ■

Corollary 5.8 *We have $\eta_\varepsilon \in W^{1,1}_{loc}$, uniformly in $\varepsilon > 0$. In addition, $\forall t \geq 0$,*

$$\int_0^t |\partial_t \eta_\varepsilon(\cdot, y)| \text{ is bounded uniformly in } \varepsilon > 0,\ y \in \mathbb{R}^n.$$

Proof Directly deduced from the definition of η_ε in (5.30) and Corollary 5.7. Note also that, differentiating (5.9), we have

$$\partial_t^2 U_\varepsilon = -\partial_t \eta_\varepsilon(t, y) \partial_\eta \Lambda(t, \eta_\varepsilon(t, y)). \tag{5.32}$$

■

5.2.4 Asymptotics

The next result deals with the asymptotics of U_ε when ε vanishes.

Proposition 5.9 *Under the assumptions of section 5.1.5, when ε vanishes, U_ε converges locally uniformly (and in $W^{1,1}_{loc}$) to a function $U(t, y) \in W^{1,\infty}_{loc}$ which is semi convex and satisfies, in the viscosity sense,*

$$\begin{cases} \partial_t U(t, y) = -\Lambda \left(y, \int_{\mathbb{R}^n} M(z) e^{\nabla_y U \cdot z} dz \right), & \forall t > 0,\ \forall y \in \mathbb{R}^n, \\ U(0, y) = u^0(y), & \forall y \in \mathbb{R}^n. \end{cases} \tag{5.33}$$

Proof See section 5.4.5. ■

We also point out that from Proposition 4.13,

$$p \mapsto -\Lambda \left(y, \int_{\mathbb{R}^n} M(z) e^{p \cdot z} dz \right) \text{ is a convex mapping, } \forall y \in \mathbb{R}^n.$$

This class of Hamiltonian has been widely studied, and numerous results on regularity as well as representation formula are available [53, 84, 151, 152].

As a direct consequence of the L^1_{loc} convergence of $\nabla_y U_\varepsilon$ to $\nabla_y U$, we have the following corollary.

Corollary 5.10 *When $\varepsilon \to 0$, η_ε converges to some η in L^1_{loc}. Consequently, $\Lambda_\varepsilon(t, y)$ converges to $\Lambda(y, \eta(t, y))$ and $Q_\varepsilon(t, x, y) := Q(x, y, \eta_\varepsilon(t, y))$ to $Q(x, y, \eta(t, y))$ in L^1_{loc}.*

As a complement, we give the following a posteriori estimate which is not used in the sequel.

Proposition 5.11 *The function $U(t, y)$ is gloablly Lipschitz.*

Proof See section 5.4.6. ■

Note that, on the contrary, U_ε may not be globally Lipschitz for $\varepsilon > 0$.

5.3 Proof of the main theorem

5.3.1 Estimates on p_ε

We define the function $p_\varepsilon(t,x,y)$ through the factorisation (5.4), according to the definition of u_ε in (5.8). As pointed out in the formal approach (section 5.1.3), the cornerstone of our method is to prove a uniform L^1 estimates on p_ε. This section is devoted to the proof of this result.

The formal approach suggests that p_ε behaves as a multiple of $Q(x,y,\eta(t,y))$ when $\varepsilon \to 0$. We also recall that we assume either that $\frac{1}{A(\cdot,y)}$ is integrable or that $b(\cdot,x)$ has a compact support, see (5.13)-(5.14). In the first case, we are able to prove a strong estimate, namely that p_ε is bounded from below and above by multiples of Q_ε.

Lemma 5.12 *Under the assumptions of section 5.1.5, if* (5.13) *holds, and for any fixed $T>0$, there exists two constants $\underline{\gamma},\overline{\gamma}>0$ such that*

$$\underline{\gamma}\ Q_\varepsilon(t,x,y) \le p_\varepsilon(t,x,y) \le \overline{\gamma}\ Q_\varepsilon(t,x,y),$$

where

$$Q_\varepsilon(t,x,y) := Q(x,y,\eta_\varepsilon(t,y)),$$

for all $\varepsilon>0$, $t\in[0,T]$, $x\ge 0$, $y\in\mathbb{R}^n$.

Proof The function $p_\varepsilon(t,x,y)$ satisfies the following equation, for $\varepsilon>0,\ t>0,\ x>0,\ y\in\mathbb{R}^n$,

$$\begin{cases} \varepsilon\partial_t p_\varepsilon + \partial_x\left[A(x,y)p_\varepsilon\right] + \left(d(x,y)-\Lambda_\varepsilon\right)p_\varepsilon = 0, \\ A(0,y)p_\varepsilon(t,0,y) = \displaystyle\iint_{x>0,\ z\in\mathbb{R}^n} M(z)e^{\frac{U_\varepsilon(t,y+\varepsilon z)-U_\varepsilon(t,y)}{\epsilon}} b(x,y+\varepsilon z)p_\varepsilon(t,x,y+\varepsilon z)dxdz. \end{cases}$$

and Q_ε satisfies

$$\begin{cases} \partial_x[A(x,y)Q_\varepsilon] - (d(x,y)-\Lambda_\varepsilon)Q_\varepsilon = 0, \\ A(0,y)Q_\varepsilon(t,0,y) = \eta_\varepsilon(t,y). \end{cases}$$

Setting

$$v_\varepsilon(t,x,y) := \frac{p_\varepsilon(t,x,y)}{Q_\varepsilon(t,x,y)},$$

we have

$$\begin{cases} \partial_t v_\varepsilon + \dfrac{A(x,y)}{\varepsilon}v_\varepsilon = -\dfrac{\partial_t Q_\varepsilon}{Q_\varepsilon}v_\varepsilon, \\ v_\varepsilon(t,0,y) = \displaystyle\iint_{x>0,\ z\in\mathbb{R}^n} J_\varepsilon(t,x,y,z)v_\varepsilon(t,x,y+\varepsilon z)dz, \end{cases} \tag{5.34}$$

where

$$J_\varepsilon(t,x,y,z) := \frac{1}{\eta_\varepsilon}M(z)e^{\frac{U_\varepsilon(t,y+\varepsilon z)-U_\varepsilon(t,y)}{\epsilon}} b(x,y+\varepsilon z)Q_\varepsilon(t,x,y+\varepsilon z).$$

Our goal is to infer some bounds on v_ε .

First, from the definition of η_ε and the normalization

$$\int_{x>0} b(x,y)Q(x,y,\eta)dx = 1, \quad \forall y\in\mathbb{R}^n,\ \eta>0,$$

we see that J is a probability kernel: $\iint_{x>0,\ z\in\mathbb{R}^n} J_\varepsilon(t,x,y,z)dxdz = 1$.

Now, let us estimate $\frac{\partial_t Q_\varepsilon}{Q_\varepsilon}$. We compute

$$\begin{aligned}\frac{\partial_t Q_\varepsilon}{Q_\varepsilon}(t,x,y) &= \partial_t\eta_\varepsilon(t,y)\frac{\partial_\eta Q}{Q}(x,y,\eta_\varepsilon(t,y))\\ &= \partial_t\eta_\varepsilon(t,y)\left(\frac{1}{\eta_\varepsilon(t,y)} + \partial_\eta\Lambda(y,\eta_\varepsilon(t,y))\int_0^x \frac{1}{A(x',y)}dx'\right).\end{aligned}$$

Using that η_ε is positively bounded (from (5.31)), and that $\partial_\eta\Lambda$ is bounded (from (5.29) and assumption (5.23)), we have

$$\left|\frac{\partial_t Q_\varepsilon}{Q_\varepsilon}(t,x,y)\right| \le K|\partial_t\eta_\varepsilon(t,y)|\left(1+\int_0^x \frac{1}{A(x',y)}dx'\right).$$

for some constant $K>0$.

Then, using assumption (5.13), we have

$$\left|\frac{\partial_t Q_\varepsilon}{Q_\varepsilon}(t,x,y)\right| \le K|\partial_t\eta_\varepsilon(t,y)|$$

for some constant still denoted K. Setting

$$\underline{v}_\varepsilon(t,y) := v_\varepsilon(t,x,y)\exp\left(-K\int_0^t |\partial_t\eta_\varepsilon(t',y)|dt'\right),$$

we have that $\underline{v}$ is a subsolution to (5.34), namely

$$\begin{cases}\partial_t\underline{v}_\varepsilon + \dfrac{A(x,y)}{\varepsilon}\underline{v}_\varepsilon \le 0\\ \underline{v}_\varepsilon(t,0,y) \le \displaystyle\iint_{x>0,\ z\in\mathbb{R}^n} J_\varepsilon(t,x,y,z)\underline{v}_\varepsilon(t,x,y+\varepsilon z)dz.\end{cases}$$

From the comparison principle, we deduce

$$\underline{v}_\varepsilon(t,x,y) \le \sup_{\substack{x\ge 0\\ y\in\mathbb{R}^n}} \underline{v}_\varepsilon(0,x,y) \le \overline{\gamma}^0,$$

where $\overline{\gamma}^0$ comes from assumption (5.19). It implies

$$p_\varepsilon(t,x,y) \le \overline{\gamma}^0 Q_\varepsilon(t,x,y)\exp\left(K\int_0^t |\partial_t\eta_\varepsilon(t',y)|dt'\right).$$

Using that $\int_0^t |\partial_t\eta_\varepsilon|$ is bounded, uniformly in $\varepsilon>0,\ y\in\mathbb{R}^n$ (see Corollary 5.8), we deduce

$$p_\varepsilon(t,x,y) \le \overline{\gamma} Q_\varepsilon(t,x,y).$$

for some constant $\overline{\gamma}$. Identically, we infer the bound from below, and the proof is achieved. ∎

We see from this result that, in our context, assumption (5.13) seems very natural from the mathematical point of view. We recall that this assumption is equivalent to the fact that $\frac{\partial_\nu Q}{Q}$ is bounded.

Note that the lemma implies that $\frac{p_\varepsilon}{Q_\varepsilon}$ is bounded and thus converges (up to extraction of a subsequence) in the L^∞ weak-$\star$ topology. Formally, it means that p_ε weakly converges to a multiple of $Q(x, y, \eta(t, y))$. Hereafter, we think that the weak convergence in x can be turned into a strong $L^1(dx)$ convergence by using a Generalized Relative Entropy estimate [225], as in Chapter 4. We leave this question for future works. Note also that, according to Theorem 5.1, the population concentrates on $\mathcal{S}$, on which $\nabla_y U \equiv 0$ and $\eta \equiv 1$. Thus, at least formally, p_ε converges to $Q(x, y, 1)$ on $\mathcal{S}$.

Now, let us turn to the other case, namely if (5.14) holds instead of (5.13). We are only able to prove a two sided estimate on p_ε, using two extremal eigenfunctions defined as

$$\overline{Q}(x, y) := \frac{\overline{\eta}}{A(x, y)} \exp\left(\int_0^x \frac{\overline{\Lambda} - d(x', y)}{A(x', y)} dx'\right),$$
$$\underline{Q}(x, y) := \frac{\underline{\eta}}{A(x, y)} \exp\left(\int_0^x \frac{\underline{\Lambda} - d(x', y)}{A(x', y)} dx'\right).$$

This weaker result will be sufficient for the sequel.

Lemma 5.13 *Under the assumptions of section 5.1.5, and for any fixed $T > 0$, there exist two constants $\underline{\gamma}, \overline{\gamma} > 0$ such that*

$$\underline{\gamma}\ \underline{Q}(x, y) \leq p_\varepsilon(t, x, y) \leq \overline{\gamma}\ \overline{Q}(x, y)$$

for all $\varepsilon > 0$, $t \in [0, T]$, $x \geq 0$, $y \in \mathbb{R}^n$.

Note that this result is weaker than Lemma 5.12 since $\frac{\eta}{\overline{\eta}}\underline{Q} \leq Q_\varepsilon \leq \frac{\overline{\eta}}{\underline{\eta}}\overline{Q}$.

Proof Noting that

$$\int_0^{\overline{x}} \frac{1}{A(x', y)} dx' \leq K$$

for some $K > 0$, and applying the same method as before for $x \in [0, \overline{x}]$, we infer, for $\varepsilon > 0,\ t \in [0, T],\ x \in [0, \overline{x}],\ y \in \mathbb{R}^n$,

$$\underline{\gamma}\ Q_\varepsilon \leq p_\varepsilon(t, x, y) \leq \overline{\gamma} Q_\varepsilon(t, x, y).$$

We deduce

$$\underline{\gamma}\ \underline{\eta} \leq A(0, y) p_\varepsilon(t, 0, y) \leq \overline{\gamma}\ \overline{\eta}.$$

Hence, we have, for $\varepsilon > 0,\ t \in [0, T],\ x > 0,\ y \in \mathbb{R}^n$,

$$\begin{cases} \varepsilon \partial_t p_\varepsilon + \partial_x \left[A(x, y) p_\varepsilon\right] + \left(d(x, y) - \overline{\Lambda}\right) p_\varepsilon \leq 0, \\ A(0, y) p_\varepsilon(t, 0, y) \leq \overline{\gamma}\ \overline{\eta}. \end{cases}$$

Setting

$$\overline{\overline{Q}}(x, y) := \overline{\gamma}\ \overline{Q}(x, y),$$

we have

$$\begin{cases} \varepsilon \partial_t \overline{\overline{Q}} + \partial_x \left[A(x, y) \overline{\overline{Q}}\right] + \left(d(x, y) - \overline{\Lambda}\right) \overline{\overline{Q}} = 0, \\ A(0, y) \overline{\overline{Q}}(0, y) = \overline{\gamma}\ \overline{\eta}. \end{cases}$$

From the comparison principle, we deduce $p_\varepsilon \leq \overline{\overline{Q}}$. We prove the lower bound similarly. ∎

We are now ready to prove a L^1-uniform estimate on p_ε, which will be crucial in the next section.

Corollary 5.14 *Under the assumptions of section 5.1.5 and for any fixed $T > 0$, there exist $\underline{I}, \overline{I} > 0$ such that*

$$\underline{I} \leq \int_{\mathbb{R}_+} p_\varepsilon(t,x,y)dx \leq \overline{I},$$

for all $\varepsilon > 0,\ t \in [0,T],\ y \in \mathbb{R}^n$.

PROOF Fom the above, we only need bounds on the integrals of $\underline{Q}$ and $\overline{Q}$. Recalling $\overline{\Lambda} < 0$, we compute

$$\begin{aligned}
\int_0^{+\infty} \overline{Q}(x,y)dx &\leq \int_0^{+\infty} \frac{\overline{\eta}}{A(x,y)} \exp\left(\int_0^x \frac{\overline{\Lambda}}{A(x',y)}dx'\right) \\
&= \left[\frac{\overline{\eta}}{\overline{\Lambda}} \exp\left(\int_0^x \frac{\overline{\Lambda}}{A(x',y)}dx'\right)\right]_{x=0}^{+\infty} \\
&= \frac{\overline{\eta}}{-\overline{\Lambda}}\left[1 - \exp\left(\int_0^{+\infty} \frac{\overline{\Lambda}}{A(x',y)}dx'\right)\right] \leq \frac{\overline{\eta}}{-\overline{\Lambda}}
\end{aligned}$$

which proves the bound from above.

For the other inequality, we note that the nondegeneracy assumption (5.11) implies $\int_{\mathbb{R}_+} Q(x,y,\eta)dx > \alpha$, for some $\alpha > 0$. ■

5.3.2 Selection of the fittest phenotype

The following result states that the total population ρ_ε is uniformly bounded and converges when $\varepsilon \to 0$. Recalling (5.4), the two first statements of Theorem 5.1 are direct consequences of the following proposition and the uniform L^1 estimate on p_ε (from Corollary 5.14).

Proposition 5.15 *Under the assumptions of section 5.1.5,*

1. *There exist two positive constants $\underline{\rho}, \overline{\rho} > 0$ such that*

$$\underline{\rho} \leq \rho_\varepsilon(t) \leq \overline{\rho}, \quad \forall \varepsilon > 0,\ t \geq 0.$$

   In addition, ρ_ε converges to some ρ in the L^∞-weak topology.*

2. *The integral $\int_{\mathbb{R}^n} e^{\frac{u_\varepsilon(t,y)}{\varepsilon}}dy$ is bounded away from 0, uniformly in $\varepsilon > 0$, $t \geq 0$. Consequently, when ε vanishes, u_ε converges locally uniformly to a function u such that*

$$\forall t \geq 0, \quad \sup_{y \in \mathbb{R}^n} u(t,y) = 0,$$

   i.e.,

$$\int_0^t \rho = \sup_{y \in \mathbb{R}^n} U(t,y).$$

Note that, according to the above result, $\rho(\cdot)$ can be seen as the Lagrange multiplier for the constraint $\sup_{\mathbb{R}^n} u(t,\cdot) = 0$. In addition, with $\mathcal{S}$ defined in (5.2), we have

$$\begin{aligned}\mathcal{S} &:= \left\{t \geq 0,\ y \in \mathbb{R}^n : U(t,y) = \sup_{\mathbb{R}^n} U(t,\cdot)\right\}\\ &= \left\{t \geq 0,\ y \in \mathbb{R}^n : U(t,y) = \int_0^t \rho\right\}\\ &= \{t \geq 0,\ y \in \mathbb{R}^n : u(t,y) = 0\}.\end{aligned}$$

Remark 5.16 *Under stronger assumptions, namely that there exists $\underline{r}, \overline{r} > 0$ such that*

$$\underline{r} \leq b(x,y) - d(x,y) \leq \overline{r}, \quad \forall x \geq 0,\ y \in \mathbb{R}^n,$$

the proof of Proposition 5.15 becomes much simpler. Indeed, integrating (5.1) *and using an integration by part, we obtain*

$$\begin{aligned}\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\rho_\varepsilon(t) &= \iint\limits_{\mathbb{R}^n\times\mathbb{R}_+} \left[\left(\frac{1}{\varepsilon^n}\int_{\mathbb{R}^n} M(\frac{y'-y}{\varepsilon})dy\right) b(x,y') - d(x,y')\right] m_\varepsilon(t,x,y')dxdy' - \rho_\varepsilon^2(t)\\ &\leq \overline{r}\rho_\varepsilon(t) - \rho_\varepsilon^2(t).\end{aligned}$$

which implies $0 \leq \rho_\varepsilon(t) \leq \max(\overline{r}, \rho_\varepsilon^0)$ and provide an a priori upper bound on ρ_ε. With the same method, we also infer a positive lower bound on ρ_ε.

Then, using the uniform L^1 estimate on p_ε from Corollary 5.14, we directly deduce $\sup\limits_{y\in\mathbb{R}^n} u(t,y) = 0$.

Proof (of Proposition 5.15) We recall that

$$\rho_\varepsilon(t) = \iint\limits_{x,y} p_\varepsilon(t,x,y) e^{\frac{u_\varepsilon(t,y)}{\varepsilon}} dxdy, \tag{5.37}$$

with $u_\varepsilon(t,y) = U_\varepsilon(t,y) - \int_0^t \rho_\varepsilon$. Multiplying by $e^{\int_0^t \frac{\rho_\varepsilon}{\varepsilon}}$ we have

$$\rho_\varepsilon(t) e^{\frac{\int_0^t \rho_\varepsilon}{\varepsilon}} = \int_{\mathbb{R}^n} e^{\frac{U_\varepsilon(t,y)}{\varepsilon}} \int_{\mathbb{R}_+} p_\varepsilon(t,x,y)dxdy,$$

and integrating over $(0,t)$, we deduce

$$\varepsilon\left(e^{\frac{\int_0^t \rho_\varepsilon}{\varepsilon}} - 1\right) = \int_0^t \int_{\mathbb{R}^n} e^{\frac{U_\varepsilon(t,y)}{\varepsilon}} \int_{\mathbb{R}_+} p_\varepsilon(t,x,y)dxdydt.$$

From $0 < -\overline{\Lambda} \leq \partial_t U_\varepsilon \leq -\underline{\Lambda}$ (Proposition 5.3) and the $L^1(dx)$ estimate on p_ε (Corollary 5.14), we have

$$\begin{aligned}\varepsilon\left(e^{\frac{\int_0^t \rho_\varepsilon}{\varepsilon}} - 1\right) &\geq \underline{I}\int_0^t \int_{\mathbb{R}^n} \frac{\varepsilon}{-\underline{\Lambda}} \frac{\partial_t U_\varepsilon(t,y)}{\varepsilon} e^{\frac{U_\varepsilon(t,y)}{\varepsilon}} dy\ dt\\ &\geq \frac{\varepsilon \underline{I}}{-\underline{\Lambda}} \int_{\mathbb{R}^n} e^{\frac{U_\varepsilon(t,y)}{\varepsilon}} - e^{\frac{u_\varepsilon^0(y)}{\varepsilon}} dy.\end{aligned}$$

Dividing by $\varepsilon e^{\frac{\int_0^t \rho_\varepsilon}{\varepsilon}}$ on both sides we find

$$\left(1 - e^{-\frac{\int_0^t \rho_\varepsilon}{\varepsilon}}\right) \geq \frac{I}{-\underline{\Lambda}} \int_{\mathbb{R}^n} \left(e^{\frac{u_\varepsilon(t,y)}{\varepsilon}} - e^{\frac{u_\varepsilon^0(y)}{\varepsilon}} e^{-\frac{\int_0^t \rho_\varepsilon}{\varepsilon}}\right) dy,$$

that we rewrite

$$\int_{\mathbb{R}^n} e^{\frac{u_\varepsilon(t,y)}{\varepsilon}} dy \leq \frac{-\underline{\Lambda}}{\underline{I}} \left(1 - e^{-\frac{\int_0^t \rho_\varepsilon}{\varepsilon}}\right) + e^{-\frac{\int_0^t \rho_\varepsilon}{\varepsilon}} \int_{\mathbb{R}^n} e^{\frac{u_\varepsilon^0(y)}{\varepsilon}} dy.$$

Then, from assumption (5.18), we deduce

$$\int_{\mathbb{R}^n} e^{\frac{u_\varepsilon(t,y)}{\varepsilon}} dy \leq \frac{-\underline{\Lambda}}{\underline{I}} \left(1 - e^{-\frac{\int_0^t \rho_\varepsilon}{\varepsilon}}\right) + \overline{J}^0 e^{-\frac{\int_0^t \rho_\varepsilon}{\varepsilon}},$$

and

$$\int_{\mathbb{R}^n} e^{\frac{u_\varepsilon(t,y)}{\varepsilon}} dy \leq \overline{K} := \max\left(\frac{-\underline{\Lambda}}{\underline{I}}, \overline{J}^0\right).$$

Identically, we infer

$$\int_{\mathbb{R}^n} e^{\frac{u_\varepsilon(t,y)}{\varepsilon}} dy \geq \underline{K} := \min\left(\frac{-\overline{\Lambda}}{\overline{I}}, \underline{J}^0\right) > 0.$$

Now, from the definition of ρ_ε (recalled in (5.37)) and the L^1 estimate on p_ε (Corollary 5.14) we find

$$\underline{I}\ \underline{K} \leq \rho_\varepsilon(t) \leq \overline{I}\ \overline{K},$$

for all $\varepsilon > 0,\ t \geq 0$.

Since $\rho_\varepsilon(t)$ is uniformly bounded, there exists a sequence $\varepsilon_k \to 0$ such that ρ_{ε_k} converges to some ρ in the $\mathcal{L}^\infty$-weak* topology when $k \to +\infty$. Since, in addition, U_ε converges locally uniformly to U (Proposition 5.9), we deduce that u_{ε_k} converges locally uniformly to some u.

Now, from $\underline{K} \leq \int_{\mathbb{R}^n} e^{\frac{u_\varepsilon(t,\cdot)}{\varepsilon}} \leq \overline{K}$, at the limit $k \to +\infty$, we have

$$\forall t > 0,\ \sup_{y \in \mathbb{R}^n} u(t,y) = 0.$$

We deduce

$$\int_0^t \rho = \sup_{y \in \mathbb{R}^n} U(t,y).$$

Therefore, $\int_0^t \rho$ does not depend on the extracted subsequence, and the convergence occurs for the whole sequence $\varepsilon \to 0$, which achieves the proof. ∎

Corollary 5.17 *Incidentally, we have*

$$\forall t > 0, \quad -\overline{\Lambda} \leq \frac{1}{t} \int_0^t \rho(s) ds \leq -\underline{\Lambda}.$$

Proof From Proposition 5.3, $\int_0^t \rho = \sup_{\mathbb{R}^n} U(t,\cdot) \leq \sup_{\mathbb{R}^n} u^0(\cdot) - \underline{\Lambda} t = -\underline{\Lambda} t$. The bound from below is proved similarly. ∎

5.3.3 Adaptive dynamics

In this section, we prove the third statement of Theorem 5.1. We need further assumptions on the initial conditions:

$$\exists !\ \bar{y}_\varepsilon^0 \in \mathbb{R}^n,\ u_\varepsilon^0(\bar{y}_\varepsilon^0) = \max_{y\in\mathbb{R}^n} u_\varepsilon^0(y) = 0, \tag{5.38}$$

$$\bar{y}_\varepsilon^0 \text{ converges to some } \bar{y}^0 \in \mathbb{R}^n \text{ when } \varepsilon \text{ vanishes}, \tag{5.39}$$

$$\nabla_y^2 u^0(\bar{y}^0) < 0. \tag{5.40}$$

Proposition 5.18 *Under the assumptions of section 5.1.5 and* (5.38)-(5.39)-(5.40)*, for a short time interval $t \in [0,T]$, there exists a unique $\bar{y}(t) \in \mathbb{R}^n$ on which $U(t,\cdot)$ reaches its maximum. Moreover, $t \mapsto \bar{y}(t) \in \mathcal{C}^1$ and satisfies the Canonical Equation*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\bar{y}(t) = \left(\nabla_y^2 U(t,\bar{y}(t))\right)^{-1} \cdot \nabla_y \Lambda(\bar{y}(t),1) + \partial_\eta \Lambda(\bar{y}(t),1) \displaystyle\int_{\mathbb{R}^n} M(z) z dz, \\ \bar{y}(0) = \bar{y}^0. \end{cases} \tag{5.41}$$

.

Note that (5.41) features a drift term $\partial_\eta \Lambda \int_{\mathbb{R}^n} M(z) z dz$. If the mutation kernel $M(\cdot)$ is even, this term vanishes and we recover the standard Canonical Equation.

Proof Since U (defined in 5.9) satisfies (5.33) with smooth initial datum, it is uniformly C^2 in the y variable for short times $t \in [0,T]$, $T > 0$ (this can be proved with the method of the characteristics, see Chapter 3.2 in [138]).

Let us consider such a time interval $[0,T]$, and $V \subset \mathbb{R}^n$ a neighborhood of y^0. We are interested in the solutions $(t,\bar{y}) \in [0,T] \times V$ of

$$\nabla_y U(t,\bar{y}) = 0. \tag{5.42}$$

From (5.38) we know that at initial time there exists a unique solution $\bar{y}^0$ of (5.42). Besides, $\nabla_y^2 u^0(\bar{y}^0)$ is invertible. From the implicit functions theorem, there exists a unique $\bar{y}(t) \in \mathbb{R}^n$ satisfying (5.42), for t in a certain time interval still denoted $[0,T]$. We can again choose a smaller T to ensure that $\bar{y}(t)$ remains in V.

From (5.40), we can also choose T and V small enough to guarantee, $\forall t \in [0,T]$,

$$\bar{y}(t) \in V, \quad U(t,\cdot) \text{ is striclty concave in } V,$$
$$\max_{y\in V} U(t,y) = \max_{y\in\mathbb{R}^n} U(t,y).$$

Hence, for all $t \in [0,T]$, the solution $\bar{y}(t)$ of (5.42) must satisfy

$$U(t,\bar{y}(t)) = \max_{y\in V} U(t,y) = \max_{y\in\mathbb{R}^n} U(t,y) = 0, \tag{5.45}$$

which proves the first part of the theorem.

Moreover, $t \mapsto \bar{y}(t)$ is a $\mathcal{C}^1$ function. Differentiating (5.42) with respect to t, and noting that

$$\eta(t,\bar{y}(t)) = \int_{\mathbb{R}^n} M(z) e^{\nabla_y U(t,\bar{y}(t))\cdot z} dz = 1,$$

we obtain

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left[\nabla_y U(t,\bar{y}(t))\right] \\ &= -\nabla_y \Lambda(\bar{y}(t),1) - \nabla_y^2 u(t,\bar{y}(t)) \cdot \partial_\eta \Lambda(\bar{y}(t),1) \int_{\mathbb{R}^n} M(z) z dz + \nabla_y^2 u(t,\bar{y}(t)) \cdot \dot{\bar{y}}(t), \end{aligned}$$

and (5.41) follows, for $t \in [0,T]$. ∎

Corollary 5.19 *Under the same assumptions, $t \mapsto \rho(t) \in C^1([0,T])$ and for all $t \in [0,T]$,*

$$\rho(t) = -\Lambda(\bar{y}(t), 1)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) = -\nabla_y\Lambda \cdot \nabla_y^2 U \cdot \nabla_y \Lambda - \partial_\eta\Lambda \left(\int_{\mathbb{R}^n} M(z) z dz\right) \cdot \left(\nabla_y^2 U\right)^2 \cdot \nabla_y\Lambda.$$

where $\nabla_y^2 U$ is evaluated in $(t, \bar{y}(t))$, and the derivatives of Λ in $(\bar{y}(t), 1)$.

In particular, if $M(\cdot)$ is even, then $\frac{\mathrm{d}}{\mathrm{d}t}\rho(t) \geq 0$ and $-\overline{\Lambda} \leq \rho \leq -\underline{\Lambda}$.

Proof Follows from $\partial_t u(t, \bar{y}(t)) = 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\left[\partial_t u(t, \bar{y}(t))\right] = 0$. ∎

Remark 5.20 *The limitation of Proposition 5.18 to a short time interval is merely due to three independant phenomena. First, the possible loss of concavity, or apparition of singularities for U, coming from the Hamilton-Jacobi equation (5.33). Secondly, the possible "jump" of the point where U reaches its maximum, contradicting $\max_{y \in V} u(t,y) = \max_{y\in\mathbb{R}^n} U(t,y)$ in (5.45). Finally, the possible explosion in finite time of $\bar{y}(t)$ from the dynamics of the Canonical Equation (5.41). Regarding the last point, we point out that Λ can sometimes be used as a Lyapunov function. Indeed, we have*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\Lambda(\bar{y}(t),1)\right] = \nabla_y\Lambda(\bar{y}(t),1) \cdot \dot{\bar{y}}(t) \\
&= \nabla_y\Lambda(\bar{y}(t),1) \cdot \nabla_y^2 U^{-1} \cdot \nabla_y\Lambda(\bar{y}(t),1) + \partial_\eta\Lambda(\bar{y}(t),1)\nabla_y\Lambda(\bar{y}(t),1) \cdot \int_{\mathbb{R}^n} M(z) z dz \\
&\leq \partial_\eta\Lambda(\bar{y}(t),1)\nabla_y\Lambda(\bar{y}(t),1) \cdot \int_{\mathbb{R}^n} M(z) z dz.
\end{aligned}$$

In particular, if $M(\cdot)$ is even, then $\frac{\mathrm{d}}{\mathrm{d}t}\left[\Lambda(\bar{y}(t),1)\right] \leq 0$. Thus, if $\bar{y}^0$ belongs to a "well" of Λ, then $\bar{y}(t)$ remains "trapped", which prevents from an explosion in finite time. It also implies, at least formally, that $\bar{y}(t)$ converges to a local minimum of $\Lambda(\cdot, 1)$ when $t \to +\infty$.

