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Algèbres de Hall et localisation des catégories

Par Mikhail GORSKY

THÈSE DE DOCTORAT DE MATHÉMATIQUES

Dirigée par Bernhard KELLER

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Résumé

Cette thèse concerne les algèbres de Hall. Nous définissons des algèbres de Hall étendues tordues de catégories triangulées et montrons que, dans certains cas, elles sont bien définies même lorsque leurs contreparties non étendues ne le sont pas. Nous montrons que chaque catégorie exacte aux équivalences faibles avec une structure supplémentaire appropriée donne naturellement lieu à une algèbre de Hall étendue tordue de sa catégorie homotopique. Nous montrons que cette construction récupère la catégorification par Bridgeland des groupes quantiques via les algèbres de Hall de complexes et sa généralisation par Lu et Peng. Nous montrons que les algèbres ainsi définies sont fonctorielles par rapport aux foncteurs exacts respectant les équivalences faibles. Cela nous permet de prouver l'invariance par basculement des algèbres de Bridgeland et de catégorifier les symétries de Lusztig des groupes quantiques. Sous des conditions de finitude appropriées, pour deux structures exactes différentes sur la même catégorie additive, l'une ayant strictement moins de conflations que l'autre, nous définissons une filtration sur l'algèbre de Hall de cette dernière dont le gradué associé est l'algèbre de Hall de la première. Cette construction généralise les filtrations de type PBW quantiques.

Mots-clés

Algèbres de Hall, localisation des catégories, catégorification, groupes quantiques, dégénerescences de PBW

Hall algebras and localization of categories

Abstract

This thesis concerns Hall algebras. We define twisted extended Hall algebras of triangulated categories and show that in some cases they are well-defined even when their non-extended counterparts are not. We show that each exact category with weak equivalences with an appropriate extra structure naturally gives rise to a twisted extended Hall algebra of its homotopy category. We prove that this construction recovers Bridgeland's categorification of quantum groups via Hall algebras of complexes and its generalization due to Lu and Peng. We prove that the algebras thus defined are functorial under exact functors respecting weak equivalences. This allows us to prove the tilting invariance of Bridgeland's algebras and to categorify Lusztig's symmetries of quantum groups. Under suitable finiteness conditions, for two different exact structures on the same additive category with one having strictly less conflations than the other, we define a filtration on the Hall algebra of the latter whose associate graded is the Hall algebra of the former. This construction generalizes quantum PBW-type filtrations.

Keywords

Hall algebras, localization of categories, categorification, quantum groups, PBW degenerations.

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Introduction

0.1 Contexte historique

Les algèbres de Hall fournissent l'un des premiers exemples connus de catégorification additive. Elles sont apparues pour la première fois dans les travaux de Steinitz [74] et Hall [27] sur des $p-$groupes finis commutatifs. Plus tard, elles sont réapparues dans les travaux de Ringel [64] sur les groupes quantiques. Il a introduit la notion de l'algèbre de Hall d'une catégorie abélienne avec des espaces de dimension finie Hom - et Ext^1. En tant qu'espace vectoriel, elle a une base paramétrisée par les classes d'isomorphisme des objets de la catégorie. La multiplication capte des informations sur les extensions entre les objets. On peut considérer cela comme une algèbre de fonctions constructibles sur le champ d'objets de la catégorie, avec le produit de convolution donné par les correspondances de Hecke.

Ringel a construit un isomorphisme entre l'algèbre de Hall torduée de la catégorie de représentations d'un carquois de Dynkin simplement lacé Q sur le corps fini $\mathbb{F}_q$ et la partie nilpotente du groupe quantique correspondant, spécialisé à la racine carré de q :

$$U_{\sqrt{q}}(\mathfrak{n}_+) \xrightarrow{\sim} \mathcal{H}_{tw}(\mathrm{rep}_{\mathbb{F}_q}(Q)).$$

Plus tard, Green [26] a généralisé ce résultat à un carquois valu Q arbitraire en montrant un isomorphisme entre la partie nilpotente de l'algèbre enveloppante universelle quantifiée de l'algèbre de Kac-Moody correspondante et la "sous-algèbre de composition" dans $\mathcal{H}_{tw}(\mathrm{rep}_{\mathbb{F}_q}(Q))$ engendrée par les classes des objets simples. À l'aide du groupe de Grothendieck de la catégorie des représentations de carquois, il introduit une version étendue de l'algèbre de Hall qui récupère la partie de Borel du groupe quantique. De plus, Green [loc. cit.] a construit la comultiplication et Xiao [78] a donné l'antipode de cette algèbre de Hall étendue et tordue qui en a fait une algèbre de Hopf auto-duale.

Lusztig [53] a étudié la version géométrique d'une sous-algèbre de composition dans l'algèbre de Hall $\mathcal{H}_{tw}(\mathrm{rep}_{\mathbb{F}_q}(Q))$, en utilisant des faisceaux pervers sur le espaces de modules de représentations de carquois. Ceci est un exemple de catégorification monoïdale, où le produit tensoriel d'une certaine catégorie monoïdale induit la multiplication dans l'algèbre. Cette approche l'a conduit à la découverte de la base canonique dans $U_{\sqrt{q}}(\mathfrak{n}_+)$ satisfaisant des propriétés de positivité très agréables.

D'autres exemples intéressants d'algèbres de Hall sont ceux de catégories de faisceaux cohérents sur des schémas. Kapranov [34] les a examinées pour la première fois, en reliant les algèbres de Hall de faisceaux cohérents sur des courbes à l'étude des formes automorphes. Depuis lors, les algèbres de Hall des faisceaux cohérents ont été étudiées de manière intensive et se sont révélées être liées à la conjecture géométrique de Langlands, aux algèbres de Cherednik, aux invariants de noeuds, etc.

Les résultats de Ringel et de Green ont soulevé une question naturelle : est-il possible de réaliser le groupe quantique entier $U_{\sqrt{q}}(\mathfrak{g})$ comme une algèbre de Hall ? On a vite compris que cette algèbre devrait être associée d'une manière ou d'une autre à la catégorie dérivée 2-périodique ou $\mathbb{Z}/2$ de la catégorie abelienne de représentations du carquois. Dans cette construction hypothétique, deux copies de $\mathrm{rep}_{\mathbb{F}_q}(Q)$ devraient fournir les deux parties nilpotentes du groupe quantique, tandis que la partie Cartan devrait être récupérée le groupe de Grothendieck. L'obstacle réside dans le fait que cette catégorie $2-$périodique n'est pas abélienne, mais triangulée. Il s'est avéré que la définition de Ringel aurait dû être modifiée afin de fournir des algèbres de Hall *associatives* associées, d'une certaine manière, à des catégories triangulées.

Ces idées ont motivé plusieurs généralisations de la construction de Ringel. Peng-Xiao [59] a récupéré des algèbres de Lie Kac-Moody à partir de catégories dérivées $2-$périodiques et, plus généralement, des algèbres de Hall de Lie à partir de catégories triangulées $2-$périodiques. Malheureusement, dans leur approche, la partie Cartan et la règle de sa commutation avec les parties nilpotentes apparaissent de manière assez ad hoc. Hubery [33] a prouvé que l'algèbre définie de la même manière que celle de Ringel, mais pour une catégorie exacte, est également unitaire et associative. Kapranov [35] a introduit une version de l'algèbre de Hall pour la catégorie dérivée liée à une catégorie abélienne héréditaire et pour sa partie avec des cohomologies concentrées en degrés 0 et 1. Cette dernière fournit un double de Heisenberg de $U_{\sqrt{q}}(\mathfrak{b}_+)$ qui est étroitement liée à $U_{\sqrt{q}}(\mathfrak{g})$ mais ne coïncide pas avec elle ; en particulier, ce double de Heisenberg n'a pas de structure d'algèbre de Hopf ce qui est une propriété très importante de $U_{\sqrt{q}}(\mathfrak{g})$.

Toën [75] a donné une construction de ce qu'il a appelé des *algèbres de Hall dérivées* pour les catégories triangulées muniées des DG-modeles satisfaisant certaines conditions de finitude. Xiao et Xu ont montré que cette construction donne une algèbre unitaire associative utilisant seulement les axiomes des catégories triangulées. La construction duale , définie sur les fonctions motiviques (au lieu de constructibles) sur le champs d'objets de modules dans une catégorie triangulée A_∞ a été introduite par Kontsevich et Soibelman. Dans ces travaux, la "correction" qui convertit le produit de Ringel en un produit associatif est obtenue en le multipliant par la forme d'Euler tronquée des facteurs. Malheureusement, les conditions de finitude imposées à une catégorie pour définir son algèbre de Hall dérivée (motivique) sont assez restrictives : elles sont valables pour les catégories dérivées bornées de catégories abéliennes Hom -finies exactes, mais elles ne sont pas satisfaites pour toute catégorie triangulée périodique. Plus précisément, cette forme d'Euler tronquée est un produit infini qui ne se stabilise pas dans le cas périodique. Par conséquent, aucune de ces techniques ne peut donner une construction satisfaisante de $U_{\sqrt{q}}(\mathfrak{g})$ sous la forme d'une

algèbre de Hall.

La première solution a été trouvée par Bridgeland [5]. Il a examiné la localisation d'une algèbre de Hall, tordue de manière appropriée, de la catégorie des complexes 2-périodiques avec des composantes projectives (en $\mathrm{rep}_{\mathbb{F}_q}(Q)$) dans les classes de tous les complexes contractibles. Il a défini une certaine réduction de cette localisation et l'a notée $\mathcal{DH}_{red}(\mathrm{rep}_{\mathbb{F}_q}(Q))$. Il a construit une intégration à partir de $U_{\sqrt{q}}(\mathfrak{g})$ dans $\mathcal{DH}_{red}(\mathcal{A})$, où $\mathcal{A}$ est la catégorie des représentations de carquois ; c'est un isomorphisme exactement dans le cas de Dynkin. Il a supposé que cette construction fournit le double de Drinfeld de l'algèbre de Hall étendue tordue $\mathcal{H}_{tw}(\mathcal{A})$ pour toute catégorie héréditaire $\mathcal{A}$ ayant assez de projectifs et satisfaisant conditions naturelles de finitude. Cela a été montré par Yanagida [82].

0.2 Algèbres de Hall étendues et localisation des catégories

Nous donnons une vaste généralisation de la construction de Bridgeland et la relions aux algèbres de Hall dérivées de Toën. Elle unifie les approches de [24] et de [23] qui forment les chapitres 2 et 4. La philosophie générale est la suivante. Nous remarquons que la forme d'Euler tronquée peut n'est pas le seul choix possible du facteur de correction : on peut formuler la restriction précise sur les facteurs possibles. En nous permettant de considérer les algèbres de Hall étendues, nous obtenons plus de liberté dans ce choix. Au niveau des algèbres, nous obtenons un module libre sur une algèbre de groupe tordue de façon appropriée de certains groupes G. L'idée est de considérer les algèbres de Hall dérivées tordues et étendues, même lorsque leurs équivalents non étendus ou leurs torsions correspondantes ne sont pas bien définis. Plus précisément, à chaque extension abélienne $\widetilde{\mathcal{N}}$ du groupe de Grothendieck additif d'une catégorie triangulée $\mathcal{T}$ par un groupe abélien G et chaque forme bilinéaire ϕ sur la pré-image $\mathcal{N}$ du cône positif satisfaisant certaines conditions naturelles, nous associons une algèbre $\mathcal{H}(\mathcal{T}, \mathcal{N}, \phi)$. Nous montrons qu'elle est associative, unitaire et se comporte naturellement sous le changement de $\mathcal{N}$ et ϕ. Nous montrons qu'après avoir tordu sa multiplication, $\mathcal{H}(\mathcal{T}, \mathcal{N}, \phi)$ se décompose comme une déformation plate de l'algèbre de Hall dérivée de $\mathcal{T}$ sur l'algèbre de groupe de G lorsque l'algèbre de Hall et le twist sont bien définis. La modification cruciale de la construction classique des algèbres de Hall étendues (de catégories abéliennes) est que l'algèbre non étendue, même bien définie, ne forme qu'un quotient de $\mathcal{H}_{tw}(\mathcal{T}, \mathcal{N}, \phi)$ et pas nécessairement une sous-algèbre : $\mathcal{H}_{tw}(\mathcal{T}, \mathcal{N}, \phi)$ est une déformation plate non triviale.

Notre preuve est une modification ad hoc de la preuve de l'associativité des algèbres de Hall dérivées présentée par Xiao et Xu dans [80]. Dans leur construction, la forme d'Euler tronquée de $\mathcal{T}$ apparaît d'abord de la manière suivante : leur produit alternatif pour les éléments d'un triangle distingué $A \to B \to C \to \Sigma A$ en tant que premier, resp. le deuxième argument compte le nombre d'éléments d'un certain ensemble. Nous notons que cet ensemble est fini lorsque la catégorie triangulée en question est Hom -finie, il n'est pas nécessaire d'exiger la finitude homologique locale à gauche (ou à droite). Il s'avère que si on considère les relèvements de classes de A, B

et C dans $\mathcal{N}$, cet ensemble peut être compté comme le produit alterné des valeurs de la forme ϕ. Le reste du démonstration de Xiao-Xu fonctionne parfaitement, jusqu'au changement des formes d'Euler tronquées dans $\mathcal{T}$ par la forme ϕ. Les éléments essentiels de la preuve sont les relèvements des carrés cartésiens homotopiques.

Dans une certaine généralité, nous pouvons définir la forme idempotente d'une algèbre de Hall étendue tordue. Dans le cas de catégories dérivées $2-$périodiques de représentations de carquois, elle est étroitement liée à l'algèbre $\dot{U}$ de Lusztig.

Il s'avère que la considération d'une catégorie triangulée comme la localisation d'une catégorie exacte (ou d'une autre catégorie triangulée) donne une source naturelle de ces versions étendues des algèbres de Hall. La catégorie des complexes 2-périodiques avec des composantes projectives est *de Frobenius* lorsqu'elle est munie d'une structure exacte naturelle. Cela signifie que cette catégorie exacte a assez de projectifs, d'injectifs et que ces deux classes d'objets coïncident. Sa catégorie *stable* (la catégorie ayant le même ensemble d'objets et les morphismes étant des quotients de morphismes dans la catégorie de Frobenius par ceux que se factorisent par projectif-injectif) est la catégorie d'homotopie 2-périodique de la sous-catégorie complète $\mathcal{P}(\mathcal{A})$ de projectives dans $\mathcal{A}$. Si $\mathcal{A}$ a assez de projectifs et est de dimension globale finie, cette dernière catégorie est équivalente à la catégorie dérivée 2-périodique de $\mathcal{A}$. La construction de Bridgeland peut donc être vue sous l'angle suivant : nous avons une catégorie triangulée $\mathcal{T} = \mathcal{D}_{\mathbb{Z}/2}(\mathcal{A})$ pour laquelle l'algèbre de Hall dérivée n'est pas définie car les conditions de finitude ne sont pas remplies. Alors le substitut correct, dans un certain sens, est donné par la règle suivante : on trouve une catégorie de Frobenius $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{P}(\mathcal{A}))$, dont la catégorie stable est équivalente à $\mathcal{T}$, et pour laquelle l'algèbre de Hall classique (en tant que de catégorie exacte) est bien définie. Ensuite, on prend cette algèbre de Hall et on la localise dans les classes de tous les objets projectifs-injectifs.

Dans [24], nous considérons une catégorie de Frobenius arbitraire $\mathcal{F}$ satisfaisant certaines conditions de finitude. Nous définissons l'algèbre de Hall semi-dérivée $\mathcal{SH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$ comme la localisation de $\mathcal{H}(\mathcal{F})$ dans les classes de tous les objets projectifs-injectifs et établissons les propriétés suivantes.

Theorem 0.2.1. *[24]*

(i) Les algèbres $\mathcal{SH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$ sont fonctorielles sous les morphismes pleinement fidèles des catégories de Frobenius.

(ii) $\mathcal{SH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$ est un module libre sur l'algèbre des groupes du groupe Grothendieck de la sous-catégorie complète $\mathcal{P}(\mathcal{F})$ desprojectifs-injectifs tordue par la forme d'Euler. Tout choix de représentants dans $\mathcal{F}$ des classes d'isomorphisme d'objets de la catégorie stable $\underline{\mathcal{F}}$ donne une base de ce module.

(iii) $\mathcal{SH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$ avec une multiplication convenablement tordue est une déformation plate de l'algèbre de Hall dérivée de $\underline{\mathcal{F}}$ sur l'algèbre de groupe de $K_0(\mathcal{P}(\mathcal{F}))$, lorsque cette derniere est bien définie.

La construction de Bridgeland montre que, dans certaines situations, $\mathcal{SH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$ est bien

défini, tandis que l'algèbre de Hall dérivée de $\underline{\mathcal{F}}$ ne l'est pas. La catégorie de complexes à composantes projectives n'est pas la plus appropriée : elle n'est pas une invariante dérivée. Au chapitre 4, nous montrons l'invariance dérivée des algèbres de Bridgeland (c'est le matériau de [23]). En particulier, nous obtenons une interprétation catégorique des foncteurs de réflexion de Bernstein-Gelfand-Ponomarev. Pour étudier les doubles de Drinfeld des algèbres de Hall de faisceaux cohérents, il faut travailler avec des catégories dérivées périodiques de catégories abéliennes sans assez de projectifs. Nous montrons comment généraliser l'analogue de la construction de Bridgeland pour les catégories dérivées bornées en une classe comprenant des catégories de faisceaux cohérentes. Cette construction a été modifiée par Lu et Peng dans [49] : ils l'ont adaptée pour définir des algèbres de Hall pour les catégories dérivées à deux périodiques de catégories abéliennes héréditaires, généralisant ainsi le travail de Bridgeland.

