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Baptiste Trey

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THÈSE

Pour obtenir le grade de

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Présentée par

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préparée au sein du **Laboratoire Institut Fourier et du LJK**
dans l'**École Doctorale Mathématiques, Sciences et Technologies de l'Information, Informatique**

Existence et régularité des formes optimales pour des problèmes d'optimisation spectrale

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Résumé

Dans cette thèse, on étudie l'existence et la régularité des formes optimales pour certains problèmes d'optimisation spectrale qui font intervenir un opérateur elliptique avec condition de Dirichlet. On s'intéresse d'abord au problème de la minimisation de la valeur propre principale d'un opérateur avec un terme de transport borné. Que le terme de transport soit fixé ou non, ce problème admet une solution parmi les quasi-ouverts, et si le terme de transport est en outre le gradient d'une fonction Lipschitzienne, alors les solutions sont des ouverts localement de classe $C^{1,\alpha}$ en dehors de points exceptionnels. On étudie ensuite en dimension deux la régularité des solutions à un problème d'optimisation à plusieurs phases pour la première valeur propre du Laplacien de Dirichlet. Enfin, on s'intéresse aux ensembles optimaux pour la somme des k premières valeurs propres d'un opérateur elliptique sous forme divergence. On montre que les k premières fonctions propres sur un ensemble optimal sont lipschitziennes de sorte que les ensembles optimaux sont ouverts, et on étudie ensuite la régularité de la frontière des ensembles optimaux.

Mots-clés: problème d'optimisation spectrale, problème d'optimisation à plusieurs phases, régularité des frontières libres, problèmes à une phase et à deux phases, quasi-minimiseur, terme de transport, quasi-ouvert, γ -convergence.

Abstract

In this thesis, we study the existence and the regularity of optimal shapes for some spectral optimization problems involving an elliptic operator with Dirichlet boundary condition. First of all, we consider the problem of minimizing the principal eigenvalue of an operator with bounded drift under inclusion and volume constraints. Whether the drift is fixed or not, this problem admits solutions among the class of quasi-open sets, and if the drift is furthermore the gradient of a Lipschitz continuous function, then the solutions are open sets and $C^{1,\alpha}$ -regular except on a set of exceptional points. Next, we study in dimension two the regularity of the solutions to a multi-phase optimization problem for the first eigenvalue of the Dirichlet Laplacian. Finally, we focus on the optimal sets for the sum of the first k eigenvalues of an operator in divergence form. We prove that the first k eigenfunctions on an optimal set are Lipschitz continuous so that the optimal sets are open sets, and we then study the regularity of the boundary of the optimal sets.

Keywords: spectral optimization problem, multi-phase optimization problem, regularity of free boundaries, one-phase and two-phase problems, almost-minimizer, drift, quasi-open set, γ -convergence.

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Introduction

Cette thèse s'inscrit dans le domaine de l'optimisation de forme. Il s'agit de trouver la forme optimale d'un ensemble qui minimise une fonctionnelle tout en satisfaisant à des contraintes données. Les problèmes d'optimisation de forme ont été abondamment étudiés et modélisent différents phénomènes : que ce soit par exemple dans l'industrie pour concevoir des objets manufacturés plus rigides et plus fiables, en physique pour maximiser la conductivité électrique ou thermique d'un matériau, en biologie pour prédire l'évolution de la forme des cellules d'un tissu, ou encore en ingénierie pour déterminer la meilleure forme aérodynamique d'une aile d'avion, les domaines concernés sont variés et les applications nombreuses. Ces problèmes d'optimisation sont probablement plus intuitifs que ceux portant sur des fonctions ou des paramètres mais sont aussi nettement plus délicats à traiter. Une difficulté par exemple, qui fait aussi toute la richesse de ces problèmes, vient de ce qu'il n'existe pas de structure d'espace vectoriel sur l'ensemble des formes et que la plupart des outils d'analyse fonctionnelle sont par conséquent inutilisables.

Un problème d'optimisation de forme s'écrit souvent comme

$$\min \{J(\Omega) : \Omega \in \mathcal{A}\},$$

où J est une fonctionnelle à valeurs réelles définie sur un ensemble \mathcal{A} de parties de \mathbb{R}^d , dites formes admissibles, qui sont généralement soumises à des contraintes géométriques de diverses natures qui peuvent être de volume, de périmètre, de convexité etc. Les questions qui apparaissent avec ce type de problèmes sont variées :

- Existence ou non d'une forme optimale.
- Obtention de conditions nécessaires ou suffisantes d'optimalité.
- Propriétés géométriques des formes optimales : régularité de la frontière, caractère borné des formes optimales, finitude de leur périmètre, connexité, convexité...

Souvent, la fonctionnelle est calculée par l'intermédiaire d'une solution, appelée fonction d'état, d'une certaine équation aux dérivées partielles modélisant par exemple la conductivité ou l'élasticité du matériau. Dans cette situation il existe alors deux familles principales de problèmes d'optimisation, ceux pour lesquels la fonctionnelle est une énergie sous forme intégrale dont l'intégrande dépend de la fonction d'état, et ceux pour lesquels la fonctionnelle est une fonction en les valeurs propres d'un opérateur elliptique. Les problèmes abordés dans cette thèse correspondent à ce deuxième type de problèmes dits d'optimisation spectraux. À titre d'exemple, l'évolution de la chaleur dans un matériau est déterminée par les valeurs propres du Laplacien de Dirichlet, le matériau pour lequel la vitesse de décroissance de la chaleur est la plus lente correspondant en première approximation à celui qui possède la plus petite première valeur propre (à volume fixé). Ces problèmes interviennent aussi en acoustique, puisque les fréquences pour lesquelles un instrument de musique peut vibrer, par exemple la membrane d'un tambour, dépendent de sa forme et sont directement liées aux valeurs propres du Laplacien de Dirichlet. Le premier problème d'optimisation spectrale étudié et le plus classique est probablement celui de la minimisation de la première valeur propre du Laplacien de Dirichlet parmi les ouverts de volume fixé. Dès 1877, Lord Rayleigh a conjecturé dans son livre *The theory of sound* que la boule est optimale pour ce problème, ce qui a ensuite été démontré dans les années 20 indépendamment

par Faber et Krahn. De manière assez surprenante toutefois, certains problèmes d'optimisation spectrale même en apparence simples sont encore mal compris. Ainsi, l'existence d'un ouvert minimisant la k -ième valeur du Laplacien de Dirichlet pour $k \geq 3$ avec une contrainte de volume est toujours un problème ouvert. Mentionnons néanmoins que de récents résultats ont permis des avancées significatives dans cette direction (voir [12, 76] pour l'existence parmi les quasi-ouverts, et [18] pour l'existence d'une fonction propre lipschitzienne sur un ensemble optimal).

La première partie de ce travail concerne l'existence de formes optimales. Bien sûr, l'existence ou non d'une forme optimale dépend de l'ensemble des formes admissibles que l'on s'est fixé. Bien qu'il soit naturel de considérer la classe des ouverts, cette dernière s'avère souvent trop restrictive en vue d'obtenir un résultat d'existence. En général, on considère plutôt un problème relaxé avec une classe plus grande de formes admissibles appelées les quasi-ouverts. Il s'agit ensuite d'introduire une topologie adéquate sur l'ensemble des quasi-ouverts qui fournisse à la fois la compacité des suites minimisantes ainsi que la continuité (ou plus généralement la semi-continuité inférieure) de la fonctionnelle, ces deux exigences étant antagonistes. Il existe ainsi deux topologies qui interviennent fréquemment en optimisation de forme appelées les topologies de la γ -convergence et de la γ -convergence faible. Mentionnons par exemple que l'ensemble des quasi-ouverts (inclus dans un ouvert borné fixé) est compact pour la γ -convergence faible et que cette topologie rend aussi semi-continues inférieurement les valeurs propres du Laplacien de Dirichlet.

En optimisation de forme, la classe des formes admissibles considérée est souvent celle des ouverts. Par conséquent, l'existence d'une solution relaxée parmi les quasi-ouverts, qui est déjà un résultat intéressant en soit, n'est toutefois pas pleinement satisfaisante. Ainsi, la seconde partie de ce travail de thèse est dédiée à l'étude de la régularité des ensembles optimaux pour des problèmes d'optimisation spectrale. Plus précisément, il s'agit de montrer que les ensembles optimaux sont des ouverts et ensuite d'étudier la régularité de leur frontière. Quand l'opérateur considéré est autoadjoint, on dispose alors d'une caractérisation variationnelle des valeurs propres ce qui permet de se ramener à un problème à frontière libre. L'intérêt est qu'un tel problème porte sur des fonctions (plutôt que sur des formes), qui sont des objets plus simples. Pour montrer que les ensembles optimaux sont bien des ouverts, on étudie généralement la continuité des fonctions propres définies sur lesdits ensembles optimaux. Bien sûr, un tel résultat de régularité fournit en particulier l'existence d'une solution du problème où la classe des formes admissibles est celle des ouverts. Précisons qu'il existe toutefois des problèmes d'optimisation spectrale qui possèdent des solutions très irrégulières, au sens où certaines solutions sont des quasi-ouverts qui ne sont pas des ouverts. Nous abordons ensuite la question de la régularité de la frontière des ensembles optimaux. Pour cela, il s'agit dans un premier temps de prouver que les fonctions propres sur un ensemble optimal sont Lipschitziennes et non-dégénérées. Cette dernière propriété signifie que les fonction propres ne prennent pas de trop petites valeurs près du bord de l'ensemble optimal, au sens où elles se comportent comme la fonction distance au bord de cet ensemble. Ceci permet ensuite d'étudier les propriétés des blow-ups des fonctions propres sur un ensemble optimal et d'en déduire une condition d'optimalité. En un sens faible, appelé sens de la viscosité, cette condition d'optimalité prescrit essentiellement la valeur du gradient des fonctions propres sur la frontière de l'ensemble optimal. Avec l'équation vérifiée par les fonctions propres, cette condition d'optimalité implique une certaine régularité de la frontière des ensembles optimaux. Plus précisément, on montre que la frontière est localement le graphe d'une fonction régulière à un petit ensemble singulier près qui peut être estimé en termes de dimension de Hausdorff.

Le lecteur intéressé par les problèmes d'optimisation de forme pourra consulter les livres et les références qu'ils contiennent [56, 61, 57, 13] ainsi que les revues [20, 55].

Précisons maintenant l'organisation de ce manuscrit. Le premier chapitre a pour vocation d'introduire certains outils classiques en optimisation de forme et de présenter les résultats obtenus et exposés dans les chapitres suivants. Les quatre autres chapitres sont des reproductions de publications ou prépublications et abordent les questions d'existence et de régularité des

ensembles optimaux pour des problèmes d'optimisation spectrale avec des contraintes d'inclusion (c'est-à-dire que les formes admissibles sont incluses dans un ouvert borné $D \subset \mathbb{R}^d$ généralement appelé «boîte») et de volume (ou bien simplement une pénalisation en volume), pour des opérateurs elliptiques avec condition de Dirichlet. Voici un résumé succinct par chapitre de ce manuscrit de thèse.

- Le chapitre 1 introduit dans un premier temps en section 1.1 les notions de capacité, de quasi-continuité, de quasi-ouverts, les topologies de la γ -convergence forte et faible, ainsi que quelques propriétés des valeurs propres d'opérateurs elliptiques définis sur des quasi-ouverts. La section 1.2 est ensuite dédiée à des problèmes d'optimisation faisant intervenir la première valeur propre et présente les résultats des chapitres 2 et 3, tandis que la dernière section 1.3 est consacrée à des problèmes avec des valeurs propres d'ordre supérieur et résume les résultats des chapitres 4 et 5.
- Le chapitre 2 est dédié à la minimisation de la première valeur propre de l'opérateur avec terme de transport $-\Delta + V \cdot \nabla$ où $V \in L^\infty(D, \mathbb{R}^d)$ est un champ de vecteurs, ce qui nécessite dans un premier temps de définir la première valeur propre sur les quasi-ouverts. Nous prouvons ensuite l'existence d'une forme optimale Ω^* parmi les quasi-ouverts de volume fixé inclus dans D avec un terme de transport V fixé, ainsi que d'un couple optimal (Ω^*, V^*) quand le terme de transport varie aussi avec une contrainte de norme infinie (voir Théorème 2.1.1 ou 1.2.3). La deuxième partie de ce travail aborde la question de la régularité de la frontière des ensembles optimaux Ω^* dans le cas où le terme de transport est le gradient d'une fonction Lipschitzienne. Plus précisément, la frontière libre $\partial\Omega^* \cap D$ est localement le graphe d'une fonction de classe $C^{1,\alpha}$ à un ensemble de dimension de Hausdorff au plus $d - 5$ près. De plus, $\partial\Omega^*$ est de classe $C^{1,1/2}$ au voisinage des points de $\partial\Omega^* \cap \partial D$ pourvu que la boîte D soit suffisamment régulière (voir Théorème 2.1.5 ou 1.2.12).
- Dans le chapitre 3 nous étudions, en dimension 2, la régularité des solutions $(\Omega_1, \dots, \Omega_n)$ à un problème d'optimisation de forme à plusieurs phases faisant intervenir les valeurs propres du Laplacien de Dirichlet. Si D est régulier, alors la frontière de chaque phase Ω_i est en tout point de classe $C^{1,\alpha}$ (voir Théorème 3.1.11 ou 1.2.14). Pour cela, nous prouvons un résultat de régularité pour les quasi-minimiseurs des fonctionnelles avec des coefficients variables à une phase (avec contrainte d'inclusion) et à deux phases, dont la preuve repose en particulier sur une inégalité épipérimétrique qui n'est valable qu'en dimension 2 (voir Théorèmes 3.1.1 et 3.1.5, ou 1.2.15).
- Le chapitre 4 est consacré à l'étude des formes optimales parmi les quasi-ouverts inclus dans D pour la somme des k premières valeurs propres avec une pénalisation en volume, où l'opérateur considéré est elliptique, sous forme divergence, avec des coefficients Höldériens et avec des conditions de Dirichlet. Nous prouvons que les k premières fonctions propres sur un ensemble optimal sont Lipschitziennes, ce qui en conséquence implique l'existence d'une forme optimale parmi les ouverts (voir Théorème 4.1.1 ou 1.3.11). La régularité Lipschitzienne des fonctions propres, qui est une première étape importante pour l'analyse de la régularité de la frontière des ensembles optimaux, passe par l'étude du caractère Lipschitzien des fonctions à valeurs vectorielles qui quasi-minimisent une fonctionnelle à deux phases avec des coefficients variables (voir Théorème 4.1.2 ou 1.3.12).
- Nous continuons au chapitre 5 l'étude de la régularité des formes optimales pour le problème considéré au chapitre précédent. Nous montrons que la frontière $\partial\Omega^* \cap D$ est localement le graphe d'une fonction de classe $C^{1,\alpha}$, et même de classe C^∞ pourvu que les coefficients de l'opérateurs soient aussi de classe C^∞ , à un ensemble singulier de dimension de Hausdorff au plus $d - 5$ près (voir Théorème 5.1.1 ou 1.3.11).

Chapitre 1

Présentation des résultats principaux

Dans la section 1.1 nous introduisons des notions classiques en optimisation de forme et nous définissons les valeurs propres d'opérateurs elliptiques sur les quasi-ouverts. La section 1.2 aborde des problèmes d'optimisation spectrale pour la première valeur propre et présente les résultats des chapitres 2 et 3, tandis que la section 1.3 est dédiée à des problèmes faisant intervenir les valeurs propres d'ordre supérieur et présente les résultats des chapitres 4 et 5. Pour finir, la section 1.4 regroupe quelques perspectives de recherche.

1.1 Préliminaires

Dans cette section nous introduisons la classe des formes admissibles appelées les quasi-ouverts, ce qui nécessite de définir au préalable la notion de capacité associée à la norme H^1 . Nous présentons ensuite quelques propriétés des valeurs propres d'opérateurs elliptiques (symétriques ou non symétriques) définis sur les quasi-ouverts. Nous introduisons ensuite une topologie classique définie sur les quasi-ouverts appelée la γ -convergence dont l'une des propriétés remarquables est de rendre continues les valeurs propres du Laplacien de Dirichlet par rapport au domaine. Nous terminons en énonçant le théorème général d'existence de Buttazzo et Dal Maso pour des problèmes de minimisation dans une boîte faisant intervenir les valeurs propres du Laplacien.

1.1.1 La capacité

La capacité est essentielle pour une bonne compréhension des propriétés fines de régularité des fonctions Sobolev et joue par conséquent un rôle important dans l'étude des solutions d'équations aux dérivées partielles. Elle permet, par exemple, une meilleure compréhension du comportement ponctuel des fonctions Sobolev, qui ne sont au premier abord définies qu'à un ensemble de mesure de Lebesgue nulle près. De nombreuses références existent pour la capacité, mentionnons par exemple les livres [61], [44] et [54].

L'espace de Sobolev $H^1(\mathbb{R}^d)$ est défini par $H^1(\mathbb{R}^d) = \{u \in L^2(\mathbb{R}^d) : \nabla u \in L^2(\mathbb{R}^d)^d\}$, où $\nabla u \in L^2(\mathbb{R}^d)^d$ est compris au sens des distributions, et est muni de la norme

$$\|u\|_{H^1(\mathbb{R}^d)}^2 = \int_{\mathbb{R}^d} (u^2 + |\nabla u|^2) dx.$$

Définition 1.1.1. La capacité (associée à H^1) d'un ensemble quelconque $E \subset \mathbb{R}^d$ est définie par

$$\text{cap}(E) = \inf \left\{ \|u\|_{H^1(\mathbb{R}^d)} : u \in \mathcal{A}(E) \right\},$$

où $\mathcal{A}(E) = \{u \in H^1(\mathbb{R}^d), u \geq 1 \text{ p.p. sur un voisinage de } E\}$. Si $\mathcal{A}(E) = \emptyset$, alors on pose $\text{cap}(E) = +\infty$.

On dit qu'une propriété $P(x)$ est vraie quasi-partout (q.p.) sur E si elle est vraie pour tout $x \in E$ sauf pour un ensemble $Z \subset E$ de capacité nulle. Comme d'habitude, l'expression presque partout (p.p.) renvoie à la mesure de Lebesgue.

Il est aussi possible de définir la capacité relative d'un ensemble $E \subset \mathbb{R}^d$ par rapport à un ouvert borné $D \subset \mathbb{R}^d$ par

$$\text{cap}_D(E) = \inf \left\{ \int_D |\nabla u|^2 dx : u \in \mathcal{A}_D(E) \right\},$$

où $\mathcal{A}_D(E) = \{u \in H_0^1(D), u \geq 1 \text{ p.p. sur un voisinage de } E\}$ et $\text{cap}_D(E) = +\infty$ si $\mathcal{A}(E) = \emptyset$. Nous serons surtout intéressés dans la suite par les ensembles de capacité nulle, de sorte qu'il revient souvent au même de considérer l'une ou l'autre de ces deux définitions de la capacité. En effet, il est possible de montrer que pour tout $E \subset D$, $\text{cap}(E) = 0$ si et seulement si $\text{cap}_D(E) = 0$ (notons cependant que pour un ouvert borné D , il vient $\text{cap}(D) < +\infty$ alors que $\text{cap}_D(D) = +\infty$). Cette deuxième définition est en particulier utile pour des calculs explicites de la capacité (voir par exemple [61, Section 3.3.3]).

La capacité possède de nombreuses propriétés désirables comme celle d'être une mesure extérieure, c'est-à-dire qu'elle est positive ou nulle et vérifie

- (i) $\text{cap}(\emptyset) = 0$,
- (ii) si $E_1 \subset E_2$, alors $\text{cap}(E_1) \leq \text{cap}(E_2)$,
- (iii) $\text{cap}(\bigcup_{i=1}^{\infty} E_i) \leq \sum_{i=1}^{\infty} \text{cap}(E_i)$ pour tout $(E_i)_{i \geq 1} \subset \mathbb{R}^d$.

La capacité est plus fine que la mesure de Lebesgue, au sens où pour tout ensemble mesurable $E \subset \mathbb{R}^d$ il vient $|E| \leq \text{cap}(E)$, où $|E|$ désigne la mesure de Lebesgue de E . En particulier, la capacité est non triviale puisque tout ouvert non vide a une capacité strictement positive. D'un autre côté, il existe une constante dimensionnelle $C_d > 0$ telle que pour toute boule $B_r(x) \subset \mathbb{R}^d$ de rayon $0 < r \leq 1$, $\text{cap}(B_r(x)) \leq C_d r^{d-2}$, de sorte que les ensembles bornés sont de capacité finie.

Cette estimation suggère aussi que la dimension naturelle de Hausdorff associée à la capacité est $d-2$. En fait, la capacité et la mesure de Hausdorff $(d-2)$ -dimensionnelle \mathcal{H}^{d-2} sont reliées par les propriétés suivantes :

- Si $\mathcal{H}^{d-2}(E) < +\infty$, alors $\text{cap}(E) = 0$.
- Si $\text{cap}(E) = 0$, alors $\mathcal{H}^s(E) = 0$ pour tout $s > d-2$.

(voir [44, Section 4.7.2]). À titre d'exemple, la capacité d'un point est strictement positive en dimension 1 et nulle en dimension $d \geq 2$, alors que la capacité d'un segment est strictement positive en dimension 2 et nulle pour tout $d \geq 3$.

Remarque 1.1.2. 1) Un des intérêts de la capacité est de mesurer la finesse des ensembles et de détecter, par exemple, des compacts fins de mesure de Lebesgue nulle, ce qui ne serait pas possible avec la seule contrainte " $u \geq 1$ p.p. sur E " dans la définition 1.1.1.

2) Il est possible d'étendre la définition de la capacité, alors appelée p -capacité, aux espaces de Sobolev $W^{1,p}(\mathbb{R}^d)$ pour $1 < p < +\infty$ (voir par exemple les livres [44] et [54]).

Les fonctions de $H^1(\mathbb{R}^d)$ n'étant pas nécessairement continues (sauf en dimension 1), la notion pertinente à considérer est celle de quasi-continuité.

Définition 1.1.3. Une fonction $u : \mathbb{R}^d \rightarrow \mathbb{R}$ est dite quasi-continue s'il existe une suite décroissante d'ouverts $\omega_n \subset \mathbb{R}^d$ tels que

$$\lim_{n \rightarrow +\infty} \text{cap}(\omega_n) = 0, \quad \text{la restriction de } u \text{ à } \mathbb{R}^d \setminus \omega_n \text{ est continue.}$$

Précisons qu'une fonction continue quasi-partout est quasi-continue, mais que la réciproque est fautive en général.

Toute fonction $u \in H^1(\mathbb{R}^d)$ possède un représentant quasi-continu $\tilde{u} \in H^1(\mathbb{R}^d)$ qui est unique à un ensemble de capacité nulle près. De plus, le représentant quasi-continu \tilde{u} de u est donné par

$$\tilde{u}(x) = \lim_{r \rightarrow 0} \int_{B_r(x)} u(y) dy \quad \text{pour quasi-tout } x \in \mathbb{R}^d \quad (1.1.1)$$

(voir [61, Théorème 3.3.29]). Dans la suite, nous identifierons toujours une fonction $u \in H^1(\mathbb{R}^d)$ avec son représentant quasi-continu $\tilde{u} \in H^1(\mathbb{R}^d)$, de sorte que toute fonction Sobolev, qui n'est a priori définie que presque partout, admet une définition ponctuelle quasi-partout (et possède des points de Lebesgue quasi-partout).

Enfin, rappelons que pour toute suite $(u_n)_{n \geq 1} \subset H^1(\mathbb{R}^d)$ qui converge fortement dans H^1 vers une fonction $u \in H^1(\mathbb{R}^d)$, il existe une sous-suite qui converge vers u quasi-partout.

1.1.2 Les quasi-ouverts

La classe des ouverts étant en général trop restrictive pour obtenir un résultat d'existence en optimisation de forme, nous considérons maintenant une classe plus grande d'ensembles mesurables appelés quasi-ouverts pour lesquels les problèmes aux valeurs propres sont bien définis (voir sous-section 1.1.3).

Définition 1.1.4. Un ensemble $\Omega \subset \mathbb{R}^d$ est dit quasi-ouvert s'il existe une suite décroissante d'ouverts $\omega_n \subset \mathbb{R}^d$ tels que

$$\forall n \in \mathbb{N}, \Omega \cup \omega_n \text{ est ouvert}, \quad \lim_{n \rightarrow +\infty} \text{cap}(\omega_n) = 0.$$

En faisant la réunion d'un quasi-ouvert avec un ensemble de capacité nulle on obtient encore un quasi-ouvert, mais ceci ne permet pas vraiment de créer de nouveaux ensembles. En d'autres termes, les quasi-ouverts ne sont vraiment définis que quasi-partout, de sorte que l'on dira souvent, de manière un peu abusive, que deux quasi-ouverts sont égaux s'ils sont égaux à un ensemble de capacité nulle près.

Remarque 1.1.5. 1) De manière équivalente, les quasi-ouverts peuvent être définis comme les ensembles $\Omega \subset \mathbb{R}^d$ tels que, pour tout $\varepsilon > 0$, il existe un ouvert $\Omega_\varepsilon \subset \mathbb{R}^d$ vérifiant $\text{cap}(\Omega \Delta \Omega_\varepsilon) < \varepsilon$, où Δ désigne la différence symétrique d'ensembles. Notons que l'on définit encore les mêmes ensembles en imposant aussi dans cette définition la contrainte $\Omega \subset \Omega_\varepsilon$.

2) Les quasi-ouverts sont des ensembles mesurables. En effet, si Ω est un quasi-ouvert, l'ensemble $\Omega \cup (\cap_n \omega_n) = \cap_n (\Omega \cup \omega_n)$ est un G_δ , c'est-à-dire une intersection dénombrable d'ouverts, qui est égal quasi-partout à Ω puisque $\cap_n \omega_n$ est de capacité nulle.

3) Il existe des exemples de quasi-ouverts qui ne sont pas ouverts, et même qui ne sont pas égaux presque partout à un ouvert (voir [61, exercice 3.6]). De plus, les quasi-ouverts peuvent être des ensembles très irréguliers et être d'intérieur vide, comme par exemple l'ensemble $]0, 1[\setminus (\cup_{k=1}^{+\infty} \overline{B}_{r_k}(x_k)) \subset \mathbb{R}^2$ où $(x_k)_k$ est une suite dense de $]0, 1[$ et les $r_k \in]0, 1[$ sont tels que

$$\sum_{k=1}^{+\infty} \text{cap}(\overline{B}_{r_k}(x_k)) < +\infty \quad \text{et} \quad \sum_{k=1}^{+\infty} \pi r_k^2 < 1.$$

Une conséquence de l'existence d'un représentant quasi-continu est que les quasi-ouverts sont exactement les ensembles de sur-niveau de fonctions Sobolev. Précisément, cela signifie qu'un ensemble $\Omega \subset \mathbb{R}^d$ est un quasi-ouvert si et seulement s'il existe une fonction $u \in H^1(\mathbb{R}^d)$ telle que $\Omega = \{u > 0\}$.

Pour un ouvert $\Omega \subset \mathbb{R}^d$, l'espace de Sobolev $H_0^1(\Omega)$ est défini comme l'adhérence de $C_c^\infty(\Omega)$ pour la norme $\|\cdot\|_{H^1}$, où $C_c^\infty(\Omega)$ désigne l'ensemble des fonctions de classe C^∞ qui sont à support compact dans Ω . Il est possible d'étendre cette définition à tout ensemble $E \subset \mathbb{R}^d$ en posant

$$H_0^1(E) = \left\{ u \in H^1(\mathbb{R}^d) : u = 0 \text{ q.p. sur } \mathbb{R}^d \setminus E \right\}.$$

Les deux définitions de l'espace $H_0^1(\Omega)$ coïncident pour les ouverts (voir, par exemple, [61, Théorème 3.3.42]). En fait, il est suffisant de se restreindre aux quasi-ouverts puisque pour tout ensemble $E \subset \mathbb{R}^d$, il existe un quasi-ouvert $\Omega \subset \mathbb{R}^d$ tel que $\text{cap}(\Omega \setminus E) = 0$ et $H_0^1(E) = H_0^1(\Omega)$ (voir [61, Proposition 3.3.44]).

Si $\Omega \subset \mathbb{R}^d$ est un quasi-ouvert, alors nous prolongerons toujours les fonctions $u \in H_0^1(\Omega)$ de l'espace de Sobolev usuel par 0 sur $\mathbb{R}^d \setminus \Omega$ de sorte que, pour tout ouvert ou quasi-ouvert Ω , $H_0^1(\Omega)$ est vu comme un sous-espace de $H^1(\mathbb{R}^d)$.

Pour tout quasi-ouvert $\Omega \subset \mathbb{R}^d$, l'espace $H_0^1(\Omega)$ est un sous-espace fermé de $H^1(\mathbb{R}^d)$ et est dense dans $L^2(\Omega)$. De plus, d'après le théorème de Rellich–Kondrachov, l'injection $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ est compacte pour tout quasi-ouvert borné $\Omega \subset \mathbb{R}^d$ (ou, plus généralement, pour tout quasi-ouvert de mesure de Lebesgue finie).

Pour deux quasi-ouverts $\Omega_1, \Omega_2 \subset \mathbb{R}^d$, $H_0^1(\Omega_1) \subset H_0^1(\Omega_2)$ si et seulement si $\Omega_1 \subset \Omega_2$ quasi-partout. Par conséquent, si Ω est un quasi-ouvert de capacité non nulle alors il vient $H_0^1(\Omega) \neq \{0\}$ (car $\Omega \not\subset \emptyset$ quasi-partout), et en particulier $|\Omega| > 0$. Ainsi, pour tout quasi-ouvert $\Omega \subset \mathbb{R}^d$, $\text{cap}(\Omega) = 0$ si et seulement si $|\Omega| = 0$.

Si $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ sont deux quasi-ouverts tels que $\text{cap}(\Omega_1 \cap \Omega_2) = 0$ et $u \in H_0^1(\Omega_1 \cup \Omega_2)$, alors les restrictions $u|_{\Omega_1}$ et $u|_{\Omega_2}$ appartiennent respectivement aux espaces $H_0^1(\Omega_1)$ et $H_0^1(\Omega_2)$.

Remarque 1.1.6. Une autre manière de définir l'espace de Sobolev $H_0^1(E)$ pour tout ensemble $E \subset \mathbb{R}^d$ est de poser

$$\tilde{H}_0^1(E) = \left\{ u \in H^1(\mathbb{R}^d) : u = 0 \text{ p.p. sur } \mathbb{R}^d \setminus E \right\}. \quad (1.1.2)$$

Cependant, cette définition ne coïncide pas avec l'espace de Sobolev usuel $H_0^1(\Omega)$ pour les ouverts. En fait, pour tout quasi-ouvert $\Omega \subset \mathbb{R}^d$, l'inclusion $H_0^1(\Omega) \subset \tilde{H}_0^1(\Omega)$ est vérifiée mais peut être stricte pour des ensembles peu réguliers. Notons toutefois que l'égalité $H_0^1(\Omega) = \tilde{H}_0^1(\Omega)$ se produit pour les ouverts Ω à bord Lipschitzien.

1.1.3 Les valeurs propres d'opérateurs elliptiques

Nous exposons brièvement la construction de l'opérateur elliptique $L = -\text{div}(A\nabla \cdot) + V \cdot \nabla + b$ avec condition de Dirichlet ainsi que certaines propriétés des valeurs propres de L sur un quasi-ouvert $\Omega \subset D$, où $D \subset \mathbb{R}^d$ un ouvert borné (ou de mesure de Lebesgue finie). Dans cette sous-section nous utiliserons les notations suivantes :

- $A = (a_{ij})_{ij} : D \rightarrow \text{Sym}_d^+$ possède des coefficients $a_{ij} \in L^\infty(D)$ mesurables bornés et vérifie la condition d'ellipticité

$$\lambda_A |\xi|^2 \leq \xi \cdot A_x \xi = \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j, \quad \text{pour tout } x \in D \text{ et } \xi \in \mathbb{R}^d,$$

pour un $\lambda_A > 0$ (Sym_d^+ désigne l'ensemble des matrices de taille $d \times d$, symétriques, réelles et définies positives) ;

- $V \in L^\infty(D, \mathbb{R}^d)$ est un champ de vecteurs borné ;
- $b \in L^\infty(D)$ est une fonction bornée ;
- $\tau > 0$ est une constante telle que

$$\|a_{ij}\|_{L^\infty(D)}, \|V\|_{L^\infty(D, \mathbb{R}^d)}, \|b\|_{L^\infty(D)} \leq \tau.$$

Précisons que l'on écrit, de manière un peu abusive, $\|V\|_{L^\infty(D, \mathbb{R}^d)}$ au lieu de $\| |V| \|_{L^\infty(D, \mathbb{R}^d)}$, où $|V|$ est la norme euclidienne de V .

Pour un quasi-ouvert $\Omega \subset D$ et $f \in L^2(\Omega, \mathbb{C})$, on dit qu'une fonction complexe u est solution de l'équation

$$Lu = -\operatorname{div}(A\nabla u) + V \cdot \nabla u + bu = f \quad \text{dans } \Omega, \quad u \in H_0^1(\Omega, \mathbb{C})$$

si l'équation est vérifiée au sens des distributions, c'est-à-dire si $u \in H_0^1(\Omega, \mathbb{C})$ et

$$\int_{\Omega} (A\nabla u \cdot \nabla \bar{\varphi} + V \cdot \nabla u \bar{\varphi} + bu\bar{\varphi}) dx = \int_{\Omega} f\bar{\varphi} dx, \quad \text{pour tout } \varphi \in H_0^1(\Omega, \mathbb{C}).$$

Soit $B : H_0^1(\Omega, \mathbb{C}) \times H_0^1(\Omega, \mathbb{C}) \rightarrow \mathbb{C}$ la forme sesquilinéaire associée à l'opérateur L et définie pour toutes fonctions complexes $u, v \in H_0^1(\Omega, \mathbb{C})$ par

$$B(u, v) = \int_{\Omega} (A\nabla u \cdot \nabla \bar{v} + V \cdot \nabla u \bar{v} + bu\bar{v}) dx.$$

Il existe une constante $\alpha = \alpha(d, \tau) > 0$ telle que pour tout $u, v \in H_0^1(\Omega, \mathbb{C})$ on a

$$|B(u, v)| \leq \tau \int_{\Omega} (d^2 |\nabla u| |\nabla v| + |\nabla u| |v| + |u| |v|) dx \leq \alpha \|u\|_{H^1} \|v\|_{H^1}.$$

De plus, en utilisant que A est uniformément elliptique et l'inégalité $2ab \leq \varepsilon a^2 + \varepsilon^{-1} b^2$ ($a, b, \varepsilon > 0$) avec $\varepsilon = \lambda_A / (2\tau)$, il vient pour tout $u \in H_0^1(\Omega, \mathbb{C})$

$$\begin{aligned} \operatorname{Re}(B(u, u)) &\geq \lambda_A \int_{\Omega} |\nabla u|^2 dx - 2\tau \int_{\Omega} |\nabla u| |u| dx - \tau \int_{\Omega} |u|^2 dx \\ &\geq (\lambda_A - \tau\varepsilon) \int_{\Omega} |\nabla u|^2 dx - \tau(1 + \varepsilon^{-1}) \int_{\Omega} |u|^2 dx \geq \frac{\lambda_A}{2} \|u\|_{H^1}^2 - c \|u\|_{L^2}^2, \end{aligned}$$

pour une constante convenable $c = c(\lambda_A, \tau) > 0$. Puisque la forme bilinéaire B n'est a priori pas coercive si $c \neq 0$, on définit l'opérateur $L_c = L + c$ ainsi que sa forme bilinéaire associée $B_c(u, v) = B(u, v) + c \int_{\Omega} uv dx$, qui vérifie pour tout $u, v \in H_0^1(\Omega, \mathbb{C})$

- (continuité) $|B_c(u, v)| \leq |B(u, v)| + c \int_{\Omega} |u| |v| dx \leq (\alpha + c) \|u\|_{H^1} \|v\|_{H^1}$,
- (coercivité) $\operatorname{Re}(B_c(u, u)) = \operatorname{Re}(B(u, u)) + c \|u\|_{L^2}^2 \geq \frac{\lambda_A}{2} \|u\|_{H^1}^2$.

Ainsi, d'après le théorème de Lax-Milgram, pour tout $f \in L^2(\Omega, \mathbb{C})$ il existe un unique $u \in H_0^1(\Omega, \mathbb{C})$ tel que $L_c u = f$ dans Ω (car la forme linéaire $\varphi \mapsto \int_{\Omega} f\varphi dx$ est continue sur $L^2(\Omega, \mathbb{C})$ et donc aussi sur $H^1(\Omega, \mathbb{C})$). Notons $R_{\Omega}^{L_c} : L^2(\Omega, \mathbb{C}) \rightarrow L^2(\Omega, \mathbb{C})$ la résolvante de L_c définie par $R_{\Omega}^{L_c}(f) = u$. C'est un opérateur continu et compact car l'injection $H_0^1(\Omega, \mathbb{C}) \hookrightarrow L^2(\Omega, \mathbb{C})$ est compacte. Puisque $\lambda \in \mathbb{C}$ est une valeur propre de L si et seulement si $(\lambda + c)^{-1}$ est une valeur propre de $R_{\Omega}^{L_c}$, alors il suit du théorème 1.A.3 que l'ensemble des valeurs propres de l'opérateur L consistent ou bien en un nombre fini de nombres complexes, ou bien en une suite de nombres complexes dont le module tend vers $+\infty$.

Néanmoins, il est remarquable de constater que la première valeur propre de l'opérateur L sur un ouvert Ω , que l'on notera $\lambda_1(\Omega, L)$ dans la suite, est en fait réelle. Dans le cas où Ω et les coefficients de L sont réguliers, nous avons le résultat suivant.

Théorème 1.1.7. Soit $\Omega \subset \mathbb{R}^d$ un ouvert connexe de classe $C^{2,\alpha}$ et supposons que les coefficients de L sont dans $C^{0,\alpha}(\bar{\Omega})$. Alors $\lambda_1(\Omega, L)$ est réelle, simple et telle que $u_1 > 0$ dans Ω ; c'est la seule valeur propre qui possède une fonction propre de signe constant dans Ω . De plus, $\lambda_1(\Omega, L) < \operatorname{Re}(\lambda)$ pour toute valeur propre $\lambda \neq \lambda_1(\Omega, L)$ de L dans Ω .

Ce résultat est une conséquence du théorème de Krein-Rutman (voir par exemple [94, Théorem 1.2]).

Théorème 1.1.8 (Krein-Rutman). Soient X un espace de Banach et $K \subset X$ un cône (i.e. un ensemble convexe, fermé tel que $tK \subset K$ pour tout $t \geq 0$ et $K \cap (-K) = \{0\}$) et supposons que l'intérieur $\text{int}(K)$ de K est non vide. Soit $T : X \rightarrow X$ un opérateur compact tel que $T(K \setminus \{0\}) \subset \text{int}(K)$. Alors

1. $r(T) > 0$, où $r(T)$ désigne le rayon spectral de T . De plus, $r(T)$ est une valeur propre simple de T avec un vecteur propre dans $\text{int}(K)$ et c'est la seule valeur propre de T qui possède un vecteur propre dans $\text{int}(K)$.
2. $|\lambda| < r(T)$ pour toute valeur propre $\lambda \neq r(T)$ de T .

Le théorème 1.1.7 se déduit du théorème de Krein-Rutman en posant $X = C_0^{1,\alpha}(\overline{\Omega})$, $K = \{u \in X : u \geq 0 \text{ dans } \Omega\}$ et $T = R_\Omega^{L,c}$. Le fait que T est un opérateur dans X provient d'un résultat de régularité elliptique (si $f \in C_0^{0,\alpha}(\overline{\Omega})$, alors la solution de $L_c u = f$ dans Ω vérifie $u \in C_0^{2,\alpha}(\overline{\Omega})$, voir [48, Theorem 6.14]), et l'hypothèse $T(K \setminus \{0\}) \subset \text{int}(K)$ est vérifiée grâce au principe du maximum fort. Notons qu'il existe une preuve différente du théorème 1.1.7 qui repose sur le théorème de point fixe de Schaefer (voir [43, Theorem 3, §6.5.2]).

Dans [6], Berestycki, Nirenberg et Varadhan étendent ce résultat au cas des ouverts Ω connexes bornés (non réguliers) pour des solutions dans $W_{\text{loc}}^{2,d}(\Omega, \mathbb{C})$. Plus précisément, $\lambda \in \mathbb{C}$ est une valeur propre de L dans Ω s'il existe une fonction propre $u \in W_{\text{loc}}^{2,d}(\Omega, \mathbb{C})$, $u \neq 0$ et u bornée telle $Lu = \lambda u$ dans Ω et $u = 0$ sur $\partial\Omega$ (voir [6, Section 3] pour la définition de la condition de Dirichlet au bord). Ils définissent ensuite la valeur propre principale de L dans Ω par

$$\lambda_1(\Omega, L) = \sup \left\{ \lambda \in \mathbb{R} : \exists \phi \in W_{\text{loc}}^{2,d}(\Omega) \text{ tel que } \phi > 0 \text{ et } -L\phi + \lambda\phi \leq 0 \text{ dans } \Omega \right\}, \quad (1.1.3)$$

et montrent qu'elle possède les propriétés suivantes (voir [6, Theorems 2.1 et 2.3]) :

Théorème 1.1.9. Soient $\Omega \subset \mathbb{R}^d$ un ouvert borné et connexe et $L = -\text{div}(A\nabla \cdot) + V \cdot \nabla + b$ tel que $a_{ij} \in C(\Omega)$ et $V, b \in L^\infty(\Omega)$.

1. Il existe une fonction propre $u_1 : \Omega \rightarrow \mathbb{R}$ telle que $u_1 \in W_{\text{loc}}^{2,p}(\Omega)$, pour tout $p \in [1, +\infty)$, et

$$Lu_1 = \lambda_1(\Omega, L) u_1 \text{ dans } \Omega, \quad u_1 = 0 \text{ sur } \partial\Omega \quad \text{et} \quad u_1 > 0 \text{ dans } \Omega.$$

2. $\lambda_1(\Omega, L) < \text{Re}(\lambda)$ pour toute valeur propre $\lambda \neq \lambda_1(\Omega, L)$ de L dans Ω . De plus, une fonction propre réelle de L associée à $\lambda \neq \lambda_1(\Omega, L)$ change de signe dans Ω .
3. Toute fonction propre bornée associée à $\lambda_1(\Omega, L)$ est un multiple de u_1 .
4. La valeur propre principale est strictement décroissante par rapport à l'inclusion : pour tous ouverts bornés connexes Ω_1, Ω_2 tels que $\Omega_1 \subsetneq \Omega_2$ il vient $\lambda_1(\Omega_2, L) < \lambda_1(\Omega_1, L)$.

La preuve passe par un argument d'approximation où ils montrent que $\lambda_1(\Omega, L)$ peut être obtenue comme la limite des valeurs propres principales sur une suite croissante d'ouverts réguliers Ω_n tels que $\cup_{n \geq 1} \Omega_n = \Omega$. Précisons qu'ils obtiennent aussi une caractérisation des domaines qui possèdent une valeur propre principale strictement positive. Plus précisément, ils montrent que $\lambda_1(\Omega, V) > 0$ si et seulement si L vérifie une certaine version du principe du maximum dans Ω (voir [6, Theorem 1.1]). Enfin, mentionnons qu'une conséquence directe de l'égalité (1.1.3) est que la valeur propre principale est caractérisée par

$$\lambda_1(\Omega, L) = \sup_{\phi \in W_{\text{loc}}^{2,d}(\Omega), \phi > 0} \inf_{\Omega} \frac{L\phi}{\phi}.$$

Remarque 1.1.10. 1) Par un résultat classique de régularité elliptique (voir [48, Theorem 9.11]) les fonctions propres $u \in H_0^1(\Omega)$ de $L = -\Delta + V \cdot \nabla$, $V \in L^\infty$, dans un ouvert borné Ω sont dans $W_{\text{loc}}^{2,p}(\Omega, \mathbb{C})$ pour tout $p < \infty$, et sont bornées. En particulier, les deux définitions précédentes des valeurs propres coïncident pour cet opérateur.

2) Si Ω est un ouvert borné et connexe, lisse (de classe $C^{2,\alpha}$) et $V \in L^\infty$, alors le principe du maximum est vérifié pour l'opérateur $L = -\Delta + V \cdot \nabla$ dans Ω . En conséquence, $\lambda_1(\Omega, L) > 0$.

Dans la suite de cette sous-section nous supposons que $V = 0$, ce qui correspond au cas où l'opérateur $L = -\operatorname{div}(A\nabla\cdot) + b$ est autoadjoint.

Commençons par rappeler l'inégalité de Poincaré. Pour tout quasi-ouvert $\Omega \subset D$, il existe une constante $C(\Omega) > 0$ dépendant uniquement de Ω telle que, pour toute fonction $u \in H_0^1(\Omega)$ on a

$$\int_{\Omega} u^2 dx \leq C(\Omega) \int_{\Omega} |\nabla u|^2 dx.$$

En particulier, les normes $\|\nabla\cdot\|_{L^2}$ et $\|\cdot\|_{H^1}$ sont équivalentes sur $H_0^1(\Omega)$. Notons que la constante optimale dans l'inégalité de Poincaré est $C(\Omega) = 1/\lambda_1(\Omega)$, où $\lambda_1(\Omega)$ désigne la première valeur propre du Laplacien de Dirichlet sur Ω (définie plus loin, voir (1.1.6)).

Soit $B : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$ la forme bilinéaire (réelle) associée à l'opérateur L et définie par

$$B(u, v) = \int_{\Omega} (A\nabla u \cdot \nabla v + buv) dx.$$

On pose $c = -\inf_{x \in \Omega} b(x)$ et on considère l'opérateur $L_c = L + c$ ainsi que sa forme bilinéaire associée $B_c(u, v) = B(u, v) + c \int_{\Omega} uv dx$, qui est continue et coercive puisque, par ellipticité de A et l'inégalité de Poincaré, il vient

$$B_c(u, u) \geq \lambda_A \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} (b+c)u^2 dx \geq \frac{\lambda_A}{1+C(\Omega)} \|u\|_{H^1}^2.$$

Par le théorème Lax-Milgram, la résolvante $R_{\Omega}^{L_c} : L^2(\Omega) \rightarrow L^2(\Omega)$ de L_c , définie par $R_{\Omega}^{L_c}(f) = u$, est un opérateur continu, positif, autoadjoint et compact. Par conséquent, d'après le théorème 1.A.1, l'ensemble des valeurs propres de L dans Ω consiste en une suite de réels que l'on note (en prenant en compte la multiplicité)

$$\inf_{x \in \Omega} b(x) < \lambda_1(\Omega, L) \leq \lambda_2(\Omega, L) \leq \dots \leq \lambda_k(\Omega, L) \leq \dots \rightarrow +\infty.$$

Les fonctions propres normalisés (pour la norme L^2) associés aux valeurs propres $\lambda_k(\Omega, L)$, notés $u_k \in H_0^1(\Omega)$, vérifient

$$\begin{cases} Lu_k = \lambda_k(\Omega, L) u_k & \text{dans } \Omega \\ u_k = 0 & \text{sur } \partial\Omega. \end{cases}$$

et forment un système orthonormal de L^2 , c'est-à-dire

$$\int_{\Omega} u_i u_j dx = \delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

On peut exprimer les valeurs propres à l'aide d'un principe min-max (aussi connu sous le nom de formules de Courant-Fischer, voir [29]) de la façon suivante

$$\lambda_k(\Omega, L) = \min_{\substack{V \text{ sous-espace de} \\ \text{dimension } k \text{ de } H_0^1(\Omega)}} \max_{v \in V \setminus \{0\}} \frac{\int_{\Omega} (A\nabla v \cdot \nabla v + b v^2) dx}{\int_{\Omega} v^2 dx}, \quad (1.1.4)$$

le minimum étant atteint par le sous-espace engendré par les k premières fonctions propres sur Ω . En particulier, la première valeur est variationnellement caractérisée par

$$\lambda_1(\Omega, L) = \min_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} (A\nabla v \cdot \nabla v + b v^2) dx}{\int_{\Omega} v^2 dx}.$$

Mentionnons maintenant quelques propriétés des valeurs propres et des fonctions propres. Dans ce qui suit, l'hypothèse $b \geq 0$ signifie que la preuve passe par le principe du maximum fort.

- (monotonie) Les valeurs propres sont décroissantes par rapport à l'inclusion : pour tous quasi-ouverts $\Omega_1, \Omega_2 \subset D$ et tout entier $k \geq 1$,

$$\Omega_1 \subset \Omega_2 \Rightarrow \lambda_k(\Omega_2, L) \leq \lambda_k(\Omega_1, L).$$

C'est une conséquence directe des formules min-max pour les valeurs propres. Notons que la décroissante n'est pas stricte, puisque si Ω_1 et Ω_2 sont deux quasi-ouverts disjoints tels que $\lambda_1(\Omega_1, L) = \lambda_1(\Omega_2, L)$, alors $\lambda_1(\Omega_1 \cup \Omega_2, L) = \lambda_1(\Omega_1, L)$. Par contre, on peut montrer que si $b \geq 0$ et $\Omega_1, \Omega_2 \subset D$ sont ouverts avec Ω_2 connexe, alors il vient

$$\Omega_1 \subsetneq \Omega_2 \Rightarrow \lambda_k(\Omega_2, L) < \lambda_k(\Omega_1, L).$$

- (spectre de l'union) Si $\Omega_1, \Omega_2 \subset D$ sont deux quasi-ouverts disjoints, alors l'ensemble des valeurs propres sur $\Omega_1 \cup \Omega_2$ correspond à l'union des ensembles des valeurs propres sur Ω_1 et Ω_2 , en comptant les multiplicités. Ceci découle du fait que si $u \in H_0^1(\Omega_1 \cup \Omega_2)$ est une valeur propre sur $\Omega_1 \cup \Omega_2$, alors la restriction $u|_{\Omega_1} \in H_0^1(\Omega_1)$ est une valeur propre sur Ω_1 . En particulier, il vient

$$\lambda_1(\Omega_1 \cup \Omega_2, L) = \min\{\lambda_1(\Omega_1, L), \lambda_1(\Omega_2, L)\}.$$

- (simplicité) Si $b \geq 0$, alors la première valeur propre d'un ouvert connexe est simple (de multiplicité 1). L'hypothèse de connexité est bien sûr nécessaire. Si par exemple $\lambda_1(\Omega_1, L) = \lambda_1(\Omega_2, L)$ avec Ω_1 et Ω_2 deux ouverts connexes et disjoints, alors la première valeur propre sur $\Omega = \Omega_1 \cup \Omega_2$ est de multiplicité 2 puisque $\lambda_1(\Omega, L) = \lambda_1(\Omega_1, L) = \lambda_1(\Omega_2, L)$.
- (signe de la première fonction propre) Si $b \geq 0$ et $\Omega \subset D$ est ouvert, alors la première fonction propre u_1 est de signe constant sur chaque composante connexe Ω_0 de Ω , c'est-à-dire $u_1 > 0$ dans Ω_0 ou $u_1 = 0$ dans Ω_0 (quitte à changer u_1 en $-u_1$). Ceci est une conséquence de la caractérisation variationnelle de la première valeur propre et du principe du maximum. Bien sûr, si Ω est un ouvert connexe, alors les fonctions propres u_k associées à $\lambda_k(\Omega, L)$, $k \geq 2$, changent de signe puisqu'elles sont orthogonales à u_1 pour le produit scalaire de L^2 .
- (ensembles nodaux) La k -ième fonction propre u_k sur un ouvert connexe Ω possède au plus k ensembles nodaux, les ensembles nodaux de u_k étant définis comme les composantes connexes des ouverts $\{u_k > 0\}$ et $\{u_k < 0\}$.

Remarque 1.1.11. Il est possible de définir de la même manière l'opérateur L sur l'espace $\tilde{H}_0^1(\Omega)$ (voir (1.1.2)). Les valeurs propres de L sur Ω pour cet espace, notées $\tilde{\lambda}_k(\Omega, L)$, vérifient $\tilde{\lambda}_k(\Omega, L) \leq \lambda_k(\Omega, L)$ puisque $H_0^1(\Omega) \subset \tilde{H}_0^1(\Omega)$ (et que $\tilde{\lambda}_k(\Omega, L)$ est caractérisée par la formule (1.1.4) avec $H_0^1(\Omega)$ est remplacé par $\tilde{H}_0^1(\Omega)$).

Pour $A = \text{Id}$ et $b = 0$ l'opérateur L correspond au Laplacien de Dirichlet $L = -\Delta$. Les valeurs propres du Laplacien, qui seront plus simplement notées $\lambda_k(\Omega) = \lambda_k(\Omega, -\Delta)$ (sauf dans la section 1.3.2 où cette notation désignera les valeurs propres d'un autre opérateur), forment une suite croissante de réels strictement positifs tendant vers $+\infty$ et sont variationnellement caractérisées par

$$\lambda_k(\Omega) = \min_{\substack{V \text{ sous-espace de} \\ \text{dimension } k \text{ de } H_0^1(\Omega)}} \max_{v \in V \setminus \{0\}} \frac{\int_{\Omega} |\nabla v|^2 dx}{\int_{\Omega} v^2 dx}, \quad (1.1.5)$$

et en particulier pour $k = 1$

$$\lambda_1(\Omega) = \min_{v \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla v|^2 dx}{\int_{\Omega} v^2 dx}. \quad (1.1.6)$$

Mentionnons enfin que les valeurs propres $\lambda_k(\Omega)$ du Laplacien sont invariantes par translation et sont (-2) -homogènes, c'est-à-dire, pour tous $x \in \mathbb{R}^d$ et $t > 0$,

$$\lambda_k(\Omega + x) = \lambda_k(\Omega), \quad \lambda_k(t\Omega) = \frac{1}{t^2} \lambda_k(\Omega).$$

1.1.4 La γ -convergence

En optimisation de forme, les preuves d'existence s'obtiennent souvent en considérant une topologie adéquate sur l'ensemble des formes et en utilisant des arguments de compacité et de continuité pour la fonctionnelle considérée. Bien sûr, il n'existe pas vraiment de topologie naturelle sur les domaines de \mathbb{R}^d , et la difficulté concernant le choix d'une bonne topologie réside dans le fait qu'elle doit être à la fois, la moins fine possible (c'est-à-dire contenant peu d'ouverts) pour obtenir la compacité des suites minimisantes, et la plus fine possible pour obtenir la semi-continuité inférieure de la fonctionnelle. La topologie qui nous intéressera est celle de la γ -convergence, qui est définie sur les quasi-ouverts et qui exprime, par définition, la continuité de la solution du problème de Dirichlet par rapport au domaine. Fixons comme précédemment un ouvert borné $D \subset \mathbb{R}^d$. Pour un quasi-ouvert $\Omega \subset D$, notons $w_\Omega = R_\Omega^{-\Delta}(1)$ la solution de l'équation

$$-\Delta w_\Omega = 1 \quad \text{dans } \Omega, \quad w_\Omega \in H_0^1(\Omega).$$

La fonction w_Ω , habituellement appelée fonction torsion, possède les propriétés suivantes : pour tout quasi-ouvert $\Omega \subset D$,

1. il existe une constante dimensionnelle $C_d > 0$ telle que

$$\|\nabla w_\Omega\|_{L^2} \leq C_d |\Omega|^{\frac{d+2}{2d}} \quad \text{et} \quad \|w_\Omega\|_{L^\infty} \leq C_d |\Omega|^{2/d}.$$

La première inégalité se déduit facilement de l'inégalité de Gagliardo-Nirenberg-Sobolev, et on pourra se référer à [89, Corollary 3.52] pour la deuxième.

2. $H_0^1(\Omega) = H_0^1(\{w_\Omega > 0\})$ et, en particulier, $\Omega = \{w_\Omega > 0\}$ (à un ensemble de capacité nulle près). Cette propriété peut être vue comme une version du principe du maximum fort (voir [89, Proposition 3.72]).

Définition 1.1.12. On dit qu'une suite de quasi-ouverts $(\Omega_n)_{n \geq 1}$ contenus dans D γ -converge vers un quasi-ouvert $\Omega \subset D$ si la suite w_{Ω_n} converge dans $L^2(D)$ vers w_Ω .

Remarque 1.1.13. 1) Si w_{Ω_n} converge dans $L^2(D)$ vers w_Ω , alors $w_{\Omega_n, f} := R_{\Omega_n}^{-\Delta}(f)$ converge fortement dans $H_0^1(D)$ vers $w_{\Omega, f} := R_\Omega^{-\Delta}(f)$ pour tout $f \in L^2(D)$ (voir [61, Théorème 3.2.5]). En particulier, il est possible de supposer dans la définition 1.1.12 que la convergence est forte dans $H^1(D)$.

2) La γ -convergence est métrisable sur l'ensemble des quasi-ouverts, si on munit par exemple cet ensemble de la distance définie pour tous quasi-ouverts $\Omega_1, \Omega_2 \subset D$ par

$$d_\gamma(\Omega_1, \Omega_2) = \|w_{\Omega_1} - w_{\Omega_2}\|_{L^2(D)}.$$

3) La γ -convergence n'est pas compacte sur l'ensemble des quasi-ouverts. Dans [27], Cioranescu et Murat définissent une suite d'ouverts (réguliers) $(\Omega_n)_{n \geq 1}$ en perforant un ouvert D avec des boules périodiquement distribuées et de même rayon $r_n > 0$. En choisissant $r_n \rightarrow 0$ de manière convenable, ils montrent que la suite w_{Ω_n} converge faiblement dans $H_0^1(D)$ vers une limite $w \in H_0^1(D)$ qui est solution du problème

$$-\Delta w + cw = 1 \quad \text{dans } D, \quad w \in H_0^1(D),$$

pour une certaine constante $c > 0$. En particulier, ceci montre qu'aucune sous-suite de $(\Omega_n)_{n \geq 1}$ ne peut γ -converger vers un quasi-ouvert $\Omega \subset D$.

4) Le compactifié de l'ensemble des quasi-ouverts contenus dans D pour la γ -convergence correspond à l'ensemble des mesures, dites capacitaires, qui sont boréliennes, positives et absolument continues par rapport à la capacité. Ce résultat est dû à Dal Maso et Mosco (voir [30]).

La γ -convergence est caractérisée par la convergence des résolvantes du Laplacien pour la topologie uniforme d'opérateur (voir [61, Lemme 4.7.3]).

Théorème 1.1.14. Soient $(\Omega_n)_{n \geq 1}$ une suite de quasi-ouverts inclus dans D et $\Omega \subset D$ un quasi-ouvert. Les assertions suivantes sont équivalentes :

1. La suite $(\Omega_n)_{n \geq 1}$ γ -converge vers Ω .
2. Pour toute suite $(f_n)_{n \geq 1} \subset L^2(D)$ qui converge faiblement dans $L^2(D)$ vers $f \in L^2(D)$, la suite $(R_{\Omega_n}^{-\Delta}(f_n))_{n \geq 1}$ converge vers $R_{\Omega}^{-\Delta}(f)$ dans $L^2(D)$.
3. La suite d'opérateurs $(R_{\Omega_n}^{-\Delta})_{n \geq 1} \subset \mathcal{L}(L^2(D))$ converge vers $R_{\Omega}^{-\Delta}$ pour la norme d'opérateur $\|\cdot\|_{\mathcal{L}(L^2(D))}$.

La continuité des valeurs propres du Laplacien par rapport à la γ -convergence est alors une conséquence de la continuité du spectre d'opérateurs autoadjoints pour la convergence en norme d'opérateur (voir Théorème 1.A.2 en annexe).

Corollaire 1.1.15. Pour tout $k \geq 1$, la fonctionnelle $\Omega \mapsto \lambda_k(\Omega)$ définie sur l'ensemble des quasi-ouverts inclus dans D est continue par rapport à la γ -convergence. Précisément, si $(\Omega_n)_{n \geq 1} \subset D$ est une suite de quasi-ouverts qui γ -converge vers un quasi-ouvert $\Omega \subset D$, alors pour tout $k \geq 1$

$$\lambda_k(\Omega) = \lim_{n \rightarrow +\infty} \lambda_k(\Omega_n).$$

Précisons que si $\Omega = \emptyset$ (ou, de manière équivalente, si Ω est un quasi-ouvert de capacité nulle), alors $H_0^1(\Omega) = \{0\}$ et par conséquent $\lambda_k(\Omega)$ n'est pas bien définie. Il est alors naturel de poser $\lambda_k(\Omega) = +\infty$ pour $\Omega = \emptyset$, de sorte que l'application $\Omega \mapsto \lambda_k(\Omega)$ est effectivement continue (pour la γ -convergence) dans tous les cas, y compris celui où $(\Omega_n)_{n \geq 1}$ est une suite de quasi-ouverts qui converge vers un quasi-ouvert Ω de capacité nulle.

En vue d'obtenir un résultat d'existence, il est naturel de considérer une topologie plus faible que celle de la γ -convergence afin d'obtenir un résultat de compacité pour les suites minimisantes.

Définition 1.1.16. On dit qu'une suite de quasi-ouverts $(\Omega_n)_{n \geq 1}$ contenus dans D γ -converge faiblement vers un quasi-ouvert $\Omega \subset D$ si la suite w_{Ω_n} converge dans $L^2(D)$ vers une fonction $w \in H_0^1(D)$ telle que $\Omega = \{w_{\Omega} > 0\}$.

Remarque 1.1.17. 1) La fonction $w \in H_0^1(\Omega)$ de la définition 1.1.16 ne coïncide en général pas avec w_{Ω} puisque la γ -convergence n'est pas compacte. En fait, ceci se produit seulement si la suite $(\Omega_n)_{n \geq 1}$ γ -converge vers Ω .

2) La γ -convergence faible est compacte sur l'ensemble des quasi-ouverts inclus dans D . En effet, la suite w_{Ω_n} est bornée dans $H_0^1(D)$ et donc, à une sous-suite près, converge faiblement dans $H_0^1(D)$ et fortement dans $L^2(D)$ vers une fonction $w \in H_0^1(D)$. Ainsi la suite $(\Omega_n)_{n \geq 1}$ γ -converge faiblement vers le quasi-ouvert Ω défini par $\Omega := \{w > 0\}$.

3) La mesure de Lebesgue est semi-continue inférieurement par rapport à la convergence faible, c'est-à-dire, si $(\Omega_n)_{n \geq 1}$ est une suite de quasi-ouverts inclus dans D qui γ -converge faiblement vers un quasi-ouvert $\Omega \subset D$, alors $|\Omega| \leq \liminf_{n \rightarrow +\infty} |\Omega_n|$.

Le prochain lemme énonce qu'il est possible, à partir d'une suite $(\Omega_n)_{n \geq 1}$ qui γ -converge faiblement vers un quasi-ouvert, d'obtenir une suite qui γ -converge vers la même limite en agrandissant les quasi-ouverts Ω_n (voir, par exemple [61, Lemme 4.7.11] ou [20, Lemma 4.10] pour une preuve).

Lemme 1.1.18. Soit $(\Omega_n)_{n \geq 1}$ une suite de quasi-ouverts inclus dans D qui γ -converge faiblement vers un quasi-ouvert $\Omega \subset D$. Alors il existe une sous-suite de $(\Omega_n)_{n \geq 1}$, encore notée $(\Omega_n)_{n \geq 1}$, et une suite $(\tilde{\Omega}_n)_{n \geq 1}$ de quasi-ouverts qui γ -converge vers Ω avec $\Omega_n \subset \tilde{\Omega}_n \subset D$.

En conséquence du Corollaire 1.1.15 et du Lemme 1.1.18, les valeurs propres du Laplacien sont semi-continues inférieurement par rapport à la γ -convergence faible.

Corollaire 1.1.19. Pour tout $k \geq 1$, la fonctionnelle $\Omega \mapsto \lambda_k(\Omega)$ définie sur l'ensemble des quasi-ouverts inclus dans D est semi-continue inférieurement par rapport à la γ -convergence faible, c'est-à-dire, si $(\Omega_n)_{n \geq 1} \subset D$ γ -converge faiblement vers $\Omega \subset D$, alors

$$\lambda_k(\Omega) \leq \liminf_{n \rightarrow +\infty} \lambda_k(\Omega_n).$$

Il découle aussi du Lemme 1.1.18 que la semi-continuité inférieure par rapport à la γ -convergence et par rapport à la γ -convergence faible coïncident pour les fonctionnelles décroissantes par rapport à l'inclusion. Ainsi, la γ -convergence faible étant compacte, il s'en déduit le théorème d'existence suivant, dû à Buttazzo et Dal Maso (voir [21]).

Théorème 1.1.20 (Buttazzo-Dal Maso). Soient $D \subset \mathbb{R}^d$ un ouvert borné et $0 < m \leq |D|$. Soit \mathcal{F} une fonctionnelle réelle définie sur les quasi-ouverts qui est

- décroissante par rapport à l'inclusion,
- semi-continue inférieurement par rapport à la γ -convergence.

Alors le problème d'optimisation

$$\min \{ \mathcal{F}(\Omega) : \Omega \subset D \text{ quasi-ouvert, } |\Omega| \leq m \} \quad (1.1.7)$$

a une solution.

Preuve. Soit $(\Omega_n)_{n \geq 1}$ une suite minimisante pour (1.1.7). À une sous-suite près, $(\Omega_n)_{n \geq 1}$ γ -converge faiblement vers un quasi-ouvert $\Omega \subset D$ et, par la Remarque 1.1.17 3), il suit que $|\Omega| \leq m$. De plus, la suite $(\tilde{\Omega}_n)_{n \geq 1}$ donnée par le Lemme 1.1.18 γ -converge vers Ω et vérifie $\Omega_n \subset \tilde{\Omega}_n$, de sorte qu'il vient

$$\mathcal{F}(\Omega) \leq \lim_{n \rightarrow +\infty} \mathcal{F}(\tilde{\Omega}_n) \leq \lim_{n \rightarrow +\infty} \mathcal{F}(\Omega_n).$$

□

Remarque 1.1.21. Dans le Théorème 1.1.7, la contrainte sur le volume peut être remplacée par $|\Omega| = m$ puisqu'en agrandissant Ω la fonctionnelle décroît.

Terminons cette section avec un résultat d'existence concernant les valeurs propres du Laplacien qui est une conséquence immédiate du théorème de Buttazzo et Dal Maso 1.1.20, les valeurs propres étant décroissantes par rapport à l'inclusion (formules de Courant-Fischer (1.1.5)) et continues par rapport à la γ -convergence (Corollaire 1.1.15).

Théorème 1.1.22. Soient $D \subset \mathbb{R}^d$ un ouvert borné et $0 < m \leq |D|$. Soit $\mathcal{F} : \mathbb{R}^k \rightarrow \mathbb{R}$ une fonctionnelle croissante et semi-continue inférieurement en chaque variable. Alors le problème d'optimisation

$$\min \{ \mathcal{F}(\lambda_1(\Omega), \lambda_2(\Omega), \dots, \lambda_k(\Omega)) : \Omega \subset D \text{ quasi-ouvert, } |\Omega| = m \} \quad (1.1.8)$$

a une solution.

Remarque 1.1.23. 1) Le caractère borné de la boîte D dans le théorème 1.1.22 est une hypothèse cruciale. Si D n'est pas bornée, alors l'injection $H_0^1(D) \hookrightarrow L^2(D)$ n'est plus compacte et un phénomène de perte de «masse» à l'infini peut se produire comme le montre l'exemple suivant. Le problème

$$\min \{ \lambda_1(\Omega) : \Omega \subset D, |\Omega| = \pi \} \quad (1.1.9)$$

où $D \subset \mathbb{R}^2$ est la boîte définie par

$$D = \left\{ (x, y) \in (1, +\infty) \times \mathbb{R} : \frac{1}{x} - 1 < y < 1 - \frac{1}{x} \right\},$$

n'a pas de solution. En effet, les ensembles optimaux pour la première valeur propre parmi les ouverts de \mathbb{R}^2 de mesure π sont exactement les boules de rayon 1 (d'après l'inégalité de Faber-Krahn, voir Théorème 1.2.1 plus loin) et ne sont pas contenus dans D . Or, en notant $x_n = (n, 0)$ et r_n le plus grand rayon tel que $B_{r_n}(x_n) \subset D$, on obtient une suite minimisante pour (1.1.9) qui converge vers $\lambda_1(B_1)$, puisque $r_n \rightarrow 1$ et donc $\lambda_1(B_{r_n}(x_n)) = r_n^2 \lambda_1(B_1) \rightarrow \lambda_1(B_1)$.

2) La preuve du théorème 1.1.22 repose aussi fortement sur la monotonie de la fonctionnelle, mais nous ne savons pas encore si cette hypothèse est nécessaire. Précisons toutefois que pour une fonctionnelle ne faisant intervenir que λ_1 et λ_2 , l'existence d'une forme optimale peut être obtenue sans cette hypothèse (en supposant seulement, en plus de la semi-continuité inférieure, que \mathcal{F} tend vers $+\infty$ à l'infini, voir [13, Section 6.4]).

3) Il existe une preuve différente du théorème 1.1.22 qui ne passe pas par la γ -convergence et qui repose sur l'utilisation de la formule min-max pour les valeurs propres (voir [57, Section 2.3]).

1.2 La valeur propre principale d'un opérateur elliptique

Dans cette section nous présentons les résultats des chapitres 2 et 3 qui correspondent aux articles [84] et [85] respectivement. Nous commençons par énoncer l'inégalité de Faber-Krahn ainsi qu'une extension de ce résultat à des opérateurs elliptiques non symétriques plus généraux que l'opérateur avec un terme de transport que nous considérons par la suite. Nous présentons ensuite des résultats d'existence concernant des problèmes de minimisation pour la valeur propre principale d'un opérateur avec un terme de transport sous des contraintes de volume et d'inclusion. La régularité des ensembles optimaux quand le terme de transport est le gradient d'une fonction lipschitzienne est aussi abordée, et nous présentons au préalable quelques résultats de régularité qui ont inspiré ce travail. Enfin, nous terminons avec un résultat de régularité en dimension deux pour un problème à plusieurs phases qui fait intervenir les valeurs propres du Laplacien de Dirichlet.

1.2.1 Inégalité de Faber-Krahn

Pour la première valeur propre du Laplacien de Dirichlet dans un domaine borné de \mathbb{R}^d , il est bien connu que les minimiseurs à une mesure du domaine fixée sont des boules. Ce résultat, appelé inégalité de Faber-Krahn, a d'abord été conjecturé par Lord Rayleigh pour $d = 2$ dans [83] (première édition en 1877), puis prouvé pour $d = 2$ indépendamment par Faber ([45]) en 1923 et Krahn ([68]) en 1925, et ensuite pour $d \geq 2$ par Krahn ([69]) en 1926.

Théorème 1.2.1 (Faber-Krahn). Soit $m > 0$. La boule $\Omega^* \subset \mathbb{R}^d$ centrée en 0 et de volume m est solution du problème

$$\min \left\{ \lambda_1(\Omega) : \Omega \subset \mathbb{R}^d \text{ ouvert, } |\Omega| = m \right\}. \quad (1.2.1)$$

De plus, Ω^* est l'unique minimiseur pour (1.2.1) à translation près (et à un ensemble de capacité nulle près).

La preuve repose sur la caractérisation variationnelle de la première valeur propre et sur l'utilisation d'un réarrangement, dit de Schwarz, qui fait décroître le quotient de Rayleigh. Pour un ouvert borné $\Omega \subset \mathbb{R}^d$ et une fonction positive $u \in H_0^1(\Omega)$, le réarrangement de Schwarz u^* de u est défini par

$$u^*(x) = \sup\{c \in \mathbb{R}_+ : x \in \Omega(c)^*\},$$

où $\Omega(c) = \{x \in \Omega : u(x) \geq c\}$ et où $\Omega^*, \Omega(c)^*$ sont les boules centrées en 0 de même volume que Ω et $\Omega(c)$ respectivement. Précisons qu'en désignant par $\rho(c)$ le rayon de la boule $\Omega(c)^*$, la fonction $\rho : \mathbb{R} \rightarrow \mathbb{R}_+$ est décroissante, et que si la décroissance de ρ est stricte alors il vient $u^*(x) = \rho^{-1}(|x|)$ pour tout $x \in \Omega$. Il est possible de montrer que $u^* \in H_0^1(\Omega^*)$, et que u^* est radiale décroissante et vérifie

$$\|u^*\|_{L^2(\Omega^*)} = \|u\|_{L^2(\Omega)} \quad \text{et} \quad \|\nabla u^*\|_{L^2(\Omega^*)} \leq \|\nabla u\|_{L^2(\Omega)}.$$

Si maintenant $u \in H_0^1(\Omega)$ désigne la première fonction propre du Laplacien sur Ω , alors il vient en utilisant (1.1.6)

$$\lambda_1(\Omega^*) \leq \frac{\int_{\Omega^*} |\nabla u^*|^2 dx}{\int_{\Omega^*} (u^*)^2 dx} \leq \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} u^2 dx} = \lambda_1(\Omega).$$

Dans [51], Hamel, Nadirashvili et Russ prouvent une inégalité de type Faber-Krahn pour un opérateur de la forme $-\operatorname{div}(A\nabla \cdot) + V \cdot \nabla + c$ avec condition de Dirichlet, où $A : \mathbb{R}^d \rightarrow \operatorname{Sym}_d^+$ vérifie une condition d'ellipticité et est à coefficients dans $W^{1,\infty}(\mathbb{R}^d)$, $V \in L^\infty(\mathbb{R}^d, \mathbb{R}^d)$ et $c \in L^\infty(\mathbb{R}^d)$. La principale difficulté est que l'opérateur n'est pas symétrique pour $V \neq 0$, et par conséquent qu'il n'existe pas de caractérisation variationnelle pour la première valeur propre. Étant donné que nous considérerons dans la suite l'opérateur avec terme de transport $L = -\Delta + V \cdot \nabla$ (voir sous-sections 1.2.2 et 1.2.4), nous n'énonçons ici leur résultat que pour cet opérateur (voir aussi [52] et [50]). Nous noterons souvent $\lambda_1(\Omega, V)$ (au lieu de $\lambda_1(\Omega, L)$) la première valeur propre de L dans Ω afin de faire apparaître la dépendance par rapport au terme de transport V .

Théorème 1.2.2. Soient $m, \tau > 0$. Notons \mathcal{C} l'ensemble des ouverts de \mathbb{R}^d qui sont bornés, connexes et de classe $C^{2,\alpha}$ pour un $\alpha \in (0, 1)$. Alors le problème

$$\min \left\{ \lambda_1(\Omega, V) : \Omega \in \mathcal{C}, |\Omega| = m, V \in L^\infty(\mathbb{R}^d, \mathbb{R}^d), \|V\|_{L^\infty} \leq \tau \right\}, \quad (1.2.2)$$

a une solution (Ω^*, V^*) , où Ω^* est la boule centrée en 0 de volume m et $V^* \in L^\infty(\mathbb{R}^d, \mathbb{R}^d)$ est défini par $V^*(x) = \tau \frac{x}{|x|}$. De plus, le couple (Ω^*, V^*) est unique à translation près, c'est-à-dire, pour tout couple (Ω, V) minimisant (1.2.2), il existe $x_0 \in \mathbb{R}^d$ tel que $(\Omega, V) = (x_0 + \Omega^*, \tau \frac{x-x_0}{|x-x_0|})$.

La preuve de ce résultat repose sur un réarrangement sphérique différent de celui de Schwarz qui permet, en utilisant l'inégalité isopérimétrique, d'obtenir une inégalité différentielle ponctuelle comparant la première fonction propre sur Ω avec sa fonction réarrangée sur Ω^* . Hamel, Nadirashvili et Russ montrent dans un premier temps, essentiellement grâce au principe du maximum et au lemme de Hopf, que pour $\Omega \in \mathcal{C}$ fixé le minimum

$$\underline{\lambda}(\Omega, \tau) := \min \left\{ \lambda_1(\Omega, V) : V \in L^\infty(\Omega, \mathbb{R}^d), \|V\|_{L^\infty} \leq \tau \right\}$$

est atteint par un (unique) $V \in L^\infty(\Omega, \mathbb{R}^d)$ donné par

$$V(x) = -\tau \frac{\nabla u(x)}{|\nabla u(x)|} \quad \text{si} \quad \nabla u(x) \neq 0 \quad \text{et} \quad V(x) = 0 \quad \text{si} \quad \nabla u(x) = 0, \quad (1.2.3)$$

où $u \in C^{2,\alpha}(\overline{\Omega})$ est la fonction propre principale sur Ω associée à V . Notons que u est alors solution de l'équation non linéaire

$$-\Delta u - \tau |\nabla u| = \underline{\lambda}(\Omega, \tau) u \quad \text{dans} \quad \Omega, \quad u \in H_0^1(\Omega).$$

Ainsi, le théorème 1.2.2 se réduit à prouver l'inégalité $\underline{\lambda}(\Omega^*, \tau) \leq \underline{\lambda}(\Omega, \tau)$. Pour cela, commençons par préciser d'une part, que leur stratégie ne s'applique pas à la fonction u , qui n'est pas suffisamment régulière, mais à une approximation analytique de u et d'autre part, que les égalités suivantes pour $x \in \Omega^*$ ne sont en fait valables qu'en dehors d'un nombre fini de sphères centrées en 0. Les auteurs de [51] définissent alors le réarrangement de u comme étant la fonction radiale décroissante $\tilde{u} \in H_0^1(\Omega^*)$ qui s'annule sur $\partial\Omega^*$ et qui vérifie, pour tout $0 \leq c < \|u\|_{L^\infty}$,

$$\int_{B_{\rho(c)}} \Delta \tilde{u}(x) dx = \int_{\Omega(c)} \Delta u(x) dx.$$

Ils montrent ensuite que \tilde{u} vérifie les deux propriétés clés suivantes :

- Pour tout $x \in \Omega^*$, il existe $y \in \Omega$ vérifiant $u(y) = \rho^{-1}(|x|)$ et tel que, pour tout $\varepsilon > 0$,

$$\Delta u(y) + (\tau + \varepsilon)|\nabla u(y)| \leq \Delta \tilde{u}(x) + (\tau + \varepsilon)|\nabla \tilde{u}(x)|.$$

- Pour tout $x \in \Omega^*$, $\tilde{u}(x) \geq \rho^{-1}(|x|)$ (le réarrangement \tilde{u} est supérieur à celui de Schwarz).

Ainsi, pour tout $x \in \Omega^*$ et tout $\varepsilon > 0$ ils obtiennent l'inégalité suivante

$$-\Delta \tilde{u}(x) - (\tau + \varepsilon)|\nabla \tilde{u}(x)| \leq (\lambda(\Omega, \tau) + \varepsilon)u(y) \leq (\lambda(\Omega, \tau) + \varepsilon)\tilde{u}(x),$$

à partir de laquelle ils prouvent que $\lambda(\Omega^*, \tau + \varepsilon) \leq \lambda(\Omega, \tau) + \varepsilon$ et concluent en passant à la limite $\varepsilon \rightarrow 0$.

Récemment, un résultat de type Faber-Krahn a été obtenu dans le cadre des variétés Riemanniennes par Lamboley et Sicbaldi (voir [73] et Théorème 1.2.10 plus loin).

Mentionnons aussi que des inégalités de Faber-Krahn ont été prouvées pour des conditions de Robin (voir, par exemple, [31] pour le Laplacien et [15] pour le p -Laplacien). Enfin, il existe également un résultat analogue concernant la maximisation de la deuxième valeur propre du Laplacien avec condition de Neumann (voir par exemple la section 7.1.2 du livre [56]).

1.2.2 Existence d'une forme optimale pour un opérateur avec un terme de transport et avec une condition de boîte

Les résultats présentés dans cette section correspondent à la première partie du chapitre 2 et ont été obtenus dans [84] en collaboration avec Emmanuel Russ et Bozhidar Velichkov.

Cette sous-section est consacrée à l'existence de formes optimales pour des problèmes de minimisation de la première valeur propre $\lambda_1(\Omega, V)$ de l'opérateur $L = -\Delta + V \cdot \nabla$, où le terme de transport $V \in L^\infty$ est borné, avec des contraintes de volume et d'inclusion.

La première étape consiste à définir la valeur propre principale de l'opérateur L sur un quasi-ouvert Ω . En effet, les valeurs propres de l'opérateur L sont complexes puisque L n'est pas autoadjoint, de sorte qu'il n'existe pas a priori de définition naturelle pour la première valeur propre. Néanmoins, nous montrons qu'il existe une valeur propre réelle $\lambda_1(\Omega, V)$ de L sur Ω , que nous appelons valeur propre principale, qui a la propriété d'être plus petite que la partie réelle de n'importe quelle autre valeur propre de L . Pour un ouvert connexe, l'existence d'une telle valeur propre a été démontrée par Berestycki, Nirenberg et Varadhan dans [6] (voir aussi Théorème 1.1.9), et nous montrons qu'il est possible d'étendre leur définition aux quasi-ouverts. Une fois la première valeur propre $\lambda_1(\Omega, V)$ bien définie, nous considérons le problème non variationnel suivant

$$\min \{ \lambda_1(\Omega, V) : \Omega \subset D \text{ quasi-ouvert, } |\Omega| \leq m, \|V\|_{L^\infty} \leq \tau \}, \quad (1.2.4)$$

où la boîte $D \subset \mathbb{R}^d$ est un ouvert borné.

Théorème 1.2.3. Soient $D \subset \mathbb{R}^d$ un ouvert borné, $0 < m < |D|$ et $\tau \geq 0$. Alors le problème d'optimisation (1.2.4) a une solution (Ω^*, V^*) , où $\Omega^* \subset D$ est un quasi-ouvert et $V^* \in L^\infty(D, \mathbb{R}^d)$ est donné par

$$V^*(x) = -\tau \frac{\nabla u(x)}{|\nabla u(x)|} \text{ si } |\nabla u(x)| \neq 0 ; \quad V^*(x) = 0 \text{ si } |\nabla u(x)| = 0 , \quad (1.2.5)$$

où u est la fonction propre de $L = -\Delta + V^* \cdot \nabla$ dans Ω^* associée à la valeur propre $\lambda_1(\Omega^*, V^*)$.

L'existence d'un minimiseur quand le terme de transport V est fixé, ainsi que quand V varie uniquement parmi les champs de vecteurs qui sont le gradient d'une fonction lipschitzienne, est aussi prouvée.

Définition de la valeur propre principale sur un quasi-ouvert

Rappelons la définition donnée dans [6] (et déjà énoncée en (1.1.3)) de la première valeur propre de L sur un ouvert connexe $\Omega \subset D$,

$$\lambda_1(\Omega, V) = \sup \left\{ \lambda \in \mathbb{R} : \exists \phi \in W_{\text{loc}}^{2,d}(\Omega) \text{ tel que } \phi > 0 \text{ et } -L\phi + \lambda\phi \leq 0 \text{ dans } \Omega \right\}.$$

Si $\Omega \subset D$ possède un nombre fini de composantes connexes, alors la première valeur propre sur Ω correspond à la première valeur propre d'une des composantes connexes de Ω . Ainsi, pour un ouvert $\Omega \subset D$, il est classique de poser $\lambda_1(\Omega, V) = \inf \lambda_1(\Omega_n, V)$, où l'infimum porte sur toutes les composantes connexes de Ω . De plus, en vue d'obtenir une valeur propre décroissante par rapport à l'inclusion, il est naturel de définir pour tout quasi-ouvert $\Omega \subset D$

$$\lambda_1(\Omega, V) := \sup \{ \lambda_1(\tilde{\Omega}, V) : \tilde{\Omega} \text{ ouvert, } \Omega \subset \tilde{\Omega} \subset D \}. \quad (1.2.6)$$

Avec cette définition, nous montrons que la plupart des propriétés de la valeur propre principale sur un ouvert restent vraies pour un quasi-ouvert $\Omega \subset D$:

1. $0 < \lambda_1(\Omega, V) < +\infty$ si $\Omega \neq \emptyset$.
2. $\lambda_1(\Omega, V)$ est une valeur propre de L dans Ω , associée à une fonction propre réelle $u \in H_0^1(\Omega)$.
3. $\lambda_1(\Omega, V) \leq \text{Re}(\lambda)$ pour toute valeur propre $\lambda \in \mathbb{C}$ de L dans Ω .

Précisons les idées principales de la preuve. Pour cela, considérons une suite d'ouverts $(\Omega_n)_{n \geq 1}$ maximisante pour (1.2.6), c'est-à-dire telle que $\Omega \subset \Omega_n \subset D$ et $\lambda_1(\Omega, V) = \lim_{n \rightarrow +\infty} \lambda_1(\Omega_n, V)$. Sans perte de généralité, il est possible de supposer que $(\Omega_n)_{n \geq 1}$ γ -converge vers Ω . Pour 1), nous montrons que si la suite $\lambda_1(\Omega_n, V)$ diverge, alors la suite des premières valeurs propres $\lambda_1(\Omega_n)$ du Laplacien sur Ω_n diverge aussi, de sorte que $\lambda_1(\Omega) = +\infty$ et donc $\Omega = \emptyset$. Pour 2) et 3), le point clé est d'appliquer le théorème 1.1.9 à Ω_n et de montrer que le théorème 1.1.14 s'étend à l'opérateur $L_c = L + c$, de sorte que $R_{\Omega_n}^{L_c}$ converge en norme vers $R_\Omega^{L_c}$. Puisque la suite $(u_n)_{n \geq 1}$ des fonctions propres de L dans Ω_n est uniformément bornée dans $H_0^1(D)$, elle converge, à une sous-suite près, faiblement dans $H_0^1(D)$ et fortement dans $L^2(D)$ vers une fonction $u \in H_0^1(D)$, qui est une fonction propre de L dans Ω associée à $\lambda_1(\Omega, V)$ car

$$u = \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} (\lambda_1(\Omega_n, V) + c) R_{\Omega_n}^{L_c}(u_n) = (\lambda_1(\Omega, V) + c) R_\Omega^{L_c}(u).$$

Le point 3) découle de la continuité du spectre d'opérateurs compacts pour la convergence en norme d'opérateur (puisque $R_{\Omega_n}^{L_c}$ converge vers $R_\Omega^{L_c}$).

Enfin, il est possible de montrer que l'application $\Omega \mapsto \lambda_1(\Omega, V)$ est continue par rapport à la γ -convergence (en posant $\lambda_1(\Omega, V) = +\infty$ si $\Omega = \emptyset$). Puisqu'elle est aussi décroissante par rapport à l'inclusion, l'existence d'un minimiseur au problème avec $V \in L^\infty(D, \mathbb{R}^d)$ fixé

$$\min \{ \lambda_1(\Omega, V) : \Omega \subset D \text{ quasi-ouvert, } |\Omega| \leq m \}$$

est une conséquence immédiate du théorème 1.1.20.

Existence d'une forme optimale pour la valeur propre principale

La preuve du théorème (1.2.4) nécessite de considérer au préalable le problème d'optimisation où la forme Ω est fixée.

Théorème 1.2.4. Soient $\Omega \subset D$ un quasi-ouvert et $\tau > 0$. Alors le problème

$$\min \left\{ \lambda_1(\Omega, V) : V \in L^\infty(\Omega, \mathbb{R}^d), \|V\|_{L^\infty} \leq \tau \right\}.$$

a une solution $V_* \in L^\infty(\Omega, D)$, où V_* est donné par (1.2.5).

L'idée est d'approcher Ω par une suite d'ouverts lisses $(\Omega_n)_{n \geq 1}$ qui γ -converge vers Ω . Précisons qu'une telle suite existe (voir Lemme 2.2.6), mais qu'il n'est alors pas possible de supposer que $\Omega \subset \Omega_n$ (voir Remarque 2.2.7 pour un contre-exemple). Puisque Ω_n est un ouvert lisse, le terme de transport optimal sur Ω_n est connu (voir [52] et (1.2.1)) et est donné par

$$V_n(x) = -\tau \frac{\nabla u_n(x)}{|\nabla u_n(x)|} \quad \text{si } |\nabla u_n(x)| \neq 0; \quad V_n(x) = 0 \quad \text{si } |\nabla u_n(x)| = 0,$$

où u_n désigne la fonction propre de $L = -\Delta + V_n \cdot \nabla$ dans Ω_n associée à $\lambda_1(\Omega, V_n)$. Il s'agit alors de montrer, à l'aide du théorème 1.1.14, que la suite $(u_n)_{n \geq 1}$ converge fortement dans $H_0^1(D)$ vers une fonction $u \in H_0^1(D)$, puis que u est une fonction propre dans Ω associée à $\lambda_1(\Omega, V_*)$. Précisons qu'une difficulté vient du fait que la limite faible dans $L^2(D)$ de $V_n \cdot \nabla u_n = -\tau |\nabla u_n|$, qui existe puisque, à une sous-suite près, u_n converge faiblement dans $H_0^1(D)$, n'est a priori pas égale à $-\tau |\nabla u|$.

Nous sommes maintenant en mesure de prouver le théorème 1.2.3. Pour cela, considérons une suite minimisante $(\Omega_n, V_n)_{n \geq 1}$ pour (1.2.4). À une sous-suite près, $(\Omega_n)_{n \geq 1}$ γ -converge faiblement vers un quasi-ouvert $\Omega \subset D$, et soit $(\tilde{\Omega}_n)_{n \geq 1}$ une suite qui γ -converge vers Ω et telle que $\Omega_n \subset \tilde{\Omega}_n$ (Lemme 1.1.18). En notant \tilde{V}_n le terme de transport optimal sur Ω_n (Théorème 1.2.4), il vient

$$0 < \lambda_1(\tilde{\Omega}_n, \tilde{V}_n) \leq \lambda_1(\tilde{\Omega}_n, V_n) \leq \lambda_1(\Omega_n, V_n) \quad \text{pour tout } n \geq 1.$$

Par suite, $\lambda_1(\tilde{\Omega}_n, \tilde{V}_n)$ converge (quitte à extraire une sous-suite) vers une limite $\tilde{\lambda}$ telle que $\tilde{\lambda} \leq \lim_{n \rightarrow +\infty} \lambda_1(\Omega_n, V_n)$, de sorte que, pour prouver le théorème 1.2.3, il suffit de montrer que $\tilde{\lambda}$ est une valeur propre de L sur Ω , ce qui peut être vérifié grâce au théorème 1.1.14.

La preuve de l'existence d'un minimiseur est plus aisée quand le terme de transport V est le gradient d'une fonction lipschitzienne $\Phi \in W^{1,\infty}(D)$. Dans ce cas, l'opérateur est autoadjoint pour la mesure $e^{-\Phi} dx$ puisqu'il peut s'écrire sous la forme $L = -e^\Phi \operatorname{div}(e^{-\Phi} \nabla \cdot)$, de sorte que la première valeur propre est variationnellement caractérisée par

$$\lambda_1(\Omega, \nabla \Phi) = \min_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_D e^{-\Phi} |\nabla u|^2 dx}{\int_D e^{-\Phi} u^2 dx}. \quad (1.2.7)$$

Théorème 1.2.5. Soient $D \subset \mathbb{R}^d$ un ouvert borné, $0 < m < |D|$ et $\tau \geq 0$. Alors le problème

$$\min \left\{ \lambda_1(\Omega, \nabla \Phi) : \Omega \subset D \text{ quasi-ouvert, } |\Omega| \leq m, \Phi \in W^{1,\infty}(D), \|\nabla \Phi\|_{L^\infty} \leq \tau \right\}. \quad (1.2.8)$$

a une solution.

C'est une conséquence de (1.2.7). En effet, soit $(\Omega_n, \Phi_n)_{n \geq 1}$ une suite minimisante pour (1.2.8) et u_n la fonction propre de $-\Delta + \nabla \Phi_n \cdot \nabla$ sur Ω_n associée à $\lambda_1(\Omega_n, \Phi_n)$. À une sous-suite près, Φ_n converge uniformément dans \overline{D} vers une fonction $\Phi \in W^{1,\infty}(D)$ (en supposant $\Phi_n(x_0) = 0$ pour un $x_0 \in D$), et u_n converge faiblement dans $H_0^1(D)$ et fortement dans $L^2(D)$ vers une fonction $u \in H_0^1(D)$. Ainsi, pour $\Omega := \{u > 0\}$ il vient

$$\lambda_1(\Omega, \Phi) \leq \frac{\int_D e^{-\Phi} |\nabla u|^2 dx}{\int_D e^{-\Phi} u^2 dx} \leq \liminf_{n \rightarrow \infty} \frac{\int_D e^{-\Phi_n} |\nabla u_n|^2 dx}{\int_D e^{-\Phi_n} u_n^2 dx} = \liminf_{n \rightarrow \infty} \lambda_1(\Omega_n, \Phi_n).$$

1.2.3 Résultats de régularité

Nous rappelons brièvement quelques théorèmes de régularité concernant les minimiseurs de la première valeur propre du Laplacien de Dirichlet. Le premier résultat dû à Briançon, Hayouni et Pierre concerne l'existence d'un minimiseur ouvert dans une boîte (voir [10]). Précisons que l'on dira que deux ensembles mesurables $\Omega_1, \Omega_2 \subset \mathbb{R}^d$ sont égaux presque partout si leur différence symétrique est de mesure nulle, c'est à dire si $|\Omega_1 \Delta \Omega_2| = 0$. nulle

Théorème 1.2.6. Soient $D \subset \mathbb{R}^d$ un ouvert connexe et borné, et $0 < m < |D|$. Alors, toute solution Ω^* au problème

$$\min \{ \lambda_1(\Omega) : \Omega \subset D \text{ quasi-ouvert, } |\Omega| \leq m \}, \quad (1.2.9)$$

est égale presque partout à un ouvert. De plus, la première fonction propre sur Ω^* est localement lipschitzienne dans D .

Rappelons que l'existence (parmi les quasi-ouverts) d'un minimiseur pour (1.2.9) provient du théorème de Buttazzo et Dal Maso (Théorème 1.1.22). Bien sûr, si u désigne la première fonction propre sur un ensemble optimal Ω^* pour (1.2.9), alors $\Omega_u := \{u > 0\}$ est aussi une solution de (1.2.9) et est un ouvert puisque u est continue. Ainsi le théorème 1.2.6 fournit l'existence d'un minimiseur au problème

$$\min \{ \lambda_1(\Omega) : \Omega \subset D \text{ ouvert, } |\Omega| \leq m \}. \quad (1.2.10)$$

La preuve de [10] s'inspire des travaux d'Alt, Caffarelli et Friedman dans [2, 3]. En vue de montrer qu'une fonction (positive) u est lipschitzienne, il s'avère que le point clé est d'obtenir une borne uniforme pour la quantité $\frac{1}{r} \int_{\partial B_r(x_0)} u \, d\mathcal{H}^{d-1}$. Par exemple, si u est harmonique dans $B_r(x_0)$, alors une telle borne implique que u est lipschitzienne puisque, en appliquant la formule de la moyenne à $\partial_i u$ (qui est aussi harmonique dans $B_r(x_0)$) et en intégrant par parties, il vient $|\partial_i u(x_0)| = \left| \int_{B_r(x_0)} \partial_i u \, d\mathcal{H}^{d-1} \right| \leq \frac{d}{r} \int_{\partial B_r(x_0)} u \, d\mathcal{H}^{d-1}$. Pour la première fonction propre u sur Ω^* , l'idée est d'utiliser la formule (où Δu est vu comme une mesure)

$$\int_{\partial B_r(x_0)} u \, d\mathcal{H}^{d-1} - u(x_0) = \frac{1}{d\omega_d} \int_0^r s^{1-d} \Delta u(B_s(x_0)) \, ds, \quad (1.2.11)$$

qui s'obtient essentiellement en dérivant $r \mapsto \int_{\partial B_r(x_0)} u \, d\mathcal{H}^{d-1}$, et ensuite de prouver l'estimation $|\Delta u|(B_r(x_0)) \leq Cr^{d-1}$ pour tout $x_0 \in D$ tel que $u(x_0) = 0$. C'est cette estimation qui repose sur le fait que u est solution d'un problème à frontière libre puisque Ω^* est solution de (1.2.9). Précisons que dans [10], les auteurs traitent aussi le cas des minimiseurs de l'énergie de Dirichlet qui ne sont pas de signe constant, et pour lesquels la preuve est par conséquent plus délicate et utilise en particulier par la formule de monotonie d'Alt-Caffarelli-Friedmann (voir [3]). De plus, ils montrent qu'une conséquence de la caractérisation de la première valeur propre est que les minimiseurs Ω^* pour (1.2.9) saturent la contrainte, au sens où $|\Omega^*| = m$.

Remarque 1.2.7 (Sur la connexité de la boîte). Si D n'est pas connexe, alors il est possible que les minimiseurs Ω^* pour (1.2.9) vérifient $|\Omega^*| < m$. Dans ce cas, les minimiseurs ne sont ni ouverts ni réguliers en général puisque tout quasi-ouvert Ω^{**} tel que $\Omega^* \subset \Omega^{**} \subset D$ et $|\Omega^{**}| = m$ est encore solution de (1.2.9). Ce phénomène se produit, par exemple, si $D = D_1 \cup D_2 \subset \mathbb{R}^d$ est l'union de deux boules disjointes de rayons $r_1 < r_2$ et si $|D_1| < m < |D_2|$, puisqu'alors, la première fonction propre u sur un ensemble optimal étant de signe constant sur chaque composante connexe de D , il vient $\Omega_u = D_1$.

La régularité de la frontière libre d'un ensemble optimal pour le problème (1.2.10) a ensuite été étudiée par Briançon et Lamboley dans [11]. Ils obtiennent le résultat suivant.

Théorème 1.2.8. Soient $D \subset \mathbb{R}^d$ un ouvert borné et connexe, et $0 < m < |D|$. Alors toute solution Ω^* au problème (1.2.10) vérifie :

1. Ω^* est de périmètre fini et

$$\mathcal{H}^{d-1}(\partial\Omega^* \setminus \partial^*\Omega^*) = 0, \quad (1.2.12)$$

où $\partial^*\Omega^*$ désigne le bord réduit de Ω^* .

2. Le bord réduit $\partial^*\Omega^*$ est analytique.
3. Si $d = 2$, alors la frontière libre $\partial\Omega^* \cap D$ est analytique.

La principale difficulté concernant ce problème est de traiter la contrainte de mesure et de montrer que la première fonction propre u sur Ω^* minimise localement une certaine fonctionnelle pénalisée (ce qui sera aussi le cas du problème considéré dans la sous-section 1.2.4). Ils montrent ensuite que pour un certain multiplicateur de Lagrange $\Lambda_u > 0$, u est une solution faible d'un problème à frontière libre (en un sens défini dans [2]), ce qui signifie en particulier que u satisfait l'équation

$$\Delta u + \lambda_1(\Omega^*)u = \sqrt{\Lambda_u} \mathcal{H}^{d-1} \llcorner \partial\Omega^*$$

au sens des distributions, où $\mathcal{H}^{d-1} \llcorner \partial\Omega^*$ désigne la restriction de la mesure de Hausdorff $(d-1)$ -dimensionnelle à $\partial\Omega^*$. Ils appliquent ensuite les résultats de régularité obtenus dans [2] pour de telles solutions faibles. Précisons qu'ils montrent aussi des bornes uniformes sur la densité de Ω^* aux points de la frontière libre $\partial\Omega^*$. Ainsi, l'estimation de l'ensemble singulier (1.2.12) est une conséquence du théorème de Federer pour les ensembles de périmètre fini (voir, par exemple, [74, Theorem 16.2]).

Remarque 1.2.9. Si la boîte D possède une boule $B \subset D$ de volume $|B| = m$, alors B est une solution de (1.2.10) par l'inégalité de Faber-Krahn (Théorème 1.2.1).

Récemment, Lamboley et Sicbaldi ont prouvé dans le cadre des variétés Riemanniennes un résultat d'existence et de régularité pour les minimiseurs de la première valeur propre de l'opérateur Laplace-Beltrami (voir [73]). Leur résultat, dont la preuve s'inspire des techniques développées dans [77] et [84], est le suivant.

Théorème 1.2.10. Soient (M, g) une variété Riemannienne lisse (sans bord) de dimension d et $0 < m < \text{Vol}_g(M)$, où Vol_g désigne le volume de M .

1. (Existence) Supposons que M est compacte. Alors le problème

$$\min \{ \lambda_1(\Omega) : \Omega \subset M \text{ quasi-ouvert, } \text{Vol}_g(\Omega) = m \} \quad (1.2.13)$$

a une solution (ici, $\lambda_1(\Omega)$ désigne la première valeur propre de l'opérateur Laplace-Beltrami $-\Delta_g$ dans Ω avec condition de Dirichlet).

2. (Régularité) Supposons que M est connexe (mais pas forcément compacte) et soit Ω^* une solution du problème (1.2.13). Alors Ω^* est un ouvert de périmètre fini. De plus, la frontière $\partial\Omega^* = \Sigma_{\text{reg}} \cup \Sigma_{\text{sing}}$ se décompose en deux ensembles disjoints tels que

(a) Σ_{reg} est un ouvert de $\partial\Omega$ et est une hypersurface lisse de M (C^∞ si M est C^∞ , analytique si M est analytique).

(b) pour une constante universelle $d^* \in \{5, 6, 7\}$, Σ_{sing} est :

- vide si $d < d^*$,
- discret si $d = d^*$,
- de dimension de Hausdorff au plus $(d - d^*)$ si $d > d^*$.

Remarque 1.2.11. Comme dans le cas euclidien, si M n'est pas compacte alors le problème (1.2.13) peut ne pas avoir de solution, et si M n'est pas connexe, alors les minimiseurs pour (1.2.13) peuvent être aussi irréguliers que le sont les quasi-ouverts (voir aussi Section 3.2 et Remark 1.3 dans [73]).

1.2.4 Régularité pour un opérateur avec un terme de transport de la forme $V = \nabla\Phi$

Les résultats présentés dans cette section correspondent à la deuxième partie du chapitre 2 et ont été obtenus dans [84] en collaboration avec Emmanuel Russ et Bozhidar Velichkov.

Nous nous concentrons ici sur la régularité des ensembles optimaux qui minimisent la première valeur propre de l'opérateur $L = -\Delta + V \cdot \nabla$ dans le cas où le terme de transport $V = \nabla\Phi$ est le gradient d'une fonction lipschitzienne $\Phi \in W^{1,\infty}(D)$, et sous des contraintes de volume et d'inclusion. Précisément, notre résultat principal est le suivant.

Théorème 1.2.12. Soient $D \subset \mathbb{R}^d$ un ouvert borné et connexe et $0 < m < |D|$. Soient $V \in L^\infty(D, \mathbb{R}^d)$ un champ de vecteurs de la forme $V = \nabla\Phi$ où $\Phi \in W^{1,\infty}(D)$ et Ω^* une solution du problème

$$\min \left\{ \lambda_1(\Omega, \nabla\Phi) : \Omega \subset D \text{ quasi-ouvert, } |\Omega| \leq m \right\}. \quad (1.2.14)$$

Alors Ω^* est un ouvert de périmètre fini. De plus, la frontière libre $\partial\Omega^* \cap D$ se décompose en l'union disjointe d'une partie régulière $\text{Reg}(\partial\Omega^* \cap D)$ et d'une partie singulière $\text{Sing}(\partial\Omega^* \cap D)$, où :

1. $\text{Reg}(\partial\Omega^* \cap D)$ est localement le graphe d'une fonction de classe $C^{1,\alpha}$ pour tout $\alpha < 1$;
2. pour une constante universelle $d^* \in \{5, 6, 7\}$, $\text{Sing}(\partial\Omega^* \cap D)$ est :
 - vide si $d < d^*$,
 - discret si $d = d^*$,
 - de dimension de Hausdorff au plus $(d - d^*)$ si $d > d^*$.

En outre, si D est de classe $C^{1,1}$, alors la frontière $\partial\Omega^*$ est de classe $C^{1,1/2}$ au voisinage de tout point $x \in \partial\Omega^* \cap \partial D$.

Remarque 1.2.13. La régularité d'une forme optimale pour le problème d'optimisation

$$\min \left\{ \lambda_1(\Omega, \nabla\Phi) : \Omega \subset D \text{ quasi-ouvert, } |\Omega| \leq m, \Phi \in W^{1,\infty}(D), \|\nabla\Phi\|_{L^\infty} \leq \tau \right\} \quad (1.2.15)$$

est une conséquence du Théorème 1.2.12, puisque si le couple $(\Omega^*, \nabla\Phi^*)$ est optimal pour (1.2.15), alors Ω^* est optimal pour (1.2.14) avec $V = \nabla\Phi^*$.

Rappelons que l'opérateur $L = -\Delta + \nabla\Phi \cdot \nabla$ est autoadjoint pour la mesure $e^{-\Phi} dx$ puisque qu'il peut s'écrire sous la forme $L = -e^\Phi \text{div}(e^{-\Phi} \nabla \cdot)$, et que la première valeur propre de L sur un quasi-ouvert $\Omega \subset D$ est par conséquent variationnellement caractérisée par

$$\lambda_1(\Omega, \nabla\Phi) = \min_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_D |\nabla u|^2 e^{-\Phi} dx}{\int_D u^2 e^{-\Phi} dx}. \quad (1.2.16)$$

Précisons que cette formulation variationnelle de la première valeur propre est un outil essentiel dans la preuve du Théorème 1.2.12. Dans le cas où le terme de transport $V \in L^\infty(D, \mathbb{R}^d)$ est quelconque, une telle caractérisation n'existe pas, et la régularité d'un ensemble optimal pour

$$\min \left\{ \lambda_1(\Omega, V) : \Omega \subset D \text{ quasi-ouvert, } |\Omega| \leq m \right\}$$

est encore un problème ouvert.

La première conséquence de la caractérisation (1.2.15) est qu'il est possible de se ramener à l'étude d'un problème à frontière libre. En effet, le problème d'optimisation (1.2.14) est équivalent au problème à frontière libre suivant

$$\min \left\{ \int_D |\nabla v|^2 e^{-\Phi} dx : v \in H_0^1(D), |\{v > 0\}| \leq m, \int_D e^{-\Phi} v^2 dx = 1 \right\}, \quad (1.2.17)$$

au sens où, si u est une fonction propre associée à la première valeur propre $\lambda_1(\Omega^*, \nabla\Phi)$ sur un ensemble Ω^* optimal pour (1.2.14), alors u est solution de (1.2.17) ; réciproquement, si u est une solution de (1.2.17), alors l'ensemble $\{u > 0\}$ est une solution du problème (1.2.14).

Alt et Caffarelli ont été les pionniers de l'étude du problème à frontière libre (à une phase) suivant

$$\min \left\{ \int_{D_0} |\nabla v|^2 dx + \Lambda |\{v > 0\}| : v \in H^1(D_0), v - u_0 \in H_0^1(D_0) \right\}, \quad (1.2.18)$$

où $D_0 \subset \mathbb{R}^d$ est un ouvert connexe et régulier fixé, et $u_0 \in H^1(D_0)$ est une fonction positive fixée (voir [2], ainsi que [91] pour des résultats récents concernant ce problème). Ils prouvent en particulier que si u est un minimiseur pour (1.2.18), alors la partie régulière de $\partial\{u > 0\} \cap D_0$ (ici, le bord réduit) est analytique, et que la partie singulière est de mesure de Hausdorff $(d-1)$ -dimensionnelle nulle. Les arguments pour aborder de tels problèmes à frontière libre sont locaux, et sont souvent fondés sur une comparaison entre une solution u pour le problème variationnel considéré avec une perturbation v de u dans une (petite) boule $B_r(x_0)$, c'est-à-dire avec une fonction $v \in H^1(B_r(x_0))$ telle que $v - u \in H_0^1(B_r(x_0))$. Il est ainsi clair que le problème (1.2.17) est similaire à celui de (1.2.18), les différences principales étant :

1. la présence de la fonction $e^{-\Phi}$ dans la fonctionnelle,
2. la présence de la contrainte $\int_D e^{-\Phi} v^2 dx = 1$,
3. la présence de la contrainte $|\{v > 0\}| \leq m$ (à la place du terme de mesure dans la fonctionnelle).

Toutefois, la stratégie suivie dans la preuve du Théorème 1.2.12 est générale et n'utilise pas les résultats de régularité de [2], comme c'est le cas, par exemple, dans [11], [9] et [1].

Précisons que, si la présence du coefficient $e^{-\Phi}$ et de la contrainte sur la norme L^2 amènent essentiellement à des difficultés techniques, la contrainte de mesure, par contre, est un problème délicat à traiter. L'approche suivie ici pour surmonter cette difficulté est différente de celle suivie dans l'article [1] et est inspirée des travaux de Briançon (voir [9]) et de Briançon et Lamboley (voir [11] et Théorème 1.2.8). Il s'agit, pour une solution u au problème (1.2.17), de prouver l'existence d'un multiplicateur de Lagrange et ensuite de montrer que u est, en un certain sens, un minimiseur local d'une fonctionnelle pénalisée avec un terme de mesure faisant intervenir ce multiplicateur de Lagrange. Cette approche exige que le multiplicateur de Lagrange ne soit pas nul, ce qui n'est pas trivial à montrer. Dans [11], la difficulté a été surmontée en utilisant un résultat tiré de [9], qui repose fortement la minimalité de u . Ici, notre stratégie est fondée sur la formule de monotonie d'Algreem et est plus générale puisqu'elle utilise uniquement l'équation satisfaite par u dans $\{u > 0\}$. Ceci permet en particulier d'obtenir la non-dégénérescence de la fonction u . Nous montrons aussi que u est lipschitzienne en adaptant l'approche suivie dans [10]. On obtient ensuite la régularité du bord d'un ensemble optimal Ω^* pour (1.2.14) en prouvant que u satisfait une condition d'optimalité au sens de la viscosité et en appliquant un résultat de De Silva (voir [36]) pour les points plats sur $\partial\Omega^* \cap D$, et un résultat de Chang-Lara et Savin (voir [26]) pour les points sur $\partial\Omega^* \cap \partial D$. Enfin, l'estimation de la dimension de l'ensemble singulier est une conséquence de la quasi-monotonie de la fonctionnelle de Weiss et des résultats de [92].

Pénalisation de la contrainte de volume

Commençons par remarquer qu'une fonction Sobolev $v \in H_0^1(D)$ n'est pas définie partout et que, bien qu'elle possède un représentant quasi-continu (voir (1.1.1)), il peut encore exister un ensemble de capacité nulle sur lequel elle n'est pas définie. Ainsi, le bord de l'ensemble $\Omega_v = \{v > 0\}$ n'est a priori pas bien défini. Néanmoins, pour une solution u au problème (1.2.17), il est possible de montrer que *tous* les points sont des points de Lebesgue pour u , et donc de choisir un représentant pour u défini *partout*. Pour cela, nous montrons qu'à certains termes négligeables près, un analogue de la formule (1.2.11) existe pour l'opérateur avec un

terme de transport. La fonction $r \mapsto \int_{\partial B_r(x_0)} u d\mathcal{H}^{d-1}$ est alors (presque) croissante et admet par conséquent une limite en 0.