5.4 Construction, estimates, and asymptotics of U_ε

5.4.1 The eigenproblem - proof of Proposition 5.2

An elementary formal calculation on (5.7) gives (5.28). Multiplying by $b(x,y)$ and integrating in x, we obtain (5.26).

From (5.21) we have, for $y \in \mathbb{R}^n$, $\forall \eta \in (\underline{\eta}(y), \overline{\eta}(y))$

$$F(y, \underline{\Lambda}) \leq \frac{1}{\eta} \leq F(y, \overline{\Lambda}).$$

As $\partial_\lambda F > 0$, we prove existence and uniqueness for Λ as the unique solution of (5.26). Now, using (5.28), we are provided with existence and uniqueness for Q.

From $\partial_\lambda F > 0$ and

$$F(y, \underline{\Lambda}) \leq F(y, \Lambda(y,\eta)) \leq F(y, \overline{\Lambda}),$$

we show (5.27).

5.4.2 Construction of U_ε - proof of Proposition 5.3

The proof of Proposition 5.3 is divided into three parts. First, we construct U_ε on a truncated problem. Then, we prove a uniform a priori estimate on $\partial_t U_\varepsilon$, which allows to remove the truncation.

Extending Λ. Proposition 5.2 defines $\Lambda(y,\eta)$ only for $y \in \mathbb{R}^n$ and $\eta \in \left(\underline{\eta}(y), \overline{\eta}(y)\right)$. We first need to artificially extend $\Lambda(y,\eta)$ for $\eta \in (0,+\infty)$. For $y \in \mathbb{R}^n$, we set

$$\tilde{\Lambda}(y,\eta) = \begin{cases} \underline{\Lambda} - \underline{B_y}(\eta) & \text{if } \eta < \underline{\eta}(y), \\ \Lambda(y,\eta) & \text{if } \underline{\eta}(y) \leq \eta \leq \overline{\eta}(y), \\ \overline{\Lambda} + \overline{B_y}(\eta) & \text{if } \eta > \overline{\eta}(y), \end{cases}$$

where $\underline{B_y}$ and $\overline{B_y}$ are chosen to be positive, increasing, bounded by 1, and such that $\tilde{\Lambda}$ is smooth. Note that the extension of Λ is completely arbitrary, since we will see a posteriori that $\eta_\varepsilon \in \left(\underline{\eta}(y), \overline{\eta}(y)\right)$. We consider the following problem

$$\begin{cases} \partial_t \tilde{U}_\varepsilon(t,y) = -\tilde{\Lambda}\left(y, \displaystyle\int_{\mathbb{R}^n} M(z) e^{\frac{\tilde{U}_\varepsilon(t,y+\varepsilon z) - \tilde{U}_\varepsilon(t,y)}{\varepsilon}} dz\right), & \forall t \geq 0,\ \forall y \in \mathbb{R}^n, \\ \tilde{U}_\varepsilon(0,y) = u_\varepsilon^0(y), & \forall y \in \mathbb{R}^n. \end{cases} \tag{5.46}$$

Solution for the truncated problem. For a fixed $R > 0$, we consider a truncation function $\phi_R : \mathbb{R} \to \mathbb{R}$ which is smooth, increasing and satisfies the following conditions:

- $\phi_R(r) = r$ for $r \in [-\frac{R}{2}, \frac{R}{2}]$,
- $\phi_R(r) = R$ for $r \geq 2R$,
- $\phi_R(r) = -R$ for $r \leq -2R$,
- $\phi_R' \geq 0$ is uniformly bounded.

For $\varepsilon > 0$, we consider the Cauchy problem

$$\begin{cases} \partial_t \tilde{U}_\varepsilon^R(t,y) = \phi_R\left(-\tilde{\Lambda}\left(y, \displaystyle\int_{\mathbb{R}^n} M(z) e^{\frac{\tilde{U}_\varepsilon^R(t,y+\varepsilon z) - \tilde{U}_\varepsilon^R(t,y)}{\varepsilon}} dz\right)\right), \\ \tilde{U}_\varepsilon^R(0,\cdot) = u_\varepsilon^0. \end{cases} \tag{5.47}$$

for which the classical Cauchy-Lipschitz theorem provides existence and uniqueness of a solution $\tilde{U}_\varepsilon^R$, defined globally in time.

Estimate on the time derivative.

Lemma 5.21 *We have*

$$\inf_{y \in \mathbb{R}^n} \partial_t u_\varepsilon^0(y) \leq \partial_t \tilde{U}_\varepsilon^R(t,y) \leq \sup_{y \in \mathbb{R}^n} \partial_t u_\varepsilon^0(y), \quad \forall \varepsilon > 0,\ t > 0,\ y \in \mathbb{R}^n,$$

for all $t \geq 0$, $y \in \mathbb{R}^n$, where $\partial_t u_\varepsilon^0 := -\Lambda\left(y, \eta_\varepsilon^0(y)\right)$ and η_ε^0 is defined in (5.20).

The full proof of this statement, which is technical, can be found in section 4.4.4. We give the formal idea of the method.

Let us fixe $\varepsilon > 0$, $R > 0$ and set $V(t,y) := \partial_t \tilde{U}_\varepsilon^R(t,y)$. Differentiating (5.47) with respect to t, we obtain

$$\partial_t V(t,y) = \int_{\mathbb{R}^n} K(t,y,z) \left(\frac{V(t,y+\varepsilon z) - V(t,y)}{\varepsilon} \right) dz,$$

where $K(t,y,z) := -\phi_R' \; \partial_\eta \tilde{\Lambda} \; M(z) e^{\frac{\tilde{U}_\varepsilon^R(t,y+\varepsilon z) - \tilde{U}_\varepsilon^R(t,y)}{\varepsilon}}$. Since $\partial_\eta \Lambda < 0$, we have $K \geq 0$. Then, if for some $t > 0$, $V(t,\cdot)$ reaches its maximum at $\bar{y} \in \mathbb{R}^n$, we obtain the inequality

$$\partial_t V(t,\bar{y}) = \int_{\mathbb{R}^n} K(t,\bar{y},z) \left(\frac{V(t,\bar{y}+\varepsilon z) - V(t,\bar{y})}{\varepsilon} \right) dz \leq 0.$$

Formally, it shows that the maximum value of V is decreasing with time, that is,

$$\sup_y V(t,y) \leq \sup_y V(0,y) = \sup_y \partial_t u_\varepsilon^0.$$

With the same method we show $\inf_y V \geq \inf_y \partial_t u_\varepsilon^0$, which conclude the proof of Lemma 5.21.

Hereafter, from assumption (5.21) and $\partial_\lambda F > 0$, we have

$$-\overline{\Lambda} \leq \partial_t u_\varepsilon^0(y) \leq -\underline{\Lambda}.$$

Using Lemma 5.21, we infer

$$-\overline{\Lambda} \leq \phi_R \left(-\tilde{\Lambda} \left(y, \int_{\mathbb{R}^n} M(z) e^{\frac{\tilde{U}_\varepsilon^R(t,y+\varepsilon z) - \tilde{U}_\varepsilon^R(t,y)}{\epsilon}} \right) \right) \leq -\underline{\Lambda}. \tag{5.48}$$

Removing the truncation. From (5.48) and the choice of ϕ_R, for R large enough, we have

$$-\phi_R \left(\tilde{\Lambda} \left(y, \int_{\mathbb{R}^n} M(z) e^{\frac{\tilde{U}_\varepsilon^R(t,y+\varepsilon z) - \tilde{U}_\varepsilon^R(t,y)}{\epsilon}} \right) \right) = \tilde{\Lambda} \left(y, \int_{\mathbb{R}^n} M(z) e^{\frac{\tilde{U}_\varepsilon^R(t,y+\varepsilon z) - \tilde{U}_\varepsilon^R(t,y)}{\epsilon}} \right).$$

Besides, since $\partial_\eta \tilde{\Lambda} < 0$, we have

$$\underline{\eta}(y) \leq \int_{\mathbb{R}^n} M(z) e^{\frac{\tilde{U}_\varepsilon^R(t,y+\varepsilon z) - \tilde{U}_\varepsilon^R(t,y)}{\epsilon}} \leq \overline{\eta}(y),$$

for all R large enough, $\varepsilon > 0$, $t \geq 0$, $y \in \mathbb{R}^n$. Thus, from the definition of $\tilde{\Lambda}$ in (5.46), we have

$$-\tilde{\Lambda} \left(y, \int_{\mathbb{R}^n} M(z) e^{\frac{\tilde{U}_\varepsilon^R(t,y+\varepsilon z) - \tilde{U}_\varepsilon^R(t,y)}{\varepsilon}} dz \right) = -\Lambda \left(y, \int_{\mathbb{R}^n} M(z) e^{\frac{\tilde{U}_\varepsilon^R(t,y+\varepsilon z) - \tilde{U}_\varepsilon^R(t,y)}{\varepsilon}} dz \right),$$

that is, $\tilde{U}_\varepsilon^R$ is a solution of (5.9). The proof is thereby achieved.

5.4.3 A priori Lipschitz estimate - proof of Proposition 5.5

We follow the same idea as for Lemma 5.21. However, there are some technical difficulties. First, we have to deal with a "source term" $\nabla_y \Lambda(y, \eta_\varepsilon(t,y))$ which is bounded by a constant $\frac{L}{l\underline{\eta}^2}$, using (5.23)-(5.24)-(5.29) and (5.31). In addition, we first need to prove the estimate on a truncated function, then to remove the truncation.

We fix $i \in \{1, \dots, n\}$, $T > 0$, and we set

$$W_\varepsilon(t, y) := \partial_{y_i} U_\varepsilon,$$

for all $t \in [0, T], \forall y \in \mathbb{R}^n$. Differentiating (5.9), we obtain

$$\begin{aligned}
&\partial_t W_\varepsilon(t, y) \\
&= -\partial_{y_i}\Lambda(y, \eta_\varepsilon) - \partial_\eta \Lambda(y, \eta_\varepsilon) \left(\int M(z) e^{\frac{U_\varepsilon(t,y+\varepsilon z) - U_\varepsilon(t,y)}{\varepsilon}} \left[\frac{W_\varepsilon(t, y + \varepsilon z) - W_\varepsilon(t, y)}{\varepsilon} \right] dz \right) \\
&:= \mathcal{F}(t, y, W_\varepsilon(t, \cdot)).
\end{aligned}$$

We formally define a truncated problem, for $R > 0$, and its solution W_ε^R satisfying

$$W_\varepsilon^R(t, y) = \phi_R \left(\partial_{y_i} u_\varepsilon^0(y) + \int_0^t \mathcal{F}(s, y, W_\varepsilon^R(s, \cdot)) ds \right), \tag{5.49}$$

where $\mathcal{F}$ is defined above and ϕ_R is a truncation function as in (5.47). We can prove existence and uniqueness of a global solution of (5.49) by a direct application of the Cauchy-Lipschitz theorem.

We set $\bar{W}_\varepsilon^R := W_\varepsilon^R - Ct$ with $C := \frac{L}{l\underline{\eta}^2}$. Our goal is to show

$$\forall t \in [0, T], \forall y \in \mathbb{R}^n, \quad \bar{W}_\varepsilon^R(t, y) \leq \sup \partial_{y_i} u_\varepsilon^0. \tag{5.50}$$

By contradiction, assume (5.50) does not hold, i.e., there exist $y_0 \in \mathbb{R}^n, t_0 \in [0, T]$ such that

$$\bar{W}_\varepsilon^R(t_0, y_0) - \sup \partial_{y_i} u_\varepsilon^0 > 0. \tag{5.51}$$

For $\beta > 0$, $\alpha > 0$ small enough, $t \in [0, T], y \in \mathbb{R}^n$ we introduce

$$\varphi_{\alpha,\beta}(t, y) := \bar{W}_\varepsilon^R(t, y) - \alpha t - \beta |y - y_0|.$$

As $\bar{W}_\varepsilon^R$ is bounded, $\varphi_{\alpha,\beta}$ reaches its maximum on $[0, T] \times \mathbb{R}^n$ at a point $(\bar{t}, \bar{y})$. We have

$$\forall z \in \mathbb{R}^n, \; \varphi_{\alpha,\beta}(\bar{t}, \bar{y} + \varepsilon z) \leq \varphi_{\alpha,\beta}(\bar{t}, \bar{y}).$$

Then, we obtain the inequality

$$\forall z \in \mathbb{R}^n, \; \frac{\bar{W}_\varepsilon^R(\bar{t}, \bar{y} + \varepsilon z) - \bar{W}_\varepsilon^R(\bar{t}, \bar{y})}{\varepsilon} \leq \beta \frac{|\bar{y} + z - y_0| - |\bar{y} - y_0|}{\varepsilon} \leq \beta |z|.$$

We choose α small enough so that $\varphi_{\alpha,\beta}(t_0, y_0) > \varphi_{\alpha,\beta}(0, y_0) = \partial_{y_i} u_\varepsilon^0(y_0)$, which is possible thanks to (5.51). It implies $\bar{t} > 0$. Hence $\partial_t \varphi_{\alpha,\beta}(\bar{t}, \bar{y}) \geq 0$, i.e. $\partial_t \bar{W}^R(\bar{t}, \bar{y}) \geq \alpha$ (if $\bar{t} = T$, then ∂_t stands for the left derivative). Differentiating (5.49) at $(\bar{t}, \bar{y})$, we have

$$\begin{aligned}
\alpha &\leq \partial_t \bar{W}_\varepsilon^R(\bar{t}, \bar{y}) \\
&\leq -\sup \phi_R' \times \partial_{y_i}\Lambda(y, \eta_\varepsilon(t, y)) - C \\
&\quad + \sup \phi_R' \times (-\partial_\eta \Lambda(y, \eta)) \int M(z) e^{\frac{U_\varepsilon(\bar{t},\bar{y}+\varepsilon z) - U(\bar{t},\bar{y})}{\varepsilon}} \left[\frac{W_\varepsilon^R(\bar{t}, \bar{y} + \varepsilon z) - W_\varepsilon^R(\bar{t}, \bar{y})}{\varepsilon} \right] dz.
\end{aligned}$$

Now, from $|\partial_{y_i}\Lambda(y,\eta_\varepsilon(t,y))| \leq \frac{L}{l\underline{\eta}^2} = C$ and $0 \leq -\partial_\eta\Lambda(y,\eta) \leq \frac{L}{\underline{\eta}^2}$, we have

$$\begin{aligned}\alpha &\leq \frac{L}{\underline{\eta}^2}\left(\int M(z)e^{\frac{U_\varepsilon(\bar{t},\bar{y}+\varepsilon z)-U(\bar{t},\bar{y})}{\varepsilon}}|z|\mathrm{d}z\right)\times\beta \\ &\leq \frac{L}{\underline{\eta}^2}\left(\int M(z)e^{\frac{-2\Lambda T}{\varepsilon}+k_0|z|}|z|\mathrm{d}z\right)\times\beta.\end{aligned}$$

Then, passing to the limit $\beta \to 0$ we obtain $\alpha \leq 0$: contradiction. Thus, we have

$$\bar{W}_\varepsilon^R \leq \sup|\partial_{y_i}u_\varepsilon^0| = k^0.$$

We proceed similarily to obtain the reverse inequality $\bar{W}_\varepsilon^R \geq k^0$. We have, for all $R>0,\ \varepsilon>0,\ t\in[0,T],\ y\in\mathbb{R}^n$

$$|W_\varepsilon^R(t,y)| \leq k^0 + Ct.$$

Finally, the bound on W_ε^R is uniform in R so we can remove the truncation, as detailed in section 5.4.2. Thus, $W_\varepsilon^R = W_\varepsilon$ for R large enough and

$$|\partial_{y_i}U_\varepsilon(t,y)| \leq k^0 + Ct.$$

5.4.4 Semi convexity - proof of Proposition 5.6

Note that in the case of a convex Hamiltonian, the semi convexity of the solution is classical. Here, we have to deal with a nonlocal operator, and the method needs to be adjusted.

The proof is divided into several lemmas. We first focus on the proof that $\partial_t^2 U_\varepsilon$ is bounded from below, which is more straightforward, and then explain how to adapt the method for $\partial_{y_i}^2 U_\varepsilon$. We also prove Corollary 5.7 at the end of this section.

Semiconvexity in t

We set $V_\varepsilon := \partial_t^2 U_\varepsilon$. For simplicity, we also define the notation

$$J_\varepsilon(t,y,z) := M(z)e^{\frac{U_\varepsilon(t,y+\varepsilon z)-U_\varepsilon(t,y)}{\varepsilon}}.$$

Differentiating (5.9) twice, we find

$$\begin{aligned}\partial_t V_\varepsilon = &-\partial_\eta\Lambda\int_{\mathbb{R}^n}\frac{V_\varepsilon(t,y+\varepsilon z)-V_\varepsilon(t,y)}{\varepsilon}J_\varepsilon dz \\ &-\partial_\eta^2\Lambda\left(\partial_t\eta_\varepsilon\right)^2 - \partial_\eta\Lambda\int_{\mathbb{R}^n}\left(\frac{\partial_t U_\varepsilon(t,y+\varepsilon z)-\partial_t U_\varepsilon(t,y)}{\varepsilon}\right)^2 J_\varepsilon dz.\end{aligned} \tag{5.52}$$

The first result states somehow that the Hamiltonian is convex.

Lemma 5.22 *For all $y\in\mathbb{R}^n$, $\eta\in(\underline{\eta}(y),\overline{\eta}(y))$, we have*

$$\partial_\eta^2\Lambda(y,\eta) + \frac{\partial_\eta\Lambda(y,\eta)}{\eta} \leq 0.$$

Proof We simply write $\Lambda := \Lambda(y,\eta)$ and $F := F(y,\Lambda(y,\eta))$ (where F is defined in (5.15)). Differentiationg (5.29) with respect to η, we find

$$\partial_\lambda^2 F(\partial_\eta \Lambda)^2 + \partial_\lambda F \partial_\eta^2 \Lambda = \frac{2}{\eta^3}.$$

Then

$$\partial_\eta \Lambda = -\frac{1}{\partial_\lambda F \eta^2}, \quad \partial_\eta^2 \Lambda = \frac{2}{\partial_\lambda F \eta^3} - \frac{1}{\eta^4 (\partial_\lambda F)^3} \partial_\lambda^2 F.$$

We deduce

$$\partial_\eta^2 \Lambda + \frac{\partial_\eta \Lambda}{\eta} = \frac{1}{(\partial_\lambda F)^3 \eta^3} \left((\partial_\lambda F)^2 - \frac{\partial_\lambda^2 F}{\eta} \right)$$

and from (5.26),

$$\partial_\eta^2 \Lambda + \frac{\partial_\eta \Lambda}{\eta} = \frac{1}{(\partial_\lambda F)^3 \eta^3} \left((\partial_\lambda F)^2 - F \partial_\lambda^2 F \right).$$

Using the Cauchy-Schwarz inequality and the definition of F (5.15), we have

$$(\partial_\lambda F)^2 - F \partial_\lambda^2 F \leq 0$$

and the proof is achieved. ∎

The following result states the somehow convexity of $U_\varepsilon \mapsto \eta_\varepsilon$.

Lemma 5.23 *For all $\varepsilon > 0$, $t > 0$, $y \in \mathbb{R}^n$, we have*

$$(\partial_t \eta_\varepsilon)^2 \leq \eta_\varepsilon \int_{\mathbb{R}^n} \left(\frac{\partial_t U_\varepsilon(t, y + \varepsilon z) - \partial_t U_\varepsilon(t,y)}{\varepsilon} \right)^2 J_\varepsilon dz.$$

Proof Use the Cauchy-Schwarz inequality and the definition of η_ε (5.30). ∎

Combining the two previous lemmas, we have

$$\partial_\eta^2 \Lambda(y,\eta_\varepsilon) \left(\partial_t \eta_\varepsilon\right)^2 + \partial_\eta \Lambda(y,\eta_\varepsilon) \int_{\mathbb{R}^n} \left(\frac{\partial_t U_\varepsilon(t, y + \varepsilon z) - \partial_t U_\varepsilon(t,y)}{\varepsilon} \right)^2 J_\varepsilon dz \leq 0,$$

for all $\varepsilon > 0$, $t > 0$, $y \in \mathbb{R}^n$. Using the above inequality and (5.52), we find

$$\partial_t V_\varepsilon \geq -\partial_\eta \Lambda(y,\eta_\varepsilon) \int_{\mathbb{R}^n} \frac{V_\varepsilon(t, y + \varepsilon z) - V_\varepsilon(t,y)}{\varepsilon} J_\varepsilon dz. \tag{5.53}$$

From this inequation, we deduce

Lemma 5.24 *V_ε is bounded from below, uniformly in $\varepsilon > 0$, $t \geq 0$, $y \in \mathbb{R}^n$.*

The proof follows closely the method of section 5.4.3. The formal idea is the following. If, for some $t > 0$, $V_\varepsilon(t,\cdot)$ reaches its minimum at $\bar{y} \in \mathbb{R}^n$, from (5.53) we obtain $\partial_t V_\varepsilon(t,\bar{y}) \geq 0$. Formally, it shows that the minimum value of V_ε is increasing with time, that is, $\inf_y V_\varepsilon(t,y) \geq \inf_y V_\varepsilon(0,y)$. Then, we conclude with the fact that $\inf_y V_\varepsilon(0,y)$ is bounded, uniformly in $\varepsilon > 0$.

Proof Our goal is to show

$$V_\varepsilon(t,y) \geq \inf_{y\in\mathbb{R}^n} V_\varepsilon^0(y), \quad \forall \varepsilon > 0,\ \forall t > 0,\ \forall y \in \mathbb{R}^n, \tag{5.54}$$

for $V_\varepsilon^0(y) := V_\varepsilon(t=0,y)$. Differentiating (5.32) in t, we obtain

$$V_\varepsilon(t,y) = -\partial_\eta \Lambda(y,\eta_\varepsilon) \int_{\mathbb{R}^n} \frac{\Lambda_\varepsilon(t,y+\varepsilon z) - \Lambda_\varepsilon(t,y)}{\varepsilon} J_\varepsilon dz. \tag{5.55}$$

In particular, our assumptions imply $\inf_{y\in\mathbb{R}^n} V_\varepsilon^0 > -\infty$ uniformly in $\varepsilon > 0$, thus (5.54) allows us to conclude that V_ε is bounded from below, uniformly in ε.

We prove (5.54) by contradiction. We assume that there exists $(T, y_0) \in (0,+\infty) \times \mathbb{R}^n$ such that

$$V_\varepsilon(T,y_0) - \inf_{y\in\mathbb{R}^n} V_\varepsilon^0(y) < 0. \tag{5.56}$$

For $\beta > 0$, $\alpha > 0$ small and for $t \in [0,T], y \in \mathbb{R}^n$, we also introduce

$$\varphi_{\alpha,\beta}(t,y) := V_\varepsilon(t,y) + \alpha t + \beta |y - y_0|.$$

From (5.55) and for a fixed $\varepsilon > 0$, we have $V_\varepsilon(t,y)$ is bounded from below uniformly in $t \in [0,T]$, $y \in \mathbb{R}^n$. Therefore, $\varphi_{\alpha,\beta}$ goes to $+\infty$ as $|y| \to +\infty$ and reaches its minimum on $[0,T] \times \mathbb{R}^n$ at a point $(\bar t, \bar y)$. We have

$$\varphi_{\alpha,\beta}(\bar t, \bar y + \varepsilon z) \geq \varphi_{\alpha,\beta}(\bar t, \bar y), \quad \forall z \in \mathbb{R}^n,$$

thus

$$\frac{V_\varepsilon(\bar t, \bar y + \varepsilon z) - V_\varepsilon(\bar t, \bar y)}{\varepsilon} \geq \beta \frac{|\bar y - y_0| - |\bar y - y_0 + \varepsilon z|}{\varepsilon} \geq -\beta |z|, \quad \forall z \in \mathbb{R}^n. \tag{5.57}$$

We choose α small enough to ensure $\varphi_{\alpha,\beta}(T,y_0) < \varphi_{\alpha,\beta}(0,y_0)$, which is possible thanks to assumption (5.56). It implies $\bar t > 0$. Hence $\partial_t \varphi_{\alpha,\beta}(\bar t, \bar y) \leq 0$, that is $\partial_t V_\varepsilon(\bar t, \bar y) \leq -\alpha$ (if $\bar t = T$ then $\partial_t V_\varepsilon^R(\bar t, \bar y)$ stands for the left-derivative). From (5.53) at $(\bar t, \bar y)$, using (5.57),

$$\begin{aligned}
-\alpha \geq \partial_t V_\varepsilon(\bar t, \bar y) &= -\partial_\eta \Lambda \int_{\mathbb{R}^n} M(z) e^{\frac{U_\varepsilon(t,y+\varepsilon z) - U_\varepsilon(t,y)}{\varepsilon}} \frac{V_\varepsilon(t,y+\varepsilon z) - V_\varepsilon(t,y)}{\varepsilon} dz \\
&\geq \beta \inf\left[-\partial_\eta \Lambda\right] \int M(z) e^{\frac{U_\varepsilon(\bar t,\bar y+\varepsilon z) - U_\varepsilon(\bar t,\bar y)}{\varepsilon}} |z| dz \\
&\geq \beta \frac{1}{L\overline{\eta}} \int M(z) e^{\left(k_0 + \frac{L}{l\underline{\eta}^2} T\right)|z|} |z| dz
\end{aligned}$$

where, in the last step, we used Proposition 5.5 and $-\partial_\eta \Lambda \geq \frac{1}{L\overline{\eta}}$. As β goes to 0, we obtain $\alpha \leq 0$, which is absurd. The proof is thereby achieved. ∎

Semi convexity in y

Let us show how the method can be adapted to prove that $\partial_{y_i}^2 U_\varepsilon$ is bounded from below. We set $W_\varepsilon := \partial_{y_i}^2 U_\varepsilon$. Differentiating (5.9) twice, we find

$$\begin{aligned}
\partial_t W_\varepsilon = &- \partial_\eta \Lambda \int_{\mathbb{R}^n} \frac{W_\varepsilon(t,y+\varepsilon z) - W_\varepsilon(t,y)}{\varepsilon} J_\varepsilon dz \\
&- \partial_\eta^2 \Lambda \left(\partial_{y_i} \eta_\varepsilon\right)^2 - \partial_\eta \Lambda \int_{\mathbb{R}^n} \left(\frac{\partial_{y_i} U_\varepsilon(t,y+\varepsilon z) - \partial_{y_i} U_\varepsilon(t,y)}{\varepsilon}\right)^2 J_\varepsilon dz \\
&- \partial_{y_i}^2 \Lambda - 2\partial_{y_i,\eta}^2 \Lambda \partial_{y_i} \eta_\varepsilon.
\end{aligned} \tag{5.58}$$

In contrast with the equation on $\partial_t^2 U_\varepsilon$ (5.52), we have to deal with a source term and a linear term in the last line.

For any constant $K > 0$, Young's inequality implies

$$-2\partial^2_{y_i,\eta}\Lambda\partial_{y_i}\eta_\varepsilon \geq -K^2|\partial^2_{y_i,\eta}\Lambda|^2 - \frac{1}{K^2}(\partial_{y_i}\eta_\varepsilon)^2.$$

Applying this inequality and Lemma 5.23 in (5.58), we obtain

$$\begin{aligned}\partial_t W_\varepsilon \geq &- \partial_\eta\Lambda \int_{\mathbb{R}^n} \frac{W_\varepsilon(t,y+\varepsilon z) - W_\varepsilon(t,y)}{\varepsilon} J_\varepsilon dz \\ &- \left(\partial^2_\eta\Lambda + \frac{\partial_\eta\Lambda}{\eta_\varepsilon} + \frac{1}{K^2}\right)(\partial_{y_i}\eta_\varepsilon)^2 \\ &- \partial^2_{y_i}\Lambda - K^2|\partial^2_{y_i,\eta}\Lambda|^2.\end{aligned}$$

Now, let us state a strong version of Lemma 5.22.

Lemma 5.25 *There exists $\beta > 0$ such that, for all $y \in \mathbb{R}^n$, $\eta \in (\underline{\eta}(y), \overline{\eta}(y))$, we have*

$$\partial^2_\eta\Lambda(y,\eta) + \frac{\partial_\eta\Lambda(y,\eta)}{\eta} \leq -\beta.$$

The proof is postponned at the end of the section.

Using the above lemma, and choosing $K \geq \frac{1}{\sqrt{\beta}}$, we have

$$\begin{aligned}\partial_t W_\varepsilon \geq &- \partial_\eta\Lambda \int_{\mathbb{R}^n} \frac{W_\varepsilon(t,y+\varepsilon z) - W_\varepsilon(t,y)}{\varepsilon} J_\varepsilon dz \\ &- \partial^2_{y_i}\Lambda - K^2|\partial^2_{y_i,\eta}\Lambda|^2.\end{aligned}$$

From assumptions (5.24)-(5.25), the source term $-\partial^2_{y_i}\Lambda - K^2|\partial^2_{y_i,\eta}\Lambda|^2$ is bounded from below by some constant $-K' < 0$. We end up with

$$\partial_t W_\varepsilon \geq -\partial_\eta\Lambda \int_{\mathbb{R}^n} \frac{W_\varepsilon(t,y+\varepsilon z) - W_\varepsilon(t,y)}{\varepsilon} J_\varepsilon dz - K'.$$

Then, applying the same method as in the proof of Lemma 5.24 (see also the proof of Proposition 5.3), we show

$$W_\varepsilon(t,y) \geq \inf_{y\in\mathbb{R}^n} W_\varepsilon(t=0,y) - K't, \quad \forall\varepsilon > 0,\ t \geq 0,\ y \in \mathbb{R}^n.$$

Finally, since $W_\varepsilon(t=0,y) \geq -C$ (from assumption (5.17)), we have

$$W_\varepsilon(t,y) \geq -C - K't, \quad \forall\varepsilon > 0,\ t \geq 0,\ y \in \mathbb{R}^n,$$

which achieves the proof.

Proof (of Lemma 5.25) From the proof of Lemma 5.22, we have

$$\partial^2_\eta\Lambda + \frac{\partial_\eta\Lambda}{\eta} = \frac{1}{(\partial_\lambda F)^3\eta^3}\left((\partial_\lambda F)^2 - F\partial^2_\lambda F\right).$$

Since $\frac{1}{(\partial_\lambda F)^3\eta^3} \geq \frac{1}{L^3\overline{\eta}^3}$, our goal to show that

$$(\partial_\lambda F)^2 - F\partial^2_\lambda F \leq -\beta,$$

for all $y \in \mathbb{R}^n$, $\eta \in (\underline{\eta}(y), \overline{\eta}(y))$.