Dans le cas des catégories dérivées des catégories abéliennes héréditaires, nous observons une observation non triviale : l'algèbre de Hall semi-dérivée (ou l'algèbre de Lu et Peng) est, modulo une torsion, un quotient d'une algèbre de Hall localisée d'une catégorie de complexes. Cette application de quotient peut être comprise comme une version relative de l'application d'intégration définie par Reineke [63].

En général, les modèles de Frobenius de catégories triangulées ne sont souvent pas les plus pratiques. Au chapitre 3, nous généralisons cette construction à d'autres modèles de catégories trangulées. Étant donné une catégorie exacte $\mathcal{E}$ aux équivalences faibles W, certaines structures supplémentaires peuvent assurer que la catégorie localisée $\mathcal{E}[W^{-1}]$ est triangulée. En particulier, cela se produit si W peut être complété à la structure d'une catégorie de modèle exacte héréditaire $\mathcal{M}$ sur $\mathcal{E}$, ou lorsque la catégorie sous-jacente de $\mathcal{E}$ a une structure exacte de Frobenius plus petite et la sous-catégorie complète d'objets faiblement triviaux $\mathcal{W}$ forme une sous-catégorie Frobenius. Ces deux classes de catégories exactes aux équivalences faibles généralisent naturellement le cas des catégories de Frobenius. Lorsque $\mathcal{E}$ et $\mathcal{W}$ satisfont à certaines conditions de finitude, nous construisons une algèbre associative unitaire $\mathcal{H}(\mathcal{E}, \mathcal{W})$ et nous montrons le théorème suivant.

Theorem 0.2.2. *(i) $\mathcal{H}(\mathcal{E}, \mathcal{W})$ est un module libre sur l'algèbre de groupe du groupe de Grothendieck numérique de la sous-catégorie complète des objets faiblement triviaux $K_0^{num}(\mathcal{W})$ tordue par la forme d'Euler. Tout choix de représentants dans $\mathcal{E}$ des classes d'isomorphismes d'objets de la catégorie $\mathcal{E}[\mathcal{W}^{-1}]$ donne une base de ce module.*

(ii) $\mathcal{H}(\mathcal{E}, \mathcal{W})$ avec une multiplication correctement tordue est une déformation plate de l'algèbre de Hall dérivée de $\mathcal{E}[\mathcal{W}^{-1}]$ sur le algèbre de groupe de $K_0^{num}(\mathcal{W})$, lorsque l'algèbre de Hall dérivée est bien définie.

(iii) Les algèbres $\mathcal{H}(\mathcal{E}, \mathcal{W})$ sont fonctorielles sous des foncteurs exacts induisant des équivalences dans $\mathcal{E}[\mathcal{W}^{-1}]$.

(iv) Quand $\mathcal{E}$ est Frobenius et $\mathcal{W}$ est la sous-catégorie complète des objets injectifs-projectifs, nous récupérons la définition précédente : $\mathcal{H}(\mathcal{E}, \mathcal{W}) = \mathcal{SH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$.

La torsion est donnée par le quotient des formes d'Euler sur $\mathcal{E}$ et sur $\mathcal{E}[\mathcal{W}^{-1}]$, lorsque celles-ci sont bien définies. Nous obtenons une algèbre sur l'anneau de groupe de $K_0^{num}(\mathcal{W})$.

L'idée derrière la définition de l'algèbre $\mathcal{H}(\mathcal{E}, \mathcal{W})$ est de considérer la sous-catégorie complète des objets faiblement triviaux $\mathcal{W}$ comme si elle était scindée, ce qui se produit lorsque $\mathcal{E}$ est Frobenius et $\mathcal{W}$ est la sous-catégorie complète des objets projectifs-injectifs. Lorsque nous localisons la catégorie exacte $\mathcal{E}$, ces objets deviennent isomorphes à 0. Mais nous savons que la classe de l'objet 0 est l'unité de notre algèbre. Cela suggère que les objets faiblement triviaux devraient être inversibles dans notre algèbre et ne pas disparaître comme on pourrait s'y attendre.

En présence d'un certain type de "remplacements relatives", nous avons une autre preuve d'associativité de notre $\mathcal{H}(\mathcal{E}, \mathcal{W})$. En remplaçant les facteurs par ces "remplacements", nous pouvons réduire le problème d'associativité de l'algèbre de Hall habituelle de la catégorie (exacte) $\mathcal{E}$. Historiquement, il s'agissait de la première construction d'algèbres de Hall semi-dérivées définies dans [23] pour les catégories de complexes. Même si les algèbres ainsi définies sont, en principe, couvertes par la nouvelle définition, nous présentons la deuxième construction pour souligner de plus l'interaction entre les algèbres de Hall de catégories exactes et triangulées et leur associativité.

0.3 Propriété du coeur et son substitut

Une des propriétés importantes des algèbres de Hall dérivées est la "propriété du coeur". Soit $\mathcal{T}$ une catégorie triangulée dont l'algèbre de Hall dérivée $\mathcal{DH}(\mathcal{T})$ est bien définie. Si $\mathcal{T}$ admet une $t-$structure bornée avec un coeur $\mathcal{A}$, Toën [75] montre que le foncteur de plongement $\mathcal{A} \to \mathcal{T}$ induit une plongement de l'algèbre de Hall du coeur en $\mathcal{DH}(\mathcal{T})$:

$$\mathcal{H}(\mathcal{A}) \hookrightarrow \mathcal{DH}(\mathcal{T}).$$

Le théoreme 0.7.2 implique que les algèbres de Hall de catégories exactes avec des équivalences faibles peuvent être considérées comme des algèbres de Hall étendues tordues de catégories triangulées. Nous avons un analogue naturel de la propriété de coeur dans ce cadre.

Corollary 0.3.1. *Soit $\mathcal{A}$ le coeur d'une $t-$structure bornée d'une catégorie $\mathcal{E}[\mathcal{W}^{-1}]$. Soit $\widetilde{\mathcal{A}}$ la sous-catégorie exacte de $\mathcal{E}$, composée de tous les objets dont l'image sous la localisation appartient à $\mathcal{A}$.. Nous avons ensuite une plongement*

$$\mathcal{H}(\widetilde{\mathcal{A}}) \hookrightarrow \mathcal{H}(\mathcal{E}, \mathcal{W}),$$

et l'algèbre $\mathcal{H}_{tw}(\widetilde{\mathcal{A}})$ est une déformation plate de $\mathcal{H}(\mathcal{A})$ sur $\mathbb{Q}K_0^{num}(\mathcal{W})$, où la torsion est donnée par la forme

$$\frac{\langle -, - \rangle_{\widetilde{\mathcal{A}}}}{\langle -, - \rangle_{\mathcal{A}}} = \frac{\langle -, - \rangle_{\mathcal{E}}}{\langle -, - \rangle_{\mathcal{E}[\mathcal{W}^{-1}]}}.$$

Plus généralement, nous avons le résultat suivant.

Corollary 0.3.2. *Soit $\mathcal{E}'$ une sous-catégorie entièrement exacte de $\mathcal{E}$ telle que $\mathcal{E}'[\mathcal{W}^{-1}]$ soit une*

sous-catégorie fermée par rapport aux extensions de $\mathcal{E}[\mathcal{W}^{-1}]$ *Ensuite, nous avons un plongement*

$$\mathcal{H}(\mathcal{E}', \mathcal{W}) \hookrightarrow \mathcal{H}(\mathcal{E}, \mathcal{W}),$$

et l'algèbre $\mathcal{H}_{tw}(\mathcal{E}', \mathcal{W})$ *est une déformation plate de* $\mathcal{H}(\mathcal{E}'[\mathcal{W}^{-1}])$ *sur* $(\mathbb{Q}K_0^{num}(\mathcal{E}' \cap \mathcal{W})$, *où la torsion est donnée par la forme*

$$\frac{\langle -, - \rangle_{\mathcal{E}'}}{\langle -, - \rangle_{\mathcal{E}'[\mathcal{W}^{-1}]}} = \frac{\langle -, - \rangle_{\mathcal{E}}}{\langle -, - \rangle_{\mathcal{E}[\mathcal{W}^{-1}]}}.$$

Si $\mathcal{T}$ est périodique, elle n'a pas de $t-$structure bornée. Pour les catégories dérivées périodiques des catégories abéliennes $\mathcal{D}_m(\mathcal{A})$, nous sommes toujours intéressés par la relation entre les algèbres de Hall étendues tordues de $\mathcal{D}_m(\mathcal{A})$ et de $\mathcal{A}$. Nous prouvons que pour une catégorie abélienne $\mathcal{A}$, les classes de $m-$ complexes périodiques quasi-isomorphes à complexes concentrées en degré 0 forment une sous-catégorie fermée par rapport aux extension de $\mathcal{D}_m(\mathcal{A})$ si et seulement si $\operatorname{gl\,dim}(\mathcal{A}) \leq m$. Ils forment toujours une sous-catégorie pleinement exacte de $\mathcal{C}_m(\mathcal{A})$ que nous désignons $\widetilde{\mathcal{A}}$.. Nous avons des cohomologies bien définies $H^i, i = 0, \ldots, m-1$, avec des valeurs dans $\mathcal{A}$, et à partir de la suite exacte longue on obtiens que les complexes à cohomologies concentrés dans le degré 0 forment toujours une sous-catégorie exacte $\mathcal{C}_m^0(\mathcal{A})$ de $\mathcal{C}_m(\mathcal{A})$ fermé sous quasi-isomorphismes et leurs classes forment une sous-catégorie fermé par rapport aux extensions de $\mathcal{D}_m(\mathcal{A})$. Ainsi, nous avons le résultat suivant.

Theorem 0.3.3. *Soit* $\mathcal{A}$ *une catégorie abélienne* $\mathbb{F}_q-$ *linéaire de dimension globale finie avec des espaces finis* $\operatorname{Hom}-$ *et* Ext^i-, *pour tout* $i > 0$. *Alors, les algèbres de Hall* $\mathcal{H}(\mathcal{A}), \mathcal{H}(\mathcal{C}_m(\mathcal{A}), \mathrm{qis}), \mathcal{H}(\mathcal{C}_m^0(\mathcal{A}), \mathrm{qis})$ *sont bien définies.*

(i) Nous avons une plongement

$$\mathcal{H}(\widetilde{\mathcal{A}}) \hookrightarrow \mathcal{H}(\mathcal{C}_m(\mathcal{A}), qis),$$

et l'algèbre

$$\mathcal{H}_{tw}(\widetilde{\mathcal{A}})$$

est une déformation plate de $\mathcal{H}(\mathcal{A})$ *sur*

$$\mathbb{Q}K_0^{num}(\mathcal{C}_{m,ac}(\mathcal{A})) \xrightarrow{\sim} \bigotimes_{i=1}^{m} K_0^{num}(\mathcal{A}),$$

si et seulement si $\operatorname{gl\,dim}(\mathcal{A}) \leq m$.

(ii) Nous avons un plongement

$$\mathcal{H}(\mathcal{C}_m^0(\mathcal{A}), \mathrm{qis}) \hookrightarrow \mathcal{H}(\mathcal{C}_m(\mathcal{A}), qis)$$

avec des isomorphismes

$$\mathcal{H}_{tw}(\mathcal{C}_m^0(\mathcal{A}), \mathrm{qis}) \xrightarrow{\sim} \mathcal{H}(\mathcal{D}_m^0(\mathcal{A})) \otimes \mathbb{Q}K_0^{num}(\mathcal{C}_{m,ac}(\mathcal{A})) \xrightarrow{\sim} \mathcal{H}(\mathcal{A}) \otimes \bigotimes_{i=1}^{m} K_0^{num}(\mathcal{A}),$$

pour tout $\mathcal{A}$.

Geng et Peng [21] ont prouvé l'existence du plongemnet et des isomorphismes de la partie (i) pour $\mathcal{A}$ ayant assez de projectifs et de $\mathrm{gl\,dim} \leq 2$ et m égal à 2 . Ils ont utilisé la construction de Bridgeland d'algèbres de Hall étendues tordues de catégories dérivées périodiques.

Le théorème 0.3.3 suggère que les algèbres $\mathcal{H}(\mathcal{C}_m^0(\mathcal{A}), \mathrm{qis})$ et leurs versions tordues peuvent être considérées comme des substituts naturels de $\mathcal{H}(\mathcal{A})$ lorsque on songe à périodique catégories dérivées. La multiplication ici implique des extensions plus hautes dans la catégorie $\mathcal{A}$. Le cas des catégories dérivées 2-périodiques semble être le plus important pour de nombreuses raisons. Il semble possible, bien que pas nécessairement probable, que les algèbres $\mathcal{H}(\mathcal{C}_2(\mathcal{A}), \mathrm{qis})$ et $\mathcal{H}(\mathcal{C}_2^0(\mathcal{A}), \mathrm{qis})$ avec des torsions justes puissent avoir une structure d'algèbre de Hopf au-delà du cas héréditaire.

0.4 Réduction des structures exactes et dégénérescence des algèbres de Hall

Au chapitre 5.1, nous présentons les résultats du travail conjoint avec Xin Fang [25].

Une catégorie additive $\mathcal{A}$ peut être munie de nombreuses structures exactes différentes. Il est donc naturel de se demander si les algèbres de Hall de ces différentes structures exactes sont liées les unes aux autres. Le théorème suivant donne une réponse pour les catégories de type fini (c'est-à-dire ayant un nombre fini d'indécomposables par rapport aux isomorphismes).

Theorem 0.4.1. *Soit* $\mathcal{A}$ *une catégorie additive* $k-$*linéaire à idempotents scindés,* $\mathrm{Hom}-$*finie et de type fini. Supposons que* $\mathcal{A}$ *soit munie de deux structures exactes* $\mathcal{E}' < \mathcal{E}$*.. Alors l'algèbre de Hall* $\mathcal{H}(\mathcal{E}')$ *de* $\mathcal{E}'$ *est une dégénérescence de l'algèbre de Hall* $\mathcal{H}(\mathcal{E})$ *de* $\mathcal{E}$*, par rapport à une filtration donnée par une fonction sur l'ensemble* $\mathrm{Iso}(\mathcal{A})$*. Cette fonction est définie comme* $w(M) = \dim \mathrm{Hom}(\bigoplus_{P \in \mathrm{ind}(\mathcal{P}(\mathcal{E}'))} P, M)$.

Grâce aux travaux d'Enomoto [16] [17], il est possible de classer des structures exactes dans de nombreuses situations lorsque $\mathcal{A}$ contient un nombre fini d'objets indécomposables. Cela couvre les cas de la catégorie de représentations d'un carquois de Dynkin et de sa catégorie de complexes 2-périodiques. Dans le premier cas, nous montrons que les dégénérescences de l'algèbre de Hall de la catégorie abélienne données par des structures exactes plus petites sont précisément les dégénérescences définies dans [14]. En particulier, la structure exacte scindée correspond à une algèbre polynomiale $q-$commutative, et nous retrouvons le théorème de PBW dans cette cas quantique.

Ainsi, le théorème 0.9.1 peut être considéré comme une généralisation du théorème de PBW dans le cadre des algèbres de Hall.

Au section 5.2, nous discutons de ses applications hypothetiques aux groupes quantiques et au double de Drinfeld en modifiant les structures exactes de la catégorie des complexes 2-périodiques.

0.5 Remarques supplémentaires

Il est intéressant de noter qu'une catégorie additive $\mathcal{A}$ avec des équivalences faibles $\mathcal{W}$ peut être munie de nombreuses structures exactes $\mathcal{E}$ compatibles avec $\mathcal{W}$. Ainsi, elle peut a priori donner lieu à de nombreuses algèbres de Hall étendues tordues de $\mathcal{E}[\mathcal{W}^{-1}] = \mathcal{A}[\mathcal{W}^{-1}]$. Pour une telle algèbre à définir, une catégorie exacte $\mathcal{E}$ doivent remplir certaines conditions, restreignant notre choix. Considérons le cas de la catégorie de complexes sur une catégorie abelienne de dimension globale finie, les quasi-isomorphismes étant considérés comme des équivalences faibles. La structure exacte scindée par composants (Frobenius) ne remplit pas ces conditions, mais ils sont valable pour la structure exacte par composants (abélienne). Plus généralement, dans tous les exemples connus de l'auteur, il existe en réalité un choix naturel d'une structure exacte répondant à ces conditions.

Kontsevich [42] a suggéré la construction d'une algèbre de Hall étendue tordue pour une classe de catégories d'orbites triangulées n'utilisant pas leurs modèles exacts (des détails peuvent apparaître ailleurs). Il semble que chacune de ces catégories triangulées a une algèbre unique et que, dans le cas des catégories dérivées 2-périodiques, on récupère les algèbres construites à l'aide de catégories de complexes. Ceci suggère l'existence du choix canonique d'une extension et d'une forme bilinéaire, définissant ainsi une algèbre de Hall étendue tordue pour chaque catégorie triangulée (satisfaisant des conditions de finitude appropriées) de manière canonique.