Soit u une solution du problème (1.2.17). Afin de s'affranchir de la contrainte en norme L^2 , nous définissons la fonctionnelle

$$J(v) := \int_D |\nabla v|^2 e^{-\Phi} dx - \lambda_m \int_D v^2 e^{-\Phi} dx, \quad v \in H_0^1(D),$$

où $\lambda_m := \int_D |\nabla u|^2 e^{-\Phi} dx$ est la valeur du minimum dans le problème (1.2.17). Il est alors aisé de vérifier que u est une solution du problème suivant :

$$\min \{ J(v) : v \in H_0^1(D), |\Omega_v| \leq m \}. \quad (1.2.19)$$

Comme mentionné ci-dessus, il est par contre plus difficile de traiter la contrainte de mesure. La première étape consiste à dériver la fonctionnelle J par rapport au domaine. Plus précisément, il s'agit de calculer la dérivée en $t = 0$ de l'application $t \mapsto J(u_t)$, qui sera notée $\delta J(u)[\xi]$, où $u_t(x) = u(x + t\xi(x))$ et $\xi \in C_c^\infty(D, \mathbb{R}^d)$. En utilisant la minimalité de u , il est alors possible d'obtenir l'équation d'Euler-Lagrange suivante

$$\delta J(u)[\xi] = \Lambda_u \int_{\Omega_u} \operatorname{div} \xi dx, \quad \xi \in C_c^\infty(D, \mathbb{R}^d), \quad (1.2.20)$$

pour un certain multiplicateur de Lagrange $\Lambda_u \geq 0$. Précisons qu'un point crucial est maintenant de montrer que $\Lambda_u > 0$, ce que l'on admettra temporairement. Une première conséquence est qu'un calcul précis de la variation $\delta J(u)[\xi]$ amène, au moins de manière formelle, à la condition d'optimalité $|\nabla u| = \sqrt{\Lambda_u} e^\Phi$ sur $\partial\Omega_u \cap D$. Précisons que ce calcul passe par une intégration par parties et n'est donc rigoureux que si Ω_u est de classe C^1 (ce que l'on cherche en fait à prouver...). Une autre conséquence est que la variation de J est inversement proportionnelle à la variation de la mesure, puisque $\frac{d}{dt} \Big|_{t=0} |\Omega_{u_t}| = - \int_{\Omega_u} \operatorname{div} \xi dx$. Heuristiquement il vient (pour $\varepsilon > 0$ petit)

$$J(u) - J(u_t) \ (\simeq \Lambda_u (|\Omega_{u_t}| - |\Omega_u|)) \leq \begin{cases} (\Lambda_u + \varepsilon) (|\Omega_{u_t}| - |\Omega_u|) & \text{si } |\Omega_{u_t}| \geq m, \\ (\Lambda_u - \varepsilon) (|\Omega_{u_t}| - |\Omega_u|) & \text{si } |\Omega_{u_t}| \leq m \end{cases}$$

(précisons que $|\Omega_u| = m$). Ainsi, ceci suggère de définir des coefficients de Lagrange supérieur et inférieur, mais il est pour cela nécessaire de considérer des perturbations locales de u (il est possible de choisir $v \in H_0^1(D)$ tel que $|\Omega_v| > m$ et $J(v) < J(u) = 0$, de sorte que pour tout $t > 0$, $tv \in H_0^1(D)$ et $J(tv) = t^2 J(v) \rightarrow -\infty$!). Pour $x_0 \in D$ et $h > 0$, nous définissons un ensemble des fonctions admissibles par

$$\mathcal{A}(u, x_0, r) := \{ v \in H_0^1(D) : u - v \in H_0^1(B_r(x_0)) \}.$$

Les multiplicateurs de Lagrange supérieur $\mu_+(h, x_0, r) \geq 0$ et inférieur $\mu_-(h, x_0, r) \geq 0$ sont alors définis par

$$\begin{aligned} \mu_+(h, x_0, r) &= \inf \{ \mu \geq 0 : J(u) + \mu |\Omega_u| \leq J(v) + \mu |\Omega_v|, \forall v \in \mathcal{A}(u, x_0, r), m \leq |\Omega_v| \leq m + h \}, \\ \mu_-(h, x_0, r) &= \sup \{ \mu \geq 0 : J(u) + \mu |\Omega_u| \leq J(v) + \mu |\Omega_v|, \forall v \in \mathcal{A}(u, x_0, r), m - h \leq |\Omega_v| \leq m \}, \end{aligned}$$

L'idée est maintenant d'étudier les comportements de $\mu_+(h, x_0, r)$ et $\mu_-(h, x_0, r)$ quand $h \rightarrow 0$. Il est alors possible de montrer qu'ils convergent vers le multiplicateur de Lagrange Λ_u pour tout $x_0 \in \partial\Omega_u$ et tout r_0 suffisamment petit (ne dépendant que de τ, λ_m et d pourvu que $B_{r_0}(x_0) \subset D$), c'est-à-dire

$$\lim_{h \rightarrow 0} \mu_+(h, x_0, r_0) = \lim_{h \rightarrow 0} \mu_-(h, x_0, r_0) = \Lambda_u.$$

La preuve de ce résultat est relativement technique et nécessite d'étudier des problèmes où la fonctionnelle J est pénalisée et où la contrainte de mesure est donnée par $m + h$. Ainsi, un des arguments clés consiste à prouver la continuité des multiplicateurs de Lagrange par rapport à une petite variation de la contrainte de mesure. Plus précisément, une solution u_n au problème (1.2.19) avec maintenant une contrainte de mesure donnée par m_n possède un multiplicateur de Lagrange Λ_{u_n} (donné par (1.2.20)), de sorte que si m_n converge vers m quand $n \rightarrow +\infty$, alors il est possible de montrer que $\lim_{n \rightarrow \infty} \Lambda_{u_n} = \Lambda_u$. Finalement, il ressort de cette analyse qu'une solution u au problème (1.2.19) est localement solution d'un problème où la fonctionnelle J est pénalisée avec un terme de mesure au sens suivant :

- pour tout $\mu > \Lambda_u$, il existe $r_0 > 0$ tel que pour tout $x_0 \in \partial\Omega_u$,

$$J(u) + \mu|\Omega_u| \leq J(v) + \mu|\Omega_v| \quad \text{pour tout } v \in \mathcal{A}(u, x_0, r_0) \quad \text{tel que } |\Omega_v| \geq m. \quad (1.2.21)$$

- pour tout $\mu < \Lambda_u$, il existe $r_0 > 0$ tel que pour tout $x_0 \in \partial\Omega_u$,

$$J(u) + \mu|\Omega_u| \leq J(v) + \mu|\Omega_v| \quad \text{pour tout } v \in \mathcal{A}(u, x_0, r_0) \quad \text{tel que } |\Omega_v| \leq m. \quad (1.2.22)$$

Comme précisé ci-dessus, un point clé de cette stratégie repose sur le fait que le multiplicateur de Lagrange Λ_u n'est pas nul. Intuitivement, une solution u au problème (1.2.19) étant solution de l'équation $-\operatorname{div}(e^{-\Phi}\nabla u) = \lambda_m u e^{-\Phi}$ dans Ω_u , le gradient de u ne peut s'annuler partout sur $\partial\Omega_u \cap D$. Or l'équation d'Euler-Lagrange (1.2.20) permet d'obtenir (formellement) la condition d'optimalité $|\nabla u| = \sqrt{\Lambda_u e^\Phi}$ sur $\partial\Omega_u \cap D$, ce qui doit ainsi impliquer, ou bien que $\Lambda_u > 0$, ou bien que $\partial\Omega_u \cap D = \emptyset$ (ce qui est impossible puisque $|\Omega_u| = m$). En fait, nous montrons que si u vérifie la condition d'extrémalité $\delta J(u)[\xi] = 0$ pour tout $\xi \in C^\infty(D, \mathbb{R}^d)$, alors nous obtenons la contradiction $|D \setminus \Omega_u| = 0$. La preuve peut se résumer comme suit. Premièrement, nous montrons qu'il est possible de se ramener au cas où $\lambda_m = 0$. Pour cela, nous considérons une fonction $\varphi > 0$ solution de $-\operatorname{div}(e^{-\Phi}\nabla\varphi) = \lambda_m \varphi e^{-\Phi}$ dans $B_r(x_0)$ pour un $x_0 = 0 \in D$. En posant $\tilde{a} = \varphi^2 e^{-\Phi}$ et $\tilde{u} = u/\varphi$, il vient $\operatorname{div}(\tilde{a}\nabla\tilde{u}) = 0$ dans $\Omega_{\tilde{u}} \cap B_r$, et en posant de plus $\tilde{J}(v) = \int_{B_r} |\nabla v|^2 \tilde{a}(x) dx$, un calcul montre que $\delta\tilde{J}(\tilde{u})[\xi] = 0$ pour tout $\xi \in C^\infty(B_r, \mathbb{R}^d)$. Par conséquent, quitte à raisonner avec \tilde{u}, \tilde{a} et \tilde{J} , il est possible de supposer que $\lambda_m = 0$. Ensuite, nous montrons que la condition d'extrémalité $\delta J(u)[\xi] = 0$ implique que la fonction $r \mapsto e^{2\tau r} N(r)$ est croissante, où $N(r)$ est défini par (et souvent appelé la fréquence d'Almgren pour $\Phi = 0$)

$$N(r) = \frac{r \int_{B_r} |\nabla u|^2 e^{-\Phi} dx}{\int_{\partial B_r} u^2 e^{-\Phi} d\mathcal{H}^{d-1}}.$$

Cette formule de monotonie d'Almgren permet d'obtenir une condition de doublement qui exprime essentiellement que la fonction u ne peut pas décroître trop vite près de x_0 . D'autre part, l'équation $\operatorname{div}(e^{-\Phi}\nabla u) = 0$ dans Ω_u (nous avons supposé que $\lambda_m = 0$) permet d'obtenir une inégalité de Caccioppoli, à partir de laquelle il est possible de montrer que si la densité de Ω_u en $x_0 \in D$ est très petite, alors u doit décroître très vite. En conséquence, ceci fournit une borne inférieure sur la densité de Ω_u en $x_0 \in D$, et montre en particulier que $|D \setminus \Omega_u| = 0$.

Régularité de la frontière libre

La première étape pour l'étude de la régularité de la frontière libre $\partial\Omega_u$ est de prouver que la fonction u est lipschitzienne, ce qui en particulier assure la convergence des suites de blow-ups (définis plus bas). La preuve repose d'une part, sur l'existence d'une inégalité similaire à (1.2.11) qui fait intervenir la mesure $\operatorname{div}(e^{-\Phi}\nabla u)$ et d'autre part, sur l'estimation $|\operatorname{div}(e^{-\Phi}\nabla u)(B_r(x_0))| \leq Cr^{d-1}$ pour tout $x_0 \in \partial\Omega_u \cap D$, cette dernière estimation étant une conséquence de la condition de minimalité (1.2.21). Précisons qu'en adaptant la preuve du Théorème 2.9 dans [10], il existe une autre preuve du caractère Lipschitzien de u qui est plus directe et qui passe par une pénalisation globale (et moins précise) de la fonctionnelle.

Un autre élément essentiel réside dans la non-dégénérescence de la fonction u près de la frontière $\partial\Omega_u$, qui assure que les blow-ups de u ne sont pas identiquement nuls. Cette condition exprime que u ne décroît pas trop vite près de $\partial\Omega_u$, au sens où pour toute (petite) boule $B_r(x_0)$ centrée en $x_0 \in \partial\Omega_u$, il existe $x \in B_r(x_0)$ tel que $u(x) \geq cr$. L'idée (standard) est que si u prend de très petites valeurs dans $B_r(x_0)$, alors en considérant une fonction cut-off dans $B_{r/2}(x_0)$ et en utilisant la minimalité de u pour (1.2.22), il est possible d'obtenir un compétiteur qui fait décroître la fonctionnelle $J(v) + \mu|\Omega_v|$ (le gain en mesure est supérieur à la perte en gradient).

Nous étudions ensuite les propriétés des blow-ups de u en $x_0 \in \partial\Omega_u$, c'est-à-dire les limites des suites de fonctions u_{x_0, r_n} quand $n \rightarrow +\infty$, où $(r_n)_{n \geq 1}$ est une suite de rayons qui tend vers 0 et $u_{x_0, r}$ est la fonction définie par

$$u_{x_0, r}(x) := \frac{1}{r} u(x_0 + rx), \quad x \in \mathbb{R}^d \text{ et } r > 0.$$

Précisons que l'existence des fonctions limites $u_{x_0} : \mathbb{R}^d \rightarrow \mathbb{R}$ est assurée par le théorème d'Arzelà-Ascoli puisque u est lipschitzienne. Les blow-ups u_{x_0} en $x_0 \in \partial\Omega_u \cap D$ sont alors des solutions du problème (1.2.18) pour toute boule $D_0 \subset \mathbb{R}^d$ où $\Lambda = \Lambda_u e^{\Phi(x_0)}$ et où la condition au bord est donnée par $v - u_{x_0} \in H_0^1(D_0)$. Si $x_0 \in \partial\Omega_u \cap \partial D$, ceci reste vrai à condition d'ajouter la contrainte supplémentaire d'inclusion $\Omega_v \subset \{x_d > 0\}$. Ce résultat pour les points $x_0 \in \partial\Omega_u \cap \partial D$ passe par un changement de variables appliqué à la fonction u qui permet de redresser la boîte D . De plus, les blow-ups u_{x_0} en tout point $x_0 \in \partial\Omega_u$ sont des fonctions (positivement) 1-homogènes. Pour cette dernière propriété, il s'agit de montrer une formule de quasi-monotonie pour une fonctionnelle de type Weiss définie par

$$W(v, \Phi, x_0, r) := \frac{1}{r^d} \int_{B_r(x_0)} |\nabla v|^2 e^{-\Phi} dx - \frac{1}{r^{d+1}} \int_{\partial B_r(x_0)} v^2 e^{-\Phi} d\mathcal{H}^{d-1} + \frac{\Lambda_u}{r^d} |\Omega_v \cap B_r(x_0)|,$$

où Λ_u est le multiplicateur de Lagrange de u . Cette formule de quasi-monotonie permet de montrer que la fonction $r \mapsto W(u_{x_0}, \Phi(0), 0, r)$ est constante, de sorte que l'homogénéité de u_{x_0} se déduit de l'inégalité

$$\frac{d}{dr} W(u_{x_0}, \Phi(0), 0, r) \geq \frac{C}{r^{d+2}} \int_{\partial B_r} |\nabla u_{x_0} \cdot x - u_{x_0}|^2 d\mathcal{H}^{d-1}.$$

Une fois ces propriétés établies (les blow-ups sont non triviaux, solutions de (1.2.18) et 1-homogènes), il est possible de donner un sens rigoureux à la condition d'optimalité mentionnée plus haut. Plus précisément, nous montrons que u vérifie, au sens de la viscosité, les conditions d'optimalité suivantes :

$$|\nabla u| = \sqrt{\Lambda_u e^{\Phi}} \quad \text{sur } \partial\Omega_u \cap D \quad \text{et} \quad |\nabla u| \geq \sqrt{\Lambda_u e^{\Phi}} \quad \text{sur } \partial\Omega_u \cap \partial D.$$

Maintenant, la régularité du bord $\partial\Omega_u \cap D$ pour les points plats, c'est-à-dire les points $x_0 \in \partial\Omega_u \cap D$ pour lesquels il existe un blow-up u_{x_0} tel que $\Omega_{u_{x_0}}$ est un demi-espace, est une conséquence du résultat de régularité concernant les problèmes à frontière libre à une phase de De Silva dans [36]. De plus, la régularité (optimale) du bord $\partial\Omega_u \cap \partial D$ quand la boîte D est régulière provient du résultat de Chang-Lara et Savin dans [26].

Enfin, l'estimation de l'ensemble singulier est une conséquence d'un résultat de régularité de Weiss basé sur un principe de réduction de la dimension. Précisons que d^* est défini comme la plus petite dimension pour laquelle il existe une solution 1-homogène de (1.2.18) qui possède une unique singularité en 0. L'idée de Weiss dans [92] est que, pour une solution 1-homogène de (1.2.18) en dimension d , les blow-ups sont constants dans une direction et encore solutions de (1.2.18). En considérant alors la restriction d'un tel blow-up à un hyperplan orthogonal à cette direction, on obtient alors une autre solution de (1.2.18) en dimension $d - 1$. Il s'agit alors de montrer que la dimension de Hausdorff de l'ensemble singulier de cette dernière solution

décroît, ce qui passe en particulier par la convergence Hausdorff des ensembles de blow-ups, puis d'appliquer ce procédé jusqu'à atteindre une dimension pour laquelle les solutions de (1.2.18) n'ont pas de singularité. Mentionnons pour terminer que la valeur exacte de d^* n'est pas encore connue, mais que l'on a $d^* \in \{5, 6, 7\}$ (voir [62] pour $d^* \geq 5$ et [37] pour $d^* \leq 7$).

1.2.5 Régularité pour un problème d'optimisation de forme avec plusieurs phases

Les résultats présentés dans cette section correspondent au chapitre 3 et ont été obtenus dans [85] en collaboration avec Luca Spolaor et Bozhidar Velichkov.

La motivation première de ce travail est l'étude, en dimension 2, de la régularité des solutions du problème d'optimisation avec plusieurs phases suivant

$$\min \left\{ \sum_{i=1}^n (\lambda_1(\Omega_i) + q_i |\Omega_i|) : \Omega_1, \dots, \Omega_n \text{ ouverts disjoints inclus dans } D \right\}, \quad (1.2.23)$$

où λ_1 désigne la première valeur propre du Laplacien (avec condition de Dirichlet), $D \subset \mathbb{R}^2$ est un ouvert borné de classe $C^{1,\beta}$ pour un certain $\beta \in]0, 1[$, et $q_i > 0$ pour tout $i = 1, \dots, n$. L'existence d'un n -uplet optimal $(\Omega_1, \dots, \Omega_n)$ pour le problème (1.2.23) a été établie dans [7] et nous nous intéressons ici à la régularité des phases Ω_i , $i = 1, \dots, n$.

Commençons par résumer les principaux résultats déjà connus pour ce problème. Dans la suite, $(\Omega_1, \dots, \Omega_n)$ désignera toujours une solution du problème (1.2.23).

1. (Absence de points triples) $\partial\Omega_i \cap \partial\Omega_j \cap \partial\Omega_k = \emptyset$ pour tous $1 \leq i < j < k \leq n$ (voir [19], [89] et [7]).
2. (Absence de points doubles sur le bord de la boîte) $\partial\Omega_i \cap \partial\Omega_j \cap \partial D = \emptyset$ pour tous $1 \leq i < j \leq n$ (voir [7]).
3. (Régularité de la frontière libre à une phase) Pour tout $i = 1, \dots, n$, la frontière libre à une phase $\Gamma_{\text{op}}(\Omega_i) \subset \partial\Omega_i$ de Ω_i est régulière (analytique), où $\Gamma_{\text{op}}(\Omega_i)$ désigne l'ensemble des points $x_0 \in \partial\Omega_i \cap D$ pour lesquels il existe $r > 0$ tel que $B_r(x_0) \cap (\cup_{j \neq i} \Omega_j) = \emptyset$ (voir [11]).

Ainsi, l'étude de la régularité d'une phase Ω_i se ramène à celle de la régularité du bord $\partial\Omega_i$ au voisinage, d'une part, des points de contact avec la boîte $x_0 \in \partial\Omega_i \cap \partial D$ et, d'autre part, des points à deux phases $x_0 \in \partial\Omega_i \cap \Omega_j$, $j \neq i$. Nous prouvons que, pour une solution du problème (1.2.23), toute la frontière de chaque phase est régulière.

Théorème 1.2.14. Soit $(\Omega_1, \dots, \Omega_n)$ une solution du problème (1.2.23). Alors, les phases Ω_i sont des ouverts de classe $C^{1,\alpha}$ pour tout $i = 1, \dots, n$.

Nous montrons que la preuve du Théorème 1.2.14 peut se ramener à l'étude de la régularité des quasi-minimiseurs pour des problèmes à frontière libre à une phase et à deux phases (Théorème 1.2.15) En fait, ce second résultat a en lui même son propre intérêt puisque la définition des quasi-minimiseurs que nous considérons est générale et apparaît naturellement dans l'étude des problèmes d'optimisation de forme qui font intervenir les valeurs propres du Laplacien de Dirichlet.

La régularité de la frontière libre des minimiseurs de la fonctionnelle à une phase a été étudiée pour la première fois par Alt et Caffarelli dans [2]. Ils prouvent, en toute dimension, que la frontière libre d'un minimiseur est régulière à un petit ensemble singulier près et, qu'en dimension 2, toute la frontière libre est régulière (de classe C^∞). Dans [3], Alt, Caffarelli et Friedman étudient le problème à deux phases et prouvent qu'en dimension 2, la frontière libre des minimiseurs est de classe C^1 . La régularité de la frontière libre a aussi été étudiée pour les quasi-minimiseurs de ces fonctionnelles dans [34] et [33]. Les auteurs prouvent que, pour le

problème à une phase, la frontière libre est de classe $C^{1,\alpha}$ à un petit ensemble singulier près qui est vide en dimension 2 (voir aussi [40]) et, pour le problème à deux phases, que la frontière libre est rectifiable.

Nous considérons ici la fonctionnelle à une phase avec des coefficients variables et contrainte d'inclusion ainsi que la fonctionnelle à deux phases avec des coefficients variables, et nous prouvons que, en dimension 2, la frontière libre des quasi-minimiseurs pour ces fonctionnelles est de classe $C^{1,\alpha}$. Précisons que, si notre stratégie permet de traiter facilement des fonctionnelles avec des coefficients variables, il n'est pas possible de se ramener directement d'un opérateur à coefficients variables à un opérateur à coefficients constants. En fait, la preuve passe par un changement de coordonnées en chaque point x_0 . Ainsi, un quasi-minimiseur pour une fonctionnelle à coefficients variables, dans ces nouvelles coordonnées, est alors un quasi-minimiseur pour la fonctionnelle avec $A = \text{Id}$, mais le changement de variables dépend bien sûr du point x_0 . Précisons aussi que notre preuve, pour le problème à une phase, permet d'obtenir un résultat de régularité pour des quasi-minimiseurs u avec une contrainte d'inclusion sans supposer que la fonction u vérifie une équation dans Ω_u .

Après la publication de ce papier, deux résultats récents ont été obtenus dans cette direction. Dans [32], les auteurs prouvent que les quasi-minimiseurs pour les fonctionnelles à une phase et deux phases avec des coefficients variables sont Lipschitziens. D'autre part, la question de la régularité pour le problème à deux phases a été résolue en toute dimension dans [35], où les auteurs prouvent que la frontière libre des minimiseurs de la fonctionnelle à deux phases (pour le Laplacien) est de classe $C^{1,\alpha}$ à un ensemble singulier de dimension au plus $d - 5$. En conséquence, ils obtiennent avec une preuve différente la généralisation du Théorème 1.2.14 à toute dimension.

Précisons quelques notations avant d'énoncer notre résultat. Soit $A = (a_{ij})_{ij} : B_2 \rightarrow \text{Sym}_2^+$ une fonction uniformément elliptique avec des coefficients δ_A -Höldériens, et soient $Q_{\text{op}}, Q_{\text{TP}}^+$ et Q_{TP}^- des fonctions δ_Q -Höldériennes sur B_2 et minorées par une constante strictement positive (rappelons que pour $r > 0$, $B_r = B(0, r)$ désigne la boule centrée en 0 et de rayon r). Pour $u \in H^1(B_2)$, $x_0 \in B_1$ et $r \in (0, 1)$, nous définissons la fonctionnelle à une phase par

$$J_{\text{op}}(u, x_0, r) = \int_{B_r(x_0)} \left(\sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + Q_{\text{op}}(x) \mathbb{1}_{\{u>0\}} \right) dx,$$

et la fonctionnelle à deux phases par

$$J_{\text{TP}}(u, x_0, r) = \int_{B_r(x_0)} \left(\sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + Q_{\text{TP}}^+(x) \mathbb{1}_{\{u>0\}} + Q_{\text{TP}}^-(x) \mathbb{1}_{\{u<0\}} \right) dx.$$

L'ensemble des fonctions admissibles dans le demi-disque supérieur $B_r^+ := B_r \cap \{x_2 > 0\}$ est défini par

$$\mathcal{A}^+(B_r) = \{u \in H^1(B_r) : u \geq 0 \text{ dans } B_r, u = 0 \text{ sur } B_r \setminus B_r^+\}.$$

Nous dirons qu'une fonction $u \in H^1(B_2)$ est

- un quasi-minimiseur pour J_{op} dans $\mathcal{A}^+(B_2)$ si $u \in \mathcal{A}^+(B_2)$ et s'il existe des constantes $r_1 > 0$, $C_1 > 0$ et $\delta_1 > 0$ telles que, pour tout $x_0 \in B_1 \cap \partial\Omega_u$ et $r \in (0, r_1)$,

$$J_{\text{op}}(u, x_0, r) \leq (1 + C_1 r^{\delta_1}) J_{\text{op}}(v, x_0, r) + C_1 r^{2+\delta_1}$$

pour tout $v \in \mathcal{A}^+(B_2)$ tel que $u = v$ dans $B_2 \setminus B_r(x_0)$.

- un quasi-minimiseur pour J_{TP} dans B_2 s'il existe des constantes $r_2 > 0$, $C_2 > 0$ et $\delta_2 > 0$ telles que, pour tout $x_0 \in B_1 \cap \partial\Omega_u$ et $r \in (0, r_2)$,

$$J_{\text{TP}}(u, x_0, r) \leq (1 + C_2 r^{\delta_2}) J_{\text{TP}}(v, x_0, r) + C_2 r^{2+\delta_2}$$

pour tout $v \in H^1(B_2)$ tel que $u = v$ dans $B_2 \setminus B_r(x_0)$.

Enfin, pour $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ nous posons $\Omega_u = \{u \neq 0\}$, $\Omega_u^+ = \{u > 0\}$ et $\Omega_u^- = \{u < 0\}$.

Théorème 1.2.15. Soit $u \in H^1(B_2)$ une fonction lipschitzienne.

1. Si u est une fonction positive et est un quasi-minimiseur pour J_{op} dans $\mathcal{A}^+(B_2)$, alors la frontière libre $\partial\Omega_u \cap B_1$ est localement le graphe d'une fonction de classe $C^{1,\alpha}$.
2. Si u est un quasi-minimiseur pour J_{tp} dans B_2 tel que les fonctions u_{\pm} sont solutions de l'équation

$$-\operatorname{div}(A\nabla u_{\pm}) = f_{\pm} \quad \text{dans} \quad \Omega_u^{\pm} \cap B_2, \quad (1.2.24)$$

où

- (a) $f_{\pm} : \overline{\Omega}_u^{\pm} \rightarrow \mathbb{R}$ sont des fonctions bornées et continues,
- (b) $A : B_2 \rightarrow \operatorname{Sym}_2^+$ est uniformément elliptique avec des coefficients de classe C^1 ,

alors les frontières libres $\partial\Omega_u^+ \cap B_1$ et $\partial\Omega_u^- \cap B_1$ sont localement le graphe d'une fonction de classe $C^{1,\alpha}$.

La preuve du Théorème 1.2.15 repose sur une formule de (quasi)-monotonie pour la fonctionnelle de Weiss et sur une inégalité épipérimétrique (voir [86]). Précisons que la seule obstruction à une généralisation des Théorèmes 1.2.15 et 1.2.14 à toute dimension provient du fait que l'inégalité épipérimétrique n'est connue qu'en dimension 2.

Régularité pour le problème d'optimisation (1.2.23)

Rappelons que pour prouver le Théorème 1.2.14, il suffit de montrer que le bord d'une phase Ω_i est de classe $C^{1,\alpha}$ au voisinage des points de $\partial\Omega_i \cap \partial D$ et des points de $\partial\Omega_i \cap \partial\Omega_j$, $j \neq i$. Notons u_i la première fonction propre (normalisée pour la norme L^2) du Laplacien de Dirichlet dans Ω_i , qui est une fonction lipschitzienne d'après [14] (voir aussi [7]).

Pour $x_0 \in \partial\Omega_i \cap D$ et un (petit) rayon $r > 0$ tel que $B_r(x_0) \cap \Omega_j = \emptyset$ pour tout $j \neq i$, la caractérisation de la première valeur propre (1.1.6) implique que u_i est solution de

$$\min \left\{ J(v) := \int_D |\nabla v|^2 dx + q_i |\Omega_v| : v \in H_0^1(D), v = u \text{ dans } D \setminus B_r(x_0), \int_D v^2 dx = 1 \right\}.$$

Il apparaît que ce problème de minimisation possède une contrainte en norme L^2 et une contrainte d'inclusion imposée par la boîte D . Il est alors possible, en utilisant que u_i est lipschitzienne, de contrôler la norme L^2 des fonctions test v et ainsi de montrer que u_i est en un certain sens un quasi-minimiseur pour J dans $B_r(x_0)$. L'idée est ensuite de changer les coordonnées afin de redresser le bord de la boîte D près de x_0 . Puisque D est régulier, le bord ∂D est localement donné par le graphe d'une fonction $g : (-\delta, \delta) \rightarrow \mathbb{R}$ de classe $C^{1,\beta}$ de sorte que la fonction $\tilde{u}(x_1, x_2) := u_i(x_1, x_2 + g(x_1))$ est telle que $\Omega_{\tilde{u}} \subset \{x_2 > 0\}$. De plus, un calcul montre que \tilde{u} est un quasi-minimiseur pour la fonctionnelle J_{op} pour un choix convenable de $A_{(x_1, x_2)}$ dépendant uniquement de $g'(x_1)$ et pour $Q_{\text{op}} = q_i$. Par conséquent, la régularité de $\partial\Omega_i$ près de x_0 est une conséquence du Théorème 1.2.15.

Pour $x_0 \in \partial\Omega_i \cap \partial\Omega_j$ et $r > 0$ suffisamment petit, $u := u_i - u_j$ est solution du problème à deux phases

$$\min \left\{ \int_D |\nabla v|^2 dx + q_i |\Omega_v^-| + q_j |\Omega_v^+| : v \in H_0^1(D), v = u \text{ dans } D \setminus B_r(x_0), \int_D v_+^2 = \int_D v_-^2 = 1 \right\}.$$

Comme précédemment, en contrôlant les normes L^2 de v_+, v_- , il est possible de montrer que u est un quasi-minimiseur pour J_{tp} avec $A = \operatorname{Id}$, $Q_{\text{tp}}^+ = q_i$ et $Q_{\text{tp}}^- = q_j$, de sorte que la régularité de $\partial\Omega_i$ près de x_0 est encore une conséquence du Théorème 1.2.15.

Régularité de la frontière libre pour les quasi-minimiseurs

Il s'agit maintenant de prouver le Théorème 1.2.15. Commençons par considérer le cas d'un quasi-minimiseur u pour J_{op} . En vue d'étudier les blow-ups u_{x_0} en $x_0 \in \partial\Omega_u \cap B_1$, c'est-à-dire les limites des suites de fonctions $u_{x_0,r}$ définies par $u_{x_0,r}(x) = \frac{1}{r}u(x_0 + rx)$, nous allons nous servir d'une formule de monotonie pour une fonctionnelle de type Weiss ainsi que d'une égalité épipérimétrique. Ainsi, la première étape consiste à effectuer un changement de coordonnées pour se ramener au cas où l'opérateur de la fonctionnelle J_{op} est $A = \text{Id}$. Pour cela, nous définissons la fonction $\bar{u} := u \circ F_{x_0}$, où $F_{x_0}(x) := x_0 + A_{x_0}^{1/2}(x)$, $x \in \mathbb{R}^2$. Ce changement de variables est motivé par le fait que pour $x \in B_r$, $|\nabla \bar{u}(x)|^2 = A_{x_0} \nabla u(y) \cdot \nabla u(y)$, où $y = F_{x_0}(x)$. Puisque les coefficients de A et Q_{op} sont Höldériens, il est alors possible de montrer que \bar{u} est, en un certain sens, un quasi-minimiseur pour la fonctionnelle

$$J_{\text{op}}^{x_0}(v, r) := \int_{B_r} \left(|\nabla v|^2 + Q_{\text{op}}(x_0) \mathbb{1}_{\{v>0\}} \right) dx. \quad (1.2.25)$$

Précisons que ce changement de coordonnées, ainsi que la fonction \bar{u} , dépendent du point $x_0 \in \partial\Omega_u \cap B_1$. Nous considérons ensuite une fonctionnelle W_{op} de type Weiss pour le problème à une phase définie par

$$W_{\text{op}}(v) := \int_{B_1} |\nabla v|^2 dx - \int_{\partial B_1} v^2 d\mathcal{H}^{d-1} + Q_{\text{op}}(x_0) |\Omega_v \cap B_1|,$$

et dont la dérivée pour la fonction $\bar{u}_r(x) = \frac{1}{r}\bar{u}(rx)$ vaut

$$\frac{\partial}{\partial r} W_{\text{op}}(\bar{u}_r) = \frac{2}{r} (W_{\text{op}}(\bar{z}_r) - W_{\text{op}}(\bar{u}_r)) + \frac{1}{r} \int_{\partial B_1} |\nabla \bar{u}_r \cdot x - \bar{u}_r|^2 d\mathcal{H}^{d-1}, \quad (1.2.26)$$

où \bar{z}_r désigne l'extension 1-homogène de la trace de \bar{u}_r dans B_1 . En conséquence de la quasi-minimalité de \bar{u}_r pour $J_{\text{op}}^{x_0}$, il est possible de minorer de la différence $W_{\text{op}}(\bar{z}_r) - W_{\text{op}}(\bar{u}_r)$ et ainsi d'obtenir une formule de (quasi) monotonie pour la fonctionnelle W_{op} . Plus précisément, il vient que la fonction $r \mapsto W_{\text{op}}(\bar{u}_r) + Cr^\delta$ est croissante pour un certain $\delta > 0$ (dépendant de δ_A, δ_Q et δ_1). En fait, il est possible de prouver une formule de (quasi) monotonie plus forte que celle-ci qui permettra d'obtenir une estimation de la vitesse de convergence des suites de blow-ups de la fonction \bar{u} . L'idée est d'obtenir une minoration plus précise de la différence $W_{\text{op}}(\bar{z}_r) - W_{\text{op}}(\bar{u}_r)$ en fonction de $W_{\text{op}}(\bar{u}_r)$ à l'aide de l'inégalité épipérimétrique établie dans [86]. Cette inégalité, qui constitue un point clé de la preuve, énonce qu'il existe un compétiteur dont l'énergie est strictement inférieure à celle de l'extension 1-homogène dans B_1 d'une fonction positive (dont la moyenne sur ∂B_1 est strictement positive). Précisément, l'inégalité épipérimétrique pour le problème à une phase appliquée à la fonction \bar{u}_r est

$$W_{\text{op}}(h_r) - \Theta_{\text{op}}(x_0) \leq (1 - \varepsilon) (W_{\text{op}}(\bar{z}_r) - \Theta_{\text{op}}(x_0)),$$

pour un certain $\varepsilon > 0$. Le terme d'ajustement $\Theta_{\text{op}}(x_0) := Q_{\text{op}}(x_0) \frac{\pi}{2}$ correspond en fait à la limite de $W_{\text{op}}(\bar{u}_r)$ quand $r \rightarrow 0$. Ainsi, en combinant l'inégalité épipérimétrique avec la quasi-minimalité de \bar{u}_r , il est possible de montrer que l'application $r \mapsto r^{-\gamma} [W_{\text{op}}(\bar{u}_r) - \Theta_{\text{op}}(x_0) + Cr^\delta]$ est croissante pour un certain $\gamma > 0$ dépendant de ε . Cette formule de monotonie permet d'obtenir une estimation plus précise du membre de droite dans (1.2.26). Par suite, en utilisant essentiellement que $\frac{d}{dr} \bar{u}_r = \frac{1}{r} (\nabla \bar{u}_r \cdot x - \bar{u}_r)$, nous obtenons une estimation de la vitesse de convergence de la suite \bar{u}_r vers un blow-up \bar{u}_0 , et donc aussi de $u_{x_0,r} = \bar{u}_r \circ A_{x_0}^{-1/2}$ vers $u_{x_0} = \bar{u}_0 \circ A_{x_0}^{-1/2}$. Plus précisément, nous avons

$$\|u_{x_0,r} - u_{x_0}\|_{L^\infty(B_1)} \leq Cr^{\gamma/4} \quad \text{pour tout } r \in (0, r_0). \quad (1.2.27)$$

Cette estimation est très utile pour l'étude de la régularité de la frontière libre puisqu'elle implique par exemple l'unicité des blow-ups, ce qui n'est en général pas connu (comme par exemple dans [84] et [88]).

D'un autre côté, la quasi-minimalité de \bar{u} pour $J_{\text{op}}^{x_0}$ permet de montrer que, pour tout $x_0 \in \partial\Omega_u \cap B_1$, les blow-ups $\bar{u}_0 := u_{x_0} \circ A_{x_0}^{1/2}$ sont des minimiseurs de la fonctionnelle (1.2.25) pour tout $r > 0$ (avec une contrainte d'inclusion), et sont des fonctions 1-homogènes, ce dernier point étant aussi une conséquence de la formule de (quasi) monotonie de Weiss. Nous obtenons ainsi la classification suivante des blow-ups en tout point $x_0 \in \partial\Omega_u \cap B_1$

$$u_{x_0}(x) = \mu(x_0) \max\{0, x \cdot \nu_{x_0}\} \quad \text{où} \quad \begin{cases} \mu(x_0) = Q_{\text{op}}(x_0) & \text{si } x_0 \in \partial\Omega_u \cap B_1 \cap \{x_2 > 0\}, \\ \mu(x_0) \geq Q_{\text{op}}(x_0) & \text{si } x_0 \in \partial\Omega_u \cap B_1 \cap \{x_2 = 0\}, \end{cases}$$

et où $\nu_{x_0} \in \mathbb{R}^2$ est de la forme $\nu_{x_0} = A_{x_0}^{-1/2}[\tilde{\nu}_{x_0}]$ pour un certain vecteur unitaire $\tilde{\nu}_{x_0} \in \partial B_1$. Comme première conséquence de la forme des blow-ups et de l'estimation de la vitesse de convergence uniforme des blow-ups (1.2.27) il suit que u vérifie la condition d'optimalité

$$|A_{x_0}^{1/2} \nabla u(x_0)| = \mu(x_0) \quad \text{sur} \quad \partial\Omega_u \cap B_1.$$

Il est aussi possible de déduire de la forme des blow-ups, de (1.2.27) et de la non-dégénérescence de u que la frontière libre $\partial\Omega_u \cap B_1$ vérifie une certaine condition de platitude, ainsi que les estimations suivantes concernant les oscillations de ν et μ :

$$|\nu_{x_0} - \nu_{y_0}| \leq C|x_0 - y_0|^\alpha \quad \text{et} \quad |\mu(x_0) - \mu(y_0)| \leq C|x_0 - y_0|^\alpha, \quad (1.2.28)$$

pour tous $x_0, y_0 \in \partial\Omega_u \cap B_{r_0}$ et pour un certain $\alpha > 0$. La régularité de la frontière libre découle maintenant du fait que la condition de platitude implique que $\partial\Omega_u$ est dans un voisinage de 0 le graphe d'une fonction différentiable, qui est nécessairement de classe $C^{1,\alpha}$ puisque la normale ν est α -Höldérienne.

Il reste à prouver la régularité de la frontière libre des quasi-minimiseurs pour la fonctionnelle à deux phases J_{TP} . Pour un tel quasi-minimiseur u , la frontière libre $\partial\Omega_u \cap B_1$ se décompose en $\partial\Omega_u \cap B_1 = \Gamma_+ \cup \Gamma_- \cup \Gamma_{\text{TP}}$, où $\Gamma_+ := (\partial\Omega_u^+ \setminus \partial\Omega_u^-) \cap B_1$ et $\Gamma_- := (\partial\Omega_u^- \setminus \partial\Omega_u^+) \cap B_1$ correspondent respectivement aux points à une phase de Ω_u^+ et Ω_u^- , et où $\Gamma_{\text{TP}} := \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1$ correspond aux points à deux phases de Ω_u . L'étude précédente des points à une phase implique donc que les bords Γ_+ et Γ_- sont de classe $C^{1,\alpha}$, de sorte qu'il ne reste plus qu'à prouver la régularité des phases Ω_u^+ et Ω_u^- au voisinage des points à deux phases de Γ_{TP} . Comme précédemment, en effectuant le changement de coordonnées $\bar{u} = u \circ F_{x_0}$ près d'un point $x_0 \in \partial\Omega_u \cap B_1$ il vient que \bar{u} est un quasi-minimiseur pour la fonctionnelle à deux phases

$$J_{\text{TP}}^{x_0}(v, r) := \int_{B_r} \left(|\nabla v|^2 + Q_{\text{TP}}^+(x_0) \mathbb{1}_{\{v>0\}} + Q_{\text{TP}}^-(x_0) \mathbb{1}_{\{v<0\}} \right) dx,$$

de sorte que les blow-ups $\bar{u}_0 = u_{x_0} \circ A_{x_0}^{1/2}$ sont des minimiseurs de $J_{\text{TP}}^{x_0}$ (pour tout $r > 0$), et sont 1-homogènes, grâce à une formule de (quasi) monotonie pour la fonctionnelle de Weiss à deux phases W_{TP} . Avec un argument de dérivée de formes, il en résulte que les blow-ups de u en $x_0 \in \partial\Omega_u \cap B_1$ sont donnés par

$$u_{x_0}(x) = \begin{cases} \mu_\pm(x_0) \max\{0, x \cdot \nu_{x_0}\} & \text{si } x_0 \in \Gamma_\pm, \\ \mu_+(x_0) \max\{0, x \cdot \nu_{x_0}\} + \mu_-(x_0) \min\{0, x \cdot \nu_{x_0}\} & \text{si } x_0 \in \Gamma_{\text{TP}}, \end{cases}$$

où $\nu_{x_0} = A_{x_0}^{-1/2}[\tilde{\nu}_{x_0}]$ pour un $\tilde{\nu}_{x_0} \in \partial B_1$, et où $\mu_\pm^2(x_0) = Q_{\text{TP}}^\pm(x_0)$ si $x_0 \in \Gamma_\pm$, et $\mu_\pm^2(x_0) \geq Q_{\text{TP}}^\pm(x_0)$ et $\mu_+^2(x_0) - \mu_-^2(x_0) = Q_{\text{TP}}^+(x_0) - Q_{\text{TP}}^-(x_0)$ si $x_0 \in \Gamma_{\text{TP}}$. Avec l'inégalité épipérimétrique, comme dans le cas de la fonctionnelle à une phase, nous obtenons l'estimation de la vitesse de convergence

(1.2.27) pour tous les points à deux phases $x_0 \in \Gamma_{\text{TP}}$. Précisons que cette estimation est aussi vérifiée pour les points à une phase $x_0 \in \Gamma_+$ (ou $x_0 \in \Gamma_-$), mais que le rayon r_0 n'est alors plus uniforme et dépend de la distance $\text{dist}(x_0, \Gamma_- \cup \Gamma_{\text{TP}})$. Ainsi, la fonction u_+ est différentiable (jusqu'au bord $\partial\Omega_u^+$) en tout point $x_0 \in \partial\Omega_u^+ \cap B_1$ et $\nabla u_+(x_0) = \mu_+(x_0) \nu_{x_0}$, ce qui montre en particulier que u vérifie la condition d'optimalité

$$|A_{x_0}^{1/2} \nabla u_{\pm}(x_0)| = \mu_{\pm}(x_0) \quad \text{sur} \quad \partial\Omega_u^{\pm} \cap B_1. \quad (1.2.29)$$

Par suite, Γ_{TP} satisfait à une condition de platitude et les estimations (1.2.28) sont vérifiées pour tout $x_0, y_0 \in \Gamma_{\text{TP}}$, de sorte que le sous ensemble fermé $\Gamma_{\text{TP}} \subset \partial\Omega_u$ est ainsi localement inclus dans une courbe de classe $C^{1,\alpha}$. Malheureusement, ceci n'est pas suffisant pour conclure que les phases Ω_u^+ et Ω_u^- sont régulières. Soulignons le fait que jusqu'à présent nous n'avons pas eu besoin de nous servir des équations (1.2.24) vérifiées par u_{\pm} dans Ω_u^{\pm} . Cette hypothèse sert une première fois pour montrer que les fonctions $\mu_{\pm} : \partial\Omega_u^{\pm} \rightarrow \mathbb{R}$ sont Höldériennes. En fait, la preuve fait intervenir une suite $u_n := u_{x_n, r_n}^+$ convergeant vers une fonction u_{∞} , et nécessite de passer à la limite dans la condition vérifiée par u_n sur le bord $\partial\{u_n > 0\}$, ce qui est justifié par l'hypothèse (1.2.24) et la régularité de classe C^1 de A . Enfin, puisque la fonction u_+ est solution de l'équation (1.2.24) et satisfait à la condition d'optimalité (1.2.29) avec μ Höldérien, le Théorème 1.2.15 est une conséquence du résultat de régularité pour les problèmes à frontière libre à une phase de De Silva (voir [36]).

1.3 Les valeurs propres d'ordre supérieur

Dans cette section nous introduisons succinctement quelques résultats d'existence et de régularité pour des problèmes de minimisation qui concernent les valeurs propres du Laplacien de Dirichlet. Nous présentons ensuite un résultat de régularité pour un problème avec contrainte d'inclusion où la fonctionnelle considérée fait intervenir la somme des premières valeurs propres d'un opérateur sous forme divergence.

1.3.1 Résultats d'existence et de régularité

Nous considérons ici des problèmes d'optimisation pour les valeurs propres du Laplacien, un exemple typique étant

$$\min \left\{ \lambda_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ quasi-ouvert, } |\Omega| = m \right\}. \quad (1.3.1)$$

Pour $k = 1$, l'inégalité de Faber-Krahn énonce que les ensembles optimaux pour (1.3.1) sont des boules (Théorème 1.2.1). Concernant la deuxième valeur propre du Laplacien, le résultat d'existence suivant est attribué à Szegö (voir [81]).

Théorème 1.3.1 (Krahn-Szegö). Soit $m > 0$. Le problème d'optimisation

$$\min \left\{ \lambda_2(\Omega) : \Omega \subset \mathbb{R}^d \text{ ouvert, } |\Omega| = m \right\} \quad (1.3.2)$$

est atteint par l'union de deux boules disjointes de même volume.

C'est une conséquence de l'inégalité de Faber-Krahn. Pour un ouvert connexe $\Omega \subset \mathbb{R}^d$, notons $\Omega_+ = \{u > 0\}$ et $\Omega_- = \{u < 0\}$, où u désigne la deuxième fonction propre sur Ω (si $\Omega = \Omega_1 \cup \Omega_2$ n'est pas connexe, la preuve est identique en raisonnant alors avec Ω_1, Ω_2 au lieu de Ω_+, Ω_-). En notant Ω_+^* et Ω_-^* deux boules disjointes de même volume que Ω_+^* et Ω_-^* respectivement, l'ensemble $\Omega^* = \Omega_+^* \cup \Omega_-^*$ vérifie

$$\lambda_2(\Omega^*) \leq \max\{\lambda_1(\Omega_+^*), \lambda_1(\Omega_-^*)\} \leq \max\{\lambda_1(\Omega_+), \lambda_1(\Omega_-)\} \leq \lambda_2(\Omega).$$

Ainsi, un minimiseur pour (1.3.2) est une union de deux boules disjointes, ces dernières étant nécessairement de même volume car sinon, en rétrécissant la plus grande et en agrandissant la plus petite (sans changer le volume total), on obtiendrait un meilleur compétiteur pour λ_2 .

Remarque 1.3.2. Il est naturel de considérer (1.3.2) avec une contrainte supplémentaire de connexité, mais ce problème n'admet pas de solution (voir [56, Section 4.1.2]). Plus intéressant est le problème (1.3.2) avec une contrainte de convexité, qui admet un minimiseur mais dont nous ne connaissons pas la forme. Mentionnons qu'en dimension 2, Henrot et Oudet ont en particulier montré que le stade, enveloppe convexe de deux disques tangents, n'est pas un minimiseur parmi les ouverts convexes du plan (voir [59], [60] et [72, Section 3.3]).

Rappelons que dans le cas où les formes admissibles sont incluses dans une boîte bornée $D \subset \mathbb{R}^d$, l'existence d'un quasi-ouvert optimal pour le problème (1.3.1) est une conséquence du théorème de Buttazzo et Dal Maso (Théorème 1.1.20). Sans contrainte de boîte (i.e. $D = \mathbb{R}^d$), l'existence d'un minimiseur pour la k -ième valeur propre du Laplacien n'a été démontrée qu'assez récemment, le passage d'un problème de minimisation dans une boîte bornée D à celui dans \mathbb{R}^d étant loin d'être trivial.

Pour $k = 3$, Bucur et Henrot prouvent dans [58] l'existence d'un minimiseur au problème (1.3.1) grâce à un principe de concentration-compacité introduit par P.L. Lions. Appliqué à une suite minimisante $(\Omega_n)_{n \geq 1}$ pour (1.3.1) (et $k \geq 3$), ce principe énonce que nécessairement l'un des deux cas suivants se produit : ou bien il existe $(x_n)_{n \geq 1} \subset \mathbb{R}^d$ telle que la suite $(x_n + \Omega_n)_{n \geq 1}$ est γ -compacte et l'existence d'un minimiseur est assuré, ou bien la suite $(\Omega_n)_{n \geq 1}$ peut être choisie de telle sorte que $\Omega_n = \Omega_n^1 \cup \Omega_n^2$ est disconnexe avec $d(\Omega_n^1, \Omega_n^2) \rightarrow +\infty$ et $\liminf_{n \rightarrow +\infty} |\Omega_n^i| \geq \varepsilon > 0$, $i = 1, 2$. Dans cette seconde situation, le problème se ramène à trouver les minimiseurs pour $\lambda_1, \lambda_2, \dots, \lambda_{k-1}$. Ainsi pour $k = 3$, quitte à permuter les indices $i = 1, 2$ il vient $\lambda_3(\Omega_n) = \lambda_1(\Omega_n^1)$ ou $\lambda_3(\Omega_n) = \lambda_2(\Omega_n^1)$ (le cas $\lambda_3(\Omega_n) = \lambda_3(\Omega_n^1)$ étant en fait impossible pour n assez grand). Or si par exemple $\lambda_3(\Omega_n) = \lambda_2(\Omega_n^1)$, alors $\lambda_1(\Omega_n^2) \leq \lambda_2(\Omega_n^1)$ et en notant Ω^1 un minimiseur pour λ_2 de volume $|\Omega_n^1|$ et Ω^2 un minimiseur pour λ_1 de volume $|\Omega_n^2|$, il suit que $\lambda_3(\Omega^1 \cup \Omega^2) \leq \lambda_3(\Omega_n)$ (car $\lambda_1(\Omega^1), \lambda_2(\Omega^1), \lambda_1(\Omega^2)$ sont trois valeurs propres sur $\Omega^1 \cup \Omega^2$ majorées par $\lambda_2(\Omega_n^1) = \lambda_3(\Omega_n)$). En conséquence, par un passage à la limite quand $n \rightarrow +\infty$ et un argument de γ -compacité, il suit qu'une union de trois boules disjointes minimise λ_3 . Précisons que cet argument ne fonctionne qu'en choisissant Ω^1 et Ω^2 disjoints, ce qui est possible ici car les minimiseurs pour λ_1 et λ_2 sont connus et, en particulier, bornés. Par conséquent, Bucur et Henrot obtiennent avec cette stratégie l'existence d'un minimiseur pour λ_3 ainsi que le résultat suivant pour $k \geq 4$: s'il existe un minimiseur borné pour $\lambda_3, \dots, \lambda_{k-1}$, alors il existe un minimiseur pour λ_k (possiblement non borné).

Plus tard, l'existence au problème (1.3.1) pour tout $k \geq 4$ est obtenue par Bucur dans [12]. En introduisant une notion de sous solution pour l'énergie de torsion, il montre qu'un minimiseur pour le problème (1.3.1), s'il existe, est nécessairement borné, ce qui combiné avec le résultat précédent prouve l'existence d'un minimiseur. De plus, il prouve aussi que les minimiseurs pour (1.3.1) sont de périmètre fini.

Au même moment, ce résultat est aussi obtenu par Mazzoleni et Pratelli avec une technique différente pour une classe plus générale de fonctionnelles monotones en les valeurs propres. Plus précisément, leur résultat est le suivant (voir [76]).

Théorème 1.3.3. Soit $\mathcal{F} : \mathbb{R}^k \rightarrow \mathbb{R}$ une fonctionnelle semi-continue inférieurement et croissante en chaque variable et soit $m > 0$. Alors il existe un minimiseur borné au problème

$$\min \left\{ \mathcal{F}(\lambda_1(\Omega), \lambda_2(\Omega), \dots, \lambda_k(\Omega)) : \Omega \subset \mathbb{R}^d \text{ quasi-ouvert, } |\Omega| = m \right\}. \quad (1.3.3)$$

De plus, ce minimiseur est contenu dans un cube de côté R , où R ne dépend que de k et d (et non de la fonctionnelle \mathcal{F}).

Leur stratégie est de montrer que, pour le problème (1.3.3), il est suffisant de ne considérer que les quasi-ouverts uniformément bornés, et ensuite d'appliquer le Théorème 1.1.22. Dans cette direction, ils prouvent que partant d'un ouvert $\Omega \subset \mathbb{R}^d$, il existe un autre ouvert uniformément borné, de même volume que Ω , et dont les k premières valeurs propres sont toutes plus petites que celles sur Ω . L'idée est que si Ω a un grand diamètre, alors il doit exister des parties fines de Ω , ce qui implique que les valeurs propres doivent être grandes au vu de leur caractérisation en terme de quotient de Rayleigh.

Remarque 1.3.4. 1) Si dans le théorème 1.3.3 la fonctionnelle \mathcal{F} est de plus strictement croissante en au moins une variable, alors les minimiseurs pour (1.3.3) sont bornés et de périmètre fini (voir [17]).

2) Pour $\Lambda > 0$, le problème

$$\min \left\{ \lambda_k(\Omega) + \Lambda|\Omega| : \Omega \subset \mathbb{R}^d \text{ quasi-ouvert} \right\} \quad (1.3.4)$$

a une solution, et les ensembles optimaux sont bornés et de périmètre fini. En fait, il suffit de remarquer que les problèmes (1.3.1) et (1.3.4) sont équivalents, au sens où ils possèdent les mêmes minimiseurs pour de bonnes valeurs de m et Λ : toute solution Ω^* de (1.3.4) est clairement une solution de (1.3.1) pour $m = |\Omega^*|$; et si Ω^* est maintenant une solution de (1.3.1), alors, en utilisant le simple fait que les valeurs propres du Laplacien sont -2 -homogènes, il suit que Ω^* est une solution de (1.3.4) pour $\Lambda = 2\lambda_k(\Omega^*)/(md)$ (voir [57, Proposition 3.33]).

3) Les boules, ou unions de boules, ne sont pas toujours optimales pour le problème (1.3.1). En dimension 2, Wolf et Keller ont prouvé que la treizième valeur propre d'un carré est strictement plus petite que celle de n'importe quelle union de disques de même aire (voir [93]). De plus, des résultats numériques montrent qu'une union de boules n'est pas optimale en dimension 2 pour $5 \leq k \leq 15$ (voir [80] et [4]).

Une autre stratégie pour pallier une éventuelle perte de masse à l'infini et ainsi obtenir un résultat d'existence consiste à considérer une mesure avec poids qui pénalise les compétiteurs trop éloignés de l'origine. Cette idée est développée dans [8] pour l'opérateur avec terme de transport $-\Delta - x \cdot \nabla$ et nous énonçons leur résultat dans le théorème qui suit.

Théorème 1.3.5. Pour tout $m > 0$, le problème d'optimisation

$$\min \left\{ \lambda_k(\Omega, -x) : \Omega \subset \mathbb{R}^d \text{ quasi-ouvert, } \int_{\Omega} e^{|x|^2/2} dx \leq m \right\}$$

a une solution.

Pour ce problème, les auteurs de [8] considèrent l'espace $H_0^1(\Omega, m_d)$, où m_d est la mesure de Lebesgue avec poids définie par $m_d = \prod_{i=1}^d e^{x_i^2/2} dx_i$, et montrent en particulier que l'injection $H_0^1(D, m_d) \hookrightarrow L^2(D, m_d)$ est compacte. L'existence d'un minimiseur repose alors sur une formule min-max pour les valeurs propres (adaptée avec la mesure m_d), qui découle du fait que l'opérateur $-\Delta - x \cdot \nabla$ est autoadjoint puisque le terme de transport $V = -x$ est le gradient d'une fonction.

Dans [18], les auteurs prouvent l'existence d'un ouvert optimal pour des fonctionnelles $\mathcal{F}(\lambda_1(\Omega), \dots, \lambda_k(\Omega))$ strictement croissantes en les valeurs propres $\lambda_k(\Omega)$ du Laplacien de Dirichlet, un exemple typique étant $\mathcal{F}(\Omega) = \lambda_1(\Omega) + \dots + \lambda_k(\Omega)$. Plus précisément, ils obtiennent le résultat suivant (voir [18, Theorem 6.1]).

Théorème 1.3.6. Soit $\mathcal{F} : \mathbb{R}^k \rightarrow \mathbb{R}$ une fonctionnelle strictement croissante et bi-lipschitzienne en chaque variable. Alors, toute solution Ω^* au problème d'optimisation

$$\min \left\{ \mathcal{F}(\lambda_1(\Omega), \dots, \lambda_k(\Omega)) : \Omega \subset \mathbb{R}^d \text{ quasi-ouvert, } |\Omega| = m \right\}, \quad (1.3.5)$$

est égale presque partout à un ouvert. De plus, les fonctions propres sur Ω^* , associées aux valeurs propres $\lambda_1(\Omega^*), \dots, \lambda_k(\Omega^*)$, sont lipschitziennes dans \mathbb{R}^d .

En vue d'obtenir un résultat de régularité, la caractérisation (1.1.5) de λ_k à travers une formule min-max est bien plus délicate à traiter que la caractérisation (1.1.6) de λ_1 , qui s'exprime comme un minimum sur $H_0^1(\Omega)$. De plus, la valeur propre $\lambda_k(\Omega^*)$ n'est a priori pas simple, de sorte que la fonctionnelle $\Omega \mapsto \lambda_k(\Omega)$ n'est alors pas différentiable en Ω^* par rapport à une variation du domaine (voir par exemple la section 2.5 dans [56] pour plus de précisions sur ce sujet). En fait, les auteurs dans [18] montrent que si $\lambda_{k-l}(\Omega^*) < \lambda_{k-l+1}(\Omega^*) = \lambda_k(\Omega^*)$ pour un certain $l \geq 1$, alors une perturbation $u_k + v$, $v \in H_0^1(B_r(x))$, de la fonction propre u_k dans (1.1.5) ne donne pas une information sur u_k seulement, mais une information qui implique toutes les fonctions propres u_{k-l+1}, \dots, u_k . Toutefois, si $\lambda_{k-1}(\Omega^*) < \lambda_k(\Omega^*)$, alors il découle de cette stratégie que u_k est un quasi-minimiseur pour l'énergie de Dirichlet, de sorte qu'en utilisant un résultat de Briançon, Hayouni et Pierre développé dans [10] pour la première valeur propre, ils obtiennent que u_k est lipschitzienne. L'idée est alors d'approcher Ω^* par des solutions Ω_ε d'un problème convenable en vue de se ramener au cas où $\lambda_{k-1}(\Omega_\varepsilon) < \lambda_k(\Omega_\varepsilon)$, puis de montrer que les fonctions propres u_k^ε sur Ω_ε , qui sont alors lipschitziennes, convergent vers u_k , prouvant ainsi que u_k est lipschitzienne. En fait, la limite de la suite Ω_ε étant égale presque partout à Ω^* , les auteurs considèrent le problème (1.3.5) avec les valeurs propres $\tilde{\lambda}_k(\Omega)$ définies avec les espaces $\tilde{H}_0^1(\Omega)$ (voir les remarques 1.1.11 et 1.1.6). Ils montrent donc que les fonctions propres u_1, \dots, u_k sur un ensemble optimal Ω^* pour $\mathcal{F}(\tilde{\lambda}_1(\Omega), \dots, \tilde{\lambda}_k(\Omega))$ sont lipschitziennes, et par conséquent que l'ensemble $\Omega^{**} := \cup_{i=1}^k \{u_i \neq 0\}$ est un ouvert. Au vu de la définition de Ω^{**} , il apparaît alors que $\tilde{\lambda}_i(\Omega^*) = \tilde{\lambda}_i(\Omega^{**}) = \lambda_i(\Omega^{**})$, pour tous $i = 1, \dots, k$, de sorte que Ω^* est égal presque partout à l'ouvert Ω^{**} qui est une solution aux deux problèmes (1.3.5) définis avec ou bien les valeurs propres $\tilde{\lambda}_i$, ou bien les valeurs propres λ_i .

Le cas d'une fonctionnelle ne faisant intervenir que certaines valeurs propres est plus délicat à traiter. Néanmoins, les auteurs prouvent aussi dans [18] que les ensembles optimaux possèdent au moins une fonction propre qui est lipschitzienne (voir [18, Theorem 6.1]).

Théorème 1.3.7. Soit $\mathcal{F} : \mathbb{R}^k \rightarrow \mathbb{R}$ une fonctionnelle strictement croissante en chaque variable et bi-lipschitzienne, et soient $0 < k_1 < \dots < k_p$ des entiers. Soit Ω^* une solution du problème

$$\min \left\{ \mathcal{F}(\tilde{\lambda}_{k_1}(\Omega), \dots, \tilde{\lambda}_{k_p}(\Omega)) : \Omega \subset \mathbb{R}^d \text{ quasi-ouvert, } |\Omega| \leq m \right\}.$$

Alors il existe une famille orthonormale de fonctions propres u_{k_1}, \dots, u_{k_p} associées aux valeurs propres $\tilde{\lambda}_{k_1}(\Omega^*), \dots, \tilde{\lambda}_{k_p}(\Omega^*)$ qui sont lipschitziennes dans \mathbb{R}^d . De plus, si $\tilde{\lambda}_{k_i}(\Omega^*) > \tilde{\lambda}_{k_{i-1}}(\Omega^*)$ pour tout $i \in \{1, \dots, p\}$, alors Ω^* est égal presque partout à un ouvert et toutes les fonctions propres associées à $\tilde{\lambda}_{k_i}(\Omega^*)$, $i = 1, \dots, p$, sont lipschitziennes dans \mathbb{R}^d .

Toutefois, des simulations numériques tendent à montrer que la relation $\lambda_k(\Omega^*) = \lambda_{k-1}(\Omega^*)$ est vérifiée pour un ensemble optimal Ω^* au problème (1.3.1) (voir [80] et [56, Section 2.5.3]). Mentionnons aussi que l'existence d'un ouvert Ω^* solution du problème (1.3.1), ainsi que le caractère Lipschitzien de l'ensemble des fonctions propres associées à $\lambda_k(\Omega^*)$, sont encore des problèmes ouverts.

Concernant la régularité de la frontière libre des ensembles optimaux, le résultat suivant dû à Mazzoleni, Terracini et Velichkov (voir [77]) a été d'une grande source d'inspiration pour le travail que nous présentons dans la sous-section suivante (Théorème 1.3.11) qui traite aussi de la somme des premières valeurs propres mais pour un opérateur plus général sous forme divergence.

Théorème 1.3.8. Soit Ω^* une solution du problème

$$\min \left\{ \lambda_1(\Omega) + \dots + \lambda_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ ouvert, } |\Omega| = 1 \right\}.$$

Alors Ω^* est connexe et la frontière libre $\partial\Omega^*$ se décompose en l'union disjointe d'une partie régulière $\text{Reg}(\partial\Omega^*)$ et d'une partie singulière $\text{Sing}(\partial\Omega^*)$, où :

1. $\text{Reg}(\partial\Omega^*)$ est localement le graphe d'une fonction de classe C^∞ .

2. pour une constante universelle $d^* \in \{5, 6, 7\}$, $\text{Sing}(\partial\Omega^*)$ est :

- vide si $d < d^*$;
- discret si $d = d^*$;
- de dimension de Hausdorff au plus $(d - d^*)$ si $d > d^*$.

La preuve passe par un principe de Harnack pour les fonctions propres à la frontière d'un ensemble optimal et permet de se ramener d'un problème à frontière libre vectoriel à un problème scalaire pour lequel la régularité est déjà connue.

Entre temps, Kriventsov et Lin obtiennent de manière indépendante et avec une preuve différente un résultat similaire pour des fonctionnelles en les valeurs propres plus générales que la somme (voir [70]).

Théorème 1.3.9. Soit $\mathcal{F} : \mathbb{R}^k \rightarrow \mathbb{R}$ une fonctionnelle de classe C^1 telle que $\partial_i \mathcal{F} \geq c > 0$, $i = 1, \dots, k$. Soit Ω^* une solution du problème

$$\min \left\{ \mathcal{F}(\lambda_1(\Omega), \dots, \lambda_k(\Omega)) + |\Omega| : \Omega \subset \mathbb{R}^d \text{ ouvert} \right\}. \quad (1.3.6)$$

Alors le bord réduit $\partial^*\Omega^*$ de Ω^* est un ouvert de $\partial\Omega^*$ qui est localement le graphe d'une fonction analytique. De plus, l'ensemble singulier $\partial\Omega^* \setminus \partial^*\Omega^*$ est de dimension de Hausdorff au plus $d - 3$.

Puisque la fonctionnelle $\mathcal{F}(\Omega) := \mathcal{F}(\lambda_1(\Omega), \dots, \lambda_k(\Omega)) + |\Omega|$ n'est pas différentiable par rapport au domaine, les auteurs dans [70] ont recours à des arguments d'approximation basés sur une idée tirée de [82]. Plus précisément, pour une solution Ω^* au problème (1.3.6), ils introduisent des fonctionnelles \mathcal{F}_p telles que les minimiseurs Ω_p^* de \mathcal{F}_p ont des valeurs propres $\lambda_i(\Omega_p^*)$ simples pour tout $i = 1, \dots, k$, et ils étudient ensuite la convergence de Ω_p^* vers Ω^* . Les fonctionnelles \mathcal{F}_p étant différentiables par rapport au domaine, ils montrent qu'il existe une équation d'Euler-Lagrange sur $\partial\Omega_p^*$ qui, par passage à la limite, leur permet d'obtenir une équation d'Euler-Lagrange sur $\partial\Omega^*$ alors même que \mathcal{F} n'est a priori pas différentiable. Plus précisément, ils obtiennent la condition $\sum_{i=1}^k \xi_i \partial_\nu(u_i^2) = 1$ sur $\partial^*\Omega^*$, pour une certaine famille orthonormale $(u_i)_{i=1}^k$ de fonctions propres sur Ω^* et pour certaines constantes strictement positives $(\xi_i)_{i=1}^k$ (ν est la dérivée normale extérieure). Le lien entre les constantes ξ_i et les dérivées partielles de \mathcal{F} n'est toutefois pas connu, sauf si la fonctionnelle \mathcal{F} est différentiable auquel cas il suit $\xi_i = \partial_i \mathcal{F}(\Omega^*)$. Précisons que dans la situation favorable où les valeurs propres $\lambda_1(\Omega^*), \dots, \lambda_k(\Omega^*)$ sont simples, ils prouvent aussi l'estimation plus précise de l'ensemble singulier obtenue dans le Théorème 1.3.8.

Peu après, les mêmes auteurs considèrent dans [71] des fonctionnelles qui ne font intervenir que certaines valeurs propres. Par souci de simplicité, nous énonçons ici leur résultat pour $\lambda_k(\Omega)$ qui correspond au théorème 1.1 dans [71], mais leur résultat s'applique en fait à des fonctionnelles plus générales (voir [71, Theorem 1.2]).

Théorème 1.3.10. Soit Ω^* une solution du problème

$$\min \left\{ \lambda_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ quasi-ouvert}, |\Omega| = 1 \right\}.$$

Alors la frontière libre $\partial\Omega^* = \partial^*\Omega^* \cup Z_{AC} \cup Z_C$ se décompose en l'union disjointe de trois ensembles qui possèdent les propriétés suivantes :

1. Le bord réduit $\partial^*\Omega^*$ est un ouvert de $\partial\Omega^*$ et est localement le graphe d'une fonction analytique.
2. L'ensemble Z_{AC} correspond aux points sur $\partial\Omega^* \setminus \partial^*\Omega^*$ en lesquels la densité de Ω^* est strictement inférieur à 1. C'est un ouvert de $\partial\Omega^* \setminus \partial^*\Omega^*$ qui est de dimension de Hausdorff au plus $d - 3$.
3. L'ensemble Z_C correspond aux points sur $\partial\Omega^*$ en lesquels la densité de Ω^* est 1.

4. Il existe une famille orthonormale $(u_i)_i$ de fonctions propres associées à $\lambda_k(\Omega^*)$ sur un sous ensemble ouvert de Ω^* qui sont lipschitziennes. De plus, il existe des constantes $\xi_i > 0$ telles que $\sum_i \xi_i = 1$ et

$$\sum_i \xi_i \partial_\nu(u_i^2) = 1 \quad \text{sur} \quad \partial^*\Omega^*.$$

La principale différence avec [70] est l'apparition d'un ensemble Z_C formé de points "cusp", qui est principalement due à l'absence d'une propriété de non-dégénérescence pour les fonctions propres u_i . Ainsi, le théorème 1.3.10 énonce que les points de la frontière libre sont réguliers, mais ne fournit pas d'estimation de l'ensemble singulier, l'ensemble des points cusp Z_C étant encore mal compris. Précisons que dans [70] et [77], ainsi que dans [88] (voir le théorème 1.3.11), la non-dégénérescence des fonctions propres repose de manière essentielle sur le fait que la première fonction propre est strictement positive dans Ω^* .

1.3.2 La somme des premières valeurs propres d'un opérateur sous forme divergence

Nous présentons ici les résultats des chapitres 4 et 5 qui correspondent à [87] et [88].

Dans cette section nous étudions la régularité des ensembles optimaux pour la somme des premières valeurs propres, comptées avec multiplicités, d'un opérateur sous forme divergence avec une pénalisation en volume et sous une contrainte d'inclusion. Plus précisément, nous fixons un ouvert borné $D \subset \mathbb{R}^d$, la boîte, et nous considérons pour un quasi-ouvert $\Omega \subset D$ les valeurs propres $\lambda_i(\Omega)$ de l'opérateur $-b(x)^{-1} \operatorname{div}(A\nabla \cdot)$ sur Ω avec condition de Dirichlet. En d'autres termes, pour chaque valeur propre $\lambda_i(\Omega)$ il existe une fonction propre $u_i \in H_0^1(\Omega)$ telle que

$$\begin{cases} -\operatorname{div}(A\nabla u_i) = \lambda_i(\Omega) b u_i & \text{dans } \Omega \\ u_i = 0 & \text{sur } \partial\Omega. \end{cases}$$

Précisons que cet opérateur est plus général que celui avec un terme de transport considéré dans le Théorème 1.2.12 puisque ce dernier correspond au cas particulier où $A = e^{-\Phi} \operatorname{Id}$ et $b = e^{-\Phi}$. Nous avons le résultat de régularité suivant.

Théorème 1.3.11. Soient $D \subset \mathbb{R}^d$ un ouvert borné, $A : D \rightarrow \operatorname{Sym}_d^+$ une fonction à valeurs dans les matrices symétriques qui est uniformément elliptique et dont les coefficients sont Höldériens, $b \in L^\infty(D)$ une fonction bornée telle que $c_b^{-1} \leq b \leq c_b$ pour une constante $c_b > 0$ et $\Lambda > 0$. Alors le problème

$$\min \{ \lambda_1(\Omega) + \cdots + \lambda_k(\Omega) + \Lambda |\Omega| : \Omega \subset D \text{ quasi-ouvert} \} \quad (1.3.7)$$

admet une solution. De plus, les solutions Ω^* sont des ouverts de périmètre fini. Par ailleurs, si $b \in W^{1,\infty}(D)$ est une fonction lipschitzienne, alors la frontière libre $\partial\Omega^* \cap D = \operatorname{Reg}(\partial\Omega^* \cap D) \cup \operatorname{Sing}(\partial\Omega^* \cap D)$ se décompose en deux ensembles disjoints tels que :

1. $\operatorname{Reg}(\partial\Omega^* \cap D)$ est localement le graphe d'une fonction de classe $C^{1,\alpha}$ pour un $\alpha \in (0, 1)$. Si, en outre, $a_{i,j} \in C^{k,\delta}(D)$ et $b \in C^{k-1,\delta}(D)$ pour certains $\delta \in (0, 1)$ et $k \geq 1$, alors $\operatorname{Reg}(\partial\Omega^* \cap D)$ est localement le graphe d'une fonction de classe $C^{k+1,\alpha}$.
2. pour une constante universelle $d^* \in \{5, 6, 7\}$, $\operatorname{Sing}(\partial\Omega^* \cap D)$ est :
 - vide si $d < d^*$;
 - discret si $d = d^*$;
 - de dimension de Hausdorff au plus $(d - d^*)$ si $d > d^*$.

Nous montrons dans un premier temps que les fonctions propres sur un ensemble optimal pour (1.3.7) sont localement lipschitziennes dans D , ce qui constitue une première étape

importante pour l'étude de la régularité de la frontière libre des ensembles optimaux. Concer-
nant la première valeur propre du Laplacien, la régularité lipschitzienne de la première fonction
propre sur un ensemble optimal a été obtenue par Briançon Hayouni et Pierre dans [10] (voir
aussi Théorème 1.2.6). Dans [18] (Théorème 1.3.6), les auteurs considèrent des fonctionnelles
 $\mathcal{F}(\lambda_1(\Omega), \dots, \lambda_k(\Omega))$ bi-lipschitziennes par rapport aux valeurs propres $\lambda_i(\Omega)$ du Laplacien, un
exemple modèle étant la somme $\lambda_1(\Omega) + \dots + \lambda_k(\Omega)$. En particulier, ils prouvent le caractère
Lipschitzien des fonctions propres sur un ensemble minimisant la somme des k premières valeurs
propres du Laplacien parmi les quasi-ouverts $\Omega \subset \mathbb{R}^d$ de mesure fixée. Ces résultats s'inspire des
travaux d'Alt, Caffarelli et Friedman pour la régularité de problèmes à frontière libre ([2, 3]). Plus
précisément, une fonction $u \in H^1(\mathbb{R}^d)$ est lipschitzienne si elle harmonique dans $\Omega_u = \{u \neq 0\}$
et qu'elle vérifie une condition de quasi-minimalité pour l'énergie de Dirichlet $\int_{\mathbb{R}^d} |\nabla v|^2 dx$. La
preuve, qui repose de manière cruciale sur la combinaison de la formule (1.2.11) et de l'estimation
 $|\Delta|u|| (B_r(x)) \leq Cr^{d-1}$, ne permet par contre pas de traiter le cas de l'opérateur $-b^{-1} \operatorname{div}(A\nabla \cdot)$
puisque, par exemple, la formule (1.2.11) ne possède pas de généralisation pour cet opérateur
(comme c'était néanmoins le cas pour l'opérateur avec un terme de transport considéré dans le
Théorème 1.2.12).

En s'inspirant des résultats de [77] (Théorème 1.3.8), il est possible de montrer que le vecteur
 $U = (u_1, \dots, u_k)$ des k premières fonctions propres sur un ensemble optimal pour (1.3.7) est un
quasi-minimiseur pour la fonctionnelle

$$H_0^1(D, \mathbb{R}^k) \ni V = (v_1, \dots, v_k) \mapsto \int_D A\nabla V \cdot \nabla V dx + \Lambda|\{|V| > 0\}|,$$

où on utilise la convention $A\nabla V \cdot \nabla V = A\nabla v_1 \cdot \nabla v_1 + \dots + A\nabla v_k \cdot \nabla v_k$. L'idée pour montrer que
de tels quasi-minimiseurs U sont Lipschitziens est, comme dans le Théorème 1.2.15, de geler les
coefficients afin que la fonction U , dans de nouvelles coordonnées près de x , devienne un quasi-
minimiseur pour une fonctionnelle à coefficients constants. Bien sûr, ce changement de variables
dépend du point x . Ensuite, nous adaptons la stratégie de David et Toro développée dans [34]
où les auteurs prouvent que les quasi-minimiseurs pour les fonctionnelles à une et deux phases
à coefficients constants sont des fonctions lipschitziennes (voir aussi [33] pour des résultats de
régularité de la frontière libre). Nous obtenons le résultat suivant.

Théorème 1.3.12. Soient $D \subset \mathbb{R}^d$ un ouvert borné et $\Omega \subset D$ un quasi-ouvert. Soit $A : D \rightarrow \operatorname{Sym}_d^+$ uniformément elliptique et à coefficients δ_A -Höldériens et soit $f = (f_1, \dots, f_k) \in L^\infty(D, \mathbb{R}^k)$. Supposons que $U = (u_1, \dots, u_k) \in H_0^1(\Omega, \mathbb{R}^k)$ est une fonction qui vérifie

- U est solution de l'équation

$$-\operatorname{div}(A\nabla U) = f \quad \text{dans } \Omega, \quad (1.3.8)$$

- U satisfait la condition de quasi-minimalité suivante : pour tout $C_1 > 0$, il existe des constantes $\varepsilon \in (0, 1)$ et $C > 0$ telles que

$$\int_D A\nabla U \cdot \nabla U dx + \Lambda|\{|U| > 0\}| \leq (1 + C\|U - \tilde{U}\|_{L^1}) \int_D A\nabla \tilde{U} \cdot \nabla \tilde{U} dx + \Lambda|\{|\tilde{U}| > 0\}|,$$

pour tout $\tilde{U} \in H_0^1(D, \mathbb{R}^k)$ tel que $\|U - \tilde{U}\|_{L^1} \leq \varepsilon$ et $\|\tilde{U}\|_{L^\infty} \leq C_1$.

Alors U est une fonction localement lipschitzienne dans D .

Précisons que presque au même moment, David, Engelstein, Smit Vega Garcia et Toro ont
prouvé dans [32] la régularité lipschitzienne des quasi-minimiseurs pour une fonctionnelle à deux
phases avec des coefficients variables sans supposer que les quasi-minimiseurs satisfassent une
équation. La question d'écarter l'hypothèse (1.3.8), qui n'est utilisée dans la preuve du Théorème
1.3.12 que pour montrer que U est bornée et pour appliquer la formule de monotonie de Mate-
vosyan et Petrosyan dans [75], n'est ici pas abordée puisque nous sommes surtout intéressés par

la régularité des fonctions propres sur un ensemble optimal au problème (1.3.8) pour lesquelles l'équation (1.3.8) est satisfaite.

La régularité de la frontière libre d'un ensemble minimisant la somme des premières valeurs propres du Laplacien de Dirichlet a été démontrée dans [77] (Théorème 1.3.8) qui constitue la principale source d'inspiration pour la preuve du Théorème 1.3.11. Précisons que le problème (1.3.7) n'est pas invariant par translation à cause de la contrainte imposée par la boîte ainsi que la présence des coefficients variables, ce qui implique en particulier que, contrairement à [77], les ensembles optimaux Ω^* pour (1.3.7) ne sont a priori pas connexes. Néanmoins, il est possible de montrer que les composantes connexes de Ω^* ne se touchent pas dans D . Puisqu'elles sont aussi solution d'un problème similaire à (1.3.7), il suffit en fait de prouver la régularité d'une composante connexe de Ω^* , notée Ω_1^* , où la première fonction propre u_1 est strictement positive. La preuve consiste ensuite à montrer que Ω_1^* est non-tangentiellement accessible (NTA) près des points réguliers de Ω_1^* , puis à prouver que les fonctions propres $U = (u_1, \dots, u_k)$ vérifient un principe de Harnack aux points réguliers de la frontière $\partial\Omega_1^*$. Ceci permet alors de montrer que u_1 satisfait une condition d'optimalité sur la partie régulière de Ω_1^* et ainsi d'appliquer un résultat de De Silva pour la régularité des problèmes à frontière libre à une phase (voir [36]).

Les fonctions propres sont lipschitziennes

Commençons par préciser que l'opérateur $-b^{-1} \operatorname{div}(A\nabla \cdot)$ étant autoadjoint dans $L^2(\Omega, bdx)$, ses valeurs propres sur un quasi-ouvert $\Omega \subset D$ forment une suite de réels strictement positifs et sont caractérisées par

$$\lambda_k(\Omega) = \min_{\substack{V \text{ sous-espace de} \\ \text{dimension } k \text{ de } H_0^1(\Omega)}} \max_{v \in V \setminus \{0\}} \frac{\int_{\Omega} A\nabla v \cdot \nabla v \, dx}{\int_{\Omega} v^2 b \, dx}.$$

De plus, les fonctions propres $(u_k)_{k \geq 1}$ sur Ω forment une famille orthonormale de $L^2(\Omega; bdx)$, c'est-à-dire $\int_{\Omega} u_i u_j b dx = \delta_{ij}$. De cette formulation variationnelle des valeurs propres il découle l'existence d'un quasi-ouvert Ω^* optimal pour (1.3.7) et, par ailleurs, que le vecteur $U = (u_1, \dots, u_k)$ des k premières fonctions propres sur Ω^* est une solution du problème suivant

$$\min \left\{ \int_D A\nabla V \cdot \nabla V + \Lambda \{ |V| > 0 \} : V = (v_1, \dots, v_k) \in H_0^1(D, \mathbb{R}^k), \int_D v_i v_j b \, dx = \delta_{ij} \right\}. \quad (1.3.9)$$

Il est alors possible de s'affranchir de la contrainte d'orthogonalité en utilisant une idée tirée de [77, Lemma 2.5] qui est basée sur la procédure d'orthonormalisation de Gram-Schmidt. Plus précisément, si $\tilde{U} \in H_0^1(D, \mathbb{R}^k)$ est une petite perturbation de U (en norme L^1), alors le vecteur V obtenu en orthonormalisant \tilde{U} a une énergie proche de celle de \tilde{U} , de sorte qu'en utilisant (1.3.9) il vient que U vérifie la condition de quasi-minimalité du Théorème 1.3.12. Le caractère Lipschitzien des fonctions propres sur Ω^* , et par conséquent l'existence d'un ensemble optimal ouvert pour (1.3.7), sont donc une conséquence du Théorème 1.3.12.

Il s'agit maintenant de montrer qu'une fonction coordonnée $u := u_i$, pour $i = 1, \dots, k$, de $U = (u_1, \dots, u_k)$ donné par le Théorème 1.3.12 est localement lipschitzienne dans D . Pour cela, on effectue comme au Théorème 1.2.15 le changement de variables en $x \in D$ donné par la fonction $F_x(\xi) := x + A_x^{1/2}[\xi]$, $\xi \in \mathbb{R}^d$. On note $u_x := u \circ F_x$ afin de faire apparaître la dépendance en le point x (ici, u_x n'est donc pas une fonction blow-up de u ...). La fonction u_x est alors un quasi-minimiseur pour l'énergie de Dirichlet, au sens où pour tout rayon $r \leq r_0$,

$$\int_{B_r} |\nabla u_x|^2 \leq (1 + Cr^{\delta_A}) \int_{B_r} |\nabla \tilde{u}|^2 + Cr^d, \quad (1.3.10)$$

pour tout $\tilde{u} \in H^1(\mathbb{R}^d)$ tel que $u_x - \tilde{u} \in H_0^1(B_r)$ et $\|\tilde{u}\|_{L^\infty} \leq \|u_x\|_{L^\infty}$. Cette condition de quasi-minimalité permet en particulier de montrer que la moyenne $\omega(u_x, r) := \left(\int_{B_r} |\nabla u_x|^2 \right)^{1/2}$ ne

peut pas croître trop vite quand $r \rightarrow 0$. Plus précisément, en comparant u_x avec son extension harmonique dans des boules de rayon $2^{-k}r$, on obtient par un argument itératif l'estimation $\omega(u_x, s) \leq C\omega(u_x, r) + C \log\left(\frac{r}{s}\right)$ pour tous $0 < s \leq r \leq r_0$. Pour $y, z \in B_r(x)$ à une distance $\delta = |x - y|$, en utilisant en particulier l'inégalité de Poincaré il vient $|u(x) - u(y)| \leq C\delta(1 + \omega(u_y, c\delta))$ de sorte qu'avec l'estimation précédente on obtient

$$|u(y) - u(z)| \leq C\left(1 + \omega(u_x, r) + \log\frac{r}{|y - z|}\right)|y - z|.$$

La fonction u est donc continue dans D et même localement Höldérienne dans D .

Cette stratégie permet aussi de montrer que u est lipschitzienne dans un voisinage de x où u ne change pas de signe. Ceci est dû au fait que la preuve de l'estimation précédente sur $\omega(u_x, r)$ repose essentiellement sur une comparaison entre u_x et son extension harmonique h_r à B_r . Or si u_x est strictement positive dans B_r , alors h_r l'est aussi par le principe du maximum, de sorte que quand on compare u_x avec h_r à l'aide de la condition de quasi-minimalité du Théorème 1.3.12, les termes de mesures se simplifient. De plus, u étant α -Höldérienne, le terme d'erreur en norme L^1 est contrôlé par $r^{d+\alpha}$. Par conséquent, si u_x est strictement positive dans B_r , alors u_x vérifie (1.3.10) avec maintenant $\tilde{u} = h_r$ et un terme d'erreur en $r^{d+\alpha}$ (au lieu de r^d). En reprenant la stratégie précédente avec ce terme d'erreur plus précis, on obtient en particulier l'estimation $\omega(u_x, s) \leq C\omega(u_x, r) + Cr^\alpha$. Par conséquent, il vient que u est lipschitzienne dans $B_{r/2}(x)$ avec $\|\nabla u\|_{L^\infty(B_{r/2}(x))} \leq C(1 + \omega(u_x, r))$. En fait, u est dans ce cas de classe $C^{1,\beta}$ pour un certain $\beta \in (0, 1)$.

Ensuite, on considère la situation où $\omega(u_x, r) \geq \kappa$ pour une certaine constante $\kappa > 0$ suffisamment grande et on distingue alors deux cas. Sous une hypothèse supplémentaire, qui exprime que la quantité $\frac{1}{r}|\int_{B_r} u_x|$ est grande, il est possible de montrer que la fonction u_x est de signe constant dans une petite boule B_{cr} , de sorte que l'on est ramené au cas précédent ; en particulier, $|\nabla u(x)| \leq C(1 + \omega(u_x, r))$. Si cette hypothèse n'est pas satisfaite, alors on montre que la moyenne $\omega(u_x, r)$ décroît pour un plus petit rayon. Plus précisément, il vient $\omega(u_x, r/3) \leq \frac{1}{2}\omega(u_x, r)$. La preuve de ce résultat utilise en particulier une formule de monotonie tirée de [75] pour des opérateurs sous forme divergence.

Pour conclure que u est lipschitzienne, on fixe $x \in D$, $r > 0$ et on considère la fonction u_x dans des boules B_{r_k} de rayon $r_k = 3^{-k}r$. D'après ce qui précède, pour tout $k \geq 0$, l'une des trois assertions suivantes est vérifiée :

- (i) u_x est de signe constant dans B_{cr_k} et en particulier $|\nabla u(x)| \leq C(1 + \omega(u_x, r_k))$,
- (ii) $\omega(u_x, r_{k+1}) \leq \frac{1}{2}\omega(u_x, r_k)$,
- (iii) $\omega(u_x, r_k) \leq \kappa$.

Par suite, on considère le plus petit entier k_0 pour lequel la condition (i) est satisfaite. Puisque pour tout $k < k_0$ l'une des conditions (ii) ou (iii) est vérifiée, on obtient par récurrence que $\omega(u_x, r_k) \leq \max\{2^{-k}\omega(u_x, r), \kappa\}$ pour tout $0 \leq k \leq k_0$. Or la condition (i) étant satisfaite pour k_0 , il vient $|\nabla u(x)| \leq C(1 + \omega(u_x, r_{k_0})) \leq C(1 + \omega(u_x, r))$. Précisons que si k_0 n'existe pas (i.e. le cas (i) n'est jamais vérifié), alors $\omega(u_x, r_k) \leq \kappa$ pour tout k suffisamment grand, de sorte que l'on obtient $|\nabla u(x)| \simeq |\nabla u_x(0)| = \liminf_{k \rightarrow +\infty} \omega(u_x, r_k) \leq \kappa$ pour tout point x de Lebesgue pour $\nabla u \in L^2$ (puisque 0 est alors un point de Lebesgue pour ∇u_x).

Régularité de la frontière libre

On commence par étudier les blow-ups du vecteur $U = (u_1, \dots, u_k) \in H_0^1(D, \mathbb{R}^k)$ des k premières fonctions propres sur un ensemble optimal Ω^* pour (1.3.7). Pour $x_0 \in D$ et $r_n \downarrow 0$, on définit la suite de fonctions $B_n(\xi) = \frac{1}{r_n}U(x_0 + r_n\xi)$ qui converge vers une fonction limite $B_0 \in H_{\text{loc}}^1(\mathbb{R}^d, \mathbb{R}^k)$ puisque U est localement lipschitzienne dans D . Afin d'étudier les blow-ups B_0 , on effectue le changement de variables précédent $F_{x_0}(\xi) = x + A_{x_0}^{1/2}[\xi]$ de sorte que la fonction

$U_{x_0} := U \circ F_{x_0}$ vérifie une condition de quasi-minimalité pour la fonctionnelle

$$J(\tilde{U}, r) = \int_{B_r} |\nabla \tilde{U}|^2 dx + \Lambda |\{|\tilde{U}| > 0\} \cap B_r|. \quad (1.3.11)$$

On en déduit que U_{x_0} est non-dégénérée en s'inspirant de [34], et on montre une formule de monotonie de Weiss pour U_{x_0} . Ainsi, la suite $B_n \circ A_{x_0}^{1/2}(\xi) = \frac{1}{r_n} U_{x_0}(r_n \xi)$ converge vers un blow-up $B_0 \circ A_{x_0}^{1/2}$ de U_{x_0} en 0 qui est non-trivial (non-dégénérescence de U_{x_0}), 1-homogène (formule de monotonie de Weiss) et qui est un minimiseur global de la fonctionnelle (1.3.11). De plus, il existe un vecteur unitaire $\xi \in \partial B_1 \subset \mathbb{R}^d$ tel que $B_0 \circ A_{x_0}^{1/2} = |B_0 \circ A_{x_0}^{1/2}| \xi$, de sorte que la fonction $|B_0 \circ A_{x_0}^{1/2}|$ est alors une solution globale de la fonctionnelle d'Alt-Caffarelli $\int_{\mathbb{R}^d} |\nabla u|^2 dx + \Lambda |\{u > 0\}|$. En conséquence, le vecteur U satisfait au sens de la viscosité la condition d'optimalité suivante

$$|A^{1/2} \nabla |U|| = \sqrt{\Lambda} \quad \text{sur} \quad \partial \Omega^* \cap D. \quad (1.3.12)$$

Ensuite, la stratégie est, comme dans [77], de se ramener à un problème à frontière libre à une phase. Rappelons que les composantes connexes de Ω^* ne se touchent pas dans D et qu'il est donc suffisant de prouver la régularité d'une composante connexe Ω_1^* de Ω^* où la première fonction propre u_1 est strictement positive. En reprenant une idée de [70] on montre que u_1 est non-dégénérée près de $\partial \Omega_1^*$. Plus précisément, pour une certaine constante $C_1 > 0$ il vient $C_1 u_1 \geq |U|$ dans Ω_1^* , de sorte que la non-dégénérescence de U implique celle de u_1 . En conséquence, on en déduit une estimation de la densité de Ω_1^* . Par suite, un point $x_0 \in \partial \Omega_1^* \cap D$ est dit régulier si $\frac{1}{r}(\Omega_1^* - x_0)$ converge vers un demi-espace quand $r \rightarrow 0$. On montre alors, en utilisant la formule de monotonie de Weiss, que les points réguliers correspondent aux points $x_0 \in \partial \Omega_1^* \cap D$ tels que la densité de $\{|U_{x_0}| > 0\}$ en 0 vaut $\frac{1}{2}$. Avec le fait que les frontières $\partial\{|B_n| > 0\}$ convergent vers $\partial\{|B_0| > 0\}$ pour la distance de Hausdorff, on en déduit que Ω_1^* est Reifenberg plat au voisinage des points réguliers, ce qui signifie en un certain sens que la frontière $\partial \Omega_1^*$ près d'un point régulier peut être approchée par des hyperplans. En particulier, la partie régulière de Ω_1^* est tangentiellement accessible (NTA) par un résultat de Kenig et Toro dans [66]. On montre alors un principe de Harnack pour les fonctions propres près des points réguliers x_0 de la frontière $\partial \Omega_1^*$. Plus précisément, on montre que les limites $g_i(x_0) = \lim_{x \rightarrow x_0} \frac{u_i(x)}{u_1(x)}$, $i = 2, \dots, k$, existent et définissent des fonctions $g_i : \partial \Omega_1^* \cap B_r(x_0) \rightarrow \mathbb{R}$ qui sont Höldériennes. La preuve de ce résultat utilise un principe de Harnack à la frontière pour des fonctions harmoniques par rapport à un opérateur sous forme divergence sur un ensemble NTA (voir [63]). Ici, les fonctions propres u_i sont solutions de l'équation $-\operatorname{div}(A \nabla u_i) = \lambda_i(\Omega^*) b u_i$ dans Ω_1^* avec un membre de droite qui fait en particulier intervenir la fonction b . La preuve s'inspire d'une idée dans [82, Appendix A] pour les fonctions propres du Laplacien avec $b = 1$. Précisons que c'est le seul résultat dans la preuve du Théorème 1.3.11 où le caractère lipschitzien de b est utilisé. Ce principe de Harnack permet ensuite d'obtenir une condition d'optimalité satisfaite par u_1 . De manière heuristique, en écrivant $u_1 = g|U|$ avec $g = (1 + g_2 + \dots + g_k)^{-1/2}$ on obtient formellement que $A^{1/2} \nabla u_1 = g A^{1/2} \nabla |U| + |U| A^{1/2} \nabla g$, de sorte qu'en utilisant la condition d'optimalité (1.3.12), et puisque $U = 0$ sur Ω_1^* , il vient $|A^{1/2} \nabla u_1| = g \sqrt{\Lambda}$ sur $\partial \Omega_1^* \cap B_r(x_0)$. On montre alors que cette condition d'optimalité est en fait satisfaite au sens de la viscosité. Il suit alors, puisque l'on a prouvé que la fonction g est Höldérienne sur $\partial \Omega_1^* \cap B_r(x_0)$, que la partie régulière de Ω_1^* est de classe $C^{1,\alpha}$ par un résultat de De Silva dans [36]. Il est aussi possible de montrer un principe de Harnack d'ordre supérieur pour les fonctions propres, ce qui avec le résultat de Kinderlehrer et Nirenberg dans [67] implique en particulier que la partie régulière est de classe C^∞ si les coefficients de l'opérateur sont de classe C^∞ . Enfin, on montre qu'il est possible d'appliquer les résultats de Weiss dans [92] afin d'obtenir l'estimation de l'ensemble singulier.

1.4 Perspectives

Nous listons dans cette section quelques remarques et perspectives de recherche liées aux problèmes abordés dans cette thèse.

- Dans [84] nous avons prouvé l'existence d'un couple optimal pour le problème

$$\min \{ \lambda_1(\Omega, V) : \Omega \subset D \text{ quasi-ouvert, } |\Omega| \leq m, \|V\|_{L^\infty} \leq \tau \}, \quad (1.4.1)$$

ainsi qu'un résultat d'existence et de régularité si l'on se restreint de surcroît aux termes de transport de la forme $V = \nabla\Phi$, $\Phi \in W^{1,\infty}(D)$. Il est alors naturel d'espérer que ce résultat de régularité est aussi valable pour le problème (1.4.1). Rappelons que quand $V = \nabla\Phi$, la preuve repose de manière cruciale sur l'existence d'une caractérisation variationnelle de la première valeur propre qui permet de se ramener à un problème à frontière libre, et qu'une telle caractérisation n'existe pas pour un terme de transport $V \in L^\infty(D, \mathbb{R}^d)$ quelconque. Une idée réside dans la décomposition de Helmholtz-Hodge, qui énonce que tout champ de vecteur $V \in L^2(D, \mathbb{R}^d)$ peut s'écrire comme $V = V_0 + \nabla\Phi$, où $V_0 \in L^2(D)$ est un champ de vecteurs de divergence nulle et $\Phi \in H_0^1(D)$. Il est alors possible de montrer que $\lambda_1(\Omega, -\Delta) \leq \lambda_1(\Omega, V_0)$ pour tout quasi-ouvert $\Omega \subset D$ et tout terme de transport V_0 de divergence nulle. Ainsi, le problème (1.4.1) restreint aux termes de transport de divergence nulle possède les mêmes solutions que celui de la minimisation de la première valeur propre du Laplacien de Dirichlet dans une boîte, pour lequel la régularité est connue d'après [11] (ou [84]). Bien sûr, la régularité pour un terme de transport $V \in L^\infty(D, \mathbb{R}^d)$ fixé, qui est un résultat plus fort, est aussi un problème ouvert.

- Une autre perspective est de montrer une inégalité de type Faber-Krahn pour l'opérateur $L = -\Delta + V \cdot \nabla$ avec condition de Robin. Dans [51] (voir aussi Théorème 1.2.2), les auteurs montrent que $\lambda_1(\Omega^*, \tau \frac{x}{|x|}) \leq \lambda_1(\Omega, V)$ pour tout ouvert (lisse) $\Omega \subset \mathbb{R}^d$ tel que $|\Omega| \leq m$ et tout champ de vecteurs $V \in L^\infty(\mathbb{R}^d, \mathbb{R}^d)$ tel que $\|V\|_{L^\infty} \leq \tau$, où Ω^* désigne la boule centrée en 0 telle que $|\Omega^*| = m$ (et $\lambda_1(\Omega, V)$ la première valeur propre de L dans Ω avec condition de Dirichlet). D'un autre côté, la boule minimise la première valeur propre du Laplacien avec condition de Robin parmi les ouverts de \mathbb{R}^d de mesure fixée (voir [31]). Ainsi il est naturel d'espérer que la boule est aussi un minimiseur pour la première valeur propre de L avec condition de Robin. L'idée serait d'adapter la stratégie développée dans [31] (voir aussi [16]) à l'opérateur L , qui est basée sur une fonctionnelle adaptée au ratio $|\nabla u|/u$ où u désigne la première fonction propre du Laplacien avec condition de Robin. Cette fonctionnelle s'étend en fait au cas de l'opérateur L puisque l'on connaît l'expression du terme de transport optimal sur un domaine fixé (voir [51] ou Théorème 1.2.4).
- Il serait intéressant de faire des simulations numériques pour déterminer la forme des ensembles optimaux qui minimisent la première valeur propre de l'opérateur $L = -\Delta + V \cdot \nabla$ selon le choix du terme de transport V . Par exemple, on sait que pour $V(x) = \frac{x}{|x|}$ la boule est un minimiseur ; qu'en est-il pour $V(x) = -\frac{x}{|x|}$? Une autre question est d'identifier le terme de transport optimal sur un domaine fixé ayant une géométrie simple, comme par exemple une ellipse. Mentionnons aussi que l'on connaît des résultats qualitatifs sur la première fonction propre quand la norme $\|V\|_{L^\infty}$ devient grande (voir [53]).
- Plusieurs questions naturelles émergent des résultats obtenus dans [87, 88] : le résultat de régularité est-il encore valable avec une contrainte de mesure du type $|\Omega| \leq m$ (plutôt qu'une pénalisation avec un terme de la forme $\Lambda|\Omega|$) ? Ou bien avec une fonctionnelle plus générale que la somme ? Les ensembles optimaux sont-ils réguliers jusqu'à la frontière de la boîte D ? Il s'agirait pour la première question d'adapter la stratégie suivie dans [11] (ou [84]) pour des quasi-minimiseurs à valeurs vectorielles. Concernant la deuxième question, précisons que les résultats dans [87, 88] sont en fait valables pour des fonctionnelles

strictement croissantes et interchangeables par rapport à chaque variable. Un résultat pour des fonctionnelles plus générales pourrait être obtenu en s'inspirant des idées développées dans [70] et [71] (voir aussi Théorèmes 1.3.9 et 1.3.10). Enfin, en vue d'obtenir la régularité jusqu'au bord de la boîte des ensembles optimaux, il est d'abord nécessaire de montrer que les fonctions propres sur un ensemble optimal sont lipschitziennes dans \mathbb{R}^d (le caractère lipschitzien prouvé dans [87] est seulement local dans D).

- Bucur, Mazzoleni et Pratelli ont prouvé l'existence d'un minimiseur borné parmi les quasi-ouverts de \mathbb{R}^d de mesure fixée pour des fonctionnelles générales en les valeurs propres du Laplacien de Dirichlet (voir [12, 76] et Théorème 1.3.3). Il serait intéressant d'étendre ce résultat à un opérateur sous forme divergence. La régularité des minimiseurs pour la somme des k premières valeurs propres dans \mathbb{R}^d serait alors une conséquence des résultats dans [88]. Néanmoins, les preuves suivies dans [12, 76] utilisent que les valeurs propres du Laplacien de Dirichlet sont invariantes par translation et (-2) -homogènes, ce qui n'est bien sûr pas le cas des valeurs propres d'un opérateur avec des coefficients variables.
- Terminons avec un problème à frontière libre. Il s'agit d'étudier la régularité des minimiseurs de la fonctionnelle à une phase suivante

$$\mathcal{F}(u) = \int_{B_1^+} (|\nabla u|^2 + Q(x)\mathbb{1}_{\{u>0\}}) dx + \beta \int_{H \cap B_1} u d\mathcal{H}^{d-1},$$

où Q est une fonction bornée, $\beta > 0$, B_1^+ et H désignent respectivement la demi-boule $B_1^+ = \{x_d \geq 0\} \cap B_1 \subset \mathbb{R}^d$ et l'hyperplan $H = \{x_d = 0\}$. La motivation derrière l'étude de ce problème est que les ensembles des blow-ups en un point sur $\partial(\{u > 0\} \cap H)$ ne sont pas des demi-espaces mais des cônes dont l'angle dépend du paramètre β . L'objectif est de prouver que la frontière libre $(d-2)$ -dimensionnelle $F := \partial(\{u > 0\} \cap H)$ est de classe $C^{1,\alpha}$ à un petit ensemble singulier près. Précisons que la frontière de l'ensemble $\{u > 0\}$ dans $\{x_d > 0\}$ est régulière d'après [2], et que l'intérêt de ce problème réside dans l'étude de la régularité de la frontière de $\{u > 0\}$ dans l'hyperplan H .

Annexe A : Spectre d'un opérateur compact

On rappelle brièvement quelques résultats classiques de théorie spectrale concernant les opérateurs compacts. Pour plus de détails on pourra consulter le livre de Kato [64].

Commençons par le cas des opérateurs autoadjoints sur un espace de Hilbert H séparable et de dimension infinie. On note $\mathcal{B}(H)$ l'ensemble des opérateurs bornés sur H .

Théorème 1.A.1 (cf. section 5.2.3 p.260 dans [64]). Soit $T \in \mathcal{B}(H)$ un opérateur autoadjoint et compact. Alors :

1. il existe une base hilbertienne $(u_n)_{n \in \mathbb{N}}$ de H formée de vecteurs propres pour T dont les valeurs propres associées $(\lambda_n)_{n \in \mathbb{N}}$ sont de multiplicité finie et forment une suite de réels qui converge vers 0.
2. si en outre T est positif, alors on peut réarranger les valeurs propres λ_n non nulles en une suite décroissante de réels strictement positifs tendant vers 0.

Le théorème suivant énonce que l'application qui à un opérateur autoadjoint $T \in \mathcal{B}(H)$ associe son spectre $\sigma(T)$ est 1-lipschitzienne par rapport à la norme d'opérateur et la distance de Hausdorff. Rappelons que la distance de Hausdorff entre deux compacts $K_1, K_2 \subset \mathbb{R}^d$ est définie par

$$d_{\mathcal{H}}(K_1, K_2) = \max \left\{ \sup_{x \in K_1} d(x, K_2), \sup_{y \in K_2} d(K_1, y) \right\}.$$

Théorème 1.A.2 (cf. Theorem 4.10 p.291 dans [64]). Soient $T_1, T_2 \in \mathcal{B}(H)$ deux opérateurs autoadjoints. Alors

$$d_{\mathcal{H}}(\sigma(T_1), \sigma(T_2)) \leq \|T_1 - T_2\|.$$

On considère maintenant des opérateurs compacts sur un espace de Banach complexe X de dimension infinie.

Théorème 1.A.3 (cf. Theorem 6.26 p.185 dans [64]). Le spectre $\sigma(T) \subset \mathbb{C}$ d'un opérateur compact $T \in \mathcal{B}(X)$ est fini ou formé d'une suite tendant vers 0. De plus, tout $\lambda \in \sigma(T) \setminus \{0\}$ est une valeur propre de T de multiplicité finie.

Le théorème suivant exprime la continuité du spectre d'un opérateur compact vis-à-vis de la convergence en norme. Il n'y a toutefois plus d'estimation de la vitesse de convergence comme au théorème 1.A.2.

Théorème 1.A.4. Soit $(T_n)_{n \geq 1} \subset \mathcal{B}(X)$ une suite d'opérateurs convergeant vers un opérateur compact T en norme d'opérateur. Soient $\lambda \in \mathbb{C} \setminus \{0\}$ une valeur propre de T et $r > 0$ tels que $B_r(\lambda) \cap \sigma(T) = \{\lambda\}$. Alors il existe $n_0 \geq 1$ tel que pour tout $n \geq n_0$, il existe une valeur propre $\lambda_n \in \sigma(T_n) \cap B_{r/2}(\lambda)$.

La suite de cette annexe est dédiée à la preuve de ce théorème, qui est implicite dans [64] (voir Theorem 3.16 p.212). Rappelons que pour $T \in \mathcal{B}(X)$, l'ensemble résolvant $\rho(T)$ est l'ensemble des complexes $\zeta \in \mathbb{C}$ tels que $T - \zeta I$ est inversible, tandis que le spectre $\sigma(T)$ est le complémentaire de $\rho(T)$ dans \mathbb{C} , c'est-à-dire $\sigma(T) = \mathbb{C} \setminus \rho(T)$. La résolvante $R(\zeta, T) \in \mathcal{B}(X)$ pour $\zeta \in \rho(T)$ est définie par $R(\zeta, T) = (T - \zeta I)^{-1}$. De plus, $\zeta \mapsto R(\zeta, T)$ est holomorphe sur l'ouvert $\rho(T)$.

Dans la suite $T \in \mathcal{B}(X)$ désigne un opérateur borné et Γ une courbe simple, fermée, rectifiable et orientée positivement dans $\rho(T)$. On note $\sigma_1(T)$ la partie du spectre de T qu'entoure Γ et $\sigma_2(T) = \sigma(T) \setminus \sigma_1(T)$.

Lemme 1.A.5. L'opérateur

$$P_{\Gamma}(T) = -\frac{1}{2i\pi} \int_{\Gamma} R(\zeta, T) d\zeta \tag{1.A.1}$$

définit une projection, appelée projection spectrale (ou projection de Riesz).

Preuve : Soit Γ' une courbe dans $\rho(T)$ séparant le spectre de T en $\sigma_1(T)$ et $\sigma_2(T)$ de sorte que $P_{\Gamma}(T) = P_{\Gamma'}(T)$, et supposons que Γ' entoure Γ . Par simplicité on note $R(\zeta) = R(\zeta, T)$. Alors

$$P_{\Gamma}(T)^2 = \frac{1}{(2i\pi)^2} \int_{\Gamma'} \int_{\Gamma} R(\zeta') R(\zeta) d\zeta d\zeta'.$$

En utilisant l'équation de la résolvante

$$R(\zeta') - R(\zeta) = (\zeta' - \zeta)R(\zeta')R(\zeta), \quad \forall \zeta, \zeta' \in \rho(T), \tag{1.A.2}$$

il vient

$$P_{\Gamma}(T)^2 = \frac{1}{(2i\pi)^2} \int_{\Gamma'} R(\zeta') d\zeta' \int_{\Gamma} \frac{d\zeta}{\zeta' - \zeta} - \frac{1}{(2i\pi)^2} \int_{\Gamma} R(\zeta) d\zeta \int_{\Gamma'} \frac{d\zeta'}{\zeta' - \zeta}.$$

Or, puisque Γ' entoure Γ , on a

$$\forall \zeta' \in \Gamma', \int_{\Gamma} \frac{d\zeta}{\zeta' - \zeta} = 0 \quad \text{et} \quad \forall \zeta \in \Gamma, \int_{\Gamma'} \frac{d\zeta'}{\zeta' - \zeta} = 2i\pi,$$

ce qui prouve que $P_{\Gamma}(T)^2 = P_{\Gamma}(T)$. □

En notant $M_\Gamma(T) = P_\Gamma(T)X$ et $N_\Gamma(T) = (I - P_\Gamma(T))X$, $P_\Gamma(T)$ est la projection de X sur $M_\Gamma(T)$ parallèlement à $N_\Gamma(T)$ et on a la décomposition $X = M_\Gamma(T) \oplus N_\Gamma(T)$. De plus, $P_\Gamma(T)$ commute avec $R(\zeta, T)$ au vu de l'équation de la résolvante (1.A.2), et donc commute aussi avec T . Ainsi, T induit sur les sous-espaces T -invariants $M_\Gamma(T)$ et $N_\Gamma(T)$ des opérateurs notés $T_M \in \mathcal{B}(M_\Gamma(T))$ et $T_N \in \mathcal{B}(N_\Gamma(T))$.

Lemme 1.A.6. On a $\sigma(T_M) = \sigma_1(T)$ et $\sigma(T_N) = \sigma_2(T)$.

Preuve : Notons par commodité d'écriture $P = P_\Gamma(T)$, $R(\zeta) = R(\zeta, T)$, $M = M_\Gamma(T)$ et $N = N_\Gamma(T)$, et soient $R_M(\zeta) \in \mathcal{B}(M)$ et $R_N(\zeta) \in \mathcal{B}(N)$ les opérateurs induits par $R(\zeta)$ sur M et N respectivement. Il est immédiat de vérifier que $R_M(\zeta) = (T_M - \zeta I)^{-1}$ et $R_N(\zeta) = (T_N - \zeta I)^{-1}$, ce qui implique que $\rho(T)$ est inclus dans les deux ensembles résolvants $\rho(T_M)$ et $\rho(T_N)$. De plus, en utilisant l'équation (1.A.2) on obtient pour $\zeta \in \rho(T)$

$$R(\zeta)P = -\frac{1}{2i\pi} \int_\Gamma R(\zeta)R(\zeta')d\zeta' = -\frac{1}{2i\pi} \int_\Gamma \frac{R(\zeta) - R(\zeta')}{\zeta - \zeta'} d\zeta'. \quad (1.A.3)$$

Ainsi, pour $\zeta \in \rho(T)$ à l'extérieur de la région délimitée par Γ il vient

$$R(\zeta)P = \frac{1}{2i\pi} \int_\Gamma \frac{R(\zeta')}{\zeta - \zeta'} d\zeta'. \quad (1.A.4)$$

Puisque le membre de droite dans (1.A.4) est holomorphe à l'extérieur de Γ , il suit que $R(\zeta)P$, et donc aussi $R_M(\zeta)$ (puisque $R(\zeta)P = R_M(\zeta)$ dans M), possèdent un prolongement analytique à l'extérieur de Γ , ce qui implique que $\sigma(T_M) \subset \sigma_1(T)$. De manière analogue, pour $\zeta \in \rho(T)$ à l'intérieur de la région délimitée par Γ , (1.A.3) donne

$$R(\zeta)(I - P) = \frac{1}{2i\pi} \int_\Gamma \frac{R(\zeta')}{\zeta' - \zeta} d\zeta',$$

ce qui définit un prolongement analytique de $R_N(\zeta)$ à l'intérieur de Γ et montre que $\sigma(T_N) \subset \sigma_2(T)$. Afin de conclure, il suffit donc de montrer que $\sigma(T) \subset \sigma(T_M) \cup \sigma(T_N)$. Or si $\zeta \in \rho(T_M) \cap \rho(T_N)$, alors $R_M(\zeta)P + R_N(\zeta)(I - P)$ est l'inverse de $T - \zeta I$ de sorte que $\zeta \in \rho(T)$, ce qui achève la preuve. \square

Dans la situation où Γ entoure une seule valeur propre isolée λ du spectre de T , c'est-à-dire si $\sigma_1(T) = \{\lambda\}$, alors $M_\Gamma(T)$ est appelé le sous-espace propre algébrique associé à λ . Si en outre $M_\Gamma(T)$ est de dimension finie, alors la dimension de $M_\Gamma(T)$ est appelée la multiplicité algébrique de la valeur propre λ . Bien sûr, si λ est un point isolé du spectre de T , alors λ est une valeur propre de T si $M_\Gamma(T)$ est de dimension finie, tandis que λ peut être ou non une valeur propre de T si $M_\Gamma(T)$ est de dimension infinie. Enfin, pour une valeur propre λ , on appelle $\text{Ker}(T - \lambda I)$ le sous-espace propre géométrique associé à λ , et sa dimension, si elle est finie, est appelée la multiplicité géométrique de λ . On pourra se référer à la section 3.6.5 dans [64] pour plus de précisions concernant la décomposition spectrale d'un opérateur.

On va maintenant montrer que la décomposition $X = M_\Gamma(T) \oplus N_\Gamma(T)$ est continue par rapport à T .

Lemme 1.A.7. Soient $P, Q \in \mathcal{B}(X)$ deux projections telles que $\|P - Q\| < 1$. Alors P et Q sont semblables ; en particulier, les images PX , QX d'une part, et les noyaux $(I - P)X$, $(I - Q)X$ d'autre part, sont isomorphes.

Preuve : Posons

$$U_0 = QP + (I - Q)(I - P), \quad V_0 = PQ + (I - P)(I - Q),$$

de sorte qu'il vient $U_0P = QP = QU_0$ et $PV_0 = PQ = V_0Q$. Les opérateurs U_0 et V_0 ne sont toutefois pas inverses l'un de l'autre, et un calcul montre en fait qu'en posant $R = (P - Q)^2$ on obtient

$$V_0U_0 = U_0V_0 = I - R.$$

Par suite, la série binomiale

$$(I - R)^{-1/2} = \sum_{n=0}^{+\infty} \binom{-1/2}{n} (-R)^n$$

est absolument convergente puisque $\|R\| < 1$ par hypothèse. Ainsi, en utilisant le fait que R commute avec P et Q , les opérateurs

$$\begin{aligned} U &= U_0(I - R)^{-1/2} = (I - R)^{-1/2}U_0, \\ V &= V_0(I - R)^{-1/2} = (I - R)^{-1/2}V_0, \end{aligned}$$

sont inverses l'un de l'autre, et on obtient par conséquent que $Q = UPU^{-1}$ et $P = U^{-1}QU$. \square

Lemme 1.A.8. Il existe $\delta > 0$ tel que si $S \in \mathcal{B}(X)$ vérifie $\|T - S\| < \delta$, alors $\Gamma \subset \rho(S)$ et les espaces $M_\Gamma(S) = P_\Gamma(S)X$ et $N_\Gamma(S) = (I - P_\Gamma(S))X$ sont isomorphes à $M_\Gamma(T)$ et $N_\Gamma(T)$ respectivement.

Preuve : Pour tout point $\zeta \in \Gamma \subset \rho(T)$, si $\|T - S\| < \delta := \min\{\|R(\zeta, T)\|^{-1} : \zeta \in \Gamma\}$ alors le membre de droite de

$$S - \zeta I = (I - (T - S)R(\zeta, T))(T - \zeta I) \quad (1.A.5)$$

est inversible de sorte que $\zeta \in \rho(S)$ (précisons que $\delta > 0$ puisque $\zeta \mapsto \|R(\zeta, T)\|$ est continue). De plus, (1.A.5) donne

$$R(\zeta, S) - R(\zeta, T) = R(\zeta, T) [(I - (T - S)R(\zeta, T))^{-1} - I],$$

ce qui implique que

$$\|R(\zeta, T) - R(\zeta, S)\| = \frac{\|T - S\| \|R(\zeta, T)\|^2}{1 - \|T - S\| \|R(\zeta, T)\|}.$$

Ainsi, quitte à choisir $\delta > 0$ suffisamment petit, on obtient $\|P_\Gamma(T) - P_\Gamma(S)\| < 1$, ce qui permet de conclure avec le lemme 1.A.7. En fait, ceci montre que la décomposition $X = M_\Gamma(S) \oplus N_\Gamma(S)$ est continue en S , au sens où la projection $P_\Gamma(S)$ de X sur $M_\Gamma(S)$ parallèlement à $N_\Gamma(S)$ converge pour la norme d'opérateur vers $P_\Gamma(T)$ quand $\|T - S\| \rightarrow 0$. \square

Le Théorème 1.A.4 est maintenant une conséquence du résultat suivant.

