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Syzygies: Algebra, Combinatorics and Geometry

dirigée par Marc Chardin

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Syzygies: Algèbre, Combinatoire et Géométrie

Résumé

La régularité de Castelnuovo-Mumford est l'un des principaux invariants numériques permettant de mesurer la complexité de la structure des modules gradués de type fini sur des anneaux polynomiaux. Il mesure le degré maximal des générateurs des modules de syzygies. Dans cette thèse, nous étudions la régularité de Castelnuovo-Mumford avec différents points de vue et, dans certaines parties, nous nous concentrons principalement sur les syzygies linéaires.

Dans le chapitre 2, nous étudions la régularité des homologies de Koszul et des cycles de Koszul de quotients unidimensionnels. Comme application, cela donne une limite inférieure pour le nombre d'étapes linaires dans la résolution libre du plongement de Veronese.

Dans le chapitre 3, nous étudions les propriétés de Lefschetz faibles et fortes d'une classe d'idéaux monomiaux artiniens. Nous donnons, dans certains cas, une réponse affirmative à une conjecture d'Eisenbud, Huneke et Ulrich.

Dans les chapitre 4 et 5, nous étudions deux comportements asymptotiques différents de la régularité de Castelnuovo-Mumford. Dans le chapitre 4, nous travaillons sur un quotient A d'une algèbre noethérienne standard S par suite régulière homogène. Tout d'abord, nous montrons que la régularité de $\mathrm{Ext}^i_A(M,N)$ devient une fonction linéaire de i, séparément pour i pair et pour i impair, pour M de dimension projective finie sur S. Lorsque, de plus, la dimension de $\mathrm{Tor}^A_i(M,N)$ est au plus un pour $i \gg 0$, un résultat similaire est valable pour $\mathrm{Tor}^A_i(M,N)$; de plus, nous donnons deux exemples montrant que l'hypothèse sur la dimension asymptotique des modules Tor est essentielle.

La régularité des puissances d'un idéal dans un anneau polynomial est, à partir d'un certain ordre, une fonction linéaire de la puissance. Au chapitre 5, nous étudions la régularité des puissances des idéaux monomiaux associés aux graphes en haltère. Nous donnons une valeur exacte pour la régularité de toutes les puissances et en déduisons que la régularité des puissances est une fonction linaire depuis le début.

Dans le chapitre 6, nous travaillons sur des espaces projectifs. Au début de ce chapitre, nous présentons un package pour le logiciel informatique Macaulay2 qui recherche les multidegrés minimaux en lesquels la troncature d'un module multigradé a une résolution linéaire. De plus, nous étudions les cohomologies des"intersections complètes" dans $\mathbb{P}^n \times \mathbb{P}^m$. En particulier, nous étudions la fonction de Hilbert bigraduée d'ensembles de points intersection complète (*i.e.* définis par $n+m$ polynômes) dans cet espace. Nous donnons une limite

inférieure optimale pour la stabilisation de la fonction de Hilbert (en d'autres termes, nous fournissons un contrôle précis pour le support de la cohomologie locale). De plus, nous montrons que, dans une zone spécifique et assez grande, la fonction de Hilbert ne dépend que du degré des formes définissant les points. Enfin, nous considérons le cas où les formes définissant les points sont choisies de manière générique. Dans ce cas, nous montrons que les projections naturelles sur $\mathbb{P}^n$ et $\mathbb{P}^m$ sont des isomorphismes et en tiront des conséquences pour la cohomologie.

Mots-clés

Anneaux d'intersection complète, anneaux multigradués, cohomologie locale, fonction de Hilbert, homologie de Koszul, edge ideals, propriétés de Lefschetz, régularité de Castelnuovo-Mumford, résolution libre, résolution presque linéaire, syzygies.

Syzygies: Algebra, Combinatorics and Geometry

Abstract

Castelnuovo-Mumford regularity is one of the main numerical invariants that measure the complexity of the structure of homogeneous finitely generated modules over polynomial rings. It measures the maximum degrees of generators of the syzygies. In this thesis we study the Castelnuovo-Mumford regularity with different points of view and, in some parts, we mainly focus on linear syzygies.

In Chapter 2 we study the regularity of Koszul homologies and Koszul cycles of one dimensional quotients. As an application, this gives a lower of bound for the number of linear steps in the minimal free resolution Veronese embeddings.

In Chapter 3 we study the weak and strong Lefschetz properties of a class of artinain monomial ideals. We show how the structure of the minimal free resolution could force weak or strong Lefschetz properties. We give, in some cases, an affirmative answer to a conjecture of Eisenbud, Huneke, and Ulrich.

In Chapter 4 and 5 we study two different asymptotic behavior of Castelnuovo-Mumford regularity. In Chapter 4 we work on a quotient of a standard graded Noetherian algebra by homogeneous regular sequence. First, we show that the regularity of $\mathrm{Ext}^i(M, N)$ becomes a linear function for odd and even i separately. With additional mild assumptions on the base ring we prove that, if the dimension of $\mathrm{Tor}_i(M, N)$ is at most one for $i \gg 0$, then a similar result holds for $\mathrm{Tor}_i(M, N)$. In addition, we give two examples showing that the assumption on the dimension is essential.

It is a celebrated result that the regularity of powers of an ideal in a polynomial ring becomes a linear function. In Chapter 5, we study the regularity of powers of dumbbell graphs. We give an exact value for regularity of all the powers and, in particular, we show that the regularity of the powers is a linear function from the beginning.

In Chapter 6, we work on product of projective spaces. In the begining of this chapter, we present a package for the computer software Macaulay2 which finds the minimal multidegrees where the truncation of a multigraded module has a linear resolution. Furthermore, we study the cohomologies of the "complete intersections" in $\mathbb{P}^n \times \mathbb{P}^m$. In particular, we study the bigraded Hilbert function of complete intersection sets of points in this space. We give a sharp lower bound for the stabilization of the bigraded Hilbert function (in other words we provide a sharp control for the support of local cohomology). In addition, we show that, in a specific and pretty large region, the bigraded Hilbert Function only depends upon the degree

of the forms defining the points. Finally, we consider the case where the forms defining the points are chosen generically. In this case we show that the natural projections to $\mathbb{P}^n$ and $\mathbb{P}^m$ are one-to-one and derive consequences on cohomology.

Keywords

Almost linear resolution, Castelnuovo-Mumford regularity, complete intersection rings, edge ideals, free resolution, Hilbert function, Koszul homology, Lefschetz properties, local cohomology, multigraded rings, syzygies.

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Introduction

Algebraic geometry, implicit in the name, is the relation between geometry and equations. One of the main subjects of commutative algebra is the qualitative study of systems of polynomial equations. But when we actually study a ring or a variety we often have to know a great deal about it before understanding its equations. Conversely, given a system of equations, it can be extremely difficult to analyze its qualitative properties, such as the geometry of the corresponding variety. The theory of syzygies offers a microscope for looking at systems of equations. Castelnuovo-Mumford regularity is an important invariant in commutative algebra and algebraic geometry which is strongly related to syzygies. D. Eisenbud and S. Goto in [35] showed that Castelnuovo-Mumford regularity can be obtained from the minimal free resolution. The minimal free resolution was first introduced by Hilbert in order to study Hilbert function. It is one of the finest invariant that we can associate with a finitely generated module M over a polynomial ring.

What are syzygies?

In algebraic geometry over a field k we study the geometry of varieties through properties of the polynomial ring

$$S = k[x_1, \ldots, x_n]$$

and its ideals. It turns out that to study ideals effectively we also need to study more general graded modules over S. The simplest way to describe a module is by generators and relations. We may think of a set $A \subset M$ of generators for an S-module M as a map from a free S-module $F = S^{|A|}$ onto M, sending the basis element of F corresponding to a generator $m \in A$ to the element $m \in M$.

Let M_1 be the kernel of the map $F \to M$; it is called the ***module of syzygies*** of M corresponding to the given choice of generators, and a syzygy of M is an element of M_1. Given generators of M, the module M is uniquely defined by its module of syzygies.

If $n = 1$, so that we are working over the polynomial ring in one variable, the module of syzygies is itself a free module, since over a principal ideal domain every submodule of a free module is free. But when $n \geq 2$ it may be the case that any set of generators of the module of syzygies has relations. To understand them, we proceed as before: we choose a generating set of syzygies and use them to define a map from a new free module, say F_1, onto M_1; equivalently, we give a map $\partial_1 : F_1 \to F$ whose image is M_1. Continuing in this way we get a free resolution of M, that is, a sequence of maps

$$\cdots \to F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F \to M \to 0,$$

where all the modules F_i are free and each map is a surjection onto the kernel of the following map. The image M_i of ∂_i is called the i-th module of syzygies of M.

In projective geometry we treat S as a graded ring by giving each variable x_i degree 1, and we will be interested in the case where M is a finitely generated graded S-module. In this case we can choose a minimal set of homogeneous generators for M (that is, one with as few elements as possible), and we choose the degrees of the generators of F so that the map $F \to M$ preserves degrees. The syzygy module M_1 is then a graded submodule of F, and Hilbert's Basis Theorem tells us that M_1 is again finitely generated, so we may repeat the procedure. Hilbert's Syzygy Theorem tells us that the modules M_i are free as soon as $i \geq n$.

The free resolution of M appears to depend strongly on our initial choice of generators for M, as well as the subsequent choices of generators of M_1, and so on. But if M is a finitely generated graded module and we choose a minimal set of generators for M, then M_1 is, up to isomorphism, independent of the minimal set of generators chosen. It follows that if we choose minimal sets of generators at each stage in the construction of a free resolution we get a minimal free resolution of M that is, up to isomorphism, independent of all the choices made. Since, by the Hilbert Syzygy Theorem, M_i is free for $i \geq n$, we see that in the minimal free resolution $F_i = 0$ for $i \geq n + 1$. In this sense the minimal free resolution is finite: it has length at most n. Moreover, any free resolution of M can be derived from the minimal one in a simple way.

The minimal free resolution is one of the finest invariant that we can associate to a finitely generated graded S-module M. For instance, Hilbert functions and Hilbert polynomials can be obtained from the minimal free resolution. Since studying minimal free resolution is somehow studying the modules of syzygies then it seems reasonable to study the modules of syzygies instead. In this thesis we put some assumptions on the module of syzygies and investigate how these assumptions affect the properties of M. Since any module can be represented by generators and relations, the simplest relations that minimal generators

could have are ones with linear forms as coefficients, these are called ***linear syzygies***. We study the syzygies with three points of view: algebraic, combinatorial and geometric.

What is in this thesis?

This thesis is based on the following articles. We will give a short overview below.

(1) **Castelnuovo-Mumford regularity of Koszul cycles and Koszul homologies** (with K. Lamei). Proc. Amer. Math. Soc. 146(2018), 2765-2772.

(2) **Lefschetz properties of monomial ideals with almost linear resolution** (with N. Altafi). Submitted, arXiv:1803.01388.

(3) **The (ir)regularity of Tor and Ext**(with M. Chardin and D. Ghosh). In preparation.

(4) **Regularity of bicyclic graphs and their powers** (with Y. Cid-Ruiz, S. Jafari and B. Picone). To appear in Journal of Algebra and its Applications, arXiv:1802.07202.

(5) **Cohomologies of complete intersections in $\mathbb{P}^n \times \mathbb{P}^m$** (with M. Chardin). In preparation.

(6) **Linear truncations package in Macaulay2** (with David Eisenbud). In preparation.

Regularity of Koszul cycles and Homologies: The first part of this thesis is about Castelnuovo-Mumford regularity of Koszul cycles and Koszul homologies for one dimensional quotients. The motivation of this work is studying the minimal free resolution of Veronese embedding. There is a very famous conjecture due to Ottaviani and Paoletti in [72] about the minimal free resolution of Veronese embedding. They conjectured that the minimal free resolution of Veronese embedding of degree d has linear syzygies for $3d-3$ steps. Up to now, this conjecture is widely open and the best known results are pretty far from the one in the conjecture. We have two goals in this part. First, we prove that Castelnuovo-Mumford regularity of Koszul cycles of one dimensional quotients is an additive function.

Theorem 1. *[64, Thm. 3.4] Let $S = k[x_1, \dots, x_n]$ and I be a graded ideal of S, if $\dim S/I \leqslant 1$ and characteristic of k is 0 or bigger than $s+t$, then*

$$\operatorname{reg}(Z_{s+t}(I)) \leqslant \operatorname{reg}(Z_t(I)) + \operatorname{reg}(Z_s(I)).$$

Second, by providing a relation between the regularity of Koszul cycles and Koszul homologies we prove a sharp regularity bound for the Koszul homologies of a homogeneous ideal in a polynomial ring under the same conditions.

Theorem 2. *[64, Thm. 4.3] Let S be a polynomial ring, I be a homogeneous ideal of S. If $\dim S/I \leq 1$, then for all $0 < i < \mu(I)$*

$$\operatorname{reg}(Z_i(I,S)) = \max_{0<j<n}\{\operatorname{reg}(H_{i-j}(I,S)) + j + 1\}.$$

Here $\mu(I)$ is the minimal number of generators of I.

As an application we state sharp bound for the regularity of Koszul homologies in dimension 1 which is a refinement of the result of A. Conca and S. Murai in dimension 0.

Theorem 3. *[64, Thm. 4.5] Let I be an ideal of S and $\dim S/I \leq 1$, then we have the following inequalities between Koszul homologies of I for all $i, j \geq 1$*

$$\operatorname{reg}(H_{i+j-1}(I,S)) \leq \max_{1\leq\alpha,\beta\leq n-1}\{\operatorname{reg}(H_{i-\alpha}(I,S)) + \operatorname{reg}(H_{j-\beta}(I,S)) + \alpha + \beta\}.$$

By using the above theorems we are able to give a new proof for a classical result due to M. Green on the Ottaviani and Paoletti conjecture.

Lefschetz properties: The weak Lefschetz property (WLP) of an Artinian graded algebra A, says that there exists a linear form ℓ that induces a multiplication map $\times\ell : A_i \longrightarrow A_{i+1}$ that has the maximal rank for each i, i.e. that is either surjective or injective. The strong Lefschetz property (SLP) says the map $\times\ell^t : A_i \longrightarrow A_{i+t}$ has maximal rank for each i and t. It turns out to be rather hard to determine if (SLP) or (WLP) holds, even for natural families of algebras. It is also interesting to ask for which t the map $\times\ell^t : A_i \longrightarrow A_{i+t}$ has maximal rank (see [68]). These fundamental properties have been studied by many authors from different point of views and for different families of algebras. We study the Lefschetz properties of Artinian monomial ideals generated in a single degree d with assumptions on their minimal free resolutions. In [37], Eisenbud, Huneke and Ulrich study the minimal free resolutions of Artinian ideals in the polynomial ring $S = k[x_1, \ldots, x_n]$ and the authors provide the following conjecture

Conjecture 1 (Eisenbud, Huneke and Ulrich)**.** *Let $I \subset S = k[x_1, \ldots, x_n]$ is an artinian ideal generated in degree d and its minimal free resolution is linear for $p-1$ steps, then*

$$\mathfrak{m}^d \subseteq I + (l_p, l_{p+1} \ldots, l_n)^2$$

for sufficiently general linear forms $l_p, \ldots, l_n$.

There is a strong relationship between this conjecture and Lefschetz properties of S/I. This conjecture was our motivation and we give an affirmative answer in the monomial case where I has almost linear resolution (the minimal free resolution of I is linear except in the last step). The main theorems we prove are the following:

Theorem 4. *[4, Thm. 3.9] Let $I \subset S = \mathbb{K}[x_1, \ldots, x_n]$ be an artinian monomial ideal generated in degree d with almost linear resolution, then for generic linear form ℓ, multiplication map $\times \ell^2 : (S/I)_a \to (S/I)_{a+2}$ has maximal rank for any a.*

Theorem 5. *[4, Thm. 4.6] Let $I \subset S = \mathbb{K}[x_1, \ldots, x_n]$ be a monomial ideal generated in degree d such that $\mathfrak{m}^{d+1} \subset I$. If the minimal free resolution of I is linear except in the last two steps, then S/I satisfies the WLP.*

We show by example that both conditions in the above theorem are necessary.

Asymptotic behavior of regularity: The study of homological invariants of powers of ideals goes back, at least, to the work of Brodmann in the 70's which has attracted a lot of attention over the last two decades. Given a projective variety $X \subseteq \mathbb{P}^r$, let $\mathcal{I}_X$ be the ideal sheaf of the embedding of X. Let d_X denotes the minimum of the degrees d such that X is a scheme-theoretic intersection of hypersurfaces of degree at most d. For a smooth complex projective variety, Bertram, Ein and Lazarsfeld [9] have shown that there is a number e such that

$$H^i(\mathbb{P}^r, \mathcal{I}_X^n(a)) = 0$$

for all $a \geq nd_X + e$, $i \geqslant 1$.

One of the most important results in this area is the asymptotic linearity of the Castelnuovo-Mumford regularity of an ideal I over a polynomial ring, obtained by Kodiyalam [63] and Cutkosky, Herzog and Trung [30], independently. There exist d, c, n_0 such $\operatorname{reg}(I^s) = sd + c$ for every $s \geq n_0$. There are series of results about the value of d, c, n_0. An interesting question in this regard is to give an explicit description of these numbers in some special cases. In [8], Banerjee, Beyarslan and Há gave a conjecture describing c, n_0 for graphs.

In [27], we studied the regularity of bicyclic graphs and the regularity of powers of dumbbell graphs. For any bicyclic graphs, we provide an exact formula for the regularity of their edge ideals in terms of induced matching number of the graph. We also study the asymptotic behavior of the regularity of powers. Let G be a graph obtained by attaching two cycles with a path of length ≤ 1 then we prove

$$\operatorname{reg}(I(G)^s) = 2s + \operatorname{reg} I(G) - 2.$$

The above theorem implies that regularity of $I(G)^s$ is a linear function in terms of s when $s \geq 1$ and answers the mentioned conjecture.

As we mentioned before the fundamental theorem on asymptotic linearity of regularity is for an ideal over a polynomial ring. The next step is to studying modules over complete

intersections. Recently D. Eisenbud and I. Peeva in [39] stdudied the structure of minimal free resolutions over complete intersections. In [22] we focus on the asymptotic behavior of regularity of $\operatorname{Ext}^i(M,N)$ and $\operatorname{Tor}_i(M,N)$ in terms of i for modules over a quotient of standard graded Noetherian algebra by regular sequence. The special, but very important, case of our theorems is for modules over a complete intersection ring.

Theorem 6. *[22, Thm. 3.2] Let Q be a standard graded Noetherian algebra, $A := Q/(\mathbf{f})$, where $\mathbf{f} := f_1, \dots, f_c$ is a homogeneous Q-regular sequence. Let M and N be finitely generated graded A-modules such that $\operatorname{Ext}^i_Q(M,N) = 0$ for all $i \gg 0$.*

Then, for every $\ell \in \{0,1\}$, there exist $a_\ell \in \{\deg(f_j) : 1 \leqslant j \leqslant c\}$ and $e_\ell \in \mathbb{Z} \cup \{-\infty\}$ such that

$$\operatorname{reg}\left(\operatorname{Ext}^{2i+\ell}_A(M,N)\right) = -a_\ell \cdot i + e_\ell \quad \text{for all } i \gg 0.$$

Theorem 7. *[22, Th. 3.3] Let Q be a standard graded Noetherian algebra, $A := Q/(\mathbf{f})$, where $\mathbf{f} := f_1, \dots, f_c$ is a homogeneous Q-regular sequence. Assume Q is *local or the epimorphic image a Gorenstein ring. Let M and N be finitely generated graded A-modules such that,*

(i) *M has finite projective dimension over Q,*

(ii) $\dim(\operatorname{Tor}^A_i(M,N)) \leqslant 1$ *for any $i \gg 0$.*

Then, for every $\ell \in \{0,1\}$, there exist $a_\ell \in \{\deg(f_j) : 1 \leqslant j \leqslant c\}$ and $e_\ell \in \mathbb{Z} \cup \{-\infty\}$ such that

$$\operatorname{reg}\left(\operatorname{Tor}^A_{2i+\ell}(M,N)\right) = a_\ell \cdot i + e_\ell, \ \forall i \gg 0.$$

In addition, we provide examples showing that the behavior of the regularity of Tor modules could be very different without the assumption in the result above.

Example 1. [22, Example 4.1] Let $Q := K[Y,Z,V,W]$ be a polynomial ring with usual grading over a field K and $A := Q/(Y^2, Z^2)$. Write $A = K[y,z,v,w]$, where y, z, v and w are the residue classes of Y, Z, V and W respectively. Fix an integer $m \geqslant 1$. Set

$$M := \operatorname{Coker}\left(\begin{bmatrix} y & z & 0 & 0 \\ -v^m & -w^m & y & z \end{bmatrix} : \begin{matrix} A(-m)^2 \\ \bigoplus \\ A(-1)^2 \end{matrix} \longrightarrow \begin{matrix} A(-m+1) \\ \bigoplus \\ A \end{matrix} \right)$$

and $N := A/(y,z)$. Then, for every $i \geqslant 1$, we have

(i) $\operatorname{indeg}\left(\operatorname{Ext}^i_A(M,N)\right) = -i - m + 1$ and $\operatorname{reg}\left(\operatorname{Ext}^i_A(M,N)\right) = -i$.

(ii) $\operatorname{indeg}\left(\operatorname{Tor}^A_i(M,N)\right) = i$ and $\operatorname{reg}\left(\operatorname{Tor}^A_i(M,N)\right) = (m+1)i + (2m-2)$.

In this example, $\operatorname{Tor}_i^A(M,N)$ is supported in dimension 2 for $i \gg 0$, its regularity is eventually linear, but the leading term depends on the module M and could be arbitrary big. This shows that the finiteness result for the Tor-algebra that we prove under the condition that $\operatorname{Tor}_i^A(M,N)$ is supported in dimension 1 for $i \gg 0$ can fail if this hypothesis is removed.

The following example that we develop in the last section shows that the eventual regularity of Tor could be very far from being linear,

Example 2. [22, Example 5.1] Let $Q := K[X,Y,Z,U,V,W]$ be a standard graded polynomial ring over a field K of characteristic 2 and $A := Q/(X^2,Y^2,Z^2)$. We write $A = K[x,y,z,u,v,w]$, where x,y,z,u,v and w are the residue classes of X,Y,Z,U,V and W respectively. Set

$$M := \operatorname{Coker}\left(\begin{bmatrix} x & y & z & 0 & 0 & 0 \\ u & v & w & x & y & z \end{bmatrix} : A(-1)^6 \longrightarrow A^2\right) \quad \text{and} \quad N := A/(x,y,z).$$

Then, for every $n \geqslant 1$, we have

(i) $\operatorname{indeg}(\operatorname{Ext}_A^n(M,N)) = \operatorname{reg}(\operatorname{Ext}_A^n(M,N)) = -n$.

(ii) $\operatorname{indeg}\left(\operatorname{Tor}_n^A(M,N)\right) = n$ and $\operatorname{reg}\left(\operatorname{Tor}_n^A(M,N)\right) = n + f(n)$, where

$$f(n) := \begin{cases} 2^{l+1}-2 & \text{if } n = 2^l - 1 \\ 2^{l+1}-1 & \text{if } 2^l \leqslant n \leqslant 2^{l+1}-2 \end{cases} \qquad \text{for all integers } l \geqslant 1.$$

As a consequence, in this example,

$$\{\operatorname{reg}(\operatorname{Tor}_{2n}^A(M,N))/2n : n \geqslant 1\} \quad \text{and} \quad \{\operatorname{reg}(\operatorname{Tor}_{2n+1}^A(M,N))/2n+1 : n \geqslant 1\}$$

are dense sets in $[2,3]$ and

$$\liminf_{n\to\infty} \frac{\operatorname{reg}(\operatorname{Tor}_n^A(M,N))}{n} = 2 \quad \text{and} \quad \limsup_{n\to\infty} \frac{\operatorname{reg}(\operatorname{Tor}_n^A(M,N))}{n} = 3.$$

Multigraded Castelnuovo-Mumford regularity. An extension of Castelnuovo-Mumford regularity for a multigraded case was first introduced by Hoffman and Wang in a special case [57], and later by Maclagan and Smith in [65] and Botbol and Chardin in [12] in a more general setting. In [38] and [25], we studied the multigraded regularity. An interesting question to ask is the relation between Castelnuovo-Mumford regularity of M in the multigraded case and the degrees $\mathbf{d}$ where the truncation $M_{\geq \mathbf{d}}$ has a linear resolution. Note that, if $M_{\geq \mathbf{d}}$ has

a linear resolution so does $M_{\geq \mathbf{d}'}$ for all $\mathbf{d}' \geq \mathbf{d}$. Therefore these multidegrees form a region and we call it *linear truncations*. Also, it is enough to find minimal generators for the region where the truncation has a linear resolution. The first non-trivial thing is to show the linear truncations is a non-empty set. Indeed, in [34] the authors provide a degree $\mathbf{d}$ where $M_{\geq \mathbf{d}}$ has a linear resolution. Unfortunately, this degree, in general, is greater than the generators of a linear truncation region. In [38], we refine this theorem and give a better bound in the bigraded case. In addition, we implemented a Macaulay2 package which computes all the minimal generators of linear truncation in general.

In [25] we studied the multigraded regularity of complete intersection points in $\mathbb{P}^n \times \mathbb{P}^m$. We find a bound for the multigraded regularity region and this bound is sharp in some cases. We show that this region is not convex and in particular, this result provides a negative answer to a question asked by D. Eisenbud.

In addition, we relate this result to the bigraded Hilbert function of these points. In this regard, we find a lower bound for the region of the stabilization of the bigraded Hilbert function.

Example 3. Let $S = k[x_0, x_1, x_2, y_0, y_1, y_2]$, $I = (x_0^2y_0^2, x_1^2y_1^2, x_2^2y_2^2, (x_0+x_1+x_2)^2(y_0+y_1+y_2)^2)$ and V be the complete intersection scheme of points defined by I. For $(0,0) \leq \mu \leq (8,8)$ the bigraded Hilbert function $\mathrm{HF}_V(\mu)$ is

$$\begin{array}{cccccccc}
\underline{24} & \underline{72} & 96 & 96 & 96 & 96 & 96 & 96 \\
\underline{24} & \underline{72} & 96 & 96 & 96 & 96 & 96 & 96 \\
21 & 63 & 86 & 90 & 93 & 95 & 96 & 96 \\
15 & 45 & 66 & 78 & 87 & 93 & 96 & 96 \\
10 & 30 & 48 & 64 & 78 & 90 & 96 & 96 \\
6 & 18 & 32 & 48 & 66 & 86 & 96 & 96 \\
3 & 9 & 18 & 30 & 45 & 63 & \underline{72} & \underline{72} \\
1 & 3 & 6 & 10 & 15 & 21 & \underline{24} & \underline{24}.
\end{array}$$

By Theorem 6.2.12 we show all points except the underline ones can be computed explicitly and they only depend on the degree of the generators of I which in this case is $(2,2)$.

The following theorem deals with the case where the scheme of points is defined by generic forms. In this case, the natural projections to $\mathbb{P}^n$ and $\mathbb{P}^m$ are one to one.

Theorem 8. *Let $S = k[x_0, \ldots, x_n, y_0, \ldots, y_m]$. If I is generated by $n+m$ generic forms of bidegree (d,e), then the scheme V defined by I is reduced set of points and*

$$|\{\mu \in \mathbb{N}^2 | \, \mathrm{HF}_V(\mu) \neq \deg(V)\}| < \infty.$$

Chapter 1

Preliminaries

In this chapter we collect the basic definitions and techniques that we will use in the next chapters. In fact, most of the materials in this chapter is well known. We refer to [16] and [32] as standard text books, in particular for the discussion of Koszul complex and basic theorems in commutative algebra. The reader can consult the notations and general facts about spectral sequences in [79].

1.1 Castelnuovo-Mumford regularity

We start by recalling Castelnuovo-Mumford regularity which is a fundamental notion in this thesis. There are two classical definitions for Castelnuovo-Mumford regularity, one with graded Betti numbers and one with local cohomology.

1.1.1 Minimal free resolution

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over a field k and $M = \oplus_{d\in\mathbb{Z}} M_d$ be a finitely generated graded S-module. Given homogeneous elements $m_i \in M$ of degree a_i that generate M as an S-module, we may define a map from the graded free module $F_0 = \oplus_i S(-a_i)$ onto M by sending the i-th generator to m_i. Let $M_1 \subset F_0$ be the kernel of this map $F_0 \to M$. By the Hilbert Basis Theorem, M_1 is also a finitely generated module. The elements of M_1 are called ***syzygies*** on the generators m_i, or simply syzygies of M. Choosing finitely many homogeneous syzygies that generate M_1, we may define a map from a graded free module F_1 to F_0 with image M_1. Continuing in this way we construct a sequence of maps of graded

free modules, called a ***graded free resolution*** of M:

$$\cdots \longrightarrow F_i \xrightarrow{\partial_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\partial_1} F_0.$$

The above sequence is minimal if for each i the image of ∂_i is contained in $\mathfrak{m}F_{i-1}$ where $\mathfrak{m} = (x_1, \ldots, x_n)$ is the unique homogeneous maximal ideal of S.

Theorem 1.1.1. *Let M be a finitely generated graded S-module. If $\boldsymbol{F}$ and $\boldsymbol{G}$ are minimal graded free resolutions of M, then there is a graded isomorphism of complexes $\boldsymbol{F} \to \boldsymbol{G}$ inducing the identity map on M. Any free resolution of M contains the minimal free resolution as a direct summand.*

Proof. See [73, Theorem 7.5] □

The above theorem not only says minimal free resolutions of M are unique up to isomorphism, but also we can obtain it from any non-minimal resolution by deleting trivial complexes of the form

$$0 \to S(-a) \xrightarrow{c} S(-a) \to 0$$

for a nonzero scalar c.

Suppose that $\mathbf{F} : 0 \to F_s \to \cdots \to F_m \to \cdots \to F_0$ is a minimal free resolution of M where $F_i = \oplus_j S(-a_{i,j})$, we can rewrite it as $F_i = \oplus_j S(-j)^{\beta_{i,j}(M)}$; that is F_i requires $\beta_{i,j}(M)$ minimal generators of degree j. Numbers $\beta_{i,j}(M)$ are called ***graded Betti numbers*** of M. There are other interpretations of graded Betti numbers via Tor and Ext modules

$$\beta_{i,j}(M) = \dim_k \left(\mathrm{Tor}_i(M,k)_j\right) = \dim_k \left(\mathrm{Ext}^i(M,k)_{-j}\right).$$

Castelnuovo-Mumford regularity is a measure of the maximal degrees of generators of the syzygy modules. We denote Castelnuovo-Mumford regularity of M by $\mathrm{reg}(M)$ and it is defined as follows:

$$\mathrm{reg}(M) := \max\{j - i | \beta_{i,j}(M) \neq 0\}.$$

1.1.2 Čech complex and local cohomology

In this section we adopt more general setting. If R is a Noetherian ring, $Q \subset R$ is an ideal, and M is an R-module, then the ***zeroth local cohomology module*** of M is

$$H^0_Q(M) := \{m \in M | Q^d m = 0 \text{ for some } d\}.$$

H^0_Q is a functor in an obvious way: if $\phi : M \to N$ is a map, the induced map $H^0_Q(\phi)$ is the restriction of ϕ to $H^0_Q(M)$. One sees immediately from this that the functor H^0_Q is left exact, so it is natural to study its derived functors, which we call H^i_Q.

Čech complex

Another useful expression for the local cohomology is obtained from a Čech complex: suppose that Q is generated by elements $(x_1, \ldots, x_t)$. We write $[t] = \{1, \ldots, t\}$ for the set of integers from 1 to t, and for any subset $J \subset [t]$ we let $x_J = \prod_{j \in J} x_j$. We denote by $M[x_J^{-1}]$ the localization of M by inverting x_J.

Definition 1.1.2 (Čech complex)**.** Let R be a commutative ring, $Q = (x_1, \ldots, x_t)$ be an ideal of R and M be a finitely generated R-module. The Čech complex $\mathbf{C}(Q, M)$ is

$$\mathbf{C}(Q, M) : 0 \to M \xrightarrow{d} \bigoplus_{1}^{t} M[x_i^{-1}] \xrightarrow{d} \cdots \longrightarrow \bigoplus_{\sharp J = s} M[x_J^{-1}] \xrightarrow{d} \cdots \longrightarrow M[x_{1,\ldots,t}^{-1}] \longrightarrow 0$$

where the differential maps are the natural ones.

Theorem 1.1.3. *Suppose R is a Noetherian ring and $Q = (x_1, \ldots, x_t)$. For any R-module M,*

$$H_Q^i(M) = H^i(\boldsymbol{C}(Q, M)).$$

Here we give two simple examples of local cohomologies that we will use later

Example 1.1.4. [33, Corollary A1.5] Let $S = k[x_1, \ldots, x_n]$. I is an $\mathfrak{m}$-primary ideal if and only if

$H_{\mathfrak{m}}^0(S/I) = S/I$ and $H_{\mathfrak{m}}^i(S/I) = 0$ for $i \neq 0$.

Example 1.1.5. [59, Example 7.16] Let $S = k[x_1, \ldots, x_n]$, then $H_{\mathfrak{m}}^i(S) = 0$ for $i \neq n$ and $H_{\mathfrak{m}}^n(S) = y_1 \cdots y_n k[y_1, \ldots, y_n]$ where $k[y_1, \ldots, y_n]$ is the polynomial ring over k, in variables $y_1, \ldots, y_n$ of degree -1, and the S-module structure on it is defined by

$$x_i \cdot (y_1^{a_1} \cdots y_n^{a_n}) = \begin{cases} y_1^{a_1} \cdots y_i^{a_i - 1} \cdots y_n^{a_n} & \text{if } a_i \geq 2 \\ 0 & \text{otherwise.} \end{cases}$$

There are two fundamental results. First, Grothendieck's theorem asserts the vanishing of $H_{\mathfrak{m}}^i(M)$ for $i > \dim(M)$ and $i < \operatorname{depth}(M)$, as well as the nonvanishing of these modules for $i = \operatorname{depth}(M)$ and $i = \dim(M)$. Second is Serre's vanishing theorem that implies the vanishing of graded pieces $H_{\mathfrak{m}}^i(M)_\mu$ for any i if μ is big enough. The Castelnuovo-Mumford regularity is a measure of this vanishing degree.

Theorem 1.1.6. *[20, Corollary 1.2.2] If M is a finitely generated graded S-module, then for any l*

$$\max_{p\leq l}\{a_p(M)+p\} = \max_{q\geq n-l}\{b_q(M)-q\}$$

where $a_i(M) := \max\{\mu | H^i_{\mathfrak{m}}(M)_\mu \neq 0\}$ and $b_j(M) := \max\{\mu | \operatorname{Tor}_i^S(M,K)_\mu \neq 0\}$. As a consequence,

$$\operatorname{reg}(M) = \max_p\{a_p(M)+p\} = \max_q\{b_q(M)-q\}.$$

1.2 Lefschetz Properties

The Weak and Strong Lefschetz properties are strongly connected to many topics in algebraic geometry, commutative algebra and combinatorics. Some of these connections are quite surprising and still not completely understood, and much work remains to be done.

Let $S = k[x_1,\ldots,x_n]$ be the polynomial ring in n variables over k. Let

$$A = S/I = \bigoplus_{i=0}^{n} A_i$$

be a graded artinian algebra. Note that A is finite dimensional over k.

Definition 1.2.1. Let ℓ be a general linear form. We say that A has the ***Weak Lefschetz Property (WLP)*** if the homomorphism induced by multiplication by ℓ,

$$\times\ell : A_i \longrightarrow A_{i+1}$$

has maximal rank for all i (i.e. is injective or surjective). We say that A has the ***Strong Lefschetz Property (SLP)*** if

$$\times\ell^d : A_i \longrightarrow A_{i+d}$$

has maximal rank for all i and d.

How do we determine if S/I fails to have the WLP? Let ℓ be a general linear form and fix an integer i. Then we have an exact sequence

$$(S/I)_{i-1} \xrightarrow{\times\ell} (S/I)_i \longrightarrow (S/(I,\ell))_i \longrightarrow 0.$$

Thus $\times\ell$ fails to have maximal rank from degree $i-1$ to degree i if and only if

$$\dim(S/(I,\ell))_i > \max\{0, \dim(S/I)_i - \dim(S/I)_{i-1}\}.$$

More precisely, if we want to show that the WLP fails, it is enough to identify a degree i for which we can produce one of the following two pieces of information:

(1) $\dim(S/I)_{i-1} \leq \dim(S/I)_i$ and $\dim\left(S/(I,\ell)\right)_i > \dim(S/I)_i - \dim(S/I)_{i-1}$; in this case we loosely say that WLP fails because of injectivity; or

(2) $\dim(S/I)_{i-1} \geq \dim(S/I)_i$ and $\dim\left(S/(I,\ell)\right)_i > 0$; in this case we loosely say that WLP fails because of surjectivity.

In general, even identifying which i is the correct place to look can be difficult. Then determining which of (1) or (2) holds, and establishing both inequalities, is often very challenging. This is where computer algebra programs have been very useful, in suggesting where to look and what to look for! On the other hand, to prove that S/I does have the WLP, the following result is helpful:

Proposition 1.2.2. *If I is a monomial ideal in $S = k[x_1, \ldots, x_n]$ then the linear form $\ell = x_1 + \cdots + x_n$ is "general enough" to determine S/I has the WLP or SLP.*

The above Proposition has been extremely useful in simplifying calculations to show the existence or failure of the WLP or SLP.