Notons que nous pouvons associer des algèbres de Hall à des localisations de catégories triangulées de la même manière que notre construction pour des localisations de catégories exactes. Récemment, Nakaoka et Palu [58] ont introduit une généralisation commune des catégories exactes et triangulées :*catégories extriangulées*. Ces catégories sont définies de manière à ce que de nombreuses propriétés liées à la localisation dans des catégories exactes ou triangulées s'appliquent également à elles. En particulier, il existe une version de la correspondance de Hovey entre les structures de modèles et les paires de cotorsions pour les catégories extriangulées. Il semble que les catégories extriangulées devraient constituer le cadre le plus naturel pour étudier des algèbres de Hall étendues tordues via la localisation de catégories additives. Notamment, les carrés cartésiens homotopiques qui sont les ingrédients essentiels de notre construction apparaissent dans le cadre de catégories extriangulées faisant partie de leurs axiomatiques.

Il semble désormais bien établi que les algèbres de Hall contiennent de nombreux phénomènes profonds de nature catégorique supérieure capturées par les espaces dits *2-Segal* définis par Dyckerhoff et Kapranov [12]. Il est associé par la $S_\bullet-$ construction de Waldhausen à une $\infty-$catégorie exacte. Cependant, comme indiqué dans [11], les algèbres de Hall ne dépendent que de la catégo-

rie additive sous-jacente. Nous voyons que les diagrammes les plus importants que nous utilisons dans les catégories triangulées sont des carrés cartésiens homotopiques qui sont plutôt de nature $1-$catégorique. C'est l'une des raisons pour lesquelles l'auteur a fortement préféré garder cette thèse dans le domaine des catégories additives.

Sans changer notre optique, nous pouvons également associer des algèbres de Hall étendues tordues à des paires de localisation de catégories différentielles graduées pré-triangulées avec des idempotents scindés en utilisant leurs catégories sous-jacentes.

En interprétant l'application de quotient de la section 3.3.5 comme une version de l'application d'intégration, il est possible de construire des versions motiviques des algèbres de Hall semi-dérivées, du moins dans une certaine généralité. Pour une catégorie abélienne faiblement de Gorenstein de la dimension de Gorenstein 1, nous trouvons que l'algèbre de Hall de sa structure de modèle de Gorenstein projective est l'image sous cette application d'intégration de l'algèbre de Hall (classique) de la catégorie abélienne. L'algèbre de Hall motivique de la catégorie moèle est l'image de l'application d'intégration de l'algèbre de Hall motivique de la catégorie abélienne, lorsque cette dernière est bien définie. Nous ne présentons pas les détails dans la thèse.

Cela couvre le cas des catégories singulières de Nakajima introduites par Keller-Scherotzke [40] et Scherotzke [67]. Ils les ont introduites afin d'étudier les variétés carquois graduées et généralisées via les méthodes de l'algèbre homologique de Gorenstein, généralisant les travaux de Kimura-Qin [41] et de Leclerc-Plamondon [47]. Les versions géométriques des algèbres de Hall correspondantes (en termes de faisceaux perverses) ont été étudiées dans les travaux de Qin [61] et de Scherotzke-Sibilla [66], voir aussi [77] [31] [47] pour la relation avec l'anneau de Grothendieck dual (tordue) d'une catégorie tensorielle de certaines représentations de l'algèbre affine quantique dans le cas gradué.

En généralisant la définition des groupes quantiques via des anneaux quantifiés de fonctions sur des groupes de Poisson-Lie, les fonctions régulières sur certaines variétés de Poisson peuvent être comprises en termes d'algèbres amassées, qui sont à leur tour liées aux algèbres de Hall. Récemment, des groupes quantiques ont été plongés dans les anneaux de fonctions régulières sur certains espaces de modules de systèmes locaux sur des surfaces, voir [22] et les références qui y figurent. Il semble probable que [22, Theorem 2.19] puisse être généralisée au cas des m points spéciaux en remplaçant $U_{\sqrt{q}}(\mathfrak{g})$ par les algèbres de Hall semi-dérivées $\mathbb{Z}/m-$graduées de représentations du carquois correspondant.

Introduction

0.6 Historical context

Hall algebras provide one of the first known examples of additive categorification. They first appeared in works of Steinitz [74] and Hall [27] on commutative finite p-groups. Later, they reappeared in the work of Ringel [64] on quantum groups. He introduced the notion of the Hall algebra of an abelian category with finite dimensional Hom- and Ext^1-spaces. As a vector space, it has a basis parametrized by the isomorphism classes of objects in the category. The multiplication captures information about the extensions between objects. One can consider this as an algebra of constructible functions on the moduli stack of objects in the category, with the convolution product given by the Hecke correspondences.

Ringel constructed an isomorphism between the twisted Hall algebra of the category of representations of a simply-laced Dynkin quiver Q over the finite field $\mathbb{F}_q$ and the nilpotent part of the corresponding quantum group, specialized at the square root of q :

$$U_{\sqrt{q}}(\mathfrak{n}_+) \xrightarrow{\sim} \mathcal{H}_{tw}(\mathrm{rep}_{\mathbb{F}_q}(Q)).$$

Later Green [26] generalized this result to an arbitrary valued quiver Q by providing an isomorphism between the nilpotent part of the quantized universal enveloping algebra of the corresponding Kac-Moody algebra and the so-called "composition" subalgebra in $\mathcal{H}_{tw}(\mathrm{rep}_{\mathbb{F}_q}(Q))$ generated by the classes of simple objects. Using the Grothendieck group of the category of quiver representations, he introduced an extended version of the Hall algebra which recovers the Borel part of the quantum group. Moreover, Green [loc. cit.] constructed the comultiplication and Xiao [78] gave the antipode in this twisted extended Hall algebra that made it a self-dual Hopf algebra.

Lusztig [53] investigated the geometric version of a composition subalgebra in the Hall algebra $\mathcal{H}_{tw}(\mathrm{rep}_{\mathbb{F}_q}(Q))$, using perverse sheaves on moduli spaces of quiver representations. This is an example of monoidal categorification, where the tensor product in a certain monoidal category gives rise to the multiplication in the algebra. This approach led him to the discovery of the canonical basis in $U_{\sqrt{q}}(\mathfrak{n}_+)$ satisfying very pleasant positivity properties.

Other interesting examples of Hall algebras are those of categories of coherent sheaves on schemes. They were first considered by Kapranov in [34], where he linked Hall algebras of coherent sheaves on curves to the study of automorphic forms. Since then, Hall algebras of coherent

sheaves have been studied intensively and turned out to be related to the geometric Langlands conjecture, Cherednik algebras, knot invariants, etc.

The results of Ringel and Green gave rise to a natural question : whether one can realize the whole quantum group $U_{\sqrt{q}}(\mathfrak{g})$ as a certain Hall algebra ? It was soon understood that this algebra should be somehow associated to the 2-periodic, or $\mathbb{Z}/2$-graded, derived category of the abelian category of quiver representations. In this hypothetical construction, two copies of $\mathrm{rep}_{\mathbb{F}_q}(Q)$ should provide the two nilpotent parts of the quantum group, while the Cartan part should be recovered from the Grothendieck group. The obstacle was that this 2-periodic category is not abelian, but rather triangulated. It turned out that the definition of Ringel should have been modified in order to provide *associative* Hall algebras associated, in some way, to triangulated categories.

These ideas motivated several generalizations of Ringel's construction. Peng-Xiao [59] recovered Lie Kac-Moody algebras from 2-periodic derived categories and, more generally, Hall Lie algebras from 2-periodic triangulated categories. Unfortunately, in their approach, the Cartan part and the rule of its commutation with the nilpotent parts appear in a quite ad hoc way. Hubery [33] proved that the algebra defined in the same way as by Ringel, but for an exact category, is also unital and associative. Kapranov [35] introduced a version of the Hall algebra for the bounded derived category of a hereditary abelian category and for its part with cohomologies concentrated in degrees 0 and 1. The latter provided a Heisenberg double of $U_{\sqrt{q}}(\mathfrak{b}_+)$ that is closely related to $U_{\sqrt{q}}(\mathfrak{g})$ but does not coincide with it ; in particular, this Heisenberg double does not have a Hopf algebra structure which is a very important property of $U_{\sqrt{q}}(\mathfrak{g})$.

Toën [75] gave a construction of what he called *derived Hall algebras* for DG-enhanced triangulated categories satisfying certain finiteness conditions. Xiao and Xu showed that this construction provides an associative unital algebra using only the axioms of triangulated categories. The dual construction, defined on motivic functions (instead of constructible ones) on the moduli stack of objects in a triangulated A_∞ category was introduced by Kontsevich and Soibelman. In these works, the "correction" that turns the Ringel product into an associative one is given by multiplying it by the truncated Euler form of the factors. Unfortunately, the finiteness conditions one imposes on a category in order to define its (motivic) derived Hall algebra are quite restrictive : they do hold for bounded derived categories of Hom-finite abelian or exact categories, but they are not satisfied for any periodic triangulated category. More precisely, this truncated Euler form is an infinite product which does not stabilize in the periodic case. Therefore, none of this techniques can give a satisfactory construction of $U_{\sqrt{q}}(\mathfrak{g})$ as a Hall algebra of some kind.

The first solution was found by Bridgeland [5]. He considered the localization of an appropriately twisted Hall algebra of the category of 2-periodic complexes with projective (in $\mathrm{rep}_{\mathbb{F}_q}(Q)$) components at the classes of all contractible complexes. He defined a certain reduction of this localization and denoted it by $\mathcal{DH}_{red}(\mathrm{rep}_{\mathbb{F}_q}(Q))$. He constructed an embedding from $U_{\sqrt{q}}(\mathfrak{g})$ into $\mathcal{DH}_{red}(\mathcal{A})$, where $\mathcal{A}$ is the category of quiver representations ; it is an isomorphism exactly in the Dynkin case. He conjectured that this construction provides the Drinfeld double of the twisted extended Hall algebra $\mathcal{H}_{tw}(\mathcal{A})$ for any hereditary category $\mathcal{A}$ having enough projectives and

satisfying natural finiteness conditions. This was proved by Yanagida [82].

0.7 Extended Hall algebras and localization of categories

We give a vast generalization of Bridgeland's construction and relate it to the derived Hall algebras of Toën. It unifies the approaches from [24] and [23] that form chapters 2 and 4. The general philosophy is the following. We notice that the truncated Euler form might not be the only possible choice of the correction factor : one can formulate the precise restriction on possible factors. Permitting ourselves to consider *extended* Hall algebras, we get more freedom in this choice. On the level of algebras, we get a free module over an appropriately twisted group algebra of some group G. In Bridgeland's algebras, the twisted group algebras realize quantum Cartan subalgebras in $U_t(\mathfrak{g})$. Thus, the idea is to consider twisted and extended derived Hall algebras, even when their non-extended counterparts or corresponding twists are not well-defined. More precisely, to each abelian extension $\widetilde{\mathcal{N}}$ of the additive Grothendieck group of a triangulated category $\mathcal{T}$ by an abelian group G and each bilinear form ϕ on the pre-image $\mathcal{N}$ of the positive cone satisfying certain natural conditions, we associate an algebra $\mathcal{H}(\mathcal{T}, \mathcal{N}, \phi)$. We prove that it is associative, unital, and behaves naturally under the change of $\mathcal{N}$ and ϕ. We show that after twisting its multiplication, $\mathcal{H}(\mathcal{T}, \mathcal{N}, \phi)$ decomposes as a flat deformation of the derived Hall algebra of $\mathcal{T}$ over the group algebra of G when the derived Hall algebra and the twist are well-defined. The crucial modification of the classical construction of extended Hall algebras (of abelian categories) is that the non-extended algebra, even if it is well-defined, forms only a quotient of $\mathcal{H}_{tw}(\mathcal{T}, \mathcal{N}, \phi)$ and not necessarily a subalgebra : $\mathcal{H}_{tw}(\mathcal{T}, \mathcal{N}, \phi)$ is a non-trivial flat deformation.

Our proof is an ad hoc modification of the proof of the associativity of the derived Hall algebras as presented by Xiao and Xu in [80]. In their construction, the truncated Euler form of $\mathcal{T}$ first appears in the following way : their alternating product for elements of a distinguished triangle $A \to B \to C \to \Sigma A$ as the first, resp. the second argument counts the number of elements of a certain set. We note that this set is finite whenever the triangulated category in question is Hom-finite, one does not need to require the left (or right) local homological finiteness. It turns out that if we consider lifts of classes of A, B and C in $\mathcal{N}$, this set can be counted as the alternating product of values of the form ϕ. The rest of the Xiao-Xu proof works perfectly, up to the change of truncated Euler forms in $\mathcal{T}$ by the form ϕ. The crucial elements in the proof are the lifts of homotopy cartesian squares.

In a certain generality, we can define the idempotented form of a twisted extended Hall algebra. In the case of $2-$periodic derived categories of quiver representations, it is closely related to the algebra $\dot{U}$ of Lusztig.

It turns out that considering a triangulated category as a localization of an exact (or another triangulated) category gives a natural source of such extended versions of Hall algebras. The category of 2-periodic complexes with projective components is *Frobenius* when endowed with a natural exact structure. That means that this exact category has enough projectives, enough injectives, and

these two classes of objects coincide. Its *stable* category (the category with the same set of objects and with morphisms being quotients of morphisms in the Frobenius category by those factoring through projective-injectives) is the 2-periodic homotopy category of the full subcategory $\mathcal{P}(\mathcal{A})$ of projectives in $\mathcal{A}$. If $\mathcal{A}$ has enough projectives, the latter category is equivalent to the 2-periodic derived category of $\mathcal{A}$. Thus, Bridgeland's construction can be seen from the following perspective : we have a triangulated category $\mathcal{T} = \mathcal{D}_{\mathbb{Z}/2}(\mathcal{A})$ for which the derived Hall algebra is not defined, as the finiteness conditions are not satisfied. Then the correct substitute, in some sense, is given by the following rule : one finds a Frobenius category $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{P}(\mathcal{A}))$, whose stable category is equivalent to $\mathcal{T}$, and for which the classical Hall algebra (as of an exact category) is well-defined. Then one takes this Hall algebra and localizes it at the classes of all projective-injective objects.

In [24], we consider an arbitrary Frobenius category $\mathcal{F}$ satisfying some finiteness conditions. We define the *semi-derived Hall algebra* $\mathcal{SH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$ as the localization of $\mathcal{H}(\mathcal{F})$ at the classes of all projective-injective objects and establish the following properties.

Theorem 0.7.1. *[24]*

(i) The algebras $\mathcal{SH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$ are functorial under fully faithful morphisms of Frobenius categories.

(ii) $\mathcal{SH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$ is a free module over the group algebra of the Grothendieck group of the full subcategory $\mathcal{P}(\mathcal{F})$ of projective-injective objects twisted by the Euler form. Any choice of representatives in $\mathcal{F}$ of the isomorphism classes of objects in the stable category $\underline{\mathcal{F}}$ yields a basis of this module.

(iii) $\mathcal{SH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$ with an appropriately twisted multiplication is a flat deformation of the derived Hall algebra of $\underline{\mathcal{F}}$ over the group algebra of $K_0(\mathcal{P}(\mathcal{F}))$, when the latter is well-defined.

Bridgeland's construction demonstrates that in some situations $\mathcal{SH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$ is well-defined while the derived Hall algebra of $\underline{\mathcal{F}}$ and the twist are not. The category of complexes with projective components is not the most appropriate though : it is not derived invariant. We prove the derived invariance of Bridgeland's algebras. In particular, we get a categorical interpretation of Bernstein-Gelfand-Ponomarev reflection functors. In order to study Drinfeld doubles of Hall algebras of coherent sheaves, one has to work with periodic derived categories of abelian categories without enough projectives. We show how to generalize the analogue of Bridgeland's construction for bounded derived categories to a class including categories of coherent sheaves. This construction was modified by Lu and Peng in [49] : they adapt it to define Hall algebras for the 2-periodic derived categories of hereditary abelian categories, generalizing the work of Bridgeland.

In the case of derived categories of hereditary abelian categories, we have a non-trivial observation : the semi-derived Hall algebra (or the algebra of Lu and Peng) is, up to a twist, a quotient of a localized Hall algebra of the category of complexes. This quotient map can be understood as a relative version of the integration map defined by Reineke [63].

In general, Frobenius models of triangulated categories are often not the most convenient to work with. In chapter 3, we generalize this construction to other models, or enhancements, of

triangulated categories. Given an exact category $\mathcal{E}$ with weak equivalences W, some additional structures may ensure that the localized category $\mathcal{E}[W^{-1}]$ is triangulated. In particular, this happens if W can be completed to the structure of a hereditary exact model category $\mathcal{M}$ on $\mathcal{E}$, or when the underlying category of $\mathcal{E}$ has a smaller Frobenius exact structure and the full subcategory $\mathcal{W}$ of weakly trivial objects forms a Frobenius subcategory. Both of these classes of exact categories with weak equivalences naturally generalize the case of Frobenius categories. When $\mathcal{E}$ and $\mathcal{W}$ satisfy some finiteness conditions, we construct a unital associative algebra $\mathcal{H}(\mathcal{E}, \mathcal{W})$ and prove the following theorem.