Proposition 1.A.9. Supposons que $\sigma_1(T) = \{\lambda_1, \dots, \lambda_k\}$ est fini, $k \geq 1$. Soit $(T_n)_{n \geq 1}$ une suite d'opérateurs convergeant vers T en norme d'opérateur. Soient Γ_j des courbes entourant λ_j , deux à deux disjointes et contenues dans la région délimitée par Γ . Alors il existe $n_0 \geq 1$ tel que pour tout $n \geq n_0$, les espaces $M_{\Gamma_j}(T)$ et $M_{\Gamma_j}(T_n)$ sont isomorphes, et la région dans Γ excluant celles délimitées par les Γ_j ne contient pas de point du spectre de T_n .

Preuve : En appliquant le Lemme 1.A.8 à Γ et Γ_j , il vient d'une part que $M_\Gamma(T)$ et $M_\Gamma(T_n)$ sont isomorphes, et d'autre part que $M_{\Gamma_j}(T)$ et $M_{\Gamma_j}(T_n)$ sont isomorphes. Par ailleurs, on a

$$P_\Gamma(T) = \sum_{j=1}^k P_{\Gamma_j}(T) \quad \text{et} \quad P_{\Gamma_j}(T)P_{\Gamma_l}(T) = \delta_{jl}P_{\Gamma_j}(T), \quad \forall j, l \in \{1, \dots, k\},$$

la première égalité étant une conséquence du théorème des résidus tandis que la seconde s'obtient comme au Lemme (1.A.5) (puisque $\Gamma_j \cap \Gamma_l = \emptyset$ si $j \neq l$). Ainsi, on obtient les isomorphismes

$$M_\Gamma(T_n) \cong M_\Gamma(T) \cong \bigoplus_{j=1}^k M_{\Gamma_j}(T) \cong \bigoplus_{j=1}^k M_{\Gamma_j}(T_n),$$

ce qui conclut la preuve. □

En particulier, si dans la Proposition 1.A.9 λ_j est une valeur propre de multiplicité algébrique finie notée m_j , alors T_n ne contient que des valeurs propres dans Γ_j et leur multiplicité algébrique totale est égale à m_j . Précisons que la multiplicité algébrique est conservée mais que ce n'est en général pas le cas de la multiplicité géométrique.

Chapter 2

Existence and regularity of optimal shapes for elliptic operators with drift

This chapter is based on [84], which is joint work with Emmanuel Russ and Bozhidar Velichkov.

Abstract

This paper is dedicated to the study of shape optimization problems for the first eigenvalue of the elliptic operator with drift $L = -\Delta + V(x) \cdot \nabla$ with Dirichlet boundary conditions, where V is a bounded vector field. In the first instance, we prove the existence of a principal eigenvalue $\lambda_1(\Omega, V)$ for a bounded quasi-open set Ω which enjoys similar properties to the case of open sets. Then, given $m > 0$ and $\tau \geq 0$, we show that the minimum of the following non-variational problem

$$\min \left\{ \lambda_1(\Omega, V) : \Omega \subset D \text{ quasi-open, } |\Omega| \leq m, \|V\|_{L^\infty} \leq \tau \right\}.$$

is achieved, where the box $D \subset \mathbb{R}^d$ is a bounded open set. The existence when V is fixed, as well as when V varies among all the vector fields which are the gradient of a Lipschitz function, are also proved.

The second interest and main result of this paper is the regularity of the optimal shape Ω^* solving the minimization problem

$$\min \left\{ \lambda_1(\Omega, \nabla\Phi) : \Omega \subset D \text{ quasi-open, } |\Omega| \leq m \right\},$$

where Φ is a given Lipschitz function on D . We prove that the optimal set Ω^* is open and that its topological boundary $\partial\Omega^*$ is composed of a *regular part*, which is locally the graph of a $C^{1,\alpha}$ function, and a *singular part*, which is empty if $d < d^*$, discrete if $d = d^*$ and of locally finite \mathcal{H}^{d-d^*} Hausdorff measure if $d > d^*$, where $d^* \in \{5, 6, 7\}$ is the smallest dimension at which there exists a global solution to the one-phase free boundary problem with singularities. Moreover, if D is smooth, we prove that, for each $x \in \partial\Omega^* \cap \partial D$, $\partial\Omega^*$ is $C^{1,1/2}$ in a neighborhood of x .

2.1 Introduction and main results

Let D be a bounded connected open set in \mathbb{R}^d , $d \geq 2$. For any bounded vector field $V : D \rightarrow \mathbb{R}^d$ and any connected open set $\Omega \subset D$, we consider the elliptic operator with drift

$L = -\Delta + V(x) \cdot \nabla$. In this paper we study variational optimization problems in which the variables are both the domain Ω and the drift V , and the cost functional is defined through the operator L . The aim of the present paper is twofold. From one side, we develop an existence theory for shape optimization problems for operators with drift. On the other hand, we study the regularity of the optimal shapes for vector fields V that are gradients of potentials $\Phi : D \rightarrow \mathbb{R}$. We focus on the model problem

$$\min \{ \lambda_1(\Omega, V) : \Omega \subset D, |\Omega| \leq m, \|V\|_{L^\infty} \leq \tau \}, \quad (2.1.1)$$

where $m > 0$ and $\tau \geq 0$ are fixed constants, and $\lambda_1(\Omega, V)$ is the principal eigenvalue of the operator L with Dirichlet boundary conditions on $\partial\Omega$. Our main results are the following.

Theorem 2.1.1. *Let $D \subset \mathbb{R}^d$ be a bounded open set, and $0 < m < |D|$ and $\tau \geq 0$ be fixed constants. Then, there exist a quasi-open set $\Omega \subset D$ and a vector field $V : D \rightarrow \mathbb{R}^d$ such that the couple (Ω, V) is a solution to the shape optimization problem (2.1.1).*

In particular, we prove in Theorem 2.3.3 below, that the principal eigenvalue $\lambda_1(\Omega, V)$ of the (non-self-adjoint) operator L is well-defined on any quasi-open set $\Omega \subset D$. Precisely, we will show that for any quasi-open set Ω , there is a real eigenvalue $\lambda_1(\Omega, V)$ of the operator L such that $\lambda_1(\Omega, V) \leq \operatorname{Re} \lambda$, for any other eigenvalue $\lambda \in \mathbb{C}$ of L .

Theorem 2.1.2. *Let $D \subset \mathbb{R}^d$ be a bounded open set, and $0 < m < |D|$ and $\tau \geq 0$ be fixed constants. Then the shape optimization problem*

$$\min \{ \lambda_1(\Omega, \nabla\Phi) : \Omega \subset D \text{ quasi-open}, |\Omega| \leq m, \Phi \in W^{1,\infty}(D), \|\nabla\Phi\|_{L^\infty(D)} \leq \tau \} \quad (2.1.2)$$

admits a solution $(\Omega^, \nabla\Phi^*)$. Moreover, if D is connected, then any optimal set Ω^* has the following properties:*

1. Ω^* is an open set;
2. Ω^* has finite perimeter;
3. Ω^* saturates the constraint, that is, $|\Omega^*| = m$;

The free boundary $\partial\Omega^ \cap D$ can be decomposed in the disjoint union of a regular part $\operatorname{Reg}(\partial\Omega^* \cap D)$ and a singular part $\operatorname{Sing}(\partial\Omega^* \cap D)$, where:*

- (4) *$\operatorname{Reg}(\partial\Omega^* \cap D)$ is locally the graph of a $C^{1,\alpha}$ -regular function for any $\alpha < 1$;*
- (5) *for a universal constant $d^* \in \{5, 6, 7\}$ (see Definition 2.5.39), $\operatorname{Sing}(\partial\Omega^* \cap D)$ is:*
 - *empty if $d < d^*$;*
 - *discrete if $d = d^*$;*
 - *of Hausdorff dimension at most $(d - d^*)$ if $d > d^*$.*

If the boundary ∂D is $C^{1,1}$, then the boundary $\partial\Omega^$ can be decomposed in the disjoint union of a regular part $\operatorname{Reg}(\partial\Omega^*)$ and a singular part $\operatorname{Sing}(\partial\Omega^*)$, where:*

- (6) *$\operatorname{Reg}(\partial\Omega^*)$ is an open subset of $\partial\Omega^*$ and locally the graph of a $C^{1,1/2}$ function; moreover, $\operatorname{Reg}(\partial\Omega^*)$ contains both $\operatorname{Reg}(\partial\Omega^* \cap D)$ and $\partial\Omega^* \cap \partial D$;*
- (7) *$\operatorname{Sing}(\partial\Omega^*) = \operatorname{Sing}(\partial\Omega^* \cap D)$.*

In fact, our result is more general. Precisely, we prove the regularity of the optimal sets for $\lambda_1(\cdot, \nabla\Phi)$ with fixed vector field $\nabla\Phi$ (see Theorem 2.1.5 and Remark 2.1.7).

For m, τ, Ω and V as in (2.1.1), Hamel, Nadirashvili and Russ [51], proved the lower bound

$$\lambda_1 \left(B, \tau \frac{x}{|x|} \right) \leq \lambda_1(\Omega, V), \quad (2.1.3)$$

where B is the ball of Lebesgue measure m centered in zero; moreover, there is an equality in (2.1.3), if and only if, up to translation, $\Omega = B$ and $V(x) = \tau \frac{x}{|x|}$. In other words, the couple $(B, \tau \frac{x}{|x|})$ is (up to translation) the unique solution of the shape optimization problem

$$\min \{ \lambda_1(\Omega, V) : \Omega \subset \mathbb{R}^d, |\Omega| = m, \|V\|_{L^\infty} \leq \tau \}. \quad (2.1.4)$$

We notice that a symmetrization technique in the spirit of [51] cannot be applied to the problem (2.1.1). In fact, the presence of the constraint D makes it impossible to determine explicitly the shape of the optimal domains or the precise analytic expression of the optimal vector fields, except in the trivial case when a ball of measure m fits into D . Thus, we first establish the existence of an optimal domain Ω in the larger (relaxed) class of *quasi-open* sets and we then study the regularity of the optimal shapes through variational free boundary techniques. We stress that, in the case of a generic vector field V , the principal eigenvalue $\lambda_1(\Omega, V)$ does *not* have a variational formulation but is only determined through the solution of a certain PDE on Ω . In particular, the shape cost functional in (2.1.1) cannot be written in terms of a variational minimization problem involving integral cost functionals on Ω . This makes the extension of the functional $\lambda_1(\cdot, V)$ to a (γ) -continuous functional on the class of quasi-open sets a non trivial problem.

In the case $\tau = 0$, (2.1.1) and (2.1.2) are reduced to the classical shape optimization problem

$$\min \{ \lambda_1(\Omega) : \Omega \subset D, |\Omega| \leq m \}, \quad (2.1.5)$$

where $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet Laplacian on Ω . For the problem (2.1.5), the existence of an optimal (quasi-open) set was proved by Buttazzo and Dal Maso in [21], the fact that the optimal sets are open (Theorem 2.1.2 (1)) was proved by Briançon and Lamboley in [11], the estimate on the perimeter of the optimal set (Theorem 2.1.2 (2)) is due to Bucur (see [12]), the regularity of the free boundary $\text{Reg}(\partial\Omega^* \cap D)$ was again proved in [11]; the estimate on the dimension of the singular set $\text{Sing}(\partial\Omega^* \cap D)$ was obtained in [77]. Even for the classical problem (2.1.5) the regularity up to the boundary of the box D (Theorem 2.1.2 (6) and (7)) is new.

Remark 2.1.3 (On the regularity of the optimal shapes for spectral functionals). The regularity of the optimal shapes for the eigenvalues of the Laplacian was an object of an intense study in the last years. As mentioned above, a regularity result, for the optimal sets for the first eigenvalue of the Laplacian, was proved by Briançon and Lamboley in [11]. The regularity of the optimal sets for more general spectral functionals was studied in [18], [77], [71] and [70]. An alternative approach in dimension two, based on the epiperimetric inequality from [86], was recently introduced in [85], where Theorem 2.1.2 (6) is proved in the case $\tau = 0$ and $d = 2$. We notice that the method from [85] can be applied to give an alternative proof of Theorem 2.1.2 (6) in the case $\tau > 0$, but the restriction on the dimension is required by the epiperimetric inequality and for now cannot be removed.

Remark 2.1.4 (On the existence of optimal shapes). The existence of optimal shapes in a bounded open set (box) $D \subset \mathbb{R}^d$ is a consequence of the theory of Buttazzo and Dal Maso (see [21] and the books [13] and [61]) for general shape optimization problems of the form

$$\min \{ \mathcal{F}(\Omega) : \Omega \subset D \text{ quasi-open}, |\Omega| \leq m \}, \quad (2.1.6)$$

for *shape cost functionals* \mathcal{F} with the following properties:

- \mathcal{F} is decreasing with respect to the set inclusion;
- \mathcal{F} is lower semi-continuous with respect to the (γ) -convergence of sets.

We notice that in the case when \mathcal{F} is a function of the spectrum of the Dirichlet Laplacian on Ω , the existence of an optimal set can be obtained directly (see for instance [90]). In fact, if $\mathcal{F}(\Omega) = \lambda_1(\Omega)$, then given a minimization sequence of quasi-open sets Ω_n for (2.1.6), and setting u_n to be the first eigenfunction of the Dirichlet Laplacian on Ω_n , it is not hard to check that, up to a subsequence, u_n converges weakly in $H_0^1(D)$ to a function $u \in H_0^1(D)$ and that the (quasi-open) set $\Omega := \{u > 0\}$ is a solution to (2.1.6). This elementary argument works not only for λ_1 , but can also be reproduced for general spectral functionals of the form $\mathcal{F}(\Omega) = F(\lambda_1(\Omega), \dots, \lambda_k(\Omega))$, and also for most of the shape cost functionals present in the literature. We stress that this is not the case of the functional $\mathcal{F}(\Omega) = \lambda_1(\Omega, V)$. Even if $\lambda_1(\cdot, V)$ is still monotone and γ -continuous (as we will prove in Section 2.4), its non-variational nature does not allow to use the elementary argument described above; thus, the only way to obtain the existence of an optimal set is through the Buttazzo-Dal Maso theory.

Optimal shapes for a fixed vector field. In this paper, we also study the case in which only the shape Ω is variable, while the vector field V is fixed. Precisely, we consider the shape optimization problem

$$\min \left\{ \lambda_1(\Omega, V) : \Omega \subset D, |\Omega| \leq m \right\}, \quad (2.1.7)$$

where both the upper bound m of the Lebesgue measure of the domain Ω and the vector field V are fixed. In this case the geometry of the optimal sets is affected both by the geometric constraint $\Omega \subset D$ and the vector field V . We notice that in this case it is the inclusion constraint that provides the compactness necessary for the existence of an optimal set. We show that the shape functional $\Omega \mapsto \lambda_1(\Omega, V)$ is lower semi-continuous with respect to the so-called γ -convergence of sets and then we obtain the existence of optimal sets by the general result discussed in Remark 2.1.4. Furthermore, when the vector field is the gradient of a Lipschitz function, we prove a regularity result for the optimal sets. Our main result is the following.

Theorem 2.1.5 (Existence and regularity of optimal shapes for a fixed vector field). *Let D be a bounded open set in \mathbb{R}^d . Let $m \in (0, |D|)$ and let the vector field $V : D \rightarrow \mathbb{R}^d$ be such that $\|V\|_{L^\infty} = \tau < +\infty$. Then the shape optimization problem*

$$\min \left\{ \lambda_1(\Omega, V) : \Omega \subset D \text{ quasi-open, } |\Omega| \leq m \right\} \quad (2.1.8)$$

admits a solution $\Omega^ \subset D$. Moreover, if D is connected and the vector field V is of the form $V = \nabla\Phi$, where $\Phi : D \rightarrow \mathbb{R}$ is a given Lipschitz function, then any solution Ω^* of (2.1.8) has the following properties:*

1. Ω^* is an open set;
2. Ω^* has finite perimeter;
3. Ω^* saturates the constraint, that is, $|\Omega^*| = m$;

The free boundary $\partial\Omega^ \cap D$ can be decomposed in the disjoint union of a regular part $\text{Reg}(\partial\Omega^* \cap D)$ and a singular part $\text{Sing}(\partial\Omega^* \cap D)$, where:*

- (4) *$\text{Reg}(\partial\Omega^* \cap D)$ is locally the graph of a $C^{1,\alpha}$ -regular function for any $\alpha < 1$;*
- (5) *for a universal constant $d^* \in \{5, 6, 7\}$ (see Definition 2.5.39), $\text{Sing}(\partial\Omega^* \cap D)$ is:*
 - *empty if $d < d^*$;*
 - *discrete if $d = d^*$;*
 - *of Hausdorff dimension at most $(d - d^*)$ if $d > d^*$.*

If D is $C^{1,1}$ -regular, then the boundary $\partial\Omega^$ can be decomposed in the disjoint union of a regular part $\text{Reg}(\partial\Omega^*)$ and a singular part $\text{Sing}(\partial\Omega^*)$, where:*

- (6) *$\text{Reg}(\partial\Omega^*)$ is an open subset of $\partial\Omega^*$ and locally the graph of a $C^{1,1/2}$ function; moreover, $\text{Reg}(\partial\Omega^*)$ contains both $\text{Reg}(\partial\Omega^* \cap D)$ and $\partial\Omega^* \cap \partial D$;*

$$(7) \text{ Sing}(\partial\Omega^*) = \text{Sing}(\partial\Omega^* \cap D).$$

Remark 2.1.6 (On the optimal regularity of the free boundary). The regularity of the boundary of an optimal set Ω^* to the problem (2.1.8) at contact points of the free boundary with the box cannot exceed $C^{1,1/2}$ even if the vector field is smooth. Indeed, Chang-Lara and Savin proved in [26] that the boundary of Ω_u , where u is a solution of the free boundary problem (2.5.41) in $\Omega_u = \Omega^*$, is at most $C^{1,1/2}$ regular.

Remark 2.1.7 (Regularity of the optimal shapes for variable vector field). We notice that if the couple (Ω, V) is a solution to the shape optimization problem (2.1.1) or (2.1.2), then fixing V , we obtain that Ω is a solution to (2.1.8). In particular, the regularity part of Theorem 2.1.2 is a consequence of Theorem 2.1.5.

Outline of the proof and plan of the paper. Throughout the paper the bounded open set $D \subset \mathbb{R}^d$ is fixed and is assumed to be (at least $C^{1,1}$) smooth.

In the sections 2.2, 2.3 and 2.4, we prove our main existence results (Theorem 2.1.1) and the existence of an optimal domain for a fixed vector field (Theorem 2.1.5), as well as the existence of an optimal domain in Theorem 2.1.2.

In Section 2.2 we recall several central definitions and results in the γ -convergence theory of quasi-open sets. In particular, we show that the (classical) γ -convergence of a sequence of quasi-open sets is equivalent to the strong convergence of the sequence of resolvent operators for $L = -\Delta + V(x) \cdot \nabla$ on each of the sets.

In Section 2.3, Theorem 2.3.3 and Corollary 2.3.9, we prove that the principal eigenvalue is well-defined on every quasi-open set $\Omega \subset D$, that is, there exists a (real) eigenvalue $\lambda_1(\Omega, V) \in \mathbb{R}$ of the operator $L = -\Delta + V(x) \cdot \nabla$, such that for any other (complex) eigenvalue $\lambda \in \mathbb{C}$ we have $\lambda_1(\Omega, V) \leq \text{Re } \lambda$. In the same section, we establish the continuity of the functional $\Omega \mapsto \lambda_1(\Omega, V)$ with respect to the γ -convergence (Proposition 2.3.7) and the fact that the principal eigenvalue is decreasing with respect to the set inclusion (Remark 2.3.2).

In Section 2.4 we prove our main existence results. The existence of the optimal set for a fixed vector field V (Theorem 2.1.5) follows by the classical Buttazzo-Dal Maso theorem (Theorem 2.2.5). We give the precise statement in Theorem 2.4.1. The proof of Theorem 2.1.1 requires a more refined argument. The reason is the following: consider a (minimizing) sequence (V_n, Ω_n) of vector fields V_n and quasi-open sets Ω_n with eigenfunctions $u_n \in H_0^1(\Omega_n)$ of $L_n = -\Delta + V_n \cdot \nabla$, solutions of

$$-\Delta u_n + V_n \cdot \nabla u_n = \lambda_1(\Omega_n, V_n) u_n \quad \text{in } \Omega_n, \quad \int_D u_n^2 dx = 1, \quad u_n \in H_0^1(\Omega_n).$$

Let us suppose for simplicity that: Ω_n γ -converge to a quasi-open set Ω ; u_n converge to a function $u \in H_0^1(\Omega)$ both strongly $L^2(D)$ and weakly $H_0^1(D)$; V_n converge weakly (in $L^2(D)$) to some $V \in L^\infty(D; \mathbb{R}^d)$. Now, the limit function u solves a PDE in Ω , which involves the (weak) limit of the term $V_n \cdot \nabla u_n$, but a priori this might be different from $V \cdot \nabla u$. In order to solve this issue, in Section 2.4, we first prove that, on any fixed quasi-open set Ω , there exists an optimal vector field (see Theorem 2.4.2). We then replace the vector fields V_n of the minimizing sequence (V_n, Ω_n) by the optimal vector field V_n^* on each domain. Finally, we use the precise expression of V_n^* to prove that the limit function u is an eigenfunction of $-\Delta + V \cdot \nabla$ on Ω and we obtain that $\lambda_1(\Omega_n, V_n^*)$ converges to $\lambda_1(\Omega, V^*)$, which concludes the proof (see Theorem 2.4.3). We cannot apply the same argument for Theorem 2.1.2, since the optimal vector field might not be a gradient. On the other hand, for gradient vector fields the first eigenvalue is a variational functional, namely

$$\lambda_1(\Omega, \nabla\Phi) = \min_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_D e^{-\Phi} |\nabla u|^2 dx}{\int_D e^{-\Phi} u^2 dx},$$

and the existence of an optimal set can be obtained directly (see Theorem 2.4.5).

In Section 2.5, for a fixed drift $V = \nabla\Phi$, we prove the regularity of the optimal sets for $\lambda_1(\cdot, \nabla\Phi)$ (Theorem 2.1.5). In particular, this implies the regularity of the optimal sets in the case when both the set Ω and the vector field $\nabla\Phi$ may vary (see Theorem 2.1.2). Our argument relies in an essential way on the variational formulation of $\lambda_1(\Omega, V)$. More precisely, we show (see Lemma 2.5.1) that, if $V = \nabla\Phi$ is fixed and $\Omega \subset D$ is a solution of (2.1.8), then the corresponding eigenfunction solves the free boundary problem

$$\min \left\{ \int_D e^{-\Phi} |\nabla u|^2 dx : u \in H_0^1(D), u \geq 0, |\{u \neq 0\}| \leq m, \int_D e^{-\Phi} u^2 dx = 1 \right\}. \quad (2.1.9)$$

This is a one-phase free boundary problem, similar to the one studied in the seminal paper of Alt and Caffarelli [2] on the local minimizers of the one-phase functional

$$u \mapsto \int |\nabla u|^2 dx + |\{u > 0\}|.$$

Nevertheless, there are four differences with respect to the classical one-phase problem [2].

- (i) the presence of the variable coefficient $e^{-\Phi}$ in the functional;
- (ii) the presence of the integral constraint $\int e^{-\Phi} u^2 dx = 1$;
- (iii) the presence of the measure constraint $|\{u > 0\}| \leq m$;
- (iv) the presence of the inclusion constraint $\{u > 0\} \subset D$ (equivalent to $u \in H_0^1(D)$).

The variable coefficient $e^{-\Phi}$ introduces several technical difficulties, but does not have an influence on the overall strategy. The issues with the integral constraint are of similar nature. In fact, we are able to deal with this term (see Subsection 2.5.3 and Remark 2.5.9) by reformulating the free boundary problem (2.1.9) in terms of the functional

$$J(v) := \int_D |\nabla v|^2 e^{-\Phi} dx - \lambda_m \int_D v^2 e^{-\Phi} dx,$$

where λ_m is the value of the minimum in (2.1.9). In fact, one easily checks that, if u is a solution of (2.1.9), then u is also a solution to the free boundary problem

$$J(u) \leq J(v) \quad \text{for every } v \in H_0^1(D) \quad \text{such that} \quad |\Omega_v| \leq m, \quad (2.1.10)$$

where, for any function v , we set $\Omega_v := \{v > 0\} = \{x \in \Omega : v(x) > 0\}$.

The measure constraint in free boundary problems first appeared in the work of Aguilera, Alt and Caffarelli [1]. In fact, it is not hard to check that, at least formally, the solution u should satisfy the optimality condition

$$|\nabla u| = \sqrt{\Lambda_u e^{-\Phi}} \quad \text{on the free boundary} \quad \partial\Omega_u \cap D,$$

where Λ_u is a Lagrange multiplier formally arising in the minimization of the functional $J(u)$ under the constraint $|\Omega_u| = m$ (see Subsection 2.5.4). Thus, at least formally, there is no difference between the classical one-phase free boundary problem and the problem with a measure constraint. In practice, the measure constraint significantly complicates the situation. In fact, the Lagrange multiplier Λ_u arises by applying internal variation to the function u , which by itself cannot be used to deduce even the basic qualitative properties of the solution u as, for instance, the Lipschitz continuity and the non-degeneracy (in other words, at the moment, the regularity of the stationary free boundaries is not known). Our approach is different from the one in [1] and is inspired by the works of Briançon-Lamboley [11] and Briançon [9]. In fact, we aim to transform the problem (2.1.10) into

$$J(u) + \Lambda_u |\Omega_u| \leq J(v) + \Lambda_u |\Omega_v| \quad \text{for every } v \in H_0^1(D). \quad (2.1.11)$$

Now, it is not possible to re-write (2.1.10) precisely in this form. Instead, we prove that

$$J(u) - J(v) \leq \begin{cases} (\Lambda_u + \varepsilon)(|\Omega_v| - |\Omega_u|) & \text{for every } v \in H_0^1(D) \text{ such that } |\Omega_v| \geq m; \\ (\Lambda_u - \varepsilon)(|\Omega_v| - |\Omega_u|) & \text{for every } v \in H_0^1(D) \text{ such that } |\Omega_v| \leq m; \end{cases} \quad (2.1.12)$$

where the constant ε improves at small scales, that is, if we consider competitors v that differ from u only in a small ball of radius r , then ε can be chosen in a function of r , $\varepsilon = \varepsilon(r)$, which is such that $\varepsilon(r) \rightarrow 0$ as $r \rightarrow 0$. In this part of the proof (Subsection 2.5.5) we follow the analysis of [11], except in one fundamental point. In fact, the approach of Briançon and Lamboley requires that the Lagrange multiplier Λ_u is not vanishing, which is not a priori known (see Proposition 2.5.12); in [11] the issue is solved by the method in [9]. In this paper, we give a different argument to prove that the Lagrange multiplier is non trivial. Our approach is based on the Almgren monotonicity formula, and the fact that it implies the non-degeneracy of the solution u . We give the proof in the appendix, since the argument is very general (based only on the stationarity condition) and might be of independent interest. We also notice that this simplifies the proof of (2.1.12) and reduces it to three fundamental steps (see Theorem 2.5.16).

Our proof of Theorem 2.1.5 is general and can be applied to the classical one-phase problem [2], to the one-phase problem with measure constraint [1] and to shape optimization problems as for instance the one of [11]. Our approach is different from (and alternative to) the one of [2], [1] and [11], as we do not use the regularity result of Alt and Caffarelli [2]. In fact, in order to prove the regularity of the flat free boundaries (Subsection 2.5.9), we prove that the optimality condition on the free boundary holds in viscosity sense (see Lemma 2.5.30) and then we apply the general results of De Silva [36] (for the regularity of the free boundary $\partial\Omega_u \cap D$) and the recent result of Chang-Lara and Savin [26] (for the regularity at the contact points $\partial\Omega_u \cap \partial D$). Finally, the estimate on the dimension of the singular set (Subsection 2.5.10) is a consequence of the Weiss' (quasi-)monotonicity formula (Lemma 2.5.37).

2.2 Preliminaries

In this section we recall the main definitions and the properties of the quasi-open sets, the γ -convergence and the weak- γ -convergence.

2.2.1 Capacity, quasi-open sets and quasi-continuous functions

The *capacity* of a set $E \subset \mathbb{R}^d$ is defined as

$$\text{cap}(E) := \inf \{ \|u\|_{H^1}^2 : u \in H^1(\mathbb{R}^d), u \geq 1 \text{ in a neighborhood of } E \},$$

where $H^1(\mathbb{R}^d)$ is the Sobolev space equipped with the norm $\|u\|_{H^1}^2 = \int_{\mathbb{R}^d} (|\nabla u|^2 + u^2) dx$.

We say that a property holds *quasi-everywhere* (q.e.) if it holds on the complementary of a set of zero capacity.

A set $\Omega \subset \mathbb{R}^d$ is said to be *quasi-open* if there exists a decreasing sequence $(\omega_n)_{n \geq 1}$ of open sets such that, for every $n \geq 1$, $\Omega \cup \omega_n$ is an open set and $\lim_{n \rightarrow \infty} \text{cap}(\omega_n) = 0$.

A function $u : \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be *quasi-continuous* if there exists a decreasing sequence $(\omega_n)_{n \geq 1}$ of open sets such that $\lim_{n \rightarrow \infty} \text{cap}(\omega_n) = 0$ and the restriction of u to $\mathbb{R}^d \setminus \omega_n$ is continuous.

It is well-known (see for instance [44, Theorem 1, Section 4.8]) that, for every $u \in H^1(\mathbb{R}^d)$, there exists a quasi-continuous representative \tilde{u} of u , which is unique up to a set of zero capacity. From now on we will identify a function $u \in H^1(\mathbb{R}^d)$ with its quasi-continuous representative. We note that, by definition of a quasi-open set and a quasi-continuous function, for every $u \in$

$H^1(\mathbb{R}^d)$, the set $\Omega_u := \{u > 0\} = \{x \in \mathbb{R}^d \mid u(x) > 0\}$ is a quasi-open set ([61, Proposition 3.3.41]). On the other hand, for every quasi-open set Ω , there exists a function $u \in H^1(\mathbb{R}^d)$ such that $\Omega = \Omega_u$ up to a set of zero capacity that is, the quasi-open sets are superlevel sets of Sobolev functions.

For any set $E \subset \mathbb{R}^d$, the Sobolev space $H_0^1(E) \subset H^1(\mathbb{R}^d)$ is defined as

$$H_0^1(E) := \{u \in H^1(\mathbb{R}^d) : u = 0 \text{ q.e. in } \mathbb{R}^d \setminus E\}.$$

Note that, whenever E is open, this definition coincides with the usual definition of $H_0^1(E)$ as the closure of $C_c^\infty(E)$ with respect to the norm $\|\cdot\|_{H^1}$, $C_c^\infty(E)$ being the set of smooth functions compactly supported in E (see for instance [61, Theorem 3.3.42]). For any set $E \subset \mathbb{R}^d$ there is a quasi-open set $\tilde{E} \subset \mathbb{R}^d$ such that $\text{cap}(\tilde{E} \setminus E) = 0$ and $H_0^1(\tilde{E}) = H_0^1(E)$. Roughly speaking, the quasi-open sets are the natural domains for the Sobolev space H_0^1 . We notice that, for every quasi-open set E , $H_0^1(E)$ is a closed subspace of $H^1(\mathbb{R}^d)$; if $E_1 \subset E_2$ are two quasi-open sets, then $H_0^1(E_1) \subset H_0^1(E_2)$ and the two sets E_1 and E_2 coincide q.e. if and only if $H_0^1(E_1) = H_0^1(E_2)$.

2.2.2 PDEs on quasi-open sets

Let $D \subset \mathbb{R}^d$ be a given open set and $\Omega \subset D$ be a quasi-open set of finite Lebesgue measure. For every quasi-open set $\Omega \subset D$ and every function $f \in L^2(\Omega)$, the Lax-Milgram theorem and the Poincaré inequality ensure that there is a unique solution $u \in H_0^1(\Omega)$ of the problem

$$-\Delta u = f \quad \text{in } \Omega, \quad u \in H_0^1(\Omega),$$

where the PDE is intended in the weak sense

$$\int_{\Omega} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} f \varphi \, dx, \quad \text{for every } \varphi \in H_0^1(\Omega).$$

In particular, taking $u = \varphi$, we notice that $\|\nabla u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)}$. Now since Ω has a finite Lebesgue measure, there is a constant C_Ω such that $\|u\|_{H^1} \leq C_\Omega \|\nabla u\|_{L^2}$ for every $u \in H_0^1(\Omega)$. Thus, we get that $\|u\|_{H^1} \leq C_\Omega \|f\|_{L^2}$.

The resolvent operator $R_\Omega^{-\Delta} : L^2(D) \rightarrow L^2(D)$ is defined as $R_\Omega^{-\Delta}(f) := u$ and is a linear, continuous, self-adjoint, positive operator such that $R_\Omega^{-\Delta}(L^2(D)) \subset H_0^1(\Omega)$. Moreover, thanks to the compact embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, the resolvent $R_\Omega^{-\Delta}$ is also compact.

The usual comparison and weak maximum principles hold in this setting. Precisely, we have:

- if $f \in L^2(D)$ is a positive function and $\Omega_1 \subset \Omega_2 \subset D$ are two quasi-open sets, then $w_{\Omega_1} \leq w_{\Omega_2}$.
- if Ω is a quasi-open set and $f, g \in L^2(\Omega)$ are such that $f \leq g$ in Ω , then $R_\Omega^{-\Delta}(f) \leq R_\Omega^{-\Delta}(g)$.

In the sequel we denote by w_Ω (and sometimes also by $R_\Omega^{-\Delta}(1)$) the solution of

$$-\Delta w_\Omega = 1 \quad \text{in } \Omega, \quad w_\Omega \in H_0^1(\Omega).$$

This function is sometimes called *torsion* or *energy* function and is useful, in particular, to define the topology of the γ -convergence on the family of quasi-open sets, which is the purpose of the next section. In the following proposition we summarize the main properties of the function w_Ω (see for instance [90, Proposition 3.50, Remark 3.53, Lemma 3.125, Proposition 3.72]).

Proposition 2.2.1 (Properties of the torsion function w_Ω).

1. There is a dimensional constant $C_d > 0$ such that

$$\|\nabla w_\Omega\|_{L^2} \leq C_d |\Omega|^{\frac{d+2}{2d}} \quad \text{and} \quad \|w_\Omega\|_{L^\infty} \leq C_d |\Omega|^{2/d}. \quad (2.2.1)$$

2. Let $\Omega_1, \Omega_2 \subset D$ be two quasi-open sets. Then we have the estimate

$$\int_D (w_{\Omega_1} - w_{\Omega_1 \setminus \Omega_2}) dx \leq \text{cap}(\Omega_2) \|w_{\Omega_1}\|_{L^\infty(\Omega_1)}^2. \quad (2.2.2)$$

3. $H_0^1(\Omega) = H_0^1(\{w_\Omega > 0\})$. In particular, $\Omega = \{w_\Omega > 0\}$ up to a set of zero capacity.

In the sequel we make the convention to extend to D any vector field $V \in L^\infty(\Omega, \mathbb{R}^d)$ and any function $u \in H_0^1(\Omega)$ by letting it equal to 0 on $D \setminus \Omega$ so that $V \in L^\infty(D, \mathbb{R}^d)$ and $u \in H_0^1(D)$.

We notice that, given a drift $V \in L^\infty(\Omega, \mathbb{R}^d)$, the bilinear form associated to the operator $L = -\Delta + V \cdot \nabla$ may not be coercive on $H_0^1(\Omega)$. Thus, in order to define the resolvent of $L = -\Delta + V \cdot \nabla$, we consider a large enough constant $c > 0$ (depending only on $\|V\|_{L^\infty(\Omega)}$), for which there exists a positive constant $\delta > 0$ such that

$$\delta \int_D (|\nabla u|^2 + u^2) dx \leq \int_D (|\nabla u|^2 + (V \cdot \nabla u)u + cu^2) dx, \quad \text{for every } u \in H_0^1(\Omega). \quad (2.2.3)$$

The bilinear form associated to the operator $L_c = L + c$ is hence coercive on $H_0^1(\Omega)$. Note that

$$\text{if } \|V\|_{L^\infty} \leq \tau, \quad \text{then we can take any } 0 < \delta < 1 \quad \text{and} \quad c \geq \delta + \frac{\tau^2}{4(1-\delta)}.$$

Therefore, thanks to Lax Milgram theorem, we define the resolvent $R_\Omega^{L_c} : L^2(D) \rightarrow L^2(D)$ as the compact (non self-adjoint) operator, which maps $f \in L^2(\Omega)$ to the unique solution of the problem

$$L_c u = f \quad \text{in } \Omega, \quad u \in H_0^1(\Omega),$$

which is intended in the weak sense

$$\int_\Omega (\nabla u \cdot \nabla \varphi + (V \cdot \nabla u)\varphi + cu\varphi) dx = \int_\Omega f\varphi dx, \quad \text{for every } \varphi \in H_0^1(\Omega).$$

2.2.3 The γ -convergence and the weak- γ -convergence

In this subsection we briefly recall the definition and the main properties of the γ -convergence of (quasi-open) sets.

Definition 2.2.2 (γ -convergence and weak- γ -convergence). Let $D \subset \mathbb{R}^d$ be a given open set of finite Lebesgue measure, $(\Omega_n)_{n \geq 1}$ be a sequence of quasi-open sets and let Ω be a quasi-open set, all included in D . We say that

- Ω_n γ -converges to Ω , if w_{Ω_n} converges to w_Ω strongly in $L^2(D)$;
- Ω_n weak- γ -converges to Ω , if there exists $w \in H_0^1(D)$ such that $\Omega = \{w > 0\}$ and w_{Ω_n} converges to w in $L^2(D)$.

Though the γ -convergence is not compact on the family of quasi-open sets (see for instance [27] and [61, § 3.2.6] for an example), it is easy to see that the weak- γ -convergence is: by (2.2.1), up to a subsequence, w_{Ω_n} weakly converges in $H_0^1(D)$ to some $w \in H_0^1(D)$ and hence Ω_n weak- γ -converges to the quasi-open set $\Omega := \{w > 0\}$. To deal with the non-compactness of the γ -convergence we will use the following Lemma (see for example [20] and [61, Lemma 4.7.11]).

Lemma 2.2.3. Let $(\Omega_n)_{n \geq 1} \subset D$ be a sequence of quasi-open sets that weak- γ -converges to the quasi-open set $\Omega \subset D$. Then there exists a subsequence of $(\Omega_n)_{n \geq 1}$, still denoted by $(\Omega_n)_{n \geq 1}$, and a sequence $(\tilde{\Omega}_n)_{n \geq 1} \subset D$ of quasi-open sets satisfying $\Omega_n \subset \tilde{\Omega}_n$, such that $\tilde{\Omega}_n$ γ -converges to Ω .

The following lemma is a direct consequence of the definition of the weak- γ -convergence and the fact that for every quasi-open set $\Omega = \{w_\Omega > 0\}$ (the detailed proof can be found for example in [20] and [90, Lemma 2.2.21]).

Lemma 2.2.4 (Lower semi-continuity of the Lebesgue measure). *Let $(\Omega_n)_{n \geq 1}$ be a sequence of quasi-open sets in D weak- γ -converging to $\Omega \subset D$, then $|\Omega| \leq \liminf_{n \rightarrow +\infty} |\Omega_n|$.*

As was shown in [13] and [20], the following theorem, first proved in [21], is an immediate consequence of Lemma 2.2.3 and Lemma 2.2.4.

Theorem 2.2.5 (Buttazzo-Dal Maso [21]). *Let \mathcal{F} be a functional on the quasi-open sets, which is:*

- *decreasing with respect to the inclusion of sets;*
- *lower semi-continuous with respect to the γ -convergence.*

Then, for every bounded open set $D \subset \mathbb{R}^d$ and every $0 < m \leq |D|$, the shape optimization problem

$$\min \left\{ \mathcal{F}(\Omega) : \Omega \text{ quasi-open, } \Omega \subset D, |\Omega| \leq m \right\}$$

has a solution.

We will not be able to apply directly Theorem 2.2.5 to establish the existence of optimal sets for both the problems (2.1.7) and (2.1.1) in the class of quasi-open sets. Instead, in Section 2.4, we will use an argument based only on Lemma 2.2.3 and Lemma 2.2.4, but before that we will need to extend the definition of $\lambda_1(\Omega, V)$ to the class of quasi-open sets. We do this in Section 2.3, where we will use several times the following approximation result.

Lemma 2.2.6 (Approximation with open and smooth sets). *Let $\Omega \subset D$ be a quasi-open set. Then:*

- (1) *there is a sequence of open sets $(\Omega_n)_{n \geq 1}$ that γ -converges to Ω and is such that $\Omega \subset \Omega_n \subset D$ and $\lim_{n \rightarrow +\infty} |\Omega_n| = |\Omega|$;*
- (2) *there is a sequence $(\Omega_n)_{n \geq 1}$ of smooth (C^∞) open sets contained in D , that γ -converges to Ω .*

Proof. The result is well-known; here we give the proof for the readers' convenience.

(1) Let $(\omega_n)_{n \geq 1}$ be a sequence of open sets such that $\lim_{n \rightarrow \infty} \text{cap}(\omega_n) = 0$ and $\Omega_n = (\Omega \cup \omega_n) \cap D$ is an open set. Then, (2.2.2) applied to the sets Ω_n and $\omega_n \setminus \Omega$ together with the second estimate in (2.2.1) show that w_{Ω_n} converges to w_Ω in $L^1(D)$. Moreover, up to a subsequence, w_{Ω_n} weakly converges in $H^1(D)$ thanks to the first estimate in (2.2.1). Since the embedding $H_0^1(D) \hookrightarrow L^2(D)$ is compact, there is a subsequence which converges strongly in $L^2(D)$. By uniqueness of the limit in $L^1(D)$, it has to be w_Ω . Thus, w_{Ω_n} converges in $L^2(D)$ to w_Ω and so, Ω_n γ -converges to Ω . Observe also that one has $\lim_{n \rightarrow +\infty} |\Omega_n| = |\Omega|$ since $\lim_{n \rightarrow +\infty} |\omega_n| = 0$.

(2) Firstly, assume that Ω is an open set. Let $(\Omega_n)_{n \geq 1}$ be an increasing sequence of smooth open sets included in Ω which Hausdorff converges to Ω . Then, up to a subsequence, $w_n := w_{\Omega_n}$ weakly converges in $H_0^1(D)$ to some $w \in H_0^1(D)$. But Ω_n, Ω are open sets such that $\Omega_n \subset \Omega$, and since the convergence of Ω_n to Ω is Hausdorff, we can pass to the limit in the equation

$$-\Delta w_n = 1 \quad \text{in } \Omega_n$$

to see that w satisfies

$$-\Delta w = 1 \quad \text{in } \Omega.$$

This also shows that the sequence of norms $\|w_n\|_{H^1(D)}$ converges to $\|w\|_{H^1(D)}$, so that the convergence of w_n to w is strong in $H^1(D)$. Finally, since $\Omega_n \subset \Omega$, we get that $w \in H_0^1(\Omega)$ and hence that $w = w_\Omega$. Therefore, the sequence of smooth open sets Ω_n γ -converges to Ω .

If now Ω is merely a quasi-open set, we can approximate Ω by a sequence of open sets which γ -converges to Ω thanks to (1). Hence, by approximating these open sets by open smooth sets as above, we get a sequence of smooth open sets which γ -converges to Ω . Recall that the topology of the γ -convergence is metrizable (see for example [13]). \square

Remark 2.2.7 (The quasi-open sets cannot be γ -approximated with bigger smooth open sets). In general, we cannot approximate a quasi-open set (or even an open set) $\Omega \subset D$ by a sequence of smooth (say of class C^1) open sets $(\Omega_n)_{n \geq 1}$ which γ -converges to Ω and such that $\Omega_n \supset \Omega$. Indeed, let $(\xi_n)_{n \geq 1}$ be a dense sequence in $\bar{D} = (0, 1)^2 \subset \mathbb{R}^2$ and pick a sequence $(r_n)_{n \geq 1}$ of positive numbers such that $\sum_{n \geq 1} \pi r_n^2 < 1$. Set $\Omega := \cup_{n \geq 1} B_{r_n}(\xi_n) \subset D$. We now claim that if $\Omega_n \supset \Omega$ is a smooth open set, then necessarily $\Omega_n \supset D$. To see this, let $x_0 \in D \subset \bar{\Omega} \subset \bar{\Omega}_n$. Then if $x_0 \in \partial\Omega_n$, there exist $r > 0$ and a smooth, say of class C^1 , function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ such that, up to reorienting the axis, we have $\Omega_n \cap B_r(x_0) = \{x \in B_r(x_0) : x_d > f(x_1, \dots, x_{d-1})\}$. It follows that $B_r(x_0) \setminus \bar{\Omega}_n \subset D$ is a nonempty open set which does not intersect Ω_n . This is in contradiction with $\Omega_n \supset \Omega$ since Ω is a dense open set in D . Hence $x \in \Omega_n$ and this shows that $D \subset \Omega_n$. Now, suppose that $(\Omega_n)_{n \geq 1}$ is a sequence of smooth sets such that $D \supset \Omega_n \supset \Omega$. Then $\Omega_n = D$ for every $n \geq 1$. Furthermore, the weak maximum principle implies $w_\Omega < w_D = w_{\Omega_n}$ in D , where the first inequality is strict since $|\Omega| < |D| = 1$. Therefore, w_{Ω_n} cannot strongly converge to w_Ω in $L^2(D)$.

We now give a characterization of the γ -convergence in terms of convergence of resolvent operators. The following theorem is a generalization of [61, Lemma 4.7.3] for the operator $L_c = L + c$.

Theorem 2.2.8 (γ -convergence and operator convergence). *Let $D \subset \mathbb{R}^d$ be a bounded open set, $(\Omega_n)_{n \geq 1} \subset D$ be a sequence of quasi-open sets and $\Omega \subset D$ be a quasi-open set. The following assertions are equivalent :*

- (1) *the sequence $(\Omega_n)_{n \geq 1}$ γ -converges to Ω ;*
- (2) *for every sequence $(f_n)_{n \geq 1} \in L^2(D)$ weakly converging in $L^2(D)$ to $f \in L^2(D)$, the sequence $(R_{\Omega_n}^{L_c}(f_n))_{n \geq 1}$ converges to $R_\Omega^{L_c}(f)$ strongly in $L^2(D)$;*
- (3) *the sequence of operators $(R_{\Omega_n}^{L_c})_{n \geq 1} \in \mathcal{L}(L^2(D))$ converges to $R_\Omega^{L_c}$ in the operator norm $\|\cdot\|_{\mathcal{L}(L^2(D))}$.*

Proof. It is plain to see that the equivalence between (2) and (3) holds for all sequence of compact operators defined on Hilbert spaces. It then remains to prove that (1) and (2) are equivalent.

(1) \Rightarrow (2). Let $f_n \in L^2(D)$ be a sequence $L^2(D)$ -weakly converging to $f \in L^2(D)$. Then $\|f_n\|_{L^2}$ is uniformly bounded. Moreover, writing $u_n = R_{\Omega_n}^{L_c}(f_n)$ we have

$$\int_D f_n u_n \, dx = \int_D (|\nabla u_n|^2 + (V \cdot \nabla u_n) u_n + c u_n^2) \, dx.$$

Thanks to (2.2.3) this gives

$$\frac{1}{2} \int_D (f_n^2 + u_n^2) \, dx \geq \delta \int_D (|\nabla u_n|^2 + u_n^2) \, dx,$$

and therefore

$$\int_D f_n^2 \, dx \geq (2\delta - 1) \int_D (|\nabla u_n|^2 + u_n^2) \, dx.$$

Taking $\delta \in (1/2, 1)$, this shows that the sequence $\|u_n\|_{H^1(D)}$ is bounded.

Assume now that the conclusion of (2) does not hold. Then there exists $\varepsilon > 0$ such that, up to a subsequence, $\|R_{\Omega_n}^{L_c}(f_n) - R_\Omega^{L_c}(f)\|_{L^2(D)} \geq \varepsilon$. Moreover, up to a subsequence, u_n weakly

converges in $H^1(D)$ to some $u \in H_0^1(D)$, and therefore $g_n = f_n - V \cdot \nabla u_n - cu_n$ weakly converges in $L^2(D)$ to $g = f - V \cdot \nabla u - cu$. Theorem 2.2.8 being true for the Laplacian (see [22, Proposition 3.4]), we conclude that $R_{\Omega_n}^{-\Delta}(g_n)$ strongly converges in $L^2(D)$ to $R_{\Omega}^{-\Delta}(g)$. Thus $R_{\Omega_n}^{Lc}(f_n) = R_{\Omega_n}^{-\Delta}(g_n)$ and $R_{\Omega}^{Lc}(f) = R_{\Omega}^{-\Delta}(g)$ imply that $R_{\Omega_n}^{Lc}(f_n)$ strongly converges in $L^2(D)$ to $R_{\Omega}^{Lc}(f)$, which yields a contradiction and therefore proves (2).

(2) \Rightarrow (1). Let $(f_n)_{n \geq 1} \in L^2(D)$ be a sequence weakly converging in $L^2(D)$ to $f \in L^2(D)$. Set $w_n := R_{\Omega_n}^{-\Delta}(f_n)$ and $w := R_{\Omega}^{-\Delta}(f)$. We claim that $w_n \rightarrow w$ strongly in $L^2(D)$, which, according to [22] and [61, Lemma 4.7.3], implies that Ω_n γ -converges to Ω . Assume by contradiction that it is not the case, and pick up $\varepsilon > 0$ and an increasing function $\varphi : \mathbb{N}^* \rightarrow \mathbb{N}^*$ such that

$$\|w_{\varphi(n)} - w\|_{L^2(D)} \geq \varepsilon \quad \text{for every } n \geq 1. \quad (2.2.4)$$

Since the sequence $(w_n)_{n \geq 1}$ is bounded in $H_0^1(D)$, up to a subsequence, there exists a function $z \in H_0^1(D)$ such that $w_{\varphi(n)}$ converges to z weakly in $H_0^1(D)$ and strongly in $L^2(D)$. Now, since

$$L_c w_n = f_n + V \cdot \nabla w_n + cw_n := g_n \quad \text{in } \Omega_n,$$

and $w_n \in H_0^1(\Omega_n)$, $w_n = R_{\Omega_n}^{Lc}(g_n)$. But $g_{\varphi(n)} \rightharpoonup g := f + V \cdot \nabla z + cz$ weakly in $L^2(D)$, so that, by assumption (2), $w_{\varphi(n)} \rightarrow R_{\Omega}^{Lc}(g)$ strongly in $L^2(D)$. Then the convergence of $w_{\varphi(n)}$ to z yields that $z = R_{\Omega}^{Lc}(g)$, is a solution of

$$L_c z = f + V \cdot \nabla z + cz \quad \text{in } \Omega, \quad z \in H_0^1(\Omega),$$

or, in other words, $z = R_{\Omega}^{-\Delta}(f) = w$. Thus, (2.2.4) provides a contradiction, therefore showing that $w_n \rightarrow w$ strongly in $L^2(D)$, which means that (1) holds. \square

2.3 The principal eigenvalue on quasi-open sets

For a bounded domain $\Omega \subset \mathbb{R}^d$ and $V \in L^\infty(\Omega, \mathbb{R}^d)$, the principal eigenvalue $\lambda_1(\Omega, V)$, of the (non self-adjoint) elliptic operator $L = -\Delta + V \cdot \nabla$ on Ω with Dirichlet boundary condition on $\partial\Omega$, was defined in [6] by

$$\lambda_1(\Omega, V) = \sup \{ \lambda \in \mathbb{R} : \exists \phi \in W^{2,d}(\Omega) \text{ such that } \phi > 0 \text{ and } -L\phi + \lambda\phi \leq 0 \text{ in } \Omega \},$$

where it was proved that $\lambda_1(\Omega, V) \in \mathbb{R}$ has the following properties:

(i) There is a positive eigenfunction $u : \Omega \rightarrow \mathbb{R}$ such that $u \in W_{loc}^{2,p}(\Omega)$, for all $p \in [1, +\infty)$, and

$$Lu = \lambda_1(\Omega, V)u \quad \text{in } \Omega, \quad u \in H_0^1(\Omega), \quad \int_{\Omega} u^2 dx = 1,$$

(see [6, Theorem 2.1]).

(ii) $\lambda_1(\Omega, V) < \text{Re}(\lambda)$ for every eigenvalue $\lambda \neq \lambda_1(\Omega, V)$ of L in Ω (see [6, Theorem 2.3]).

(iii) The functional $\Omega \mapsto \lambda_1(\Omega, V)$ is decreasing with respect to the domain inclusion.

In the sequel we extend the definition of $\lambda_1(\Omega, V)$ to quasi-open sets. We first recall that the definition can be extended to an arbitrary open set $\Omega \subset D$ by

$$\lambda_1(\Omega, V) = \inf \lambda_1(\Omega_n, V),$$

where the infimum is taken over all the connected component Ω_n of Ω . Now, in view of property (iii) above, for any quasi-open set $\Omega \subset D$, we define

$$\lambda_1(\Omega, V) := \sup \{ \lambda_1(\tilde{\Omega}, V) : \tilde{\Omega} \text{ open, } \Omega \subset \tilde{\Omega} \subset D \}. \quad (2.3.1)$$

Remark 2.3.1. Notice that, these two definitions coincide for open sets.

Remark 2.3.2. The functional $\Omega \mapsto \lambda_1(\Omega, V)$, defined on the family of quasi-open sets, is still non-increasing with respect to the set inclusion, that is $\lambda_1(\Omega_2, V) \leq \lambda_1(\Omega_1, V)$, whenever $\Omega_1 \subset \Omega_2$.

We will show that $\lambda_1(\Omega, V)$ is finite and is an eigenvalue of L in Ω satisfying the minimality property (ii). Recall that, for a quasi-open set of finite Lebesgue measure $\Omega \subset D$, we say that $\lambda \in \mathbb{C}$ is an eigenvalue of the operator $L = -\Delta + V \cdot \nabla$ in Ω if there is an eigenfunction $u : \mathbb{R}^d \rightarrow \mathbb{C}$, (weak) solution to the problem

$$-\Delta u + V \cdot \nabla u = \lambda u \quad \text{in } \Omega, \quad u \in H_0^1(\Omega; \mathbb{C}), \quad \int_{\Omega} |u|^2 dx = 1. \quad (2.3.2)$$

Let now $c > 0$ be the constant from Subsection 2.2.2 and $L_c = L + c$. Note that $\lambda \in \mathbb{C}$ is an eigenvalue of L in Ω , if and only if, $\lambda + c$ is an eigenvalue of L_c in Ω . By the argument from Subsection 2.2.2, we have that the bilinear form associated to the operator L_c is coercive and so, $R_{\Omega}^{L_c}$ is a compact operator on $L^2(D)$. In particular, the spectrum is a discrete set of eigenvalues with no accumulation points except zero and $\lambda \in \mathbb{C}$ is an eigenvalue of L in the sense of (2.3.2) if and only if $(\lambda + c)^{-1}$ is an eigenvalue of $R_{\Omega}^{L_c}$.

The following theorem shows that most of the properties of the principal eigenvalue on an open set still hold for $\lambda_1(\Omega, V)$ if $\Omega \subset D$ is merely a quasi-open set.

Theorem 2.3.3 (Definition of the principal eigenvalue on quasi-open sets). *Let D be a bounded open set, $V \in L^\infty(D, \mathbb{R}^d)$ and $\Omega \subset D$ be a non-empty quasi-open set. Then*

(1) $\lambda_1(\Omega, V)$ is well-defined that is, $\lambda_1(\Omega, V) < +\infty$.

(2) $\lambda_1(\Omega, V)$ is an eigenvalue of L in Ω ; there is a (non-trivial) real-valued eigenfunction u such that

$$Lu = \lambda_1(\Omega, V)u \quad \text{in } \Omega, \quad u \in H_0^1(\Omega), \quad \int_{\Omega} u^2 dx = 1.$$

(3) If $\lambda \in \mathbb{C}$ is an eigenvalue of L in Ω , then $\lambda_1(\Omega, V) \leq \text{Re}(\lambda)$.

In order to prove Theorem 2.3.3 we will need the following two lemmas. The key estimate for the proof of Theorem 2.3.3 (1) is contained in the following lemma inspired by [6, Proposition 5.1].

Lemma 2.3.4. *Let $V \in L^\infty(D, \mathbb{R}^d)$ and $\Omega \subset D$ be an open set. Suppose that there is $\tau > 0$ such that $\|V\|_{L^\infty(\Omega)} \leq \tau < 2\sqrt{\lambda_1(\Omega, V)}$. Then*

$$\lambda_1(\Omega, 0) \geq \lambda_1(\Omega, V) - \tau\sqrt{\lambda_1(\Omega, V)}. \quad (2.3.3)$$

Proof. Let us first suppose that Ω is connected. For convenience, set $\lambda := \lambda_1(\Omega, V)$. By the definition of the first eigenvalue of $-\Delta$ on domains, it is enough to find some $\phi > 0$ in Ω such that $-\Delta\phi \geq (\lambda - \tau\sqrt{\lambda})\phi$ in Ω . Since Ω is an open set, from [6, Theorem 2.1], there exists a positive eigenfunction ϕ_V for the first eigenvalue of L in Ω , that is, $\phi_V > 0$ in Ω and $L\phi_V = \lambda\phi_V$. Set $\phi := \phi_V^\alpha$ for some $\alpha \in (0, 1)$ to be chosen later. Then, in Ω , we have

$$\begin{aligned} -\Delta\phi - \lambda\phi &= -\alpha(\Delta\phi_V)\phi_V^{\alpha-1} - \alpha(\alpha-1)|\nabla\phi_V|^2\phi_V^{\alpha-2} - \lambda\phi_V^\alpha \\ &= \left[\lambda(\alpha-1) - \alpha V \cdot \frac{\nabla\phi_V}{\phi_V} + \alpha(1-\alpha)\frac{|\nabla\phi_V|^2}{\phi_V^2} \right] \phi_V^\alpha \\ &\geq \left[\lambda(\alpha-1) - \alpha\tau\frac{|\nabla\phi_V|}{\phi_V} + \alpha(1-\alpha)\frac{|\nabla\phi_V|^2}{\phi_V^2} \right] \phi_V^\alpha. \end{aligned}$$

The function $x \mapsto -\alpha\tau x + \alpha(1-\alpha)x^2$ reaches its minimum at $x = \tau/(2(1-\alpha))$. Therefore, we get

$$-\Delta\phi - \lambda\phi \geq \left[\lambda(\alpha-1) - \alpha\frac{\tau^2}{4(1-\alpha)} \right] \phi_V^\alpha = \left[\lambda(\alpha-1) - \alpha\frac{\tau^2}{4(1-\alpha)} \right] \phi.$$

Since $\alpha \in (0, 1)$ is arbitrary, we can choose it so that it maximizes the term in the brackets of the above estimate, that is, such that $1 - \alpha = \tau/(2\sqrt{\lambda})$. Note that, by hypothesis on τ , we have $\alpha \in (0, 1)$. It follows

$$-\Delta\phi - \lambda\phi \geq \left[-\tau\sqrt{\lambda} + \frac{\tau^2}{4} \right] \phi \geq -\tau\sqrt{\lambda}\phi,$$

which proves the claim in the case when Ω is connected.

In the general case, let $(\Omega_n)_{n \geq 1}$ be the connected components of Ω . Then, for every V , we have

$$\lambda_1(\Omega, V) = \inf_n \lambda_1(\Omega_n, V).$$

Then, we have, for all n ,

$$\lambda_1(\Omega_n, 0) \geq \lambda_1(\Omega_n, V) - \tau\sqrt{\lambda_1(\Omega_n, V)} \geq \lambda_1(\Omega, V) - \tau\sqrt{\lambda_1(\Omega, V)},$$

where the last inequality is due to the fact that $x \mapsto x - \tau\sqrt{x}$ is a non-increasing function on the interval $[\lambda_1(\Omega, V), +\infty)$. \square

The next lemma is a direct consequence of the classical result [64, Theorem 3.16] on the convergence of a spectrum of closed operators with suitable properties. We will use it in the proof of Theorem 2.3.3 (3).

Lemma 2.3.5 (Convergence of the spectra). *Let H be a separable Hilbert space and $(T_n)_{n \geq 1} \in \mathcal{L}(H)$ a sequence of compact operators converging to the compact operator $T \in \mathcal{L}(H)$ in the operator norm $\|\cdot\|_{\mathcal{L}(H)}$. Suppose that $\lambda \in \mathbb{C} \setminus \{0\}$ is an (isolated) eigenvalue of T and let $r > 0$ be such that $B_r(\lambda) \cap \sigma(T) = \{\lambda\}$. Then, there is $n_0 \geq 1$ such that for every $n \geq n_0$ there is an eigenvalue $\lambda_n \in \sigma(T_n) \cap B_{r/2}(\lambda)$.*

We are now in position to prove Theorem 2.3.3.

Proof of Theorem 2.3.3. Consider a maximizing sequence $(\Omega_n)_{n \geq 1}$ for (2.3.1), that is, a sequence of open sets $(\Omega_n)_{n \geq 1}$ such that

$$\lambda_1(\Omega, V) = \lim_{n \rightarrow \infty} \lambda_1(\Omega_n, V) \quad \text{and} \quad \Omega \subset \Omega_n \subset D \quad \text{for every } n \geq 1.$$

We first show that we can assume that Ω_n γ -converges to Ω . Let ω_n be a sequence of open sets such that $\Omega \cup \omega_n$ is open and $\text{cap}(\omega_n) \rightarrow 0$. We set $\tilde{\Omega}_n := \Omega_n \cap (\Omega \cup \omega_n) = \Omega \cup (\omega_n \cap \Omega_n)$. By (2.3.1) and the inclusion $\Omega \subset \tilde{\Omega}_n \subset \Omega_n$ we have $\lambda_1(\Omega_n, V) \leq \lambda_1(\tilde{\Omega}_n, V) \leq \lambda_1(\Omega, V)$, so we get

$$\lambda_1(\Omega, V) = \lim_{n \rightarrow \infty} \lambda_1(\tilde{\Omega}_n, V) \quad \text{and} \quad \Omega \subset \tilde{\Omega}_n \subset D \quad \text{for every } n \geq 1.$$

Thus, we may consider $\tilde{\Omega}_n$ in place of Ω_n as a maximizing sequence for (2.3.1). Finally, as in Lemma 2.2.6, $\tilde{\Omega}_n$ γ -converges to Ω thanks to the estimate (2.2.2) applied to the sets $\tilde{\Omega}_n$ and $\Omega_n \cap \omega_n$.

We now prove claim (1). Indeed, suppose by contradiction that

$$\lambda_1(\Omega, V) = \lim_{n \rightarrow \infty} \lambda_1(\Omega_n, V) = +\infty.$$

Then, by Lemma 2.3.4 we have that

$$\lim_{n \rightarrow \infty} \lambda_1(\Omega_n, 0) = +\infty.$$

Now, since $\Omega \mapsto \lambda_1(\Omega, 0)$ is decreasing and $\Omega \subset \Omega_n$, we get that $\lambda_1(\Omega, 0) = +\infty$. By the variational characterization

$$\lambda_1(\Omega, 0) = \min_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_{\Omega} |\nabla u|^2 dx}{\int_{\Omega} u^2 dx},$$

we get that $H_0^1(\Omega) = \{0\}$, which implies that $\Omega = \emptyset$ (or, equivalently, $\text{cap } \Omega = 0$), which is absurd.

We now prove (2). Let $u_n \in H_0^1(\Omega_n) \subset H_0^1(D)$ be the (normalized) eigenfunction associated to $\lambda_1(\Omega_n, V)$. Then we have

$$L_c u_n = (\lambda_1(\Omega_n, V) + c) u_n \quad \text{in } \Omega_n, \quad u_n \in H_0^1(\Omega_n), \quad \int_{\Omega_n} u_n^2 dx = 1.$$

Multiplying the above equation by u_n , integrating over Ω_n and using the estimate (2.2.3) we get

$$\delta \|u_n\|_{H^1}^2 \leq \lambda_1(\Omega_n, V) + c \quad \text{for every } n \geq 1.$$

In particular, since $\lambda_1(\Omega, V) < \infty$, we get that $(u_n)_{n \geq 1}$ is uniformly bounded in $H_0^1(D)$ and so, up to a subsequence, we may assume that u_n converges, weakly in $H_0^1(D)$ and strongly in $L^2(D)$, to a function $u \in H_0^1(D)$. Moreover, Ω_n γ -converges to Ω and so, $R_{\Omega_n}^{L_c}$ converges in norm to $R_{\Omega}^{L_c}$. Thus,

$$u = \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} (\lambda_1(\Omega_n, V) + c) R_{\Omega_n}^{L_c}(u_n) = (\lambda_1(\Omega, V) + c) R_{\Omega}^{L_c}(u),$$

which concludes the proof of (2).

Proof of (3). Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue of L on Ω such that $\text{Re}(\lambda) < \lambda_1(\Omega, V)$. Then, $(\lambda + c)^{-1} \in \mathbb{C}$ is a (non-zero) eigenvalue of the compact operator $R_{\Omega}^{L_c}$. Applying Lemma 2.3.5, we can assume that for n large enough, there is an eigenvalue λ_n of L on Ω_n such that $\text{Re}(\lambda_n) < \lambda_1(\Omega_n, V)$, which is a contradiction with [6, Theorem 2.3]. \square

Remark 2.3.6 (On the sign of the first eigenfunction). In particular, as a consequence of the proof of Theorem 2.3.3 (2), there is an eigenfunction u of L on the quasi-open set Ω , which is non-negative, being the limit of non-negative functions. We notice that u does not need to be strictly positive as Ω might be disconnected.

We conclude this section with a proposition on the continuity of $\lambda_1(\cdot, V)$ with respect to the γ -convergence.

Proposition 2.3.7 (γ -continuity of $\lambda_1(\cdot, V)$). *Let $D \subset \mathbb{R}^d$ be a bounded open set, $V \in L^\infty(D; \mathbb{R}^d)$ be a fixed vector field, and $(\Omega_n)_{n \geq 1} \subset D$ be a sequence of quasi-open sets that γ -converges to the quasi-open set $\Omega \subset D$. Then*

$$\lambda_1(\Omega, V) = \begin{cases} \lim_{n \rightarrow \infty} \lambda_1(\Omega_n, V), & \text{if } \Omega \neq \emptyset, \\ +\infty, & \text{if } \Omega = \emptyset. \end{cases}$$

Proof. Let $\tau = \|V\|_{L^\infty(D)}$ and δ and c be as in (2.2.3). Set $L_c = L + c$.

Suppose first that the sequence $(\lambda_1(\Omega_n, V))_{n \geq 1}$ is bounded. Reasoning as in the proof of Theorem 2.3.3 (2) we get that, up to a subsequence, $\lambda_1(\Omega_n, V)$ converges to an eigenvalue $\lambda \in \mathbb{R}$ of L on Ω . Now, by the argument of Theorem 2.3.3 (3) and Lemma 2.3.5, we have that λ satisfies the property (3) of Theorem 2.3.3, so $\lambda = \lambda_1(\Omega, V)$, which concludes the proof since the sequence $(\lambda_1(\Omega_n, V))_{n \geq 1}$ is bounded.

Next, suppose that the sequence $(\lambda_1(\Omega_n, V))_{n \geq 1}$ is unbounded. Applying Lemma 2.3.4, we get that, up to a subsequence, $\lim_{n \rightarrow \infty} \lambda_1(\Omega_n, 0) = +\infty$. Since $R_{\Omega_n}^{-\Delta}$ are self-adjoint compact operators, we get that

$$\lim_{n \rightarrow \infty} \|R_{\Omega_n}^{-\Delta}\|_{\mathcal{L}(L^2(D))} = \lim_{n \rightarrow \infty} \frac{1}{\lambda_1(\Omega_n, 0)} = 0.$$

Finally, the γ -convergence gives that $R_{\Omega}^{-\Delta} \equiv 0$ and so, $H_0^1(\Omega) = \{0\}$ and $\text{cap}(\Omega) = 0$. \square

Remark 2.3.8. In view of Proposition 2.3.7 we set $\lambda_1(\emptyset, V) = +\infty$.

Putting together Theorem 2.3.3 and Proposition 2.3.7 we obtain the following result.

Corollary 2.3.9 (Equivalent definition of the principal eigenvalue on quasi-open sets). *Let Ω be a bounded quasi-open set and $V \in L^\infty(\Omega; \mathbb{R}^d)$. Then, there is an eigenvalue $\lambda_1(\Omega, V) \in \mathbb{R}$ of $L = -\Delta + V \cdot \nabla$ in Ω such that:*

$$\begin{aligned} \lambda_1(\Omega, V) &= \min \{ \operatorname{Re} \lambda : \lambda \in \mathbb{C} \text{ is an eigenvalue of } L \text{ on } \Omega \} \\ &= \sup \{ \lambda_1(\tilde{\Omega}, V) : \tilde{\Omega} \text{ is an open set containing } \Omega \} \\ &= \lim_{n \rightarrow \infty} \lambda_1(\Omega_n, V), \text{ where } (\Omega_n)_{n \geq 1} \text{ is any sequence} \\ &\quad \text{of (smooth) open sets } \gamma\text{-converging to } \Omega. \end{aligned}$$

Proof. The first two inequalities are due to Theorem 2.3.3. For the third one it is sufficient to note that for every quasi-open set Ω there is a sequence of smooth open sets γ -converging to Ω and to apply Proposition 2.3.7. \square

Remark 2.3.10 (Faber-Krahn with drift for quasi-open sets). As further consequence of Corollary 2.3.9 we can extend the Hamel-Nadirashvili-Russ inequality to the class of (bounded) quasi-open sets. Precisely, for every bounded quasi-open set $\Omega \subset \mathbb{R}^d$ with $|\Omega| > 0$ and every $\tau > 0$, we have

$$\lambda_1 \left(B, \tau \frac{x}{|x|} \right) \leq \lambda_1(\Omega, V) \quad \text{for every } V \in L^\infty(\Omega; \mathbb{R}^d) \quad \text{with } \|V\|_{L^\infty} \leq \tau, \quad (2.3.4)$$

where B is the ball centered in zero of the same Lebesgue measure as Ω . Indeed, let $\Omega \subset \mathbb{R}^d$ be a bounded quasi-open set and $V \in L^\infty(\Omega; \mathbb{R}^d)$ be such that $\|V\|_{L^\infty} \leq \tau$ (in what follows we assume that V is extended by zero outside Ω). Let $(\Omega_n)_{n \geq 1}$ be a sequence of bounded open sets which γ -converges to Ω and such that $|\Omega_n|$ converges to $|\Omega|$ (see Lemma approx qo). Denote by B_{r_n} (resp. B) the ball centred at 0 whose Lebesgue measure is $|B_{r_n}| = |\Omega_n|$ (resp. $|B| = |\Omega|$). Then, since Ω_n is an open set, we have $\lambda_1(B_{r_n}, \tau e_r) \leq \lambda_1(\Omega_n, v)$ thanks to [52, Remark 6.10]. Moreover, B_{r_n} γ -converges to B (since $|B_{r_n}| \rightarrow |B|$ and hence B_{r_n} converges to B in the sense of Hausdorff; see [61, Proposition 3.4.2]). Therefore, Corollary 2.3.9 implies that $\lambda_1(B_{r_n}, \tau e_r)$ converges to $\lambda_1(B, \tau e_r)$ and similarly, $\lambda_1(\Omega_n, V) \rightarrow \lambda_1(\Omega, V)$. Passing to the limit we get (2.3.4).

2.4 Existence of optimal domains

In this section we prove the existence of optimal domains for the cost functional $\lambda_1(\Omega, V)$. We first consider the case when the drift V is fixed, for which the existence follows by the result of the previous section and a classical theorem in shape optimization. The case when both the domain Ω and the drift V may vary requires more careful analysis and the rest of the section is dedicated to the proof of Theorem 2.4.3. In the end of the section (Theorem 2.4.5) we also prove that a solution (Ω, V) exists also in the class of vector fields V obtained as gradients of Lipschitz continuous functions.

Theorem 2.4.1 (Existence of optimal sets for a fixed vector field). *Let $D \subset \mathbb{R}^d$ be a bounded open set and $V \in L^\infty(D; \mathbb{R}^d)$. Then, for every $0 < m \leq |D|$, there is an optimal domain, solution of the problem (2.1.8).*

Proof. By Remark 2.3.2 and Proposition 2.3.7 we get that $\Omega \mapsto \lambda_1(\Omega, V)$ is γ -continuous and decreasing with respect to the set inclusion. The claim follows by Theorem 2.2.5. \square

2.4.1 Optimal drifts on a fixed domain

Let $\Omega \subset \mathbb{R}^d$ be a fixed bounded quasi-open set and $\tau > 0$ be given. We consider the following variational minimization problem

$$\min \{ \lambda_1(\Omega, V) : V \in L^\infty(\Omega, \mathbb{R}^d), \|V\|_{L^\infty} \leq \tau \}. \quad (2.4.1)$$

Theorem 2.4.2 (Optimal vector field on a fixed quasi-open set). *The problem (2.4.1) has a solution, which satisfies*

$$V_*(x) = -\tau \frac{\nabla u(x)}{|\nabla u(x)|} \quad \text{if } |\nabla u(x)| \neq 0; \quad V_*(x) = 0 \quad \text{if } |\nabla u(x)| = 0, \quad (2.4.2)$$

where u is the eigenfunction of $L = -\Delta + V_* \cdot \nabla$ in Ω , associated to the eigenvalue $\lambda_1(\Omega, V_*)$.

Proof. Let $(\Omega_n)_{n \geq 1}$ be a sequence of smooth, say of class $C^{2,\alpha}$ for some $0 < \alpha < 1$, open sets which γ -converges to Ω (see Remark 2.2.6). Since Ω_n is smooth, we already know (see [51, theorem 1.5]) that the problem (2.4.1) for the fixed domain Ω_n has a solution V_n . Moreover, if u_n is the associated eigenfunction of $-\Delta + V_n \cdot \nabla$ in Ω_n , that is, u_n is defined by

$$-\Delta u_n + V_n \cdot \nabla u_n = \lambda_1(\Omega_n, V_n) u_n \quad \text{in } \Omega_n, \quad u_n \in H_0^1(\Omega_n), \quad \int_{\Omega_n} u_n^2 dx = 1,$$

then the optimal vector field V_n is unique and is given by

$$V_n(x) = \begin{cases} -\tau \frac{\nabla u_n(x)}{|\nabla u_n(x)|} & \text{if } |\nabla u_n(x)| \neq 0, \\ 0 & \text{if } |\nabla u_n(x)| = 0. \end{cases}$$

In particular, u_n is a solution of

$$-\Delta u_n - \tau |\nabla u_n| = \lambda_1(\Omega_n, V_n) u_n \quad \text{in } \Omega_n, \quad u_n \in H_0^1(\Omega_n), \quad \int_{\Omega_n} u_n^2 dx = 1.$$

We first claim that the sequence $(\lambda_1(\Omega_n, V_n))_{n \geq 1}$ is bounded. Indeed, by optimality of V_n , one has $\lambda_1(\Omega_n, V_n) \leq \lambda_1(\Omega_n, 0)$, which is nothing but the principal eigenvalue of $-\Delta$ on Ω_n with Dirichlet boundary condition. But since Ω_n γ -converges to Ω , Proposition 2.3.7 yields that $\lambda_1(\Omega_n, 0) \rightarrow \lambda_1(\Omega, 0)$ so that the sequence $(\lambda_1(\Omega_n, 0))_{n \geq 1}$ is bounded, proving our claim. Therefore, up to a subsequence, $\lambda_1(\Omega_n, V_n)$ converges to some $\lambda \in \mathbb{R}$ and u_n has a uniformly bounded norm in $H_0^1(D)$, which yields a function $u \in H_0^1(D)$ such that, up to a subsequence,

$$u_n \rightharpoonup u \text{ weakly in } H_0^1(D) \text{ and } u_n \rightarrow u \text{ strongly in } L^2(D). \quad (2.4.3)$$

Since the sequence $|\nabla u_n|$ is bounded in $L^2(D)$, up to a subsequence, $-\tau |\nabla u_n| \rightharpoonup z$ weakly in $L^2(D)$ for some function $z \in L^2(D)$. Therefore, $f_n := \lambda_1(\Omega_n, V_n) u_n + \tau |\nabla u_n|$ weakly converges in $L^2(D)$ to $f := \lambda u - z$. Thanks to theorem 2.2.8 (applied to $-\Delta$), $u_n = R_{\Omega_n}^{-\Delta}(f_n)$ strongly converges in $L^2(D)$ to $R_{\Omega}^{-\Delta}(f)$. By (2.4.3), we have $u = R_{\Omega}^{-\Delta}(f)$ and hence $u \in H_0^1(\Omega)$. Furthermore

$$\begin{aligned} \int_D |\nabla u|^2 dx &= \int_D (-zu + \lambda u^2) dx \\ &= \lim_{n \rightarrow +\infty} \int_D (\tau |\nabla u_n| u_n + \lambda_1(\Omega_n, V_n) u_n^2) dx = \lim_{n \rightarrow +\infty} \int_D |\nabla u_n|^2 dx, \end{aligned}$$

where the first line is due to the fact that $u \in H_0^1(\Omega)$ and $-\Delta u = \lambda u - z$ in Ω . This proves that u_n converges strongly in $H^1(D)$ to u , that $|\nabla u_n|$ strongly converges in $L^2(D)$ to $|\nabla u|$, and hence that $z = -\tau |\nabla u|$. Therefore u satisfies

$$-\Delta u + V_* \cdot \nabla u = -\Delta u - \tau |\nabla u| = \lambda u \quad \text{in } \Omega, \quad u \in H_0^1(\Omega), \quad \int_{\Omega} u^2 dx = 1,$$

where $V_* \in L^\infty(D, \mathbb{R}^d)$ is given by (2.4.2). This shows that λ is an eigenvalue of the operator $L = -\Delta + V_* \cdot \nabla$ in Ω . In particular, we have $\|V_*\|_\infty \leq \tau$ and $\lambda_1(\Omega, V_*) \leq \lambda$. On the other hand, by the minimality of V_n , we have $\lambda_1(\Omega_n, V_n) \leq \lambda_1(\Omega_n, V_*)$. Hence, letting $n \rightarrow \infty$, we get that $\lambda \leq \lambda_1(\Omega, V_*)$, which yields $\lambda = \lambda_1(\Omega, V_*)$ and concludes the proof of the theorem. \square

2.4.2 Shape optimization problem over domains and vector fields

Let $D \subset \mathbb{R}^d$ be a bounded open set, $0 < m \leq |D|$ and $\tau > 0$. We consider the shape optimization problem

$$\min \{ \lambda_1(\Omega, V) : \Omega \subset D \text{ quasi-open, } |\Omega| \leq m, \|V\|_{L^\infty} \leq \tau \}. \quad (2.4.4)$$

Theorem 2.4.3 (Existence of optimal sets and optimal vector fields). *Let $\tau \geq 0$ and $m \in (0, |D|)$. Then the problem (2.4.4) has a solution (Ω^*, V^*) , where V^* is given by (2.4.2).*

Proof. Let (Ω_n, V_n) be a minimizing sequence for (2.4.4) and let

$$\underline{\lambda} := \inf \{ \lambda_1(\Omega, V) : \Omega \subset D \text{ quasi-open, } |\Omega| \leq m, \|V\|_{L^\infty} \leq \tau \} = \lim_{n \rightarrow \infty} \lambda_1(\Omega_n, V_n),$$

Since the topology of the weak γ -convergence is compact, we can assume that, up to a subsequence, Ω_n weakly γ -converges to a quasi-open set $\Omega \subset D$. Then, let $\tilde{\Omega}_n$ be a sequence of quasi-open sets as in Lemma 2.2.3. Denote by \tilde{V}_n the optimal vector field given by Theorem 2.4.2 on $\tilde{\Omega}_n$, and let $u_n \in H_0^1(\tilde{\Omega}_n)$ be a solution of

$$-\Delta u_n + \tilde{V}_n \cdot \nabla u_n = \lambda_1(\tilde{\Omega}_n, \tilde{V}_n) u_n \quad \text{in } \tilde{\Omega}_n, \quad u_n \in H_0^1(\tilde{\Omega}_n), \quad \int_D u_n^2 dx = 1.$$

By the minimality of \tilde{V}_n and the inclusion $\Omega_n \subset \tilde{\Omega}_n$, we have

$$0 < \lambda_1(\tilde{\Omega}_n, \tilde{V}_n) \leq \lambda_1(\tilde{\Omega}_n, V_n) \leq \lambda_1(\Omega_n, V_n) \quad \text{for every } n \geq 1.$$

Therefore, up to a subsequence, $\lambda_1(\tilde{\Omega}_n, \tilde{V}_n)$ converges to some $\tilde{\lambda}$ such that $\tilde{\lambda} \leq \underline{\lambda}$. In particular, $(u_n)_{n \geq 1}$ is uniformly bounded in $H_0^1(D)$ and so, up to a subsequence, u_n weakly converges in $H_0^1(D)$ to some $u \in H_0^1(D)$. Now, since $\tilde{\Omega}_n$ γ -converges to Ω , we can argue as in the end of the proof of Theorem 2.4.2 to conclude that the convergence of u_n to u is strong in $H^1(D)$. This yields that u is not identically zero and satisfies

$$-\Delta u + V \cdot \nabla u = \Delta u - \tau |\nabla u| = \tilde{\lambda} u \quad \text{in } \Omega, \quad u \in H_0^1(\Omega), \quad \int_D u^2 dx = 1,$$

where $V \in L^\infty(D, \mathbb{R}^d)$ is given by (2.4.2). Furthermore, thanks to Lemma 2.2.4, we have that $|\Omega| \leq m$. Hence, $\underline{\lambda} \leq \tilde{\lambda}$. Thus, we get that $\tilde{\lambda} = \underline{\lambda}$ and hence that $\underline{\lambda} = \lambda_1(\Omega, V)$, which proves that the couple (Ω, V) is a solution of (2.4.4). \square

Remark 2.4.4. If the box D contains a ball $B \subset D$ such that $|B| = m$, then by Remark 2.3.10 a solution of (2.4.4) is given by $\lambda_1(B, \tau \frac{x}{|x|})$.

We now consider a shape optimization problem in the more restrictive class of couples (Ω, V) , in which the vector field V is a gradient of a Lipschitz function. Precisely, given a bounded open set $D \subset \mathbb{R}^d$, $\tau \geq 0$ and $m \in (0, |D|)$, we consider the shape optimization problem

$$\min \{ \lambda_1(\Omega, \nabla \Phi) : \Omega \subset D \text{ quasi-open, } \Phi \in W^{1,\infty}(D), |\Omega| \leq m, \|\nabla \Phi\|_{L^\infty} \leq \tau \}. \quad (2.4.5)$$

In this case the argument from Theorem 2.4.3 does not apply since the optimal vector field from Theorem 2.4.2 may not be the gradient of a Lipschitz function. On the other hand, the

functional $\lambda_1(\Omega, \nabla\Phi)$ is variational so we can use a more direct approach. Indeed, for every $\lambda \in \mathbb{R}$ and $u \in H_0^1(\Omega)$ we have

$$-\Delta u + \nabla\Phi \cdot \nabla u = \lambda u \quad \text{in } \Omega \quad \Leftrightarrow \quad -\operatorname{div}(e^{-\Phi}\nabla u) = \lambda e^{-\Phi}u \quad \text{in } \Omega,$$

and since the operator $A = -\operatorname{div}(e^{-\Phi}\nabla \cdot)$ is self-adjoint, we get that

$$\lambda_1(\Omega, \nabla\Phi) = \min_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_D e^{-\Phi} |\nabla u|^2 dx}{\int_D e^{-\Phi} u^2 dx}. \quad (2.4.6)$$

Theorem 2.4.5 (Existence of optimal sets and optimal potentials). *Let $D \subset \mathbb{R}^d$ be a bounded open set, $\tau \geq 0$ and $m \in (0, |D|)$. Then the problem (2.4.5) has a solution.*

Proof. Suppose that (Ω_n, Φ_n) is a minimizing sequence for (2.4.5) and let $\lambda_n = \lambda_1(\Omega_n, \nabla\Phi_n)$. Given $x_0 \in D$, we may suppose that $\Phi_n(x_0) = 0$ for every $n \geq 1$. Thus, up to a subsequence, Φ_n converges uniformly in \overline{D} to a function $\Phi \in W^{1,\infty}(D)$ such that $\Phi(x_0) = 0$ and $\|\nabla\Phi\|_{L^\infty} \leq \tau$. Let u_n be the solution of

$$-\Delta u_n + \nabla\Phi_n \cdot \nabla u_n = \lambda_n u_n \quad \text{in } \Omega_n, \quad u_n \in H_0^1(\Omega_n), \quad \int_D u_n^2 dx = 1.$$

Then, u_n is uniformly bounded in $H_0^1(D)$ and so, up to a subsequence, u_n converges weakly in $H_0^1(D)$ and strongly in $L^2(D)$ to a function $u \in H_0^1(D)$. Thus, we have

$$\int_D e^{-\Phi} u^2 dx = \lim_{n \rightarrow \infty} \int_D e^{-\Phi_n} u_n^2 dx \quad \text{and} \quad \int_D e^{-\Phi} |\nabla u|^2 dx \leq \liminf_{n \rightarrow \infty} \int_D e^{-\Phi_n} |\nabla u_n|^2 dx.$$

Now, choosing $\Omega := \{u > 0\}$ and applying (2.4.6), we get

$$\lambda_1(\Omega, \Phi) \leq \frac{\int_D e^{-\Phi} |\nabla u|^2 dx}{\int_D e^{-\Phi} u^2 dx} \leq \liminf_{n \rightarrow \infty} \frac{\int_D e^{-\Phi_n} |\nabla u_n|^2 dx}{\int_D e^{-\Phi_n} u_n^2 dx} = \liminf_{n \rightarrow \infty} \lambda_1(\Omega_n, \Phi_n).$$

Now, in order to conclude, it is sufficient to notice that by choosing a subsequence, we may assume that u_n converges to u pointwise a.e., so we get

$$|\Omega| = |\{u > 0\}| \leq \liminf_{n \rightarrow \infty} |\{u_n > 0\}| \leq \liminf_{n \rightarrow \infty} |\Omega_n| \leq m,$$

which proves that (Ω, Φ) is a solution of (2.4.5). \square

2.5 Regularity of the optimal sets

In this section we prove Theorem 2.1.5. We prove the regularity of the boundary $\partial\Omega$ of the optimal sets Ω from Theorem 2.1.5. We only consider the case $V = \nabla\Phi$, with $\Phi \in W^{1,\infty}(D)$, since in this case the optimization problem (2.1.8) is equivalent to a free boundary problem for the first eigenfunction u on the optimal set Ω . The regularity for a generic vector field $V \in L^\infty(D)$ remains an open problem essentially due to the lack of variational characterization of the eigenvalue $\lambda_1(\Omega, V)$. We start with the following lemma.

Lemma 2.5.1 (Reduction to a free boundary problem). *Let $D \subset \mathbb{R}^d$ be a bounded open set, $0 < m < |D|$, $\tau > 0$, $\Phi \in W^{1,\infty}(D)$, with $\|\nabla\Phi\|_{L^\infty} \leq \tau$, and $V = \nabla\Phi$. Suppose that the quasi-open set $\Omega \subset D$ is a solution of (2.1.8). Then every corresponding first eigenfunction u_Ω of the operator $-\Delta + V \cdot \nabla$ on Ω is a solution to the variational problem*

$$\lambda_m := \min \left\{ \int_D |\nabla u|^2 e^{-\Phi} dx : u \in H_0^1(D), |\{u \neq 0\}| \leq m, \int_D e^{-\Phi} u^2 dx = 1 \right\}. \quad (2.5.1)$$

Conversely, if u is a solution of (2.5.1), then the quasi-open set $\{u \neq 0\}$ is a solution of (2.1.8).

Proof. The proof is a straightforward consequence of the variational formula (2.4.6). \square

Remark 2.5.2. It turns out that if u is a solution of (2.5.1), then $u \geq 0$ in D (see Lemma 2.5.8 below).

The rest of this section is dedicated to the regularity of the free boundary $\partial\Omega_u \cap D$ and of the whole boundary $\partial\Omega_u$ if D is smooth, of a solution u of (2.5.1), where we recall that, for any function $v \in H_0^1(D)$ we denote by Ω_v the (quasi-open) set $\{v > 0\}$.

This section is organized as follows.

In Subsection 2.5.1 we prove that the solutions of (2.5.1) are bounded. This is important due to the fact that u solves the equation

$$-\operatorname{div}(e^{-\Phi}\nabla u) = \lambda_1(\Omega_u, \nabla\Phi)e^{-\Phi}u \quad \text{in } \Omega_u,$$

and in the rest of the section we will often use the fact that the right-hand side is bounded.

In Subsection 2.5.2, we prove that the solution u is essentially a subharmonic function on D with respect to the operator $\operatorname{div}(e^{-\Phi}\nabla)$ (see Lemma 2.5.5). In particular, this implies that u and the set Ω_u are well-defined everywhere (not just up to a set of measure zero). The free boundary is thus defined as the topological boundary of the set Ω_u . In the same subsection, in Lemma 2.5.8, we prove that the measure constraint $|\{u > 0\}| \leq m$ is saturated, that is, $|\Omega_u| = m$. This proves Theorem 2.1.5 (3).

In Subsection 2.5.3 we get rid of the integral constraint $\int e^{-\Phi}u^2 dx = 1$ and we rewrite the problem (2.5.1) in terms of the functional

$$J(u) = \int_D (|\nabla u|^2 - \lambda_m u^2) e^{-\Phi} dx.$$

In Subsection 2.5.4, we write the Euler-Lagrange equation that arises in the minimization of the functional J under the measure constraint $|\Omega_u| \leq m$. We consider only internal variations, that is, test functions of the form $\tilde{u}(x) = u(x + t\xi(x))$ for smooth vector fields ξ . In Subsection 2.5.5, we prove that at small scales u is a solution (in the sense of (2.1.12)) to the minimization problem for the functional $v \mapsto J(v) + \Lambda_u |\Omega_v|$, where Λ_u is the Lagrange multiplier from Subsection 2.5.4.

In Subsection 2.5.6 and Subsection 2.5.7, we use the result from Subsection 2.5.5 to prove that the solutions of (2.5.1) are Lipschitz continuous and non-degenerate at the free boundary; we also prove that the set Ω_u has finite perimeter. This proves Theorem 2.1.5 (1) and (2).

Subsection 2.5.8 is dedicated to the compactness of the blow-up sequences and the optimality of the blow-up limits. In Subsection 2.5.10 we prove a (quasi-)monotonicity formula for (a variant of) the Weiss' boundary adjusted energy. As a consequence, we obtain that the blow-up limits are one-homogeneous.

In Subsection 2.5.9, we prove that the solution u satisfies the optimality condition $|\nabla u| = \sqrt{\Lambda_u}e^{\Phi}$ on the free boundary $\partial\Omega_u \cap D$ in viscosity sense, we deduce the regularity of the regular part $\operatorname{Reg}(\partial\Omega_u)$ and we show that the remaining singular set has zero $(d-1)$ -dimensional Hausdorff measure. In this subsection, we complete the proof of Theorem 2.1.5 (4), (6) and (7). Finally, in Subsection 2.5.10, we give some further estimates on the Hausdorff dimension of the singular set, which complete the proof of Theorem 2.1.5 (5).

2.5.1 Boundedness of the eigenfunctions

In this subsection we give a bound on the L^∞ norm of the eigenfunctions on generic bounded quasi-open sets. We first prove that if u is a solution of a PDE with sufficiently integrable right-hand side, then u is bounded. Then we use and iterate an interpolation argument to improve the integrability of the eigenfunctions.

Lemma 2.5.3 (Boundedness of the solutions of PDEs on quasi-open sets). *Let $D \subset \mathbb{R}^d$ be a bounded open set, $\Omega \subset D$ be a quasi-open set and $\Phi \in W^{1,\infty}(D)$. Let $f \in L^p(D)$ for some $p \in (d/2, +\infty]$ and let $u \in H_0^1(\Omega)$ be the solution of*

$$-\operatorname{div}(e^{-\Phi}\nabla u) = f \quad \text{in } \Omega, \quad u \in H_0^1(\Omega). \quad (2.5.2)$$

Then, there is a dimensional constant C_d such that

$$\|u\|_{L^\infty} \leq \frac{C_d e^{M_\Phi}}{2/d - 1/p} |\Omega|^{2/d - 1/p} \|f\|_{L^p},$$

where $M_\Phi := \|\Phi\|_{L^\infty(D)}$.

Proof. We first assume that f is a non-negative function. We notice that $u \geq 0$ on Ω and that u is a minimum in $H_0^1(\Omega)$ of the functional

$$J(u) := \frac{1}{2} \int_{\Omega} e^{-\Phi} |\nabla u|^2 dx - \int_{\Omega} f u dx.$$

The rest of the proof follows precisely as in [90, Lemma 3.51]. For every $0 < t < \|u\|_{L^\infty}$ and $\varepsilon > 0$, we consider the test function $u_{t,\varepsilon} = u \wedge t + (u - t - \varepsilon)_+$. The inequality $J(u) \leq J(u_{t,\varepsilon})$ gives that

$$\frac{1}{2} \int_{\{t < u \leq t + \varepsilon\}} e^{-\Phi} |\nabla u|^2 dx \leq \int_{\mathbb{R}^d} f(u - u_{t,\varepsilon}) dx \leq \varepsilon \int_{\{u > t\}} f dx \leq \varepsilon \|f\|_{L^p} |\{u > t\}|^{\frac{p-1}{p}},$$

and, using the co-area formula and passing to the limit as $\varepsilon \rightarrow 0$, we get

$$\int_{\{u=t\}} |\nabla u| d\mathcal{H}^{d-1} \leq 2e^{M_\Phi} \|f\|_{L^p} |\{u > t\}|^{\frac{p-1}{p}}. \quad (2.5.3)$$

Now, setting $\varphi(t) := |\{u > t\}|$ and using the co-area formula again as well as the Cauchy-Schwarz inequality, we obtain

$$\varphi'(t) = - \int_{\{u=t\}} \frac{1}{|\nabla u|} d\mathcal{H}^{d-1} \leq - \left(\int_{\{u=t\}} |\nabla u| d\mathcal{H}^{d-1} \right)^{-1} \operatorname{Per}(\{u > t\})^2,$$

which, together with the isoperimetric inequality $|\{u > t\}|^{\frac{d-1}{d}} \leq C_d \operatorname{Per}(\{u > t\})$ and (2.5.3), gives

$$\varphi'(t) \leq - \frac{C_d}{e^{M_\Phi} \|f\|_{L^p}} \varphi(t)^{\frac{d-2}{d} + \frac{1}{p}},$$

where we recall that the dimensional constant C_d may change from line to line.

Setting $\alpha = \frac{d-2}{d} + \frac{1}{p} < 1$ and $C = C_d \|f\|_{L^p}^{-1} e^{-M_\Phi}$, we have $\varphi' \leq -C\varphi^\alpha$. If

$$t_{\max} := \sup \{t > 0; \varphi(s) > 0 \text{ for all } s \in [0, t)\} \leq +\infty,$$

then $\varphi'(t)\varphi(t)^{-\alpha} \leq -C$ for all $t \in [0, t_{\max})$, so that

$$0 \leq \varphi(t) \leq (|\Omega|^{1-\alpha} - (1-\alpha)Ct)^{\frac{1}{1-\alpha}} \quad \text{for all } t \in [0, t_{\max}).$$

This shows that $t_{\max} < +\infty$ and that

$$\|u\|_{L^\infty} \leq t_{\max} \leq \frac{1}{C} \frac{|\Omega|^{2/d - 1/p}}{2/d - 1/p},$$

which concludes the proof when f is non-negative. For a general function f , the proof now follows by applying the estimate in Lemma 2.5.3 to both the positive and the negative parts of f . \square

Lemma 2.5.4 (Boundedness of the eigenfunctions). *Let $D \subset \mathbb{R}^d$ be a bounded open set, $\Omega \subset D$ be a quasi-open set, $\Phi \in W^{1,\infty}(D)$ and $V = \nabla\Phi$. Let $R : L^2(\Omega) \rightarrow L^2(\Omega)$ be the resolvent operator of $-\Delta + V \cdot \nabla$ on Ω . Then, there are constants $n \in \mathbb{N}$, depending only on d , and $C \in \mathbb{R}$, depending on d , $|\Omega|$ and $\|\Phi\|_{L^\infty}$, such that*

$$R^n(L^2(\Omega)) \subset L^\infty(\Omega) \quad \text{and} \quad \|R^n\|_{\mathcal{L}(L^2(\Omega); L^\infty(\Omega))} \leq C.$$

In particular, if u is a first eigenfunction of $-\Delta + V \cdot \nabla$ on Ω normalized by $\|u\|_{L^2} = 1$, then $u \in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty} \leq C\lambda_1^n(\Omega, V).$$

Proof. Let us first notice that if $d \leq 3$, then $d/2 < 2$ and so, taking $n = 1$, the claim follows directly by Lemma 2.5.3. If $d > 3$, then setting $2^* = \frac{2d}{d-2}$, we have

$$R : L^2(\Omega) \rightarrow L^{2^*}(\Omega) \quad \text{and} \quad R : L^d(\Omega) \rightarrow L^\infty(\Omega).$$

Thus, interpolating between 2 and d , we get

$$\|R\|_{\mathcal{L}(L^p; L^q)} \leq C, \quad \text{where } p \in [2, d] \quad \text{and} \quad q = \frac{pd}{d-p} \geq \frac{2d}{d-2}, \quad (2.5.4)$$

where C depends only on d , $|\Omega|$ and $\|\Phi\|_{L^\infty}$. Now, it is sufficient to notice that $R^k \in \mathcal{L}(L^2; L^{q_k})$, where $q_k = \left(\frac{2d}{d-2}\right)^k$. For k big enough we have that $q_k > d/2$ and so, $R^{k+1} \in \mathcal{L}(L^2; L^\infty)$, which proves the first part of the claim. Finally, in order to get the estimate on u , it is sufficient to notice that $R(u) = \lambda_1^{-1}(\Omega, V)u$ and $R^n(u) = \lambda_1^{-n}(\Omega, V)u$. \square

2.5.2 Pointwise definition of the solutions

When we deal with Sobolev functions we usually reason up to a choice of certain representative of the function. Even if this representative is defined quasi-everywhere, there still might be a set of zero capacity where the function is not defined. Of course, this interferes with the notion of a free boundary in the sense that we cannot just consider the topological boundary of Ω_u without specifying the representative of u that we work with. Fortunately, the eigenfunctions of the quasi-open sets are defined pointwise everywhere, that is *every* point is a Lebesgue point.

Lemma 2.5.5 (Subharmonicity and a mean-value formula for positive solutions of PDEs). *Let $D \subset \mathbb{R}^d$ be a bounded open set, $\Omega \subset D$ a quasi-open set and $\Phi \in W^{1,\infty}(D)$ a given Lipschitz function. Let $f \in L^\infty(D)$ and $u \geq 0$ be a solution to the problem (2.5.2) in Ω .*

1. *Then, $\text{div}(e^{-\Phi}\nabla u) + f \geq 0$ in \mathbb{R}^d , in the sense of distributions. In particular, $\text{div}(e^{-\Phi}\nabla u)$ is a (signed) Radon measure on \mathbb{R}^d .*
2. *For any $x_0 \in \mathbb{R}^d$, we can define the value of u at x_0 by*

$$u(x_0) = \lim_{r \rightarrow 0} \int_{\partial B_r(x_0)} u(x) d\mathcal{H}^{d-1}(x) = \lim_{r \rightarrow 0} \int_{B_r(x_0)} u(x) dx.$$

Moreover, we have the identity

$$\begin{aligned} \int_{\partial B_r(x_0)} u e^{-\Phi} d\mathcal{H}^{d-1} - u(x_0) e^{-\Phi(x_0)} &= \frac{1}{d\omega_d} \int_0^r s^{1-d} \text{div}(e^{-\Phi}\nabla u)(B_s(x_0)) ds \\ &\quad - \frac{1}{d\omega_d} \int_0^r s^{1-d} ds \int_{\partial B_s} (\nabla\Phi \cdot \nu) u e^{-\Phi} d\mathcal{H}^{d-1}, \end{aligned} \quad (2.5.5)$$

where ν denotes the normal to ∂B_s pointing outwards.

3. Let $x_0 \in \mathbb{R}^d$ and $R > 0$. Suppose that there is a constant $C > 0$ such that

$$|\operatorname{div}(e^{-\Phi}\nabla u)(B_r(x_0))| \leq Cr^{d-1} \quad \text{for every } 0 < r \leq R. \quad (2.5.6)$$

Then we have the estimate

$$\left| u(x_0) - \int_{\partial B_r(x_0)} u \, d\mathcal{H}^{d-1} \right| \leq e^{M_\Phi} \left(\frac{C}{d\omega_d} + 2L_\Phi M_u e^{M_\Phi} \right) r \quad \text{for every } 0 < r \leq R, \quad (2.5.7)$$

where $L_\Phi := \|\nabla\Phi\|_{L^\infty(D)}$, $M_u := \|u\|_{L^\infty(D)}$ and $M_\Phi := \|\Phi\|_{L^\infty(D)}$.

Proof. (1) For $n \in \mathbb{N}$ define $p_n : \mathbb{R} \rightarrow \mathbb{R}$ by

$$p_n(s) = 0, \quad \text{for } s \leq 0; \quad p_n(s) = ns, \quad \text{for } s \in [0, 1/n]; \quad p_n(s) = 1, \quad \text{for } s \geq 1/n.$$

Since p_n is Lipschitz continuous, we have $p_n(u) \in H_0^1(\Omega)$ and $\nabla p_n(u) = p_n'(u)\nabla u$. Let $\varphi \in C_0^\infty(D)$, $\varphi \geq 0$ in D . Using $\varphi p_n(u)$ as a test function in (2.5.2), we get

$$\int_D p_n(u)\nabla u \cdot \nabla \varphi e^{-\Phi} dx \leq \int_D (p_n(u)\nabla u \cdot \nabla \varphi + \varphi p_n'(u)|\nabla u|^2) e^{-\Phi} dx = \int_D \varphi p_n(u) dx.$$

which, letting $n \rightarrow \infty$, gives the first claim.

In order to prove (2), we suppose that $x_0 = 0$ and we calculate

$$\begin{aligned} \frac{d}{ds} \int_{\partial B_s} u e^{-\Phi} d\mathcal{H}^{d-1} &= \frac{d}{ds} \int_{\partial B_1} u(s\xi) e^{-\Phi(s\xi)} d\mathcal{H}^{d-1} \\ &= \int_{\partial B_1} [\xi \cdot \nabla u(s\xi) - u(s\xi)\xi \cdot \nabla \Phi(s\xi)] e^{-\Phi(s\xi)} d\mathcal{H}^{d-1} \\ &= \frac{s^{1-d}}{d\omega_d} \operatorname{div}(e^{-\Phi}\nabla u)(B_s) - \frac{s^{1-d}}{d\omega_d} \int_{\partial B_s} (\nabla \Phi \cdot \nu) u e^{-\Phi} d\mathcal{H}^{d-1}. \end{aligned}$$

Then, integrating from ρ to r ($\rho < r$), using the inequality from (1) and the fact that $u \in L^\infty(D)$ by Lemma 2.5.4, we get

$$\begin{aligned} \int_{\partial B_r} u e^{-\Phi} d\mathcal{H}^{d-1} - \int_{\partial B_\rho} u e^{-\Phi} d\mathcal{H}^{d-1} &= \frac{1}{d\omega_d} \int_\rho^r s^{1-d} \operatorname{div}(e^{-\Phi}\nabla u)(B_s(x_0)) ds \\ &\quad - \frac{1}{d\omega_d} \int_\rho^r s^{1-d} ds \int_{\partial B_s} (\nabla \Phi \cdot \nu) u e^{-\Phi} d\mathcal{H}^{d-1} \\ &\geq -\frac{1}{2d} \|f\|_{L^\infty} (r^2 - \rho^2) - e^{-\min \Phi} \|u\|_{L^\infty} \|\nabla \Phi\|_{L^\infty} (r - \rho) \\ &:= -A(r^2 - \rho^2) - B(r - \rho), \end{aligned} \quad (2.5.8)$$

where $A, B > 0$. This shows that the function $r \mapsto \int_{\partial B_r} u e^{-\Phi} d\mathcal{H}^{d-1} + Ar^2 + Br$ is non-decreasing.

In particular, the limit $\ell(x_0) = \lim_{r \rightarrow 0} \int_{\partial B_r(x_0)} u e^{-\Phi} d\mathcal{H}^{d-1}$ exists and we set $u(x_0) := e^{\Phi(x_0)} \ell(x_0)$.

Now, (2.5.5) follows by letting $\rho \rightarrow 0$ in (2.5.8). Finally, in order to prove the claim (3), we notice that (2.5.5) implies

$$\left| \int_{\partial B_r(x_0)} u e^{-\Phi} d\mathcal{H}^{d-1} - u(x_0) e^{-\Phi(x_0)} \right| \leq \left(\frac{C}{d\omega_d} + L_\Phi M_u e^{M_\Phi} \right) r \quad \text{for every } 0 < r \leq R, \quad (2.5.9)$$

Now, by the triangular inequality we have

$$\begin{aligned} \left| \int_{\partial B_r(x_0)} u d\mathcal{H}^{d-1} - u(x_0) \right| &= e^{\Phi(x_0)} \left| \int_{\partial B_r(x_0)} u e^{-\Phi(x_0)} d\mathcal{H}^{d-1} - u(x_0) e^{-\Phi(x_0)} \right| \\ &\leq e^{\Phi(x_0)} \left| \int_{\partial B_r(x_0)} u e^{-\Phi} d\mathcal{H}^{d-1} - u(x_0) e^{-\Phi(x_0)} \right| \\ &\quad + e^{\Phi(x_0)} \int_{\partial B_r(x_0)} u \left| e^{-\Phi(x_0)} - e^{-\Phi(x)} \right| d\mathcal{H}^{d-1}. \end{aligned}$$

Thus, the claim follows since, by the Lipschitz continuity of Φ , we have that for every $x \in \partial B_r(x_0)$,

$$\left| e^{-\Phi(x_0)} - e^{-\Phi(x)} \right| \leq e^{\|\Phi\|_{L^\infty(D)}} \|\nabla\Phi\|_{L^\infty(D)} |x - x_0|.$$

(2.5.7) is a direct consequence of (2.5.5). \square

As a direct consequence of Lemma 2.5.5 and (2.5.5) we get the following strong maximum principle.

Lemma 2.5.6 (Strong maximum principle). *Let $D \subset \mathbb{R}^d$ be an open connected set and $u \in H_0^1(D)$ be non-negative, $u \geq 0$ in D . Assume that $\operatorname{div}(e^{-\Phi}\nabla u) \in L^\infty(D)$ satisfies $\operatorname{div}(e^{-\Phi}\nabla u) \leq 0$. Then, if u is not identically vanishing in D , then u is strictly positive in D .*

Proof. Set $A := \{x_0 \in D; u(x_0) = 0\}$. If $x_0 \in A$, then (2.5.5) implies that $u(x) = 0$ for almost every $x \in B_r(x_0)$ whenever $B_r(x_0) \subset D$. Therefore, for all $x \in B_r(x_0)$, since x is a Lebesgue point for u , $u(x) = 0$. Thus, A is open.

Consider now a sequence $(x_n)_{n \geq 1} \in A$ converging to $x_0 \in D$. For some n large enough, there exists a ball $B_r(x_n) \subset D$ containing x_0 . Since u vanishes everywhere in $B_r(x_n)$, $u(x_0) = 0$, which proves that A is closed in D . We conclude by the connectedness of D . \square

A consequence of Lemma 2.5.5 is the fact that the set $\Omega_u = \{u > 0\}$ and the (topological) free boundary $\partial\Omega_u \cap D$ are well defined. Below we prove that the topological boundary coincides with the measure theoretic one.

Lemma 2.5.7 (The topological boundary coincides with the measure-theoretic one). *Let $u \in H_0^1(D)$, $u \geq 0$ in D , be a solution of (2.5.1), $x_0 \in \partial\Omega_u$ and let $r > 0$ be such that $D_r(x_0) := B_r(x_0) \cap D$ is connected. Then we have $0 < |\Omega_u \cap B_r(x_0)|$. Moreover, if $x_0 \in \partial\Omega_u \cap D$, we have $|\Omega_u \cap B_r(x_0)| < |D_r(x_0)|$.*

Proof. In order to prove the first inequality, suppose that $x_0 \in \partial\Omega_u$ and $|B_r(x_0) \cap \Omega_u| = 0$ for some $r > 0$. Since every point $x \in B_r(x_0)$ is a Lebesgue point for u and $u = 0$ almost everywhere in $B_r(x_0)$ we have that $u \equiv 0$ in $B_r(x_0)$, but this contradicts the fact that $x_0 \in \partial\Omega_u$.

In order to prove the second inequality, we assume by contradiction that $|\Omega_u \cap B_r(x_0)| = |D_r(x_0)|$ for some $r > 0$. We claim that u is a solution of

$$-\operatorname{div}(e^{-\Phi}\nabla u) = \lambda_m u e^{-\Phi} \quad \text{in } D_r(x_0), \quad \text{where } \lambda_m := \int_D |\nabla u|^2 e^{-\Phi} dx.$$

Indeed, let v be the solution of

$$-\operatorname{div}(e^{-\Phi}\nabla v) = \lambda_m u e^{-\Phi} \quad \text{in } D_r(x_0), \quad v = u \quad \text{in } D \setminus B_r(x_0).$$

Then, Lemma 2.5.6 implies that $v > 0$ in $D_r(x_0)$. Since $|\Omega_v| = |\Omega_u|$, the optimality of u gives

$$\frac{\int_D |\nabla v|^2 e^{-\Phi} dx}{\int_D v^2 e^{-\Phi} dx} \geq \int_D |\nabla u|^2 e^{-\Phi} dx = \frac{\int_D |\nabla u|^2 e^{-\Phi} dx}{\int_D v^2 e^{-\Phi} dx} + \lambda_m \left(1 - \frac{\int_D u^2 e^{-\Phi} dx}{\int_D v^2 e^{-\Phi} dx} \right),$$

which implies

$$0 \geq \int_D (|\nabla u|^2 - |\nabla v|^2) e^{-\Phi} dx + \lambda_m \int_D (v^2 - u^2) e^{-\Phi} dx = \int_D (|\nabla(u-v)|^2 + \lambda_m(u-v)^2) e^{-\Phi} dx,$$

where the last equality follows by the definition of v and the fact that $v - u \in H_0^1(D_r(x_0))$. This implies that $u = v$ almost everywhere and hence, by Lemma 2.5.5, that $u = v$ everywhere. Therefore, we have $u > 0$ in $B_r(x_0) \cap D$, which is in contradiction with $x_0 \in \partial\Omega_u \cap D$. \square

Lemma 2.5.8 (Saturation of the constraint). *Let $D \subset \mathbb{R}^d$ be a connected bounded open set, and let $\Phi \in W^{1,\infty}(D)$, m and τ be as in Lemma 2.5.1. Then every solution u of (2.5.1) is such that $u \geq 0$ on D and $|\Omega_u| = m$ (up to a change of sign). In particular, every solution Ω of (2.1.8) saturates the measure constraint, that is, $|\Omega| = m$.*

Proof. Let u be a solution of (2.5.1). If $u = u_+$ or $u = u_-$, the claim is obvious. Otherwise, we have that u changes sign, that is, $u_+ \neq 0$ and $u_- \neq 0$. We set

$$u_1 = \frac{u_+}{\left(\int_D u_+^2 e^{-\Phi}\right)^{1/2}} \quad \text{and} \quad u_2 = \frac{u_-}{\left(\int_D u_-^2 e^{-\Phi}\right)^{1/2}}.$$

Now, the claim follows from the estimate

$$\inf \left(\frac{\int_D |\nabla u_+|^2 e^{-\Phi} dx}{\int_D u_+^2 e^{-\Phi} dx}, \frac{\int_D |\nabla u_-|^2 e^{-\Phi} dx}{\int_D u_-^2 e^{-\Phi} dx} \right) \leq \frac{\int_D (|\nabla u_+|^2 + |\nabla u_-|^2) e^{-\Phi} dx}{\int_D (u_+^2 + u_-^2) e^{-\Phi} dx} = \frac{\int_D |\nabla u|^2 e^{-\Phi} dx}{\int_D u^2 e^{-\Phi} dx}.$$

This means that at least one of the functions u_1 and u_2 is also a solution of (2.5.1). Up to changing u into $-u$, we assume that u_1 is a solution of (2.5.1). Now, suppose by contradiction that $|\Omega_u| < m$. Then, for every ball $B_r(x_0) \subset D$ such that $|\Omega_u| + |B_r| \leq m$, writing that

$$\int_D |\nabla u_1|^2 e^{-\Phi} dx \leq \int_D |\nabla(u_1 + t\varphi)|^2 e^{-\Phi} dx$$

for all functions $\varphi \in H_0^1(B_r(x_0))$ and all $t \in \mathbb{R}$, we easily get that u_1 is a solution of

$$-\operatorname{div}(e^{-\Phi} \nabla u_1) = \lambda_m e^{-\Phi} u_1 \quad \text{in } B_r(x_0).$$

By the strong maximum principle, we get $u > 0$ in $B_r(x_0)$, which is a contradiction. This proves both the saturation of the constraint and the positivity of u . \square

2.5.3 A free-boundary problem with measure constraint

We now follow the strategy adopted in [11, 9]. In particular, the proof of Theorem 2.5.16 below is very close to the one of Theorem 1.5 in [11]. Note that the approach is local and that a result analogous to Theorem 2.5.16 with perturbations in D is vain (see Remark 1.6 in [11]).