In the case of one variable, the WLP and SLP are trivial since all ideals are principal. The case of two variables also has a nice result, at least in characteristic 0:

Theorem 1.2.3. *If $\mathrm{char}(k) = 0$ and I is any homogeneous ideal in $S = k[x, y]$ then S/I has the SLP.*

The above theorem is also true for the case of WLP in any characteristic. However, the characteristic zero assumption cannot be omitted for guaranteeing the SLP. In fact, also the WLP may fail if there are at least three variables.

Theorem 1.2.4. *Let $S = k[x, y, z]$ and $I = (f_1, f_2, f_3)$ be a complete intersection. Then S/I has the WLP.*

1.3 Edge ideals

In this part we recall a notion of edge ideals and some important theorems about regularity of powers of them. Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, \ldots, v_l\}$. For a vertex u in a graph $G = (V, E)$, let $N_G(u) = \{v \in V | \{u, v\} \in E\}$ be the set of *neighbors* of u, and set $N_G[u] := N_G(u) \cup \{u\}$. An edge e is *incident* to a vertex u if $u \in e$. The degree of a vertex $u \in V$, denoted by $\deg_G(u)$, is the number of edges incident to u. When there is no confusion, we will omit G and write $N(u), N[u]$ and $\deg(u)$. For an edge e in a graph

$G = (V, E)$, we define $G \setminus e$ to be the subgraph of G obtained by deleting e from E (but the vertices are remained). For a subset $W \subseteq V$ of the vertices in G, we define $G \setminus W$ to be the subgraph of G deleting the vertices of W and their incident edges. When $W = \{u\}$ consists of a single vertex, we write $G \setminus u$ instead of $G \setminus \{u\}$. For an edge $e = \{u, v\} \in E$, let $N_G[e] = N_G[u] \cup N_G[v]$ and define G_e to be the induced subgraph of G over the vertex set $V \setminus N_G[e]$.

One can think of the vertices of $G = (V, E)$ as the variables of the polynomial ring $S = k[x_1, \ldots, x_r]$ for convenience. Similarly, the edges of G can be considered as square free monomials of degree two. We define edge ideal of G as an ideal generated by $x_i x_j$ where $\{x_i, x_j\}$ is an edge of G and we denote it by $I(G)$. By abuse of notation, we use e to refer to both the edge $e = \{x_i, x_j\}$ and the monomial $e = x_i x_j \in I(G)$.

Theorem 1.3.1. *[51, Lemma 3.1, Theorems 3.4 and 3.5] Let $G = (V, E)$ be a graph.*

(i) If H is an induced subgraph of G, then $\operatorname{reg} I(H) \leq \operatorname{reg} I(G)$;

(ii) Let $x \in V$, then

$$\operatorname{reg} I(G) \leq \max\{\operatorname{reg} I(G \setminus x), \operatorname{reg} I(G \setminus N[x]) + 1\};$$

(iii) Let $e \in E$, then

$$\operatorname{reg} I(G) \leq \max\{2, \operatorname{reg} I(G \setminus e), \operatorname{reg} I(G_e) + 1\}.$$

Now we recall the concept of even-connection introduced by Banerjee in [7].

Definition 1.3.2 ([7])**.** Let $G = (V, E)$ be a graph with edge ideal $I = I(G)$. Two vertices x_i and x_j in G are called *even-connected* with respect to an s-fold product $M = e_1 \cdots e_s$, where $e_1, \ldots, e_s$ are edges in G, if there is a path $p_0, \ldots, p_{2l+1}$, for some $l \geq 1$, in G such that the following conditions hold:

(i) $p_0 = x_i$ and $p_{2l+1} = x_j$;

(ii) for all $0 \leq j \leq l - 1$, $\{p_{2j+1}, p_{2j+2}\} = e_i$ for some i;

(iii) for all i, $\big|\{j \mid \{p_{2j+1}, p_{2j+2}\} = e_i\}\big| \leq \big|\{t \mid e_t = e_i\}\big|$.

Theorem 1.3.3. *[7, Theorems 6.1 and 6.5] Let $M = e_1 e_2 \cdots e_s$ be a minimal generator of I^s. Then $(I^{s+1} \colon M)$ is minimally generated by monomials of degree 2, and uv (u and v may be the same) is a minimal generator of $(I^{s+1} \colon M)$ if and only if either $\{u, v\} \in E$ or u and v are even-connected with respect to M.*

Remark 1.3.4. [7, Lemma 6.11] Let $(I^{s+1}\colon M)^{\mathrm{pol}}$ be the polarization of the ideal $(I^{s+1}\colon M)$ (see e.g. [55, §1.6]). From the previous theorem we can construct a graph G' whose edge ideal is given by $(I^{s+1}\colon M)^{\mathrm{pol}}$. The new graph G' is given by:

(i) All the vertices and edges of G.

(ii) Any two vertices u, v, $u \neq v$ that are even-connected with respect to M are connected by an edge in G'.

(iii) For every vertex u which is even-connected to itself with respect to M, there is a new vertex u' which is connected to u by an edge and not connected to any other vertex (so uu' is a whisker).

Theorem 1.3.5. *[7, Theorem 5.2] Let G be a graph and $\{m_1, \ldots, m_r\}$ be the set of minimal monomial generators of $I(G)^q$ for all $q \geq 1$, then*

$$\operatorname{reg} I(G)^{q+1} \leq \max\{\operatorname{reg}(I(G)^q\colon m_l) + 2q, 1 \leq l \leq r, \operatorname{reg} I(G)^q\}.$$

The aforementioned theorem is our main tool to provide an upper bound for the regularity of powers of edge ideals.

The decycling number of a graph is an important combinatorial invariant which can be used to obtain an upper bound for the regularity of the edge ideal of a graph.

In [10] Beyarslan, Hà and Trung provided a formula for the regularity of the powers of edge ideals of forests and cycles in terms of the induced matching number.

Theorem 1.3.6. *[10, Theorem 4.7] Let G be a forest, then*

$$\operatorname{reg} I(G)^q = 2q + \nu(G) - 1$$

for all $q \geqslant 1$.

Theorem 1.3.7. *[10, Theorem 5.2]. Let C_n be a cycle with n vertices, then*

$$\operatorname{reg} I(C_n) = \begin{cases} \nu(C_n) + 1 & \text{if } n \equiv 0, 1 \pmod 3, \\ \nu(C_n) + 2 & \text{if } n \equiv 2 \pmod 3, \end{cases}$$

where $\nu(C_n) = \lfloor \frac{n}{3} \rfloor$ denote the induced matching number of C_n. Moreover,

$$\operatorname{reg} I(C_n)^q = 2q + \nu(C_n) - 1$$

for all $q \geqslant 2$.

In addition, the authors of [10] provided a lower bound for the regularity of the powers of the edge ideal of an arbitrary graph, and an upper bound for the regularity of the edge ideal of a graph containing a Hamiltonian path.

Theorem 1.3.8. *[10, Theorem 4.5] Let G be a graph and let $\nu(G)$ denote its induced matching number. Then, for all $q \geq 1$, we have*

$$\operatorname{reg} I(G)^q \geq 2q + \nu(G) - 1.$$

Theorem 1.3.9. *[10, Theorem 3.1] Let G be a graph on n vertices. Assume G contains a Hamiltonian path, then*

$$\operatorname{reg} I(G) \leq \lfloor \frac{n+1}{3} \rfloor + 1.$$

Chapter 2

Regularity of Koszul homologies and Koszul cycles

We extend to one dimensional quotients the result of A. Conca and S. Murai on the convexity of the regularity of Koszul cycles. By providing a relation between the regularity of Koszul cycles and Koszul homologies we prove a sharp regularity bound for the Koszul homologies of a homogeneous ideal in a polynomial ring under the same conditions.

A classic way to describe projective variety and its properties is by means of its defining equations and syzygies among them. In this regard, M. Green and R. Lazarsfeld defined the property N_p which, roughly speaking, refers to the simplicity of syzygies of the homogeneous coordinate ring of a smooth projective variety embedded by a very ample line bundle. M. Green in [48] proved that the coordinate ring of the image of Veronese embedding of degree d satisfies the property N_d. W. Bruns, A. Conca and T. Römer in [15] improved this result so that the d-th Veronese subring of a polynomial ring has Green-Lazarsfeld index larger than or equal to $d+1$, their approach is based on investigation of the homological invariants of the Koszul cycles and Koszul homologies of d-th power of the maximal ideal.

With the aforementioned motivation A. Conca and S. Murai studied the Castelnuovo-Mumford regularity of the Koszul cycles $Z_t(I,S)$ of a homogeneous ideal in a polynomial ring S in n variables. Under mild assumptions on the base field A. Conca and S. Murai proved that regularity of Koszul cycles $Z_i(I,S)$ as a function of i is subadditive when $\dim S/I = 0$ as follows:

$$\operatorname{reg}(Z_{s+t}(I,S)) \leq \operatorname{reg}(Z_t(I,S)) + \operatorname{reg}(Z_s(I,S)).$$

We make a generalization showing that with the same assumptions on the base field the same formula holds when $\dim S/I \leq 1$.

From the convexity of the regularity of Koszul cycles in dimension 0, A. Conca and S. Murai [29, Corollary 3.3] obtained a bound on the regularity of Koszul homologies. Inspired by the remarkable result of M.Chardin and P. Symonds [26] on the regularity of cycles and homologies of a general complex, first we determine the regularity of Koszul cycles by the regularity of the previous Koszul homologies. Let S be a polynomial ring, I be a homogeneous ideal of S. If $\dim S/I \leq 1$, then for all $0 < i < \mu(I)$

$$\operatorname{reg}(Z_i(I,S)) = \max_{0<j<n}\{\operatorname{reg}(H_{i-j}(I,S)) + j + 1\}.$$

Here $\mu(I)$ is the minimal number of generators of I.

As an application we state sharp bounds for the regularity of Koszul homologies in dimension 1 which is a refinement of the result of A. Conca and S. Murai in dimension 0. Let I be an ideal of S and $\dim S/I \leq 1$, then we have the following inequalities between Koszul homologies of I for all $i, j \geq 1$

$$\operatorname{reg}(H_{i+j-1}(I,S)) \leq \max_{1\leq\alpha,\beta\leq n-1}\{\operatorname{reg}(H_{i-\alpha}(I,S)) + \operatorname{reg}(H_{j-\beta}(I,S)) + \alpha + \beta\}.$$

2.1 Regularity of Koszul cycles

In this section we will present a generalization of a result about convexity of regularity of Koszul cycles of A. Conca and S. Murai.

Let $I = (f_1, \ldots, f_r)$ be a graded S-ideal minimally generated in degrees $d_1, \cdots, d_r$. Define $K(I,S) = \oplus K_t(I,S)$ as the Koszul complex associated to the S-linear map $\phi : F_0 = \oplus S(-d_i) \to S$ in which $\phi(e_i) = f_i$. Let $K(I,M) = K(I,S) \otimes M$ and denote $Z_t(I,M)$, $B_t(I,M)$ and $H_t(I,M)$ the cycles, boundaries and homologies of $K(I,M)$ respectively at the homological position t. We use $Z_t(I), B_t(I)$ and $H_t(I)$ whenever $M = S$. We set $K_t(I,S) = 0$ for $t < 0$.

Remark 2.1.1. The Koszul complex does depend on the choice of the generators, but it is unique up to isomorphism if we choose minimal set of generators. Since we only deal with the case the set of generators is minimal, we use $K(I)$ instead of $K(f_1, \ldots, f_r)$.

The following result is due to W. Bruns, A. Conca and T. Römer in [14]

Lemma 2.1.2. *[14, Lemma 2.4] Let S be a polynomial ring, I be a homogeneous ideal of S and M be a finitely generated graded S-module. Suppose that the element $\begin{pmatrix} s+t \\ s \end{pmatrix}$ is invertible in S. Then $Z_{s+t}(I,M)$ is a direct summand of $Z_s(I, Z_t(I,M))$*

The following lemma allows us to compare regularities of different terms of exact sequences and basically it plays the main role in the generalization of the result of A. Conca and S. Murai[29] on the convexity of regularity of Koszul cycles.

Lemma 2.1.3. *Let* $L : 0 \longrightarrow L_4 \xrightarrow{d_4} L_3 \xrightarrow{d_3} L_2 \xrightarrow{d_2} L_1 \longrightarrow 0$ *be an exact sequence of finitely generated graded* S*-modules such that* L_1 *and* L_4 *have dimension* ≤ 1*, and* $\operatorname{depth} L_2 \geq 2$ *then*

$$\operatorname{reg}(L_3) = \max\{\operatorname{reg}(L_4), \operatorname{reg}(L_2), \operatorname{reg}(L_1) - 1\},$$

in particular $\operatorname{reg}(L_2) \leq \operatorname{reg}(L_3)$.

Proof. First we decompose the complex L into the following short exact sequences

$$0 \longrightarrow L_4 \xrightarrow{d_4} L_3 \xrightarrow{can.} \operatorname{coker}(d_4) \longrightarrow 0,$$

$$0 \longrightarrow \operatorname{coker}(d_4) \xrightarrow{\bar{d_3}} L_2 \xrightarrow{d_2} L_1 \longrightarrow 0.$$

Given the above short exact sequences, one can obtain the following induced long exact sequences on local cohomology:

$$\text{(I)} \qquad \cdots \longrightarrow H^i_{\mathfrak{m}}(L_4) \to H^i_{\mathfrak{m}}(L_3) \to H^i_{\mathfrak{m}}(\operatorname{coker}(d_4)) \to H^{i+1}_{\mathfrak{m}}(L_4) \longrightarrow \cdots$$

$$\text{(II)} \qquad \cdots \longrightarrow H^i_{\mathfrak{m}}(L_2) \to H^i_{\mathfrak{m}}(L_1) \to H^{i+1}_{\mathfrak{m}}(\operatorname{coker}(d_4)) \to H^{i+1}_{\mathfrak{m}}(L_2) \longrightarrow \cdots$$

$H^i_{\mathfrak{m}}(L_4) = 0$ for $i \geq 2$ as $\dim L_4 \leq 1$ thus (I) gives

$$H^2_{\mathfrak{m}}(L_3) \cong H^2_{\mathfrak{m}}(\operatorname{coker}(d_4)). \tag{2.1.1}$$

As $\dim L_1 \leq 1$, by (II) we have

$$H^i_{\mathfrak{m}}(\operatorname{coker}(d_4)) \cong H^i_{\mathfrak{m}}(L_2) \ \forall i \geq 3.$$

As $\operatorname{depth} L_2 \geq 2$ and $H^i_{\mathfrak{m}}(L_1) = 0$ for $i = 0, 1$, by (II)

$$H^0_{\mathfrak{m}}(L_1) \cong H^1_{\mathfrak{m}}(\operatorname{coker}(d_4)) \text{ and } H^0_{\mathfrak{m}}(\operatorname{coker}(d_4)) = 0. \tag{2.1.2}$$

From the exact sequences (I) and (2.1.2) we get the following short exact sequences

$$0 \to H^1_{\mathfrak{m}}(L_4) \to H^1_{\mathfrak{m}}(L_3) \to H^0_{\mathfrak{m}}(L_1) \to 0,$$

also the exact sequences (II) and (2.1.1) give

$$0 \to H^1_{\mathfrak{m}}(L_1) \to H^2_{\mathfrak{m}}(L_3) \to H^2_{\mathfrak{m}}(L_2) \to 0.$$

As a result we have

$$a^i_{\mathfrak{m}}(L_3) = \begin{cases} a^0_{\mathfrak{m}}(L_4) & \text{if } i = 0 \\ \max\{a^1_{\mathfrak{m}}(L_4), a^0_{\mathfrak{m}}(L_1)\} & \text{if } i = 1 \\ \max\{a^2_{\mathfrak{m}}(L_2), a^1_{\mathfrak{m}}(L_1)\} & \text{if } i = 2 \\ a^i_{\mathfrak{m}}(L_2) & \text{if } i \geq 3 \end{cases}$$

which proves the statement. □

Proposition 2.1.4. *Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring, let M be a finitely generated graded S-module with* $\operatorname{depth} M \geq 2$ *and let I be a graded ideal of S such that* $\dim S/I \leqslant 1$, *then*

$$\operatorname{reg}(Z_t(Z_s(I, M))) \leqslant \operatorname{reg}(Z_t(I)) + \operatorname{reg}(Z_s(I, M)).$$

Proof. By definition one has the following exact sequences,

$$(\dagger) \qquad 0 \to Z_s(I, M)) \to K_s(I, M) \xrightarrow{d_t} K_{s-1}(I, M)$$

$$(\ddagger) \qquad 0 \to Z_t(I, Z_s(I, M)) \to K_t(I, Z_s(I, M)) \xrightarrow{d_t} K_{t-1}(I, Z_s(I, M))$$

Note that $K_s(I, M)$ and $K_{s-1}(I, M)$ are direct sums of copies of M, (†) then implies that $\operatorname{depth} Z_s(I, M) \geqslant \min\{2, \operatorname{depth} M\} = 2$. Using (‡),

$$\operatorname{depth} Z_t(I, Z_s(I, M)) \geq \min\{2, \operatorname{depth} Z_s(I, M)\} = 2.$$

For the canonical map in [14, Section 5]

$$u_{s,t} : Z_t(I) \otimes Z_s(I, M) \to Z_t(I, Z_s(I, M))$$

Proposition 5.1 in [14] gives an exact sequence,

$$0 \to \ker(u_{s,t}) \to Z_t(I) \otimes Z_s(I, M) \to Z_t(Z_s(I, M)) \to \operatorname{Tor}^S_1(\frac{K_{s-1}(I, M)}{B_{s-1}(I, M)}, Z_t(I)) \to 0.$$

Notice that after localization at prime ideals not in the support of S/I all the Koszul cycles become direct sum of copies of M and the map $u_{s,t}$ becomes an isomorphism. Therefore

$\mathrm{Tor}_1^S(\frac{K_{s-1}(I,M)}{B_{s-1}(I,M)}, Z_t(I))$ and $\ker(u_{s,t})$ are supported in S/I, hence have dimension at most 1.

Thus the conditions of Lemma 2.1.3 are fulfilled, and this lemma gives:

$$\mathrm{reg}(Z_t(Z_s(I,M))) \leqslant \mathrm{reg}(Z_t(I,M) \otimes Z_s(I,M)).$$

Notice that $\mathrm{Tor}_1^R(Z_t(I), Z_s(I,M))$ has dimension at most 1 because $Z_t(I)$ is free when we localize at prime ideals not in the support of S/I, so we apply Corollary 3.1 in [37] to get

$$\mathrm{reg}(Z_t(I,M) \otimes Z_s(I,M)) \leqslant \mathrm{reg}(Z_t(I)) + \mathrm{reg}(Z_s(I,M)).$$

As a result we get

$$\mathrm{reg}(Z_t(Z_s(I,M))) \leqslant \mathrm{reg}(Z_t(I)) + \mathrm{reg}(Z_s(I,M)).$$

□

Theorem 2.1.5. *Let $S = k[x_1, \ldots, x_n]$ and I be a graded ideal of S, if $\dim S/I \leqslant 1$ and characteristic of k is 0 or bigger than $s+t$, then*

$$\mathrm{reg}(Z_{s+t}(I)) \leqslant \mathrm{reg}(Z_t(I)) + \mathrm{reg}(Z_s(I)).$$

Proof. The theorem follows from Proposition 2.1.4 and Lemma 2.1.2. □

2.2 Regularity of Koszul homologies

We start this section by a fact which is likely part of folklore but we did not find it in the classical references.

Proposition 2.2.1. *Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring and I be an ideal of S minimally generated by $f_1 \ldots, f_r$, then $Z_i(I) \subset \mathfrak{m}K_i(I)$ for all i.*

Proof. Suppose it is not, then there exists $z \in Z_i(I)$ that is not in $\mathfrak{m}K_i(I)$. By symmetry we may assume it has the form:

$$z = e_1 \wedge \cdots \wedge e_i + \sum_{j>i} c_j e_1 \wedge \cdots \wedge e_{i-1} \wedge e_j + \text{terms without } e_1 \wedge \cdots \wedge e_{i-1}.$$

Since $\partial(z) = 0$ it follows that $(-1)^i f_i + \sum_{j>i}(-1)^j c_j f_j = 0$, as it is the coefficient of $e_1 \wedge \cdots \wedge e_{i-1}$ in the expression of $\partial(z)$, which is a contradiction with the fact that $f_1, \ldots, f_r$ is a minimal set of generators for I. □

Corollary 2.2.2. *Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring and $I = (f_1, \ldots, f_r)$ be a homogeneous ideal of S. Let $f_1, \ldots, f_r$ be a minimal generating set of I and $\deg(f_i) = d_i$ where $d_1 \geq d_2 \geq \cdots \geq d_r$, then $\operatorname{reg}(Z_i(I)) > d_1 + \cdots + d_i$ for $i \leq r$.*

Proof. Fix a basis element $e_1 \wedge \cdots \wedge e_i \in K_i$. Since $K_\bullet(I)$ is a complex,

$$\partial(e_1 \wedge \cdots \wedge e_i \wedge e_{i+1}) = (-1)^{i+1} f_{i+1} e_1 \wedge \cdots \wedge e_i + \sum_{0<j<i+1} (-1)^j f_j e_1 \wedge \cdots \wedge \hat{e_j} \wedge \cdots \wedge e_{i+1} \in Z_i(I).$$

Therefore an element of the form $ge_1 \wedge \cdots \wedge e_i$ should appear as a summand in a minimal generating element of $Z_i(I)$. By Proposition 2.2.1, $g \in \mathfrak{m}$. So there exists minimal generator of degree at least $d_1 + \cdots + d_i + 1$. Hence $\operatorname{reg}(Z_i(I)) > d_1 + \cdots + d_i$. □

M. Chardin and P. Symonds in [26] presented a new approach to the study of the regularity of cycles of a general complex by the regularity of previous homologies. Here we determine a concrete relation between regularity of cycles and homologies of a Koszul complex.

Theorem 2.2.3. *Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring and I be a homogeneous ideal of S minimally generated by $f_1, \ldots, f_r$. If $\dim S/I \leq 1$, then for $0 < i < r$:*

$$\operatorname{reg}(Z_i(I)) = \max_{0<j\leq \min\{n-1,i\}} \{\operatorname{reg}(H_{i-j}(I)) + j + 1\}. \tag{2.2.1}$$

Proof. Let $I = (f_1, \ldots, f_r)$ and $\deg(f_i) = d_i$ where $d_1 \geq d_2 \geq \cdots \geq d_r$. Let $K^i_\bullet(I)$ be the i-th truncated Koszul complex of I as follows:

$$K^i_\bullet(I) : 0 \to Z_i(I) \xrightarrow{\partial'_i} K_i(I) \xrightarrow{\partial_i} K_{i-1}(I) \xrightarrow{\partial_{i-1}} \cdots \xrightarrow{\partial_1} K_0(I) \to 0$$

and $C^\bullet$ be the Čech complex. Consider double complex $X = C^\bullet \bigotimes K^r_\bullet(I)$ where $X_{p,q} = C^{\bullet-p} \otimes K^r_\bullet(I)_q$, and its associated spectral sequence. We first compute homology vertically

and we get

$$\begin{array}{ccccc}
H^0_{\mathfrak{m}}(Z_i(I)) & 0 & 0 & \cdots & 0 \\
H^1_{\mathfrak{m}}(Z_i(I)) & 0 & 0 & \cdots & 0 \\
H^2_{\mathfrak{m}}(Z_i(I)) & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
H^n_{\mathfrak{m}}(Z_i(I)) \xrightarrow{H^n_{\mathfrak{m}}(\partial'_i)} & H^n_{\mathfrak{m}}(K_i(I)) \xrightarrow{H^n_{\mathfrak{m}}(\partial_i)} & H^n_{\mathfrak{m}}(K_{i-1}(I)) \xrightarrow{H^n_{\mathfrak{m}}(\partial_{i-1})} & \cdots \xrightarrow{H^n_{\mathfrak{m}}(\partial_1)} & H^n_{\mathfrak{m}}(K_0(I)).
\end{array}$$

By continuing the process we have:

$$E^{\infty}_{p,q} = E^2_{p,q} \begin{cases} H^p_{\mathfrak{m}}(Z_q(I)) & \text{if } q = i+1, p < n \\ H_q(H^n_{\mathfrak{m}}(K^i_{\bullet}(I))) & \text{if } p = n, q \leq i \\ \ker(H^n_{\mathfrak{m}}(\partial'_i)) & \text{if } (p,q) = (n, i+1) \\ 0 & \text{Otherwise.} \end{cases}$$

Notice that since $a^n_{\mathfrak{m}}(K_j(I)) = d_1 + \cdots + d_j - n$, it follows that for all $0 \leq q \leq i$ we have $\operatorname{end}(E^{\infty}_{n,q}) \leq \operatorname{end}(E^1_{n,q}) = d_1 + \cdots + d_q - n$.

On the other hand, if we start taking homology horizontally we have ${E'}^2_{p,q} = H^p_{\mathfrak{m}}(H_q(I))$ for all p and $q < i$ and ${E'}^2_{p,q} = 0$ for $q = i, i+1$. Notice that $\dim H_i(I) \leq \dim S/I \leq 1$, therefore spectral sequence collapses in the second page and we have:

$${E'}^{\infty}_{p,q} = {E'}^2_{p,q} \begin{cases} H^p_{\mathfrak{m}}(H_q(I)) & \text{if } p = 0, 1 \text{ and } q < i \\ 0 & \text{otherwise.} \end{cases}$$

The comparison of two spectral sequences gives

$$\begin{aligned}
&H^0_{\mathfrak{m}}(Z_i(I)) = H^1_{\mathfrak{m}}(Z_i(I)) = 0 \\
&a^2_{\mathfrak{m}}(Z_i(I)) = a^0_{\mathfrak{m}}(H_{i-1}(I)) \\
&a^j_{\mathfrak{m}}(Z_i(I)) = \max\{a^1_{\mathfrak{m}}(H_{i-j+2}(I)), a^0_{\mathfrak{m}}(H_{i-j+1}(I))\}, \ \forall\ 2 < j < n
\end{aligned}$$

In addition, for the last local cohomology we have

$$a^n_{\mathfrak{m}}(Z_i(I)) \leq \max\{a^1_{\mathfrak{m}}(H_{i-n+2}(I)), a^0_{\mathfrak{m}}(H_{i-n+1}(I)), d_1 + \cdots d_i - n\},$$

furthermore

$$a^n_{\mathfrak{m}}(Z_i(I)) = \max\{a^1_{\mathfrak{m}}(H_{i-n+2}(I)), a^0_{\mathfrak{m}}(H_{i-n+1}(I))\}$$

if $a^n_{\mathfrak{m}}(Z_i(I)) > d_1 + \cdots d_i - n$. By Corollary 2.2.2, we can deduce that

$$a^n_{\mathfrak{m}}(Z_i(I)) = \max\{a^1_{\mathfrak{m}}(H_{i-n+2}(I)), a^0_{\mathfrak{m}}(H_{i-n+1}(I))\} \text{ or } a^n_{\mathfrak{m}}(Z_i(I)) + n < \operatorname{reg}(Z_i(I))$$

In addition, the comparison of the two spectral sequences and Corollary 2.2.2 give

$$a^1_{\mathfrak{m}}(H_{i-n+1}(I)) \leq \operatorname{end}(E^{\infty}_{n,i-1}) \leq d_1 + \cdots + d_{i-1} - n < \operatorname{reg}(Z_i(I)) - n.$$

As a result we have:

$$\begin{aligned}
\operatorname{reg}(Z_i(I)) &= \max_{0\leq j\leq n}\{a^j_{\mathfrak{m}}(Z_i(I))) + j\} \\
&= \max_{3\leq j\leq \max\{n,i+2\}}\{a^0_{\mathfrak{m}}(H_{i-1}(I)) + 2, a^1_{\mathfrak{m}}(H_{i-j+2}(I)) + j, a^0_{\mathfrak{m}}(H_{i-j+1}(I)) + j, d_1 + \cdots d_i\} \\
&= \max_{2\leq j\leq \max\{n,i+1\}}\{\operatorname{reg}(H_{i-j+1}(I)) + j\}.
\end{aligned}$$

□

Remark 2.2.4. From the proof of the Theorem 2.2.3, the following equality also holds

$$\operatorname{reg}(Z_i(I)) = \max_{j>0}\{\operatorname{reg}(H_{i-j}(I)) + j + 1\}.$$

As a consequence of the Theorem 2.2.3 and Theorem 2.1.5 we give a regularity bound for Koszul homologies in dimension at most 1.

Theorem 2.2.5. *Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring and I be a homogeneous ideal of S. If $\dim S/I \leq 1$, then for all $i, j \geq 1$ we have the following regularity bound for the Koszul homologies of I.*

$$\operatorname{reg}(H_{i+j-1}(I)) \leq \max_{0<\alpha,\beta<n}\{\operatorname{reg}(H_{i-\alpha}(I)) + \operatorname{reg}(H_{j-\beta}(I)) + \alpha + \beta\}. \tag{2.2.2}$$

Proof. By Theorem 2.1.5 we have the following inequality for all i, j

$$\operatorname{reg}(Z_{i+j}(I)) \leq \operatorname{reg}(Z_i(I)) + \operatorname{reg}(Z_j(I)).$$

By using Theorem 2.2.3 we have

$$\begin{aligned}\operatorname{reg}(H_{i+j-1}(I)) + 2 &\leq \operatorname{reg}(Z_{i+j}(I)) \\ &\leq \operatorname{reg}(Z_i(I)) + \operatorname{reg}(Z_j(I)) \\ &= \max_{0<\alpha<n}\{\operatorname{reg}(H_{i-\alpha}(I)) + \alpha + 1\} + \max_{0<\beta<n}\{\operatorname{reg}(H_{j-\beta}(I)) + \beta + 1\} \\ &= \max_{0<\alpha,\beta<n}\{\operatorname{reg}(H_{i-\alpha}(I)) + \operatorname{reg}(H_{j-\beta}(I)) + \alpha + \beta + 2\}.\end{aligned}$$

□

The following example shows the deviation degree of our bound comparing to the bound provided by A. Conca and S. Murai in dimension 0.

Example 2.2.6. Let $S = k[x, y, z]$ be a polynomial ring and $I = (x, y, z)^4$. We compare our bound for the regularity of $H_{12}(I)$ for different i, j by the bound in the [29]. By using Macaulay2 [47] one can see that the $\operatorname{reg}(H_{12}(I)) = 57$. For bounding regularity of $H_{12}(I)$ we should choose i, j such that $i + j = 13$. By choosing $(i, j) = (1, 12)$ (respectively $(2, 11), (3, 10), (4, 9), (5, 8), (6, 7)$) the right hand side of Equation 2.2.2 is 57 (respectively $58, 58, 59, 59, 58$). On the other hand in the bound proposed by A. Conca and S. Murai the best possible estimate is 61.

Corollary 2.2.7. *Let $S = k[x_1, \ldots, x_n]$ be a polynomial ideal and I be an ideal of S. If $\dim S/I \leq 1$, then*

$$\operatorname{reg}(H_i(I)) \leq (i + 1)\operatorname{reg}(H_0(I)) + 2i.$$

In particular, $\operatorname{reg}(H_i(I)) \leq (i + 1)(\operatorname{reg}(I) - 1) + 2i$.

Proof. We prove by induction. For $i = 1$ by Theorem 2.2.5 we have

$$\operatorname{reg}(H_1(I)) \leq \{\operatorname{reg}(H_0(I)) + 1 + \operatorname{reg}(H_0(I)) + 1\} = 2\operatorname{reg}(H_0(I)) + 2.$$

Let $\operatorname{reg}(H_i(I)) \leq (i + 1)\operatorname{reg}(H_0(I)) + 2i$ for all $i \leq r$, by choosing $i = 1$ and $j = r + 1$ in Equation 2.2.2 we have

$$\operatorname{reg}(H_{r+1}(I)) \leq \max_{0<\beta<n}\{\operatorname{reg}(H_0(I)) + \operatorname{reg}(H_{r+1-\beta}(I)) + \beta + 1\}$$

For all $0 < \beta < n$ we have

$$\begin{aligned}reg(H_{r+1-\beta}(I)) + \beta + 1 &\leq (r - \beta + 2)\operatorname{reg}(H_0(I)) + 2(r + 1 - \beta) + \beta + 1 \\ &\leq (r + 1)\operatorname{reg}(H_0(I)) + 2(r + 1).\end{aligned}$$

Therefore $\operatorname{reg}(H_{r+1}(I)) \leq (r + 2)\operatorname{reg}(H_0(I)) + 2(r + 1)$. □

2.3 Green-Lazarsfeld index of Veronese embedding

Let X be a smooth projective variety with a very ample line bundle L which sets up an embedding into projective space $\mathbb{P}^r$ where $r = h^0(X, L) - 1$. Let $S = \operatorname{Sym} H^0(X, L)$ be the homogeneous coordinate ring of $\mathbb{P}^r$ and if we define $R := \oplus H^0(X, \mathcal{O}(kL))$, then R can be viewed as a finitely generated graded S-module. The syzygies of R as an S-module is investigated by M. Green. Let X be a curve of genus g and let $\mathcal{L}$ be a very ample line bundle on X M. Green proved that if $\deg \mathcal{L} = 2g + p + 1$ then the embedding defined by $\mathcal{L}$ has property N_p. In the case of Veronese embedding of projective spaces $\varphi_c : \mathbb{P}^{n-1} \to \mathbb{P}^N$ M. Green proved that the Veronese subring $S^{(c)} = \bigoplus_{i\in\mathbb{N}} S_{ic}$ satisfies property N_c and then W. Bruns A., Conca and T. Römer extend the lower bound to $c + 1$.

G. Ottaviani and R. Paolletti [72] proved that the Veronese embedding φ for $n \geqslant 2$ and $c \geqslant 3$ does not satisfy property N_{3c-2}. In zero characteristic therefore one can deduce that

$$c + 1 \leqslant \operatorname{index}(S^{(c)}) \leqslant 3c - 3.$$

G. Ottaviani and R. Paolletti showed that if $n = 3$ then $\operatorname{index}(S^{(c)}) = 3c - 3$ and they conjectured that the equality holds for arbitrary $n \geq 3$. Recently, in an unpublished work T. Vu proved the conjecture of G. Ottaviani and R. Paoletti in the case $c = 4$. The following theorem provides an equivalence between the study of the syzygies of Veronese embedding and the study of Koszul homologies of powers of the maximal ideal.

Theorem 2.3.1. *Let $S = k[x_1, \ldots, x_n]$. For $i \in \mathbb{N}$ and $j \in \mathbb{Z}$ we have:*

$$\beta_{i,j}(V_S(c)) = \dim_K H_i(\mathfrak{m}^c, R)_{jc} \tag{2.3.1}$$

Proof. See 4.1 in [14] for the proof. □

Now we use our results on regularity of Koszul cycles and Koszul homologies to find a lower bound for the Green-Lazarsfeld index of Veronese embedding. In this regard, we are able to reproof the statement of M. Green in [48].

Lemma 2.3.2. *Let $S = k[x_1, \ldots, x_n]$ and $V_s(c)$ be the c-th Veronese subring of S, then*

$$\operatorname{reg}(Z_{i+2}(\mathfrak{m}^c)) = \operatorname{reg}(H_{i+1}(\mathfrak{m}^c)) + 2, \text{ for } i \leq \min\{\operatorname{index}(V_S(c)), 2c\}$$

Proof. By Theorem 2.2.3, it suffices to prove that the regularity of Koszul homologies are increasing as a function of i. Since $i \leq \operatorname{index}(V_S(c))$, by Theorem 2.3.1

$$H_i(\mathfrak{m}^c)_{(i+2)c} = 0 \text{ and } \operatorname{reg}(H_{i+1}(\mathfrak{m}^c)) \geq (i + 2)c.$$

As $H_i(\mathfrak{m}^c)_{(i+2)c} = 0$, $\operatorname{reg}(H_i(\mathfrak{m}^c)) < (i+2)c$ if and only if $H_i(\mathfrak{m}^c)$ has no generator of degree greater than $(i+2)c$. Hence it suffices to show that $\operatorname{reg}(Z_i(\mathfrak{m}^c)) \leq (i+2)c$. By Theorem 2.1.5 and Theorem 2.2.3 we have

$$\operatorname{reg}(Z_i(\mathfrak{m}^c)) \leq i \operatorname{reg}(Z_1(\mathfrak{m}^c)) = i(\operatorname{reg}(H_0(\mathfrak{m}^c)) + 2) = i(c+1)$$

Since $i \leq 2c$, then $\operatorname{reg}(Z_i(\mathfrak{m}^c)) \leq (i+2)c$. □

As a consequence of Theorem 2.3.1 we know that $\operatorname{reg}(H_i(\mathfrak{m}^c)) = (i+1)c + r_i$. From the definition of Green-Lazarsfeld index one can see $i \leq \operatorname{index}(V_R(c))$ if and only if $r_i \leq c-1$. In order to find a bound for the index of Veronese embedding we can study the behavior of r_i's. Notice that $r_i \in \mathbb{Z}$ for instance $r_0 = -1$.