We set, for all $x \geq 0$, $y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$,

$$f(x, y, \lambda) := \frac{b(x, y)}{A(x, y)} \exp\left(\int_0^x \frac{\lambda - d(x', y)}{A(x', y)} dx'\right).$$

According to (5.15), we have $F(y, \lambda) = \int_{\mathbb{R}_+} f(x, y, \lambda) dx$. We also define the probability measure $\tilde{f}(x, y, \lambda) := \frac{f(x,y,\lambda)}{F(y,\lambda)}$. Now, setting $\mathcal{A}(x, y) := \int_0^x \frac{1}{A(x',y)} dx'$, we have

$$\partial_\lambda F(y, \lambda) = \int_{\mathbb{R}_+} \mathcal{A}(x, y) f(x, y, \lambda) dx, \quad \partial_\lambda^2 F(y, \lambda) = \int_{\mathbb{R}_+} \mathcal{A}(x, y)^2 f(x, y, \lambda) dx.$$

It gives,

$$(\partial_\lambda F)^2 - F \partial_\lambda^2 F = -F^2 \int_{\mathbb{R}_+} \left(\mathcal{A}(x, y) - \int_{\mathbb{R}_+} \mathcal{A} \tilde{f}\right)^2 \tilde{f}$$

(the integrals are on x and the function are evaluated on y and $\lambda = \Lambda(y, \eta)$). We have $F^2 \leq \frac{1}{\underline{\eta}^2}$, and we let the reader be convinced that assumption (5.11) implies that the above term is negative uniformly in $y \in \mathbb{R}^n$, $\eta \in (\underline{\eta}(y), \overline{\eta}(y))$. ■

Proof of Corollary 5.7

We recall that, from Proposition 5.3 and Proposition 5.5, U_ε is Lipschitz continuous, uniformly in $\varepsilon > 0$, $y \in \mathbb{R}^n$, locally uniformly in t. Besides, from Proposition 5.6 and for a fixed $T > 0$, there exists a constant $K > 0$ such that $(\partial_t^2 U_\varepsilon(t, y))^- \leq K$ for all $(t, y) \in [0, T] \times \mathbb{R}^n$ (where $^-$ denotes the negative part). We deduce

$$\begin{aligned}\int_0^T |\partial_t^2 U_\varepsilon(t, y)| dt = \int_0^T \partial_t^2 U_\varepsilon(t, y) dt + 2 \int_0^T (\partial_t^2 U_\varepsilon(t, y))^- dt &\leq \partial_t U_\varepsilon(t, y) + 2Kt \\ &\leq -\underline{\Lambda} + 2Kt,\end{aligned}$$

which proves the claim. With the same method, we show $\partial_{y_i}^2 U_\varepsilon \in L^1_{loc}$.

5.4.5 Asymptotics of U_ε - proof of Proposition 5.9

Extraction of a subsequence

From the a priori estimate of Proposition 5.3 and Ascoli's theorem, we know that U_ε converges locally uniformly to some U, up to extraction of a subsequence. Incidentally, this convergence also occurs in $W^{1,1}$, from the $W^{2,1}$ estimate in Corollary 5.7 and a standard compact embedding. In addition, we know from Proposition 5.3 and Proposition 5.5 that U is locally Lipschitz continuous:

$$|U(t, y) - U(t', y')| \leq -\underline{\Lambda}(t - t') + \left(k^0 + \frac{L}{l\underline{\eta}^2} t\right) |y - y'|,$$

for all $t \geq t' \geq 0$ and $y, y' \in \mathbb{R}^n$. We also mention that Proposition 5.6 implies that U is semiconvex, uniformly in $y \in \mathbb{R}^n$, locally uniformly in t.

Viscosity solution

We are going to show that U is a viscosity solution of (5.33), i.e U satisfies

$$\partial_t U = H(y, \nabla_y U), \quad U(0,y) = u^0(y), \tag{5.59}$$

with

$$H(y,p) := -\Lambda\left(y, \int_{\mathbb{R}^n} M(z)e^{p\cdot z}dz\right).$$

The proof is adapted from classical stability results for viscosity solutions of Hamilton-Jacobi equations (see [31]). However, this case is not completely standard because of the nonlocal term $\int_{\mathbb{R}^n} M(z)e^{\frac{U_\varepsilon(t,y+\varepsilon z)-U_\varepsilon(t,y)}{\varepsilon}}dz$.

Lemma 5.26 *The function U is a viscosity solution of* (5.59) *in* $(0,\infty)\times\mathbb{R}^n$*. Also, for all $T>0$, the viscosity inequalities stand for $t\in(0,T]$.*

Proof We are going to prove that U is a subsolution of (5.59). Let us consider a test function φ and a point (t_0,y_0) such that $U-\varphi$ reaches a global maximum at (t_0,y_0). From classical results, there exists $(t_\varepsilon,y_\varepsilon)$ such that

$$\begin{cases} (t_\varepsilon,y_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} (t_0,y_0), \\ \max\limits_{t,y} U_\varepsilon - \varphi = (U_\varepsilon-\varphi)(t_\varepsilon,y_\varepsilon). \end{cases}$$

For all $z\in\mathbb{R}^n$, $\varphi(t_\varepsilon,y_\varepsilon+\varepsilon z)-U_\varepsilon(t_\varepsilon,y_\varepsilon+\varepsilon z)\geq\varphi(t_\varepsilon,y_\varepsilon)-U_\varepsilon(t_\varepsilon,y_\varepsilon)$, thus we have

$$\frac{\varphi(t_\varepsilon,y_\varepsilon+\varepsilon z)-\varphi(t_\varepsilon,y_\varepsilon)}{\varepsilon} \geq \frac{U_\varepsilon(t_\varepsilon,y_\varepsilon+\varepsilon z)-U_\varepsilon(t_\varepsilon,y_\varepsilon)}{\varepsilon}.$$

Since $\partial_\eta\Lambda<0$, equation (5.59) gives

$$\begin{aligned}\partial_t\varphi(t_\varepsilon,y_\varepsilon) &= -\Lambda\left(y_\varepsilon, \int_{\mathbb{R}^n} M(z)e^{\frac{U_\varepsilon(t_\varepsilon,y_\varepsilon+\varepsilon z)-U_\varepsilon(t_\varepsilon,y_\varepsilon)}{\varepsilon}}dz\right) \\ &\leq -\Lambda\left(y_\varepsilon, \int_{\mathbb{R}^n} M(z)e^{\frac{\varphi(t_\varepsilon,y_\varepsilon+\varepsilon z)-\varphi(t_\varepsilon,y_\varepsilon)}{\varepsilon}}dz\right).\end{aligned}$$

As ε goes to 0,

$$\partial_t\varphi(t_0,y_0) \leq -\Lambda\left(y_0, \int_{\mathbb{R}^n} M(z)e^{\nabla_y\varphi(t_0,y_0)\cdot z}\right) = H(y_0,\nabla_y\varphi(t_0,y_0)),$$

then U is a viscosity subsolution of (5.59). With the same method, we prove that U is also a viscosity supersolution. It completes the first part of the proof. The second part of the statement is a well-known result, and proof can be found in [31]. ∎

Uniqueness

We point out that the Hamiltonian H is Lipschitz continuous in the y variable. We introduce a truncated Hamiltonian

$$\tilde{H}(y,p) := \begin{cases} H(y,p) & \text{if } \int_{\mathbb{R}^n} M(z)e^{p\cdot z}dz \in [\underline{\eta},\overline{\eta}], \\ 0 & \text{otherwise.} \end{cases}$$

Since $\underline{\eta}\leq\eta_\varepsilon(t,y)\leq\overline{\eta}$ (from (5.31)), we have

$$\partial_t U = \tilde{H}(y,\nabla_y U).$$

For this equation, a classical uniqueness result is in order (see e.g [31, 243]). We deduce that U_ε converges to U for the whole sequence $\varepsilon\to 0$ (and not for an extracted subsequence).

5.4.6 A posteriori Lipschitz estimate - proof of Proposition 5.11

From Proposition 5.3 and Proposition 5.5, we know that U is Lipschitz, globally in t and locally in y. Our goal is to show that U is globally Lipschitz continuous, i.e., that there exists a constant $C > 0$ such that

$$\forall t \geq 0,\ \forall (y, y') \in (\mathbb{R}^n)^2, \quad U(t, y) - U(t, y') \leq C|y - y'|. \tag{5.60}$$

Let us fix $t \geq 0$ and $(y, y') \in (\mathbb{R}^n)^2$.

With η_ε defined in (5.30) and the bound $\eta_\varepsilon \leq \overline{\eta}$ from (5.31), we have, for all $\varepsilon > 0$, $z \in \mathbb{R}^n$

$$\int_{\mathbb{R}^n} M(z) e^{\frac{U_\varepsilon(t,y+\varepsilon z) - U(t,y)}{\varepsilon}} dz \leq \eta_\varepsilon(t, y) \leq \overline{\eta}.$$

From the assumption that $M(\cdot)$ is not degenerate (5.12), we deduce that, for some $r_0 > 0$, and for all $z \in \mathbb{R}^n$ such that $|z| = r_0$, then

$$\frac{U_\varepsilon(t, y + \varepsilon z) - U_\varepsilon(t, y)}{\varepsilon} \leq C$$

for some constant C (independant of ε, t, y and z). Then, chosing z and ε such that $y - y' = \varepsilon z$, we have $U_\varepsilon(t, y) - U_\varepsilon(t, y') \leq C|y - y'|$. As $\varepsilon \to 0$, we prove the goal (5.60).

Chapter 6

Horizontal Gene Transfer

In collaboration with Vincent Calvez, Susely Figueroa Iglesias, Hélène Hivert, Sylvie Méléard, and Anna Melnykova.

Horizontal Gene Transfer (HT) denotes the transmission of genetic material between two living organisms, while the vertical transmission refers to a DNA transfer from parents to their offspring. Consistent experimental evidence report that this phenomenon plays an essential role in the evolution of certain bacterias. In particular, HT is believed to be the main instrument for developing antibiotic resistance. In this work, we consider several models which describe this phenomenon: a stochastic jump process (individual-based) and the deterministic nonlinear integrodifferential equation obtained as a limit for large populations. We also consider a Hamilton-Jacobi equation, obtained as a limit of the deterministic model under the assumption of small mutations. The goal of this paper is to compare these models with the help of numerical simulations. More specifically, our goal is to understand to which extent the Hamilton-Jacobi model reproduces the qualitative behavior of the stochastic model, devoting a special attention to the phenomenon of evolutionary rescue.

Contents

6.1 Introduction

Accurate mathematical description of the evolutionary mechanism is an open question in biology, medicine, and industry. In particular, transmission of pathogens, or antibiotic resistance of bacteria is directly linked to the ability of the bacteria population to mutate and exchange genetic material either vertically (from parents to offspring), or horizontally (from the interaction between non-parental individuals).

Horizontal Gene Transfer was first described in bacteria when the antibiotic resistance was discovered. This resistance occurs when one bacterial cell becomes resistant to an antibiotic due to mutation, and then transfers resistance genes to other species of bacteria. However, the Horizontal Transfer of biologic information is not restricted to genes, but also occurs for the transfer of plasmids and endosymbionts [208, 215]. Some artificial applications of horizontal transfer include forms of genetic engineering (Gene Delivery) that result in an organism with its genes changed in some way, and, consequently, possessing new properties or functions (see for instance [188]). These applications are particularly useful for "Gene Therapy", which is an experimental procedure that may help treat or prevent genetic disorders and some types of cancer.

The primary goal of our work is to describe the mechanism of the transfer itself and how it affects the population dynamics. Since we will not focus on horizontal transfer's object, throughout the paper, we will refer to it as HT.

Our study starts with finding a good model of a bacteria population. Several mathematical models for describing a population dynamics were proposed in the literature. The first model we consider is a stochastic birth and death process (see, for reference, [56, 156]), which describes the dynamics of reproduction, competition, and exchange of genetic material between individuals in a population. A numerical parameter, called *trait*, describes the phenotype of each individual. Numerical experiments show that the effect of a unilateral horizontal gene transfer may lead to a cyclic behavior of the population. Roughly speaking, while HT drives individuals towards a non-fit phenotype — and, consequently, to extinction, very few not affected by transfer fit individuals may eventually repopulate the environment, before being driven again to deleterious phenotypes. This phenomenon is called an *evolutionary rescue of a small population*.

However, within a framework of stochastic jump processes, it is hard to define and study the observed cycling phenomena accurately. The second drawback of the stochastic system is that it is costly to compute, especially for large populations. In the case of a large population, it is more practical to work with a deterministic PDE model, obtained as a limit for a stochastic system [54, 55, 146]. In specific settings, the population dynamics involve concentration phenomena (i.e., the convergence of the population density to singular solutions, such as Dirac masses). In that case, the PDE formulation is not suitable. Applying a limiting procedure for small mutations and time rescaling to the PDE model, we pass to a Hamilton-Jacobi type equation.

The primary goal of our work is thus to conduct a numerical analysis of the population dynamics on a macroscopic individual-based model and to compare it with the deterministic system which is obtained as a limit for a large population. We are especially interested in determining to which extent the limiting Hamilton-Jacobi equation can grasp qualitative properties of the stochastic model. This framework has already been successfully used to understand the concentration phenomena, and the location of the dominant trait (see for instance [36,213]). We aim to understand if the Hamilton-Jacobi approach is also well suited to describe the evolutionary rescue phenomena which crucially rely on an accurate description of the small populations.

On this step, the choice of an approximation scheme for simulating solutions of the PDE model is of tremendous importance. As we further explain in Section 6.3, classical explicit schemes do not preserve the asymptotic behavior of the solution if the time rescaling step goes to 0. From a numerical point of view, it involves operations with exponentially large values, which lead to non-negligible errors for explicit numerical schemes. We address this question by proposing an asymptotic preserving scheme for a Hamilton-Jacobi equation, adapting an approach proposed in [103]. More generally, the numerical approximation problem for solutions of Hamilton-Jacobi equations is treated in [1].

This paper is structured as follows: in Section 6.2 we introduce the model both in a stochastic and deterministic setting, and formally derive the limiting Hamilton-Jacobi equation. Then, we simulate a jump process, describing the bacteria population, and study its properties for different values of parameters. Numerical experiments are gathered in Section 6.3. We aim to numerically determine the critical HT rate, which leads to an almost sure extinction of the whole population. On the next step, we conduct the same analysis for a Hamilton-Jacobi equation with the help of an asymptotic preserving scheme and compare it with the stochastic model on an appropriate timescale, and explain why the classical scheme fails to work. We end our study with conclusions and discussion of yet unsolved numerical and theoretical questions.

6.2 Model

6.2.1 Stochastic model

We consider a stochastic model describing the evolution of a population structured by phenotype, which is described at each time t by the point measure

$$\nu_t^K(dx) = \frac{1}{K} \sum_{i=0}^{N_t^K} \delta_{X_i(t)}(dx), \tag{6.1}$$

where K is a scale parameter, referred to as the *carrying capacity*. It stands for the maximal number of individuals that the underlying environment is able to host (K can represent, for example, the amount of available resources). $N_t^K = K \int \nu_t^K(dx)$ is the size of the population at time t, and $X_i(t) \in \mathbb{R}^d$ is the trait of i-th individual living at t, which summarizes all the informations on phenotype.

The demography of the population is regulated, first of all, by birth and death. An individual with trait x gives birth to a new individual with rate $b(x)$. The trait y of the offspring is chosen from a probability distribution $m(x-y)dy$, referred to as the mutation kernel. An individual with trait x dies according to an intrinsic death rate $d(x)$ plus an additional death rate $C\dfrac{N_t^K}{K}$ (independent of x) which stands for the competition between individuals.

Finally, an individual with trait x induces a *unilateral* Horizontal Transfer to an individual with trait y at rate $h_K(x, y, \nu)$, such that the pair (x, y) becomes

(x, x). This kind of transfer is sometimes referred to as *conjugation* in the biological literature. For simplicity, we assume $h_K(x, y, \nu)$ to be in the particular form

$$h_K(x, y, \nu) = h_K(x - y, N) = \tau_0 \frac{\alpha(x - y)}{N/K}, \tag{6.2}$$

where $N = K \int_{\mathbb{R}^d} \nu(dx)$ is the number of individuals, $\tau_0 > 0$ is a constant and α is either a Heaviside, or a smooth bounded function, such that for a small $\delta > 0$:

$$\alpha(z) = \begin{cases} 0 & \text{if } z < -\delta \\ 1 & \text{if } z > +\delta \end{cases}, \quad \alpha'(0) = \frac{1}{2\delta}, \tag{6.3}$$

where δ is the stiffness parameter. We introduce δ to have the advantage of working with smooth function (which will be useful in the following parts), while mimicking the binary nature of the Heaviside function.

For a population $\nu = \frac{1}{K} \sum_{i=1}^{N} \delta_{x_i}$ and a generic measurable bounded function F, the generator of the process is then given by:

$$\begin{aligned} L^K F(\nu) = \sum_{i=1}^{N} b(x_i) \int_{\mathbb{R}^d} \left(F\left(\nu + \frac{1}{K}\delta_y\right) - F(\nu)\right) m(x_i, dy) \\ + \sum_{i=1}^{N} \left(d(x_i) + C\frac{N}{K}\right) \left(F\left(\nu - \frac{1}{K}\delta_{x_i}\right) - F(\nu)\right) \\ + \sum_{i,j=1}^{N} h_K(x_i, x_j, \nu) \left(F\left(\nu + \frac{1}{K}\delta_{x_i} - \frac{1}{K}\delta_{x_j}\right) - F(\nu)\right). \end{aligned}$$

It is standard to construct the measure-valued process ν^K as the solution of a stochastic differential equation driven by Poisson point measures and to derive moment and martingale properties (see for instance [156]).

6.2.2 The PDE model

It is proven (see, in particular [54, 92]) that for $K \to +\infty$ the stochastic process defined by a sequence of point measures given by (6.1) converges in probability to a non-linear integro-differential equation, whose solution exists and is unique. This equation is given by: for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^d$,

$$\begin{cases} \partial_t f(t, x) = -(d(x) + C\rho_1(t)) f(t, x) + \int_{\mathbb{R}^d} m(x - y) b(y) f(t, y) dy \\ \qquad\qquad + f(t, x) \int_{\mathbb{R}^d} \tau(x - y) \frac{f(t, y)}{\rho_1(t)} dy, \\ \rho_1(t) = \int_{\mathbb{R}^d} f(t, x) dx, \\ f(0, x) = f^0(x) > 0, \end{cases}$$

where $f(t, x)$ is the macroscopic density of the population with trait x at time t and, accordingly to the previous section, $b(x)$, $d(x)$ and C are the birth, death and competition rate respectively, m is the mutation kernel, and

$$\tau(y - x) := \tau_0 \left[\alpha(x - y) - \alpha(y - x)\right] \tag{6.4}$$

is the horizontal transfer flux.

Now, our goal is to pass from micro- to macroscopic scale using a rescaling. On the one hand, we consider the case of small mutations: for a small parameter $\varepsilon > 0$ we define

$$m_\varepsilon(x-y) = \frac{1}{\varepsilon^d} m\left(\frac{x-y}{\varepsilon}\right).$$

With a change of variable $z = \frac{x-y}{\varepsilon}$ we can rewrite the mutation term at (t,x) as

$$\int_{\mathbb{R}^d} m_\varepsilon(x-y) b(y) f(t,y) dy = \int_{\mathbb{R}^d} m(z) b(x+\varepsilon z) f(t, x+\varepsilon z) dz.$$

On the other hand, when ε is small, the effect of mutations can only be observed in a larger time scale. Thus, we rescale time with $t \mapsto \frac{t}{\varepsilon}$.

We end up with the following system, for $\varepsilon > 0$, and $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$:

$$\begin{cases} \varepsilon \partial_t f_\varepsilon(t,x) = -(d(x) + C\rho_\varepsilon(t)) f_\varepsilon(t,x) + \displaystyle\int_{\mathbb{R}^d} m(z) b(x+\varepsilon z) f_\varepsilon(t, x+\varepsilon z) dz \\ \qquad\qquad + f_\varepsilon(t,x) \displaystyle\int_{\mathbb{R}^d} \tau(x-y) \frac{f_\varepsilon(t,y)}{\rho_\varepsilon(t)} dy, \\ \rho_\varepsilon(t) = \displaystyle\int_{\mathbb{R}^d} f_\varepsilon(t,x) dx, \\ f_\varepsilon(0,x) = f_\varepsilon^0(x) > 0. \end{cases} \tag{6.5}$$

6.2.3 The Hamilton-Jacobi limit

We now derive the limiting problem (6.5) when $\varepsilon \to 0$. As we will see, the limiting problem allows us to give a rigorous mathematical framework and to perform useful formal calculations.

Equations in the form of (6.5) often give rise to concentration phenomena, i.e., the convergence of f_ε towards a Dirac mass when $\varepsilon \to 0$ (see [36, 125]). The usual way to deal with these asymptotics is to perform a Hopf-Cole transformation (or WKB ansatz), i.e., to consider

$$u_\varepsilon(t,x) := \varepsilon \ln(f_\varepsilon(t,x)). \tag{6.6}$$

This change of variable comes from the intuition that f_ε should behave like a Gaussian of variance ε, when $\varepsilon \to 0$. Accordingly, we expect u_ε to have a nonsingular limit when $\varepsilon \to 0$. Incidentally, this substitution also gives insights on the convenient scheme to use for numerical simulations, as we will see in the following section.

Now, let us explain how to identify and derive some properties about the asymptotics of u_ε when $\varepsilon \to 0$. The following computations are only formal since rigorous proofs are often intricate in this context. Substituting (6.6) into (6.5) we deduce that u_ε satisfies

$$\begin{aligned} \partial_t u_\varepsilon = &-(d(x) + C\rho_\varepsilon(t)) + \int_{\mathbb{R}^d} m(z) b(x+\varepsilon z) e^{\frac{u_\varepsilon(t,x+\varepsilon z) - u_\varepsilon(t,x)}{\varepsilon}} dz \\ &+ \int_{\mathbb{R}^d} \tau(x-y) \frac{f_\varepsilon(t,y)}{\rho_\varepsilon(t)} dy. \end{aligned} \tag{6.7}$$

Formally, at the limit $\varepsilon \to 0$, u_ε converges to a continuous function u which satisfies the following Hamilton-Jacobi equation in the *viscosity* sense:

$$\partial_t u = -(d(x) + C\rho(t)) + b(x) \int_{\mathbb{R}^d} m(z) e^{z \cdot \nabla_x u} dz + \tau(x - \overline{x}(t)), \tag{6.8}$$

where $\rho(t) \geq 0$ is the weak limit of $\rho_\varepsilon(t)$ and

$$\bar{x}(t) = \text{argmax } u(t, \cdot). \tag{6.9}$$

We formally assume here and in the following that the definition of $\bar{x}(t)$ is unambiguous, i.e that u reaches its maximum on a single point. Note that the limiting function u is not expected to be C^1 for all time. We thus need to deal with a generalized notion of solutions, namely *viscosity solution* (see [31]).

This framework is convenient because most of the information is contained in the dynamics of $\bar{x}(t)$. See the next section for further formal analysis.

6.2.4 Formal analysis on the Hamilton-Jacobi equation

Hamilton-Jacobi equations are particularly known in mathematical biology to be a good model to describe how a population concentrates around the dominant trait(s) when the mutations are small. However, here we are interested in using this model to describe a phenomenon of *evolutionary rescue*, which relies on the presence of individuals with non-dominant phenotypes. In this section, we perform a formal analysis of the equation. We point out that the calculations are only formal since rigorous proofs are intricate and beyond the scope of this paper.

Generality

From an integration of (6.5) with respect to x and classical computations (under the assumptions of bounded functions for the birth, death and transfer rates), we deduce that our model satisfies a *saturation property*, i.e. $\rho_\varepsilon(t)$ is bounded from above, uniformly in $t \geq 0$ and $\varepsilon > 0$.

From this, and $\rho_\varepsilon(t) = \int_{\mathbb{R}^d} e^{\frac{u_\varepsilon(t,x)}{\varepsilon}} dx$, we deduce that for all $t > 0$, $\sup_{x \in \mathbb{R}^d} u(t, x) \leq 0$ and the following constraint holds:

$$\sup_{x \in \mathbb{R}^d} u(t, x) = 0 \quad \text{when} \quad \rho(t) > 0. \tag{6.10}$$

Note that our model allows the population to get extinct, thus we cannot expect ρ to be positive at all times. As a byproduct, we derive the *concentration property*, i.e., the formal weak convergence of measures

$$f_\varepsilon(t, x) \rightharpoonup \rho(t)\delta_{\bar{x}(t)}(dx), \quad \text{when } \varepsilon \to 0,$$

where $\delta_{\bar{x}(t)}$ denotes, as usually, the Dirac measure centered in $\bar{x}(t)$. From (6.10), it is possible to formally derive a formula for ρ. Indeed, either $\rho(t) = 0$, or $\rho(t) > 0$ and

$$\partial_t u(t, \bar{x}(t)) = 0,$$

which implies

$$\rho(t) = \frac{b(\bar{x}(t)) - d(\bar{x}(t)) + \tau(0)}{C} = \frac{b(\bar{x}(t)) - d(\bar{x}(t))}{C}, \tag{6.11}$$

for τ defined in (6.4).

Having above definitions in hand, we can now perform a formal analysis on the dynamics of $\bar{x}(t)$, defined below in (6.16). For simplicity, we fix all constants but τ_0. We assume

$$b(x) = b_r > 0, \tag{6.12}$$

$$d(x) = d_r x^2, \quad d_r > 0, \tag{6.13}$$

$$m(z) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{z^2}{2\sigma^2}}, \tag{6.14}$$

and the transfer function $h_K(x, y, \nu)$ as in (6.2). We assume that the initial condition is given by

$$f_\varepsilon^0(x) = \frac{1}{\sqrt{\varepsilon}} e^{-\frac{x^2}{2\varepsilon}}, \tag{6.15}$$

In this section, we make the following further assumptions

$$\begin{aligned} u(t,\cdot) \text{ reaches its maximum on a single point } \bar{x}(t), \text{ for almost every } t \geq 0, \\ \bar{x}(t) \text{ is a non-degenerate maximum, i.e., } \nabla_x^2 u(t,x) < 0, \end{aligned} \tag{6.16}$$

and the formal assumption

$$\bar{x}(t) \text{ is smooth with respect to } t. \tag{6.17}$$

Smooth dynamics $\bar{x}(t)$.

The following statement deals with the smooth dynamics of $\bar{x}(t)$, that is, when no jump of occurs in the dynamics of $\bar{x}(t)$.

Statement 6.1 *See Figure 6.1. Under assumptions* (6.12)-(6.16)*, the function $t \mapsto \bar{x}(t)$ is an increasing function which satisfies the following inequality for every $t \geq 0$,*

$$0 \leq \bar{x}(t) \leq x_\star := \frac{\tau_0}{2d\delta}.$$

More precisely, $\bar{x}(t)$ satisfies the canonical equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{x}(t) = \left[-\nabla_x^2 u(t,\bar{x}(t))\right]^{-1} \cdot \left(\nabla_x r(\bar{x}(t)) + \nabla_x \tau(0)\right), \tag{6.18}$$

where

$$r(x) := b(x) - d(x), \tag{6.19}$$

and $\nabla_x^2 u$ denotes the Hessian of u with respect to the x variable.

Equation *canonical equation* is refered to as the *canonical equation* in the literature (see, for instance, [232]).

Proof.

Starting from

$$\nabla_x u(t, \bar{x}(t)) = 0,$$

a differentiation with respect to t gives (6.18). Equation (6.18) has a unique singular point $x_\star$, which satisfies $r'(x_\star) + \tau'(0) = 0$, with τ defined in (6.2) and r in (6.19). We find

$$x_\star = \frac{\tau_0}{2d_r\delta}. \tag{6.20}$$

Note that $t \mapsto \bar{x}(t)$ is increasing when $\bar{x}(t) < x_\star$ and decreasing when $\bar{x}(t) > x_\star$. Besides, from the initial condition (6.15), we have $\bar{x}(0) = 0$, and consequently $0 \leq \bar{x}(t) \leq x_\star \quad \forall t \geq 0$.

■

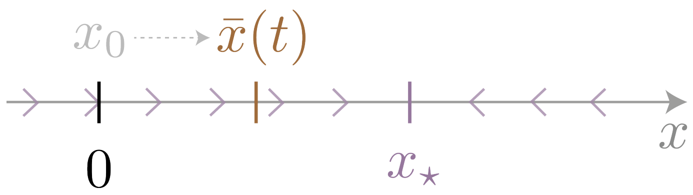


Figure 6.1 – The attractor $x_\star$

Evolutionary rescue.

In general, the canonical equation (6.18) does not hold in every point of time. Indeed, a new maximum of u can arise in finite time, which would cause a "jump" in the dynamics of $\bar{x}(t)$: this is what we call an *evolutionary rescue*. Formally, this is what happens (periodically in time) in the case of cycles, see Figure 6.8b. We thus expect $\bar{x}(t)$ to possibly jump periodically and to follow (6.18) between two jumps. We now try to characterize the possible jumps. For $T > 0$, we denote

$$\bar{x}(T^-) := \lim_{\substack{t \to T \\ t < T}} \bar{x}(t), \quad \bar{x}(T^+) := \lim_{\substack{t \to T \\ t > T}} \bar{x}(t).$$

Statement 6.2 *See Figure 6.2. We assume* (6.12)-(6.16)*, and that* (6.17) *holds until a time* $T > 0$*, such that* $u(T, \cdot)$ *reaches its maximum on* $\bar{x}(T^-)$ *and on another point* $\tilde{x}$*. Then* $\tilde{x} = 0$ *and* $\bar{x}(t)$ *will jump towards 0 at time* T*, i.e* $\bar{x}(T^+) = 0$*.*

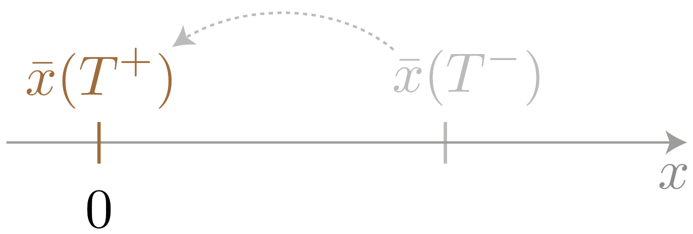


Figure 6.2 – Jump of $barx(t)$

Proof.
From assumption (6.16), we have for all $t \in [0, T]$ that $u(t, \cdot)$ is concave non-degenerate on $[\bar{x}(t) \pm \theta]$, with $\theta > 0$. For simplicity, we further assume $\delta \leq \theta$, where δ is defined in (6.3).