Theorem 0.7.2. *(i) $\mathcal{H}(\mathcal{E}, \mathcal{W})$ is a free module over the group algebra of the numerical Grothendieck group of the full subcategory of weakly trivial objects $K_0^{num}(\mathcal{W})$ twisted by the Euler form. Any choice of representatives in $\mathcal{E}$ of the isomorphism classes of objects in the category $\mathcal{E}[\mathcal{W}^{-1}]$ yields a basis of this module.*

(ii) $\mathcal{H}(\mathcal{E}, \mathcal{W})$ with an appropriately twisted multiplication is a flat deformation of the derived Hall algebra of $\mathcal{E}[\mathcal{W}^{-1}]$ over the group algebra of $K_0^{num}(\mathcal{W})$, when the derived Hall algebra is well-defined.

(iii) The algebras $\mathcal{H}(\mathcal{E}, \mathcal{W})$ are functorial under exact functors inducing equivalences in $\mathcal{E}[\mathcal{W}^{-1}]$.

(iv) When $\mathcal{E}$ is Frobenius and $\mathcal{W}$ is the full subcategory of projective-injective objects, we recover the previous definition : $\mathcal{H}(\mathcal{E}, \mathcal{W}) = \mathcal{SH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$.

The twist is given by the quotient of the Euler forms on $\mathcal{E}$ and on $\mathcal{E}[\mathcal{W}^{-1}]$, when those are well-defined. We get an algebra over the group ring of $K_0^{num}(\mathcal{W})$.

The idea behind the definition of the algebra $\mathcal{H}(\mathcal{E}, \mathcal{W})$ is to consider the full subcategory of weakly trivial objects $\mathcal{W}$ as if it were split exact, which happens when $\mathcal{E}$ is Frobenius and $\mathcal{W}$ is the full subcategory of projective-injective objects. When we localize the exact category $\mathcal{E}$, these objects become isomorphic to 0. But we know that the class of the 0 object is the unit of our algebra. This suggests that weakly trivial objects should be invertible in our algebra, and not vanish as one might expect.

In the presence of some type of “relative replacements”, we have another proof of associativity of our $\mathcal{H}(\mathcal{E}, \mathcal{W})$. By replacing the factors with these “replacements”, we can reduce the associativity problem of the usual Hall algebra of the (exact) category $\mathcal{E}$. Historically, this was the first construction of semi-derived Hall algebras defined in [23] for the categories of complexes. Even if the algebra thus defined is, in principle, covered by the new definition, we present the second construction to emphasize the interaction between the Hall algebras of exact and of triangulated categories and their associativity.

0.8 Heart property and its substitute

One of the important properties of the derived Hall algebras is the so-called "heart property". Let $\mathcal{T}$ be a triangulated category whose derived Hall algebra $\mathcal{DH}(\mathcal{T})$ is well-defined. If $\mathcal{T}$ admits a

bounded $t-$structure with a heart $\mathcal{A}$, Toën [75] proved that the embedding functor $\mathcal{A} \to \mathcal{T}$ induces an embedding of the Hall algebra of the heart into $\mathcal{DH}(\mathcal{T})$:

$$\mathcal{H}(\mathcal{A}) \hookrightarrow \mathcal{DH}(\mathcal{T}).$$

Theorem 0.7.2 implies that Hall algebras of exact categories with weak equivalences can be seen as twisted extended Hall algebras of triangulated categories. We have a natural analogue of the heart property in this setting.

Corollary 0.8.1. *Let $\mathcal{A}$ be a heart of a bounded $t-$structure on a category $\mathcal{E}[\mathcal{W}^{-1}]$. Let $\widetilde{\mathcal{A}}$ be the full exact subcategory of $\mathcal{E}$ consisting of all objects whose image under the localization belongs to $\mathcal{A}$. Then we have an embedding*

$$\mathcal{H}(\widetilde{\mathcal{A}}) \hookrightarrow \mathcal{H}(\mathcal{E}, \mathcal{W}),$$

and the algebra $\mathcal{H}_{tw}(\widetilde{\mathcal{A}})$ is a flat deformation of $\mathcal{H}(\mathcal{A})$ over $\mathbb{Q}K_0^{num}(\mathcal{W})$, where the twist is given by the form

$$\frac{\langle -,- \rangle_{\widetilde{\mathcal{A}}}}{\langle -,- \rangle_{\mathcal{A}}} = \frac{\langle -,- \rangle_{\mathcal{E}}}{\langle -,- \rangle_{\mathcal{E}[\mathcal{W}^{-1}]}}.$$

More generally, we have the following result.

Corollary 0.8.2. *Let $\mathcal{E}'$ be a fully exact subcategory of $\mathcal{E}$ such that $\mathcal{E}'[\mathcal{W}^{-1}]$ is an extension closed subcategory of $\mathcal{E}[\mathcal{W}^{-1}]$. Then we have an embedding*

$$\mathcal{H}(\mathcal{E}', \mathcal{W}) \hookrightarrow \mathcal{H}(\mathcal{E}, \mathcal{W}),$$

and the algebra $\mathcal{H}_{tw}(\mathcal{E}', \mathcal{W})$ is a flat deformation of $\mathcal{H}(\mathcal{E}'[\mathcal{W}^{-1}])$ over $(\mathbb{Q}K_0^{num}(\mathcal{E}' \cap \mathcal{W})$, where the twist is given by the form

$$\frac{\langle -,- \rangle_{\mathcal{E}'}}{\langle -,- \rangle_{\mathcal{E}'[\mathcal{W}^{-1}]}} = \frac{\langle -,- \rangle_{\mathcal{E}}}{\langle -,- \rangle_{\mathcal{E}[\mathcal{W}^{-1}]}}.$$

If $\mathcal{T}$ is periodic, it does not have any bounded $t-$structure. For periodic derived categories of abelian categories $\mathcal{D}_m(\mathcal{A})$, we are still interested in the relation between the twisted extended Hall algebras of $\mathcal{D}_m(\mathcal{A})$ and of $\mathcal{A}$. We prove that for an abelian category $\mathcal{A}$, the classes of $m-$periodic complexes quasi-isomorphic to stalk complexes concentrated in degree 0 form an extension-closed subcategory of $\mathcal{D}_m(\mathcal{A})$ if and only if $\operatorname{gl\,dim}(\mathcal{A}) \leq m$. They always form a fully exact subcategory of $\mathcal{C}_m(\mathcal{A})$ that we denote $\widetilde{\mathcal{A}}$, as in the bounded case. We have well-defined cohomologies $H^i, i = 0, \ldots, m-1$, with values in $\mathcal{A}$, and from the long exact sequence it follows that the complexes with cohomologies concentrated in degree 0 always form a fully exact subcategory $\mathcal{C}_m^0(\mathcal{A})$ of $\mathcal{C}_m(\mathcal{A})$ closed under quasi-isomorphisms and their classes form an extension-closed subcategory $\mathcal{D}_m^0(\mathcal{A})$ of $\mathcal{D}_m(\mathcal{A})$. Thus, we have the following result.

Theorem 0.8.3. *Let $\mathcal{A}$ be an $\mathbb{F}_q-$linear abelian category of finite global dimension with finite* $\mathrm{Hom}-$ *and* Ext^i-*space, for all $i > 0$. Then the Hall algebras $\mathcal{H}(\mathcal{A}), \mathcal{H}(\mathcal{C}_m(\mathcal{A}), \mathrm{qis}), \mathcal{H}(\mathcal{C}_m^0(\mathcal{A}), \mathrm{qis})$ are well-defined.*

(i) We have an embedding

$$\mathcal{H}(\widetilde{\mathcal{A}}) \hookrightarrow \mathcal{H}(\mathcal{C}_m(\mathcal{A}), qis),$$

and the algebra

$$\mathcal{H}_{tw}(\widetilde{\mathcal{A}})$$

is a flat deformation of $\mathcal{H}(\mathcal{A})$ over

$$\mathbb{Q}K_0^{num}(\mathcal{C}_{m,ac}(\mathcal{A})) \xrightarrow{\sim} \bigotimes_{i=1}^{m} K_0^{num}(\mathcal{A}),$$

if and only if $\mathrm{gl\,dim}(\mathcal{A}) \leq m$.

(ii) We have an embedding

$$\mathcal{H}(\mathcal{C}_m^0(\mathcal{A}), \mathrm{qis}) \hookrightarrow \mathcal{H}(\mathcal{C}_m(\mathcal{A}), qis)$$

together with isomorphisms

$$\mathcal{H}_{tw}(\mathcal{C}_m^0(\mathcal{A}), \mathrm{qis}) \xrightarrow{\sim} \mathcal{H}(\mathcal{D}_m^0(\mathcal{A})) \otimes \mathbb{Q}K_0^{num}(\mathcal{C}_{m,ac}(\mathcal{A})) \xrightarrow{\sim} \mathcal{H}(\mathcal{A}) \otimes \bigotimes_{i=1}^{m} K_0^{num}(\mathcal{A}),$$

for all $\mathcal{A}$.

Geng and Peng [21] proved the existence of the embedding and the isomorphisms in part (i) for $\mathcal{A}$ having enough projectives and of $\mathrm{gl\,dim} \leq 2$ and m being equal to 2. They used Bridgeland's construction of twisted extended Hall algebras of periodic derived categories.

Theorem 4.4.3 suggests that the algebras $\mathcal{H}(\mathcal{C}_m^0(\mathcal{A}), \mathrm{qis})$ and their twisted versions can be seen as natural substitutes of $\mathcal{H}(\mathcal{A})$ when thinking of periodic derived categories. The multiplication there involves higher extensions in the category $\mathcal{A}$. The case of $2-$periodic derived categories seems to be the most important by many reasons. It seems possible, although not necessarily probable, that the algebras $\mathcal{H}(\mathcal{C}_2(\mathcal{A}), \mathrm{qis})$ and $\mathcal{H}(\mathcal{C}_2^0(\mathcal{A}), \mathrm{qis})$ with aproppriate twists may have a Hopf algebra structure beyond the hereditary case.

0.9 Reduction of exact structures and degeneration of Hall algebras

In Section 5.1, we present results of joint work with Xin Fang [25].

An additive category $\mathcal{A}$ can be endowed with many different exact structures. It is then a natural question to ask whether the Hall algebras of these different exact structures are related between each other. The following theorem gives an answer for the categories of finite type (that is, having finitely many indecomposables up to isomorphism).

Theorem 0.9.1. *Let $\mathcal{A}$ be a* Hom *−finite idempotent complete $k-$linear additive category of finite type. Suppose $\mathcal{A}$ is endowed with two exact structures $\mathcal{E}' < \mathcal{E}$. Then the Hall algebra $\mathcal{H}(\mathcal{E}')$ of $\mathcal{E}'$ is a degeneration of the Hall algebra $\mathcal{H}(\mathcal{E})$ of $\mathcal{E}$, with respect to a filtration given by a function on the set* $\mathrm{Iso}(\mathcal{A})$. *This function is defined as* $w(M) = \dim \mathrm{Hom}(\bigoplus_{P \in \mathrm{ind}(\mathcal{P}(\mathcal{E}'))} P, M)$.

Thanks to works of Enomoto [16][17], one can classify exact structures in many situations when $\mathcal{A}$ has finitely many indecomposable objects. This covers the cases of the category of representations of a Dynkin quiver and its category of 2-periodic complexes. In the first case, we prove that the degenerations of the Hall algebra of the abelian category given by smaller exact structures are precisely the degenerations defined in [14]. In particular, the split exact structure corresponds to a $q-$commutative polynomial algebra, and we recover the famous PBW theorem in this quantized setting. Thus, Theorem 0.9.1 can be though of as a generalization of PBW theorem in the framework of Hall algebras.

In Section 5.2, we discuss the possible applications to quantum groups and Drinfeld doubles by means of changing exact structures on the category of 2-periodic complexes.

0.10 Further remarks

It is worth to note that an additive category $\mathcal{A}$ with weak equivalences $\mathcal{W}$ can be endowed with many exact structures $\mathcal{E}$ compatible with $\mathcal{W}$. Thus, it can, a priori, give rise to many twisted extended Hall algebras of $\mathcal{E}[\mathcal{W}^{-1}] = \mathcal{A}[\mathcal{W}^{-1}]$. For such an algebra to be defined, an exact category $\mathcal{E}$ must satisfy certain conditions, restricting our choice. Let us consider the case of the category of complexes over an abelian category of finite global dimension, with quasi-isomorphisms taken as weak equivalences. The component-wise split (Frobenius) exact structure does not satisfy these conditions, but they hold for the component-wise (abelian) exact structure. More generally, in all the examples known to the author, there is actually a natural choice of an exact structure satisfying these conditions.

Kontsevich [42] suggested a construction of a twisted extended Hall algebra for a class of triangulated orbit categories that does not use their exact models (details may appear elsewhere). It seems that to each such triangulated category one associates a unique algebra, and in the case of 2-periodic derived categories one recovers the algebras constructed via categories of complexes. This suggests the existence of the canonical choice of an extension and a bilinear form on it, thus defining a twisted extended Hall algebra for every triangulated category (satisfying aproppriate finiteness conditions) in a canonical way.

Let us note that we can associate Hall algebras to localizations of triangulated categories similarly to our construction for localizations of exact categories. Recently, Nakaoka and Palu [58] introduced a common generalization of exact and triangulated categories : *extriangulated categories*. These categories are defined in such a way that many properties related to localization in exact or triangulated categories hold for them as well. In particular, one has a version of Hovey's cor-

respondence between model structures and cotorsion pairs for extriangulated categories. It seems that extriangulated categories should give the most natural setting to study twisted extended Hall algebras via localization of additive categories. Notably, homotopy cartesian squares that are the crucial ingredients of our construction appear in the setting of extriangulated categories as part of their axiomatics.

It seems well-established by now that there is a lot of deep phenomena of higher categorical nature behind the Hall algebras captured by so-called *2-Segal spaces* defined by Dyckerhoff and Kapranov [12]. It is associated by Waldhausen $S_\bullet-$construction to an exact $\infty-$category. However, as noted in [11], Hall algebras themselves depend only on the underlying additive category. We see that the most important diagrams that we use in triangulated categories are homotopy cartesian squares that are rather of $1-$categorical nature. This is one of the reasons why the author strongly preferred to keep this thesis in the realm of additive categories.

Without changing our optique, we can also associate twisted extended Hall algebras to localization pairs of pre-triangulated differential graded categories with split idempotents by using their underlying categories.

By interpreting the quotient map from section 3.3.5 as a version of the integration map, one can construct motivic versions of the semi-derived Hall algebras at least in some generality. For a weakly Gorenstein abelian category of Gorenstein dimension 1, we find that the Hall algebra of its Gorenstein projective model structure is the image under this integration map of the (classical) Hall algebra of the abelian category. Then the motivic Hall algebra of the model category is the image under the integration map of the motivic Hall algebra of the abelian category, whenever the latter is well-defined. We do not present the details in the thesis.

This covers the case of singular Nakajima categories introduced by Keller-Scherotzke [40] and Scherotzke [67]. They introduced them in order to study graded and generalized quiver varieties via the methods of Gorenstein homological algebra methods, generalizing works of Kimura-Qin [41] and Leclerc-Plamondon [47]. The geometric versions of the corresponding Hall algebras (in terms of perverse sheaves) in some cases have been studied in works of Qin [61] and Scherotzke-Sibilla [66], see also [77][31][47] for the relation to the (twisted) dual Grothendieck ring of a tensor category of certain representations of the quantum loop algebra in the graded case.

Generalizing the definition of quantum groups via quantized rings of functions on Poisson-Lie groups, regular functions on certain Poisson varieties can be understood in terms of cluster algebras that are, in turn, related to the (dual) Hall algebras. Recently, quantum groups were embedded into rings of regular functions on certain moduli spaces of local systems on surfaces, see [22] and references therein. It seems probable that [22, Theorem 2.19] can be generalized to the case of boundary components with m special points by replacing $U_{\sqrt{q}}(\mathfrak{g})$ with the $\mathbb{Z}/m-$graded semi-derived Hall algebras of representations of the corresponding quiver.

Chapitre 1

Preliminaries

All the categories in this thesis are assumed to be additive.

1.1 Exact categories

In an additive category $\mathcal{A}$ a pair of morphisms

$$A \overset{i}{\rightarrowtail} B \overset{p}{\twoheadrightarrow} C,$$

is said to be *exact* or a *kernel-cokernel pair*, if i is a kernel of p and p is a cokernel of i. An *exact category* in the sense of Quillen is an additive category $\mathcal{A}$ endowed with a class $\mathcal{E}$ of exact pairs, closed under isomorphism and satisfying the following axioms :

[E0] For each object $E \in \mathcal{A}$, the identity morphism 1_E is an inflation.

[E0^{op}] For each object $E \in \mathcal{A}$, the identity morphism 1_E is a deflation.

[E1] The class of inflations is closed under composition.

[E1^{op}] The class of deflations is closed under composition.

[E2] The push-out of an inflation along an arbitrary morphism exists and yields an inflation.

[E2^{op}] The pull-back of a deflation along an arbitrary morphism exists and yields a deflation.