Proposition 2.3.3. *With the above notations, for all $i \leq \min\{\operatorname{index}(V_S(c)) + 1, 2c+1\}$ we have $r_{i+1} \leq r_i + 1$.*

Proof. By Theorem 2.1.5 we have a triangle inequality between the regularity of Koszul cycles, in particular for $Z_1(\mathfrak{m}^c)$ and $Z_i(\mathfrak{m}^c)$. By using Corollary 2.3.2 and Theorem 2.2.3

$$\begin{aligned}\operatorname{reg}(H_{i+1}(\mathfrak{m}^c)) + 2 &= \operatorname{reg}(Z_{i+2}(\mathfrak{m}^c)) \\ &\leq \operatorname{reg}(Z_{i+1}(\mathfrak{m}^c)) + \operatorname{reg}(Z_1(\mathfrak{m}^c)) \\ &= \operatorname{reg}(H_i(\mathfrak{m}^c)) + 2 + \operatorname{reg}(H_0(\mathfrak{m}^c)) + 2.\end{aligned}$$

By the above notation we have that $(i+2)c + r_{i+1} \leq (i+1)c + r_i + c + 1$. In particular $r_{i+1} \leq r_{i-1} + 1$. □

Corollary 2.3.4. *The Green-Lazarsfeld index of Veronese embedding $V_S(c)$ is at least c.*

Proof. As we mention above, for finding the Green-Lazarsfeld index of Veronese embedding we should control r_i's. Proposition 2.3.3 shows that in each step, r_i's can be increased only be one. Since $r_0 = -1$ so $r_c \leq c-1$ that means $\operatorname{index}(V_S(c)) \geq c$. □

Chapter 3

Lefschetz properties of monomial ideals

3.1 Background and introduction

In this chapter we study the WLP and SLP of artinian monomial ideals in $S = k[x_1, \ldots, x_n]$ via studying their minimal free resolutions. We study the Lefschetz properties of such ideals where the minimal free resolution of S/I is linear for at least $n-2$ steps. We give an affirmative answer to a conjecture of Eisenbud, Huneke and Ulrich for artinian monomial ideals with almost linear resolutions.

The weak Lefschtez property (WLP) of an artinian graded algebra A, says that there exists a linear form ℓ that induces a multiplication map $\times\ell : A_i \longrightarrow A_{i+1}$ that has maximal rank for each i, i.e. that is either surjective or injective. The strong Lefschetz property (SLP) says the map $\times\ell^t : A_i \longrightarrow A_{i+t}$ has maximal rank for each i and t. It may seem a simple problem to establish the algebras with this properties but it turns out to be rather hard to determine even for natural families of algebras. It is also interesting to ask for which t the map $\times\ell^t : A_i \longrightarrow A_{i+t}$ has maximal rank (see [66]). These fundamental properties have been studied by many authors from different point of views and for different families of algebras. In this chapter we study the Lefschetz properties of artinian monomial ideals generated in a single degree d with assumptions on their minimal free resolutions.

In [37], Eisenbud, Huneke and Ulrich studied the minimal free resolutions of artinian ideals in the polynomial ring $S = k[x_1, \ldots, x_n]$. They proved that for an artinian ideal $I \subset S$

generated in degree d with the minimal free resolution with $p-1$ linear steps, we have that

$$\mathfrak{m}^d \subset I + (l_p + \cdots + l_p),$$

where $l_1, \ldots, l_p$ are linearly independent linear forms and $\mathfrak{m} = (x_1, \ldots, x_n)$ is the maximal ideal of S (see [37, Corollary 5.2]). They also conjectured that under this assumption, we have that $\mathfrak{m}^d \subset I + (l_p + \cdots + l_p)^2$ for sufficiently general linear forms $l_1 \ldots, l_p$ (see [37, Conjecture 5.5]). Note that this conjecture in the case that $p = n-1$ is equivalent to the Lefschetz property of S/I. This conjecture motivated us to study the Lefschetz properties of artinian monomial ideals in $S = k[x_1, \ldots, x_n]$ generated in degree d by considering some assumptions on their minimal free resolutions.

As a corollary of the result of Eisenbud, Huneke and Ulrich [37, Corollary 5.2] we conclude that for artinian ideal $I \subset S$ generated in degree d with almost linear resolution (the minimal free resolution of S/I is linear for $n-1$ steps), S/I satisfies the WLP. In Section 3.2, we study the multiplication map by higher powers of a linear form on an artinian monomial algebra S/I where the minimal free resolution of S/I is linear for $n-1$ steps, see Theorem 3.2.8. In particular, in this case we are able to give an affirmative answer the conjecture posed by Eisenbud, Huneke and Ulrich. In the rest of this section, we prove that an artinian monomial algebra S/I satisfies the SLP, where we consider an assumption on the generators of I, see Theorem 3.2.10. For artinian monomial ideals $I \subset S$ where the minimal free resolution of S/I is linear for $n-2$ steps the WLP does not hold necessarily. In the main theorem of Section 3.3, Theorem 3.3.6, we prove for an artinian monomial ideal $I \subset S$ generated in degree d, S/I satisfies the WLP if the minimal free resolution of S/I has $n-2$ linear steps and $\mathfrak{m}^{d+1} \subset I$. The last assumption is equivalent to say Castelnuovo-Mumford regularity of I, $\operatorname{reg}(S/I)$, is d. Observe that the assumption on the Castelnuovo-Mumford regularity of I is essential. In the polynomial ring with three variables the assumption of having $n-2$ linear steps in the minimal free resolution of S/I is always fulfilled whenever I is generated in a single degree, but the Togliatti system defined by artinian monomial ideal $I = (x_1^3, x_2^3, x_3^3, x_1x_2x_3)$ fails the WLP.

We consider standard graded algebra $S/I = \oplus_{i\geq 0}(S/I)_i$, where $S = k[x_1, \ldots, x_n]$ is a polynomial ring over a field of characteristic zero and all x_i's have degree 1 and $I \subset S$ is an artinian homogeneous ideal generated in a single degree d. Let us define the weak and strong Lefschetz properties for artinian algebras.

Definition 3.1.1. Let $I \subset S$ be an artinian homogeneous ideal. We say that S/I has the *weak Lefschetz property* (WLP) if there is a linear form $\ell \in (S/I)_1$ such that, for all integers j, the multiplication map

$$\times\ell : (S/I)_j \longrightarrow (S/I)_{j+1}$$

has maximal rank, i.e. it is injective or surjective. In this case the linear form ℓ is called a *Lefschetz element* of S/I. If for general form $\ell \in (S/I)_1$ and for an integer number j the map $\times\ell$ does not have the maximal rank we will say that the ideal I fails the WLP in degree j.
We say that S/I has the *strong Lefschetz property* (SLP) if there is a linear form $\ell \in (S/I)_1$ such that, for all integers j and k the multiplication map

$$\times\ell^k : (S/I)_j \longrightarrow (S/I)_{j+k}$$

has the maximal rank, i.e. it is injective or surjective. By abusing the notation we often say that I satisfies or fails the SLP or WLP.

In the case of one variable, the WLP and SLP trivially hold since all ideals are principal. In the case of two variables there is a nice result in characteristic zero by Harima, Migliore, Nagel and Watanabe [53, Proposition 4.4].

Proposition 3.1.2. *Every artinian ideal $I \subset k[x, y]$ where k has characteristic zero, has the Strong Lefschetz property (and consequently also the Weak Lefschetz property).*

In [67, Proposition 2.2], Migliore, Miró-Roig and Nagel by using the action of a torus on monomial algebras provide the existence of the canonical Lefschetz element.

Proposition 3.1.3. *[67, Proposition 2.2] Let $I \subset S = k[x_1, \ldots, x_n]$ be an artinian monomial ideal. Then S/I has the weak Lefschetz property if and only if $x_1+x_2+\cdots+x_n$ is a Lefschetz element for S/I.*

Remark 3.1.4. Let multiplication map $\times\ell^c : (S/I)_{a-c} \to (S/I)_a$ where I is an ideal of S generated in degree d. If $\mathrm{HF}_{S/I}(d) \leq \mathrm{HF}_{S/I}(d-c)$, then $\times\ell^c : (S/I)a - c \to (S/I)_a$ has maximal rank for every a if and only if it is surjective for $a = d$. In fact if the multiplication map $\times\ell^c : (S/I)_{a-c} \to (S/I)_a$ is surjective we have that $[(S/I)/\ell^c(S/I)]_d = 0$ therefore $[(S/I)/\ell^c(S/I)]_k$ for all $k \geq d$ and so $\times\ell^c : (S/I)_{k-c} \to (S/I)_k$ is surjective for each $k \geq d$. On the other hand, since I is generated in degree d, the multiplication map by ℓ^c is injective in the degrees less than $d - c$ hence S/I.

Definition 3.1.5. Let $I \subset S = k[x_1, \ldots, x_n]$ be an ideal of S generated in degree d. We say that the minimal free resolution of S/I is *linear* for r steps if $\beta_{i,i+j}(S/I) = 0$, for all $1 \leq i \leq r$ and all $j \geq d$. We say S/I has *linear resolution* if $r = n$, and it has *almost linear resolution* if $r = n - 1$.

In [37] Eisenbud, Huneke and Ulrich studied artinian ideals in polynomial rings. They prove the following result:

Theorem 3.1.6. *[37, Corollary 5.2] Let $I \subset S$ be an artinian ideal generated in degree d and $\mathfrak{m} = (x_1, \ldots, x_n)$. If the minimal free resolution of I is linear for $p-1$ steps, then*

$$\mathfrak{m}^d \subseteq I + (l_p, \ldots, l_n)$$

for linearly independent linear forms $l_p, l_{p+1}, \ldots, l_n$.

The above result says that in terms of the minimal free resolution with these assumptions we have $\operatorname{reg}(I + (l_p, \ldots, l_n)) \leq d$. They also conjecture that under the same assumptions as Theorem 3.1.6, $\operatorname{reg}(I + (l_p, \ldots, l_n)^2) \leq d$ where $l_p, \ldots, l_n$ are sufficiently general linear forms.

Conjecture 3.1.7. *[37, Conjecture 5.4] Suppose $I \subset S$ is artinian ideal generated in degree d and its minimal free resolution is linear for $p-1$ steps then*

$$\mathfrak{m}^d \subseteq I + (l_p, , l_{p+1}, \ldots, l_n)^2$$

for sufficiently general linear forms $l_p, \ldots, l_n$.

Remark 3.1.8. Note that Theorem 3.1.6 holds for any set of linearly independent linear forms $l_p, l_{p+1} \ldots, l_n$ but this is not the case necessarily for Conjecture 3.1.7. For instance let $S = k[x, y, z]$ and $I = (x^3, y^3, z^3, xy^2, x^2y, xz^2, x^2z, y^2z, yz^2)$. The minimal free resolution of I is as follows:

$$0 \to S(-5)^3 \oplus S(-6) \to S(-4)^{12} \to S(-3)^9 \to S \to 0$$

and I has almost linear resolution. By Theorem 3.1.6 we have

$$\mathfrak{m}^3 \subseteq I + (x).$$

The statement of Conjecture 3.1.7 does not hold for the linear form $l = x$ since we have that $xyz \notin I + (x^2)$, $\mathfrak{m}^3 \nsubseteq I + (x)^2$. But if $l = x - y$, one can check that $\mathfrak{m}^3 \subseteq I + (l)^2$ and the Conjecture 3.1.7 holds in this case. Thus we need to consider sufficiently general linear forms in the conjecture.

As a consequence of Theorem 3.1.6 we have:

Corollary 3.1.9. *If $I \subset S$ is an artinian ideal generated in degree d with almost linear resolution, then S/I satisfies the WLP.*

Proof. Since the minimal free resolution of I is linear for $n-1$ steps, Theorem 3.1.6 implies that for a general linear form ℓ, we have that $\mathfrak{m}^d \subset I + (\ell)$. This is equivalent to have the surjective map $\times \ell : (S/I)_{d-1} \to (S/I)_d$ and the assertion follows from Remark 3.1.4. □

Macaulay inverse systems

Let us now recall some facts of the theory of the *inverse system*, or *Macaulay duality*, which will be a fundamental tool in this paper. For a complete introduction, we refer the reader to [43] and [58].

Let $R = k[y_1, \dots, y_n]$, and consider R as a graded S-module where the action of x_i on R is partial differentiation with respect to y_i.
There is a one-to-one correspondence between graded artinian algebras S/I and finitely generated graded S-submodules M of R, where $I = \mathrm{Ann}_S(M)$ is the annihilator of M in S and, conversely, $M = I^{-1}$ is the S-submodule of R which is annihilated by I (cf. [43, Remark 1]), p.17). By duality, the map $\circ\ell : R_{i+1} \longrightarrow R_i$ is dual to the map $\times\ell : (S/I)_i \longrightarrow (S/I)_{i+1}$. So the injectivity(resp. surjectivity) of the first map is equivalent to the surjectivity (resp. injectivity) of the second one. Here by " $\circ\,\ell$" we mean the linear form ℓ acts on R.

The inverse system module I^{-1} of an ideal I is generated by monomials in R if and only if I is a monomial ideal in S.

3.2 Lefschetz properties of monomial ideals with $n-1$ linear steps

The goal of this section is to give an affirmative answer to the Conjecture 3.1.7 in the case of monomial ideals with almost linear resolution.
Let $I \subset S$ be an artinian monomial ideal generated in degree d, in the following proposition we provide an upper bound for the Hilbert function $H_{S/I}(d) := \dim_k(S/I)_d$ in terms of the number of linear steps in the minimal free resolution of S/I.

Proposition 3.2.1. *Let $I \subset S = k[x_1, \dots, x_n]$ be an artinian monomial ideal generated in degree d. If the minimal free resolution of I is linear for r steps, then for every monomial $m \in Mon(S/I)_d$ we have $|\,\mathrm{Supp}(m)| \geq r+1$. In particular,*

$$\mathrm{HF}_{S/I}(d) \leq \binom{n}{r+1} \mathrm{HF}_{S/I}(d-r-1).$$

Proof. Let

$$0 \to F_n \xrightarrow{\varphi_n} F_{n-1} \xrightarrow{\varphi_{n-1}} \cdots F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} 0$$

be the minimal free resolution of I^{-1} which is dual to the minimal free resolution of S/I. By Macaulay duality we can consider m as an element in $(I^{-1})_d$ there exists a generator

$m' \in I^{-1}$ such that $m = h \circ m'$ for some $h \in S$. Suppose $|\operatorname{Supp}(m)| \leq r$ and variables $y_1, \ldots, y_{n-r} \notin \operatorname{Supp}(m)$. If $\varphi_1(e_1) = m'$ for a basis element e_1 of F_1, then $L_1 := (x_1 h) \circ e_1$ is a first syzygy of I^{-1}. Therefore, it corresponds to a basis element of F_2, say e_2. Observe that $L_2 := x_2 \circ e_2$ is a second syzygy and it corresponds to a basis element of F_3. By continuing this procedure $n-r$ times, we find a basis element for F_{n-r} of degree higher than $d-n-r$. Using the duality of the minimal free resolution of I^{-1} and S/I, we get $\beta_{r,d+r}(S/I) \neq 0$ which contradicts the fact that the minimal free resolution of S/I is linear for r steps. $\square$

Remark 3.2.2. In [37, Proposition 11.1] Eisenbud, Huneke and Ulrich find a lower bound for the number of generators of an ideal with almost linear resolution where the bound implies that $\mathrm{HF}_{S/I}(d) \leq \mathrm{HF}_{S/I}(d-2)$. By Proposition 3.2.1 for a monomial ideal I with almost linear resolution, $\mathrm{HF}_{S/I}(d) \leq \mathrm{HF}_{S/I}(d-n)$.

Let us define a specific class of well-known matrices with non-negative integer entries:

Definition 3.2.3. For integers n, m, k where $m \geq 1$, we define the following Toeplitz matrix $T_{n,m,k}$ as the $m \times m$ matrix

$$T_{n,m,k} := \begin{bmatrix} \binom{n}{k} & \binom{n}{k+1} & \binom{n}{k+2} & \cdots & \binom{n}{n} & 0 & \cdots & 0 \\ \binom{n}{k-1} & \binom{n}{k} & \binom{n}{k+1} & \cdots & \binom{n}{n-1} & \binom{n}{n} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \binom{n}{k-3} & \binom{n}{k-2} & \binom{n}{k-1} & \binom{n}{k} \end{bmatrix}$$

where the $(i,j)^{th}$ entry of this matrix is $\binom{n}{k+j-i}$ and we use the convention that $\binom{n}{i} = 0$ for $i \leq 0$ and $i > n$.

Determining the rank of such matrices is an open problem even in many specific cases. Here using the fact that any monomial algebra in the polynomial ring with two variables has the SLP we show that Toeplitz matrix $T_{n,m,k}$ has maximal rank.

Lemma 3.2.4. *For every integers $0 \leq k \leq n$ and $m \geq 1$, Toeplitz matrix $T_{n,m,k}$ is invertible.*

Proof. Consider ideal $I = (x^{m+n-k}, y^{m+k+1})$ in the polynomial ring $S = k[x,y]$. Choose monomial bases $\{x^{m-i}y^{i-1}\}_{i=1}^{m}$ and $\{x^{m+n-k-j}y^{k+j-1}\}_{j=1}^{m}$ for k-vector spaces $(S/I)_{m-1}$ and $(S/I)_{m+n-1}$, respectively. Observe that, $T_{n,m,k}$ is the matrix representing the multiplication map $\times(x+y)^n : (S/I)_{m-1} \to (S/I)_{m+n-1}$ with respect to the chosen monomial bases. Since

by Proposition 3.1.2, any monomial R-algebra has the SLP, and by Proposition 3.1.3, $x+y$ is a Lefschetz element for S/I, the multiplication map by $x+y$ is bijection and therefore Toeplitz matrix $T_{n,m,k}$ has nonzero determinant and therefore it is invertible. □

Remark 3.2.5. In [3] there is a more general result about this case of Toeplitz matrices using the same technique as the proof of Lemma 3.2.4.

Definition 3.2.6. Let $M = \{m_1, \dots, m_r\}$ be a set of monomials in $S = k[x_1, \dots, x_n]$ of degree d. We say M is a *line segment* with respect to (x_i, x_j) if

(1) $x_i x_j | m_t, \ \forall \ 1 \leq t \leq r,$

(2) $(x_j/x_i)m_t = m_{t+1}, \ \forall \ 1 \leq t \leq r-1.$

In addition, for a monomial ideal $I \subset S$ generated in degree d we say M is a *S/I-maximal line segment* with respect to (x_i, x_j) if in addition we have $(x_i/x_j)m_1, (x_j/x_i)m_r \in I$.

Lemma 3.2.7. *Let $M = \{m_1, \dots, m_r\}$ be a set of monomials of degree d in $S = k[x_1, \dots, x_n]$ which form a line segment w.r.t. (x_i, x_j) and let J_M be the ideal generated by all the monomials in $S_d \setminus M$. If $x_i^a x_j^b | m_t$, for every $1 \leq t \leq r$, then multiplication map $\times(x_i + x_j)^{a+b} : (S/J_M)_{k-a-b} \to (S/J_M)_k$ has maximal rank for every k.*

Proof. Without loss of generality assume $i = 1$ and $j = 2$. Since $H_{S/J_M}(d) \leq H_{S/J_M}(d-a-b)$ and J_M is generated in degree d, by Remark 3.1.4 it is suffices to show that the map $\times(x_1 + x_2)^{a+b} : (S/J_M)_{d-a-b} \to (S/J_M)_d$ is surjective. Set $f_i := m_i/x_1^a x_2^b$ and φ the restriction of multiplication map $\times(x_1 + x_2)^{a+b} : (S/J_M)_{d-a-b} \to (S/J_M)_d$ to $f_1, \dots, f_r$. Observe that the Toeplitz matrix $T_{a+b,r,a}$ is the matrix representing φ. By Lemma 3.2.4 this matrix is invertible so we can find preimage of each m_i which means $\times(x_1 + x_2)^{a+b} : (S/J_M)_{d-a-b} \to (S/J_M)_d$ is surjective. □

Using Lemma 3.2.4 and Lemma 3.2.7 we prove that the multiplication map by a power of a linear form on an specific class of artinian monomial algebra has maximal rank in every degree.

Theorem 3.2.8. *Let $I \subset S = k[x_1, \dots, x_n]$ be an artinian monomial ideal generated in degree d. If there exist integers $1 \leq i < j \leq n$ such that for every monomial $m \in (S/I)_d$, $x_i^a x_j^b | m$ for some $a, b \geq 0$, then the multiplication map $\times(x_i + x_j)^{a+b} : (S/I)_{k-a-b} \to (S/I)_k$ has maximal rank for every k.*

Proof. Note that by Remark 3.1.4 it is enough to show that $\times(x_i + x_j)^{a+b} : (S/I)_{d-a-b} \to (S/I)_d$ is surjective. Without loss of generality we assume that $i = 1$ and $j = 2$. For an $n-2$-tuple $\mathbf{a} := (a_3, \dots a_n) \in (\mathbb{N} \cup \{0\})^{n-2}$ define

$$\mathcal{M}_{\mathbf{a}} := \{x_1^{a_1} x_2^{a_2} x_3^{a_3} \dots x_n^{a_n} \in (S/I)_d \mid a_1, a_2 \geq 0\}.$$

We will show that $\mathcal{M}_{\mathbf{a}}$ is in the image of $\times(x_1 + x_2)^{a+b} : (S/I)_{d-a-b} \to (S/I)_d$ for every $\mathbf{a}$. For a fixed $n-2$-tuple $\mathbf{a}$, $\mathcal{M}_{\mathbf{a}}$ may contains different S/I-maximal line segments w.r.t (x_1, x_2) by Lemma 3.2.7 each of them is in the image. By the procedure in the proof of Lemma 3.2.7, the preimages of the elements in $\mathcal{M}_{\mathfrak{a}}$ are all distinct and this completes the proof.

□

As a consequence of the above result and Proposition 3.2.1 we prove Conjecture 3.1.7 holds for monomial ideals $I \subset S$ with almost linear resolution.

Theorem 3.2.9. *Let $I \subset S = k[x_1, \dots, x_n]$ be an artinian monomial ideal generated in degree d with almost linear resolution, then Conjecture 3.1.7 holds.*

Proof. Proposition 3.2.1 implies that for all $m \in Mon(S/I)_d$ we have $|\operatorname{Supp}(m)| \geq n$ therefore $x_1 \cdots x_n | m$. By Theorem 3.2.8 the multiplication map

$$\times(x_i + x_j)^2 : (S/I)_{d-2} \to (S/I)_d$$

is surjective for every $1 \leq i < j \leq n$. This implies that $\mathfrak{m}^d \subset I + (x_i + x_j)^2$ for every $1 \leq i < j \leq n$. □

Now in the last theorem of this section we prove that for a class of artinian monomial ideals the SLP is satisfied.

Theorem 3.2.10. *Let $I \subset S = k[x_1, \dots, x_n]$ be an artinian monomial ideal generated in degree d. If there exist integers $1 \leq i < j \leq n$ such that for every monomial $m \in S_d$, $x_i x_j | m$ is equivalent to $m \notin I$. Then S/I enjoys the SLP.*

Proof. If $n = 2$, by Proposition 3.1.2 every artinian ideal I has the SLP. Let $n \geq 3$. Without loss of generality, assume $i = 1$ and $j = 2$. Consider bigrading $\deg(x_1) = \deg(x_2) = (1, 0)$ and $\deg(x_i) = (0, 1)$ for $3 \leq i \leq n$ on S. By the assumption, if $b \geq d$ we have $(S/I)_{(*,b)} \cong 0$. For every $b < d$ module $(S/I)_{(*,b)}$ is isomorphic to some copies of $(k[x_1, x_2]/(x_1^{d-b}, x_2^{d-b}))_*$. Since every artinian algebra in two variables has the SLP, for all a, b, c multiplication map $\times \ell^c : (S/I)_{(a-c,b)} \to (S/I)_{(a,b)}$ has maximal rank for a generic linear form ℓ.

For completing the proof it is sufficient to show that if $a+b=a'+b'$ then $\times\ell^c : (S/I)_{(a-c,b)} \to (S/I)_{(a,b)}$ is injective (respectively, surjective) if and only if $\times\ell^c : (S/I)_{(a'-c,b')} \to (S/I)_{(a',b')}$ is injective (respectively, surjective). Since $(S/I)_{(*,b)}$ is a complete intersection artinian algebra, its Hilbert function (as a sequence) is symmetric and the maximum value obtained in the bidegree $(d-b-1,b)$. Now we have equivalent conditions:

$$\begin{aligned}
&\times \ell^c : (S/I)_{(a-c,b)} \to (S/I)_{(a,b)} \quad \text{is injective}\\
\Leftrightarrow &|(d-b-1)-(a-c)| \geq |(d-b-1)-a|\\
\Leftrightarrow &|(d-b'-1)-(a'-c)| \geq |(d-b'-1)-a'|\\
\Leftrightarrow &\times \ell^c : (S/I)_{(a'-c,b')} \to (S/I)_{(a',b')} \quad \text{is injective.}
\end{aligned}$$

Similar argument works for surjectivity. □

We end this section by stating a conjecture that we have observed experimentally in a large number of cases using Macaulay2 software [47].

Conjecture 3.2.11. *Let $I \subset S = k[x_1,\dots,x_n]$ and $I \subset S$ be an artinian monomial ideal generated in degree d. If for every monomial $m \in (S/I)_d$ we have $x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}|m$, then the multiplication map $\times(\ell)^a : (S/I)_{k-a} \to (S/I)_k$ has maximal rank for every k, where $\ell = x_1+\cdots+x_n$ and $a = a_1+\cdots+a_n$.*

If Conjecture 3.2.11 is true by combining with 3.2.1 we get that if I is a monomial ideal generated in a single degree d with almost linear resolution then $\mathfrak{m}^d \subset I+(\ell)^n$ for a sufficiently general linear form ℓ.

3.3 Lefschetz properties via studying Macaulay inverse systems

In this section we study the inverse system module I^{-1} for monomial ideals in S generated in degree d and prove some results about the number of generators of I satisfying the WLP. Also we study the artinian monomial ideals $I \subset S = k[x_1,\dots,x_n]$ generated in degree d where the minimal free resolution of S/I is linear for $n-2$ steps and we prove that if $\operatorname{reg}(S/I) = d$ then S/I satisfies the WLP.

Definition 3.3.1. In a polynomial ring $S = k[x_1,\dots,x_n]$, for any monomial m and variable x_i, define

$$\deg_i(m) := max\{e \mid x_i^e|m\}.$$

Proposition 3.3.2. *Let $I \subset S = k[x_1, \dots, x_n]$ be a monomial ideal of S generated in a single degree d and homogeneous form $F = \sum_{m \in (I^{-1})_d} a_m m \in (I^{-1})_d$ such that $(x_1 + \cdots + x_n) \circ F = 0$. If $a_m \neq 0$ and $y_i | m$, then for all $0 \leq j < \deg_i(m)$ there exists a monomial $m_{i,j} \in (I^{-1})_d$ with $\deg_i(m_{i,j}) = j$ such that $a_{m_{i,j}} \neq 0$.*

Proof. Let $m = y_1^{b_1} \cdots y_n^{b_n} \in (I^{-1})_d$ and $a_m \neq 0$,

$$(x_1 + \cdots + x_n) \circ m = a_m \sum_{y_i | m} b_i \frac{m}{y_i}.$$

Since $(x_1 + \cdots + x_n) \circ F = 0$ for each $1 \leq i \leq n$ where $y_i | m$ there exists a monomial $m' \in (I^{-1})_d$ with nonzero coefficient in F such that $\frac{m}{y_i} = \frac{m'}{y_k}$ for some $k \neq i$. Note that $\deg_i(m) = \deg_i(m') + 1 = b_i + 1$ and define $m_{i,b_i-1} := m'$. If $b_i - 1 \neq 0$ then we can do the same and find m_{i,b_i-2} in the support of F. The assertion follows by continuing this procedure to find distinct monomials $m_{i,b_i-3}, \dots, m_{i,0}$ in the support of F. □

Corollary 3.3.3. *Let $I \subset S$ be a monomial ideal generated in degree d. If the multiplication map $\times \ell : (S/I)_{d-1} \to (S/I)_d$ fails to be surjective for every linear form l, then $\mathrm{HF}_{S/I}(d) \geq d + 1$. In other word, if $\mathrm{HF}_{S/I}(d) \leq d$ then S/I enjoys the WLP.*

Proof. Suppose the multiplication map $\times(x_1 + \cdots + x_n) : (S/I)_{d-1} \to (S/I)_d$ is not surjective so by Macaulay duality there exists a non-zero form $F = \sum_{m \in (I^{-1})_d} a_m m \in (I^{-1})_d$ such that $(x_1 + \cdots + x_n) \circ F = 0$. Let $m = y_1^{b_1} \cdots y_n^{b_n} \in (I^{-1})_d$ be a monomial in the support of the form F, using Proposition 3.3.2, there are at least $\deg_1(m) + \cdots + \deg_n(m) = b_1 + \cdots + b_n = d$ monomials different from m with nonzero coefficients in F. Therefore we have that $\mathrm{HF}_{S/I}(d) \geq d + 1$. □

Remark 3.3.4. In [3] N. Altafi and M. Boij provide a better bound for $H_{S/I}(d)$ when I is an artinian monomial ideal in S generated in degree d and fails the WLP.

Definition 3.3.5. Let $I \subset S$ be an ideal, the *socle* elements of S/I is

$$\mathrm{soc}(S/I) = \{f \in S/I \mid \mathfrak{m} f = 0\}.$$

If $I \subset S$ is an artinian ideal with linear resolution it equals a power of maximal ideal and therefore it satisfies the WLP trivially. On the other hand in Corollary 3.1.9 we have seen that an artinian ideal $I \subset S$ with almost linear resolution satisfies the WLP. In the following result we determine whether an artinian monomial ideal $I \subset S$ has the WLP where the minimal free resolution of S/I is linear for $n - 2$.

Theorem 3.3.6. *Let $I \subset S = k[x_1, \dots, x_n]$ be a monomial ideal generated in degree d and $\mathfrak{m}^{d+1} \subset I$. If the minimal free resolution of S/I is linear for $n-2$ linear steps, then S/I satisfies the WLP.*

Proof. Since I has linear resolution for $n-2$ steps by Proposition 3.2.1, for all $m \in \operatorname{soc}(S/I)$ we have $|\operatorname{Supp}(m)| \geq n-1$. If for all $m \in \operatorname{soc}(S/I)$ we have $|\operatorname{Supp}(m)| = n$ then clearly we have the WLP. Suppose that there exists $m \in \operatorname{soc}(S/I)$ such that $|\operatorname{Supp}(m)| = n-1$. Since I is generated in degree d, to prove that S/I has the WLP it is enough to show that the multiplication map

$$\times(x_1 + \cdots + x_n) : (S/I)_{d-1} \to (S/I)_d$$

is surjective, or equivalently the differentiation map

$$\circ(x_1 + \cdots + x_n) : (I^{-1})_d \to (I^{-1})_{d-1}$$

is injective. Suppose not, so there exists a non-zero form $F = \sum_{m \in I^{-1}} a_m m$ such that

$$(x_1 + \cdots + x_n) \circ F = 0. \tag{3.3.1}$$

Observe that, there exists a monomial $m \in I^{-1}$ with non-zero coefficient in F such that $|\operatorname{Supp}(m)| = n-1$. Let $m = y_1^{b_1}, \dots, y_{n-1}^{b_{n-1}}$ hence

$$(x_1 + \cdots + x_n) \circ m = b_1 \frac{m}{y_1} + \cdots + b_{n-1} \frac{m}{y_{n-1}}.$$

If Equation (3.3.1) holds there must exist $m_1 \in I^{-1}$ and integer $1 \leq i \leq n$ such that $\dfrac{m_1}{y_i} = \dfrac{m}{y_1}$. Suppose $y_n \nmid m_1$ then $|\operatorname{Supp}(m_1)| = |\operatorname{Supp}(m)| = n-1$. Let

$$F_1 := (b_i + 1)x_1 \circ m - (b_1)x_i \circ m_1, \quad F_2 := x_n \circ m \ \text{ and } \ F_3 := x_n \circ m_1$$

are the linear first syzygies for the inverse system module. In addition,

$$x_n \circ F_1 - (b_i + 1)x_1 \circ F_2 + (b_1)x_i \circ F_3$$

is a linear second syzygy for the inverse system module which contradicts the fact that the minimal free resolution of I is linear for $n-2$ steps. Therefore, $y_n \mid m_1$ and since $y_n \nmid m$ we conclude that $i = n$.

By duality, if for $m \in \operatorname{soc}(S/I)$ we have $x_n \nmid m$, then there exist monomials $m_1, \dots, m_{n-1} \in \operatorname{soc}(S/I)$ such that $\frac{m_i}{x_n} = \frac{m}{x_i}$ for all $1 \leq i \leq n-1$. This implies that $x_n m \in \mathfrak{m}^{d+1}$ but $x_n m \notin I$ which contradicts the assumption that $\mathfrak{m}^{d+1} \subset I$. $\square$

Next example illustrates that the assumption, $\mathfrak{m}^{d+1} \subset I$ in Theorem 3.3.6 is necessary.

Example 3.3.7. The artinian monomial ideal $I = (x_1^3, x_2^3, x_3^3, x_1x_2x_3)$ in $S = k[x_1, x_2, x_3]$ defines a *Togliatti system* and therefore fails the WLP. Note that the minimal free resolution of S/I is linear for 1 step but $\mathfrak{m}^4 \nsubseteq I$.

Chapter 4

The (ir)regularity of Tor and Ext

4.1 Introduction

There has been a keen interest in understanding the behavior of $\operatorname{reg}(I^n)$ as a function of n, where I is a homogeneous ideal in a polynomial ring $S = K[x_1, \ldots, x_d]$ over a field. Geramita, Gimigliano and Pitteloud [44] and Chandler [18] proved that if $\dim(S/I) \leqslant 1$, then $\operatorname{reg}(I^n) \leqslant n \cdot \operatorname{reg}(I)$ for all $n \geqslant 1$. This bound need not be true for higher dimension, due to an example of Sturmfels [76]. However, in [77, Thm. 3.6], Swanson showed that $\operatorname{reg}(I^n) \leqslant cn$ for all $n \geqslant 1$, where c is some constant. Thereafter, Cutkosky, Herzog and Trung [30, 1.1] and Kodiyalam [63] independently showed that asymptotically $\operatorname{reg}(I^n)$ is a linear function of n. Later, in [78, 3.2], Trung and Wang generalized this result over Noetherian standard graded ring. This behavior also has been studied for powers of more than one ideals in [6], [45] and [13].

One notices that $\operatorname{Tor}_1^S(S/I^p, S/I^q) = I^p/I^{p+q}$ if $p \geq q$, which relates this question to more general results for finitely generated graded S-modules M and N. The following results are known in this case

(1) [37, Cor. 3.1] and [19, Thm. 5.7] If $\dim(\operatorname{Tor}_1^S(M, N)) \leqslant 1$, then

$$\max_{0 \leqslant i \leqslant d} \{\operatorname{reg}\left(\operatorname{Tor}_i^S(M, N)\right) - i\} = \operatorname{reg}(M) + \operatorname{reg}(N).$$

This generalizes results of Sidman [75], Conca-Herzog [28], and Caviglia [17] and Eisenbud-Huneke-Ulrich. The equality in (1) extends to the case when S is standard graded, and M or N has finite projective dimension, replacing the right hand side by $\operatorname{reg}(M) + \operatorname{reg}(N) - \operatorname{reg}(S)$.

(2) [21, 3.2 and 4.6]

$$\min_{0\leqslant i\leqslant d}\{\operatorname{indeg}(\operatorname{Ext}^i_S(M,N))+i\}=\operatorname{indeg}(N)-\operatorname{reg}(M)$$

and if $\dim(M\otimes_S N)\leqslant 1$, then

$$\max_{0\leqslant i\leqslant d}\left\{\operatorname{reg}\left(\operatorname{Ext}^i_S(M,N)\right)+i\right\}=\operatorname{reg}(N)-\operatorname{indeg}(M),$$

where $\operatorname{indeg}(W):=\inf\{n\in\mathbb{Z}:W_n\neq 0\}$.

(3) [23, Thm. 2.4(2) and 3.5] An upper bound of $\operatorname{reg}(\operatorname{Ext}^i_S(M,N))+i$ is given in terms of certain invariants of M and N.

When working over standard graded algebras that are not regular (*i. e.* not a polynomial ring over a regular ring), one can also bound regulariy of Tor modules under the same kind of hypothesis, for instance

Theorem 4.1.1. *[19, Thm. 5.7] Suppose Q is a standard graded ring over a field, but Q is not a polynomial ring. Let M and N be finitely generated graded Q-modules, and $d:=\min\{\dim(M),\dim(N)\}$. If $\dim\left(\operatorname{Tor}^Q_i(M,N)\right)\leqslant 1$ for all $i\geqslant i_0$, then*

$$\operatorname{reg}\left(\operatorname{Tor}^Q_i(M,N)\right)-i\leqslant\operatorname{reg}(M)+\operatorname{reg}(N)+\left\lfloor\frac{i+d}{2}\right\rfloor(\operatorname{reg}(Q)-1),\ \forall i\geqslant i_0-1.$$

This implies that if $\operatorname{Proj}(Q)$ has isolated singularities, then the estimate in Theorem 4.1.1 holds true for $i\geqslant\dim(Q)-2$.