First, let us show that $\tilde{x} = 0$. We define the fitness function of trait x in an population concentrated in $\bar{x}$:

$$F_{\bar{x}}(x) := r(x) + \tau(x - \bar{x}),$$

where τ and r are respectively defined in (6.19) and (6.2)-(6.3). Note that we have $\partial_t u(t, x) = F_{\bar{x}(t)}(x) - C\rho(t)$, for $t < T$. But $\tilde{x} \notin [\bar{x}(t) \pm \delta]$ and the choice of parameters (6.12)-(6.13)-(6.3) implies that $\tilde{x}$ must maximize $F_{\bar{x}(T^-)}(\cdot)$, hence $\tilde{x} = 0$.

The second step is to prove that there will be an actual jump towards 0, i.e., $\bar{x}(T^+) = 0$. First, note that there exists a small $\eta > 0$ such that $\forall t \in (T - \eta, T)$, $u(t, \bar{x}(t)) = 0$ and $u(t, 0) < 0$. Let us fix $t \in (T - \eta, T)$. We have $F_{\bar{x}(t)}(0) \geq F_{\bar{x}(t)}(\bar{x}(t))$, and we claim that the inequality is strict. Indeed, since $t \mapsto x(t)$ is increasing,

$F_{\bar{x}(t)}(\bar{x}(t))$ is decreasing, whereas $F_{\bar{x}(t)}(0)$ is constant (as long as η is small enough such that $\bar{x}(T-\eta) > \delta$). We end up with

$$F_{\bar{x}(t)}(0) > F_{\bar{x}(t)}(\bar{x}(t)).$$

The above inequality expresses the fact that 0 is fitter than $\bar{x}(t)$ in a population with trait $\bar{x}(t)$. In general, this does not allow to conclude that 0 will invade and become the new dominant trait (i.e., that the jump will occur) because it does not imply that 0 will remain fitter during all the process of invasion. But the particular form of our problem, especially the fact that τ is an odd function, implies

$$F_0(0) > F_0(\bar{x}(t)).$$

Indeed we have from the definition of $F_{\bar{x}}(x)$ that

$$F_0(0) - F_0(\bar{x}(t)) = r(0) - r(\bar{x}(t)) + \tau(\bar{x} - \bar{x}) - \tau(0) = d_r \bar{x}(t)^2 > 0.$$

Consequently, for all $\lambda \in [0,1]$,

$$\lambda F_0(0) + (1-\lambda) F_{\bar{x}(t)}(0) > \lambda F_0(\bar{x}(t)) + (1-\lambda) F_{\bar{x}(t)}(\bar{x}(t)).$$

It shows that 0 remains the fittest trait during all the process of invasion, and therefore that 0 will actually invade, i.e., that $\bar{x}(t)$ will actually jump towards 0 at time T^+.

■

Threshold for cycles

In the previous section, we described the possible *evolutionary rescue*, i.e the possible jumps in the dynamics of $\bar{x}(t)$ towards $x = 0$. When a jump occurs, a new cycle begins: it leads to a periodical behavior of $\bar{x}(t)$, hence the cycling phenomena.

We recall that a jump corresponds to a rescue of the population concentrated at $\bar{x}(t)$ by the small population with trait $x = 0$. It is possible only if $\bar{x}(t) > \delta$ and if 0 is fitter than $\bar{x}(t)$ during a sufficiently large interval of time (which is the time needed for the small population at $x = 0$ to regrow). Note that 0 is fitter than $\bar{x}(t)$ if and only if

$$\begin{aligned} F_{\bar{x}(t)}(0) \geq F_{\bar{x}(t)}(\bar{x}(t)) \quad &\text{iff} \quad b_r - \tau_0 \geq b_r - d_r \bar{x}(t)^2, \\ &\text{iff} \quad \bar{x}(t) \geq x_{resc} := \sqrt{\frac{\tau_0}{d_r}}. \end{aligned} \tag{6.21}$$

But if no jump occurs, $\bar{x}(t)$ formally follows (6.18), thus $\bar{x}(t) < x_\star$ and $\bar{x}(t)$ converges to $x_\star$ when $t \to +\infty$ (with $x_\star$ is defined in (6.20)).

Statement 6.3 *See Figure 6.3a-6.3b. Under assumptions* (6.12)-(6.16)*, the evolutionary rescue phenomena occurs if and only if*

$$\tau_0 > \tau_{cyc} := 4 d_r \delta^2. \tag{6.22}$$

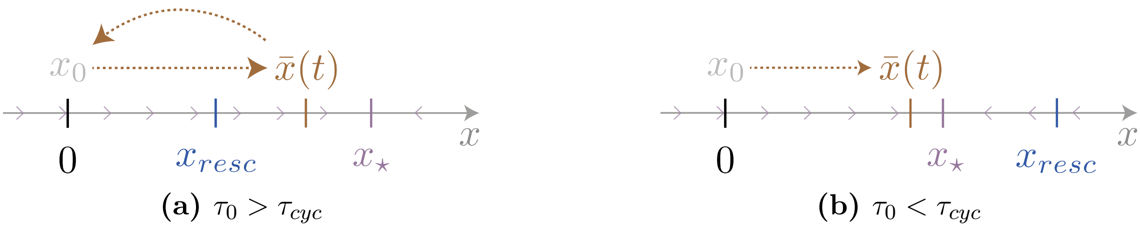


Figure 6.3 – Threshold for cycles

Threshold for extinction.

The population is said to be "extinct" at time t if $\rho(t) = 0$. According to (6.11), we define x_{ext} as the solution of $r(x_{ext}) = 0$, i.e.,

$$x_{ext} := \sqrt{\frac{b_r}{d_r}}. \tag{6.23}$$

A population concentrated at trait $\bar{x}$ is extinct iff $\bar{x} \geq x_{ext}$.

The picture is simple in the case of stabilization without cycles, i.e., when $\tau_0 \leq \tau_{cyc}$ (see (6.22)). In this case, we recall that $\bar{x}(t)$ formally follows (6.18) for all $t > 0$, thus $\bar{x}(t) < x_\star$ and $\bar{x}(t)$ converges to $x_\star$ when $t \to +\infty$ (where $x_\star$ is defined in (6.20)). Thus, if $x_\star \leq x_{ext}$, we have $\rho(t) > 0$ for all $t > 0$; on the contrary, if $x_\star > x_{ext}$, there exists a time $t_{ext} > 0$ for which $\rho(t) = 0$ for all $t \geq t_{ext}$. It gives a sharp threshold for extinction of the population: indeed, the population eventually gets extinct if and only if $x_\star > x_{ext}$, which is equivalent to

Statement 6.4 *Under assumptions* (6.12)-(6.16)*, if* $\tau_0 \leq \tau_{cyc}$*, then the population eventually gets extinct if and only if*

$$\tau_0 > \tau_{ext} := 2\sqrt{b_r d_r}\delta.$$

We point out that, surprisingly enough, τ_{ext} is an increasing function of the death rate d_r, meaning that under a higher death rate, the population can survive to a higher HT rate. The interpretation we propose is that if d_r is high, the population driven outward $x = 0$ dies rapidly; thus the population that remained closer to 0 undergoes a milder HT, which makes the overall population more resistant to a high HT rate.

Let us now focus on the case where the cycling phenomenon occurs, i.e when $\tau_0 > \tau_{cyc}$. In this case, $\bar{x}(t)$ will follow (6.18) and will periodically jump to $x = 0$. First, note that if $x_\star < x_{ext}$, $\bar{x}(t)$ remains below x_{ext} for all t and the population does not get extinct:

$$\text{if} \quad \tau \leq \tau_{ext}, \quad \text{then} \quad \rho(t) > 0, \quad \forall t > 0.$$

The most intricate case is when $x_\star > x_{ext}$, which contains case of extinction and non-extinction, depending on whether the jump of $\bar{x}(t)$ towards 0 happens before or after $\bar{x}(t)$ has passed beyond x_{ext}. In other words, extinction can be avoided if the evolutionary rescue happens before the dominant trait is led to extinction, i.e if $\bar{x}(T^-) \leq x_{ext}$, where T is the time where a jump of $\bar{x}(t)$ towards 0 occurs. However, we are not able to give a satisfactory formula or estimate on T.

Besides, when the jump of $\bar{x}(t)$ occurs, it can happen that the trait $x = 0$ is not fit enough to avoid extinction: in this case the evolutionary rescue does not manage to sustain the population. It corresponds to the case $x_{resc} > x_{ext}$. We have the following threshold: the evolutionary rescue is able to sustain the population iff $r(0) + \tau_0 > 0$, which is equivalent to

$$\tau_0 < \tau_{sus} := b_r.$$

If $\tau \geq \tau_{sus}$, the population eventually gets extinct. If $\tau < \tau_{sus}$, the population is effectively rescued by the evolutionary rescue, even in the case where it passed through an episode of extinction during the previous cycle: in some cases, the population can regrow after being extinct, see Figure 6.8c. We think this is an interesting feature that the Hamilton-Jacobi approach is able to grasp. Regarding the stochastic model, an episode of extinction on Hamilton-Jacobi corresponds to an interval of time where the population reaches extremely small values (of order $e^{-\frac{1}{\varepsilon}}$, with ε the variance of the mutation kernel), and thus on which there is a nonzero probability that every individual dies.

Statement 6.5 *Assume* (6.12)-(6.16) *and* $\tau_0 > \tau_{cyc}$.
— *if* $\tau_0 \leq \tau ext$, *the population never gets extinct.*
— *the evolutionary rescue effectively manages to sustain the population if and only if* $\tau_0 < \tau_{sus} := b_r$.

Characteristics of a Hamilton-Jacobi equation

Denoting

$$-H(t,x,p) := -(d(x) + \rho(t)) + b(x)\int_{\mathbb{R}} m(z)e^{pz}dz + \tau(x - \bar{x}(t)),$$

from (6.8) we have $\partial_t u(t,x) + H(t,x,\nabla_x u(t,x)) = 0$. Since H is convex in the p variable, we have the following representation formula (see [209]):

$$u(t,x) = \inf_{\substack{\gamma\in C^0(\mathbb{R}_+,\mathbb{R})\\ \gamma(t)=x}} \left[\int_0^t L\left(s,\gamma(s),\dot{\gamma}(s)\right)ds + u^0(\gamma(0))\right], \tag{6.24}$$

where $L(t,x,v)$ is the Lagrangian of the equation, obtained through a Legendre transform (or a convex conjugate) of H.

Every γ which is admissible as a minimizer in (6.24) is called a *characteristic* of the Hamilton-Jacobi equation (6.8). Note that every characteristic γ formally satisfies the condition

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[\partial_v L\left(s,\gamma,\dot{\gamma}(s)\right)\right] = \partial_x L\left(s,\gamma(s),\dot{\gamma}(s)\right). \tag{6.25}$$

This equation is obtained from the fact that γ must be a critical point of the functional in (6.24).

Let us consider a simpler but comparable case, where we replace H by

$$\tilde{H}(x,p) = -\frac{x^2}{2} + \frac{p^2}{2} + 1.$$

In this case, the Legendre transform of $\tilde{H}$ can be computed explicitly, we obtain

$$\tilde{L}(x,v) = \frac{x^2}{2} + \frac{v^2}{2} - 1.$$

Then (6.25) becomes

$$\ddot{\gamma}(s) = \gamma(s).$$

6.3 Numerical tests

In this section we perform several numerical tests for the presented models considering different values of parameters, replicating different scenarios: stabilization around an optimal value, cycles (occurring through the evolutionary rescue phenomenon) and extinction. Then, we compare the numerical results obtained for the stochastic and deterministic approaches, using, in particular, an asymptotic-preserving scheme which allows us to observe the population dynamics on the passage from the integrodifferential equation (6.5) to a limit (6.7). Throughout this section, we define the birth rate, death rate, and mutation kernel as in (6.12)-(6.14), with fixed parameters $b \equiv 1$, $d_r \equiv 1$, $C \equiv 0.5$.

6.3.1 Stochastic model

The scheme

We aim to simulate the population dynamics over a fixed interval $[0, T]$. We begin by simulating an initial population of size N^0. We assume that the population is normally distributed around a mean trait x^0_{mean} with a standard deviation σ^0 so that the resulting vector X^0 belongs to $\mathbb{R}^{N^0}$. In a time step Δ, an individual can die, give birth, or be a subject to HT. Each event happens according to a certain probability that we compute from the rates. A more detailed description of the simulations is provided in Algorithm 1.

Note that in our setting it is possible that $1, 2$ or 3 events happen within the time step. Keeping a discretization time step small helps us to keep a biological sense in our simulation: even if the event of horizontal transfer with an "already dead" individual is possible in our setting (if $T_d \leq T_{HT} \leq \Delta$), this event is extremely rare.

We simulate the population of initial size $N_0 = 10000$ up to time $T = 1000$ with $\Delta = 0.01$, with the parameters being defined at the beginning of the section, and α a Heaviside function. Even if a Heaviside function is not the easiest to analyze when we pass to the deterministic limit of the system (see Subsections 6.2.2 and 6.2.3), we use it for the stochastic simulation, since it is the most straightforward model for HT in biological context, and is much faster to compute than a smooth function. We fix all constants but τ_0, which regulates the Horizontal Transfer, and study how it affects the dynamics. Then, we plot the density of the population at each time (left side of each Figure): brighter colors on the plot mean that there is a significant amount of individuals with very similar traits. On the right top and right bottom, we plot the normalized population size (ratio between the actual size and the carrying capacity of the system), and the mean trait.

Algorithm 1: Population dynamics on time interval $[0, T]$

```
Random initialization of a population X^0 := N(x^0_mean, σ^0) × N^0 ;
while iΔ ≤ T do
    X^i = X^{i-1}, N^{i-1} = size(X^{i-1});
    for ∀x ∈ X^i do
        R_b := b(x), R_d := d(x)+C N^{i-1}, R_HT := Σ_{y∈X^i} h_K(x − y, N^{i-1});
        T_b := λ(R_b), T_d := λ(R_d), T_HT := λ(R_HT), where λ denotes an
          exponential random law;
        if T_b ≤ Δ then
            pick up a new trait z from N(x, σ);
            add a new individual with trait z to X^i;
        end
        if T_HT ≤ Δ then
            pick a trait y ∈ X^{i-1} according to the law h_K(x−y, N^{i-1}) / Σ_{y∈X^i} h_K(x−y, N^{i-1});
            remove individual with trait x and add individual with trait y;
        end
        if T_d ≤ Δ then
            remove the individual with trait x from X^i
        end
    end
    return X^i
end
```

Depending on the parameters, we may observe three types of behavior, (see Figure 6.4). First possibility, for small values of τ_0, is the stabilization (Figure 6.4a). In this case, the population rapidly reaches the equilibrium and concentrates around the optimal trait, which is close to 0.1 for $\tau_0 = 0.02$. Note that in this case, the mean trait is shifted in comparison to the optimal trait without HT (which is $x = 0$).

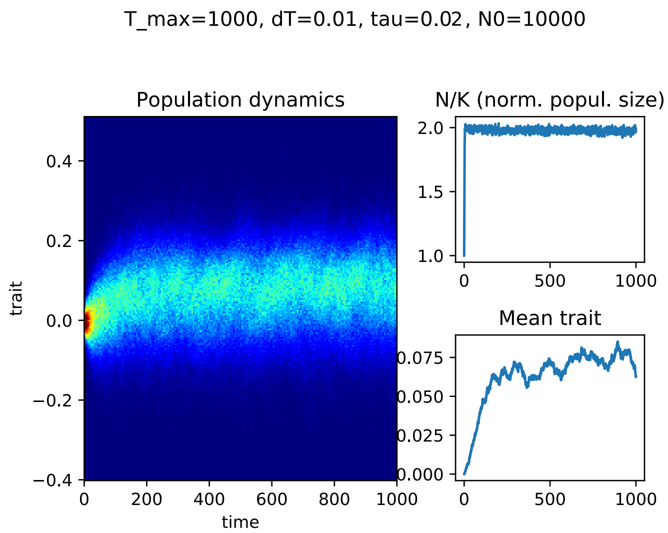
Second option, for intermediate values of τ_0, is the cycling behavior (Figure 6.4b). Since the transfer rate is sufficiently large, the population is driven towards a deleterious trait, which is eventually less fit than the trait $x = 0$. If the drift is not too strong, the very few individuals who were not affected by HT and remained fit (with x close to 0) manage to regrow and eventually repopulate the environment, which relaunches the cycle.

The last possibility, for large values of tτ_0, is extinction of the population (Figure 6.4c). It occurs because too many individuals were affected by HT.

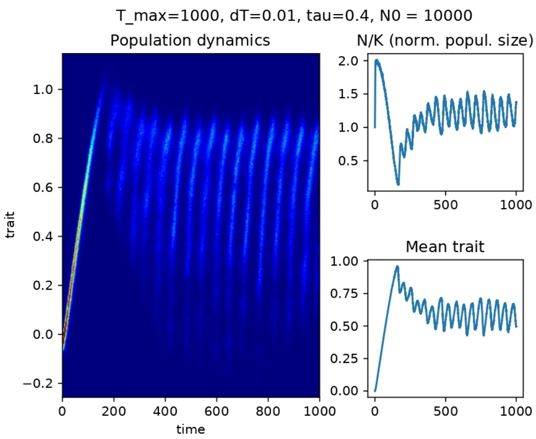
To understand better this phenomenon, we have to give a precise definition of what we mean by "the critical value" of the transfer rate. In the stochastic setting, the answer is not trivial, and that is where the individual-based model reaches its limit. What we observe experimentally is the following.

These observations suggest that there exist two threshold values τ_{cyc}, τ_{ext} such that,

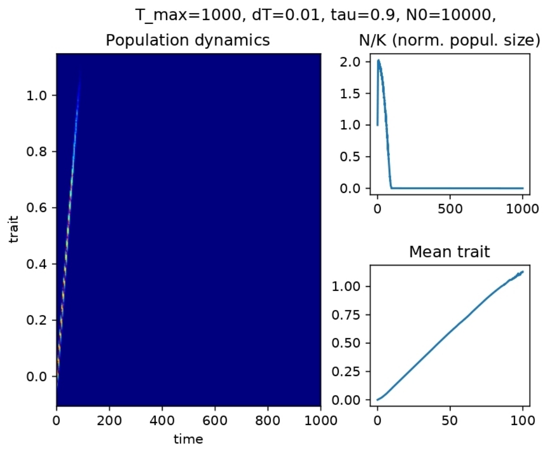
$$
\begin{aligned}
\tau_0 < \tau_{cyc} &\quad\longleftrightarrow\quad \text{stabilization,}\\
\tau_{cyc} < \tau_0 < \tau_{ext} &\quad\longleftrightarrow\quad \text{cycles,}\\
\tau_{cyc} < \tau_0 &\quad\longleftrightarrow\quad \text{extinction.}
\end{aligned}
$$



(a) Stabilization: $\tau_0 = 0.02$



(b) Cycles: $\tau_0 = 0.4$



(c) Extinction: $\tau_0 = 0.9$

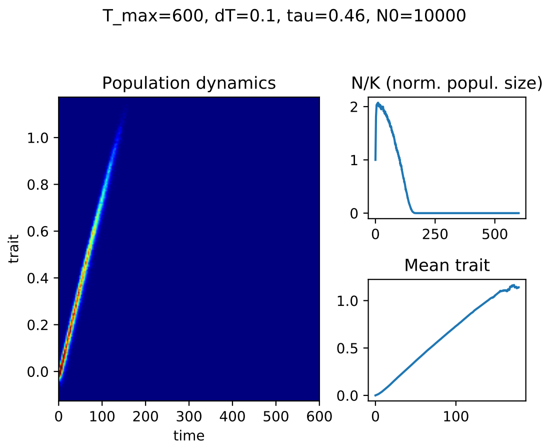
Figure 6.4 – Behavior of the population dynamics as the mutation rate τ_0 is changing, ($b_r = d_r = 1$, $\sigma = 10^{-2}$, $K = 10^4$, $\sigma^0 = 10^{-2}$, $x^0_{mean} = 0$, $N^0 = 10^4$).

However, It is not easy from the numerical simulation to clearly grasp the threshold values. First, it is not always clear how to differentiate between stabilization and cycles when τ_0 is close to τ_{cyc}. Secondly, when the value of τ_0 is getting closer to τ_{ext}, (we find $\tau_{ext} \approx 0.49$ with our parameters), we may observe either cycles, extinctions, or both in different repetitions of the same experiment. It is illustrated in Figure 6.5, where the computations are launched with the same set of parameters.

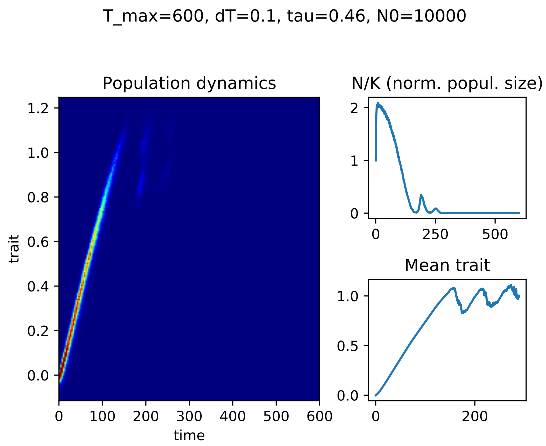
This constraint of an individual-based model naturally leads us to study a limiting system described in Subsection 6.2.2.

Lineages

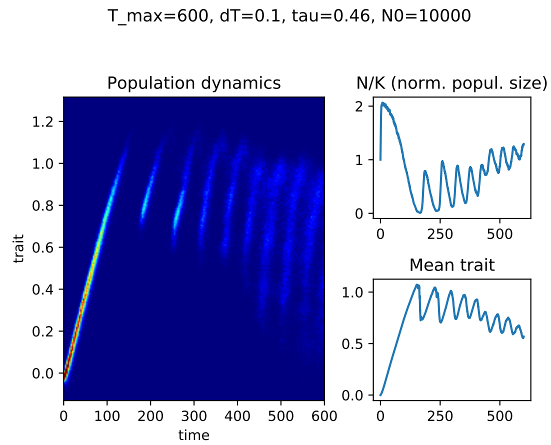
In the stochastic model, we can keep track of the lineage of an individual i which lives at a final observed time T. We illustrate some numerical experiments on Figure 6.6. The four simulations are done with the same parameters. In the background, every point with coordinates (t, x) represents an individual with trait x living at time t (as in Figure 1). The solid lines represent the lineages of the



(a) Extinction



(b) Cycle and extinction



(c) Cycles

Figure 6.5 – Different behaviors for $\tau_0 = 0.46$ (and the other parameters as in Figure 6.4).

individuals that live at final time. Small fluctuations are the results of birth with mutation, while the large upwards jumps correspond to an occurrence of a HT.

First of all, we can see on the plot that all the lineages gather into one line up to $t = 400$. It means that all individuals that live at final time $t = 700$ emanate from one single ancestor of the initial population. This phenomenon is well known and referred to as *coalescence* in the literature (see for instance [193], or [13, 14] for a mathematical description of a classical population genetics theory).

Besides, we see that the lineages remain centered around $x = 0$ during almost all the observed time. It is explained by the fact that every lineage that goes to a high value of x (corresponding to deleterious phenotype) cannot recover (since the mutations are small) and eventually goes extinct. It illustrates that the population manages to sustain because of the very few individuals that were not affected by HT throughout history.

6.3.2 Numerical scheme for the PDE model

In this subsection, a numerical scheme for (6.5) is presented, and its properties are numerically investigated. For the discretization of (6.5), we consider a bounded

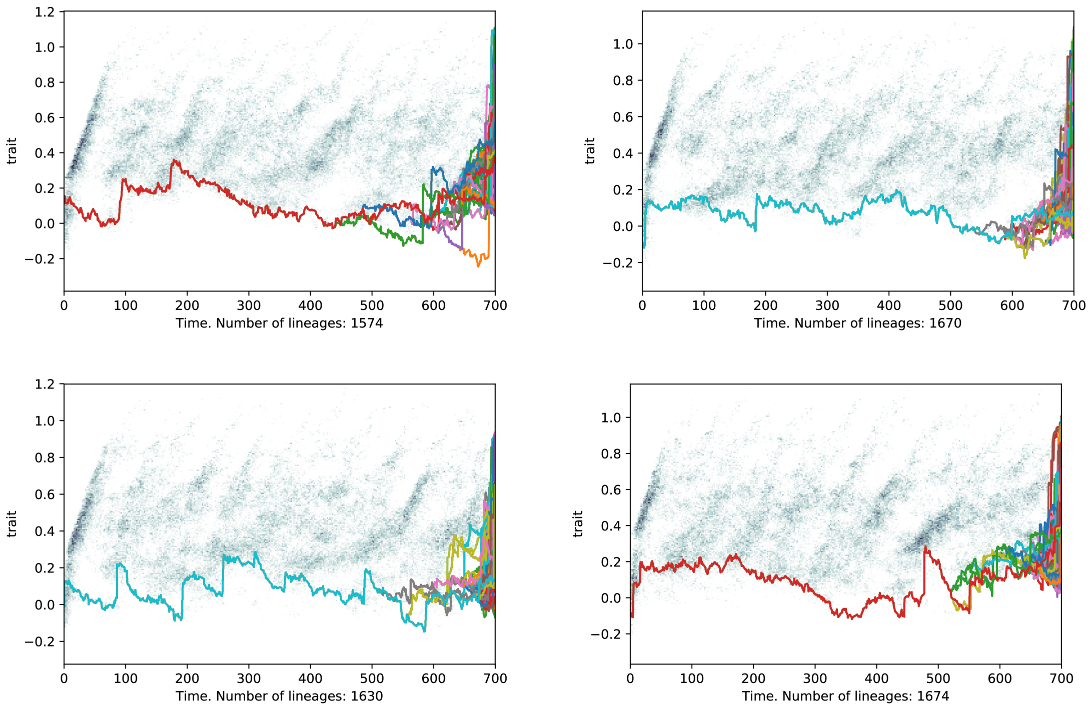


Figure 6.6 – Simulations on the stochastic model with lineages. $\tau_0 = 0.4$, $T_{max} = 700$, $dT = 0.1$, $K = N_0 = 1000$ and other parameters as in Figure 6.4.

space of traits $[X_{\min}, X_{\max}]$, discretized with N_x points. Denoting N_x the number of discretization points of the interval $[X_{\min}, X_{\max}]$, we define

$$\Delta x = \frac{X_{\min} - X_{\max}}{N_x - 1},$$

and

$$x_i = X_{\min} + i\Delta x, \quad 0 \le i \le N_x - 1.$$

We consider the time interval $[0, T_{\max}]$, discretized with N_t points $t_n = n\Delta t$, for $0 \le n \le N_t - 1$, and where Δt is defined as

$$\Delta t = \frac{T_{\max}}{N_t - 1}.$$

The approximations of the solution f of (6.5) at (t_n, x_i), and of its density ρ at t_n are denoted f_i^n and ρ^n respectively.

We recall that he initial condition f^0 is a smooth function of x given in (6.15) and the initial density ρ^0 is computed using a left-point quadrature rule for f^0 as follows:

$$\rho^0 = \Delta x \sum_{i=0}^{N_x - 1} f^0(x_i).$$

The scheme is written with an explicit Euler scheme, in which the integrals are computed with a left-point quadrature rule. For $n \ge 1$ and $0 \le i \le N_x - 1$, it reads

$$\varepsilon \frac{f_i^{n+1} - f_i^n}{\Delta t} = \left(d(x_i) + C\rho^n\right) f_i^n + [m * (bf)]_i^n + f_i^n \Delta x \sum_{j=0}^{N_x - 1} \tau(x_i - x_j) \frac{f_j^n}{\rho^n}. \quad (6.26)$$

In (6.26), the convolution product $[m * (bf)]_i^n$ is computed with a left-point quadrature rule, as well of the other integrals. To do so, a grid in the z variable is defined as for the x variable. Let $Z_{\min}$ and $Z_{\max}$, and the number N_z of discretization points be given. The grid in z is defined as

$$\forall 0 \leq k \leq N_z - 1, z_k = Z_{\min} + k\Delta z,$$

where $\Delta z = (Z_{\max} - Z_{\min}) / (N_z - 1)$. When $x_i + \varepsilon z_k \in [X_{\min}, X_{\max}]$, the value of $f(t_n, x_i + \varepsilon z_k)$ is approximated by linear interpolation of the $(f_i^n)_{0\leq i\leq N_x-1}$. When $x_i + \varepsilon z_k < X_{\min}$, or $x_i + \varepsilon z_k > X_{\max}$, it is computed with a linear extrapolation of the $(f_i^n)_{0\leq i\leq N_x-1}$, using the slope at the corresponding end of the X domain. Using the notation $f^n(x_i + \varepsilon z_k)$ for the approximation of $f(t_n, x_i + \varepsilon z_k)$, we then define

$$[m * (bf)]_i^n = \Delta z \sum_{k=0}^{N_z-1} m(z_k) b(x_i + \varepsilon z_k) f^n(x_i + \varepsilon z_k).$$

Case $\varepsilon = 1$: comparison with stochastic model

The first thing that we are interested in is whether, under identical parameters and initial conditions ,we may reproduce the same behavior as in the stochastic model. Thus, we conduct several experiments, fixing $\varepsilon = 1$ and all the other parameters as in stochastic simulation case.

As we may see in Figure 6.7, simulations correspond in overall to those of the stochastic model. Indeed, when the HT rate τ_0 is small enough the population rapidly stabilizes around its equilibrium state (see Figure 6.7a)

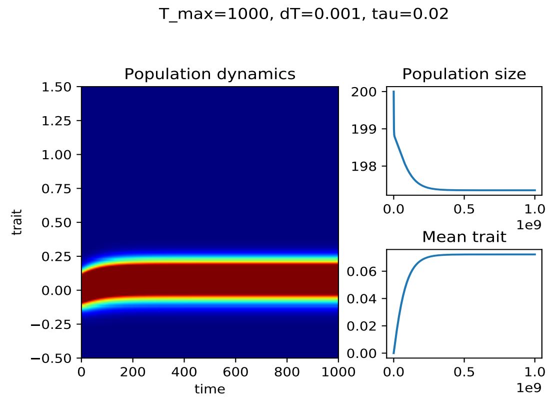
For larger values of τ_0 we observe damped oscillations, see Figure 6.7b. This can be put in contrast with the un-damped oscillations observed in the stochastic model (Figure 6.4b). The way we understand the damping in the oscillations is that our numerical algorithm is not designed to have a precise grasp on the exponentially small values of f, on which the cycling phenomenon relies. This limitation suggests to perform the change of variable (6.6), and to write a numerical scheme which converges uniformly when $\varepsilon \to 0$. It is what the next section is devoted to.

On Figure 6.7c, we observe that as τ_0 becomes larger the population gets extinct, and then, surprisingly enough, "reborns" after a period of extinction. On Figure 6.7d we observe a full extinction of the population without report. We will give further insights into those two cases in the next section.