Let $u \in H_0^1(D)$ be a solution of (2.5.1). For any $v \in H_0^1(D)$ we set

$$J(v) := \int_D |\nabla v|^2 e^{-\Phi} dx - \lambda_m \int_D v^2 e^{-\Phi} dx, \quad (2.5.10)$$

where it is recalled that $\lambda_m = \int_D |\nabla u|^2 e^{-\Phi} dx$.

Remark 2.5.9 (Removal of the integral constraint). It is plain to see that, when $u \in H_0^1(D)$ is a solution of (2.5.1), we have

$$J(u) = \min \{ J(v) : v \in H_0^1(D), |\Omega_v| \leq m \}. \quad (2.5.11)$$

In what follows, we will make the following assumptions on the domain D and the function u :

$$\left\{ \begin{array}{l} D \text{ is a connected bounded open set of } \mathbb{R}^d, \\ u \in H_0^1(D) \text{ is non-negative, } u \geq 0 \text{ in } D, \\ J(u) = \min \{ J(v) : v \in H_0^1(D), |\Omega_v| \leq m \}, \\ \text{the constraint is saturated, that is, } |\Omega_u| = |\{u > 0\}| = m. \end{array} \right. \quad (2.5.12)$$

For a ball $B_r(x_0) \subset \mathbb{R}^d$ we define the admissible set

$$\mathcal{A}(u, x_0, r) := \{v \in H_0^1(D) : u - v \in H_0^1(B_r(x_0))\}.$$

Remark 2.5.10 (Coercivity of J). We notice that the set $\{v \in \mathcal{A}(u, x_0, r) : J(v) < C\}$ is weakly compact in $H_0^1(D)$. Precisely, if $u \in H_0^1(D)$, $\Phi \in W^{1,\infty}(D)$ and J be given by (2.5.10), then there is a constant $r_0 > 0$, depending on d , Φ , λ_m and D such that for all $r \leq r_0$,

$$\int_{B_r(x_0)} |\nabla v|^2 dx \leq 2e^{\max \Phi} J(v) + (1 + 4\lambda_m e^{\max \Phi - \min \Phi}) \|u\|_{H^1(D)}^2, \quad \forall v \in \mathcal{A}(u, x_0, r). \quad (2.5.13)$$

Indeed, let $v \in \mathcal{A}(u, x_0, r)$ with $r \leq r_0$. We have

$$\begin{aligned} \int_D v^2 dx &\leq 2 \int_{B_r(x_0)} (v - u)^2 dx + 2 \int_D u^2 dx \leq \frac{2}{\lambda_1(B_r(x_0))} \int_{B_r(x_0)} |\nabla(v - u)|^2 dx + 2 \int_D u^2 dx \\ &\leq \frac{4r_0^2}{\lambda_1(B_1)} \int_{B_r(x_0)} (|\nabla v|^2 + |\nabla u|^2) dx + 2 \int_D u^2 dx, \end{aligned}$$

where the last inequality is due to the (-2) -homogeneity of $\lambda_1(B_r)$ and the fact that $r \leq r_0$. Choosing r_0 small enough (depending only on d , λ_m , $\|\nabla \Phi\|_{L^\infty}$ and the diameter of D) we get

$$\begin{aligned} \int_D |\nabla v|^2 dx &\leq e^{\max \Phi} J(v) + \lambda_m e^{\max \Phi - \min \Phi} \int_D v^2 dx \\ &\leq e^{\max \Phi} J(v) + \frac{1}{2} \int_{B_r(x_0)} (|\nabla v|^2 + |\nabla u|^2) dx + 2\lambda_m e^{\max \Phi - \min \Phi} \int_D u^2 dx. \end{aligned}$$

which concludes the proof of (2.5.13).

As a consequence, we obtain the following result, which gives us the existence of a solution to a local version of the minimization problem (2.5.11) with some different measure constraint.

Lemma 2.5.11 (Existence of local minimizers). *Let $D \subset \mathbb{R}^d$ and $u \in H_0^1(D)$ satisfy (2.5.12). Let $B_r(x_0) \subset \mathbb{R}^d$ be a fixed ball and let \tilde{m} be a real constant such that $\tilde{m} > |\Omega_u \setminus B_r(x_0)|$. Then:*

1. *the problem*

$$\min \left\{ J(v) : v \in \mathcal{A}(u, x_0, r), |\Omega_v| \leq \tilde{m} \right\} \quad (2.5.14)$$

has a solution provided that $r \leq r_0$ with r_0 given by Remark 2.5.10,

2. *if $D_r(x_0) := B_r(x_0) \cap D$ is connected and $|\Omega_u \cup D_r(x_0)| > \tilde{m}$, then $|\Omega_v| = \tilde{m}$;*

3. *there exists $r_0 > 0$ such that, for every $r < r_0$, every solution v of (2.5.14) is non-negative.*

Proof. In order to prove (1), it is enough to notice that, by Remark 2.5.10, J is bounded from below in $\mathcal{A}(u, x_0, r)$. Then, by (2.5.13), any minimizing sequence $(v_n)_{n \geq 1}$ for (2.5.14) is bounded in H^1 . The second claim (2) follows as in the proof of Lemma 2.5.8. For (3), let v be a solution of (2.5.14). Then, by the optimality of v and the fact that $v^+ \in \mathcal{A}(u, x_0, r)$ and $\Omega_{v^+} \subset \Omega_v$, one has

$$J(v^+) + J(v^-) = J(v) \leq J(v^+),$$

which means that $J(v^-) \leq 0$. Therefore,

$$\begin{aligned} \int_{B_r(x_0)} |\nabla v^-|^2 e^{-\Phi} dx &\leq \lambda_m \int_{B_r(x_0)} |v^-|^2 e^{-\Phi} dx \\ &\leq \lambda_m e^{2r\tau} C_d r^2 \int_{B_r(x_0)} |\nabla v^-|^2 e^{-\Phi} dx, \end{aligned}$$

where the second inequality is due to the fact that $\max_{B_r(x_0)} \Phi - \min_{B_r(x_0)} \Phi \leq 2r\tau$ and the variational characterization and the scaling of $\lambda_1(B_r, 0) = C_d r^{-2}$. Thus, for r small enough ($r \leq r_0$ with r_0 depending only on τ , λ_m and d), $v^- = 0$. \square

2.5.4 An internal variation optimality condition

Let $D \subset \mathbb{R}^d$ be a bounded open set, $u \in H_0^1(D)$ and $\xi \in C_c^\infty(D; \mathbb{R}^d)$. The first variation $\delta J(u)[\xi]$, of J at u in the direction ξ , is given by

$$\delta J(u)[\xi] := \lim_{t \rightarrow 0} \frac{J(u_t) - J(u)}{t}, \quad \text{where } u_t(x) := u(x + t\xi(x)).$$

A straightforward computation gives that

$$\delta J(u)[\xi] := \int_D \left[2D\xi(\nabla u) \cdot \nabla u + (|\nabla u|^2 - \lambda_m u^2)(\nabla \Phi \cdot \xi - \operatorname{div} \xi) \right] e^{-\Phi} dx. \quad (2.5.15)$$

In Proposition 2.5.12 we prove the existence of an Euler-Lagrange multiplier for every solution u of (2.5.11). We will use this result several times in this section and, in particular, in Lemma 2.5.30, where we will derive an optimality condition for u on the free boundary $\partial\Omega_u$.

Proposition 2.5.12 (Euler-Lagrange equation). *Let $D \subset \mathbb{R}^d$ and $u \in H_0^1(D)$ satisfy (2.5.12). Then, there exists $\Lambda_u > 0$ such that*

$$\delta J(u)[\xi] = \Lambda_u \int_{\Omega_u} \operatorname{div} \xi dx \quad \text{for every } \xi \in C_c^\infty(D; \mathbb{R}^d). \quad (2.5.16)$$

Moreover, for every $x_0 \in \partial\Omega_u \cap \partial D$ and every $r > 0$, we have

$$\delta J(u)[\xi] \geq \Lambda_u \int_{\Omega_u} \operatorname{div} \xi dx,$$

for every $\xi \in C_c^\infty(B_r(x_0), \mathbb{R}^d)$ such that $(\operatorname{Id} + \xi)^{-1}(D_r(x_0)) \subset D_r(x_0)$.

Proof. Let $\xi \in C_c^\infty(D; \mathbb{R}^d)$ and $u_t(x) = u(x + t\xi(x))$. Then we have

$$|\Omega_{u_t}| = |\Omega_u| - t \int_{\Omega_u} \operatorname{div} \xi dx + o(t). \quad (2.5.17)$$

Step 1. We first notice that if $B_r(x_0) \subset \mathbb{R}^d$ is a ball such that

$$D_r(x_0) := B_r(x_0) \cap D \quad \text{is connected} \quad \text{and} \quad 0 < |D_r(x_0) \cap \Omega_u| < |D_r(x_0)|,$$

then there is a vector field $\xi_0 \in C_c^\infty(D_r(x_0); \mathbb{R}^d)$ such that $\int_{\Omega_u} \operatorname{div} \xi_0 dx = 1$. Indeed, if this is not the case, then we have

$$\int_{\Omega_u} \operatorname{div} \xi dx = 0 \quad \text{for every } \xi \in C_c^\infty(D_r(x_0); \mathbb{R}^d).$$

For every ball $B_\rho(x_1) \subset D_r(x_0)$, take a vector field of the form $\xi(x) = (x - x_1)\phi_\varepsilon(x)$ with $0 \leq \phi_\varepsilon \leq 1$ on $B_\rho(x_1)$, ϕ radially decreasing in $B_\rho(x_1)$ with $|\nabla \phi_\varepsilon| \leq C(\rho\varepsilon)^{-1}$, $\phi_\varepsilon = 1$ on

$B_{\rho(1-\varepsilon)}(x_1)$ and $\phi_\varepsilon = 0$ on $\partial B_\rho(x_1)$. Then we have $\int_{\Omega_u} (d\phi_\varepsilon(x) + (x - x_1) \cdot \nabla \phi_\varepsilon(x)) dx = 0$ and, passing to the limit as $\varepsilon \rightarrow 0$, we get

$$d|\Omega_u \cap B_\rho(x_1)| - \rho \mathcal{H}^{d-1}(\Omega_u \cap \partial B_\rho(x_1)) = 0.$$

In particular, we get that the map $\rho \mapsto \rho^{-d}|\Omega_u \cap B_\rho(x_1)|$ is constant. Since the above identity holds for all balls $B_\rho(x_1) \subset D_r(x_0)$, we get that $|\Omega_u \cap D_r(x_0)| = 0$ or $|\Omega_u \cap D_r(x_0)| = |D_r(x_0)|$, which concludes the proof of the claim.

Step 2. We now prove the first statement of the proposition. Let $\xi_0 \in C_c^\infty(D; \mathbb{R}^d)$ be as in Step 1 and $\xi \in C_c^\infty(D; \mathbb{R}^d)$. There are two cases:

If $\int_{\Omega_u} \operatorname{div} \xi dx = 0$, define $\xi_1 = \xi + \eta \xi_0$ with $\eta > 0$ so that $\int_{\Omega_u} \operatorname{div} \xi_1 dx = \eta$.

Set $u_t(x) = u(x + t\xi_1(x))$. Then, for t small enough, $u_t \in H_0^1(D)$, $|\Omega_{u_t}| \leq |\Omega_u| = m$ and

$$J(u_t) = J(u) + t \delta J(u)[\xi_1] + o(t).$$

By the minimality of u we have $J(u) \leq J(u_t)$ and so, $\delta J(u)[\xi_1] \geq 0$. Therefore,

$$\delta J(u)[\xi] \geq -\eta \delta J(u)[\xi_0] \quad \text{for every } \eta > 0,$$

and hence, we get $\delta J(u)[\xi] \geq 0$. Taking $-\xi$ instead of ξ we have that $\delta J(u)[\xi] = 0$, and hence (2.5.16) holds for any $\Lambda_u \geq 0$.

If $\int_{\Omega_u} \operatorname{div} \xi dx \neq 0$, define $\xi_2 := \xi - \xi_0 \int_{\Omega_u} \operatorname{div} \xi dx$. Then $\int_{\Omega_u} \operatorname{div} \xi_2 dx = 0$ and, by the preceding case, we have $\delta J(u)[\xi_2] = 0$. On the other hand,

$$\delta J(u)[\xi_2] = \delta J(u)[\xi] - \delta J(u)[\xi_0] \int_{\Omega_u} \operatorname{div} \xi dx,$$

which proves (2.5.16) with $\Lambda_u := \delta J(u)[\xi_0]$. Moreover, for t small enough, $u_t(x) = u(x + t\xi(x)) \in H_0^1(D)$ and, by the minimality of u , we have

$$J(u) \leq J(u_t) = J(u) + t\Lambda_u + o(t),$$

which proves that $\Lambda_u \geq 0$. The strict inequality follows by a general result (Proposition 2.A.1) for minimizers of J with respect to internal perturbations.

Step 3. Let $x_0 \in \partial\Omega_u \cap \partial D$, $r > 0$ and $\xi_0 \in C_c^\infty(D; \mathbb{R}^d)$ be as in Step 1 so that we have $\delta J(u)[\xi_0] = \Lambda_u$. For any $\xi \in C_c^\infty(B_r(x_0), \mathbb{R}^d)$ such that $(Id + \xi)^{-1}(D_r(x_0)) \subset D_r(x_0)$, we set $\xi_1 = \xi - (1 - \eta)\xi_0 \int_{\Omega_u} \operatorname{div} \xi dx$ where η is some positive constant. Note that the vector field ξ_1 is such that $u_t(x) = u(x + t\xi_1(x)) \in H_0^1(D)$ for small $t > 0$ and $\int_{\Omega_u} \operatorname{div} \xi_1 dx = \eta > 0$. Therefore, using the minimality of u , we have for every $t > 0$ small enough

$$J(u) \leq J(u_t) = J(u) + t\delta J(u)[\xi_1] + o(t),$$

so that we get $\delta J(u)[\xi_1] \geq 0$. It follows that $\delta J(u)[\xi] \geq (1 - \eta)\Lambda_u$ for every $\eta > 0$, which concludes the proof. \square

In the following lemma we show that the Lagrange multipliers, associated to the solutions of variational problems with measure constraint in a fixed ball $B_r(x_0)$, are continuous with respect to variations of the measure constraint around m . This lemma will be used several times in the proof of the optimality of the blow-up limits.

Lemma 2.5.13 (Convergence of the Lagrange multipliers). *Let $D \subset \mathbb{R}^d$ and $u \in H_0^1(D)$ be as in (2.5.12), and let Λ_u be the constant from (2.5.16). Let $B_r(x_0) \subset \mathbb{R}^d$ be a ball such that*

$$D_r(x_0) := B_r(x_0) \cap D \quad \text{is connected} \quad \text{and} \quad 0 < |D_r(x_0) \cap \Omega_u| < |D_r(x_0)|.$$

Let the sequence $(m_n)_{n \geq 1}$ be such that $\lim_{n \rightarrow \infty} m_n = m$. Then, for n big enough, there is a solution $u_n \in \mathcal{A}(u, x_0, r)$ of the problem

$$\min \left\{ J(v) : v \in \mathcal{A}(u, x_0, r), |\Omega_v| \leq m_n \right\}. \quad (2.5.18)$$

Moreover, up to a subsequence, we have:

(a) for every n there is a Lagrange multiplier $\Lambda_{u_n} > 0$ for which (2.5.16) holds for u_n in $D_r(x_0)$;

(b) for every n there is a vector field $\xi_n \in C_c^\infty(D_r(x_0); \mathbb{R}^d)$ such that

$$\frac{d}{dt} \Big|_{t=0} J(u_n^t) = \Lambda_{u_n} \quad \text{and} \quad \frac{d}{dt} \Big|_{t=0} |\Omega_{u_n^t}| = -1 \quad \text{where} \quad u_n^t(x) := u_n(x + t\xi_n(x)); \quad (2.5.19)$$

(c) u_n converges strongly in $H_0^1(D)$ and pointwise almost everywhere to a function $u_\infty \in \mathcal{A}(u, x_0, r)$ which is a solution of (2.5.14);

(d) the sequence of characteristic functions $\mathbb{1}_{\Omega_{u_n}}$ converges to $\mathbb{1}_{\Omega_{u_\infty}}$ pointwise almost everywhere and strongly in $L^2(D)$;

(e) if we have $0 < |\Omega_u \setminus B_r(x_0)| < |D \setminus B_r(x_0)|$, then $\lim_{n \rightarrow \infty} \Lambda_{u_n} = \Lambda_u$.

Furthermore, if D is of class C^1 , $\partial D \cap B_r(x_0) \neq \emptyset$ and $m_n < m$, for every n large enough, then all these properties still hold even if the assumption $|\Omega_u \cap D_r(x_0)| < |D_r(x_0)|$ is not satisfied.

Proof. First of all, we notice that since $|\Omega_u \setminus D_r(x_0)| < m < |\Omega_u \cup D_r(x_0)|$, we may assume that the same holds for every m_n , for n large enough. Thus, by Lemma 2.5.11, the problem (2.5.18) has a solution u_n such that $|\Omega_{u_n}| = m_n$. Then, it follows that u_n satisfies

$$0 < |\Omega_{u_n} \cap D_r(x_0)| < |D_r(x_0)|. \quad (2.5.20)$$

Therefore, by Step 1 of the proof of Proposition 2.5.12, there is a vector field $\xi_n \in C_c^\infty(D_r(x_0); \mathbb{R}^d)$ such that $\int_{\Omega_{u_n}} \operatorname{div} \xi_n dx = 1$, and, reasoning as in Proposition 2.5.12, there exists $\Lambda_{u_n} > 0$ such that

$$\delta J(u_n)[\xi] = \Lambda_{u_n} \int_{\Omega_{u_n}} \operatorname{div} \xi dx \quad \text{for every} \quad \xi \in C_0^\infty(D_r(x_0), \mathbb{R}^d). \quad (2.5.21)$$

Moreover, taking $u_n^t(x) = u_n(x + t\xi_n(x))$, we obtain (2.5.19). This proves (a) and (b). We notice that the only difference with Proposition 2.5.12 is that in the present case, u_n is only a solution of a variational problem in $B_r(x_0)$.

Let now n be fixed and $\xi_0 \in C_c^\infty(B_r(x_0); \mathbb{R}^d)$ be the vector field, from the proof of Proposition 2.5.12, associated to u . Then, taking $u_t(x) := u(x + t\xi_0(x))$, we have that

$$\frac{d}{dt} \Big|_{t=0} |\Omega_{u_t}| = - \int_{\Omega_u} \operatorname{div} \xi_0 dx = -1,$$

and so, for n large enough, there is a unique $t_n \in \mathbb{R}$ such that $|\Omega_{u_n}| = m_n = |\Omega_{u_{t_n}}|$. In particular, there are constants C and n_0 , depending on u and ξ_0 , but not on n , such that

$$J(u_n) \leq J(u_{t_n}) \leq C \quad \text{for every} \quad n \geq n_0.$$

Then, by Remark 2.5.10, $(u_n)_{n \geq 1}$ is uniformly bounded in $H_0^1(D)$, so up to a subsequence, u_n converges weakly in H^1 , strongly in L^2 and pointwise a.e. to a function $u_\infty \in \mathcal{A}(u, x_0, r)$. Now, since the pointwise convergence implies $\mathbb{1}_{\Omega_{u_\infty}} \leq \liminf \mathbb{1}_{\Omega_{u_n}}$, we get that $|\Omega_{u_\infty}| \leq \liminf m_n = m$. In particular, $J(u) \leq J(u_\infty)$. On the other hand, the weak H^1 convergence of u_n gives that

$$J(u_\infty) \leq \liminf_{n \rightarrow \infty} J(u_n) \leq \liminf_{n \rightarrow \infty} J(u_{t_n}) = J(u),$$

so, we get $J(u_\infty) = J(u)$, u_∞ is a solution of (2.5.14), $|\Omega_{u_\infty}| = m$ (by the saturation of the constraint). Moreover, $J(u_n) \rightarrow J(u_\infty)$ since we have

$$\limsup_{n \rightarrow \infty} J(u_n) \leq \limsup_{n \rightarrow \infty} J(u_{t_n}) = J(u) \leq J(u_\infty) \leq \liminf_{n \rightarrow \infty} J(u_n).$$

But u_n strongly converges in $L^2(D)$ to u_∞ so that it gives $\int_D e^{-\Phi} |\nabla u_n|^2 dx \rightarrow \int_D e^{-\Phi} |\nabla u_\infty|^2 dx$, which means that the convergence of u_n to u is strong in $H_0^1(D)$.

We now check that the convergence of $\mathbb{1}_{\Omega_{u_n}}$ to $\mathbb{1}_{\Omega_{u_\infty}}$ is strong in L^2 . Indeed, for all non-negative function $\varphi \in L^2(D)$, the Fatou lemma shows that

$$\int_D \mathbb{1}_{\Omega_{u_\infty}} \varphi \leq \int_D \liminf_{n \rightarrow \infty} \mathbb{1}_{\Omega_{u_n}} \varphi \leq \liminf_{n \rightarrow \infty} \int_D \mathbb{1}_{\Omega_{u_n}} \varphi. \quad (2.5.22)$$

Up to a subsequence, there exists $h \in L^2(D)$ such that $\mathbb{1}_{\Omega_{u_n}} \rightharpoonup h$ weakly in $L^2(D)$. Thus, (2.5.22) yields $\mathbb{1}_{\Omega_{u_\infty}} \leq h$. Moreover, $\|h\|_2 \leq \liminf \|\mathbb{1}_{\Omega_{u_n}}\|_2$. As a consequence, $\|h\|_2 = m^{1/2}$, which entails that $\mathbb{1}_{\Omega_{u_n}} \rightarrow h$ strongly in $L^2(D)$. Since $\mathbb{1}_{\Omega_{u_\infty}} \leq h$, we conclude that $\mathbb{1}_{\Omega_{u_n}} \rightarrow \mathbb{1}_{\Omega_{u_\infty}}$ strongly in $L^2(D)$, and so, up to a subsequence $\mathbb{1}_{\Omega_{u_n}}$ converges to $\mathbb{1}_{\Omega_{u_\infty}}$ pointwise almost everywhere. This proves (c) and (d).

In order to prove (e), we first notice that u and u_∞ are both solutions of (2.5.11). This is due to the fact that $J(u_\infty) = J(u)$. Therefore, there is a Lagrange multiplier Λ_∞ such that

$$\delta J(u_\infty)[\xi] = \Lambda_\infty \int_{\Omega_{u_\infty}} \operatorname{div} \xi dx \quad \text{for every } \xi \in C_c^\infty(D; \mathbb{R}^d), \quad (2.5.23)$$

Moreover, by (c) and (d), we get that

$$\delta J(u_\infty)[\xi] = \lim_{n \rightarrow \infty} \delta J(u_n)[\xi] \quad \text{and} \quad \int_{\Omega_{u_\infty}} \operatorname{div} \xi dx = \lim_{n \rightarrow \infty} \int_{\Omega_{u_n}} \operatorname{div} \xi dx,$$

for every $\xi \in C_c^\infty(D_r(x_0); \mathbb{R}^d)$. Now, choosing $\xi \in C_c^\infty(D_r(x_0); \mathbb{R}^d)$ such that $\int_{\Omega_{u_\infty}} \operatorname{div} \xi dx \neq 0$ and using (2.5.23) and (2.5.21) we get that Λ_{u_n} converges to Λ_∞ . Finally, if we have $0 < |\Omega_u \setminus B_r(x_0)| < |D \setminus B_r(x_0)|$, there exists $\xi \in C_c^\infty(D \setminus B_r(x_0); \mathbb{R}^d)$ such that $\int_{\Omega_{u_\infty}} \operatorname{div} \xi dx \neq 0$, so that $\Lambda_\infty = \Lambda_u$ since $u = u_\infty$ outside the ball $B_r(x_0)$.

The proof of the last statement of the Proposition is very similar. We have $|\Omega_u \setminus D_r(x_0)| < m = |\Omega_u \cup D_r(x_0)|$ so that, since $m_n < m$, we have $|\Omega_u \setminus D_r(x_0)| < m_n < |\Omega_u \cup D_r(x_0)|$ for every n large enough. It follows from Lemma 2.5.11 that the problem (2.5.18) has a solution u_n with $|\Omega_{u_n}| = m_n$ and such that (2.5.20) holds. Note also that there exists a vector field $\xi_0 \in C_0^\infty(B_r(x_0), \mathbb{R}^d)$ such that $(Id + t\xi_0)^{-1}(D_r(x_0)) \subset D_r(x_0)$ for every small $t > 0$ and $\int_{\Omega_u} \operatorname{div} \xi_0 dx = 1$. Indeed, since D is C^1 -regular and $\partial D \cap B_r(x_0)$ is not empty, we can take a point $x_1 \in \partial D \cap B_r(x_0)$ and a sufficiently small neighborhood of x_1 in which ∂D is the graph of a C^1 function in the direction $e \in \mathbb{R}^d$. It is now sufficient to take ξ_0 of the form $e\phi$, where $\phi \in C_c^\infty(\mathbb{R}^d)$ is a smooth nonnegative function with (sufficiently small) compact support and such that $\phi(x_1) = 1$. Moreover, we have $t_n > 0$ (since $m_n < m$) and hence $u_{t_n} \in H_0^1(D)$. The rest of the proof is unchanged. \square

2.5.5 Almost optimality of u at small scales

Let u be a solution of (2.5.1) in $D \subset \mathbb{R}^d$. For $x_0 \in \mathbb{R}^d$ and $h > 0$, we define the upper and the lower Lagrange multipliers, $\mu_-(h, x_0, r) \geq 0$ and $\mu_+(h, x_0, r) \geq 0$, by

$$\begin{aligned} \mu_+(h, x_0, r) &= \inf\{\mu \geq 0 : J(u) + \mu|\Omega_u| \leq J(v) + \mu|\Omega_v|, \forall v \in \mathcal{A}(u, x_0, r), m \leq |\Omega_v| \leq m+h\}, \\ \mu_-(h, x_0, r) &= \sup\{\mu \geq 0 : J(u) + \mu|\Omega_u| \leq J(v) + \mu|\Omega_v|, \forall v \in \mathcal{A}(u, x_0, r), m-h \leq |\Omega_v| \leq m\}. \end{aligned}$$

Remark 2.5.14 ($\mu_- \leq \Lambda_u \leq \mu_+$). We notice that if $B_r(x_0) \subset \mathbb{R}^d$ is a ball such that $D_r(x_0) := D \cap B_r(x_0)$ is connected and $0 < |D_r(x_0) \cap \Omega_u| < |D_r(x_0)|$, then

$$\mu_-(h, x_0, r) \leq \Lambda_u \leq \mu_+(h, x_0, r) \quad \text{for every } h > 0.$$

Indeed, by Step 1 of the proof of Proposition 2.5.12, there is a vector field $\xi \in C_c^\infty(D_r(x_0); \mathbb{R}^d)$ such that $\int_{\Omega_u} \operatorname{div} \xi \, dx = 1$. Let $u_t(x) = u(x + t\xi(x))$. Then for $|t|$ small enough $u_t \in \mathcal{A}(u, x_0, r)$ and $m - h < |\Omega_{u_t}| < m + h$. Moreover, for every $\mu \geq 0$ we have

$$J(u_t) + \mu|\Omega_{u_t}| = J(u) + t\Lambda_u + \mu(|\Omega_u| - t) + o(t). \quad (2.5.24)$$

Now, if $t > 0$ is small enough and $\Lambda_u < \mu$, then $m > |\Omega_{u_t}|$ and, by (2.5.24), $J(u_t) + \mu|\Omega_{u_t}| < J(u) + \mu|\Omega_u|$, which proves that $\Lambda_u \geq \mu_-(h, x_0, r)$. Analogously, if $t < 0$ and $\Lambda_u > \mu$, then $m < |\Omega_{u_t}|$ and again $J(u_t) + \mu|\Omega_{u_t}| < J(u) + \mu|\Omega_u|$, which gives that $\Lambda_u \leq \mu_+(h, x_0, r)$.

Remark 2.5.15 (Monotonicity of μ_+ and μ_-). We notice that the following inclusion holds:

$$\mathcal{A}(u, x, r) \subseteq \mathcal{A}(u, x_0, r_0) \quad \text{for every } B_r(x) \subset B_{r_0}(x_0).$$

In particular, for every $0 < h \leq h_0$ and every $B_r(x) \subset B_{r_0}(x_0)$, we have

$$\mu_-(h_0, x_0, r_0) \leq \mu_-(h, x, r) \quad \text{and} \quad \mu_+(h, x, r) \leq \mu_+(h_0, x_0, r_0).$$

Theorem 2.5.16 (Convergence of the upper and the lower Lagrange multipliers). *Let u be a solution of (2.5.1) in the connected bounded open set $D \subset \mathbb{R}^d$ and let Λ_u be given by Proposition 2.5.12. Then there exists a constant $r_0 > 0$, which depends only on τ, λ_m and d , with the following property: for every ball $B_r(x_0) \subset \mathbb{R}^d$ centred at $x_0 \in \partial\Omega_u$ with $r \leq r_0$ and such that*

$$D_r(x_0) := B_r(x_0) \cap D \text{ is connected} \quad \text{and} \quad 0 < |\Omega_u \cap D_r(x_0)| < |D_r(x_0)|, \quad (2.5.25)$$

we have

$$\lim_{h \rightarrow 0} \mu_+(h, x_0, r_0) = \lim_{h \rightarrow 0} \mu_-(h, x_0, r_0) = \Lambda_u.$$

If, moreover, D is of class C^1 , then there exists a constant $r_1 > 0$, which depends only on τ, λ_m, d and D , such that, for every ball $B_r(x_0)$ centred at $x_0 \in \partial\Omega_u \cap \partial D$ with $r \leq r_1$, we have

$$\lim_{h \rightarrow 0} \mu_-(h, x_0, r_0) = \Lambda_u.$$

Proof of Theorem 2.5.16: Let $x_0 \in \partial\Omega_u$ be such that (2.5.25) holds and let $h > 0$ be small. We set for simplicity $r = r_0$, $B_r(x_0) = B_r$, $\mu_+(h) := \mu_+(h, x_0, r)$ and $\mu_-(h) := \mu_-(h, x_0, r)$. We proceed in three steps.

Step 1. We first prove that $\mu_+(h)$ is finite. Let, for any $n \in \mathbb{N}$, $v_n \in \mathcal{A}(u, x_0, r)$ be a solution of the variational problem

$$\min \{ J(v) + n(|\Omega_v| - m)_+ : v \in \mathcal{A}(u, x_0, r), |\Omega_v| \leq m + h \}. \quad (2.5.26)$$

If there exists n such that $|\Omega_{v_n}| \leq m$, then $\mu_+(h) \leq n$ and hence $\mu_+(h)$ is finite. Indeed, by the minimality of u and the definition of v_n , we have for every $v \in \mathcal{A}(u, x_0, r)$ such that $m \leq |\Omega_v| \leq m + h$

$$J(u) + n|\Omega_u| \leq J(v_n) + n|\Omega_u| \leq J(v) + n|\Omega_v|,$$

so that $\mu_+(h) \leq n$ and the inequality $\mu_+(h) < \infty$ holds.

Suppose, by contradiction, that $|\Omega_{v_n}| > m$ for every n . First notice that since $J(v_n)$ is bounded from below (see Remark 2.5.10) and $J(v_n) + n(|\Omega_{v_n}| - m) \leq J(u)$, we have that $|\Omega_{v_n}| \rightarrow m$ as $n \rightarrow \infty$. Since v_n is a solution of (2.5.18) with $m_n := |\Omega_{v_n}|$, there is a Lagrange multiplier

Λ_{v_n} such that (2.5.16) holds for v_n and a vector field ξ_n such that (2.5.19) holds for $v_n^t(x) = v_n(x + t\xi_n(x))$. For $t > 0$ small enough, $v_n^t \in \mathcal{A}(u, x_0, r)$ and $m < |\Omega_{v_n^t}| < m + h$. Then, by the minimality of v_n we have

$$J(v_n) + n(|\Omega_{v_n}| - m) \leq J(v_n^t) + n(|\Omega_{v_n^t}| - m) = J(v_n) + t\Lambda_{v_n} + n(|\Omega_{v_n}| - t - m) + o(t),$$

which implies $n \leq \Lambda_{v_n}$, in contradiction with $\lim_{n \rightarrow \infty} \Lambda_{u_n} = \Lambda_u$ from Lemma 2.5.13.

Step 2. $\lim_{h \rightarrow 0} \mu_+(h) = \Lambda_u$. Let $(h_n)_{n \geq 1}$ be a decreasing sequence such that $h_n \rightarrow 0$. Since $\Lambda_u \leq \mu_+(h)$ and $h \mapsto \mu_+(h)$ is non-decreasing, it is sufficient to prove that $\lim_{n \rightarrow \infty} \mu_+(h_n) = \Lambda_u$. Fix $\varepsilon \in (0, \Lambda_u)$ and let $0 < \alpha_n := \mu_+(h_n) - \varepsilon < \mu_+(h_n)$. Let u_n be the solution of the problem

$$\min \{ J(v) + \alpha_n(|\Omega_v| - m)^+ : v \in \mathcal{A}(u, x_0, r), |\Omega_v| \leq m + h_n \}.$$

Notice that $|\Omega_{u_n}| > m$, since otherwise we would have $J(u) \leq J(u_n) + \alpha_n(|\Omega_{u_n}| - m)^+$, which contradicts the definition of $\mu_+(h_n)$. For n large enough, (2.5.25) holds with u_n , and since u_n is solution of (2.5.18) with $m_n = |\Omega_{u_n}|$, by Proposition 2.5.12, there is a Lagrange multiplier $\Lambda_{u_n} \geq 0$ and a vector field ξ_n such that (2.5.19) holds for $u_n^t(x) := u_n(x + t\xi_n(x))$. By the minimality of u_n , for $t > 0$ small enough, we have

$$J(u_n) + \alpha_n(|\Omega_{u_n}| - m) \leq J(u_n^t) + \alpha_n(|\Omega_{u_n^t}| - m) = J(u_n) + t\Lambda_{u_n} + \alpha_n(|\Omega_{u_n}| - t - m) + o(t),$$

which shows that $\Lambda_{u_n} \geq \alpha_n$. By Lemma 2.5.13 we have

$$\lim_{n \rightarrow \infty} \mu_+(h_n) - \varepsilon = \lim_{n \rightarrow \infty} \alpha_n \leq \lim_{n \rightarrow \infty} \Lambda_{u_n} = \Lambda_u,$$

which proves the claim since $\varepsilon > 0$ is arbitrary.

Step 3. $\lim_{h \rightarrow 0} \mu_-(h) = \Lambda_u$. We prove this result for any $x_0 \in \partial\Omega_u$, which will conclude the proof of the Theorem. Note that the smoothness of D implies that there exists a constant $c_D > 0$ such that $D_r(x_0)$ is connected for every $r \leq r_D$ and every $x_0 \in \partial\Omega_u \cap \partial D$.

Let $\varepsilon > 0$ and $(h_n)_{n \in \mathbb{N}}$ be a decreasing infinitesimal sequence. We will show that $\Lambda_u - \varepsilon \leq \lim_{n \rightarrow \infty} \mu_-(h_n)$. Let u_n be a solution of the problem

$$\min \{ J(v) + (\mu_-(h_n) + \varepsilon)(|\Omega_v| - (m - h_n))^+ : v \in \mathcal{A}(u, x_0, r), |\Omega_v| \leq m \}. \quad (2.5.27)$$

Up to replacing u_n by u_n^+ , we can assume that $u_n \geq 0$ in B_r (the argument is similar to the proof of Lemma 2.5.11). We claim that

$$m - h_n \leq |\Omega_{u_n}| < m. \quad (2.5.28)$$

Suppose that $|\Omega_{u_n}| = m$. By the minimality of u and u_n we get

$$J(u) + (\mu_-(h_n) + \varepsilon)|\Omega_u| \leq J(u_n) + (\mu_-(h_n) + \varepsilon)|\Omega_{u_n}| \leq J(v) + (\mu_-(h_n) + \varepsilon)|\Omega_v|,$$

for every $v \in \mathcal{A}(u, x_0, r)$ such that $m - h_n \leq |\Omega_v| \leq m$, which contradicts the definition of $\mu_-(h_n)$. Now, if $|\Omega_{u_n}| < m - h_n$, we have $J(u_n) \leq J(u_n + t\varphi)$ for every $\varphi \in C_c^\infty(D_r(x_0))$ with sufficiently small compact support. Thus u_n solves the PDE $-\operatorname{div}(e^{-\Phi}\nabla u_n) = \lambda_m e^{-\Phi} u_n$ in $D_r(x_0)$. Since $u_n \geq 0$ in $D_r(x_0)$, by the strong maximum principle, we have that either $u_n \equiv 0$ or $u_n > 0$ in $D_r(x_0)$, in contradiction with (2.5.20). Thus, we proved (2.5.28).

We have that u_n is solution of (2.5.18) with $m_n := |\Omega_{u_n}|$ which converges to m as $n \rightarrow \infty$. By Lemma 2.5.13, we have an Euler-Lagrange equation for u_n in B_r for some Λ_{u_n} . Let $\xi_n \in C_c^\infty(D_r(x_0); \mathbb{R}^d)$ be the vector field from Lemma 2.5.13 (b) and let $u_n^t(x) = u_n(x + t\xi_n(x))$. For negative $t < 0$ and $|t|$ small enough, $u_n^t \in \mathcal{A}(u, x_0, r)$ and $|\Omega_{u_n^t}| \leq |\Omega_{u_n}| < m$. Thus, by the minimality of u_n , we get

$$J(u_n) + (\mu_-(h_n) + \varepsilon)(|\Omega_{u_n}| - (m - h_n)) \leq J(u_n) + \Lambda_{u_n} t + (\mu_-(h_n) + \varepsilon)(|\Omega_{u_n}| - t - (m - h_n)) + o(t),$$

which implies that $\Lambda_{u_n} \leq \mu_-(h_n) + \varepsilon$. Now, by Lemma 2.5.13, we get

$$\Lambda_u = \lim_{n \rightarrow \infty} \Lambda_{u_n} \leq \lim_{n \rightarrow \infty} \mu_-(h_n) + \varepsilon,$$

which concludes the proof. \square

Remark 2.5.17 (Quasi-minimality at small scales). Suppose that $D \subset \mathbb{R}^d$ is just a bounded open set. By the monotonicity of μ_+ and μ_- with respect to the inclusion (Remark 2.5.15) and a covering argument we get that for every compact set $\mathcal{K} \subset D$ there is $r(\mathcal{K}) > 0$ such that: for every $\varepsilon > 0$ there is $h > 0$ such that

$$\mu_+(h, x, r) - \varepsilon \leq \Lambda_u \leq \mu_-(h, x, r) + \varepsilon \quad \text{for every } x \in \mathcal{K} \cap \partial\Omega_u \quad \text{and every } 0 < r \leq r(\mathcal{K}).$$

If, moreover, D is of class $C^{1,1}$, then there exists $r_D > 0$ such that, for every $\varepsilon > 0$ there exists $h > 0$ such that: for every $0 < r \leq r_D$ and every $x_0 \in \partial\Omega_u$ we have

$$\begin{aligned} \mu_+(h, x, r) - \varepsilon \leq \Lambda_u \leq \mu_-(h, x, r) + \varepsilon & \quad \text{if } |\Omega_u \cap D_r(x_0)| < |D_r(x_0)|, \\ \Lambda_u \leq \mu_-(h, x, r) + \varepsilon & \quad \text{otherwise.} \end{aligned}$$

2.5.6 Lipschitz continuity of the eigenfunctions on the optimal sets

In this subsection we prove that the solutions of (2.5.1) are (locally) Lipschitz continuous in D . For $\delta > 0$ we set $D_\delta = \{x \in D : d(x, \partial D) > \delta\}$ and let $\mu > 0$ be fixed. By Theorem 2.5.16 and Remark 2.5.17 we get that if u is a solution of (2.5.1) and $\mu > \Lambda_u$, then there is $r_0 > 0$ such that, for every $x_0 \in \partial\Omega_u \cap D_\delta$, we have

$$J(u) + \mu|\Omega_u| \leq J(v) + \mu|\Omega_v| \quad \text{for every } v \in \mathcal{A}(u, x_0, r_0) \quad \text{such that } |\Omega_v| \geq |\Omega_u|. \quad (2.5.29)$$

Note that the condition $|\Omega_v| \leq |\Omega_u| + h$ can be dropped by choosing r_0 such that $|B_{r_0}| \leq h$. We will prove that if $u \in H^1(B_{r_0})$ is bounded, nonnegative and satisfies (2.5.5) and (2.5.29), then u is Lipschitz in D_δ .

Proposition 2.5.18 (Lipschitz continuity of the eigenfunctions on the optimal sets). *Let $D \subset \mathbb{R}^d$ be a connected bounded open set. Let $\tau \geq 0$, $m \in (0, |D|)$ and $\Phi \in W^{1,\infty}(D)$. Then, every solution of (2.5.1) is locally Lipschitz continuous in D . More precisely, it is Lipschitz in D_δ for all $\delta > 0$. Moreover, if the box D is of class $C^{1,\beta}$, for some $\beta > 0$, then u (extended by 0 in $\mathbb{R}^d \setminus D$) is Lipschitz in \mathbb{R}^d .*

The proof is based on the following lemma, whose (more general) two-phase counterpart can be found for instance in [3], [10] and [18].

Lemma 2.5.19 (A bound on the measure $\text{div}(e^{-\Phi}\nabla u)$). *Let D and u be as in (2.5.12) and let $r_0 > 0$ be such that u satisfies (2.5.29) for some $\mu > \Lambda_u$. Then, there is a constant $C > 0$ such that for every $x_0 \in \partial\Omega_u \cap D_\delta$ we have*

$$|\text{div}(e^{-\Phi}\nabla u)|(B_r(x)) \leq Cr^{d-1} \quad \text{for every ball } B_{2r}(x) \subset B_{r_0}(x_0). \quad (2.5.30)$$

Proof. Let $x = 0$ and $\eta \in C_c^\infty(B_{2r})$ be such that

$$0 \leq \eta \leq 1 \quad \text{in } B_{2r}, \quad \eta = 1 \quad \text{in } B_r, \quad \|\nabla\eta\|_{L^\infty} \leq \frac{C_d}{r}.$$

Using $u + t\eta$ as a test function for J , and setting $\langle f, g \rangle := \int_D fg \, dx$, we get

$$2\langle \text{div}(e^{-\Phi}\nabla u) + \lambda_m u e^{-\Phi}, \eta \rangle \leq tJ(\eta) + \frac{\mu}{t}|B_{2r}| \leq C \left(t\|\nabla\eta\|_{L^2}^2 + \frac{r^d}{t} \right)$$

where the constant $C > 0$ depends on d , Φ and μ . Now, minimizing over $t > 0$ and using the estimate $\|\nabla\eta\|_{L^2} \leq Cdr^{\frac{d}{2}-1}$, we get

$$\langle \operatorname{div}(e^{-\Phi}\nabla u) + \lambda_m u e^{-\Phi}, \eta \rangle \leq Cr^{d-1}.$$

By Lemma 2.5.5, we have that $\operatorname{div}(e^{-\Phi}\nabla u) + \lambda_m u e^{-\Phi}$ is a positive Radon measure. Thus, the inequality $\eta \geq \mathbb{1}_{B_r}$ and the boundedness of u imply

$$|\operatorname{div}(e^{-\Phi}\nabla u)|(B_r) \leq \lambda_m \int_{B_r} u e^{-\Phi} dx + \langle \operatorname{div}(e^{-\Phi}\nabla u) + \lambda_m u e^{-\Phi}, \mathbb{1}_{B_r} \rangle \leq Cr^{d-1}. \quad \square$$

The main ingredients of the proof of Proposition 2.5.18 will be Lemma 2.5.19 and the classical gradient estimate that we recall in the lemma below.

Lemma 2.5.20 (Gradient estimate). *Let $p > d$. Let \mathcal{U} be an open subset of \mathbb{R}^d and let $u \in W_{loc}^{2,p}(\mathcal{U}) \cap L^p(\mathcal{U})$ be a (strong) solution to the equation*

$$\sum_{i,j=1}^d a_{ij}(x) \partial_{ij} u + \sum_{i=1}^d b_i(x) \partial_i u = f \quad \text{in } \mathcal{U},$$

where $f \in L^p(\mathcal{U})$ and we suppose that:

- (a) the functions $a_{ij} : \mathcal{U} \rightarrow \mathbb{R}$ are Hölder continuous, that is, there are constants $C_a > 0$ and $\delta_a > 0$ such that

$$|a_{ij}(x) - a_{ij}(y)| \leq C_a |x - y|^{\delta_a} \quad \text{for every } x, y \in \mathcal{U};$$

- (b) there is a constant $M > 0$ such that

$$\|a_{ij}\|_{L^\infty(\mathcal{U})} \leq M \quad \text{and} \quad \|b_i\|_{L^\infty(\mathcal{U})} \leq M \quad \text{for every } 1 \leq i, j \leq d;$$

- (c) the matrix $(a_{ij})_{ij}$ is uniformly elliptic, that is, there is a constant $c_a > 0$ such that

$$\sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \geq c_a |\xi|^2 \quad \text{for every } x \in \mathcal{U} \quad \text{and} \quad \xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d.$$

Then, for any domain $\mathcal{U}' \subset\subset \mathcal{U}$, we have

$$\|u\|_{C^{1,\gamma}(\mathcal{U}')} \leq C (\|u\|_{L^p(\mathcal{U})} + \|f\|_{L^p(\mathcal{U})}) \quad \text{where } \gamma = 1 - \frac{d}{p},$$

and C is a constant depending on d , p , M , C_a , δ_a , c_a , \mathcal{U}' and \mathcal{U} .

Proof. First notice that by [48, Theorem 9.11], there is a constant C' such that

$$\|u\|_{W^{2,p}(\mathcal{U}')} \leq C' (\|u\|_{L^p(\mathcal{U})} + \|f\|_{L^p(\mathcal{U})}).$$

Now, the claim follows by the Sobolev inequality (see [43, Section 5.6, Theorem 6]). \square

Proof of Proposition 2.5.18. Let u be a solution of (2.5.1). We proceed in four steps.

Step 1. Ω_u is open. Let $\bar{x} \in \partial\Omega_u \cap D$. We will prove that $u(\bar{x}) = 0$. Let $r_1 > 0$ be such that $B_{r_1}(\bar{x}) \subset D$ and let $x_n \in B_{r_1/2}(\bar{x})$ be a sequence converging to \bar{x} such that $u(x_n) = 0$ (such a sequence exists by Lemma 2.5.7). By Lemma 2.5.19 and Lemma 2.5.5 (2.5.7), for every n and every $r \leq r_1/2$ we have

$$\int_{\partial B_r(x_n)} u d\mathcal{H}^{d-1} \leq u(x_n) + Cr = Cr,$$

where the constant C does not depend on n . Passing to the limit as $n \rightarrow \infty$, we get that

$$\int_{\partial B_r(\bar{x})} u d\mathcal{H}^{d-1} \leq Cr \quad \text{for every } r \leq r_1/2,$$

which, passing to the limit as $r \rightarrow 0$, proves that $u(\bar{x}) = 0$.

Step 2. Gradient estimate in Ω_u . We claim that, for every ball $B_r(\bar{x}) \subset \Omega_u$, there is a constant C_2 , depending only on Φ , d and λ_m , such that

$$\|\nabla u\|_{L^\infty(B_{r/2}(\bar{x}))} \leq \frac{C_2}{r} \|u\|_{L^\infty(B_r(\bar{x}))}. \quad (2.5.31)$$

Indeed, suppose that $\bar{x} = 0$ and set $\Phi_r(x) := \Phi(rx)$ and $u_r(x) = u(rx)$. Then u_r is a solution of

$$\operatorname{div}(e^{-\Phi_r} \nabla u_r) + \lambda_m e^{-\Phi_r} u_r = 0 \quad \text{in } B_1,$$

which can be re-written as

$$\Delta u_r - \nabla \Phi \cdot \nabla u_r = -\lambda_m u_r \quad \text{in } B_1.$$

Applying Lemma 2.5.20 with $\mathcal{U} = B_1$, $\mathcal{U}' = B_{1/2}$, $a_{ij} = \delta_{ij}$, $b = \nabla \Phi$, $M = 1 + \|\nabla \Phi\|_{L^\infty}$, $f = -\lambda_m u$ and any $p > d$, we get

$$\|\nabla u_r\|_{L^\infty(B_{1/2})} \leq \|u_r\|_{C^{1,\alpha}(B_{1/2})} \leq C_2 \|u_r\|_{L^\infty(B_1)},$$

which, after rescaling, is precisely (2.5.31).

Step 3. Proof of the local Lipschitz continuity. Let $\bar{x} \in \Omega_u \cap D_\delta$ and set $r := \operatorname{dist}(\bar{x}, \partial\Omega_u)$. Let $r_0 \in (0, \delta/2)$ be such that u satisfies (2.5.29) for every point x_0 on $\partial\Omega_u \cap D_{\delta/2}$ (such an r_0 exists by a standard covering argument). We now consider two cases

Case 1. If $r \geq r_0/6$, then the estimate (2.5.31) gives $|\nabla u(\bar{x})| \leq C_{r_0}$.

Case 2. If $r \leq r_0/6$, let \bar{y} be the projection of \bar{x} on $\partial\Omega_u$, that is, $\bar{y} \in \partial\Omega_u$ and $r = |\bar{x} - \bar{y}|$. Notice that in this case we have that $\bar{y} \in \partial\Omega_u \cap D_{\delta/2}$. Now, take any $\bar{z} \in B_r(\bar{x})$. By Lemma 2.5.19 and by the estimate (2.5.7) of Lemma 2.5.5, we have

$$u(\bar{z}) \leq \int_{\partial B_s(\bar{z})} u d\mathcal{H}^{d-1} + Cs \quad \text{for every } 0 < s \leq r,$$

where C is the constant in the right-hand side of (2.5.7). Now, multiplying by s^{d-1} and then integrating from 0 to r the above inequality, we get

$$\begin{aligned} u(\bar{z}) &\leq \int_{B_r(\bar{z})} u dx + Cr \leq 3^d \int_{B_{3r}(\bar{y})} u dx + Cr \\ &= \frac{d}{r^d} \int_0^{3r} s^{d-1} ds \int_{\partial B_s(\bar{y})} u d\mathcal{H}^{d-1} + Cr. \end{aligned}$$

Using again (2.5.7), this time for \bar{y} (at which $u(\bar{y}) = 0$ by Step 1 of the proof), we get that

$$u(\bar{z}) \leq 3^{d+1} Cr \quad \text{for every } \bar{z} \in B_r(\bar{x}).$$

Finally, using the estimate (2.5.31) this gives

$$|\nabla u(\bar{x})| \leq \|\nabla \bar{u}\|_{L^\infty(B_{r/2}(\bar{x}))} \leq \frac{C_2}{r} \|u\|_{L^\infty(B_r(\bar{x}))} \leq 3^{d+1} C_2 C. \quad (2.5.32)$$

This proves that $|\nabla u|$ is bounded in D_δ without assuming any regularity of D .

Step 4. Global Lipschitz estimate. We first notice that, since D is $C^{1,\beta}$ regular, the radius r_0 , for which (2.5.29) holds, does not depend on the point $x_0 \in \partial\Omega_u$. Now, let $\bar{x} \in \Omega_u \setminus D_{r_0}$ and set

$r := \text{dist}(\bar{x}, \partial\Omega_u \cap D)$. We consider the projection \bar{y} of \bar{x} on $\partial\Omega_u$ and we distinguish two cases. If $6r \leq \text{dist}(\bar{x}, \partial D)$, then we apply the estimate from Step 3 and we get that $|\nabla u(\bar{x})| \leq C$. If $6r \geq \text{dist}(\bar{x}, \partial D)$, we consider the solution w to the problem

$$-\text{div}(e^{-\Phi}\nabla w) = 1 \quad \text{in } D, \quad w \in H_0^1(D),$$

which is Lipschitz continuous in \mathbb{R}^d since D is of class $C^{1,\beta}$ (see [47, Theorem A]). Moreover, by the strong maximum principle, we have that $u \leq Cw$ for some constant C depending on λ_m , d and Φ . Therefore, setting $r_1 = \text{dist}(\bar{x}, \partial D)$, we have for every $\bar{z} \in B_{r_1}(\bar{x})$,

$$u(\bar{z}) \leq Cw(\bar{z}) \leq C|\bar{z} - \bar{y}| \leq Cr_1,$$

and we conclude by the gradient estimate (2.5.31). \square

2.5.7 Non-degeneracy of the eigenfunctions and finiteness of the perimeter of Ω_u

Let u be a solution of (2.5.1) in the connected bounded open set $D \subset \mathbb{R}^d$. Let $x_0 \in \partial\Omega_u$ and $r(x_0)$ be such that for every $0 < r \leq r(x_0)$ the set $D_r(x_0) := B_r(x_0) \cap D$ is connected. Notice that such an $r(x_0)$ trivially exists if $x_0 \in \partial\Omega_u \cap D$, while in the general case it is sufficient to assume some a priori regularity of the box D (C^1 is sufficient). Then, by Remark 2.5.17, for every $\mu < \Lambda_u$ there is some $r_0 > 0$ such that, for every $x_0 \in \partial\Omega_u$, we have

$$J(u) + \mu|\Omega_u| \leq J(v) + \mu|\Omega_v| \quad \text{for every } v \in \mathcal{A}(u, x_0, r_0) \quad \text{such that } |\Omega_v| \leq |\Omega_u|. \quad (2.5.33)$$

This property was first exploited by Alt and Caffarelli to prove the non-degeneracy of the solutions. More recently, it was exploited by Bucur who introduced the notion of a *shape subsolution* which found application to several shape optimization problems (see for example [12] and [19]).

Lemma 2.5.22 below is a fundamental step in the proof of the regularity of the free boundary since it allows to prove that the blow-up limits (see Subsection 2.5.8) are non trivial. It is the analogue of the non-degeneracy estimate from [2] and the proof is based on the same idea. Before we state it, we recall the following boundary estimate for solutions to elliptic PDEs.

Lemma 2.5.21 (Boundary gradient estimate). *Let $p > d$. Let \mathcal{U} be a bounded connected open subset of \mathbb{R}^d with $C^{1,1}$ boundary. Let T_1, \dots, T_k be the connected components of the boundary $\partial\mathcal{U}$ and let c_1, \dots, c_k are given constants. Let $u \in W_{loc}^{2,p}(\mathcal{U}) \cap L^p(\mathcal{U})$ be a (strong) solution to the problem*

$$\sum_{i,j=1}^d a_{ij}(x)\partial_{ij}u + \sum_{i=1}^d b_i(x)\partial_i u = f \quad \text{in } \mathcal{U}, \quad u = c_i \quad \text{in } T_i, \quad i = 1, \dots, k,$$

where $f \in L^p(\mathcal{U})$ and we suppose that $A = (a_{ij})_{ij}$ and $b = (b_1, \dots, b_d)$ satisfy the conditions (a), (b) and (c) of Lemma 2.5.21. Then, we have

$$\|u\|_{C^{1,\gamma}(\mathcal{U})} \leq C (\|u\|_{L^p(\mathcal{U})} + \|f\|_{L^p(\mathcal{U})}) \quad \text{where} \quad \gamma = 1 - \frac{d}{p},$$

and C is a constant depending on d , p , M , C_a , δ_a , c_a (defined in Lemma 2.5.20), and \mathcal{U} .

Proof. By [48, Theorem 9.13], there is a constant C' such that

$$\|u\|_{W^{2,p}(\mathcal{U})} \leq C' (\|u\|_{L^p(\mathcal{U})} + \|f\|_{L^p(\mathcal{U})}).$$

The claim follows by the Sobolev inequality (see for instance [43, Section 5.6, Theorem 6]). \square

Lemma 2.5.22 (Non-degeneracy of the eigenfunctions on the optimal sets). *Let u be a non-negative solution of (2.5.1) in the connected bounded open set $D \subset \mathbb{R}^d$. Suppose that $x_0 \in \partial\Omega_u$, $0 < \mu < \Lambda_u$ and $r_0 > 0$ are such that (2.5.33) holds. Then there are constants $c > 0$ and $r_1 > 0$ which depend only on τ, λ_m, μ and d , such that for every ball $B_{2r}(x) \subset B_{r_0}(x_0)$ with $r \leq r_1$, we have that:*

$$\text{If } \|u\|_{L^\infty(B_{2r}(x))} \leq cr, \text{ then } u = 0 \text{ in } B_r(x).$$

Proof. Let r, x be such that $B_{2r}(x) \subset B_{r_0}(x_0)$ and $\|u\|_{L^\infty(B_{2r}(x))} < cr$. Assume for simplicity that $x = 0$. Let $\eta \in H^1(B_{2r})$ be the solution of the problem

$$-\operatorname{div}(e^{-\Phi}\nabla\eta) = \beta e^{-\Phi} \text{ in } B_{2r} \setminus B_r, \quad \eta = 0 \text{ in } B_r, \quad \eta = cr \text{ in } D \setminus B_{2r},$$

where $\beta > 0$ will be chosen later. Consider the test function $\tilde{u} \in H_0^1(D)$ defined as

$$\tilde{u} = u \wedge \eta \text{ in } B_{2r}, \quad \tilde{u} = u \text{ in } D \setminus B_{2r}.$$

By (2.5.33), we get

$$\int_D |\nabla u|^2 e^{-\Phi} dx - \lambda_m \int_D u^2 e^{-\Phi} dx + \mu |\Omega_u| \leq \int_D |\nabla \tilde{u}|^2 e^{-\Phi} dx - \lambda_m \int_D \tilde{u}^2 e^{-\Phi} dx + \mu |\Omega_{\tilde{u}}|. \quad (2.5.34)$$

Let $E(u, r) := \int_{B_r} |\nabla u|^2 e^{-\Phi} dx + \mu |\Omega_u \cap B_r|$. Since $\tilde{u} \equiv 0$ in B_r we have $|\Omega_u| - |\Omega_{\tilde{u}}| = |\Omega_u \cap B_r|$ and $\int_{B_r} \tilde{u}^2 dx = \int_{B_r} |\nabla \tilde{u}|^2 dx = 0$. Thus, we can rewrite (2.5.34) in the form

$$E(u, r) \leq \int_{B_{2r} \setminus B_r} (|\nabla \tilde{u}|^2 - |\nabla u|^2) e^{-\Phi} dx + 4cr\lambda_m \int_{B_{2r} \setminus B_r} (u - \tilde{u}) e^{-\Phi} dx + \lambda_m \int_{B_r} u^2 e^{-\Phi} dx, \quad (2.5.35)$$

where in the estimate of the second term we used that in B_r

$$u^2 - \tilde{u}^2 = (u + \tilde{u})(u - \tilde{u}) \leq 2u(u - \tilde{u}) \leq 2cr(u - \tilde{u}).$$

Next, we estimate the first term of the right-hand side of (2.5.35). We have

$$|\nabla \tilde{u}|^2 - |\nabla u|^2 = -|\nabla(\tilde{u} - u)|^2 + 2\nabla \tilde{u} \cdot \nabla(\tilde{u} - u) \leq 2\nabla \tilde{u} \cdot \nabla(\tilde{u} - u). \quad (2.5.36)$$

Integrating by parts and using that $(u - \eta)_+ = 0$ on ∂B_{2r} , we get

$$\begin{aligned} \int_{B_{2r} \setminus B_r} \nabla \tilde{u} \cdot \nabla(\tilde{u} - u) e^{-\Phi} dx &= - \int_{B_{2r} \setminus B_r} \nabla \eta \cdot \nabla[(u - \eta)_+] e^{-\Phi} dx \\ &\leq -\beta \int_{B_{2r} \setminus B_r} (u - \eta)_+ e^{-\Phi} dx + \|\nabla \eta\|_{L^\infty(\partial B_r)} \int_{\partial B_r} u e^{-\Phi} d\mathcal{H}^{d-1}. \end{aligned} \quad (2.5.37)$$

We now set $\beta = 2cr\lambda_m$ so that, combining (2.5.35), (2.5.36) and (2.5.37) we have

$$E(u, r) \leq 2\|\nabla \eta\|_{L^\infty(\partial B_r)} \int_{\partial B_r} u e^{-\Phi} d\mathcal{H}^{d-1} + \lambda_m \int_{B_r} u^2 e^{-\Phi} dx.$$

Now, for every $s \in (0, r]$, we have by the $W^{1,1}$ trace inequality in B_s

$$\begin{aligned} \int_{\partial B_s} u e^{-\Phi} d\mathcal{H}^{d-1} &\leq e^{-\min \Phi} C_d \left(\int_{B_s} |\nabla u| dx + \frac{1}{s} \int_{B_s} u dx \right) \\ &\leq e^{-\min \Phi} C_d \left(\frac{1}{2} \int_{B_s} |\nabla u|^2 dx + \frac{1}{2} |\Omega_u \cap B_s| + c |\Omega_u \cap B_s| \right) \\ &\leq C \left(\int_{B_s} |\nabla u|^2 e^{-\Phi} dx + \mu |\Omega_u \cap B_s| \right) \leq CE(u, s) \leq CE(u, r), \end{aligned}$$

where we have set $C = e^{-\min \Phi} C_d \max \left\{ e^{\max \Phi}, \frac{1}{\mu}(1 + 2c) \right\}$. Moreover, since the above inequality holds for every $s \in (0, r]$, we have

$$\int_{B_r} u e^{-\Phi} dx = \int_0^r ds \int_{\partial B_s} u e^{-\Phi} d\mathcal{H}^{d-1} \leq r C E(u, r).$$

Finally, using the bound (2.5.38), we get

$$E(u, r) \leq (2\|\nabla \eta\|_{L^\infty(\partial B_r)} + r^2 c \lambda_m) C E(u, r).$$

Thus, the claim will follow, if we can choose c such that

$$(2\|\nabla \eta\|_{L^\infty(\partial B_r)} + r^2 c \lambda_m) C < 1.$$

We now estimate $\|\nabla \eta\|_{L^\infty(\partial B_r)}$. Notice that in $B_{2r} \setminus B_r$

$$\eta(x) = r^2 w(x/r) + c r h(x/r),$$

where $w : B_2 \setminus B_1 \rightarrow \mathbb{R}$ and $h : B_2 \setminus B_1 \rightarrow \mathbb{R}$ are the solutions to

$$-\Delta h + \nabla \Phi_r \cdot \nabla h = 0 \quad \text{in } B_2 \setminus B_1, \quad h = 0 \quad \text{on } \partial B_1, \quad h = 1 \quad \text{on } \partial B_2,$$

$$-\Delta w + \nabla \Phi_r \cdot \nabla w = \beta \quad \text{in } B_2 \setminus B_1, \quad w = 0 \quad \text{on } \partial B_1 \cup \partial B_2,$$

where $\Phi_r(x) = \Phi(rx)$. Thus, we have

$$\|\nabla \eta\|_{L^\infty} \leq r \|\nabla w\|_{L^\infty} + c \|\nabla h\|_{L^\infty},$$

and so, it is sufficient to estimate ∇w and ∇h . First, applying Lemma 2.5.21 to h , we have

$$\|\nabla h\|_{L^\infty(B_2 \setminus B_1)} \leq C \|h\|_{L^\infty(B_2 \setminus B_1)} \leq C,$$

where the last inequality follows by the maximum principle ($0 \leq h \leq 1$ in $B_2 \setminus B_1$). Next, applying Lemma 2.5.21 to w , we get

$$\|\nabla w\|_{L^\infty(B_2 \setminus B_1)} \leq C (\|w\|_{L^\infty(B_2 \setminus B_1)} + \beta) \leq C \beta,$$

where the last inequality follows by Lemma 2.5.3. Combining the above estimates, we get

$$\|\nabla \eta\|_{L^\infty(B_r)} \leq C (r\beta + c), \tag{2.5.38}$$

which, for c and r small enough, implies that $E(u, r) = 0$ and concludes the proof. \square

Another consequence of property (2.5.33) is that the optimal sets have finite perimeter. This fact is of independent interest but it can also be used to estimate the dimension of the singular set of the free boundary (see Subsection 2.5.9). The local finiteness of the perimeter was also obtained in [2] in the case of the Laplacian by a different argument. Here we use the more direct approach from [78], which is also the local version of an estimate that was used in [12] to prove that some optimal shapes have finite perimeter.

Lemma 2.5.23 (Local finiteness of the perimeter). *Let $D \subset \mathbb{R}^d$ be a connected bounded open set and u be a nonnegative solution of (2.5.1). Then Ω_u is a set of locally finite perimeter in D . Moreover, if D is C^1 -regular, then Ω_u is a set of finite perimeter.*

Proof. Let $x_0 \in \partial\Omega_u$ and $0 < \mu < \Lambda_u$ be fixed. Let $r > 0$ be such that (2.5.33) holds in $D_r(x_0) := B_r(x_0) \cap D$. Assume $x_0 = 0$ and $r_0 = r$. In the sequel we denote by $C > 0$ any constant, which does not depend on t or x_0 . Let $t \in (0, 1)$ and $\eta \in C_c^\infty(B_r)$ be such that

$$0 \leq \eta \leq 1, \quad \eta = 1 \quad \text{in} \quad B_{r/2}, \quad \eta = 0 \quad \text{in} \quad \mathbb{R}^d \setminus B_r, \quad |\nabla \eta| \leq \frac{C}{r}.$$

We set

$$u_t := \eta(u - t)_+ + (1 - \eta)u = \begin{cases} (1 - \eta)u, & \text{if } u < t, \\ u - t\eta, & \text{if } u \geq t. \end{cases}$$

We can now compute on $\{u \geq t\}$

$$\begin{aligned} |\nabla u|^2 - |\nabla u_t|^2 &= |\nabla u|^2 - |\nabla(u - t\eta)|^2 = t(2\nabla u \cdot \nabla \eta - t|\nabla \eta|^2); \\ u^2 - u_t^2 &= u^2 - (u - t\eta)^2 = t(2u\eta - t\eta^2). \end{aligned}$$

Next, on the set $\{u < t\}$, we compute

$$\begin{aligned} u^2 - u_t^2 &= (2\eta - \eta^2)u^2 \leq t^2(2\eta - \eta^2); \\ |\nabla u|^2 - |\nabla u_t|^2 &= |\nabla u|^2 - |\nabla(u - u\eta)|^2 = 2\nabla u \cdot \nabla(u\eta) - |\nabla(u\eta)|^2 \\ &= (2\eta - \eta^2)|\nabla u|^2 + 2u(1 - \eta)\nabla u \cdot \nabla \eta - |\nabla \eta|^2 u^2 \\ &\geq \mathbb{1}_{\{\eta=1\}}|\nabla u|^2 - 2t(1 - \eta)|\nabla u \cdot \nabla \eta| - |\nabla \eta|^2 u^2. \end{aligned}$$

Notice that $u_t \in \mathcal{A}(u, x_0, r)$. Thus, by the optimality of u , we have

$$\int_{B_r} (|\nabla u|^2 - \lambda_m u^2) e^{-\Phi} dx + \mu |\Omega_u \cap B_r| \leq \int_{B_r} (|\nabla u_t|^2 - \lambda_m u_t^2) e^{-\Phi} dx + \mu |\Omega_{u_t} \cap B_r|.$$

By the above estimates, there is a constant C , depending only on $\mu, r, \lambda_m = \int_D |\nabla u|^2 e^{-\Phi} dx$ and $\|\Phi\|_{L^\infty(D)}$ such that, for every $t \leq 1$, we have

$$\begin{aligned} \int_{\{0 < u < t\} \cap B_{r/2}} |\nabla u| dx &\leq \int_{\{0 < u < t\} \cap B_{r/2}} (|\nabla u|^2 + 1) dx \leq \max\{1, 1/\mu\} \int_{\{0 < u < t\} \cap B_{r/2}} (|\nabla u|^2 + \mu) dx \\ &= \max\{1, 1/\mu\} \left(\int_{B_{r/2}} (|\nabla u|^2 - |\nabla u_t|^2) dx + \mu(|\Omega_u \cap B_r| - |\Omega_{u_t} \cap B_r|) \right) \leq Ct. \end{aligned}$$

We now use the co-area formula to rewrite the above inequality as

$$\frac{1}{t} \int_0^t \text{Per}(\{u > s\}; B_{r/2}) ds \leq C.$$

Hence, there is a sequence $t_n \rightarrow 0$ such that $\text{Per}(\{u > t_n\}; B_{r/2}) ds \leq C$, which implies that $\text{Per}(\Omega_u; B_{r/2}) ds \leq C$. The last claim of the lemma follows by a standard covering argument. \square

2.5.8 Blow-up sequences and blow-up limits

Let u be a solution of (2.5.1) in the bounded open set $D \subset \mathbb{R}^d$. For $r > 0$ and $x_0 \in \partial\Omega_u$, we define the rescaled function

$$u_{x_0, r}(x) := \frac{1}{r} u(x_0 + rx).$$

Now since u is Lipschitz continuous in some ball $B_{r_0}(x_0)$ (assume some regularity of the box if $x_0 \in \partial D$) we get that every sequence $(u_{x_0, r_n})_{n \geq 1}$ such that $r_n \rightarrow 0$ admits a subsequence

(still denoted by r_n) that converges to a function $u_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ uniformly on every compact set $K \subset \mathbb{R}^d$. We say that u_0 is a blow-up limit of u at x_0 and we use the notation $\mathcal{BU}_u(x_0)$ for the family of all blow-up limits of u at x_0 . We notice that, due to the non-degeneracy of u , the blow-up limits are non-trivial. Precisely, $u_0 \neq 0$ and there is a constant $c > 0$ such that $\|u_0\|_{L^\infty(B_r)} \geq cr$.

The following proposition is standard. For a detailed proof we refer for example to [77, Proposition 4.5].

Proposition 2.5.24 (Convergence of the blow-up sequences). *Let u be a solution of (2.5.1) and let $x_0 \in \partial\Omega_u$. If $x_0 \in \partial D$, then we also assume that D is of class $C^{1,\beta}$, for some $\beta > 0$. Let $u_0 \in \mathcal{BU}_u(x_0)$ and $u_n := u_{x_0, r_n}$ be a blow-up sequence such that $u_n \rightarrow u_0$ locally uniformly in \mathbb{R}^d as $n \rightarrow \infty$. Then*

1. *The sequence $(u_n)_{n \geq 1}$ converges to u_0 strongly in $H_{loc}^1(\mathbb{R}^d)$.*
2. *The sequence of characteristic functions $(\mathbb{1}_{\Omega_{u_n}})_{n \geq 1}$ converges to $\mathbb{1}_{\Omega_{u_0}}$ in $L_{loc}^1(\mathbb{R}^d)$.*
3. *The sequences of closed sets $(\bar{\Omega}_n)_{n \geq 1}$ and $(\Omega_n^c)_{n \geq 1}$ Hausdorff converge locally in \mathbb{R}^d to $\bar{\Omega}_0$ and Ω_0^c , respectively.*
4. *If $x_0 \in \partial\Omega_u \cap D$, then u_0 is a non-trivial global minimizer of the one-phase Alt-Caffarelli functional with $\Lambda = \Lambda_u e^{\Phi(x_0)}$ (see Definition 2.5.25 below).*
If $x_0 \in \partial\Omega_u \cap \partial D$, then, up to a rotation, u_0 is a non-trivial global minimizer of the one-phase constrained Alt-Caffarelli functional with $\Lambda = \Lambda_u e^{\Phi(x_0)}$.

Definition 2.5.25 (Global minimizers of the one-phase problem). Let $\Lambda > 0$ and $u \in H_{loc}^1(\mathbb{R}^d)$ be a non-negative function.

- We say that u is a global minimizer of the one-phase Alt-Caffarelli functional with Λ , if

$$\int_B |\nabla u|^2 dx + \Lambda |\{u > 0\} \cap B| \leq \int_B |\nabla v|^2 dx + \Lambda |\{v > 0\} \cap B|, \quad (2.5.39)$$

for every ball $B \subset \mathbb{R}^d$ and every function $v \in H^1(B)$ such that $u - v \in H_0^1(B)$.

- We say that u is a global minimizer of the one-phase constrained Alt-Caffarelli functional with Λ , if $\Omega_u \subset \{x_d > 0\}$ and (2.5.39) holds for every ball $B \subset \mathbb{R}^d$ and every function $v \in H^1(B)$ such that $u - v \in H_0^1(B)$ and $\Omega_v \subset \{x_d > 0\}$.

The optimality of the blow-up limit at points $x_0 \in D$ (Proposition 2.5.24, claim (4)) follows by a standard argument based on our analysis in Subsection 2.5.5. Below, we give the proof in the case when x_0 lies on the boundary of D . The idea is to straighten out the boundary of the box and to show that the function u in the new coordinates satisfies an almost-minimality condition. We only give the proof of Proposition 2.5.24 (4) in order to show how to deal with the fact that on different scales r the inclusion constraint on the set Ω_{u_r} changes and that at the limit the box D becomes the half-space $\{x_d > 0\}$.

Let $x_0 = 0 \in \partial\Omega_u \cap \partial D$. Since D is $C^{1,\beta}$ -regular, there exist $\delta > 0$ and a $(C^{1,\beta}$ -regular) function $g : (-\delta, \delta)^{d-1} \rightarrow \mathbb{R}$ such that

$$D \cap SQ_\delta = \{(x', x_d) \in SQ_\delta : g(x') < x_d\},$$

where $SQ_\delta = (-\delta, \delta)^d \subset B_{r_0}$. Moreover, up to a rotation, we can assume that the differential Dg_0 of g at 0 is zero. Let $\psi : SQ_\delta \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$ be the function that straightens out the boundary of D and let $\phi := \psi^{-1} : \psi(SQ_\delta) \subset \mathbb{R}^d \rightarrow \mathbb{R}^d$ be its inverse:

$$\psi(x', x_d) = (x', x_d - g(x')), \quad \phi(x', x_d) = (x', x_d + g(x')).$$

We define the matrix-valued function $A = (a_{ij})_{ij} : SQ_\delta \rightarrow \text{Sym}_d^+(\mathbb{R})$ by

$$A_x = (D\phi_x)^{-1} ({}^t D\phi_x)^{-1}, \quad \text{for every } x \in SQ_\delta,$$

where ${}^tD\phi_x$ stands for the transpose of the Jacobian matrix of ϕ at x . Note that the coefficients a_{ij} are Hölder continuous functions and that A_x are symmetric positive definite matrices since they are small variations of $A_0 = Id$. For $v \in H^1(\mathbb{R}^d)$ and $r > 0$ we define the functional

$$\tilde{J}(v, r) = \int_{B_r} \left(a_{ij}(x) \frac{\partial v}{\partial x_i} \frac{\partial v}{\partial x_j} - \lambda_m v^2 \right) e^{-\tilde{\Phi}} dx,$$

where we have set $\tilde{\Phi} = \Phi \circ \phi$. Moreover, we set $H = \{(x', x_d) \in \mathbb{R}^d : x_d > 0\}$. With an elementary change of variables we get the following result.

Lemma 2.5.26 (Minimality of u in the straightened coordinates). *Let u be a solution of (2.5.1) and $x_0 = 0 \in \partial\Omega_u \cap \partial D$. Let $h > 0$. There exist $c > 0$ and $r_0 > 0$ such that $B_{2r_0} \subset \psi(SQ_\delta)$ and the function $\tilde{u} = u \circ \phi$ satisfies the minimality condition: for every $r \in (0, r_0)$ we have*

$$\tilde{J}(\tilde{u}, r) + \mu |\Omega_{\tilde{u}} \cap B_r| \leq \tilde{J}(\tilde{v}, r) + \mu |\Omega_{\tilde{v}} \cap B_r|$$

for every $\tilde{v} \in H^1(B_{2r_0})$ such that $\tilde{u} = \tilde{v}$ on $B_{2r_0} \setminus B_r$, $\Omega_{\tilde{v}} \subset H$ and where

$$\mu = \begin{cases} \mu_+(h, 0, cr) & \text{if } |\Omega_{\tilde{u}} \cap B_r| \leq |\Omega_{\tilde{v}} \cap B_r| \leq |\Omega_{\tilde{u}} \cap B_r| + h, \\ \mu_-(h, 0, cr) & \text{if } |\Omega_{\tilde{u}} \cap B_r| - h \leq |\Omega_{\tilde{v}} \cap B_r| \leq |\Omega_{\tilde{u}} \cap B_r|. \end{cases} \quad (2.5.40)$$

Proof. Let $r_0 > 0$ be such that $B_{2r_0} \subset \psi(SQ_\delta)$, $r \in (0, r_0)$ and \tilde{v} such that $\tilde{u} = \tilde{v}$ on $B_{2r_0} \setminus B_r$, $\Omega_{\tilde{v}} \subset H$. Assume that $|\Omega_{\tilde{u}} \cap B_r| \leq |\Omega_{\tilde{v}} \cap B_r| \leq |\Omega_{\tilde{u}} \cap B_r| + h$. We define $v \in H_0^1(D)$ by $v = \tilde{v} \circ \psi$ in $\phi(B_{2r_0})$ and $v = u$ otherwise. Let c be a positive constant depending only on ϕ such that $\phi(B_r) \subset B_{cr}$. Then, it follows that $u = v$ on $D \setminus B_{cr}$. Moreover, since $\det(D\phi_x) = 1$ we have $|\Omega_u| \leq |\Omega_v| \leq |\Omega_u| + h$. Therefore, up to choosing $r_0 > 0$ smaller (depending on c), we get

$$J(u) + \mu_+(h, 0, cr) |\Omega_u| \leq J(v) + \mu_+(h, 0, cr) |\Omega_v|.$$

Since we have $u = v$ on $\phi(B_r(x_0))$, this can rewrite as

$$\int_{\phi(B_r)} (|\nabla u|^2 - \lambda_m u^2) e^{-\Phi} dx + \mu |\Omega_u \cap \phi(B_r)| \leq \int_{\phi(B_r)} (|\nabla v|^2 - \lambda_m v^2) e^{-\Phi} dx + \mu |\Omega_v \cap \phi(B_r)|,$$

where we have set $\mu = \mu_+(h, 0, cr)$. Now, a change of variables gives

$$\begin{aligned} \tilde{J}(\tilde{u}, r) + \mu |\Omega_{\tilde{u}} \cap B_r| &= \int_{\phi(B_r)} (|\nabla u|^2 - \lambda_m u^2) e^{-\Phi} dx + \mu |\Omega_u \cap \phi(B_r)| \\ &\leq \int_{\phi(B_r)} (|\nabla v|^2 - \lambda_m v^2) e^{-\Phi} dx + \mu |\Omega_v \cap \phi(B_r)| = \tilde{J}(\tilde{v}, r) + \mu |\Omega_{\tilde{v}} \cap B_r|. \quad \square \end{aligned}$$

The next Lemma states that \tilde{u} is an almost-minimizer also of the functional J .

Lemma 2.5.27. *Let $D \subset \mathbb{R}^d$ be a connected bounded open set of class $C^{1,\beta}$, for some $\beta > 0$. Let u be a nonnegative solution of (2.5.1) and $x_0 = 0 \in \partial\Omega_u \cap \partial D$. Let $h > 0$ be a given constant and let $c > 0$ and $r_0 > 0$ be as in Lemma 2.5.26. Then there exists a constant $C > 0$ such that $\tilde{u} = u \circ \phi$ satisfies the following almost-minimality condition: for every $r \in (0, r_0)$ we have*

$$J(\tilde{u}, r) + \mu |\Omega_{\tilde{u}} \cap B_r| \leq (1 + Cr^\beta) J(\tilde{v}, r) + \mu |\Omega_{\tilde{v}} \cap B_r| + Cr^\beta \int_{B_r} \tilde{v}^2 dx + Cr^{d+\beta}$$

for every $\tilde{v} \in H^1(B_{2r_0})$ such that $\tilde{u} = \tilde{v}$ on $B_{2r_0} \setminus B_r$, $\Omega_{\tilde{v}} \subset H$ and where μ is as in (2.5.40).

Proof. Using the $C^{0,\beta}$ continuity of A and the Lipschitz continuity of Φ , we estimate

$$\begin{aligned} J(\tilde{u}, r) &= \tilde{J}(\tilde{u}, r) + \int_{B_r} (a_{ij}(x_0) - a_{ij}(x)) \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial \tilde{u}}{\partial x_j} e^{-\Phi} dx \\ &\quad + \int_{B_r} (a_{ij}(x) \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial \tilde{u}}{\partial x_j} - \lambda_m \tilde{u}^2) (e^{-\Phi} - e^{-\tilde{\Phi}}) dx \leq \tilde{J}(\tilde{u}, r) + Cr^{d+\beta}, \end{aligned}$$

for some positive constant C that does not depend on r . Analogously, we get the following estimate from below

$$\begin{aligned} J(\tilde{v}, r) &= \tilde{J}(\tilde{v}, r) + \int_{B_r} (a_{ij}(x_0) - a_{ij}(x)) \frac{\partial \tilde{v}}{\partial x_i} \frac{\partial \tilde{v}}{\partial x_j} e^{-\Phi} dx \\ &\quad + \int_{B_r} (a_{ij}(x) \frac{\partial \tilde{v}}{\partial x_i} \frac{\partial \tilde{v}}{\partial x_j} - \lambda_m \tilde{v}^2) (e^{-\Phi} - e^{-\tilde{\Phi}}) dx \\ &\geq \tilde{J}(\tilde{v}, r) - Cr^\beta \left(\int_{B_r} a_{ij}(x) \frac{\partial \tilde{v}}{\partial x_i} \frac{\partial \tilde{v}}{\partial x_j} e^{-\Phi} dx + \int_{B_r} \tilde{v}^2 dx \right) \\ &\geq (1 - Cr^\beta) \tilde{J}(\tilde{v}, r) - Cr^\beta \int_{B_r} \tilde{v}^2 dx. \end{aligned}$$

Now, using Lemma 2.5.26 and then combining the above estimates we get

$$\begin{aligned} J(\tilde{u}, r) + \mu |\Omega_{\tilde{u}} \cap B_r| &\leq \tilde{J}(\tilde{u}, r) + \mu |\Omega_{\tilde{u}} \cap B_r| + Cr^{d+\beta} \leq \tilde{J}(\tilde{v}, r) + \mu |\Omega_{\tilde{v}} \cap B_r| + Cr^{d+\beta} \\ &\leq \frac{1}{1 - Cr^\beta} \left(\tilde{J}(\tilde{v}, r) + Cr^\beta \int_{B_r} \tilde{v}^2 dx \right) + \mu |\Omega_{\tilde{v}} \cap B_r| + Cr^{d+\beta}, \end{aligned}$$

which concludes the proof. \square

We are now in position to prove the claim (4) of Proposition 2.5.24 in the constrained case.

Proof of Proposition 2.5.24 (4). Let $x_0 = 0 \in \partial\Omega_u \cap \partial D$ and let $u_0 \in \mathcal{BU}_u(x_0)$ be the blow-up limit of the sequence $u_n(x) = r_n^{-1}u(r_nx)$, where $(r_n)_n$ is some fixed sequence decreasing to 0. Let $v \in H^1(B_r)$ be such that $u_0 - v \in H_0^1(B_r)$ and $\Omega_v \subset H$. We define $v_n = v + \tilde{u}_n - u_0 \in H^1(\mathbb{R}^d)$, where we have set $\tilde{u}_n(x) = r_n^{-1}\tilde{u}(r_nx)$. Note that the sequence $(u_n)_n$ and $(\tilde{u}_n)_n$ converge to the same limit u_0 since the function ϕ is $C^{1,\beta}$ regular. Moreover, since $u_0 - v \in H_0^1(B_r)$ we have $\tilde{u}_n - v_n \in H_0^1(B_r)$ and hence $\tilde{u} - v_n^{r_n} \in H_0^1(B_{rr_n})$, where we write $v_n^{r_n}(x) = r_n v_n(x/r_n)$. Note that we have $\Omega_{v_n^{r_n}} \subset H$. We set $h_n = |B_{rr_n}|$ and assume that $|\Omega_{\tilde{u}} \cap B_{rr_n}| \leq |\Omega_{v_n^{r_n}} \cap B_{rr_n}| \leq |\Omega_{\tilde{u}} \cap B_{rr_n}| + h_n$. Now we set $\mu = \mu_+(h_n, 0, r)$ and $\Phi_n(x) = \Phi(r_nx)$ and we apply Lemma 2.5.27 to the test function $v_n^{r_n}$ to estimate

$$\begin{aligned} \int_{B_r} (|\nabla \tilde{u}_n|^2 - r_n^2 \lambda_m \tilde{u}_n^2) e^{-\Phi_n(x)} dx + \mu |\Omega_{\tilde{u}_n} \cap B_r| &= \frac{1}{r_n^d} \left(J(\tilde{u}, rr_n) + \mu |\Omega_{\tilde{u}} \cap B_{rr_n}| \right) \\ &\leq \frac{1}{r_n^d} \left((1 + C(rr_n)^\beta) J(v_n^{r_n}, rr_n) + \mu |\Omega_{v_n^{r_n}} \cap B_{rr_n}| + C(rr_n)^\beta \int_{B_{rr_n}} (v_n^{r_n})^2 dx + C(rr_n)^{d+\beta} \right) \\ &= (1 + C(rr_n)^\beta) \int_{B_r} (|\nabla v_n|^2 - r_n^2 \lambda_m v_n^2) e^{-\Phi_n(x)} dx + \mu |\Omega_{v_n} \cap B_r| \\ &\quad + C(rr_n)^\beta r_n^2 \int_{B_r} v_n^2 dx + Cr^{d+\beta} r_n^\beta. \end{aligned}$$

By Proposition 2.5.24 (1) the sequence \tilde{u}_n (resp. v_n) strongly converges in $H^1(B_r)$ to u_0 (resp. v) and the sequence of characteristic functions $(\mathbb{1}_{\Omega_{\tilde{u}_n}})_{n \geq 1}$ (resp. $(\mathbb{1}_{\Omega_{v_n}})_{n \geq 1}$) converges to $\mathbb{1}_{\Omega_{u_0}}$ (resp. $\mathbb{1}_{\Omega_v}$). Moreover, $\mu = \mu_+(h_n, 0, r)$ tends to Λ_u as $r_n \rightarrow 0$ by Theorem 2.5.16. Therefore, passing at the limit in the above inequality and then multiplying by $e^{\Phi(x_0)}$ gives the claim. \square

Remark 2.5.28 (Lebesgue density on the free boundary). For every $\gamma \in [0, 1]$ we define

$$\Omega_u^{(\gamma)} := \left\{ x \in \mathbb{R}^d : \lim_{r \rightarrow 0} \frac{|\Omega_u \cap B_r(x)|}{|B_r|} = \gamma \right\}.$$

We notice that, as a consequence of Proposition 2.5.24, we get that

$$\partial\Omega_u \cap D \cap \Omega_u^{(0)} = \emptyset \quad \text{and} \quad \partial\Omega_u \cap D \cap \Omega_u^{(1)} = \emptyset.$$

The first equality follows by the non-degeneracy of u , while the second one follows from the fact that all the blow-up limits vanish in zero and are global solutions of the Alt-Caffarelli problem.

2.5.9 Regularity of the free boundary

In this section we prove Theorem 2.1.5 (4) and (6), and Theorem 2.1.2 (4) and (6). We first show that the optimality condition $|\nabla u|^2 = \Lambda_u e^\Phi$ on the free boundary $\partial\Omega_u \cap D$ and $|\nabla u|^2 \geq \Lambda_u e^\Phi$ on $\partial\Omega_u \cap \partial D$ holds in the viscosity sense (Lemma 2.5.30). We will then decompose the free boundary into regular and singular parts (Definition 2.5.34) and we will show that the regular part is $C^{1,\alpha}$ regular (Proposition 2.5.35).

Definition 2.5.29 (Optimality condition in viscosity sense). Let Ω be an open set and $u : \Omega \rightarrow \mathbb{R}$ be a continuous function, that is, $u \in C(\Omega)$.

- We say that $\varphi \in C(\Omega)$ touches u by below (resp. by above) at $x_0 \in \Omega$ if $\varphi(x_0) = u(x_0)$ and $\varphi \leq u$ (resp. $\varphi \geq u$) in a neighborhood of x_0 .

- Let $\Lambda : \bar{D} \rightarrow \mathbb{R}$ and $u : \bar{D} \rightarrow \mathbb{R}$ be continuous non-negative functions on \bar{D} , and let $\Omega_u = \{u > 0\}$. We say that u satisfies the boundary condition

$$|\nabla u| = \sqrt{\Lambda} \quad \text{on} \quad \partial\Omega_u \cap D$$

in viscosity sense if, for every $x_0 \in \partial\Omega_u \cap D$ and $\varphi \in C^2(D)$ such that $\varphi(x_0) = 0$ and $\varphi_+ := \max\{\varphi, 0\}$ touches u by below (resp. by above) at x_0 , we have

$$|\nabla\varphi|(x_0) \leq \sqrt{\Lambda(x_0)} \quad \left(\text{resp. } |\nabla\varphi|(x_0) \geq \sqrt{\Lambda(x_0)} \right).$$

Analogously, we say that u satisfies the boundary condition

$$|\nabla u| \geq \sqrt{\Lambda} \quad \text{on} \quad \partial\Omega_u \cap \partial D$$

in viscosity sense if, for every $x_0 \in \partial\Omega_u \cap D$ and $\varphi \in C^2(D)$ such that $\varphi(x_0) = 0$ and φ_+ touches u from above at x_0 , we have $|\nabla\varphi|(x_0) \geq \sqrt{\Lambda(x_0)}$.

Lemma 2.5.30 (Optimality condition on the free boundary). *Let $D \subset \mathbb{R}^d$ be a bounded open set of class $C^{1,\beta}$, for some $\beta > 0$, and let u be a solution of (2.5.1). Then u is a solution of the problem*

$$\begin{cases} -\operatorname{div}(e^{-\Phi}u) = \lambda_m u e^{-\Phi} & \text{in } \Omega_u, \\ |\nabla u| = \sqrt{\Lambda_u e^\Phi} & \text{on } \partial\Omega_u \cap D, \\ |\nabla u| \geq \sqrt{\Lambda_u e^\Phi} & \text{on } \partial\Omega_u \cap \partial D, \end{cases} \quad (2.5.41)$$

where the boundary conditions hold in viscosity sense.

To prove the optimality condition we will need the following result.

Lemma 2.5.31. *Let $u \in H_{loc}^1(\mathbb{R}^d)$ be a non-trivial, continuous and one-homogeneous function, harmonic in the set $\Omega_u = \{u > 0\}$. Then the following holds. If $\Omega_u \subset \{x_d > 0\}$ then $\Omega_u = \{x_d > 0\}$, while if $\Omega_u \supset \{x_d > 0\}$ then either $\Omega_u = \{x_d > 0\}$ or $\Omega_u = \{x_d \neq 0\}$.*

Proof. Set $S = \Omega_u \cap \partial B_1$ and denote by $C_S = \{r\theta : \theta \in S, r > 0\}$ the cone generated by S . Since u is a one-homogeneous function and is solution of

$$\Delta u = 0 \quad \text{in } C_S, \quad u = 0 \quad \text{on } \partial C_S,$$

it follows that the trace $\varphi = u|_{\partial B_1}$ is a solution of

$$-\Delta_{\mathbb{S}^{d-1}} \varphi = (d-1)\varphi \quad \text{in } S, \quad \varphi = 0 \quad \text{on } \partial S.$$

Therefore φ is an eigenfunction of $-\Delta_{\mathbb{S}^{d-1}}$ in S , corresponding to the eigenvalue $(d-1)$. Hence $\lambda_1(S) \leq d-1$, where $\lambda_1(S)$ is the first eigenfunction on S . We also notice that the function $\varphi_1(x) = x_d^+ (= \max\{x_d, 0\})$ is, up to a multiplicative constant, the first eigenfunction on the spherical set $S_+ := \{x_d > 0\} \cap \partial B_1$ with eigenvalue $\lambda_1(S_+) = d-1$.

Assume that $S \subset S_+$. Then $\varphi \in H_0^1(S_+)$ and by the variational characterization of $\lambda_1(S_+)$ we have

$$\frac{\int_{\partial B_1} |\nabla \varphi|^2 d\mathcal{H}^{d-1}}{\int_{\partial B_1} \varphi^2 d\mathcal{H}^{d-1}} = d-1 = \lambda_1(S_+) \leq \frac{\int_{\partial B_1} |\nabla(\varphi + t\psi)|^2 d\mathcal{H}^{d-1}}{\int_{\partial B_1} (\varphi + t\psi)^2 d\mathcal{H}^{d-1}},$$

for every $\psi \in H_0^1(S_+)$ and $t \in \mathbb{R}$. This gives that φ is solution of

$$-\Delta_{\mathbb{S}^{d-1}} \varphi = \lambda_1(S_+) \varphi \quad \text{in } S_+, \quad \varphi = 0 \quad \text{on } \partial S_+,$$

that is, φ is the first eigenfunction on S_+ . Since $\lambda_1(S_+)$ is simple (because S_+ is connected) it follows that $\varphi = c x_d^+$ for some constant $c > 0$. In particular, $\{\varphi > 0\} = S_+$ and hence $\Omega_u = \{x_d > 0\}$ by the one-homogeneity of u .

Assume now that $S \supset S_+$ and write S as a disjoint union $S = S_0 \cup S_1$, where S_0 is the connected component of S which contains S_+ . If $S_1 \neq \emptyset$, then it follows by the previous step that $S_1 = S_- := \{x_d < 0\} \cap \partial B_1$; hence $S = S_+ \cup S_-$ and $\Omega_u = \{x_d \neq 0\}$. Now, if $S_1 = \emptyset$, then $S = S_0$ is connected. Moreover, $\varphi_1 \in H_0^1(S)$ and using the variational characterization of $\lambda_1(S)$ it follows that φ_1 is the first eigenfunction in S . Then $\varphi_1 > 0$ in S (since S is connected) which proves that $S = S_+$. \square

Proof of Lemma 2.5.30. From Proposition 2.5.18 it follows that u is continuous in D . We only have to prove that u satisfies the two boundary conditions in the viscosity sense. We first show that $|\nabla u| = \sqrt{\Lambda_u} e^{\Phi}$ holds on $\partial \Omega_u \cap D$. Let $\varphi \in C^2(D)$ a function such that $\varphi(x_0) = 0$ and φ^+ touches u by below at $x_0 \in \partial \Omega_u \cap D$. Let r_n be an infinitesimal sequence and

$$u_n(x) = \frac{1}{r_n} u(x_0 + r_n x) \quad \text{and} \quad \varphi_n(x) = \frac{1}{r_n} \varphi(x_0 + r_n x). \quad (2.5.42)$$

Up to a subsequence, u_n converges locally uniformly to some $u_0 \in \mathcal{BU}_u(x_0)$, while φ_n converges to $\varphi_0(x) := x \cdot \nabla \varphi(x_0)$. Up to a change of coordinates, we may suppose that $\nabla \varphi(x_0) = |\nabla \varphi(x_0)| e_d$. If $|\nabla \varphi(x_0)| = 0$, then $|\nabla \varphi(x_0)| \leq \sqrt{\Lambda}$, where we have set $\Lambda = \Lambda_u e^{\Phi(x_0)}$, and we are done. Otherwise, we have $u_0 > 0$ in the half-space $\{x_d > 0\}$ since $u_0 \geq \varphi_0$. Moreover, u_0 is a one-homogeneous function by Lemma 2.5.38 and it follows that $\Omega_{u_0} = \{x_d > 0\}$ by Lemma 2.5.31, because the case $\Omega_{u_0} = \{x_d \neq 0\}$ is ruled out (by (4) in Proposition 2.5.24 or Remark 2.5.28). Moreover, u_0 is a local minimizer of the Alt-Caffarelli functional for Λ by Proposition 2.5.24 and hence satisfies (in the classical sense) the optimality condition

$$|\nabla u_0| = \sqrt{\Lambda} \quad \text{on } \{x_d = 0\}$$

(see [2, Theorem 2.5]). This implies that $u_0 = \sqrt{\Lambda} x_d^+$. To see this, note that the boundary condition implies that v defined by $v = u_0$ in $\{x_d > 0\}$ and $v = \sqrt{\Lambda} x_d$ in $\{x_d \leq 0\}$ is harmonic in \mathbb{R}^d ; hence $v = \sqrt{\Lambda} x_d$ by uniqueness of the solution to Cauchy problem for the Laplacian. Finally, since $u_0 \geq \varphi_0$ we have $\sqrt{\Lambda} \geq |\nabla \varphi_0|(0) = |\nabla \varphi|(x_0)$. The case when φ^+ touches u from

above is similar; since $u_0 \leq \varphi_0^+$ we have $\Omega_{u_0} \subset \{x_d > 0\}$ and hence $\Omega_u = \{x_d > 0\}$ by Lemma 2.5.31. Notice that u_0 is a non trivial function by the non-degeneracy property in Lemma 2.5.22.

Suppose now that φ^+ touches u from above at $x_0 \in \partial\Omega_u \cap \partial D$ and consider u_n and φ_n defined in (2.5.42). By Proposition 2.5.24, u_n converges to a local minimizer u_0 of the Alt-Caffarelli functional with $\Lambda = \Lambda_u e^{\Phi(x_0)}$ in the half-space $\{x_d > 0\}$ (up to a change of coordinates). Therefore, the sequence $u_{0n} := r_n^{-1} u_0(r_n x)$ converges to a limit u_{00} which is a solution of the constrained Alt-Caffarelli problem in $\{x_d > 0\}$ and is one-homogenous (see Remark 2.5.32). Therefore, $\Omega_{u_{00}} = \{x_d > 0\}$ by Lemma 2.5.31 and u_{00} satisfies the optimality condition

$$|\nabla u_{00}| \geq \sqrt{\Lambda} \quad \text{on} \quad \{x_d = 0\}.$$

This implies that $u_{00}(x) = \alpha x_d^+$, for some $\alpha \geq \sqrt{\Lambda}$, and thus that $\sqrt{\Lambda} \leq \alpha \leq |\nabla \varphi_0|(0) = |\nabla \varphi|(x_0)$ (since $u_{00} \leq \varphi_0^+$). \square

Remark 2.5.32. The homogeneity of the blow-up limits of the (local) minimizers of the Alt-Caffarelli functional was first obtained by Weiss in [92]. In the case of the constrained problem, when the solution u is optimal only among the functions with support in $\{x_d > 0\}$, the Weiss formula can still be applied because the one-homogeneous extensions of u are admissible competitors. Thus, the blow-up limits in this case are still one-homogeneous. We refer for instance to [85, Proposition 4.3] and Lemma 2.5.37 below.

Remark 2.5.33 (On the Alt-Caffarelli optimality condition). Using an argument based on an internal variation of the boundary as in [2, Theorem 2.5] we can get in a weak sense the optimality boundary condition given in Lemma 2.5.30, namely: for every $x_0 \in \partial\Omega_u \cap D$, $r > 0$ such that $B_r(x_0) \subset D$ and $\xi \in C_0^\infty(B_r(x_0), \mathbb{R}^d)$ we have

$$\lim_{\varepsilon \downarrow 0} \int_{\partial\{u > \varepsilon\}} (|\nabla u|^2 - \Lambda_u e^\Phi) e^{-\Phi} \xi \cdot \nu d\mathcal{H}^{d-1} = 0,$$

while for every $x_0 \in \partial\Omega_u \cap \partial D$, $r > 0$ such that $D_r(x_0)$ is connected and every $\xi \in C_0^\infty(B_r(x_0), \mathbb{R}^d)$ such that $(Id + \xi)^{-1}(D_r(x_0)) \subset D_r(x_0)$ we have

$$\lim_{\varepsilon \downarrow 0} \int_{\partial\{u > \varepsilon\}} (|\nabla u|^2 - \Lambda_u e^\Phi) e^{-\Phi} \xi \cdot \nu d\mathcal{H}^{d-1} \geq 0.$$

Definition 2.5.34 (Regular and singular parts of the free boundary). We say that $x_0 \in \partial\Omega_u$ is a regular point if there exists a blow-up $u_0 \in \mathcal{BU}_u(x_0)$ of the form

$$u_0(x) = \sqrt{\Lambda_u e^{\Phi(x_0)}} (x \cdot \nu)_+ \quad \text{if} \quad x_0 \in \partial\Omega_u \cap D, \quad (2.5.43)$$

$$u_0(x) = q (x \cdot \nu)_+ \quad \text{if} \quad x_0 \in \partial\Omega_u \cap \partial D, \quad (2.5.44)$$

where $\nu \in \partial B_1$ is some unit vector and q is a constant such that $q \geq \sqrt{\Lambda_u e^{\Phi(x_0)}}$.

We denote by $Reg(\partial\Omega_u \cap D)$ the set of regular points (the regular part of the free boundary) in D , and by $Reg(\partial\Omega_u)$ the set of all regular points of $\partial\Omega_u$. We define the singular part of the boundary as $Sing(\partial\Omega_u) := \partial\Omega_u \setminus Reg(\partial\Omega_u)$ and $Sing(\partial\Omega_u \cap D) = \partial\Omega_u \setminus Reg(\partial\Omega_u \cap D)$.

Proposition 2.5.35 (Regularity of the free boundary). *Suppose that u is a solution of (2.5.1) in the bounded open set $D \subset \mathbb{R}^d$. Then, we have:*

1. $Reg(\partial\Omega_u \cap D)$ is locally the graph of a $C^{1,\alpha}$ function for any $\alpha < 1$;
2. the reduced boundary $\partial^* \Omega_u \cap D$ is contained in $Reg(\partial\Omega_u \cap D)$;
3. $\mathcal{H}^{d-1}(Sing(\partial\Omega_u \cap D)) = 0$; moreover, if $d \leq 4$, then $Reg(\partial\Omega_u \cap D) = \emptyset$.

If D is a $C^{1,\beta}$ regular domain, for some $\beta > 1/2$, then:

- (4) $Reg(\partial\Omega_u)$ is locally the graph of a $C^{1,1/2}$ regular function;

- (5) $\partial\Omega_u \cap \partial D \subset \text{Reg}(\partial\Omega_u)$;
 (6) $\text{Reg}(\partial\Omega_u \cap D) \subset \text{Reg}(\partial\Omega_u)$ and $\text{Sing}(\partial\Omega_u) = \text{Sing}(\partial\Omega_u \cap D)$.

Proof. By Lemma 2.5.30, u is a viscosity solution of (2.5.41). Let $x_0 \in \partial\Omega_u$ be a regular point. Then, for some $r > 0$ small enough the function $u_{r,x_0} = \frac{1}{r}u(x_0 + rx)$ is also a viscosity solution and is ε -flat in the sense of [36]. Applying the results of De Silva [36] (in the case when $x_0 \in D$) and Chang-Lara-Savin [26] (if $x_0 \in \partial D$), we get the claims (1) and (4).

We next prove (2) and (3). Let $x_0 \in \partial\Omega_u \cap D$ and $u_n := u_{x_0, r_n}$ be a blow-up sequence at x_0 converging to some $v \in \mathcal{BU}_u(x_0)$ such that $\mathbb{1}_{\Omega_{u_n}}$ converges in $L^1_{loc}(\mathbb{R}^d)$ to $\mathbb{1}_{\Omega_v}$. If $x_0 \in \partial^*\Omega_u \cap D$, then Ω_v is a half-plane of the form $H = \{x \in \mathbb{R}^d : x \cdot \nu > 0\}$ for some $\nu \in \partial B_1$. Without loss of generality, we assume $\nu = e_d$. On the other hand v is a solution to the Alt-Caffarelli problem with $\Lambda = \Lambda_u e^{\Phi(x_0)}$. Thus v is harmonic in H and zero on ∂H , so it is smooth up to the boundary of H . Now, the optimality condition $|\nabla v| = \sqrt{\Lambda}$ on ∂H and the unique continuation of harmonic functions in the half-plane imply that $v(x) = \sqrt{\Lambda}x_d^+$, which proves (2). Now, since Ω_u has (locally) finite perimeter in D , the Federer's Theorem and Remark 2.5.28 give that

$$\mathcal{H}^{d-1}(\partial\Omega_u \cap D \setminus (\partial^*\Omega_u)) = \mathcal{H}^{d-1}(\partial\Omega_u \cap D \setminus (\partial^*\Omega_u \cup \Omega_u^{(0)} \cup \Omega_u^{(1)})) = 0,$$

which proves that $\mathcal{H}^{d-1}(\text{Sing}(\partial\Omega_u \cap D)) = 0$. We now prove the second claim of (3). As above, let $x_0 \in \partial\Omega_u \cap D$ and $v = \lim_{n \rightarrow \infty} u_{r_n, x_0}$ be a blow-up limit of u at x_0 . Then v is a solution of the Alt-Caffarelli problem. Let $\rho_n \rightarrow 0$ and $v_{\rho_n}(x) = \frac{1}{\rho_n}v(x\rho_n)$ be a sequence that converges locally uniformly to a function v_0 . Since $d \leq 4$, we have that the free boundary $\partial\Omega_v$ is $C^{1,\alpha}$ and v_0 is of the form $v_0(x) = \sqrt{\Lambda}(x \cdot \nu)_+$ for some $\nu \in \partial B_1$ (see [2] for $d = 2$, [24] for $d = 3$ and [62] for $d = 4$). Now since, for fixed $n > 0$, we have that $v_{\rho_n} = \lim_{m \rightarrow \infty} u_{\rho_n r_m, x_0}$, we can choose a diagonal sequence u_{R_n, x_0} , where $R_n = \rho_n r_{m(n)}$, such that $v_0 = \lim_{n \rightarrow \infty} u_{R_n, x_0}$. This proves that x_0 is a regular point.

The claim (5) follows by the same diagonal sequence argument. This time $x_0 \in \partial D$ and the blow-up v is a solution of the constrained Alt-Caffarelli problem in $\{x_d > 0\}$. Thus, the blow-up v_0 of v is one-homogeneous solution of the constrained problem for $\Lambda = \Lambda_u e^{\Phi(x_0)}$. This implies (in any dimension) that $v_0(x) = qx_d^+$ for some $q \geq \sqrt{\Lambda}$ (see [85, Proposition 4.3]).

Finally, (6) follows by the definition of the regular part and claim (5). \square

Remark 2.5.36 (On the higher regularity of the free boundary). The smoothness of the free boundary can be improved under an additional regularity assumption on Φ . Indeed, if $\Phi \in C^{k,\alpha}(D)$ for some $k \geq 1$ and $\alpha \in (0, 1)$, then by [67, Theorem 1], $\text{Reg}(\partial\Omega^* \cap D)$ is locally a graph of a $C^{k+1,\alpha}$ function.

2.5.10 Monotonicity formula and some further estimates on the dimension of the singular set

This section is dedicated to the estimates on the dimension of the singular set (Theorem 2.1.5 (5) and Theorem 2.1.2 (5)). The main ingredient is a monotonicity formula that implies the homogeneity of the blow-up limits at any free boundary point $x_0 \in \partial\Omega_u \cap D$.

Let u be a solution (2.5.1) and Λ_u be the constant given by Theorem 2.5.12. We define the Weiss-type boundary adjusted energy as

$$W(u, \Phi, x_0, r) = \frac{1}{r^d} \int_{B_r(x_0)} |\nabla u|^2 e^{-\Phi} dx - \frac{1}{r^{d+1}} \int_{\partial B_r(x_0)} u^2 e^{-\Phi} d\mathcal{H}^{d-1} + \frac{\Lambda_u}{r^d} |\Omega_u \cap B_r(x_0)|.$$

Lemma 2.5.37 (Weiss monotonicity formula). *Let u be a solution (2.5.1) in the bounded open set D . Then, for every $x_0 \in \partial\Omega_u \cap D$ and every $0 < r < \text{dist}(x_0, \partial D)$, the function W satisfies the differential inequality*

$$\frac{d}{dr} W(u, \Phi, x_0, r) \geq \frac{2e^{-\max \Phi}}{r^{d+2}} \int_{\partial B_r(x_0)} |\nabla u \cdot x - u|^2 d\mathcal{H}^{d-1} - C, \quad (2.5.45)$$

where $C > 0$ is a constant depending only on $\lambda_m, \Phi, L := \|\nabla u\|_{L^\infty}$ and the dimension d .

Proof. We first prove the claim when $x_0 \in \partial\Omega_u \cap D$. Assume $x_0 = 0$. We set

$$H(r) := \int_{\partial B_r} u^2 e^{-\Phi} d\mathcal{H}^{d-1} \quad \text{and} \quad D(r) := \int_{B_r} |\nabla u|^2 e^{-\Phi} dx,$$

$$H_\Phi(r) := \int_{\partial B_r} (\nu \cdot \nabla \Phi) u^2 e^{-\Phi} d\mathcal{H}^{d-1} \quad \text{and} \quad D_\Phi(r) := \int_{B_r} (|\nabla u|^2 - \lambda_m u^2) (x \cdot \nabla \Phi) e^{-\Phi} dx,$$

where $\nu(x) = x/r$ is the exterior normal to the sphere ∂B_r at x . As in Proposition 2.A.1 (notice that in Proposition 2.A.1 D_Φ is defined differently) we have

$$D'(r) = \int_{\partial B_r} |\nabla u|^2 e^{-\Phi} d\mathcal{H}^{d-1} \quad \text{and} \quad H'(r) = \frac{d-1}{r} H(r) + 2D(r) - 2\lambda_m \int_{B_r} u^2 e^{-\Phi} dx - H_\Phi(r).$$

Let ϕ_ε be a radially decreasing function such that

$$0 \leq \phi_\varepsilon \leq 1 \text{ in } B_r, \quad \phi_\varepsilon = 1 \text{ in } B_{r(1-\varepsilon)}, \quad \phi_\varepsilon = 0 \text{ on } \partial B_r \quad \text{and} \quad |\nabla \phi_\varepsilon| \leq C(r\varepsilon)^{-1}. \quad (2.5.46)$$

As in Step 2 of the proof of Proposition 2.A.1, the optimality condition $\delta J(u)[\xi] = \Lambda_u \int_{\Omega_u} \operatorname{div} \xi dx$, applied to the vector field $\xi(x) = x\phi_\varepsilon(x)$, gives that

$$\begin{aligned} \Lambda_u \left(d|\Omega_u \cap B_r| - r\mathcal{H}^{d-1}(\Omega_u \cap \partial B_r) \right) &= -(d-2)D(r) + rD'(r) - 2r \int_{\partial B_r} (\partial_\nu u)^2 e^{-\Phi} d\mathcal{H}^{d-1} \\ &\quad + \lambda_m \left(d \int_{B_r} u^2 e^{-\Phi} dx - r \int_{\partial B_r} u^2 e^{-\Phi} d\mathcal{H}^{d-1} \right) + rD_\Phi(r), \end{aligned}$$

where $\partial_\nu u := \nu \cdot \nabla u$. We now calculate

$$\begin{aligned} \frac{d}{dr} W(u, \Phi, x_0, r) &= \frac{1}{r^d} D'(r) - \frac{d}{r^{d+1}} D(r) - \frac{1}{r^{d+1}} H'(r) + \frac{d+1}{r^{d+2}} H(r) \\ &\quad + \frac{\Lambda_u}{r^{d+1}} \left(r\mathcal{H}^{d-1}(\Omega_u \cap \partial B_r) - d|\Omega_u \cap B_r| \right) \\ &= \frac{2}{r^{d+2}} \int_{\partial B_r} |\nabla u \cdot x - u|^2 e^{-\Phi} d\mathcal{H}^{d-1} + \frac{1}{r^{d+1}} H_\Phi(r) - \frac{1}{r^d} D_\Phi(r) \\ &\quad - \frac{\lambda_m}{r^{d+1}} \left((d+2) \int_{B_r} u^2 e^{-\Phi} dx - r \int_{\partial B_r} u^2 e^{-\Phi} d\mathcal{H}^{d-1} \right) \\ &\geq \frac{2e^{-\max \Phi}}{r^{d+2}} \int_{\partial B_r} |\nabla u \cdot x - u|^2 d\mathcal{H}^{d-1} - C, \end{aligned}$$

which gives the claim if $x_0 \in \partial\Omega_u \cap D$ and $r < \operatorname{dist}(x_0, \partial D)$. \square

Lemma 2.5.38 (Homogeneity of the blow-up limits). *Let u be a solution (2.5.1) in the bounded open set D and let $x_0 \in \partial\Omega_u$. Then every blow-up limit $u_0 \in \mathcal{BU}_u(x_0)$ is one-homogeneous.*

Proof. Let $x_0 = 0$ and $W(u, \Phi, r) := W(u, \Phi, x_0, r)$. Recall that $u_r(x) = \frac{1}{r}u(rx)$ and $\Phi_r(x) = \Phi(rx)$. We first notice that for every $r > 0$ and $s > 0$ such that $rs \leq \operatorname{dist}(x_0, \partial D)$ we have

$$W(u_r, \Phi_r, s) = W(u, \Phi, rs).$$

Moreover, since the function $r \mapsto W(u, \Phi, r) + Cr$ is monotone, the limit

$$W(u, \Phi, 0) := \lim_{r \rightarrow 0^+} W(u, \Phi, r)$$

exists (and is finite due to the Lipschitz continuity of u). On the other hand, for every blow-up sequence u_{r_n} with blow-up limit u_0 , we have

$$W(u_0, \Phi(0), s) = \lim_{n \rightarrow \infty} W(u_{r_n}, \Phi_{r_n}(0), s) = \lim_{n \rightarrow \infty} W(u, \Phi(0), r_n s) = W(u, \Phi(0), 0).$$

Thus, the function

$$s \mapsto \frac{1}{s^d} \int_{B_s} |\nabla u_0|^2 dx - \frac{1}{s^{d+1}} \int_{\partial B_s} u_0^2 d\mathcal{H}^{d-1} + \frac{\Lambda_u e^{\Phi(0)}}{s^d} |\Omega_{u_0} \cap B_s|,$$

is constant. Now, by [92] (or, simply by applying (2.5.45) to $u = u_0$, $\lambda_m = 0$ and $\Phi = 0$), we have that u_0 is one-homogeneous. \square

Definition 2.5.39. We define d^* as the smallest dimension which admits one-homogeneous global minimizers of the one-phase Alt-Caffarelli problem with (isolated) singularity in zero.