Over complete intersection ring, the following result controls the asymptotic behavior with respect to both a power of an ideal and the homological degree,

Theorem 4.1.2. *[46, Thm. 5.4] Set $A:=Q/(\mathbf{f})$, where Q is a polynomial ring over a field, and $\mathbf{f}=f_1,\dots,f_c$ is a homogeneous Q-regular sequence. Let M and N be finitely generated graded A-modules, and I be a homogeneous ideal of A. Then,*

(i) $\operatorname{reg}\left(\operatorname{Ext}^i_A(M,I^nN)\right)\leqslant\rho_N(I)\cdot n-w\cdot\left\lfloor\frac{i}{2}\right\rfloor+e,\quad\forall i,n\geqslant 0,$

(ii) $\operatorname{reg}\left(\operatorname{Ext}^i_A(M,N/I^nN)\right)\leqslant\rho_N(I)\cdot n-w\cdot\left\lfloor\frac{i}{2}\right\rfloor+e',\quad\forall i,n\geqslant 0,$

where $e,e'\in\mathbb{Z}$, $w:=\min\{\deg(f_j):1\leqslant j\leqslant c\}$, and $\rho_N(I)$ is an invariant defined in terms of reduction ideals of I with respect to N.

Moreover, in [46], the authors raised the following question.

Question 4.1.3. *For $\ell \in \{0,1\}$, do there exist $a_\ell, a'_\ell \in \mathbb{Z}_{>0}$ and $e_\ell, e'_\ell \in \mathbb{Z} \cup \{-\infty\}$ such that*

(i) $\operatorname{reg}\left(\operatorname{Ext}_A^{2i+\ell}(M,N)\right) = -a_\ell \cdot i + e_\ell$ *for all* $i \gg 0$ *?*

(ii) $\operatorname{reg}\left(\operatorname{Tor}_{2i+\ell}^A(M,N)\right) = a'_\ell \cdot i + e'_\ell$ *for all* $i \gg 0$ *?*

In this text we are addressing this question. We prove that the answer to (i) is positive, even in a more general situation, while the answer to (ii) is negative. However, if $\dim\left(\operatorname{Tor}_i^A(M,N)\right) \leqslant 1$ for all $i \gg 0$, the second question does have a positive answer.

Our main positive result on these questions is the following,

Theorem 4.1.4. *Let Q be a standard graded Noetherian algebra, $A := Q/(\mathbf{f})$, where $\mathbf{f} := f_1, \ldots, f_c$ is a homogeneous Q-regular sequence. Let M and N be finitely generated graded A-modules such that $\operatorname{Ext}_Q^i(M,N) = 0$ for all $i \gg 0$.*

Then,

(i) *for $\ell \in \{0,1\}$, there exist $a_\ell \in \{\deg(f_j) : 1 \leqslant j \leqslant c\}$ and $e_\ell \in \mathbb{Z} \cup \{-\infty\}$ such that*

$$\operatorname{reg}\left(\operatorname{Ext}_A^{2i+\ell}(M,N)\right) = -a_\ell \cdot i + e_\ell \quad \text{for all } i \gg 0.$$

(ii) *if further Q is *local or the epimorphic image a Gorenstein ring, M has finite projective dimension over Q and*

$$\dim\left(\operatorname{Tor}_i^A(M,N)\right) \leqslant 1, \ \forall i \gg 0,$$

then, for $\ell \in \{0,1\}$, there exist $a'_\ell \in \{\deg(f_j) : 1 \leqslant j \leqslant c\}$ and $e'_\ell \in \mathbb{Z} \cup \{-\infty\}$ such that

$$\operatorname{reg}\left(\operatorname{Tor}_{2i+\ell}^A(M,N)\right) = a'_\ell \cdot i + e'_\ell, \ \forall i \gg 0.$$

On the negative side, we provide exemples showing that the behavior of the regularity of Tor modules could be very different without the assumptions as in the result above.

Example 4.1.5. Let $Q := K[Y,Z,V,W]$ be a polynomial ring with usual grading over a field K and $A := Q/(Y^2, Z^2)$. Write $A = K[y,z,v,w]$, where y, z, v and w are the residue classes of Y, Z, V and W respectively. Fix an integer $m \geqslant 1$. Set

$$M := \operatorname{Coker}\left(\begin{bmatrix} y & z & 0 & 0 \\ -v^m & -w^m & y & z \end{bmatrix} : \begin{matrix} A(-m)^2 \\ \bigoplus \\ A(-1)^2 \end{matrix} \longrightarrow \begin{matrix} A(-m+1) \\ \bigoplus \\ A \end{matrix} \right)$$

and $N := A/(y,z)$. Then, for every $i \geqslant 1$, we have

(i) $\operatorname{indeg}\left(\operatorname{Ext}_A^i(M,N)\right) = -i - m + 1$ and $\operatorname{reg}\left(\operatorname{Ext}_A^i(M,N)\right) = -i$.

(ii) $\operatorname{indeg}\left(\operatorname{Tor}_i^A(M,N)\right) = i$ and $\operatorname{reg}\left(\operatorname{Tor}_i^A(M,N)\right) = (m+1)i + (2m-2)$.

In this example, $\operatorname{Tor}_i^A(M,N)$ is supported in dimension 2 for $i \gg 0$, its regularity is eventually linear, but the leading term depends on the module M and could be arbitrary big, opposite to the case where $\operatorname{Tor}_i^A(M,N)$ is supported in dimension 1 for $i \gg 0$ – in that case we showed the leading term would then be $\frac{2}{2} = 1$, as compared to $m+1$ here.

This shows that the finiteness result for the Tor-algebra that we prove under the condition that $\operatorname{Tor}_i^A(M,N)$ is supported in dimension 1 for $i \gg 0$ can fail if this hypothesis is removed. Additional results around the hypothesis on the asymptotic dimension of Tor are given in Remark 4.3.8 and in Proposition 4.3.9.

The following example that we develop in the last section shows that the eventual regularity of Tor could be very far from being linear,

Example 4.1.6. Let $Q := K[X,Y,Z,U,V,W]$ be a standard graded polynomial ring over a field K of characteristic 2 and $A := Q/(X^2,Y^2,Z^2)$. We write $A = K[x,y,z,u,v,w]$, where x,y,z,u,v and w are the residue classes of X,Y,Z,U,V and W respectively. Set

$$M := \operatorname{Coker}\left(\begin{bmatrix} x & y & z & 0 & 0 & 0 \\ u & v & w & x & y & z \end{bmatrix} : A(-1)^6 \longrightarrow A^2\right) \quad \text{and} \quad N := A/(x,y,z).$$

Then, for every $n \geqslant 1$, we have

(i) $\operatorname{indeg}\left(\operatorname{Ext}_A^n(M,N)\right) = \operatorname{reg}\left(\operatorname{Ext}_A^n(M,N)\right) = -n$.

(ii) $\operatorname{indeg}\left(\operatorname{Tor}_n^A(M,N)\right) = n$ and $\operatorname{reg}\left(\operatorname{Tor}_n^A(M,N)\right) = n + f(n)$, where

$$f(n) := \begin{cases} 2^{l+1} - 2 & \text{if } n = 2^l - 1 \\ 2^{l+1} - 1 & \text{if } 2^l \leqslant n \leqslant 2^{l+1} - 2 \end{cases} \qquad \text{for all integers } l \geqslant 1.$$

As a consequence, in this example,

$$\{\operatorname{reg}(\operatorname{Tor}_{2n}^A(M,N))/2n : n \geqslant 1\} \quad \text{and} \quad \{\operatorname{reg}(\operatorname{Tor}_{2n+1}^A(M,N))/2n+1 : n \geqslant 1\}$$

are dense sets in $[2,3]$ and

$$\liminf_{n\to\infty} \frac{\operatorname{reg}(\operatorname{Tor}_n^A(M,N))}{n} = 2 \quad \text{and} \quad \limsup_{n\to\infty} \frac{\operatorname{reg}(\operatorname{Tor}_n^A(M,N))}{n} = 3.$$

4.2 Module structures on Ext and Tor

Most of our results are proved under the following hypothesis.

Hypothesis 4.2.1. *The ring Q is a standard graded Noetherian algebra, $A = Q/(\mathbf{f})$, where $\mathbf{f} := f_1, \dots, f_c$ is a homogeneous Q-regular sequence with $w_j = \deg(f_j)$, and M, N are finitely generated graded A-modules such that $\operatorname{Ext}^i_Q(M, N) = 0$ for all $i \gg 0$.*

4.2.2. *Write $A = A_0[x_1, \dots, x_d]$, where $\deg(x_i) = 1$ for $1 \leqslant i \leqslant d$. When A_0 is local, then following the terminologies in [16, pp. 141], A is *local, i.e., it has a unique maximal homogeneous ideal $\mathfrak{m} = \mathfrak{m}_0 + A_+$. Setting $E_0 := E_{A_0}(A_0/\mathfrak{m}_0)$, the Matlis dual of M is defined to be $M^\vee := {}^*\operatorname{Hom}_{A_0}(M, E_0)$, where $(M^\vee)_n = \operatorname{Hom}_{A_0}(M_{-n}, E_0)$ for every $n \in \mathbb{Z}$. In view of [16, 3.6.16 and 3.6.17], the contravariant functor $(-)^\vee$ from the category of finitely generated graded A-modules to itself is exact, and $M^{\vee\vee} \cong M$.*

4.2.3 (Eisenbud operators). *We need to remind facts about Eisenbud operators [31, Section 1] in the graded setting. By a homogeneous homomorphism, we mean a graded homomorphism of degree zero. Let $\mathbb{F} : \cdots \to F_n \to \cdots \to F_1 \to F_0 \to 0$ be a graded free resolution of M over A. In view of the construction of Eisenbud operators [31, p. 39, (b)], one may choose homogeneous A-module homomorphisms $t'_j : F_{i+2} \to F_i(-w_j)$ (for every i) corresponding to f_j.*

Thus the Eisenbud operators corresponding to $\mathbf{f} = f_1, \dots, f_c$ are given by $t'_j : \mathbb{F}[2] \to \mathbb{F}(-w_j)$, $1 \leqslant j \leqslant c$, where $[-]$ and $(-)$ denote respectively shift in homological degree and internal degree.

4.2.4 (Graded module structures on Ext and Tor). *The homogeneous chain maps t'_j are determined uniquely up to homotopy; see [31, 1.4]. Therefore the maps*

$$\begin{aligned} \operatorname{Hom}_A(t'_j, N) : \operatorname{Hom}_A(\mathbb{F}^{(j)}, N) &\longrightarrow \operatorname{Hom}_A(\mathbb{F}[2], N) \\ t'_j \otimes_A 1_N : \mathbb{F}[2] \otimes_A N &\longrightarrow \mathbb{F}^{(j)} \otimes_A N \end{aligned}$$

induce well-defined homogeneous A-module homomorphisms

$$s_j : \operatorname{Ext}^i_A(M, N) \longrightarrow \operatorname{Ext}^{i+2}_A(M, N)(-w_j) \quad \text{for all } i \text{ and } 1 \leqslant j \leqslant c, \tag{4.2.1}$$

$$t_j : \operatorname{Tor}^A_{i+2}(M, N) \longrightarrow \operatorname{Tor}^A_i(M, N)(-w_j) \quad \text{for all } i \text{ and } 1 \leqslant j \leqslant c. \tag{4.2.2}$$

Hence, for every $l \geqslant 0$, applying the functors $H^l_{A_+}(-)$ and $(-)^\vee$ successively on (4.2.2), one obtains the homogeneous A-module homomorphisms

$${}^+t^l_j := H^l_{A_+}(t_j)^\vee : H^l_{A_+}\big(\operatorname{Tor}^A_i(M, N)\big)^\vee \longrightarrow H^l_{A_+}\big(\operatorname{Tor}^A_{i+2}(M, N)\big)^\vee(-w_j) \tag{4.2.3}$$

$$ {}^{\mathfrak{m}}t_j^l := H_{\mathfrak{m}}^l(t_j)^\vee : H_{\mathfrak{m}}^l\big(\mathrm{Tor}_i^A(M,N)\big)^\vee \longrightarrow H_{\mathfrak{m}}^l\big(\mathrm{Tor}_{i+2}^A(M,N)\big)^\vee(-w_j) \tag{4.2.4} $$

for all i and $1 \leqslant j \leqslant c$. These coincide whenever A_0 is artinian. By [31, 1.5], since the chain maps t'_j $(1 \leqslant j \leqslant c)$ commute up to homotopy,

$$ \mathrm{Ext}_A^\star(M,N),\ H_{A_+}^l\big(\mathrm{Tor}_\star^A(M,N)\big)^\vee \text{ and } H_{\mathfrak{m}}^l\big(\mathrm{Tor}_\star^A(M,N)\big)^\vee $$

turn into graded $T := A[y_1,\dots,y_c]$-modules, where T is a graded polynomial ring over A with $\deg(y_j) = 2$ for $1 \leqslant j \leqslant c$. The actions of y_j on these three graded T-modules are defined by the maps s_j, ${}^+t_j^l$ and ${}^{\mathfrak{m}}t_j^l$, respectively.

These structures depend only on $\mathbf{f}$, are natural in both module arguments and commute with the connecting maps induced by short exact sequences.

*Choosing a graded epimorphism $B \to Q$, such that B is *local and Cohen-Macaulay of dimension b, with canonical module ω_B, local duality provides a commutative diagram,*

$$ \begin{tikzcd} H_{\mathfrak{m}}^l\big(\mathrm{Tor}_i^A(M,N)\big)^\vee \arrow[r, "{}^{\mathfrak{m}}t_j^l"] \arrow[d, "\simeq"] & H_{\mathfrak{m}}^l\big(\mathrm{Tor}_{i+2}^A(M,N)\big)^\vee(-w_j) \arrow[d, "\simeq"] \\ \mathrm{Ext}_B^{b-l}(\mathrm{Tor}_i^A(M,N),\omega_B) \arrow[r, "\mathrm{Ext}_B^{b-l}(t_j^l,1_{\omega_B})"] & \mathrm{Ext}_B^{b-l}(\mathrm{Tor}_{i+2}^A(M,N),\omega_B)(-w_j) \end{tikzcd} $$

where map on the top row identifies to the one in 4.2.3, whenever A_0 is artinian.

Theorem 4.2.5. *[50, 3.1] The graded module $\mathrm{Ext}_A^\star(M,N)$ is finitely generated over $A[y_1,\dots,y_c]$ provided $\mathrm{Ext}_Q^i(M,N) = 0$ for all $i \gg 0$.*

For instance, when Q is a polynomial ring over a field, $\mathrm{Ext}_A^\star(M,N)$ is finitely generated over $A[y_1,\dots,y_c]$, but $H_{A_+}^l\big(\mathrm{Tor}_\star^A(M,N)\big)^\vee$ is not necessarily finitely generated by Remark 4.4.4. Nevertheless, we prove that if $\dim(\mathrm{Tor}_i^A(M,N)) \leqslant 1$ for all $i \gg 0$, then the modules $H_{\mathfrak{m}}^l\big(\mathrm{Tor}_\star^A(M,N)\big)^\vee$ are finitely generated over $A[y_1,\dots,y_c]$; see Theorem 4.3.6. In order to prove our results, we use the canonical bigraded structures on these graded modules.

4.2.6 (Bigraded structures). *We make $T = A[y_1,\dots,y_c]$ a $\mathbb{Z}^2$-graded ring as follows. Write*

$$ T = A[y_1,\dots,y_c] = A_0[x_1,\dots,x_d,y_1,\dots,y_c], \tag{4.2.5} $$

and set $\deg(x_i) = (0,1)$ for $1 \leqslant i \leqslant d$ and $\deg(y_j) = (2,-w_j)$ for $1 \leqslant j \leqslant c$. We give $\mathbb{Z}^2$-grading structures on $E^\star := \mathrm{Ext}_A^\star(M,N)$, ${}^+D_\star^l := H_{A_+}^l\big(\mathrm{Tor}_\star^A(M,N)\big)^\vee$ and ${}^{\mathfrak{m}}D_\star^l := H_{\mathfrak{m}}^l\big(\mathrm{Tor}_\star^A(M,N)\big)^\vee$ by setting their (i,a)th graded components as the ath graded components of $\mathbb{Z}$-graded modules $\mathrm{Ext}_A^i(M,N)$, $H_{A_+}^l\big(\mathrm{Tor}_i^A(M,N)\big)^\vee$ and $H_{\mathfrak{m}}^l\big(\mathrm{Tor}_i^A(M,N)\big)^\vee$ respectively,

for $(i,a) \in \mathbb{Z}^2$*. Hence, in view of Section 4.2.4,* $E^\star\ {}^+D^l_\star$ *and* ${}^{\mathfrak{m}}D^l_\star$ *are* $\mathbb{Z}^2$*-graded* T*-modules. We consider the corresponding graded submodules corresponding to direct sums of even and odd components :*

$$E^{2\star} := \bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}_A^{2i}(M,N), \quad E^{2\star+1} := \bigoplus_{i\in\mathbb{Z}} \operatorname{Ext}_A^{2i+1}(M,N), \tag{4.2.6}$$

that we will also refer to as $\operatorname{Ext}_A^{\mathrm{even}}(M,N)$ *and* $\operatorname{Ext}_A^{\mathrm{odd}}(M,N)$*, respectively, depending on the context. Similarly one defines*

$${}^+D^l_{2\star}, {}^+D^l_{2\star+1}, {}^{\mathfrak{m}}D^l_{2\star}, {}^{\mathfrak{m}}D^l_{2\star+1} \tag{4.2.7}$$

by taking direct sums over even or odd homological degree components.

In view of (4.2.5)*, set a polynomial ring* $S := Q_0[X_1,\dots,X_d,Y_1,\dots,Y_c]$*, where* $\deg(X_i) = (0,1)$ *for* $1 \leqslant i \leqslant d$ *and* $\deg(Y_j) = (1,-w_j)$ *for* $1 \leqslant j \leqslant c$*. The modules stated in* (4.2.6) *and* (4.2.7) *are canonically* $\mathbb{Z}^2$*-graded* S*-modules. For instance, the* (i,a)*th graded component of* $E^{2\star}$ *is defined to be* $\operatorname{Ext}_A^{2i}(M,N)_a$ *for* $(i,a) \in \mathbb{Z}^2$*, while the actions of* $X_1,\dots,X_d,Y_1,\dots,Y_c$ *on* $E^{2\star}$ *are defined by* $x_1,\dots,x_d,y_1,\dots,y_c$ *respectively. Note that* $Y_j \cdot \operatorname{Ext}_A^{2i}(M,N) \subseteq \operatorname{Ext}_A^{2(i+1)}(M,N)$ *for* $i \in \mathbb{Z}$ *and* $1 \leqslant j \leqslant c$*.*

Thus, in bigraded setup, we have the following result on Ext modules

Proposition 4.2.7. *If* $\operatorname{Ext}_Q^i(M,N) = 0$ *for all* $i \gg 0$*, then* $\operatorname{Ext}_A^{\mathrm{even}}(M,N)$ *and* $\operatorname{Ext}_A^{\mathrm{odd}}(M,N)$ *are finitely generated* $\mathbb{Z}^2$*-graded over* $S = Q_0[X_1,\dots,X_d,Y_1,\dots,Y_c]$*, where* $\deg(X_l) = (0,1)$ *for* $1 \leqslant l \leqslant d$ *and* $\deg(Y_j) = (1,-w_j)$ *for* $1 \leqslant j \leqslant c$*.*

Recall that for every $i,a \in \mathbb{Z}$,

$$\operatorname{Ext}_A^{\mathrm{even}}(M,N)_{(i,a)} = \operatorname{Ext}_A^{2i}(M,N)_a \quad \text{and} \quad \operatorname{Ext}_A^{\mathrm{odd}}(M,N)_{(i,a)} = \operatorname{Ext}_A^{2i+1}(M,N)_a,$$

where $L_{(i,*)} := \bigoplus_{a\in\mathbb{Z}} L_{(i,a)}$ for a $\mathbb{Z}^2$-graded S-module L.

Proof. By virtue of Theorem 4.2.5, $\operatorname{Ext}_A^\star(M,N)$ is a finitely generated graded module over $T = A[y_1,\dots,y_c]$. Therefore the graded submodules $\operatorname{Ext}_A^{\mathrm{even}}(M,N)$ and $\operatorname{Ext}_A^{\mathrm{odd}}(M,N)$ are also finitely generated. Since we are only extending the grading, the proposition now follows from 4.2.6. □

4.3 Linearity of regularity of Ext and Tor

In this section, we show that $\operatorname{reg}(\operatorname{Ext}_A^{2i}(M,N))$ and $\operatorname{reg}(\operatorname{Ext}_A^{2i+1}(M,N))$ are asymptotically linear in i, where M and N are finitely generated graded modules over a complete intersection ring A. Moreover, a similar result for Tor modules is proved when $\dim(\operatorname{Tor}_i^A(M,N)) \leqslant 1$ for all $i \gg 0$. We use the following result, which is a consequence of a theorem due to Bagheri, Chardin and Hà.

Proposition 4.3.1. *[6, Thm. 4.6] Let Q_0 be a commutative Noetherian ring. Set $R := Q_0[X_1,\dots,X_d,Z_1,\dots,Z_c]$, where $\deg(X_i) = (0,1)$ for $1 \leqslant i \leqslant d$ and $\deg(Z_j) = (1,g_j)$ for some $g_j \in \mathbb{Z}$, $1 \leqslant j \leqslant c$. Let L be a finitely generated $\mathbb{Z}^2$-graded R-module. Set $Q := Q_0[X_1,\dots,X_d]$, where $\deg(X_i) = 1$ for $1 \leqslant i \leqslant d$.*

Then, for every $l \geqslant 0$, there exist $a_l, a_l' \in \{g_j : 1 \leqslant j \leqslant c\}$ and $e_l, e_l' \in \mathbb{Z} \cup \{-\infty,+\infty\}$ such that

$$\operatorname{end}\left(\operatorname{Tor}_l^Q(L_{(t,*)},Q_0)\right) = t\cdot a_l + e_l \quad \text{for all } t \gg 0, \tag{4.3.1}$$

$$\operatorname{indeg}\left(\operatorname{Tor}_l^Q(L_{(t,*)},Q_0)\right) = t\cdot a_l' + e_l' \quad \text{for all } t \gg 0. \tag{4.3.2}$$

Hence, there exist $a, a' \in \{g_j : 1 \leqslant j \leqslant c\}$, $e \in \mathbb{Z} \cup \{-\infty\}$ and $e' \in \mathbb{Z} \cup \{+\infty\}$ such that

$$\operatorname{reg}\left(L_{(t,*)}\right) = \max\left\{\operatorname{end}\left(\operatorname{Tor}_l^Q(L_{(t,*)},Q_0)\right) - l : 0 \leqslant l \leqslant d\right\} = t\cdot a + e \text{ for all } t \gg 0,$$

$$\operatorname{indeg}\left(L_{(t,*)}\right) = \operatorname{indeg}\left(\operatorname{Tor}_0^Q(L_{(t,*)},Q_0)\right) = t\cdot a' + e' \text{ for all } t \gg 0.$$

Proof. The same proof as of [6, Thm. 4.6] works if one considers L in place of $M\mathcal{R}$.

So there exists a finite collection of integers $\{\delta_p^l, t_{p,1}^l : 1 \leqslant p \leqslant m\}$, and a subset $E_{p,1}^l$ of $\Gamma_1 = \{g_j : 1 \leqslant j \leqslant c\}$ such that $\Delta E_{p,1}^l$ is linearly independent for every $1 \leqslant p \leqslant m$, satisfying:

$$\operatorname{Supp}_{\mathbb{Z}}\left(\operatorname{Tor}_l^Q(L_{(t,*)},Q_0)\right) = \bigcup_{p=1}^{m}\left(\delta_p^l + \bigcup_{c_1 \in \mathbb{Z}_{\geqslant 0}^{|E_{p,1}^l|},\, |c_1| = t - t_{p,1}^l} c_1 \cdot E_{p,1}^l\right) \tag{4.3.3}$$

for all $t \geqslant \max_p\{t_{p,1}^l\}$, where $\Delta E_{p,1}^l = \{h_2 - h_1, \dots, h_r - h_{r-1}\}$ if $E_{p,1}^l = \{h_1,\dots,h_r\}$. So the cardinality of each $E_{p,1}^l$ must be at most 2. It can be observed that the equalities (4.3.1) and (4.3.2) follow from (4.3.3) once we set

$$a_l := \max\{h : h \in E_{p,1}^l, 1 \leqslant p \leqslant m\}, \quad a_l' := \min\{h : h \in E_{p,1}^l, 1 \leqslant p \leqslant m\},$$
$$e_l := \max\{\delta_p^l - a_l \cdot t_{p,1}^l : 1 \leqslant p \leqslant m \text{ for which } a_l \in E_{p,1}^l\} \quad \text{and}$$
$$e_l' := \min\{\delta_p^l - a_l \cdot t_{p,1}^l : 1 \leqslant p \leqslant m \text{ for which } a_l \in E_{p,1}^l\}.$$

Finally, one obtains the last part from (4.3.1) and (4.3.2) by choosing suitable a, a', e and e'. □

Here are our results on the linearity of regularity for Ext and Tor modules.

Theorem 4.3.2. *Let Q be a standard graded Noetherian algebra, $A := Q/(\mathbf{f})$, where $\mathbf{f} := f_1, \dots, f_c$ is a homogeneous Q-regular sequence. Let M and N be finitely generated graded A-modules such that $\operatorname{Ext}^i_Q(M, N) = 0$ for all $i \gg 0$.*

Then, for every $\ell \in \{0, 1\}$, there exist $a_\ell \in \{\deg(f_j) : 1 \leqslant j \leqslant c\}$ and $e_\ell \in \mathbb{Z} \cup \{-\infty\}$ such that

$$\operatorname{reg}\left(\operatorname{Ext}^{2i+\ell}_A(M, N)\right) = -a_\ell \cdot i + e_\ell \quad \textit{for all } i \gg 0.$$

Proof. The theorem follows from Propositions 4.2.7 and 4.3.1. □

If Q is regular, then the assumption on vanishing of Ext modules over Q is superfluous.

The asymptotic linearity of regularity for Tor modules holds in certain cases.

Theorem 4.3.3. *Let Q be a standard graded Noetherian algebra, $A := Q/(\mathbf{f})$, where $\mathbf{f} := f_1, \dots, f_c$ is a homogeneous Q-regular sequence. Assume Q is *local or the epimorphic image a Gorenstein ring. Let M and N be finitely generated graded A-modules such that,*

(i) *M has finite projective dimension over Q,*

(ii) $\dim(\operatorname{Tor}^A_i(M, N)) \leqslant 1$ *for any $i \gg 0$.*

Then, for every $\ell \in \{0, 1\}$, there exist $a_\ell \in \{\deg(f_j) : 1 \leqslant j \leqslant c\}$ and $e_\ell \in \mathbb{Z} \cup \{-\infty\}$ such that

$$\operatorname{reg}\left(\operatorname{Tor}^A_{2i+\ell}(M, N)\right) = a_\ell \cdot i + e_\ell, \ \forall i \gg 0.$$

We postpone the proof of the above theorem until presenting ingredients of the proof.

Remark 4.3.4. In Sections 4.4 and 4.5 we show that the condition (ii) in the Theorem 4.3.3 is necessary.

Lemma 4.3.5. *Let $B \to A$ be a graded epimorphism of *local rings. Assume B is Cohen-Macaulay. Let W be a finitely generated graded A-module. Set $\operatorname{reg}_{\mathfrak{m}}(W) := \max_j\{\operatorname{end}(H^j_{\mathfrak{m}}(W)) + j\}$. Then one has*

(1) $\operatorname{end}(H^i_{\mathfrak{m}}(W)) = -\operatorname{indeg}(\operatorname{Ext}^{\dim B - j}_B(W, \omega_B))$.

(2) $\operatorname{reg}(W) \leqslant \operatorname{reg}_{\mathfrak{m}}(W) \leqslant \operatorname{reg}(W) + \dim A_0$.

(3) *If* $\dim W \leqslant 1$, *then* $H^p_{\mathfrak{m}_0}(H^q_{A_+}(W)) = 0$ *for* $p+q>1$ *and*

$$\begin{aligned}\operatorname{reg}_{\mathfrak{m}}(W) &= \max\{\operatorname{end}(H^0_{\mathfrak{m}}(W)), \operatorname{end}(H^1_{\mathfrak{m}_0}(H^0_{A_+}(W)))+1, \operatorname{end}(H^0_{\mathfrak{m}_0}(H^1_{A_+}(W)))+1\}\\ \operatorname{reg}(W) &= \max\{\operatorname{end}(H^0_{\mathfrak{m}}(W)), \operatorname{end}(H^1_{\mathfrak{m}_0}(H^0_{A_+}(W))), \operatorname{end}(H^0_{\mathfrak{m}_0}(H^1_{A_+}(W)))+1\}\end{aligned}$$

Proof. Part (1) follows from [16, Thm. 3.6.19]. Parts (2) and (3) follow from [54, Prop. 3.4] by considering the composed functor spectral sequence $H^p_{\mathfrak{m}_0}(H^q_{A'_+}(-)) \Rightarrow H^{p+q}_{\mathfrak{m}}(-)$. □

Theorem 4.3.6. *Let* $B \to Q$ *be a graded epimorphism,* $A := Q/(\mathbf{f})$, *where* $\mathbf{f} := f_1, \dots, f_c$ *is a homogeneous* Q*-regular sequence. Let* P *be a finitely generated graded* B*-module and* M *and* N *be finitely generated* A*-modules such that*

(i) $\operatorname{Ext}^q_B(N,P) = 0$ *for* $q \gg 0$,

(ii) M *has finite projective dimension over* Q,

(iii) $\exists r$, $\operatorname{Ext}^q_B(\operatorname{Tor}^A_i(M,N),P) = 0$ *for* $q \notin \{r-1, r\}$ *and* $i \gg 0$.

Then, for any q,

$$\operatorname{Ext}^q_B\big(\operatorname{Tor}^A_\star(M,N),P\big)$$

is a finitely generated graded $A[y_1,\dots,y_c]$*-module.*

Recall that whenever B is Cohen-Macaulay and $P = \omega_B$, then the modules $\operatorname{Ext}^q_B\big(\operatorname{Tor}^A_\star(M,N),\omega_B\big)$ only depend upon $\operatorname{Tor}^A_\star(M,N)$ as, in the local case, these are Matlis dual to local cohomologies of $\operatorname{Tor}^A_\star(M,N)$.

With such a choice for B and P, condition (i) is satisfied and condition (iii) with $r = \dim B$ is equivalent to $\dim\big(\operatorname{Tor}^A_i(M,N)\big) \leqslant 1$. This will be the main case of application of this result.

Also condition (i) is always satisfied if B is regular.

Proof. Let $\mathbb{F}^M_\bullet$ be a graded minimal free resolution of M over A, and $\mathbb{I}^\bullet_P$ be a graded minimal injective resolution of P over B. Consider the double complex $\mathbb{K}^{\bullet,\bullet}$ defined by

$$\mathbb{K}^{p,q} := \operatorname{Hom}_B\big(\mathbb{F}^M_p \otimes_A N, \mathbb{I}^q_P\big) \cong \operatorname{Hom}_A\big(\mathbb{F}^M_p, \operatorname{Hom}_B(N, \mathbb{I}^q_P)\big) \tag{4.3.4}$$

and its associated spectral sequences. The double complexes in (4.3.4) are equalized by the natural isomorphism. Since $\operatorname{Hom}_A\big(\mathbb{F}^M_p, -\big)$ is an exact functor, by computing cohomology vertically,

$${}^vE_1^{p,q} = \operatorname{Hom}_A\big(\mathbb{F}^M_p, \operatorname{Ext}^q_B(N,P)\big) \text{ and } {}^vE_2^{p,q} = \operatorname{Ext}^p_A(M, \operatorname{Ext}^q_B(N,P)).$$

According to Theorem 4.2.5, condition (ii) implies that the graded $A[y_1,\dots,y_c]$-modules $\operatorname{Ext}^\star_A(M,\operatorname{Ext}^q_B(N,P))$ are finitely generated for every q. As these are not zero for all but finitely many q by (i), ${}^vE^{\star,q}_\infty$, for any q, as well as the homology $H^\star$ of the totalization of $\mathbb{K}^{\bullet,\bullet}$ are finitely generated graded $A[y_1,\dots,y_c]$-modules.

On the other hand, since $\operatorname{Hom}_B(-,\mathbb{I}^q_P)$ is an exact functor, if we start taking cohomology horizontally, then we obtain the first pages of the spectral sequence:

$$ {}^hE^{p,q}_1 = \operatorname{Hom}_B\big(\operatorname{Tor}^A_p(M,N),\mathbb{I}^q_P\big)\,,{}^hE^{p,q}_2 = \operatorname{Ext}^q_B\big(\operatorname{Tor}^A_p(M,N),P\big)$$

and condition (iii) implies that there exists p_0 such that ${}^hE^{p,q}_2=0$ unless $q=r$ or $q=r-1$, if $p\geq p_0$. Hence ${}^hE^{p,q}_2={}^hE^{p,q}_\infty$ for $p\geqslant p_0$.

Taking direct sum over $p\geqslant p_0+r$ and using the naturality of Eisenbud operators, as in 4.2.4, we obtain a short exact sequence of graded $A[y_1,\dots,y_c]$-modules:

$$\begin{aligned} 0\longrightarrow \bigoplus_{p\geqslant p_0+r}\operatorname{Ext}^r_B\big(\operatorname{Tor}^A_{p-r}(M,N),P\big)&\longrightarrow \bigoplus_{p\geqslant p_0+r}H^p\\ &\longrightarrow \bigoplus_{p\geqslant p_0+r}\operatorname{Ext}^{r-1}_B\big(\operatorname{Tor}^A_{p-r+1}(M,N),P\big)\longrightarrow 0.\end{aligned}$$

The middle term is a finitely generated graded $A[y_1,\dots,y_c]$-module, as $H^\star$ is so. Hence the assertion follows. □

Proof of Theorem 4.3.3. Set $W_i:=\operatorname{Tor}^A_{2i}(M,N)$, We will show the linearity of $\operatorname{reg}(W_i)$ for $i\gg 0$. The result for $\operatorname{Tor}^A_{2i+1}(M,N)$ follows similarly. We adopt the notations of the proof of 4.3.6 after choosing a graded epimorphism $B\to Q$ with B equidimensional Cohen-Macaulay and $P=\omega_B$ in this statement and choose i_0 such that $\dim(W_i)\leqslant 1$ for $i\geqslant i_0$. Notice that $\operatorname{Ext}^{\dim B-j}_B(W_i,\omega_B)=0$ for $i\geqslant i_0$ and $j\neq 0,1$. Set $H^0_{[0]}(M):=\cup_{\mathfrak{m}_0\in\operatorname{Specmax}(A_0)}H^0_{\mathfrak{m}_0}(M)$ as in [24, Section 7]. Let $D_i:=\operatorname{Ext}^{\dim B-1}_B(W_i,\omega_B)$, $E_i:=H^0_{[0]}(D_i)$, F_i be defined by the exact sequence

$$0\to\oplus_{i\geqslant i_0}E_i\to\oplus_{i\geqslant i_0}\operatorname{Ext}^{\dim B-1}_B(W_i,\omega_B)\to\oplus_{i\geqslant i_0}F_i\to 0, \tag{4.3.5}$$

and $G_i:=\operatorname{Ext}^{\dim B}_B(W_i,\omega_B)$. By Theorem 4.3.6, $\oplus_iG_i$ and $\oplus_iD_i$ are finitely generated graded $A[y_1,\dots,y_c]$- modules. Hence by 4.3.5 so are $\oplus_iE_i$ and $\oplus_iF_i$. Then Proposition 4.3.1 shows that there exist $a,a',a''\in\{\deg(f_j):1\leqslant j\leqslant c\}$, $e,e',e''\in\mathbb{Z}\cup\{-\infty\}$ and $i_0'\geqslant i_0$, such that for all $i\geqslant i_0'$

$$\begin{aligned}\operatorname{indeg}(G_i)&=-ai-e\\ \operatorname{indeg}(F_i)&=-a'i-e'\\ \operatorname{indeg}(E_i)&=-a''i-e''\end{aligned}$$

We will now show that for $i \geqslant i_0'$

$$\operatorname{reg}(W_i) = r(i) := \max\{ai + e, a'i + e', a''i + e'' + 1\}.$$

By [24, Lemma 7.2], for any graded A-module M, $H^0_{[0]}(M)_\mu = H^0_{[0]}(M_\mu)$ and for $\mathfrak{m}_0 \in \operatorname{Specmax}(A_0)$

$$H^0_{[0]}(M) \otimes_{A_0} (A_0)_{\mathfrak{m}_0} = H^0_{\mathfrak{m}_0}(M \otimes_{A_0} (A_0)_{\mathfrak{m}_0}). \tag{4.3.6}$$

Recall that $\operatorname{reg}(W_i) = \max\{\operatorname{reg}(W_i \otimes_{A_0} (A_0)_{\mathfrak{m}_0}) | \mathfrak{m}_0 \in \operatorname{Specmax}(A_0)\}$. Let $\mathfrak{m}_0 \in \operatorname{Specmax}(A_0)$, $\mathfrak{m} := \mathfrak{m}_0 + A_+$ and write

$$-' := - \otimes_{A_0} (A_0)_{\mathfrak{m}_0} \quad \text{and} \quad -^\vee := {}^*\operatorname{Hom}_{A_0'}(-, E_{A_0'}(A_0'/\mathfrak{m}_0)).$$

Applying $-'$ to the sequence 4.3.5 we get by 4.3.6 for $i \geqslant i_0$ exact sequences

$$0 \to H^0_{\mathfrak{m}_0}(D_i') \to D_i' \to F_i' \to 0. \tag{4.3.7}$$

With $(D_i')^\vee \cong H^1_{\mathfrak{m}}(W_i')$ by [16, Cor. 3.5.9]. With the notations as in Lemma 4.3.5 and considering the composed functor spectral sequence $H^p_{\mathfrak{m}_0}(H^q_{A'_+}(-)) \Rightarrow H^{p+q}_{\mathfrak{m}}(-)$ as in the proof of [54, Prop. 3.4] for $i \geqslant i_0$ we have the following exact sequences of graded A'-modules

$$0 \to H^1_{\mathfrak{m}_0}(H^0_{A'_+}(W_i')) \to H^1_{\mathfrak{m}}(W_i') \to H^0_{\mathfrak{m}_0}(H^1_{A'_+}(W_i')) \to 0. \tag{4.3.8}$$

Since $H^0_{A'_+}(W_i')_\mu$ is a finitely generated A_0-module of dimension at most 1 for any μ, $H^1_{\mathfrak{m}_0}(H^0_{A'_+}(W_i'))^\vee = \oplus_\mu H^1_{\mathfrak{m}_0}(H^0_{A'_+}(W_i')_\mu)^\vee$ has no $\mathfrak{m}_0$-torsion, it follows that

$$H^0_{\mathfrak{m}_0}(H^1_{A'_+}(W_i'))^\vee \cong H^0_{\mathfrak{m}_0}(D_i').$$

It shows that 4.3.8 is the Matlis dual of 4.3.7 and the Matlis dual of G_i' is $H^0_{\mathfrak{m}}(W_i')$. In particular, we get

$$\begin{aligned} \operatorname{end}(H^0_{\mathfrak{m}}(W_i')) &= -\operatorname{indeg}(G_i') \\ \operatorname{end}(H^1_{\mathfrak{m}_0}(H^0_{A'_+}(W_i'))) &= -\operatorname{indeg}(F_i') \\ \operatorname{end}(H^0_{\mathfrak{m}_0}(H^1_{A'_+}(W_i'))) &= -\operatorname{indeg}(E_i'). \end{aligned}$$

As for any graded A-module M, $\operatorname{indeg}(M') \geqslant \operatorname{indeg}(M)$ with equality for some $\mathfrak{m}_0 \in \operatorname{Specmax}(A_0)$ if $\operatorname{indeg}(M) \neq -\infty$, it follows from Lemma 4.3.5 that $\operatorname{reg}(W_i) = r(i)$ for all $i \geqslant i_0'$. □

Remark 4.3.7. Notice that whenever A is *local Cohen-Macaulay (equivalently Q), one may apply the same line of proof with $B = A$, $P = \omega_A$ and N replaced by a high syzygy to assume that N is maximal Cohen-Macaulay. In this particular (but important) situation, the vertical spectral sequence abuts on step 2.