Here an *inflation* is a morphism i for which there exists p such that (i, p) belongs to $\mathcal{E}$. It is also called an *admissible monic*, or an *admissible monomorphism*. *Deflations* (or *admissible epics*, or *admissible epimorphisms*) are defined dually. We depict admissible monics by $\rightarrowtail$ and admissible epics by $\twoheadrightarrow$ in diagrams. Exact pairs belonging to $\mathcal{E}$ are called *conflations*, or *admissible exact sequences*. An inflation which is simultaneously a deflation is an isomorphism. By abuse of notation, we will denote an exact category $(\mathcal{A}, \mathcal{E})$ simply by $\mathcal{E}$ and call $\mathcal{A}$ its *underlying additive category*.

Remark 1.1.1. This set of axioms is not minimal, cf. [37], [7]. There are also slightly different notions of exact categories, but in case when $\mathcal{A}$ is *weakly idempotent complete*, i.e. retractions have kernels, all of them coincide, cf. [7]. We will always assume that $\mathcal{A}$ is *idempotent complete*,

i.e. idempotents have kernels, which is an even stronger condition, cf. below.

Any abelian category has a canonical structure of an exact category. In this case, the class of conflations coincides with the class of all exact pairs. If $\mathcal{E}$ and $\mathcal{E}'$ are exact categories, an exact functor $\mathcal{E} \to \mathcal{E}'$ is an additive functor taking conflations of $\mathcal{E}$ to conflations of $\mathcal{E}'$. A fully exact subcategory of an exact category $\mathcal{E}$ is a full additive subcategory $\mathcal{E}' \subset \mathcal{E}$ which is closed under extensions, i.e. if it contains the end terms of a conflation of $\mathcal{E}$, it also contains the middle term. Then $\mathcal{E}'$ endowed with the conflations of $\mathcal{E}$ having their terms in $\mathcal{E}'$ is an exact category, and the inclusion $\mathcal{E}' \hookrightarrow \mathcal{E}$ is a fully faithful exact functor.

Note that one additive category may be endowed with a lot of different exact structures. One can consider any additive category $\mathcal{A}$ as an exact category with a split exact structure : namely, one can take all split exact sequences as the set of conflations. If $\mathcal{A}$ is an abelian category that is not semi-simple, its split exact structure differs from the abelian one. All possible exact structures are between these two : the class of conflations is necessarily contained in the class of all exact pairs and contains all split exacts pairs. In other words, the split exact structure is the minimal one and the abelian exact structure is the maximal one. More discussion and the precise statement can be found in [13], for the case with no abelian structure see [65] and [10] and references therein.

We say that $\mathcal{E}$ is *idempotent complete*, or that *idempotents split* in $\mathcal{E}$, if Ker$e$ exists in $\mathcal{E}$ for each idempotent $e : \mathcal{E} \to \mathcal{E}$. Any abelian category is idempotent complete. Note that the idempotent completeness is a property of an additive category rather than of its exact structure. In particular, if we take an idempotent complete exact category and endow it with a bigger exact structure (by enlarging its class of conflations), the new exact category still will be idempotent complete. Also, if an additive category can be endowed with an abelian structure, it is idempotent complete. One can find a classification of exact structures on an idempotent complete additive category $\mathcal{A}$ in terms of Serre subcategories of the category of finitely generated $\mathcal{A}-$modules and its opposite category in [16].

We will assume that $\mathcal{E}$ is idempotent complete, linear over some field k and essentially small. Suppose that for all objects $A, B \in \mathcal{E}$, we have $\dim(\mathrm{Hom}(A, B)) < \infty$. Then it is known that $\mathcal{E}$ is a Krull-Schmidt category. That is, each object decomposes into a finite direct sum of indecomposables in a unique way, up to permutation, and each of the indecomposables has a local endomorphism ring.

To each essentially small exact category $\mathcal{E}$, one associates its Grothedieck group $K_0(\mathcal{E})$ defined in the same way as in the case of abelian categories : it is the free abelian group on the set of isomorphism classes $\mathrm{Iso}(\mathcal{E})$ modulo the relations $[B] = [A]+[C]$ for all conflations $A \rightarrowtail B \twoheadrightarrow C$. In particular, to each additive category $\mathcal{A}$, one associates the Grothendieck group of its split exact structure. It is called the *additive*, or *split* Grothendieck group of $\mathcal{A}$. We denote it $K_0^{add}(\mathcal{A})$, it is also often denoted $K_0^{split}(\mathcal{A})$ or $K_0(\mathcal{A}, \oplus)$. Each exact (or triangulated) category $\mathcal{E}$ (resp. $\mathcal{T}$) is additive. Thus, we have an additive Grothendieck group of its underlying additive category. By abuse of notation, we denote it $K_0^{add}(\mathcal{E})$ (resp. $K_0^{add}(\mathcal{T})$). It is known that for a Krull-Schmidt category, its additive Grothendieck group is freely generated by the classes of indecomposable

objects. When an additive category is endowed with two different exact structures $\mathcal{E}$ and $\mathcal{E}'$, where $\mathcal{E}'$ is larger than $\mathcal{E}$ (i.e. the class of conflations in $\mathcal{E}'$ contains the class of conflations in $\mathcal{E}$), it follows that the Grothendieck group $K_0(\mathcal{E}')$ is a quotient of the Grothendieck group $K_0(\mathcal{E})$. In particular, for each essentially small exact category $\mathcal{E}$, its Grothendieck group $K_0(\mathcal{E})$ is a quotient of $K_0^{add}(\mathcal{E})$. If $\mathcal{E}$ can be endowed with an abelian structure $\mathcal{A}$, then $K_0(\mathcal{A})$ is a quotient of $K_0(\mathcal{E})$.

One can also associate Grothendieck monoids to exact and triangulated categories. The Grothendieck monoid $M_0(\mathcal{E})$ is defined as the quotient of the free commutative monoid on the set $\mathrm{Iso}(\mathcal{E})$ by the congruence $[B] \sim [A] + [C]$ for all conflations $A \rightarrowtail B \twoheadrightarrow C$. For a triangulated category, the definition is similar. The Grothendieck group is the group completion of the Grothendieck monoid. The positive cone (or *effective submonoid*) in a Grothendieck group consists of classes of objects. We will denote it by $K_{0,+}(\mathcal{E})$. For $\mathcal{E}$ non-abelian, $M_0(\mathcal{E})$ is not necessarily cancellative, and $K_{0,+}(\mathcal{E})$ is isomorphic to its maximal cancellative quotient, see [18]. For any additive category $\mathcal{A}$, we have an isomorphism $M_0^{add}(\mathcal{A}) \xrightarrow{\sim} K_{0,+}^{add}(\mathcal{A})$.

Various diagram lemmas hold in exact categories, e.g. the $3-$lemma and the five-lemma do, [7]. We will use the second one, in the following form.

Lemma 1.1.2 ([7, Corollary 3.2])**.** *Consider a morphism of conflations*

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, "a"'] & B \arrow[r, two heads] \arrow[d, "b"'] & C \arrow[d, "c"'] \\ A' \arrow[r, tail] & B' \arrow[r, two heads] & C'. \end{tikzcd}$$

If a and c are isomorphisms (or admissible monics, or admissible epics) then so is b.

Given an exact category $\mathcal{E}$, we define various additive categories of complexes $\mathcal{C}(\mathcal{E}), \mathcal{C}^{\pm}(\mathcal{E}), \mathcal{C}^b(\mathcal{E})$ over it as in the abelian case. Each of these categories has various exact structures, but two of them are the most natural : component-wise exact and component-wise split. If not specified, we will use the first one and denote the corresponding exact categories simply by $\mathcal{C}(\mathcal{E}), \mathcal{C}^{\pm}(\mathcal{E}), \mathcal{C}^b(\mathcal{E})$. These exact categories are abelian. The categories with the component-wise split exact structure will be denoted by $\mathcal{C}_{cs}(\mathcal{E}), \mathcal{C}_{cs}^{\pm}(\mathcal{E}), \mathcal{C}_{cs}^b(\mathcal{E})$. The third interesting exact structure is defined only for $\mathcal{E} = \mathcal{A}$ abelian. Its class of conflations was introduced in order to construct the *Cartan-Eilenberg resolutions*. The fact this defines an exact category is proved in [45, Remark 8, Remark 9]. Following [15], we call it the *Cartan-Eilenberg*, or the CE exact structure [1].

Definition 1.1.3. We say that a short component-wise exact sequence $A^{\cdot} \rightarrowtail B^{\cdot} \twoheadrightarrow C^{\cdot}$ of complexes over an abelian category is CE exact, if at any degree i, the sequence $Z^i(A^\bullet) \xrightarrow{f^i} Z^i(B^\bullet) \xrightarrow{g^i} Z^i(C^\bullet)$ is short exact, where Z^i are the $i-$th cycles.

1. The properties of the conflations in this structure were also considered by Maclane [54] under the name of *proper* short exact sequences. Thus, calling it the *Maclane exact structure* would also make sense.

We define homotopies between morphisms of complexes and the *(bounded) homotopy category* $\mathbf{K}^b(\mathcal{E})$ of an exact category in the natural way. Null-homotopic complexes are also called *contractible*. The category $\mathbf{K}^b(\mathcal{E})$ has a canonical triangulated structure, cf. [38].

A complex $K^\bullet$ over $\mathcal{E}$ is *acyclic* in degree n if d_K^{n-1} factors as

$$\begin{tikzcd} K^{n-1} \arrow[rr, "d_K^{n-1}"] \arrow[dr, two heads, "p^{n-1}"'] & & K^n \\ & Z^{n-1}, \arrow[ur, tail, "i^{n-1}"'] & \end{tikzcd}$$

where p^{n-1} is a cokernel for d^{n-2} and a deflation, and i^{n-1} is a kernel for d^n and an inflation. The complex K is *acyclic* if it is acyclic in each degree. Since $\mathcal{E}$ is idempotent complete, any contractible complex is acyclic. A morphism is a *quasi-isomorphism* if its mapping cone is acyclic. Given this notion, we define the *(bounded) derived category* $\mathcal{D}^b(\mathcal{E})$ as in the abelian case. We can also define quasi-isomorphisms in $\mathbf{K}^b(\mathcal{E})$ as the morphisms $\overline{s}$ arising in triangles $A \to B \xrightarrow{\overline{s}} C \to \Sigma A$, where Σ is the suspension (or shift) functor and A is an acyclic complex in $\mathbf{K}^b(\mathcal{E})$. Using the category $\mathcal{D}^b(\mathcal{E})$, we define *derived functors* as in the abelian case. The Yoneda construction of higher extensions generalizes from the case of abelian categories to the exact case. This definition of higher extensions agrees with that as of derived functors of the functor Hom, as in the abelian case. An exact category $\mathcal{E}$ is *hereditary*, if all higher extensions vanish in $\mathcal{E}$:

$$\mathrm{Ext}^i(A, B) = 0, \quad \forall i > 1, \quad \forall A, B \in \mathcal{E}.$$

1.2 Exact categories with weak equivalences

An *exact category with weak equivalences* is an exact category $\mathcal{E}$ together with a class $W \subset \mathrm{Mor}\,\mathcal{E}$ of morphisms in $\mathcal{E}$. The morphisms in W are called *weak equivalences*. The set of weak equivalences is required to contain all identity morphisms ; to be closed under isomorphisms, retracts, push-outs along inflations, pull-backs along deflations ; and to satisfy the "two out of three" property for composition : if two of the three maps among a, b, ab are weak equivalences, then so is the third.

For an exact category with weak equivalences $(\mathcal{E}, W)$, write $\mathcal{W} \subset \mathcal{E}$ for the full subcategory of *weakly acyclic*, or *weakly trivial* objects, that is, those objects E of $\mathcal{E}$ for which the unique map $0 \to \mathcal{E}$ is a weak equivalence. The category $\mathcal{W}$ is closed under extensions in $\mathcal{E}$, and thus inherits an exact structure from $\mathcal{E}$ such that the inclusion $\mathcal{W} \subset \mathcal{E}$ is fully exact.

We will consider two classes of exact categories with weak equivalences and some extra structure : Frobenius pairs with larger exact structures and hereditary exact model categories (these notions in the following sections). In both of these cases, the category $\mathcal{W}$ is known to be closed

under retracts and, in particular, under isomorphisms. Moreover, such an extra structure ensures that the category $\mathcal{E}[W^{-1}]$ is triangulated.

1.3 Frobenius and stable categories

Recall that in an exact category $\mathcal{F}$ an object P is called projective, if $\mathrm{Ext}^1_{\mathcal{F}}(P, X) = 0$ for any object $X \in \mathcal{F}$. Injective objects are defined in the dual way. An exact category $\mathcal{F}$ is *Frobenius*, if it has enough projectives and enough injectives and if, moreover, an object is projective if and only if it is injective. Let $\mathcal{F}$ be a Frobenius category, $\mathcal{P}(\mathcal{F})$ the full subcategory of projective-injective objects in $\mathcal{F}$. We define the *stable category* $\underline{\mathcal{F}}$ of $\mathcal{F}$. The objects of $\underline{\mathcal{F}}$ are the same as the objects of $\mathcal{F}$, and the morphism spaces $\mathrm{Hom}_{\underline{\mathcal{F}}}(X, Y)$ are the morphism spaces in $\mathcal{F}$ modulo morphisms factoring through projective-injective objects. The stable category $\underline{\mathcal{F}}$ is a triangulated category in a natural way [28]. By declaring weak equivalences to be morphisms that induce isomorphisms in the stable category, we get a structure of an exact category with weak equivalences on $\mathcal{F}$.

A triangulated category $\mathcal{T}$ is called *algebraic* if it is equivalent to the stable category of a Frobenius category $\mathcal{F}$. In this case, one says also that $\mathcal{F}$ is a *Frobenius model* for $\mathcal{T}$.

The first and the main example of a Frobenius category is the category of complexes $\mathcal{C}^\sharp_{cs}$ with the component-wise split exact structure, where $\sharp \in \{b, -, +, _\}$. Its stable category is the corresponding homotopy category $\mathbf{K}^\sharp(\mathcal{E})$. Indeed, one can easily see that contractible complexes and only they are projective (and injective).

1.4 Hereditary exact model categories and Hovey triples

In this subsection, we recall the definition of exact model categories (in particular, hereditary) and formulate the Hovey correspondence relating them to cotorsion pairs in this setting. The concept is mainly due to Hovey [32] for abelian categories, and it was generalized to exact categories by Gillespie [19][20]. We also follow works of Becker [2][3].

Let $\mathcal{A}$ be an additive category. It is called *bicomplete* if it posesses all small limits and colimits. A *model structure* on a category $\mathcal{A}$ is a triple of classes of morphisms $(\mathrm{Cof}, W, \mathrm{Fib})$, called cofibrations, weak equivalences and fibrations, satisfying certain axioms. A *model category* is a bicomplete category endowed with a model structure. An object X is called weakly trivial if $0 \to X \in W$ (equivalently, $X \to 0 \in W$). Similarly, it is called cofibrant if $0 \to X \in \mathrm{Cof}$, and it is called fibrant if $X \to 0 \in \mathrm{Fib}$. It is called bifibrant if it is both fibrant and cofibrant. The classes of cofibrant, weakly trivial, and fibrant objects will be denoted $\mathcal{C}, \mathcal{W}$ and $\mathcal{F}$, respectively.

There is a notion of a reasonable compatibility between exact and model structures :

Definition 1.4.1. A model structure on a (weakly) idempotent complete exact category is called *exact* if cofibrations equal inflations with cofibrant cokernel and fibrations equal deflations with fibrant kernel. An *exact model category* is a bicomplete exact category equipped with an exact model structure.

Hovey's correspondence proved by Hovey and Gillespie for abelian, resp. weakly idempotent complete exact categories, implies that for exact model categories, the axioms on the classes of morphisms $(\mathrm{Cof}, W, \mathrm{Fib})$ are equivalent to certain conditions on the classes of objects $(\mathcal{C}, \mathcal{W}, \mathcal{F})$. This explains why we did not present here the set of axioms on $(\mathrm{Cof}, W, \mathrm{Fib})$. Instead, we will formulate the correspondence and the conditions on $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ providing exact model structures. To describe those, we need to recall the notion of *cotorsion pairs*.

Definition 1.4.2. A *cotorsion pair* in $\mathcal{E}$ is a pair of classes $(\mathcal{X}, \mathcal{Y})$ of objects in $\mathcal{E}$ satisfying the following two conditions :

(1) $\mathcal{X} = {}^{\perp}\mathcal{Y} =: \{X \in \mathcal{E} \quad | \quad \mathrm{Ext}^1(X, \mathcal{Y}) = 0\}$;

(2) $\mathcal{Y} = \mathcal{X}^{\perp} =: \{Y \in \mathcal{E} \quad | \quad \mathrm{Ext}^1(\mathcal{X}, Y) = 0\}$.

The cotorsion pair is called *complete* if for any $E \in \mathcal{E}$, there exists a conflation $Y \rightarrowtail X \twoheadrightarrow E$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, and another conflation $E \rightarrowtail Y' \twoheadrightarrow X'$ with $X' \in \mathcal{X}$ and $Y' \in \mathcal{Y}$.

The first sequence in the definition of complete cotorsion pairs generalizes the concept of having enough projectives while the second generalizes the concept of having enough injectives.

To state Hovey's correspondence, we call a class of objects *thick* if it is closed under direct summands and satisfies that whenever two out of three terms in a short exact sequence are in it, then so is the third.