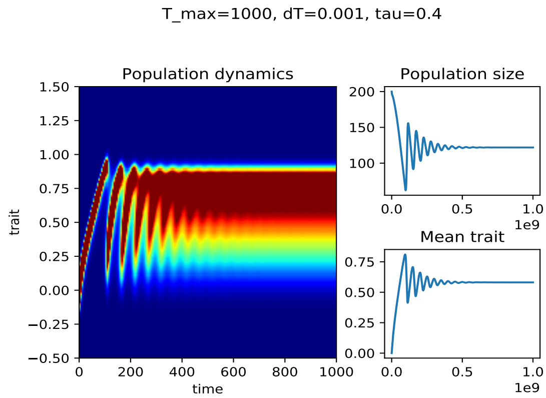
6.3.3 The scheme for the Hamilton-Jacobi equation

Case $\varepsilon \to 0$: description of the numerical scheme

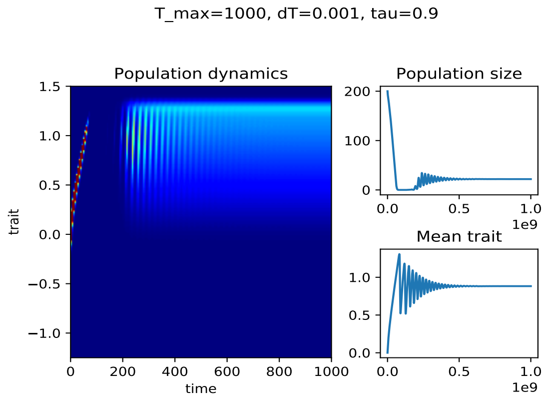
As the rescaling parameter ε goes to 0, the model given by (6.7) gets closer to its limiting state (6.8). However, numerical approximation of the (6.5) for $\varepsilon \ll 1$ is not a trivial task. Indeed, for small ε, the solution f^ε of (6.5) is expected to concentrate at a dominant trait. To be able to catch its stiffness numerically, one then has to refine the grid in x, to ensure enough precision in the computation of f. As a consequence, the computational cost of the numerical simulations increases when $\varepsilon \to 0$, and reaching the asymptotic regime with this scheme is not possible. In this part, we present a numerical scheme for (6.5) which enjoys stability properties in the limit $\varepsilon \to 0$.



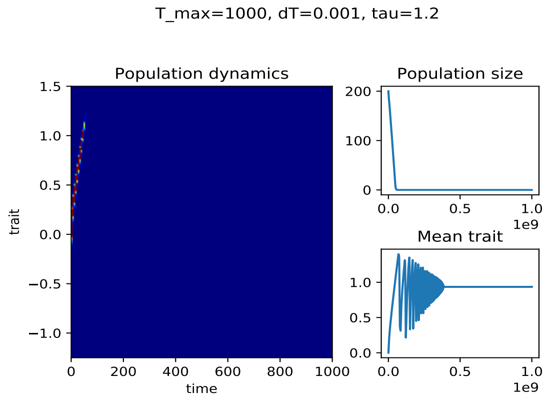
(a) Stabilization: $\tau_0 = 0.02$



(b) Cycles: $\tau_0 = 0.4$



(c) Extinction and cycles: $\tau_0 = 0.9$



(d) Full Extinction $\tau_0 = 1.2$

Figure 6.7 – Behavior of the population dynamics described by a PDE model as the mutation rate τ_0 is changing, ($b_r = d_r = 1$, $\sigma = 0.01$, $\varepsilon = 1$).

To avoid the increase of computational cost when reaching the asymptotics, and to ensure the scheme approaches the limit Hamilton-Jacobi equation for small ε, a scheme for the solution u^ε of (6.7) which enjoys the Asymptotic Preserving (AP) property is proposed here. Such schemes have been introduced in [187, 195, 196], their properties are often summarized by the following diagram

$$\begin{tikzcd}
P_\varepsilon \arrow[r, "\varepsilon\to 0"] & P_0 \\
S_\varepsilon^h \arrow[u, "h\to 0"] \arrow[r, "\varepsilon\to 0"] & S_0^h \arrow[u, "h\to 0"]
\end{tikzcd}$$

that should be understood as follows: when the parameter $\varepsilon > 0$ is fixed, the scheme S_ε^h is consistent with the ε-dependent problem P_ε. When ε goes to 0, the solution of P_ε converges to the solution of the limit problem P_0. The AP scheme S_ε^h is stable along the transition to the asymptotic regime. It means that, when ε goes to 0 with fixed discretization parameters h, the scheme becomes a limit scheme S_0^h, which is consistent with the limit problem P_0.

As an AP scheme is required to enjoy stability properties when ε is going to 0, one has to ensure that all the quantities that have to be computed enjoy this property. In the case we are considering, the main concerns are the computation of the integral containing the birth term, the computation of the integral containing the transfer term and the computation of ρ. If all of them are correctly defined, the scheme we propose reads

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -(d(x_i) + \rho^{n+1}) + B_i^n + T_i^n, \tag{6.27}$$

where B_i^n stands for an approximation of

$$\int_{\mathbb{R}} m(z) b(x_i + \varepsilon z) \mathrm{e}^{(u^\varepsilon(t^n, x_i+\varepsilon z) - u^\varepsilon(t^n, x_i))/\varepsilon} \mathrm{d}z, \tag{6.28}$$

and T_i^n is for

$$\int_{\mathbb{R}} \tau(x_i - y) \frac{f(t^n, y)}{\rho(t^n)} \mathrm{d}y. \tag{6.29}$$

Here, we used the notations and discretization grids defined in the beginning of Section 6.3.2, and the dependences in ε are omitted to simplify the notations. In what follows, we present how T_i^n, B_i^n and ρ^{n+1} can be computed in a way that ensures they are consistent with their definition for fixed ε, that they can be computed with a constant computational cost with respect to ε, and that their asymptotic behavior when ε goes to 0 is meeting the continuous one (6.8).

— **Computation of T_i^n.** The direct approximation of (6.29) with a quadrature rule is consistent for $\varepsilon \sim 1$. However, since f is expected to concentrate when $\varepsilon \to 0$, it lacks precision in the asymptotic regime. Especially, the convergence of f/ρ to a Dirac is not ensured when the integral is approximated directly. Remarking that

$$\frac{f^\varepsilon(t^n, y)}{\rho^\varepsilon(t^n)} = \frac{\mathrm{e}^{u^\varepsilon(t^n, y)/\varepsilon}}{\int_{\mathbb{R}} \mathrm{e}^{u^\varepsilon(t^n, z)/\varepsilon} \mathrm{d}z} = \frac{\mathrm{e}^{(u^\varepsilon(t^n, y) - \max\limits_x u^\varepsilon(t^n, x))/\varepsilon}}{\int_{\mathbb{R}} \mathrm{e}^{(u^\varepsilon(t^n, z) - \max\limits_x u^\varepsilon(t^n, x))/\varepsilon} \mathrm{d}z},$$

(6.29) is computed with a left-point quadrature rule in the integrals of the previous expression. It reads

$$T_i^n = \Delta x \sum_{j=1}^{N_x - 1} \tau(x_i - y_j) \frac{\mathrm{e}^{(u_j^n - \max\limits_l u_l^n)/\varepsilon}}{\Delta x \sum\limits_{k=0}^{N_x-1} \mathrm{e}^{(u_k^n - \max\limits_l u_l^n)/\varepsilon}} = \frac{\sum\limits_{j=1}^{N_x-1} \tau(x_i - x_j) \mathrm{e}^{(u_j^n - \max\limits_l u_l^n)/\varepsilon}}{\sum\limits_{k=0}^{N_x-1} \mathrm{e}^{(u_k^n - \max\limits_l u_l^n)/\varepsilon}}. \tag{6.30}$$

For fixed ε, (6.30) is consistent with (6.29). Since all the arguments of the exponentials are nonpositive, the limit of (6.30) for small ε can be read on that expression. Denoting j_0 the index such that

$$u_{j_0}^n = \max_l u_l^n,$$

and supposing that there exists a unique such j_0, the limit of (6.30) for small ε is

$$\tau(x_i - x_{j_0}).$$

This is consistent with the last term in the limit Hamilton-Jacobi equation (6.8).

— **Computation of B_i^n.** Once again, the numerical approximation of (6.28) is done with a quadrature in the integral. Using the notations of Section 6.3.2, a grid in z is defined. The functions m and b are respectively evaluated at z_k and $x_i+\varepsilon z_k$, but the interpolation of u^n at $x_i+\varepsilon z_k$ has to be done with special care to make the scheme enjoy the expected asymptotic behavior. Using a left-point quadrature rule, (6.28) is approximated by

$$\Delta z \sum_{\substack{k=0 \\ \varepsilon|z|\leq dx}}^{N_z-1} m(z_k) b(x_i+\varepsilon z_k) \mathrm{e}^{z_k \nabla_{n,i,k}^{\varepsilon,small}} + \Delta z \sum_{\substack{k=0 \\ \varepsilon|z|> dx}}^{N_z-1} m(z_k) b(x_i+\varepsilon z_k) \mathrm{e}^{z_k \nabla_{n,i,k}^{\varepsilon,large}},$$

where $\nabla_{n,i,k}^{\varepsilon}$ stands for an approximation of

$$\frac{u^\varepsilon(t^n, x_i+\varepsilon z_k) - u^\varepsilon(t^n, x_i)}{\varepsilon z_k}.$$

In both cases, it is computed with a linear interpolation of the values u_i^n. Hence, $\nabla_{n,i,k}^{\varepsilon,large}$ is given by

$$\nabla_{n,i,k}^{\varepsilon,large} = \frac{\tilde{u}_{i,k}^n - u_i^n}{\varepsilon z_k},$$

where $\tilde{u}_{i,k}^n$ is computed as the linear interpolation of $(u_i^n)_{1\leq i\leq N_x}$ at $x_i+\varepsilon z_k$. If $x_i+\varepsilon z_k < X_{\min}$ or $x_i+\varepsilon z_k > X_{\max}$, the extrapolation is done linearly using the slope at the first or last point of the interval. Since $\varepsilon z_k > \Delta x$, no stability issue is faced in this computation. Still using a linear interpolation, when $0 < \varepsilon z_k \leq \Delta x$, it is worth noticing that

$$\frac{\tilde{u}_{i,k}^n - u_i^n}{\varepsilon z_k} = \frac{u_{i+1}^n - u_i^n}{\Delta z},$$

and when $0 > \varepsilon z_k \geq -\Delta x$,

$$\frac{\tilde{u}_{i,k}^n - u_i^n}{\varepsilon z_k} = \frac{u_i^n - u_{i-1}^n}{\Delta x}.$$

as a consequence, we define:

$$\nabla_{n,i,k}^{\varepsilon,small} = \begin{cases} \dfrac{u_{i+1}^n - u_i^n}{\Delta x}, & \text{if } \ 0 < \varepsilon z_k \leq \Delta x \\ \dfrac{u_i^n - u_{i-1}^n}{\Delta x}, & \text{if } \ -\Delta x \leq \varepsilon z_k < 0 \\ 0, & \text{if } \ z_k = 0. \end{cases}$$

This definition of B_i^n is consistent with (6.28). Moreover, when ε goes to 0 with fixed numerical parameters, such as $Z_{\min}$ and $Z_{\max}$, the expression $\nabla_{n,i,k}^{\varepsilon,large}$ is not used at all, and

$$\begin{aligned} B_i^n \underset{\varepsilon\to 0}{=} B_i^{n,0} = \Delta z \sum_{\substack{k=0 \\ z_k<0}}^{N_z-1} m(z_k) b(x_i) \mathrm{e}^{z_k \frac{u_i^n - u_{i-1}^n}{\Delta x}} + \Delta z m(0) b(x_i) \\ + \Delta z \sum_{\substack{k=0 \\ z_k>0}}^{N_z-1} m(z_k) b(x_i) \mathrm{e}^{z_k \frac{u_{i+1}^n - u_i^n}{\Delta x}}. \end{aligned} \tag{6.31}$$

— **Computation of ρ^{n+1}.** In (6.27), ρ^{n+1} is considered in an implicit way, to make the limit scheme be consistent with the limit equation (6.8). Since

$$\rho(t) = \int_{\mathbb{R}} \mathrm{e}^{u(t,x)/\varepsilon} \mathrm{d}x,$$

for $\varepsilon > 0$, we define

$$\rho^{n+1} = \Delta x \sum_{i=0}^{N_x-1} \mathrm{e}^{u_i^{n+1}/\varepsilon}.$$

A closed equation on ρ^{n+1} can be deduced from (6.27). Indeed, (6.27) yields

$$\mathrm{e}^{u_i^{n+1}/\varepsilon} = \mathrm{e}^{-\Delta t \rho^{n+1}/\varepsilon} \mathrm{e}^{\left(u_i^n + \Delta t\left[-d(x_i) + B_i^n + T_i^n\right]\right)/\varepsilon},$$

and so

$$\rho^{n+1} = \Delta x \ \mathrm{e}^{-\Delta t \rho^{n+1}/\varepsilon} \sum_{i=0}^{N_x-1} \mathrm{e}^{A_i^n/\varepsilon}, \tag{6.32}$$

where A_i^n denotes $u_i^n + \Delta t \left(-d(x_i) + B_i^n + T_i^n\right)$ to simplify the notations. Eventually, ρ^{n+1} is solution of $h(y) = 0$, where

$$h(y) = y \mathrm{e}^{\Delta t y/\varepsilon} - \Delta x \mathrm{e}^{A_{i_0}^n/\varepsilon} \sum_{i=0}^{N_x-1} \mathrm{e}^{(A_i^n - A_{i_0}^n)/\varepsilon}, \tag{6.33}$$

where $A_{i_0}^n = \max_i A_i^n$ has been taken apart to get an uniform estimate with respect to ε on the remaining sum. It is also solution of the equivalent equation $g(y) = 0$, with

$$g(y) = -\varepsilon \ln(y) - \Delta t y + \varepsilon \ln(\Delta x) + A_{i_0}^n + \varepsilon \ln\left(\sum_{i=0}^{N_x-1} \mathrm{e}^{(A_i^n - A_{i_0}^n)/\varepsilon}\right). \tag{6.34}$$

To find ρ^{n+1}, a Newton's method is applied on expression (6.33) or on (6.34). Both expressions are smooth convex functions of ρ, and are equivalent. Hence, the Newton's method converges whatever is used. Nevertheless, it must be chosen with care. Indeed, because of numerical phenomena, (6.33) is to be chosen when ρ^{n+1} is close to 0, whereas (6.34) is more adapted when ρ^{n+1} is not small. In the effective implementation of the method, either one formulation or the other is chosen, depending on the values reached during the iterations of the algorithm. Eventually, to ensure the stability of the numerical resolution of (6.32) when $\varepsilon \to 0$, the inverse of the derivatives of h and g are analytically computed and implemented as

$$\frac{1}{h'(y)} = \frac{\varepsilon}{\varepsilon + \Delta t} \mathrm{e}^{-\Delta t y/\varepsilon}, \qquad \frac{1}{g'(y)} = -\frac{y}{\varepsilon + \Delta t}.$$

Since $y > 0$, these two expressions are uniformly bounded with respect to ε when Δt is fixed. As a consequence, the cost of the numerical resolution of (6.32) does not increase with ε.

When $\varepsilon > 0$ is fixed, the scheme (6.27) is consistent with (6.7), since only quadrature formula and interpolation methods have been used to write it. The way all the terms are computed, as well as the numerical resolution of the non-linear equation (6.32), ensures the stability of the numerical computations in the small ε regime. Hence, when $\varepsilon \to 0$ with fixed discretization parameters, the scheme (6.27) becomes

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} = -\Big(d(x_i) + \rho^{n+1}\Big) + B_i^{n,0} + \tau(x_i - x_{j_0}),$$

where j_0 is such that $u_{j_0}^n = \max_i u_i^n$, and $B_i^{n,0}$ has been defined in (6.31).

We do not give a strict proof of the consistency of this scheme for the limiting Hamilton-Jacobi equation (6.8) since it is out of the scope of the project. However, we draw attention to a few important points which need to be taken into account while working with the scheme. In particular, the behavior of the quantity $\rho(t)$ is not well understood in the case of extinction. The problem is that intuitively, $\rho(t)$ must represent the density of the population — so that when it goes to zero, we expect an extinction. However, in a Hamilton-Jacobi case even when the $\rho(t)$ reaches zero, the population can still regrow after some time. This can be explained by the fact that after two limiting procedures (passing first to infinite system size and then to the infinite time horizon), the "size" of the population cannot be described straightforwardly. An accurate link between the quantities obtained as a result of stochastic and PDE simulation is also a question which requires further investigation when $\rho(t) \ll 1$.

Case $\varepsilon \to 0$: the numerical results

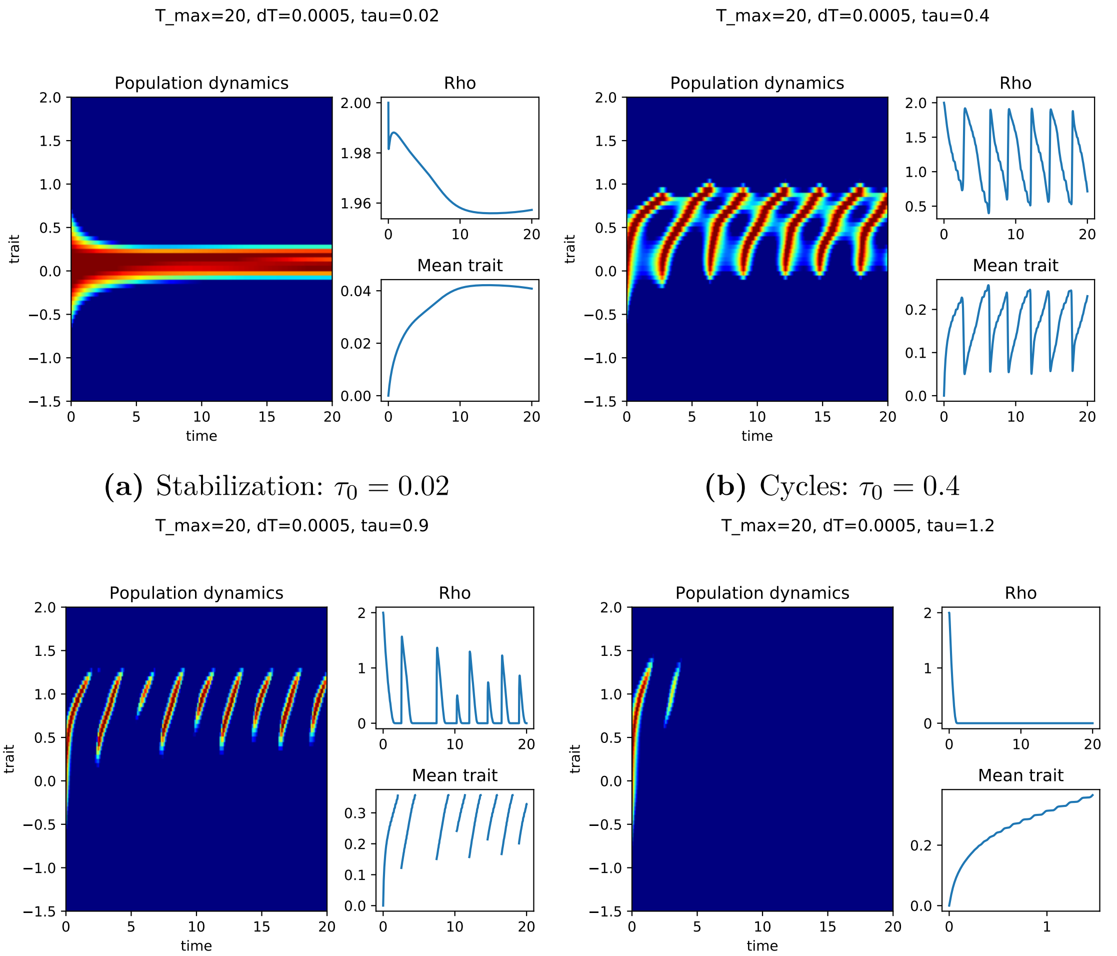
In this subsection, we simulate the dynamics of the population by considering a small value of ε and discuss the obtained results in order to compare them with previous simulations. Note that, in order to compare both, the stochastic and the Hamilton-Jacobi behaviors, the first thing to do is to obtain the simulations for the stochastic model also in the case where the HT rate is a smooth function as we do for the Hamilton-Jacobi case. We recall that, in Subsection 6.3.1, simulations for the stochastic model are done with a Heaviside function as HT rate since it is a more natural choice for simulation of a jump process.

On Figure 6.8 we simulate the population dynamics for $\varepsilon = 0.01$. Upon rescaling time (for chosen ε time scale $T = 10$ correspond, in fact, to $\frac{T}{\varepsilon} = 1000$ in previous simulations) and the variance parameter, we see the same patterns, with few differences.

On Figure 6.8a, we observe a stabilization of the mean trait, as in Figure 6.4a. Similarly, in Figure 6.8b, we observe cycles, but on the contrary to the PDE model, oscillations are not damped. Moreover, it is worth pointing out that the duration of a cycle here corresponds to what we observe in the corresponding stochastic plot (on Figure 6.4b) multiplied by $\varepsilon = 0.01$. On Figure 6.8c, we also observe a cycling behavior, but the population goes periodically extinct (i.e., the population reaches exponentially small value, of order $e^{1/\varepsilon}$), and then reborn. On the stochastic model, it corresponds to what is illustrated in Figure 6.5. It is not surprising that this behavior is difficult to observe on the stochastic model since very small populations are likely to go extinct.

On Figure 6.8d, we see a full extinction. On 6.8c, we see that the population undergoes periodic episodes of extinction and rebirth.

To finish with, let us give some flavor on the computational cost of the simulations for each type. In Table 6.1 we give a short overview of the elapsed time for the same



(a) Stabilization: $\tau_0 = 0.02$

(b) Cycles: $\tau_0 = 0.4$

(c) Cyclic extinction: $\tau_0 = 0.9$

(d) Full extinction: $\tau_0 = 1.2$

Figure 6.8 – Behavior of the population dynamics described by a PDE model for $\varepsilon = 0.01$ as the mutation rate τ is changing, $(b_r = d_r = 1,\ \sigma = 1,\ \varepsilon = 1)$.

values of parameters, but for different schemes. As expected, the individual-based model is the most expensive to compute. All the computations were performed in `numpy` library of Python on MacBook Pro (Intel Core i5 processor, 2,7GHz).

6.3.4 Comparison of the theoretical analysis of the Hamilton-Jacobi equation and the numerical simulations of the stochastic model

Formal computations

In this section, we conduct formal computations on the stochastic model, based on the analysis of the Hamilton-Jacobi equation performed in section 6.2.4. To fix ideas, we assume $d = 1$, (6.12)-(6.13), (6.3)-(6.4), and we fix all constants but τ_0, as in the previous section. We choose the function α as a Heaviside function (this is what has been used in the simulations), which is not a smooth function, and thus will lead to modifications compared to section 6.2.4.

We make a strong formal assumption: taking $K \gg 1$, we assume that the popu-

	$\Delta = 0.1,\ T = 10$	$\Delta = 0.01,\ T = 10$
SM ($N = 1000$)	3.883s	38.145s
SM ($N = 10000$)	15.805s	153.255s
PDE ($\varepsilon = 1$)	0.186s	1.673s
HJ ($\varepsilon = 10^{-2}$)	0.191s	1.636s
HJ ($\varepsilon = 10^{-6}$)	0.195s	1.656s

Table 6.1 – Elapsed time for simulation of population dynamics for different models (other parameters are fixed to values used throughout all the other simulations, $\tau = 0.5$).

lation behaves like a normally distributed random variable all the time, i.e.,

$$\nu_t^K(dx) = \rho(t)\frac{1}{\sqrt{2\pi}s(t)}e^{-\frac{|x-\bar{x}(t)|^2}{2s(t)^2}}dx,$$

for some standard deviation $s(t)$ and for $\bar{x}(t)$ is defined in (6.9). We expect $s(t)$ to be of the same order as σ, but giving a general estimate for $s(t)$ in function of $\bar{x}(t)$ seems intricate. The normalized size of the population $\rho(t) := \frac{N_t^K}{K}$ is approximately given by (see (6.11))

$$\rho(t) = \frac{1}{C}r(\bar{x}(t)), \tag{6.35}$$

where r is defined in (6.19).

We now formally compute the evolutionary singular state $x_\star$. Since α is a Heaviside function (which formally corresponds to the case when $\delta \to 0$ in (6.20)), our derivations must be slightly adapted. In particular, $\tau(x - \bar{x}(t))$ in (6.8) has to be replaced by

$$\int_{\mathbb{R}} \tau(x-y)\frac{\nu_t^K(dy)}{\rho(t)},$$

and accordingly, recalling that the weak derivative of a Heaviside is a Dirac mass at 0, $\tau'(0)$ in (6.20) has to be replaced by

$$\int_{\mathbb{R}} \tau'(\bar{x}(t)-y)\frac{\nu(dy)}{\rho(t)} = \frac{2\tau_0}{\sqrt{2\pi}s(t)}.$$

We find

$$x_\star = \frac{\tau_0}{\sqrt{2\pi}s_\star d_r},$$

where $s_\star$ is unknown, and corresponds to the standard deviation of the population at equilibrium concentrated at $x = x_\star$. Note that it corresponds to (6.20) with $\tilde{\delta} := s_\star\sqrt{\pi/2}$.

We now try to estimate $s_\star$. Formally, $s_\star$ should be such that $u_\star(x) := \frac{-(x-x_\star)^2}{2s_\star^2}$ is a stationary solution of (6.8). Differentiating twice, and applying at $x = x_\star$ we find

$$0 = b_r\sigma^2\left(u_\star''(x_\star)\right)^2 - 2d_r,$$

(with the reasonable assumption $\tau''(0) = 0$), which gives

$$s_\star = \sqrt{\sigma\sqrt{\frac{b_r}{2d_r}}}.$$

Numerically, we find $s_\star = 0.12$. We end up with the following formula:

$$x_\star = \frac{\tau_0}{\sqrt{2\pi\sigma} d_r} \sqrt[4]{\frac{2d_r}{b_r}}. \tag{6.36}$$

Stabilization

We run a numerical test on the stochastic model corresponding to stabilization, for $\tau_0 = 0.02$, and the other parameters as in Figure (6.4a). In this case, $x_\star$ corresponds to the mean trait of the population for large time. From, (6.36) we find $x_\star = 0.067$, and from (6.35), we obtain $\rho_\star = 1.99$, which corresponds to what we can see on Figure (6.4a).

Threshold for cycles

Since equation (6.21) remains unchanged, we obtain the following threshold for cycles (corresponding to (6.22)):

$$\tau_{cyc} = 2\pi d_r \sigma \sqrt{\frac{b_r}{2d_r}}.$$

With our choice of parameters, we obtain $\tau_{cyc} = 0.09$. This threshold corresponds to the numerical simulations.

Threshold for extinction

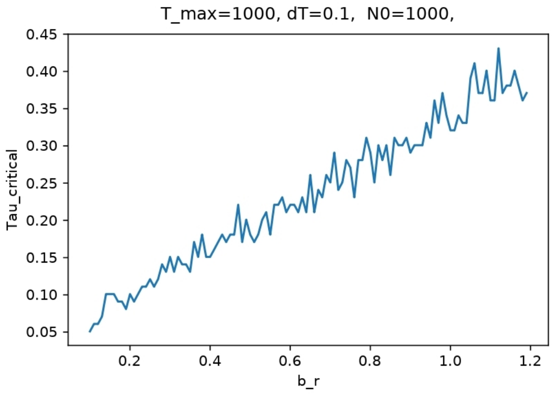
Using (6.23), we can also find a threshold for extinction:

$$\tau_{ext} := \sqrt{2\pi b_r d_r \sigma} \sqrt[4]{\frac{b_r}{2d_r}}.$$

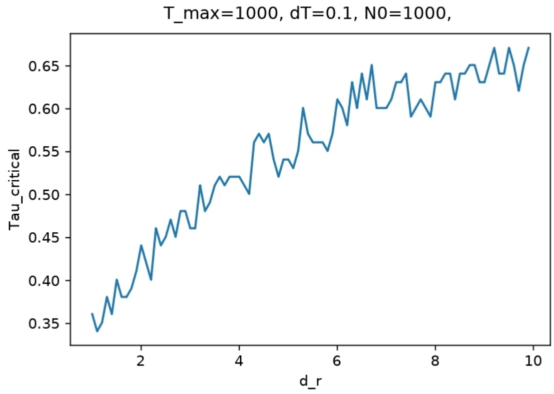
For our choice of parameters, we obtain $\tau_{ext} = 0.30$. This formula is qualitatively confirmed by the numerical experiments on the stochastic model illustrated in Figure 6.9a-6.9b. The numerics are performed as follows: we fix the birth b_r or the death rate d_r and save the first value of τ_0 under which the extinction occurs. Then, we increase the rate and save the next HT rate under which we have an extinction. Resulting curve for the birth rate is saved on Figure 6.9a (for death rate: Figure 6.9b). Non-concerned parameters remain fixed as in Subsection 6.3.1.

6.4 Conclusions

The first achievement of the paper consists in an accurate numerical study conducted on the stochastic model given by a point measure (6.1). To the best of our knowledge, in-depth analysis of the influence of the HT rate on the evolutionary dynamics has not been yet attempted. Along with its accuracy, the stochastic model reveals its limitation: an accurate theoretical description of what happens in each observed scenario from a mathematical point of view seems to be out of reach. However, we show that this model can be used for tracing back the lineage of the survived individuals through several generations.



(a) Birth dependency



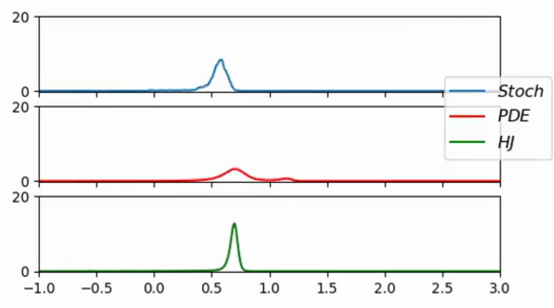
(b) Death dependency

Figure 6.9 – Dependency on the threshold for extinction τ_{ext} with respect to the birth rate b_r and death rate d_r

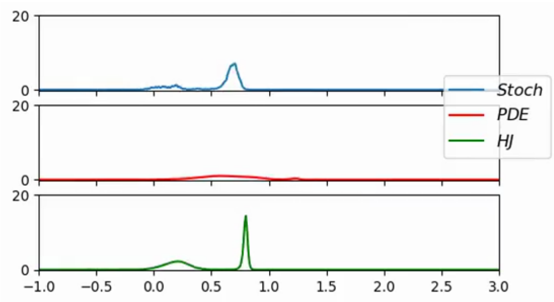
On the next step, in a comparative numerical study between the stochastic (individual-based) and the PDE (density) model both models exhibit the same behavior for a given set of parameters, which illustrates theoretical results from [54,56]. Minor differences — in particular, the presence of damping oscillations – can be explained by choice of a numerical scheme. However, further analysis shows that the classical PDE model defined by (6.2.2) leads to instabilities if we try to pass to an asymptotic setting under the small mutation assumption. Those instabilities are then resolved by a transformation of an initial model to a Hamilton-Jacobi type equation and using an asymptotic-preserving scheme. A further advantage of this approach is that the resulting equation (6.7) makes a more accessible subject of theoretical analysis.