Remark 4.3.8. Theorem 4.3.6 with B *local Cohen-Macaulay shows that if there exits an integer $r \geqslant 1$ such that $r - 1 \leqslant \operatorname{depth}\big(\operatorname{Tor}_i^A(M,N)\big)$ and $\dim\big(\operatorname{Tor}_i^A(M,N)\big) \leqslant r$ for all $i \gg 0$, then for any q,

$$\operatorname{Ext}_B^q\big(\operatorname{Tor}_\star^A(M,N),\omega_B\big)$$

is a finitely generated graded $A[y_1,\dots,y_c]$-module.

Proposition 4.3.9. *In Theorem 4.3.6, assume B is Cohen-Macaulay, $P = \omega_B$ and replace the hypothesis* (iii) *by the following weaker assumption*

(iii)' $\dim\big(\operatorname{Tor}_i^A(M,N)\big) \leqslant 2$ *for* $i \gg 0$.

Then, for any $q \neq \dim B, \dim B - 2$, $\operatorname{Ext}_B^q\big(\operatorname{Tor}_\star^A(M,N),\omega_B\big)$ is a finitely generated graded $A[y_1,\dots,y_c]$-module and the following are equivalent :

(a) $\operatorname{Ext}_B^{\dim B}\big(\operatorname{Tor}_\star^A(M,N),\omega_B\big)$ *is a finitely generated graded $A[y_1,\dots,y_c]$-module,*

(b) $\operatorname{Ext}_B^{\dim B-2}\big(\operatorname{Tor}_\star^A(M,N),\omega_B\big)$ *is a finitely generated graded $A[y_1,\dots,y_c]$-module.*

Proof. Using the same argument as in the proof of Theorem 4.3.6, the abutment of the spectral sequence is obtained in the third page for the following components:

$$ {}^hE_\infty^{p,q} = {}^hE_3^{p,q} = \begin{cases} \operatorname{Coker}(\Phi_{p+1}) & \text{if } p \geqslant p_0 - 2 \text{ and } q = b, \\ \operatorname{Ext}_B^q\big(\operatorname{Tor}_p^A(M,N),\omega_B\big) & \text{if } p \geqslant p_0 - 1 \text{ and } q = b-1, \\ \operatorname{Ker}(\Phi_p) & \text{if } p \geqslant p_0 - 1 \text{ and } q = b-2, \\ 0 & \text{if } p \geqslant p_0 \text{ and } q \notin \{b, b-1, b-2\}, \end{cases} \tag{4.3.9}$$

where $\Phi_p : \operatorname{Ext}_B^{b-2}\big(\operatorname{Tor}_p^A(M,N),\omega_B\big) \longrightarrow \operatorname{Ext}_B^{b}\big(\operatorname{Tor}_{p-1}^A(M,N),\omega_B\big)$ are the induced maps in the second page of the spectral sequence. For every q, the graded $A[y_1,\dots,y_c]$-module $\bigoplus_p {}^hE_\infty^{p,q}$ is finitely generated, because the spectral sequence identifies it as a quotient of two graded submodules of $H^\star$. Thus, according to (4.3.9), it shows that

$$\bigoplus_{p\geqslant p_0-1} \operatorname{Coker}(\Phi_p), \quad \bigoplus_{p\geqslant p_0-1} \operatorname{Ext}_B^q\big(\operatorname{Tor}_p^A(M,N),\omega_B\big) \quad \text{and} \quad \bigoplus_{p\geqslant p_0-1} \operatorname{Ker}(\Phi_p) \tag{4.3.10}$$

are finitely generated over $A[y_1,\dots,y_c]$. For completing the proof, we use (4.3.10) and the

exact sequence

$$0 \longrightarrow \bigoplus_p \operatorname{Ker}(\Phi_p) \longrightarrow \bigoplus_p \operatorname{Ext}_B^{b-2}\big(\operatorname{Tor}_p^A(M,N),\omega_B\big)$$
$$\longrightarrow \bigoplus_p \operatorname{Ext}_B^{b}\big(\operatorname{Tor}_{p-1}^A(M,N),\omega_B\big) \longrightarrow \bigoplus_p \operatorname{Coker}(\Phi_p) \longrightarrow 0$$

of graded modules over $A[y_1,\dots,y_c]$. □

Remark 4.3.10. Whenever B is a standard graded Gorenstein ring over a field and W or P has finite projective dimension over B, the regularity of W is provided by the formula [21, 3.2] :

$$\operatorname{reg}(W) = \operatorname{reg}(B) + \operatorname{indeg}(P) - \min_j\{\operatorname{indeg}(\operatorname{Ext}_R^j(W,P)) + j\}.$$

Hence Theorem 4.3.6 offers other choices of P that could be used to deduce the linearity of the regularity for high Tor modules in specific situations, or to derive its value. To emphasize this remark, we recall now what Theorem 4.3.6 and this fact says whenever Q is a polynomial ring over a field.

Proposition 4.3.11. *Let Q be a polynomial ring over a field, $A := Q/(\mathbf{f})$, where $\mathbf{f} := f_1,\dots,f_c$ is a homogeneous Q-regular sequence. Let M, N and P be finitely generated graded A-modules and $r \in \mathbb{N}$. If*

$$\operatorname{Ext}_Q^q(\operatorname{Tor}_i^A(M,N),P) = 0, \forall i \gg 0 \text{ if } q \notin \{r-1,r\},$$

then

(i) $\operatorname{Ext}_Q^j\big(\operatorname{Tor}_\star^A(M,N),P\big)$ *is a finitely generated graded $A[y_1,\dots,y_c]$-module, for any j.*

(ii) $\operatorname{reg}(\operatorname{Tor}_i^A(M,N)) = \operatorname{indeg}(P) - \min_j\{\operatorname{indeg}(\operatorname{Ext}_Q^j(\operatorname{Tor}_i^A(M,N),P)) + j\}$, *for any i.*

Remark 4.3.12. Along with the same proof as of Corollary 4.3.3, Remark 4.3.8 yields the following. With Hypothesis 4.2.1, further assume that $r-1 \leqslant \operatorname{depth}\big(\operatorname{Tor}_i^A(M,N)\big)$ and $\dim\big(\operatorname{Tor}_i^A(M,N)\big) \leqslant r$ for all $i \gg 0$, where $r \geqslant 1$ is an integer. Then, for every $l \in \{0,1\}$, there exist $a_l \in \{w_j : 1 \leqslant j \leqslant c\}$ and $e_l \in \mathbb{Z}\cup\{-\infty\}$ such that $\operatorname{reg}\big(\operatorname{Tor}_{2i+l}^A(M,N)\big) = a_l\cdot i + e_l$ for all $i \gg 0$.

4.4 Examples on linearity of regularity

Here we construct an example, which shows that the result in Theorem 4.3.3 does not necessarily hold true for higher dimension. In this example, though $\operatorname{reg}\big(\operatorname{Tor}_i^A(M,N)\big)$ is

asymptotically linear in i, but unlike Ext modules, the leading term of the linear function for Tor depends on the modules M and N.

Example 4.4.1. Let $Q := K[Y, Z, V, W]$ be a polynomial ring with usual grading over a field K and $A := Q/(Y^2, Z^2)$. Write $A = K[y, z, v, w]$, where y, z, v and w are the residue classes of Y, Z, V and W respectively. Fix an integer $m \geqslant 1$. Set

$$M := \operatorname{Coker}\left(\begin{bmatrix} y & z & 0 & 0 \\ -v^m & -w^m & y & z \end{bmatrix} : \begin{matrix} A(-m)^2 \\ \bigoplus \\ A(-1)^2 \end{matrix} \longrightarrow \begin{matrix} A(-m+1) \\ \bigoplus \\ A \end{matrix}\right)$$

and $N := A/(y, z)$. Then, for every $i \geqslant 1$, we have

(i) $\operatorname{indeg}\left(\operatorname{Ext}^i_A(M, N)\right) = -i - m + 1$ and $\operatorname{reg}\left(\operatorname{Ext}^i_A(M, N)\right) = -i$.

(ii) $\operatorname{indeg}\left(\operatorname{Tor}^A_i(M, N)\right) = i$ and $\operatorname{reg}\left(\operatorname{Tor}^A_i(M, N)\right) = (m+1)i + (2m-2)$.

We postpone the proof of Example 4.4.1 until the end of this section.

Remark 4.4.2. In Example 4.4.1(ii), though $\operatorname{reg}\left(\operatorname{Tor}^A_i(M, N)\right)$ is linear in i, but the leading term is $(m+1)$, which can be as large as possible depending on M. In particular, it shows that the result in Corollary 4.3.3 is not necessarily true for higher dimension of $\operatorname{Tor}^A_i(M, N)$. In the proof of Example 4.4.1(ii), since $\dim(\operatorname{Ker}(\Phi_i)) = 2$, it follows that $\dim\left(\operatorname{Tor}^A_i(M, N)\right) = 2$ for all $i \geqslant 1$.

Remark 4.4.3. In view of Theorem 4.1.1 and Example 4.4.1(ii), by comparing the coefficients of i from both sides, we can conclude that the inequalities in Theorem 4.1.1 do not necessarily hold true for higher dimension of Tor modules.

Remark 4.4.4. With Setup 4.4.5, the graded modules

$$\bigoplus_{i \geqslant 0} H^0_{A_+}\left(\operatorname{Tor}^A_i(M, N)\right)^\vee \quad \text{and} \quad \bigoplus_{i \geqslant 0} H^2_{A_+}\left(\operatorname{Tor}^A_i(M, N)\right)^\vee$$

are not finitely generated over $A[y_1, \ldots, y_c]$. Otherwise, using Proposition 4.3.9, as in Corollary 4.3.3, one obtains that $\operatorname{reg}\left(\operatorname{Tor}^A_{2i}(M, N)\right)$ is linear in i with leading coefficient 2, which is a contradiction because $\operatorname{reg}\left(\operatorname{Tor}^A_{2i}(M, N)\right) = 2(m+1)i + (2m-2)$.

Setup 4.4.5. *Along with the hypotheses of Example 4.4.1, for every integer $n \geqslant 1$, we set the matrices B_{2n} and C_{2n} of order $2n \times (2n+1)$ as follows:*

$$B_{2n} := \begin{bmatrix} y & -z & 0 & 0 & \cdots & 0 \\ 0 & y & z & 0 & \cdots & 0 \\ 0 & 0 & y & -z & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y & z \end{bmatrix}, C_{2n} := \begin{bmatrix} v^m & w^m & 0 & 0 & \cdots & 0 \\ 0 & -v^m & w^m & 0 & \cdots & 0 \\ 0 & 0 & v^m & w^m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -v^m & w^m \end{bmatrix}$$

while for $n \geqslant 0$, we set the matrices B_{2n+1} and C_{2n+1} of order $(2n+1) \times (2n+2)$ as follows:

$$B_{2n+1} := \begin{bmatrix} y & z & 0 & 0 & \cdots & 0 \\ 0 & y & -z & 0 & \cdots & 0 \\ 0 & 0 & y & z & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y & z \end{bmatrix}, \; C_{2n+1} := \begin{bmatrix} -v^m & -w^m & 0 & 0 & \cdots & 0 \\ 0 & v^m & -w^m & 0 & \cdots & 0 \\ 0 & 0 & -v^m & -w^m & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -v^m & -w^m \end{bmatrix}$$

Note that B_n and C_n are matrices over A both of order $n \times (n+1)$ for every $n \geqslant 1$. Finally, we set a block matrix D_n of order $2n \times (2n+2)$ as follows:

$$D_n := \begin{bmatrix} B_n & O_n \\ C_n & B_n \end{bmatrix} \quad \textit{for every } n \geqslant 1,$$

where O_n denotes the matrix of order $n \times (n+1)$ with all entries 0.

The following relations of B_n and C_n $(n \geqslant 1)$ help us to build minimal free resolution of M.

Proposition 4.4.6. *With Setup 4.4.5, for every $n \geqslant 1$, $B_n C_{n+1} + C_n B_{n+1} = 0$.*

Proof. We use induction on n. It can be verified that $B_1C_2 + C_1B_2 = 0$ and $B_2C_3 + C_2B_3 = 0$. Assuming the equality $B_pC_{p+1} + C_pB_{p+1} = 0$ for $p \leqslant n$, we verify it for $n+1$. We may assume that n is even, say $2q$. The case when n is odd can be treated in a similar way. Note that

$$B_{2q+1}C_{2q+2} = \left[\begin{array}{cc|cccccc} yv^m & yw^m - zv^m & zw^m & 0 & 0 & \cdots & 0 \\ 0 & -yv^m & yw^m - zv^m & -zw^m & 0 & \cdots & 0 \\ \hline 0 & 0 & & & & & \\ \vdots & \vdots & & & B_{2q-1}C_{2q} & & \\ 0 & 0 & & & & & \end{array}\right] \text{ and}$$

$$C_{2q+1}B_{2q+2} = \left[\begin{array}{cc|cccccc} -yv^m & zv^m - yw^m & -zw^m & 0 & 0 & \cdots & 0 \\ 0 & yv^m & zv^m - yw^m & zw^m & 0 & \cdots & 0 \\ \hline 0 & 0 & & & & & \\ \vdots & \vdots & & & C_{2q-1}B_{2q} & & \\ 0 & 0 & & & & & \end{array}\right].$$

Hence induction hypothesis yields that $B_{2q+1}C_{2q+2} + C_{2q+1}B_{2q+2} = 0$. $\square$

Here we construct graded minimal free resolutions of M and N over A.

Lemma 4.4.7. *With Setup 4.4.5, the following statements hold true.*
(i) *A graded minimal free resolution of N over A is given by $\mathbb{F}^N_\bullet$:*

$$0 \longleftarrow A \xleftarrow{B_1} A(-1)^2 \xleftarrow{B_2} \cdots \longleftarrow A(-n+1)^n \xleftarrow{B_n} A(-n)^{n+1} \longleftarrow \cdots$$

(ii) *A graded minimal free resolution of M over A is given by $\mathbb{F}^M_\bullet$:*

$$0 \longleftarrow \begin{matrix} A(-m+1) \\ \bigoplus \\ A \end{matrix} \xleftarrow{D_1} \cdots \longleftarrow \begin{matrix} A(-m-n+2)^n \\ \bigoplus \\ A(-n+1)^n \end{matrix} \xleftarrow{D_n} \begin{matrix} A(-m-n+1)^{n+1} \\ \bigoplus \\ A(-n)^{n+1} \end{matrix} \xleftarrow{D_{n+1}} \cdots$$

Proof. (i) Set $N_1 := A/(y)$ and $N_2 := A/(z)$. Clearly,

$$\begin{aligned} \mathbb{F}^{N_1}_\bullet :&\ \ 0 \leftarrow A \xleftarrow{y} A(-1) \xleftarrow{y} A(-2) \xleftarrow{y} \cdots \quad \text{and} \\ \mathbb{F}^{N_2}_\bullet :&\ \ 0 \leftarrow A \xleftarrow{z} A(-1) \xleftarrow{z} A(-2) \xleftarrow{z} \cdots \end{aligned}$$

are graded minimal A-free resolutions of N_1 and N_2 respectively. Since $\mathbb{F}^{N_1}_\bullet \otimes_A N_2$ is acyclic, it follows that $\operatorname{Tor}^A_i(N_1, N_2) = 0$ for all $i \geqslant 1$. Let $\mathbb{F}_\bullet$ be the tensor product of $\mathbb{F}^{N_1}_\bullet$ and $\mathbb{F}^{N_2}_\bullet$ over A; see [74, pp 614]. Note that the homology $H_i(\mathbb{F}_\bullet) = \operatorname{Tor}^A_i(N_1, N_2)$ (cf. [74, 10.22]). Thus, since $H_i(\mathbb{F}_\bullet) = 0$ for all $i \geqslant 1$, $\mathbb{F}_\bullet$ provides a free resolution of $N_1 \otimes_A N_2 = A/(y, z) = N$. It follows from the definition of tensor product of complexes that $\mathbb{F}_\bullet$ is same as the desired free resolution $\mathbb{F}^N_\bullet$.

(ii) Set $\mathbb{G} := \mathbb{F}^N_\bullet$, the resolution shown in (i), and $\mathbb{H} := \mathbb{G}[1](-m+1)$, i.e.,

$$\mathbb{H}_n = \mathbb{G}_{n+1}(-m+1) \quad \text{and} \quad d^{\mathbb{H}}_n = (-1)d^{\mathbb{G}}_{n+1} \quad \text{for every } n;$$

see [79, 1.2.8]. We construct a map $f : \mathbb{H} \to \mathbb{G}$ as follows: the nth component $f_n : \mathbb{H}_n \to \mathbb{G}_n$ of f is defined by $(-1)C_{n+1}$. By virtue of Proposition 4.4.6, f is a homogeneous map of chain complexes. We consider the mapping cone $\operatorname{Cone}(f)$; see [79, 1.5.1] for its definition. Note that $\operatorname{Cone}(f)_n = \mathbb{H}_{n-1} \oplus \mathbb{G}_n$ with the nth differential

$$\begin{bmatrix} -d^{\mathbb{H}}_{n-1} & 0 \\ -f_{n-1} & d^{\mathbb{G}}_n \end{bmatrix} : \begin{matrix} \mathbb{H}_{n-1} \\ \bigoplus \\ \mathbb{G}_n \end{matrix} \begin{matrix} \longrightarrow \\ \searrow \\ \longrightarrow \end{matrix} \begin{matrix} \mathbb{H}_{n-2} \\ \bigoplus \\ \mathbb{G}_{n-1} \end{matrix}$$

which is nothing but D_n as given in the desired resolution. Since $H_n(\mathbb{G}) = 0 = H_{n-1}(\mathbb{H})$ for every $n \geqslant 1$, in view of [79, 1.5.2], we have $H_n(\operatorname{Cone}(f)) = 0$ for every $n \geqslant 1$. Hence $\operatorname{Cone}(f)$ provides the desired free resolution $\mathbb{F}^M_\bullet$. $\square$

4.4.8 (Computations of $\operatorname{Tor}_i^A(M,N)$ and $\operatorname{Ext}_A^i(M,N)$ with Setup 4.4.5)**.** *In view of Lemma 4.4.7(ii), we obtain that the complex $\mathbb{F}_\bullet^M \otimes_A N$ is given by*

$$0 \longleftarrow \bigoplus_{N}^{N(-m+1)} \xleftarrow{D'_1} \cdots \longleftarrow \bigoplus_{N(-n+1)^n}^{N(-m-n+2)^n} \xleftarrow{D'_n} \bigoplus_{N(-n)^{n+1}}^{N(-m-n+1)^{n+1}} \xleftarrow{D'_{n+1}} \cdots$$

where

$$D'_n := D_n \otimes_A A/(y,z) = \begin{bmatrix} O_n & O_n \\ C_n & O_n \end{bmatrix} \quad \text{for every } n \geqslant 1.$$

This yields that

$$\operatorname{Tor}_n^A(M,N) = \begin{array}{c} \operatorname{Ker}\left(N(-m-n+1)^{n+1} \xrightarrow{C_n} N(-n+1)^n\right) \\ \bigoplus \\ \operatorname{Coker}\left(N(-m-n)^{n+2} \xrightarrow{C_{n+1}} N(-n)^{n+1}\right) \end{array} \quad \text{for } n \geqslant 1. \tag{4.4.1}$$

It follows that $\operatorname{Tor}_n^A(M,N)_\mu = 0$ for every $\mu < n$, and $\operatorname{Tor}_n^A(M,N)_n \neq 0$. Therefore

$$\operatorname{indeg}\left(\operatorname{Tor}_n^A(M,N)\right) = n \quad \text{for every } n \geqslant 1. \tag{4.4.2}$$

To compute Ext modules, consider the complex $\operatorname{Hom}_A(\mathbb{F}_\bullet^M, N)$, which is given by

$$0 \longrightarrow \bigoplus_{N}^{N(m-1)} \xrightarrow{(D'_1)^t} \cdots \longrightarrow \bigoplus_{N(n-1)^n}^{N(m+n-2)^n} \xrightarrow{(D'_n)^t} \bigoplus_{N(n)^{n+1}}^{N(m+n-1)^{n+1}} \xrightarrow{(D'_{n+1})^t} \cdots$$

where $(-)^t$ stands for the transpose of a matrix. Hence it can be observed that

$$\operatorname{Ext}_A^n(M,N) = \begin{array}{c} \operatorname{Coker}\left(N(n-1)^n \xrightarrow{C_n^t} N(m+n-1)^{n+1}\right) \\ \bigoplus \\ \operatorname{Ker}\left(N(n)^{n+1} \xrightarrow{C_{n+1}^t} N(m+n)^{n+2}\right) \end{array} \quad \text{for } n \geqslant 1. \tag{4.4.3}$$

We are now able to provide a proof for the example.

Proof of Example 4.4.1. In view of (4.4.1) and (4.4.3), it suffices to study the regularity of kernel and cokernel of

$$\Phi_n := N(-m-n+1)^{n+1} \xrightarrow{C_n} N(-n+1)^n, \ \Psi_n := N(n-1)^n \xrightarrow{C_n^t} N(m+n-1)^{n+1}$$

for all $n \geqslant 1$. Since N is annihilated by (y, z), we can substitute N with $R := K[V, W]$, and v^m, w^m in the entries of the matrices C_n with V^m, W^m respectively.

(i) Since $I_n(C_n^t)$, the ideal of maximal minors of C_n^t, has depth $= 2$, by the Hilbert-Burch Theorem (cf. [16, 1.4.17]), we have a graded minimal R-free resolution of $\mathrm{Coker}(\Psi_n)$:

$$0 \longrightarrow R(n-1)^n \xrightarrow{C_n^t} R(m+n-1)^{n+1} \xrightarrow{\pi} \mathrm{Coker}(\Psi_n) = I_n(C_n^t) \longrightarrow 0, \tag{4.4.4}$$

where π sends the standard basis element e_i to $(-1)^i\delta_i$, and δ_i denotes the $n \times n$ minor of C_n^t with the ith row deleted for $1 \leqslant i \leqslant n+1$. Therefore, for every $n \geqslant 1$, one obtains that $\mathrm{Ker}(\Psi_n) = 0$, $\mathrm{indeg}(\mathrm{Coker}(\Psi_n)) = -m-n+1$ and $\mathrm{reg}(\mathrm{Coker}(\Psi_n)) = -n$. Thus it follows from (4.4.3) that for every $n \geqslant 1$,

$$\begin{aligned}\mathrm{indeg}(\mathrm{Ext}_A^n(M,N)) &= \min\{\mathrm{indeg}(\mathrm{Coker}(\Psi_n)), \mathrm{indeg}(\mathrm{Ker}(\Psi_{n+1}))\}\\ &= -n-m+1 \quad \text{and}\\ \mathrm{reg}(\mathrm{Ext}_A^n(M,N)) &= \max\{\mathrm{reg}(\mathrm{Coker}(\Psi_n)), \mathrm{reg}(\mathrm{Ker}(\Psi_{n+1}))\} = -n.\end{aligned}$$

(ii) By (4.4.4), since $\deg(\delta_i) = mn$, we get an exact sequence of graded R-modules:

$$0 \longrightarrow R(n-1)^n \xrightarrow{C_n^t} R(m+n-1)^{n+1} \xrightarrow{E_n := \left[-\delta_1\ \delta_2\ \cdots\ (-1)^{n+1}\delta_{n+1}\right]} R(mn+m+n-1).$$

Applying $\mathrm{Hom}_R(-, R)$, we obtain a complex

$$0 \longrightarrow R(-mn-m-n+1) \xrightarrow{E_n^t} R(-m-n+1)^{n+1} \xrightarrow{C_n} R(-n+1)^n \longrightarrow 0 \tag{4.4.5}$$

which is acyclic, due to Buchsbaum-Eisenbud acyclicity criterion [16, 1.4.13]. Thus (4.4.5) is a graded minimal R-free resolution of $\mathrm{Coker}(\Phi_n)$, and $\mathrm{Ker}(\Phi_n) \cong R(-mn-m-n+1)$.

Hence it follows from (4.4.1) that for every $n \geqslant 1$,

$$\begin{aligned}\mathrm{reg}\left(\mathrm{Tor}_n^A(M,N)\right) &= \max\{\mathrm{reg}(\mathrm{Ker}(\Phi_n)), \mathrm{reg}(\mathrm{Coker}(\Phi_{n+1}))\}\\ &= \max\{(m+1)n+m-1, (m+1)(n+1)+m-3\}\\ &= (m+1)n+(2m-2).\end{aligned} \tag{4.4.6}$$

Thus (4.4.2) and (4.4.6) yield the assertion (ii). □

4.5 Examples on nonlinearity of regularity

The aim of this section is to show that $\mathrm{reg}\left(\mathrm{Tor}_{2i}^A(M,N)\right)$ and $\mathrm{reg}\left(\mathrm{Tor}_{2i+1}^A(M,N)\right)$ need not be asymptotically linear in i even over a complete intersection ring A. We give the following example over a codimension three complete intersection ring in positive characteristic.

Example 4.5.1. Let $Q := K[X, Y, Z, U, V, W]$ be a standard graded polynomial ring over a field K of characteristic 2 and $A := Q/(X^2, Y^2, Z^2)$. We write $A = K[x, y, z, u, v, w]$, where x, y, z, u, v and w are the residue classes of X, Y, Z, U, V and W respectively. Set

$$M := \operatorname{Coker}\left(\begin{bmatrix} x & y & z & 0 & 0 & 0 \\ u & v & w & x & y & z \end{bmatrix} : A(-1)^6 \longrightarrow A^2\right) \quad \text{and} \quad N := A/(x, y, z).$$

Then, for every $n \geqslant 1$, we have

(i) $\operatorname{indeg}(\operatorname{Ext}^n_A(M, N)) = -n$ and $\operatorname{reg}(\operatorname{Ext}^n_A(M, N)) = -n$.

(ii) $\operatorname{indeg}\left(\operatorname{Tor}^A_n(M, N)\right) = n$ and $\operatorname{reg}\left(\operatorname{Tor}^A_n(M, N)\right) = f(n) + n$, where

$$f(n) := \begin{cases} 2^{l+1} - 2 & \text{if } n = 2^l - 1 \\ 2^{l+1} - 1 & \text{if } 2^l \leqslant n \leqslant 2^{l+1} - 2 \end{cases} \qquad \text{for all integers } l \geqslant 1.$$

Remark 4.5.2. Example 4.5.1(ii) shows that $\operatorname{reg}(\operatorname{Tor}^A_{2n}(M, N))$ and $\operatorname{reg}(\operatorname{Tor}^A_{2n+1}(M, N))$ are not asymptotically linear as functions of n. Moreover, one obtains that $n + 1 \leqslant f(n) \leqslant 2n$ for every $n \geqslant 1$, while $f(n) = n + 1$ if $n = 2^{l+1} - 2$, and $f(n) = 2n$ if $n = 2^l - 1$ for $l \geqslant 1$. Therefore

$$\liminf_{n\to\infty} \frac{\operatorname{reg}(\operatorname{Tor}^A_n(M, N))}{n} = 2 \quad \text{and} \quad \limsup_{n\to\infty} \frac{\operatorname{reg}(\operatorname{Tor}^A_n(M, N))}{n} = 3.$$

Furthermore, for any $\alpha \in (2, 3)$, by choosing any subsequence $n_\alpha(l)$ such that $|n_\alpha(l) - \lfloor 2^l/(\alpha - 1)\rfloor|$ is bounded for all $l \geqslant 1$,

$$\lim_{l\to\infty} \frac{\operatorname{reg}(\operatorname{Tor}^A_{n_\alpha(l)}(M, N))}{n_\alpha(l)} = \alpha.$$

In particular, $n_\alpha(l)$ can be a sequence of even (resp. odd) integers. Thus both

$$\{\operatorname{reg}(\operatorname{Tor}^A_{2n}(M, N))/2n : n \geqslant 1\} \quad \text{and} \quad \{\operatorname{reg}(\operatorname{Tor}^A_{2n+1}(M, N))/2n + 1 : n \geqslant 1\}$$

are dense sets in $[2, 3]$.

Before proving the claims in Example 4.5.1, we need to setup some notations and provide some preliminary lemmas.

Setup 4.5.3. *Along with the hypotheses of Example 4.5.1, for every integer $n \geqslant 1$, we set the matrices B_n and C_n of order $n \times (n+1)$ as follows:*

$$B_n := \begin{bmatrix} y & z & 0 & 0 & \cdots & 0 \\ 0 & y & z & 0 & \cdots & 0 \\ 0 & 0 & y & z & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & y & z \end{bmatrix} \text{ and } C_n := \begin{bmatrix} v & w & 0 & 0 & \cdots & 0 \\ 0 & v & w & 0 & \cdots & 0 \\ 0 & 0 & v & w & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & v & w \end{bmatrix}.$$

Setting I_n as the $n \times n$ identity matrix, we construct the block matrices E_n and F_n both of order $\binom{n+1}{2} \times \binom{n+2}{2}$ as follows:

$$E_n := \left[\begin{array}{c|c|c|c|c} xI_1 & B_1 & & & \\ \hline & xI_2 & B_2 & & \\ \hline & & \ddots & \ddots & \\ \hline & & & xI_n & B_n \end{array}\right] \text{ and } F_n := \left[\begin{array}{c|c|c|c|c} uI_1 & C_1 & & & \\ \hline & uI_2 & C_2 & & \\ \hline & & \ddots & \ddots & \\ \hline & & & uI_n & C_n \end{array}\right]$$

Finally, we set the block matrix

$$D_n := \left[\begin{array}{c|c} E_n & \\ \hline F_n & E_n \end{array}\right] \quad \text{of order } 2\binom{n+1}{2} \times 2\binom{n+2}{2}.$$

Here the empty blocks in E_n, F_n and D_n are filled with zero matrices of suitable order.

In view of Proposition 4.4.6, replacing v^m and w^m by v and w respectively, since $\operatorname{char}(K) = 2$, one obtains the following relations.

Remark 4.5.4. With Setup 4.5.3, $B_n C_{n+1} + C_n B_{n+1} = 0$ for every $n \geqslant 1$.

A similar relation holds for E_n and F_n, which helps us to build minimal free resolution of M.

Proposition 4.5.5. *With Setup 4.5.3, $E_n F_{n+1} + F_n E_{n+1} = 0$ for every $n \geqslant 1$.*

Proof. For every $n \geqslant 1$, the block matrix multiplication yields that

$$E_nF_{n+1} = \left[\begin{array}{c|c|c|c|c|c} xuI_1 & xC_1+uB_1 & B_1C_2 & & & \\ \hline & xuI_2 & xC_2+uB_2 & B_2C_3 & & \\ \hline & & \ddots & \ddots & \ddots & \\ \hline & & & xuI_n & xC_n+uB_n & B_nC_{n+1} \end{array}\right],$$

$$F_nE_{n+1} = \left[\begin{array}{c|c|c|c|c|c} uxI_1 & uB_1+xC_1 & C_1B_2 & & & \\ \hline & uxI_2 & uB_2+xC_2 & C_2B_3 & & \\ \hline & & \ddots & \ddots & \ddots & \\ \hline & & & uxI_n & uB_n+xC_n & C_nB_{n+1} \end{array}\right].$$

Hence 'char$(K) = 2$' and Remark 4.5.4 yield that $E_nF_{n+1} + F_nE_{n+1} = 0$ for every $n \geqslant 1$. □

We compute $\operatorname{Tor}_n^A(M,N)$ $(n \geqslant 1)$ by constructing a graded minimal free resolution of M.

Lemma 4.5.6. *With Setup 4.5.3, the following statements hold true.*
(i) *A graded minimal free resolution of N over A is given by $\mathbb{F}_\bullet^N$:*

$$0 \longleftarrow A \xleftarrow{E_1} A(-1)^3 \xleftarrow{E_2} \cdots \longleftarrow A(-n+1)^{\binom{n+1}{2}} \xleftarrow{E_n} A(-n)^{\binom{n+2}{2}} \longleftarrow \cdots$$

(ii) *A graded minimal free resolution of M over A is given by $\mathbb{F}_\bullet^M$:*

$$0 \longleftarrow \begin{matrix} A \\ \bigoplus \\ A \end{matrix} \xleftarrow{D_1} \begin{matrix} A(-1)^3 \\ \bigoplus \\ A(-1)^3 \end{matrix} \xleftarrow{D_2} \cdots \longleftarrow \begin{matrix} A(-n+1)^{\binom{n+1}{2}} \\ \bigoplus \\ A(-n+1)^{\binom{n+1}{2}} \end{matrix} \xleftarrow{D_n} \begin{matrix} A(-n)^{\binom{n+2}{2}} \\ \bigoplus \\ A(-n)^{\binom{n+2}{2}} \end{matrix} \xleftarrow{D_{n+1}} \cdots$$

Proof. The proof is almost same as that of Lemma 4.4.7. So we just mention the steps here.

(i) Set $N_1 := A/(x)$ and $N_2 := A/(y,z)$. Then

$$\mathbb{F}_\bullet^{N_1} : \quad 0 \leftarrow A \xleftarrow{x} A(-1) \xleftarrow{x} A(-2) \xleftarrow{x} \cdots \quad \text{and}$$
$$\mathbb{F}_\bullet^{N_2} : \quad 0 \leftarrow A \xleftarrow{B_1} A(-1)^2 \xleftarrow{B_2} A(-2)^3 \xleftarrow{B_3} \cdots$$

are graded minimal A-free resolutions of N_1 and N_2 respectively, where $\mathbb{F}_\bullet^{N_2}$ is obtained as in Lemma 4.4.7(i). Set $\mathbb{F}_\bullet := \mathbb{F}_\bullet^{N_1} \otimes_A \mathbb{F}_\bullet^{N_2}$. Hence $H_i(\mathbb{F}_\bullet) = \operatorname{Tor}_i^A(N_1,N_2) = 0$ for all $i \geqslant 1$ (since $\mathbb{F}_\bullet^{N_1} \otimes_A N_2$ is acyclic). Therefore $\mathbb{F}_\bullet$ is a free resolution of $N_1 \otimes_A N_2 = A/(x,y,z) = N$. The assertion follows because $\mathbb{F}_\bullet$ is same as the given free resolution $\mathbb{F}_\bullet^N$.