Theorem 1.4.3. *(Hovey's Correspondence [32, Theorem 2.2][19][20]). Let $\mathcal{E}$ be a weakly idempotent complete exact category with the classes $(\mathcal{C}, \mathcal{W}, \mathcal{F})$ of cofibrant, weakly trivial and fibrant objects. Then $\mathcal{W}$ is a thick class in $\mathcal{E}$ and both $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion pairs in $\mathcal{E}$. Conversely, given a thick class $\mathcal{W}$ and classes $\mathcal{C}$ and $\mathcal{F}$ such that both $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are complete cotorsion pairs, then there is an exact model structure on $\mathcal{E}$ where $\mathcal{C}$ are the cofibrant objects, $\mathcal{W}$ are the trivial, and $\mathcal{F}$ are the fibrant objects.*

An exact model structure is called *projective* if every object is fibrant and *injective* if every object is cofibrant.

In model categories, there is a concept generalizing projecive and injective resolutions : *cofibrant* and *fibrant replacements*. For an object E in an exact model category, those are given by CE and FE, where $W \rightarrowtail CE \twoheadrightarrow E$ and $E \rightarrowtail FE \twoheadrightarrow W'$ are any conflations with $W \in \mathcal{W} \cap \mathcal{F}, CE \in \mathcal{C}, W' \in \mathcal{C} \cap \mathcal{W}, FE \in \mathcal{F}$. FCE is called a *bifibrant replacement* of E.

Model categories were introduced as a class of categories with weak equivalences where the localization behaves reasonably well. To each model category $\mathcal{M}$, one associates its *homotopy category* $\mathrm{Ho}(\mathcal{M})$, defined via bifibrant replacements of objects and equivalent to the localization of the underlying category of $\mathcal{M}$ at weakly trivial objects. Whenever a model category is *stable*, its homotopy category is triangulated in a natural way. For exact model categories, the natural source of triangulated homotopy categories is defined as follows.

Definition 1.4.4. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is *hereditary* if $\mathcal{X}$ is closed under taking kernels of epimorphisms between objects in $\mathcal{X}$ and $\mathcal{Y}$ is closed under taking cokernels of monomorphisms

between objects in $\mathcal{Y}$. An exact model structure is *hereditary* when the two associated cotorsion pairs in Hovey's correspondence are hereditary.

Gillespie proved that each hereditary exact model category is stable. Therefore, its homotopy category is triangulated in a natural way. Moreover, one has the following description of this triangulated category.

Proposition 1.4.5. *[19][20] Let $\mathcal{M}$ be a hereditary exact model structure on a weakly idempotent exact category $\mathcal{E}$. Then the full subcategory of bifibrant objects $\mathcal{C} \cap \mathcal{F}$, endowed with the exact structure inherited from $\mathcal{E}$, is Frobenius. Its class of projective-injective objects coincides with the class of weakly trivial bifibrant objects $\mathcal{C} \cap \mathcal{W} \cap \mathcal{F}$. The embedding of the stable category $\underline{\mathcal{C} \cap \mathcal{F}} \hookrightarrow \mathrm{Ho}(\mathcal{M}) \xrightarrow{\sim} \mathcal{E}[\mathcal{W}^{-1}]$ induced by the embedding $\mathcal{C} \cap \mathcal{F} \hookrightarrow \mathcal{E}$ is a triangle equivalence. Its inverse is given by the bifibrant replacement.*

The following result is proved by Becker in the abelian case, but the proof works in the case of (weakly) idempotent complete exact categories as well.

Theorem 1.4.6. *[3, Proposition II.C.1.1, Corollary II.C.1.2, Corollary II.C.1.5] Each conflation $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{E}$ induces a distinguished triangle $A \to B \to C \to \Sigma A$ in the homotopy category $\mathrm{Ho}(\mathcal{M})$, and any distinguished triangle is isomorphic to such a triangle.*

By using the calculus of fractions if necessary, for any $A, C \in \mathcal{E}[\mathcal{W}^{-1}]$, we can represent any equivalence class of distinguished triangles in $\mathrm{Ext}^1_{\mathcal{E}[\mathcal{W}^{-1}]}(C, A)$ by a diagram of the form

$$\begin{tikzcd} A \arrow[r, tail, two heads] & B' \arrow[r, tail, two heads] & C' \arrow[d, two heads] \\ & & C, \end{tikzcd} \tag{1.1}$$

with $A \rightarrowtail B' \twoheadrightarrow C'$ a conflation in $\mathcal{E}$, $\quad C' \twoheadrightarrow C$ a deflation weak equivalence.

1.5 Frobenius pairs with larger exact structure

A *morphism*[2] of Frobenius categories is an exact functor that preserves projective-injective objects. A pair $(\mathcal{F}, \mathcal{F}_0)$ of Frobenius categories $\mathcal{F}$ and $\mathcal{F}_0$ together with a fully exact embedding $\mathcal{E}_0 \hookrightarrow \mathcal{E}$ is called a *Frobenius pair*. Without loss of generality, we always assume that $\mathcal{F}_0$ contains all projective-injective objects of $\mathcal{F}$. A map of Frobenius pairs $(\mathcal{F}, \mathcal{F}_0) \to (\mathcal{F}', \mathcal{F}'_0)$ is defined similarly : it is an exact functor $\mathcal{E} \to \mathcal{E}'$ that preserves projective-injective objects and that maps $\mathcal{F}_0$ into $\mathcal{F}'_0$. Given a Frobenius pair $(\mathcal{F}, \mathcal{F}_0)$, we have a fully faithful triangulated functor $\underline{\mathcal{F}_0} \hookrightarrow \underline{\mathcal{F}}$. The corresponding Verdier quotient is called *the derived category* $\mathcal{D}(\mathcal{E}, \mathcal{E}_0)$ of the Frobenius pair $(\mathcal{F}, \mathcal{F}_0)$. The full subcategory in $\mathcal{F}$ of objects that become isomorphic to 0 in $\mathcal{D}(\mathcal{F}, \mathcal{F}_0)$ can be

2. It is called a *map* of Frobenius categories by Schlichting [70]

easily shown to be Frobenius with $\mathcal{P}(\mathcal{F})$ as projective-injective objects, and it contains $\mathcal{F}_0$, so under our assumption it is $\mathcal{F}_0$. Similarly to the case of Frobenius categories, we can consider $\mathcal{F}$ as an exact category with weak equivalences $\mathcal{W}$ whose localization is $\mathcal{D}(\mathcal{F}, \mathcal{F}_0)$. The category of weakly trivial objects coincides with $\mathcal{F}_0$.

Endow the underlying category of $\mathcal{F}$ with *larger*, that is, having more conflations, exact structure $\mathcal{E}$ in such a way that the objects of $\mathcal{F}_0$ form a full exact subcategory $\mathcal{E}_0$. Then the class of weak equivalences $\mathcal{W}$ is compatible with the exact structure $\mathcal{E}$, i.e $(\mathcal{E}, \mathcal{W})$ is an exact category with weak equivalences as defined earlier. The category $\mathcal{E}[\mathcal{W}^{-1}] = \mathcal{F}[\mathcal{W}^{-1}]$ is triangulated : it is the derived category $\mathcal{D}(\mathcal{F}, \mathcal{F}_0)$.

Note that for each object A, we have an inflation $A \rightarrowtail W$ and a deflation $W' \twoheadrightarrow A$, with W, W' weakly trivial : by definition of Frobenius categories, we can find W injective and W' projective with respect to $\mathcal{F}$.

Theorem 1.4.6 naturally translates to this setting.

Theorem 1.5.1. *Each conflation $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{E}$ induces a distinguished triangle $A \to B \to C \to \Sigma A$ in the category $\mathcal{E}[\mathcal{W}^{-1}]$, and every distinguished triangle is isomorphic to such a triangle.*

Proof. As we noted earlier, A fits into a conflation $A \rightarrowtail W \twoheadrightarrow A'$, with W weakly trivial. We can form the push-out and complete it to a commutative diagram of the form

$$\begin{tikzcd} A \arrow[r, tail] \arrow[d, tail] & B \arrow[r, two heads] \arrow[d, tail] & C \arrow[d, equal] \\ W \arrow[r, tail] \arrow[d, two heads] & D \arrow[r, two heads] \arrow[d, two heads] & C \\ A' \arrow[r, equal] & A' & \end{tikzcd} \tag{1.2}$$

Then the morphism $D \twoheadrightarrow C$ becomes invertible in the localized category, and by composing its inverse with the morphism $D \twoheadrightarrow A'$, we get a morphism $C \to A'$ in $\mathcal{E}[\mathcal{W}^{-1}]$. In $\mathcal{E}[\mathcal{W}^{-1}]$, the object A' is isomorphic to the shift of A. Thus, the diagram 1.2 induces a triangle $A \to B \to C \to A'$ in $\mathcal{E}[\mathcal{W}^{-1}]$.

By calculus of fractions applied to $\underline{\mathcal{F}_0} \hookrightarrow \underline{\mathcal{F}}$, we see that each triangle in $\mathcal{E}[\mathcal{W}^{-1}] = \mathcal{D}(\mathcal{F}, \mathcal{F}_0)$ is isomorphic to a triangle that arises in this way, for a conflation $A \rightarrowtail B \twoheadrightarrow C$ in $\mathcal{F}$ (and, therefore, in $\mathcal{E}$). $\square$

As in the case of model categories, for any $A, C \in \mathcal{E}[\mathcal{W}^{-1}]$, we can represent any equivalence class of distinguished triangles in $\mathrm{Ext}^1_{\mathcal{E}[\mathcal{W}^{-1}]}(C, A)$ by a diagram of the form (1.1). We can actually take $A \rightarrowtail B' \twoheadrightarrow C'$ to be a conflation in the smaller exact structure $\mathcal{F}$.

1.6 Hall algebras and Euler forms

Let $\mathcal{E}$ be an essentially small exact category, linear over a finite field k. Assume that $\mathcal{E}$ has finite morphism and (first) extension spaces :

$$|\mathrm{Hom}(A,B)| < \infty, \quad |\operatorname{Ext}^1(A,B)| < \infty, \quad \forall A, B \in \mathcal{E}.$$

Given objects $A, B, C \in \mathcal{A}$, define $\operatorname{Ext}^1(A,C)_B \subset \operatorname{Ext}^1(A,C)$ as the subset parameterizing extensions whose middle term is isomorphic to B. We define the Hall algebra $\mathcal{H}(\mathcal{E})$ to be the $\mathbb{Q}-$vector space whose basis is formed by the isomorphism classes $[A]$ of objects A of $\mathcal{E}$, with the multiplication defined by

$$[A] \diamond [C] = \sum_{B \in \mathrm{Iso}(\mathcal{E})} \frac{|\operatorname{Ext}^1(A,C)_B|}{|\mathrm{Hom}(A,C)|}[B].$$

The following result was proved by Ringel [64] for $\mathcal{E}$ abelian, and later by Hubery [33] for $\mathcal{E}$ exact. The definition of $\mathcal{H}(\mathcal{E})$ is also due to Ringel.

Theorem 1.6.1. *The algebra $\mathcal{H}(\mathcal{E})$ is associative and unital. The unit is given by $[0]$, where 0 is the zero object of $\mathcal{E}$.*

Remark 1.6.2. The choice of the structure constants $\frac{|\operatorname{Ext}^1_{\mathcal{E}}(A,B)_C|}{|\operatorname{Hom}_{\mathcal{E}}(A,B)|}$ is the one that was used by Bridgeland [5]. This choice is equivalent to that of the usual structure constants $|\{B' \subset C | B' \cong B, C/B' \cong A\}|$, called the *Hall numbers* and appearing in [64],[68] and [33]. See [5, §2.3] for the details. More precisely, the usual Hall algebra carries a natural coalgebra structure. Our choice of the structure constants actually defines the dual algebra of the Hall coalgebra. It is known that this dual algebra and the usual Hall algebra are naturally isomorphic, see [80] for the details.

For twisted and extended Hall algebras defined below, one has to tensor $\mathcal{H}(\mathcal{E})$ with $\mathbb{Q}(\nu)$, for $\nu = \sqrt{q}$, and consider it as an algebra over $\mathbb{Q}(\nu)$. By abuse of notation, we will use the same notation $\mathcal{H}(\mathcal{E})$ for this new algebra, and we will not usually specify which of the two we consider.

Assume that $\mathcal{E}$ is locally homologically finite and that all higher extension spaces are finite :

$$\forall A, B \in \mathcal{E} \quad \exists p_0 : \quad \operatorname{Ext}^p(A,B) = 0, \quad \forall p > p_0;$$

$$|\operatorname{Ext}^p(A,B)| < \infty, \quad \forall p \geq 0, \quad \forall A, B \in \mathcal{E}.$$

For objects $A, B \in \mathcal{E}$, we define the *Euler form*

$$\langle A, B \rangle := \prod_{i \in \mathbb{Z}} |\operatorname{Ext}^i_{\mathcal{E}}(A,B)|^{(-1)^i}.$$

It is well known and easy to check that this form descends to a bilinear form on the Grothendieck

group $K_0(\mathcal{E})$ of $\mathcal{E}$, denoted by the same symbol :

$$\langle \cdot, \cdot \rangle \, K_0(\mathcal{E}) \times K_0(\mathcal{E}) \to \mathbb{Q}^{\times}.$$

Given this bilinear form, one defines a $\nu-$deformation of the Hall algebra. The *twisted Hall algebra* $\mathcal{H}_{tw}(\mathcal{E})$ is the same vector space over $\mathbb{Q}(\nu)$ as $\mathcal{H}(\mathcal{E})$, with the twisted multiplication

$$[A] * [B] := \sqrt{\langle A, B \rangle} \cdot [A] \diamond [B], \quad \forall A, B \in \mathrm{Iso}(\mathcal{E}). \tag{1.3}$$

The *twisted extended Hall algebra* $\mathcal{H}^e_{tw}(\mathcal{E})$ is defined as the extension of $\mathcal{H}_{tw}(\mathcal{E})$ obtained by adjoining symbols K_α for all classes $\alpha \in K_0(\mathcal{E})$, and imposing relations

$$K_\alpha * K_\beta = K_{\alpha+\beta}, \quad K_\alpha * [B] = \sqrt{(\alpha, B)} \cdot [B] * K_\alpha,$$

for $\alpha, \beta \in K_0(\mathcal{E})$ and $B \in \mathrm{Iso}(\mathcal{E})$. Of course, we can construct $\mathcal{H}^e_{tw}(\mathcal{E})$ from $\mathcal{H}(\mathcal{E})$ in a different order : first adjoin symbols K_α and impose relations

$$K_\alpha \diamond K_\beta = \frac{1}{\sqrt{\langle \alpha, \beta \rangle}} K_{\alpha+\beta}, \quad K_\alpha \diamond [B] = [B] \diamond K_\alpha,$$

defining the *extended Hall algebra* $\mathcal{H}^e(\mathcal{E})$, and then twist the multiplication by the Euler form. Note that $\mathcal{H}^e(\mathcal{E})$ has by definition a module structure over the twisted group algebra of $K_0(\mathcal{E})$, where the multiplication in the last one is twisted by the inverse of the Euler form.

Famous theorems due to Green and Xiao state that if $\mathcal{A}$ is abelian and hereditary, then $\mathcal{H}^e_{tw}(\mathcal{A})$ admits a structure of a self-dual Hopf algebra (see [68, Sections 1.4-1.7] and references therein).

1.7 Derived Hall algebras and Euler forms

In [75], Toën defined a version of Hall algebras for certain class of dg-enhanced triangulated categories. He called these objects *derived Hall algebras*. This work was further generalized by Xiao and Xu [79] for triangulated categories without assumptions on the existence of a dg-enhancement. The construction is as follows. Let $\mathcal{T}$ be an essentially small triangulated category, linear over a finite field k. We also assume that $\mathcal{T}$ has finite morphism spaces. We denote the shift functor in $\mathcal{T}$ by Σ. As usual, the space of $i-$th extensions of X by Y is defined as $\mathrm{Ext}^i_{\mathcal{T}}(X, Y) = \mathrm{Hom}_{\mathcal{T}}(X, \Sigma^i Y)$, for $X, Y \in \mathcal{T}$ and for $i \in \mathbb{Z}$. Assume that $\mathcal{T}$ is *left locally homologically finite*, that is, it satisfies the following property : for each pair of objects $A, B \in \mathcal{F}$, there exists $N \in \mathbb{N}$, such that for each $i > N$, we have

$$|\,\mathrm{Ext}^{-i}(A, B)| = 0.$$

The derived Hall algebra $\mathcal{DH}(\mathcal{T})$ is the $\mathbb{Q}-$vector space whose basis is formed by the isomor-

phism classes $[A]$ of objects A of $\mathcal{T}$, with the multiplication defined by

$$[A] \diamond [C] = \sum_{B \in \mathrm{Iso}(\mathcal{T})} \frac{|\operatorname{Ext}^1_{\mathcal{T}}(A,C)_B|}{|\mathrm{Hom}_{\mathcal{T}}(A,C)|} \prod_{i>0} |\operatorname{Ext}^{-i}_{\mathcal{T}}(A,C)|^{(-1)^{(i-1)}} [B].$$

Here the set $\operatorname{Ext}^1_{\mathcal{T}}(A,C)_B$ is defined as in the exact case.