Finally, in a Hamilton-Jacobi setting, we manage to numerically replicate the evolutionary rescue of a small population which we observe in the stochastic model. This phenomenon is illustrated for stochastic, PDE and HJ simulation on Figure 6.10. On Figures 6.10a-6.10c we trace the moment of the regrowth for different models. Figure 6.10a show the state of the population at a certain time: we see how the individuals are centered around a mean trait. For PDE and HJ model (red and green line respectively) we plot the density function, and on the first plot (blue), we approximate a histogram which describes ratio $\frac{N_t}{K}$ sorted by traits in the stochastic model. Stochastic simulations show the evolutionary rescue more distinctly: we see how the small number of non-mutated individuals rescues the whole population from extinction (transition from 6.10b to 6.10c). On the contrary, the transition on the PDE model is damped, and the regrowth is not clearly visible. It is due to, again, numerical instability of the PDE scheme for small values of the density function. Finally, HJ explicitly shows how the cycle occurs: the regrow of the "fit" individuals we see in the stochastic plot is reproduced by a change of the maximum point (see Figure 6.10b-6.10c).

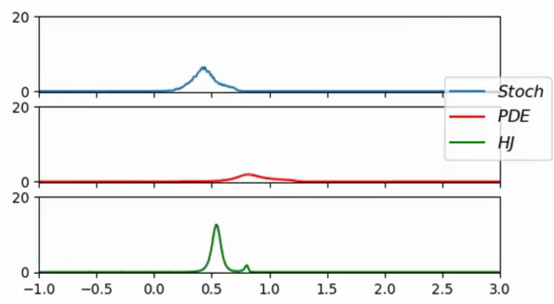
Even though a rigorous theoretical analysis is not provided, the formal computations suggest that the Hamilton-Jacobi framework is well suited to describe the dynamics accurately.



(a) $t = 167$



(b) $t = 198$



(c) $t = 216$

Figure 6.10 – Comparison of numerical simulations between the different models. $\tau_0 = 0.4$, $\varepsilon = 0.1$, $\delta = 0.001$ and other parameters as in Figure 6.4.

Part III

Systems of Reaction-Diffusion Equations

Chapter 7

A generalization of the SI epidemic model

Contents

7.1 Introduction

7.1.1 General Framework

Consider the following system of two Reaction-Diffusion equations, for $t \geq 0$, $x \in \mathbb{R}$

$$\begin{cases} \partial_t u(t,x) = d_1 \Delta_x u(t,x) + \Phi(u(t,x), v(t,x)), \\ \partial_t v(t,x) = d_2 \Delta_x v(t,x) + \Psi(u(t,x), v(t,x)), \end{cases} \tag{7.1}$$

with initial conditions $v(0,x) = v_0 > 0$ and $u(0,x) = u_0(x) \gneqq 0$, $d_1 > 0$, $d_2 \geq 0$. We aim to study the asymptotic behavior of (u, v) and the existence of transition waves, under our main assumption

$$\Psi(0, v) = 0. \tag{7.2}$$

This assumption means that, if $u = 0$, the dynamics of v consists only in pure diffusion. Apart from some other technical assumptions, we keep the approach quite general. In particular, we do not assume any monotonicity.

Note that assumption (7.2) implies that $(0, v)$ is a steady state for all $v > 0$. We are mainly interested in solutions which emanate from a small perturbation of such a steady state. Thus, we mostly consider initial datum $u_0(\cdot) \gneqq 0$ with compact support and a small L^∞ norm. For simplicity, we always assume that v_0 is constant. The spirit of our approach is to consider v_0 as a bifurcation parameter to describe the dynamics of (7.1). As we shall see in the sequel, many qualitative properties are deduced from the sign of

$$K_0 := \partial_u \Phi(0, v_0).$$

We point out that this quantity does not depend on the equation on v (i.e., d_2 and Ψ).

Our set of assumptions encompasses the *SI* (*Susceptible*, *Infected*) epidemic model [191], with $S(t,x) \equiv v(t,x)$ and $I(t,x) \equiv u(t,x)$, $\Phi(u,v) = \beta SI - \gamma I$ and $\Psi(u,v) = -uv$. In its simplest version (with space), it reads

$$\begin{cases} \partial_t I = \Delta I + \beta SI - \gamma I, \\ \partial_t S = -\beta SI, \end{cases} \tag{7.3}$$

with $\beta, \gamma > 0$. There is a vast literature on this model and variants (see, for example, [16, 120, 122, 123, 132, 133, 180, 284]). In this setting, we have $\Psi \leq 0$, thus $v(t, \cdot) \leq v_0$ for all times, and

$$\Phi(u(t,x), v(t,x)) \leq \partial_u \Phi(0, v_0) u(t,x).$$

This observation suggests to devote a special attention to the socalled *inhibiting case*, when

$$\Phi(u,v) \leq \partial_u \Phi(0, v_0) u, \quad \forall u, v \geq 0. \tag{7.4}$$

The term *inhibiting* comes from the idea that, as soon as the system leaves the steady state $(0, v_0)$, the reaction term $\Phi(u,v)$ decreases and the dynamics of u is *inhibited*. This assumption is somehow analogous to the KPP condition for scalar equation. Assumption (7.4) can seem very restrictive at first sight. However, if for example $\Psi \leq 0$, we know that $v(t,x) \leq v_0$. If, in addition, $v \mapsto \Phi(u,v)$ increases and $u \mapsto \frac{\Phi(u,v)}{u}$ decreases (i.e. $\Phi(\cdot, v)$ is *KPP*), then we can restrict $\Phi(u, \cdot)$ to $[0, v_0]$, and (7.4) is fullfilled .

In the inhibiting case, that is, under assumptions (7.2) and (7.4), the dynamics of (7.1) is governed, in some aspects, by the linearization of the first equation around $u = 0$, $v = v_0$:

$$\partial_t u = d_1 \Delta_x u + \partial_u \Phi(0, v_0) u. \tag{7.5}$$

Loosely speaking, we will see that u behaves as a solution of a scalar *KPP* equation. In particular, the sign of K_0 plays a crucial role. On the one hand, if $K_0 < 0$, the steady state $(0, v_0)$ attracts the dynamics. On the other hand, if $K_0 > 0$ then $(0, v_0)$ is repulsive and a propagation phenomenon occurs at speed $c_0 := 2\sqrt{d_1 \partial_u \Phi(0, v_0)}$.

In the general case, i.e., without assumption (7.4), we shall see that u behaves like a solution of a scalar *monostable* equation.

Our work can be seen as generalization of some results on the *SI* model to a broader class of systems. Although the *SI* model is widely studied, most of the methods available rely on the specific form of (7.3) and can hardly be adapted to a general class. Indeed, the methods often consist in performing a particular change of variable which reduces the system to a single equation. More precisely, from the equation on S in (7.3), we derive

$$S(t,x) = S_0 e^{-\beta R(t,x)}, \quad \text{with} \quad R(t,x) = \int_0^t I(\cdot, x).$$

Injecting this formula in the first equation and integrating on $(0,t)$ we find

$$\partial_t R = \Delta_x R + f(R) + I_0, \quad \text{where} \quad f(z) := S_0 \left(1 - e^{-\beta z}\right) - \gamma z.$$

Assuming $S_0 > \frac{\gamma}{\beta}$, f satsfies the (weak) KPP assumption, namely

$$\exists A > 0 : f(0) = f(A) = 0, \quad f > 0 \text{ in } (0,A), \quad \forall z \in (0,A), f(z) \leq f'(0)z.$$

Thus, in $\mathbb{R}^n \backslash \text{supp } I_0$, the system (7.3) reduces to a scalar KPP-equation, hence many classical results are available. Roughly speaking, we know that the dynamics of R is almost fully characterized by the linearization near $R = 0$ (which turns out to coincide with equation (7.5), with our notations).

This point of view on the *SI* model has both the advantage and disadvantage to rely on explicit calculations: the method is simple but not well suited for a general class of models. Being more general, our method provides a better insight into the classical *SI* model itself, and suprisingly, simpler proofs than those available in the litterature.

7.1.2 Assumptions and notations

General assumptions. Throughout the paper, we assume $d_1 > 0$, $d_2 \geq 0$ and

$$\begin{cases} \Psi(0,v) = 0 & \text{(Main assumption)} \\ \exists M > 0,\ \forall v \geq M,\ \Psi(\cdot, v) \leq 0 & \text{(Saturation in } v) \\ \Phi(0,\cdot) = 0 \quad \text{and} \quad \Psi(\cdot, 0) = 0, & \text{(Homogeneity)} \\ \|\Phi\|_{C^2}, \|\Psi\|_{C^2} < +\infty & \text{(Regularity).} \end{cases}$$

We also assume without loss of generality that $v_0 \leq M$. In particular, these assumptions imply $u(t,x) \geq 0$ and $0 \leq v(t,x) \leq M$.

Typically, the nonlinearities are of the form

$$\Phi(u,v) = vu(1-u) - \gamma u,$$
$$\Psi(u,v) = uvf(u,v),$$

where $\gamma > 0$ and f is smooth and bounded.

Notations. Set

$$K_0 := \partial_u \Phi(0, v_0), \quad \overline{K} := \sup_{\substack{u \geq 0 \\ v \in (0,M)}} \frac{\Phi(u,v)}{u},$$

and, if $K_0 > 0$,

$$c_0 := 2\sqrt{d_1 K_0}, \quad \bar{c} := 2\sqrt{d_1 \overline{K}}. \tag{7.6}$$

Note that the *inhibiting case* (7.4) corresponds to $K_0 = \overline{K}$ and $c_0 = \bar{c}$.

Linearized equation. For any $c > c_0$, we consider two scalar linear equations,

$$cU' - d_1 U'' - K_0 = 0,$$

and

$$cU' - d_1 U'' - \overline{K} = 0.$$

It suggests to define

$$\mu_0 := \frac{c - \sqrt{c^2 - c_0^2}}{2d_1} \quad \text{and} \quad \mu_0' := \frac{c + \sqrt{c^2 - c_0^2}}{2d_1}$$

the roots of $P_0(X) = d_1 X^2 - cX + K_0$. Respectively, for any $c > \overline{c}$, we define

$$\overline{\mu} := \frac{c - \sqrt{c^2 - \overline{c}^2}}{2d_1} \quad \text{and} \quad \overline{\mu}' := \frac{c + \sqrt{c^2 - \overline{c}^2}}{2d_1}$$

the roots of $\overline{P}(X) = d_1 X^2 - cX + \overline{K}$.

7.2 Main results

We introduce the main results of our work, first for the *inhibiting case*, then for the general case.

7.2.1 The inhibiting case

In this section, we always assume (7.4). We shall see that, in this case, the system shares some qualitative properties with a scalar *KPP* whose linearization is (7.5). We also recall that our set of assumptions encompasses the *SI* model.

Stability analysis

The first result establishes that $(0, v_0)$ is attractive when $K_0 < 0$ and repulsive when $K_0 > 0$.

Theorem 7.1 *Let $v_0 \in (0, M)$, $u_0(\cdot) \in L^\infty(\mathbb{R}^n)$ and assume (7.4).*

— *If $K_0 < 0$,*

$$\begin{cases} \lim\limits_{t \to +\infty} \left(\sup\limits_{x \in \mathbb{R}^n} |u(t,x)| \right) = 0, \\ \sup\limits_{x \in \mathbb{R}^n} |v(t,x) - v_0| \leq C \sup u_0, \quad \forall t \geq 0 \end{cases}$$

with a constant C independent of u_0.
If, in addition, $d_2 > 0$ and $u_0(\cdot) \geq 0$ is compactly supported, then

$$\lim_{t \to +\infty} \left(\sup_{x \in \mathbb{R}^n} |(u(t,x), v(t,x)) - (0, v_0)| \right) = 0.$$

— *If $K_0 > 0$ and $u_0(\cdot) \gneqq 0$, there exists $\delta_0 > 0$ (which only depends on $\Phi(u,v)$ near $(0, v_0)$) such that*

$$\limsup_{t \to +\infty} \left(\sup_{x \in \mathbb{R}^n} |(u(t,x), v(t,x)) - (0, v_0)| \right) \geq \delta_0.$$

This result expresses the fact that there exists a threshold phenomenon on v_0, through the sign of K_0. We stress out that the above theorem holds for arbitrarily large or small initial conditions $u_0(\cdot)$, as long as $u_0(\cdot) \in L^\infty(\mathbb{R}^n)$, $u_0(\cdot) \gneqq 0$ and $u_0(\cdot)$ is compactly supported. In particular, the case $K_0 > 0$ enjoys a *Hair-trigger effect*.

Note that in the particular case of the *SI*, the condition "$K_0 > 0$" reduces to the classical condition "$S_0 > \frac{\gamma}{\beta}$" for the outbreak of an epidemic.

In the degenerate case $K_0 = 0$, no general answer holds for the stability of $(0, v_0)$. Nevertheless, the inhibiting assumption (7.4) formally implies that the degenerate case $K_0 = 0$ corresponds to the case $K_0 < 0$. If, for example, $\Phi(u, v) = vu(1-u) - u$, $\Psi(u, v) = -uv$ and $v_0 = 1$, then $\partial_t u - \Delta u \leq -u^2$ which implies that u vanishes as fast as $\frac{1}{t}$ (uniformly in space).

Speed of propagation

While the solution vanishes uniformly in space when $K < 0$, the above result does not say anything on how the solution propagates through space when $K_0 > 0$. We give the following result.

Theorem 7.2 *Assume* (7.4)*,* $K_0 > 0$ *and* $u_0 \gneqq 0$ *has a compact support. Then, for* c_0 *defined in* (7.6)*,*

$$\forall c < c_0, \quad \limsup_{t \to +\infty} \left(\sup_{|x| \geq ct} |(u(t,x), v(t,x)) - (0, v_0)| \right) > 0, \tag{7.7}$$

$$\forall c > c_0, \quad \lim_{t \to +\infty} \left(\sup_{|x| \geq ct} |(u(t,x), v(t,x)) - (0, v_0)| \right) = 0.$$

Note that (7.7) is a weak information because it does not specify if either u, v or both u and v propagate. Nevertheless, we think that all three cases are possible under our quite general assumptions. Besides, since we only have a "$\limsup$" and not a "$\liminf$" in (7.7), this results only implies that the propagation phenomenon occurs at a speed which $\limsup_{t \to +\infty}$ is equal to c_0.

Transition waves

We now focus on the study of transition wave solutions propagating at constant speed, i.e solutions of (7.1) of the form $u(t,x) = U(x \cdot e + ct)$, $v(t,x) = V(x \cdot e + ct)$, with $c \geq 0$, $e \in \mathbb{S}^n$ and prescribed values in $-\infty$. Namely, setting $\xi := x \cdot e + ct \in \mathbb{R}$, we consider the system

$$\begin{cases} cU'(\xi) = d_1 U''(\xi) + \Phi(U(\xi), V(\xi)), \\ cV'(\xi) = d_2 V''(\xi) + \Psi(U(\xi), V(\xi)), \\ U, \ V \text{ smooth positive and bounded} \end{cases} \tag{7.8}$$

along with

$$\begin{cases} U(-\infty) = 0, \\ V(-\infty) = v_0. \end{cases} \tag{7.9}$$

We state the following result on existence and nonexistence of transition wave.

Theorem 7.3 *Assume* (7.4)*.*

1. *If $K_0 < 0$, there exists no transition wave.*
2. *If $K_0 > 0$, there exists no transition wave with speed $c < c_0$ and there exists a transition wave for any speed $c > c_0$.*

Even for the *SI* model, our proof of the nonexistence for $K_0 > 0$ and $c < c_0$ seems simpler that the current ones avalaible in the litterature which often rest on Tauberian methods with Laplace transforms, see e.g [126, 293, 296, 297].

Note that we do not impose any condition at $+\infty$ and thus the transition wave can be of various form. For example, in the *SI* model, a transition wave (U, V) must be such that V is monotone decreasing whereas U has the shape of a "bump" with $U(-\infty) = U(+\infty) = 0$.

We do not know if a transition wave exists in general for the critical velocity $c = c_0$. However, a positive answer holds for the *SI* model, see [68].

Another open question is the uniqueness of transition waves, which has also been proved for the *SI* model, see [29]. See also [134].

7.2.2 The general case

Now, we drop assumption (7.4) and focus on the general case. As for the inhibiting case, we will see that K_0 contains information on

1. the stability of the steady state $(0, v_0)$,
2. the speed of propagation of the solution,
3. the existence of transition waves.

We also state more precise results in the socalled *enhancing case*

$$\Phi(u, v_0) \leq \partial_u \Phi(0, v) u, \quad \forall u, v \geq 0, \tag{7.10}$$

which is somehow opposite to the *inhibiting case.*

Stability Analysis

The following theorem states that the stability of $(0, v_0)$ is determined by the sign of $K_0 > 0$.

Theorem 7.4 *Let $v_0 \in (0, M)$.*

— *If $K_0 < 0$, there exists $\varepsilon_0 > 0$ (which only depends on $\Phi(u, v)$ and $\Psi(u, v)$ near $(0, v_0)$) and $C > 0$ such that, for any $0 \leq u_0(\cdot) \leq \varepsilon_0$,*

$$\begin{cases} \lim\limits_{t \to +\infty} \left(\sup\limits_{x \in \mathbb{R}^n} |u(t, x)| \right) = 0, \\ \sup\limits_{x \in \mathbb{R}^n} |v(t, x) - v_0| \leq C \sup u_0, \quad \forall t \geq 0 \end{cases}$$

In addition, if $d_2 > 0$ and $u_0(\cdot) \leq \varepsilon_0$ is compatcly supported,

$$\lim_{t \to +\infty} \left(\sup_{x \in \mathbb{R}^n} |(u(t, x), v(t, x)) - (0, v_0)| \right) = 0.$$

— *If $K_0 > 0$, there exists $\delta_0 > 0$ (which only depends on $\Phi(u,v)$ near $(0,v_0)$) such that for any $u_0(\cdot) \gneqq 0$,*

$$\limsup_{t\to+\infty}\left(\sup_{x\in\mathbb{R}^n}|(u(t,x),v(t,x))-(0,v_0)|\right)\geq\delta_0.$$

As in Theorem 7.1 in the inhibiting case, the stability is fully determined by the sign of K_0. In addition, a *Hair-trigger* effect occurs when $K_0 > 0$. The difference with the inhibiting case lies in the first statement, where admissible initial conditions u_0 must have a small L^∞ norm.

In the enhancing case (7.10), if $K_0 > 0$, we have

$$\liminf_{t\to+\infty}\left(\sup_{x\in\mathbb{R}^n}|(u(t,x),v(t,x))-(0,v_0)|\right)\geq\delta_0.$$

In the degenerate case $K_0 = 0$, no general answer holds. For example, take $\Psi \equiv 0$ and $\Phi^{\pm}(u,v) = \pm u^{1+\varepsilon}(1-u))$. Let us also say that, in the inhibiting case (7.4), $K_0 = 0$ is typically similar to $K_0 < 0$. On the contrary, in the enhancing case (7.10), $K_0 = 0$ is typically similar to $K_0 > 0$. Note also that the case $K_0 = 0$ contains the case where u and v play symmetric roles.

Speed of propagation

Now let us establish an upper-lower bound on the asymptotic speed of propagation.

Theorem 7.5 *Assume (7.4), $K_0 > 0$ and $u_0 \gneqq 0$ has a compact support. Then, there exists $c_\star \in [c_0, \overline{c}]$ (where c_0 and $\overline{c}$ are defined in (7.6)) such that*

$$\forall c < c_\star,\quad \limsup_{t\to+\infty}\left(\sup_{|x|\geq ct}|(u(t,x),v(t,x))-(0,v_0)|\right)>0,$$
$$\forall c > c_\star,\quad \lim_{t\to+\infty}\left(\sup_{|x|\geq ct}|(u(t,x),v(t,x))-(0,v_0)|\right)=0.$$

Of course, there exist cases where $c_\star > c_0$. Note, for example, that $\Psi \equiv 0$ and $\Phi(u,v) = \Phi(u)$ reduces to a scalar *monostable* equation.

In the enhancing case (7.10), we have

$$\forall c < c_\star,\quad \liminf_{t\to+\infty}\left(\sup_{|x|\geq ct}|(u(t,x),v(t,x))-(0,v_0)|\right)>0.$$

Thus, $c_\star$ is the asymptotic speed of propagation in the classical sense, that is, a runner at speed $c < c_\star$ is eventually behind de propagation front.

Transition waves

We turn to the study of transition wave solutions, i.e., solutions of (7.8)-(7.9). We first give a result of non-existence.

Theorem 7.6 *If $K_0 > 0$, there exists no transition wave with speed $c < c_0$.*

Let us now state a result of existence. We assume

$$\exists M' > 0,\ \forall u \geq M', \quad \Phi(u,\cdot). \leq 0 \quad \text{(Saturation in } u) \tag{7.11}$$

We also define

$$g(c) := \frac{\sqrt{c^2 - c_0^2} - \sqrt{c^2 - \overline{c}^2}}{2c}, \quad \forall c > \overline{c},$$

Note that $g(\cdot)$ is decreasing, and that $g(+\infty) = 0$. This function somehow mesures *the lack of inhibition.*

We first deal with the case when $\Phi(\cdot, v)$ is *KPP*.

Theorem 7.7 *Assume* (7.11) *and*

$$\Phi(u,v) \leq \partial_u \Phi(0,v) u. \quad (\Phi(\cdot,v)\ \textit{KPP}) \tag{7.12}$$

If $K_0 > 0$, $c > \overline{c}$ *and*

$$d_1 \geq d_2 g(c), \tag{7.13}$$

there exists a transition wave with speed c*.*

Note that the range of c for which (7.13) holds is of the form $[\tilde{c}, +\infty)$, with $\tilde{c} = g^{-1}\left(\frac{d_1}{d_2}\right)$. Formally, the quantity $\overline{c}$ can be interpreted as the speed of propagation of the lack of inhibition. In particular, if

$$d_1 \geq d_2 \sqrt{1 - \frac{K_0}{\overline{K}}},$$

the above result prove the existence of transition waves for all $c > \overline{c}$. We also point out that (7.13) is equivalent to the condition

$$\overline{\mu} - \mu_0 < \frac{c}{d_2},$$

with μ_0, $\overline{\mu}$ defined in section 7.1.

Let us now drop the *KPP* assumption and deal with the general case.

Theorem 7.8 *Assume* (7.11) *and* $K_0 > 0$*. If* $c > \overline{c}$ *satisfies* (7.13) *and*

$$h(c) := c + \sqrt{c^2 - \overline{c}} - 2\sqrt{c^2 - c_0^2} > 0, \tag{7.14}$$

there exists a transition wave with speed c*.*

Assumption (7.14) somehow focuses on the lack of inhibition due to the fact that $\Phi(\cdot, v)$ is not *KPP*. When $c \to +\infty$, it reduces to

$$\overline{K} < 2K_0.$$

We also point out that (7.14) is equivalent to the condition

$$\overline{\mu} - \mu_0 < \mu_0,$$

with μ_0, $\overline{\mu}$ defined in section 7.1.

7.3 Stability analysis

This section is devoted to the analysis of the stability of the steady state $(0, v_0)$ and the proof of Theorem 7.1 and Theorem 7.4.

Note that, because of the continuum of steady states $\{(0, v) : v \geq 0\}$, the stability of $(u = 0, v = v_0)$ cannot be anything but degenerate. For simplicity, we only consider stability with respect to small L^∞ perturbations on u.

7.3.1 Stability when $K_0 < 0$

First, we show the following proposition on non-compactly-supported perturbations.

Proposition 7.9 *Assume $K_0 < 0$. There exists $\varepsilon_0 > 0$ and $C > 0$ such that if $\sup_{\mathbb{R}^n} u_0 < \varepsilon_0$, then*

$$\lim_{t\to+\infty} \left(\sup_{x\in\mathbb{R}^n} u(t, x) \right) = 0,$$
$$\sup_{\substack{t\geq 0\\ x\in\mathbb{R}^n}} |v(t, x) - v_0| \leq C \sup_{\mathbb{R}^n} u_0.$$

In the inhibiting case (7.4)*, $\varepsilon_0 = +\infty$.*

This result was not included in Theorem 7.1 and Theorem 7.4 since, here, we do not suppose that $u_0(\cdot)$ is compactly supported.

The above proposition implies that, if $K_0 < 0$, the steady state $(0, v_0)$ is stable degenerate. Namely, fix V an arbitrary small L^∞ neighborhood V of $(0, v_0)$. Then for any small L^∞ intial condition u_0, the solution remains trapped in V during all the dynamics.

If $u_0(\cdot)$ is not compactly supported, for example, if u_0 is constant, then the stability of $(0, v_0)$ is indeed degenerate, that is, it does not attract the solution.

Proof (of Proposition 7.9) For simplicity, we denote by a C all positive constants. As $K_0 < 0$, there exists a small $\delta > 0$ such that

$$-K_\delta := \sup_{\substack{u\in(0,\delta)\\ v\in(v_0-\delta,v_0+\delta)}} \frac{\Phi(u, v)}{u} < 0.$$

We define

$$T_\delta := \sup \left\{ T \geq 0 : \forall t \in (0, T),\ \sup_{\mathbb{R}^n} \Big| (u(t, \cdot), v(t, \cdot)) - (0, v_0) \Big| \leq \delta \right\}.$$

We have $T_\delta > 0$. In addition, u satsfies

$$\partial_t u - d_1 \Delta_x u + K_\delta u \leq 0, \quad \forall t \in (0, T_\delta),\ x \in \mathbb{R}^n. \tag{7.15}$$

We deduce

$$u(t, x) \leq \varepsilon e^{-K_\delta t}, \quad \forall t \in (0, T_\delta),\ x \in \mathbb{R}^n.$$

where $\varepsilon := \sup_{\mathbb{R}^n} u_0$. This estimates implies the uniform convergence of $u(t, \cdot)$ to 0.

Using this estimates in the equation on v gives

$$-C\varepsilon e^{-K_\delta t} \leq \partial_t v - d_2 \Delta_x v \leq C\varepsilon e^{-K_\delta t}, \quad \forall t \in (0, T_\delta),\ x \in \mathbb{R}^n.$$

We deduce

$$|v(t,x) - v_0| \leq \frac{C\varepsilon}{K_\delta}, \quad \forall t \in (0, T_\delta),\ x \in \mathbb{R}^n.$$

Combining the above inequalities, we see that if ε is smaller than some $\varepsilon_0 > 0$, then $T_\delta = +\infty$ and we deduce the first statement.

Note that in the inhibiting case (7.4), we directly have (7.15) for all $t \geq 0$ and we can conclude directly for any bounded u_0, i.e $\varepsilon_0 = +\infty$. ∎

Now, let us turn to compactly supported perturbations.

Proposition 7.10 *If $\sup_{\mathbb{R}^n} u_0 < \varepsilon_0$ (defined in Proposition 7.9) and $u_0(\cdot)$ has a compact support, then*

$$\lim_{t \to +\infty} \left(\sup_{x \in \mathbb{R}^n} |v(t,x) - v_0| \right) = \sup_{x \in \mathbb{R}^n} \left(\lim_{t \to +\infty} |v(t,x) - v_0| \right) = 0.$$

Proof We use the same notations as in the proof of Proposition 7.9. From equation (7.15), and the fact that $\sup_{\mathbb{R}^n} u_0 < \varepsilon_0$, we have

$$\partial_t u - d_1 \Delta_x u + K_\delta u \leq 0, \quad \forall t > 0,\ x \in \mathbb{R}^n. \tag{7.16}$$

We construct a supersolution to this equation. Fix $L \gg 1$, $0 < a \ll 1$ and $0 < b \ll 1$ real parameters (to be determined later) and a direction $e \in \mathbb{S}^n$. We define

$$\overline{u}_e(t,x) := L e^{-ax \cdot e - bt}, \quad \forall t \geq 0,\ x \in \mathbb{R}^n.$$

First, we choose L large enough and a small enough to ensure $u_0(\cdot) \leq \overline{u}(0, \cdot)$, which is possible since $u_0(\cdot)$ is bounded and compactly supported. Now, we compute

$$\partial_t \overline{u}_e(t,x) - d_1 \Delta_x \overline{u}_e(t,x) + K_\delta u_e(t,x) = \left(-b - d_1 a^2 + \frac{K_\delta}{L} \right) u_e(t,x),$$

which is positive if $L \gg 1$, $a \ll 1$ and $b \ll 1$. Thus, with a good choice of parameters, $\overline{u}_e$ is a supersolution of (7.16), and we have $u \leq \overline{u}_e$. Applying the same procedure in all directions $e \in \mathbb{S}^n$, we deduce

$$u(t,x) \leq L e^{-a|x| - bt}, \quad \forall t \geq 0,\ x \in \mathbb{R}^n. \tag{7.17}$$

Setting

$$f(t,x) := \mathbb{1}_{t>0} \left(\frac{1}{\alpha \pi t} \right)^{\frac{n}{2}} e^{\frac{-|x|^2}{\alpha t}}, \quad \alpha := \frac{4}{d_2}$$

the heat kernel in $\mathbb{R}^n$, we can write

$$v(t,x) = v_0 + \int_0^{+\infty} \int_{\mathbb{R}^n} f(t-s,z)[\Psi(u,v)](s, x-z) dz ds.$$

From $\Psi(u,v) \le Cu$ and (7.17), we deduce

$$|v(t,x) - v_0| \le C \int_0^{+\infty} \int_{\mathbb{R}^n} \mathbb{1}_{\{t-s>0\}} \frac{1}{(\alpha\pi(t-s))^{\frac{n}{2}}} e^{\frac{-|z|^2}{\alpha(t-s)}} e^{-a|x-z|-bs} dz ds.$$

This expression reaches its supremum at $x = 0$. With a change variable we have

$$\le C \int_0^{+\infty} \int_{\mathbb{R}^n} \mathbb{1}_{\{t-s>0\}} e^{-|z|^2 - a\sqrt{\alpha(t-s)}|z| - bs} dz ds$$

Using the dominate convergence theorem, we conclude that $v(t,\cdot)$ converges uniformly to v_0 when $t \to +\infty$. ∎

7.3.2 Unstability when $K_0 > 0$

Assume $K_0 > 0$. Our goal is to prove

$$\limsup_{t\to+\infty} \left(\sup_{x\in\mathbb{R}^n} |(u(t,x), v(t,x)) - (0,v_0)| \right) \ge \delta_0. \tag{7.18}$$

By contradiction, assume

$$\limsup_{t\to+\infty} \left(\sup_{x\in\mathbb{R}^n} |(u(t,x), v(t,x)) - (0,v_0)| \right) \le \delta_0.$$

We define the set

$$E := \left\{ (u,v) \in [0,+\infty)^2 : \partial_u \Phi(u,v) > 0 \right\}.$$

It is an open set of $[0,+\infty)^2$ which contains $(0,v_0)$, thus E contains a ball centered on $(0,v_0)$ of radius $\delta_0 > 0$. Note that δ_0 does not depend on $u_0(\cdot)$. Then, there exists $T > 0$ such that, for all $t > T$, $x \in \mathbb{R}^n$, $u(t,x)$ satisfies

$$\partial_t u - d_1 \Delta u - \alpha u \ge 0,$$

with some $\alpha > 0$. From classical results, we deduce

$$\inf_{x\in\mathbb{R}^n} \left(\liminf_{t\to+\infty} u(t,x) \right) = +\infty :$$

contradiction. We have proved (7.18)

7.4 Asymptotic speed of propagation

This section is devoted to the proof of Theorem 7.5. We show the following proposition.