(ii) Set $\mathbb{G} := \mathbb{F}^N_\bullet$ and $\mathbb{H} := \mathbb{G}[1]$, i.e., $\mathbb{H}_n = \mathbb{G}_{n+1}$ and $d^{\mathbb{H}}_n = (-1)d^{\mathbb{G}}_{n+1}$ for every n. We construct a map $f : \mathbb{H} \to \mathbb{G}$ as follows: the nth component $f_n : \mathbb{H}_n \to \mathbb{G}_n$ of f is defined by $(-1)F_{n+1}$. By virtue of Proposition 4.5.5, f is a homogeneous map of chain complexes. As in the proof of Lemma 4.4.7(ii), the mapping cone of f provides the desired free resolution $\mathbb{F}^M_\bullet$. □

4.5.7 (Computations of $\operatorname{Tor}^A_i(M,N)$ and $\operatorname{Ext}^i_A(M,N)$ with Setup 4.5.3)**.** *In view of Lemma 4.5.6(ii), by considering the complex $\mathbb{F}^M_\bullet \otimes_A N$ as in 4.4.8, we compute that*

$$\operatorname{Tor}^A_n(M,N) = \begin{array}{l} \operatorname{Ker}\left(N(-n)^{\binom{n+2}{2}} \xrightarrow{F_n} N(-n+1)^{\binom{n+1}{2}}\right) \\ \bigoplus \\ \operatorname{Coker}\left(N(-n-1)^{\binom{n+3}{2}} \xrightarrow{F_{n+1}} N(-n)^{\binom{n+2}{2}}\right) \end{array} \quad \text{for } n \geqslant 1. \tag{4.5.1}$$

It follows that $\operatorname{Tor}^A_n(M,N)_\mu = 0$ for every $\mu < n$, and $\operatorname{Tor}^A_n(M,N)_n \neq 0$. Therefore

$$\operatorname{indeg}\left(\operatorname{Tor}^A_n(M,N)\right) = n \quad \text{for every } \ n \geqslant 1. \tag{4.5.2}$$

To compute Ext modules, we consider the complex $\operatorname{Hom}_A(\mathbb{F}^M_\bullet, N)$, which yields that

$$\operatorname{Ext}^n_A(M,N) = \begin{array}{l} \operatorname{Coker}\left(N(n-1)^{\binom{n+1}{2}} \xrightarrow{F^t_n} N(n)^{\binom{n+2}{2}}\right) \\ \bigoplus \\ \operatorname{Ker}\left(N(n)^{\binom{n+2}{2}} \xrightarrow{F^t_{n+1}} N(n+1)^{\binom{n+3}{2}}\right) \end{array} \quad \text{for } n \geqslant 1, \tag{4.5.3}$$

where F^t_n is the transpose of F_n. It follows from (4.5.3) that

$$\operatorname{indeg}\left(\operatorname{Ext}^n_A(M,N)\right) = -n \quad \text{for every } \ n \geqslant 1. \tag{4.5.4}$$

In order to compute regularity of $\operatorname{Tor}^A_n(M,N)$ and $\operatorname{Ext}^n_A(M,N)$, we interpret the matrix maps F_n and F^t_n in different ways.

Definition 4.5.8. For a ring S, we denote by $\delta_X : S[X] \to S[X]$ the S-linear map defined by

$$X^a \longmapsto \begin{cases} X^{a-1} & \text{if } a \geqslant 1, \\ 0 & \text{else.} \end{cases}$$

4.5.9 (Interpretations of F_n and F^t_n)**.** *Set $R := K[U,V,W]$, polynomial ring over a field K of characteristic 2. Consider the sequences of graded R-linear maps (which are not complexes):*

$$\mathbb{F}^U_\bullet : \quad \cdots \longrightarrow R(-3) \xrightarrow{U} R(-2) \xrightarrow{U} R(-1) \xrightarrow{U} R \longrightarrow 0,$$

similarly $\mathbb{F}_\bullet^V$ *and* $\mathbb{F}_\bullet^W$. *Set* $\mathbb{F}_\bullet := \mathbb{F}_\bullet^U \otimes_R \mathbb{F}_\bullet^V \otimes_R \mathbb{F}_\bullet^W$, *which can be defined exactly in the same way as tensor product of complexes is defined. In view of Lemma 4.5.6(i) and its proof, the* n*th map of the sequence* $\mathbb{F}_\bullet$ *is given by*

$$R(-n)^{\binom{n+2}{2}} \xrightarrow{F_n} R(-n+1)^{\binom{n+1}{2}},$$

where F_n *is obtained from Setup 4.5.3 by replacing* u, v, w *with* U, V, W *respectively. Identifying the free summand* $R(-n)$ *corresponding to* $\mathbb{F}_{a_1}^U \otimes \mathbb{F}_{a_2}^V \otimes \mathbb{F}_{a_3}^W$ *with* $RX^{a_1}Y^{a_2}Z^{a_3} \subseteq R[X,Y,Z]_n$, *where* $a_1 + a_2 + a_3 = n$ *and* $a_i \geqslant 0$, *one obtains an* R*-module isomorphism* $R(-n)^{\binom{n+2}{2}} \xrightarrow{\cong} R[X,Y,Z]_n$. *On the other hand, labeling the basis elements of* $R(-n)^{\binom{n+2}{2}}$ *by* $e_{(a_1,a_2,a_3)}$, *the action of* F_n *on* $e_{(a_1,a_2,a_3)}$ *can be described as follows:*

$$F_n\left(e_{(a_1,a_2,a_3)}\right) = \epsilon_1 U e_{(a_1-1,a_2,a_3)} + \epsilon_2 V e_{(a_1,a_2-1,a_3)} + \epsilon_3 W e_{(a_1,a_2,a_3-1)},$$

where $\epsilon_i = 1$ *if* $a_i \geqslant 1$, *else* $\epsilon_i = 0$. *Hence it can be checked that the diagram*

$$\begin{tikzcd} R(-n)^{\binom{n+2}{2}} \arrow[r, "F_n"] \arrow[d, "\cong"'] & R(-n+1)^{\binom{n+1}{2}} \arrow[d, "\cong"] \\ R[X,Y,Z]_n \arrow[r, "\delta"] & R[X,Y,Z]_{n-1} \end{tikzcd} \tag{4.5.5}$$

is commutative, where $\delta := U\delta_X + V\delta_Y + W\delta_Z$, *which is an* R*-linear map. Dualizing the commutative diagram (4.5.5), or dualizing the above notion, one obtains another commutative diagram*

$$\begin{tikzcd} R(n)^{\binom{n+2}{2}} & R(n-1)^{\binom{n+1}{2}} \arrow[l, "F_n^t"'] \\ R[X,Y,Z]_n \arrow[u, "\cong"] & R[X,Y,Z]_{n-1} \arrow[l, "\mu_n"'] \arrow[u, "\cong"'] \end{tikzcd} \tag{4.5.6}$$

where μ_n *is an* R*-linear map defined by multiplication with* $UX + VY + WZ$. *Since* μ_n *is injective, it follows that the map given by* F_n^t *is an injective map.*

The origin of the nonlinear behavior of regularity in Example 4.5.1(ii) rely on the behavior of coefficient ideals in positive characteristic.

Lemma 4.5.10. *Set* $R := K[U,V,W]$, *where* $\operatorname{char}(K) = 2$. *For every* $n \geqslant 1$, *let* $\mathcal{B}_n$ *be the set of all monomials in* U, V, W *which are the coefficients of* $(UX + VY + WZ)^n$, *and* I_n *be the ideal of* R *generated by* $\mathcal{B}_n$. *Then* $\operatorname{reg}(R/I_n) = 3(2^l - 1)$ *if* $2^l \leqslant n \leqslant 2^{l+1} - 1$ *for some* $l \geqslant 0$.

Proof. Writing n in base 2, $n = \sum_{i\geqslant 0} a_i 2^i$ with $a_i \in \{0,1\}$, set $l := \max\{i : a_i \neq 0\}$ and $S_n := \{i : a_i \neq 0\}$. Since $\mathrm{char}(K) = 2$,

$$\begin{aligned}\sum_{U^aV^bW^c\in\mathcal{B}_n} U^aV^bW^cX^aY^bZ^c &= (UX+VY+WZ)^n \\ &= \prod_{i\in S_n}\left(U^{2^i}X^{2^i}+V^{2^i}Y^{2^i}+W^{2^i}Z^{2^i}\right).\end{aligned}$$

Since $\sum_{0\leqslant j\leqslant r} 2^j < 2^{r+1}$ for any $r \geqslant 0$, the above equalities show that the map $\prod_{i\in S_n}\mathcal{B}_{2^i} \to \mathcal{B}_n$ sending a tuple of monomials to their product is a bijection. Therefore $I_n = \prod_{a_i=1} I_{2^i}$. It shows that the minimal number of generators of I_n is $3^{|S_n|}$, a fact that we will not use for the proof.

We now use induction on l. Since $\mathrm{reg}(R/I_1) = 0$, for $l = 0$, the assertion holds. Suppose $\mathrm{reg}(R/I_n) = 3(2^l-1)$ if $2^l \leqslant n \leqslant 2^{l+1}-1$ for some $l \geqslant 0$. Since R/I_n is Artinian, and the regularity is given by the shifts in the last component of the minimal free resolution $\mathbb{F}_\bullet^{R/I_n}$, applying the Frobenius map, we get $\mathbb{F}_\bullet^{R/I_{2n}}$. So

$$\mathrm{reg}(R/I_{2n}) = 2(\mathrm{reg}(R/I_n)+3)-3 = 3(2^{l+1}-1) \text{ if } 2^{l+1} \leqslant 2n \leqslant 2^{l+2}-2. \tag{4.5.7}$$

Note that $I_{2n+1} = \mathfrak{m}I_{2n}$, where $\mathfrak{m} = (U,V,W)$. Considering the exact sequence

$$0 \longrightarrow I_{2n}/\mathfrak{m}I_{2n} \longrightarrow R/\mathfrak{m}I_{2n} \longrightarrow R/I_{2n} \longrightarrow 0,$$

for every $2^{l+1}+1 \leqslant 2n+1 \leqslant 2^{l+2}-1$,

$$\begin{aligned}\mathrm{reg}(R/I_{2n+1}) &= \max\{\mathrm{reg}(R/I_{2n}), \mathrm{reg}(I_{2n}/\mathfrak{m}I_{2n})\} \\ &= \max\{3(2^{l+1}-1), 2n\} = 3(2^{l+1}-1).\end{aligned} \tag{4.5.8}$$

Thus the assertion for $l+1$ follows from (4.5.7) and (4.5.8). □

Using the interpretation of F_n given in 4.5.9, we now prove the following facts.

Lemma 4.5.11. *Set $R := K[U,V,W]$, where $\mathrm{char}(K) = 2$. Then the R-linear map $\Phi_n : R(-n)^{\binom{n+2}{2}} \xrightarrow{F_n} R(-n+1)^{\binom{n+1}{2}}$ has the following properties.*

(i) *For every $n \geqslant 1$, $\mathrm{Coker}(\Phi_n)$ is an Artinian R-module.*

(ii) *For every $n \geqslant 1$, $\mathrm{reg}(\mathrm{Coker}(\Phi_n)) \leqslant \mathrm{reg}(\mathrm{Coker}(\Phi_{n+1})) - 1$.*

(iii) $\mathrm{reg}(\mathrm{Coker}(\Phi_1\Phi_2\cdots\Phi_n)) = 3(2^l-1)$ *if $2^l \leqslant n \leqslant 2^{l+1}-1$ for some $l \geqslant 0$.*

(iv) $\operatorname{reg}(\operatorname{Coker}(\Phi_n)) = 2(n-1)$ *if* $n = 2^l - 1$ *for some* $l \geqslant 1$.

(v) $\operatorname{reg}(\operatorname{Coker}(\Phi_n)) = 2(2^l - 1) + n - 1$ *if* $2^l \leqslant n \leqslant 2^{l+1} - 1$ *for some* $l \geqslant 0$.

Proof. (i) Let $I(F_n)$ be the ideal of maximal minors of F_n. By construction, and changing the role of U, V and W, one can see that $\left(U^{\binom{n+1}{2}}, V^{\binom{n+1}{2}}, W^{\binom{n+1}{2}}\right) \subset I(F_n)$. Therefore the assertion follows from the fact that $\operatorname{Supp}(\operatorname{Coker}(\Phi_n)) \subseteq \operatorname{Supp}(R/I(F_n))$, which is shown in [32, 20.4 and 20.7.a].

(ii) By virtue of (i), $\operatorname{reg}(\operatorname{Coker}(\Phi_n))$ is the smallest number r such that Φ_n is surjective on the graded components $\geqslant r+1$. Set $\Psi_n : R(-1)^{\binom{n+2}{2}} \xrightarrow{F_n} R^{\binom{n+1}{2}}$, which is same as Φ_n but the grading is shifted by $n-1$. So $\operatorname{reg}(\operatorname{Coker}(\Phi_n)) = \operatorname{reg}(\operatorname{Coker}(\Psi_n)) + n - 1$. It can be derived from

$$F_{n+1} = \left[\begin{array}{cccc|c} & & F_n & & 0 \\ \hline 0 & \cdots & 0 & UI_{n+1} & C_{n+1} \end{array}\right] : \begin{array}{ccc} R(-1)^{\binom{n+2}{2}} & \xrightarrow{F_n} & R^{\binom{n+1}{2}} \\ \bigoplus & \searrow & \bigoplus \\ R(-1)^{n+2} & \longrightarrow & R^{n+1} \end{array}$$

that $\operatorname{reg}(\operatorname{Coker}(\Psi_n)) \leqslant \operatorname{reg}(\operatorname{Coker}(\Psi_{n+1}))$, and hence

$$\operatorname{reg}(\operatorname{Coker}(\Phi_n)) \leqslant \operatorname{reg}(\operatorname{Coker}(\Phi_{n+1})) - 1.$$

(iii) In view of the diagram (4.5.5), the composition $\Phi_1\Phi_2\cdots\Phi_n$ can be interpreted by the map $\delta^n : R[X,Y,Z]_n \to R$, where $\delta = U\delta_X + V\delta_Y + W\delta_Z$. Therefore $\operatorname{Image}(\Phi_1\Phi_2\cdots\Phi_n)$ is equal to the coefficient ideal of $(U\delta_X + V\delta_Y + W\delta_Z)^n$. Hence the result follows from Lemma 4.5.10.

(iv) Let $\mathcal{B}'_n := \left\{m_1, \ldots, m_{\binom{n+2}{2}}\right\}$ be the set of monomial generators of $R[X,Y,Z]_n$ ordered by lex with $X \succ Y \succ Z$. Let $\mathcal{A}_n$ be the R-submodule of $R(X,Y,Z)$ generated by the ordered set

$$\mathcal{B}''_n := \left\{\frac{X^nY^nZ^n}{m_i} : 1 \leqslant i \leqslant \binom{n+2}{2}\right\}.$$

Clearly, $R[X,Y,Z]_n$ and $\mathcal{A}_n$ both are free R-modules of same rank with ordered bases $\mathcal{B}'_n$ and $\mathcal{B}''_n$ respectively. Consider the R-linear map $\delta^n : \mathcal{A}_n \to R[X,Y,Z]_n$ defined by acting δ^n on the basis elements of $\mathcal{A}_n$, where $\delta = U\delta_X + V\delta_Y + W\delta_Z$. Let G_n be the matrix representation of δ^n with respect to the described bases. Thus we have a commutative diagram

$$\begin{array}{ccccc} R(-2n)^{\binom{n+2}{2}} & \xrightarrow{G_n} & R(-n)^{\binom{n+2}{2}} & \xrightarrow{F_n} & R(-n+1)^{\binom{n+1}{2}} \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ \mathcal{A}_n & \xrightarrow{\delta^n} & R[X,Y,Z]_n & \xrightarrow{\delta} & R[X,Y,Z]_{n-1}. \end{array} \tag{4.5.9}$$

Since $n = 2^l - 1$, the composition $\delta^{n+1} : \mathcal{A}_n \to R[X, Y, Z]_{n-1}$ is a zero map. It follows that the top row of (4.5.9) is also a complex. Writing $m_i = X^{a_{i_1}} Y^{a_{i_2}} Z^{a_{i_3}}$ for $1 \leqslant i \leqslant \binom{n+2}{2}$, the matrix G_n can be expressed as

$$(G_n)_{(i,j)} = \epsilon_{(i,j)} U^{n-a_{i_1}-a_{j_1}} V^{n-a_{i_2}-a_{j_2}} W^{n-a_{i_3}-a_{j_3}},$$

where $\epsilon_{(i,j)} = 1$ if $U^{n-a_{i_1}-a_{j_1}} V^{n-a_{i_2}-a_{j_2}} W^{n-a_{i_3}-a_{j_3}} \in \mathcal{B}_n$ as defined in Lemma 4.5.10, and $\epsilon_{(i,j)} = 0$ else. Therefore G_n is a symmetric matrix. Hence

$$\begin{aligned} 0 \longrightarrow R(-2n-1)^{\binom{n+1}{2}} &\xrightarrow{F_n^t} R(-2n)^{\binom{n+2}{2}} \\ &\xrightarrow{G_n} R(-n)^{\binom{n+2}{2}} \xrightarrow{F_n} R(-n+1)^{\binom{n+1}{2}} \longrightarrow 0 \end{aligned} \tag{4.5.10}$$

is a complex. Note that the ideal of maximal minors of F_n has depth 3. On the other hand, choosing the $(n+1)$ rows and columns of G_n indexed by

$$\left\{X^{n-i}Y^i : 0 \leqslant i \leqslant n\right\} \quad \text{and} \quad \left\{X^nY^nZ^n/X^{n-j}Y^j : 0 \leqslant j \leqslant n\right\}$$

respectively, the corresponding submatrix is antidiagonal with entries W^n on the antidiagonal. Similarly, one may consider suitable minors for U and V. Thus the ideal $I_{n+1}(G_n)$ of all $(n+1)$ minors of G_n contains pure powers of U, V and W. So $\operatorname{depth}(I_{n+1}(G_n), R) = 3$. Therefore, by Buchsbaum-Eisenbud acyclicity criterion [16, 1.4.13], (4.5.10) is acyclic. So $\operatorname{reg}(\operatorname{Coker}(\Phi_n)) = 2(n-1)$.

(v) Set $g(n) := \operatorname{reg}(\operatorname{Coker}(\Phi_1\Phi_2\cdots\Phi_n))$. It follows from (i) that every Φ_n is surjective on all high enough graded components. Let $n = 2^l$. Then, by (iii), $g(n) > g(n-1)$, which implies that the component $[\Phi_1\Phi_2\cdots\Phi_{n-1}]_{g(n)}$ is onto, but $[\Phi_1\Phi_2\cdots\Phi_n]_{g(n)}$ is not onto. Therefore $[\Phi_n]_{g(n)}$ is not onto, and hence $\operatorname{reg}(\operatorname{Coker}(\Phi_n)) \geqslant g(n) = 3(n-1)$ by (iii). Along with this inequality, the statements (ii) and (iv) yield that

$$\begin{aligned} 3(n-1) &\leqslant \operatorname{reg}(\operatorname{Coker}(\Phi_n)) \\ &\leqslant \operatorname{reg}(\operatorname{Coker}(\Phi_{n+1})) - 1 \\ &\ \ \vdots \\ &\leqslant \operatorname{reg}(\operatorname{Coker}(\Phi_{2n-1})) - (n-1) \\ &= 2(2n-2) - (n-1) = 3(n-1). \end{aligned}$$

Therefore all the above inequalities must be equalities, and it follows that

$$\operatorname{reg}(\operatorname{Coker}(\Phi_n)) = 2(2^l - 1) + n - 1 \text{ if } 2^l \leqslant n \leqslant 2^{l+1} - 1 \text{ for some } l \geqslant 0.$$

□

With all the ingredients in Lemma 4.5.11, we are now able to compute the regularity of Ext and Tor modules in Example 4.5.1.

Proof of Example 4.5.1. The expressions for indeg are shown in (4.5.2) and (4.5.4). In view of (4.5.1) and (4.5.3), it requires to compute the regularity of kernel and cokernel of

$$\begin{aligned} \Phi_n &:= N(-n)^{\binom{n+2}{2}} \xrightarrow{F_n} N(-n+1)^{\binom{n+1}{2}} \quad \text{and} \\ \Psi_n &:= N(n-1)^{\binom{n+1}{2}} \xrightarrow{F_n^t} N(n)^{\binom{n+2}{2}} \end{aligned} \tag{4.5.11}$$

for all $n \geqslant 1$. Since N is annihilated by (x, y, z), we can substitute N with $R := K[U, V, W]$, and the entries u, v, w in the matrices F_n with U, V, W respectively.

(i) By the observations made in 4.5.9, the complex

$$0 \longrightarrow R(n-1)^{\binom{n+1}{2}} \xrightarrow{F_n^t} R(n)^{\binom{n+2}{2}} \longrightarrow 0$$

is acyclic, and it provides a graded minimal R-free resolution of $\operatorname{Coker}(\Psi_n)$. Therefore, $\operatorname{Ker}(\Psi_n) = 0$ and $\operatorname{reg}(\operatorname{Coker}(\Psi_n)) = -n$ for every $n \geqslant 1$. Hence the assertion follows from (4.5.3).

(ii) It follows from the Koszul complex of U, V, W over R that regularities of $\operatorname{Coker}(\Phi_1)$ and $\operatorname{Ker}(\Phi_1)$ are 0 and 2 respectively. So we need to focus on $n \geqslant 2$. By virtue of Lemma 4.5.11(v),

$$\operatorname{reg}(\operatorname{Coker}(\Phi_n)) = 2(2^l - 1) + n - 1 \quad \text{if } 2^l \leqslant n \leqslant 2^{l+1} - 1. \tag{4.5.12}$$

Thus, for every $n \geqslant 2$, since $\operatorname{reg}(\operatorname{Coker}(\Phi_n)) > n - 1$, in view of (4.5.11),

$$\operatorname{reg}(\operatorname{Coker}(\Phi_n)) = \max\{n - 1, \operatorname{reg}(\operatorname{Ker}(\Phi_n)) - 2\} = \operatorname{reg}(\operatorname{Ker}(\Phi_n)) - 2. \tag{4.5.13}$$

Therefore (4.5.12) and (4.5.13) yield that

$$\operatorname{reg}(\operatorname{Ker}(\Phi_n)) = 2(2^l - 1) + n + 1 \quad \text{if } 2^l \leqslant n \leqslant 2^{l+1} - 1. \tag{4.5.14}$$

It follows from (4.5.1), (4.5.12) and (4.5.14) that

$$\begin{aligned} &\operatorname{reg}\left(\operatorname{Tor}_n^A(M, N)\right) = \max\left\{\operatorname{reg}(\operatorname{Ker}(\Phi_n)), \operatorname{reg}(\operatorname{Coker}(\Phi_{n+1}))\right\} = \\ &\begin{cases} \max\left\{2(2^l - 1) + n + 1, 2(2^l - 1) + n\right\} = 2^{l+1} - 1 + n & \text{if } 2^l \leqslant n \leqslant 2^{l+1} - 2 \\ \max\left\{2(2^l - 1) + n + 1, 2(2^{l+1} - 1) + n\right\} = 2^{l+2} - 2 + n & \text{if } n = 2^{l+1} - 1. \end{cases} \end{aligned}$$

Hence, computing $\operatorname{reg}(\operatorname{Tor}_1^A(M, N)) = 3$ separately, the assertion follows. □

Remark 4.5.12. Note that by (4.5.1) and Lemma 4.5.11(i),

$$H^0_{A_+}\big(\mathrm{Tor}^A_\star(M,N)\big)^\vee = \bigoplus_{n\geqslant 1}(\mathrm{Coker}(\Phi_n))^\vee.$$

Hence Lemma 4.5.11(i) and (v) yield that

$$\mathrm{indeg}\left(H^0_{A_+}\big(\mathrm{Tor}^A_n(M,N)\big)^\vee\right) = -2(2^l-1)-n+1 \text{ if } 2^l \leqslant n \leqslant 2^{l+1}-1.$$

Therefore, by Proposition 4.3.1, one cannot make $H^0_{A_+}\big(\mathrm{Tor}^A_\star(M,N)\big)^\vee$ a finitely generated module over any Noetherian $\mathbb{Z}$-graded algebra $A[z_1,\ldots,z_r]$.

Chapter 5

Regularity of powers of bicyclic graphs

One of the most important results on the behavior of the regularity of powers of ideals was given independently by Cutkosky, Herzog, and Trung in [30], and by Kodiyalam in [63]. In both papers, it is proved that for all $q \geq q_0$, the regularity of powers of I is asymptotically a linear function $\operatorname{reg}(I^q) = dq + b$, where q_0 is the so-called stabilizing index, and b is the so-called constant. The value of d in the above formula is well understood. For example, d is equal to the degree of the generators of I when I is equigenerated. However, their method does not give precise information on q_0 and b.

Since then, many researchers have tried to compute q_0 and b for special families of ideals. The most simple case, yet interesting, is when I is the edge ideal of a finite simple graph. Let $G = (V(G), E(G))$ denote a finite simple undirected graph. Let R be the polynomial ring $k[x_i \mid x_i \in V(G)]$ where K is any field. The edge ideal $I(G)$ of G is the ideal

$$I(G) = (x_i x_j \mid \{x_i, x_j\} \in E(G)).$$

Several authors have settled the problem of determining the stabilizing index and the constant for special families of graphs. Banerjee proved that $\operatorname{reg} I(G)^q = 2q$, for all $q \geq 2$, when G is a gap-free and cricket-free graph (see [7]). E. Nevo and I. Peeva give a conjecture on regularity of powers of C_4-free edge ideals (see [70]). Moghimian, Fakhari, and Yassemi answered the question for the family of whiskered graphs (see [69]). Beyarslan, Hà, and Trung settled the problem for the family of forests and cycles (see [10]). Their results were expanded to the family of unicyclic graphs by Alilooee, Beyarslan, and Selvaraja (see [2]). Moreover, Alilooee and Banerjee determined the stabilizing index and the constant for the family of bipartite graphs with regularity equal to three (see [1]). Jayanthan and Selvaraja

settled the problem for the family of very well-covered graphs (see [60]). Recently, Erey proved that if G is a gap-free and diamond-free graph, then $\operatorname{reg} I(G)^q = 2q$ for all $q \geq 2$ (see [40]). The approach is focused on the relations between the combinatorics of graphs and algebraic properties of edge ideals. We refer the reader to see [62], [52], [11], [56], [5], [80] and [71] for more information on this topic.

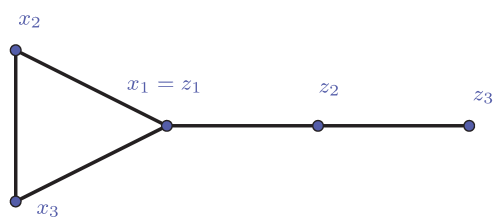
5.1 Castelnuovo-Mumford regularity of bicyclic graphs

In this section we recall some theorems about regularity of bicyclic graphs from [27].

Notation 5.1.1. Let ξ_3 be the function defined as below

$$\xi_3(n) = \begin{cases} 1 & \text{if } n \equiv 0, 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Let $C_n \cdot P_l$ be the graph given by connecting the path P_l to the cycle C_n. For instance, the graph $C_3 \cdot P_3$ can be illustrated as the following:



Proposition 5.1.2. *[27, Proposition 2.3] Let $n \geqslant 3$ and $l \geqslant 1$, then*

$$\nu(C_n \cdot P_l) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{l - \xi_3(n) + 1}{3} \right\rfloor.$$

Theorem 5.1.3. *[27, Theorem 2.4] Let $n, m \geqslant 3$ and $l \geqslant 1$, then*

$$\nu(C_n \cdot P_l \cdot C_m) = \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{m}{3} \right\rfloor + \left\lfloor \frac{l - \xi_3(n) - \xi_3(m) + 1}{3} \right\rfloor.$$

Theorem 5.1.4. *[27, Theorem 2.16] Let $m, n \geqslant 3$ and $l \geqslant 1$, then*

(i) if $l \equiv 0, 1 \pmod 3$, then

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & \text{if } n, m \equiv 2 \pmod 3, \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise;} \end{cases}$$

(ii) if $l \equiv 2 \pmod 3$*, then*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \begin{cases} \nu(C_n \cdot P_l \cdot C_m) + 2 & n \equiv 0, 1 \pmod 3,\ m \equiv 2 \pmod 3; \\ \nu(C_n \cdot P_l \cdot C_m) + 1 & \text{otherwise.} \end{cases}$$

5.2 Castelnuovo-Mumford regularity of powers

In this section, we study the regularity of the powers of $I(C_n \cdot P_l \cdot C_m)$ when $l \leq 2$. Our strategy to compute $\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q$ for $q \geqslant 1$ relies on finding an upper bound and a lower bound on $\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q$ where these bounds coincide and are equal to

$$2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2.$$

In order to obtain an upper bound, we follow the even-connection argument given in [7, Theorem 5.2]. Then we proceed by looking at "nice" induced subgraphs of $C_n \cdot P_l \cdot C_m$ and we find a lower bound on $\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q$ which is equal to the found upper bound.

Let I be an arbitrary ideal generated in degree d and let $b_q := \operatorname{reg}(I^q) - dq$ for $q \geq 1$. An interesting question is to study of the sequence $\{b_i\}_{i\geq 1}$. In [36] Eisenbud and Harris proved that if $\dim(R/I) = 0$, then $\{b_i\}_{i\geq 1}$ is a weakly decreasing sequence of non-negative integers. In [8] Banerjee, Beyarslan and Hà conjectured that for any edge ideal, $\{b_i\}_{i\geq 1}$ is a weakly decreasing sequence (see [8, Conjecture 7.11]). For the edge ideal of any dumbbell graph with $l \leq 2$, we prove $b_i = b_1$ for all $i \geq 1$. However, we expect $b_i \leq b_1$ for all $i \geq 1$ for any graph.

Remark 5.2.1. From Theorem 5.1.3 and Theorem 5.1.4, for any $l \leqslant 2$ we have that

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m) \geqslant \left\lfloor \frac{n+m+l+1}{3} \right\rfloor.$$

The previous inequality is not satisfied when $l \geqslant 3$, because $\operatorname{reg} I(C_4 \cdot P_3 \cdot C_4) = 3$ and $\lfloor \frac{4+4+3+1}{3} \rfloor = 4$.

As recalled earlier, we use the notation of even-connection from Banerjee [7, Theorem 5.2]. The following lemma is important in our treatment of the even-connected vertices, and its proof is similar to [7, Lemma 6.13].

Lemma 5.2.2. *Let G be a graph. As in Remark 1.3.4, let G' be the graph associated to $(I(G)^{q+1} \colon e_1 \cdots e_q)^{pol}$. Suppose $u = p_0, p_1, \ldots, p_{2s+1} = v$ is a path that even-connects u and v with respect to the q-fold $e_1 \cdots e_q$. Then we have*

$$\bigcup_{i=0}^{2s+1} N_{G'}[p_i] \subset N_{G'}[u] \cup N_{G'}[v].$$

Proof. Let U be the set of vertices $U = \{p_0, p_1, \ldots, p_{2s+1}\}$. For each $1 \leqslant k \leqslant s$ we have that $p_{2k-1}p_{2k} = e_{j_k}$ for some $1 \leqslant j_k \leqslant q$, i.e. u and v are even connected with respect to the s-fold $e_{j_1}e_{j_2}\cdots e_{j_s}$.

Let w be a vertex even-connected to some vertex $z \in U$ with respect to the q-fold $e_1 \cdots e_q$. Then, there exists a path $z = r_0, r_1, \ldots, r_{2t+1} = w$ that even-connects z and w with respect to the q-fold $e_1 \cdots e_q$. Let i be the largest integer such that $r_i \in U$. From the fact that $r_0 = z \in U$, we have that the integer i is well defined and $i \geqslant 0$. Let k be an integer such that $p_k = r_i$.

The proof is now divided into four different cases depending on $i \bmod 2$ and $k \bmod 2$. When i and k are both odd integers, we have that $r_i r_{i+1}$ is equal to some edge of $\{e_1, e_2, \ldots, e_q\}$ and that $p_{k-1}p_k$ is not equal to any edge of $\{e_{j_1}, e_{j_2}, \ldots e_{j_s}\}$. By the definition of i we have

$$\{r_{i+1}, r_{i+2}, \ldots, r_{2t+1}\} \cap U = \emptyset.$$

So, in this case, it follows that

$$u = p_0, \ldots, p_{k-1}, p_k = r_i, r_{i+1}, \ldots, r_{2t+1} = w$$

is a path that even-connects u and w with respect to the q-fold $e_1 \cdots e_q$.

The other three cases follow in a similar way. Therefore, we have that if w even-connected to some $z \in U$, then w is even-connected to either u or v.

Now, we only need to prove that any $w \in N_G[z]$ for some $z \in U$ is even-connected to either u or v. This part is simple, if $z = p_{2j}$ then $u = p_0, \ldots, p_{2j} = z, w$ is a path that even-connects u and w, otherwise, if $z = p_{2j-1}$ then $w, z = p_{2j-1}, \ldots, p_{2s+1} = v$ is a path that even-connects w and v.

So we are done. □

The next lemma is similar to [10, Lemma 5.1], but adapted to the current setting of a dumbbell.

Lemma 5.2.3. *Let $G = C_n \cdot P_l \cdot C_m$. If $(I(G)^{q+1} \colon e_1 \cdots e_q)$ is not a square-free monomial ideal and G' is the associated graph, then there exists a vertex z which is even-connected to itself. Then, G' has a leaf and $N_{G'}[z]$ contains one of the two cycles. In particular, if we denote the corresponding leaf by e, then G'_e is an induced subgraph of a unicyclic graph.*

Proof. Suppose $z = p_0, p_1, \ldots, p_{2l+1} = z$ is an even-connection of z with itself. Let $0 \leqslant a < b \leqslant 2l+1$ be integers such that $p_a, p_{a+1}, \ldots, p_b = p_a$ is an even-connection and $b - a$ is

minimal. Then, $p_a, p_{a+1}, \ldots, p_b = p_a$ is a simple closed path lying on $C_n \cdot P_l \cdot C_m$ and so it is necessarily equal to either C_n or C_m.

Finally, Lemma 5.2.2 implies that $N_{G'}[z]$ contains either C_n or C_m. □

Lemma 5.2.4. *Let $G = C_n \cdot P_l \cdot C_m$ with $l \leqslant 2$ and H be a graph such that G is a subgraph of H with the same set of vertices (i.e., $V(H) = V(G)$ and $E(H) \supseteq E(G)$). For any two vertices $u, v \in H$ such that $\{u, v\} \notin E(G)$, we have that*

$$\operatorname{reg} I\left(H \setminus (N_H[u] \cup N_H[v])\right) \leqslant \operatorname{reg} I(G) - 1.$$

Proof. Let $K = N_G[u] \cap N_G[v]$. We divide the proof according to the cardinality $|K|$ of K. Notice that for the dumbbell G we always have $0 \leqslant |K| \leqslant 2$.

Since $H \setminus (N_H[u] \cup N_H[v])$ is an induced subgraph of $H \setminus (N_G[u] \cup N_G[v])$, from Theorem 1.3.1$(i)$, it is enough to prove that $\operatorname{reg} I\left(H \setminus (N_G[u] \cup N_G[v])\right) \leqslant \operatorname{reg} I(G) - 1$.

Step 1. Suppose that $|K| = 0$. Then, the graph $H \setminus (N_G[u] \cup N_G[v])$ is obtained by deleting at least 6 vertices, and so $|H \setminus (N_G[u] \cup N_G[v])| \leqslant |G| - 6 \leqslant n + m + l - 8$. Note that we can add two vertices to $H \setminus (N_G[u] \cup N_G[v])$ and connect them in such a way that we obtain a Hamiltonian path. Let L be a graph obtained by adding two vertices and certain edges connecting these two new vertices, such that L has a Hamiltonian path. Since $|L| \leqslant n + m + l - 6$, Theorem 1.3.9 yields

$$\operatorname{reg} I(L) \leq \left\lfloor \frac{n + m + l - 5}{3} \right\rfloor + 1 = \left\lfloor \frac{n + m + l + 1}{3} \right\rfloor - 1,$$

and by applying Remark 5.2.1, we get $\operatorname{reg} I(L) \leq \operatorname{reg} I(G) - 1$. Since $H \setminus (N_G[u] \cup N_G[v])$ is an induced subgraph of L, Theorem 1.3.1(i) implies that $\operatorname{reg} I\left(H \setminus (N_G[u] \cup N_G[v])\right) \leq \operatorname{reg} I(G) - 1$.

Step 2. Suppose that $|K| = 1$. Here the proof follows along the same lines of Step 1. In this case the graph $H \setminus (N_G[u] \cup N_G[v])$ is obtained by deleting at least 5 vertices. Now, note that we can add one vertex to $H \setminus (N_G[u] \cup N_G[v])$ and connect it in such a way that we obtain a Hamiltonian path. Let L be a graph obtained by adding one vertex and certain edges connecting this new vertex, such that L has a Hamiltonian path. Since $|L| \leqslant (n + m + l - 2) - 5 + 1 = n + m + l - 6$, then the rest of the proof follows as in Step 1.

Step 3. Suppose that $|K| = 2$. In this case, note that one of the cycles is necessarily equal to C_4, say $C_n = C_4$, and that $u, v \in C_4$ with $\{u, v\} \notin E(G)$. Hence, it follows that $H \setminus (N_G[u] \cup N_G[v])$ has a Hamiltonian path with $\leqslant m$ vertices if $l = 2$ and $\leqslant m - 1$ vertices if $l = 1$. From Theorem 1.3.9 and Remark 5.2.1, then we have $\operatorname{reg} I(H \setminus (N_G[u] \cup N_G[v])) \leq \operatorname{reg} I(G) - 1$.