Theorem 1.7.1. *[75][79][80] The algebra $\mathcal{DH}(\mathcal{T})$ is associative and unital. The unit is given by $[0]$, where 0 is the zero object of $\mathcal{T}$.*

As in the subsection 2.1, our choice of structure constants is not the one given by Toën but an equivalent and slightly renormalized one. It is due to Kontsevich-Soibelman [43], cf. also [80]. In this dual point of view, the elements of $\mathcal{DH}(\mathcal{T})$ are interpreted as finite (signed) measures instead of finitely supported functions.

1.8 Flat deformations

For an abelian group G, its group algebra $\mathbb{Q}G$ is an augmented $\mathbb{Q}-$algebra. Its *augmentation ideal* $I(G)$ is the subpace spanned by $\{g-1|g \in G\}$. It is naturally a two-sided ideal. A *flat G-deformation* of A, or a *flat deformation of A over* $\mathbb{Q}G$ is an algebra B over $\mathbb{Q}G$ which is a free $\mathbb{Q}G$-module, together with an isomorphism $B/I(G) \to A$ of algebras. Any $\mathbb{Q}-$algebra has a trivial flat deformation $A \otimes_{\mathbb{C}} \mathbb{Q}G$.

Chapitre 2

Semi-derived Hall algebras of Frobenius categories

2.1 Semi-derived Hall algebras for stable categories

Assume that a Frobenius category $\mathcal{F}$ satisfies the following conditions :

(C1) $\mathcal{F}$ is essentially small, idempotent complete and linear over some ground field k;

(C2) $\mathcal{F}$ is Hom *–finite*. That is, for each pair of objects $A, B \in \mathcal{F}$, we have

$$|\operatorname{Hom}(A, B)| < \infty.$$

Note that these assumptions ensure that $\mathcal{F}$ is Krull-Schmidt. Moreover, it is known that if $\mathcal{F}$ is Krull-Schmidt, then its stable category $\underline{\mathcal{F}}$ is also Krull-Schmidt. We prove the following lemma similar to [28, 2.3].

Lemma 2.1.1. *All extension spaces in the category $\mathcal{F}$ coincide with those in the stable category $\underline{\mathcal{F}}$. Explicitly, for any $M, N \in \mathcal{F}$, the canonical map :*

$$\operatorname{Ext}^p_{\mathcal{F}}(M, N) \to \operatorname{Ext}^p_{\underline{\mathcal{F}}}(M, N)$$

is bijective for all $p > 0$.

Proof. By definition of the triangulated structure on the stable category $\underline{\mathcal{F}}$, we have a family of conflations

$$\Sigma^{-i}M \rightarrowtail P(\Sigma^{-i-1}M) \twoheadrightarrow \Sigma^{-i-1}M, \qquad i \in \mathbb{Z},$$

where $P(\Sigma^{-i-1}M)$ belongs to $\mathcal{P}(\mathcal{F})$, $\quad \Sigma$ is the suspension functor in $\underline{\mathcal{F}}$. Thus, the complex

$$R(M)^\bullet = \ldots \to P(\Sigma^{-2}M) \to P(\Sigma^{-1}M) \to P(M) \to 0$$

is a projective resolution of M in $\mathcal{F}$. Therefore, extensions of M by N are exactly the homologies

of the complex $\mathrm{Hom}_{\mathcal{F}}(R(M)^{\bullet}, N)$, i.e. $\mathrm{Ext}^p_{\mathcal{F}}(M,N)$ is the quotient of the set of morphisms $\mathrm{Hom}_{\mathcal{F}}(\Sigma^{-(p)}M, N)$ by the subset of morphisms which factor through $P(\Sigma^{-p-1}M)$. This last subset is, by definition, the full subset of $\mathrm{Hom}_{\mathcal{F}}(\Sigma^{-(p)}M, N)$ containing morphisms which factor through a projective-injective object. Thus, we have an isomorphism

$$\mathrm{Ext}^p_{\mathcal{F}}(M,N) \xrightarrow{\sim} \mathrm{Hom}_{\underline{\mathcal{F}}}(\Sigma^{-(p)}M, N).$$

The right hand side is nothing but $\mathrm{Hom}_{\underline{\mathcal{F}}}(M, \Sigma^p N)$, that is equal to $\mathrm{Ext}^p_{\underline{\mathcal{F}}}(M,N)$. $\square$

Corollary 2.1.2. *If a Frobenius category $\mathcal{F}$ satisfies condition (C2), it is also* Ext^p *−finite, for any $p > 0$. That is, for any $M, N \in \mathcal{F}$ and for any $p > 0$, we have*

$$|\,\mathrm{Ext}^p_{\mathcal{F}}(M,N)| < \infty.$$

Proof. By (the proof of) Lemma 2.1.1, we know that the set $\mathrm{Ext}^p_{\mathcal{F}}(M,N)$ is a subquotient of the set $\mathrm{Hom}_{\mathcal{F}}(\Sigma^{-(p)}M, N)$. The latter being finite, so is the former. $\square$

It follows that the classical Hall algebra $\mathcal{H}(\mathcal{C})$ of the exact category $\mathcal{F}$ satisfying conditions (C1) and (C2) is well-defined.

A morphism of Frobenius categories $F : \mathcal{F}' \to \mathcal{F}$ induces an exact functor between the stable categories $\underline{F} : \underline{\mathcal{F}'} \to \underline{\mathcal{F}}$ [28, 2.6]. If F is fully faithful then $\underline{F}$ is fully faithful as well, see, e.g., [46, Remark 7]. Following [68], we say that an exact functor $G : \mathcal{A} \to \mathcal{B}$ between exact categories is *extremely faithful*, if it induces isomorphisms $\mathrm{Ext}^i(M,N) \xrightarrow{\sim} \mathrm{Ext}^i(G(M), G(N))$, for all $M, N \in \mathcal{A}$ and all $i \geq 0$. By Lemma 2.1.1, extensions in Frobenius categories are certain morphisms in their stable categories. Therefore, we can make the following observation.

Lemma 2.1.3. *Any fully faithful morphism $F : \mathcal{F}' \to \mathcal{F}$ between Frobenius categories is extremely faithful.*

Let $F : \mathcal{F}' \to \mathcal{F}$ be an exact functor. It induces a natural linear map of vector spaces

$$F_* : \mathcal{H}(\mathcal{F}') \to \mathcal{H}(\mathcal{F}), \quad [M] \mapsto [F(M)].$$

In general, it is not a morphism of algebras. Nonetheless, by [68, Corollary 1.16] and Lemma 2.1.3, we get the following result about the functoriality of Hall algebras.

Corollary 2.1.4. *If $F : \mathcal{F}' \to \mathcal{F}$ is a fully faithful morphism between essentially small* Hom *−finite Frobenius categories, then F_* is an embedding of algebras.*

Since the exact structure on the category $\mathcal{P}(\mathcal{F})$ of projective-injectives in $\mathcal{F}$ splits, the Euler form on $K_0(\mathcal{P}(\mathcal{F})) = K_0^{split}(\mathcal{P}(\mathcal{F}))$ is well-defined and is given on classes of objects simply as the cardinality of morphism spaces :

$$\langle \cdot, \cdot \rangle : K_0(\mathcal{P}(\mathcal{F})) \times K_0(\mathcal{P}(\mathcal{F})) \to \mathbb{Q}^{\times}, \quad \langle A, B \rangle = |\,\mathrm{Hom}(A,B)|, \qquad \text{for} \quad A, B \in \mathcal{P}(\mathcal{F}).$$

Similarly, the Euler form is well-defined and given by the same formula on $K_0(\mathcal{P}(\mathcal{F})) \times K_0(\mathcal{F})$ and on $K_0(\mathcal{F}) \times K_0(\mathcal{P}(\mathcal{F}))$. We define the *quantum torus of projective-injectives* $\mathbb{T}(\mathcal{P}(\mathcal{F}))$ as the group algebra of $K_0(\mathcal{P}(\mathcal{F}))$ with the multiplication twisted by the Euler form. For any $P \in \mathcal{P}(\mathcal{F}), C \in \mathcal{F}$, their products in the Hall algebra $\mathcal{H}(\mathcal{F})$ take very simple form :

$$[P] \diamond [C] = \frac{1}{|\operatorname{Hom}(P, C)|}[P \oplus C] = \frac{1}{\langle P, C\rangle}[P \oplus C];$$

$$[C] \diamond [P] = \frac{1}{|\operatorname{Hom}(C, P)|}[P \oplus C] = \frac{1}{\langle C, P\rangle}[P \oplus C].$$

It follows that the set of all classes of the form $[P]$, for $P \in \mathcal{P}(\mathcal{F})$, satisfies the Ore conditions in this algebra. Therefore, we can localize $\mathcal{H}(\mathcal{F})$ at these classes.

Definition 2.1.5. The *semi-derived Hall algebra of the pair* $(\mathcal{F}, \mathcal{P}(\mathcal{F}))$ is the localization of $\mathcal{H}(\mathcal{C})$ at the classes of all projective-injective objects :

$$\mathcal{SDH}(\mathcal{F}, \mathcal{P}(\mathcal{F})) := \mathcal{H}(\mathcal{F})[[P]^{-1} | P \in \mathcal{P}(\mathcal{F})].$$

By definition, $\mathcal{SDH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$ is an associative unital algebra, where the unit is given by $[0]$, 0 being the zero object of $\mathcal{F}$. Moreover, by its definition, it satisfies the following functoriality property.

Proposition 2.1.6. *If $F : \mathcal{F}' \to \mathcal{F}$ is a fully faithful morphism between essentially small* $\operatorname{Hom}-$*finite Frobenius categories, then F_* induces an embedding of algebras $\mathcal{SDH}(\mathcal{F}', \mathcal{P}(\mathcal{F}')) \hookrightarrow \mathcal{SDH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$.*

We have natural left and right actions of $\mathbb{T}(\mathcal{P}(\mathcal{F}))$ on $\mathcal{SDH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$ given by the Hall product. Let us denote by $\mathcal{M}(\mathcal{F})$ this bimodule structure on $\mathcal{SDH}(\mathcal{F}, \mathcal{P}(\mathcal{F}))$.

Theorem 2.1.7. *Assume that $\mathcal{F}$ satisfies conditions (C1) and (C2). Then $\mathcal{M}(\mathcal{F})$ is a free right (resp. left) module over $\mathbb{T}(\mathcal{P}(\mathcal{F}))$. Each choice of representatives in $\mathcal{F}$ of the isomorphism classes of the stable category $\underline{\mathcal{F}}$ yields a basis.*

Proof. Assume that the images in $\underline{\mathcal{F}}$ of two objects M, M' from $\mathcal{F}$ are isomorphic. Since the image $\underline{\mathcal{P}(\mathcal{F})}$ of $\mathcal{P}(\mathcal{F})$ in $\underline{\mathcal{F}}$ is contained in the isomorphism class of 0, we have

$$\underline{\mathcal{F}} = \underline{\mathcal{F}}/\underline{\mathcal{P}(\mathcal{F})}.$$

This means that there is a sequence of objects $M_0 = M, M_1, M_2, \ldots, M_n = M'$ in $\mathcal{F}$, such that for each $i = 1, 2, \ldots, n$ there is either a conflation

$$P \rightarrowtail M_{i-1} \overset{\text{qis}}{\twoheadrightarrow} M_i,$$

or a conflation

$$P \rightarrowtail M_i \overset{\text{qis}}{\twoheadrightarrow} M_{i-1},$$

with P projective. Therefore, we either have

$$[M_i] = [P \oplus M_{i-1}] = |\operatorname{Hom}(P, M_{i-1})|[P] \diamond [M_{i-1}],$$

or

$$[M_{i-1}] = [P \oplus M_i] = |\operatorname{Hom}(K, M_i)|[K] \diamond [M_i] \Rightarrow [M_i] = \frac{1}{|\operatorname{Hom}(K, M_i)|}[K]^{-1} \diamond [M_{i-1}].$$

It follows that $[M'] \in \mathbb{T}(\mathcal{P}(\mathcal{F})) \diamond [M]$. Therefore, (the representatives of) the isomorphism classes in the stable category $\underline{\mathcal{F}}$ generate $\mathcal{M}(\mathcal{F})$ over $\mathbb{T}(\mathcal{P}(\mathcal{F}))$. It remains to prove that they are independent over this quantum torus.

One can decompose $\mathcal{M}(\mathcal{F})$ into the direct sum

$$\mathcal{M}(\mathcal{F}) = \bigoplus_{\alpha \in \operatorname{Iso}(\underline{\mathcal{F}})} \mathcal{M}_\alpha(\mathcal{F}),$$

where $\mathcal{M}_\alpha(\mathcal{F})$ is the component containing the classes of all objects whose isomorphism class in $\underline{\mathcal{F}}$ is α. We claim that for each α, the $\mathbb{T}(\mathcal{P}(\mathcal{F}))-$submodule $\mathcal{M}_\alpha(\mathcal{F})$ is free of rank one. Let M be an object of $\mathcal{F}$. By the above argument, the map

$$\mathbb{T}(\mathcal{P}(\mathcal{F})) \to \mathcal{M}_{[M]}(\mathcal{F}), \quad [K] \mapsto [K] \diamond [M] \tag{2.1}$$

is surjective. Since $\mathbb{T}(\mathcal{P}(\mathcal{F}))$ is the (twisted) group algebra of $K_0(\mathcal{P}(\mathcal{F}))$, Lemma 2.1.8 below shows that its composition with the natural map

$$\mathcal{M}_{[M]}(\mathcal{F}) \to \mathcal{M}(\mathcal{F}) \to \mathbb{Q}[K_0^{split}(\mathcal{F})]$$

is injective. Here, the last map is the identity on objects; it is well-defined, since $\mathcal{M}(\mathcal{F})$ has a natural grading by the group $K_0^{split}(\mathcal{F})$. Therefore, the map (2.1) is bijective, q.e.d. $\square$

Lemma 2.1.8. *Under conditions of Theorem 2.1.7, the natural map*

$$i : K_0(\mathcal{P}(\mathcal{F})) \to K_0^{split}(\mathcal{F}), \quad [M] \mapsto [M]$$

is injective.

Proof. Since $\mathcal{F}$ is Krull-Schmidt, one can define the "projective part" of an object in $\mathcal{E}$: each object $M \in \mathcal{F}$ can be decomposed in a unique way (up to a permutation of factors) into a finite direct sum of indecomposables :

$$M = \bigoplus_{i=1}^{m(M)} M_i \oplus \bigoplus_{j=1}^{k(M)} M'_j,$$

where all M_i belong to $\mathcal{P}(\mathcal{F})$ while the M'_j do not. Then

$$\phi : K_0^{split}(\mathcal{F}) \twoheadrightarrow K_0^{split}(\mathcal{P}(\mathcal{F})) = K_0(\mathcal{P}(\mathcal{F})), [M] \mapsto \bigoplus_{i=1}^{m(M)} M_i$$

is a well-defined group epimorphism, and we get

$$\phi \circ i = \mathrm{Id}_{K_0(\mathcal{P}(\mathcal{F}))}.$$

□

Theorem 2.1.9. *Let $\mathcal{F}', \mathcal{F}$ be two Frobenius categories satisfying assumptions (C1) and (C2), $\mathcal{P}', \mathcal{P}$ their full subcategories of projective-injective objects. Assume that*

$$F : \mathcal{F}' \to \mathcal{F}$$

is a fully faithful exact functor inducing an equivalence of the stable categories

$$\underline{\mathcal{F}'} \overset{F}{\xrightarrow{\sim}} \underline{\mathcal{F}}$$

and an isomorphism of the Grothendieck groups of projective-injectives

$$K_0(\mathcal{P}(\mathcal{F}')) \overset{F}{\xrightarrow{\sim}} K_0(\mathcal{P}(\mathcal{F})). \tag{2.2}$$

Then F induces an isomorphism of algebras

$$F_* : \mathcal{SDH}(\mathcal{F}', \mathcal{P}(\mathcal{F}')) \xrightarrow{\sim} \mathcal{SDH}(\mathcal{F}, \mathcal{P}(\mathcal{F})).$$

Proof. Since F is fully faithful and induces the isomorphism (2.2), it also induces an isomorphism of the quantum tori of projective-injectives :

$$\mathbb{T}(\mathcal{P}(\mathcal{F}')) \overset{F}{\xrightarrow{\sim}} \mathbb{T}(\mathcal{P}(\mathcal{F})).$$

Therefore, by Theorem 2.1.7, F induces an isomorphism

$$\mathcal{M}(\mathcal{F}') \xrightarrow{\sim} \mathcal{M}(\mathcal{F})$$

of the free modules over isomorphic quantum tori with bases which are in bijection by the stable equivalence. By the full faithfulness and Lemma 2.1.1, the multiplication is preserved as well, i.e. F induces the desired isomorphism of algebras. □

2.2 Semi-derived vs. derived Hall algebras

Let $\mathcal{F}$ be a Frobenius category satisfying conditions (C1), (C2). As before, we denote by $\underline{\mathcal{F}}$ its stable category. It is evidently $\mathrm{Hom}-$finite, since its morphism spaces are subquotients of the morphism spaces in $\mathcal{F}$. Assume that $\underline{\mathcal{F}}$ is left locally homologically finite.