Proposition 7.11 *Assume $K_0 > 0$ and $u_0(\cdot) \gneqq 0$ has a compact support.*

$$\forall c < c_0, \quad \limsup_{t\to+\infty} \left(\sup_{|x|\ge ct} |(u(t,x), v(t,x)) - (0,v_0)| \right) > 0, \tag{7.19}$$

$$\forall c > \overline{c}, \quad \lim_{t\to+\infty} \left(\sup_{|x|\ge ct} |(u(t,x), v(t,x)) - (0,v_0)| \right) = 0. \tag{7.20}$$

This result directly implies the existence and the bounds of the asymptotic speed $c_\star \in [c_0, \overline{c}]$. Theorem 7.2 is a consequence of the fact that, in the inhibiting case (7.4), $c_0 = \overline{c}$.

7.4.1 Upper bound

The upper bound (7.20) on the asymptotic speed of propagation is a consequence of the following estimate.

Proposition 7.12 *Assume $K_0 > 0$ and $u_0(x) \geq 0$ has a compact support. For all $e \in \mathbb{S}^n$, $t \geq 0$, $x \in \mathbb{R}^n$,*

$$u(t,x) \leq Ce^{\sqrt{\frac{\overline{K}}{d_1}}(x\cdot e+\overline{c}t)},$$

and

$$|v(t,x) - v_0| \leq C'e^{\varepsilon(x\cdot e+\overline{c}t)},$$

for some constants $C, C', \varepsilon > 0$.

Proof First, let us show the bound on $u(t,x)$. We set

$$\overline{u}(t,x) := Ce^{\sqrt{\frac{\overline{K}}{d_1}}(x\cdot e+\overline{c}t)}$$

for some $C > 0$ to be determined. Note that $\overline{\mu}(c = \overline{c}) = \sqrt{\frac{\overline{K}}{d_1}}$. Using $\Phi(u,v) \leq \overline{K}u$, and from elementary computations (detailed in section 7.5), we find

$$\partial_t\overline{u} - d_1\Delta_x\overline{u} - \Phi(\overline{u}, v) \geq 0.$$

Since u_0 has a compact support, choosing C large enough implies $u_0(\cdot) < \overline{u}(0,\cdot)$. From the parabolic comparison principle, we have $u(t,\cdot) \leq \overline{u}(t,\cdot)$ for all $t \geq 0$, which proves the bound on u.

Now, set

$$\underline{v}(t,x) := \max\left(v_0(1 - C'e^{\varepsilon(x\cdot e+\overline{c}t)}), 0\right)$$
$$\overline{v}(t,x) := \min\left(v_0(1 + C'e^{\varepsilon(x\cdot e+\overline{c}t)}), M\right)$$

with $0 < \varepsilon < \min\left(\sqrt{\frac{\overline{K}}{d_1}}, \frac{c}{d_2}\right)$. Choosing C' large enough, from the estimate on u and elementary computations (detailed in section 7.5), we find

$$\partial_t\overline{v} - d_2\Delta_x\overline{v} - \Psi(u, \overline{v}) \geq 0,$$
$$\partial_t\underline{v} - d_2\Delta_x\underline{v} - \Psi(u, \underline{v}) \leq 0.$$

Since $\underline{v}(0,\cdot) \leq v_0 \leq \overline{v}(0,\cdot)$, the parabolic comparison principle implies $\underline{v} \leq v \leq \overline{u}$. ■

7.4.2 Lower bound

Let $c < c_0$, $e \in \mathbb{S}^n$ and set

$$A(t) := \sup_{x\cdot e \geq -ct} |(u(t,x), v(t,x)) - (0, v_0)|, \quad \forall t > 0.$$

Our goal is to show $\limsup_{t\to+\infty} A(t) > 0$, which implies (7.19). By contradiction, we assume

$$\lim_{t\to+\infty} A(t) = 0. \tag{7.21}$$

We are going to construct a subsolution. For any $\varepsilon > 0$ and $\delta <$ small enough such that

$$\omega_\delta := \sqrt{\frac{c_0^2 - c^2}{4d_1^2} - \frac{\delta}{d_1}} = \sqrt{\frac{K_0 - \delta}{d_1} - \frac{c^2}{4d_1^2}}$$

is well defined, we set

$$\underline{U}(\xi) := \begin{cases} -\varepsilon \sin(\omega_\delta \xi) e^{\frac{c}{2d_1}\xi} & \text{if } \xi \in (-\frac{\pi}{\omega_\delta}, 0) \\ 0 & \text{if } \xi \in \mathbb{R}\backslash[-\frac{\pi}{\omega_\delta}, 0], \end{cases}$$

which satisfies

$$c\underline{U}' - d_2\underline{U}'' - (K_0 - \delta)\underline{U} = 0 \quad \text{on } (-\frac{\pi}{\omega_\delta}, 0).$$

As the sup of two solutions, $\underline{U}$ is in fact a (generalized) subsolution in all $\mathbb{R}$:

$$c\underline{U}' - d_2\underline{U}'' - (K_0 - \delta)\underline{U} \leq 0 \quad \text{on } \mathbb{R}$$

in a weak sense. Then, define

$$\underline{u}(t, x) := \underline{U}(x \cdot e + ct), \forall t > 0, x \in \mathbb{R}^n.$$

It is a (generalized) subsolution:

$$\partial_t \underline{u} - d_1 \Delta_x \underline{u} - (K_0 - \delta)\underline{u} \leq 0$$

in a weak sense.

From (7.21), we have $\frac{\Phi(u(t,\cdot),v(t,\cdot))}{u(t,\cdot)} \underset{t\to+\infty}{\longrightarrow} K_0$, uniformly in $\mathbb{R}^n$. Thus, there exists $T > 0$ such that $\forall t \geq T$, $\forall x \in \mathbb{R}^n$,

$$\partial_t u - d_1 \Delta_x u - (K_0 - \delta)u \geq 0.$$

Now, we choose $\varepsilon > 0$ small enough to ensure $\underline{u}(T, \cdot) < u(T, \cdot)$, which is possible since $\underline{u}(T, \cdot)$ has a compact support and $u(T, \cdot) > 0$. From the parabolic comparison principle, we deduce

$$u(t, \cdot) > \underline{u}(t, \cdot) \quad \forall t \geq T.$$

Then, from

$$\sup_{x \cdot e > -ct} \underline{u}(t, x) = \varepsilon,$$

we deduce $\liminf_{t\to+\infty} A(t) > \varepsilon$: contradiction.

7.5 Transition waves

In this section, we prove the results of existence and nonexistences of transition waves, namely Theorem 7.3 and Theorem 7.8. We recall that, by transition wave, we mean solutions of (7.8) for some speed $c \geq 0$.

7.5.1 Non-existence

We first prove Theorem 7.6, that is, the non existence of transition wave for $K_0 > 0$ and $c < c_0$. The proof relies on a quite simple argument dealing with the principal eigenvalue.

PROOF (OF THEOREM 7.6) Let $c \geq 0$ be such that there exists a solution (U, V) of (7.8). Take an arbitrary $\delta > 0$. Then, there exists $\xi_1 \in \mathbb{R}$ such that

$$\frac{\Phi(U(\xi), V(\xi))}{U(\xi)} - \partial_u \Phi(0, v_0) + \delta \geq 0, \quad \forall \xi < \xi_1.$$

The function

$$\tilde{U}(\xi) := U(\xi) e^{-\frac{c}{2d_1}\xi}$$

satisfies

$$-d_1 \tilde{U}'' = \left(K_0 - \delta - \frac{c^2}{4d_1} \right) \tilde{U}.$$

This implies that $K_0 - \delta - \frac{c^2}{4d_1}$ is smaller than the Dirichlet principal eigenvalue of the operator $z \mapsto -d_1 z''$ in any interval $(\xi_1 - a, \xi_1)$, $a > 0$. The latter is equal to $\frac{\pi^2}{a^2}$. That is, we have found that $c^2 > 4d_1 \left(K_0 - \delta - \frac{\pi^2}{a^2} \right)$. By arbitrariness of $\delta, a > 0$, we deduce that $c^2 \geq 4d_1 K_0$, i.e., $c \geq c_0$. ∎

Now, let us prove the non-existence of transition waves for $K_0 < 0$ in the inhibiting case.

Proposition 7.13 *Assume* (7.4). *If* $K_0 < 0$, *there exists no transition wave for any speed* $c \geq 0$.

We do not know whether this result extends to the general case. We also point out that, in the inhibiting case, if $K_0 < 0$, it is possible to construct transition waves with negative speed $c < 0$.

PROOF Assume $K_0 < 0$. By contraction, assume there exists (U, V) a transition wave with speed $c \geq 0$. From (7.4), U satisfies

$$cU' - d_1 U'' - K_0 U < 0.$$

1st case: $c = 0$ We have $-U'' \leq 0$ and U bounded. From the classical Liouville theorem, we find $U \equiv 0$: contradiction.

2nd case: $c > 0$ We have

$$\frac{\mathrm{d}}{\mathrm{d}\xi} \left(U'(\xi) e^{-\frac{c}{d_1}\xi} \right) \geq C \frac{e^{-\frac{c}{d_1}\xi}}{d_1} U(\xi),$$

and integrating from ξ to $+\infty$,

$$U'(\xi) \leq -\frac{C}{d_1} \int_{\xi}^{+\infty} e^{-\frac{c}{d_1}(z-\xi)} U(z) dz < 0.$$

We deduce that U is decreasing. As U is positive and $U(-\infty) = 0$, we deduce $U \equiv 0$: contradiction. ∎

7.5.2 Existence

Let us assume $K_0 > 0$ and fix $c > \overline{c}$. We aim to apply Schauder's theorem in some well chosen function set. For this, we construct a system of sub and supersolutions of (7.8). Namely, set

$$\begin{cases} \overline{U}(\xi) := \min\left(\dfrac{1}{B}e^{\mu_0\xi} + e^{(\overline{\mu}+\delta)\xi}, M'\right), \\ \underline{U}(\xi) := \max\left(e^{\mu_0\xi}\left[\dfrac{1}{B} - e^{\delta\xi}\right], 0\right), \\ \overline{V}(\xi) := \min\left(v_0 + Ce^{\varepsilon\xi}, M\right), \\ \underline{V}(\xi) = \max\left(v_0 - Ce^{\varepsilon\xi}, 0\right). \end{cases}$$

with $B, C, \varepsilon, \delta$ to be determined. Set

$$\Gamma := \Gamma_U \times \Gamma_V,$$

where

$$\Gamma_U := \left\{u \in C^2(\mathbb{R}) : \underline{U} \leq u \leq \overline{U}\right\}, \quad \Gamma_V := \left\{v \in C^2(\mathbb{R}) : \underline{V} \leq v \leq \overline{V}\right\}.$$

Lemma 7.14 *Under the conditions of Theorem 7.7 or Theorem 7.8, the quadruplet* $(\overline{U}, \underline{U}, \overline{V}, \underline{V})$ *is a system of sub and supersolutions, namely* $\Gamma \neq \emptyset$ *and for all* $(u, v) \in \Gamma$*,*

$$\begin{aligned} c\overline{U}' - d_1\overline{U}'' - \Phi(\overline{U}, v) &\geq 0, \\ c\underline{U}' - d_1\underline{U}'' - \Phi(\underline{U}, v) &\leq 0, \\ c\overline{V}' - d_2\overline{V}'' - \Psi(u, \overline{V}) &\geq 0, \\ c\underline{V}' - d_2\underline{V}'' - \Psi(u, \underline{V}) &\leq 0, \end{aligned}$$

provided:

$$0 < \varepsilon < \min\left(\mu_0, \frac{c}{d_2}\right), \quad 0 < \delta < \min(\varepsilon, \overline{\mu}' - \overline{\mu}), \quad \textit{and} \quad B \gg C \gg 1.$$

To prove the existence result of Theorem 7.3 in the inhibiting case, we replace $\overline{U}(\xi)$ with $\frac{1}{B}e^{\mu_0\xi}$ and apply the same method. We omit the details.

Before proving the lemma, let us show how we can construct a solution of (7.9) from such a system of sub and supersolutions. For $N > 0$, set

$$\Gamma_N := \left\{(u, v)\Big|_{[-N,N]} : (u, v) \in \Gamma\right\}$$

and the evolution operator

$$\mathcal{L}_N : \begin{array}{ccc} \Gamma_N & \to & \left(C^2([-N, N], \mathbb{R})\right)^2 \\ (u, v) & \mapsto & (\tilde{u}, \tilde{v}) \end{array}$$

where $\tilde{u}$ is the unique solution of

$$\begin{cases} c\tilde{u}' - d_1\tilde{u}'' - C\tilde{u} = \Phi(u, v) - Cu, \\ \tilde{u}(\pm N) = \overline{U}(\pm N), \end{cases}$$

and $\tilde{v}$ is the unique solution of

$$\begin{cases} c\tilde{v}' - d_2\tilde{v}'' - C\tilde{v} = \Psi(u,v) - Cv, \\ \tilde{v}(\pm N) = \overline{V}(\pm N). \end{cases}$$

where $\sup|\partial_u \Phi|,\ \sup|\partial_v \Psi| < C$. It is classical that this operator is well defined (see Theorems 4.3 and 6.8 in [162]). The functional space Γ_N is nonempty, closed, and convex. Besides, by standard elliptic estimates, the application $\mathcal{L}_N$ is compact and continuous. From Lemma 7.14 and the Maximum Principle, we have $\mathcal{L}_N(\Gamma) \subset \Gamma$. From Schauder fixed-point theorem, $\mathcal{L}_N$ admits a fixed point, i.e., there exists $(U_N, V_N) \in \Gamma_N$ solution of

$$\begin{cases} c\tilde{U}_N' - d_1\tilde{U}_N'' - \Phi(U_N, V_N) = 0, \\ c\tilde{V}_N' - d_2\tilde{V}_N'' - \Psi(U_N, V_N) = 0, \end{cases}$$

$$\begin{cases} \tilde{U}_N(\pm N) = \overline{U}(\pm N), \\ \tilde{V}_N(\pm N) = \overline{V}(\pm N). \end{cases}$$

From classical Schauder estimates, we have that $U_{N'}$, $V_{N'}$ are bounded in $C^{2,\alpha}([-N,N])$, uniformly in $N' \geq N$, and thus converge (up to an extraction) to some U, V in $C^2([-N,N])$. Using a standard diagonal argument, we are provided with $(U,V) \in \Gamma$ solution of (7.8)-(7.9).

Proof (of Lemma 7.14) It is immediate that $\Gamma \neq \emptyset$.
Supersolution $\overline{V}$. Note that $\exists\ \xi_{\overline{V}} > 0$ such that

$$\overline{V}(\xi) < M \quad \text{iff} \quad \xi < \xi_{\overline{V}}.$$

Besides, from $\Psi(u,0) = \partial_v \Psi(0,v) = 0$, there exists a constant L_1 such that

$$\Psi(u,v) \leq L_1 u, \quad \forall u \in [0, M'],\ v \in [0, M].$$

Hence, for all $\xi < \xi_{\overline{V}}$ and $u \in \Gamma_u$, we have

$$\begin{aligned} c\overline{V}'(\xi) - d_2\overline{V}''(\xi) - \Psi(u(\xi), \overline{V}(\xi)) &\geq c\overline{V}'(\xi) - d_2\overline{V}''(\xi) - L_1\overline{U}(\xi) \\ &= e^{\varepsilon\xi}\left[C(c\varepsilon - d_2\varepsilon^2) - L_1\frac{1}{B}e^{(\mu_0-\varepsilon)\xi} - L_1 e^{(\overline{\mu}+\delta-\varepsilon)\xi}\right], \end{aligned}$$

and, choosing $\varepsilon < \mu_0$ $(\leq \overline{\mu})$,

$$\geq e^{\varepsilon\xi}\left[C(c\varepsilon - d_2\varepsilon^2) - L_1\frac{1}{B}e^{(\mu_0-\varepsilon)\xi_{\overline{V}}} - L_1 e^{(\overline{\mu}+\delta-\varepsilon)\xi_{\overline{V}}}\right].$$

Noticing that $\xi_{\overline{V}} \underset{C\to+\infty}{\longrightarrow} -\infty$, the above expression is negative as $\varepsilon < \frac{c}{d_2}$, $B \gg C \gg 1$.

Subsolution $\underline{V}$. Similar as for $\overline{V}$.

Subsolution $\underline{U}$.
There exists $\xi_{\underline{U}} > 0$ such that

$$\underline{U}(\xi) > 0 \quad \text{iff} \quad \xi < \xi_{\underline{U}},$$

and that

$$\xi_{\underline{U}} \underset{B\to+\infty}{\longrightarrow} -\infty. \tag{7.22}$$

From a Taylor expansion, using $K_0 = \partial_u \Phi(0, v_0)$ and $\Phi(0, \cdot) = 0$, we have, for all $(u, v) \in \Gamma$

$$\Phi(u, v) \geq K_0 u - L_4 u^2 - L_5 u|v_0 - v|, \quad \text{with } L_4,\ L_5 > 0. \tag{7.23}$$

For all $\xi < \xi_{\underline{U}}$, we have

$$\underline{U}'(\xi) = \left[\frac{1}{B}\mu_0 - (\mu_0 + \delta)e^{\delta\xi}\right] e^{\mu_0\xi}, \quad \underline{U}''(\xi) = \left[\frac{1}{B}\mu_0^2 - (\mu_0 + \delta)^2 e^{\delta\xi}\right] e^{\mu_0\xi},$$

thus, for all $v \in \Gamma_v$, $\xi < \xi_{\underline{U}}$,

$$\begin{aligned}
&c\underline{U}' - d_1\underline{U}'' - \Phi(\underline{U}(\xi), v(\xi)) \\
&= \left[\frac{1}{B}\left(c\mu_0 - d_1\mu_0^2\right) - \left(c(\mu_0+\delta) - d_1(\mu_0+\delta)^2\right)e^{\delta\xi}\right] e^{\mu_0\xi} - \Phi(\underline{U}(\xi), \underline{V}(\xi)) \\
&= P_0(\mu_0+\delta)e^{(\mu_0+\delta)\xi} + K_0\underline{U}(\xi) - \Phi(\underline{U}(\xi), \underline{V}(\xi)),
\end{aligned}$$

using (7.23),

$$\leq P_0(\mu_0+\delta)e^{(\mu_0+\delta)\xi} + L_4\underline{U}^2 + L_5|v_0 - v|\underline{U},$$

from $\underline{U} \leq \frac{1}{B}e^{\mu_0\xi}$,

$$\leq e^{(\mu_0+\delta)\xi}\left(P_0(\mu_0+\delta) + L_4\frac{1}{B^2}e^{(\mu_0-\delta)\xi} + L_5\frac{C}{B}v_0 e^{(\varepsilon-\delta)\xi}\right),$$

and choosing $\delta < \min(\mu_0, \varepsilon)$,

$$\leq e^{(\mu_0+\delta)\xi}\left(P_0(\mu_0+\delta) + L_4\frac{1}{B^2}e^{(\mu_0-\delta)\xi_{\underline{U}}} + L_5\frac{C}{B}v_0 e^{(\varepsilon-\delta)\xi_{\underline{U}}}\right).$$

We recall that, since $c > c_0$, $P_0(X) := d_1X^2 - cX + K_0$ has two roots $\mu_0 < \mu_0'$. Thus choosing $\delta < \mu_0' - \mu_0$, we have $P_0(\mu_0+\delta) < 0$. Recalling (7.22), the right member becomes negative as $B \gg C \gg 1$.

Supersolution $\overline{U}$.

There exists $\xi_{\overline{U}} > 0$ such that

$$\overline{U}(\xi) < M' \quad \text{iff} \quad \xi < \xi_{\overline{U}},$$

and that

$$\xi_{\overline{U}} \leq \frac{1}{\overline{\mu}+\delta}\ln M'. \tag{7.24}$$

First case - proof of Theorem 7.8. Assume (7.14). We recall that (7.13) comes from the condition

$$\overline{\mu} - \mu_0 \leq \frac{c}{d_2}$$

and (7.14) from

$$\overline{\mu} - \mu_0 < \mu_0,$$

with μ_0, $\overline{\mu}$ defined in section 7.1.2. From a Taylor expansion, using the definition of K_0, $\overline{K}$ and $\Phi(0, \cdot) = 0$, there exists $L_2, L_3 > 0$ such that for all $v \in \Gamma_V$, $\xi < \xi_{\overline{U}}$,

$$\begin{aligned}
\Phi(\overline{U}(\xi), v(\xi)) &\leq \Phi(e^{(\overline{\mu}+\delta)\xi}, v(\xi)) + \frac{1}{B}e^{\mu_0\xi} \sup_{z\in[0,\overline{U}(\xi)]} \partial_u\Phi(z, v) \\
&\leq \overline{K}e^{(\overline{\mu}+\delta)\xi} + \frac{1}{B}e^{\mu_0\xi}\left(K_0 + L_2\overline{U}(\xi) + L_3|v(\xi) - v_0|\right).
\end{aligned} \tag{7.25}$$

For all $v \in \Gamma_v$, $\xi < \xi_{\overline{U}}$ we have

$$\begin{aligned}
&c\overline{U}' - d_1\overline{U}'' - \Phi(\overline{U}, v) \\
&= \frac{1}{B}(c\mu_0 - d_2\mu_0^2)e^{\mu_0\xi} + \left(c(\overline{\mu}+\delta) - d_1(\overline{\mu}+\delta)^2\right) e^{(\overline{\mu}+\delta)\xi} - \Phi(\overline{U}(\xi), v(\xi)) \\
&= \frac{1}{B}\left(K_0 - P_0(\mu_0)\right) e^{\mu_0\xi} + \left(\overline{K} - \overline{P}(\overline{\mu}+\delta)\right) e^{(\overline{\mu}+\delta)\xi} - \Phi(\overline{U}(\xi), v(\xi))
\end{aligned}$$

using $P_0(\mu_0) = 0$ and (7.25),

$$\begin{aligned}
&\geq -\overline{P}(\overline{\mu}+\delta)e^{(\overline{\mu}+\delta)\xi} - L_2\frac{1}{B}e^{\mu_0\xi}\overline{U}(\xi) - L_3\frac{1}{B}e^{\mu_0\xi}|v(\xi) - v_0| \\
&\geq -\overline{P}(\overline{\mu}+\delta)e^{(\overline{\mu}+\delta)\xi} - L_2\frac{1}{B}e^{(\mu_0+\overline{\mu}+\delta)\xi} - L_2\frac{1}{B^2}e^{2\mu_0\xi} - L_3\frac{1}{B}Ce^{(\mu_0+\varepsilon)\xi} \\
&= e^{(\overline{\mu}+\delta)\xi}\left[-\overline{P}(\overline{\mu}+\delta) - L_2\frac{1}{B}e^{\mu_0\xi} - L_2\frac{1}{B^2}e^{(2\mu_0-\overline{\mu}-\delta)\xi} - L_3\frac{1}{B}Ce^{(\mu_0+\varepsilon-\overline{\mu}-\delta))\xi}\right].
\end{aligned}$$

Using (7.13) and (7.14), we can choose δ small enough such that $\overline{\mu} - \mu_0 + \delta < \min(\mu_0, \varepsilon)$. We have

$$\begin{aligned}
&c\overline{U}' - d_1\overline{U}'' - \Phi(\overline{U}, v) \\
&\geq e^{(\overline{\mu}+\delta)\xi}\left[-\overline{P}(\overline{\mu}+\delta) - L_2\frac{1}{B}e^{\mu_0\xi_{\overline{U}}} - L_2\frac{1}{B^2}e^{(2\mu_0-\overline{\mu}-\delta)\xi_{\overline{U}}} - L_3\frac{1}{B}Ce^{(\mu_0+\varepsilon-\overline{\mu}-\delta))\xi_{\overline{U}}}\right].
\end{aligned} \tag{7.26}$$

We recall that, since $c > \overline{c}$, $\overline{P}(X) := d_1X^2 - cX + \overline{K}$ has two roots $\overline{\mu} < \overline{\mu}$. Thus choosing $\delta < \overline{\mu}' - \overline{\mu}$, we have $\overline{P}(\overline{\mu}+\delta) < 0$. From (7.24), as $B \gg C \gg 1$, the right member of (7.26) becomes positive.

Second case - proof of Theorem 7.7. Assume (7.12). In this case, replace (7.25) by

$$\Phi(\overline{U}(\xi), v(\xi)) \leq \overline{K}e^{(\overline{\mu}+\delta)\xi} + \frac{1}{B}e^{\mu_0\xi}\left(K_0 + L_3|v(\xi) - v_0|\right).$$

It corresponds to the first case with $L_2 = 0$. Thus, we can conclude as above (in this case, we only need $\delta < \varepsilon$, thus we do not use (7.14)). ∎

Chapter 8

Modelization of social unrest

Contents

8.1 Introduction

8.1.1 The dynamics of Social Unrest

The principal motivation of this work comes from the modelization, in social sciences, of the dynamics of Social Unrests. Our approach is in the spirit of [46,48,52]. For a litterature on the subject, see [46, 58, 59, 167, 168] and references therein.

We refer to Social Unrest, denoted `SU`, as the quantity of rioting activities or civil disobedience. We can think of `SU` as the total number of illegal actions weighted by their respective importance.

We do not aim to discuss the sociological origins of `SU`. Instead, we propose a model built from simple ingredients to account for recurrent patterns observed in the field.

Our model also features a level of Social Tension. By Social Tension, denoted `ST`, we mean a quantity accounting for the resentment of a population towards society, would it be for political, economic, or sociological reasons.

The philosophy of our approach rests on the hypothesis that `SU` and `ST` follow coupled dynamics. A key assumption is to *discard the intrinsic dynamics of* `ST` in absence of `SU`. In other words, we assume that in a normal situation, the evolution of `ST` occurs on a larger time scale than episodes of `SU`. This assumption allows focussing more clearly on the interplay between `SU` and `ST`.

Let us now review the most common characteristics of the dynamics of `SU` and define some vocabulary.

First, movements of `SU` often occur as episodic *bursts*, commonly called *riots*, *revolutions*, etc.

However, all episodes of `SU` are commonly considered to have been triggered by a single *exogenous event* (or triggering event). One can think of this *exogenous event* as the straw that broke the camel's back. Whether such a single event can initiate a *burst* of `SU` depends on the level of `ST`.

`ST` plays the role of an *activator*. Namely, if `ST` is high enough, a small *exogenous event* triggers a *burst* of `SU`; whereas if `ST` is low, the same event is followed by a prompt *resumption of calm*.

These observations suggest, from the modelization point of view, that an intrinsic mechanism of *relaxation* occurs on `SU` in a context of low `ST`. The *relaxation* rate accounts for various sociological features, such as fatigue, police repression, incarceration, etc.

On the other hand, a high `ST` activates an *endogeneous growth* of `SU`. In other words, if `ST` is above a threshold level, then a mechanism of self-reinforcement occurs on `SU`. This is analogous to a flame propagation: an *endogenous* growth is activated when the temperature is high enough. One can think of this *endogeneous* feature as the gregarious dimension of social movements: the larger the movement, the more prone an individual is to join it.

Naturally, this self-reinforcement mechanism has to be counterbalanced with a *saturation* effect, which accounts for the limited number of individuals, resources, goods to be damaged, etc.

Another important features usually observed during *bursts* of `SU` is the *geographical spread*. This phenomenon can somehow be interpreted as the result of the rioters' movement. A striking example is the case of the 2005 riots in France (see [58, 267]), which was triggered by the death of two young men trying to escape the police in Clichy-sous-Bois, a poor suburb of Paris. This event occurred in a context of high social tension and was the spark for the riots that spread throughout the country and lasted over three weeks.

Even if social movements can take many different forms, a first naive classification would be to distinguish a *riot*, which lasts a couple of weeks and then fades, from a *revolution*, which lasts longer and can result in significant political or sociological changes (think of the French Revolution, or the Arab Spring [201, 214]. See also [15]).

A *riot* can be interpreted as a *burst* of `SU` which *decreases* `ST`. Once `ST` falls below a threshold value, `SU` fades and eventually stops. This case is called *tension inhibiting*. It is qualitatively comparable to the outburst of a disease, which propagates until the number of susceptible individuals falls below a certain threshold. This behavior is well captured by the famous *SI* epidemic model [191], with $S =$ `ST` and $I =$ `SU`.

A *revolution* can be seen as a *burst* of SU which *increases* ST. This dynamics of positive feedback escalates towards a sustainable state of high SU. This case is called *tension enhancing*. From the modelization point of view, it is comparable to a cooperative system.

The dynamics of SU suggests different classes of mathematical models: epidemic models on the one hand and monotone systems on the other hand. Those two classes of model are studied quite separately in the literature. Our aim here is to propose a single framework that encompasses both behaviors.

8.1.2 Construction of the model

In the spirit of [46, 48, 52], and according to the previous section, we propose a mathematical model to account for the dynamics of SU and ST. We denote SU by $u(t,x) \geq 0$, depending on time $t \geq 0$ and location $x \in \mathbb{R}^n$, and ST by $v(t,x) \geq 0$. Our model takes the general form of a system of two Reaction-Diffusion equations

$$\begin{cases} \partial_t u(t,x) = d_1 \Delta_x u(t,x) + \Phi(u(t,x), v(t,x)), \\ \partial_t v(t,x) = d_2 \Delta_x v(t,x) + \Psi(u(t,x), v(t,x)), \\ u(0,x) := u_0(x), \quad v(0,x) := v_0(x). \end{cases}$$

with $d_1 > 0$, $d_2 > 0$ and Φ, Ψ that we will specify.

For homogeneity, we assume $\Phi(0,v) = 0$. In other words, we normalize at 0 the base value of u in the absence of any unusual event.

We model the *single exogenous event* as a small perturbation of $u = 0$. We include it in the initial condition $u_0(x) \gneqq 0$, that we shall take of small intensity and with compact support.