So we are through. □

Theorem 5.2.5. *Let $G = C_n \cdot P_l \cdot C_m$ with $l \leqslant 2$ and $I = I(G)$ be its edge ideal, then*

$$\operatorname{reg}(I^{q+1} \colon e_1 \cdots e_q) \leq \operatorname{reg} I$$

for any $1 \leq q$ and any edges $e_1, \ldots, e_q \in E(G)$.

Proof. We split the proof into two cases.

Case 1. First, suppose $(I^{q+1} \colon e_1 \cdots e_q)$ is a square-free monomial ideal. In this case $(I^{q+1} \colon e_1 \cdots e_q) = I(G')$ where G' is a graph with $V(G) = V(G')$ and $E(G) \subseteq E(G')$. Let $E(G') = E(G) \cup \{a_1, \ldots, a_r\}$, then each edge a_i is induced from even-connecting two different vertices (i.e., each a_i is not a whisker). By Theorem 1.3.1, we have

$$\operatorname{reg} I(G') \leq \max\{\operatorname{reg} I(G' \setminus a_1), \operatorname{reg} I(G'_{a_1}) + 1\}$$

Since $a_1 \notin E(G)$, Lemma 5.2.4 implies that $\operatorname{reg} I(G'_{a_1}) + 1 \leqslant \operatorname{reg} I(G)$.

In the same way, for any subgraph $H = G' \setminus \{a_1, \ldots, a_i\}$, since $V(H) = V(G)$ and $E(H) \supseteq E(G)$, Lemma 5.2.4 also gives us that

$$\operatorname{reg}(I(H_{a_{i+1}})) + 1 \leqslant \operatorname{reg}(I(G)).$$

By continuing this process, we get $\operatorname{reg} I(G') \leq \operatorname{reg} I(G)$.

Case 2. Suppose $(I^{q+1} \colon e_1 \cdots e_q)$ is not square-free and G' is the graph associated to $(I^{q+1} \colon e_1 \cdots e_q)^{\mathrm{pol}}$. Let $\{b_1, b_2, \ldots, b_s\}$ be the subset of edges of $E(G') \setminus E(G)$ that are generated by square monomials (i.e., each b_i is a whisker).

From Theorem 1.3.1 we have the inequality

$$\operatorname{reg} I(G') \leqslant \max\{\operatorname{reg} I(G' \setminus b_1), 1 + \operatorname{reg} I(G'_{b_1})\}.$$

Remark 5.2.3 implies that one of the cycles is deleted from G'_{b_1}, then there exists an edge $e \in G$ such that $d(e, G'_{b_1}) \geqslant 2$. So, for such an edge e we get that the disjoint union $G'_{b_1} \cup e$ is an induced subgraph of $G' \setminus b_1$. Thus, Theorem 1.3.1 and [8, Corollary 3.10] yield that

$$\operatorname{reg}(I(G'_{b_1})) + 1 = \operatorname{reg}(I(G'_{b_1} \cup e)) \leqslant \operatorname{reg}(I(G' \setminus b_1)).$$

Therefore, we obtain that $\operatorname{reg} I(G') \leqslant \operatorname{reg} I(G' \setminus b_1)$.

By applying the same argument, it follows that

$$\operatorname{reg} I(G') \leqslant \operatorname{reg} I(G' \setminus b_1) \leqslant \operatorname{reg} I(G' \setminus \{b_1, b_2\}) \leqslant \cdots \leqslant \operatorname{reg} I(G' \setminus \{b_1, \ldots, b_s\}).$$

Since the graph $G' \setminus \{b_1, \ldots, b_s\}$ has no whiskers, then Step 1 implies that

$$\operatorname{reg} I(G') \leqslant \operatorname{reg} I(G' \setminus \{b_1, \ldots, b_s\}) \leqslant \operatorname{reg} I(G).$$

Therefore, the proof is completed. □

Remark 5.2.6. The previous theorem is a generalization of a work done by Gu in [49] for the case $l = 1$.

Theorem 5.2.7. *For the dumbbell graph $C_n \cdot P_l \cdot C_m$ with $l \leqslant 2$, we have*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q \geq 2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2,$$

for any $q \geqslant 1$.

Proof. Using the inequality $\operatorname{reg} I(C_n \cdot P_2 \cdot C_m)^q \geqslant 2q + \nu(C_n \cdot P_2 \cdot C_m) - 1$ of Theorem 1.3.8, for the cases where $\operatorname{reg} I(C_n \cdot P_l \cdot C_m) = \nu(C_n \cdot P_l \cdot C_m) + 1$ we get the expected inequality. We divide the proof in two halves, the cases $l = 1$ and $l = 2$.

Case 1. Let $l = 1$. We only need to focus on the case where $n, m \equiv 2 \pmod 3$. Let $H = (C_n \cdot P_1 \cdot C_m) \setminus \{x_n\} = P_{n-1} \cdot C_m$ be an induced subgraph of $C_n \cdot P_1 \cdot C_m$ Using Theorem 5.1.3, Proposition 5.1.2 and the modularity $n, m \equiv 2 \pmod 3$, we can check that

$$\nu(H) = \nu(C_n \cdot P_1 \cdot C_m)$$

and that

$$\nu(H) = \nu(H \setminus \Gamma_H(C_m)).$$

From Theorem 5.1.4 and [2, Theorem 1.2] we get

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m) = \nu(C_n \cdot P_1 \cdot C_m) + 2 = \nu(H) + 2 = \operatorname{reg} I(H).$$

Since H is an induced subgraph of $C_n \cdot P_1 \cdot C_m$, then from [2, Theorem 1.1] and [10, Corollay 4.3] we get the inequality

$$\operatorname{reg} I(C_n \cdot P_1 \cdot C_m)^q \geqslant \operatorname{reg} I(H)^q = 2q + \operatorname{reg} I(H) - 2 = 2q + \operatorname{reg} I(C_n \cdot P_1 \cdot C_m) - 2.$$

Case 2. Let $l = 2$. We only need to focus on the cases where $n \equiv 0, 1 \pmod 3$ and $m \equiv 2 \pmod 3$. We take $H = (C_n \cdot P_2 \cdot C_m) \setminus \{x_1\}$ of $C_n \cdot P_2 \cdot C_m$ is given as the union of a path of length $n - 1$ and the cycle C_m, i.e., $H = P_{n-1} \cup C_m$.

By Theorem 5.1.4, for the cases $n \equiv 0, 1 \pmod 3$ and $m \equiv 2 \pmod 3$, we have

$$\operatorname{reg} I(C_n \cdot P_2 \cdot C_m) = \nu(C_n \cdot P_2 \cdot C_m) + 2 = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2,$$

and from [8, Corollary 3.10], [81, Theorem 2.18] and Theorem 1.3.7 we obtain

$$\operatorname{reg} I(H) = \operatorname{reg}(I(P_{n-1})) + \operatorname{reg}(I(C_m)) - 1 = \nu(P_{n-1}) + \nu(C_m) + 2 = \lfloor \frac{n}{3} \rfloor + \lfloor \frac{m}{3} \rfloor + 2.$$

Hence, we get $\operatorname{reg} I(C_n \cdot P_2 \cdot C_m) = \operatorname{reg} I(H)$. Finally, using [2, Theorem 1.1] and [10, Corollary 4.3], we get the inequality

$$\operatorname{reg} I(C_n \cdot P_2 \cdot C_m)^q \geqslant \operatorname{reg} I(H)^q = 2q + \operatorname{reg} I(H) - 2 = 2q + \operatorname{reg} I(C_n \cdot P_2 \cdot C_m) - 2.$$

Therefore, the proof is completed. □

Theorem 5.2.8. *For the dumbbell graph $C_n \cdot P_l \cdot C_m$ with $l \leq 2$, we have*

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q = 2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2$$

for all $q \geq 1$.

Proof. It follows by Theorem 5.2.5, Theorem 1.3.5 and Theorem 5.2.7. □

Remark 5.2.9. One may ask whether

$$\operatorname{reg} I(C_n \cdot P_l \cdot C_m)^q = 2q + \operatorname{reg} I(C_n \cdot P_l \cdot C_m) - 2$$

always holds for given n, m, l and q. Unfortunately, it is no longer true for any n, m, l and q as it can be seen from the following example:

$$6 = \operatorname{reg} I(C_5 \cdot P_3 \cdot C_5)^2 < 4 + \operatorname{reg} I(C_5 \cdot P_3 \cdot C_5) - 2 = 7.$$

Chapter 6

Cohomologies of complete intersection in $\mathbb{P}^n \times \mathbb{P}^m$

6.1 linearTruncations Package

In the begining of this section we present a Macaulay2 package, entitled *linearTruncations*, which computes the multigraded truncations that give linear resolutions.

Castelnuovo-Mumford regularity is a fundamental invariant in commutative algebra and algebraic geometry. Roughly speaking, it measures the complexity of a module or a sheaf. There is a folklore result that Castelnuovo-Mumford regularity is the smallest number where truncation of the module has a linear resolution. An extension of Castelnuovo-Mumford regularity for a multigraded case was first introduced by Hoffman and Wang in a special case [57], and later by Maclagan and Smith in [65] and Botbol and Chardin in [12] in a more general setting. An interesting question is to ask about the relation between Castelnuovo-Mumford regularity of M in the multigraded case and the degrees $\mathbf{d}$ where the truncation $M_{\geq \mathbf{d}}$ has a linear resolution. Note that, If $M_{\geq \mathbf{d}}$ has a linear resolution so does $M_{\geq \mathbf{d}'}$ for all $\mathbf{d}' \geq \mathbf{d}$. Therefore these multidegrees form a region and we call it *linear truncations*. Also, it is enough to find minimal generators for the region where the truncation has a linear resolution.

The first question is to show that the linear truncations is a non-empty set. In Subsection 6.1.1 we will answer to this question. Indeed, in [34] the authors provide a degree $\mathbf{d}$ where $M_{\geq \mathbf{d}}$ has a linear resolution. Unfortunately, this degree, in general, is greater than the generators of a linear truncation region. Moreover, we refine this theorem and give a better bound in the bigraded case in Theorem 6.1.3.

In the last section, we provide some interesting examples and give an answer to some initial questions that may arise about linear truncations.

6.1.1 Linear Truncations

Throughout, we shall use the following notations. Let k be a field and $S = k[x_1, \ldots, x_n]$ be a $\mathbb{Z}^r$-graded polynomial ring over k. Let M be a finitely generated S-module, let $\mathbf{d} = (d_1, \ldots, d_r) \in \mathbb{Z}^r$, define $\bar{\mathbf{d}} = d_1 + \cdots + d_r$ to be the total degree of $\mathbf{d}$ and $M_{\geq \mathbf{d}} := \oplus_{\mathbf{d}' \geq \mathbf{d}} M_{\mathbf{d}'}$ is the truncation of M at $\mathbf{d}$. We define the *linear truncations* of M to be

$$\{\mathbf{d} \in \mathbb{Z}^r \mid M_{\geq \mathbf{d}} \text{ has a linear resolution}\} \subset \mathbb{Z}^r .$$

As we mentioned before, in [34] the authors proved the linear truncations is a non-empty set.

Proposition 6.1.1. *[34, Proposition 1.7] Let M be finitely generated $\mathbb{Z}^r$-graded S-module. Suppose M has a finite free multi-homogeneous resolution*

$$0 \leftarrow M \leftarrow G_0 \leftarrow G_1 \leftarrow \cdots \leftarrow G_N \leftarrow 0.$$

Write $G_k = \oplus S(-a)^{\beta_{k,a}}$ and set $b_i = \max\{a_i \mid \exists \beta_{k,a} \neq 0\}$ and $b = (b_1, \ldots, b_r)$, then $M_{\geq b}$ has a linear resolution.

The code `CoarseMultigradedRegularity` in the package is implemented to find an element in the linear truncations of a finitely generated module. While $r \geq 3$, this code is implemented by using Proposition 6.1.1.

```
i0 : S= QQ[x,y,z,Degrees=>{{1,0,0},{0,1,0}, {0,0,1}}];
i1 : I = ideal(x*y*z, x*y^2, y*z^2);
i2 : M = S^1/I;
i3 : coarseMultigradedRegularity M
o3 :  {1, 2, 2}
```

In the above example one can check truncation of M at $(0,0,2), (0,1,1), (1,0,1)$ and $(1,1,0)$ has a linear resolution.

Bigraded Case

In this part, let $S = k[x_0, \dots, x_n, y_0, \dots, y_m]$ be a polynomial ring and $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. for any $\mathbf{d} \in \mathbb{Z}^2$ we have

$$0 \to M_{\geq \mathbf{d}} \to M \to E \to 0$$

where $E = M/M_{\geq \mathbf{d}}$. The above short exact sequence yields the following long exact sequence on Tor modules

$$\cdots \to \operatorname{Tor}_{i+1}^S(M, k) \to \operatorname{Tor}_{i+1}^S(E, k) \to \operatorname{Tor}_i^S(M_{\geq \mathbf{d}}, k) \to \operatorname{Tor}_i^S(M, k) \to \cdots .$$

Therefore, the vanishing of $\operatorname{Tor}_i^S(M_{\geq \mathbf{d}}, k)$ in degree μ is deduced from vanishing of $\operatorname{Tor}_i^S(M, k)$ and $\operatorname{Tor}_{i+1}^S(E)$ in that degree.

Definition 6.1.2. Let M be a finitely generated bigraded S-module. Define

$$b_i^x(M) := \max\{p \mid \exists\, q\,;\, \operatorname{Tor}_i^S(M, k)_{p,q} \neq 0\} = \max\{p \mid \operatorname{Tor}_i^S(M, k[y])_{(p,\star)\neq 0}\}$$
$$b_i^y(M) := \max\{q \mid \exists\, p\,;\, \operatorname{Tor}_i^S(M, k)_{p,q} \neq 0\} = \max\{q \mid \operatorname{Tor}_i^S(M, k[x])_{(\star,q)\neq 0}\}.$$

The partial regularities of M up to i-th step are defined as follows:

$$\operatorname{reg}_x^i(M) := \max_{j \leq i}\{b_j^x(M) - j\}, \operatorname{reg}_y^i(M) = \max_{j \leq i}\{b_j^y(M) - j\}$$

and partial regularities of M are:

$$\operatorname{reg}_x(M) := \max_j\{b_j^x(M) - j\}, \operatorname{reg}_y(M) = \max_j\{b_j^y(M) - j\}.$$

Theorem 6.1.3. *Let $S = k[x_0, \dots, x_n, y_0, \dots, y_m]$ be a bigraded polynomial ring and M be a finitely generated bigraded S-module. Let $\boldsymbol{d} \in \mathbb{Z}^2$, if $\bar{\boldsymbol{d}} \geq \operatorname{reg}(M)$ and $\boldsymbol{d} \geq (\operatorname{reg}_x^t(M), \operatorname{reg}_y^t(M))$, then $M_{\geq \boldsymbol{d}}$ has a linear resolution for t steps. In particular, truncation of M at $(\operatorname{reg}_x(M), \operatorname{reg}_y(M))$ has a linear resolution.*

Proof. For simplicity, we replace M with $M(\mathbf{d})$ and we show $M_{\geq 0}$ has a linear resolution.

We have $\operatorname{Tor}_i^S(M, k)_{(a,b)} = 0$ for any $a > 0$ or $b > 0$ and $i \leq t$. Hence, it is suffices to show $\operatorname{Tor}_{i+1}^S(E, k)_{(a,b)} = 0$ if $a+b \geq i+1$. Note that, $\operatorname{Tor}_\star^S(E, k)$ can be computed by the homologies of the Koszul complex $\mathbf{K}_\bullet((\mathbf{x}, \mathbf{y}), E)$, where $\mathbf{x} = (x_0, \dots, x_n)$ and $\mathbf{y} = (y_0, \dots, y_m)$. On the other hand, homologies of $\mathbf{K}_\bullet((\mathbf{x}, \mathbf{y}), E)$ is the total homologies of the double Koszul complex $\mathbf{K}_{\bullet,\bullet}(\mathbf{x}; \mathbf{y}, E)$. Therefore,

$$\begin{aligned}\operatorname{Tor}_{i+1}^S(E, k)_{(a,b)} &\cong H_{i+1}\mathbf{K}_\bullet((\mathbf{x}, \mathbf{y}), E)_{(a,b)} \\ &\cong H_{i+1}(\mathbf{Tot}(\mathbf{K}_{\bullet,\bullet}(\mathbf{x}; \mathbf{y}, E))_{(a,b)}.\end{aligned}$$

In the spectral sequence, $\mathbf{K}_{p,q}(\mathbf{x};\mathbf{y},E)_{(a,b)} \cong E(a-p,b-q)^{\binom{n}{p}\binom{m}{q}}$. Let

$$z \in Z_{i+1}(\mathbf{Tot}(\mathbf{K}_{\bullet,\bullet}(\mathbf{x};\mathbf{y},E))_{(a,b)} \subseteq \oplus_{p+q=i+1}\mathbf{K}_{p,q}(\mathbf{x};\mathbf{y},E)_{(a,b)}$$

be a cycle. Decompose z into $z = z_1 \oplus z_2$, where $z_1 \in \oplus_{p>a}\mathbf{K}_{p,q}(\mathbf{x};\mathbf{y},E)_{(a,b)}$,$z_2 \in \oplus_{q>b}\mathbf{K}_{p,q}(\mathbf{x};\mathbf{y},E)_{(a,b)}$ and $p+q=i+1$. It is suffices to show that

$$z_1 \oplus z_2 \in B_{i+1}(\mathbf{Tot}(\mathbf{K}_{\bullet,\bullet}(\mathbf{x};\mathbf{y},E)))_{(a,b)}.$$

Truncate $\mathbf{K}_{p,q}$ where $p > a$ and denote it by $(\mathbf{K}^{p>a}_{\bullet,\bullet})$ and denote the total complex by $(\mathbf{K}^{p>a}_{\bullet}, \partial^{p>a})$. Note that

$$\mathbf{K}^{p>a}_{\bullet,\bullet}(\mathbf{x};\mathbf{y},E)_{(a,b)} \cong \mathbf{K}^{p>a}_{\bullet,\bullet}(\mathbf{x};\mathbf{y},M)_{(a,b)}.$$

Denote the corresponding spectral sequence to $\mathbf{K}^{p>a}_{\bullet,\bullet}(\mathbf{x};\mathbf{y},E)_{(a,b)}$ by $\mathcal{E}$, therefore $\mathcal{E}^1_{p,q} = 0$ for $p \leq a$ and if $p > a$ then

$$\mathcal{E}^1_{p,q_{(a,b)}} \cong \mathrm{Tor}^S_q(M, k[\mathbf{x}])^{\binom{n}{p}}_{(a-p,b)}.$$

In particular, $(\mathcal{E}^1_{p,q})_{(a,b)} = 0$ if $b-q > b^y_q(M) - q$. This holds as $b-q > \mathrm{reg}^q_y(M)$. Indeed, $q \leq i$ since $p \geq 1$ and $b > q$ as $p+q \leq a+b$.

Hence there exists $c_1 \in \mathbf{K}^{p>a}_{i+2}(\mathbf{x};\mathbf{y},E)_{(a,b)}$ such that $\partial^{p>a}(c_1) = z_1$. Write $c_1 = \oplus_{p>a}c_{p,i+2-p}$

$$\partial(c_1) = \partial^{p>a}(c_1) + \partial^x(c_{a+1,i+1-a})$$

$\partial^x(c_{a+1,i+1-a}) \in \mathbf{K}_{a,i+1-a}(\mathbf{x};\mathbf{y},E)_{(a,b)} \cong E_{(0,a+b-(i+1))}{}^{\binom{n}{a}\binom{m}{i+1-a}} = 0$ since $a+b \geq i+1$. The argument for z_2 is the same. □

Remark 6.1.4. The assumption $\bar{\mathbf{d}} \geq \mathrm{reg}\, M$ is necessary for the Theorem 6.1.3. Let $S = k[x_1,y_1,y_2]$ and $I = (x_1^2y_1, x_1y_2^2)$, $\mathrm{reg}(S/I) = 3$. The partial regularities of S/I is $(1,1)$, but one can see the truncation of M at $(1,1)$ does not have a linear resolution.

Proposition 6.1.1 and Theorem 6.1.3 give a single degree in the linear truncations of M . In general, this degree could be far from the minimal generators. Because of that, we implemented `findAllLinearTruncation`, which is useful to find all minimal generators of

linear truncations of M.

Input : A module M and a range (a, b)
Output: Minimal generators of linear truncation with total degree between a and b
$A, L := \emptyset \subset \mathbb{Z}^r$;
while $a \leq i \leq b$ **do**
  for *all* $d \in \mathbb{Z}^r$ *with* $\bar{d} = i$ **do**
    if $d \notin L$ *and* $M_{\geq d}$ *has a linear resolution* **then**
      $L := L \cup (d + \mathbb{Z}^r)$ $A := A \cup \{d\}$
    end
  end
end
return A.

Algorithm 1: Algorithm for implementing `findAllLinearTruncation`

6.1.2 Some Examples

In this section, we provide some interesting examples. In the following example ,the minimal generators of linear truncations have two different total degrees.

```
i0 : S = QQ[x_1..x_6,Degrees=>{{1,0,0},{1,0,0},{0,1,0},{0,1,0},{0,0,1},{0,0,1}}];
i1 : I = ideal(x_1*x_4*x_6,x_1*x_3^2,x_3^2*x_4*x_5,x_2^2*x_5^2,x_1*x_4^2*x_5,x_1*x_2^2*x_4);
i2 : M = S^1/I;
i3 : c = coarseMultigradedRegularity M
o3 = {3, 4, 2}
i4 : findAllLinearTruncations({regularity M, sum coarseMultigradedRegularity M}, M)
o4 = {{2, 2, 1}, {1, 3, 2}}
```

Question 6.1.5. *Let $S = k[x_1, \ldots, x_n]$ be polynomial ring and M be a finitely generated S-module. Is it possible for a degree $\boldsymbol{d}$ where $\bar{\boldsymbol{d}} < \operatorname{reg}(M)$ to be in a linear truncations of M?*

The answer to this question is yes. Let M be an Artinian module, the regularity of M is equal to the maximum degree of the socle elements. On the other hand, truncating at each socle element has a linear resolution. Hence, it is possible to find that $\mathbf{d}$ if the two socle elements of M have different degrees. Here is a simple example:

```
i0 : S = QQ[x,y,Degrees=>{{1,0},{0,1}}];
i1 : I = ideal(x^3, x*y,y^7 );
i2 : M = S^1/I;
i3 : regularity M
o3 = 6
i4 : isLinearComplex (res truncate({2,0},M))
o4 = true
```

Since M is an Artinian module, it is are not very interesting in the algebraic geometry's point of view. Moreover, there is an interesting example by studying 96 complete intersection points in $\mathbb{P}^2 \times \mathbb{P}^2$.

```
i0 : S = QQ[x_0,x_1,x_2,y_0,y_1,y_2,Degrees=>{{1,0},{1,0},{1,0},{0,1},{0,1},{0,1}}];
i1 : I = ideal(x_0^2*y_0^2,x_1^2*y_1^2,x_2^2*y_2^2,(x_0+x_1+x_2)^2*(y_0+y_1+y_2)^2);
i2 : B = intersect(ideal(x_0,x_1,x_2), ideal(y_0, y_1,y_2));
i3 : J = saturate (I,B);
i4 : M = S^1/J;
i5 : regularity M
o5 = 11
i6 : findAllLinearTruncations({0,11},M)
o6 = {{2, 6}, {6, 2}}
```

This example is also interesting because the minimal generators of the linear truncations of M is not a convex set. Furthermore, by looking at the bigraded Hilbert function of M and applying [65, Proposition 6.7], one can see the multigraded regularity and linear truncations in this example are the same.

6.2 Complete intersections in $\mathbb{P}^n \times \mathbb{P}^m$

Notation 6.2.1. Let $S = k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ be a bigraded polynomial ring where $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. Define $B_1 = (x_0, \ldots, x_n), B_2 = (y_0, \ldots, y_m)$ and $B = (x_0, \ldots, x_n) \cdot (y_0, \ldots, y_m)$ the irrelevant ideals of $\mathbb{P}^n, \mathbb{P}^m$ and $\mathbb{P}^n \times \mathbb{P}^m$.

Definition 6.2.2. A subscheme $V \subseteq \mathbb{P}^n \times \mathbb{P}^m$, is a *complete intersection* if $V = \mathrm{Proj}(S/I)$ where I is generated by $\mathrm{codim}(V)$ bihomogeneous elements.

Proposition 6.2.3. *[42, Example 8.4.2] Let $S = k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ be a bigraded polynomial ring where $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. Let $I = (f_1, \ldots, f_{n+m})$ generated by $n + m$ forms of degree (d_i, e_i) and V be the scheme defined by I. If I is a complete intersection, then*

$$\deg(V) = \sum d_{i_1} \cdots d_{i_n} \cdot e_{j_1} \cdots e_{j_m},$$

where the sum is over all permutations $(d_{i_1}, \ldots, d_{i_n}, e_{j_1}, \ldots, e_{j_m})$ of $(1, \ldots, n + m)$ with $i_1 < i_2 < \cdots < i_n$ and $j_1 < j_2 < \cdots < j_m$. In particular if $d_i = d$ and $e_i = e$ for all i, then

$$\deg(V) = \binom{n+m}{n} d^n e^m.$$

Proposition 6.2.4. *Let $I = (f_1, \dots, f_r)$ be a bigraded ideal defining a complete intersection of codimension r and let $\deg(f_i) = \boldsymbol{d}_i$. For $\mu \in \cap_i \operatorname{reg}_B(H_i(\boldsymbol{K}(\boldsymbol{f}, S))$,*

$$HF_{S/I}(\mu) = P(\mu),$$

where P is a polynomial that only depends upon $\boldsymbol{d}_1, \dots, \boldsymbol{d}_r$.

Proof. By the Serre Grothendieck formula

$$HP_{S/I}(\mu) = HF_{S/I}(\mu) + \sum_i (-1)^i \dim H^i_B(S/I)_\mu. \tag{6.2.1}$$

Hence, if $\mu \in \operatorname{reg}_B(S/I)$ then $HP_{S/I}(\mu) = HF_{S/I}(\mu)$. Set

$$\chi(\mu) := \sum_i (-1)^i \dim(H_i(\mathbf{K}(\mathbf{f}, S))_\mu = \sum_i (-1)^i \dim(K_i)_\mu. \tag{6.2.2}$$

Note that the second equality shows that χ is a function that only depends upon the degrees. Since I is complete intersection, $H_i(\mathbf{K}(\mathbf{f}, S) = H^0_B(H_i(\mathbf{K}(\mathbf{f}, S))$. Hence, $\operatorname{Supp}(H_i(\mathbf{K}(\mathbf{f}, S)) \cap \operatorname{reg}_B(H_i(\mathbf{K}(\mathbf{f}, S)) = \emptyset$. Therefore, for $\mu \in \cap_{i>0} \operatorname{reg}_B(H_i(\mathbf{K}(\mathbf{f}, S))$, $\chi(\mu) = HF_{S/I}(\mu)$. Hence, for $\mu \in \cap_i \operatorname{reg}_B(H_i(\mathbf{K}(\mathbf{f}, S))$,

$$HP_{S/I}(\mu) = HF_{S/I}(\mu) = \chi(\mu).$$

Setting $P := HP_{S/I}$ the first equality shows that $HF_{S/I}(\mu) = P(\mu)$ and the second equality shows this function only depends upon the degrees. □

Proposition 6.2.5. *With the Notation 6.2.1, for any bigraded free S-module M,*

$$H^{n+m+1}_B(M) \cong H^{n+m+2}_{\mathfrak{m}}(M).$$

Furthermore, If $n = m$ then

$$H^{n+1}_B(M) \cong H^{n+1}_{B_1}(M) \oplus H^{n+1}_{B_2}(M),$$

else if $n < m$ then,

$$H^{n+1}_B(M) \cong H^{n+1}_{B_1}(M) \quad \textit{and} \quad H^{m+1}_B(M) \cong H^{m+1}_{B_2}(M).$$

Proof. Note that $H^a_{B_1}(M) = H^b_{B_2}(M) = H^c_{\mathfrak{m}}(M) = 0$ if $a \neq n+1$, $b \neq m+1$ and $c \neq n+m+2$. The Meyer-Vietoris exact sequence

$$\cdots \to H^i_{\mathfrak{m}}(M) \to H^i_{B_1}(M) \oplus H^i_{B_2}(M) \to H^i_B(M) \to H^{i+1}_{\mathfrak{m}}(M) \to \cdots .$$

gives the results. □

Theorem 6.2.6. *Let $S = k[x_0, \dots, x_n, y_0, \dots, y_m]$ be a bigraded polynomial ring where $\deg(x_i) = (1,0)$ and $\deg(y_i) = (0,1)$. Define $B = (x_0, \dots, x_n) \cdot (y_0, \dots, y_m)$ the irrelevant ideal of $\mathbb{P}^n \times \mathbb{P}^m$. Assume $I = (f_1, \dots, f_r)$ with $\deg(f_i) = (d_i, e_i)$ and V be the scheme defined by I. If $\operatorname{codim}(V) = r$ then*

- $\operatorname{Supp}(H_i(\boldsymbol{K}(\boldsymbol{f}, S)) = \operatorname{Supp}(V_{n+1+i}) \cup \operatorname{Supp}(W_{m+1+i})$ *for* $i > 0$
- $\operatorname{Supp}(H_B^i(S/I)) = \operatorname{Supp}(V_{n+1-i}) \cup \operatorname{Supp}(W_{m+1-i})$ *for* $0 \le i \le \dim V$, *and*
- $\operatorname{Supp}(H_B^{\dim V+1}(S/I)) = \{(\mu,\nu) | H^{\dim V}(V, \mathcal{O}_V(\mu,\nu)) \neq 0\}$
  $\subseteq \operatorname{Supp}(V_{r-m}) \cup \operatorname{Supp}(W_{r-n}) \cup \{\sigma - \operatorname{Supp}(S/I)\}$

where V_i (resp. W_i) is a subquotient of $H_{B_1}^{n+1}(\boldsymbol{K}_i(\boldsymbol{f}, S))$ (resp. $H_{B_2}^{m+1}(\boldsymbol{K}_i(\boldsymbol{f}, S))$) and $\sigma := \sum_i (d_i, e_i) - (n+1, m+1)$.

Proof. Consider the double complex $\mathbf{C}_B^\bullet(\mathbf{K}_\bullet(\mathbf{f}, S))$ and suppose $n \le m$. If we start taking homologies vertically, by 6.2.5 in the third page we have:

$$\begin{array}{ccccc}
0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
V_r' & V_{r-1}' & \cdots & V_1' & V_0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
W_r' & W_{r-1}' & \cdots & W_1' & W_0' \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
M & \cdots & 0 & 0 & 0.
\end{array}$$

where

$$\begin{aligned}
M &\cong H_r(H_B^{n+m+1}(\mathbf{K}(\mathbf{f}, S)) \\
&\cong H_r\left(H_{\mathfrak{m}}^{n+m+2}\left(\mathbf{K}(\mathbf{f}, S)\right)\right) \\
&\cong H_r(\mathbf{K}(\mathbf{f}, S)\left(\mathbf{d} - (n+1, m+1)\right)^\star) \\
&\cong (S/I)^\star\left(\mathbf{d} - (n+1, m+1)\right).
\end{aligned}$$

If we start taking homology horizontally, in the second page we have

$$\begin{array}{cccccc}
\cdots & 0 & H_\gamma & \cdots & H_1 & H_B^0(S/I) \\
\cdots & 0 & 0 & 0 & 0 & H_B^1(S/I) \\
\cdots & 0 & 0 & 0 & 0 & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & H_B^{\dim(V)+1}(S/I) \\
\vdots & \vdots & \vdots & \vdots & \vdots & 0
\end{array}.$$

where $\gamma := \max\{0, r-n-1, r-m-1\}$. The result is obtained by comparing the two abutments. □

Definition 6.2.7. Let $\mathbf{d} = (d, e) \in \mathbb{N}^2$, define

$$\begin{aligned} v_i &:= i \cdot \mathbf{d} - (n+1, 0), \\ w_i &:= i \cdot \mathbf{d} - (0, m+1), \\ \sigma &:= (n+m) \cdot \mathbf{d} - (n+1, m+1). \end{aligned}$$

Lemma 6.2.8. *Let $I = (f_1, \ldots, f_r) \subseteq S$ and $\deg(f_i) = \boldsymbol{d} = (d, e)$ and V be the scheme defined by I. If $\operatorname{codim}(V) = r$ then*

$$\begin{aligned} \operatorname{Supp}(H_i(\boldsymbol{K}(\boldsymbol{f}, S)) \subseteq & v_{n+1+i} + (-\mathbb{N}, \mathbb{N}) \cup w_{m+1+i} + (\mathbb{N}, -\mathbb{N}) \quad \textit{for } i > 0, \\ \operatorname{Supp}(H_B^i(S/I)) \subseteq & v_{n+1-i} + (-\mathbb{N}, \mathbb{N}) \cup w_{m+1-i} + (\mathbb{N}, -\mathbb{N}) \quad \textit{for } 0 \le i \le \dim V. \end{aligned}$$

In addition, the inclusions are sharp.

Proof. With the proof of Theorem 6.2.6, in the second page of the spectral sequence we have for $p < n+m+1$

$$ {}^vE^{\infty}_{p,q} = {}^v E^2_{p,q} = \begin{cases} H_q(H^{n+1}_{B_1} \mathbf{K}(\mathbf{f}, S)) & \text{if } p = n+1, \\ H_q(H^{n+1}_{B_2} \mathbf{K}(\mathbf{f}, S)) & \text{if } p = m+1, \\ 0 \quad \text{else.} \end{cases} $$

Combining with the Theorem 6.2.6 shows the inclusions. The sharpness follows from the fact that for any q, v_q (resp. w_q) is in the support of $H^{n+1}_{B_1} K_q(\mathbf{f}, S)$ (resp. $H^{m+1}_{B_2} K_q(\mathbf{f}, S)$) and it is not in the support of $H^{n+1}_{B_1} K_{q-1}(\mathbf{f}, S)$ and $H^{n+1}_{B_1} K_{q+1}(\mathbf{f}, S)$ (resp. $H^{m+1}_{B_2} K_{q-1}(\mathbf{f}, S)$ and $H^{m+1}_{B_2} K_{q-1}(\mathbf{f}, S)$). □

Corollary 6.2.9. *Let $I = (f_1, \ldots, f_r) \subseteq S$ and $\deg(f_i) = \boldsymbol{d} = (d, e)$ and V be the scheme defined by I. If $\operatorname{codim}(V) = r$, then*

(1) *if $\mu \notin v_i + (-\mathbb{N}, \mathbb{N}) \cup w_j + (\mathbb{N}, -\mathbb{N})$ for $i = r-m, r-m-1$ and $j = r-n, r-n-1$ then*

$$H^{\dim V}(V, \mathcal{O}_V(\mu)) \cong H^r_B(S/I))_\mu = (S/I)^\star_{\sigma-\mu}.$$

Proof. Note thatSupp $\left(H^{n+1}_{B_1}(K_i(\mathbf{f}, S))\right) = v_i + (-\mathbb{N}, \mathbb{N})$ and $\operatorname{Supp} H^{m+1}_{B_2}(K_i(\mathbf{f}, S)) = w_i + (\mathbb{N}, +\mathbb{N})$ for all i. Consider the double complex $\mathbf{C}^\bullet_B(\mathbf{K}_\bullet(\mathbf{f}, S))$ and suppose $n \le m$ as in the proof of Theorem 6.2.6. Since $\operatorname{Supp}(V_i) \cap \operatorname{Supp}(W_j) = \emptyset$, there will be no nonzero

map in the spectral sequence among them. In this case, in the $n+2$-th page we have an induced map $\psi_1 : M \to W_{r-n-1}$, and in the $m+2$-th page we have an induced map $\psi_2 : \ker \psi_1 \to V_{r-m-1}$. If $\mu \notin v_{r-m} + (-\mathbb{N}, \mathbb{N}) \cup w_{r-n} + (\mathbb{N}, -\mathbb{N})$, $H^r_B(S/I)_\mu \cong (\ker \psi_2)_\mu$. In addition, if $\mu \notin v_{r-m-1} + (-\mathbb{N}, \mathbb{N}) \cup w_{r-n-1} + (\mathbb{N}, -\mathbb{N})$

$$H^{\dim V}(V, \mathcal{O}_V(\mu)) \cong H^r_B(S/I)_\mu \cong (\ker \psi_2)_\mu = M_\mu = (S/I)^\star_{\sigma-\mu}.$$

□

6.2.1 Complete intersection points in $\mathbb{P}^n \times \mathbb{P}^m$

In this section we apply the results in the previous section to the case of complete intersection points in $\mathbb{P}^n \times \mathbb{P}^m$. In the rest of this section, $I = (f_1, \ldots, f_{n+m})$ and V be the complete intersection scheme of points defined by I with $\mathbf{d} := \deg(f_i) = (d, e)$.