By [62] and [37] we know that $d^* \in \{5, 6, 7\}$. Weiss was first to prove that the monotonicity formula implies the dimension estimate

$$\dim_{\mathcal{H}} \text{Sing}(\partial\Omega_u) := \inf\{\alpha \geq 0 : \mathcal{H}^\alpha(\text{Sing}(\partial\Omega_u)) = 0\} \leq d - d^*,$$

for every $d > d^*$ (see also [77] for an argument using only the monotonicity of W). Thus, as a consequence of Lemma 2.5.37, Lemma 2.5.38 and the results from [92] and [77], we get

Proposition 2.5.40 (On the dimension of the singular set). *Let u be a solution of (2.5.1) in the bounded open set $D \subset \mathbb{R}^d$. Then*

- $\text{Sing}(\partial\Omega_u) = \emptyset$ if $d < d^*$;
- $\text{Sing}(\partial\Omega_u)$ is a discrete (locally finite) set if $d = d^*$;
- $\dim_{\mathcal{H}} \text{Sing}(\partial\Omega_u) \leq d - d^*$ if $d > d^*$.

Proof. The proof of this proposition is standard, once we have the monotonicity of W (Lemma 2.5.37) and the consequent homogeneity of the blow-up limits (Lemma 2.5.38). We refer to [92, Section 4] and [77, Section 5.5]. \square

Remark 2.5.41. Recently, using the innovative approach of Naber and Valtorta [79], Edelen and Engelstein [42] showed the monotonicity formula of Weiss can be used to obtain the (local) estimate $\mathcal{H}^{d-d^*}(\text{Sing}(\partial\Omega_u)) < \infty$, which in particular implies that $\dim_{\mathcal{H}} \text{Sing}(\partial\Omega_u) \leq d - d^*$.

Appendix A: Extremality conditions and Lebesgue density

In this section we prove Proposition 2.A.1, which we use in Proposition 2.5.12 to show that the Lagrange multiplier Λ_u is strictly positive, but the result is of independent interest. For instance, it applies to optimal partition problems (see, for example, [28] and [25]). We first show that a function which is critical for the functional

$$J(u) := \int_D |\nabla u|^2 e^{-\Phi} dx - \lambda \int_D u^2 e^{-\Phi} dx, \quad (2.A.1)$$

with respect to internal variations that is

$$\delta J(u)[\xi] := \lim_{t \rightarrow 0} J(u(x + t\xi(x))) = 0 \quad \text{for every vector field } \xi \in C_c^\infty(D; \mathbb{R}^d),$$

satisfies a monotonicity formula for the associated Almgren frequency function $N(r)$. Now, by the argument of Garofalo and Lin (see [46]) the monotonicity of the frequency function implies that u cannot decay too fast around the free boundary points. If, in addition, u is a solution of

$-\operatorname{div}(e^{-\Phi}\nabla u) = \lambda u e^{-\Phi}$ on the positivity set $\Omega_u = \{u > 0\}$, we can use a Caccioppoli inequality to show that if the Lebesgue density of Ω_u is too small, then the decay of u on the balls of radius r should be very fast. This, in combination with the monotonicity of the Almgren's frequency function, shows that the Lebesgue density of Ω_u should be bounded from below *everywhere* (and not only on the boundary of Ω_u). In particular, there cannot be points of zero Lebesgue density for Ω_u in D .

Proposition 2.A.1. *Let $D \subset \mathbb{R}^d$ be a bounded open set and $\Phi \in W^{1,\infty}(D)$. Suppose that $\lambda \geq 0$ and $u \in H^1(D)$ is a nonnegative (non-identically-zero) function such that*

(a) *u is a solution of the equation*

$$-\operatorname{div}(e^{-\Phi}\nabla u) = \lambda e^{-\Phi}u \quad \text{in } \Omega_u = \{u > 0\}; \quad (2.A.2)$$

(b) *u satisfies the extremality condition*

$$\delta J(u)[\xi] = 0 \quad \text{for every } \xi \in C_c^\infty(D; \mathbb{R}^d),$$

where J is given by (2.A.1) and its first variation in the direction ξ is given by

$$\delta J(u)[\xi] := \int_D \left[2D\xi(\nabla u) \cdot \nabla u + (|\nabla u|^2 - \lambda u^2)(\nabla\Phi \cdot \xi - \operatorname{div}\xi) \right] e^{-\Phi} dx. \quad (2.A.3)$$

Then, $|D \setminus \Omega_u| = 0$.

2.A.1 Reduction to the case $\lambda = 0$

In this section we will show that it is sufficient to prove Proposition 2.A.1 for $\lambda = 0$. The general case will then follow by an elementary substitution argument. In the next lemma we deal with the first variation of the functional J .

Lemma 2.A.2. *Suppose that $D \subset \mathbb{R}^d$ is a bounded open set, $a : D \rightarrow \mathbb{R}$ is a given Lipschitz function such that $0 < \varepsilon \leq a \leq \varepsilon^{-1}$ on D . Let $\lambda > 0$ and let $\varphi \in H^2(D)$ be such that*

$$-\operatorname{div}(a\nabla\varphi) = \lambda a\varphi \quad \text{in } D, \quad \varphi \geq \varepsilon > 0 \quad \text{on } D.$$

For any $u \in H^1(D)$, we set $\tilde{a}(x) := \varphi^2(x)a(x)$, $\tilde{u} := u/\varphi$,

$$J(u) := \int_D (|\nabla u|^2 - \lambda u^2) a(x) dx \quad \text{and} \quad \tilde{J}(u) := \int_D |\nabla u|^2 \tilde{a}(x) dx,$$

$$\delta J(u)[\xi] := \int_D \left[2aD\xi(\nabla u) \cdot \nabla u - (|\nabla u|^2 - \lambda u^2) \operatorname{div}(a\xi) \right] dx,$$

$$\delta \tilde{J}(u)[\xi] := \int_D \left[2\tilde{a}D\xi(\nabla u) \cdot \nabla u - |\nabla u|^2 \operatorname{div}(\tilde{a}\xi) \right] dx \quad \text{for any } \xi \in C_c^\infty(D; \mathbb{R}^d).$$

Then, for every $u \in H^1(D)$ and every $\xi \in C_c^\infty(D; \mathbb{R}^d)$, we have

$$\delta \tilde{J}(\tilde{u})[\xi] = \delta J(u)[\xi] - 2 \int_D \nabla(u\xi \cdot \nabla(\ln\varphi)) \cdot \nabla u a dx + 2 \int_D (u\xi \cdot \nabla(\ln\varphi)) \lambda u a dx. \quad (2.A.4)$$

Proof. Notice that we may assume $u \in C^\infty(D)$. First we notice that an integration by parts gives

$$\begin{aligned} \delta \tilde{J}(\tilde{u})[\xi] &= \int_D 2\partial_i \xi_j \partial_i \tilde{u} \partial_j \tilde{u} \tilde{a} dx - \int_D |\nabla \tilde{u}|^2 \operatorname{div}(\tilde{a}\xi) dx \\ &= - \int_D 2\xi_j \partial_i (\tilde{a} \partial_i \tilde{u}) \partial_j \tilde{u} dx - \int_D 2\xi_j \partial_i \tilde{u} \partial_{ij} \tilde{u} \tilde{a} dx - \int_D |\nabla \tilde{u}|^2 \operatorname{div}(\tilde{a}\xi) dx \\ &= - \int_D 2\xi_j \partial_i (\tilde{a} \partial_i \tilde{u}) \partial_j \tilde{u} dx - \int_D \operatorname{div}(\tilde{a} |\nabla \tilde{u}|^2 \xi) dx = - \int_D 2\xi_j \partial_i (\tilde{a} \partial_i \tilde{u}) \partial_j \tilde{u} dx \\ &= - \int_D 2(\xi \cdot \nabla \tilde{u}) \operatorname{div}(\tilde{a} \nabla \tilde{u}) dx. \end{aligned}$$

and, analogously,

$$\delta J(u)[\xi] = - \int_D 2(\xi \cdot \nabla u) \operatorname{div}(a \nabla u) \, dx + \lambda \int_D u^2 \operatorname{div}(a \xi) \, dx.$$

Now, since

$$\operatorname{div}(\tilde{a} \nabla \tilde{u}) = \operatorname{div}(a(\varphi \nabla u - u \nabla \varphi)) = \varphi \operatorname{div}(a \nabla u) - u \operatorname{div}(a \nabla \varphi) = \varphi(\operatorname{div}(a \nabla u) + \lambda a u),$$

we get

$$\begin{aligned} \delta \tilde{J}(\tilde{u})[\xi] &= -2 \int_D \xi \cdot (\nabla u - \frac{u}{\varphi} \nabla \varphi) (\operatorname{div}(a \nabla u) + \lambda a u) \, dx \\ &= 2 \int_D \xi \cdot \nabla \varphi \frac{u}{\varphi} (\operatorname{div}(a \nabla u) + \lambda a u) \, dx - 2 \int_D (\xi \cdot \nabla u) (\operatorname{div}(a \nabla u) + \lambda a u) \, dx \\ &= -2 \int_D \nabla \left(\frac{\xi \cdot \nabla \varphi}{\varphi} u \right) \cdot \nabla u \, a \, dx + 2 \int_D \left(\frac{\xi \cdot \nabla \varphi}{\varphi} u \right) \lambda a u \, dx + \delta J(u)[\xi], \end{aligned}$$

which is precisely (2.A.4). \square

Let now $D \subset \mathbb{R}^d$ and $u \in H^1(D)$ be as in Proposition 2.A.1 for some $\lambda > 0$. In order to prove that $|D \setminus \Omega_u| = 0$, it is sufficient to prove that $|(D \cap B) \setminus \Omega_u| = 0$ for any (small) ball $B \subset D$. Let now $x_0 \in D$ and let $R > 0$ be such that $\lambda_1(B_R(x_0), \nabla \Phi) = \lambda$. Such a radius exists, since the map $f(r) := \lambda_1(B_r(x_0), \nabla \Phi)$ is continuous, $f(0) = +\infty$ and $f(+\infty) = 0$. Notice also that we may assume Φ to be defined on the entire space \mathbb{R}^d . Let φ be the first eigenfunction on $B_R(x_0)$ and let $r = R/2$. Then, we can apply Lemma 2.A.2 in the set $D \cap B_r(x_0)$ with $a = e^{-\Phi}$. Moreover, since u satisfies (2.A.2), we get that

$$\delta \tilde{J}(\tilde{u})[\xi] = \delta J(u)[\xi] = 0, \quad \text{for every } \xi \in C_c^\infty(D \cap B_r(x_0); \mathbb{R}^d),$$

which proves that $\tilde{u} = u/\varphi$ satisfies hypothesis (b) for $\lambda = 0$. Finally, in order to prove that \tilde{u} satisfies hypothesis (a), we notice that on $\Omega_u = \Omega_{\tilde{u}}$ we have (in a weak sense)

$$\operatorname{div}(\tilde{a} \nabla \tilde{u}) = \varphi \operatorname{div}(a \nabla u) - u \operatorname{div}(a \nabla \varphi) = \varphi(\operatorname{div}(a \nabla u) + \lambda a u) = 0.$$

2.A.2 Proof of Proposition 2.A.1 in the case $\lambda = 0$

Let $\lambda = 0$. Then we have

$$J(u) := \int_D |\nabla u|^2 e^{-\Phi} \, dx, \tag{2.A.5}$$

$$\delta J(u)[\xi] := \int_D \left[2D\xi(\nabla u) \cdot \nabla u + |\nabla u|^2 (\nabla \Phi \cdot \xi - \operatorname{div} \xi) \right] e^{-\Phi} \, dx. \tag{2.A.6}$$

Let $x_0 = 0 \in D$ and $\tau = \|\nabla \Phi\|_{L^\infty(D)}$. We set

$$H(r) := \int_{\partial B_r} u^2 e^{-\Phi} d\mathcal{H}^{d-1}, \quad D(r) := \int_{B_r} |\nabla u|^2 e^{-\Phi} \, dx \quad \text{and} \quad N(r) := \frac{rD(r)}{H(r)}.$$

Step 1. Derivative of H . We calculate

$$\begin{aligned} H'(r) &= \frac{d-1}{r} H(r) + r^{d-1} \frac{d}{dr} \int_{\partial B_1} u^2(rx) e^{-\Phi(rx)} d\mathcal{H}^{d-1}(x) \\ &= \frac{d-1}{r} H(r) + 2 \int_{\partial B_r} u \frac{\partial u}{\partial n} e^{-\Phi} d\mathcal{H}^{d-1} - \int_{\partial B_r} u^2 (n \cdot \nabla \Phi) e^{-\Phi} d\mathcal{H}^{d-1} \\ &= \frac{d-1}{r} H(r) + 2 \int_{B_r} |\nabla u|^2 e^{-\Phi} \, dx - \int_{\partial B_r} u^2 (n \cdot \nabla \Phi) e^{-\Phi} d\mathcal{H}^{d-1}, \end{aligned}$$

which we rewrite as

$$H'(r) = \frac{d-1}{r}H(r) + 2D(r) - H_\Phi(r). \quad (2.A.7)$$

where we have set

$$H_\Phi(r) := \int_{\partial B_r} u^2(n \cdot \nabla \Phi) e^{-\Phi} d\mathcal{H}^{d-1} \quad \text{and} \quad |H_\Phi(r)| \leq \tau H(r).$$

Step 2. Equidistribution of the energy. Let ϕ_ε be a radially decreasing function such that $0 \leq \phi_\varepsilon \leq 1$ on B_r , $\phi_\varepsilon = 1$ on $B_{r(1-\varepsilon)}$, $\phi_\varepsilon = 0$ on ∂B_r and $|\nabla \phi_\varepsilon| \leq C(r\varepsilon)^{-1}$. The vector field $\xi(x) := x\phi_\varepsilon(x)$ satisfies $\operatorname{div} \xi(x) = d\phi_\varepsilon(x) + x \cdot \nabla \phi_\varepsilon$ and $\partial_i \xi_j = \delta_{ij}\phi_\varepsilon(x) + x_j \partial_i \phi_\varepsilon(x)$. Since $\lambda = 0$ we have

$$\begin{aligned} \delta J(u)[\xi] &= \int_D \left[2D\xi(\nabla u) \cdot \nabla u + |\nabla u|^2(\nabla \Phi \cdot \xi - \operatorname{div} \xi) \right] e^{-\Phi} dx \\ &= \int_D \left[2|\nabla u|^2 \phi_\varepsilon + 2(x \cdot \nabla u)(\nabla \phi_\varepsilon \cdot \nabla u) - |\nabla u|^2(d\phi_\varepsilon(x) + x \cdot \nabla \phi_\varepsilon) \right] e^{-\Phi} dx \\ &\quad + \int_D |\nabla u|^2(\nabla \Phi \cdot x) \phi_\varepsilon e^{-\Phi} dx, \end{aligned}$$

and passing to the limit as $\varepsilon \rightarrow 0$, rearranging the terms and using the property (b), we get

$$\begin{aligned} 0 &= -(d-2) \int_{B_r} |\nabla u|^2 e^{-\Phi} dx + r \int_{\partial B_r} |\nabla u|^2 e^{-\Phi} d\mathcal{H}^{d-1} \\ &\quad - 2r \int_{\partial B_r} \left(\frac{\partial u}{\partial n} \right)^2 e^{-\Phi} d\mathcal{H}^{d-1} + \int_{B_r} |\nabla u|^2(\nabla \Phi \cdot x) e^{-\Phi} dx, \end{aligned}$$

which we rewrite as

$$-(d-2)D(r) + rD'(r) = 2r \int_{\partial B_r} \left(\frac{\partial u}{\partial n} \right)^2 e^{-\Phi} d\mathcal{H}^{d-1} - rD_\Phi(r),$$

where

$$D_\Phi(r) := \frac{1}{r} \int_{B_r} |\nabla u|^2(\nabla \Phi \cdot x) e^{-\Phi} dx \quad \text{and} \quad |D_\Phi(r)| \leq \tau D(r).$$

Step 3. The derivative of N . We notice that $N(r)$ is only defined for r such that $H(r) > 0$. In what follows we fix $r_0 > 0$ such that $B_{r_0}(x_0) \subset D$ and $H(r_0) > 0$. Since $u \in H^1(D)$, there is an interval $(a, b) \ni r_0$, on which $H > 0$.

$$\begin{aligned} N'(r) &= \frac{D(r)H(r) + rD'(r)H(r) - rD(r)H'(r)}{H^2(r)} \\ &= \frac{D(r)H(r) + rD'(r)H(r) - rD(r) \left(\frac{d-1}{r}H(r) + 2D(r) - H_\Phi(r) \right)}{H^2(r)} \\ &= \frac{-(d-2)D(r)H(r) + rD'(r)H(r) - 2rD^2(r) + rD(r)H_\Phi(r)}{H^2(r)} \\ &= \frac{2r}{H^2(r)} \left(H(r) \int_{\partial B_r} \left(\frac{\partial u}{\partial n} \right)^2 e^{-\Phi} d\mathcal{H}^{d-1} - D^2(r) \right) + \frac{r(D(r)H_\Phi(r) - D_\Phi(r)H(r))}{H^2(r)} \end{aligned} \quad (2.A.8)$$

Now we notice that, since u solves (2.A.2) on Ω_u , we have

$$D(r) = \int_{B_r} |\nabla u|^2 e^{-\Phi} dx = \int_{\partial B_r} u \frac{\partial u}{\partial n} e^{-\Phi} d\mathcal{H}^{d-1},$$

and so, by the Cauchy-Schwarz inequality and (2.A.8) we obtain

$$N'(r) \geq \frac{r(D(r)H_\Phi(r) - D_\Phi(r)H(r))}{H^2(r)} \geq -2\tau N(r). \quad (2.A.9)$$

Step 4. A bound on $N(r)$. Using the estimate (2.A.9) from the previous step we get that the function $r \mapsto e^{2\tau r} N(r)$ is non-decreasing in r and so

$$N(r) \leq e^{2\tau(r_0-r)} N(r_0) \leq e^{2\tau r_0} N(r_0) \quad \text{for every } a < r \leq r_0.$$

Step 5. Strict positivity and doubling inequality for $H(r)$. By the step 4 we have

$$\frac{d}{dr} \left[\log \left(\frac{H(r)}{r^{d-1}} \right) \right] = 2 \frac{N(r)}{r} - \frac{H_\Phi(r)}{H(r)} \leq \frac{2e^{2\tau r_0} N(r_0)}{r} + \tau, \quad (2.A.10)$$

and integrating we get

$$\log \left(\frac{H(r_0)}{r_0^{d-1}} \right) - \log \left(\frac{H(r)}{r^{d-1}} \right) \leq \log \left(\frac{r_0}{r} \right) 2e^{2\tau r_0} N(r_0) + \tau r_0, \quad \text{for every } a < r \leq r_0.$$

In particular, $H > 0$ on every interval $[\varepsilon r_0, r_0]$ and so, $H > 0$ on $(0, r_0]$ and we might take $a = 0$. Moreover, integrating once again the inequality (2.A.10) from $r < r_0/2$ to $2r$, we get

$$\log \left(\frac{H(2r)}{H(r)} \right) \leq ((d-1) \log 2 + \tau r_0) + 2 \log 2 e^{2\tau r_0} N(r_0) \quad \text{for every } 0 < r \leq \frac{r_0}{2}.$$

Taking $r_0 \leq 1$, there is a constant C , depending only on d and τ , such that

$$H(2r) \leq C \exp(CN(r_0)) H(r) \quad \text{for every } 0 < r \leq \frac{r_0}{2}. \quad (2.A.11)$$

Integrating once more in r we get

$$\int_{B_{2r}} u^2 e^{-\Phi} dx \leq C \exp(CN(r_0)) \int_{B_r} u^2 e^{-\Phi} dx \quad \text{for every } 0 < r \leq \frac{r_0}{2}. \quad (2.A.12)$$

Step 6. Caccioppoli inequality and conclusion. Let $r \in (0, r_0/2]$ and let $\phi \in C_0^\infty(B_{2r})$ be such that $\phi = 1$ in B_r , $\phi = 0$ on ∂B_{2r} , $0 \leq \phi \leq 1$ and $|\nabla \phi| \leq 2/r$ on $B_{2r} \setminus B_r$. Using the fact that u is a solution of $-\operatorname{div}(e^{-\Phi} \nabla u) = 0$ in Ω_u , we get the following Caccioppoli inequality:

$$\begin{aligned} \int_{B_r} |\nabla u|^2 e^{-\Phi} dx &\leq \int_{B_{2r}} |\nabla(u\phi)|^2 e^{-\Phi} dx = \int_{B_{2r}} (u^2 |\nabla \phi|^2 + \nabla u \cdot \nabla(u\phi^2)) e^{-\Phi} dx \\ &= \int_{B_{2r}} u^2 |\nabla \phi|^2 e^{-\Phi} dx - \int_{B_{2r}} u\phi^2 \operatorname{div}(e^{-\Phi} \nabla u) dx = \int_{B_{2r}} u^2 |\nabla \phi|^2 e^{-\Phi} dx. \\ &\leq \frac{4}{r^2} \int_{B_{2r}} u^2 e^{-\Phi} dx. \end{aligned} \quad (2.A.13)$$

On the other hand, there are dimensional constants C_d and $\varepsilon_d > 0$ such that, if $|\Omega_u \cap B_r| \leq \varepsilon_d |B_r|$, then the following inequality does hold (see [22, Lemma 4.4])

$$\int_{B_r} u^2 dx \leq C_d r^2 \left(\frac{|\Omega_u \cap B_r|}{|B_r|} \right)^{2/d} \int_{B_r} |\nabla u|^2 dx,$$

which, taking $C := C_d \exp(\max \Phi - \min \Phi)$, implies

$$\int_{B_r} u^2 e^{-\Phi} dx \leq C r^2 \left(\frac{|\Omega_u \cap B_r|}{|B_r|} \right)^{2/d} \int_{B_r} |\nabla u|^2 e^{-\Phi} dx.$$

This, together with (2.A.13) and the doubling inequality (2.A.12), gives that there are constants C_1 and C_2 , depending only on d and τ such that

$$\min \{ \varepsilon_d, C_1 \exp(-C_2 N(r_0)) \} \leq \frac{|\Omega_u \cap B_r|}{|B_r|} \quad \text{for every } 0 < r \leq \frac{r_0}{2},$$

where to be precise we recall that we assumed $r_0 \leq 1$. In particular, we have a lower density bound for Ω_u at every point of D , which implies that $|D \setminus \Omega_u| = 0$ and concludes the proof.

Chapter 3

Free boundary regularity for a multiphase shape optimization problem

This chapter is based on [85], which is joint work with Luca Spolaor and Bozhidar Velichkov.

Abstract

In this paper we prove a $C^{1,\alpha}$ regularity result in dimension two for almost-minimizers of the constrained one-phase Alt-Caffarelli and the two-phase Alt-Caffarelli-Friedman functionals for an energy with variable coefficients. As a consequence, we deduce the complete regularity of solutions of a multiphase shape optimization problem for the first eigenvalue of the Dirichlet Laplacian, up to the boundary of a fixed domain that acts as a geometric inclusion constraint. One of the main ingredients is a new application of the isoperimetric inequality of [86] up to the boundary of the constraint. While the framework that leads to this application is valid in every dimension, the isoperimetric inequality is known only in dimension two, thus the restriction on the dimension.

3.1 Introduction

In this paper we prove the regularity of the free boundary of solutions to variational one-phase and two-phase free boundary problems in dimension two. In particular, we consider the case when the support of the solution is constrained in certain region $D \subset \mathbb{R}^2$ with smooth ($C^{1,\beta}$ -regular, for some $\beta > 0$) boundary and we show that the free boundaries must be $C^{1,\alpha}$ -regular up to the points of contact with the boundary of the region D . Our arguments strongly rely on the isoperimetric inequality for the one-phase and the two-phase Bernoulli free boundary functionals (see [86]). The only obstruction to the generalization of our results to any dimension comes from the fact that the isoperimetric inequality is only known in dimension two. Since our methods are of purely variational nature, we are able to obtain the regularity of the free boundaries of almost-minimizers (even in the presence of a geometric constraint D). Precisely, we prove the following results.

(OP) The $C^{1,\alpha}$ -regularity of the boundaries of the almost-minimizers of the one-phase Alt-Caffarelli functional for an operator with variable coefficients, which may also satisfy a further geometric inclusion constraint (Theorem 3.1.1 and Corollary 3.1.3);

(**TP**) The $C^{1,\alpha}$ -regularity of the boundaries (of each phase) of the almost-minimizers of the two-phase Alt-Caffarelli-Friedman functional for an operator with variable coefficients (Theorem 3.1.5). In particular, this is a problem left open in [33].

In the second part of the paper, we apply our regularity results for almost-minimizers to the solutions of multiphase shape optimization problems, where the variables are n -uples of different disjoint domains (phases). Precisely, we consider the model case in which the variational cost functional is given by the sum of the first eigenvalue (of the Dirichlet Laplacian) and the area of each domain. In Theorem 3.1.11, we show that if the family of domains $(\Omega_1, \dots, \Omega_n)$ is a solution to such a multiphase problem, then the boundaries $\partial\Omega_i$ are $C^{1,\alpha}$ -regular; thus we solve the (regularity) problem left open in [7].

The regularity of the free boundaries arising in the context of shape optimization problems involving Dirichlet eigenvalues is a topic that received a lot of attention recently (we refer for instance to the recent papers [77, 70, 71, 23]). The multiphase shape optimization problem we consider is part of this general framework and was studied in [19] and [7]. It is related to the regularity of the optimal sets (in a box) for the second Dirichlet eigenvalue, which is expected to have common qualitative properties with the solutions to the multiphase problem in the case $n = 2$. We also notice that our almost-minimality conditions naturally arise in shape optimization problems involving Dirichlet eigenvalues (see for instance [77], [18] and Section 3.7). The geometric inclusion constraints are also very common in this framework (see for instance [14], [11] and the books [61], [13]). For instance, they are often used to provide the compactness necessary for the existence of optimal shapes. Thus, our regularity results and techniques can be applied not only to the solutions of the multiphase problem studied in [19] and [7], but to a variety of shape optimization problems, for instance, on manifolds or for operators with variable coefficients.

The regularity of the one-phase free boundaries was first studied by Alt and Caffarelli in [2] who prove that in any dimension the (local) minimizers of the one-phase functional have smooth free boundaries up to a small singular set, while in dimension two, they show that the entire free boundaries are C^∞ smooth. The regularity of almost-minimizers of the one-phase functional was addressed by David and Toro in [34] and by David, Engelstein and Toro in [33], where they prove the $C^{1,\alpha}$ regularity of the free boundary up to a singular set, which is empty in dimension two; the same result was recently obtained by De Silva and Savin [40] by a different method based on a non-infinitesimal notion of viscosity solution. In [41] and [40], the authors consider only the case of the Laplacian, but it is clear that their proof can be generalized to an operator with variable coefficients.

With our approach, working with a functional with variable coefficients, instead of constant ones, leads only to minor variations in the proof. We also stress that there is no way to reduce the non-constant-coefficients case to the constant-coefficients one. In fact, if u is a minimizer (or almost-minimizer) of a functional with variable coefficients, then for any point x , there is a change of coordinates, for which the new function becomes an almost-minimizer for a functional involving only the Dirichlet energy (see Lemma 3.3.2). This change of coordinates, on the other hand, depends on the point. It is also responsible for the anisotropic optimality condition on the free boundary (see (3.1.5)).

The regularity of minimizers for the one-phase functional subjected to a geometric inclusion constraint was studied by Chang-Lara and Savin in [26], where, using the approach of De Silva [36], the authors prove that in a neighborhood of the contact set (with the boundary of the constraint) the free boundary is $C^{1,\alpha}$ regular in any dimension. In Theorem 3.1.1 we consider almost-minimizers satisfying an inclusion constraint, which combines the difficulties from [33] and [26]: the lack of equation ([33]) and the presence of a geometric constraint ([26]). In fact, our approach allows to treat these two situations at once.

The two-phase problem was first studied by Alt, Caffarelli and Friedman in [3] (see Remark

3.1.6) but the regularity of the free boundary (for minimizers) was achieved only recently in [86]. In [33], David, Engelstein and Toro address the question of the regularity of almost-minimizers of the two-phase functional and prove the rectifiability of the free boundary in any dimension. In Theorem 3.1.5 we prove the $C^{1,\alpha}$ regularity of the free boundaries of almost-minimizers for two-phase functional. We notice that, since each phase acts as a geometric obstacle for the other one, and in view of the Chang-Lara-Savin result ([26]), the $C^{1,\alpha}$ -regularity is optimal even for minimizers, the best α being $1/2$. We also notice that, in the two-phase case, an additional (smooth) geometric inclusion constraint does not affect the regularity of the free boundaries. In fact, a two-phase free boundary cannot touch the boundary of the constraint. This is a simple consequence of the three-phase monotonicity formula introduced in [89] and [19], just as in the case of the multiphase shape optimization problem (3.1.10) (see [7] and Section 3.7).

3.1.1 Regularity for almost-minimizers

Throughout this paper we will use the following notations.

Let Sym_k^+ be the family of the real positive symmetric $k \times k$ matrices. We fix a matrix-valued function $A = (a_{ij})_{ij} : B_2 \rightarrow Sym_2^+$, for which there are constants $\delta_A, C_A, M_A > 0$ such that

$$|a_{ij}(x) - a_{ij}(y)| \leq C_A |x - y|^{\delta_A} \quad \text{for every } i, j \quad \text{and } x, y \in B_2;$$

$$M_A^{-1} |\xi|^2 \leq \xi \cdot A(x) \xi = \sum_{i,j=1}^2 \xi_i \xi_j a_{ij}(x) \leq M_A |\xi|^2 \quad \text{for every } x \in B_2 \quad \text{and } \xi \in \mathbb{R}^2. \quad (3.1.1)$$

We fix $Q_{\text{op}}, Q_{\text{TP}}^+$ and Q_{TP}^- to be Hölder continuous functions on B_2 , for which there are constants $\delta_Q, C_Q, M_Q > 0$ such that for any $Q = Q_{\text{op}}, Q_{\text{TP}}^+$ or Q_{TP}^- , we have

$$|Q(x) - Q(y)| \leq C_Q |x - y|^{\delta_Q} \quad \text{for every } x, y \in B_2;$$

$$M_Q^{-1} \leq Q(x) \leq M_Q \quad \text{for every } x \in B_2.$$

Finally, for every function $u : \mathbb{R}^2 \rightarrow \mathbb{R}$, we will use the following standard notations

$$u_{\pm}(x) := \max\{\pm u(x), 0\}, \quad \Omega_u := \{u \neq 0\}, \quad \Omega_u^+ := \{u > 0\} \quad \text{and} \quad \Omega_u^- := \{u < 0\}.$$

We are now in position to state our main free boundary regularity results.

The one-phase free boundaries. For every $u \in H^1(B_2)$, $x_0 \in B_1$ and $r \in (0, 1)$, we define the one-phase functional

$$J_{\text{op}}(u, x_0, r) = \int_{B_r(x_0)} \left(\sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + Q_{\text{op}}(x) \mathbb{1}_{\{u > 0\}} \right) dx.$$

Here and after $B_r(x)$ denotes the ball with center $x \in \mathbb{R}^2$ and radius $r > 0$ and we will write $B_r := B_r(0)$. Let $\mathcal{A}^+(B_r)$ be the admissible set

$$\mathcal{A}^+(B_r) = \{u \in H^1(B_r) : u \geq 0 \text{ in } B_r, u = 0 \text{ on } B_r \setminus B_r^+\},$$

where H stands for the upper half-plane $H := \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and $B_r^+ := B_r \cap H$. We say that the nonnegative function $u : B_2 \rightarrow \mathbb{R}$ is a *almost-minimizer of the one-phase functional* J_{op} in the upper half-disk B_2^+ , if $u \in \mathcal{A}^+(B_2)$ and there are constants $r_1 > 0$, $C_1 > 0$ and $\delta_1 > 0$ such that, for every $x_0 \in B_1 \cap \partial\Omega_u$ and $r \in (0, r_1)$, we have

$$J_{\text{op}}(u, x_0, r) \leq (1 + C_1 r^{\delta_1}) J_{\text{op}}(v, x_0, r) + C_1 r^{2+\delta_1}, \quad (3.1.2)$$

for every $v \in \mathcal{A}^+(B_2)$ such that $u = v$ on $B_2 \setminus B_r(x_0)$.

We have the following result for the almost-minimizers of the one-phase Alt-Caffarelli functional J_{op} constrained in the upper half-disk B_2^+ .

Theorem 3.1.1 (Regularity of the constrained one-phase free boundaries). *Let $B_2 \subset \mathbb{R}^2$ and $u : B_2 \rightarrow \mathbb{R}$ be a non-negative and Lipschitz continuous function. If u is a almost-minimizer of the functional J_{op} in $\mathcal{A}^+(B_2)$, then the free boundary $B_1 \cap \partial\Omega_u$ is locally the graph of a $C^{1,\alpha}$ function. Moreover, u satisfies the optimality condition*

$$\begin{cases} |A_{x_0}^{1/2} \nabla u|(x_0) = Q_{\text{op}}^{1/2}(x_0) & \text{for every } x_0 \in \partial\Omega_u^+ \cap B_2 \cap \{x_2 > 0\}, \\ |A_{x_0}^{1/2} \nabla u|(x_0) \geq Q_{\text{op}}^{1/2}(x_0) & \text{for every } x_0 \in \partial\Omega_u^+ \cap B_2 \cap \{x_2 = 0\}. \end{cases} \quad (3.1.3)$$

Remark 3.1.2. The Hölder continuity of the (exterior) normal vector n_Ω is the best regularity result that one can expect. Indeed, recently Chang-Lara and Savin [26] showed that even for minimizers the regularity of the constrained free boundaries cannot exceed $C^{1,1/2}$. Moreover, we notice that the result analogous to Theorem 3.1.1 was proved in any dimension in [26], by a viscosity approach, but only for minimizers of the functional J_{op} .

Analogously, we say that the nonnegative function $u : B_2 \rightarrow \mathbb{R}$ is a *almost-minimizer of the one-phase functional J_{op} in B_2* , if $u \in H^1(B_2)$ and there are constants $r_1 > 0$, $C_1 > 0$ and $\delta_1 > 0$ such that, for every $x_0 \in B_1 \cap \partial\Omega_u$ and $r \in (0, r_1)$, we have

$$J_{\text{op}}(u, x_0, r) \leq (1 + C_1 r^{\delta_1}) J_{\text{op}}(v, x_0, r) + C_1 r^{2+\delta_1}, \quad (3.1.4)$$

for every $v \in H^1(B_2)$ such that $u = v$ on $B_2 \setminus B_r(x_0)$.

The regularity of the unconstrained one-phase free boundary $\partial\Omega_u$ follows directly by Theorem 3.1.1. For the sake of completeness, we give the precise statement in Corollary 3.1.3 below.

Corollary 3.1.3 (Regularity of the unconstrained one-phase free boundaries). *Let $B_2 \subset \mathbb{R}^2$ and $u : B_2 \rightarrow \mathbb{R}$ be a non-negative and Lipschitz continuous function. If u is a almost-minimizer of the functional J_{op} in B_2 , then the free boundary $B_1 \cap \partial\Omega_u$ is locally the graph of a $C^{1,\alpha}$ function. Moreover, u satisfies the optimality condition*

$$|A_{x_0}^{1/2} \nabla u|(x_0) = Q_{\text{op}}^{1/2}(x_0) \quad \text{for every } x_0 \in \partial\Omega_u^+ \cap B_2. \quad (3.1.5)$$

Remark 3.1.4. The regularity of the free boundaries of the one-phase (unconstrained) almost-minimizers was proved in [33] in every dimension, by a different approach.

The two-phase free boundaries. For every $u \in H^1(B_2)$, $x_0 \in B_1$ and $r \in (0, 1)$, we define the two-phase functional

$$J_{\text{TP}}(u, x_0, r) = \int_{B_r(x_0)} \left(\sum_{i,j} a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} + Q_{\text{TP}}^+(x) \mathbb{1}_{\{u>0\}} + Q_{\text{TP}}^-(x) \mathbb{1}_{\{u<0\}} \right) dx.$$

We say that the function $u \in H^1(B_2)$ is a *almost-minimizer of the two-phase functional J_{TP} in B_2* , if there are constants $r_2 > 0$, $C_2 > 0$ and $\delta_2 > 0$ such that, for every $x_0 \in B_1 \cap \partial\Omega_u$ and $r \in (0, r_2)$, we have

$$J_{\text{TP}}(u, x_0, r) \leq (1 + C_2 r^{\delta_2}) J_{\text{TP}}(v, x_0, r) + C_2 r^{2+\delta_2}, \quad (3.1.6)$$

for every $v \in H^1(B_2)$ such that $u = v$ on $B_2 \setminus B_r(x_0)$.

Then, we have the following result:

Theorem 3.1.5 (Regularity of the two-phase free boundaries). *Let $B_2 \subset \mathbb{R}^2$ and let $u : B_2 \rightarrow \mathbb{R}$ be Lipschitz continuous and such that the functions u_\pm are solutions of the PDEs*

$$-\operatorname{div}(A \nabla u_\pm) = f_\pm \quad \text{in } \Omega_u^\pm \cap B_2, \quad (3.1.7)$$

where:

(a) the functions $f_{\pm} : \overline{\Omega}_u^{\pm} \rightarrow \mathbb{R}$ are bounded and continuous;

(b) the matrix-valued function $A : B_2 \rightarrow \text{Sym}_2^+$ satisfies (3.1.1) and has C^1 -regular coefficients.

Under these conditions, if u is a almost-minimizer of the two-phase functional J_{TP} in B_2 , then the free boundaries $B_1 \cap \partial\Omega_u^+$ and $B_1 \cap \partial\Omega_u^-$ are locally graphs of $C^{1,\alpha}$ functions, for some $\alpha > 0$. Moreover, u satisfies the optimality condition (on $\partial\Omega_u^+$)

$$\begin{cases} |A_{x_0}^{1/2} \nabla u_+|(x_0) = (Q_{\text{op}}^+(x_0))^{1/2} & \text{for every } x_0 \in \partial\Omega_u^+ \setminus \partial\Omega_u^- \cap B_2, \\ |A_{x_0}^{1/2} \nabla u_+|(x_0) \geq (Q_{\text{op}}^+(x_0))^{1/2} & \text{for every } x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_2. \end{cases} \quad (3.1.8)$$

Remark 3.1.6 (Regularity of the two-phase free boundaries for minimizers). Alt, Caffarelli and Friedman were the first to study the two-phase free boundaries in (see [3]). Precisely, they studied the local minimizers of the functional

$$J_{\text{ACF}}(u, x_0, r) = \int_{B_r(x_0)} \left(|\nabla u|^2 + q^2(x) \lambda^2(u(x)) \right) dx,$$

where q is Hölder continuous and bounded away from zero and λ is the function

$$\lambda^2(u) = \lambda_1^2 \quad \text{if } u > 0 \quad \text{and} \quad \lambda^2(u) = \lambda_2^2 \quad \text{if } u < 0.$$

In the case, $\lambda(0) = \lambda_1$, they prove that the free boundary $\partial\{u > 0\}$ is C^1 and that the two free boundaries $\partial\{u > 0\}$ and $\partial\{u < 0\}$ coincide (this means that the set $\{u = 0\}$ has empty interior). On the other hand, the case

$$0 \leq \lambda^2(0) < \min\{\lambda_1^2, \lambda_2^2\}, \quad (3.1.9)$$

where branching points may appear, was left completely open in [3]. We notice that our Theorem 3.1.5 covers this case, by setting

$$Q_+(x) = q^2(x)(\lambda_1^2 - \lambda^2(0)) \quad \text{and} \quad Q_-(x) = q^2(x)(\lambda_2^2 - \lambda^2(0)).$$

Precisely, Theorem 3.1.5 implies that, if u is a minimizer of J_{ACF} in a ball B_r and (3.1.9) holds, then the free boundaries $\partial\Omega_u^+ \cap B_r$ and $\partial\Omega_u^- \cap B_r$ are $C^{1,\alpha}$ -regular. This result (for minimizers in the constant-coefficient case and in dimension two) was first proved by the first and the third author in [86]. In particular, it concludes the analysis of the free boundary for minimizers of the functional J_{ACF} , which was started in [3].

Remark 3.1.7 (Remark on the Lipschitz continuity of u). In Theorem 3.1.3 and Theorem 3.1.5 we assume that the function u is Lipschitz continuous. In the case of the Laplacian, David and Toro [34] proved that the Lipschitz continuity is a consequence of the almost-minimality condition. It is, of course, natural to expect that the same will hold if the operator involved has variable coefficients. We will not address this question in the present paper since our main motivation comes from the application to shape optimization problems as (3.1.10), for which the Lipschitz continuity is often already known. Actually, in the case of (3.1.10), the Lipschitz continuity of the eigenfunctions is used to deduce the almost-minimality (see Section 3.7).

We notice that, just from the fact that u is almost-minimizer of the two-phase functional J_{TP} , without using the additional assumption (3.1.7), we can still deduce that u is differentiable at points of the free boundary and that the optimality condition (3.1.8) holds. We summarize the regularity properties, that can be obtained just from the almost-minimality, in the following proposition.

Proposition 3.1.8. *Let $B_2 \subset \mathbb{R}^2$ and let $u : B_2 \rightarrow \mathbb{R}$ be Lipschitz continuous almost-minimizer of the two-phase functional J_{TP} in B_2 . Then, for every free boundary point $x_0 \in \partial\Omega_u^+ \cap B_1$ (the case $x_0 \in \partial\Omega_u^- \cap B_1$ is symmetric), the following claims hold true.*

- (i) For every $x_0 \in \partial\Omega_u^+$, there is a unique blow-up limit $u_{x_0}^+ = \lim_{r \rightarrow 0} u_{r,x_0}^+$, where u_{r,x_0}^+ is the rescaling $u_{r,x_0}^+(x) = \frac{1}{r}u_+(x_0 + rx)$ and the convergence is uniform on every ball $B_R \subset \mathbb{R}^2$.
- (ii) The blow-up limit is of the form

$$u_{x_0}^+(x) = \mu_+(x_0) \max \{0, x \cdot A_{x_0}^{-1/2}[\tilde{\nu}_{x_0}]\},$$

where $\mu_+(x_0)$ is a positive real number and $\tilde{\nu}_{x_0} \in \mathbb{R}^2$ is a unit vector.

- (iii) There are universal constants $C > 0$ and $\alpha > 0$, and a radius $r(x_0) > 0$, such that

$$\|u_{r,x_0}^+ - u_{x_0}^+\|_{L^\infty(B_1)} \leq Cr^\alpha \quad \text{for every } 0 < r < r(x_0).$$

- (iv) u is differentiable at x_0 , up to the boundary of Ω_u^+ , that is,

$$u(x) = (x - x_0) \cdot \nabla u_+(x_0) + O(|x - x_0|^\alpha) \quad \text{for every } x \in \Omega_u^+,$$

and the gradient at x_0 is given by

$$\nabla u_+(x_0) = \mu_+(x_0) A_{x_0}^{-1/2}[\tilde{\nu}_{x_0}],$$

where $\tilde{\nu}_{x_0}$ is the unit vector from ((ii)).

- (v) If x_0 is a one-phase point, $x_0 \in \partial\Omega_u^+ \setminus \partial\Omega_u^-$, then

$$\mu_+(x_0)^2 = Q_{\text{tp}}^+(x_0) \quad \text{and} \quad r(x_0) \text{ depends on the point } x_0.$$

- (vi) If x_0 is a two-phase point, $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^-$, then

$$\mu_+(x_0)^2 \geq Q_{\text{tp}}^+(x_0) \quad \text{and} \quad r(x_0) = r_0,$$

where r_0 may depend on u , but not on x_0 .

This statement is contained in Proposition 3.4.3, Lemma 3.6.1, Lemma 3.6.2 and Lemma 3.6.3. As a corollary of Proposition 3.1.8, we obtain the following corollary, whose proof is standard and is, in fact, similar to the proof of Theorem 3.1.1.

Corollary 3.1.9. *Let $B_2 \subset \mathbb{R}^2$ and let $u : B_2 \rightarrow \mathbb{R}$ be Lipschitz continuous almost-minimizer of the two-phase functional J_{tp} in B_2 . Then, for every $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1$, there exists a neighborhood \mathcal{U}_{x_0} such that the two-phase free boundary $\partial\Omega_u^+ \cap \partial\Omega_u^- \cap \mathcal{U}_{x_0}$ is contained in a single $C^{1,\alpha}$ -regular embedded curve.*

We notice that the result from Theorem 3.1.5 is stronger (but requires the additional technical assumption (3.1.7)). In fact, if the set Ω_u^+ , then the two-phase boundary $\partial\Omega_u^+ \cap \partial\Omega_u^-$ is (obviously) contained in a $C^{1,\alpha}$ curve. We believe that the $C^{1,\alpha}$ regularity of the sets Ω_u^+ and Ω_u^- still holds without the additional assumption (3.1.7), but at the moment, we cannot remove it from our argument. This is mainly due to the fact that we use a combination of viscosity and variational techniques. We give more details on the use of (3.1.7) in the following remark.

Remark 3.1.10 (On the role of the additional assumption (3.1.7) in Theorem 3.1.5). In order to conclude the proof of Theorem 3.1.5, we will use Proposition 3.1.8 and the assumption (3.1.7). Indeed, the $C^{1,\alpha}$ regularity of $\partial\Omega_u^+ \cap B_1$ follows from the following two claims:

- (1) The function $\mu_+ : \partial\Omega_u \rightarrow \mathbb{R}$ is Hölder continuous.
- (2) Suppose that $u_+ : B_1 \rightarrow \mathbb{R}$ is also a (viscosity) solution to

$$-\operatorname{div}(A\nabla u_+) = f \quad \text{in } \Omega_u^+ \cap B_1, \quad |A^{1/2}\nabla u_+| = Q \quad \text{on } \partial\Omega_u^+ \cap B_1,$$

where f is bounded, A is C^1 and Q is Hölder continuous and bounded from below and above. Then $\partial\Omega_u^+ \cap B_1$ is $C^{1,\alpha}$ regular for some $\alpha > 0$.

The first claim is proved in Lemma 3.6.4. We notice that in order to prove the continuity of μ_+ , we use an argument by contradiction, in which we consider a sequence of the form $u_n := u_{x_n, r_n}^+$, which converges to some function u_∞ . At this point, we need that the boundary condition on $\partial\{u_n > 0\}$ passes to the limit (in viscosity sense). It is not at the present clear how to overcome this difficulty if u is just an almost-minimizer. The second claim follows by the De Silva ε -regularity theorem (see Theorem 3.A.1). Also in this case we need that u solves an elliptic equation inside the positivity set Ω_u^+ .

3.1.2 Multiphase shape optimization problem for the first eigenvalue

As a consequence of Theorem 3.1.1 and Theorem 3.1.5, we prove a regularity result for the solutions of the following multiphase shape optimization problem:

$$\min \left\{ \sum_{i=1}^n (\lambda_1(\Omega_i) + q_i |\Omega_i|) : \Omega_1, \dots, \Omega_n \text{ are disjoint open subsets of } \mathcal{D} \right\}, \quad (3.1.10)$$

where, we will use the following notations:

- $1 \leq n \in \mathbb{N}$, and $0 < q_i \in \mathbb{R}$, for every $i = 1, \dots, n$;
- $\mathcal{D} \subset \mathbb{R}^2$ is a bounded open set with $C^{1,\beta}$ -regular boundary, for some $\beta > 0$;
- $|\Omega|$ denotes the Lebesgue measure of Ω ;
- $\lambda_1(\Omega)$ is the first eigenvalue of the Dirichlet Laplacian on Ω .

Theorem 3.1.11. *Let $(\Omega_1, \dots, \Omega_n)$ be a solution of (3.1.10). Then, each set Ω_i , $i = 1, \dots, n$, is a bounded open set with $C^{1,\alpha}$ regular boundary, for some $\alpha > 0$.*

We notice that, in the above theorem, we prove that the *entire* boundary $\partial\Omega_i$, $i = 1, \dots, n$, is $C^{1,\alpha}$ -regular. In particular, this holds at the contact points of Ω_i with the other phases Ω_j , $j \neq i$, and also at the contact points of $\partial\Omega_i$ with the boundary of the box $\partial\mathcal{D}$.

In the special case $n = 1$, (3.1.10) reduces to the classical (one-phase) shape optimization problem

$$\min \{ \lambda_1(\Omega) + \Lambda |\Omega| : \Omega \text{ open, } \Omega \subset \mathcal{D} \}. \quad (3.1.11)$$

The existence of a solution in the class of open sets and the regularity of the free boundary (precisely, of the part contained in the open set \mathcal{D}) was proved by Briançon and Lamboley in [11]. As a direct corollary of our Theorem 3.1.11, we obtain that the entire boundary is $C^{1,\alpha}$ regular.

Corollary 3.1.12 (Regularity of the optimal sets for the first eigenvalue). *Let $\mathcal{D} \subset \mathbb{R}^2$ be a bounded open set of class $C^{1,\beta}$, for some $\beta > 0$, and let $\Lambda > 0$. Then, there is $\alpha \in (0, 1)$ such that every solution $\Omega \subset \mathcal{D}$ of (3.1.11) is $C^{1,\alpha}$ regular.*

3.1.3 Organization of the paper

In Section 3.2 we recall the definitions of the Weiss' boundary adjusted energies and the statements of the epiperimetric inequalities. Moreover, we show how to use the monotonicity formula and the epiperimetric inequality to deduce the rate of convergence of the blow-up sequences and the uniqueness of the blow-up limits. In Section 3.3 we prove a technical lemma that reduces the one-phase and two-phase problems to the case of the Laplacian, which allows us to apply the results of Section 3.2. Section 3.4 is dedicated to the classification of the blow-up limits for the one-phase and the two-phase problems. In Section 3.5 and Section 3.6 we prove Theorem 3.1.1 and Theorem 3.1.5, respectively. In Section 3.7 we prove that the (eigenfunctions associated to the) solutions of the multiphase problem (3.1.10) are locally almost-minimizers of the one-phase or the two-phase problems, and we prove Theorem 3.1.11.

3.2 Boundary adjusted energy and epiperimetric inequality

All the arguments in this section hold in every dimension $d \geq 2$, except the epiperimetric inequalities Theorem 3.2.2 and Theorem 3.2.3, which are known only in dimension two.

3.2.1 One-homogeneous rescaling and excess

Let $d \geq 2$ and $u \in H_{loc}^1(\mathbb{R}^d)$. For $r > 0$ and $x_0 \in \mathbb{R}^d$, we define the one-homogeneous rescaling of u as

$$u_{x_0,r}(x) := \frac{u(x_0 + rx)}{r} \quad \text{for every } x \in \mathbb{R}^d. \quad (3.2.1)$$

Then, $u_{x_0,r} \in H_{loc}^1(\mathbb{R}^d)$ and for almost every $r > 0$, $E(u_{x_0,r})$ is well defined, where we set

$$E(v) := \int_{\partial B_1} |x \cdot \nabla v - v|^2 d\mathcal{H}^{d-1}, \quad (3.2.2)$$

where $x \in \partial B_1$ is the exterior normal derivative to ∂B_1 at the point $x \in \mathbb{R}^d$ and \mathcal{H}^{d-1} stands for the $(d-1)$ -dimensional Hausdorff measure. The excess function $e(r) = E(u_{x_0,r})$ controls the asymptotic behavior, as $r \rightarrow 0^+$, of the one parameter family $u_{x_0,r} \in L^2(\partial B_1)$. Precisely, we have the following elementary estimate.

Lemma 3.2.1. *Let $u \in H_{loc}^1(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$. Suppose that there are constants $r_0 > 0$, $\gamma \in (0, 1)$ and $I > 0$ such that*

$$\int_0^{r_0} \frac{E(u_{x_0,r})}{r^{1+\gamma}} dr \leq I. \quad (3.2.3)$$

Then, there is a unique function $u_{x_0} \in L^2(\partial B_1)$ such that

$$\|u_{r,x_0} - u_{x_0}\|_{L^2(\partial B_1)}^2 \leq \gamma^{-1} I r^\gamma \quad \text{for every } r \in (0, r_0).$$

Proof. We set for simplicity, $x_0 = 0$ and $u_r := u_{x_0,r}$. Let $0 < r < R \leq r_0$. Notice that, for any $x \in \partial B_1$, we have

$$\frac{u(Rx)}{R} - \frac{u(rx)}{r} = \int_r^R \left(\frac{x \cdot (\nabla u)(sx)}{s} - \frac{u(sx)}{s^2} \right) ds = \frac{1}{s} \int_r^R (x \cdot \nabla u_s(x) - u_s(x)) ds.$$

Thus, by the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} \int_{\partial B_1} |u_R - u_r|^2 d\mathcal{H}^{d-1} &\leq \int_{\partial B_1} \left(\int_r^R \frac{1}{s} |x \cdot \nabla u_s - u_s| ds \right)^2 d\mathcal{H}^{d-1} \\ &\leq \int_{\partial B_1} \left(\int_r^R s^{\gamma-1} ds \right) \left(\int_r^R \frac{1}{s^{1+\gamma}} |x \cdot \nabla u_s - u_s|^2 ds \right) d\mathcal{H}^{d-1} \\ &\leq \frac{R^\gamma - r^\gamma}{\gamma} \int_r^R \frac{E(u_s)}{s^{1+\gamma}} ds \leq \frac{R^\gamma}{\gamma} \int_0^{r_0} \frac{E(u_s)}{s^{1+\gamma}} ds, \end{aligned}$$

which implies the claim by a standard argument. \square

3.2.2 The one-phase boundary adjusted energy

Let $d \geq 2$ and $u \in H^1(B_1)$. For any $\Lambda > 0$, we define the one-phase Weiss' boundary adjusted energy as

$$W_{\text{op}}(u) := \int_{B_1} |\nabla u|^2 dx - \int_{\partial B_1} u^2 d\mathcal{H}^{d-1} + \Lambda |\{u > 0\} \cap B_1|. \quad (3.2.4)$$

Let $r > 0$, $x_0 \in \mathbb{R}^2$ and $u \in H_{loc}^1(\mathbb{R}^d)$. The relation between W_{OP} and the excess E is given by the following formula, which holds for any function u and can be obtained by a direct computation (see [92] and [77]).

$$\frac{\partial}{\partial r} W_{\text{OP}}(u_{x_0,r}) = \frac{d}{r} (W_{\text{OP}}(z_{x_0,r}) - W_{\text{OP}}(u_{x_0,r})) + \frac{1}{r} E(u_{x_0,r}), \quad (3.2.5)$$

where $z_{x_0,r}$ denotes the one-homogeneous extension of the trace of $u_{x_0,r}$ in B_1 , that is,

$$z_{x_0,r}(x) := |x| u_{x_0,r}(x/|x|) = \frac{|x|}{r} u(rx/|x|), \quad \text{for every } x \in B_1. \quad (3.2.6)$$

In [86], the first and the third author proved the following *epiperimetric inequality* for the Weiss' energy W_{OP} .

Theorem 3.2.2 (Epiperimetric inequality for W_{OP}). *Let $d = 2$. Let $C_0 > 0$ be a given constant. There exists a constant $\varepsilon > 0$ such that: for every non-negative $c \in H^1(\partial B_1)$ satisfying $\int_{\partial B_1} c d\mathcal{H}^1 \geq C_0$, there exists a non-negative function $h \in H^1(B_1)$ such that $h = c$ on ∂B_1 and*

$$W_{\text{OP}}(h) - \Lambda \frac{\pi}{2} \leq (1 - \varepsilon) \left(W_{\text{OP}}(z) - \Lambda \frac{\pi}{2} \right), \quad (3.2.7)$$

where W_{OP} is given by (3.2.4) and $z \in H^1(B_1)$ denotes the one-homogeneous extension of c into B_1 . Moreover, the competitor h has the following properties:

- (a) There is a universal numerical constant $C > 0$ such that $\|h\|_{H^1(B_1)} \leq C \|c\|_{H^1(\partial B_1)}$.
- (b) If $H_{x_0,\nu} := \{x \in \mathbb{R}^2 : (x - x_0) \cdot \nu \geq 0\}$, for some $x_0 \in \mathbb{R}^2$ and $\nu \in \partial B_1$, is a half-plane such that

$$0 \in H_{x_0,\nu} \quad \text{and} \quad z = 0 \quad \text{on} \quad \mathbb{R}^2 \setminus H_{x_0,\nu}, \quad (3.2.8)$$

then we can choose the competitor $h : B_1 \rightarrow \mathbb{R}$ such that $h = 0$ on $\mathbb{R}^2 \setminus H_{x_0,\nu}$.

3.2.3 The two-phase boundary adjusted energy

For every $\Lambda_1, \Lambda_2 > 0$ and $v \in H^1(B_1)$, we define the two-phase Weiss' boundary adjusted energy as

$$W_{\text{TP}}(v) = \int_{B_1} |\nabla v|^2 dx - \int_{\partial B_1} v^2 d\mathcal{H}^{d-1} + \Lambda_1 |\{v > 0\} \cap B_1| + \Lambda_2 |\{v < 0\} \cap B_1|. \quad (3.2.9)$$

As in the one phase case, we have

$$\frac{\partial}{\partial r} W_{\text{TP}}(u_{x_0,r}) = \frac{d}{r} (W_{\text{TP}}(z_{x_0,r}) - W_{\text{TP}}(u_{x_0,r})) + \frac{1}{r} E(u_{x_0,r}), \quad (3.2.10)$$

where $z_{x_0,r}$ is given by (3.2.6).

Theorem 3.2.3 (Epiperimetric inequality for W_{TP}). *Let $d = 2$. For every $C_0 > 0$ there is $\varepsilon > 0$ such that: for every $c \in H^1(\partial B_1)$ such that $\int_{\partial B_1} c^+ d\mathcal{H}^1 \geq C_0$ and $\int_{\partial B_1} c^- d\mathcal{H}^1 \geq C_0$, there exists a function $h \in H^1(B_1)$ with $h = c$ on ∂B_1 such that*

$$W_{\text{TP}}(h) - (\Lambda_1 + \Lambda_2) \frac{\pi}{2} \leq (1 - \varepsilon) \left(W_{\text{TP}}(z) - (\Lambda_1 + \Lambda_2) \frac{\pi}{2} \right), \quad (3.2.11)$$

where $z \in H^1(B_1)$ is the one-homogeneous extension of the trace of c to B_1 . Moreover, there is a universal numerical constant $C > 0$ such that $\|h\|_{H^1(B_1)} \leq C \|c\|_{H^1(\partial B_1)}$.

3.2.4 Almost-monotonicity and almost-minimality

Let $u \in H_{loc}^1(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$. For any $r > 0$, the function $u_{x_0,r}$ and $z_{x_0,r}$ are defined as in (3.2.1) and (3.2.6), respectively. In the next lemma we will show that a almost-minimality of u , with respect to radial perturbations, implies that the function $r \mapsto W_{\square}(u_{x,r})$ is monotone up to a small error term (\square stands for OP or TP).

Lemma 3.2.4 (Monotonicity of W_{\square}). *Let $u \in H_{loc}^1(\mathbb{R}^d)$ and $x_0 \in \mathbb{R}^d$. Suppose that there are constants $r_0 > 0$, $C > 0$ and $\delta > 0$ such that*

$$W_{\square}(u_{x_0,r}) \leq W_{\square}(z_{x_0,r}) + Cr^{\delta} \quad \text{for every } r \in (0, r_0), \quad (3.2.12)$$

where \square stands for OP or TP. Then, the function

$$r \mapsto W_{\square}(u_{x_0,r}) + \frac{Cd}{\delta}r^{\delta}, \quad (3.2.13)$$

is non-decreasing on the interval $(0, r_0)$.

Proof. Using (3.2.5) for $\square = \text{OP}$ (resp. (3.2.10) for $\square = \text{TP}$), and the condition (3.2.12) we get

$$\frac{\partial}{\partial r} W_{\square}(u_{x_0,r}) \geq \frac{d}{r} (W_{\square}(z_{x_0,r}) - W_{\square}(u_{x_0,r})) \geq Cdr^{\delta-1},$$

which gives (3.2.13). \square

3.2.5 Epiperimetric inequality and energy decay

In this section we show how to use the epiperimetric inequality to obtain at once the decay for the energy $W_{\square}(u_{x_0,r})$ and the convergence of $u_{x_0,r}$ in $L^2(\partial B_1)$. The argument is very general and we treat the cases $\square = \text{OP}$ and $\square = \text{TP}$ simultaneously.

Lemma 3.2.5. *Let $u \in H_{loc}^1(\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$ and W_{\square} be as in (3.2.4), if $\square = \text{OP}$, and (3.2.9), if $\square = \text{TP}$. Suppose that there are constants $r_0 \in (0, 1)$, $C > 0$, $\delta > 0$ and $\varepsilon \in (0, \frac{\delta}{2d+\delta})$ such that:*

- (a) (3.2.12) holds and the limit $\Theta_{\square} := \lim_{r \rightarrow 0} W_{\square}(u_{x_0,r})$ (which exists due to Lemma 3.2.4) is finite;
- (b) for every $r \in (0, r_0)$ there is a function $h_{x_0,r} \in H^1(B_1)$ such that

$$W_{\square}(u_{x_0,r}) \leq W_{\square}(h_{x_0,r}) + Cr^{\delta}, \quad (3.2.14)$$

and we have the epiperimetric inequality

$$W_{\square}(h_{x_0,r}) - \Theta_{\square} \leq (1 - \varepsilon)(W_{\square}(z_{x_0,r}) - \Theta_{\square}). \quad (3.2.15)$$

Then, there is a unique function $u_{x_0} \in L^2(\partial B_1)$ such that

$$\|u_{r,x_0} - u_{x_0}\|_{L^2(\partial B_1)}^2 \leq \gamma^{-1} I r^{\gamma} \quad \text{for every } r \in (0, r_0),$$

where $\gamma = \frac{d\varepsilon}{1-\varepsilon}$ and $I = r_0^{-\gamma} (W_{\square}(u_{x_0,r_0}) - \Theta_{\square}) + \frac{dC}{\delta - \gamma} r_0^{\delta - \gamma}$.

Proof. We use (3.2.5) for $\square = \text{OP}$ (resp. (3.2.10) for $\square = \text{TP}$), then the epiperimetric inequality (3.2.15) and the almost-minimality condition (3.2.14).

$$\begin{aligned} \frac{\partial}{\partial r} (W_{\square}(u_{x_0,r}) - \Theta_{\square}) &\geq \frac{d}{r} \left((W_{\square}(z_{x_0,r}) - \Theta_{\square}) - (W_{\square}(u_{x_0,r}) - \Theta_{\square}) \right) \\ &\geq \frac{d}{r} \left(\frac{1}{1-\varepsilon} (W_{\square}(h_{x_0,r}) - \Theta_{\square}) - (W_{\square}(u_{x_0,r}) - \Theta_{\square}) \right) \\ &\geq \frac{d}{r} \left(\frac{\varepsilon}{1-\varepsilon} (W_{\square}(u_{x_0,r}) - \Theta_{\square}) - Cr^{\delta} \right), \end{aligned}$$

which implies that the function

$$f(r) = \frac{W_{\square}(u_{x_0,r}) - \Theta_{\square}}{r^{\gamma}} + \frac{dC}{\delta - \gamma} r^{\delta - \gamma}$$

is non-decreasing on $(0, r_0)$ for $\gamma = \frac{d\varepsilon}{1-\varepsilon}$, where we notice that $\gamma \leq \frac{\delta}{2}$ due to the choice $\varepsilon \leq \frac{\delta}{2d+\delta}$. In particular, using again (3.2.5) (resp. (3.2.10)), we get

$$f'(r) \geq \frac{1}{r^{\gamma+1}} E(u_{x_0,r}),$$

which integrated gives

$$f(r_0) - f(s) \geq \int_s^{r_0} \frac{1}{r^{\gamma+1}} E(u_{x_0,r}) dr,$$

for every $s \in (0, r_0)$. Now, notice that, up to choosing a bigger constant C in (3.2.14), Lemma 3.2.4 implies that $f(s) \geq 0$ for every $s > 0$. Thus, we get

$$f(r_0) \geq \int_0^{r_0} \frac{1}{r^{\gamma+1}} E(u_{x_0,r}) dr,$$

which is precisely (3.2.3) with $I := f(r_0)$. \square

3.3 Change of variables and freezing of the coefficients

The arguments of the previous section, the monotonicity formula and the decay of the blow-up sequences, can be applied only in the case when the operator in J_{op} (resp. J_{TP}) is the identity. Thus, in order to prove the regularity results Theorem 3.1.1 and Theorem 3.1.5 we need to change the coordinates and reduce to the case $A = Id$. We prove the main estimate of this section in Lemma 3.3.2 below, but before we will introduce several notations.

Let $A = (a_{ij})_{ij} : B_2 \rightarrow \text{Sym}_2^+$ and $Q_{\text{op}}, Q_{\text{TP}}^+, Q_{\text{TP}}^- : B_2 \rightarrow \mathbb{R}^+$ be as in the Introduction and note that we have

$$\|A_x^{1/2}\| \leq M_A^{1/2} \quad \text{and} \quad \|A_x^{-1/2}\| \leq M_A^{1/2} \quad \text{for every } x \in B_2,$$

where $\|A\| = \sup \{|Au| : u \in \mathbb{R}^2, |u| = 1\}$ and M_A is a constant (as in Subsection 3.1.1).

Remark 3.3.1. We recall that if the (real) matrix A is symmetric and positive ($A \in \text{Sym}_d^+$), then there is an orthogonal matrix P such that $PAP^t = \text{diag}(\lambda_1, \dots, \lambda_d)$, where P^t is the transpose of P and $\text{diag}(\lambda_1, \dots, \lambda_d)$ is the diagonal matrix with eigenvalues $\lambda_1, \dots, \lambda_d$. We set $D = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d})$ and define $A^{1/2} := P^t D P$.

We now fix $x_0 \in B_2$ and, for any $r > 0$, we define the functionals

$$J_{\text{op}}^{x_0}(v, r) := \int_{B_r} \left(|\nabla v|^2 + Q_{\text{op}}(x_0) \mathbb{1}_{\{v>0\}} \right) dx;$$

$$J_{\text{TP}}^{x_0}(v, r) := \int_{B_r} \left(|\nabla v|^2 + Q_{\text{TP}}^+(x_0) \mathbb{1}_{\{v>0\}} + Q_{\text{TP}}^-(x_0) \mathbb{1}_{\{v<0\}} \right) dx.$$

For every $x_0 \in B_1$, we define the function

$$F_{x_0}(x) := x_0 + A_{x_0}^{1/2}(x) \tag{3.3.1}$$

and the half-plane $\mathcal{H}_{x_0} := \{x \in \mathbb{R}^2 : F_{x_0}(x) \cdot e_2 > 0\}$, where $e_2 = (0, 1)$.

Lemma 3.3.2. *Let $L > 0$. There are constants $C > 0$ and $r_0 \in (0, 1)$ (depending only on the constants $C_A, C_Q, M_A, M_Q, \delta_A, \delta_Q, \delta_1, C_1$ and L defined in Subsection 3.1.1) and $\delta = \min\{\delta_A, \delta_Q, \delta_1\}$ such that: if $u \in H^1(B_1)$ is a nonnegative L -Lipschitz continuous function and a almost-minimizer of J_{op} in B_2^+ , $x_0 \in B_{r_0} \cap \partial\Omega_u$ and $\bar{u} = u \circ F_{x_0}$ (F_{x_0} is defined in (3.3.1) above), then for every $r \in (0, r_0)$,*

$$J_{\text{op}}^{x_0}(\bar{u}, r) \leq (1 + Cr^\delta)J_{\text{op}}^{x_0}(\bar{v}, r) + Cr^{2+\delta}, \quad (3.3.2)$$

for every $\bar{v} \in H^1(B_r)$ such that $\bar{u} - \bar{v} \in H_0^1(B_r)$ and $\bar{v} = 0$ on $\mathbb{R}^2 \setminus \mathcal{H}_{x_0}$.

Moreover, there is a numerical constant $C_0 > 0$, such that

$$W_{\text{op}}(\bar{u}_r) \leq \begin{cases} W_{\text{op}}(\bar{z}_r) + C_0(M_A L^2 + M_Q)Cr^\delta, \\ W_{\text{op}}(\bar{h}_r) + C_0(M_A L^2 + M_Q)Cr^\delta, \end{cases}$$

for every $r \in (0, r_0)$, where C is the constant from (3.3.2), $\bar{u}_r(x) := \frac{1}{r}\bar{u}(rx)$, \bar{z}_r is the 1-homogeneous extension of \bar{u}_r in B_1 , \bar{h}_r is the competitor from Theorem 3.2.2 and $\Lambda = Q_{\text{op}}(x_0)$, as in (3.2.4).

Proof. Let $x_0 \in \partial\Omega_u \cap B_1$ be fixed. For $r > 0$, we set $\rho = M_A^{1/2}r$. Notice that we have the inclusion $F_{x_0}(B_r) \subset B_\rho(x_0)$. Let $\bar{u} = u \circ F_{x_0}$ and $\bar{v} = v \circ F_{x_0}$. Then, using the Hölder continuity of A and $Q := Q_{\text{op}}$, and the ellipticity of A , we estimate

$$\begin{aligned} \tilde{J}_{\text{op}}(u, x_0, \rho) &:= \int_{B_\rho(x_0)} \left(a_{ij}(x_0) \partial_i u \partial_j u + Q(x_0) \mathbb{1}_{\{u>0\}} \right) dx \leq J_{\text{op}}(u, x_0, \rho) \\ &\quad + C_A M_A \rho^{\delta_A} \int_{B_\rho(x_0)} a_{ij}(x) \partial_i u \partial_j u dx + C_Q M_Q \rho^{\delta_Q} \int_{B_\rho(x_0)} Q(x) \mathbb{1}_{\{u>0\}} dx \\ &\leq (1 + Cr^\delta) J_{\text{op}}(u, x_0, \rho), \end{aligned}$$

for some positive constant $C > 0$. Analogously, we get the following estimate from below:

$$\tilde{J}_{\text{op}}(v, x_0, \rho) \geq (1 - Cr^\delta) J_{\text{op}}(v, x_0, \rho). \quad (3.3.3)$$

Putting the two estimates together and using the almost-minimality of u , we get

$$\tilde{J}_{\text{op}}(u, x_0, \rho) \leq \frac{1 + Cr^\delta}{1 - Cr^\delta} (1 + C_1 \rho^{\delta_1}) \tilde{J}_{\text{op}}(v, x_0, \rho) + C_1 (1 + Cr^\delta) \rho^{2+\delta_1}.$$

Now, notice that by the choice of the function F_{x_0} we have the identity

$$|\nabla \bar{u}|^2(x) = a_{ij}(x_0) \partial_i u(F_{x_0}(x)) \partial_j u(F_{x_0}(x)) \quad \text{for every } x \in B_{M_A^{-1/2}}.$$

Therefore, a change of coordinates and the estimate (3.3.3) give

$$\begin{aligned} \int_{F_{x_0}^{-1}(B_\rho(x_0))} \left(|\nabla \bar{u}|^2 + Q(x_0) \mathbb{1}_{\{\bar{u}>0\}} \right) dx &= \det(A_{x_0}^{-1/2}) \tilde{J}_{\text{op}}(u, x_0, \rho) \\ &\leq (1 + Cr^\delta) \int_{F_{x_0}^{-1}(B_\rho(x_0))} \left(|\nabla \bar{v}|^2 + Q(x_0) \mathbb{1}_{\{\bar{v}>0\}} \right) dx + Cr^{2+\delta}, \end{aligned}$$

for some other positive constant $C > 0$. Finally, since $B_r \subset F_{x_0}^{-1}(B_\rho(x_0))$ and $\bar{u} = \bar{v}$ outside B_r , we can rearrange the terms of the above estimate to obtain

$$J_{\text{op}}^{x_0}(\bar{u}, r) \leq (1 + Cr^\delta) J_{\text{op}}^{x_0}(\bar{v}, r) + Cr^{2+\delta} + Cr^\delta J_{\text{op}}^{x_0}(\bar{u}, M_A^{-1/2}r),$$

which gives (3.3.2) since \bar{u} is Lipschitz continuous with Lipschitz constant $\|\nabla \bar{u}\|_{L^\infty} = M_A^{1/2}L$.

We next notice that we have the scaling

$$J_{\text{op}}^{x_0}(\bar{u}_r, 1) = \frac{1}{r^2} J_{\text{op}}^{x_0}(\bar{u}, r).$$

Thus, the almost-minimality inequality (3.3.2) translates in

$$J_{\text{op}}^{x_0}(\bar{u}_r, 1) \leq (1 + Cr^\delta) J_{\text{op}}^{x_0}(\bar{v}_r, 1) + Cr^\delta. \quad (3.3.4)$$

Let $C_E > 0$ be the constant from Theorem 3.2.2. Then, since \bar{u} is Lipschitz continuous, we have

$$\int_{B_1} |\nabla \bar{h}_r|^2 dx \leq C_E \int_{\partial B_1} (|\nabla \bar{u}_r|^2 + \bar{u}_r^2) dx \leq C_0 M_A L^2,$$

where C_0 is a numerical constant and \bar{h}_r is the competitor from Theorem 3.2.2. Taking \bar{h}_r as a competitor in (3.3.4), we obtain

$$\begin{aligned} J_{\text{op}}^{x_0}(\bar{u}_r, 1) &\leq J_{\text{op}}^{x_0}(\bar{h}_r, 1) + Cr^\delta \left(\int_{B_1} |\nabla \bar{h}_r|^2 dx + Q(x_0) |B_1| \right) + Cr^\delta \\ &\leq J_{\text{op}}^{x_0}(\bar{h}_r, 1) + Cr^\delta \left(C_0 M_A L^2 + M_Q |B_1| \right) + Cr^\delta, \end{aligned}$$

which concludes the proof, the case $\bar{v}_r = \bar{z}_r$ being analogous. \square

An analogous result, with essentially the same proof holds in the two-phase case.

Lemma 3.3.3. *Let $L > 0$. There are constants $C > 0$ and $r_0 \in (0, 1)$ (depending only on $C_A, C_Q, M_A, M_Q, \delta_A, \delta_Q, \delta_1, C_2$ and L) and $\delta = \min\{\delta_A, \delta_Q, \delta_2\}$ such that: if $u \in H^1(B_1)$ is a L -Lipschitz continuous function and a almost-minimizer of J_{TP} in B_2 , $x_0 \in B_{r_0} \cap \partial\Omega_u$ and $\bar{u} = u \circ F_{x_0}$, then we have that for every $r \in (0, r_0)$,*

$$\begin{aligned} J_{\text{TP}}^{x_0}(\bar{u}, r) &\leq (1 + Cr^\delta) J_{\text{TP}}^{x_0}(\bar{v}, r) + Cr^{2+\delta}, \\ &\text{for every } \bar{v} \in H^1(B_r) \text{ such that } \bar{u} - \bar{v} \in H_0^1(B_r). \end{aligned} \quad (3.3.5)$$

Moreover, there is a numerical constant $C_0 > 0$ such that

$$W_{\text{TP}}(\bar{u}_r) \leq \begin{cases} W_{\text{TP}}(\bar{z}_r) + C_0(M_A L^2 + M_Q) Cr^\delta, \\ W_{\text{TP}}(\bar{h}_r) + C_0(M_A L^2 + M_Q) Cr^\delta, \end{cases}$$

for every $r \in (0, r_0)$, where C is the constant from (3.3.5), $\bar{u}_r(x) := \frac{1}{r} \bar{u}(rx)$, \bar{z}_r is the one homogeneous extension of \bar{u}_r in B_1 , \bar{h}_r is the competitor given by Theorem 3.2.3 and $\Lambda_1 = Q_{\text{TP}}^+(x_0)$, $\Lambda_2 = Q_{\text{TP}}^-(x_0)$ are as in (3.2.9).

Remark 3.3.4 (On the non-degeneracy). In [34] David and Toro proved that Lipschitz continuous almost-minimizers to the one-phase and the two-phase functionals for the Laplacian are non-degenerate (see [34, Theorem 10.1]). Note that their definition of almost-minimizer is slightly different from ours. However, their proof still holds in our case with small changes which come from the additional term $Cr^{2+\delta}$ of our definition. It follows from Lemma 3.3.2 and Lemma 3.3.3 that if u is a almost-minimizer of the functional J_{op} (resp. u is a almost-minimizer of J_{TP}) then u (resp. u_\pm) is non-degenerate with respect to A in the sense of the following definition.

Definition 3.3.5 (Non-degeneracy). Let $d \geq 2$ and $A : \mathbb{R}^d \rightarrow \text{Sym}_d^+$ be a given function. We say that the non-negative function $u \in H^1(B_2)$ is non-degenerate (with respect to A), if there are constants $\eta > 0$, $\varepsilon \in (0, 1)$ and $r_0 > 0$ such that, for every $x_0 \in B_1$ and $r \in (0, r_0)$, the following implication holds:

$$\int_{\partial B_r} u \circ F_{x_0} d\mathcal{H}^{d-1} < \eta r^d \quad \Rightarrow \quad u \circ F_{x_0} \equiv 0 \quad \text{in } B_{\varepsilon r}(x_0),$$

where $F_{x_0}(x) := x_0 + A_{x_0}^{1/2}(x)$.

3.4 Blow-up sequences and blow-up limits

Let $u \in H^1(B_2)$ be a Lipschitz continuous function. Let $(x_n)_{n \geq 1}$ be a sequence of points in $B_1 \cap \partial\Omega_u$ converging to some $x_0 \in B_1 \cap \partial\Omega_u$, and $(r_n)_{n \geq 1}$ be an infinitesimal sequence in $(0, 1)$. Then, the sequence u_{x_n, r_n} is uniformly Lipschitz in every compact subset of \mathbb{R}^2 . Thus, up to extracting a subsequence, there is a Lipschitz continuous function $u_0 : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$\lim_{n \rightarrow \infty} u_{x_n, r_n} = u_0, \quad (3.4.1)$$

where u_{r_n, x_n} is defined in (3.2.1) and the convergence is uniform on every compact subset of \mathbb{R}^2 .

Definition 3.4.1. If (3.4.1) holds, we will say that u_{x_n, r_n} is a blow up sequence (with fixed center, if $x_n = x_0$, for every $n \geq 1$). If the center is fixed, we will say that u_0 is a blow-up limit at x_0 .

We summarize the main properties of the blow-up sequences and the blow-up limits in the following two propositions. We notice that Proposition 3.4.2 below holds in every dimension $d \geq 2$, while Proposition 3.4.3 is known to hold only for $2 \leq d \leq 4$.

Proposition 3.4.2 (Convergence of the blow-up sequences). *Let $u \in H^1(B_2)$ be as in Theorem 3.1.1 or Theorem 3.1.5 and let $u_n := u_{r_n, x_n}$ be a blow-up sequence converging to some $u_0 \in H_{loc}^1(\mathbb{R}^2)$. Then:*

- i) *the sequence u_n converges to u_0 strongly in $H_{loc}^1(\mathbb{R}^2)$;*
- ii) *the sequences of characteristic functions $\mathbb{1}_{\{u_n > 0\}}$ and $\mathbb{1}_{\{u_n < 0\}}$ converge strongly in $L_{loc}^1(\mathbb{R}^2)$ to the characteristic functions $\mathbb{1}_{\{u_0 > 0\}}$ and $\mathbb{1}_{\{u_0 < 0\}}$, respectively.*

Proposition 3.4.3 (Classification of the blow-up limits). *Let $u \in H^1(B_2)$ be as in Theorem 3.1.1 or Theorem 3.1.5. Let $x_0 \in \partial\Omega_u \cap B_1$ and $u_0 \in H_{loc}^1(\mathbb{R}^2)$ be a blow-up limit of u at x_0 .*

(OP) *If u is as in Theorem 3.1.1 and $x_0 \in \partial\Omega_u \cap B_1^+$, then u_0 is of the form*

$$u_0(x) = Q_{op}^{1/2}(x_0) \max \{0, x \cdot A_{x_0}^{-1/2}[\nu]\}, \quad \text{where } \nu \in \partial B_1. \quad (3.4.2)$$

(OP-c) *If u is as in Theorem 3.1.1 and $x_0 \in \partial\Omega_u \cap \partial H \cap B_1$, then u_0 is of the form*

$$u_0(x) = \mu \max \{0, x \cdot A_{x_0}^{-1/2}[\nu]\}, \quad (3.4.3)$$

where $\mu \geq Q_{op}^{1/2}(x_0)$ and $\nu \in \partial B_1$ is such that $A_{x_0}^{-1/2}[\nu]$ is the (interior) normal to ∂H .

(TP) *If u is as in Theorem 3.1.5 and $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1$, then u_0 is of the form*

$$u_0(x) = \mu_+ \max \{0, x \cdot A_{x_0}^{-1/2}[\nu]\} + \mu_- \min \{0, x \cdot A_{x_0}^{-1/2}[\nu]\}, \quad (3.4.4)$$

for some $\nu \in \partial B_1$ and some $\mu_+, \mu_- > 0$ such that

$$\mu_+^2 \geq Q_{tp}^+(x_0) \quad \mu_-^2 \geq Q_{tp}^-(x_0) \quad \text{and} \quad \mu_+^2 - \mu_-^2 = Q_{tp}^+(x_0) - Q_{tp}^-(x_0).$$

The proof of Proposition 3.4.2 follows by a standard variational argument that only uses the almost-minimality of u ; for more details, we refer to [2] (see also [77]). Proposition 3.4.3 follows by the optimality of the blow-up limits and the Weiss' monotonicity formula (Lemma 3.2.4). We will need the following definition.

Definition 3.4.4 (Global solutions). Let $u : \mathbb{R}^2 \rightarrow \mathbb{R}$, $u \in H_{loc}^1(\mathbb{R}^2)$ be given.

(OP) We say that u is a global solution of the **one-phase** Bernoulli problem, if: $u \geq 0$ and, for every ball $B := B_R(x_0) \subset \mathbb{R}^2$, we have

$$\int_B |\nabla u|^2 dx + \Lambda |\{u > 0\} \cap B| \leq \int_B |\nabla v|^2 dx + \Lambda |\{v > 0\} \cap B|, \quad (3.4.5)$$

for every $v \in H^1(B)$ such that $u - v \in H_0^1(B)$.

(OP-c) We say that u is a global solution of the **one-phase constrained** Bernoulli problem in the half-plane H , if $u \geq 0$ on H , $u = 0$ on $\mathbb{R}^2 \setminus H$ and (3.4.5) holds, for every ball $B := B_R(x_0) \subset \mathbb{R}^2$ and every $v \in H^1(B)$ such that $u - v \in H_0^1(B)$ and $\{v > 0\} \subset H$.

(TP) We say that u is a global solution of the **two-phase** Bernoulli problem if, for every ball $B := B_R(x_0) \subset \mathbb{R}^2$, we have

$$\int_B \left(|\nabla u|^2 + \Lambda_1 \mathbb{1}_{\{u > 0\}} + \Lambda_2 \mathbb{1}_{\{u < 0\}} \right) dx \leq \int_B \left(|\nabla v|^2 + \Lambda_1 \mathbb{1}_{\{v > 0\}} + \Lambda_2 \mathbb{1}_{\{v < 0\}} \right) dx, \quad (3.4.6)$$

for every $v \in H^1(B)$ such that $u - v \in H_0^1(B)$.

Lemma 3.4.5 (Optimality of the blow-up limits). *Let $u \in H^1(B_2)$ be as in Theorem 3.1.1 or Theorem 3.1.5 and let $u_n := u_{r_n, x_0}$ be a blow-up sequence converging to the blow-up limit $u_0 \in H_{loc}^1(\mathbb{R}^2)$. Then, we have:*

(OP) *If u is as in Theorem 3.1.1 and $x_0 \in \partial\Omega_u \cap B_1^+$, then $u_0 \circ A_{x_0}^{1/2}$ is a global solution of the one-phase problem with $\Lambda = Q_{op}(x_0)$.*

(OP-c) *If u is as in Theorem 3.1.1 and $x_0 \in \partial\Omega_u \cap \partial H \cap B_1$, then, up to a rotation, $u_0 \circ A_{x_0}^{1/2}$ is a global solution of the constrained one-phase problem with $\Lambda = Q_{op}(x_0)$.*

(TP) *If u is as in Theorem 3.1.5 and $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1$, then $u_0 \circ A_{x_0}^{1/2}$ is a global solution of the two-phase problem with $\Lambda_1 = Q_{tp}^+(x_0)$ and $\Lambda_2 = Q_{tp}^-(x_0)$.*

Recall that the function $\bar{u} = u \circ F_{x_0}$, where F_{x_0} is as in (3.3.1), is an almost-minimizer of the functional $J_{op}^{x_0}$ (Lemma 3.3.2). We then refer to Lemma 4.6 in [77] applied to \bar{u} for the proof of Lemma 3.4.5. It is also worth mentioning that the strong convergence of the blow-up sequences and the optimality of the blow-up limits are equivalent.

Lemma 3.4.6 (Homogeneity of the blow-up limits). *Let $u \in H^1(B_2)$ be as in Theorem 3.1.1 or Theorem 3.1.5. Let $x_0 \in B_1 \cap \partial\Omega_u$ and let u_{x_0, r_n} be a blow-up sequence converging to a blow-up limit u_0 . Then, u_0 is one-homogeneous.*

Proof. Assume that $x_0 = 0$ and set $\bar{u} = u \circ F_{x_0}$. Then

$$u_{x_0, r} = \bar{u}_r \circ A_{x_0}^{-1/2}, \quad \text{where} \quad \bar{u}_r(x) := \frac{\bar{u}(rx)}{r}.$$

We first notice that by Lemma 3.3.2, Lemma 3.2.4 and the Lipschitz continuity of u , we get that the limit $\Theta_{\square} := \lim_{r \rightarrow 0} W_{\square}(\bar{u}_r)$, $\square = \text{OP, TP}$, exists and is finite. Now the strong convergence of \bar{u}_{r_n} to $\bar{u}_0 := u_0 \circ A_{x_0}^{1/2}$ (Proposition 3.4.2) implies that, for every $s > 0$, we have

$$\Theta_{\square} := \lim_{r \rightarrow 0} W_{\square}(\bar{u}_r) = \lim_{n \rightarrow \infty} W_{\square}(\bar{u}_{r_n}) = \lim_{n \rightarrow \infty} W_{\square}(\bar{u}_{r_n s}) = \lim_{n \rightarrow \infty} W_{\square}((\bar{u}_{r_n})_s) = W_{\square}((\bar{u}_0)_s).$$

In particular, $s \mapsto W_{\square}(\bar{u}_0, s)$ is constant. Now, since \bar{u}_0 is a global solution (Lemma 3.4.5), (3.2.5) and (3.2.10) imply that $E((\bar{u}_0)_s) = 0$, for every $s > 0$. Thus we have $x \cdot \nabla \bar{u}_0 = \bar{u}_0$ in \mathbb{R}^2 , which implies that \bar{u}_0 (and thus, u_0) is one-homogeneous. \square

Proof of Proposition 3.4.3. We now notice that $\bar{u}_0 = u_0 \circ A_{x_0}^{1/2} : B_1 \rightarrow \mathbb{R}$ is one-homogeneous and harmonic on the cone $B_1 \cap \{\bar{u}_0 \neq 0\}$. Thus, the trace of \bar{u}_0 on the sphere satisfies the equation

$$-\Delta_{\mathbb{S}} \bar{u}_0 = (d-1)\bar{u}_0 \quad \text{on} \quad \mathbb{S}^{d-1} \cap \{\bar{u}_0 \neq 0\},$$

where in dimension two the spherical Laplacian $\Delta_{\mathbb{S}}$ is simply the second derivative and $d-1 = 1$. Thus, \bar{u}_0 is of the form $\bar{u}_0(\theta) = \sin(\theta + \theta_0)$, $\theta \in \mathbb{S}^1$, for some constant θ_0 . This implies that $\{\bar{u}_0 \neq 0\}$ is a union of intervals of length π . In the one-phase case, since u is non-degenerate (see Remark 3.3.4), this implies that \bar{u}_0 is of the form (3.4.2), for some constant $\mu(x_0)$. Now,

an internal variation argument (see [2]) implies that $\mu(x_0) = Q_{\text{op}}^{1/2}(x_0)$, if $x_0 \in H \cap B_1^+$, and $\mu(x_0) \geq Q_{\text{op}}^{1/2}(x_0)$, if $x_0 \in \partial H \cap B_1$. The two-phase case follows again by an internal variation argument (see [3]). \square

Finally, we prove a uniqueness result for the one- and two-phase (Theorem 3.1.1 and Theorem 3.1.5) blow-up limits. This is the only result of this section that cannot be immediately extended to higher dimension. This is due to the fact that the epiperimetric inequality (Theorem 3.2.2 and Theorem 3.2.3) is known (for the moment) only in dimension two.

Proposition 3.4.7 (Uniqueness of the blow-up and rate of convergence of the blow-up sequences). *Let $u : B_2 \rightarrow \mathbb{R}$ be as in Theorem 3.1.1 or Theorem 3.1.5. There are constants $C > 0$, $\gamma > 0$ and $r_0 > 0$ such that the following claims do hold.*

(OP) *If u is as in Theorem 3.1.1, then for every $x_0 \in \partial\Omega_u \cap B_1$, there is a unique blow-up $u_{x_0} : \mathbb{R}^2 \rightarrow \mathbb{R}$ (of the form (3.4.2) or (3.4.3)) such that*

$$\|u_{x_0,r} - u_{x_0}\|_{L^\infty(B_1)} \leq Cr^\gamma \quad \text{for every } r \in (0, r_0). \quad (3.4.7)$$

(TP) *If u is as in Theorem 3.1.5, then for every $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1$, there is a unique blow-up $u_{x_0} : \mathbb{R}^2 \rightarrow \mathbb{R}$ (of the form (3.4.4)) such that*

$$\|u_{x_0,r} - u_{x_0}\|_{L^\infty(B_1)} \leq Cr^\gamma \quad \text{for every } r \in (0, r_0). \quad (3.4.8)$$

Proof. Let u be as in **(OP)** and $x_0 \in \partial\Omega_u \cap B_1$. We set $\bar{u} = u \circ F_{x_0}$ and $\bar{u}_r(x) := \frac{\bar{u}(rx)}{r}$, and we notice that $\bar{u}_r = u_{x_0,r} \circ A_{x_0}^{1/2}$. By Lemma 3.3.2 and Lemma 3.2.4, $r \mapsto W_{\text{op}}(\bar{u}_r) + Cr^\delta$ is monotone. On the other hand, the homogeneity of the blow-up limits, imply that

$$\Theta_{\text{op}} := \lim_{r \rightarrow 0} W_{\text{op}}(\bar{u}_r) = \frac{\pi}{2} Q_{\text{op}}(x_0).$$

Thus, by the epiperimetric inequality (Theorem 3.2.2), Lemma 3.3.2 and Lemma 3.2.5, we have that there exists a one-homogeneous function \bar{u}_0 such that, for $r > 0$ small enough,

$$\|\bar{u}_r - \bar{u}_0\|_{L^2(\partial B_1)} \leq Cr^{\gamma_0/2},$$

where γ_0 is the constant from Lemma 3.2.5. Integrating in r , we get that

$$\|\bar{u}_r - \bar{u}_0\|_{L^2(B_1)} \leq Cr^{\gamma_0/2}.$$

Now, since $\bar{u}_r = u_{x_0,r} \circ A_{x_0}^{1/2}$ and $A_{x_0}^{1/2}$ is invertible, we get

$$\|u_{x_0,r} - u_{x_0}\|_{L^2(B_1)} \leq Cr^{\gamma_0/2},$$

where $u_{x_0} = \bar{u}_0 \circ A_{x_0}^{1/2}$. Finally, we notice that the Lipschitz continuity of u implies that there is an universal bound on $\|\nabla u_{x_0,r}\|_{L^\infty(B_1)}$ and $\|\nabla u_{x_0}\|_{L^\infty(B_1)}$. Thus, we get (3.4.7) with $\gamma = \gamma_0/4$. The proof of **(TP)** is analogous. \square

Remark 3.4.8. We notice that the above result does not hold at the one-phase points $x_0 \in \partial\Omega_u^+ \setminus \partial\Omega_u^-$ of the solutions u of the two-phase problem (Theorem 3.1.5). This is due to the fact that the positive part u_+ is not a solution of the one-phase problem in the balls $B_r(x_0)$ that have non-empty intersection with the negative phase Ω_u^- . In fact, the blow-up limit u_{x_0} (of u at x_0) is still unique, but the decay estimate (3.4.7) holds only for $r < \frac{1}{2} \text{dist}(x_0, \Omega_u^-)$.

3.5 Regularity of the one-phase free boundaries. Proof of Theorem 3.1.1

Let $u \in H^1(B_2)$, $u \geq 0$, be as in Theorem 3.1.1. By Proposition 3.4.7 we have that, for every $x_0 \in \partial\Omega_u \cap B_1$, there is a unique blow-up limit of u at x_0 . We denote it by

$$u_{x_0}(x) = \mu(x_0) \max\{0, \nu_{x_0} \cdot x\},$$

where ν_{x_0} is of the form $A_{x_0}^{1/2}[\nu]$, for some $\nu \in \partial B_1$; and $\mu(x_0)$ satisfies the inequalities

$$Q_{\text{op}}(x_0) \leq \mu^2(x_0) \leq M_A L^2,$$

where L is the Lipschitz constant of u . We also notice that

$$\mu(x_0) = Q_{\text{op}}^{1/2}(x_0) \quad \text{whenever } x_0 \in \partial\Omega_u \cap B_1^+.$$

Moreover, for every point $x_0 \in \partial\Omega_u \cap B_1$, we define the half-plane

$$H_{x_0} := \{x \in \mathbb{R}^2 : x \cdot \nu_{x_0} > 0\}.$$

We first prove the following:

Lemma 3.5.1. *Let u be as in Theorem 3.1.1. There are constants $C > 0$, $\gamma > 0$ and $r_0 > 0$ such that, for every $x_0 \in \partial\Omega_u \cap B_1$, we have*

$$\Omega_{x_0,r} \cap B_1 \supset \{x \in B_1 : x \cdot \nu_{x_0} > Cr^\gamma\} \quad \text{and} \quad \Omega_{x_0,r} \cap \{x \in B_1 : x \cdot \nu_{x_0} < -Cr^\gamma\} = \emptyset, \quad (3.5.1)$$

for every $r \in (0, r_0)$, where $\Omega_{x_0,r} := \{u_{x_0,r} > 0\}$.

Proof. The first part of (3.5.1) follows by the uniform convergence of the blow-up sequence $u_{x_0,r}$ (Proposition 3.4.7, equation (3.4.7)) and the form of the blow-up limit u_{x_0} . The second part of (3.5.1) follows again by (3.4.7), the fact that $u_{x_0} \equiv 0$ on $B_1 \setminus H_{x_0}$ and by the non-degeneracy of u , which can be written as

$$\text{If } u_{x_0,r}(y_0) > 0, \quad \text{then } \|u_{x_0,r}\|_{L^\infty(B_s(y_0))} \geq Cs, \quad \text{for every } s \in (0, 1),$$

for some $C > 0$. □

Lemma 3.5.2. *Let u be as in Theorem 3.1.1. There are constants $R, \alpha \in (0, 1)$ and $C > 0$ such that, for every $x_0, y_0 \in \partial\Omega_u \cap B_R$, we have*

$$|\nu_{x_0} - \nu_{y_0}| \leq C|x_0 - y_0|^\alpha \quad \text{and} \quad |\mu(x_0) - \mu(y_0)| \leq C|x_0 - y_0|^\alpha. \quad (3.5.2)$$

Proof. Let $\gamma \in (0, 1)$ be the exponent from Proposition 3.4.7 and let $\alpha := \frac{\gamma}{1+\gamma}$. Let $x_0, y_0 \in B_R \cap \partial\Omega_u$, where we choose R such that $(2R)^{1-\alpha} \leq r_0$, where r_0 is the constant from Proposition 3.4.7. We set $r := |x_0 - y_0|^{1-\alpha}$. Recall that u is Lipschitz continuous and set $L = \|\nabla u\|_{L^\infty}$. Then, for every $x \in B_1$, we have

$$|u_{x_0,r}(x) - u_{y_0,r}(x)| = \frac{1}{r} |u(x_0 + rx) - u(y_0 + rx)| \leq L \frac{|x_0 - y_0|}{r} = L|x_0 - y_0|^\alpha.$$

and then, by an integration on B_1 , we get

$$\|u_{x_0,r} - u_{y_0,r}\|_{L^2(B_1)} \leq |B_1|^{1/2} L|x_0 - y_0|^\alpha.$$

On the other hand, by the choice of R , we have that $r \leq r_0$; applying Proposition 3.4.7, we get

$$\|u_{x_0,r} - u_{x_0}\|_{L^2(B_1)} \leq Cr^\gamma \quad \text{and} \quad \|u_{y_0,r} - u_{y_0}\|_{L^2(B_1)} \leq Cr^\gamma.$$

Thus, by the triangular inequality and the fact that $r^\gamma = |x_0 - y_0|^\alpha$, we obtain

$$\|u_{x_0} - u_{y_0}\|_{L^2(B_1)} \leq (|B_1|^{1/2}L + 2C)|x_0 - y_0|^\alpha. \quad (3.5.3)$$

The conclusion now follows by a general argument. Indeed, for any pair of vectors $v_1, v_2 \in \mathbb{R}^2$, we have

$$\begin{aligned} |v_1 - v_2| &= \left(\frac{2}{\pi} \int_{B_1} |v_1 \cdot x - v_2 \cdot x|^2 dx \right)^{1/2} \\ &\leq \left(\int_{B_1} |(v_1 \cdot x)_+ - (v_2 \cdot x)_+|^2 dx \right)^{1/2} + \left(\int_{B_1} |(v_1 \cdot x)_- - (v_2 \cdot x)_-|^2 dx \right)^{1/2} \\ &= 2 \left(\int_{B_1} |(v_1 \cdot x)_+ - (v_2 \cdot x)_+|^2 dx \right)^{1/2}. \end{aligned} \quad (3.5.4)$$

Applying the above estimate to $v_1 = \mu(x_0)\nu_{x_0}$ and $v_2 = \mu(y_0)\nu_{y_0}$, and using (3.5.3), we get (3.5.2). \square

Proof of Theorem 3.1.1. We first claim that, for every $\varepsilon > 0$, there exists $\rho > 0$ such that, for $x_0 \in \partial\Omega_u \cap B_\rho$ we have

$$u > 0 \text{ on } C^+(x_0, \varepsilon) \cap B_\rho(x_0) \quad \text{and} \quad u = 0 \text{ on } C^-(x_0, \varepsilon) \cap B_\rho(x_0), \quad (3.5.5)$$

where

$$C^\pm(x_0, \varepsilon) := \{x \in \mathbb{R}^2 \setminus \{0\} : \pm \nu_{x_0} \cdot (x - x_0) \geq \varepsilon|x - x_0|\}.$$

Indeed, the flatness estimate (3.5.1) implies (3.5.5) by taking ρ such that $C\rho^\gamma \leq \varepsilon$, where C and γ are the constants from Lemma 3.5.1.

We now fix $x_0 \in B_1 \cap \partial\Omega_u$. Without loss of generality we can suppose that $x_0 = 0$ and $H_{x_0} = \{(s, t) \in \mathbb{R}^2 : t > 0\}$. Now, let $\varepsilon \in (0, 1)$ and $\rho > 0$ as in (3.5.5) and set $\delta = \rho\sqrt{1 - \varepsilon^2}$. By (3.5.5) we have for every $s \in (-\delta, \delta)$

- the set $\mathcal{S}_+^s := \{t \in (-\delta, \delta) : u(s, t) > 0\}$ contains the interval $(\rho\varepsilon, \delta)$;
- the set $\mathcal{S}_0^s := \{t \in (-\delta, \delta) : u(s, t) = 0\}$ contains the interval $(-\delta, -\rho\varepsilon)$.

This implies that the function

$$g(s) := \max\{t \in \mathbb{R} : u(s, t) > 0\}$$

is well defined and such that

$$SQ_\delta \cap \Omega_u = \{(s, t) \in SQ_\delta : g(s) < t\} \quad \text{and} \quad SQ_\delta \setminus \Omega_u = \{(s, t) \in SQ_\delta : g(s) \geq t\},$$

where $SQ_\delta = (-\delta, \delta) \times (-\delta, \delta)$. Now, the flatness condition (3.5.1) implies that g is differentiable on $(-\delta, \delta)$. Furthermore, since ν is Hölder continuous, we deduce that g is a function of class $C^{1, \alpha}$. This concludes the proof. \square

3.6 Regularity of the two-phase free boundaries. Proof of Theorem 3.1.5

Let u be as in Theorem 3.1.5. Then, by Proposition 3.4.7, at every point $x_0 \in \partial\Omega_u \cap B_1$ there is a unique blow-up limit u_{x_0} given by

$$\begin{aligned} u_{x_0}(x) &= \mu_+(x_0) \max\{0, x \cdot \nu_{x_0}\}, & \text{if } x_0 \in \Gamma_+ &:= (\partial\Omega_u^+ \setminus \partial\Omega_u^-) \cap B_1; \\ u_{x_0}(x) &= \mu_-(x_0) \min\{0, x \cdot \nu_{x_0}\}, & \text{if } x_0 \in \Gamma_- &:= (\partial\Omega_u^- \setminus \partial\Omega_u^+) \cap B_1; \\ u_{x_0}(x) &= \mu_+(x_0) \max\{0, x \cdot \nu_{x_0}\} + \mu_-(x_0) \min\{0, x \cdot \nu_{x_0}\}, & \text{if } x_0 \in \Gamma_{\text{TP}} &:= \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1, \end{aligned}$$

where $\nu_{x_0} \in \mathbb{R}^2$ is of the form $A_{x_0}^{-1/2}[\nu]$, for some $\nu \in \partial B_1$, and $\mu_+(x_0)$ and $\mu_-(x_0)$ are positive and such that $Q_{\text{TP}}^\pm(x_0) \leq \mu_\pm^2(x_0) \leq M_A L^2$, where $L = \|\nabla u\|_{L^\infty(B_2)}$ is the Lipschitz constant of u , and

$$\begin{aligned} \mu_\pm^2(x_0) &= Q_{\text{TP}}^\pm(x_0), \quad \text{if } x_0 \in \Gamma_\pm \\ \mu_+^2(x_0) - \mu_-^2(x_0) &= Q_{\text{TP}}^+(x_0) - Q_{\text{TP}}^-(x_0), \quad \text{if } x_0 \in \Gamma_{\text{TP}}. \end{aligned}$$

Notice that Corollary 3.1.3 already implies that the one-phase free boundaries Γ_+ and Γ_- are $C^{1,\alpha}$ regular. Thus, it remains to prove that $\partial\Omega_u^+$ and $\partial\Omega_u^-$ are smooth in a neighborhood of Γ_{TP} .

Lemma 3.6.1 (Flatness of the free boundary at the two-phase points). *Let u be as in Theorem 3.1.5. There are constants $C > 0$, $\gamma > 0$ and $r_0 > 0$ such that, for every $x_0 \in \partial\Gamma_{\text{TP}}$, we have*

$$\Omega_{x_0,r}^+ \cap B_1 \supset \{x \in B_1 : x \cdot \nu_{x_0} > Cr^\gamma\} \quad \text{and} \quad \Omega_{x_0,r}^- \cap B_1 \supset \{x \in B_1 : x \cdot \nu_{x_0} < -Cr^\gamma\}, \quad (3.6.1)$$

for every $r \in (0, r_0)$, where $\Omega_{x_0,r}^+ := \{u_{x_0,r} > 0\}$ and $\Omega_{x_0,r}^- := \{u_{x_0,r} < 0\}$.

Proof. Both the inclusions of (3.6.1) follow by the uniform convergence of $u_{x_0,r}$ (Proposition 3.4.7, equation (3.4.8)) to the blow-up limit u_{x_0} . \square

Lemma 3.6.2. *Let u be as in Theorem 3.1.5. There are constants $R, \alpha \in (0, 1)$ and $C > 0$ such that, for every $x_0, y_0 \in \partial\Gamma_{\text{TP}} \cap B_R$, we have*

$$|\nu_{x_0} - \nu_{y_0}| \leq C|x_0 - y_0|^\alpha \quad \text{and} \quad |\mu_\pm(x_0) - \mu_\pm(y_0)| \leq C|x_0 - y_0|^\alpha. \quad (3.6.2)$$

Proof. The proof follows step by step the one of Lemma 3.5.2. \square

Reasoning as in the one-phase case, and using Lemma 3.6.1 and Lemma 3.6.2, one can prove that the two-phase free boundary Γ_{TP} is contained in a $C^{1,\alpha}$ curve. Unfortunately, this result by itself is not sufficient to deduce that $\partial\Omega_u^\pm$ are smooth. We now prove that the function u_+ (resp. u_-) is a solution of the one-phase free boundary problem

$$-\operatorname{div}(A\nabla u_+) = f_+ \quad \text{in } \Omega_u^+, \quad |A_{x_0}^{1/2}\nabla u_+(x_0)| = \mu_+(x_0) \quad \text{for every } x_0 \in \partial\Omega_u^+ \quad (3.6.3)$$

where the boundary equation is understood in a classical sense. This is an immediate consequence of the following lemma which states that u_+ is differentiable in Ω_u^+ up to the boundary.

Lemma 3.6.3 (Differentiability at points of the free boundary). *Let u be as in Theorem 3.1.5. We consider two cases.*

(OP). *For every $x_0 \in (\partial\Omega_u^+ \setminus \partial\Omega_u^-) \cap B_1$, u_+ is differentiable at x_0 and there is $r(x_0) > 0$ such that*

$$|u_+(x) - \mu_+(x_0)(x - x_0) \cdot \nu_{x_0}| \leq C|x - x_0|^{1+\gamma} \quad \text{for every } x \in B_{r(x_0)}(x_0) \cap \Omega_u^+.$$

(TP). *There exists a universal constant $r_0 > 0$ such that, for every $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1$, the function u_+ is differentiable in $B_{r_0}(x_0) \cap \Omega_u^+$ and*

$$|u_+(x) - \mu_+(x_0)(x - x_0) \cdot \nu_{x_0}| \leq C|x - x_0|^{1+\gamma} \quad \text{for every } x \in B_{r_0}(x_0) \cap \Omega_u^+. \quad (3.6.4)$$

In particular, for every $x_0 \in \partial\Omega_u^+ \cap B_1$, we have $\nabla u_+(x_0) = \mu_+(x_0)\nu_{x_0}$.

Proof. The two cases are analogous. We will prove **(TP)**. By Proposition 3.4.7, for every $r < r_0$, we have

$$\|\max\{0, u_{x_0,r}\} - \max\{0, u_{x_0}\}\|_{L^\infty(B_1)} \leq \|u_{x_0,r} - u_{x_0}\|_{L^\infty(B_1)} \leq Cr^\gamma.$$

Thus, using the flatness of the free boundary (Lemma 3.6.1), we get for every $x \in B_1 \cap \{u_{x_0,r} > 0\}$

$$\begin{aligned} \left| \max\{0, u_{x_0,r}(x)\} - \mu_+(x_0)x \cdot \nu_{x_0} \right| &\leq \left| \max\{0, u_{x_0,r}(x)\} - \max\{0, u_{x_0}(x)\} \right| \\ &\quad + \left| \max\{0, u_{x_0}(x)\} - \mu_+(x_0)x \cdot \nu_{x_0} \right| \leq Cr^\gamma. \end{aligned}$$

Now, taking $r = |x - x_0|$ and rescaling the above inequality, we obtain (3.6.4) \square

We notice that at the two-phase free boundary point the estimate (3.6.4) holds in a ball whose radius does not depend on the point. Moreover, on the two-phase free boundary the gradient has a universal modulus of continuity (see Lemma 3.6.2). This concludes the proof of Proposition 3.1.8.

We next show that μ_+ is continuous on $\partial\Omega_u^+$.

Lemma 3.6.4. *The function $\mu_+ : \partial\Omega_u^+ \rightarrow \mathbb{R}$ is continuous.*

Proof. We start noticing that:

- on the set $\partial\Omega_u^+ \setminus \partial\Omega_u^-$, we have $\mu_+ = Q_+^{1/2}$, where we set $Q_+ := Q_{\text{TP}}^+$.
- for every $y_1, y_2 \in \partial\Omega_u^+ \cap \partial\Omega_u^-$, we have $|\mu_+(y_1) - \mu_+(y_2)| \leq C|y_1 - y_2|^\alpha$.

Thus, it is sufficient to prove that if $(x_n)_{n \geq 1}$ is a sequence of one-phase points, $x_n \in \partial\Omega_u^+ \setminus \partial\Omega_u^-$, converging to a two-phase point $y_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^-$, then $\mu_+(y_0) = Q_+^{1/2}(y_0)$. Up to a linear change of coordinates we may suppose that $A_{y_0} = Id$.

Denote by y_n the projection of x_n on the closed set $\partial\Omega_u^+ \cap \partial\Omega_u^-$ and let $r_n := |x_n - y_n|$. Since u is Lipschitz continuous, up to a subsequence, $u_n := u_{x_n, r_n}^+$ converges locally uniformly to some function u_∞ . The absence of two-phase points in $B_{r_n}(x_n)$ implies that u_n is a solution of

$$-\operatorname{div}(A_n \nabla u_n) = r_n f_n \quad \text{in } \{u_n > 0\} \cap B_1, \quad |\nabla u_n| = q_n \quad \text{on } \partial\{u_n > 0\} \cap B_1,$$

where $A_n(x) := A(x_n + r_n x)$, $f_n(x) := f_+(x_n + r_n x)$ and $q_n(x) = Q_+^{1/2}(x_n + r_n x) |\nu_{x_n + r_n x}|$, where we recall that $\nu_{x_n + r_n x}$ is of the form $A_{x_n + r_n x}^{1/2}[\tilde{\nu}]$, for some $\tilde{\nu} \in \partial B_1$. Passing to the limit as $n \rightarrow \infty$, we obtain that u_∞ is a viscosity solution to

$$-\Delta u_\infty = 0 \quad \text{in } \{u_\infty > 0\} \cap B_1, \quad |\nabla u_\infty| = Q_+^{1/2}(y_0) |\nu_{y_0}| \quad \text{on } \partial\{u_\infty > 0\} \cap B_1.$$

On the other hand, for every $\xi \in B_1$, we have

$$u_{x_n, r_n}(\xi) = u_{y_n, r_n}(\xi + \xi_n), \quad \text{where } \xi_n := \frac{x_n - y_n}{r_n} \in \partial B_1,$$

and, up to a subsequence, we can assume that ξ_n converges to some $\xi_\infty \in \partial B_1$. Since $y_n \in \partial\Omega_u^+ \cap \partial\Omega_u^-$, Lemma 3.6.3 implies that, for every $x \in B_{2r_n}(y_n) \cap \{u > 0\}$, we have

$$|u(x) - \mu_+(y_n) \max\{0, (x - y_n) \cdot \nu_{y_n}\}| \leq C|x - y_n|^{1+\gamma} \leq Cr_n^{1+\gamma}.$$

After rescaling, this gives

$$|u_{y_n, r_n}(\xi + \xi_n) - \mu_+(y_n) \max\{0, (\xi + \xi_n) \cdot \nu_{y_n}\}| \leq Cr_n^\gamma \quad \text{for every } \xi \in B_1 \cap \{u_{x_n, r_n} > 0\}.$$

Moreover, by the continuity of μ_+ on $\partial\Omega_u^+ \cap \partial\Omega_u^-$, we have that, for every $\xi \in B_1$,

$$\lim_{n \rightarrow \infty} |\mu_+(y_n) \max\{0, (\xi + \xi_n) \cdot \nu_{y_n}\} - \mu_+(y_0) \max\{0, (\xi + \xi_\infty) \cdot \nu_{y_0}\}| = 0.$$

Therefore, it follows that $u_{x_n, r_n}(\xi) = u_{y_n, r_n}(\xi + \xi_n)$ converges to

$$u_\infty(\xi) = \mu_+(y_0) \max\{0, (\xi + \xi_\infty) \cdot \nu_{y_0}\} \quad \text{for every } \xi \in B_1.$$

Next we claim that $\xi_\infty \cdot \nu_{y_0} = 0$. Indeed, if $\xi_\infty \cdot \nu_{y_0} > 0$, then $u_\infty(0) > 0$ which is in contradiction with the uniform convergence of u_n ; on the other hand, if $\xi_\infty \cdot \nu_{y_0} < 0$, then $u_\infty \equiv 0$ in a neighborhood of zero, which is in contradiction with the non-degeneracy of u_n . Thus, we get

$$u_\infty(\xi) = \mu_+(y_0) \max\{0, \xi \cdot \nu_{y_0}\} \quad \text{for every } \xi \in B_1.$$

Now since $|\nabla u_\infty| = \mu_+(y_0)$, we get that $\mu_+(y_0) = Q_+^{1/2}(y_0)$. □

In the next lemma we establish the Hölder continuity of μ_+ .