The diffusion terms $d_1\Delta_x u(t,x)$ and $d_2\Delta_x v(t,x)$ describe the influence that one location has on its *geographical neighbors*.

The term Φ can be chosen of the form

$$\Phi(u,v) := r(v)G(u) - \omega u$$

The *endogeneous* factor (or self-reinforcement/saturation mechanism) is modeled by the function $G(\cdot)$ that we choose of the KPP type, for example $G(z) = z(1-z)$ for $z \in (0,1)$.

The paramater $\omega > 0$ is the natural rate of *relaxation* of SU in absence of self-reinforcement.

The endogenous factor is regulated by $r(v(t,x))$, which models the role of *activator* played by ST. We choose $r(\cdot)$ to be nonnegative and nondecreasing. We can think of this term as an on-off switch. For example, $r(\cdot)$ can be linear $r(v) = v$, or take the form

$$r(v) := \frac{1}{1 + e^{(v-\alpha)\beta}}.$$

Here, $\alpha \geq 0$ is a threshold value while $\beta > 0$ measures the stiffness of the transition between the relaxed state and the excited state. The case $\beta = +\infty$ corresponds to $r(v) := \mathbb{1}_{v>\alpha}$.

Denoting $v_\star$ the threshold on v above which a *burst* of u occurs from a small *exogenous event*, we have $v_\star := r^{-1}\left(\frac{\omega}{G'(0)}\right)$.

The term Ψ is of particular importance since it models the feedback of u on v. Since we chose to *neglect the intrinsic dynamics of* ST in absence of SU, we assume $\Psi(0, v) = 0$. This can be considered as our main assumption.

In the same spirit, as long as $d_2 > 0$, we shall take v_0 constant so that $(u = 0, v = v_0)$ is a state of equilibrium.

In addition, we assume homogeneity, i.e $\Phi(u, 0) = 0$, and a saturation phenomenon on v, i.e $\forall v \geq 1$, $\Psi(\cdot, v) \leq 0$.

Of course, we want the threshold $v_\star$ to lie in $(0, 1)$ in order to cover both possibilities of a *burst* or a *resumption of calm*. Namely, we assume

$$\frac{\omega}{G'(0)} \in (r(0), r(1)).$$

Taking the above into account we choose Ψ of the form

$$\Psi(u, v) := uvf(u, v)$$

for a function f to be determined. We will mainly focus on the two following particular choices of f. Each case illustrates a different qualitative behavior.

1. *The tension inhibiting case:* $f \leq 0$. In this case, a *burst* of u will *decrease* v. We expect this case to describe a *riot*, and to behave comparably to the *SI* epidemic model. As an example, we can take

$$f(u, v) := -1.$$

   This particular choice could even be called *strongly inhibiting* since, for all $v > 0$, $u \mapsto \Psi(u, v) = uvf(u, v)$ is decreasing.
2. *The tension enhancing case:* $f \geq 0$. In this case, a *burst* of u will *increase* v. We expect this case to describe a *revolution*, and to behave comparably to a monotone system. As an example, we can take

$$f(u, v) := (1 - v).$$

   This particular choice could even be called *strongly enhancing* since, for all $v > 0$, $u \mapsto \Psi(u, v) = uvf(u, v)$ is increasing.

We will perform a more detailed analysis on those two cases in the sequel.

8.2 General framework

8.2.1 Assumptions and notations

We consider $u(t, x)$, which stands for the level of SU at time $t \geq 0$ and location $x \in \mathbb{R}^n$, and $v(t, x)$, which stands for the level of ST, solution of

$$\begin{cases} \partial_t u = d_1 \Delta_x u + r(v)G(u) - \omega u, \\ \partial_t v = d_2 \Delta_x v + uvf(u, v), \\ u(0, x) := u_0(x), \quad v(0, x) := v_0, \end{cases} \tag{8.1}$$

under the following assumptions:

— $d_1, d_2 > 0$; $0 \lneqq u_0(x) < 1$; v_0 is constant and $0 < v_0 < 1$.

— $G(\cdot)$ is of the KPP type, i.e

$$u \mapsto \frac{G(u)}{u} \text{ decreases} \quad \text{and} \quad G(0) = G(1) = 0.$$

For example, $G(u) = u(1-u)$.

— $r(\cdot)$ is smooth, nonnegative and increasing.

— $f(\cdot,\cdot)$ is smooth and features a saturation effect, namely

$$\forall v \geq 1, \quad f(\cdot, v) \leq 0.$$

— We assume

$$\frac{\omega}{G'(0)} \in (r(0), r(1)).$$

We define

$$v_\star := r^{-1}\left(\frac{\omega}{G'(0)}\right) \in (0,1).$$

We will see $v_\star$ is the threshold on v_0 above which an outburst of u occurs. This can be seen intuitively by studying the constant steady states of (8.1). For any fixed $v \in (0,1)$, consider the scalar equation

$$r(v)G(u) - \omega u = 0, \quad u \geq 0. \tag{8.2}$$

— If $v \leq v_0$, then (8.2) has exactly one solution $u = 0$ (calm state).

— If $v > v_0$, then (8.2) has exactly two solutions, $u = 0$ (calm state) and $u = u_\star(v)$ (excited state), defined by

$$u_\star(v) := \left(z \mapsto \frac{G(z)}{z}\right)^{-1}\left(\frac{\omega}{r(v)}\right).$$

Note that $v \mapsto u_\star(v)$ is increasing and $\lim_{v \to v_\star} u_\star(v) = 0$.

We shall see in the sequel that the steady state $(0, v_0)$ is stable if $v_0 < v_\star$ and unstable if $v_0 > v_\star$. Considering v_0 as a parameter, $v_\star$ is a bifurcation point.

An immediate consequence of our assumptions along with the parabolic comparison principle is the following

Lemma 8.1 *Let $u(t,x)$ and $v(t,x)$ a solution of* (8.1). *We have, for all $t > 0$, $x \in \mathbb{R}^n$,*

$$0 < u(t,x) < 1 \quad \text{and} \quad 0 < v(t,x) < 1$$

In the sequel, we are also interested in qualitative properties of transition waves, i.e solutions of (8.1) of the form $u(t,x) = U(x \cdot e + ct)$, $v(t,x) = V(x \cdot e + ct)$, with $c \geq 0$, $e \in \mathbb{S}^n$ and prescribed values in $-\infty$. Namely, setting $\xi := x \cdot e + ct \in \mathbb{R}$, we consider $U(\xi)$ and $V(\xi)$ bounded positive solutions of

$$\begin{cases} cU' = d_1 U'' + r(V)G(U) - \omega U, \\ cV' = d_2 V'' + UVf(U,V), \\ U, V \text{smooth, bounded and positive} \end{cases} \tag{8.3}$$

along with

$$\begin{cases} U(-\infty) = 0, \\ V(-\infty) = v_0. \end{cases} \tag{8.4}$$

8.2.2 Main results

In our attempt to propose a single model to account for both *riots* and *revolutions*, we give an example which combines the *tension inhibiting* and the *tension enhancing* case. Consider our main equation (8.1) with

$$f(u,v) := (1-v)(v-1/2),$$

$$\frac{\omega}{G'(0)} \in (r(0), r(1/2)) .$$

The latter assumption implies $v_\star < \frac{1}{2}$. This model features a double threshold phenomenon. From the modelization point of vue, a small *exogenous event* can lead to three different situations:

- For small initial ST: *resumption of calm.*
- For intermediate ST: *burst of a riot.*
- For higher ST: *burst of a revolution.*

Each situation is studied in details in the sequel on model cases. The following theorem somehow sums up all our results.

Theorem 8.2

1. *Case $v_0 \in (0, v_\star)$. There exists no transition wave. If $d_2 > 0$, $(u(t,\cdot), v(t,\cdot))$ converges uniformly to $(0, v_0)$ when $t \to +\infty$.*
2. *Case $v_0 \in (v_\star, 1/2)$. See Figure 8.1. There exists no transition wave for speeds $c < c_0 := 2\sqrt{d_1 \left(r(v_0)G'(0) - \omega\right)}$, and there exists a transition wave for any speed $c > c_0$. Moreover, any transition wave (U, V) is such that V is decreasing, $U(+\infty) = 0$ and $V(+\infty) = V_\infty$ for some $V_\infty \le v_\star$.*
3. *Case $v_0 \in (1/2; 1)$. See Figure 8.2. There exists no transition wave for speeds $c < c_0$ and there exists a transition wave for any speed $c > \overline{c} := 2\sqrt{d_1 \left(r(1)G'(0) - \omega\right)}$ satisfying* (7.13)*. Moreover, any transition wave (U, V) is such that V is increasing, $V(+\infty) = 1$ and $U(+\infty) = u_\star(1)$.*
   The solution $(u(t,\cdot), v(t,\cdot))$ converges locally uniformly to $(u_\star, 1)$ when $t \to +\infty$, and propagates with asymptotic speed $c_\star \in [c_0, \overline{c}]$.

8.2.3 Numerical simulations

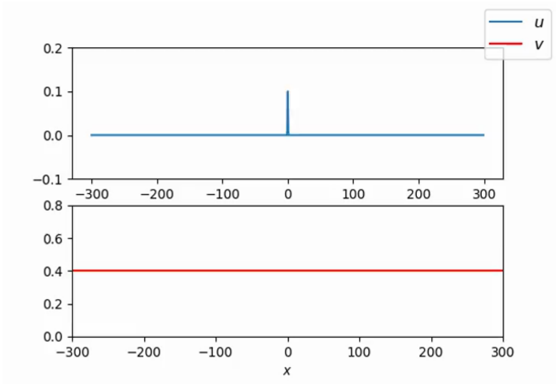
We illustrate Theorem 8.2 with numerical simulations. Equation (8.1) is solved by a standard explicit finite-difference method, for $n = 1$. The plots show $u(t,\cdot)$ (in blue) and $v(t,\cdot)$ (in red) at three different times, $t = 0$, $t = 150$, and $t = 270$. Figure 8.1 illustrates the propagation of a *riot*, with $v_0 = 0.4$; Figure 8.2 illustrates the propagation of a *revolution*, with $v_0 = 0.6$.

The other parameters are identical and chosen as follows. $r(v) = v$, $G(u) = u(1-u)$, $\omega = 1/3$, $f(u,v) = (1-v)(v-1/2)$, $d_1 = d_2 = 1$, $u_0(x) := \sup(0,\ 0.1 - x^2)$.

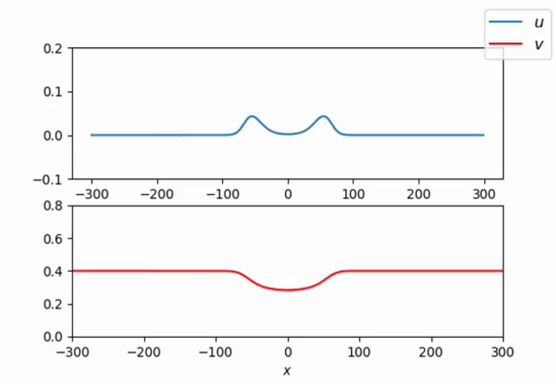
Note that, with those parameters, we have $v_\star = 1/3$, $c_0(v_0 = 0,4) \simeq 0.52$, $c_0(v_0 = 0.6) \simeq 1.03$, $\overline{c}(v_0 = 0.6) \simeq 1.63$, $u_\star(1) = 2/3$.

8.2.4 Comparison with previous models

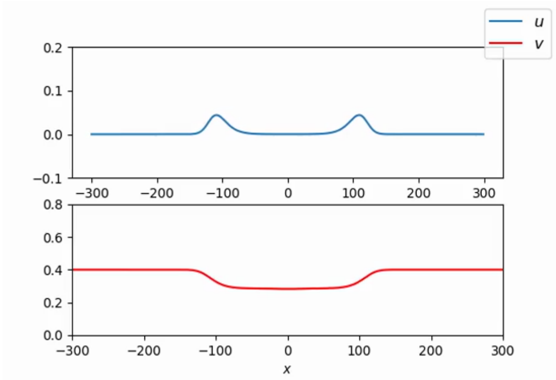
Our model is inspired by a series of papers [46, 48, 52] which proposes a model to account for the dynamics of riots. As before, the quantity $u(t,x)$, depending on



(a) $t = 0$



(b) $t = 150$



(c) $t = 270$

Figure 8.1 – $v_0 = 0.4$

time $t \geq 0$ and location $x \in \mathbb{R}^n$, represents the level of Social Unrest SU, while the level of Social Tension ST is denoted by $v(t,x)$. In most cases, the model reduces to the following system

$$\begin{cases} \partial_t u(t,x) = d\Delta_x u(t,x) + r(v(t,x))G(u(t,x)) - \omega(u(t,x) - u_b(x)), \\ \partial_t v(t,x) = d\Delta_x v(t,x) + S(t,x) - \theta\left(\frac{1}{(1+u(t,x))^p} v(t,x) - v_b(x)\right). \end{cases} \tag{8.5}$$

The parameters $r(\cdot)$, $G(\cdot)$ and ω are the same as described in the previous section. Let us describe the other parameters and discuss the main differences between this model and ours (8.1).

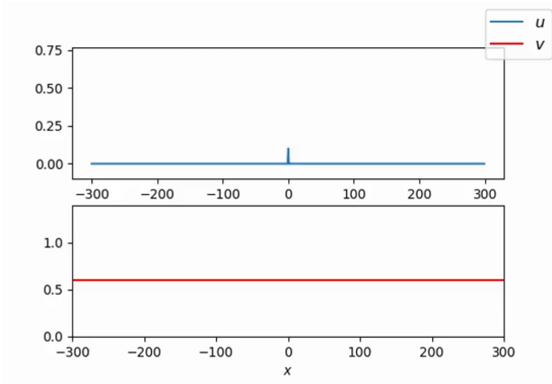
A substential difference with our model is that, in (8.5), the *intrinsic dynamics of v is not discarded*. In other word, our assumption $\Psi(0,v) \equiv 0$ is not fullfilled.

In [46, 48, 52], the source term $S(t,x)$ accounts for the *exogenous events*. It can be of the form

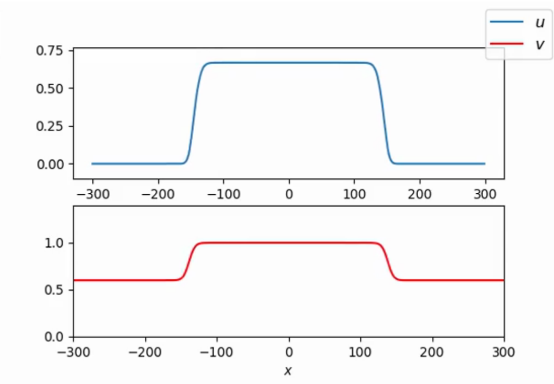
$$S(t,x) = \sum_{i=1}^{n} A_i \delta_{t=t_i, x=x_i}$$

in the case of isolated events, occuring at time t_i and location x_i, of intensity A_i. On the contrary, our model assume a *single exogenous event*, which is modeled by a perturbation on u (not on v) and is included in u_0.

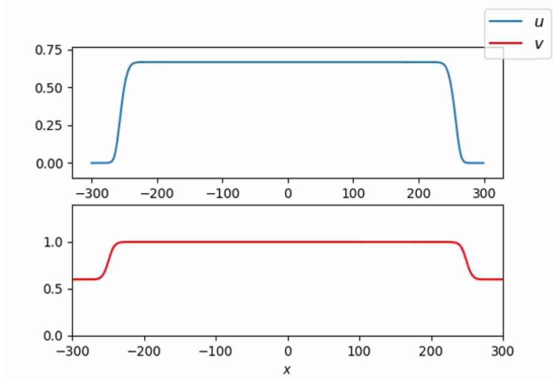
The function $u_b(x)$ stands for the low recurrent SU in the absence of any unusual factors. Accordingly, $v_b(x)$ denotes the base level of ST in absence of any rioting activity. Our model (8.1) corresponds to the case $u_b \equiv v_b \equiv 0$, or more generally to the case where $u_b(\cdot)$ and $v_b(\cdot)$ are constant, using the change of variable $\tilde{u} := u - u_b$



(a) $t = 0$



(b) $t = 150$



(c) $t = 270$

Figure 8.2 – $v_0 = 0.6$

and $\tilde{v} := v - v_b$. We point out that, with general $u_b(x)$ and $v_b(x)$, one cannot guarantee $u(t,x) \geq u_b(x)$ and $v(t,x) \geq v_b(x)$ for all $t \geq 0$.

The quantity $\theta > 0$ is the natural relaxation rate of `ST` to the base rate v_b in absence of any rioting activity.

The parameter $p \in \mathbb{R}$ models the feedback of u on v. The sign of this parameter is crucial. If $p > 0$, then a *burst* of u will *slow* the relaxation of v; if $p < 0$, then a *burst* of u will *accelerate* the relaxation of v. Of course, if $p = 0$, the system is decoupled. In [48, 52], the cases $p > 0$ and $p < 0$ are called respectively *tension enhancing* and *tension inhibiting*, however it does not correspond to what we call *tension enhancing* and *tension inhibiting* in the present work. Indeed, for both cases $p > 0$ and $p < 0$, a burst of u will *decrease* v, thus they are both *tension inhibiting*, according to our vocabulary.

The model (8.5) has been first introduced [46]. See also [49] for a similar approach for criminal activity. In [52], the authors focus on the effect of a restriction of information, which is modeled by substituting the KPP term G with the combustion term $\tilde{G}_\alpha$, $\alpha \in (0,1)$ such that

$$\tilde{G}_\alpha(u) = 0 \text{ for } u \in (0,\alpha), \quad \tilde{G}_\alpha(u) > 0 \text{ for } u \in (\alpha,1) \quad \text{and} \quad G_\alpha(1) = 0.$$

The paper [48] consider (8.5) without space and study the dynamics of the system for a periodic source term

$$S(t) := A \sum_{i \geq 0} \delta_{t=iT}.$$

8.3 Analysis

8.3.1 Resumption of calm

In this section, we shall see that, if $v_0 < v_\star$, the steady state $(u = 0, v = v_0)$ is stable with respect to a small perturbation on u. From the modelization point of vue, it means that an *exogenous event* is followed by a *resumption of calm*. From Theorem 7.4, if $d_2 > 0$ and $v_0 < v_\star$ (which corresponds to $K_0 < 0$ with the notations of Chapter 7), there exists $\varepsilon_0 > 0$ such that, for $\sup_{\mathbb{R}^n} u_0(\cdot) < \varepsilon_0$ and $u_0(\cdot)$ compactly supported,

$$\lim_{t\to+\infty} \left(\sup_{x\in\mathbb{R}^n} |(u(t,x), v(t,x)) - (0, v_0)| \right) = 0 \tag{8.6}$$

This has two implications from the modelization point of view. First, it means that an *exogenous event* with small intensity has an effect of small intensity on the system. In addition, it means that a localized *exogenous event* has a localized effect.

8.3.2 Burst of Social Unrest

We are now interested in the case $v_0 > v_\star$. We shall see that, in contrast with the case $v_0 < v_\star$, a small *exogenous event* triggers a *burst* of SU. From Theorem 7.4, if $v_0 > v_\star$ and $u_0(\cdot) \gneqq 0$, we have

$$\limsup_{t\to+\infty} \left(\sup_{x\in\mathbb{R}^n} |(u(t,x), v(t,x)) - (0, v_0)| \right) \geq \delta_0,$$

for some $\delta_0 > 0$ (which does not depend on $u_0(\cdot)$). It means that even a small *exogenous event* is sufficient to trigger a *burst* of SU, which will lead the system far from the initial condition. This can be put in contrast with (8.6) in the case $v_0 < v_\star$.

8.3.3 Geographical propagation

An important question is to understand whether the *burst* described in the previous section will remain localized or will have a long-range effect, geographically speaking. From Theorem 7.5, if $v_0 > v_\star$ and $u_0 \gneqq 0$ has a compact support, then, there exists

$$c_\star \in \left[2\sqrt{d_1 \left(r(v_0)G'(0) - \omega \right)}, 2\sqrt{d_1 \left(r(1)G'(0) - \omega \right)} \right]$$

such that

$$\forall c < c_\star, \quad \limsup_{t\to+\infty} \left(\sup_{|x|\geq ct} |(u(t,x), v(t,x)) - (0, v_0)| \right) > 0,$$

$$\forall c > c_\star, \quad \lim_{t\to+\infty} \left(\sup_{|x|\geq ct} |(u(t,x), v(t,x)) - (0, v_0)| \right) = 0.$$

It means that the *burst* of SU propagates through all space with an asymptotic speed $c_\star$. This should also be put in contrast with (8.6) in the case $v_0 < v_\star$, where the *resumption of calm* occurs uniformly in space.

We also point out that the lower bound on the asymptotic speed is increasing with respect to v_0. It suggests that the higher the initial level of ST, the faster the propagation occurs.

8.4 Tension Inhibiting - dyanmics of a riot

We consider equation (8.1) with

$$f(u,v) = -1, \tag{8.7}$$

which corresponds to the *tension inhibiting case*. As described in the previous section, this case is designed to grasp the dynamics of a *riot*, and the model should behaves comparably to the SI epidemic model.

It is reasonable to think that, when $v_0 > v_\star$, the solution of (8.1) should converge to a transition wave (but we do not prove such a result). It is thus interesting to understand the shape of transition waves, i.e solutions of (8.3)-(8.4) under (8.7). Setting $c_0 := 2\sqrt{d_1\,(r(v_0)G'(0) - \omega)}$, Theorem 7.3 states that

— If $v_0 < v_\star$, there exists no transition wave.

— If $v_0 > v_\star$, there exists no transition wave with speed $c < c_0$ and there exists a transition wave for any speed $c > c_0$, where $c_0 := 2\sqrt{d_1\,(r(v_0)G'(0) - \omega)}$.

We give the following complement.

Proposition 8.3 *Assume $v_0 > v_\star$ and let (U, V) be a transition wave with speed c. We have that $V(\cdot)$ is decreasing and*

$$U(+\infty) = 0, \quad V(+\infty) = V_\infty,$$

for some $V_\infty \leq v_\star$.

Proof We first show that V is decreasing. We have

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(V'(\xi)e^{-\frac{c}{d_2}\xi}\right) = \frac{e^{-\frac{c}{d_2}\xi}}{d_2}U(\xi)V(\xi),$$

and integrating from ξ to $+\infty$,

$$V'(\xi) = -\frac{1}{d_2}\int_\xi^{+\infty} e^{-\frac{c}{d_2}(z-\xi)}U(z)V(z)dz < 0.$$

As $V(\cdot)$ is decreasing and bounded from below by zero, it converges to a constant $V_\infty < v_0$. From classical global a priori estimates, we deduce $\lim\limits_{+\infty} V' = \lim\limits_{+\infty} V'' = 0$. Passing to the limit in the equation on V, we obtain $\lim\limits_{+\infty} UV = 0$.

Let us show $V_\infty \leq v_\star$. By contradiction, assume $V_\infty > v_\star$. Then, there exists $\xi_0 > 0$ such that

$$cU' - d_1U'' - r(\alpha)\,G(U) - \omega U \geq 0, \quad \text{on } (\xi_0, +\infty),$$

with $\alpha := \frac{V_\infty + v_\star}{2}$. But since $r(\alpha)G'(0) - \omega > 0$, from a simple ODE argument we deduce

$$\limsup_{\xi\to+\infty} U(\xi) > 0,$$

which contradicts $\lim\limits_{+\infty} UV = 0$ and $V_\infty = 0$. Hence $V_\infty \leq v_\star$.

Finally, let us show $\lim\limits_{+\infty} U = 0$. If $V_\infty > 0$, the result follows immediately from $\lim\limits_{+\infty} UV = 0$. Let us assume $V_\infty = 0$. Hereafter, there exists $\xi_0 > 0$ such that

$$cU' - d_1U'' + CU \leq 0, \quad \text{on } (\xi_0, +\infty), \tag{8.8}$$

with $C := \omega - r\left(\frac{v_\star}{2}\right)G'(0) > 0$. We have

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(U'(\xi)e^{-\frac{c}{d_1}\xi}\right) \geq C\frac{e^{-\frac{c}{d_1}\xi}}{d_1}U(\xi),$$

and integrating from ξ to $+\infty$,

$$U'(\xi) \leq -\frac{C}{d_1}\int_\xi^{+\infty} e^{-\frac{c}{d_1}(z-\xi)}U(z)dz < 0.$$

We deduce that U is decreasing, thus U converges to some U_∞. From classical a priori estimates, we have $\lim_{+\infty} U' = \lim_{+\infty} U'' = 0$. Then, passing to the limit in (8.8), we obtain $U_\infty = 0$. ■

8.5 Tension Enhancing - dynamics of a revolution

We consider equation (8.1) with

$$f(u,v) = (1-v). \tag{8.9}$$

It corresponds to the *tension enhancing case*. This case is designed to grasp the dynamics of a *revolution*. More precisely, we will see that if ST is above a certain threshold (i.e $v_0 > v_\star$) then any *exogenous event* (i.e $u_0 \gneqq 0$) triggers a *burst* of SU, which propagates everywhere in space at constant speed. The level of ST then *increases* and converge to 1, while the level of SU converge to an excited state, uniform in space.

8.5.1 Asymptotic behavior of solutions

The following result states that the solution converges to a sustainable excited state.

Proposition 8.4 *Assume $v_0 > v_\star$ and let (u,v) be the solution of* (8.1) *with* (8.7). *Assume $0 \lneqq u_0(x) \leq 1$ and $0 < v_0 < 1$. Then, $v \geq v_0$ and*

$$\lim_{t\to+\infty} u(t,x) = u_\star(1), \quad \lim_{t\to+\infty} v(t,x) = 1,$$

locally uniformly in $x \in \mathbb{R}^n$.

Proof (of Proposition 8.4) We deduce $v \geq v_0$ from the fact that v_0 is a sub-solution, namely, for all $u > 0$,

$$-d_2\Delta v_0 - uv_0 = -uv_0(1-v_0) < 0,$$

and the parabolic comparison principle. In addition, u satisfies

$$\partial_t u - d_1\Delta_x u - r(v_0)G(u) - \omega u \geq 0,$$

which is a standard scalar KPP equation. From classical results, we deduce that

$$\inf_{x\in\mathbb{R}^n}\left(\liminf_{t\to+\infty} u(t,x)\right) \geq u_\star(v_0) > 0.$$

Then, for t large enough, v satisfies

$$\partial_t v - d_2 v - \alpha v(1-v) \geq 0.$$

Again, from classical results on KPP equations, we deduce that $v(t,\cdot)$ converges locally uniformly to 1.

Now, let us show that $u(t,\cdot)$ converges to $u_\star(1)$. First, $u(t,x)$ satisfies, for all $t \geq 0,\ x \in \mathbb{R}^n$,

$$\partial_t u - d_1 \Delta_x u - r(1)G(u) - \omega u \leq 0,$$

therefore, from classical KPP arguments,

$$\sup_{x\in\mathbb{R}^n} \left(\limsup_{t\to+\infty} u(t,x) \right) \leq u_\star(1).$$

Let us fix $\varepsilon > 0$. There exists $T_\varepsilon > 0$ such that for all $t > T_\varepsilon$, $u(t,\cdot)$ satisfies

$$\partial_t u - d_1 \Delta_x u - r(1-\varepsilon)G(u) - \omega u \geq 0,$$

Identically, we deduce that

$$\inf_{x\in\mathbb{R}^n} \left(\liminf_{t\to+\infty} u(t,x) \right) \geq u_\star(1-\varepsilon).$$

From the continuity of $v \mapsto u_\star(v)$, at the limit $\varepsilon \to 0$ we have

$$\inf_{x\in\mathbb{R}^n} \left(\liminf_{t\to+\infty} u(t,x) \right) \geq u_\star(1),$$

which achieves the proof ∎

8.5.2 Transition waves

Let $v_0 > v_\star$. Setting $\bar{c} := 2\sqrt{d_1\left(r(1)G'(0) - \omega\right)}$ and , $\bar{\bar{c}}$ as the unique positive solution of $d_1 = d_2 g(c)$ (where $g(\cdot)$ is defined in (7.13), then Theorem 7.3 states that

— there exists no transition wave with speed $c < c_0$,

— there exists a transition wave with any speed $c > \max(\bar{c}, \bar{\bar{c}}$.

We give the following complement.

Proposition 8.5 *Assume $v_0 > v_\star$, let (U,V) be a transition wave with speed $c > 0$, i.e a solution of* (8.3)-(8.4) *under* (8.9). *We have that $V(\cdot)$ is increasing, and*

$$U(+\infty) = u_\star(1), \quad V(+\infty) = 1.$$

Proof The first step is to show $V < 1$. Assume that V reaches a local maximum at $\bar{\xi} \in \mathbb{R}$. We have $V'(\bar{\xi}) = 0$, $V''(\bar{\xi}) \leq 0$, and from the equation on V we obtain

$$U(\bar{\xi})V(\bar{\xi})\Big(1 - V(\bar{\xi})\Big) \geq 0,$$

thus $V(\bar{\xi}) \leq 1$. As $V \not\equiv 1$, and that 1 is a solution of the equation, we deduce $V(\bar{\xi}) < 1$. Hereafter, assume by contradiction that $\sup V > 1$. Given the above, the supremum must be reached at $+\infty$, and V must be nondecreasing in some interval $(A,+\infty)$, with $A > 0$. We deduce $V'' \geq 0$ in $(A,+\infty)$. From a classical Liouville property, we get a contradiction with the fact that V is bounded. Thus, we have proved $V < 1$.

Let us show that V is increasing. We have

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(V'(\xi)e^{-\frac{c}{d_2}\xi}\right) = -\frac{e^{-\frac{c}{d_2}\xi}}{d_2}U(\xi)V(\xi)(1-V(\xi)),$$

and integrating from ξ to $+\infty$,

$$V'(\xi) = \frac{1}{d_2}\int_{\xi}^{+\infty} e^{-\frac{c}{d_2}(z-\xi)}U(z)V(z)(1-V(z))dz > 0.$$

As $V(\cdot)$ is increasing and bounded from above by 1, it converges to a constant $V_\infty > 0$. Let us show that U converges to $u_\star(1)$. First, U satisfies

$$cU' - d_1U'' < r(1)G(U) - \omega U.$$

Using the same method as the proof of $V < 1$, we have $U < u_\star(1)$. Now, fix $\varepsilon > 0$. There exists $\overline{\xi} \in \mathbb{R}$ such that

$$cU' - d_1U'' \geq r(1-\varepsilon)G(U) - \omega U, \quad \text{on } (\overline{\xi}, +\infty).$$

We deduce

$$U'(\xi) \geq \frac{1}{d_1}\int_{\xi}^{+\infty} e^{-\frac{c}{d_1}(z-\xi)}\Big[r(1-\varepsilon)G(U(z)) - \omega U(z)\Big]dz,$$

and

$$\liminf_{+\infty} U \geq u_\star(1-\varepsilon),$$

which, by continuity of $v \mapsto u_\star(v)$, implies $U(+\infty) = u_\star(1)$. ∎

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