Definition 6.2.10. Set $\Gamma_i := \mathrm{Supp}(H^{n+1}_{B_1}(\mathbf{K}_{n+i}(\mathbf{f}, S))) \cup \mathrm{Supp}(H^{m+1}_{B_2}(\mathbf{K}_{m+i}(\mathbf{f}, S)))$. Notice that $\Gamma_i = \emptyset$ if and only if $i \geq \max\{n, m\}$. If $i \leq m$ then

$$\mathrm{Supp}(H^{n+1}_{B_1}(\mathbf{K}_{n+i}(\mathbf{f}, S))) = v_{n+i} + (-\mathbb{N}, \mathbb{N}),$$

and similarly for $\mathrm{Supp}(H^{m+1}_{B_2}(\mathbf{K}_{m+i}(\mathbf{f}, S)))$ if $i \leq n$.

Definition 6.2.11. For a function $F : \mathbb{Z}^2 \to \mathbb{Z}$ define

$$\begin{aligned} F^\star(a, b) &:= F(-a, -b) \\ F'(a, b) &:= F(-a, b) \\ F''(a, b) &:= F(a, -b). \end{aligned}$$

Theorem 6.2.12. *Let $S = k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ be a bigraded polynomial ring where $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. Assume V be a complete intersection scheme of points defined by $I = (f_1, \ldots, f_{n+m})$ with $\deg(f_i) = (d, e)$.*

(1) *If $\mu \notin \cup_{i \geq 2} \Gamma_i$ then*

$$HF_{S/I}(\mu) = \chi(\mu)$$

where $\chi((a, b)) = \sum_i (-1)^i \binom{n+m}{i} \binom{n+a-id}{n} \binom{m+b-ie}{m}$.

(2) *If $\mu \notin \Gamma_1$ then*

$$HF_{S/I}(\mu) = HF_V(\mu)$$

(3) *If* $\mu \notin \Gamma_0$ *then*

$$\mathrm{HF}_V(\mu) + \mathrm{HF}_V(\sigma - \mu) = \deg(V) = \binom{n+m}{n} d^n e^m.$$

In particular, if $\mu \in (nd - n, (n+m)e - m) + \mathbb{N}^2 \cup ((n+m)d - n, me - m) + \mathbb{N}^2$ *then* $\mathrm{HF}_V(\mu) = \deg(V)$ *and* $\mu \in \mathrm{reg}_B(V)$

(4) *If* $\mu \in \Gamma_0 \setminus \{\Gamma_{-1} \cup \Gamma_1\}$ *then*

$$\mathrm{HF}_V(\mu) + \mathrm{HF}_V(\sigma - \mu) = \binom{n+m}{n} (d^n e^m - \epsilon(\mu))$$

where $\epsilon(\mu) = HF'_S(\mu - v_n) + HF''_S(\mu - w_n)$ *as in the Definition 6.2.11.*

(5) *If* $\mu \in \cup_{i \geq 1} \Gamma_i \setminus \{\Gamma_0\}$ *then*

$$\mathrm{HF}_V(\mu) = \binom{n+m}{n} d^n e^m - \chi(\sigma - \mu).$$

In particular, if $\mu \in \Gamma_1 \setminus \{\Gamma_0 \cup \Gamma_2\}$ *then*

$$\dim(I_V/I)_\mu = \binom{n+m}{n+1} \mathrm{HF}'_S(\mu - v_{n+1}) + \binom{n+m}{m+1} \mathrm{HF}''_S(\mu - w_{m+1}).$$

Proof. Denote the saturation of I with respect to B by J. In this case $\mathrm{HF}_V(\mu) = \dim(S/J)_\mu$ for all $\mu \in \mathbb{Z}^2$. In the proof we use these two simple fact that $\mu \in \Gamma_{i-1} \cup \Gamma_{i+1}$ yields $\mu \in \Gamma_i$ and if $\mu \in \Gamma_i$ then $\sigma - \mu \in \Gamma_{-i}$. By Proposition 6.2.3 the Hilbert polynomial of V is $D := \binom{n+m}{n} d^n e^m$ which in this case is equal to the $\deg(V)$. By Serre Grothendieck formula,

$$\mathrm{HF}_{S/J}(\mu) + \dim(H^1_B(S/J)_\mu) = D \tag{6.2.3}$$

(1) If $\mu \notin \cup_{i \geq 2} \Gamma_i$, by Lemma 6.2.8 $(H_i(\mathbf{K}(\mathbf{f}, S))_\mu = 0$ for all $i \geq 1$. Therefore by Equation 6.2.2, $(S/I)_\mu = \chi(\mu)$.
(2) If $\mu \notin \Gamma_1$, by Lemma 6.2.8, $(H^0_B(S/I))_\mu = 0$ therefore $(S/I)_\mu = (S/J)_\mu$.
(3) First note that $\mu, \sigma - \mu \notin \Gamma_0$. We claim that $\mu \notin \Gamma_{-1}$ or $\sigma - \mu \notin \Gamma_{-1}$. Suppose not, then $\mu \in \Gamma_{-1} \cap \Gamma_1$ which yields $\mu \in \Gamma_0$ which is a contradiction. So assume $\mu \notin \Gamma_{-1}$. By Corollary 6.2.9,

$$\dim(H^1_B(S/I)_\mu) = \mathrm{HF}_{S/I}(\sigma - \mu).$$

Since $\sigma - \mu \notin \Gamma_1$, by part (2) and Equation 6.2.1

$$\binom{n+m}{n} d^n e^m - \mathrm{HF}_V(\mu)\,\mathrm{HF}_V(\sigma - \mu).$$

The same argument works if we assume $\sigma - \mu \notin \Gamma_{-1}$.
(4) By part (1) $\mathrm{HF}_{S/I}(\mu) = \mathrm{HF}_V(\mu)$. By considering the double complex as in the proof of Theorem 6.2.6 and Lemma 6.2.8,

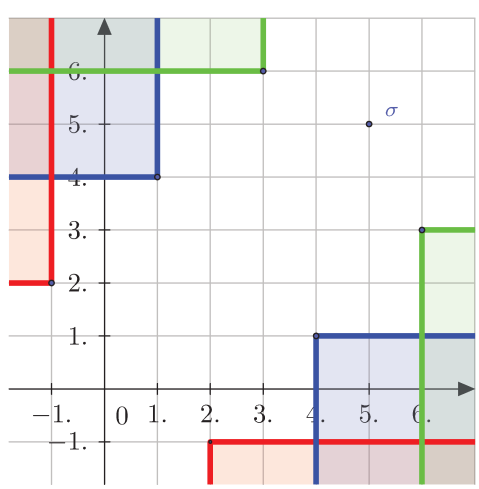
$$\left(H_n(H^{n+1}_{B_1}\mathbf{K}(\mathbf{f}, S))\right)\mu = (H^{n+1}_{B_1}(K_n(\mathbf{f}, S))_\mu \text{ and } \left(H_m(H^{m+1}_{B_2}\mathbf{K}(\mathbf{f}, S))\right)_\mu = \left(H^{m+1}_{B_2}(K_m(\mathbf{f}, S)\right)_\mu.$$

By the abutment of the spectral sequence

$$\left(H^1_B(S/J)\right)_\mu = HF_{S/I}(\sigma - \mu) + \dim(H^{n+1}_{B_1}(K_n(\mathbf{f}, S))_\mu + .\left(H^{m+1}_{B_2}(K_m(\mathbf{f}, S)\right)_\mu.$$

Since $\sigma - \mu \notin \Gamma_1$, by parts (1) and (2), $HF_{S/I}(\mu) = HF_V(\mu)$ and by the definition, $\dim(H^{n+1}_{B_1}(K_n(\mathbf{f}, S))_\mu + \left(H^{m+1}_{B_2}(K_m(\mathbf{f}, S)\right)_\mu = \binom{n+m}{n}\epsilon(\mu)$. The assertion follows by $(H^1_B(S/J))_\mu = \binom{n+m}{n} d^n e^m - HF_V(\mu)$.
(5) By part (3), $\mathrm{HF}_V(\mu) = \binom{n+m}{n} d^n e^m - \mathrm{HF}_V(\sigma - \mu)$. Because $\mu \notin \Gamma_{-1}$ ($\mu \in \Gamma_{-1}$ yields $\mu \in \Gamma_0$) therefore $\sigma - \mu \notin \Gamma_1$, which by part (2), yields $\mathrm{HF}_V(\sigma - \mu) = HF_{S/I}(\sigma - \mu)$. On the other hand, since $\mu \notin \cup_{i \geq 2}\Gamma_i$, by part (1), $HF_{S/I}(\sigma - \mu) = \chi(\sigma - \mu)$. □

Let $S = [x_0, x_1, x_2, y_0, y_1, y_2]$ and $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. Let $I = (f_1, \ldots, f_4)$ where $\deg(f_i) = (2, 2)$. Suppose scheme V defined by I is complete intersection. The following picture demonstrate the regions Γ_i in the Theorem 6.2.12.



Γ_{-1} is the red region, Γ_0 is the blue region and Γ_1 is the green region. By Theorem 6.2.12, Hilbert function of V at the bidegrees except the intersection of blue and green are independent from the choices of f_i's and they can be computed via χ and ϵ defined in the Theorem 6.2.12. On the other hand, the rest do depends upon the f_i's. Here we computed two example via computer system Macaulay2 [47].

Example 6.2.13. Let $S = k[x_0, x_1, x_2, y_0, y_1, y_2]$, $I = (x_0^2y_0^2, x_1^2y_1^2, x_2^2y_2^2, (x_0 + x_1 + x_2)^2(y_0 + y_1 + y_2)^2)$ and V be the complete intersection scheme of points defined by I. For $(0, 0) \leq \mu \leq (7, 7)$ the bigraded Hilbert function $\mathrm{HF}_V(\mu)$ is

$$\begin{matrix}
24 & 72 & 96 & 96 & 96 & 96 & 96 & 96 \\
24 & 72 & 96 & 96 & 96 & 96 & 96 & 96 \\
21 & 63 & 86 & 90 & 93 & 95 & 96 & 96 \\
15 & 45 & 66 & 78 & 87 & 93 & 96 & 96 \\
10 & 30 & 48 & 64 & 78 & 90 & 96 & 96 \\
6 & 18 & 32 & 48 & 66 & 86 & 96 & 96 \\
3 & 9 & 18 & 30 & 45 & 63 & 72 & 72 \\
1 & 3 & 6 & 10 & 15 & 21 & 24 & 24
\end{matrix}.$$

Where blue corresponds to the bidegrees in Γ_0, green corresponds to Γ_1 and orange to their intersections. Also red indicate the $HF_V(\sigma)$. For $\mu \notin \Gamma_0$ by Theorem 6.2.12 parts (1), (2) and (3) one can compute the bigraded Hilbert function and for blue points by part (4). For the orange points, Theorem 6.2.12 does not say anything. In addition, every points except orange ones only depend on the degree of the generator of I which in this case is $(2, 2)$.

Assume I' is generated by 4 random forms, in a sense of computer software Macaulay2, of bidegree $(2, 2)$ and V' be the complete intersection scheme of points defined by I. For $(0, 0) \leq \mu \leq (13, 7)$ the bigraded Hilbert function $\mathrm{HF}_{V'}(\mu)$ is

$$\begin{matrix}
33 & 84 & 96 & 96 & 96 & 96 & 96 & 96 & 96 & 96 & 96 & 96 & 96 & 96 \\
27 & 81 & 96 & 96 & 96 & 96 & 96 & 96 & 96 & 96 & 96 & 96 & 96 & 96 \\
21 & 63 & 86 & 90 & 93 & 95 & 96 & 96 & 96 & 96 & 96 & 96 & 96 & 96 \\
15 & 45 & 66 & 78 & 87 & 93 & 96 & 96 & 96 & 96 & 96 & 96 & 96 & 96 \\
10 & 30 & 48 & 64 & 78 & 90 & 96 & 96 & 96 & 96 & 96 & 96 & 96 & 96 \\
6 & 18 & 32 & 48 & 66 & 86 & 96 & 96 & 96 & 96 & 96 & 96 & 96 & 96 \\
3 & 9 & 18 & 30 & 45 & 63 & 81 & 84 & 96 & 96 & 96 & 96 & 96 & 96 \\
1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 & 55 & 66 & 78 & 91 & 96
\end{matrix}$$

The only differences are in the orange spots. In addition, in this case $\mathrm{HF}_{V'}(0, 13) = \mathrm{HF}_{V'}(13, 0) = 96$ which means the natural projection of V' to each $\mathbb{P}^n$ and $\mathbb{P}^m$ is one to one.

6.2.2 Generic complete intersection points in $\mathbb{P}^n \times \mathbb{P}^m$

In this subsection we adopt the Notation 6.2.1 as well.

Proposition 6.2.14. *Let k be a field of characteristics zero and $V \subset \mathbb{P}^n_k \times \mathbb{P}^m_k$ be a reduced scheme and $(d, e) \in \mathbb{N}^2$ with $d, e \neq 0$. Let f_U be a form of bidegree (d, e) with indeterminate coefficients $U_{\alpha,\beta}$. There exists a non empty open set $\Omega \subset \operatorname{Spec}(k[U_{\alpha,\beta}])$ such that for any $\mathfrak{p} \in \Omega$, the corresponding form $f_\mathfrak{p}$ is such that*

$$V \cap Z(f_\mathfrak{p}) \subseteq \mathbb{P}^n_{k_\mathfrak{p}} \times \mathbb{P}^m_{k_\mathfrak{p}},$$

where $k_\mathfrak{p}$ is the residue field of $k[U_{\alpha,\beta}]_\mathfrak{p}$, is reduced of dimension equal $\dim V - 1$, unless the $\dim V = 0$ in which case $V \cap Z(f_\mathfrak{p}) = \emptyset$.

Proof. Consider the Segre-Veronese map $\Psi : \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^N$ with $N = \binom{n+d}{d} \times \binom{m+e}{e} - 1$. Let $V' = \Psi(V)$. Under the map Ψ, f_U is mapped to a linear form ℓ_U in $N + 1$ variables and this correspondence is one to one. By Bertini theorem (see [41, Corollary 3.4.9]) there exists Ω as claimed such that for $\mathfrak{p} \in \Omega$, $V' \cap Z(\ell_\mathfrak{p}) \cong V \cap Z(f_\mathfrak{p})$ is a reduced scheme with the asserted dimension. □

Definition 6.2.15. Forms of bidegree (d, e) are in one to one correspondence with closed points in $\mathbb{A}_k^{|S_{(d,e)}|}$.

Remark 6.2.16. Notice that $\Omega \subset \operatorname{Spec} k[U_{\alpha,\beta}] \setminus Z(F)$ for some form $F \neq 0$. In particular, for $c = (c_{\alpha,\beta}) \in k^N$ such that $F(c) \neq 0$, Proposition 6.2.14 with $\mathfrak{p}$ generated by the elements $u_{\alpha,\beta} - c_{\alpha,\beta}$, shows that $Z(f_c) \cap V$ is reduced.

Theorem 6.2.17. *Let $S = k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ be a bigraded polynomial ring over a field k of characteristic zero where $\deg(x_i) = (1, 0)$ and $\deg(y_i) = (0, 1)$. Let $(d_i, e_i) \geq (1, 1)$ for $i = 1, \ldots, n + m$ be bidegrees. Then, there exists a non-empty open set*

$$\Omega \subset \mathbb{A}_k^N := \prod_{i=1}^{n+m} \mathbb{A}_k^{|S_{(d_i,e_i)}|}$$

such that, for any $(f_1, \ldots, f_{n+m})$ corresponding to a point in Ω,

(1) *$Z := \operatorname{Proj}(S/(f_1, \ldots, f_{n+m}))$ is reduced of dimension zero.*

(2) *The natural projections of Z to the factors $\mathbb{P}^n$ and $\mathbb{P}^m$ are isomorphism.*

Proof. Let $V := Z(g_1, \ldots, g_{n+m}) \subset \mathbb{A}^N_k \times \mathbb{P}^n_k \times \mathbb{P}^m_k$ where $\mathbb{A}^N_k = \operatorname{Spec} k[U_{i,\alpha,\beta}]$ for $1 \leq i \leq n+m$, $|\alpha| = d_i$, $|\beta| = e_i$ and $g_i := \sum U_{i,\alpha,\beta}\mathbf{x}^\alpha \mathbf{y}^\beta$. Set $\pi : \mathbb{A}^N_k \times \mathbb{P}^n_k \times \mathbb{P}^m_k \to \mathbb{P}^n_k \times \mathbb{P}^m_k$ be the natural projection, then V is a vector bundle over $\mathbb{P}^n_k \times \mathbb{P}^m_k$ via π. For any point $p \in \mathbb{P}^n_k \times \mathbb{P}^m_k$ the fiber of p is a linear space of dimension $N - n - m$. Hence V is geometrically irreducible scheme and $\dim(V) = n + m + (N - n - m) = N$.

(1) Set $W := (g_1, \ldots, g_{n+m}, \operatorname{Jac}^{n+m}(g_1, \ldots, g_{n+m})) \subset V$ where Jac^{n+m} is the Jacobian of order $n + m$. By using Proposition 6.2.14 inductively, there exist a point $u \in \mathbb{A}^N_k$ and $(f_1, \ldots, f_{n+m})$ corresponding to u such that $\operatorname{Jac}^{n+m}(f_1, \ldots, f_{n+m}) \neq 0$ which yields $W \subsetneq V$ and $\dim(W) < \dim(V) = N$. Consider the natural projection $p : \mathbb{A}^N_k \times \mathbb{P}^n_k \times \mathbb{P}^m_k \to \mathbb{A}^N_k$. Since $\dim(W) < N$, $p(W)$ is a non empty closed subset of $\mathbb{A}^N_k$. Set $\Omega_0 := \left(\mathbb{A}^N_k \setminus \pi(W)\right)$. Therefore for any $(f_1, \ldots, f_{n+m})$ corresponding to a point in Ω_0, $\operatorname{Proj}(S/(f_1, \ldots, f_{n+m}))$ is geometrically reduced and is of dimension zero.

(2) Set $p_1 : \mathbb{A}^N_k \times \mathbb{P}^n_k \times \mathbb{P}^m_k \to \mathbb{A}^N_k \times \mathbb{P}^n_k$ and $q_2 : \mathbb{A}^N_k \times \mathbb{P}^n_k \to \mathbb{A}^N_k$ the natural projections. First we show that it is enough to show that the restriction $p'_1 : V \to p_1(V)$ is birational. Indeed if p'_1 is birational then there exists a closed subset $Z_1 \subsetneq p_1(V)$ such that setting $W_1 := (p'_1)^{-1}(Z_1)$ then $p'_1|_{V \setminus W_1} : V \setminus W_1 \to p_1(V) \setminus Z_1$ is an isomorphism. As $\dim(Z_1) < N$, $q_2(Z_1) \subsetneq \mathbb{A}^N_k$. Set $\Omega_1 := \mathbb{A}^N_k \setminus (q_2(Z_1) \cup \pi(W))$. For any point $u \in \Omega_1$ and $(f_1, \ldots, f_{n+m})$ corresponding to u,

$$q_2^{-1}(u) = p_1(u, Z(f_1, \ldots, f_{n+m})) \cong (u, Z(f_1, \ldots, f_{n+m}))$$

showing that the natural projection of Z onto $\mathbb{P}^n$ is an isomorphism. By replacing p_1 and q_2 by $p_2 : \mathbb{A}^N_k \times \mathbb{P}^n_k \times \mathbb{P}^m_k \to \mathbb{A}^N_k \times \mathbb{P}^m_k$ and $q_1 : \mathbb{A}^N_k \times \mathbb{P}^m_k \to \mathbb{A}^N_k$, along the same lines it shows the existence of $Z_2 \in p_2(V)$ and $\Omega_2 = \mathbb{A}^N_k \setminus (q_1(Z_2) \cup \pi(W))$ such that for $u \in \Omega_2$ and $(f_1, \ldots, f_{n+m})$ corresponding to u,

$$q_1^{-1}(u) = p_2(u, Z(f_1, \ldots, f_{n+m})) \cong (u, Z(f_1, \ldots, f_{n+m}))\,.$$

Consider the following diagram

$$\begin{tikzcd}
V \arrow[r, "\subset"] \arrow[d, "p'_1"] & \mathbb{A}^N_k \times \mathbb{P}^n_k \times \mathbb{P}^m_k \arrow[d, "p_1"] \\
V_1 := p_1(V) \arrow[r, "\subset"] \arrow[d, "q_2|_{V_1}"] & \mathbb{A}^N_k \times \mathbb{P}^n_k \arrow[d, "q_2"] \\
\mathbb{A}^N_k \arrow[r, "="] & \mathbb{A}^N_k .
\end{tikzcd}$$

We now show that p_1' is birational.

Let $I := (g_1, \ldots, g_{n+m)}$ and $J := I : B^\infty$. By part (1), V_1 is a geometrically irreducible scheme. Hence J and $I_{V_1} = J \cap k[\mathbf{x}, \mathbf{U}]$ are prime and remain prime under any extension of k.

As V_1 is generically smooth, there exists $G \in I_{V_1}$ such that $D := \dfrac{\partial G}{\partial U_{i,\alpha,\beta}} \notin I_{V_1}$ for some i, α, and β unless $G \in k[\mathbf{y}]$ which is impossible: indeed $I_{V_1} \cap k[\mathbf{y}] = (0)$ because the projection $V \to \mathbb{P}^m$ is onto.

Set $Z_1 := Z(D) \subset V_1$. By [61, Lemma 4.6.1],

$$\mathbf{x}^\alpha \mathbf{y}^\beta \frac{\partial G}{\partial U_{i,\alpha',\beta'}} - \mathbf{x}^{\alpha'} \mathbf{y}^{\beta'} \frac{\partial G}{\partial U_{i,\alpha,\beta}} \in I_V$$

for any choices of α' and β'. Let $\mathbf{y}^\beta = y_p \mathbf{y}^{\beta_0}$ for some i. For any $0 \le j \le m$ put $\alpha' := \alpha$ and $\beta' := y_j \beta_0$. As $\mathbf{x}^\alpha$ and $\mathbf{y}^{\beta_0}$ are not in I_V it follows that

$$y_p \frac{\partial G}{\partial U_{i,\alpha',\beta'}} - y_j \frac{\partial G}{\partial U_{i,\alpha,\beta}} \in I_V$$

for all $0 \le j \le m$. Therefore by localizing at D, we get an isomorphism

$$\phi : \frac{(k[\mathbf{U}, \mathbf{x}]_D[\mathbf{y}])}{I_V} \xrightarrow{\cong} \frac{(k[\mathbf{U}, \mathbf{x}]_D[y_p])}{I_{V_1}}.$$

Therefore the natural maps

$$\left(\frac{k[\mathbf{U}, \mathbf{x}]}{I_{V_1}}\right)_D \xhookrightarrow{\iota} \frac{(k[\mathbf{U}, \mathbf{x}]_D[\mathbf{y}])}{I_V} \xrightarrow{\phi} \frac{(k[\mathbf{U}, \mathbf{x}]_D[y_p])}{I_{V_1}}$$

are such that $\phi \circ \iota$ induces the identity from $\operatorname{Proj}\left(\left(k[\mathbf{U}, \mathbf{x}]_D[y_p]\right)/I_{V_1}\right)$ to $\operatorname{Spec}\left(\left(k[\mathbf{U}, \mathbf{x}]_D\right)/I_{V_1}\right)$. Hence ϕ provides the inverse φ

$$V_1 \setminus Z_1 \xrightarrow{\varphi} V \setminus (Z_1 \times \mathbb{P}^m_k) \xrightarrow{p_1'} V_1 \setminus Z_1.$$

□

Theorem 6.2.18. *Let $S = k[x_0, \ldots, x_n, y_0, \ldots, y_m]$ be a bigraded polynomial ring over a field k of characteristics zero. If I is generated by $n + m$ generic forms of bidegree (d, e), then the scheme V defined by I is a set of reduced points and*

$$|\{\mu \in \mathbb{N}^2 | \operatorname{HF}_V(\mu) \neq \deg(V)\}| < \infty$$

Proof. By Theorem 6.2.17 part (1), V is a set of reduced points and by part (2), $\operatorname{HF}_V(0, i) = \operatorname{HF}_V(j, 0) = \deg(V)$ for any $i, j \gg 0$. □

Chapter 7

Bibliography

[1] Alilooee, A., and Banerjee, A. Powers of edge ideals of regularity three bipartite graphs. *J. Commut. Algebra 9*, 4 (2017), 441–454.

[2] Alilooee, A., Beyarslan, S., and Selvaraja, S. Regularity of Powers of Unicyclic Graphs. arXiv:1702.00916.

[3] Altafi, N., and Boij, M. The Weak Lefschtez Property of Equigenerated Monomial Ideals. arXiv:1807.02138.

[4] Altafi, N., and Nemati, N. Lefschetz properties of monomial ideals with almost linear resolution. arXiv:1803.01388.

[5] Altafi, N., Nemati, N., Seyed Fakhari, S. A., and Yassemi, S. Free resolution of powers of monomial ideals and Golod rings. *Math. Scand. 120*, 1 (2017), 59–67.

[6] Bagheri, A., Chardin, M., and Hà, H. T. The eventual shape of Betti tables of powers of ideals. *Math. Res. Lett. 20*, 6 (2013), 1033–1046.

[7] Banerjee, A. The regularity of powers of edge ideals. *J. Algebraic Combin. 41*, 2 (2015), 303–321.

[8] Banerjee, A., Beyarslan, S., and Ha, H. T. Regularity of Edge Ideals and Their Powers. arXiv:1712.00887.

[9] Bertram, A., Ein, L., and Lazarsfeld, R. Vanishing theorems, a theorem of Severi, and the equations defining projective varieties. *Journal of the American Mathematical Society 4*, 3 (sep 1991), 587–587.

[10] Beyarslan, S., Hà, H. T., and Trung, T. N. Regularity of powers of forests and cycles. *J. Algebraic Combin. 42*, 4 (2015), 1077–1095.

[11] Biyikoglu, T., and Civan, Y. Castelnuovo-mumford regularity of graphs. arXiv:1503.06018.

[12] Botbol, N., and Chardin, M. Castelnuovo Mumford regularity with respect to multigraded ideals. *Journal of Algebra 474* (mar 2017), 361–392.

[13] Bruns, W., and Conca, A. A remark on regularity of powers and products of ideals. *Journal of Pure and Applied Algebra 221*, 11 (2017), 2861 – 2868.

[14] Bruns, W., Conca, A., and Römer, T. Koszul cycles. In *Combinatorial aspects of commutative algebra and algebraic geometry, Abel Symp.* 2011, pp. 17–33.

[15] Bruns, W., Conca, A., and Römer, T. Koszul homology and syzygies of Veronese subalgebras. *Mathematische Annalen 351*, 4 (dec 2011), 761–779.

[16] Bruns, W., and Herzog, J. *Cohen-Macaulay rings*, vol. 39 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1993.

[17] Caviglia, G. Bounds on the Castelnuovo-Mumford regularity of tensor products. *Proceedings of the American Mathematical Society 135*, 07 (jul 2007), 1949–1958.

[18] Chandler, K. A. Regularity of the powers of an ideal. *Communications in Algebra 25*, 12 (1997), 3773–3776.

[19] Chardin, M. On the behavior of Castelnuovo-Mumford regularity with respect to some functors. arXiv:0706.2731.

[20] Chardin, M. Some Results and Questions on Castelnuovo-Mumford Regularity. In *Syzygies and Hilbert Functions*, I. Peeva, Ed., 254 ed. Lect. Notes Pure Appl. Math., 2007.

[21] Chardin, M., and Divaani-Aazar, K. Generalized local cohomology and regularity of Ext modules. *J. Algebra 319*, 11 (2008), 4780–4797.

[22] Chardin, M., Ghosh, D., and Nemati, N. The (ir)regularity of Tor and Ext. In preparation.

[23] Chardin, M., Ha, D. T., and Hoa, L. T. Castelnuovo-Mumford regularity of Ext modules and homological degree. *Trans. Amer. Math. Soc. 363*, 7 (2011), 3439–3456.

[24] Chardin, M., Jouanolou, J.-P., and Rahimi, A. The eventual stability of depth, associated primes and cohomology of a graded module. *J. Commut. Algebra 5*, 1 (2013), 63–92.

[25] Chardin, M., and Nemati, N. Multigraded regularity of complete intersection points in $\mathbb{P}^n \times \mathbb{P}^m$. In preparation.

[26] Chardin, M., and Symonds, P. Degree bounds on homology and a conjecture of Derksen. *Compositio Mathematica 152*, 10 (oct 2016), 2041–2049.

[27] Cid-Ruiz, Y., Jafari, S., Nemati, N., and Picone, B. Regularity of bicyclic Graphs and their powers. To appear in the Journal of Algebra and its Application, arXiv:1802.07202.

[28] Conca, A., and Herzog, J. Castelnuovo-mumford regularity of products of ideals. *Collectanea Mathematica 54*, 2 (2003), 137–152.

[29] Conca, A., and Murai, S. Regularity bounds for Koszul cycles. *Proceedings of the American Mathematical Society 143*, 2 (oct 2014), 493–503.

[30] Cutkosky, S. D., Herzog, J., and Trung, N. V. Asymptotic behaviour of the Castelnuovo-Mumford regularity. *Compositio Mathematica 118*, 3 (1999), 243–261.

[31] Eisenbud, D. Homological algebra on a complete intersection, with an application to group representations. *Trans. Amer. Math. Soc. 260*, 1 (1980), 35–64.

[32] Eisenbud, D. *Commutative Algebra: With a View Toward Algebraic Geometry*. Graduate Texts in Mathematics. Springer, 1995.

[33] Eisenbud, D. *The Geometry of Syzygies*, vol. 229 of *Graduate Texts in Mathematics*. Springer New York, 2005.

[34] Eisenbud, D., Erman, D., and Schreyer, F.-O. Tate Resolutions for Products of Projective Spaces. *Acta Mathematica Vietnamica 40*, 1 (mar 2015), 5–36.

[35] Eisenbud, D., and Goto, S. Linear free resolutions and minimal multiplicity. *Journal of Algebra 88*, 1 (may 1984), 89–133.

[36] Eisenbud, D., and Harris, J. Powers of ideals and fibers of morphisms. *Math. Res. Lett. 17*, 2 (2010), 267–273.

[37] Eisenbud, D., Huneke, C., and Ulrich, B. The Regularity of Tor and Graded Betti Numbers. *Amer. J. Math. 128*, 3 (2006), pp. 573–605.

[38] Eisenbud, D., and Nemati, N. Linear truncations package in Macaulay2. In Preparation.

[39] Eisenbud, D., and Peeva, I. *Minimal Free Resolutions over Complete Intersections*, vol. 2152 of *Lecture Notes in Mathematics*. Springer International Publishing, Cham, 2016.

[40] Erey, N. Powers of Edge Ideals with Linear Resolutions. arXiv:1703.01561.

[41] Flenner, H., O'Carroll, L., and Vogel, W. *Joins and intersections*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1999.

[42] Fulton, W. *Intersection theory*, second ed., vol. 2 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1998.

[43] Geramita, A. V. Inverse systems of fat points: Waring's problem, secant varieties of Veronese varieties and parameter spaces for Gorenstein ideals. *The curves seminar at Queen's 10* (1996), 2–114.

[44] Geramita, A.V., G. A. Graded betti numbers of some embedded rational n-folds. *Mathematische Annalen 301*, 2 (1995), 363–380.

[45] Ghosh, D. Asymptotic linear bounds of castelnuovomumford regularity in multigraded modules. *Journal of Algebra 445* (2016), 103 – 114.

[46] Ghosh, D., and Puthenpurakal, T. J. Asymptotic bound for Castelnuovo-Mumford regularity of certain Exts over complete intersections. arXiv:1801.01864.

[47] Grayson, D. R., and Stillman, M. E. Macaulay2, a software system for research in algebraic geometry. Available at `https://faculty.math.illinois.edu/Macaulay2/`.

[48] Green, M. L. Koszul cohomology and the geometry of projective varieties. *Journal of Differential Geometry 19*, 1 (1984), 125–171.

[49] Gu, Y. Regularity of powers of edge ideals of some graphs. *Acta Math. Vietnam. 42*, 3 (2017), 445–454.

[50] Gulliksen, T. H. A change of ring theorem with applications to Poincaré series and intersection multiplicity. *Math. Scand. 34* (1974), 167–183.

[51] Hà, H. T. Regularity of squarefree monomial ideals. In *Connections between algebra, combinatorics, and geometry*, vol. 76 of *Springer Proc. Math. Stat.* Springer, New York, 2014, pp. 251–276.

[52] Hà, H. T., and Van Tuyl, A. Monomial ideals, edge ideals of hypergraphs, and their graded Betti numbers. *J. Algebraic Combin. 27*, 2 (2008), 215–245.

[53] Harima, T., Migliore, J. C., Nagel, U., and Watanabe, J. The Weak and Strong Lefschetz properties for Artinian K-algebras. *Journal of Algebra 262*, 1 (apr 2003), 99–126.

[54] Hassanzadeh, S. H. Cohen-Macaulay residual intersections and their Castelnuovo-Mumford regularity. *Trans. Amer. Math. Soc. 364*, 12 (2012), 6371–6394.

[55] Herzog, J., and Hibi, T. *Monomial ideals*, vol. 260 of *Graduate Texts in Mathematics*. Springer-Verlag London, Ltd., London, 2011.

[56] Hibi, T., Higashitani, A., Kimura, K., and Tsuchiya, A. Dominating induced matchings of finite graphs and regularity of edge ideals. *J. Algebraic Combin. 43*, 1 (2016), 173–198.

[57] Hoffman, J. W., and Wang, H. H. CastelnuovoMumford regularity in biprojective spaces. *Advances in Geometry 4*, 4 (jan 2004).

[58] Iarrobino, A., and Kanev, V. *Power Sums, Gorenstein Algebras, and Determinantal Loci*, vol. 1721 of *Lecture Notes in Mathematics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1999.

[59] Iyengar, S., Leuschke, G., Leykin, A., Miller, C., Miller, E., Singh, A., and Walther, U. *Twenty-Four Hours of Local Cohomology*, vol. 87 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, Rhode Island, nov 2007.

[60] Jayanthan, A., and Selvaraja, S. Asymptotic behavior of Castelnuovo-Mumford regularity of edge ideals of very well-covered graphs. *ArXiv e-prints* (Aug. 2017).

[61] Jouanolou, J.-P. Le formalisme du résultant. *Adv. Math. 90*, 2 (1991), 117–263.

[62] Katzman, M. Characteristic-independence of Betti numbers of graph ideals. *J. Combin. Theory Ser. A 113*, 3 (2006), 435–454.

[63] Kodiyalam, V. Asymptotic Behaviour of Castelnuovo-Mumford Regularity. *Proceedings of the American Mathematical Society 128*, 2 (2000), 407–411.

[64] Lamei, K., and Nemati, N. Castelnuovo-Mumford regularity of Koszul cycles and Koszul homologies. *Proceedings of the American Mathematical Society 146*, 7 (2018), 2765–2772.

[65] Maclagan, D., and Smith, G. G. Multigraded Castelnuovo-Mumford regularity. *J. Reine Angew. Math 14*, 1 (2005), 137–164.

[66] Migliore, J., and Nagel, U. The Lefschetz question for ideals generated by powers of linear forms in few variables. arXiv:1703.07456.

[67] Migliore, J. C., Miró-Roig, R. M., and Nagel, U. Monomial ideals, almost complete intersections and the Weak Lefschetz property. *Transactions of the American Mathematical Society 363*, 01 (jan 2011), 229–229.

[68] Migliore, J. C., and Nagel, U. The Lefschetz question for ideals generated by powers of linear forms in few variables. arXiv:1703.07456.

[69] Moghimian, M., Fakhari, S. A. S., and Yassemi, S. Regularity of powers of edge ideal of whiskered cycles. *Comm. Algebra 45*, 3 (2017), 1246–1259.

[70] Nevo, E., and Peeva, I. C4-free edge ideals. *Journal of Algebraic Combinatorics 37*, 2 (mar 2013), 243–248.

[71] Norouzi, P., Seyed Fakhari, S. A., and Yassemi, S. Regularity of Powers of edge ideal of very well-covered graphs. *ArXiv e-prints* (July 2017).

[72] Ottaviani, G., and Paoletti, R. No Title. *Compositio Mathematica 125*, 1 (2001), 31–37.

[73] Peeva, I. *Graded Syzygies*. Springer London, London, 2011.

[74] Rotman, J. J. *An introduction to homological algebra*, second ed. Universitext. Springer, New York, 2009.

[75] Sidman, J. On the CastelnuovoMumford regularity of products of ideal sheaves. *Advances in Geometry 2*, 3 (jan 2002).

[76] Sturmfels, B. Four counterexamples in combinatorial algebraic geometry. *Journal of Algebra 230*, 1 (2000), 282 – 294.

[77] Swanson, I. Powers of ideals. Primary decompositions, Artin-Rees lemma and regularity. *Math. Ann. 307*, 2 (1997), 299–313.

[78] Trung, N. V., and Wang, H.-J. On the asymptotic linearity of castelnuovomumford regularity. *Journal of Pure and Applied Algebra 201*, 1 (2005), 42 – 48.

[79] Weibel, C. A. *An introduction to homological algebra*, vol. 38 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1994.

[80] Woodroofe, R. Matchings, coverings, and Castelnuovo-Mumford regularity. *J. Commut. Algebra 6*, 2 (2014), 287–304.

[81] Zheng, X. Resolutions of facet ideals. *Comm. Algebra 32*, 6 (2004), 2301–2324.