Lemma 3.6.5. *The function $\mu_+ : \partial\Omega_u^+ \rightarrow \mathbb{R}$ is Hölder continuous.*

Proof. It is sufficient to show that μ_+ is Hölder continuous in a neighborhood of every two-phase point. Let $x_0 \in \partial\Omega_u^+ \cap \partial\Omega_u^-$. Without loss of generality, we can assume that $x_0 = 0$ and $\nu_{x_0} = (0, 1)$. For every $x = (s, t) \in \mathbb{R}^2$, we denote by $SQ_\delta(x)$ the square $(s-\delta, s+\delta) \times (t-\delta, t+\delta)$. By Lemma 3.6.1, for every $\varepsilon > 0$, there exists $\delta_0 > 0$ such that the following flatness condition holds:

For every two-phase point $x = (x_1, x_2) \in \partial\Omega_u^+ \cap \partial\Omega_u^-$ in the strip $(-\delta_0, \delta_0) \times (-\varepsilon\delta_0, \varepsilon\delta_0)$ and every $\delta \leq \delta_0$ we have:

$$\begin{cases} (x_1 - \delta, x_1 + \delta) \times (x_2 - \delta, x_2 - \varepsilon\delta) \subset \Omega_u^- \cap SQ_\delta(x), \\ (x_1 - \delta, x_1 + \delta) \times (x_2 + \varepsilon\delta, x_2 + \delta) \subset \Omega_u^+ \cap SQ_\delta(x). \end{cases} \quad (3.6.5)$$

Notice that the flatness condition (3.6.5) implies that for every two-phase point $x = (x_1, x_2) \in \partial\Omega_u^+ \cap \partial\Omega_u^-$ and every $y = (y_1, y_2) \in \partial\Omega_u^+$, both lying in the strip $(-\delta_0, \delta_0) \times (-\varepsilon\delta_0, \varepsilon\delta_0)$, we have

$$|x_1 - y_1| \leq |x - y| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2} \leq \sqrt{1 + \varepsilon^2} |x_1 - y_1|. \quad (3.6.6)$$

Next, for every $t \in (-\delta_0, \delta_0)$ we define the vertical sections

$$\mathcal{S}^t := \{(x_1, x_2) \in SQ_{\delta_0}(0) : x_1 = t\}, \quad \mathcal{S}_+^t := \Omega_u^+ \cap \mathcal{S}^t, \quad \mathcal{S}_-^t := \Omega_u^- \cap \mathcal{S}^t.$$

Let \mathcal{U}_{tp} be the set of points $t \in (-\delta_0, \delta_0)$ such that there is a two-phase point $x \in SQ_{\delta_0}(0)$ lying on the section \mathcal{S}^t . It is immediate to check that \mathcal{U}_{tp} is a closed subset of $(-\delta_0, \delta_0)$ and that, due to the flatness condition (3.6.5), for every $t \in \mathcal{U}_{\text{tp}}$, there is at most one two-phase point on the section \mathcal{S}^t . We will denote this point by x^t .

Let now $x, y \in \partial\Omega_u^+ \cap SQ_{\delta_0}(0)$. We have three possibilities:

- (i) $x \in \partial\Omega_u^+ \cap \partial\Omega_u^-$ and $y \in \partial\Omega_u^+ \cap \partial\Omega_u^-$;
- (ii) $x \in \partial\Omega_u^+ \setminus \partial\Omega_u^-$ and $y \in \partial\Omega_u^+ \setminus \partial\Omega_u^-$;
- (iii) $x \in \partial\Omega_u^+ \setminus \partial\Omega_u^-$ and $y \in \partial\Omega_u^+ \cap \partial\Omega_u^-$.

In each of this cases we will show that

$$|\mu_+(x) - \mu_+(y)| \leq C|x - y|^\alpha. \quad (3.6.7)$$

We set $Q_+ := Q_{\text{tp}}^+$. In the case (i), (3.6.7) follows directly by Lemma 3.6.2. In the case (ii), we have that $\mu_+(x) = Q_+^{1/2}(x)$ and $\mu_+(y) = Q_+^{1/2}(y)$, so (3.6.7) follows by the Hölder continuity of Q_+ . It remains to prove (3.6.7) in the case (iii). Let $x = (x_1, x_2)$, $y = (y_1, y_2)$ and, without loss of generality, suppose that $x_1 < y_1$. Let the open interval (a, t) be the connected component of $(-\delta_0, \delta_0) \setminus \mathcal{U}_{\text{tp}}$ containing x_1 . Then, we have that $t \in \mathcal{U}_{\text{tp}}$ and $t \leq y_1$. Let $x^t = (t, x_2^t)$ be the two-phase point lying in the section \mathcal{S}^t . Then, by construction of x^t , there exists (at least) one point $x_s \in \mathcal{S}^s \cap \Gamma^+$ for every $s \in (x_1, t)$. Moreover, since x^t is a two-phase point, by the flatness condition (3.6.5) we have that $u(t, s) > 0$ for every $s > x_2^t$ and $u(t, s) < 0$ for every $s < x_2^t$. Therefore, the sequence of one-phase point x_s converges as $s \rightarrow t$ to x^t and so, $\mu_+(x^t) = Q_+^{1/2}(x^t)$. Thus, using (i), the Hölder continuity of Q_+ and (3.6.6), we have

$$\begin{aligned} |\mu_+(x) - \mu_+(y)| &\leq |\mu_+(x) - \mu_+(x^t)| + |\mu_+(x^t) - \mu_+(y)| \\ &= |Q_+^{1/2}(x) - Q_+^{1/2}(x^t)| + |\mu_+(x^t) - \mu_+(y)| \\ &\leq C|x - x^t|^\alpha + C|x^t - y|^\alpha \\ &\leq C\left(\sqrt{1 + \varepsilon^2}\right)^\alpha \left((t - x_1)^\alpha + (y_1 - t)^\alpha\right) \\ &\leq 2C\left(\sqrt{1 + \varepsilon^2}\right)^\alpha (y_1 - x_1)^\alpha \leq 2C\left(\sqrt{1 + \varepsilon^2}\right)^\alpha |x - y|^\alpha, \end{aligned}$$

which concludes the proof. \square

Theorem 3.1.5 is now a consequence of (3.6.3), the Lemma 3.6.4 and a general result (Theorem 3.A.1) on the regularity of the one-phase flat free boundaries, which is due to De Silva (see [36]). In the appendix we state Theorem 3.A.1 in its full generality, for viscosity solutions of the problem (3.6.3), but in our case the function u_+ is a classical solution, differentiable everywhere on $\overline{\Omega}_u^+$.

3.7 Proof of Theorem 3.1.11

3.7.1 Preliminary results

In this subsection, we briefly recall the known results on the problem (3.1.10). The existence of a solution of (3.1.10) in the class of the almost-open subsets of \mathcal{D} can be proved by a general variational argument (we refer to [19] and to the book [13] for more details). In the context of open sets, the existence of an optimal n -uple was proved in [7].

From now on, $(\Omega_1, \dots, \Omega_n)$ will be a solution of (3.1.10) and $u_i : \mathbb{R}^d \rightarrow \mathbb{R}$, for $i = 1, \dots, n$, will denote the first normalized eigenfunction of the Dirichlet Laplacian on Ω_i , that is,

$$-\Delta u_i = \lambda_1(\Omega_i)u_i \quad \text{in } \Omega_i, \quad u_i = 0 \quad \text{on } \mathbb{R}^2 \setminus \Omega_i, \quad \int_{\Omega_i} u_i^2 dx = 1,$$

where, for every $i = 1, \dots, n$,

$$\lambda_1(\Omega_i) = \min_{u \in H_0^1(\Omega_i) \setminus \{0\}} \frac{\int_{\Omega_i} |\nabla u|^2 dx}{\int_{\Omega_i} u^2 dx} = \frac{\int_{\Omega_i} |\nabla u_i|^2 dx}{\int_{\Omega_i} u_i^2 dx},$$

where $H_0^1(\Omega_i) = \{u \in H^1(\mathbb{R}^d) : u = 0 \text{ on } \mathbb{R}^2 \setminus \Omega_i\}$. In particular, $u_i \geq 0$ on \mathbb{R}^2 and $\Omega_i = \{u_i > 0\}$.

Lipschitz continuity. The functions $u_i : \mathbb{R}^2 \rightarrow \mathbb{R}$ are Lipschitz continuous on \mathbb{R}^2 , that is, there is a universal constant $L > 0$ such that $\|\nabla u_i\|_{L^\infty(\mathbb{R}^2)} \leq L$, for every $i = 1, \dots, n$. We refer to [19] for the general case and to [7] for a simplified version in dimension two.

Non-degeneracy. There is a constant $C_0 > 0$ such that, for every $i = 1, \dots, n$, we have

$$\int_{\partial B_r} u_i d\mathcal{H}^1 \geq C_0 r^2 \quad \text{for every } x_0 \in \partial\Omega_i \quad \text{and } r \in (0, 1).$$

Again, we refer to [19] and [7].

Absence of triple points. For every $1 \leq i < j < k \leq n$, we have that $\partial\Omega_i \cap \partial\Omega_j \cap \partial\Omega_k = \emptyset$ (see [19] and [89], and also [7] for a more direct proof in dimension two).

Absence of two-phase points on the boundary of the box. For every $1 \leq i < j \leq n$, we have that $\partial\Omega_i \cap \partial\Omega_j \cap \partial\mathcal{D} = \emptyset$ (see [7]).

As a consequence of the above properties, we have that, for every $i \in \{1, \dots, n\}$, the boundary $\partial\Omega_i$ can be decomposed as follows:

$$\partial\Omega_i = \bigcup_{k \neq i} (\partial\Omega_i \cap \partial\Omega_k) \cup (\partial\Omega_i \cap \partial\mathcal{D}) \cup \Gamma_{\text{op}}(\Omega_i),$$

where $\Gamma_{\text{op}}(\Omega_i)$ is the one-phase free boundary of Ω_i , determined by:

$$x_0 \in \Gamma_{\text{op}}(\Omega_i) \Leftrightarrow \text{there exists } r > 0 \text{ such that } B_r(x_0) \cap \left((\mathbb{R}^2 \setminus \mathcal{D}) \cup \bigcup_{k \neq i} \Omega_k \right) = \emptyset.$$

We notice that already using the the regularity result of Briançon and Lamboley [11], the one-phase free boundary (lying inside the open set \mathcal{D}) is locally a $C^{1,\alpha}$ curve. Thus, in order to prove Theorem 3.1.11, it will be sufficient to show that $\partial\Omega_i$ is $C^{1,\alpha}$ in a neighborhood of the points of $\partial\Omega_i \cap \partial\mathcal{D}$ (Subsection 3.7.2) and $\partial\Omega_i \cap \partial\Omega_k$ (Subsection 3.7.3).

3.7.2 One-phase points at the boundary of the box

Let $1 \leq i \leq n$ and $x_0 \in \partial D \cap \partial \Omega_i$. Then, there is a neighborhood \mathcal{U} of x_0 such that $\mathcal{U} \cap \Omega_j = \emptyset$, for every $j \neq i$. For the sake of simplicity, in this subsection, we will set

$$\Omega = \Omega_i, \quad u = u_i, \quad x_0 = 0 \quad \text{and} \quad D = \Omega_i \cup (\mathcal{D} \cap \mathcal{U}).$$

It is well known that the eigenvalues of the Dirichlet Laplacian have the following variational characterization:

$$\lambda_1(\Omega) = \int_{\Omega} |\nabla u|^2 dx = \min \left\{ \int_{\Omega} |\nabla v|^2 dx : v \in H_0^1(\Omega), \int_{\Omega} v^2 dx = 1 \right\}.$$

Moreover, $\{u > 0\} = \Omega$ and u is a solution of the following minimization problem:

$$\min \left\{ \int_D |\nabla v|^2 dx + \Lambda |\{v > 0\}| : v \in H_0^1(D), \int_D v^2 dx = 1 \right\}. \quad (3.7.1)$$

We will show that the solution u of (3.7.1) is an almost-minimizer of the one-phase functional J_{op} . A result in the same spirit was proved in a more general case in [77, Proposition 2.1].

Lemma 3.7.1 (Almost-minimality of the eigenfunction). *Let $u : \mathbb{R}^d \rightarrow \mathbb{R}$ be a Lipschitz continuous function, $L = \|\nabla u\|_{L^\infty}$ be the Lipschitz constant of u and $\lambda_1(\Omega_u) = \int_D |\nabla u|^2 dx$. If u is a solution of the minimization problem (3.7.1), then there exists $r_0 > 0$ such that u satisfies the following almost-minimality condition:*

For every $r \in (0, r_0)$ and $x_0 \in \partial \Omega_u$,

$$\int_{B_r(x_0)} |\nabla u|^2 dx + \Lambda |\Omega_u \cap B_r(x_0)| \leq (1 + C_1 r^{d+2}) \int_{B_r(x_0)} |\nabla v|^2 dx + \Lambda |\Omega_v \cap B_r(x_0)| + C_2 r^{d+2},$$

for every $v \in H_0^1(D)$ such that $u = v$ on $D \setminus B_r(x_0)$, where $C_1 = 2L^2$ and $C_2 = \lambda_1(\Omega_u)2L^2$.

Proof. Let $x_0 \in \partial \Omega_u$, $r > 0$ and $v \in H_0^1(D)$ be such that $u = v$ on $D \setminus B_r(x_0)$. Then, define the renormalization $w = \|v\|_{L^2}^{-1} v \in H_0^1(D)$ and notice that we have

$$\begin{aligned} \int_D |\nabla w|^2 dx &= \left(\int_D v^2 dx \right)^{-1} \int_D |\nabla v|^2 dx \leq \left(1 - \int_{B_r(x_0)} u^2 dx \right)^{-1} \int_D |\nabla v|^2 dx \\ &\leq \frac{1}{1 - L^2 r^{d+2}} \int_D |\nabla v|^2 dx \leq (1 + 2L^2 r^{d+2}) \int_D |\nabla v|^2 dx, \end{aligned}$$

where for the last inequality, we choose r_0 such that $2L^2 r_0^{d+2} \leq 1$ and we use the inequality $\frac{1}{1-X} \leq 1+2X$, for every $X \leq 1/2$, with $X = L^2 r^{d+2}$. Now use w as a test function in (3.7.1) to get that

$$\int_D |\nabla u|^2 dx + \Lambda |\{u > 0\}| \leq (1 + 2L^2 r^{d+2}) \int_D |\nabla v|^2 dx + \Lambda |\{v > 0\}|, \quad (3.7.2)$$

from which the claim easily follows since $\int_{D \setminus B_r(x_0)} |\nabla v|^2 dx = \int_{D \setminus B_r(x_0)} |\nabla u|^2 dx \leq \lambda_1(\Omega_u)$. \square

We now notice that the C^2 regularity of $\partial \mathcal{D}$ implies that there is a constant $\delta > 0$ and a function $g : (-\delta, \delta) \rightarrow \mathbb{R}$ such that

$$\mathcal{D} \cap SQ_\delta = \{(x_1, x_2) \in SQ_\delta : g(x_1) < x_2\},$$

where $SQ_\delta = (-\delta, \delta) \times (-\delta, \delta)$. Moreover, up to a rotation of the plane, we can assume that $g'(0) = 0$. Let $\psi : SQ_\delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be the function that straightens out the boundary of \mathcal{D} and let $\phi = \psi^{-1} : \psi(SQ_\delta) \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be its inverse:

$$\psi(x_1, x_2) = (x_1, x_2 - g(x_1)), \quad \phi(x_1, x_2) = (x_1, x_2 + g(x_1)).$$

We define the matrix-valued function $A = (a_{ij})_{ij} : SQ_\delta \rightarrow M^2(\mathbb{R})$ by

$$A_x := \begin{pmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{pmatrix} = \begin{pmatrix} 1 & -g'(x_1) \\ -g'(x_1) & 1 + (g'(x_1))^2 \end{pmatrix} \quad \text{for every } x = (x_1, x_2) \in SQ_\delta.$$

We recall that $H = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\}$. By an elementary change of coordinates, we obtain the following result.

Lemma 3.7.2. *Let u and A be as above. There exist constants $C_1, C_2 > 0$ and $r_0 > 0$ such that $B_{2r_0} \subset \psi(SQ_\delta)$ and the function $\tilde{u} := u \circ \phi$ satisfies the following almost-minimality condition:*

- For every $x_0 \in \partial\Omega_{\tilde{u}} \cap B_{r_0}$ and $r \in (0, r_0)$ we have

$$\begin{aligned} \int_{B_r(x_0)} a_{ij}(x) \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial \tilde{u}}{\partial x_j} dx + \Lambda |\Omega_{\tilde{u}} \cap B_r(x_0)| \\ \leq (1 + C_1 r^{d+2}) \int_{B_r(x_0)} a_{ij}(x) \frac{\partial \tilde{v}}{\partial x_i} \frac{\partial \tilde{v}}{\partial x_j} dx + \Lambda |\Omega_{\tilde{v}} \cap B_r(x_0)| + C_2 r^{d+2}, \end{aligned}$$

for every $\tilde{v} \in H^1(B_{2r_0})$ such that $\tilde{u} = \tilde{v}$ on $B_{2r_0} \setminus B_r(x_0)$ and $\Omega_{\tilde{v}} \subset H$.

Proof. Let $x_0 \in B_{r_0}$, $r \in (0, r_0)$ and \tilde{v} such that $\tilde{u} = \tilde{v}$ on $B_{2r_0} \setminus B_r(x_0)$. Then, use $v \in H_0^1(D)$ defined by $v = \tilde{v} \circ \psi$ in $\psi^{-1}(B_{2r_0})$ and $v = u$ otherwise, as a test function in Lemma 3.7.1 to get

$$\int_{B_{c_\phi r}(y_0)} |\nabla u|^2 dx + \Lambda |\Omega_u \cap B_{c_\phi r}(y_0)| \leq (1 + C_1 r^{d+2}) \int_{B_{c_\phi r}(y_0)} |\nabla v|^2 dx + \Lambda |\Omega_v \cap B_{c_\phi r}(y_0)| + C r^{d+2},$$

where c_ϕ is a positive constant depending only on ϕ such that $\phi(B_r(x_0)) \subset B_{c_\phi r}(y_0)$ and $y_0 = \phi(x_0)$. Now, with a change of coordinates and noticing that $u = v$ on $\phi(B_r(x_0))$ we have

$$\begin{aligned} \int_{B_r(x_0)} a_{ij}(x) \frac{\partial \tilde{u}}{\partial x_i} \frac{\partial \tilde{u}}{\partial x_j} dx + \Lambda |\Omega_{\tilde{u}} \cap B_r(x_0)| &= \int_{\phi(B_r(x_0))} |\nabla u|^2 dx + \Lambda |\Omega_u \cap \phi(B_r(x_0))| \\ &\leq (1 + C_1 r^{d+2}) \int_{\phi(B_r(x_0))} |\nabla v|^2 dx + \Lambda |\Omega_v \cap \phi(B_r(x_0))| + C_2 r^{d+2} \\ &= (1 + C_1 r^{d+2}) \int_{B_r(x_0)} a_{ij}(x) \frac{\partial \tilde{v}}{\partial x_i} \frac{\partial \tilde{v}}{\partial x_j} dx + \Lambda |\Omega_{\tilde{v}} \cap B_r(x_0)| + C_2 r^{d+2}, \end{aligned}$$

where $C_2 = \lambda_1(\Omega_u)C_1 + C$. This concludes the proof. \square

Proof of Theorem 3.1.11 (the one-phase boundary points). We are now in position to conclude the regularity of the free boundary $\partial\Omega_i$ in a neighborhood of any one-phase boundary point $x_0 \in \partial\Omega_i \cap \partial\mathcal{D}$. Indeed, we may assume that $x_0 = 0$ and that $\partial\mathcal{D}$ is the graph of a function g . Reasoning as above, we have that $\tilde{u}_i(x_1, x_2) = u_i(x_1, x_2 + g(x_1))$ satisfies the almost-minimality condition from Lemma 3.7.2 in a neighborhood of the origin. On the other hand, it is immediate to check that \tilde{u}_i is still Lipschitz continuous. Thus, we can apply Theorem 3.1.1 obtaining that, in a neighborhood of zero, $\partial\Omega_i$ is the graph of a $C^{1,\alpha}$ function. \square

3.7.3 Two-phase points

Let Ω_i and Ω_j be two different sets from the optimal n -uple $(\Omega_1, \dots, \Omega_n)$, solution of (3.1.10). Let u_i and u_j be the first normalized eigenfunctions, respectively on Ω_i and Ω_j . Finally, let $x_0 \in \partial\Omega_i \cap \partial\Omega_j$. We know that there is a neighborhood $\mathcal{U} \subset \mathcal{D}$ of x_0 such that $\mathcal{U} \cap \Omega_k = \emptyset$, for every $k \notin \{i, j\}$. Setting $D := \Omega_i \cup \Omega_j \cup \mathcal{U}$, we get that the function $u := u_i - u_j$ is the solution of the two-phase problem

$$\min \left\{ \int_D |\nabla v|^2 dx + q_i |\Omega_v^+| + q_j |\Omega_v^-| : v \in H_0^1(D), \int_D v_+^2 dx = \int_D v_-^2 dx = 1 \right\}. \quad (3.7.3)$$

We next show that the solutions of (3.7.3) satisfy a almost-minimality condition.

Lemma 3.7.3. *Let $D \subset \mathbb{R}^d$, $u \in H_0^1(D)$ be a Lipschitz continuous function on \mathbb{R}^d and L its Lipschitz constant. Suppose that u is a solution of the minimization problem (3.7.3). Then, there is some $r_0 > 0$ such that u satisfies the following almost-minimality condition:*

For every $r \in (0, r_0)$ and $x_0 \in \partial\Omega_u$,

$$\begin{aligned} & \int_{B_r(x_0)} |\nabla u|^2 dx + q_i |\Omega_u^+ \cap B_r(x_0)| + q_j |\Omega_u^- \cap B_r(x_0)| \\ & \leq (1 + C_1 r^{d+2}) \int_{B_r(x_0)} |\nabla v|^2 dx + q_i |\Omega_v^+ \cap B_r(x_0)| + q_j |\Omega_v^- \cap B_r(x_0)| + C_2 r^{d+2}, \end{aligned}$$

for every $v \in H_0^1(D)$ such that $u - v \in H_0^1(B_r(x_0))$, where $C_1 = 2L^2$ and $C_2 = C_1 \int_D |\nabla u|^2 dx$.

Proof. Follows precisely as in Lemma 3.7.1. \square

We are now in position to complete the proof of Theorem 3.1.11.

Proof of Theorem 3.1.11 (the two-phase free boundary). We only need to notice that in a neighborhood of any two-phase point $x_0 \in \partial\Omega_i \cap \partial\Omega_j \cap \mathcal{D}$, Lemma 3.7.3 implies that u is a almost-minimizer of J_{TP} , where the matrix A is the identity, $Q_+ = q_i$ and $Q_- = q_j$. Thus, it is sufficient to apply Theorem 3.1.5. \square

Appendix A: The flat one-phase free boundaries are $C^{1,\alpha}$

In this section we discuss a regularity theorem for viscosity solutions of the one-phase problem (without constraint). We fix the real-valued function $f : B_2 \rightarrow \mathbb{R}$ and the matrix-valued $A : B_2 \rightarrow \text{Sym}_d^+$ to be as follows:

- $f : B_2 \rightarrow \mathbb{R}$ is bounded and continuous;
- $A : B_2 \rightarrow \text{Sym}_d^+$ is coercive, bounded and has C^1 -regular coefficients.

Before we state the result, we recall that a Lipschitz continuous nonnegative function $u : \mathbb{R}^d \supset B_1 \rightarrow \mathbb{R}$ is a viscosity solution to

$$-\text{div}(A\nabla u) = f \quad \text{in } \Omega_u \cap B_1, \quad |A^{1/2}[\nabla u]| = g \quad \text{on } \partial\Omega_u \cap B_1, \quad (3.A.1)$$

if the first equation holds in the open set Ω_u and if, for every $x_0 \in \partial\Omega_u$ and every $\varphi \in C^\infty(\mathbb{R}^d)$ touching $u \circ F_{x_0}$ from above (below) at zero, we have that $|\nabla\varphi|(0) \geq g(x_0)$ (resp. $|\nabla\varphi|(0) \leq g(x_0)$). Recall that *touching from above (below)* means that $\varphi(0) = 0$ and $\varphi \geq u \circ F_{x_0}$ (resp. $\varphi \leq u \circ F_{x_0}$) in $\Omega_u \cap B_1$. Moreover, we suppose that g is Hölder continuous and that there are constants $\eta_g > 0$, $C_g > 0$ and $\delta_g > 0$ such that

$$\begin{cases} |g(x) - g(y)| \leq C_g |x - y|^{\delta_g} & \text{for every } x, y \in \partial\Omega_u \cap B_1, \\ \eta_g \leq g(x) & \text{for every } x \in \partial\Omega_u \cap B_1. \end{cases} \quad (3.A.2)$$

The following result follows immediately from the results proved in [36].

Theorem 3.A.1 (Flat free boundaries are $C^{1,\alpha}$). *Suppose that $u : B_1 \rightarrow \mathbb{R}$ is a viscosity solution of (3.A.1) and that $g : \partial\Omega_u \rightarrow \mathbb{R}$ satisfies (3.A.2). Then, there exist $\varepsilon > 0$ and $\rho > 0$ such that if $x_0 \in \partial\Omega_u \cap B_1$ and u is such that*

$$g(x_0) \max\{0, x \cdot \nu - \varepsilon\rho\} \leq u \circ F_{x_0}(x) \leq g(x_0) \max\{0, x \cdot \nu + \varepsilon\rho\} \quad \text{for every } x \in B_\rho,$$

then $\partial\Omega_u$ is $C^{1,\alpha}$ in $B_{\rho/2}(x_0)$.

Remark 3.A.2. Notice that since in dimension two all the blow-up limits of u_+ (given by Theorem 3.1.5) are half-plane solutions (Proposition 3.4.3), we have that the flatness assumption of the above Theorem is satisfied at every point of the free boundary $\partial\Omega_u^+$. We also notice that, in our case, we have $g = \mu_+$, which is Hölder continuous by Lemma 3.6.4.

Definition 3.A.3 (Flatness). Let $u : B_1 \rightarrow \mathbb{R}$ be continuous, $u \geq 0$ and $u \in H^1(B_1)$. We say that u is (ε, ν) -flat, if there are a matrix-valued $A = (a_{ij})_{ij} : B_1 \rightarrow \text{Sym}_d^+$ with Hölder continuous coefficients, and a continuous $f : B_1 \rightarrow \mathbb{R}$ such that:

$$-\sum_{i,j} a_{ij} \partial_{ij} u = f \quad \text{in } \Omega_u \cap B_1; \quad (3.A.3)$$

$$\|f\|_{L^\infty(B_1)} \leq \varepsilon \quad \text{and} \quad \|a_{ij} - \delta_{ij}\|_{L^\infty(B_1)} \leq \varepsilon^2 \quad \text{for every } 1 \leq i, j \leq d; \quad (3.A.4)$$

$$1 - \varepsilon^2 \leq |\nabla u| \leq 1 + \varepsilon^2 \quad \text{on } \partial\Omega_u \cap B_1; \quad (3.A.5)$$

$$\max\{0, x \cdot \nu - \varepsilon\} \leq u(x) \leq \max\{0, x \cdot \nu + \varepsilon\} \quad \text{for every } x \in B_1. \quad (3.A.6)$$

Remark 3.A.4. The condition (3.A.5) is intended in a viscosity sense, that is, for any $\varphi \in C^\infty(B_1)$, we have:

- if $\varphi(x_0) = u(x_0)$ for some $x_0 \in \partial\Omega_u \cap B_1$ and $\varphi^+ \geq u$ in $\Omega_u \cap B_1$, then $|\nabla\varphi(x_0)| \geq 1 - \varepsilon^2$;
- if $\varphi(x_0) = u(x_0)$ for some $x_0 \in \partial\Omega_u \cap B_1$ and $\varphi \leq u$ in $\Omega_u \cap B_1$, then $|\nabla\varphi(x_0)| \leq 1 + \varepsilon^2$.

In order to prove Theorem 3.A.1 one has to show that the flatness improves at lower scales, that is, if u is (ε, ν) -flat, then a rescaling u_r of u is $(\varepsilon/2, \nu')$ -flat for some ν' , which is close to ν . Of course, the essential (and hardest) part of the proof is to show the improvement of the geometric flatness (3.A.6). This was proved by De Silva in [36, Lemma 4.1].

Lemma 3.A.5 (Improvement of the geometric flatness). *There are universal constants $C > 0$, $r_0 > 0$ and $\varepsilon_0 > 0$ such that if u is ε -flat in the direction ν , for some $\varepsilon \in (0, \varepsilon_0)$ and $\nu \in \partial B_1$, then, for every $r \in (0, r_0)$ there is some $\nu' \in \partial B_1$ such that $|\nu - \nu'| \leq C\varepsilon^2$ and*

$$\max\left\{0, x \cdot \nu' - \frac{\varepsilon}{2}\right\} \leq u_r(x) \leq \max\left\{0, x \cdot \nu' + \frac{\varepsilon}{2}\right\} \quad \text{for every } x \in B_1,$$

where $u_r : B_1 \rightarrow \mathbb{R}$ is the one-homogeneous rescaling $u_r(x) = \frac{u(rx)}{r}$.

Proof of Theorem 3.A.1. We will first prove that the flatness condition (3.A.3)-(3.A.6) improves at smaller scales. We fix $x_0 \in \partial\Omega_u \cap B_1$ and we consider the function $\tilde{u} = \frac{1}{g(x_0)} u \circ F_{x_0}$ (recall that $F_{x_0}(x) = x_0 + A_{x_0}^{1/2}[x]$). Let ε and r_0 be the constants from Lemma 3.A.5. We will prove that there is $r_1 \leq r_0$ such that: if \tilde{u} is (ε, ν) -flat, then for every $r \leq r_1$, \tilde{u}_r is $(\varepsilon/2, \nu')$ -flat, for ν' given again by Lemma 3.A.5. It is sufficient that the conditions (3.A.3), (3.A.4) and (3.A.5) are satisfied for \tilde{u}_r with the flatness parameter $\varepsilon/2$. We notice that \tilde{u} is a viscosity solution of

$$-\text{div}(\tilde{A}\nabla\tilde{u}) = \tilde{f} \quad \text{in } \Omega_{\tilde{u}}, \quad |\tilde{A}^{1/2}[\nabla\tilde{u}]| = \tilde{g} \quad \text{on } \partial\Omega_{\tilde{u}}, \quad (3.A.7)$$

where $\tilde{A}_x = A_{x_0}^{-1/2} A_{F_{x_0}(x)} A_{x_0}^{-1/2}$, $\tilde{f} = \frac{1}{g(x_0)} f \circ F_{x_0}$, $\tilde{g} = \frac{1}{g(x_0)} g \circ F_{x_0}$ and $\tilde{A}_x^{1/2} = A_{F_{x_0}(x)}^{1/2} \circ A_{x_0}^{-1/2}$.

Notice that $0 \in \partial\Omega_{\tilde{u}}$ and set $\tilde{u}_r(x) := \frac{\tilde{u}(rx)}{r}$. Thus, for small enough $r > 0$, \tilde{u}_r is a viscosity solution of

$$-\text{div}(\tilde{A}_r \nabla \tilde{u}_r) = \tilde{f}_r \quad \text{in } \Omega_{\tilde{u}} \cap B_1, \quad |\tilde{A}_r^{1/2}[\nabla \tilde{u}_r]| = \tilde{g}_r \quad \text{on } \partial\Omega_{\tilde{u}} \cap B_1, \quad (3.A.8)$$

where $\tilde{A}_r(x) := \tilde{A}(rx)$, $\tilde{f}_r(x) = r\tilde{f}(rx)$, $\tilde{g}_r(x) = \tilde{g}(rx)$ and $\tilde{A}_r^{1/2}(x) = A_{F_{x_0}(rx)}^{1/2} \circ A_{x_0}^{-1/2}$.

We set $\tilde{a}_{ij}^r(x)$ to be the coefficients of $\tilde{A}_r(x)$ and \tilde{b}^r to be the vector with coefficients $\tilde{b}_i^r = \sum_j \partial_j \tilde{a}_{ij}^r(x)$. Then, (3.A.8) can be equivalently written as

$$-\sum_{i,j} \tilde{a}_{ij}^r \partial_{ij} \tilde{u}_r = \tilde{b}^r \cdot \nabla \tilde{u}_r + \tilde{f}_r \quad \text{in } \Omega_{\tilde{u}} \cap B_1, \quad |\tilde{A}_r^{1/2}[\nabla \tilde{u}_r]| = \tilde{g}_r \quad \text{on } \partial\Omega_{\tilde{u}} \cap B_1, \quad (3.A.9)$$

Now, if \tilde{u} is (ε, ν) -flat, then the Hölder continuity of the coefficients a_{ij} and the boundedness of f imply that (3.A.4) holds with $\varepsilon/2$ and \tilde{u}_r , for any $r \leq r_1$, where $r_1 \leq r_0$, is a universal constant depending on the Hölder norm of a_{ij} . Now, in order to get (3.A.5) for $\varepsilon/2$ and \tilde{u}_r , we suppose that $\varphi \in C^\infty(B_1)$ touches \tilde{u}_r from below at a point $y_0 \in B_1 \cap \partial\{\tilde{u}_r > 0\}$. Thus, we have that

$$\left| A_{F_{x_0}(ry_0)}^{1/2} \circ A_{x_0}^{-1/2} [\nabla \varphi(y_0)] \right| \leq \frac{g(F_{x_0}(ry_0))}{g(x_0)},$$

and so, if $\|\cdot\| = \|\cdot\|_{\mathcal{L}(\mathbb{R}^d)}$ stands for the space of $d \times d$ matrices, we have

$$|\nabla \varphi(y_0)| \leq \left\| A_{x_0}^{1/2} \circ A_{F_{x_0}(ry_0)}^{-1/2} \right\| \frac{g(F_{x_0}(ry_0))}{g(F_{x_0}(0))}.$$

Now, by the Hölder continuity (and the uniform boundedness from below) of g , we can choose r_1 such that

$$\frac{g(F_{x_0}(ry_0))}{g(F_{x_0}(0))} \leq 1 + \frac{\varepsilon^2}{10}.$$

On the other hand, there are universal constants C and $\delta > 0$, depending only on the Hölder exponent δ_A and the norm C_A , of the matrix-valued function A , such that

$$\|A_{x_0}^{1/2} \circ A_{F_{x_0}(ry_0)}^{-1/2} - Id\| \leq \|A_{x_0}^{1/2} - A_{F_{x_0}(ry_0)}^{1/2}\| \cdot \|A_{F_{x_0}(ry_0)}^{-1/2}\| \leq C|ry_0|^\delta \leq Cr_1^\delta.$$

Choosing r_1 such that $Cr_1^\delta \leq \frac{\varepsilon^2}{10}$ and using the triangular inequality, we get

$$|\nabla \varphi(y_0)| \leq \left\| A_{x_0}^{1/2} \circ A_{F_{x_0}(ry_0)}^{-1/2} \right\| \frac{g(F_{x_0}(ry_0))}{g(F_{x_0}(0))} \leq \left(1 + \frac{\varepsilon^2}{10}\right)^2 \leq 1 + (\varepsilon/2)^2,$$

which completes the proof of the improvement of flatness for \tilde{u} , the case when φ touches from above being analogous. Now, the claim follows by a standard argument, similar to the one we used in Section 3.5. \square

Chapter 4

Lipschitz continuity of the eigenfunctions on optimal sets for functionals with variable coefficients

This chapter is based on [87].

Abstract

This paper is dedicated to the spectral optimization problem

$$\min \{ \lambda_1(\Omega) + \cdots + \lambda_k(\Omega) + \Lambda|\Omega| : \Omega \subset D \text{ quasi-open} \}$$

where $D \subset \mathbb{R}^d$ is a bounded open set and $0 < \lambda_1(\Omega) \leq \cdots \leq \lambda_k(\Omega)$ are the first k eigenvalues on Ω of an operator in divergence form with Dirichlet boundary condition and Hölder continuous coefficients. We prove that the first k eigenfunctions on an optimal set for this problem are locally Lipschitz continuous in D and, as a consequence, that the optimal sets are open sets. We also prove the Lipschitz continuity of vector-valued functions that are almost-minimizers of a two-phase functional with variable coefficients.

4.1 Introduction and main results

Let D be a bounded open subset of \mathbb{R}^d and Λ be a positive constant. We consider the spectral optimization problem

$$\min \{ \lambda_1(\Omega) + \cdots + \lambda_k(\Omega) + \Lambda|\Omega| : \Omega \subset D \text{ quasi-open} \} \quad (4.1.1)$$

where $0 < \lambda_1(\Omega) \leq \cdots \leq \lambda_k(\Omega)$ denote the first k eigenvalues, counted with the due multiplicity, of the operator in divergence form $-b(x)^{-1} \operatorname{div}(A_x \nabla \cdot)$. This means that for every $\lambda_i(\Omega)$ there is an eigenfunction $u_i \in H_0^1(\Omega)$ such that

$$\begin{cases} -\operatorname{div}(A \nabla u_i) = \lambda_i(\Omega) b u_i & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.1.2)$$

The aim of the present paper is twofold. From one side, we prove a Lipschitz regularity result for vector-valued functions which are almost-minimizers of a two-phase functional with

variable coefficients (Theorem 4.1.2). On the other hand, we show that if Ω^* is an optimal set for (4.1.1), then the vector $U = (u_1, \dots, u_k)$ of the first k eigenfunctions on Ω^* satisfies the almost-minimality condition of Theorem 4.1.2, and hence that the eigenfunctions u_1, \dots, u_k are Lipschitz continuous.

We first state our Lipschitz regularity result for eigenfunctions on optimal sets for (4.1.1).

Theorem 4.1.1. *Let $D \subset \mathbb{R}^d$ be a bounded open set and let $\Lambda > 0$. Let $A : D \rightarrow \text{Sym}_d^+$ be a matrix valued function satisfying (4.1.5) and (4.1.6) and let $b \in L^\infty(D)$ be a function satisfying (4.1.7) (see below). Then the spectral optimization problem (4.1.1) admits a solution Ω^* . Moreover, the first k eigenfunctions on any optimal set Ω^* are locally Lipschitz continuous in D . As a consequence, every optimal set for (4.1.1) is an open set.*

In [10], Briancon, Hayouni and Pierre proved the Lipschitz continuity of the first eigenfunction on an optimal set which minimizes the first eigenvalue of the Dirichlet Laplacian among all sets of prescribed volume included in a box. Their proof, which is inspired by the pioneering work of Alt and Caffarelli in [2] on the regularity for a free boundary problem, relies on the fact that the first eigenfunction is the minimum of a variational problem. For spectral optimization problems involving higher eigenvalues, the study of the regularity of the optimal sets and the corresponding eigenfunctions is more involved due to the variational characterization of the eigenvalue λ_k through a min-max procedure. In [18] the authors considered the spectral functionals $F(\lambda_1(\Omega), \dots, \lambda_k(\Omega))$ which are bi-Lipschitz with respect to each eigenvalue $\lambda_i(\Omega)$ of the Dirichlet Laplacian, a typical example being the sum of the first k eigenvalues. In particular, they proved the Lipschitz continuity of the eigenfunctions on optimal sets minimizing the sum $\lambda_1(\Omega) + \dots + \lambda_k(\Omega)$ among all shapes $\Omega \subset \mathbb{R}^d$ of prescribed measure (see [18, Theorem 6.1]). The present paper extends this result to the case of an operator with variable coefficients, but with a completely different proof.

Concerning spectral optimization problems involving an operator with variable coefficients, a regularity result has been obtained in [84], where the authors consider the problem of minimizing the first eigenvalue of the operator with drift $-\Delta + \nabla \Phi \cdot \nabla$, $\Phi \in W^{1,\infty}(D, \mathbb{R}^d)$, under inclusion and volume constraints. We stress out that our result also applies to this operator with drift since it corresponds to the special case where $A = e^{-\Phi} \text{Id}$ and $b = e^{-\Phi}$. We would like also to mention a recent work of Lamboley and Sicbaldi in [73] where they prove an existence and regularity result for Faber-Krahn minimizers in a Riemannian setting.

Let us highlight that the Lipschitz regularity of the eigenfunctions in Theorem 4.1.1 turned out to be a quite difficult question due to both the min-max nature of the eigenvalues and the presence of the variable coefficients, but it is an important first step for the analysis of the regularity of the free boundary of the optimal shapes for (4.1.1) which we study in [88].

As already pointed out, the proof of Theorem 4.1.1 goes through the study of the Lipschitz regularity of vector-valued almost-minimisers for a two-phase functional with variable coefficients. Our approach is to reduce from the non-constant coefficients case to the constant coefficients-one by a change of variables and is inspired by [85], where the authors prove free boundary regularity of almost-minimizers of the one-phase and two-phase functionals in dimension 2 using an isoperimetric inequality. The second contribution which was a strong inspiration for our work is of David and Toro in [34]. They in particular prove the Lipschitz regularity of almost-minimizers of the one-phase and the two-phase functionals with constant coefficients (see also [33] for free boundary regularity results).

We have the following result for almost-minimizers of the two-phase functional.

Theorem 4.1.2. *Let $D \subset \mathbb{R}^d$ be a bounded open set and let $\Omega \subset D$ be a quasi-open set. Let $A : D \rightarrow \text{Sym}_d^+$ be a matrix valued function satisfying (4.1.5) and (4.1.6). Let $f = (f_1, \dots, f_k) \in L^\infty(D, \mathbb{R}^k)$. Assume that $U = (u_1, \dots, u_k) \in H_0^1(\Omega, \mathbb{R}^k)$ is a vector-valued function such that*

- U is a solution of the equation

$$-\operatorname{div}(A\nabla U) = f \quad \text{in } \Omega, \quad (4.1.3)$$

- U satisfies the following quasi-minimality condition: for every $C_1 > 0$, there exist constants $\varepsilon \in (0, 1)$ and $C > 0$ such that

$$\int_D A\nabla U \cdot \nabla U + \Lambda\{|U| > 0\} \leq (1 + C\|U - \tilde{U}\|_{L^1}) \int_D A\nabla \tilde{U} \cdot \nabla \tilde{U} + \Lambda\{|\tilde{U}| > 0\}, \quad (4.1.4)$$

for every $\tilde{U} \in H_0^1(D, \mathbb{R}^k)$ such that $\|U - \tilde{U}\|_{L^1} \leq \varepsilon$ and $\|\tilde{U}\|_{L^\infty} \leq C_1$.

Then the vector-valued function U is locally Lipschitz continuous in D .

Remark 4.1.3 (On the assumption (4.1.4) of Theorem 4.1.2). The quasi-minimality in Theorem 4.1.2 is not local but naturally arises from the shape optimization problem (4.1.1) (see Proposition 4.3.4). We stress out that our conclusion also holds, with exactly the same proof, if the quasi-minimality property (4.1.4) is replaced by its "local" version, namely: for every $C_1 > 0$, there exist constants $r_0 \in (0, 1)$ and $C > 0$ such that for every $x \in D$ and every $r \leq r_0$ such that $B_r(x) \subset D$ we have

$$\int_{B_r(x)} A\nabla U \cdot \nabla U + \Lambda\{|U| > 0\} \cap B_r(x) \leq (1 + C\|U - \tilde{U}\|_{L^1}) \int_{B_r(x)} A\nabla \tilde{U} \cdot \nabla \tilde{U} + \Lambda\{|\tilde{U}| > 0\} \cap B_r(x),$$

for every $\tilde{U} \in H_0^1(D, \mathbb{R}^k)$ such that $U - \tilde{U} \in H_0^1(B_r(x), \mathbb{R}^k)$ and $\|\tilde{U}\|_{L^\infty} \leq C_1$.

Remark 4.1.4. We point out that we will only use the assumption (4.1.3) to prove that U is bounded and to get an almost-monotonicity formula (see Proposition 4.2.15 and Corollary 4.2.16).

In [34, Theorem 6.1], David and Toro proved an almost-monotonicity formula for quasi-minimizers in the case of the Laplacian. It is natural to expect that the same holds for an operator with variable coefficients, but we will not address this question in the present paper since we are mainly interested in the Lipschitz continuity of the eigenfunctions on optimal shapes for the problem (4.1.1) for which the equation (4.1.3) is already known.

However, soon before the present paper was published online, a new preprint of the same authors, in collaboration with Engelstein and Smit Vega Garcia (see [32]), appeared on Arxiv. They prove a regularity result for functions satisfying a suitable quasi-minimality condition for operators with variable coefficients. We stress that the present paper and the work in [32] were done in a completely independent way. We notice that our main result neither directly implies nor is directly implied by the main result from [32].

Notations. Let us start by setting the assumptions on the coefficients of the operator that we will use throughout this paper. The matrix-valued function $A = (a_{ij})_{ij} : D \rightarrow \operatorname{Sym}_d^+$ has Hölder continuous coefficients and is uniformly elliptic, where Sym_d^+ denotes the family of all real positive symmetric $d \times d$ matrices. Precisely, there exist positive constants $\delta_A, c_A > 0$ and $\lambda_A \geq 1$ such that

$$|a_{ij}(x) - a_{ij}(y)| \leq c_A |x - y|^{\delta_A}, \quad \text{for every } i, j \quad \text{and } x, y \in D; \quad (4.1.5)$$

$$\frac{1}{\lambda_A^2} |\xi|^2 \leq \xi \cdot A_x \xi = \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \leq \lambda_A^2 |\xi|^2, \quad \text{for every } x \in D \quad \text{and } \xi \in \mathbb{R}^d. \quad (4.1.6)$$

The function $b \in L^\infty(D)$ is positive and bounded away from zero: there exists $c_b > 0$ such that

$$c_b^{-1} \leq b(x) \leq c_b \quad \text{for almost every } x \in D. \quad (4.1.7)$$

We now fix some notations and conventions. For $x \in \mathbb{R}^d$ and $r > 0$ we use the notation $B_r(x)$ to denote the ball centred at x of radius r and we simply write B_r if $x = 0$. We denote by $|\Omega|$ the Lebesgue measure of a generic set $\Omega \subset \mathbb{R}^d$ and by ω_d the Lebesgue measure of the unit ball $B_1 \subset \mathbb{R}^d$. The $(d - 1)$ -dimensional Hausdorff measure is denoted by \mathcal{H}^{d-1} . Moreover, we define the positive and the negative parts of a function $u : \mathbb{R} \rightarrow \mathbb{R}$ by

$$u^+ = \max(u, 0) \quad \text{and} \quad u^- = \max(-u, 0).$$

For a quasi-open set $\Omega \in \mathbb{R}^d$ we denote by $H_0^1(\Omega)$ the Sobolev space defined as the set of functions $u \in H^1(\mathbb{R}^d)$ which, up to a set of capacity zero, vanishes outside Ω ; that is

$$H_0^1(\Omega) = \{u \in H^1(\mathbb{R}^d) : u = 0 \text{ quasi-everywhere in } \mathbb{R}^d \setminus \Omega\}.$$

(see e.g. [61] for a definition of the capacity). Notice that if Ω is an open set, then $H_0^1(\Omega)$ is the usual Sobolev space defined as the closure of the smooth real-valued functions with support compact $C_c^\infty(\Omega)$ with respect to the norm $\|u\|_{H^1} = \|u\|_{L^2} + \|\nabla u\|_{L^2}$. We denote by $H_0^1(\Omega, \mathbb{R}^k)$ the space of vector-valued functions $U = (u_1, \dots, u_k) : \Omega \rightarrow \mathbb{R}^k$ such that $u_i \in H_0^1(\Omega)$ for every $i = 1, \dots, k$, and endowed with the norm

$$\|U\|_{H^1(\Omega)} = \|U\|_{L^2(\Omega)} + \|\nabla U\|_{L^2(\Omega)} = \sum_{i=1}^k (\|u_i\|_{L^2(\Omega)} + \|\nabla u_i\|_{L^2(\Omega)}).$$

We also define the following norms (whenever it makes sense)

$$\|U\|_{L^1(\Omega)} = \sum_{i=1}^k \|u_i\|_{L^1(\Omega)} \quad \text{and} \quad \|U\|_{L^\infty(\Omega)} = \sup_{1 \leq i \leq k} \|u_i\|_{L^\infty(\Omega)}.$$

Moreover, for $U = (u_1, \dots, u_k) : \Omega \rightarrow \mathbb{R}^k$ we set $|U|^2 = u_1^2 + \dots + u_k^2$, $|\nabla U|^2 = |\nabla u_1|^2 + \dots + |\nabla u_k|^2$ and $A\nabla U \cdot \nabla U = A\nabla u_1 \cdot \nabla u_1 + \dots + A\nabla u_k \cdot \nabla u_k$. For $f = (f_1, \dots, f_k) \in L^2(\Omega, \mathbb{R}^k)$ we say that $U = (u_1, \dots, u_k) \in H_0^1(\Omega, \mathbb{R}^k)$ is solution to the equation

$$-\operatorname{div}(A\nabla U) = f \quad \text{in } \Omega, \quad U \in H_0^1(\Omega, \mathbb{R}^k)$$

if, for every $i = 1, \dots, k$, the component u_i is solution to the equation

$$-\operatorname{div}(A\nabla u_i) = f_i \quad \text{in } \Omega, \quad u_i \in H_0^1(\Omega),$$

where the PDE is intended in the weak sense, that is

$$\int_{\Omega} A\nabla u_i \cdot \nabla \varphi = \int_{\Omega} f_i \varphi \quad \text{for every } \varphi \in H_0^1(\Omega).$$

Moreover, we always extend functions of the spaces $H_0^1(\Omega)$ and $H_0^1(\Omega, \mathbb{R}^k)$ by zero outside Ω so that we have the inclusions $H_0^1(\Omega) \subset H^1(\mathbb{R}^d)$ and $H_0^1(\Omega, \mathbb{R}^k) \subset H^1(\mathbb{R}^d, \mathbb{R}^k)$.

4.2 Lipschitz continuity of quasi-minimizers

This section is dedicated to the proof of Theorem 4.1.2. Our approach is to locally freeze the coefficients to reduce to the case where $A = Id$. More precisely, for every point $x \in D$, an almost-minimizer of the functional with variable coefficients becomes, in a new set of coordinates near x , an almost minimizer for a functional with constant coefficients. We stress out the dealing with the dependence of this change of variables with respect to the point x is not a trivial task. We then adapt the strategy developed by David and Toro in [34] for almost-minimizers of a functional involving the Dirichlet energy.

In this section, u will stand for a coordinate function of the vector U from Theorem 4.1.2. In subsection 4.2.1 we explicit the change of variables for which u becomes a quasi-minimizer of the Dirichlet energy (in small balls of fixed center). We then prove that u is continuous and we give an estimate of the modulus of continuity from which we deduce that u is locally Hölder continuous in D .

Subsection 4.2.2 is addressed to the Lipschitz continuity of u in some region where the function u has a given sign. We show, using in particular the Hölder continuity of u , that most of the estimates proved in Subsection 4.2.1 can be improved provided that u keeps the same sign. In this case, we prove that u is Lipschitz continuous and we provide a bound on the Lipschitz constant of u . We also show that u is $C^{1,\beta}$ -regular for some $\beta \in (0, 1)$. Next, we show that under some assumption (see the first inequality in (4.2.45) from Proposition 4.2.11), if the Dirichlet energy of u in a small ball is big enough, then u keeps the same sign in a smaller ball, which in view of the preceding analysis implies that u is Lipschitz continuous.

In subsection 4.2.3 we complete the proof of the Lipschitz continuity of u . The main missing step is to deal with the case where the Dirichlet energy is big and the first assumption of (4.2.45) in Proposition 4.2.11 fails. Using an almost-monotonicity formula for operators with variable coefficients proved by Matevosyan and Petrosyan in [75, Theorem III], we show that in this case the value of the Dirichlet energy has to decrease at some smaller scale.

Throughout this section we fix $u := u_i$, for some $i = 1, \dots, k$, a coordinate function of the vector $U = (u_1, \dots, u_k)$ from Theorem 4.1.2. We start by proving that u is a bounded function in D .

Lemma 4.2.1 (Boundedness). *Let $\Omega \subset D$ be a (non-empty) quasi-open set, $f \in L^p(D)$ for some $p \in (d/2, +\infty]$ and let $u \in H_0^1(\Omega)$ be the solution of*

$$-\operatorname{div}(A\nabla u) = f \quad \text{in } \Omega, \quad u \in H_0^1(\Omega).$$

Then, there is a dimensional constant C_d such that

$$\|u\|_{L^\infty} \leq \frac{\lambda_A^2 C_d}{2/d - 1/p} |\Omega|^{2/d - 1/p} \|f\|_{L^p}.$$

Proof. Up to arguing with the positive and the negative parts of f , we can assume that f is a non-negative function. By the maximum principle (see [48, Theorem 8.1]) we have $u \geq 0$ on Ω . Moreover, u is a minimum of the following functional

$$J(\varphi) := \frac{1}{2} \int_{\Omega} A\nabla\varphi \cdot \nabla\varphi - \int_{\Omega} f\varphi, \quad \varphi \in H_0^1(\Omega).$$

We consider, for every $0 < t < \|u\|_{L^\infty}$ and $\varepsilon > 0$, the test function $u_{t,\varepsilon} = u \wedge t + (u - t - \varepsilon)_+ \in H_0^1(\Omega)$. Then, by ellipticity of the matrices A_x and the inequality $J(u) \leq J(u_{t,\varepsilon})$ we get that

$$\begin{aligned} \frac{1}{2\lambda_A^2} \int_{\{t < u \leq t+\varepsilon\}} |\nabla u|^2 &\leq \frac{1}{2} \int_{\{t < u \leq t+\varepsilon\}} A\nabla u \cdot \nabla u \leq \int_{\mathbb{R}^d} f(u - u_{t,\varepsilon}) \\ &\leq \varepsilon \int_{\{u > t\}} f \leq \varepsilon \|f\|_{L^p} |\{u > t\}|^{\frac{p-1}{p}}, \end{aligned}$$

The end of the proof now follows precisely as in [84, Lemma 5.3]. \square

4.2.1 Continuity and Hölder continuity

We change the coordinates and reduce to the case $A = \operatorname{Id}$ using in particular the Hölder continuity of the coefficients of A , and we then prove that u is locally Hölder continuous in D .

Let us first introduce few notations that we will use throughout this section. For $x \in D$ we define the function $F_x : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by

$$F_x(\xi) := x + A_x^{1/2}[\xi], \quad \xi \in \mathbb{R}^d.$$

Moreover, we set $u_x = u \circ F_x$ for every $x \in D$.

Remark 4.2.2. For $M \in \text{Sym}_d^+$ we denote by $M^{1/2}$ the square root matrix of M . We recall that if $M \in \text{Sym}_d^+$, then there is an orthogonal matrix P such that $PM P^t = \text{diag}(\lambda_1, \dots, \lambda_d)$, where P^t is the transpose of P and $\text{diag}(\lambda_1, \dots, \lambda_d)$ is the diagonal matrix with eigenvalues $\lambda_1, \dots, \lambda_d$. The matrix $M^{1/2}$ is then defined by $M^{1/2} := P^t D P$ where $D = \text{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_d})$.

Remark 4.2.3 (Notation of the harmonic extension). One of the main ingredient in the proof of Theorem 4.1.2 is based on small variations of the function u_x . Precisely, we will often compare u_x in some ball B_r with the harmonic extension of the trace of u_x to ∂B_r . This function will often be denoted by $h_{x,r}$, or more simply h_r if there is no confusion, and is defined by $h_r = h_{x,r} \in H^1(B_r)$ and

$$\Delta h_r = 0 \quad \text{in } B_r, \quad u_x - h_r \in H_0^1(B_r).$$

We notice that h_r is a minimizer of the Dirichlet energy in the ball B_r , that is

$$\int_{B_r} |\nabla h_r|^2 \leq \int_{B_r} |\nabla v|^2 \quad \text{for every } v \in H^1(B_r) \text{ such that } h_r - v \in H_0^1(B_r).$$

We now prove that the function u_x is in some sense a quasi-minimizer for the Dirichlet energy in small balls centred at the origin.

Proposition 4.2.4. *There exist constants $r_0 \in (0, 1)$ and $C > 0$ such that, if $x \in D$ and $r \leq r_0$ satisfy $B_{\lambda_A r}(x) \subset D$, then we have*

$$\int_{B_r} |\nabla u_x|^2 \leq (1 + C r^{\delta_A}) \int_{B_r} |\nabla \tilde{u}|^2 + C r^d, \quad (4.2.1)$$

for every $\tilde{u} \in H^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ such that $u_x - \tilde{u} \in H_0^1(B_r)$ and $\|\tilde{u}\|_{L^\infty} \leq \|u\|_{L^\infty}$.

Proof. Let $v \in H_0^1(D)$ be such that $\tilde{u} = v \circ F_x$ and set $\tilde{U} = (u_1, \dots, v, \dots, u_k) \in H_0^1(D, \mathbb{R}^k)$, where v stands at the i -th position. Set $\rho = \lambda_A r$ and note that $F_x(B_r) \subset B_\rho(x) \subset D$. Then, using \tilde{U} as a test function and observing that $u - v \in H_0^1(F_x(B_r))$, we get

$$\int_{F_x(B_r)} A \nabla u \cdot \nabla u \leq \int_{F_x(B_r)} A \nabla v \cdot \nabla v + C \|u - v\|_{L^1} \int_D A \nabla \tilde{U} \cdot \nabla \tilde{U} + \Lambda |B_\rho|,$$

where C is the constant from Theorem 4.1.2. Together with

$$\begin{aligned} \int_D A \nabla \tilde{U} \cdot \nabla \tilde{U} &\leq \int_D A \nabla U \cdot \nabla U - \int_D A \nabla u \cdot \nabla u + \int_D A \nabla v \cdot \nabla v \\ &\leq \int_D A \nabla U \cdot \nabla U + \int_{F_x(B_r)} A \nabla v \cdot \nabla v \end{aligned}$$

this yields

$$\int_{F_x(B_r)} A \nabla u \cdot \nabla u \leq (1 + \tilde{C} r^d) \int_{F_x(B_r)} A \nabla v \cdot \nabla v + \tilde{C} r^d, \quad (4.2.2)$$

for some constant \tilde{C} . On the other hand, using the Hölder continuity and the ellipticity of A we estimate

$$\begin{aligned} \int_{B_r} |\nabla u_x|^2 &= \det(A_x^{-1/2}) \int_{F_x(B_r)} A_x \nabla u \cdot \nabla u \\ &\leq \det(A_x^{-1/2}) (1 + d c_A \lambda_A^2 \rho^{\delta_A}) \int_{F_x(B_r)} A \nabla u \cdot \nabla u. \end{aligned} \quad (4.2.3)$$

Similarly, we have the following estimate from below

$$\int_{B_r} |\nabla \tilde{u}|^2 \geq \det(A_x^{-1/2})(1 - dc_A \lambda_A^2 \rho^{\delta_A}) \int_{F_x(B_r)} A \nabla v \cdot \nabla v. \quad (4.2.4)$$

Now, combining (4.2.3), (4.2.2) and (4.2.4) we get

$$\int_{B_r} |\nabla u_x|^2 \leq (1 + dc_A \lambda_A^2 \rho^{\delta_A}) \left[\frac{1 + \tilde{C}r^d}{1 - dc_A \lambda_A^2 \rho^{\delta_A}} \int_{B_r} |\nabla \tilde{u}|^2 + \lambda_A^d \tilde{C}r^d \right].$$

which gives (4.2.2). \square

We now prove that the function u is continuous in D . In the sequel we will often use the following notation: for $x \in D$ and $r > 0$ we set

$$\omega(u, x, r) = \left(\int_{B_r(x)} |\nabla u|^2 \right)^{1/2} \quad \text{and} \quad \omega(u_x, r) = \left(\int_{B_r} |\nabla u_x|^2 \right)^{1/2}.$$

Proposition 4.2.5. *The function u is continuous in D . Moreover, there exist $r_0 > 0$ and $C > 0$ such that, if $x \in D$ and $r \leq r_0$ satisfy $B_r(x) \subset D$, then we have*

$$|u(y) - u(z)| \leq C \left(1 + \omega(u, x, r) + \log \frac{r}{|y - z|} \right) |y - z| \quad \text{for every } y, z \in B_{r/2}(x). \quad (4.2.5)$$

The next Lemma shows that $\omega(u_x, r)$ cannot grow too fast as r tends to zero and will be useful throughout the proof of the Lipschitz continuity of u .

Lemma 4.2.6. *There exist constants $r_0 > 0$ and $C > 0$ such that, if $x \in D$ and $r \leq r_0$ satisfy $B_{\lambda_A r}(x) \subset D$, then we have*

$$\omega(u_x, s) \leq C \omega(u_x, r) + C \log \left(\frac{r}{s} \right) \quad \text{for every } 0 < s \leq r. \quad (4.2.6)$$

If, moreover, x is a Lebesgue point for u , then we have

$$\left| u(x) - \int_{B_r} u_x \right| \leq Cr(1 + \omega(u_x, r)). \quad (4.2.7)$$

Proof. Let $t \leq r$ and use h_t as a test function in (4.2.1), where $h_t = h_{x,t}$ denotes the harmonic extension in B_t of the trace of u_x to ∂B_t , to get

$$\begin{aligned} \int_{B_t} |\nabla(u_x - h_t)|^2 &= \int_{B_t} |\nabla u_x|^2 - \int_{B_t} |\nabla h_t|^2 \\ &\leq Ct^{\delta_A} \int_{B_t} |\nabla h_t|^2 + Ct^d \leq Ct^{\delta_A} \int_{B_t} |\nabla u_x|^2 + Ct^d, \end{aligned} \quad (4.2.8)$$

where in the last inequality we have used that h_t is a minimizer of the Dirichlet energy on B_t . Moreover, since $|\nabla h_t|$ is subharmonic on B_s for every $s \leq t$, we have

$$\int_{B_s} |\nabla h_t|^2 \leq \int_{B_t} |\nabla h_t|^2 \quad \text{for every } s \leq t. \quad (4.2.9)$$

Therefore, the triangle inequality, (4.2.9) and (4.2.8) give for every $s \leq t \leq r_0$

$$\begin{aligned} \omega(u_x, s) &\leq \left(\int_{B_s} |\nabla(u_x - h_t)|^2 \right)^{1/2} + \left(\int_{B_s} |\nabla h_t|^2 \right)^{1/2} \\ &\leq \left(\frac{t}{s} \right)^{d/2} \left(\int_{B_t} |\nabla(u_x - h_t)|^2 \right)^{1/2} + \left(\int_{B_t} |\nabla h_t|^2 \right)^{1/2} \\ &\leq \left(\frac{t}{s} \right)^{d/2} C \left(t^{\delta_A/2} \omega(u_x, t) + 1 \right) + \omega(u_x, t) \\ &\leq \left(1 + C \left(\frac{t}{s} \right)^{d/2} t^{\delta_A/2} \right) \omega(u_x, t) + C \left(\frac{t}{s} \right)^{d/2}. \end{aligned} \quad (4.2.10)$$

We then use the estimate (4.2.10) with the radii $r_i = 2^{-i}r$, $i \geq 0$, and we get

$$\omega(u_x, r_i) \leq \left(1 + Cr_i^{\delta_A/2}\right)\omega(u_x, r_{i-1}) + C, \quad i \geq 1.$$

This, with an iteration, implies that for every $i \geq 1$ we have

$$\begin{aligned} \omega(u_x, r_i) &\leq \omega(u_x, r) \prod_{j=0}^{i-1} \left(1 + Cr_j^{\delta_A/2}\right) + C \sum_{j=1}^{i-1} \prod_{l=j}^{i-1} \left(1 + Cr_l^{\delta_A/2}\right) + C \\ &\leq C\omega(u_x, r) + Ci, \end{aligned} \quad (4.2.11)$$

where we used that the product $\prod_{j=0}^{\infty} (1 + Cr_j^{\delta_A/2})$ is bounded by a constant depending on r_0 . The first estimate of the Lemma now follows from (4.2.11). Indeed, choose $i \geq 0$ such that $r_{i+1} < s \leq r_i$ and note that we have $\omega(u_x, s) \leq 2^{d/2}\omega(u_x, r_i)$. If $i = 0$, this directly implies (4.2.6); otherwise, $i \geq 1$ and use also (4.2.11).

We now prove the second estimate. For $i \geq 0$ we set $m_i = \int_{B_{r_i}} u_x$. By the Poincaré inequality and (4.2.11) we have

$$\left(\int_{B_{r_i}} |u_x - m_i|^2\right)^{1/2} \leq Cr_i\omega(u_x, r_i) \leq Cr_i(\omega(u_x, r) + i). \quad (4.2.12)$$

Furthermore, 0 is a Lebesgue point for u_x since x is a Lebesgue point for u and that for every $s \leq r$ we have

$$\lambda_A^{-2d} \int_{B_{\lambda_A^{-1}s}(x)} |u - u(x)| \leq \int_{B_s} |u_x - u_x(0)| = \int_{F_x(B_s)} |u - u(x)| \leq \lambda_A^{2d} \int_{B_{\lambda_A s}(x)} |u - u(x)|.$$

In particular, it follows that m_i converges to $u_x(0) = u(x)$ as $i \rightarrow +\infty$. Therefore, this with the Cauchy-Schwarz inequality and (4.2.12) give

$$\begin{aligned} |u(x) - m_i| &\leq \sum_{j=i}^{+\infty} |m_{j+1} - m_j| \leq \sum_{j=i}^{+\infty} \int_{B_{r_{j+1}}} |u_x - m_j| \\ &\leq 2^d \sum_{j=i}^{+\infty} \int_{B_{r_j}} |u_x - m_j| \leq 2^d \sum_{j=i}^{+\infty} \left(\int_{B_{r_j}} |u_x - m_j|^2\right)^{1/2} \\ &\leq C \sum_{j=i}^{+\infty} r_j(\omega(u_x, r) + j) \leq Cr_i(\omega(u_x, r) + i + 1), \end{aligned}$$

where in the last inequality we used that $\sum_{j=i}^{+\infty} 2^{i-j}j \leq C(i+1)$. Then, observe that (4.2.7) is precisely the above inequality with $i = 0$ to conclude the proof. \square

Proof of Proposition 4.2.5. Let $y, z \in B_{r/2}(x)$ and notice that it is enough to prove (4.2.5) when y and z are Lebesgue points for u . Set $\delta = |y - z|$. We first assume that $4\lambda_A^2\delta \leq r$. Observe that we hence have the inclusions $F_z(B_{\lambda_A^{-1}\delta}) \subset F_y(B_{2\lambda_A\delta}) \subset B_r(x) \subset D$. Using a change of variables, the Poincaré inequality and then the ellipticity of A , we estimate

$$\begin{aligned} \left|\int_{B_{2\lambda_A\delta}} u_y - \int_{B_{\lambda_A^{-1}\delta}} u_z\right| &= \left|\int_{F_y(B_{2\lambda_A\delta})} u - \int_{F_z(B_{\lambda_A^{-1}\delta})} u\right| \leq \int_{F_z(B_{\lambda_A^{-1}\delta})} \left|u - \int_{F_y(B_{2\lambda_A\delta})} u\right| \\ &\leq 2^d \lambda_A^{4d} \int_{F_y(B_{2\lambda_A\delta})} \left|u - \int_{F_y(B_{2\lambda_A\delta})} u\right| \leq C\delta \left(\int_{F_y(B_{2\lambda_A\delta})} |\nabla u|^2\right)^{1/2} \\ &\leq C\delta\lambda_A \left(\int_{F_y(B_{2\lambda_A\delta})} A_y \nabla u \cdot \nabla u\right)^{1/2} \leq C\delta\omega(u_y, 2\lambda_A\delta). \end{aligned} \quad (4.2.13)$$

On the other hand, since $F_z(B_{\lambda_A^{-1}\delta}) \subset F_y(B_{2\lambda_A\delta})$ we have

$$\begin{aligned} \omega(u_z, \lambda_A^{-1}\delta) &= \left(\int_{F_z(B_{\lambda_A^{-1}\delta})} A_z \nabla u \cdot \nabla u \right)^{1/2} \leq \lambda_A \left(\int_{F_z(B_{\lambda_A^{-1}\delta})} |\nabla u|^2 \right)^{1/2} \\ &\leq 2^{d/2} \lambda_A^{2d+1} \left(\int_{F_y(B_{2\lambda_A\delta})} |\nabla u|^2 \right)^{1/2} \leq 2^{d/2} \lambda_A^{2d+2} \left(\int_{F_y(B_{2\lambda_A\delta})} A_y \nabla u \cdot \nabla u \right)^{1/2} \\ &\leq C\omega(u_y, 2\lambda_A\delta). \end{aligned} \quad (4.2.14)$$

We now apply (4.2.7) to get

$$\left| u(y) - \int_{B_{2\lambda_A\delta}} u_y \right| \leq C\delta(\omega(u_y, 2\lambda_A\delta) + 1) \quad (4.2.15)$$

and

$$\left| u(z) - \int_{B_{\lambda_A^{-1}\delta}} u_z \right| \leq C\delta(\omega(u_z, \lambda_A^{-1}\delta) + 1) \leq C\delta(\omega(u_y, 2\lambda_A\delta) + 1), \quad (4.2.16)$$

where we used (4.2.14) in the last inequality. Therefore, combining the triangle inequality, (4.2.15), (4.2.13) and (4.2.16) we get that

$$|u(y) - u(z)| \leq C\delta(\omega(u_y, 2\lambda_A\delta) + 1). \quad (4.2.17)$$

Moreover, by (4.2.6) (recall that we assumed $4\lambda_A^2\delta \leq r$) we have

$$\omega(u_y, 2\lambda_A\delta) \leq C\omega(u_y, (2\lambda_A)^{-1}r) + C \log \frac{r}{4\lambda_A^2\delta}. \quad (4.2.18)$$

By the ellipticity of A and since $F_y(B_{(2\lambda_A)^{-1}r}) \subset B_r(x)$, we have the following estimate

$$\begin{aligned} \omega(u_y, (2\lambda_A)^{-1}r) &= \left(\int_{F_y(B_{(2\lambda_A)^{-1}r})} A_y \nabla u \cdot \nabla u \right)^{1/2} \leq \lambda_A \left(\int_{F_y(B_{(2\lambda_A)^{-1}r})} |\nabla u|^2 \right)^{1/2} \\ &\leq 2^{d/2} \lambda_A^{d+1} \left(\int_{B_r(x)} |\nabla u|^2 \right)^{1/2} \leq C\omega(u, x, r). \end{aligned} \quad (4.2.19)$$

Finally, combine (4.2.17), (4.2.18) and (4.2.19) to get

$$\begin{aligned} |u(y) - u(z)| &\leq C\delta \left(1 + \omega(u, x, r) + \log \frac{r}{4\lambda_A^2\delta} \right) \\ &\leq C|y - z| \left(1 + \omega(u, x, r) + \log \frac{r}{|y - z|} \right), \end{aligned} \quad (4.2.20)$$

which is (4.2.5).

Now, if the assumption $4\lambda_A^2|y - z| \leq r$ is not satisfied, choose n points $y_1 = y, y_2, \dots, y_n = z$ in $B_r(x)$ such that $4\lambda_A^2\eta = |y - z|$, where we have set $\eta = |y_i - y_{i+1}|$, $i = 1, \dots, n$. Then we have $4\lambda_A^2\eta \leq r$. We notice that we can assume the y_i to be Lebesgue points for u . Moreover, observe that we can bound the number of points by $n \leq 16\lambda_A^4 + 2$. Therefore, applying the estimate (4.2.20) to each pair (y_i, y_{i+1}) we have

$$\begin{aligned} |u(y) - u(z)| &\leq \sum_{i=1}^{n-1} |u(y_i) - u(y_{i+1})| \leq C \sum_{i=1}^{n-1} \eta \left(1 + \omega(u, x, r) + \log \frac{r}{4\lambda_A^2\eta} \right) \\ &\leq nC \frac{|y - z|}{4\lambda_A^2} \left(1 + \omega(u, x, r) + \log \frac{r}{|y - z|} \right), \end{aligned}$$

which concludes the proof. \square

We are now in position to prove the Hölder continuity of u .

Proposition 4.2.7. *The function u is locally α -Hölder continuous in D for every $\alpha \in (0, 1)$, that is, for every compact set $K \subset D$, there exist $r_K > 0$ and $C_K > 0$ such that for every $x \in K$ we have*

$$|u(y) - u(z)| \leq C_K |y - z|^\alpha \quad \text{for every } y, z \in B_{r_K}(x). \quad (4.2.21)$$

Proof. Let $x \in K$ and set $4r_K = r_1 = \min\{r_0, \text{dist}(K, D^c)\}$ where r_0 is given by Proposition 4.2.5. Since the function $r \mapsto r^{1-\alpha} \log(r_1/r)$ is non-decreasing on $(0, c_\alpha)$ for some constant $c_\alpha > 0$ depending on α and r_1 , it follows from Proposition 4.2.5 that, if $y, z \in B_{r_1/2}(x)$ are such that $|y - z| \leq c_\alpha$, we have

$$\begin{aligned} |u(y) - u(z)| &\leq C \left(r_1^{1-\alpha} (1 + \omega(u, x, r_1)) + c_\alpha^{1-\alpha} \log \frac{r_1}{c_\alpha} \right) |y - z|^\alpha \\ &\leq C (1 + \omega(u, x, r_1)) |y - z|^\alpha \end{aligned} \quad (4.2.22)$$

If now $|y - z| > c_\alpha$, then choose n points $y_1 = y, \dots, y_n = z$ in $B_{r_1/2}(x)$ such that $|y_i - y_{i+1}| = c_\alpha r_1^{-1} |y - z|$, with n bounded by some constant depending on α and r_1 . Then apply (4.2.22) to each pair (y_i, y_{i+1}) to prove that u is α -Hölder continuous in the ball $B_{r_1/2}(x)$ with a modulus of continuity depending on $\omega(u, x, r_1)$. Now, (4.2.21) follows by a compactness argument with the constant C_K depending on $\max\{\omega(u, x_i, r_1), i = \dots, N\}$, where the x_i 's are given by some subcovering of $K \subset \cup_{i=1}^N B_{r_K}(x_i)$. \square

4.2.2 Bound of the Lipschitz constant in $\{u > 0\}$

We prove that u is Lipschitz continuous and even $C^{1,\beta}$ -regular in the regions where u keeps the same sign. We also provide in this case an estimate of the Lipschitz constant of u in terms of $\omega(u, x, r)$ (see Proposition 4.2.8). Then, we show that under suitable conditions, u keeps the same sign and is therefore Lipschitz continuous (see Proposition 4.2.11).

Proposition 4.2.8. *Let $K \subset D$ be a compact set. There exist constants $r_K > 0$ and $C_K > 0$ such that, if $x \in K$ and $r \leq r_K$ satisfy*

$$\text{either } u_x > 0 \text{ a.e. in } B_r \quad \text{or} \quad u_x < 0 \text{ a.e. in } B_r, \quad (4.2.23)$$

then u is Lipschitz continuous in $B_{r/2}(x)$ and we have

$$|u(y) - u(z)| \leq C_K (1 + \omega(u, x, r)) |y - z| \quad \text{for every } y, z \in B_{r/2}(x). \quad (4.2.24)$$

Moreover, u is $C^{1,\beta}$ in the ball $B_{r/4}(x)$ where $\beta = \frac{\delta_A}{d+\delta_A+2}$ and we have

$$|\nabla u(y) - \nabla u(z)| \leq C_K r^{-\frac{\delta_A}{d+2}} (1 + \omega(u, x, r)) |y - z|^\beta \quad \text{for every } y, z \in B_{r/4}(x). \quad (4.2.25)$$

In the next Lemma we compare the Dirichlet energy of u_x and of its harmonic extension in small balls where u_x has a given sign. The estimate (4.2.26) in Lemma 4.2.9 below is similar to (4.2.1) but with a smaller error term. Thanks to this improvement, the strategy developed in the proof of Lemma 4.2.6 will lead to a sharper result than estimate (4.2.6), namely (4.2.24).

Lemma 4.2.9. *Let $K \subset D$ be a compact set and let $\alpha \in (0, 1)$. There exist constants $r_K > 0$ and $C > 0$ such that, if $x \in K$ and $r \leq r_K$ are such that (4.2.23) holds, then the function $u_x = u \circ F_x$ satisfies*

$$\int_{B_r} |\nabla u_x|^2 \leq (1 + Cr^{\delta_A}) \int_{B_r} |\nabla h_r|^2 + Cr^{d+\alpha}, \quad (4.2.26)$$

where h_r stands for the harmonic extension of the trace of u_x to ∂B_r .

Proof. Set $\rho := \lambda_A r$ for some $r > 0$ small enough so that $B_\rho(x) \subset D$. We define $v \in H_0^1(D)$ by $h_r = v \circ F_x$ in B_r and $v = u$ elsewhere so that we have $u - v \in H_0^1(F_x(B_r))$. Set $\tilde{U} = (u_1, \dots, v, \dots, u_k) \in H_0^1(D, \mathbb{R}^k)$ and observe that $|\{|\tilde{U}| > 0\}| = |\{|U| > 0\}|$ by (4.2.23) and because $v > 0$ in $F_x(B_r)$. Then, we use \tilde{U} as a test function in (4.1.3) to get

$$\int_{F_x(B_r)} A \nabla u \cdot \nabla u \leq \int_{F_x(B_r)} A \nabla v \cdot \nabla v + C \|u - v\|_{L^1} \int_D A \nabla \tilde{U} \cdot \nabla \tilde{U},$$

where C is the constant from Theorem 4.1.2. Now, since u is locally α -Hölder continuous, we have the bound $\|u - v\|_{L^1} \leq C_d C_K r^{d+\alpha}$, where the constant C_K is given by Proposition 4.2.7. Moreover we have the estimate

$$\int_D A \nabla \tilde{U} \cdot \nabla \tilde{U} \leq \int_D A \nabla U \cdot \nabla U + \int_{F_x(B_r)} A \nabla v \cdot \nabla v.$$

Altogether this gives

$$\int_{F_x(B_r)} A \nabla u \cdot \nabla u \leq (1 + \tilde{C} r^{d+\alpha}) \int_{F_x(B_r)} A \nabla v \cdot \nabla v + \tilde{C} r^{d+\alpha},$$

for some constant \tilde{C} which involves $\int_D A \nabla U \cdot \nabla U$. Finally, using the Hölder continuity and the ellipticity of A as in the proof of Proposition 4.2.4, we get

$$\int_{B_r} |\nabla u_x|^2 \leq (1 + d c_A \lambda_A^2 \rho^{\delta_A}) \left[\frac{1 + \tilde{C} r^{d+\alpha}}{1 - d c_A \lambda_A^2 \rho^{\delta_A}} \int_{B_r} |\nabla h_r|^2 + \lambda_A^d \tilde{C} r^{d+\alpha} \right].$$

which gives (4.2.26). \square

Next Lemma is analogue to Lemma 4.2.6 with a better estimate of the error term. Its proof is quite similar but we nonetheless sketch the argument since there are small differences.

Lemma 4.2.10. *Let $K \subset D$ be a compact set and $\alpha \in (0, 1)$. There exist constants $r_K > 0$ and $C > 0$ such that, for every $x \in K$ and every $r \leq r_K$ such that (4.2.23) holds, we have*

$$\omega(u_x, s) \leq C \omega(u_x, r) + C r^{\alpha/2} \quad \text{for every } 0 < s \leq r. \quad (4.2.27)$$

If, moreover, x is a Lebesgue point for u , we have

$$\left| u(x) - \fint_{B_r} u_x \right| \leq C r (\omega(u_x, r) + r^{\alpha/2}). \quad (4.2.28)$$

Proof. For $t \leq r \leq r_K$ we have by Lemma 4.2.9

$$\int_{B_t} |\nabla(u_x - h_t)|^2 = \int_{B_t} |\nabla u_x|^2 - \int_{B_t} |\nabla h_t|^2 \leq C t^{\delta_A} \int_{B_t} |\nabla u_x|^2 + C t^{d+\alpha}, \quad (4.2.29)$$

since h_t is a minimizer of the Dirichlet energy on B_t . Now, for $s \leq t \leq r_0$ we use (4.2.9) and (4.2.29) to estimate as in (4.2.10)

$$\omega(u_x, s) \leq \left(1 + C \left(\frac{t}{s}\right)^{d/2} t^{\delta_A/2}\right) \omega(u_x, t) + C \left(\frac{t}{s}\right)^{d/2} t^{\alpha/2},$$

which, applied to $s = 2^{-i} r$ and $t = 2^{-(i-1)} r$, gives

$$\omega(u_x, r_i) \leq \left(1 + C r_{i-1}^{\delta_A/2}\right) \omega(u_x, r_{i-1}) + C r_{i-1}^{\alpha/2}, \quad i \geq 1,$$

where we have set $r_i = 2^{-i}r$. Iterating the above estimate we get for every $i \geq 1$

$$\begin{aligned}\omega(u_x, r_i) &\leq \omega(u_x, r) \prod_{j=0}^{i-1} \left(1 + Cr_j^{\delta_A/2}\right) + C \sum_{j=1}^{i-1} \left(r_{j-1}^{\alpha/2} \prod_{l=j}^{i-1} \left(1 + Cr_l^{\delta_A/2}\right)\right) + Cr_{i-1}^{\alpha/2} \\ &\leq C\omega(u_x, r) + Cr^{\alpha/2},\end{aligned}$$

since $\prod_{j=0}^{\infty} (1 + Cr_j^{\delta_A/2})$ is bounded by a constant depending on r_K . This proves (4.2.27).

Finally, (4.2.28) is proved in the same way than (4.2.7) but with (4.2.12) replaced by the estimate

$$\left(\int_{B_{r_i}} |u_x - m_i|^2\right)^{1/2} \leq Cr_i \omega(u_x, r_i) \leq Cr_i (\omega(u_x, r) + r^{\alpha/2}).$$

□

Proof of Proposition 4.2.8. Let us first prove (4.2.24). We follow the proof of Proposition 4.2.5 and we only detail the few differences. Let $y, z \in B_{r/2}(x)$ be Lebesgue points for u and set $\delta = |y - z|$. We first assume that $4\lambda_A^2 \delta \leq r$. By (4.2.28) we have

$$\left|u(y) - \int_{B_{2\lambda_A \delta}} u_y\right| \leq C\delta(\omega(u_y, 2\lambda_A \delta) + \delta^{\alpha/2}), \quad (4.2.30)$$

and, using also (4.2.14),

$$\left|u(z) - \int_{B_{\lambda_A^{-1} \delta}} u_z\right| \leq C\delta(\omega(u_z, \lambda_A^{-1} \delta) + r^{\alpha/2}) \leq C\delta(\omega(u_y, 2\lambda_A \delta) + \delta^{\alpha/2}). \quad (4.2.31)$$

Moreover, by (4.2.27) we have

$$\omega(u_y, 2\lambda_A \delta) \leq C\omega(u_y, (2\lambda_A)^{-1}r) + Cr^{\alpha/2}. \quad (4.2.32)$$

Then, combining (4.2.30), (4.2.13), (4.2.31) and then (4.2.32) and (4.2.19) we have

$$\begin{aligned}|u(y) - u(z)| &\leq C\delta(\omega(u_y, 2\lambda_A \delta) + \delta^{\alpha/2}) \leq C\delta(1 + \omega(u, x, r) + r^{\alpha/2}) \\ &\leq C(1 + \omega(u, x, r))|y - z|.\end{aligned} \quad (4.2.33)$$

Finally, if $4\lambda_A^2 \delta > r$, we argue as in the proof of Proposition 4.2.5 and choose a few number of points which connect y and z to prove (4.2.33).

We now prove the estimate (4.2.25). Let $y \in B_{r/4}(x)$ and $\bar{r} \leq \lambda_A^{-1}r/4$. We set $m(u_y, \rho) = \int_{B_\rho} \nabla u_y$ for $\rho \leq \bar{r}$ and $m = \int_{B_{\bar{r}}} \nabla h_{y, \bar{r}} = \nabla h_{y, \bar{r}}(0)$, where $h_{y, \bar{r}}$ denotes the harmonic extension of the trace of u_y to $\partial B_{\bar{r}}$. Let $\eta \in (0, 1/4)$. We want to estimate

$$\int_{B_{\eta \bar{r}}} |\nabla u_y - m(u_y, \eta \bar{r})|^2 \leq \int_{B_{\eta \bar{r}}} |\nabla u_y - m|^2 \leq 2 \int_{B_{\eta \bar{r}}} |\nabla(u_y - h_{y, \bar{r}})|^2 + 2 \int_{B_{\eta \bar{r}}} |\nabla h_{y, \bar{r}} - m|^2. \quad (4.2.34)$$

Firstly, by (4.2.29) we have

$$\begin{aligned}\int_{B_{\eta \bar{r}}} |\nabla(u_y - h_{y, \bar{r}})|^2 &\leq C(\eta \bar{r})^{-d} \int_{B_{\bar{r}}} |\nabla(u_y - h_{y, \bar{r}})|^2 \leq C(\eta \bar{r})^{-d} \left(\bar{r}^{\delta_A} \int_{B_{\bar{r}}} |\nabla u_y|^2 + \bar{r}^{d+\alpha}\right) \\ &\leq C\eta^{-d} \bar{r}^{\delta_A} \omega(u_y, \bar{r})^2 + C\eta^{-d} \bar{r}^\alpha.\end{aligned} \quad (4.2.35)$$

Moreover, (4.2.33) says that for almost every $z \in B_{r/4}(y) \subset B_{r/2}(x)$ we have $|\nabla u(z)| \leq C(1 + \omega(u, x, r))$, which implies that

$$\omega(u_y, \bar{r})^2 = \int_{B_{\bar{r}}} |\nabla u_y|^2 \leq \lambda_A^{2(d+1)} \int_{B_{r/4}(y)} |\nabla u|^2 \leq C(1 + \omega(u, x, r))^2. \quad (4.2.36)$$

On the other hand, by estimates on harmonic functions (see [48, Theorem 3.9]), the Cauchy-Schwarz inequality and (4.2.36) we have for every $\xi \in B_{\eta\bar{r}}$

$$\begin{aligned} |\nabla h_{y,\bar{r}}(\xi) - m| &= |\nabla h_{y,\bar{r}}(\xi) - \nabla h_{y,\bar{r}}(0)| \leq \eta\bar{r} \sup_{B_{\eta\bar{r}}} |\nabla^2 h_{y,\bar{r}}| \leq C\eta \sup_{B_{2\eta\bar{r}}} |\nabla h_{y,\bar{r}}| \\ &\leq C\eta \left(\int_{B_{\bar{r}}} |\nabla h_{y,\bar{r}}|^2 \right)^{1/2} \leq C\eta \left(\int_{B_{\bar{r}}} |\nabla u_y|^2 \right)^{1/2} \\ &\leq C\eta \omega(u_y, \bar{r}) \leq C\eta(1 + \omega(u, x, r)), \end{aligned} \quad (4.2.37)$$

where $\nabla^2 h_{y,\bar{r}}$ stands for the Hessian matrix of $h_{y,\bar{r}}$. Therefore, combining (4.2.34), (4.2.35), (4.2.36) and (4.2.37) we get

$$\begin{aligned} \int_{B_{\eta\bar{r}}} |\nabla u_y - m(u_y, \eta\bar{r})|^2 &\leq C\eta^{-d\bar{r}\delta_A} (1 + \omega(u, x, r))^2 + C\eta^{-d\bar{r}\alpha} + C\eta^2(1 + \omega(u, x, r))^2 \\ &\leq C(1 + \omega(u, x, r))^2 \left[\eta^{-d\bar{r}\delta_A} + \eta^{-d\bar{r}\alpha} + \eta^2 \right]. \end{aligned} \quad (4.2.38)$$

We set $\alpha = \delta_A$ (recall that $\alpha \in (0, 1)$ was arbitrary). Moreover, we set $\beta = \frac{\delta_A}{d+\delta_A+2}$ and $\eta = \frac{\delta_A}{\bar{r}^{d+2}}$ so that we have $\eta^{-d\bar{r}\delta_A} = \eta^2 = (\eta\bar{r})^{2\beta}$. Notice also that $\eta\bar{r} = \bar{r}^{1+\varepsilon}$, where $\varepsilon = \frac{\delta_A}{d+2}$. Therefore, (4.2.38) implies that for every $y \in B_{r/4}(x)$ and every $\rho \leq \left(\frac{r}{4\lambda_A}\right)^{1+\varepsilon}$ we have

$$\int_{B_\rho} |\nabla u_y - m(u_y, \rho)|^2 \leq C(1 + \omega(u, x, r))^2 \rho^{2\beta}. \quad (4.2.39)$$

Now, let $y, z \in B_{r/4}(x)$ be Lebesgue points for u and set $\delta = |x - y|$. We first assume that $2\lambda_A^2\delta \leq \left(\frac{r}{4\lambda_A}\right)^{1+\varepsilon}$. Setting $\delta_i = 2^{-i}\delta$, $i \geq 0$, we have using (4.2.39)

$$\begin{aligned} |\nabla u_y(0) - m(u_y, \delta)| &\leq \sum_{i=0}^{+\infty} |m(u_y, \delta_{i+1}) - m(u_y, \delta_i)| \leq \sum_{i=0}^{+\infty} \int_{B_{\delta_{i+1}}} |\nabla u_y - m(u_y, \delta_i)| \\ &\leq 2^d \sum_{i=0}^{+\infty} \int_{B_{\delta_i}} |\nabla u_y - m(u_y, \delta_i)| \leq 2^d \sum_{i=0}^{+\infty} \left(\int_{B_{\delta_i}} |\nabla u_y - m(u_y, \delta_i)|^2 \right)^{1/2} \\ &\leq C(1 + \omega(u, x, r))\delta^\beta. \end{aligned} \quad (4.2.40)$$

Similarly, we have

$$|\nabla u_z(0) - m(u_z, 2\lambda_A^2\delta)| \leq C(1 + \omega(u, x, r))\delta^\beta. \quad (4.2.41)$$

Moreover, using that $F_z^{-1} \circ F_y(B_\delta) \subset B_{2\lambda_A^2\delta}$ we have

$$\begin{aligned} |m(u_y, \delta) - m(u_z, 2\lambda_A^2\delta)| &\leq \int_{B_\delta} |\nabla u_y - m(u_z, 2\lambda_A^2\delta)| \\ &\leq \int_{F_z^{-1} \circ F_y(B_\delta)} |A_y^{1/2} A_z^{-1/2} \nabla u_z - m(u_z, 2\lambda_A^2\delta)| \\ &\leq (2\lambda_A^2)^{2d} \int_{B_{2\lambda_A^2\delta}} |A_y^{1/2} A_z^{-1/2} \nabla u_z - m(u_z, 2\lambda_A^2\delta)|. \end{aligned} \quad (4.2.42)$$

Notice that the matrices $A^{1/2}$ have Hölder continuous coefficients with exponent $\delta_A/2$ and hence that $|A_y^{1/2} A_z^{-1/2} - \text{Id}| \leq \lambda_A |A_y^{-1/2} - A_z^{-1/2}| \leq C\delta^{\delta_A/2} \leq C\delta^\beta$ (because $\beta \leq \delta_A/2$). Therefore, using (4.2.27) it follows that

$$\begin{aligned} \int_{B_{2\lambda_A^2\delta}} |A_y^{1/2} A_z^{-1/2} \nabla u_z - \nabla u_z| &\leq C\delta^\beta \int_{B_{2\lambda_A^2\delta}} |\nabla u_z| \leq C\delta^\beta \omega(u_z, 2\lambda_A^2\delta) \\ &\leq C\delta^\beta (\omega(u_x, \lambda_A^{-1}r) + r^{\delta_A}) \leq C(1 + \omega(u, x, r))\delta^\beta. \end{aligned} \quad (4.2.43)$$

Furthermore, the triangle inequality in (4.2.42) together with (4.2.43), (4.2.39) and Cauchy-Schwarz's inequality give

$$|m(u_y, \delta) - m(u_z, 2\lambda_A^2 \delta)| \leq C(1 + \omega(u, x, r))\delta^\beta. \quad (4.2.44)$$

Now, (4.2.40), (4.2.44) and (4.2.41) infer

$$\begin{aligned} |\nabla u_y(0) - \nabla u_z(0)| &\leq |\nabla u_y(0) - m(u_y, \delta)| + |m(u_y, \delta) - m(u_z, 2\lambda_A^2 \delta)| + |\nabla u_z(0) - m(u_z, 2\lambda_A^2 \delta)| \\ &\leq C(1 + \omega(u, x, r))\delta^\beta. \end{aligned}$$

Since $|\nabla u_y(0)| \leq \lambda_A |\nabla u(y)| \leq C(1 + \omega(u, x, r))$ for almost every $y \in B_{r/4}(x)$ by (4.2.33), we get

$$\begin{aligned} |\nabla u(y) - \nabla u(z)| &= |A_y^{-1/2} \nabla u_y(0) - A_z^{-1/2} \nabla u_z(0)| \\ &\leq |A_y^{-1/2} \nabla u_y(0) - A_z^{-1/2} \nabla u_y(0)| + |A_z^{-1/2} \nabla u_y(0) - A_z^{-1/2} \nabla u_z(0)| \\ &\leq |A_y^{-1/2} - A_z^{-1/2}| |\nabla u_y(0)| + |A_z^{-1/2}| |\nabla u_y(0) - \nabla u_z(0)| \\ &\leq C(1 + \omega(u, x, r))\delta^\beta. \end{aligned}$$

If $|y - z| \geq \left(\frac{r}{4\lambda_A}\right)^{1+\varepsilon}$, then we can connect y and z through less than $\lambda_A \left(\frac{4\lambda_A}{r}\right)^\varepsilon + 2$ points. This shows (4.2.25) and concludes the proof. \square

The strategy to prove Theorem 4.1.2 is to show that $\omega(u_x, r)$ cannot become too big as r gets small. In the next Proposition we prove, under some condition (see the first inequality in (4.2.45) below), that if $\omega(u_x, r)$ is big enough then u keeps the same sign near the point x and is hence Lipschitz continuous by Proposition 4.2.8. The case where $\omega(u_x, r)$ is big and this condition fails is treated in the next subsection. We set for $x \in D$ and $r > 0$

$$b(u_x, r) = \int_{\partial B_r} u_x d\mathcal{H}^{d-1} \quad \text{and} \quad b^+(u_x, r) = \int_{\partial B_r} |u_x| d\mathcal{H}^{d-1}.$$

Proposition 4.2.11. *Let $K \subset D$ be a compact set and let $\gamma > 0$. There exists constants $r_K, C_K > 0$ and $\kappa_1 > 0$ such that, if $x \in K$ and $r \leq r_K$ satisfy*

$$\gamma r(1 + \omega(u_x, r)) \leq |b(u_x, r)| \quad \text{and} \quad \kappa_1 \leq \omega(u_x, r), \quad (4.2.45)$$

then there exists a constant $c > 0$ (independent from x and r) such that u is Lipschitz continuous in $B_{cr/2}(x)$ and we have

$$|u(y) - u(z)| \leq C_K(1 + \omega(u, x, r))|y - z| \quad \text{for every } y, z \in B_{cr/2}(x). \quad (4.2.46)$$

Moreover, u is $C^{1,\beta}$ in $B_{cr/4}(x)$ where $\beta = \frac{\delta_\Lambda}{d + \delta_\Lambda + 2}$ and we have

$$|\nabla u(y) - \nabla u(z)| \leq C_K r^{-\frac{\delta_\Lambda}{d+2}} (1 + \omega(u, x, r))|y - z|^\beta \quad \text{for every } y, z \in B_{cr/4}(x). \quad (4.2.47)$$

Roughly speaking, the condition (4.2.45) says that the absolute value of the trace of u_x to ∂B_r is big. This will in fact ensure that u_x has, in some smaller ball, the same sign than (the average of) u_x on ∂B_r .

Lemma 4.2.12. *Let γ and τ be two positive constants. There exist $r_0, \eta \in (0, 1)$ and $\kappa_1 > 0$ such that, if $x \in D$ and $r \leq r_0$ satisfy $B_{\lambda_A r}(x) \subset D$,*

$$\gamma(1 + \omega(u_x, r)) \leq \frac{1}{r}|b(u_x, r)| \quad \text{and} \quad \kappa_1 \leq \omega(u_x, r), \quad (4.2.48)$$

then there exist $\rho \in \left(\frac{\eta r}{2}, \eta r\right)$ such that

$$\tau(1 + \rho^{\delta_\Lambda/2} \omega(u_x, \rho)) \leq \frac{1}{\rho}|b(u_x, \rho)| \quad \text{and} \quad b^+(u_x, \rho) \leq 3|b(u_x, \rho)|. \quad (4.2.49)$$

Moreover, $b(u_x, r)$ and $b(u_x, \rho)$ have the same sign.

Proof. We first prove the second inequality in (4.2.49). Let us recall that $h_r = h_{x,r}$ denotes the harmonic extension of the trace of u_x to ∂B_r . We want to estimate both $\int_{\partial B_\rho} |h_r|$ and $\int_{\partial B_\rho} |u_x - h_r|$ in terms of $|b(u_x, r)|$ for some $\rho \in (\frac{\eta r}{2}, \eta r)$ defined soon (by (4.2.52)). If $\eta \leq 1/2$, then by subharmonicity of $|\nabla h_r|$ in B_r we have that for every $\xi \in B_{\eta r}$

$$|\nabla h_r(\xi)|^2 \leq \int_{B_{r/2}(\xi)} |\nabla h_r|^2 \leq 2^d \int_{B_r} |\nabla h_r|^2 \leq 2^d \omega(u_x, r)^2.$$

Moreover, $b(u_x, r) = b(h_r, r) = h_r(0)$ since h_r is harmonic and hence, choosing η such that $\eta 2^{d/2} \leq \gamma/4$, we get

$$\begin{aligned} |b(u_x, r) - h_r(\xi)| &= |h_r(0) - h_r(\xi)| \leq \eta r \|\nabla h_r\|_{L^\infty(B_{\eta r})} \leq \eta r 2^{d/2} \omega(u_x, r) \\ &\leq \frac{\gamma r}{4} \omega(u_x, r) \leq \frac{1}{4} |b(u_x, r)|, \end{aligned} \quad (4.2.50)$$

where in the last inequality we used the first estimate of (4.2.48). This gives (because $\rho < \eta r$)

$$\frac{3}{4} |b(u_x, r)| \leq \int_{\partial B_\rho} |h_r| \leq \frac{5}{4} |b(u_x, r)|. \quad (4.2.51)$$

On the other hand, we now fix some $\rho = \rho_x \in (\frac{\eta r}{2}, \eta r)$ such that

$$\int_{\partial B_\rho} |u_x - h_r| \leq \frac{2}{\eta r} \int_{\eta r/2}^{\eta r} ds \int_{\partial B_s} |u_x - h_r|. \quad (4.2.52)$$

By Cauchy-Schwarz's inequality, Poincaré's inequality and (4.2.1) applied to the test function h_r , it follows that we have

$$\begin{aligned} \int_{\partial B_\rho} |u_x - h_r| &\leq \frac{2}{\eta r} \int_{B_{\eta r}} |u_x - h_r| \leq C(\eta r)^{\frac{d}{2}-1} \left(\int_{B_{\eta r}} |u_x - h_r|^2 \right)^{1/2} \\ &\leq C(\eta r)^{\frac{d}{2}-1} \left(\int_{B_r} |u_x - h_r|^2 \right)^{1/2} \leq C(\eta r)^{\frac{d}{2}-1} r \left(\int_{B_r} |\nabla(u_x - h_r)|^2 \right)^{1/2} \\ &\leq C(\eta r)^{\frac{d}{2}-1} r \left(r^{\delta_A} \int_{B_r} |\nabla h_r|^2 + r^d \right)^{1/2} \leq C\eta^{\frac{d}{2}-1} r^d (r^{\delta_A} \omega(u_x, r)^2 + 1)^{1/2} \\ &\leq C\eta^{\frac{d}{2}-1} r^d (r^{\delta_A/2} \omega(u_x, r) + 1) \end{aligned}$$

In view of the two hypothesis in (4.2.48) we then get

$$\begin{aligned} \int_{\partial B_\rho} |u_x - h_r| &\leq C\eta^{-\frac{d}{2}} r (r_0^{\delta_A/2} \omega(u_x, r) + 1) \leq C\eta^{-\frac{d}{2}} r \omega(u_x, r) \left(r_0^{\delta_A/2} + \frac{1}{\kappa_1} \right) \\ &\leq C\eta^{-\frac{d}{2}} \gamma^{-1} |b(u_x, r)| \left(r_0^{\delta_A/2} + \frac{1}{\kappa_1} \right) \leq \frac{1}{4} |b(u_x, r)|, \end{aligned} \quad (4.2.53)$$

where the last inequality holds if we choose r_0 small enough and $\kappa_1 > 0$ large enough (both depending on η) such that

$$C\gamma^{-1} \left(r_0^{\delta_A/2} + \frac{1}{\kappa_1} \right) \leq \frac{1}{4} \eta^{d/2}. \quad (4.2.54)$$

Now, using (4.2.51) and (4.2.53) we have

$$b^+(u_x, \rho) = \int_{\partial B_\rho} |u_x| \leq \int_{\partial B_\rho} |h_r| + \int_{\partial B_\rho} |u_x - h_r| \leq \frac{3}{2} |b(u_x, r)|,$$

and, using also that h_r keeps the same sign on ∂B_ρ by (4.2.50), we have

$$|b(u_x, \rho)| \geq \left| \int_{\partial B_\rho} h_r \right| - \left| \int_{\partial B_\rho} (u_x - h_r) \right| \geq \int_{\partial B_\rho} |h_r| - \int_{\partial B_\rho} |u_x - h_r| \geq \frac{1}{2} |b(u_x, r)|. \quad (4.2.55)$$

This proves the second inequality in (4.2.48). Moreover, (4.2.53) and (4.2.50) imply that

$$\begin{aligned} |b(u_x, \rho) - b(u_x, r)| &\leq \left| b(u_x, \rho) - \int_{\partial B_\rho} h_r \right| + \left| \int_{\partial B_\rho} h_r - b(u_x, r) \right| \\ &\leq \int_{\partial B_\rho} |u_x - h_r| + \int_{\partial B_\rho} |h_r - b(u_x, r)| \leq \frac{1}{2} |b(u_x, r)|, \end{aligned}$$

which shows that $b(u_x, r)$ and $b(u_x, \rho)$ have the same sign.

For the first estimate in (4.2.49), by (4.2.55) and the first hypothesis in (4.2.48), we have

$$\frac{1}{\rho} |b(u_x, \rho)| \geq \frac{1}{2\rho} |b(u_x, r)| \geq \frac{1}{2\eta r} |b(u_x, r)| \geq \frac{\gamma}{2\eta} (1 + \omega(u_x, r)),$$

which using (4.2.6) gives (notice that we assumed that $B_{\lambda_A r}(x) \subset D$)

$$\begin{aligned} 1 + \rho^{\delta_A/2} \omega(u_x, \rho) &\leq 1 + \omega(u_x, \rho) \leq 1 + C \left(\omega(u_x, r) + \log \frac{r}{\rho} \right) \\ &\leq C(1 + \omega(u_x, r) + |\log \eta|) \leq C(1 + |\log \eta|)(1 + \omega(u_x, r)) \\ &\leq C(1 + |\log \eta|) \frac{2\eta}{\gamma\rho} |b(u_x, \rho)|. \end{aligned}$$

Finally, observe that with η small enough (and also r_0 small enough and κ_1 large enough so that (4.2.54) still holds) we have

$$\tau C(1 + |\log \eta|) \frac{2\eta}{\gamma} \leq 1.$$

This completes the proof. \square

We continue with a self-improvement lemma whose strategy is similar to the one followed in the previous lemma, the main difference being that we now consider u_x with different points x .

Lemma 4.2.13. *There exist constants $r_0 \in (0, 1)$ and $\tau_0 \geq 1$ with the following property: if $x \in D$, $\tau \geq \tau_0$ and $\rho \leq r_0$ satisfy $B_{\lambda_A \rho}(x) \subset D$,*

$$\tau(1 + \rho^{\delta_A/2} \omega(u_x, \rho)) \leq \frac{1}{\rho} |b(u_x, \rho)| \quad \text{and} \quad b^+(u_x, \rho) \leq 3|b(u_x, \rho)|, \quad (4.2.56)$$

then for every $y \in B_{\varepsilon\rho}(x)$, where $\varepsilon = \tau^{-1/d}$, there exists $\rho_1 \in (\frac{\varepsilon\rho}{2}, \varepsilon\rho)$ such that

$$2\tau(1 + \rho_1^{\delta_A/2} \omega(u_y, \rho_1)) \leq \frac{1}{\rho_1} |b(u_y, \rho_1)| \quad \text{and} \quad b^+(u_y, \rho_1) \leq 3|b(u_y, \rho_1)|. \quad (4.2.57)$$

Moreover, $b(u_x, \rho)$ and $b(u_y, \rho_1)$ have the same sign.

Proof. Firstly, if ε is small enough so that $\bar{\varepsilon} := 2\lambda_A^2 \varepsilon \leq 1/4$, then by standard estimates on harmonic functions (see [48, Theorem 3.9]) $h_\rho = h_{x, \rho}$ satisfies

$$\|\nabla h_\rho\|_{L^\infty(B_{\bar{\varepsilon}\rho})} \leq \frac{C}{\rho} \|h_\rho\|_{L^\infty(B_{\rho/2})} \leq \frac{C}{\rho} \int_{\partial B_\rho} |h_\rho| = \frac{C}{\rho} b^+(u_x, \rho).$$

Using that $b(u_x, \rho) = b(h_\rho, \rho) = h_\rho(0)$ by harmonicity and the second hypothesis in (4.2.56), it follows that for every $\xi \in B_{\bar{\varepsilon}\rho}$ we have

$$|b(u_x, \rho) - h_\rho(\xi)| \leq \bar{\varepsilon}\rho \|\nabla h_\rho\|_{L^\infty(B_{\bar{\varepsilon}\rho})} \leq \varepsilon C b^+(u_x, \rho) \leq \tau_0^{-\frac{1}{d}} C |b(u_x, \rho)| \leq \frac{1}{4} |b(u_x, \rho)|, \quad (4.2.58)$$

where the last inequality holds if τ_0 is big enough. This implies that

$$\frac{3}{4}|b(u_x, \rho)| \leq |h_\rho(\xi)| \leq \frac{5}{4}|b(u_x, \rho)| \quad \text{for every } \xi \in B_{\bar{\varepsilon}\rho}. \quad (4.2.59)$$

Moreover, by (4.2.1) applied to h_ρ (and since $\bar{\varepsilon}\rho \leq \rho$ for τ_0 large enough) we have

$$\begin{aligned} \int_{B_{\bar{\varepsilon}\rho}} |u_x - h_\rho| &\leq C(\bar{\varepsilon}\rho)^{\frac{d}{2}} \left(\int_{B_{\bar{\varepsilon}\rho}} |u_x - h_\rho|^2 \right)^{1/2} \leq C(\varepsilon\rho)^{\frac{d}{2}} \left(\int_{B_\rho} |u_x - h_\rho|^2 \right)^{1/2} \\ &\leq C(\varepsilon\rho)^{\frac{d}{2}} \rho \left(\int_{B_\rho} |\nabla(u_x - h_\rho)|^2 \right)^{1/2} \leq C(\varepsilon\rho)^{\frac{d}{2}} \rho \left(\rho^{\delta_A} \int_{B_\rho} |\nabla h_\rho|^2 + \rho^d \right)^{1/2} \\ &\leq C\varepsilon^{\frac{d}{2}} \rho^{d+1} (\rho^{\delta_A} \omega(u_x, \rho)^2 + 1)^{1/2} \leq C\varepsilon^{\frac{d}{2}} \rho^{d+1} (\rho^{\delta_A/2} \omega(u_x, \rho) + 1). \end{aligned} \quad (4.2.60)$$

We now fix some $y \in B_{\varepsilon\rho}(x)$. Let $F : B_{\varepsilon\rho} \subset \mathbb{R}^d \rightarrow B_{\bar{\varepsilon}\rho} \subset \mathbb{R}^d$ be the function defined by $F(z) = F_x^{-1} \circ F_y(z)$. Then the coarea formula gives (and because $\partial F(B_s) = \partial\{|F^{-1}| > s\}$)

$$\begin{aligned} \lambda_A^{-2} \int_{\varepsilon\rho/2}^{\varepsilon\rho} ds \int_{\partial F(B_s)} |u_x - h_\rho| d\mathcal{H}^{d-1} &\leq \int_{\varepsilon\rho/2}^{\varepsilon\rho} ds \int_{\partial F(B_s)} \frac{|u_x - h_\rho|}{|\nabla|F^{-1}||} d\mathcal{H}^{d-1} \\ &= \int_{\{\varepsilon\rho/2 \leq |F^{-1}| \leq \varepsilon\rho\}} |u_x - h_\rho| \\ &\leq \int_{F(B_{\varepsilon\rho})} |u_x - h_\rho| \leq \int_{B_{\bar{\varepsilon}\rho}} |u_x - h_\rho|. \end{aligned} \quad (4.2.61)$$

We now choose $\rho_1 \in (\frac{\varepsilon\rho}{2}, \varepsilon\rho)$ such that

$$\int_{\partial F(B_{\rho_1})} |u_x - h_\rho| d\mathcal{H}^{d-1} \leq \frac{2}{\varepsilon\rho} \int_{\varepsilon\rho/2}^{\varepsilon\rho} ds \int_{\partial F(B_{\rho_1})} |u_x - h_\rho| d\mathcal{H}^{d-1},$$

so that (4.2.60), (4.2.61) and the first hypothesis in (4.2.56) imply

$$\begin{aligned} \int_{\partial F(B_{\rho_1})} |u_x - h_\rho| d\mathcal{H}^{d-1} &\leq C\varepsilon^{-\frac{d}{2}} \rho (\rho^{\delta_A/2} \omega(u_x, \rho) + 1) \leq C\varepsilon^{-\frac{d}{2}} \tau^{-1} |b(u_x, \rho)| \\ &\leq C\tau_0^{-1/2} |b(u_x, \rho)| \leq \frac{1}{4} \lambda_A^{-4(d-1)} |b(u_x, \rho)|, \end{aligned} \quad (4.2.62)$$

where the last inequality holds for τ_0 is large enough. Moreover, because the functions F and F^{-1} are Lipschitz continuous with Lipschitz constants bounded by λ_A^2 , we have for every set $E \subset \mathbb{R}^d$ (see [74, Proposition 3.5])

$$\lambda_A^{-2(d-1)} \mathcal{H}^{d-1}(E) \leq F_\# \mathcal{H}^{d-1}(E) \leq \lambda_A^{2(d-1)} \mathcal{H}^{d-1}(E), \quad (4.2.63)$$

where $F_\# \mathcal{H}^{d-1}$ stands for the pushforward measure of \mathcal{H}^{d-1} along F . Therefore, by (4.2.59) (and since $\partial F(B_{\rho_1}) \subset B_{\bar{\varepsilon}\rho}$), (4.2.63) and (4.2.62) we have

$$\begin{aligned} b^+(u_y, \rho_1) &= \int_{\partial B_{\rho_1}} |u_y| d\mathcal{H}^{d-1} = \frac{1}{\mathcal{H}^{d-1}(\partial B_{\rho_1})} \int_{\partial F(B_{\rho_1})} |u_x| dF_\# \mathcal{H}^{d-1} \\ &\leq \frac{1}{\mathcal{H}^{d-1}(\partial B_{\rho_1})} \left(\int_{\partial F(B_{\rho_1})} |h_\rho| dF_\# \mathcal{H}^{d-1} + \int_{\partial F(B_{\rho_1})} |u_x - h_\rho| dF_\# \mathcal{H}^{d-1} \right) \\ &\leq \frac{5}{4} |b(u_x, \rho)| + \lambda_A^{4(d-1)} \int_{\partial F(B_{\rho_1})} |u_x - h_\rho| d\mathcal{H}^{d-1} \\ &\leq \frac{3}{2} |b(u_x, \rho)|. \end{aligned} \quad (4.2.64)$$

On the other hand, we have by (4.2.62)

$$\begin{aligned} \left| b(u_y, \rho_1) - \frac{1}{\mathcal{H}^{d-1}(\partial B_{\rho_1})} \int_{\partial F(B_{\rho_1})} h_\rho dF_\# \mathcal{H}^{d-1} \right| &\leq \frac{1}{\mathcal{H}^{d-1}(\partial B_{\rho_1})} \int_{\partial F(B_{\rho_1})} |u_x - h_\rho| dF_\# \mathcal{H}^{d-1} \\ &\leq \lambda_A^{4(d-1)} \int_{\partial F(B_{\rho_1})} |u_x - h_\rho| d\mathcal{H}^{d-1} \quad (4.2.65) \\ &\leq \frac{1}{4} |b(u_x, \rho)|. \end{aligned}$$

Moreover, by (4.2.58) and since $\partial F(B_{\rho_1}) \subset B_{\varepsilon\rho}$ we have

$$\begin{aligned} \left| \frac{1}{\mathcal{H}^{d-1}(\partial B_{\rho_1})} \int_{\partial F(B_{\rho_1})} h_\rho dF_\# \mathcal{H}^{d-1} - b(u_x, \rho) \right| \\ \leq \max \left\{ \max_{\xi \in \partial F(B_{\rho_1})} h_\rho(\xi) - b(u_x, \rho), b(u_x, \rho) - \min_{\xi \in \partial F(B_{\rho_1})} h_\rho(\xi) \right\} \leq \frac{1}{4} |b(u_x, \rho)|. \quad (4.2.66) \end{aligned}$$

Therefore, using the triangle inequality, (4.2.65) and (4.2.66) we get

$$|b(u_y, \rho_1) - b(u_x, \rho)| \leq \frac{1}{2} |b(u_x, \rho)|.$$

This proves that $b(u_x, \rho)$ and $b(u_y, \rho_1)$ have the same sign and also implies that

$$|b(u_y, \rho_1)| \geq |b(u_x, \rho)| - |b(u_y, \rho_1) - b(u_x, \rho)| \geq \frac{1}{2} |b(u_x, \rho)| \quad (4.2.67)$$

Finally, (4.2.64) and (4.2.67) gives

$$b^+(u_y, \rho_1) \leq \frac{3}{2} |b(u_x, \rho)| \leq 3 |b(u_y, \rho_1)|,$$

which is the second inequality in (4.2.57).

We now prove the first inequality in (4.2.57). By (4.2.67) and the first hypothesis in (4.2.56) we have

$$|b(u_y, \rho_1)| \geq \frac{1}{2} |b(u_x, \rho)| \geq \frac{\tau\rho}{2} (1 + \rho^{\delta_A/2} \omega(u_x, \rho)), \quad (4.2.68)$$

We then apply Lemma 4.2.6 (notice that we have $B_{\lambda_A \rho}(x) \subset D$ and $2\lambda_A^2 \rho_1 \leq \rho$) and eventually choose τ_0 bigger (depending only on d and δ_A) to get

$$\begin{aligned} 1 + \rho_1^{\delta_A/2} \omega(u_y, \rho_1) &\leq 1 + \rho_1^{\delta_A/2} \lambda_A^2 \omega(u_x, 2\lambda_A^2 \rho_1) \leq 1 + \rho_1^{\delta_A/2} C(\omega(u_x, \rho) + \log(2\varepsilon^{-1})) \\ &\leq C(1 + \rho_1^{\delta_A/2} \omega(u_x, \rho) + \rho_1^{\delta_A/2} \log(2\varepsilon^{-1})) \\ &\leq C(1 + \rho^{\delta_A/2} \omega(u_x, \rho) + \tau_0^{-\delta_A/2d} \log(\tau_0)) \\ &\leq C(1 + \rho^{\delta_A/2} \omega(u_x, \rho)). \end{aligned}$$

Therefore, with (4.2.68) this gives

$$\frac{1}{\rho_1} |b(u_y, \rho_1)| \geq \frac{1}{\varepsilon\rho} \frac{\tau\rho}{2} (1 + \rho^{\delta_A/2} \omega(u_x, \rho)) \geq \frac{\tau^{1/d}}{2C} \tau (1 + \rho_1^{\delta_A/2} \omega(u_y, \rho_1)),$$

which, choosing τ_0 big enough so that $\tau_0^{1/d} \geq 4C$, completes the proof. \square

We are now in position to prove Proposition 4.2.11 using the results from Lemmas 4.2.12, 4.2.13 and Proposition 4.2.8.

Proof of Proposition 4.2.11. Set $\varepsilon = \tau_0^{-1/d}$ and $\bar{r} = \frac{\varepsilon\eta}{2}r$ where η and τ_0 are the constants given by Lemmas 4.2.12 and 4.2.13. Note that in view of the first hypothesis in (4.2.45) we have $b(u_x, r) \neq 0$. We will prove that if $b(u_x, r) > 0$ (resp. if $b(u_x, r) < 0$), then $u > 0$ almost everywhere (resp. $u < 0$ a.e.) in $B_{\bar{r}}(x)$.

Let $y \in B_{\bar{r}}(x)$ be fixed. We first apply Lemma 4.2.12. Now, we apply once Lemma 4.2.13 at x (notice that we have $y \in B_{\varepsilon\rho}(x)$) and then iteratively at the point y . It follows that there exists a sequence of raddi $\rho_i > 0$ such that

$$2^i \tau (1 + \rho_i^{\delta_\Lambda/2} \omega(u_y, \rho_i)) \leq \frac{1}{\rho_i} b(u_y, \rho_i), \quad i \geq 0, \quad (4.2.69)$$

and that $b(u_y, \rho_i)$ has the same sign than $b(u_x, r)$ for every $i \geq 0$. Assume that $b(u_x, r) > 0$, the proof in the case $b(u_x, r) < 0$ is identical. Let us denote by $h_i = h_{y, \rho_i}$ the harmonic extension of the trace of the function u_y to ∂B_{ρ_i} . With the same argument as in (4.2.58) we get

$$|b(u_y, \rho_i) - h_i(\xi)| \leq \frac{1}{4} |b(u_y, \rho_i)| \quad \text{for every } \xi \in B_{\varepsilon\rho_i}.$$

Since $b(u_y, \rho_i) > 0$, this implies that for every $\xi \in B_{\varepsilon\rho_i} \cap \{u_y \leq 0\}$ we have

$$\begin{aligned} |u_y(\xi) - h_i(\xi)| &\geq |u_y(\xi) - b(u_y, \rho_i)| - |b(u_y, \rho_i) - h_i(\xi)| \\ &\geq |b(u_y, \rho_i)| - \frac{1}{4} |b(u_y, \rho_i)| = \frac{3}{4} |b(u_y, \rho_i)|. \end{aligned}$$

By the Chebyshev inequality, the Lebesgue measure of $B_{\varepsilon\rho_i} \cap \{u_y \leq 0\}$ is estimate as

$$|B_{\varepsilon\rho_i} \cap \{u_y \leq 0\}| \leq \frac{4}{3|b(u_y, \rho_i)|} \int_{B_{\varepsilon\rho_i}} |u_y - h_i|. \quad (4.2.70)$$

On the other hand, by (4.2.60) in this context we have

$$\int_{B_{\varepsilon\rho_i}} |u_y - h_i| \leq C \varepsilon^{d/2} \rho_i^{d+1} (1 + \rho_i^{\delta_\Lambda/2} \omega(u_y, \rho_i)). \quad (4.2.71)$$

Now, combining (4.2.70), (4.2.71) and (4.2.69) we get

$$\frac{|B_{\varepsilon\rho_i} \cap \{u_y \leq 0\}|}{|B_{\varepsilon\rho_i}|} \leq (\varepsilon\rho_i)^{-d} C \varepsilon^{d/2} \rho_i^{d+1} (2^i \tau \rho_i)^{-1} \leq \varepsilon^{d/2} C 2^{-i},$$

which implies that

$$\frac{|F_y(B_{\varepsilon\rho_i}) \cap \{u \leq 0\}|}{|F_y(B_{\varepsilon\rho_i})|} = \frac{|B_{\varepsilon\rho_i} \cap \{u_y \leq 0\}|}{|B_{\varepsilon\rho_i}|} \xrightarrow{i \rightarrow +\infty} 0,$$

where $F_y(B_{\varepsilon\rho_i}) = y + \rho_i A_y^{1/2}(B_\varepsilon)$. This shows that the density of the set $\{u \leq 0\}$ at every point $y \in B_{\bar{r}}(x)$ is 0 (see [74, exercise 5.19]), and hence that $u > 0$ almost-everywhere in $B_{\bar{r}}(x)$. Now, we set $c = \lambda_\Lambda^{-1} \tau_0^{-1/d} \eta/2$, where η and τ_0 are the constants given by Lemma 4.2.12 and 4.2.13. Then (4.2.46) and (4.2.47) follow from Proposition 4.2.8 and the fact that $\omega(u, x, cr) \leq c^{-d/2} \omega(u, x, r)$. This concludes the proof. \square

4.2.3 Lipschitz continuity

In this subsection we prove Theorem 4.1.2. At this stage, the main work is to deal with the case where $\omega(u_x, r)$ is big and the first condition in (4.2.45) fails. In this case, we show in Proposition 4.2.14 below that the value of $\omega(u_x, r)$ decreases at some smaller scale. Notice that the extra hypothesis (4.2.72) is almost irrelevant in view of Proposition 4.2.8.

Proposition 4.2.14. *Let $K \subset D$ be a compact set. There exist positive constants $r_K, \gamma \in (0, 1)$ and $\kappa_2 > 0$ with the following property: if $x \in K$ and $r \leq r_K$ satisfy*

$$u_x(\xi_0) = 0 \quad \text{for some } \xi_0 \in B_{(2\lambda_A)^{-2}r}, \quad (4.2.72)$$

$$|b(u_x, r)| \leq \gamma r(1 + \omega(u_x, r)) \quad \text{and} \quad \kappa_2 \leq \omega(u_x, r), \quad (4.2.73)$$

then we have

$$\omega(u_x, r/3) \leq \frac{1}{2}\omega(u_x, r). \quad (4.2.74)$$

We will need the following almost-monotonicity formula for operators in divergence form. We refer to [75, Theorem III] for a proof (see also [3] and [24] for the case of the Laplacian). Let us set for $u_+, u_- \in H^1(B_1)$ and $r \in (0, 1)$

$$\Phi(u_+, u_-, r) = \left(\frac{1}{r^2} \int_{B_r} \frac{|\nabla u_+(\xi)|^2}{|\xi|^{d-2}} d\xi \right) \left(\frac{1}{r^2} \int_{B_r} \frac{|\nabla u_-(\xi)|^2}{|\xi|^{d-2}} d\xi \right).$$

Proposition 4.2.15. *Let $B = (b_{ij})_{ij} : B_1 \rightarrow \text{Sym}_d^+$ be a uniformly elliptic matrix-valued function with Hölder continuous coefficients, that is, for every $x, y \in B_1$ and $\xi \in \mathbb{R}^d$*

$$|b_{ij}(x) - b_{ij}(y)| \leq c_B |x - y|^{\delta_B} \quad \text{and} \quad \frac{1}{\lambda_B^2} |\xi|^2 \leq \xi \cdot B_x \xi \leq \lambda_B^2 |\xi|^2.$$

Let u_+, u_- be two non-negative and continuous functions in the unit ball B_1 such that

$$\text{div}(B\nabla u_{\pm}) \geq -1 \quad \text{in } B_1 \quad \text{and} \quad u_+ u_- = 0 \quad \text{in } B_1.$$

Then there exist $r_0 > 0$ and $C > 0$, depending only on d, c_B, δ_B and λ_B , such that for every $r \leq r_0$ we have

$$\Phi(u_+, u_-, r) \leq C(1 + \|u_+ + u_-\|_{L^2(B_1)}^2).$$

We now state this almost-monotonicity formula for the functions u_x^{\pm} .

Corollary 4.2.16. *Let $\Omega \subset D$ be a quasi-open set and $K \subset D$ be a compact set. Let A be a matrix-valued function satisfying (4.1.5) and (4.1.6). Let $f \in L^\infty(D)$. Assume that $u \in H_0^1(\Omega)$ is a continuous function solution of the equation*

$$-\text{div}(A\nabla u) = f \quad \text{in } \Omega. \quad (4.2.75)$$

Then there exists $r_K > 0$ and $C_m > 0$, depending only on $d, c_A, \delta_A, \lambda_A, \|f\|_{L^\infty}, |D|$ and $\text{dist}(K, D^c)$, such that for every $x \in K$ and every $r \leq r_K$ the function u_x satisfies

$$\Phi(u_x^+, u_x^-, r) \leq C_m.$$

Proof. We first prove that we have, in the sense of distributions,

$$\text{div}(A\nabla u^+) \geq f \mathbb{1}_{\{u>0\}} \quad \text{in } D \quad \text{and} \quad \text{div}(A\nabla u^-) \geq f \mathbb{1}_{\{u<0\}} \quad \text{in } D. \quad (4.2.76)$$

Let us define $p_n : \mathbb{R} \rightarrow \mathbb{R}^+$ for $n \in \mathbb{N}$ by

$$p_n(s) = 0, \quad \text{for } s \leq 0; \quad p_n(s) = ns, \quad \text{for } s \in [0, 1/n]; \quad p_n(s) = 1, \quad \text{for } s \geq 1/n,$$

and set $q_n(s) = \int_0^s p_n(t) dt$. Since p_n is Lipschitz continuous, we have $p_n(u) \in H_0^1(\Omega)$ and $\nabla p_n(u) = p_n'(u) \nabla u$. Let $\varphi \in C_0^\infty(D)$ be such that $\varphi \geq 0$ in D . Multiplying the equation (4.2.75) with $\varphi p_n(u) \in H_0^1(\Omega)$ we get

$$\begin{aligned} \int_D A\nabla q_n(u) \cdot \nabla \varphi &= \int_D p_n(u) A\nabla u \cdot \nabla \varphi \leq \int_D (p_n(u) A\nabla u \cdot \nabla \varphi + \varphi p_n'(u) A\nabla u \cdot \nabla u) \\ &= \int_D A\nabla u \cdot \nabla (\varphi p_n(u)) = \int_D f \varphi p_n(u). \end{aligned}$$

Now, the inequality for u^+ in (4.2.76) follows by letting n tend $+\infty$, because $p_n(u)$ converges almost-everywhere to $\mathbb{1}_{\{u>0\}}$ and $\nabla q_n(u)$ converges in L^2 to ∇u^+ . The same proof holds for u^- .

Now, set $\rho = \lambda_A^{-1} \text{dist}(K, D^c)$ and define for every $\xi \in B_1$

$$u_{\pm}(\xi) = \rho^{-2} \|f\|_{L^\infty}^{-1} u_x^\pm(\rho\xi), \quad \tilde{f}(\xi) = f \circ F_x(\rho\xi), \quad B_\xi = A_x^{-1/2} A_{F_x(\rho\xi)} A_x^{-1/2}.$$

Then the functions u_{\pm} satisfy

$$\text{div}(B\nabla u_{\pm}) \geq \|f\|_{L^\infty}^{-1} \tilde{f} \mathbb{1}_{\{u_{\pm}>0\}} \geq -1 \quad \text{in } B_1.$$

Therefore, by Proposition 4.2.15 we have for every $r \leq r_K := r_0\rho$

$$\begin{aligned} \Phi(u_x^+, u_x^-, r) &= \rho^4 \|f\|_{L^\infty}^4 \Phi(u_+, u_-, r/\rho) \leq \rho^4 \|f\|_{L^\infty}^4 C (1 + \|u_+ + u_-\|_{L^2(B_1)}^2)^2 \\ &\leq \rho^4 \|f\|_{L^\infty}^4 C \left(1 + \lambda_A^d \rho^{-d-4} \|f\|_{L^\infty}^{-2} \|u\|_{L^2(D)}^2\right)^2 \\ &\leq \rho^4 \|f\|_{L^\infty}^4 C \left(1 + \lambda_A^d \rho^{-d-4} C_d |D|^{1+4/d}\right)^2 =: C_m. \end{aligned}$$

□

Proof of Proposition 4.2.14. Let us denote as before $h_r = h_{x,r}$ the harmonic extension of the trace of u_x to ∂B_r . Then we have

$$\omega(u_x, r/3)^2 = \int_{B_{r/3}} |\nabla u_x|^2 \leq 2 \int_{B_{r/3}} |\nabla h_r|^2 + 2 \int_{B_{r/3}} |\nabla(u_x - h_r)|^2. \quad (4.2.77)$$

By the quasi-minimality property of u_x we can estimate the second term in the right hand side of (4.2.77) as we did in (4.2.8), this gives

$$\begin{aligned} \int_{B_{r/3}} |\nabla(u_x - h_r)|^2 &\leq 3^d \int_{B_r} |\nabla(u_x - h_r)|^2 \leq C r^{\delta_A} \omega(u_x, r)^2 + C \\ &\leq C(r^{\delta_A} + \kappa_2^{-2}) \omega(u_x, r)^2, \end{aligned} \quad (4.2.78)$$

where in the last inequality we have used the second hypothesis in (4.2.73). On the other hand, estimates for harmonic functions give

$$\|\nabla h_r\|_{L^\infty(B_{r/3})} \leq \frac{C}{r} \|h_r\|_{L^\infty(B_{r/2})} \leq \frac{C}{r} \int_{\partial B_r} |h_r| = \frac{C}{r} b^+(u_x, r). \quad (4.2.79)$$

We now want to estimate $b^+(u_x, r)$ in terms of $r\omega(u_x, r)$. Let us assume that $\omega(u_x^+, r) \leq \omega(u_x^-, r)$, the same proof holds if the opposite inequality is satisfied. We first prove that for $\xi_0 \in B_{r/2}$ and $\eta < 1/2$ we have

$$\int_{\partial B_r} u_x^+ - \int_{\partial B_{\eta r}(\xi_0)} u_x^+ \leq C_d \eta^{1-d} r \int_{B_r} |\nabla u_x^+|. \quad (4.2.80)$$

Notice that up to considering the function $\xi \mapsto u_x^+(r\xi)$ we can assume that $r = 1$. Let us define a one to one function $F : B_1 \setminus B_\eta \rightarrow B_1 \setminus B_\eta(\xi_0)$ by

$$F(\xi) = \xi + \frac{1 - |\xi|}{1 - \eta} \xi_0.$$

We set $v = u_x^+ \circ F$. For every $\xi \in \partial B_1$ we have by the fundamental theorem of the calculus

$$v(\xi) - v(\eta\xi) = \int_\eta^1 \frac{d}{dt} v(t\xi) dt \leq \int_\eta^1 |\nabla v(t\xi)| dt.$$

Note that F is the identity on ∂B_1 and is simply a translation on ∂B_η . Therefore, averaging on $\xi \in \partial B_1$ (and since $|\nabla v| \leq C_d |\nabla u_x^+ \circ F|$) we get

$$\begin{aligned} \int_{\partial B_1} u_x^+ - \int_{\partial B_\eta(\xi_0)} u_x^+ &= \int_{\partial B_1} v - \int_{\partial B_\eta(\xi_0)} v \leq \frac{1}{d\omega_d} \int_\eta^1 dt \int_{\partial B_1} |\nabla v(t\xi)| d\mathcal{H}^{d-1}(\xi) \\ &\leq \frac{\eta^{1-d}}{d\omega_d} \int_{B_1 \setminus B_\eta} |\nabla v| \leq C_d \eta^{1-d} \int_{B_1 \setminus B_\eta} |\nabla u_x^+ \circ F| \\ &\leq C_d \eta^{1-d} \int_{B_1 \setminus B_\eta(\xi_0)} |\nabla u_x^+| |\det \nabla F^{-1}| \leq C_d \eta^{1-d} \int_{B_1} |\nabla u_x^+|, \end{aligned}$$

which proves (4.2.80). Now, let $\xi_0 \in B_{(2\lambda_A)^{-2}r}$ be such that $u_x(\xi_0) = 0$ as in (4.2.72). By Proposition 4.2.5 we have for every $\xi \in B_{\eta r}(\xi_0)$ (and because $F_x(\xi_0), F_x(\xi) \in B_{(2\lambda_A)^{-1}r}(x)$ if η is small enough)

$$\begin{aligned} u_x^+(\xi) &\leq |u_x(\xi)| = |u_x(\xi_0) - u_x(\xi)| = |u(F_x(\xi_0)) - u_x(F_x(\xi))| \\ &\leq C |F_x(\xi_0) - F_x(\xi)| \left(1 + \omega(u, x, \lambda_A^{-1}r) + \log \frac{\lambda_A^{-1}r}{|F_x(\xi_0) - F_x(\xi)|} \right) \\ &\leq C |\xi_0 - \xi| \left(1 + \omega(u, x, \lambda_A^{-1}r) + \log \frac{r}{|\xi_0 - \xi|} \right) \\ &\leq Cr(1 + \eta \omega(u_x, r)), \end{aligned} \tag{4.2.81}$$

where the last inequality holds for η small enough and since we have

$$\begin{aligned} \omega(u, x, \lambda_A^{-1}r)^2 &= \int_{B_{\lambda_A^{-1}r}(x)} |\nabla u|^2 \leq \lambda_A^2 \int_{B_{\lambda_A^{-1}r}(x)} A_x \nabla u \cdot \nabla u \\ &= \lambda_A^2 \int_{F_x^{-1}(B_{\lambda_A^{-1}r}(x))} |\nabla u_x|^2 \leq \lambda_A^{2(d+1)} \omega(u_x, r)^2. \end{aligned}$$

Moreover, recall that we assumed that $\omega(u_x^+, r) \leq \omega(u_x^-, r)$. Using the monotonicity formula in Corollary 4.2.16 we get

$$\omega(u_x^+, r)^4 \leq \omega(u_x^+, r)^2 \omega(u_x^-, r)^2 \leq C_d \Phi(u_x^+, u_x^-, r) \leq C_d C_m,$$

which implies by Cauchy-Schwarz's inequality

$$\int_{B_r} |\nabla u_x^+| \leq \left(\int_{B_r} |\nabla u_x^+|^2 \right)^{1/2} = \omega(u_x^+, r) \leq (C_d C_m)^{1/4}. \tag{4.2.82}$$

Therefore, combining (4.2.81), (4.2.80), (4.2.82) and using the first hypothesis in (4.2.73) we have (and also since $u_x^- = u_x^+ - u_x$)

$$\begin{aligned} b^+(u_x, r) &= \int_{\partial B_r} |u_x| = 2 \int_{\partial B_r} u_x^+ - \int_{\partial B_r} u_x \\ &\leq 2 \|u_x^+\|_{L^\infty(\partial B_{\eta r}(\xi_0))} + 2 \left(\int_{\partial B_r} u_x^+ - \int_{\partial B_{\eta r}(\xi_0)} u_x^+ \right) + |b(u_x, r)| \\ &\leq C \left((\eta + \gamma) \omega(u_x, r) + 1 + \eta^{1-d} C_m^{1/4} \right) r \\ &\leq C \left((\eta + \gamma) + (1 + \eta^{1-d} C_m^{1/4}) \kappa_2^{-1} \right) r \omega(u_x, r), \end{aligned} \tag{4.2.83}$$

where in the last inequality we used the second hypothesis in (4.2.73). We now return to (4.2.77).

With (4.2.78), (4.2.79) and (4.2.83) we get

$$\begin{aligned} \omega(u_x, r/3)^2 &\leq \frac{C}{r^2} b^+(u_x, r)^2 + C(r^{\delta_A} + \kappa_2^{-2}) \omega(u_x, r)^2 \\ &\leq C \left((\eta + \gamma)^2 + r^{\delta_A} + (1 + \eta^{1-d} C_m^{1/4})^2 \kappa_2^{-2} \right) \omega(u_x, r)^2. \end{aligned}$$

Therefore, choosing first η, γ and r_K small enough and then κ_2 big enough (depending on η) we obtain (4.2.74), which concludes the proof. \square

We are now in position to prove Theorem 4.1.2 using an iterative argument and Propositions 4.2.8, 4.2.11, 4.2.14.

Proof of Theorem 4.1.2. Recall that we denote by u any coordinate function of the vector U and that we have to prove that u is locally Lipschitz continuous in D . Let $K \subset D$ be a compact set and let $x \in K$. Let $r \leq r_K$, where r_K is smaller than the constants given by Propositions 4.2.8, 4.2.11 and 4.2.14. Set $\kappa = \max\{\kappa_1, \kappa_2\}$ where κ_1, κ_2 are the constants given by Propositions 4.2.11 and 4.2.14. We consider the following four cases:

Case 1:

$$\text{either } u_x > 0 \text{ in } B_{(2\lambda_A)^{-2}r} \quad \text{or} \quad u_x < 0 \text{ in } B_{(2\lambda_A)^{-2}r} \quad (4.2.84)$$

Case 2:

$$\gamma r(1 + \omega(u_x, r)) \leq |b(u_x, r)| \quad \text{and} \quad \kappa \leq \omega(u_x, r), \quad (4.2.85)$$

Case 3:

$$|b(u_x, r)| \leq \gamma r(1 + \omega(u_x, r)) \quad \text{and} \quad \kappa \leq \omega(u_x, r), \quad (4.2.86)$$

Case 4:

$$\omega(u_x, r) \leq \kappa. \quad (4.2.87)$$

For $k \geq 0$ we set $r_k = 3^{-k}r$. We denote by k_0 , if it exists, the smallest integer $k \geq 0$ such that the pair (x, r_k) satisfies either (4.2.84) or (4.2.85), and we set $k_0 = +\infty$ otherwise. If $k_0 > 0$, then for every $k < k_0$ we have that: if (x, r_k) satisfies (4.2.86) then by Proposition 4.2.14 we have (notice that (4.2.72) holds since u is continuous and that (4.2.84) is not satisfied)

$$\omega(u_x, r_{k+1}) \leq \frac{1}{2}\omega(u_x, r_k),$$

while if (x, r_k) satisfies (4.2.87), then we have

$$\omega(u_x, r_{k+1}) \leq 3^{d/2}\omega(u_x, r_k) \leq 3^{d/2}\kappa.$$

Therefore, with an induction we get that for every $0 \leq k \leq k_0$

$$\omega(u_x, r_k) \leq \max\{2^{-k}\omega(u_x, r), 3^{d/2}\kappa\}. \quad (4.2.88)$$

Assume that $k_0 = +\infty$. If x is a Lebesgue point for ∇u , then 0 is a Lebesgue point of u_x and it follows from (4.2.88) that

$$|\nabla u(x)| \leq \lambda_A |\nabla u_x(0)| = \lambda_A \lim_{k \rightarrow +\infty} \omega(u_x, r_k) \leq \lambda_A 3^{d/2}\kappa.$$

Assume now that $k_0 < +\infty$. Then, by definition of k_0 , the pair (x, r_{k_0}) satisfies either (4.2.84) or (4.2.85). If (4.2.84) holds, then Proposition 4.2.8 infers that u is $C^{1,\beta}$ near x and that we have (using also (4.2.88))

$$\begin{aligned} |\nabla u(x)| &\leq C_K(1 + \omega(u, x, (2\lambda_A)^{-2}r_{k_0})) \leq C_K(1 + \omega(u, x, \lambda_A^{-1}r_{k_0})) \\ &\leq C_K(1 + \omega(u_x, r_{k_0})) \leq C_K(1 + \max\{2^{-k_0}\omega(u_x, r), 3^{d/2}\kappa\}) \\ &\leq C_K(1 + \omega(u_x, r)). \end{aligned}$$

Moreover, by Proposition 4.2.11 the same estimate holds if the pair (x, r_{k_0}) satisfies (4.2.85). Therefore, in all cases it follows that for almost every point $x \in K$ and every $r \leq r_K$ we have

$$|\nabla u(x)| \leq C_K(1 + \omega(u_x, r)). \quad (4.2.89)$$

Let now $x_0 \in K$. Then, for almost every $x \in B_{r_K/2}(x_0)$, it follows by (4.2.89) that

$$|\nabla u(x)| \leq C_K(1 + \omega(u_x, r_K/2)) \leq C_K(1 + \omega(u_{x_0}, r_K)).$$

With a compactness argument this proves that u is locally Lipschitz continuous in D and completes the proof. \square

4.3 Lipschitz continuity of the eigenfunctions

This section is dedicated to the proof of Theorem 4.1.1. Precisely, we prove that the vector $U = (u_1, \dots, u_k)$ of the first k eigenfunctions on an optimal set for (4.1.1) is locally Lipschitz continuous in D . Using an idea taken from [77], we show that U is a quasi-minimizer in the sense of (4.1.4), and we then apply Theorem 4.1.2 to get the Lipschitz continuity of U .

4.3.1 Preliminaries and existence of an optimal set

We start with some properties about the spectrum of the operator in divergence form defined in 4.1.2, and we then prove that the problem (4.1.1) admits a solution among the class of quasi-open sets.

Let us define the weighted Lebesgue measure $m = b dx$, where dx stands for the Lebesgue measure in \mathbb{R}^d . For a quasi-open set $\Omega \subset \mathbb{R}^d$, we define the spaces $L^2(\Omega; m) = L^2(\Omega)$ and $H_0^1(\Omega; m) = H_0^1(\Omega)$ endowed respectively with the following norms

$$\|u\|_{L^2(\Omega; m)} = \left(\int_{\Omega} u^2 dm \right)^{1/2} \quad \text{and} \quad \|u\|_{H^1(\Omega; m)} = \|u\|_{L^2(\Omega; m)} + \|\nabla u\|_{L^2(\Omega)}.$$

Moreover, if $\Omega = \mathbb{R}^d$ we will simply write $\|u\|_{L^2(m)} = \|u\|_{L^2(\mathbb{R}^d; m)}$. We notice that, by the hypothesis (4.1.7) on the function b , the norms $\|\cdot\|_{L^2(\Omega; m)}$ and $\|\cdot\|_{L^2(\Omega)}$ are equivalent. We stress out that the choice of these norms is natural in view of (4.1.2) and is motivated by the variational formulation of the sum of the first k eigenfunctions (see (4.3.1) below). Now, the Lax-Milgram theorem and the Poincaré inequality imply that for every $f \in L^2(\Omega, m)$ there exists a unique solution $u \in H_0^1(\Omega, m)$ to the problem

$$-\operatorname{div}(A\nabla u) = f b \text{ in } \Omega, \quad u \in H_0^1(\Omega, m).$$

The resolvent operator $R_{\Omega} : f \in L^2(\Omega; m) \rightarrow H_0^1(\Omega; m) \subset L^2(\Omega; m)$ defined as $R_{\Omega}(f) = u$ is a continuous, self-adjoint and positive operator. Since the Sobolev space $H_0^1(\Omega; m)$ is compactly embedded into $L^2(\Omega; m)$ (because we have assumed that $b \geq c_b > 0$, see (4.1.7)), the resolvent R_{Ω} is in addition a compact operator. We say that a complex number λ is an eigenvalue of the operator (4.1.2) in Ω if there exists a non-trivial eigenfunction $u \in H_0^1(\Omega; m)$ solution of the equation

$$-\operatorname{div}(A\nabla u) = \lambda u b \text{ in } \Omega, \quad u \in H_0^1(\Omega; m).$$

The above properties of the resolvent ensure that the spectrum of the operator (4.1.2) in Ω is given by an increasing sequence of eigenvalues which are strictly positive real numbers, non-necessarily distinct, and which we denote by

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \dots \leq \lambda_k(\Omega) \leq \dots$$

The eigenvalues $\lambda_k(\Omega)$ are variationnaly characterized by the following min-max formula

$$\lambda_k(\Omega) = \min_{\substack{V \text{ subspace of} \\ \text{dimension } k \text{ of } H_0^1(\Omega, m)}} \max_{v \in V \setminus \{0\}} \frac{\int_{\Omega} A\nabla v \cdot \nabla v dx}{\int_{\Omega} v^2 dm}.$$

Moreover, we denote by u_k the normalized (with respect to the norm $\|\cdot\|_{L^2(\Omega; m)}$) eigenfunctions corresponding to the eigenvalues $\lambda_k(\Omega)$ and note that the family $(u_k)_k$ form an orthonormal system in $L^2(\Omega; m)$, that is

$$\int_{\Omega} u_i u_j dm = \delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

As a consequence, we have the following variational formulation for the sum of the first k eigenvalues on a quasi-open set Ω

$$\sum_{i=1}^k \lambda_i(\Omega) = \min \left\{ \int_{\Omega} A \nabla V \cdot \nabla V \, dx : V = (v_1, \dots, v_k) \in H_0^1(\Omega, \mathbb{R}^k), \int_{\Omega} v_i v_j \, dm = \delta_{ij} \right\}, \quad (4.3.1)$$

for which the minimum is attained for the vector $U = (u_1, \dots, u_k)$. We now deduce from this characterization that the minimum in (4.1.1) is reached.

Proposition 4.3.1 (Existence). *The shape optimization problem (4.1.1) has a solution.*

Proof. Let $(\Omega_n)_{n \in \mathbb{N}}$ be a minimizing sequence of quasi-open sets to the problem (4.1.1) and denote by $U_n = (u_1^n, \dots, u_k^n)$ the first k eigenfunctions on Ω_n . Since the matrices A_x , $x \in D$, are uniformly elliptic, we have the following inequality

$$\lambda_A^{-2} \int_D |\nabla U_n|^2 \, dx \leq \int_D A \nabla U_n \cdot \nabla U_n \, dx = \sum_{i=1}^k \lambda_i(\Omega_n)$$

which infers that the norm $\|U_n\|_{H^1}$ is uniformly bounded. Therefore, up to a subsequence, U_n converges weakly in $H^1(D, \mathbb{R}^k)$ and strongly in $L^2(D, \mathbb{R}^k)$ to some $V \in H^1(D, \mathbb{R}^k)$. Notice that V is an orthonormal vector. Set $\Omega^* := \{|V| > 0\}$. Then using (4.3.1), the weak convergence in H^1 of U_n to V and the semi-continuity of the Lebesgue measure we have

$$\begin{aligned} \sum_{i=1}^k \lambda_i(\Omega^*) + \Lambda |\Omega^*| &\leq \int_D A \nabla V \cdot \nabla V \, dx + \Lambda |\{|V| > 0\}| \\ &\leq \liminf_n \left(\int_D A \nabla U_n \cdot \nabla U_n \, dx + \Lambda |\{|U_n| > 0\}| \right) \\ &\leq \liminf_n \left(\sum_{i=1}^k \lambda_i(\Omega_n) + \Lambda |\Omega_n| \right) \end{aligned}$$

which concludes the proof. □

In the next Lemma we prove that the eigenfunctions are bounded. This result is a consequence of Lemma 4.2.1 and we refer to [84, Lemma 5.4] for a proof which is based on an interpolation argument.

Lemma 4.3.2 (Boundedness of the eigenfunctions). *Let $\Omega \subset \mathbb{R}^d$ be a bounded quasi-open set. There exist a dimensional constant $n \in \mathbb{N}$ and a constant $C > 0$ depending only on d, λ_A, c_b and $|\Omega|$, such that the resolvent operator $R_\Omega : L^2(\Omega; m) \rightarrow L^2(\Omega; m)$ satisfies*

$$R^n(L^2(\Omega; m)) \subset L^\infty(\Omega) \quad \text{and} \quad \|R^n\|_{\mathcal{L}(L^2(\Omega; m); L^\infty(\Omega))} \leq C.$$

In particular, if u is an eigenfunction on Ω normalized by $\|u\|_{L^2(m)} = 1$, then $u \in L^\infty(\Omega)$ and

$$\|u\|_{L^\infty} \leq C \lambda(\Omega)^n,$$

where $\lambda(\Omega)$ denotes the eigenvalue corresponding to u .

To conclude this subsection, we show that the first eigenfunction on an optimal set Ω^* keeps the same sign on every connected component of Ω^* . Notice that Ω^* may not be connected and has at most k connected components.

Lemma 4.3.3 (Sign of the principal eigenfunction). *Let $\Omega \subset D$ be an open and connected set and let $u \in H_0^1(\Omega)$ be the normalized first eigenvalue on Ω , that is*

$$-\operatorname{div}(A\nabla u) = \lambda_1(\Omega)bu \quad \text{in } \Omega \quad \text{and} \quad \int_{\Omega} u^2 dm = 1.$$

Then u is non-negative in Ω (up to a change of sign).

Proof. We assume that $u^+ \neq 0$ (if not, take $-u$ instead of u) and we set

$$u_+ = u^+ / \|u^+\|_{L^2(m)} \quad \text{and} \quad u_- = u^- / \|u^-\|_{L^2(m)}.$$

Since u is variationally characterized by

$$\lambda_1(\Omega) = \int_{\Omega} A\nabla u \cdot \nabla u dx = \min \left\{ \int_{\Omega} A\nabla \tilde{u} \cdot \nabla \tilde{u} dx : \tilde{u} \in H_0^1(\Omega), \int_{\Omega} \tilde{u}^2 dm = 1 \right\}, \quad (4.3.2)$$

we have

$$\int_{\Omega} A\nabla u \cdot \nabla u dx \leq \int_{\Omega} A\nabla u_+ \cdot \nabla u_+ dx \quad \text{and} \quad \int_{\Omega} A\nabla u \cdot \nabla u dx \leq \int_{\Omega} A\nabla u_- \cdot \nabla u_- dx.$$

Then, it follows that the two above inequalities are in fact equalities since otherwise we have

$$\begin{aligned} \int_{\Omega} A\nabla u \cdot \nabla u dx &= \int_{\Omega} A\nabla u^+ \cdot \nabla u^+ dx + \int_{\Omega} A\nabla u^- \cdot \nabla u^- dx \\ &> \left(\int_{\Omega} (u^+)^2 dm + \int_{\Omega} (u^-)^2 dm \right) \int_{\Omega} A\nabla u \cdot \nabla u dx = \int_{\Omega} A\nabla u \cdot \nabla u dx, \end{aligned}$$

which is absurd. In view of the minimization characterization (4.3.2), this ensures that u_+ is solution of the equation

$$-\operatorname{div}(A\nabla u_+) = \lambda_1(\Omega)u_+b \quad \text{in } \Omega.$$

Then, the strong maximum principle (see [48, Theorem 8.19]) and the connectedness of Ω imply that u_+ is strictly positive in Ω , which completes the proof. \square

4.3.2 Quasi-minimality and Lipschitz continuity of the eigenfunctions

We prove that the vector $U = (u_1, \dots, u_k)$ of normalized eigenfunctions on an optimal set Ω^* for the problem (4.1.1) is a local quasi-minimizer of the vector-valued functional

$$H_0^1(D, \mathbb{R}^k) \ni \tilde{U} \mapsto \int_D A\nabla \tilde{U} \cdot \nabla \tilde{U} dx + \Lambda|\{|\tilde{U}| > 0\}|$$

in the sense of the Proposition below. The Lipschitz continuity of the eigenfunctions is then a consequence of Theorem 4.1.2. We notice that, in view of the variational formulation (4.3.1), the vector U is solution to the following problem

$$\min \left\{ \int_D A\nabla V \cdot \nabla V dx + \Lambda|\{ |V| > 0 \}| : V = (v_1, \dots, v_k) \in H_0^1(D, \mathbb{R}^k), \int_D v_i v_j dm = \delta_{ij} \right\}. \quad (4.3.3)$$

Proposition 4.3.4 (Quasi-minimality of U). *Let $\Omega^* \subset D$ be an optimal set for the problem (4.1.1). Then the vector of orthonormalized eigenfunctions $U = (u_1, \dots, u_k) \in H_0^1(\Omega^*, \mathbb{R}^k)$ satisfies the following quasi-minimality condition: for every $C_1 > 0$ there exist constants $\varepsilon \in (0, 1)$ and $C > 0$, depending only on $d, k, C_1, \|U\|_{L^\infty}$ and $|D|$, such that*

$$\int_D A\nabla U \cdot \nabla U dx + \Lambda|\{ |U| > 0 \}| \leq (1 + C\|U - \tilde{U}\|_{L^1}) \int_D A\nabla \tilde{U} \cdot \nabla \tilde{U} dx + \Lambda|\{ |\tilde{U}| > 0 \}|, \quad (4.3.4)$$

for every $\tilde{U} \in H_0^1(D, \mathbb{R}^k)$ such that $\|U - \tilde{U}\|_{L^1} \leq \varepsilon$ and $\|\tilde{U}\|_{L^\infty} \leq C_1$.

The next Lemma, in which we get rid of the orthogonality constraint in (4.3.3), is similar to Lemma 2.5 in [77] with only slight modifications, but we decided to recall the whole proof for a sake of completeness.

Lemma 4.3.5. *Let $\Omega \subset D$ be a quasi-open set and let $U = (u_1, \dots, u_k)$ be the vector of normalized eigenvalues on Ω . Let $\delta > 0$. Then there exist $\bar{\varepsilon}_k \in (0, 1)$ and $\bar{C}_k > 0$, depending only on d, k, δ and $|\Omega|$, such that for every $\tilde{U} = (\tilde{u}_1, \dots, \tilde{u}_k) \in H_0^1(D, \mathbb{R}^k)$ satisfying*

$$\varepsilon_k := \sum_{i=1}^k \int_D |\tilde{u}_i - u_i| dm \leq \bar{\varepsilon}_k \quad \text{and} \quad \sup_{i=1, \dots, k} \left\{ \|u_i\|_{L^\infty} + \|\tilde{u}_i\|_{L^\infty} \right\} \leq \delta$$

the following estimate holds

$$\int_D A \nabla V \cdot \nabla V dx \leq (1 + \bar{C}_k \varepsilon_k) \int_D A \nabla \tilde{U} \cdot \nabla \tilde{U} dx, \quad (4.3.5)$$

where $V = (v_1, \dots, v_k) \in H_0^1(D, \mathbb{R}^k)$ is the vector obtained by orthonormalizing \tilde{U} with the Gram-Schmidt procedure:

$$v_i = w_i / \|w_i\|_{L^2(m)} \quad \text{where} \quad w_i = \begin{cases} \tilde{u}_1 & \text{if } i = 1 \\ \tilde{u}_i - \sum_{j=1}^{i-1} \left(\int_D \tilde{u}_i v_j dm \right) v_j & \text{if } i = 2, \dots, k. \end{cases}$$

Proof. We first prove an estimate of $\|u_k - w_k\|_{L^2(m)}$ in terms of ε_k . Precisely, we prove by induction on k that there exist constants $\bar{\varepsilon}_k \in (0, 1)$ and $C_k > 0$ such that the following estimates hold whenever $\varepsilon_k \leq \bar{\varepsilon}_k$

$$\|u_k - w_k\|_{L^1(m)} \leq C_k \varepsilon_k, \quad \sum_{i=1}^k \|u_i - v_i\|_{L^1(m)} \leq C_k \varepsilon_k, \quad \max_{i=1, \dots, k} \|v_i\|_{L^\infty} \leq C_k. \quad (4.3.6)$$

For $k = 1$ the first estimate obviously holds with $C_1 \geq 1$. Moreover we have

$$\begin{aligned} \|\tilde{u}_1 - v_1\|_{L^1(m)} &= \frac{\|\tilde{u}_1\|_{L^2(m)} - 1}{\|\tilde{u}_1\|_{L^2(m)}} \|\tilde{u}_1\|_{L^1(m)} \leq \frac{\|\tilde{u}_1\|_{L^2(m)}^2 - 1}{\|\tilde{u}_1\|_{L^2(m)}^2} \|\tilde{u}_1\|_{L^1(m)} \\ &= \frac{\|u_1 + (\tilde{u}_1 - u_1)\|_{L^2(m)}^2 - 1}{\|u_1 + (\tilde{u}_1 - u_1)\|_{L^2(m)}^2} \|u_1 + (\tilde{u}_1 - u_1)\|_{L^1(m)} \\ &\leq \frac{2 \int u_1 |\tilde{u}_1 - u_1| dm + \|\tilde{u}_1 - u_1\|_{L^2(m)}^2}{1 - 2 \int u_1 |\tilde{u}_1 - u_1| dm} \left(\|u_1\|_{L^1(m)} + \|\tilde{u}_1 - u_1\|_{L^1(m)} \right) \\ &\leq \frac{(2\|u_1\|_{L^\infty} + \|\tilde{u}_1 - u_1\|_{L^\infty}) \|\tilde{u}_1 - u_1\|_{L^1(m)}}{1 - 2\|u_1\|_{L^\infty} \|\tilde{u}_1 - u_1\|_{L^1(m)}} \left(\|u_1\|_{L^1(m)} + \|\tilde{u}_1 - u_1\|_{L^1(m)} \right) \\ &\leq \frac{3\delta\varepsilon_1}{1 - 2\delta\varepsilon_1} \left(|\Omega|^{1/2} + \varepsilon_1 \right) \leq 12\delta|\Omega|^{1/2}\varepsilon_1, \end{aligned} \quad (4.3.7)$$

where the last inequality holds if $\varepsilon_1 \leq \min\{(4\delta)^{-1}, |\Omega|^{1/2}\}$. This gives the following L^1 -estimate

$$\|u_1 - v_1\|_{L^1(m)} \leq \|u_1 - \tilde{u}_1\|_{L^1(m)} + \|\tilde{u}_1 - v_1\|_{L^1(m)} \leq (1 + 12\delta|\Omega|^{1/2})\varepsilon_1.$$

Finally, we estimate the infinity norm

$$\begin{aligned} \|v_1\|_{L^\infty} &= \frac{\|\tilde{u}\|_{L^\infty}}{\|\tilde{u}_1\|_{L^2(m)}} = \frac{\|\tilde{u}\|_{L^\infty}}{\|u_1 + (\tilde{u}_1 - u_1)\|_{L^2(m)}} \leq \frac{\|\tilde{u}\|_{L^\infty}}{(1 - 2 \int u_1 |\tilde{u}_1 - u_1| dm)^{1/2}} \\ &\leq \frac{\|\tilde{u}\|_{L^\infty}}{1 - 2 \int u_1 |\tilde{u}_1 - u_1| dm} \leq \frac{\delta}{1 - 2\delta\varepsilon_1} \leq 2\delta, \end{aligned}$$

which proves the claim for $k = 1$. Suppose now that the claim holds for $1, \dots, k-1$. We first estimate $\|u_k - w_k\|_{L^1(m)}$. Since the functions u_i form an orthogonal system of $L^2(\Omega, m)$ and by the induction's hypothesis we have (and also because $\varepsilon_{k-1} \leq \varepsilon_k$)

$$\begin{aligned} \sum_{i=1}^{k-1} \left| \int_D \tilde{u}_k v_i dm \right| &\leq \sum_{i=1}^{k-1} \int_D \left(|\tilde{u}_k - u_k| u_i + |v_i - u_i| u_k + |v_i - u_i| |\tilde{u}_k - u_k| \right) dm \\ &\leq ((k-1)\delta + \delta C_{k-1} + (k-1)(C_{k-1} + \delta))\varepsilon_k =: \tilde{C}_k \varepsilon_k. \end{aligned} \quad (4.3.8)$$

Therefore, with the triangle inequality we obtain

$$\begin{aligned} \|u_k - w_k\|_{L^1(m)} &\leq \|u_k - \tilde{u}_k\|_{L^1(m)} + \sum_{i=1}^{k-1} \left| \int_D \tilde{u}_k v_i dm \right| (\|u_i\|_{L^1(m)} + \|v_i - u_i\|_{L^1(m)}) \\ &\leq (1 + \tilde{C}_k(|\Omega|^{1/2} + \varepsilon_{k-1}))\varepsilon_k \leq (1 + 2\tilde{C}_k|\Omega|^{1/2})\varepsilon_k. \end{aligned} \quad (4.3.9)$$

We now prove the second estimate in (4.3.6). Using once again (4.3.8), we have the following estimate of the L^∞ -norm of w_k

$$\|w_k\|_{L^\infty} \leq \|\tilde{u}_k\|_{L^\infty} + \sum_{i=1}^k \left| \int_D \tilde{u}_k v_i dm \right| \|v_i\|_{L^\infty} \leq \delta + C_{k-1} \tilde{C}_k. \quad (4.3.10)$$

Moreover, with the same procedure as in (4.3.7), it follows from (4.3.9) and (4.3.10) that

$$\begin{aligned} \|\tilde{u}_k - v_k\|_{L^1(m)} &\leq \frac{(3\|u_k\|_{L^\infty} + \|w_k\|_{L^\infty})\|w_k - u_k\|_{L^1(m)}}{1 - 2\|u_k\|_{L^\infty}\|w_k - u_k\|_{L^1(m)}} \left(\|u_k\|_{L^1(m)} + \|w_k - u_k\|_{L^1(m)} \right) \\ &\leq \frac{(4\delta + C_{k-1}\tilde{C}_k)(1 + 2\tilde{C}_k|\Omega|^{1/2})}{1 - 2\delta(1 + 2\tilde{C}_k|\Omega|^{1/2})\varepsilon_k} (|\Omega|^{1/2} + (1 + 2\tilde{C}_k|\Omega|^{1/2}))\varepsilon_k. \end{aligned}$$

Now, choose $\varepsilon_k \leq [4\delta(1 + 2\tilde{C}_k|\Omega|^{1/2})]^{-1}$ so that with the triangle inequality we have

$$\begin{aligned} \|u_k - v_k\|_{L^1(m)} &\leq \|u_k - \tilde{u}_k\|_{L^1(m)} + \|\tilde{u}_k - v_k\|_{L^1(m)} \\ &\leq \left[1 + 2(4\delta + C_{k-1}\tilde{C}_k)(1 + 2\tilde{C}_k|\Omega|^{1/2})(|\Omega|^{1/2} + (1 + 2\tilde{C}_k|\Omega|^{1/2})) \right] \varepsilon_k. \end{aligned}$$

We then use the inductive hypothesis to get the desired L^1 -estimate. It remains only to estimate $\|v_k\|_{L^\infty}$. Firstly, notice that we have

$$\begin{aligned} |\|w_k\|_{L^2(m)} - 1| &\leq |\|w_k\|_{L^2(m)}^2 - 1| = |\|u_k + (w_k - u_k)\|_{L^2(m)}^2 - 1| \\ &\leq \left| 2 \int_D u_k (u_k - w_k) dm + \int_D (u_k - w_k)^2 dm \right| \\ &\leq (3\|u_k\|_{L^\infty} + \|w_k\|_{L^\infty})\|u_k - w_k\|_{L^1(m)} \\ &\leq (4\delta + C_{k-1}\tilde{C}_k)(1 + 2\tilde{C}_k|\Omega|^{1/2})\varepsilon_k. \end{aligned}$$

Thus, with the extra assumption $\varepsilon_k \leq [(4\delta + C_{k-1}\tilde{C}_k)(1 + 2\tilde{C}_k|\Omega|^{1/2})]^{-1}$, it follows that $1/2 \leq \|w_k\|_{L^2(m)} \leq 3/2$. With (4.3.10) this gives the following L^∞ -norm of v_k

$$\|v_k\|_{L^\infty} = \frac{\|w_k\|_{L^\infty}}{\|w_k\|_{L^2(m)}} \leq 2(\delta + C_{k-1}\tilde{C}_k)$$

and concludes the proof of the claim.

We are now in position to prove the Lemma by induction. For $k = 1$, we ask that $\varepsilon_1 \leq (4\delta)^{-1}$, so that we have

$$\begin{aligned} \int_D A \nabla v_1 \cdot \nabla v_1 \, dx &\leq \|\tilde{u}_1\|_{L^2(m)}^{-2} \int_D A \nabla \tilde{u}_1 \cdot \nabla \tilde{u}_1 \, dx \\ &\leq (1 - 2\|u_1\|_{L^\infty} \|\tilde{u}_1 - u_1\|_{L^1(m)})^{-2} \int_D A \nabla \tilde{u}_1 \cdot \nabla \tilde{u}_1 \, dx \\ &\leq (1 + 4\delta\varepsilon_1)^2 \int_D A \nabla \tilde{u}_1 \cdot \nabla \tilde{u}_1 \, dx \leq (1 + 12\delta\varepsilon_1) \int_D A \nabla \tilde{u}_1 \cdot \nabla \tilde{u}_1 \, dx. \end{aligned}$$

Suppose now that the Lemma holds for $1, \dots, k-1$. Thanks to the first estimate in (4.3.6) of the preceding claim we have

$$\|w_k\|_{L^2(m)}^{-2} \leq (1 - 2\|u_k\|_{L^\infty} \|u_k - w_k\|_{L^1(m)})^{-2} \leq (1 + 4\delta C_k \varepsilon_k)^2 \leq 1 + 12\delta C_k \varepsilon_k,$$

where the last inequality holds if $\varepsilon_k \leq (4\delta C_k)^{-1}$. On the other hand, for every $i = 1, \dots, k-1$, we have by the inductive assumption

$$\int_D A \nabla v_i \cdot \nabla v_i \, dx \leq \sum_{j=1}^{k-1} \int_D A \nabla v_j \cdot \nabla v_j \, dx \leq (1 + \bar{C}_{k-1} \varepsilon_{k-1}) \sum_{j=1}^{k-1} \int_D A \nabla \tilde{u}_j \cdot \nabla \tilde{u}_j \, dx.$$

Therefore, using the estimate (4.3.8) we get

$$\begin{aligned} \left(\int_D A \nabla w_k \cdot \nabla w_k \, dx \right)^{1/2} &\leq \left(\int_D A \nabla \tilde{u}_k \cdot \nabla \tilde{u}_k \, dx \right)^{1/2} + \sum_{i=1}^{k-1} \left| \int_D \tilde{u}_k v_i \, dm \right| \left(\int_D A \nabla v_i \cdot \nabla v_i \, dx \right)^{1/2} \\ &\leq \left(\int_D A \nabla \tilde{u}_k \cdot \nabla \tilde{u}_k \, dx \right)^{1/2} + \tilde{C}_k \varepsilon_k (1 + \bar{C}_{k-1})^{1/2} \left(\sum_{j=1}^{k-1} \int_D A \nabla \tilde{u}_j \cdot \nabla \tilde{u}_j \, dx \right)^{1/2}. \end{aligned}$$

We then ask that $\varepsilon_k \leq (2\tilde{C}_k)^{-1} (1 + \bar{C}_{k-1})^{-1/2}$ so that we get

$$\begin{aligned} \int_D A \nabla v_k \cdot \nabla v_k \, dx &= \|w_k\|_{L^2(m)}^{-2} \int_D A \nabla w_k \cdot \nabla w_k \, dx \\ &\leq (1 + 12\delta C_k \varepsilon_k) \left((1 + \tilde{C}_k (1 + \bar{C}_{k-1})^{1/2} \varepsilon_k) \int_D A \nabla \tilde{u}_k \cdot \nabla \tilde{u}_k \, dx + \sum_{j=1}^{k-1} \int_D A \nabla \tilde{u}_j \cdot \nabla \tilde{u}_j \, dx \right). \end{aligned}$$

This, using once again the inductive hypothesis, proves (4.3.5) and concludes the proof. \square

Proof of Proposition 4.3.4. Let \tilde{U} be a vector satisfying the hypothesis of Proposition 4.3.4 and let $V \in H_0^1(D, \mathbb{R}^k)$ be the vector given by Lemma 4.3.5 and obtained by orthonormalizing \tilde{U} . Using V as a test function in (4.3.3) and then Lemma 4.3.5, we have

$$\begin{aligned} \int_D A \nabla U \cdot \nabla U \, dx + \Lambda |\{|U| > 0\}| &\leq \int_D A \nabla V \cdot \nabla V \, dx + \Lambda |\{|V| > 0\}| \\ &\leq (1 + \bar{C}_k \|U - \tilde{U}\|_{L^1}) \int_D A \nabla \tilde{U} \cdot \nabla \tilde{U} \, dx + \Lambda |\{|\tilde{U}| > 0\}|, \end{aligned}$$

where we have used in the last inequality that $\{|V| > 0\} \subset \{|\tilde{U}| > 0\}$ (which holds by construction of V). \square

We now conclude the proof of Theorem 4.1.1.

Proof of Theorem 4.1.1. The proof follows from Proposition 4.3.4 and Theorem 4.1.2 (see also Lemma 4.3.2). \square

Chapter 5

Regularity of optimal sets for some functional involving eigenvalues of an operator in divergence form

This chapter is based on [88].

Abstract

In this paper we consider minimizers of the functional

$$\min \{ \lambda_1(\Omega) + \cdots + \lambda_k(\Omega) + \Lambda|\Omega|, : \Omega \subset D \text{ open} \}$$

where $D \subset \mathbb{R}^d$ is a bounded open set and where $0 < \lambda_1(\Omega) \leq \cdots \leq \lambda_k(\Omega)$ are the first k eigenvalues on Ω of an operator in divergence form with Dirichlet boundary condition and with Hölder continuous coefficients. We prove that the optimal sets Ω^* have finite perimeter and that their free boundary $\partial\Omega^* \cap D$ is composed of a *regular part*, which is locally the graph of a $C^{1,\alpha}$ -regular function, and a *singular part*, which is empty if $d < d^*$, discrete if $d = d^*$ and of Hausdorff dimension at most $d - d^*$ if $d > d^*$, for some $d^* \in \{5, 6, 7\}$.

5.1 Introduction

This paper is dedicated to the regularity properties of the minimizers to the problem

$$\min \{ \lambda_1(\Omega) + \cdots + \lambda_k(\Omega) + \Lambda|\Omega| : \Omega \subset D \text{ open} \} \quad (5.1.1)$$

where $D \subset \mathbb{R}^d$ is a bounded open set (a box), Λ is a positive constant and $0 < \lambda_1(\Omega) \leq \cdots \leq \lambda_k(\Omega)$ stand for the first k eigenvalues (counted with the due multiplicity) of an operator in divergence form. More precisely, we consider the operator $-b(x)^{-1} \operatorname{div}(A_x \nabla \cdot)$, where the matrix-valued function $A : D \rightarrow \operatorname{Sym}_d^+$ is uniformly elliptic with Hölder continuous coefficients, and $b \in W^{1,\infty}(D)$ is a positive Lipschitz continuous function bounded away from 0. This means that for every eigenvalue $\lambda_i(\Omega)$ there exists an eigenfunction $u_i \in H_0^1(\Omega)$ such that

$$\begin{cases} -\operatorname{div}(A \nabla u_i) = \lambda_i(\Omega) b u_i & \text{in } \Omega \\ u_i = 0 & \text{on } \partial\Omega. \end{cases} \quad (5.1.2)$$

We now state in the following theorem the main result of this present paper.

Theorem 5.1.1. *Let $D \subset \mathbb{R}^d$ be a bounded open set and let $A : D \rightarrow \text{Sym}_d^+$, $b \in W^{1,\infty}(D)$ satisfying (5.1.5), (5.1.6) and (5.1.7) (see below). Then every solution Ω^* to the problem (5.1.1) has finite perimeter. Moreover, the free boundary $\partial\Omega^* \cap D$ can be decomposed into the disjoint union of a regular part $\text{Reg}(\partial\Omega^* \cap D)$ and a singular part $\text{Sing}(\partial\Omega^* \cap D)$, where:*

1. *$\text{Reg}(\partial\Omega^* \cap D)$ is locally the graph of a $C^{1,\alpha}$ -regular function. If, moreover, $a_{i,j} \in C^{k,\delta}(D)$ and $b \in C^{k-1,\delta}(D)$ for some $\delta \in (0,1)$ and $k \geq 1$, then $\text{Reg}(\partial\Omega^* \cap D)$ is locally the graph of a $C^{k+1,\alpha}$ -regular function.*
2. *for a universal constant $d^* \in \{5, 6, 7\}$ (see Definition 5.4.19), $\text{Sing}(\partial\Omega^* \cap D)$ is:*
 - *empty if $d < d^*$;*
 - *discrete if $d = d^*$;*
 - *of Hausdorff dimension at most $(d - d^*)$ if $d > d^*$.*

The problem (5.1.1) can also be considered in the class of the quasi-open sets, but we stress out that it is the same thing. Indeed, preliminary results, inspired by the work of David and Toro in [34] (see also [32]), have already been obtained in [87] in view to prove the regularity of the minimizers to (5.1.1). The main results of the paper are stated in Theorem 5.1.2 below, where the author shows that if a quasi-open set Ω^* is solution, among the class of quasi-open sets, to the problem (5.1.1), then the first k eigenfunctions on Ω^* are locally Lipschitz continuous, and hence Ω^* is an open set.

One of the main interest and difficulty of this paper is to consider an operator with variable coefficients. This case is more involved than the case of the Laplacian and has been studied only recently. We notice that our result is quite general and applies, for instance, to an operator with drift $-\Delta + \nabla\Phi \cdot \nabla$ or in the case of a manifold.

The first result concerning the regularity of the free boundary of optimal sets (for spectral functionals) was established by Briançon and Lamboley in [11], where they consider the minimization problem of the first eigenvalue of the Dirichlet Laplacian with inclusion and volume constraints. More precisely, using the strategy developed by Alt and Caffarelli in [2], they prove that the optimal sets for the problem

$$\min \{ \lambda_1(\Omega) : \Omega \subset D \text{ open, } |\Omega| \leq m \} \tag{5.1.3}$$

have C^∞ -regular boundary (inside D) up to a singular set whose $(d - 1)$ -Hausdorff measure is zero (provided that the box is bounded and connected). In [77], Mazzoleni, Terracini and Velichkov study the regularity properties of sets that minimize the sum of the first k eigenvalues of the Dirichlet Laplacian among all sets of fixed volume, that is, minimizers of

$$\min \{ \lambda_1(\Omega) + \dots + \lambda_k(\Omega) : \Omega \subset \mathbb{R}^d \text{ open, } |\Omega| = 1 \}. \tag{5.1.4}$$

They prove that the regular part of the boundary of an optimal set is C^∞ -regular and, thanks to a dimension's reduction argument due to Weiss (see [92]), that the singular set is of dimension at most $d - d^*$, hence improving the smallness estimate of the singular set. Meanwhile, Kriventsov and Lin consider in [70] a more general functional and prove that minimizers of

$$\min \{ F(\lambda_1(\Omega), \dots, \lambda_k(\Omega)) + |\Omega| : \Omega \subset \mathbb{R}^d \text{ open} \}.$$

are C^∞ -regular up to a singular set of dimension at most $d - 3$. Here, $F : \mathbb{R}^k \rightarrow \mathbb{R}$ is a function of class C^1 which is strictly increasing in each variable ($\partial_i F \geq c > 0$). In [71], they also obtain a regularity result in the case where the functional F is non-decreasing in its parameters, which hence apply to minimizers of

$$\min \{ F(\lambda_{k_1}(\Omega), \dots, \lambda_{k_n}(\Omega)) + |\Omega| : \Omega \subset \mathbb{R}^d \text{ quasi-open} \},$$

where the first eigenvalue is not necessary involved. Notice that in these problems, the main difficulty is to deal with higher eigenvalues since they have a min-max variational characterization.

On the other hand, regularity problems involving different operators have been studied only recently. In [84], the authors prove the regularity of the minimizers to (5.1.3) where λ_1 now stands for the first eigenvalue of a drifted operator $-\Delta + \nabla\Phi \cdot \nabla$ with Dirichlet boundary condition (for some $\Phi \in W^{1,\infty}(D, \mathbb{R}^d)$), and therefore extend the result of Briançon and Lamboley. We highlight that the operator considered in this paper (see (5.1.2)) is more general than the operator with drift $-\Delta + \nabla\Phi \cdot \nabla$ which corresponds to the special case where $A = e^{-\Phi}\text{Id}$ and $b = e^{-\Phi}$. Recently, Lamboley and Sicbaldi successfully treated the minimization problem (5.1.3) in the manifold setting with the Laplace-Beltrami operator (see [73]). They prove the existence of an optimal set among quasi-open set provided that the manifold M is compact and that optimal sets are C^∞ -regular if M is connected (and C^∞) up to $(d - d^*)$ -dimensional singular set.

Let us also mention that some regularity results have also been established in the context of multiphase shape optimization problems involving eigenvalues (see, for instance, [28], [25], [82], [85])

We notice that we deal with a penalized functional and that it is natural to expect that a similar result also holds with a volume constraint as in (5.1.3), but we will not address this question in this paper since our main motivation is to treat the case of an operator with variable coefficients.

5.1.1 Preliminaries and notations

We will use the following notations throughout this paper. We fix a matrix-valued function $A = (a_{ij})_{ij} : D \rightarrow \text{Sym}_d^+$, where Sym_d^+ denotes the family of the real positive symmetric $d \times d$ matrices, which is uniformly elliptic and has Hölder continuous coefficients. Precisely, there exist positive constants $\delta_A, c_A > 0$ and $\lambda_A \geq 1$ such that

$$|a_{ij}(x) - a_{ij}(y)| \leq c_A |x - y|^{\delta_A}, \quad \text{for every } i, j \quad \text{and } x, y \in D; \quad (5.1.5)$$

$$\frac{1}{\lambda_A^2} |\xi|^2 \leq \xi \cdot A_x \xi = \sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \leq \lambda_A^2 |\xi|^2, \quad \text{for every } x \in D \quad \text{and } \xi \in \mathbb{R}^d. \quad (5.1.6)$$

We also fix a Lipschitz continuous function $b \in W^{1,\infty}(D)$ which we assume to be positive and bounded away from zero: there exists $c_b > 0$ such that

$$c_b^{-1} \leq b(x) \leq c_b \quad \text{for every } x \in D. \quad (5.1.7)$$

We set $m = b dx$ and we define, for any an open set $\Omega \subset D$, the spaces $L^2(\Omega; m) = L^2(\Omega)$ and $H_0^1(\Omega; m) = H_0^1(\Omega)$ endowed respectively with the norms

$$\|u\|_{L^2(\Omega; m)} = \left(\int_{\Omega} u^2 dm \right)^{1/2} \quad \text{and} \quad \|u\|_{H^1(\Omega; m)} = \|u\|_{L^2(\Omega; m)} + \|\nabla u\|_{L^2(\Omega)}.$$

By the Lax-Milgram theorem and the Poincaré inequality, for every $f \in L^2(\Omega, m)$ there exists a unique solution $u \in H_0^1(\Omega, m)$ to the problem

$$-\text{div}(A\nabla u) = fb \text{ in } \Omega, \quad u \in H_0^1(\Omega, m).$$

The resolvent operator $R_\Omega : f \in L^2(\Omega; m) \rightarrow H_0^1(\Omega; m) \subset L^2(\Omega; m)$ defined as $R_\Omega(f) = u$ is continuous, self-adjoint, positive and compact (since $H_0^1(\Omega; m)$ is compactly embedded into $L^2(\Omega; m)$, because $b \geq c_b > 0$). Therefore, the operator $-b^{-1} \text{div}(A\nabla \cdot)$ in Ω has a discrete spectrum which consists in real and positive eigenvalues denoted by

$$0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \dots \leq \lambda_k(\Omega) \leq \dots$$

For every $\lambda_i(\Omega)$ there exists an eigenfunction $u_i \in H_0^1(\Omega; m)$ satisfying

$$-\operatorname{div}(A\nabla u_i) = \lambda_i(\Omega) b u_i \quad \text{in } \Omega,$$

where the PDE is intended in the weak sense, that is

$$\int_{\Omega} A\nabla u_i \cdot \nabla \varphi \, dx = \lambda_i(\Omega) \int_{\Omega} u_i \varphi \, dm \quad \text{for every } \varphi \in H_0^1(\Omega).$$

Moreover, the eigenfunctions $(u_i)_{i \in \mathbb{N}}$ (on an open set $\Omega \subset D$) will always be normalized with respect to the norm $\|\cdot\|_{L^2(\Omega; m)}$ and form an orthonormal system in $L^2(\Omega; m)$, that is

$$\int_{\Omega} u_i u_j \, dm = \delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

We denote by $H_0^1(\Omega, \mathbb{R}^k)$ the space of all vector-valued function $U = (u_1, \dots, u_k) : \Omega \rightarrow \mathbb{R}^k$ such that $u_i \in H_0^1(\Omega)$, endowed with the norm

$$\|U\|_{H^1(\Omega)} = \|U\|_{L^2(\Omega)} + \|\nabla U\|_{L^2(\Omega)} = \sum_{i=1}^k (\|u_i\|_{L^2(\Omega)} + \|\nabla u_i\|_{L^2(\Omega)}).$$

Similarly, we will also need the following norms for $U = (u_1, \dots, u_k) : \Omega \rightarrow \mathbb{R}^k$

$$\|U\|_{L^1(\Omega)} = \sum_{i=1}^k \|u_i\|_{L^1(\Omega)} \quad \text{and} \quad \|U\|_{L^\infty(\Omega)} = \sup_{i=1}^k \|u_i\|_{L^\infty(\Omega)}.$$

Moreover, for $U = (u_1, \dots, u_k) : \Omega \rightarrow \mathbb{R}^k$ we set $|U| = u_1^2 + \dots + u_k^2$, $|\nabla U|^2 = |\nabla u_1|^2 + \dots + |\nabla u_k|^2$ and $A\nabla U \cdot \nabla U = A\nabla u_1 \cdot \nabla u_1 + \dots + A\nabla u_k \cdot \nabla u_k$. Finally, for $f = (f_1, \dots, f_k) \in L^2(\Omega, \mathbb{R}^k)$ we say that $U = (u_1, \dots, u_k) \in H_0^1(\Omega, \mathbb{R}^k)$ is solution to the equation

$$-\operatorname{div}(A\nabla U) = f \quad \text{in } \Omega, \quad U \in H_0^1(\Omega, \mathbb{R}^k)$$

if, for every $i = 1, \dots, k$, the component u_i is solution to the equation

$$-\operatorname{div}(A\nabla u_i) = f_i \quad \text{in } \Omega, \quad u_i \in H_0^1(\Omega).$$

We summarize in the following theorem the main results obtained in [87].

Theorem 5.1.2. *Let $D \subset \mathbb{R}^d$ be a bounded open set and let $A : D \rightarrow \operatorname{Sym}_d^+$, $b \in L^\infty(D)$ satisfying (5.1.5), (5.1.6) and (5.1.7). Then the minimum*

$$\min \{ \lambda_1(\Omega) + \dots + \lambda_k(\Omega) + \Lambda|\Omega| : \Omega \subset D \text{ quasi-open} \} \quad (5.1.8)$$

is achieved. Moreover, the vector $U = (u_1, \dots, u_k) \in H_0^1(\Omega^, \mathbb{R}^k)$ of the first k normalized eigenfunctions on any optimal set Ω^* for (5.1.8) satisfies:*

1. $U \in L^\infty(D)$ and is a locally Lipschitz continuous function in D . In particular, Ω^* is an open set.
2. U satisfies the following quasi-minimality property: for every $C_1 > 0$ there exist constants $\varepsilon \in (0, 1)$ and $C > 0$, depending only on $d, k, C_1, \|U\|_{L^\infty}$ and $|D|$, such that

$$\int_D A\nabla U \cdot \nabla U \, dx + \Lambda\{|U| > 0\} \leq (1 + C\|U - \tilde{U}\|_{L^1}) \int_D A\nabla \tilde{U} \cdot \nabla \tilde{U} \, dx + \Lambda\{|\tilde{U}| > 0\}, \quad (5.1.9)$$

for every $\tilde{U} \in H_0^1(D, \mathbb{R}^k)$ such that $\|U - \tilde{U}\|_{L^1} \leq \varepsilon$ and $\|\tilde{U}\|_{L^\infty} \leq C_1$.

5.1.2 General strategy and main points of the proof

Throughout this paper we will always denote by Ω^* an optimal set to the problem (5.1.1). In section 5.2, we reduce to the case where $A = Id$ and prove that the vector $U = (u_1, \dots, u_k)$ of the first k eigenfunctions on Ω^* is, in some new set of coordinates, a quasi-minimizer of the Dirichlet energy in small balls centered at the origin (Proposition 5.2.2). We notice that we perform a change of coordinates near every point $x \in \partial\Omega^*$ and hence that one of the main issue is to deal with functions $U_x = U \circ F_x$ which depends on the point x (see (5.2.3) for the definition of F_x). We adapt the strategy developed by David and Toro in [34] to prove that U_x is non-degenerate (Proposition 5.2.3). Using an idea of Kriventsov and Lin in [70], we show that the first eigenfunction u_1 is non degenerate in Ω_1^* (Proposition 5.2.6), where Ω_1^* denotes any connected component of Ω^* where u_1 is positive. From this result we then deduce a uniform growth of u_1 near the boundary $\partial\Omega_1^*$ and a density estimate for Ω_1^* .

We notice that, unlike in [77], the optimal set Ω^* may not be connected. Indeed, the geometrical constraint imposed by the box D and the presence of variable coefficients do not allow to translate the connected components of Ω^* and hence to prove as in [77] that Ω^* is connected. However, we prove in Proposition 5.3.7 that the connected components of Ω^* cannot meet inside D . Therefore, in order to prove Theorem 5.1.1 it is enough to prove only the regularity of Ω_1^* (see also remark 5.1.3 below). This result comes from the structure of the blow-up limits studied in section 5.3, where we in particular prove that the blow-up limits are one-homogeneous functions and solution of the Alt-Caffarelli functional.

Section 5.4 is then dedicated to the regularity of Ω_1^* . Since we work with the first k eigenfunctions in a new set of coordinates, namely with U_x , we define the regular part of Ω_1^* in a different way than in [77] (see Definition 5.4.4). Then, we show as in [77] that we can reduce to a one-phase problem, for which the regularity of the free boundary was proved by De Silva (see [36] and [85, Appendix A]). To this aim, we prove that Ω_1^* is a non-tangentially accessible (NTA) domain near the regular points and we prove a boundary Harnack principle for the eigenfunctions $U = (u_1, \dots, u_k)$ on Ω_1^* . More precisely, we prove that for every x_0 on the regular part of the boundary $\partial\Omega_1^*$, the limits $g_i(x_0) = \lim_{x \rightarrow x_0} \frac{u_i(x)}{u_1(x)}$ exist and define Hölder continuous functions $g_i : \partial\Omega_1^* \cap B_r(x_0) \rightarrow \mathbb{R}$. We notice that one difficulty comes from the presence of the function b and that it is the only point in the paper where the Lipschitz continuity assumption on b is needed. As a consequence, we deduce that u_1 satisfies the following optimality condition

$$|A_x^{1/2}[\nabla u_1(x)]| = g(x)\sqrt{\Lambda} \quad \text{for every } x \in \partial\Omega_1^* \cap B_r(x_0),$$

where g is an Hölder continuous function depending on the functions g_i (see (5.4.16)). In subsection 5.4.5 we provide an estimation of the singular set by proving that we can apply the strategy developed by Weiss in [92] to the case of an operator in divergence form (see Lemmas 5.4.21 and 5.4.22).

Remark 5.1.3 (On the connected components of the optimal sets). We highlight that it is enough to prove the regularity of any connected component of Ω^* where the first eigenfunction is positive. Indeed, if Ω_0^* is a connected component of Ω^* , then there exists $k_0 > 0$ such that $\lambda_i(\Omega_0^*) \in \{\lambda_1(\Omega^*), \dots, \lambda_{k_0}(\Omega^*)\}$ for any $i \in \{1, \dots, k_0\}$ and $\lambda_i(\Omega_0^*) \notin \{\lambda_1(\Omega^*), \dots, \lambda_k(\Omega^*)\}$ for any $i > k_0$. Using that $\sigma(\Omega^*) = \sigma(\Omega_0^*) \cup \sigma(\Omega^* \setminus \Omega_0^*)$, it is straightforward to check that Ω_0^* is solution to the problem (5.1.1) with $k = k_0$ and $D = D \setminus (\overline{\Omega^*} \setminus \Omega_0^*)$. Notice also that the connected components of Ω^* cannot meet inside D (see Proposition 5.3.7).

Moreover, we notice that Ω^* has at most k connected components. Indeed, denote by Ω_i^* a connected component of Ω^* such that $\lambda_i(\Omega^*) \in \sigma(\Omega_i^*)$. Then, it turns out that the first k eigenvalues on Ω^* coincide with the first k eigenvalues on $\cup_{i=1}^k \Omega_i^*$ and therefore we have $|\cup_{i=1}^k \Omega_i^*| = |\Omega^*|$ (since otherwise the optimality of Ω^* gives a contradiction).

5.2 General properties

In this section we study some properties of the optimal sets Ω^* to the problem (5.1.1) and of its first normalized eigenfunctions $U = (u_1, \dots, u_k)$. We first prove that the optimal sets have finite perimeter and that the vector U is non degenerate. We then prove that the first eigenfunction u_1 is non degenerate on any connected component Ω_1^* of Ω^* where u_1 is positive. As a consequence, we show that Ω_1^* satisfies a density estimate. We conclude the section with an almost Weiss type formula for U .

5.2.1 Finiteness of the perimeter

We prove that the De Giorgi perimeter of any optimal set to the problem (5.1.1) is finite. We follow the strategy introduced by Bucur in [12] for the eigenvalues of the Dirichlet Laplacian (see also [78] and [84]). Together with a density estimate for the optimal sets Ω^* (Proposition 5.2.9), this provides a kind of smallness of the singular set of Ω^* (see section 5.4.5). The proof of this result will also be used to obtain a non-degeneracy property of the first eigenfunction u_1 on Ω_1^* (Lemma 5.2.4).

Proposition 5.2.1. *Let $\Omega^* \subset D$ be an optimal set for the problem (5.1.1). Then Ω^* is a set finite perimeter in \mathbb{R}^d .*

Proof. Let $U = (u_1, \dots, u_k) \in H_0^1(\Omega^*, \mathbb{R}^k)$ be the vector of normalized eigenfunctions on Ω^* . We prove that $\{|u_i| > 0\}$ is a set of locally finite perimeter in D for every $i \in \{1, \dots, k\}$. This then implies that the optimal set $\Omega^* = \{|U| > 0\}$ has finite perimeter. Let $x \in \partial\{|u_i| > 0\} \cap D$ and assume for simplicity that $x = 0$. Let $r > 0$ be small, $t \in (0, 1)$ and $\eta \in C_c^\infty(B_r)$ be such that $0 \leq \eta \leq 1$, $\{\eta = 1\} = B_{r/2}$ and $\|\nabla\eta\|_{L^\infty} \leq C/r$. We set

$$u_{i,t} = \eta(u_i - t)^+ - \eta(u_i + t)^- + (1 - \eta)u_i = \begin{cases} u_i - t\eta & \text{if } u_i \geq t, \\ (1 - \eta)u_i & \text{if } |u_i| < t, \\ u_i + t\eta & \text{if } u_i \leq -t, \end{cases}$$

and $U_t = (u_1, \dots, u_{i,t}, \dots, u_k) \in H_0^1(D, \mathbb{R}^k)$, where $u_{i,t}$ stands at the i -th position. Notice that we have $U - U_t \in H_0^1(B_r, \mathbb{R}^k)$ and $\|U - U_t\|_{L^1} \leq t|B_r|$. We denote by C any constant which does not depend on x or t . By the quasi-minimality property of the function U in Theorem 5.1.2 we have

$$\begin{aligned} \int_{B_r} (A\nabla u_i \cdot \nabla u_i - A\nabla u_{i,t} \cdot \nabla u_{i,t}) + \Lambda(|\{U\} \cap B_r| - |\{U_t\} \cap B_r|) \\ \leq C\|U - U_t\|_{L^1} \int_D A\nabla U_t \cdot \nabla U_t \leq Ct. \end{aligned} \quad (5.2.1)$$

Since $\eta = 1$ in $B_{r/2}$ we have $\nabla u_{i,t} = \nabla u_i \mathbb{1}_{\{|u_i| \geq t\}}$ in $B_{r/2}$ and hence

$$\int_{B_{r/2}} (A\nabla u_i \cdot \nabla u_i - A\nabla u_{i,t} \cdot \nabla u_{i,t}) = \int_{\{0 < |u_i| < t\} \cap B_{r/2}} A\nabla u_i \cdot \nabla u_i.$$

On the other hand, with an easy computation we get

$$\begin{aligned} \int_{B_r \setminus B_{r/2}} (A\nabla u_i \cdot \nabla u_i - A\nabla u_{i,t} \cdot \nabla u_{i,t}) &= \int_{\{u_i \geq t\} \cap (B_r \setminus B_{r/2})} (2tA\nabla u_i \cdot \nabla \eta - t^2 A\nabla \eta \cdot \nabla \eta) \\ &+ \int_{\{|u_i| < t\} \cap (B_r \setminus B_{r/2})} (\eta(2 - \eta)A\nabla u_i \cdot \nabla u_i - u_i^2 A\nabla \eta \cdot \nabla \eta + 2(1 - \eta)u_i A\nabla u_i \cdot \nabla \eta) \\ &+ \int_{\{u_i \leq -t\} \cap (B_r \setminus B_{r/2})} (-2tA\nabla u_i \cdot \nabla \eta - t^2 A\nabla \eta \cdot \nabla \eta) \geq -Ct. \end{aligned}$$

Moreover, since $\eta \neq 1$ in $B_r \setminus B_{r/2}$ and by definition of $u_{i,t}$ we have

$$\begin{aligned} |\{|U| > 0\} \cap B_r| - |\{|U_t| > 0\} \cap B_r| &= |\{|U| > 0\} \cap B_{r/2}| - |\{|U_t| > 0\} \cap B_{r/2}| \\ &= |\{0 \leq |u_i| \leq t\} \cap \{|U| > 0\} \cap B_{r/2}| \geq |\{0 < |u_i| < t\} \cap B_{r/2}|. \end{aligned}$$

Then, we now get from (5.2.1) that

$$\int_{\{0 < |u_i| < t\} \cap B_{r/2}} A \nabla u_i \cdot \nabla u_i + \Lambda |\{0 < |u_i| < t\} \cap B_{r/2}| \leq Ct \quad (5.2.2)$$

and therefore we have

$$\begin{aligned} \int_{\{0 < |u_i| < t\} \cap B_{r/2}} |\nabla u_i| &\leq \int_{\{0 < |u_i| < t\} \cap B_{r/2}} (|\nabla u_i|^2 + 1) \\ &\leq \max\{\lambda_\Lambda^2, \Lambda^{-1}\} \left(\int_{\{0 < |u_i| < t\} \cap B_{r/2}} A \nabla u_i \cdot \nabla u_i + \Lambda |\{0 < |u_i| < t\} \cap B_{r/2}| \right) \leq Ct. \end{aligned}$$

We now use the co-area formula to rewrite the above inequality as

$$\frac{1}{t} \int_0^t \text{Per}(\{|u_i| > s\}; B_{r/2}) ds \leq C.$$

Therefore, there exists a sequence $t_n \downarrow 0$ such that $\text{Per}(\{|u_i| > t_n\}; B_{r/2}) \leq C$. Passing to the limit we get that $\text{Per}(\{|u_i| > 0\}; B_{r/2}) \leq C$, which concludes the proof. \square

5.2.2 Freezing of the coefficients and non-degeneracy of the eigenfunctions

The properties of the eigenfunctions on optimal sets in the case where $A = Id$ have already been studied in [77]. Thus, we perform a change of variables in order to reduce to this case. We prove in the spirit of [85, Lemma 3.2] (see also [87, Proposition 2.4]) that the vector of the first k eigenfunctions is a local quasi-minimizer at the origin of the Alt-Caffarelli functional. We then prove a non-degeneracy property for the vector of the first k eigenfunctions at the boundary of the optimal set.

We start with some notations which will be used throughout this paper. For $U \in H^1(\mathbb{R}^d, \mathbb{R}^k)$ and $r > 0$ we set

$$J(U, r) = \int_{B_r} |\nabla U|^2 + \Lambda |\{|U| > 0\} \cap B_r|.$$

For $x \in D$ we define the function $F_x : \mathbb{R}^d \rightarrow \mathbb{R}^d$ by

$$F_x(\xi) := x + A_x^{1/2}[\xi], \quad \xi \in \mathbb{R}^d, \quad (5.2.3)$$

where $A_x^{1/2} \in \text{Sym}_d^+$ denotes the square root matrix of A_x (notice that, by assumption, the matrix A_x is positive definite). Moreover, for $U = (u_1, \dots, u_k) \in H^1(\mathbb{R}^d, \mathbb{R}^k)$ we set $U_x = U \circ F_x$ and $u_{x,i} = u_i \circ F_x$, $i = 1, \dots, k$.

Proposition 5.2.2. *Let $U \in H_0^1(D, \mathbb{R}^k)$ be the vector of the first k normalized eigenfunctions on Ω^* . There exist constants $r_0 \in (0, 1)$ and $C > 0$ such that, if $x \in D$ and $r \leq r_0$ satisfy $B_{\lambda_A r}(x) \subset D$, then*

$$J(U_x, r) \leq (1 + Cr^{\delta_A})J(\tilde{U}, r) + C\|U_x - \tilde{U}\|_{L^1} \quad (5.2.4)$$

for every $\tilde{U} \in H^1(\mathbb{R}^d, \mathbb{R}^k)$ such that $U_x - \tilde{U} \in H_0^1(B_r, \mathbb{R}^k)$ and $\|\tilde{U}\|_{L^\infty} \leq \|U_x\|_{L^\infty}$.

Proof. Let $V \in H_0^1(D, \mathbb{R}^k)$ be such that $\tilde{U} = V \circ F_x$ and set $\rho = \lambda_A r$. Observe that $U - V \in H_0^1(F_x(B_r))$ and use V as a test function in (5.1.9) to get

$$\int_{F_x(B_r)} A \nabla U \cdot \nabla U + \Lambda |\{|U| > 0\} \cap F_x(B_r)| \leq (1 + Cr^d) \int_{F_x(B_r)} A \nabla V \cdot \nabla V + \Lambda |\{|V| > 0\} \cap F_x(B_r)| + C \|U - V\|_{L^1}. \quad (5.2.5)$$

Moreover, since A has Hölder continuous coefficients and is uniformly elliptic, we have

$$J(U_x, r) \leq \det(A_x^{-1/2}) \left[(1 + dc_A \lambda_A^2 \rho^{\delta_A}) \int_{F_x(B_r)} A \nabla U \cdot \nabla U + \Lambda |\{|U| > 0\} \cap F_x(B_r)| \right]. \quad (5.2.6)$$

Similarly, we have the estimate from below

$$J(\tilde{U}, r) \geq \det(A_x^{-1/2}) \left[(1 - dc_A \lambda_A^2 \rho^{\delta_A}) \int_{F_x(B_r)} A \nabla V \cdot \nabla V + \Lambda |\{|V| > 0\} \cap F_x(B_r)| \right]. \quad (5.2.7)$$

Combining (5.2.6), (5.2.5) and (5.2.7) we get

$$J(U_x, r) \leq (1 + dc_A \lambda_A^2 \rho^{\delta_A}) \left[\frac{1 + Cr^d}{1 - dc_A \lambda_A^2 \rho^{\delta_A}} J(\tilde{U}, r) + C \|U_x - \tilde{U}\|_{L^1} \right]$$

which gives (5.2.4). \square

We now prove a non-degeneracy property of the function $U_x = U \circ F_x$ using the approach of David and Toro in [34] which is a variant of the result in [2].

Proposition 5.2.3 (Non-degeneracy of U_x). *Let $U = (u_1, \dots, u_k)$ be the vector of the k first eigenfunctions on Ω^* . Let $K \subset \Omega^*$ be a compact set. There exist constants $\eta = \eta_K > 0$ and $r_K > 0$ such that for every $x \in K$ and $r \leq r_K$ we have*

$$\sum_{i=1}^k \int_{\partial B_r} |u_{x,i}| \leq \eta r \quad \implies \quad U = 0 \quad \text{in} \quad B_{r/4\lambda_A}(x).$$

We will need the following Lemma which, loosely speaking, provides an estimate of the non-subharmonicity of U_x .

Lemma 5.2.4. *Let $U = (u_1, \dots, u_k)$ be the vector of the k first eigenfunctions on Ω^* . Let $K \subset \Omega^*$ be a compact set. There exists constants $C_K > 0$ and $r_K > 0$ such that for every $x \in K$ and $r \leq r_K$ we have*

$$\sum_{i=1}^k \int_{B_r} [(u_{x,i} - h_{r,i})^+]^2 \leq C_K r^{2+\delta_A}, \quad (5.2.8)$$

where $h_{x,i}$ denotes the harmonic extension of the trace of $u_{x,i}$ to ∂B_r .

Proof. We define the vector $\tilde{U} = (\tilde{u}_1, \dots, \tilde{u}_k) \in H^1(\mathbb{R}^d, \mathbb{R}^k)$ by

$$\tilde{u}_i = \begin{cases} \min(u_{x,i}, h_{x,i}) & \text{in } B_r \\ u_{x,i} & \text{in } D \setminus B_r. \end{cases}$$

Then, using \tilde{U} as a test function in Proposition 5.2.2 we get (since U_x is locally Lipschitz continuous in D and because we have the inclusion $\{|\tilde{U}| > 0\} \subset \{|U_x| > 0\}$ since $\tilde{u}_i \leq u_{x,i}$)

$$\begin{aligned} \int_{B_r} |\nabla U_x|^2 &\leq (1 + Cr^{\delta_A}) \int_{B_r} |\nabla \tilde{U}|^2 + Cr^{\delta_A} |\{|\tilde{U}| > 0\} \cap B_r| + C \|U_x - \tilde{U}\|_{L^1(B_r)} \\ &\leq (1 + Cr^{\delta_A}) \int_{B_r} |\nabla \tilde{U}|^2 + Cr^{d+\delta_A}. \end{aligned} \quad (5.2.9)$$

We now set $V_i = \{h_{r,i} < u_{x,i}\}$ for every $i = 1, \dots, k$, so that by (5.2.9) we have

$$\sum_{i=1}^k \int_{V_i} (|\nabla u_{x,i}|^2 - |\nabla h_{r,i}|^2) \leq Cr^{\delta_A} \int_{B_r} |\nabla \tilde{U}|^2 + Cr^{d+\delta_A}. \quad (5.2.10)$$

Moreover, we have the following equalities

$$\int_{B_r} (|\nabla \tilde{u}_i|^2 - |\nabla u_{x,i}|^2) = \int_{V_i} (|\nabla h_{x,i}|^2 - |\nabla u_{x,i}|^2) = - \int_{V_i} |\nabla(u_{x,i} - h_{r,i})|^2. \quad (5.2.11)$$

Indeed, the first equality follows from the definition of V_i . For the second one, we set $v_i = \max(u_{x,i}, h_{r,i})$ in B_r and $v_i = u_{x,i}$ elsewhere, so that by harmonicity of $h_{r,i}$ we have

$$0 = \int_{B_r} \nabla h_{r,i} \cdot \nabla(v_i - h_{r,i}) = \int_{V_i} \nabla u_{x,i} \cdot \nabla(v_i - h_{r,i}) = \int_{V_i} \nabla h_{r,i} \cdot \nabla u_{x,i} - \int_{V_i} |\nabla h_{r,i}|^2,$$

which gives (5.2.11). Finally, combining Poincaré inequality, (5.2.11), (5.2.10) and using that U_x is Lipschitz continuous we get

$$\begin{aligned} \sum_{i=1}^k \int_{B_r} [(u_{x,i} - h_{r,i})^+]^2 &\leq Cr^2 \sum_{i=1}^k \int_{B_r} |\nabla(u_{x,i} - h_{r,i})^+|^2 = Cr^{2-d} \sum_{i=1}^k \int_{V_i} |\nabla(u_{x,i} - h_{r,i})|^2 \\ &= Cr^{2-d} \sum_{i=1}^k \int_{V_i} (|\nabla u_{x,i}|^2 - |\nabla h_{r,i}|^2) \leq Cr^{2-d} \left(Cr^{\delta_A} \int_{B_r} |\nabla \tilde{U}|^2 + Cr^{d+\delta_A} \right) \\ &= Cr^{2+\delta_A} \left(\int_{B_r} |\nabla \tilde{U}|^2 + 1 \right) \leq Cr^{2+\delta_A} \left(\int_{B_r} |\nabla U_x|^2 + 1 \right) \leq Cr^{2+\delta_A}. \end{aligned}$$

□

Proof of Proposition 5.2.3. Let $\eta > 0$ be small and assume that

$$\sum_{i=1}^k \int_{\partial B_r} |u_{x,i}| \leq \eta r. \quad (5.2.12)$$

We first claim that for every $i \in \{1, \dots, k\}$ we have $|u_{x,i}| < 4^{d+1}\eta r$ in $B_{r/2}$. Suppose by contradiction that there exists $\xi_0 \in B_{r/2}$ such that $|u_{x,i}(\xi_0)| \geq 4^{d+1}\eta r$. Since $u_{x,i}$ is L -Lipschitz continuous (with L depending on K), we have for every $\xi \in B_{\eta r/L}(\xi_0)$

$$|u_{x,i}(\xi)| \geq |u_{x,i}(\xi_0)| - |u_{x,i}(\xi) - u_{x,i}(\xi_0)| \geq (4^{d+1} - 1)\eta r. \quad (5.2.13)$$

Moreover, if $\eta \leq L/4$, by Poisson formula we have for every $\xi \in B_{\eta r/L}(\xi_0) \subset B_{3r/4}$

$$|h_{r,i}(\xi)| = \frac{r^2 - |\xi|^2}{d\omega_d r} \left| \int_{\partial B_r} \frac{u_{x,i}(\tilde{\xi})}{|\xi - \tilde{\xi}|^d} d\mathcal{H}^{d-1}(\tilde{\xi}) \right| \leq \frac{r}{d\omega_d} \left(\frac{4}{r}\right)^d \int_{\partial B_r} |u_{x,i}| \leq 4^d \eta r. \quad (5.2.14)$$

Therefore, using (5.2.13) and (5.2.14) it follows that

$$\begin{aligned} \int_{B_r} [(u_{x,i} - h_{r,i})^+]^2 &\geq \left(\frac{\eta}{L}\right)^d \int_{B_{\eta r/L}(\xi_0)} (u_{x,i} - h_{r,i})^2 \geq \left(\frac{\eta}{L}\right)^d \int_{B_{\eta r/L}(\xi_0)} (|u_{x,i}| - |h_{r,i}|)^2 \\ &\geq \left(\frac{\eta}{L}\right)^d \left[(4^{d+1} - 1 - 4^d)\eta r \right]^2 \geq \frac{\eta^{d+2}}{L^d} r^2, \end{aligned}$$

which is in contradiction with (5.2.8) if r is small enough.

Now, let $\varphi \in C^\infty(B_r)$ be a smooth function such that $0 \leq \varphi \leq 1$, $\varphi = 1$ in $B_{r/2}$, $\varphi = 0$ in $B_r \setminus B_{3r/4}$ and $|\nabla\varphi| \leq Cr$. We set for $i = 1, \dots, k$

$$\tilde{u}_i = \begin{cases} (u_{x,i} - 4^{d+1}\eta r\varphi)^+ - (u_{x,i} + 4^{d+1}\eta r\varphi)^- & \text{in } B_r \\ u_{x,i} & \text{in } D \setminus B_r, \end{cases}$$

and $\tilde{U} = (\tilde{u}_1, \dots, \tilde{u}_k) \in H^1(\mathbb{R}^d, \mathbb{R}^k)$. Notice that we have $U_x - \tilde{U} \in H^1(B_r, \mathbb{R}^k)$. Moreover, by the preceding claim we have the inclusion $\{|\tilde{U}| > 0\} \cap B_r \subset \{|U_x| > 0\} \cap (B_r \setminus B_{r/2})$. Therefore, Proposition 5.2.2 applied to the vector \tilde{U} gives

$$\Lambda\{|U_x| > 0\} \cap B_{r/2} \leq \int_{B_r} (|\nabla\tilde{U}|^2 - |\nabla U_x|^2) + Cr^{\delta_A} J(\tilde{U}, r) + C\|U_x - \tilde{U}\|_{L^1(B_r)}. \quad (5.2.15)$$

By the definition of \tilde{U} and since U_x is L -Lipschitz continuous, we have in the ball B_r

$$|\nabla\tilde{U}|^2 \leq |\nabla U_x|^2 + 2 \cdot 4^{d+1} k \eta r L |\nabla\varphi| + 4^{2(d+1)} \eta^2 r^2 |\nabla\varphi|^2 \leq |\nabla U_x|^2 + C\eta \quad \text{in } B_r.$$

Since once again U_x is Lipschitz continuous, (5.2.15) now gives

$$\Lambda\{|U_x| > 0\} \cap B_{r/2} \leq C\eta r^d + Cr^{\delta_A} ((L^2 + C\eta)r^d + \Lambda\omega_d r^d) + CLr^{d+1} \leq C(\eta + r^{\delta_A})r^d. \quad (5.2.16)$$

Then, using once again the claim, we deduce that

$$\sum_{i=1}^k \int_{B_{r/2}} |u_{x,i}| = \sum_{i=1}^k \int_{B_{r/2} \cap \{|U_x| > 0\}} |u_{x,i}| \leq 4^{d+1} k \eta r |\{|U_x| > 0\} \cap B_{r/2}| \leq C(\eta + r^{\delta_A})\eta r^{d+1}. \quad (5.2.17)$$

Let $y \in B_{r/4\lambda_A}(x)$. We will find by induction a sequence of radii r_j such that the estimate (5.2.12) holds with the radius r_j and at the point y . Let us choose $r_1 \in (\frac{r}{8\lambda_A^2}, \frac{r}{4\lambda_A^2})$ such that

$$\int_{\partial B_{r_1}} \sum_{i=1}^k |u_{y,i}| \leq \frac{8\lambda_A^2}{r} \int_{r/8\lambda_A^2}^{r/4\lambda_A^2} ds \int_{\partial B_s} \sum_{i=1}^k |u_{y,i}|$$

Then, by (5.2.17) (and since $F_y^{-1} \circ F_x(B_{r/4\lambda_A^2}) \subset B_{r/2}$) we get

$$\begin{aligned} \sum_{i=1}^k \int_{\partial B_{r_1}} |u_{y,i}| &\leq Cr^{1-d} \sum_{i=1}^k \int_{\partial B_{r_1}} |u_{y,i}| \leq Cr^{-d} \sum_{i=1}^k \int_{r/8\lambda_A^2}^{r/4\lambda_A^2} ds \int_{\partial B_s} |u_{y,i}| \\ &\leq Cr^{-d} \sum_{i=1}^k \int_{B_{r/4\lambda_A^2}} |u_{y,i}| = Cr^{-d} \sum_{i=1}^k \int_{F_x^{-1} \circ F_y(B_{r/4\lambda_A^2})} |u_{x,i}| |\det(F_y^{-1} \circ F_x)| \\ &\leq Cr^{-d} \sum_{i=1}^k \int_{B_{r/2}} |u_{x,i}| \leq C(\eta + r^{\delta_A})\eta r_1 \leq \eta r_1, \end{aligned}$$

where the last inequality holds if η and r are small enough. Therefore, by the same above argument we use to get (5.2.16) and (5.2.17) we now deduce that

$$|\{|U_y| > 0\} \cap B_{r_1/2}| \leq C(\eta + r_1^{\delta_A})r_1^d$$

and

$$\sum_{i=1}^k \int_{B_{r_1/2}} |u_{y,i}| \leq C(\eta + r_1^{\delta_A})\eta r_1^{d+1}.$$

We now choose $r_2 \in (\frac{r_1}{4}, \frac{r_1}{2})$ such that

$$\sum_{i=1}^k \int_{\partial B_{r_2}} |u_{y,i}| \leq Cr_1^{-d} \sum_{i=1}^k \int_{r_1/4}^{r_1/2} ds \int_{\partial B_s} |u_{y,i}| \leq Cr_1^{-d} \sum_{i=1}^k \int_{B_{r_1/2}} |u_{y,i}| \leq C(\eta + r_1^{\delta_A})\eta r_1 \leq \eta r_1,$$

provided that η and r are small enough. By induction it follows that there exists a sequence of radii $(r_j)_j$ such that $r_{j+1} \in (\frac{r_j}{4}, \frac{r_j}{2})$ and

$$|\{|U_y| > 0\} \cap B_{r_j/2}| \leq C(\eta + r_j^{\delta_A})\eta r_j. \quad (5.2.18)$$

Now, if $|U_y|(0) > 0$, then $|U_y| > 0$ in a neighborhood of 0 since U_y is continuous, which is in contradiction with (5.2.18) for η small enough and j large enough. Hence $|U|(y) = |U_y|(0) = 0$ for every $y \in B_{r/4\lambda_A}(x)$, that is, $U = 0$ in $B_{r/4\lambda_A}(x)$. \square

Remark 5.2.5 (L^∞ non-degeneracy of U). A consequence of Proposition 5.2.3 is that U also enjoys the following non-degeneracy property: there exist $\eta = \eta_K > 0$ and $r_K > 0$ such that for every $x \in K$ and $r \leq r_K$ we have

$$\|U\|_{L^\infty(B_{\lambda_A r}(x))} \leq \eta r \quad \implies \quad U = 0 \quad \text{in} \quad B_{r/4\lambda_A}(x).$$

5.2.3 Non-degeneracy of the first eigenfunction and density estimate

We prove that the first eigenfunction u_1 on an optimal set Ω^* to (5.1.1) is non degenerate at every point of the boundary of Ω_1^* , where Ω_1^* denotes any connected component of Ω^* where u_1 is positive. The proof follows an idea of Kriventsov and Lin taken from [70]. As a consequence, we obtain that u_1 behaves like the distance function to the boundary and also a density estimate for the optimal sets. Obviously, these properties only hold in Ω_1^* , that is, where u_1 is positive. However, as pointed out in Remark 5.1.3, it is enough to restrict ourselves to this case in order to get the regularity of the whole optimal set Ω^* .

Proposition 5.2.6 (Non-degeneracy of u_1). *There exists a constant $C_1 > 0$ such that $C_1 u_1 \geq |U|$ in Ω_1^* .*

We first recall the following standard result which is a consequence of [87, Lemma 2.1].

Lemma 5.2.7. *Let $\Omega \subset D$ be a (non-empty) quasi-open set, $f \in L^\infty(D)$, $f \geq 0$, and $u \in H^1(D)$ be such that $u \geq 0$ on $\partial\Omega$ and*

$$\operatorname{div}(A\nabla u) \leq f \quad \text{in} \quad \Omega.$$

Then, there exists a constant $C > 0$, depending only on d and λ_A , such that

$$\|u^-\|_{L^\infty(\Omega)} \leq C \|\{u < 0\} \cap \Omega\|^{2/d} \|f\|_{L^\infty(\Omega)}.$$

Proof. Set $\Omega^- = \{u < 0\} \cap \Omega$ and notice that $u \in H_0^1(\Omega^-)$. Let $v \in H_0^1(\Omega^-)$ be the solution of $\operatorname{div}(A\nabla v) = f$ in Ω^- . By the weak maximum principle we have $v \leq 0$ in Ω^- (since $f \geq 0$) and $v \leq u$ in Ω^- ; in particular, $u^- \leq v^- = -v$ in Ω^- . The proof now follows from Lemma 2.1 in [87] (applied to $-v$). \square

Proof of Proposition 5.2.6. We first claim that $\operatorname{div}(A\nabla|U|) \geq -C|U|$ in Ω^* . Let $\varphi \in H_0^1(\Omega^*)$, $\varphi \geq 0$. We use an approximation by mollifiers $A^\varepsilon = (a_{ij}^\varepsilon)$ where $a_{ij}^\varepsilon = a_{ij} * \rho_\varepsilon$, and we compute

$$\begin{aligned} \langle \operatorname{div}(A^\varepsilon \nabla|U|), \varphi \rangle &= - \sum_{i,j} \int a_{ij}^\varepsilon \partial_i |U| \partial_j \varphi = - \sum_{i,j,l} \int a_{ij}^\varepsilon \partial_i u_l \frac{u_l}{|U|} \partial_j \varphi \\ &= \sum_{i,j,l} \int \partial_j (a_{ij}^\varepsilon \partial_i u_l) \frac{u_l}{|U|} \varphi + \sum_{i,j,l} \int a_{ij}^\varepsilon \partial_i u_l \partial_j \left(\frac{u_l}{|U|} \right) \varphi \\ &= - \sum_{i,j,l} \int a_{ij}^\varepsilon \partial_i u_l \partial_j \left(\frac{u_l}{|U|} \right) \varphi + \sum_{i,j,l,p} \int a_{ij}^\varepsilon \partial_i u_l \left(\frac{\partial_j u_l}{|U|} - \frac{u_l u_p}{|U|^3} \partial_j u_p \right) \varphi. \end{aligned}$$

Therefore, passing to the limit as $\varepsilon \rightarrow 0$ we get

$$\begin{aligned} \langle \operatorname{div}(A\nabla|U|), \varphi \rangle &\geq \sum_l \int \operatorname{div}(A\nabla u_l) \frac{u_l}{|U|} \varphi + \sum_{l,p} \int \frac{1}{|U|^3} (u_l^2 A\nabla u_l \cdot \nabla u_l - u_l u_p A\nabla u_l \cdot \nabla u_p) \varphi \\ &\geq - \sum_l \lambda_l(\Omega^*) \int b \frac{u_l^2}{|U|} \varphi \geq -\lambda_k(\Omega^*) c_b \int |U| \varphi, \end{aligned} \tag{5.2.19}$$

which proves the claim.

Let $r_0 > 0$ be small (to be chosen soon) and set $\Omega_r = \{x \in \Omega_1^* : |U(x)| < r\}$ for every $r > 0$. Since $u_1 > 0$ in Ω_1^* we have $m := \inf\{u_1(x) : x \in \Omega_1^*, |U(x)| = r_0\} > 0$. We set $M_0 = m^{-1}r_0$ and $v_0 = M_0 u_1 - |U|$. The claim implies that $\operatorname{div}(A\nabla v_0) \leq C|U|$ in Ω_{r_0} . Moreover, by construction of v_0 we have $v_0 \geq 0$ on $\partial\Omega_{r_0}$. Therefore, by Lemma 5.2.7 we get

$$-\inf_{\Omega_{r_0}} v_0 = \|v_0^-\|_{L^\infty(\Omega_{r_0})} \leq C|\{v_0 < 0\} \cap \Omega_{r_0}|^{2/d} \|U\|_{L^\infty(\Omega_{r_0})}.$$

Then, from (5.2.2) (and a compactness argument) we have $|\Omega_{r_0}| \leq Cr_0$ so that we deduce from the above inequality that $-\inf_{\Omega_{r_0}} v_0 \leq \bar{C}r_0^{1+2/d}$ for some $\bar{C} > 0$ independent of r_0 . Therefore, in $\Omega_{r_0} \setminus \Omega_{r_0/2}$ we have

$$M_0 u_1 = |U| + v_0 \geq |U| - \bar{C}r_0^{1+2/d} \geq (1 - 2\bar{C}r_0^{2/d})|U| \quad \text{in } \Omega_{r_0} \setminus \Omega_{r_0/2}.$$

We now choose r_0 small enough so that $4\bar{C}r_0^{2/d} \leq 1$ and set $M_1 = [1 - 2\bar{C}r_0^{2/d}]^{-1}M_0$ and $v_1 = M_1 u_1 - |U|$. It follows that $v_1 \geq 0$ in $\Omega_{r_0} \setminus \Omega_{r_0/2}$; in particular we have $v_1 \geq 0$ on $\partial\Omega_{r_0/2}$ and hence the above argument now applies to v_1 in $\Omega_{r_0/2}$. Therefore, an induction gives that $v_k \geq 0$ in $\Omega_{r_{k-1}} \setminus \Omega_{r_k}$ for every $k \geq 1$, where we have set $v_k = M_k u_1 - |U|$, $M_k = [1 - 2\bar{C}r_{k-1}^{2/d}]^{-1}M_{k-1}$ and $r_k = 2^{-k}r_0$. Moreover, we have

$$\log(M_k) = \log(M_0) - \sum_{i=1}^k \log[1 - 2\bar{C}r_{i-1}^{2/d}] \leq \log(M_0) + C \sum_{i=1}^k 2^{-2i/d} \leq C + \log(M_0)$$

and hence $M_k \leq CM_0$. It follows that $|U| \leq M_k u_1 \leq CM_0 u_1$ in $\Omega_{r_{k-1}} \setminus \Omega_{r_k}$ for every $k \geq 0$ and therefore that $|U| \leq CM_0 u_1$ in Ω_{r_0} . On the other hand, since $\inf_{\Omega_1^* \setminus \Omega_{r_0}} u_1 > 0$, there exists $M > 0$ such that $|U| \leq M u_1$ in $\Omega_1^* \setminus \Omega_{r_0}$. This completes the proof. \square

We now prove that the first eigenfunction on an optimal set has the same growth than the distance function near the boundary. This property will be useful to prove that the boundaries of blow-up sets Hausdorff convergence to the boundary of the blow-up limit set.

Proposition 5.2.8 (Uniform growth of u_1 at the boundary). *Let $K \subset D$ be a compact set. There exist constants $c_K > 0$ and $r_K > 0$ such that the following growth condition holds*

$$u_1(x) \geq c_K \operatorname{dist}(x, \partial\Omega_1^*) \quad \text{for every } x \in \Omega_1^* \cap K \text{ such that } \operatorname{dist}(x, \partial\Omega_1^*) \leq r_K.$$

Proof. We set $r = (2\lambda_A)^{-1} \operatorname{dist}(x, \partial\Omega_1^*)$ and we denote by $h_{x,1}$ the harmonic extension of the trace of $u_{x,1}$ to ∂B_r . By non degeneracy of u_1 (Propositions 5.2.6 and 5.2.3) we have (and because $h_{x,1}$ is harmonic)

$$h_{x,1}(0) = \int_{\partial B_r} h_{x,1} = \int_{\partial B_r} u_{x,1} \geq \frac{1}{C_1} \int_{\partial B_r} |U_x| \geq \frac{\eta}{\sqrt{k}C_1} r =: \eta_1 r. \tag{5.2.20}$$

Therefore, with the triangle inequality we get

$$u_1(x) = u_{x,1}(0) \geq h_{x,1}(0) - |u_{x,1}(0) - h_{x,1}(0)| \geq \eta_1 r - |u_{x,1}(0) - h_{x,1}(0)|. \tag{5.2.21}$$

We now want to estimate $|u_{x,1}(0) - h_{x,1}(0)|$ in terms of r . We apply Proposition 5.2.2 to the test function $\tilde{U} = (h_{x,1}, u_{x,2}, \dots, u_{x,k})$ and get (since $u_{x,1}$ is Lipschitz continuous and that $|u_{x,1}| > 0$ in B_r)

$$\int_{B_r} |\nabla(u_{x,1} - h_{x,1})|^2 = \int_{B_r} (|\nabla u_{x,1}|^2 - |\nabla h_{x,1}|^2) \leq Cr^{d+\delta_A}. \quad (5.2.22)$$

Now, let $\tau > 0$ be small to be chosen soon. Since $u_{x,1}$ and $h_{x,1}$ are Lipschitz continuous, we have for every $\xi \in B_{\tau r}$

$$\begin{aligned} |u_{x,1}(0) - h_{x,1}(0)| &\leq |u_{x,1}(0) - u_{x,1}(\xi)| + |u_{x,1}(\xi) - h_{x,1}(\xi)| + |h_{x,1}(\xi) - h_{x,1}(0)| \\ &\leq C\tau r + |u_{x,1}(\xi) - h_{x,1}(\xi)|. \end{aligned}$$

Moreover, using Poincaré inequality to the function $u_{x,1} - h_{x,1}$ and the estimate (5.2.22), we have

$$\begin{aligned} |u_{x,1}(0) - h_{x,1}(0)| &\leq C\tau r + \int_{B_{\tau r}} |u_{x,1}(\xi) - h_{x,1}(\xi)| \leq C\tau r + \tau^{-d} \int_{B_r} |u_{x,1}(\xi) - h_{x,1}(\xi)| \\ &\leq C\tau r + C\tau^{-d} r \int_{B_r} |\nabla(u_{x,1}(\xi) - h_{x,1}(\xi))| \\ &\leq C\tau r + C\tau^{-d} r \left(\int_{B_r} |\nabla(u_{x,1}(\xi) - h_{x,1}(\xi))|^2 \right)^{1/2} \\ &\leq (C\tau + C\tau^{-d} r^{\delta_A/2}) r \leq \frac{\eta_1}{2} r, \end{aligned}$$

where the last inequality holds by choosing first τ small enough and then r_K (depending on τ) small enough. In view of (5.2.21), Proposition 5.2.6 now follows. \square

Proposition 5.2.9 (Density estimate for Ω_1^*). *Let U be the vector of the first k normalized eigenfunctions on Ω^* and let $K \subset D$ be a compact set. There exist constants $r_K > 0$ and $c_K \in (0, 1)$ such that for every $x_0 \in \partial\Omega_1^* \cap K$ and $r \leq r_K$ we have*

$$c_K |B_r| \leq |\Omega_1^* \cap B_r(x_0)| \leq (1 - c_K) |B_r|.$$

Proof. We first prove that we have

$$c |B_r| \leq |\{|U_{x_0}| > 0\} \cap B_r| \leq (1 - c) |B_r|. \quad (5.2.23)$$

The first inequality follows from the non-degeneracy of U_{x_0} (Proposition (5.2.3)) since it implies that there exists $\xi \in \partial B_{r/2}$ such that $\sum_{i=1}^k |u_{x_0,i}(\xi)| \geq \frac{\eta r}{2}$, and hence, using that U_{x_0} is L -Lipschitz continuous, that

$$|U_{x_0}| \geq \frac{1}{\sqrt{k}} \sum_{i=1}^k |u_{x_0,i}| \geq \frac{\eta r}{4\sqrt{k}} \quad \text{in } B_{\frac{\eta r}{4L}}(\xi).$$

For the second estimate, consider the test function $\tilde{U} = (h_{r,1}, u_{x_0,2}, \dots, u_{x_0,k}) \in H^1(\mathbb{R}^d, \mathbb{R}^k)$, where $h_{r,1}$ denotes as usual the harmonic extension of $u_{x_0,1}$ to ∂B_r , and note that by the strong maximum principle we have $h_{r,1} > 0$ in B_r since $u_{x_0,1}$ is non-negative. Then, by Proposition 5.2.2 applied to \tilde{U} , and since $u_{x_0,1}$ is L -Lipschitz continuous, we get

$$\begin{aligned} \int_{B_r} (|\nabla u_{x_0,1}|^2 - |\nabla h_{r,1}|^2) &\leq \Lambda |\{|U_{x_0}| = 0\} \cap B_r| + Cr^{\delta_A} J(\tilde{U}, r) + C \|u_{x_0,1} - h_{r,1}\|_{L^1} \\ &\leq \Lambda |\{|U_{x_0}| = 0\} \cap B_r| + Cr^{d+\delta_A}. \end{aligned} \quad (5.2.24)$$

Moreover, by Proposition 5.2.6 and the harmonicity of $h_{r,1}$ (and also because $u_{x_0,1}(0) = 0$), we have

$$|u_{x_0,1}(0) - h_{r,1}(0)| = h_{r,1}(0) = \int_{\partial B_r} h_{r,1} = \int_{\partial B_r} u_{x_0,1} \geq \eta_1 r, \quad (5.2.25)$$

where η_1 is defined as in (5.2.20). Now, let $\tau > 0$ be small. Since $h_{r,1}$ is $2L$ -Lipschitz continuous we have for every $\xi \in B_{\tau r}$

$$\begin{aligned} |u_{x_0,1}(0) - h_{r,1}(0)| &\leq |u_{x_0,1}(0) - u_{x_0,1}(\xi)| + |u_{x_0,1}(\xi) - h_{r,1}(\xi)| + |h_{r,1}(\xi) - h_{r,1}(0)| \\ &\leq 3L\tau r + |u_{x_0,1}(\xi) - h_{r,1}(\xi)|. \end{aligned}$$

Then, averaging over $B_{\tau r}$ and using (5.2.25) leads to

$$\eta_1 r \leq |u_{x_0,1}(0) - h_{r,1}(0)| \leq 3L\tau r + \int_{B_{\tau r}} |u_{x_0,1} - h_{r,1}|. \quad (5.2.26)$$

Moreover, by Poincaré inequality and Cauchy-Schwarz inequality we have

$$\begin{aligned} \int_{B_{\tau r}} |u_{x_0,1} - h_{r,1}| &\leq \tau^{-d} \int_{B_r} |u_{x_0,1} - h_{r,1}| \leq \tau^{-d} r \int_{B_r} |\nabla(u_{x_0,1} - h_{r,1})| \\ &\leq \tau^{-d} r^{1-\frac{d}{2}} \left(\int_{B_r} |\nabla(u_{x_0,1} - h_{r,1})|^2 \right)^{1/2} = \tau^{-d} r^{1-\frac{d}{2}} \left(\int_{B_r} |\nabla u_{x_0,1}|^2 - |\nabla h_{r,1}|^2 \right)^{1/2} \end{aligned}$$

which combined with (5.2.24) and (5.2.26), and after some rearrangements, gives

$$2\Lambda r^{-d} |\{|U_{x_0}| = 0\} \cap B_r| \geq \eta_1^2 \tau^{2d} - C\tau^{2d+2} - Cr^{\delta_\Lambda}.$$

Then choose τ , depending only on η_1 and C , small enough so that $C\tau^{2d+2} \leq \eta_1^2 \tau^{2d}/2$ and then choose r , depending only on η_1, τ and C , such that $Cr^{\delta_\Lambda} \leq \eta_1^2 \tau^{2d}/4$ to conclude the proof.

Now, by a change of variables, the density estimate in (5.2.23) gives

$$c|A_{x_0}^{1/2}[B_r]| \leq |\{|U| > 0\} \cap A_{x_0}^{1/2}[B_r]| \leq (1-c)|A_{x_0}^{1/2}[B_r]|.$$

Then set $c_K = \lambda_\Lambda^{-2d} c$ so that (because we have the inclusions $B_{\lambda_\Lambda^{-1}r} \subset A_{x_0}^{1/2}[B_r] \subset B_{\lambda_\Lambda r}$)

$$c_K |B_{\lambda_\Lambda r}| = c |B_{\lambda_\Lambda^{-1}r}| \leq c |A_{x_0}^{1/2}[B_r]| \leq |\{|U| > 0\} \cap A_{x_0}^{1/2}[B_r]| \leq |\{|U| > 0\} \cap B_{\lambda_\Lambda r}|.$$

Similarly we have

$$\begin{aligned} |\{|U| = 0\} \cap B_{\lambda_\Lambda r}| &\geq |\{|U| = 0\} \cap A_{x_0}^{1/2}[B_r]| = |A_{x_0}^{1/2}[B_r]| - |\{|U| > 0\} \cap A_{x_0}^{1/2}[B_r]| \\ &\geq c |A_{x_0}^{1/2}[B_r]| \geq c |B_{\lambda_\Lambda^{-1}r}| = c_K |B_{\lambda_\Lambda r}|, \end{aligned}$$

which concludes the proof. \square

5.2.4 Weiss monotonicity formula

We prove a monotonicity formula for the vector of the first k eigenfunctions on an optimal set Ω^* . The proof follows the idea of [92, Theorem 1.2] (see also [77, Proposition 3.1]). For every $U \in H^1(\mathbb{R}^d, \mathbb{R}^k)$ and $r > 0$ we define

$$W(U, r) = \frac{1}{r^d} J(U, r) - \frac{1}{r^{d+1}} \int_{\partial B_r} |U|^2.$$

Proposition 5.2.10. *Let $U = (u_1, \dots, u_k)$ be the vector of the first k normalized eigenfunctions on Ω^* and let $K \subset D$ be a compact set. Then there exist constants $r_K > 0$ and $C_K > 0$ such that for every $x_0 \in \partial\Omega^* \cap K$ and every $r \leq r_K$ the function $U_{x_0} = U \circ F_{x_0} = (u_{x_0,1}, \dots, u_{x_0,k})$ satisfies*

$$\frac{d}{dr}W(U_{x_0}, r) \geq \frac{1}{r^{d+2}} \sum_{i=1}^k \int_{\partial B_r} |x \cdot \nabla u_{x_0,i} - u_{x_0,i}|^2 dx - C_K r^{\delta_\Lambda - 1}. \quad (5.2.27)$$

Moreover, the limit $\lim_{r \rightarrow 0^+} W(U_{x_0}, r)$ exists and is finite.

Proof. We first compare U_{x_0} with its one-homogeneous extension in the ball B_r , namely the one-homogeneous function $\tilde{U} = (\tilde{u}_1, \dots, \tilde{u}_k) : B_r \rightarrow \mathbb{R}^k$ defined by $\tilde{U}(\xi) = \frac{|\xi|}{r} U_{x_0} \left(\frac{r}{|\xi|} \xi \right)$. We have

$$\int_{B_r} |\nabla \tilde{U}|^2 = \int_{B_r} \left[|\nabla_\theta U_{x_0}|^2 + \frac{|U_{x_0}|^2}{r^2} \right] \left(\frac{r}{|\xi|} \xi \right) d\xi = \frac{r}{d} \int_{\partial B_r} \left[|\nabla_\theta U_{x_0}|^2 + \frac{|U_{x_0}|^2}{r^2} \right]$$

and for the measure term

$$|\{|\tilde{U}| > 0\} \cap B_r| = \frac{r}{d} \mathcal{H}^{d-1}(\{|U| > 0\} \cap \partial B_r).$$

Then, we use \tilde{U} as a test function in (5.2.4) which gives

$$\begin{aligned} J(U_{x_0}, r) &\leq J(\tilde{U}, r) + C \left(r^{\delta_\Lambda} J(\tilde{U}, r) + \|U_{x_0} - \tilde{U}\|_{L^1} \right) \\ &\leq \frac{r}{d} \int_{\partial B_r} \left[|\nabla_\theta U_{x_0}|^2 + \frac{|U_{x_0}|^2}{r^2} \right] + \Lambda \frac{r}{d} \mathcal{H}^{d-1}(\{|U| > 0\} \cap \partial B_r) + C_0 r^{d+\delta_\Lambda} \end{aligned} \quad (5.2.28)$$

for some $C_0 \geq C(2\omega_d \|\nabla U_{x_0}\|_{L^\infty}^2 + \Lambda \omega_d + 2\omega_d \|\nabla U_{x_0}\|_{L^\infty})$ where the constant C is given by Proposition 5.2.2. We now compute the derivative of $W(U_{x_0}, r)$ and use (5.2.28) to obtain

$$\begin{aligned} \frac{d}{dr}W(U_{x_0}, r) &= \frac{1}{r^d} \left(\int_{\partial B_r} |\nabla U_{x_0}|^2 + \Lambda \mathcal{H}^{d-1}(\{|U_{x_0}| > 0\} \cap \partial B_r) \right) - \frac{d}{r^{d+1}} J(U_{x_0}, r) \\ &\quad + \frac{2}{r^{d+2}} \int_{\partial B_r} |U_{x_0}|^2 - \frac{1}{r^{d+1}} \sum_{i=1}^k \int_{\partial B_r} 2u_{x_0,i} \frac{\partial u_{x_0,i}}{\partial \nu} \\ &\geq \frac{1}{r^d} \int_{\partial B_r} \left| \frac{\partial U_{x_0}}{\partial \nu} \right|^2 + \frac{1}{r^{d+2}} \int_{\partial B_r} |U_{x_0}|^2 - \frac{1}{r^{d+1}} \sum_{i=1}^k \int_{\partial B_r} 2u_{x_0,i} \frac{\partial u_{x_0,i}}{\partial \nu} - dC_0 r^{\delta_\Lambda - 1} \\ &= \frac{1}{r^{d+2}} \sum_{i=1}^k \int_{\partial B_r} \left[r^2 \left| \frac{\partial u_{x_0,i}}{\partial \nu} \right|^2 + u_{x_0,i}^2 - 2ru_{x_0,i} \frac{\partial u_{x_0,i}}{\partial \nu} \right] - dC_0 r^{\delta_\Lambda - 1} \\ &= \frac{1}{r^{d+2}} \sum_{i=1}^k \int_{\partial B_r} |x \cdot \nabla u_{x_0,i} - u_{x_0,i}|^2 - dC_0 r^{\delta_\Lambda - 1}, \end{aligned}$$

which is (5.2.27). This also proves that the function $r \mapsto W(U_{x_0}, r) + \frac{d}{\delta_\Lambda} C_0 r^{\delta_\Lambda}$ is non-decreasing and hence that the limit of $W(U_{x_0}, r)$ as r tend to 0 exists. Moreover, this limit is finite since we have the bound

$$W(U_{x_0}, r) \geq -\frac{1}{r^{d+1}} \int_{\partial B_r} |U_{x_0}|^2 \geq -d\omega_d \|\nabla U_{x_0}\|_{L^\infty}^2 \quad \text{for every } r > 0.$$

□

As a consequence of the previous result, we get a monotonicity formula for global minimizers of the Alt-Caffarelli functional.

Definition 5.2.11. We say that $U \in H^1(\mathbb{R}^d, \mathbb{R}^k)$ is a global minimizer of the (vectorial) Alt-Caffarelli functional

$$J(U) = \int_{\mathbb{R}^d} |\nabla U|^2 + \Lambda|\{|U| > 0\}|$$

if $J(U, r) \leq J(\tilde{U}, r)$ for every $r > 0$ and every $\tilde{U} \in H^1(\mathbb{R}^d, \mathbb{R}^k) \cap L^\infty(\mathbb{R}^d, \mathbb{R}^k)$ such that $U - \tilde{U} \in H_0^1(B_r, \mathbb{R}^k)$.

Proposition 5.2.12. Let $U = (u_1, \dots, u_k) \in H^1(\mathbb{R}^d, \mathbb{R}^k)$ be a global minimizer of the Alt-Caffarelli functional J such that $U(0) = 0$. Then we have

$$\frac{d}{dr} W(U, r) \geq \frac{1}{r^{d+2}} \sum_{i=1}^k \int_{\partial B_r} |x \cdot \nabla u_i - u_i|^2.$$

In particular, if $r \mapsto W(U, r)$ is constant in $(0, +\infty)$, then U is a one-homogeneous function.

Proof. Since U is a global minimizer of J , it satisfies (5.2.4) with $C = 0$ and hence the computations in the proof of Proposition 5.2.10 hold with $C_0 = 0$. The last claim of the proposition follows from the fact that $x \cdot \nabla u_i = u_i$ in \mathbb{R}^d implies that u_i is one-homogeneous. \square

5.3 Blow-ups

In this section we study the blow-up limits (at the origin) of the functions $U_{x_0} = U \circ F_{x_0}$, where $x_0 \in \partial\Omega^* \cap D$. Throughout this section, U will denote the first k normalized eigenfunctions on the optimal set $\Omega^* = \{|U| > 0\}$. We prove that the blow-up limits are one-homogeneous and global minimizers of the Alt-Caffarelli functional. As a consequence, we also prove that the boundaries of two connected components of Ω^* have an empty intersection in D .

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence of points on $\partial\Omega^* \cap D$ converging to some $x_0 \in \partial\Omega^* \cap D$ and let $(r_n)_{n \in \mathbb{N}}$ be a sequence of positive radii tending to 0. Since U is Lipschitz continuous, up to extracting a subsequence, the sequence defined by

$$B_{x_n, r_n}(\xi) = \frac{1}{r_n} U(x_n + r_n \xi), \quad \xi \in \mathbb{R}^d,$$

converges locally uniformly to a Lipschitz continuous function $B_0 \in H_{loc}^1(\mathbb{R}^d, \mathbb{R}^k)$. We will often set $B_n = B_{x_n, r_n}$ and deal with this sequence in a new set of coordinates, that is, we will consider the sequence \tilde{B}_n defined by

$$\tilde{B}_n(\xi) = B_n \circ A_{x_n}^{1/2}(\xi) = \frac{1}{r_n} U_{x_n}(r_n \xi), \quad \xi \in \mathbb{R}^d.$$

Definition 5.3.1. If B_{x_n, r_n} converges locally uniformly in \mathbb{R}^d to some B_0 , we say that B_{x_n, r_n} is a blow-up sequence (with fixed center if $x_n = x_0$ for every $n \geq 1$). If the center is fixed, we say that B_0 is a blow-up limit at x_0 . We denote by $\mathcal{BU}_U(x_0)$ the space of all blow-up limits at x_0 .

We start with a standard result on the convergence of the blow-up sequences and we give the details of the proofs for convenience of the reader. Recall that Ω_1^* stands for any connected component of Ω^* where the first eigenfunction u_1 is positive.

Proposition 5.3.2 (Convergence of the blow-up sequences). *Let $(x_n)_{n \in \mathbb{N}} \subset \partial\Omega^* \cap D$ be a sequence converging to some $x_0 \in \partial\Omega^* \cap D$, $r_n \rightarrow 0$ and assume that the blow-up sequence $B_n := B_{x_n, r_n}$ converges locally uniformly to $B_0 \in H_{loc}^1(\mathbb{R}^d, \mathbb{R}^k)$. Then, up to a subsequence, we have*

1. The sequence B_n converges to B_0 strongly in $H_{loc}^1(\mathbb{R}^d, \mathbb{R}^k)$.

2. The sequences of characteristic functions $\mathbb{1}_{\Omega_n}$ converges in $L^1_{loc}(\mathbb{R}^d)$ to the characteristic function $\mathbb{1}_{\Omega_0}$, where we have set $\Omega_n = \{|B_n| > 0\}$ and $\Omega_0 = \{|B_0| > 0\}$.
3. The function B_0 is non-degenerate: there exists a constant $\eta_0 > 0$ such that for every $y \in \bar{\Omega}_0$ we have

$$\|B_0\|_{L^\infty(B_r(y))} \geq \eta_0 r \quad \text{for every } r > 0.$$

4. If $x_0 \in \partial\Omega_1^* \cap D$, then the sequences of closed sets $\bar{\Omega}_n$ and Ω_n^c converge locally Hausdorff to $\bar{\Omega}_0$ and Ω_0^c respectively.

Proof. Notice that it is enough to prove that the sequence $\tilde{B}_n = B_n \circ A_{x_n}^{1/2}$ strongly converges to $\tilde{B}_0 := B_0 \circ A_{x_0}^{1/2}$ in $H^1_{loc}(\mathbb{R}^d, \mathbb{R}^k)$ and that $\mathbb{1}_{\{|\tilde{B}_n| > 0\}}$ converges to $\mathbb{1}_{\{|\tilde{B}_0| > 0\}}$ in $L^1_{loc}(\mathbb{R}^d)$ to prove the parts (1) and (2) of Proposition 5.3.2.

Since \tilde{B}_n is uniformly Lipschitz, \tilde{B}_n converges, up to a subsequence, weakly in $H^1_{loc}(\mathbb{R}^d, \mathbb{R}^k)$ and strongly in $L^\infty_{loc}(\mathbb{R}^d, \mathbb{R}^k)$ to \tilde{B}_0 . Moreover, the local uniform convergence of $|B_n|$ to $|B_0|$ implies that $\mathbb{1}_{\{|\tilde{B}_0| > 0\}} \leq \liminf_{n \rightarrow \infty} \mathbb{1}_{\{|\tilde{B}_n| > 0\}}$. Therefore, it is sufficient to prove that for every ball $B_r \subset \mathbb{R}^d$ we have

$$\limsup_{n \rightarrow +\infty} \left(\int_{B_r} |\nabla \tilde{B}_n|^2 + \Lambda |\{|\tilde{B}_n| > 0\} \cap B_r| \right) \leq \int_{B_r} |\nabla \tilde{B}_0|^2 + \Lambda |\{|\tilde{B}_0| > 0\} \cap B_r|. \quad (5.3.1)$$

Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ be a smooth function such that $0 \leq \varphi \leq 1$, $\{\varphi = 1\} = B_r$ and $\varphi = 0$ outside B_{2r} . We set $\tilde{U}_n = \varphi \tilde{B}_0 + (1 - \varphi) \tilde{B}_n \in H^1(\mathbb{R}^d, \mathbb{R}^k)$ and notice that we have

$$U_{x_0} - \tilde{U}_n^{r_n} \in H^1_0(B_{2rr_n}, \mathbb{R}^k) \quad \text{where} \quad \tilde{U}_n^{r_n}(\xi) = r_n \tilde{U}_n\left(\frac{1}{r_n} \xi\right), \quad \xi \in \mathbb{R}^d.$$

Then, using $\tilde{U}_n^{r_n}$ as a test function in Proposition 5.2.2 and by a change of variables we get

$$\int_{B_{2r}} |\nabla \tilde{B}_n|^2 + \Lambda |\{|\tilde{B}_n| > 0\} \cap B_{2r}| \leq (1 + C(rr_n)^{\delta_A}) \left(\int_{B_{2r}} |\nabla \tilde{U}_n|^2 + \Lambda |\{|\tilde{U}_n| > 0\} \cap B_{2r}| \right) + r_n C \|\varphi(\tilde{B}_0 - \tilde{B}_n)\|_{L^1}. \quad (5.3.2)$$

Since we have $\tilde{U}_n = \tilde{B}_n$ in $\{\varphi = 0\}$ and $\tilde{U}_n = \tilde{B}_0$ in $\{\varphi = 1\}$, it follows that

$$|\{|\tilde{U}_n| > 0\} \cap B_{2r}| \leq |\{|\tilde{B}_n| > 0\} \cap \{\varphi = 0\} \cap B_{2r}| + |\{|\tilde{B}_0| > 0\} \cap \{\varphi = 1\}| + |\{0 < \varphi < 1\}|,$$

so that (5.3.2) now gives

$$\begin{aligned} \int_{\{\varphi > 0\}} (|\nabla \tilde{B}_n|^2 - |\nabla \tilde{U}_n|^2) + \Lambda (|\{|\tilde{B}_n| > 0\} \cap \{\varphi > 0\}| - |\{|\tilde{B}_0| > 0\} \cap \{\varphi = 1\}|) &\leq \Lambda |\{0 < \varphi < 1\}| \\ &+ C(rr_n)^{\delta_A} \left(\int_{B_{2r}} |\nabla \tilde{U}_n|^2 + \Lambda |\{|\tilde{U}_n| > 0\} \cap B_{2r}| \right) + r_n C \|\varphi(\tilde{B}_0 - \tilde{B}_n)\|_{L^1}. \end{aligned} \quad (5.3.3)$$

Now, since \tilde{B}_n converges strongly in $L^2(B_{2r})$ we have that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \int_{\{\varphi > 0\}} (|\nabla \tilde{B}_n|^2 - |\nabla \tilde{U}_n|^2) &= \limsup_{n \rightarrow +\infty} \int_{\{\varphi > 0\}} (|\nabla \tilde{B}_n|^2 - |\nabla(\varphi \tilde{B}_0 + (1 - \varphi) \tilde{B}_n)|^2) \\ &= \limsup_{n \rightarrow +\infty} \int_{\{\varphi > 0\}} (|\nabla \tilde{B}_n|^2 - |(\tilde{B}_0 - \tilde{B}_n) \nabla \varphi + (1 - \varphi) \nabla \tilde{B}_n + \varphi \nabla \tilde{B}_0|^2) \\ &= \limsup_{n \rightarrow +\infty} \int_{\{\varphi > 0\}} ((1 - (1 - \varphi)^2) |\nabla \tilde{B}_n|^2 - 2\varphi(1 - \varphi) \nabla \tilde{B}_n \cdot \nabla \tilde{B}_0 - \varphi^2 |\nabla \tilde{B}_0|^2) \quad (5.3.4) \\ &= \limsup_{n \rightarrow +\infty} \int_{\{\varphi > 0\}} (1 - (1 - \varphi)^2) (|\nabla \tilde{B}_n|^2 - |\nabla \tilde{B}_0|^2), \end{aligned}$$

and since $\nabla \tilde{B}_n$ converges weakly in $L^2(\{0 < \varphi < 1\})$ to \tilde{B}_0 we have that

$$\int_{\{0 < \varphi < 1\}} (1 - (1 - \varphi)^2) |\nabla \tilde{B}_0|^2 \leq \limsup_{n \rightarrow +\infty} \int_{\{0 < \varphi < 1\}} (1 - (1 - \varphi)^2) |\nabla \tilde{B}_n|^2. \quad (5.3.5)$$

Therefore, (5.3.5) and (5.3.4) now entail that

$$\begin{aligned} \limsup_{n \rightarrow +\infty} \int_{\{\varphi=1\}} (|\nabla \tilde{B}_n|^2 - |\nabla \tilde{B}_0|^2) &\leq \limsup_{n \rightarrow +\infty} \int_{\{\varphi>0\}} (1 - (1 - \varphi)^2) (|\nabla \tilde{B}_n|^2 - |\nabla \tilde{B}_0|^2) \\ &\leq \limsup_{n \rightarrow +\infty} \int_{\{\varphi>0\}} (|\nabla \tilde{B}_n|^2 - |\nabla \tilde{U}_n|^2). \end{aligned} \quad (5.3.6)$$

Finally, combining (5.3.6) and (5.3.3) we get

$$\begin{aligned} &\limsup_{n \rightarrow +\infty} \left(\int_{\{\varphi=1\}} (|\nabla \tilde{B}_n|^2 - |\nabla \tilde{B}_0|^2) + \Lambda(|\{\tilde{B}_n| > 0\} \cap \{\varphi = 1\}| - |\{\tilde{B}_0| > 0\} \cap \{\varphi = 1\}|) \right) \\ &\leq \limsup_{n \rightarrow +\infty} \left(\int_{\{\varphi>0\}} (|\nabla \tilde{B}_n|^2 - |\nabla \tilde{U}_n|^2) + \Lambda(|\{\tilde{B}_n| > 0\} \cap \{\varphi > 0\}| - |\{\tilde{B}_0| > 0\} \cap \{\varphi = 1\}|) \right) \\ &\leq \Lambda|\{0 < \varphi < 1\}|. \end{aligned}$$

Since we can choose φ so that $|\{0 < \varphi < 1\}|$ is arbitrary small, this proves (5.3.1) and concludes the proof of parts (1) and (2) of Proposition 5.3.2.

We now prove part (3). Let $y \in \bar{\Omega}_0$ and $r > 0$. There exists $z \in B_r(y)$ such that $|B_0|(z) > 0$, and hence such that $|B_n|(z) > 0$ for n large enough. Therefore, $U \neq 0$ in $B_{rr_n}(x_n + r_n z)$ and hence, by the non-degeneracy of U (Remark 5.2.5), we get that

$$r_n \|B_n\|_{L^\infty(B_{4\lambda_A^2 r}(z))} = \|U\|_{L^\infty(B_{4\lambda_A^2 rr_n}(x_n + r_n z))} \geq 4\lambda_A \eta r r_n.$$

In particular, there exists $z_n \in B_{4\lambda_A^2 r}(z)$ such that $|B_n|(z_n) \geq 4\lambda_A \eta r$. Up to a subsequence, z_n converges to some $z_\infty \in \overline{B_{4\lambda_A^2 r}(z)}$ and, since B_n uniformly converges to B_0 , we have that

$$\|B_0\|_{L^\infty(B_{(4\lambda_A^2 + 1)r}(y))} \geq \|B_0\|_{L^\infty(B_{4\lambda_A^2 r}(z))} \geq |B_0|(z_\infty) = \lim_{n \rightarrow +\infty} |B_n|(z_n) \geq 4\lambda_A \eta r,$$

which gives (3). The proof of the Hausdorff convergence of the free boundaries is standard (see, for instance, [91, Section 6]) and follows from the non-degeneracy of U and B_0 , and the growth property of U near the boundary of Ω_1^* (see Proposition 5.2.8). \square

Lemma 5.3.3 (Optimality of the blow-up limits). *Let $(x_n)_{n \in \mathbb{N}} \subset \partial\Omega^* \cap D$ be a sequence converging to some $x_0 \in \partial\Omega^* \cap D$, $r_n \rightarrow 0$ and assume that the blow-up sequence $B_n := B_{x_n, r_n}$ converges to some $B_0 \in H_{loc}^1(\mathbb{R}^d, \mathbb{R}^k)$ in the sense of Proposition 5.3.2. Then $\tilde{B}_0 := B_0 \circ A_{x_0}^{1/2}$ is a global minimizer of the Alt-Caffarelli functional J (see definition 5.2.11).*

Proof. Let $r > 0$ and $\tilde{U} \in H_{loc}^1(\mathbb{R}^d, \mathbb{R}^k) \cap L^\infty(\mathbb{R}^d, \mathbb{R}^k)$ be such that $\tilde{B}_0 - \tilde{U} \in H_0^1(B_r, \mathbb{R}^k)$. Let $\eta \in C_c^\infty(B_r)$ be such that $0 \leq \eta \leq 1$ and set $\tilde{B}_n = B_n \circ A_{x_n}^{1/2}$ and $\bar{V}_n = \tilde{U} + (1 - \eta)(\tilde{B}_n - \tilde{B}_0)$. Consider the test function $V_n \in H^1(\mathbb{R}^d, \mathbb{R}^k)$ defined by $V_n(\xi) = r_n \bar{V}_n(r_n^{-1} \xi)$ and note that $U_{x_0} - V_n \in H_0^1(B_{rr_n}, \mathbb{R}^k)$ (since we have $\tilde{B}_n - \bar{V}_n \in H_0^1(B_r, \mathbb{R}^k)$). By Proposition 5.2.2 applied to V_n and a change of variables it follows that

$$\begin{aligned} \int_{B_r} |\nabla \tilde{B}_n|^2 + \Lambda|\{\tilde{B}_n| > 0\} \cap B_r| &\leq (1 + C(rr_n)^{\delta_A}) \left(\int_{B_r} |\nabla \bar{V}_n|^2 + \Lambda|\{\bar{V}_n| > 0\} \cap B_r| \right) \\ &\quad + Cr_n \|\tilde{B}_n - \bar{V}_n\|_{L^1(B_r)}. \end{aligned} \quad (5.3.7)$$

Note that from (1) and (2) of Proposition 5.3.2 we deduce that \bar{V}_n converges strongly in H_{loc}^1 to \tilde{U} and that $\mathbb{1}_{\{|\bar{V}_n|>0\}}$ converges strongly in L_{loc}^1 to $\mathbb{1}_{\{|\tilde{U}|>0\}}$. Moreover, since $\bar{V}_n = \tilde{U}$ in $\{\eta = 1\}$, we have the estimate

$$|\{|\bar{V}_n| > 0\} \cap B_r| \leq |\{|\tilde{U}| > 0\} \cap B_r| + |\{\eta \neq 1\} \cap B_r|.$$

Therefore, passing to the limit as $n \rightarrow \infty$ in (5.3.7) we get

$$\int_{B_r} |\nabla \tilde{B}_0|^2 + \Lambda |\{|\tilde{B}_0| > 0\} \cap B_r| \leq \int_{B_r} |\nabla \tilde{U}|^2 + \Lambda (|\{|\tilde{U}| > 0\} \cap B_r| + |\{\eta \neq 1\} \cap B_r|).$$

Since we can choose η such that that $|\{\eta \neq 1\} \cap B_r|$ is arbitrary small, this gives that $J(\tilde{B}_0, r) \leq J(\tilde{U}, r)$ and concludes the proof. \square

As a consequence of the Weiss almost-monotonicity formula we get that the blow-up sequences with fixed center converge to a one-homogeneous function.

Lemma 5.3.4 (Homogeneity of the blow-up limits). *For every $x_0 \in \partial\Omega^* \cap D$, the blow-up limits $B_0 \in \mathcal{BU}_U(x_0)$ are one-homogeneous functions.*

Proof. Let $B_n = B_{x_0, r_n}$ converging (in the sense of Proposition 5.3.2) to B_0 . In particular, \tilde{B}_n converges strongly in H_{loc}^1 and in L_{loc}^1 to \tilde{B}_0 which implies that $\lim_{n \rightarrow +\infty} W(\tilde{B}_n, r) = W(\tilde{B}_0, r)$. Moreover, by Proposition 5.2.10 the limit $\lim_{s \rightarrow 0^+} W(U_{x_0}, s)$ exists and is finite. Therefore, we have for every $r > 0$

$$W(\tilde{B}_0, r) = \lim_{n \rightarrow +\infty} W(\tilde{B}_n, r) = \lim_{n \rightarrow +\infty} W(U_{x_0}, rr_n) = \lim_{s \rightarrow 0^+} W(U_{x_0}, s), \quad (5.3.8)$$

which says that the function $r \mapsto W(\tilde{B}_0, r)$ is constant on $(0, +\infty)$. Then, it follows from Lemma 5.3.3 and Proposition 5.2.12 that \tilde{B}_0 , and hence B_0 , is one-homogeneous. \square

We now reduce to the scalar case. More precisely, we prove that for any blow-up limit $B_0 \in \mathcal{BU}_U(x_0)$, the function $|\tilde{B}_0| = |B_0 \circ A_{x_0}^{1/2}|$ is a global minimizer of the scalar Alt-Caffarelli functional

$$H_{\text{loc}}^1(\mathbb{R}^d) \ni u \rightarrow \int_{\mathbb{R}^d} |\nabla u|^2 + \Lambda |\{u > 0\}|. \quad (5.3.9)$$

Lemma 5.3.5. *Let $x_0 \in \partial\Omega^* \cap D$, $B_0 \in \mathcal{BU}_U(x_0)$ and set $\tilde{B}_0 = B_0 \circ A_{x_0}^{1/2}$. Then there exists a unit vector $\xi \in \partial B_1 \subset \mathbb{R}^k$ such that $\tilde{B}_0 = |\tilde{B}_0| \xi$.*

Proof. Set $S = \partial B_1 \cap \{|\tilde{B}_0| > 0\}$. By Lemma 5.3.4, the components of $\tilde{B}_0 = (b_1, \dots, b_k)$ are one-homogeneous functions and by Lemma 5.3.3, they are harmonic on the cone $\{|\tilde{B}_0| > 0\}$. Therefore, in polar coordinates we have $b_i(r, \theta) = r\varphi_i(r)$ where φ_i is solution of the equation

$$-\Delta_{\mathbb{S}^{d-1}} \varphi_i = (d-1)\varphi_i \quad \text{in } S, \quad \varphi_i = 0 \quad \text{on } \partial S,$$

where $\Delta_{\mathbb{S}^{d-1}}$ stands for the Laplace-Beltrami operator. By Proposition 5.2.9, the components of \tilde{B}_0 are not all zero. Therefore, at least one φ_i is non-zero and hence $d-1$ is an eigenvalue of $-\Delta_{\mathbb{S}^{d-1}}$ on S . Since the functions φ_i are non-negative, it follows that $\lambda_1(S) = d-1$, where $\lambda_1(S)$ denotes the first eigenvalue on S . Moreover, by Lemma 5.3.3 we have $|S| < |\partial B_1|$ and by [77, Remark 4.8] it follows that the first eigenvalue $\lambda_1(S)$ is simple. Then, there exists non-negative constants $\alpha_1, \dots, \alpha_k$, not all zero, such that $\varphi_i = \alpha_i \varphi$ where φ stands for the normalized eigenfunction of $-\Delta_{\mathbb{S}^{d-1}}$ on S . Now set $\alpha = (\alpha_1, \dots, \alpha_k)$ so that we have $\tilde{B}_0 = \varphi \alpha$ on ∂B_1 . Since $|\alpha| \neq 0$, setting $\xi = |\alpha|^{-1} \alpha$ we have $\tilde{B}_0 = |\tilde{B}_0| \xi$ on ∂B_1 and hence on \mathbb{R}^d by one-homogeneity. \square

Lemma 5.3.6. *Let $x_0 \in \partial\Omega^* \cap D$, $B_0 \in \mathcal{BU}_U(x_0)$ and set $\tilde{B}_0 = B_0 \circ A_{x_0}^{1/2}$. Then the function $|\tilde{B}_0|$ is a global minimizer of the (scalar) Alt-Caffarelli functional defined in (5.3.9).*

Proof. Let $r > 0$ and $\tilde{u} \in H_{\text{loc}}^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ be such that $|\tilde{B}_0| - \tilde{u} \in H_0^1(B_r)$. Since $\tilde{B}_0 = |\tilde{B}_0|\xi$ by Lemma 5.3.5, we have that $\tilde{B}_0 - \tilde{u}\xi = (|\tilde{B}_0| - \tilde{u})\xi \in H_0^1(B_r, \mathbb{R}^k)$ and hence, by optimality of \tilde{B}_0 (see Lemma 5.3.3) we have

$$\int_{B_r} |\nabla|\tilde{B}_0||^2 + \Lambda|\{|\tilde{B}_0| > 0\} \cap B_r| = J(\tilde{B}_0, r) \leq J(\tilde{u}\xi, r) = \int_{B_r} |\nabla\tilde{u}|^2 + \Lambda|\{\tilde{u} > 0\} \cap B_r|.$$

□

We conclude this section with a consequence of the one-homogeneity and the optimality of $|\tilde{B}_0|$ which states that two connected components of an optimal set cannot meet inside D . It is then enough to prove the regularity of one connected component Ω_1^* of Ω^* and hence to reduce to a one-phase free boundary problem (see Proposition 5.4.17).

Proposition 5.3.7. *Denote by $(\Omega_i^*)_{i=1}^l$ the $l \leq k$ connected components of an optimal set Ω^* for (5.1.1). Then, we have $\partial\Omega_i^* \cap \partial\Omega_j^* \cap D = \emptyset$ for every $i, j \in \{1, \dots, l\}$, $i \neq j$.*

Proof. Let $x_0 \in \partial\Omega_i^* \cap \partial\Omega_j^* \cap D$. Since $\sigma(\Omega_i^*) \subset \sigma(\Omega^*)$, there exists $k_i \in \{1, \dots, k-1\}$ such that $\lambda_s(\Omega_i^*) \in \{\lambda_1(\Omega^*), \dots, \lambda_{k_i}(\Omega^*)\}$ for every $s = 1, \dots, k_i$ and $\lambda_s(\Omega_i^*) \notin \{\lambda_1(\Omega^*), \dots, \lambda_{k_i}(\Omega^*)\}$ for every $s > k_i$. It follows that Ω_i^* is solution of the problem (5.1.1) with $k = k_i$ and $D = D \setminus (\Omega^* \setminus \Omega_i^*)$. Similarly, for some $k_j \in \{1, \dots, k-1\}$, Ω_j^* is solution of (5.1.1) with $k = k_j$. Then, we denote by $V = (v_1, \dots, v_{k_i})$ and $W = (w_1, \dots, w_{k_j})$ the eigenfunctions on Ω_i^* and Ω_j^* respectively. Let $r_n \rightarrow 0$ and define the blow-up sequences

$$B_n(\xi) = \frac{1}{r_n}U(x_0 + r_n\xi), \quad B_n^V(\xi) = \frac{1}{r_n}V(x_0 + r_n\xi), \quad B_n^W(\xi) = \frac{1}{r_n}W(x_0 + r_n\xi), \quad \xi \in \mathbb{R}^d.$$

Up to a subsequence, B_n , B_n^V and B_n^W converge to some blow-up limits $B_0 \in \mathcal{BU}_U(x_0)$, $B_0^V \in \mathcal{BU}_V(x_0)$ and $B_0^W \in \mathcal{BU}_W(x_0)$. By Lemmas 5.3.4 and 5.3.6, $|\tilde{B}_0^V|$ and $|\tilde{B}_0^W|$ are non-trivial, one-homogeneous and global solutions of the Alt-Caffarelli functional. Therefore, the density at the origin of each set $\{|\tilde{B}_0^V| > 0\}$ and $\{|\tilde{B}_0^W| > 0\}$ is at least $1/2$ (see [77, Lemma 5]) and, since all the components of \tilde{B}_0^V and \tilde{B}_0^W are among the ones of \tilde{B}_0 , it follows that $|\{|\tilde{B}_0| > 0\} \cap B_1| = |B_1|$. Hence, $|\tilde{B}_0|$ is harmonic in B_1 since it minimizes the Alt-Caffarelli functional. And since $|\tilde{B}_0|$ is also a non-trivial and non-negative function which vanishes at 0, this gives a contradiction (by the maximum principle). □

5.4 Regularity of the free boundary

This section is devoted to the proof of Theorem 5.1.1. Recall that we denote by Ω^* a solution to the problem (5.1.1) and that Ω_1^* stands for any connected component of Ω^* where the first eigenfunction is positive.

5.4.1 The optimality condition on the free boundary

We prove that the vector U of the first k eigenfunctions on Ω^* satisfies an optimality condition on the boundary $\partial\Omega^* \cap D$ in the sense of the viscosity.

Definition 5.4.1. Let $D \subset \mathbb{R}^d$ be an open set and $U : D \subset \mathbb{R}^d \rightarrow \mathbb{R}^k$ be a continuous function.

- We say that $\varphi \in C(D)$ touches $|U|$ by below (resp. by above) at $x_0 \in D$ if $\varphi(x_0) = |U(x_0)|$ and $\varphi \leq |U|$ (resp. $\varphi \geq |U|$) in a neighborhood of x_0 .

• Let $\Omega \subset D$ be an open set and let $g : D \rightarrow \mathbb{R}$ be continuous and non-negative function. We say that U satisfies the boundary condition

$$|A^{1/2}[\nabla|U||] = g \quad \text{on} \quad \partial\Omega \cap D \quad (5.4.1)$$

in the viscosity sense if, for every $x_0 \in \partial\Omega \cap D$ and every $\varphi \in C^2(D)$ such that $\varphi^+ := \max(\varphi, 0)$ touches $|U|$ by below (resp. by above) at x_0 we have

$$|A_{x_0}^{1/2}[\nabla\varphi(x_0)]| \leq g(x_0) \quad \left(\text{resp.} \quad |A_{x_0}^{1/2}[\nabla\varphi(x_0)]| \geq g(x_0) \right).$$

• Let, moreover, $\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{R}^k$ be a vector of positive coordinates. We say that the function $U = (u_1, \dots, u_k)$ is a viscosity solution of the problem

$$-\operatorname{div}(A\nabla U) = \lambda bU \quad \text{in} \quad \Omega, \quad U = 0 \quad \text{on} \quad \partial\Omega \cap D, \quad |A^{1/2}[\nabla|U||] = g \quad \text{on} \quad \partial\Omega \cap D,$$

if for every $i = 1, \dots, k$ the component u_i is a solution of the PDE

$$-\operatorname{div}(A\nabla u_i) = -\lambda_i b u_i \quad \text{in} \quad \Omega, \quad u_i = 0 \quad \text{on} \quad \partial\Omega \cap D,$$

and if the boundary condition (5.4.1) holds in the viscosity sense.

Remark 5.4.2. Another equivalent definition of the boundary condition is to say that (5.4.1) holds if for every $x_0 \in \partial\Omega \cap D$ and every $\psi \in C^2(\mathbb{R}^d)$ such that ψ^+ touches $|U_{x_0}|$ by below (resp. by above) at 0 we have $|\nabla\psi(0)| \leq g(x_0)$ (resp. $|\nabla\psi(0)| \geq g(x_0)$). Indeed, if we set $\psi = \varphi \circ F_{x_0}$ then we have $|\nabla\psi(0)| = |A_{x_0}^{1/2}[\nabla\varphi(0)]|$ (see also [85, Appendix A]).

Lemma 5.4.3 (Optimality condition on the free boundary). *Let $U = (u_1, \dots, u_k)$ be the vector of the first k normalized eigenfunctions on Ω^* and set $\lambda = (\lambda_1(\Omega^*), \dots, \lambda_k(\Omega^*))$. Then U is a viscosity solution of the problem*

$$-\operatorname{div}(A\nabla U) = \lambda bU \quad \text{in} \quad \Omega^*, \quad U = 0 \quad \text{on} \quad \partial\Omega^* \cap D, \quad |A^{1/2}[\nabla|U||] = \sqrt{\Lambda} \quad \text{on} \quad \partial\Omega^* \cap D. \quad (5.4.2)$$

Proof. Since U is Lipschitz continuous, we only have to prove that the boundary condition holds in the viscosity sense. Let $x_0 \in \partial\Omega^* \cap D$ and let $\psi \in C^2(\mathbb{R}^d)$ be a function touching $|U_{x_0}|$ by below at 0 (see Remark 5.4.2). We fix an infinitesimal sequence r_n and set for every $\xi \in \mathbb{R}^d$

$$\tilde{B}_n(\xi) = \frac{1}{r_n} U_{x_0}(r_n \xi) \quad \text{and} \quad \psi_n(\xi) = \frac{1}{r_n} \psi(r_n \xi).$$

Up to a subsequence, the blow-up sequences $(\tilde{B}_n)_{n \in \mathbb{N}}$ and $(\psi_n)_{n \in \mathbb{N}}$ converge locally uniformly in \mathbb{R}^d to some $\tilde{B}_0 \in H_{\text{loc}}^1(\mathbb{R}^d, \mathbb{R}^k)$ and to $\psi_0(\xi) := \xi \cdot \nabla\psi(0)$ respectively. We can assume that $\nabla\psi(0) = |\nabla\psi(0)|e_d$ (by a change of variables) and that $|\nabla\psi(0)| \neq 0$, since otherwise $|\nabla\psi(0)| \leq \sqrt{\Lambda}$ obviously holds. We have $\psi \leq |U_{x_0}|$ near 0 and hence $\psi_0 \leq |\tilde{B}_0|$ in \mathbb{R}^d which gives that $|\tilde{B}_0| > 0$ in the half-space $\{x_d > 0\}$. Since $|\tilde{B}_0|$ is a one-homogeneous (Lemma 5.3.4) and non-degenerate (Proposition 5.3.2) function, it follows that $\{\tilde{B}_0 > 0\} = \{x_d > 0\}$ (see [84, Lemma 5.30]). Moreover, $|\tilde{B}_0|$ is a local minimizer of the Alt-Caffarelli functional (Lemma 5.3.6) and hence satisfies the optimality condition

$$|\nabla|\tilde{B}_0|| = \sqrt{\Lambda} \quad \text{on} \quad \{x_d = 0\}.$$

Therefore we have $|\tilde{B}_0(\xi)| = \sqrt{\Lambda} \xi_d^+$ and hence $\psi_0(\xi) = |\nabla\psi(0)| \xi_d \leq |\tilde{B}_0(\xi)| = \sqrt{\Lambda} \xi_d^+$, which completes the proof when ψ touches by below. The case when ψ touches by above is similar. \square

5.4.2 Regular and singular parts of the optimal sets

In this section we prove that the regular part of an optimal set Ω^* (see Definition 5.4.4) is relatively open in $\partial\Omega^*$.

For any set $\Omega \subset \mathbb{R}^d$ we define the blow-ups sets $\Omega_{x,r}$ of Ω by

$$\Omega_{x,r} = \frac{\Omega - x}{r}, \quad x \in \mathbb{R}^d, r > 0.$$

Given Lebesgue measurable sets $(\Omega_n)_{n \in \mathbb{N}}$ and Ω in \mathbb{R}^d , we say that Ω_n locally converges to Ω , and we write $\Omega_n \xrightarrow{\text{loc}} \Omega$, if the sequence of characteristics functions $\mathbb{1}_{\Omega_n}$ converges in L^1_{loc} to $\mathbb{1}_{\Omega}$.

Definition 5.4.4. Let $\Omega \subset D$ be an open set. We define the regular part of Ω in D by

$$\text{Reg}(\partial\Omega \cap D) = \left\{ x_0 \in \partial\Omega \cap D : \exists \nu_{x_0} \in \partial B_1 \subset \mathbb{R}^d, \Omega_{x_0,r} \xrightarrow{\text{loc}} \{y \in \mathbb{R}^d : y \cdot \nu_{x_0} \leq 0\} \text{ as } r \rightarrow 0^+ \right\}.$$

The singular part of Ω in D is then define by $\text{Sing}(\partial\Omega \cap D) = (\partial\Omega \cap D) \setminus \text{Reg}(\partial\Omega \cap D)$.

Lemma 5.4.5. Let $U = (u_1, \dots, u_k)$ be the vector of the first k normalized eigenfunctions on Ω^* . Then,

1. For every $x_0 \in \partial\Omega^* \cap D$ the limit

$$\Theta_{U_{x_0}}(0) := \lim_{r \rightarrow 0^+} \frac{|\{|U_{x_0}| > 0\} \cap B_r|}{|B_r|} \quad (5.4.3)$$

exists and we have

$$\Theta_{U_{x_0}}(0) = \frac{1}{\Lambda\omega_d} \lim_{r \rightarrow 0^+} W(U_{x_0}, r). \quad (5.4.4)$$

2. There exists $\delta > 0$ such that, for every $x_0 \in \partial\Omega^* \cap D$ we have $\Theta_{U_{x_0}}(0) \in \{\frac{1}{2}\} \cup [\frac{1}{2} + \delta, 1[$.

Proof. Let $(r_n)_{n \in \mathbb{N}}$ be an infinitesimal sequence and set $\tilde{B}_n(\xi) = \frac{1}{r_n} U_{x_0}(r_n \xi)$. Up to a subsequence, \tilde{B}_n converges to some \tilde{B}_0 (in the sense of Proposition 5.3.2). Since \tilde{B}_0 is one homogeneous (Lemma 5.3.4) and harmonic in $\{|\tilde{B}_0| > 0\}$ (Lemma 5.3.3) we have

$$\int_{B_r} |\nabla \tilde{B}_0|^2 = \frac{1}{r} \int_{\partial B_r} |\tilde{B}_0|^2,$$

and hence, for every $r > 0$, we get that

$$\begin{aligned} W(\tilde{B}_0, r) &= \frac{1}{r^d} \int_{B_r} |\nabla \tilde{B}_0|^2 + \frac{\Lambda}{r^d} |\{| \tilde{B}_0 | > 0 \} \cap B_r| - \frac{1}{r^{d+1}} \int_{\partial B_r} |\tilde{B}_0|^2 \\ &= \Lambda\omega_d \frac{|\{| \tilde{B}_0 | > 0 \} \cap B_r|}{|B_r|}. \end{aligned} \quad (5.4.5)$$

On the other hand, by (5.3.8) we have that $W(\tilde{B}_0, r) = \lim_{s \rightarrow 0^+} W(U_{x_0}, s)$ for every $r > 0$ and therefore

$$\frac{1}{\Lambda\omega_d} \lim_{s \rightarrow 0^+} W(U_{x_0}, s) = \frac{|\{| \tilde{B}_0 | > 0 \} \cap B_r|}{|B_r|} \quad \text{for every } r > 0. \quad (5.4.6)$$

Then, using that \tilde{B}_n converges to \tilde{B}_0 in $L^1_{\text{loc}}(\mathbb{R}^d)$, it follows that

$$\frac{1}{\Lambda\omega_d} \lim_{s \rightarrow 0^+} W(U_{x_0}, s) = \frac{|\{| \tilde{B}_0 | > 0 \} \cap B_1|}{|B_1|} = \lim_{n \rightarrow \infty} \frac{|\{| \tilde{B}_n | > 0 \} \cap B_1|}{|B_1|} = \lim_{n \rightarrow \infty} \frac{|\{| U_{x_0} | > 0 \} \cap B_{r_n}|}{|B_{r_n}|}.$$

This proves part (1) of the Lemma since the above equalities hold for any sequence $r_n \downarrow 0$.

From (5.4.6) and (5.4.4) it follows that the density of the cone $\{|\tilde{B}_0| > 0\}$ at 0 is given by

$$\lim_{r \rightarrow 0^+} \frac{|\{|\tilde{B}_0| > 0\} \cap B_r|}{|B_r|} = \Theta_{U_{x_0}}(0) \in [0, 1].$$

Moreover, $|\tilde{B}_0|$ is a non-trivial (part (3) of Proposition 5.3.2), one-homogeneous (Lemma 5.3.4) and harmonic function in $\{|\tilde{B}_0| > 0\}$ (Lemma 5.3.6). Therefore, the density of $\{|\tilde{B}_0| > 0\}$ at 0 cannot be strictly less than $\frac{1}{2}$ (otherwise, setting $S = \{|\tilde{B}_0| > 0\} \cap \partial B_1$, the two first parts of [77, Remark 4.8] respectively give $\lambda_1(S) \leq d-1$ and $\lambda_1(S) > d-1$), cannot belong to $(\frac{1}{2}, \frac{1}{2} + \delta)$ for some universal constant $\delta > 0$ (see [77, Lemma 5.3]) and is less than $1 - c$ by Proposition 5.2.9. \square

We will also need the following characterization of the regular part.

Lemma 5.4.6. *We have*

$$\text{Reg}(\partial\Omega^* \cap D) = \left\{ x_0 \in \partial\Omega^* \cap D : \Theta_{U_{x_0}}(0) = \frac{1}{2} \right\},$$

where $\Theta_{U_{x_0}}(0)$ is define in (5.4.3).

Proof. Let $x_0 \in \partial\Omega^* \cap D$, $r_n \downarrow 0$ and $B_n = B_{x_0, r_n}$ be a blow-up sequence converging (in the sense of Proposition 5.3.2) to some B_0 ; in particular, $\Omega_{x_0, r_n}^* = \{|B_n| > 0\}$ locally converges to $\{|B_0| > 0\}$. By (5.4.4), (5.4.6) and a change of variables (because $\tilde{B}_0 = B_0 \circ A_{x_0}^{1/2}$) we have

$$\Theta_{U_{x_0}}(0) = \frac{|\{|\tilde{B}_0| > 0\} \cap B_1|}{|B_1|} = \frac{|\{|B_0| > 0\} \cap A_{x_0}^{1/2}[B_1]|}{|A_{x_0}^{1/2}[B_1]|}. \quad (5.4.7)$$

If $x_0 \in \text{Reg}(\partial\Omega^* \cap D)$, then $\{|B_0| > 0\}$ is an half-space and it follows by (5.4.7) that $\Theta_{U_{x_0}}(0) = 1/2$. Reciprocally, assume that $\Theta_{U_{x_0}}(0) = 1/2$. It is enough to prove that $\{|\tilde{B}_0| > 0\}$ is an half-space, since then $\{|B_0| > 0\}$ is also an half-space. Set $S = \{|\tilde{B}_0| > 0\} \cap \partial B_1$ and notice that $\mathcal{H}^{d-1}(S) = d\omega_d/2$ since $|\tilde{B}_0|$ is one homogeneous. Assume by contradiction that $S = S_0 \sqcup S_1$ is the disjoint union of two sets $S_0, S_1 \subset \partial B_1$. Since $|\tilde{B}_0|$ is one homogeneous and harmonic on $\{|\tilde{B}_0| > 0\}$ it follows that $\varphi = |\tilde{B}_0|_{|\partial B_1}$ is solution of

$$-\Delta_{\mathbb{S}^{d-1}}\varphi = (d-1)\varphi \quad \text{in } S_0, \quad \varphi = 0 \quad \text{on } \partial S_0,$$

which implies that $\lambda_1(S_0) \leq d-1$. On the other hand, since $\mathcal{H}^{d-1}(S_0) < d\omega_d/2$, we also have that $\lambda_1(S_0) > d-1$ (see [77, Remark 4.8]), which is a contradiction. Therefore, S is connected and hence $\lambda_1(S) = d-1$. This implies that S is, up to a rotation, the half-sphere $\partial B_1 \cap \{x_d > 0\}$ and hence that $\{|\tilde{B}_0| > 0\}$ is the half-space $\{x_d > 0\}$. \square

Proposition 5.4.7. *The regular set $\text{Reg}(\partial\Omega^* \cap D)$ is an open subset of $\partial\Omega^*$.*

Proof. Let $x_0 \in \text{Reg}(\partial\Omega^* \cap D)$ and assume by contradiction that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ of points in $\text{Sing}(\partial\Omega^* \cap D) = (\partial\Omega^* \cap D) \setminus \text{Reg}(\partial\Omega^* \cap D)$ converging to x_0 . By Lemmas 5.4.5 and 5.4.6 we have $\Theta_{U_{x_0}}(0) = 1/2$ and $\Theta_{U_{x_n}}(0) \geq 1/2 + \delta$. Since the function $\varphi_n(r) = W(U_{x_n}, r) + Cr^{\delta_A}$ is non-decreasing by Proposition 5.2.10, we have for every $r > 0$

$$\frac{1}{2} + \delta \leq \Theta_{U_{x_n}}(0) = \frac{1}{\Lambda\omega_d} \lim_{s \rightarrow 0^+} W(U_{x_n}, s) = \frac{1}{\Lambda\omega_d} \lim_{s \rightarrow 0^+} \varphi_n(s) \leq \frac{1}{\Lambda\omega_d} \varphi_n(r).$$

Passing to the limit as $n \rightarrow \infty$ and using that $\lim_{n \rightarrow \infty} W(U_{x_n}, r) = W(U_{x_0}, r)$, it follows that for every $r > 0$

$$\frac{1}{2} + \delta \leq \frac{1}{\Lambda\omega_d} \lim_{n \rightarrow \infty} \varphi_n(r) = \frac{1}{\Lambda\omega_d} [W(U_{x_0}, r) + Cr^{\delta_A}].$$

But the right hand side converges to $\Theta_{U_{x_0}}(0) = 1/2$ as $r \rightarrow 0$ which is a contradiction \square

5.4.3 The regular part is Reifenberg flat

We prove that the regular part of Ω_1^* is locally Reifenberg flat. Recall that by Proposition 5.4.7, $\text{Reg}(\partial\Omega_1^* \cap D)$ is relatively open in $\partial\Omega_1^*$. Roughly speaking, a domain is said to be Reifenberg flat if its boundary can be well approximated by hyperplanes. We give here a precise definition.

Definition 5.4.8. Let $\Omega \subset \mathbb{R}^d$ be an open set and let $\delta, R > 0$. We say that Ω is a (δ, R) -Reifenberg flat domain if:

1. For every $x \in \partial\Omega$ there exist an hyperplane $H = H_{x,R}$ containing x and a unit vector $\nu = \nu_{x,R} \in \partial B_1 \subset \mathbb{R}^d$ orthogonal to H such that

$$\begin{aligned} \{y + t\nu \in B_R(x) : y \in H, t \geq 2\delta R\} &\subset \Omega, \\ \{y - t\nu \in B_R(x) : y \in H, t \geq 2\delta R\} &\subset \mathbb{R}^d \setminus \Omega. \end{aligned}$$

2. For every $x \in \partial\Omega$ and every $r \in (0, R]$ there exists an hyperplane $H = H_{x,r}$ containing x such that

$$\text{dist}_{\mathcal{H}}(\partial\Omega \cap B_r(x), H \cap B_r(x)) < \delta r.$$

Proposition 5.4.9. Let $\delta > 0$. Then, for every $x_0 \in \text{Reg}(\partial\Omega_1^* \cap D)$ there exists $R = R(x_0) > 0$ such that Ω_1^* is (δ, R) -Reifenberg flat in a neighborhood of x_0 .

Proof. Assume by contradiction that there exists $\delta > 0$ and $x_0 \in \text{Reg}(\partial\Omega_1^* \cap D)$ such that, for every $R > 0$, Ω_1^* is not (δ, R) -Reifenberg flat in any neighborhood of x_0 . Then, there exist sequences $x_n \rightarrow x_0$, $x_n \in \partial\Omega_1^*$, and $r_n \downarrow 0$ such that one of the following assertion holds

- i) For every hyperplane H containing x_n and every $\nu \in \partial B_1$ we have either

$$\{y + t\nu \in B_{r_n}(x_n) : y \in H, t \geq 2\delta r_n\} \not\subset \Omega_1^* \quad \text{or} \quad \{y - t\nu \in B_{r_n}(x_n) : y \in H, t \geq 2\delta r_n\} \not\subset \mathbb{R}^d \setminus \Omega_1^*.$$

- ii) For every hyperplane containing x_n we have

$$\text{dist}_{\mathcal{H}}(\partial\Omega_1^* \cap B_{r_n}(x_n), H \cap B_{r_n}(x_n)) \geq \delta r_n.$$

We consider the blow-up sequence $B_n(\xi) = \frac{1}{r_n}U(x_n + r_n\xi)$ and set $\Omega_n = \{|B_n| > 0\}$. Then the above assumptions can be equivalently reformulated as

- i') For every hyperplane H containing 0 and every $\nu \in \partial B_1$ we have either

$$\{y + t\nu \in B_1 : y \in H, t \geq 2\delta\} \not\subset \Omega_n \quad \text{or} \quad \{y - t\nu \in B_1 : y \in H, t \geq 2\delta\} \not\subset \mathbb{R}^d \setminus \Omega_n.$$

- ii') For every hyperplane containing 0 we have

$$\text{dist}_{\mathcal{H}}(\partial\Omega_n \cap B_1, H \cap B_1) \geq \delta.$$

Notice that $x_n \in \text{Reg}(\partial\Omega_1^* \cap D)$ for n large enough since $\text{Reg}(\partial\Omega_1^* \cap D)$ is an open subset of $\partial\Omega_1^*$ (Proposition 5.4.7). Up to a subsequence, B_n and $\tilde{B}_n = B_n \circ A_{x_n}^{1/2}$ converge (in the sense of Proposition 5.3.2) to B_0 and $\tilde{B}_0 = B_0 \circ A_{x_0}^{1/2}$ respectively.

We first prove that

$$W(\tilde{B}_0, r) = \frac{\Lambda\omega_d}{2} \quad \text{for every } r > 0. \quad (5.4.8)$$

By Proposition 5.2.10, $\varphi_n(r) := W(U_{x_n}, rr_n) + C(rr_n)^{\delta_\Lambda}$ is a non-decreasing function. Moreover, by Lemma 5.4.5 and since we have $\Theta_{U_{x_n}}(0) = 1/2$ (Lemma 5.4.6), it follows that

$$\lim_{r \rightarrow 0^+} \varphi_n(r) = \lim_{r \rightarrow 0^+} W(U_{x_n}, r) = \Lambda\omega_d \Theta_{U_{x_n}}(0) = \frac{\Lambda\omega_d}{2}.$$

We now fix $r > 0$ and $\varepsilon > 0$. Since $\lim_{s \rightarrow 0^+} W(U_{x_0}, s) = \Lambda\omega_d \Theta_{U_{x_0}}(0) = \frac{\Lambda\omega_d}{2}$ there exists $\bar{r} > 0$ such that

$$W(U_{x_0}, \bar{r}) + C\bar{r}^{\delta_A} \leq \frac{\Lambda\omega_d}{2} + \varepsilon.$$

Moreover, since $\lim_{n \rightarrow \infty} W(U_{x_n}, \bar{r}) = W(U_{x_0}, \bar{r})$, we have for n large enough that

$$W(U_{x_n}, \bar{r}) \leq W(U_{x_0}, \bar{r}) + \varepsilon.$$

Therefore, choosing n large enough so that $rr_n \leq \bar{r}$, we get that

$$\frac{\Lambda\omega_d}{2} \leq \varphi_n(r) \leq \varphi_n\left(\frac{\bar{r}}{r_n}\right) = W(U_{x_n}, \bar{r}) + C\bar{r}^{\delta_A} \leq W(U_{x_0}, \bar{r}) + \varepsilon + C\bar{r}^{\delta_A} + \frac{\Lambda\omega_d}{2} + 2\varepsilon,$$

which proves that

$$\lim_{n \rightarrow \infty} \varphi_n(r) = \frac{\Lambda\omega_d}{2} \quad \text{for every } r > 0.$$

Since \tilde{B}_n converges strongly in H_{loc}^1 to \tilde{B}_0 and $\mathbb{1}_{\tilde{\Omega}_n}$ converges in L_{loc}^1 to $\mathbb{1}_{\tilde{\Omega}_0}$ we have that $\lim_{n \rightarrow \infty} W(\tilde{B}_n, r) = W(\tilde{B}_0, r)$. Hence we get for every $r > 0$

$$\frac{\Lambda\omega_d}{2} = \lim_{n \rightarrow \infty} \varphi_n(r) = \lim_{n \rightarrow \infty} W(U_{x_n}, rr_n) = \lim_{n \rightarrow \infty} W(\tilde{B}_n, r) = W(\tilde{B}_0, r).$$

Now, since \tilde{B}_0 is solution of the Alt-Caffarelli functional (Proposition 5.3.3) and since $W(\tilde{B}_0, r)$ is constant by (5.4.8), it follows from Proposition 5.2.12 that \tilde{B}_0 is one-homogeneous, and hence by (5.4.5) that

$$\frac{1}{2} = \frac{1}{\Lambda\omega_d} W(\tilde{B}_0, r) = \frac{|\{|B_0| > 0\} \cap B_r|}{|B_r|}.$$

Then, as in the proof of Lemma 5.4.6, we get that $\Omega_0 = \{|B_0| > 0\}$ is a half-space and hence that $\partial\Omega_0 = \partial\{|B_0| > 0\}$ is an hyperplane (containing 0). This is in contradiction with both assumptions i') and ii') since $\tilde{\Omega}_n$ and Ω_n^c converge locally Hausdorff to $\tilde{\Omega}_0$ and Ω_0^c respectively (Proposition 5.3.2). This concludes the proof. \square

5.4.4 The regular part is $C^{1,\alpha}$

We prove that the regular part of Ω_1^* is $C^{1,\alpha}$ -regular and that it is C^∞ -regular provided that $a_{ij}, b \in C^\infty$ (see Proposition 5.4.10). Using a boundary Harnak principle for non-tangentially accessible (NTA) domains proved by Jerison and Kenig in [63], we prove that the first eigenfunction satisfies an optimality condition on Ω_1^* . The proof then follows from the regularity result of De Silva for the one-phase free boundaries (see [36]).

Proposition 5.4.10. *The regular part $\text{Reg}(\partial\Omega_1^* \cap D)$ is locally the graph of a $C^{1,\alpha}$ function. Moreover, if $a_{i,j} \in C^{k,\delta}(D)$ and $b \in C^{k-1,\delta}(D)$, for some $\delta \in (0, 1)$ and $k \geq 1$, then $\text{Reg}(\partial\Omega_1^* \cap D)$ is locally the graph of a $C^{k+1,\alpha}$ function. In particular, if $a_{i,j}, b \in C^\infty(D)$, then $\text{Reg}(\partial\Omega_1^* \cap D)$ is locally the graph of a C^∞ function.*

Definition 5.4.11. A bounded open set $\Omega \subset \mathbb{R}^d$ is NTA with constants $M > 1$ and $r_0 > 0$ if the following conditions hold:

- (Corkscrew condition) For every $x \in \partial\Omega$ and $r \in (0, r_0)$ there exists $z_r(x) \in \Omega$ such that

$$M^{-1}r < d(z_r(x), \partial\Omega) < |x - z_r(x)| < r,$$

- $\mathbb{R}^d \setminus \Omega$ satisfies the corkscrew condition,

- (Harnack chain condition) If $\varepsilon > 0$, $x_1, x_2 \in \Omega$, $d(x_i, \partial\Omega) > \varepsilon$, $|x_1 - x_2| < k\varepsilon$, then there exists a sequence of Mk overlapping balls included in Ω of radius ε/M such that, the first one is centered at x_1 and the last one at x_2 , and such that the center of two consecutive balls are at most $\varepsilon/(2M)$ apart.

We now recall that any (δ, R) -Reifenberg flat set is NTA, provided that $\delta > 0$ is small enough. This result is due to Kenig and Toro, see [66, Theorem 3.1].

Theorem 5.4.12 (Reifenberg flat implies NTA). *There exists $\delta_0 > 0$ such that if $\Omega \subset \mathbb{R}^d$ is a (δ, R) -Reifenberg flat domain for some $R > 0$ and some $\delta \leq \delta_0$, then Ω is an NTA domain.*

In the following theorem we state the Boundary Harnack Principle for NTA domains and for solutions of uniformly elliptic equations in divergence form with bounded, measurable coefficients. We refer to [65, Corollary 1.3.7] or [63, Lemma 4.10] for a proof (see also [39] for operator in non-divergence form).

Theorem 5.4.13 (Boundary Harnack principle). *Let $\Omega \subset \mathbb{R}^d$ be an NTA domain and $2r \in (0, r_0)$. Let $\tilde{A} : \mathbb{R}^d \rightarrow \text{Sym}_d^+$ be uniformly elliptic (i.e. $\exists \lambda > 0$, $\lambda^{-1}|\xi|^2 \leq \xi \cdot \tilde{A}_x \xi \leq \lambda|\xi|^2 \forall x, \xi \in \mathbb{R}^d$) with bounded measurable coefficients. Let $x_0 \in \partial\Omega$ and let $u, v \in H^1(\Omega \cap B_{2r}(x_0)) \cap C(\Omega \cap B_{2r}(x_0))$ be such that $u, v = 0$ on $\partial\Omega \cap B_{2r}(x_0)$, $v > 0$ in $\Omega \cap B_{2r}(x_0)$ and*

$$\text{div}(\tilde{A}\nabla u) = \text{div}(\tilde{A}\nabla v) = 0 \quad \text{in } \Omega \cap B_{2r}(x_0).$$

Then there exists $C > 0$, depending only on d and λ and the NTA constants, such that

$$C^{-1} \frac{u(z_r(x_0))}{v(z_r(x_0))} \leq \frac{u(x)}{v(x)} \leq C \frac{u(z_r(x_0))}{v(z_r(x_0))} \quad \text{for every } x \in \Omega \cap B_r(x_0). \quad (5.4.9)$$

Since the estimate (5.4.9) holds for every harmonic functions with a uniform constant, it is standard to deduce that the quotient of two harmonics functions on an NTA domain is Hölder continuous up to the boundary. We refer to [65, Corollary 1.3.9] or [63, Theorem 7.9] (see also [5, Corollary 1]).

Corollary 5.4.14. *Let $\Omega, \tilde{A}, x_0, r$ and u, v be as in Theorem 5.4.13. Then there exist constants $\alpha \in (0, 1)$ and $C > 0$, depending only on d and λ and the NTA constants, such that*

$$\left| \frac{u(x)}{v(x)} - \frac{u(y)}{v(y)} \right| \leq C \frac{u(z_r(x_0))}{v(z_r(x_0))} \left(\frac{|x - y|}{r} \right)^\alpha \quad \text{for every } x, y \in \Omega \cap B_r(x_0).$$

In particular, for every $x \in \partial\Omega \cap B_r(x_0)$ the limit $\lim_{\Omega \ni y \rightarrow x} \frac{u(y)}{v(y)}$ exists and $\frac{u}{v} : \bar{\Omega} \cap B_r(x_0) \rightarrow \mathbb{R}$ is α -Hölder continuous.

We now prove the analogous boundary Harnack theorem for the eigenfunctions on an optimal set Ω^* to the problem (5.1.1). We notice that in the proof it is essential that the first eigenfunction u_1 is positive and non-degenerate (Proposition 5.2.6). The case of the eigenfunctions for the Laplacian is already treated in [82, Appendix A]. We extend this result to the case of the operator $-b^{-1} \text{div}(A\nabla \cdot)$. We highlight that one of the difficulty comes from the presence of the Lipschitz function b .

Theorem 5.4.15 (Boundary Harnack principle for eigenvalues). *Let $U = (u_1, \dots, u_k)$ be the first k normalized eigenfunctions on Ω^* and let $x_0 \in \text{Reg}(\partial\Omega_1^* \cap D)$. Then Ω_1^* is NTA in $B_r(x_0)$ for some $r = r(x_0) > 0$ and there exists $\alpha \in (0, 1)$, depending only on d, λ_A and the NTA constants of Ω_1^* , such that for every $i = 2, \dots, k$*

$$\frac{u_i}{u_1} \text{ is } \alpha\text{-Hölder continuous in } \bar{\Omega}_1^* \cap B_r(x_0).$$

We will need the following Lemma.

Lemma 5.4.16. *Let $\Omega \subset D$ be a quasi-open set, $u \in H_0^1(\Omega)$ and $\lambda > 0$. Then, for every $x_0 \in \partial\Omega \cap D$ there exists $r_0 > 0$ depending only on d, λ_A, c_b and λ , such that for every $r \leq r_0$ with $B_r(x_0) \subset D$, there exists a unique solution $v \in H_0^1(D)$ of*

$$\begin{cases} -\operatorname{div}(A\nabla v) = \lambda bv & \text{in } \Omega \cap B_r(x_0) \\ v = u, & \text{on } \partial(\Omega \cap B_r(x_0)). \end{cases} \quad (5.4.10)$$

If, moreover, $u \in L^\infty(D)$, then $v \in L^\infty(D)$ and we have the estimate

$$\|v\|_{L^\infty(\Omega \cap B_r(x_0))} \leq C(r\|u\|_{H^1(\Omega; m)} + \|u\|_{L^\infty(\partial B_r(x_0))}) \quad (5.4.11)$$

where the constant $C > 0$ depends only on d, λ_A, c_b and λ .

Proof. Observe that any minimizer in $\mathcal{A} := \{\varphi \in H_0^1(D) : u - \varphi \in H_0^1(\Omega \cap B_r(x_0))\}$ of the functional

$$\tilde{J}(\varphi) = \int_D A\nabla\varphi \cdot \nabla\varphi - \lambda \int_D \varphi^2 b$$

is solution of (5.4.10). Therefore, it is enough to prove that $\{\varphi \in \mathcal{A} : \tilde{J}(\varphi) \leq C\}$ is weakly compact in $H_0^1(D)$ to prove the existence of a function v solution of (5.4.10). We first compute

$$\begin{aligned} \int_D \varphi^2 b &\leq 2 \int_{\Omega \cap B_r(x_0)} (\varphi - u)^2 b + 2 \int_D u^2 b \\ &\leq \frac{2}{\lambda_1(\Omega \cap B_r(x_0))} \int_{\Omega \cap B_r(x_0)} A\nabla(\varphi - u) \cdot \nabla(\varphi - u) + 2 \int_D u^2 b \\ &\leq \frac{4\lambda_A^2}{\lambda_1(B_{r_0}(x_0))} \int_D (|\nabla\varphi|^2 + |\nabla u|^2) + 2 \int_D u^2 b. \end{aligned}$$

Then, for r_0 small enough (such that $4\lambda_A^4 \lambda \leq \lambda_1^{-\Delta}(B_1)/(2\lambda_A^2 c_b r_0^2)$ where $\lambda_1^{-\Delta}(B_1)$ stands for the first eigenvalue of the Dirichlet Laplacian on B_1) we have

$$\int_D |\nabla\varphi|^2 \leq \lambda_A^2 \tilde{J}(\varphi) + \lambda_A^2 \lambda \int_D \varphi^2 b \leq \lambda_A^2 \tilde{J}(\varphi) + \frac{1}{2} \int_D (|\nabla\varphi|^2 + |\nabla u|^2) + 2\lambda_A^2 \lambda \int_D u^2 b,$$

which gives that

$$\int_D |\nabla\varphi|^2 \leq 2\lambda_A^2 \tilde{J}(\varphi) + \int_D |\nabla u|^2 + 4\lambda_A^2 \lambda \int_D u^2 b \leq 2\lambda_A^2 \tilde{J}(\varphi) + (1 + 4\lambda_A^2 \lambda) \|u\|_{H^1(\Omega; m)}.$$

This proves the existence of v , and the uniqueness easily follows provided that $\lambda \leq \lambda_1(\Omega \cap B_r(x_0))$.

We now prove the L^∞ -estimate. We consider the functions defined by

$$\begin{cases} \operatorname{div}(A\nabla h) = 0, & -\operatorname{div}(A\nabla w) = \lambda bv & \text{in } \Omega \cap B_r(x_0) \\ h = u, & w = 0 & \text{on } \partial(\Omega \cap B_r(x_0)). \end{cases}$$

Reasoning as above this functions exist and are unique, and we have $v = h + w$. Let $R = R_{\Omega \cap B_r(x_0)}$ be the resolvent of $-b^{-1} \operatorname{div}(A\nabla \cdot)$ in $\Omega \cap B_r(x_0)$. We have the estimates $\|R\|_{\mathcal{L}(L^2, L^{2^*})} \leq C_d$ where $2^* = \frac{2d}{d-2}$ and $\|R\|_{\mathcal{L}(L^d, L^\infty)} \leq Cr$ by [87, Lemma 2.1], where the constant C depends only on d, λ_A and c_b . Notice also that we have

$$v = \lambda^n R^n(v) + \sum_{i=0}^{n-1} \lambda^i R^i(h),$$

and that $\|h\|_{L^\infty(\Omega \cap B_r(x_0))} \leq \|u\|_{L^\infty(\partial B_r(x_0))}$ by the maximum principle. Therefore, with an interpolation argument, there exists a dimensional constant $n \geq 1$ such that we have the estimate

$$\|v\|_{L^\infty(\Omega \cap B_r(x_0))} \leq C(r\|v\|_{L^2(D;m)} + \|u\|_{L^\infty(\partial B_r(x_0))}),$$

where now C also depends on λ . Hence, it remains only to estimate $\|v\|_{L^2(D;m)}$ to complete the proof. Then, for r_0 small enough, we have

$$\begin{aligned} \int_D v^2 b &\leq 2 \int_{\Omega \cap B_r(x_0)} (v-u)^2 b + 2 \int_D u^2 b \leq \frac{4}{\lambda_1(B_{r_0}(x_0))} \int_D (|\nabla v|^2 + |\nabla u|^2) + 2 \int_D u^2 b \\ &\leq \frac{1}{2\lambda} \tilde{J}(v) + \frac{1}{2} \int_D v^2 b + \frac{1}{2\lambda_A^2 \lambda} \int_D |\nabla u|^2 + 2 \int_D u^2 b, \end{aligned}$$

which implies that (since $\tilde{J}(v) \leq \tilde{J}(u)$)

$$\int_D v^2 b \leq \frac{1}{\lambda} \tilde{J}(u) + \frac{1}{\lambda_A^2 \lambda} \int_D |\nabla u|^2 + 4 \int_D u^2 b \leq \left(\frac{2\lambda_A^2}{\lambda} + 4 \right) \|u\|_{H^1(D;m)}.$$

□

Proof of Theorem 5.4.15. By Proposition 5.4.9 and Theorem 5.4.12, Ω_1^* is an NTA domain near x_0 . Let α be the constant given by Corollary 5.4.14 and set $\beta = \frac{\alpha}{1+\alpha}$. Let $x, y \in B_r(x_0)$ and set $\bar{r} = |x-y|^\beta$, $d_x = d(x, \partial\Omega_1^*)$, $d_y = d(y, \partial\Omega_1^*)$. We divide the proof in three steps.

Step 1. Assume that $d_x, d_y \geq 2\bar{r}$. By a change of variables, it follows that $\tilde{u}(z) = \bar{r}^{-1}u_1(x + \bar{r}z)$ is solution of

$$-\operatorname{div}(\tilde{A}\nabla\tilde{u}) = \bar{r}^2 \lambda_1(\Omega^*) \tilde{b}\tilde{u} \quad \text{in } B_2,$$

where we have set $\tilde{A}_z = A_{x+\bar{r}z}$ and $\tilde{b}(z) = b(x + \bar{r}z)$. By standard Schauder estimates (see [48, Theorem 8.8]) we have

$$\|\tilde{u}\|_{C^{1,\delta_A}(B_1)} \leq C(\|\tilde{u}\|_{L^\infty(B_2)} + \bar{r}^2 \lambda_1(\Omega^*) \|\tilde{b}\tilde{u}\|_{L^\infty(B_2)}),$$

where C depends only on d, c_A and λ_A . In particular,

$$\|u_1\|_{C^1(B_{\bar{r}}(x))} \leq \|\tilde{u}\|_{C^1(B_1)} \leq \|\tilde{u}\|_{C^{1,\delta_A}(B_1)} \leq C\|\tilde{u}\|_{L^\infty(B_2)} \leq \frac{C}{\bar{r}}.$$

Similarly, we have $\|u_i\|_{C^1(B_{\bar{r}}(x))} \leq C/\bar{r}$. On the other hand, by non-degeneracy of u_1 we have $u_1(x) \geq cd_x$ and $u_1(y) \geq cd_y$ for some constant $c > 0$. Therefore, since u_i is L -Lipschitz continuous (and because $y \in B_{\bar{r}}(x)$), we get

$$\begin{aligned} \left| \frac{u_i(x)}{u_1(x)} - \frac{u_i(y)}{u_1(y)} \right| &\leq \frac{|u_i(x) - u_i(y)|}{u_1(x)} + \frac{|u_1(x) - u_1(y)| |u_i(y)|}{u_1(x)u_1(y)} \leq \frac{C}{\bar{r}} |x-y| \left(\frac{1}{cd_x} + \frac{Ld_y}{c^2 d_x d_y} \right) \\ &\leq \frac{C}{\bar{r}^2} |x-y| = C|x-y|^{1-2\beta} \leq C|x-y|^\beta, \end{aligned}$$

where the last inequality holds provided that $\beta \leq 1/3$.

Step 2. Assume that $d_x \leq 2\bar{r}$. Let $\bar{x} \in \partial\Omega_1^*$ such that $d_x = |\bar{x} - x|$. We write for simplicity $\lambda_1 = \lambda_1(\Omega^*)$, $\lambda_i = \lambda_i(\Omega^*)$ and $B = B_{6\bar{r}}(\bar{x})$. Since u_i may change its sign, we consider the functions

$$\begin{cases} -\operatorname{div}(A\nabla v_i) = \lambda_i b v_i, & -\operatorname{div}(A\nabla w_i) = \lambda_i b w_i \quad \text{in } \Omega_1^* \cap B \\ v_i = u_i^+, & w_i = u_i^- \quad \text{on } \partial(\Omega_1^* \cap B). \end{cases}$$

These functions exist thanks to Lemma 5.4.16 and we have $u_i = v_i - w_i$. We now set $m = \min_{z \in B} b(z)$ and $M = \max_{z \in B} b(z)$ and $I = (-1, 1)$. Moreover, for $(z, z_{d+1}) \in (\Omega_1^* \cap B) \times I$ we define the functions

$$\begin{aligned} u_{1,m}(z, z_{d+1}) &= e^{-\sqrt{\lambda_1 m} z_{d+1}} u_1(z) & u_{1,M}(z, z_{d+1}) &= e^{-\sqrt{\lambda_1 M} z_{d+1}} u_1(z) \\ u_{i,m}(z, z_{d+1}) &= e^{-\sqrt{\lambda_i m} z_{d+1}} v_i(z) & u_{i,M}(z, z_{d+1}) &= e^{-\sqrt{\lambda_i M} z_{d+1}} v_i(z). \end{aligned}$$

We define the matrix-valued function $\tilde{A} : (\Omega_1^* \cap B) \times I \subset \mathbb{R}^{d+1} \rightarrow \text{Sym}_{d+1}^+$ by

$$\tilde{A}_{(z, z_{d+1})} = \begin{pmatrix} A_z & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for every } (z, z_{d+1}) \in (\Omega_1^* \cap B) \times I.$$

Moreover, we define the harmonic extensions of the above functions as follows

$$\begin{cases} \text{div}(\tilde{A}\nabla h_{1,m}) = \text{div}(\tilde{A}\nabla h_{1,M}) = \text{div}(\tilde{A}\nabla h_{i,m}) = \text{div}(\tilde{A}\nabla h_{i,M}) = 0 & \text{in } (\Omega_1^* \cap B) \times I \\ h_{1,m} = u_{1,m}, \quad h_{1,M} = u_{1,M}, \quad h_{i,m} = u_{i,m}, \quad h_{i,M} = u_{i,M} & \text{on } \partial[(\Omega_1^* \cap B) \times I] \end{cases}$$

Now, we get with an easy computation that

$$\text{div}(\tilde{A}\nabla(u_{1,m} - h_{1,m})) = \lambda_1 e^{-\sqrt{\lambda_1 m} x_{d+1}} (m - b(x)) u_1(x) \leq 0 \quad \text{in } (\Omega_1^* \cap B) \times I,$$

which, by the weak maximum principle, implies that $h_{1,m} \leq u_{1,m}$ in $(\Omega_1^* \cap B) \times I$. Similarly we have (since the functions $u_{i,m}, u_{i,M}$ are positive)

$$h_{1,m} \leq u_{1,m}, \quad u_{1,M} \leq h_{1,M}, \quad h_{i,m} \leq u_{i,m}, \quad u_{i,M} \leq h_{i,M} \quad \text{in } (\Omega_1^* \cap B) \times I. \quad (5.4.12)$$

Moreover, using again the maximum principle, we have the following inequalities

$$\frac{e^{\sqrt{\lambda_1 m}}}{e^{\sqrt{\lambda_1 M}}} h_{1,m} \leq h_{1,M} \leq \frac{e^{\sqrt{\lambda_1 M}}}{e^{\sqrt{\lambda_1 m}}} h_{1,m}, \quad \text{in } (\Omega_1^* \cap B) \times I, \quad (5.4.13)$$

and similarly we have

$$\frac{e^{\sqrt{\lambda_i m}}}{e^{\sqrt{\lambda_i M}}} h_{i,m} \leq h_{i,M} \leq \frac{e^{\sqrt{\lambda_i M}}}{e^{\sqrt{\lambda_i m}}} h_{i,m}, \quad \text{in } (\Omega_1^* \cap B) \times I. \quad (5.4.14)$$

Now, since $x, y \in B_{3\bar{r}}(\bar{x}) \subset B$, we can use (5.4.12), (5.4.13) and (5.4.14) to estimate

$$\begin{aligned} \frac{v_i(x)}{u_1(x)} - \frac{v_i(y)}{u_1(y)} &= \frac{u_{i,M}(x, 0)}{u_{1,m}(x, 0)} - \frac{u_{i,m}(y, 0)}{u_{1,M}(y, 0)} \leq \frac{h_{i,M}(x, 0)}{h_{1,m}(x, 0)} - \frac{h_{i,m}(y, 0)}{h_{1,M}(y, 0)} \\ &\leq \frac{e^{\sqrt{\lambda_i M}} h_{i,m}(x, 0)}{e^{\sqrt{\lambda_i m}} h_{1,m}(x, 0)} - \frac{e^{\sqrt{\lambda_1 m}} h_{i,m}(y, 0)}{e^{\sqrt{\lambda_1 M}} h_{1,m}(y, 0)} \\ &\leq \frac{h_{i,m}(x, 0)}{h_{1,m}(x, 0)} - \frac{h_{i,m}(y, 0)}{h_{1,m}(y, 0)} + C\bar{r} \frac{h_{i,m}(x, 0)}{h_{1,m}(x, 0)} + C\bar{r} \frac{h_{i,m}(y, 0)}{h_{1,m}(y, 0)} \end{aligned}$$

where the last inequality follows from the definitions of m, M and the fact that b is a Lipschitz continuous function. Now, observe that $\Omega_1^* \times I \subset \mathbb{R}^{d+1}$ is an NTA domain near $(\bar{x}, 0)$ with the same constants than Ω_1^* . By Corollary 5.4.14, setting $z_0 = z_{3\bar{r}}(\bar{x}, 0) \in \mathbb{R}^{d+1}$, we have (notice also that $x, y \in B_{3\bar{r}}(\bar{x})$)

$$\frac{h_{i,m}(x, 0)}{h_{1,m}(x, 0)} - \frac{h_{i,m}(y, 0)}{h_{1,m}(y, 0)} \leq C \frac{h_{i,m}(z_0)}{h_{1,m}(z_0)} \left(\frac{|x - y|}{3\bar{r}} \right)^\alpha = C\bar{r} \frac{h_{i,m}(z_0)}{h_{1,m}(z_0)},$$

where in the last equality we have used that $\bar{r} = |x - y|^\beta$ with $\beta = \frac{\alpha}{1+\alpha}$. Moreover, by Theorem 5.4.13 we have

$$\frac{h_{i,m}(x, 0)}{h_{1,m}(x, 0)} \leq C \frac{h_{i,m}(z_0)}{h_{1,m}(z_0)}, \quad \frac{h_{i,m}(y, 0)}{h_{1,m}(y, 0)} \leq C \frac{h_{i,m}(z_0)}{h_{1,m}(z_0)},$$

which finally gives

$$\frac{v_i(x)}{u_1(x)} - \frac{v_i(y)}{u_1(y)} \leq C\bar{r} \frac{h_{i,m}(z_0)}{h_{1,m}(z_0)}. \quad (5.4.15)$$

Then, using (5.4.13) and (5.4.12) we have the following estimate

$$h_{1,m}(z_0) \geq \frac{e^{\sqrt{\lambda_1 m}}}{e^{\sqrt{\lambda_1 M}}} h_{1,M}(z_0) \geq \frac{e^{\sqrt{\lambda_1 m}}}{e^{\sqrt{\lambda_1 M}}} u_{1,M}(z_0) \geq \left(\frac{e^{\sqrt{\lambda_1 m}}}{e^{\sqrt{\lambda_1 M}}} \right)^2 u_{1,m}(z_0) \geq C u_{1,m}(z_0).$$

Now, in view of the definition of $z_0 = z_{3\bar{r}}(\bar{x}, 0) \in \mathbb{R}^{d+1}$ we have $d(z_0, \partial(\Omega_1^* \times I)) > 3\bar{r}M^{-1}$ and by non-degeneracy of u_1 (Proposition 5.2.6) it follows that $u_{1,m}(z_0) \geq C\bar{r}$. Moreover, by (5.4.11), it follows that $\|v_i\|_{L^\infty(B)} \leq C\bar{r}$ since u_i is Lipschitz continuous. Therefore we have

$$\frac{v_i(x)}{u_1(x)} - \frac{v_i(y)}{u_1(y)} \leq C\bar{r} \frac{u_{i,m}(z_0)}{u_{1,m}(z_0)} \leq C\bar{r} \frac{\|v_i\|_{L^\infty(B)}}{u_{1,m}(z_0)} \leq C\bar{r} = C|x - y|^\beta.$$

This concludes the proof since the same estimate also holds for w_i and that we have $u_i/u_1 = v_i/u_1 - w_i/u_1$. \square

As a consequence of the optimality condition of U (Lemma 5.4.3) and of the boundary Harnack principle (Theorem 5.4.15), it follows that the first eigenfunction is solution of a one-phase free boundary problem on Ω_1^* .

Lemma 5.4.17. *For every $x_0 \in \text{Reg}(\partial\Omega_1^* \cap D)$ there exist $r = r(x_0) > 0$, $c \in (0, 1)$ and a Hölder continuous function $g : \partial\Omega_1^* \cap B_r(x_0) \rightarrow [c, 1]$ such that u_1 is a viscosity solution to the problem*

$$-\text{div}(A\nabla u_1) = \lambda_1(\Omega^*)b u_1 \text{ in } \Omega_1^*, \quad u_1 = 0 \text{ on } \partial\Omega_1^*, \quad |A^{1/2}[\nabla u_1]| = g\sqrt{\Lambda} \text{ on } \partial\Omega_1^* \cap B_r(x_0).$$

Proof. Let $U = (u_1, \dots, u_k)$ be the first k eigenfunctions on Ω^* . By Theorem 5.4.15 the functions $g_i := \frac{u_i}{u_1} : \partial\Omega_1^* \cap B_r(x_0) \rightarrow \mathbb{R}$, for $i = 2, \dots, k$, are Hölder continuous. Therefore, the function $g : \partial\Omega_1^* \cap B_r(x_0) \rightarrow [0, 1]$ defined by

$$g = \frac{1}{\sqrt{1 + g_2^2 + \dots + g_k^2}}. \quad (5.4.16)$$

is also Hölder continuous. Since $u_1 = g|U|$, it follows from the non-degeneracy of u_1 that $g \geq c := C_1^{-1}$ where C_1 is the constant from Proposition 5.2.6. Now, let $y \in \partial\Omega_1^* \cap B_r(x_0)$ and let $\varphi \in C^2(D)$ be a function touching u_1 by below at the point y . Since $1/g$ is β -Hölder continuous for some $\beta \in (0, 1)$, there exists $C > 0$ such that for $\rho > 0$ small enough we have

$$\frac{1}{g(x)} \geq \frac{1}{g(y)} - C|x - y|^\beta \geq 0 \quad \text{for every } x \in \overline{\Omega_1^*} \cap B_\rho(y).$$

Therefore, the function $\psi(x) = \varphi(x) \left(\frac{1}{g(y)} - C|x - y|^\beta \right)$ is such that $\psi(y) = |U(y)|$ and satisfies

$$\psi(x) \leq u_1(x) \left(\frac{1}{g(x_0)} - C|x - y|^\beta \right) \leq |U(x)| \quad \text{for every } x \in \overline{\Omega_1^*} \cap B_\rho(y). \quad (5.4.17)$$

This proves that ψ touches $|U|$ by below at the point y . On the other hand, ψ is differentiable at y and we have $\nabla\psi(y) = \frac{1}{g(y)}\nabla\varphi(y)$. Therefore, using that U is a viscosity solution of (5.4.2), it follows that

$$\sqrt{\Lambda} \geq |A_y^{1/2}[\nabla\psi(y)]| \geq \frac{1}{g(y)} |A_y^{1/2}[\nabla\varphi(y)]|.$$

The case when φ touches u_1 by above is similar. \square

Theorem 5.4.18 (Higher boundary Harnack principle for eigenvalues). *Let $k \geq 1$ and assume that Ω_1^* is $C^{k,\alpha}$ -regular near $x_0 \in \partial\Omega_1^* \cap D$ for some $\alpha \in (0, 1)$. If $k \geq 2$, suppose moreover that $a_{ij}, b \in C^{k-1,\alpha}(D)$. Then there exists $r > 0$ such that for every $i = 2, \dots, k$*

$$\frac{u_i}{u_1} \text{ is of class } C^{k,\alpha} \text{ in } \overline{\Omega_1^*} \cap B_r(x_0).$$

Proof. Let $R > 0$ such that there exists $\varphi \in H_0^1(B_R(x_0))$ satisfying $\varphi > 0$ in $B_R(x_0)$ and solution of the equation

$$-\operatorname{div}(A\nabla\varphi) = \lambda_1(\Omega^*)b\varphi \quad \text{in } B_R(x_0).$$

(More precisely, we extend $a_{i,j}$ and b to bounded functions in \mathbb{R}^d with $b \geq c_b$, and we choose $R > 0$ such that $\lambda_1(B_R) = \lambda_1(\Omega^*)$). Let $2r < R$ be such that Ω_1^* is $C^{k,\alpha}$ -regular in the ball $B_{2r}(x_0) \subset D$. Then we have

$$\begin{aligned} \operatorname{div}\left(\varphi^2 A\nabla\left(\frac{u_1}{\varphi}\right)\right) &= \operatorname{div}(\varphi A\nabla u_1 - u_1 A\nabla\varphi) \\ &= \varphi \operatorname{div}(A\nabla u_1) + \nabla\varphi A\nabla u_1 - \nabla u_1 A\nabla\varphi - u_1 \operatorname{div}(A\nabla\varphi) = 0 \quad \text{in } \Omega_1^* \cap B_{2r}(x_0), \end{aligned}$$

and similarly

$$\operatorname{div}\left(\varphi^2 A\nabla\left(\frac{u_i}{\varphi}\right)\right) = (\lambda_1(\Omega^*) - \lambda_i(\Omega^*))bu_i\varphi \quad \text{in } \Omega_1^* \cap B_{2r}(x_0).$$

Now, the proof follows by [38, Theorem 2.4] for $k = 1$ and by [38, Theorem 3.1] for $k \geq 2$, which say that $u_1/\varphi, u_i/\varphi \in C^{k,\alpha}(\overline{\Omega_1^*} \cap B_r(x_0))$. \square

Proof of Proposition 5.4.10. We prove the regularity by a finite induction on $l \in \{1, \dots, k\}$. For $l = 1$, by [36, Theorem 1.1] and Lemma 5.4.17 it follows that $\operatorname{Reg}(\partial\Omega_1^* \cap D)$ is locally $C^{1,\alpha}$ -regular. Now, if $\operatorname{Reg}(\partial\Omega_1^* \cap D)$ is $C^{l,\alpha}$ -regular, $l \leq k$, by Theorem 5.4.18 and the definition of g in (5.4.16), we have that g is a $C^{l,\alpha}$ function on $\operatorname{Reg}(\partial\Omega_1^* \cap D)$. Therefore, in view of Lemma 5.4.17 and by [67, Theorem 2] it follows that $\operatorname{Reg}(\partial\Omega_1^* \cap D)$ is locally $C^{l+1,\alpha}$ -regular. This completes the proof. \square

5.4.5 Dimension of the singular set

We prove in this last subsection some kind of smallness of the singular set. We recall that Ω^* denotes an optimal set to (5.1.1) and that Ω_1^* stands for any connected component of Ω^* at which the first eigenfunction is positive.

An estimate of the dimension of the singular set can be obtain as a consequence of the Federer's Theorem. Indeed, since Ω_1^* is a set of finite perimeter (Proposition 5.2.1) and in view of the density estimate (Proposition 5.2.9), it follows from the Federer's Theorem (see, for instance, [74, Theorem 16.2]) that $\mathcal{H}^{d-1}(\operatorname{Sing}(\partial\Omega_1^* \cap D)) = 0$. In Proposition 5.4.20 below we provide a more precise estimate of the dimension of the singular set.

Definition 5.4.19. We define d^* as the smallest dimension which admits a one-homogeneous global minimizer of the Alt-Caffarelli functional with exactly one singularity at zero.

The exact value of the critical dimension d^* is still unknown but we know that $d^* \in \{5, 6, 7\}$ (see [62] for $d^* \geq 5$ and [37] for $d^* \leq 7$). The following result on the smallness of the singular set is standard and was first proved in the framework of the minimal surfaces (for which the critical dimension is exactly 8, see for example [49, Chapter 11]). Later, in [92], Weiss adapted this strategy for minimizers of the Alt-Caffarelli functional by introducing a monotonicity formula. In [77], the authors prove this result in the vectorial setting. In this section we follow the same approach and we extend this result to the case of variable coefficients.

Proposition 5.4.20 (On the dimension of the singular set). *The singular part $\operatorname{Sing}(\partial\Omega_1^* \cap D)$ is:*

1. empty if $d < d^*$,
2. a discrete (locally finite) set if $d = d^*$,

3. of Hausdorff dimension at most $(d - d^*)$ if $d > d^*$, that is, $\mathcal{H}^{d-d^*+s}(\text{Sing}(\partial\Omega_1^* \cap D)) = 0$ for every $s > 0$.

We first prove two preliminary Lemmas and to this aim we extend the definition of the Weiss functional for any ball. Let $U \in H^1(\mathbb{R}^d, \mathbb{R}^k)$, $x \in \mathbb{R}^d$ and $r > 0$. We set

$$J(U, x, r) = \int_{B_r(x)} |\nabla U|^2 + \Lambda |\{|U| > 0\} \cap B_r(x)|$$

and

$$W(U, x, r) = \frac{1}{r^d} J(U, x, r) - \frac{1}{r^{d+1}} \int_{\partial B_r(x)} |U|^2.$$

Obviously we have $J(U, r) = J(U, 0, r)$ and $W(U, r) = W(U, 0, r)$.

Lemma 5.4.21. *Let $(x_n)_{n \in \mathbb{N}} \subset \partial\Omega_1^* \cap D$ be a sequence converging to $x_0 \in \partial\Omega_1^* \cap D$ and let $B_n = B_{x_0, r_n}$ be a blow-up sequence with fixed center. We set $\tilde{B}_n = B_n \circ A_{x_0}^{1/2}$ and $\tilde{\Omega}_n = \{|\tilde{B}_n| > 0\}$. Then, up to a subsequence, the sequence $y_n = A_{x_0}^{-1/2} \left[\frac{x_n - x_0}{r_n} \right] \in \partial\tilde{\Omega}_n$ converges to some y_0 and, for every small $r > 0$, there exists n_0 such that for every $n \geq n_0$ we have*

$$W(U_{x_n}, rr_n) \leq W(\tilde{B}_n, y_0, r) + C|x_0 - x_n|^{\delta_A/2} + C \frac{|y_0 - y_n|}{r}, \quad (5.4.18)$$

where the constant $C > 0$ depends only on $d, c_A, \lambda_A, \Lambda$ and the Lipschitz constant $L = \|\nabla U\|_{L^\infty(K)}$ of U in some compact neighborhood $K \subset D$ of x_0 .

Proof. We first compare $J(U_{x_n}, rr_n)$ and $J(\tilde{B}_n, y_0, r)$. Since $U_{x_n} = U \circ F_{x_n}$ by definition, we compute

$$\begin{aligned} J(U_{x_n}, rr_n) &= \int_{B_{rr_n}} (|\nabla U_{x_n}(\xi)|^2 + \Lambda \mathbb{1}_{\{|U_{x_n}(\xi)| > 0\}}) d\xi \\ &= \int_{B_{rr_n}} (A_{x_n} \nabla U \cdot \nabla U + \Lambda \mathbb{1}_{\{|U| > 0\}}) \circ F_{x_n}(\xi) d\xi \\ &\leq \int_{B_{rr_n}} (A_{x_0} \nabla U \cdot \nabla U + \Lambda \mathbb{1}_{\{|U| > 0\}}) \circ F_{x_n}(\xi) d\xi + \omega_d (rr_n)^d L^2 c_A |x_0 - x_n|^{\delta_A}, \end{aligned}$$

where in the last inequality we have used that the coefficients a_{ij} are δ_A -Hölder continuous, that is $\|A_{x_0} - A_{x_n}\| \leq c_A |x_0 - x_n|^{\delta_A}$. We perform the change of variables $\tilde{\xi} = r_n^{-1} F_{x_0}^{-1} \circ F_{x_n}(\xi)$ and set $B = y_n + A_{x_0}^{-1/2} A_{x_n}^{1/2} [B_r]$ to get

$$\frac{1}{(rr_n)^d} J(U_{x_n}, rr_n) \leq \frac{1}{r^d} \int_B (|\nabla \tilde{B}_n|^2 + \Lambda \mathbb{1}_{\{|\tilde{B}_n| > 0\}}) |\det(A_{x_n}^{-1/2} A_{x_0}^{1/2})| d\tilde{\xi} + \omega_d L^2 c_A |x_0 - x_n|^{\delta_A}. \quad (5.4.19)$$

Moreover, since the coefficients of $A^{1/2}$ are $\frac{\delta_A}{2}$ -Hölder continuous, we have the estimate of the determinant $|\det(A_{x_n}^{-1/2} A_{x_0}^{1/2})| \leq 1 + c_A |x_0 - x_n|^{\delta_A/2}$ and the following estimate of the symmetric difference

$$\begin{aligned} |B \Delta B_r(y_n)| &= |A_{x_0}^{-1/2} A_{x_n}^{1/2} [B_r] \Delta B_r| \leq \omega_d r^d \left[(1 + c_A |x_0 - x_n|^{\delta_A/2})^d - (1 - c_A |x_0 - x_n|^{\delta_A/2})^d \right] \\ &\leq \omega_d r^d \left[(1 + d2^d c_A^d |x_0 - x_n|^{\delta_A/2}) - (1 - d2^d c_A^d |x_0 - x_n|^{\delta_A/2}) \right] \leq r^d C |x_0 - x_n|^{\delta_A/2}. \end{aligned}$$

Similarly, for n large enough so that $|y_0 - y_n| \leq r/2$, we have

$$|B_r(y_0) \Delta B_r(y_n)| \leq \omega_d r^d \left[\left(1 + \frac{|y_0 - y_n|}{r}\right)^d - \left(1 - \frac{|y_0 - y_n|}{r}\right)^d \right] \leq r^d C \frac{|y_0 - y_n|}{r}.$$

Combining all these estimates (5.4.19) now gives (because \tilde{B}_n is $\lambda_A L$ -Lipschitz continuous)

$$\begin{aligned} \frac{1}{(rr_n)^d} J(U_{x_n}, rr_n) &\leq \frac{1}{r^d} J(\tilde{B}_n, y_0, r) + \omega_d L^2 c_A |x_0 - x_n|^{\delta_A} + \frac{\lambda_A^2 L^2 + \Lambda}{r^d} |B|_{c_A} |x_0 - x_n|^{\delta_A/2} + \\ &\quad + \frac{\lambda_A^2 L^2 + \Lambda}{r^d} \left[|B \Delta B_r(y_n)| + |B_r(y_0) \Delta B_r(y_n)| + |B|_{c_A} |x_0 - x_n|^{\delta_A/2} \right] \\ &\leq \frac{1}{r^d} J(\tilde{B}_n, y_0, r) + C |x_0 - x_n|^{\delta_A/2} + C \frac{|y_0 - y_n|}{r}. \end{aligned} \quad (5.4.20)$$

We now compare the boundary integral terms. Since $U_{x_n}(\xi) = r_n \tilde{B}_n(y_n + r_n^{-1} A_{x_0}^{-1/2} A_{x_n}^{1/2}(\xi))$ and by the change of variables $\tilde{\xi} = r_n^{-1} \xi + y_0$ we have

$$\begin{aligned} \int_{\partial B_{rr_n}} |U_{x_n}|^2(\xi) d\mathcal{H}^{d-1}(\xi) &= \int_{\partial B_{rr_n}} r_n^2 |\tilde{B}_n|^2(y_n + r_n^{-1} A_{x_0}^{-1/2} A_{x_n}^{1/2}(\xi)) d\mathcal{H}^{d-1}(\xi) \\ &= r_n^{d+1} \int_{\partial B_r(y_0)} |\tilde{B}_n|^2(y_n + A_{x_0}^{-1/2} A_{x_n}^{1/2}(\tilde{\xi} - y_0)) d\mathcal{H}^{d-1}(\tilde{\xi}). \end{aligned}$$

Therefore, using that \tilde{B}_n is $\lambda_A L$ -Lipschitz continuous, $\tilde{B}_n(y_n) = 0$ and that $|y_0 - y_n| \leq r/2$, we get that

$$\begin{aligned} \frac{1}{r^{d+1}} \int_{\partial B_r(y_0)} |\tilde{B}_n|^2(\xi) d\mathcal{H}^{d-1}(\xi) - \frac{1}{(rr_n)^{d+1}} \int_{\partial B_{rr_n}} |U_{x_n}|^2(\xi) d\mathcal{H}^{d-1}(\xi) &= \\ &= \frac{1}{r^{d+1}} \int_{\partial B_r(y_0)} \left(|\tilde{B}_n|^2(\xi) - |\tilde{B}_n|^2(y_n + A_{x_0}^{-1/2} A_{x_n}^{1/2}(\xi - y_0)) \right) d\mathcal{H}^{d-1}(\xi) \\ &\leq \frac{\lambda_A^2 L^2}{r^{d+1}} \int_{\partial B_r(y_0)} |A_{x_0}^{-1/2} (A_{x_0}^{1/2} - A_{x_n}^{1/2})(\xi - y_0) + y_0 - y_n| (|\xi - y_n| + \lambda_A^2 r) d\mathcal{H}^{d-1}(\xi) \\ &\leq C |x_0 - x_n|^{\delta_A/2} + C \frac{|y_0 - y_n|}{r}. \end{aligned}$$

Now, in view of (5.4.20) we get (5.4.18). This completes the proof. \square

In the following Lemma we prove that if \tilde{B}_n is a blow-up sequence with fixed center converging to \tilde{B}_0 , then locally the singular set of $\{|\tilde{B}_n| > 0\}$ must lie close to the singular set of $\{|\tilde{B}_0| > 0\}$ (see [92, Lemma 4.2] and [77, Lemma 5.20]).

Lemma 5.4.22. *Let $x_0 \in \partial\Omega_1^* \cap D$ and let $B_n = B_{x_0, r_n}$ be a blow-up sequence converging in the sense of Proposition 5.3.2 to some $B_0 \in \mathcal{BU}_U(x_0)$. We set $\tilde{B}_n = B_n \circ A_{x_0}^{1/2}$, $\tilde{B}_0 = B_0 \circ A_{x_0}^{1/2}$, $\tilde{\Omega}_n = \{|\tilde{B}_n| > 0\}$ and $\tilde{\Omega}_0 = \{|\tilde{B}_0| > 0\}$. Then, for every compact set $K \subset \mathbb{R}^d$ and every open set $\mathcal{O} \subset \mathbb{R}^d$ such that $\text{Sing}(\partial\tilde{\Omega}_0) \cap K \subset \mathcal{O}$, we have $\text{Sing}(\partial\tilde{\Omega}_n) \cap K \subset \mathcal{O}$ for n large enough.*

Proof. Arguing by contradiction there exist a compact set $K \subset \mathbb{R}^d$ and an open set $\mathcal{O} \subset \mathbb{R}^d$ such that $\text{Sing}(\partial\tilde{\Omega}_0) \cap K \subset \mathcal{O}$ and a sequence $(y_n)_{n \in \mathbb{N}} \subset \text{Sing}(\partial\tilde{\Omega}_n) \cap K \setminus \mathcal{O}$. Up to a subsequence, y_n converges to some $y_0 \in K \setminus \mathcal{O}$. Since $\partial\tilde{\Omega}_n$ locally Hausdorff converges to $\partial\tilde{\Omega}_0$ by Proposition 5.3.2, it follows that $y_0 \in \partial\tilde{\Omega}_0$ and, since $\text{Sing}(\partial\tilde{\Omega}_0) \cap K \subset \mathcal{O}$, we have that y_0 is a regular point of $\partial\tilde{\Omega}_0$, that is $y_0 \in \text{Reg}(\partial\tilde{\Omega}_0)$. Since, moreover, \tilde{B}_0 is solution of the Alt-Caffarelli functional and is one-homogeneous, it follows that $\frac{1}{\Lambda\omega_d} \lim_{r \rightarrow 0^+} W(\tilde{B}_0, y_0, r) = \frac{1}{2}$ (see [77, Lemma 5.4]). We now fix $r > 0$ such that

$$\frac{1}{\Lambda\omega_d} W(\tilde{B}_0, y_0, r) \leq \frac{1}{2} + \frac{\delta}{4},$$

where δ is the constant from Lemma 5.4.5. Now, since $\lim_{n \rightarrow \infty} W(\tilde{B}_n, y_0, r) = W(\tilde{B}_0, y_0, r)$, it follows that for every n large enough we have

$$\frac{1}{\Lambda\omega_d} W(\tilde{B}_n, y_0, r) \leq \frac{1}{\Lambda\omega_d} W(\tilde{B}_0, y_0, r) + \frac{\delta}{4} \leq \frac{1}{2} + \frac{\delta}{3}. \quad (5.4.21)$$

Set $x_n = x_0 + r_n A_{x_0}^{1/2}(y_n) \in \partial\Omega_1^* \cap D$ and notice that x_n converges to x_0 . By Lemma 5.4.21 and (5.4.21) we get that for every n large enough

$$\frac{1}{\Lambda\omega_d} W(U_{x_n}, rr_n) \leq \frac{1}{2} + \frac{\delta}{3} + C|x_0 - x_n|^{\delta_A/2} + C\frac{|y_0 - y_n|}{r}.$$

On the other hand, by Proposition 5.2.10, the function $\varphi_n(s) = W(U_{x_n}, s) + Cs^{\delta_A}$ is non-decreasing and hence

$$\begin{aligned} \Theta_{U_{x_n}}(0) &= \frac{1}{\Lambda\omega_d} \lim_{s \rightarrow 0^+} W(U_{x_n}, s) = \frac{1}{\Lambda\omega_d} \lim_{s \rightarrow 0^+} \varphi_n(s) \leq \frac{1}{\Lambda\omega_d} \varphi_n(rr_n) = \frac{1}{\Lambda\omega_d} W(U_{x_n}, rr_n) + C(rr_n)^{\delta_A} \\ &\leq \frac{1}{2} + \frac{\delta}{3} + C|x_0 - x_n|^{\delta_A/2} + C\frac{|y_0 - y_n|}{r} + C(rr_n)^{\delta_A} < \frac{1}{2} + \frac{\delta}{2}, \end{aligned}$$

where the last inequality holds for n large enough. It follows from Lemmas 5.4.5 and 5.4.6 that x_n is a regular point of Ω_1^* , in contradiction with the fact that $y_n = A_{x_0}^{-1/2} \left[\frac{x_n - x_0}{r_n} \right] \in \text{Sing}(\partial\tilde{\Omega}_n)$. \square

We are now in position to prove Proposition 5.4.20.

Proof of Proposition 5.4.20. (1) Let $x_0 \in \partial\Omega_1^* \cap D$ and $B_0 \in \mathcal{BU}_U(x_0)$ and set $\tilde{B}_0 = B_0 \circ A_{x_0}^{1/2}$ and $\tilde{\Omega}_0 = \{|\tilde{B}_0| > 0\}$. By Lemma 5.3.6, $|\tilde{B}_0|$ is a local minimizer of the scalar Alt-Caffarelli functional and since $d < d^*$, it follows that $\partial\tilde{\Omega}_0$ is the graph of a $C^{1,\alpha}$ function near 0 (see [92, Section 3]). In particular, the density of $\tilde{\Omega}_0$ at 0 is 1/2 and hence $\Theta_{U_{x_0}}(0) = 1/2$ by (5.4.6). In view of Lemma 5.4.6 we get that $x_0 \in \text{Reg}(\partial\Omega_1^* \cap D)$.

(2) Assume by contradiction that there exists a sequence $(x_n)_{n \in \mathbb{N}} \subset \text{Sing}(\partial\Omega_1^* \cap D)$ converging to some $x_0 \in \partial\Omega_1^* \cap D$. Set $r_n = |x_0 - x_n|$ and let $B_n := B_{x_0, r_n}$ be a blow-up sequence converging (in the sense of Proposition 5.3.2) to some blow-up limit $B_0 \in \mathcal{BU}_U(x_0)$. We consider two cases:

Case 1: $\text{Sing}(\partial\tilde{\Omega}_0) \setminus \{0\} \neq \emptyset$. By a rotation we may assume that $e_d \in \mathbb{R}^d$ is a singular point of $\partial\Omega_0$. Notice that $u_0 = |\tilde{B}_0|$ is solution of the scalar Alt-Caffarelli functional and is one-homogeneous. Consider a blow-up limit u_{00} of u_0 at e_d . By [92, Lemma 3.1], $\{u_{00} > 0\}$ is a minimal cone with vertex 0 such that the whole line te_d , $t \in \mathbb{R}$, consists of singular points. Then, by [92, Lemma 3.2], denoting the restriction $\bar{u} = u_{00}|_{\mathbb{R}^{d-1}}$, we have that $\{\bar{u} > 0\}$ is a minimal cone of dimension $(d-1)$ which is singular at 0. Now, either 0 is the only singular point and we have a contradiction with the definition of d^* , or we can repeat this procedure and get a contradiction since there are no three-dimensional singular minimal cones.

Case 2: $\text{Sing}(\partial\tilde{\Omega}_0) \setminus \{0\} = \emptyset$. Let $r > 0$ to be chosen later. By Lemma 5.4.21, we have for every n large enough

$$W(U_{x_n}, rr_n) \leq W(\tilde{B}_n, y_0, r) + C\frac{|y_0 - y_n|}{r} + C|x_0 - x_n|^{\delta_A/2}.$$

Now, by Proposition 5.2.10, the function $\varphi_n(s) = W(U_{x_n}, s) + Cs^{\delta_A}$ is non-decreasing and, since $x_n \in \text{Sing}(\partial\Omega_1^* \cap D)$, by Lemmas 5.4.6 and 5.4.5 we have that $\frac{1}{\Lambda\omega_d} \lim_{s \rightarrow 0^+} W(U_{x_n}, s) \geq \frac{1}{2} + \delta$. Therefore, we have

$$\begin{aligned} \frac{1}{2} + \delta &\leq \frac{1}{\Lambda\omega_d} \lim_{s \rightarrow 0^+} W(U_{x_n}, s) = \frac{1}{\Lambda\omega_d} \lim_{s \rightarrow 0^+} \varphi_n(s) \leq \varphi_n(rr_n) = \frac{1}{\Lambda\omega_d} W(U_{x_n}, rr_n) + C(rr_n)^{\delta_A} \\ &\leq \frac{1}{\Lambda\omega_d} W(\tilde{B}_n, y_0, r) + C\frac{|y_0 - y_n|}{r} + C|x_0 - x_n|^{\delta_A/2} + C(rr_n)^{\delta_A}. \end{aligned} \quad (5.4.22)$$

Now, since $y_0 \in \partial\tilde{\Omega}_0 \setminus \{0\}$ is a regular point of $\partial\tilde{\Omega}_0$ (and also because \tilde{B}_0 is solution of the Alt-Caffarelli functional and is one-homogeneous), it follows that $\frac{1}{\Lambda\omega_d} \lim_{r \rightarrow 0^+} W(\tilde{B}_0, y_0, r) = \frac{1}{2}$ (see [77, Lemma 5.4]). Using also that $\lim_{n \rightarrow \infty} W(\tilde{B}_n, y_0, r) = W(\tilde{B}_0, y_0, r)$, it follows that we can choose $r > 0$ small enough such that for every n large enough we have

$$\frac{1}{\Lambda\omega_d} W(\tilde{B}_n, y_0, r) \leq \frac{1}{\Lambda\omega_d} W(\tilde{B}_0, y_0, r) + \frac{\delta}{4} \leq \frac{1}{2} + \frac{\delta}{2}.$$

Therefore, passing to the limit $n \rightarrow \infty$ in the equation (5.4.22) gives a contradiction.

(3) Assume by contradiction that $\mathcal{H}^{d-d^*+s}(\text{Sing}(\partial\Omega_1^* \cap D)) > 0$ for some $s > 0$. By Lemma 5.4.22 and [92, Lemmas 4.3 and 4.4] there exists $x_0 \in \partial\Omega_1^* \cap D$ and a blow-up limit $B_0 \in \mathcal{BU}_U(x_0)$ such that $\mathcal{H}^{d-d^*+s}(\text{Sing}(\partial\tilde{\Omega}_0)) > 0$, where we have set $\tilde{B}_0 = B_0 \circ A_{x_0}^{1/2}$ and $\tilde{\Omega}_0 = \{|\tilde{B}_0| > 0\}$. Since $|\tilde{B}_0|$ is a minimizer of the Alt-Caffarelli functional and is one-homogeneous, the dimension reduction procedure in [92, Lemma 4.5] applies and yields to a minimizer $u : \mathbb{R}^{d^*} \rightarrow \mathbb{R}$ of the Alt-Caffarelli functional such that $\mathcal{H}^s(\text{Sing}(\partial\{u > 0\})) > 0$, in contradiction with [92, Lemma 4.1]. \square

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