We introduce the *relative Euler form*

$$\langle \cdot, \cdot \rangle_{(\mathcal{F},\underline{\mathcal{F}})} : K_0(\mathcal{F}) \times K_0(\mathcal{F}) \to \mathbb{Q}$$

by the following rule : for each pair $A, B \in \mathcal{F}$, we pose

$$\langle A, B \rangle_{(\mathcal{F},\underline{\mathcal{F}})} = \frac{|Hom_{\mathcal{F}}(A,B)|}{|Hom_{\underline{\mathcal{F}}}(A,B)} \cdot \prod_{i>0} |\operatorname{Ext}^{-i}_{\underline{\mathcal{F}}}(A,C)|^{(-1)^{(i-1)}}.$$

Lemma 2.2.1. *The form*

$$\langle \cdot, \cdot \rangle_{(\mathcal{F},\underline{\mathcal{F}})}$$

is a well-defined group homomorphism.

Proof. The statement follows from Lemma 2.1.1 and the comparison of long exact sequences of extensions in $\mathcal{F}$ and $\underline{\mathcal{F}}$. Consider an arbitrary conflation

$$A_1 \rightarrowtail A_2 \twoheadrightarrow A_3$$

in the category $\mathcal{F}$. For any $B \in \mathcal{F}$, we have two long exact sequences of extensions of elements of this conflation by B : the sequence of extensions in $\mathcal{F}$ and the one of those in $\underline{\mathcal{F}}$:

$$\begin{gathered} 0 \to \operatorname{Hom}_{\mathcal{F}}(A_3,B) \to \operatorname{Hom}_{\mathcal{F}}(A_2,B) \to \operatorname{Hom}_{\mathcal{F}}(A_1,B) \xrightarrow{f} \\ \operatorname{Ext}^1_{\mathcal{F}}(A_3,B) \to \operatorname{Ext}^1_{\mathcal{F}}(A_2,B) \to \operatorname{Ext}^1_{\mathcal{F}}(A_1,B) \to \dots; \end{gathered} \tag{2.3}$$

$$\begin{gathered} \dots \operatorname{Ext}^{-1}_{\underline{\mathcal{F}}}(A_3,B) \to \operatorname{Ext}^{-1}_{\underline{\mathcal{F}}}(A_2,B) \to \operatorname{Ext}^{-1}_{\underline{\mathcal{F}}}(A_1,B) \xrightarrow{g} \\ \operatorname{Hom}_{\underline{\mathcal{F}}}(A_3,B) \to \operatorname{Hom}_{\underline{\mathcal{F}}}(A_2,B) \to \operatorname{Hom}_{\underline{\mathcal{F}}}(A_1,B) \to \\ \operatorname{Ext}^1_{\underline{\mathcal{F}}}(A_3,B) \to \operatorname{Ext}^1_{\underline{\mathcal{F}}}(A_3,B) \to \operatorname{Ext}^1_{\underline{\mathcal{F}}}(A_3,B) \dots. \end{gathered} \tag{2.4}$$

By lemma 2.1.1, any term of the form $\operatorname{Ext}^i_{\mathcal{F}}(A_j,B)$, with $i > 0, j = 1,2,3$, is isomorphic to its counterpart in the second sequence. Thus, we have an isomorphism $Ker(f) \xrightarrow{\sim} Ker(g)$. We have two exact sequences to the left of $Ker(f)$ in (2.3), respectively of $Ker(g)$ in (2.4), and find out that the alternating products of their terms both equal 1. Hence their quotient equals 1 as well. On the other hand, it coincides with

$$\langle A_1, B \rangle_{(\mathcal{F},\underline{\mathcal{F}})} \cdot \frac{1}{\langle A_2, B \rangle_{(\mathcal{F},\underline{\mathcal{F}})}} \cdot \langle A_3, B \rangle_{(\mathcal{F},\underline{\mathcal{F}})}.$$

The statement follows from this and from the dual result concerning $\langle B, A_i\rangle_{(\mathcal{F},\underline{\mathcal{F}})}$, which has a similar proof. $\square$

It is easy to see that, by Lemma 2.2.1, one can twist the multiplication in $\mathcal{SDH}(\mathcal{F},\mathcal{P}(\mathcal{F}))$ by the rule

$$A * B := \langle A, B\rangle_{(\mathcal{F},\underline{\mathcal{F}})}\, A \diamond B.$$

We call the result *the twisted semi-derived Hall algebra* $\mathcal{SDH}(\mathcal{F},\mathcal{P}(\mathcal{F}))_{tw}$.

We are ready now to present the main result of this section comparing our construction with derived Hall algebras.

Theorem 2.2.2. *Assume that a Frobenius category $\mathcal{F}$ satisfies properties (C1) and (C2), and its stable category $\underline{\mathcal{F}}$ is left locally homologically finite. Then $\mathcal{SDH}(\mathcal{F},\mathcal{P}(\mathcal{F}))_{tw}$ is a flat $\mathbb{Q}[K_0(\mathcal{P}(\mathcal{F}))])-$deformation of $\mathcal{DH}(\underline{\mathcal{F}})$. Each choice of representatives in $\mathcal{F}$ of the isomorphism classes of the stable category $\underline{\mathcal{F}}$ yields an isomorphism*

$$\mathcal{SDH}(\mathcal{F},\mathcal{P}(\mathcal{F}))_{tw}/I(K_0(\mathcal{P}(\mathcal{F}))) \xrightarrow{\sim} \mathcal{DH}(\underline{\mathcal{F}}).$$

Proof. As a vector space, $\mathcal{DH}(\underline{\mathcal{F}})$ has the basis parameterized by the isomorphism classes of objects in $\underline{\mathcal{F}}$. By Theorem 2.1.7 and by the choice of the twist, $\mathcal{SDH}(\mathcal{F},\mathcal{P}(\mathcal{F}))_{tw}$ is a free module over $\mathbb{Q}[K_0(\mathcal{P}(\mathcal{F}))]$. Moreover, each choice of representatives in $\mathcal{F}$ of the isomorphism classes of the stable category $\underline{\mathcal{F}}$ yields a basis of this module. Therefore, $\mathcal{SDH}(\mathcal{F},\mathcal{P}(\mathcal{F}))_{tw}/I(K_0(\mathcal{P}(\mathcal{F})))$ is the same $\mathbb{Q}-$vector space as $\mathcal{DH}(\underline{\mathcal{F}})$. It remains to show that the multiplicative structures are the same on both sides. By lemma 2.1.1, the sets $\mathrm{Ext}^1_{\mathcal{F}}(A,B)_C$ and $\mathrm{Ext}^1_{\underline{\mathcal{F}}}(A,B)_C$ are isomorphic for any triple of objects $A, B, C \in \mathcal{F}$. Now the statement follows from the form of the structure constants in $\mathcal{SDH}(\mathcal{F},\mathcal{P}(\mathcal{F}))$ and in $\mathcal{DH}(\underline{\mathcal{F}})$, by the choice of the twist. $\square$

2.3 Examples

Example 2.3.1. $\mathcal{F}$ is the category of bounded (or m-periodic complexes) over an exact category $\mathcal{E}$, with the component-wise split exact structure. This is a Frobenius category, whose stable category is the bounded (resp. $m-$periodic) homotopy category $\mathcal{H}^b(\mathcal{E})$ (res. $\mathcal{H}_{\mathbb{Z}/m}(\mathcal{E})$). If $\mathcal{E}$ satisfies conditions (C1) and (C2), so do $\mathcal{H}^b(\mathcal{E})$ and $\mathcal{H}_{\mathbb{Z}/m}(\mathcal{E})$. There, our construction provides a Hall-like algebra for bounded and periodic homotopy categories.

Example 2.3.2. $\mathcal{F} = \mathcal{C}^b(\mathcal{P}(\mathcal{E}))$, where $\mathcal{P}(\mathcal{E})$ is the full subcategory of projective objects in an exact category $\mathcal{E}$ with enough projectives and where each object has finite projective resolution. Then the stable category $\underline{\mathcal{F}}$ is equivalent to the bounded derived category $\mathcal{D}^b(\mathcal{E})$. Therefore, our construction provides a version of the Hall algebra for the bounded derived category $\mathcal{D}^b(\mathcal{E})$. It was introduced by the author in [23] and called the "semi-derived Hall algebra of $\mathcal{E}$". See Chapter 4 for the detailed treatment.

Example 2.3.3. $\mathcal{F}$ is the category of $m-$periodic complexes over $\mathcal{P}(\mathcal{E})$, for $m > 1$. As in the previous example, the stable category is equivalent the $m-$periodic derived category $\mathcal{D}_{\mathbb{Z}/m}(\mathcal{E})$. As $\mathcal{D}_{\mathbb{Z}/m}(\mathcal{E})$ is an $m-$periodic triangulated category, it is never left locally homologically finite. Therefore, one cannot define its derived Hall algebra, and neither the twist nor the right-hand side in the identity in Theorem 2.2.2 are well-defined. On the other hand, the construction presented in this work provides an associative algebra. The case of $m = 2$ was first considered in the work of Bridgeland [5] that provided the main inspiration to our work, see also [23]. Yanagida [82] proved the conjecture of Bridgeland [5] that, under certain conditions and for $\mathcal{E}$ abelian, this algebra with an appropriate twist provides the Drinfeld double of the twisted extended Hall algebra of $\mathcal{E}$. The generic version of such an algebra (in the abelian case, but for an arbitrary positive m) was introduced in [9]. Zhao [83] proved that for $\mathcal{E}$ abelian, the category $\mathcal{D}_{\mathbb{Z}/m}(\mathcal{E})$ is equivalent to the *generalized root category* of $\mathcal{E}$, see references in [loc. cit]. Therefore, the algebra (that we call in [23] "the $\mathbb{Z}/m-$graded semi-derived Hall algebra of $\mathcal{E}$") is also the substitute of the non-defined derived Hall algebra of the root category. See Chapter 4.

Example 2.3.4. Dually, we can take as $\mathcal{F}$ the category of bounded or periodic complexes over the full subcategory of injectives in an exact category $\mathcal{E}$ with enough injectives and where each object has finite injective resolution. The corresponding stable category is again equivalent to the bounded (resp. periodic) derived category of $\mathcal{E}$, so we get an algebra isomorphic to the one from previous examples.

Example 2.3.5. Let $\mathcal{D}$ be a differential graded (DG) category. We freely use basic facts on DG categories that can be found, e.g., in surveys [39] and [76]. One can define the category $\mathcal{C}_{dg}(\mathcal{D})$ of DG modules over $\mathcal{D}$ and the derived category $\mathcal{D}_{dg}(\mathcal{D})$. There is a Yoneda embedding of $\mathcal{D}$ into $\mathcal{C}_{dg}(\mathcal{D})$. One says that $\mathcal{D}$ is *pretriangulated* if the image of the Yoneda embedding is closed under taking cones of morphisms and under the shift functor. One can show that $\mathcal{D}$ is pretriangulated if and only if its underlying category $Z^0(\mathcal{D})$ is Frobenius; in such a case the stable category of the latter is the homotopy category $H^0(\mathcal{D})$. Then the Yoneda embedding induces an embedding of $H^0(\mathcal{D})$ into the full subcategory $\mathrm{per}(\mathcal{D})$ of perfect (and, equivalently, compact) objects in $\mathcal{D}_{dg}(\mathcal{D})$. This perfect derived category is then the idempotent completion of $H^0(\mathcal{D})$. Thus, if $H^0(\mathcal{D})$ is idempotent complete, then we have an equivalence $H^0(\mathcal{D}) \xrightarrow{\sim} \mathrm{per}(\mathcal{D})$. When this condition holds, one says that $\mathcal{D}$ is a *triangulated DG category*, or that $\mathcal{D}$ is *saturated*, or *Morita fibrant*. The latter notion reflects the fact that there exists a model structure on the category of DG categories, s.t. triangulated DG categories are precisely the fibrant objects. This model structure is called the *Morita model structure*. It is known that each DG category has a Morita fibrant replacement, i.e. that it is Morita equivalent to a triangulated DG category, cf. [76].

For a triangulated category $\mathcal{T}$, by an *enhancement* one understands a triangulated DG category $\mathcal{D}$ with a triangulated equivalence $\mathcal{T} \xrightarrow{\sim} \mathrm{per}(\mathcal{D})$. It is known that an idempotent complete triangulated category is algebraic if and only if it has an enhancement : if for $\mathcal{T}$ there exists a Frobenius category $\mathcal{F}$, s.t. $\mathcal{T} \xrightarrow{\sim} \underline{\mathcal{F}}$, then one can endow the category of the complexes with

projective-injective components in $\mathcal{P}(\mathcal{F})$ with a natural DG category structure, such that the corresponding homotopy category $H^0(\mathcal{C}_{dg}(\mathcal{P}(\mathcal{F})))$ will be equivalent to $\mathcal{T}$. If $\mathcal{D}$ is pretriangulated and a Frobenius category $Z^0(\mathcal{D})$ is idempotent complete, then its stable category $H^0(\mathcal{D})$ is idempotent complete as well, and $\mathcal{D}$ is triangulated. We can apply our main theorems to the case of such DG categories.

Corollary 2.3.6. *Let $\mathcal{D}, \mathcal{D}'$ be a pair of triangulated DG categories whose underlying categories $Z^0(\mathcal{D}), Z^0(\mathcal{D}')$ satisfy conditions (C1) and (C2). If a DG functor $F : \mathcal{D}' \to \mathcal{D}$ induces a fully faithful map of the underlying categories, then it induces an embedding of algebras*

$$F_* : \mathcal{SDH}(Z^0(\mathcal{D}')), \mathcal{P}((Z^0(\mathcal{D}'))) \hookrightarrow \mathcal{SDH}(Z^0(\mathcal{D})), \mathcal{P}((Z^0(\mathcal{D}))).$$

If, moreover, F induces an equivalence of perfect derived categories $\mathrm{per}(\mathcal{D}') \xrightarrow{\sim} \mathrm{per}(\mathcal{D})$ and an isomorphism of the Grothendieck groups $K_0(\mathcal{P}((Z^0(\mathcal{D}')) \xrightarrow{\sim} \mathcal{P}((Z^0(\mathcal{D}))$, then F_* is an isomorphism of algebras.

Chapitre 3

Twisted extended Hall algebras of triangulated categories

3.1 Twisted extended Hall algebras of triangulated categories

Let $\mathcal{T}$ be a $\mathrm{Hom}-$finite, small, idempotent complete, $\mathbb{F}_q-$linear triangulated category. In this section, we denote its objects by underlined letters. For a pair of objects $\underline{A}, \underline{B} \in \mathcal{T}$, for each element ε of the set $\mathrm{Ext}^1_{\mathcal{T}}(\underline{A}, \underline{B})$, the isomorphism class of the cone of the correspondig morphism $\underline{A} \to \Sigma\underline{B}$ is well-defined. We denote by $\mathbf{mt}(\varepsilon)$ the isomorphism class of its shift by -1. Recall that $M_0^{add}(\mathcal{T}) \xrightarrow{\sim} K_{0,+}^{add}(\mathcal{T}) \xrightarrow{\sim} (\mathrm{Iso}(\mathcal{T}), \oplus)$. Denote the canonical projection $K_0^{add}(\mathcal{T}) \twoheadrightarrow K_0(\mathcal{T})$ by $\pi_{\mathcal{T}}$.

Suppose that we have the following datum :

(i) A short exact sequence of abelian groups

$$G \overset{i}{\hookrightarrow} \widetilde{\mathcal{N}} \overset{\pi}{\twoheadrightarrow} K_0^{add}(\mathcal{T});$$

(ii) A multiplicative bilinear form $\phi : \widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}} \to \mathbb{Q}^{\times}$.

Since i is injective, we identify the elements of the group G with their images in $\widetilde{\mathcal{N}}$.

Let $\mathcal{N}$ be the preimage $\mathcal{N} = \pi^{-1}(K_{0,+}^{add}(\mathcal{T}))$. It is a commutative monoid, and since it embeds into a group, it is cancellative. For elements of $\mathcal{N}$, we can understand their images under π as isomorphism classes in $\mathcal{T}$.

For all $A, B, C \in \mathcal{N}$, each $\varepsilon \in \mathrm{Ext}^1_{\mathcal{T}}(\pi(A), \pi(B))$ induces linear maps

$$f_\varepsilon^C : \mathrm{Hom}(\pi(B), \pi(C)) \to \mathrm{Ext}^1(\pi(A), \pi(C));$$

$$h_\varepsilon^C : \mathrm{Hom}(\pi(C), \pi(A)) \to \mathrm{Ext}^1(\pi(C), \pi(B));$$

$$k_\varepsilon^C : \mathrm{Ext}^1(\pi(A), \pi(C)) \to \mathrm{Ext}^1(\mathbf{mt}(\varepsilon), \pi(C));$$

$$l_\varepsilon^C : \mathrm{Ext}^1(\pi(C), \pi(B)) \to \mathrm{Ext}^1(\pi(C), \mathbf{mt}(\varepsilon)).$$

From the long exact sequences associated to ε and C, we get that

$$\mathrm{Im}(f_\varepsilon^C) = \mathrm{Ker}(k_\varepsilon^C); \quad \mathrm{Im}(h_\varepsilon^C) = \mathrm{Ker}(l_\varepsilon^C).$$

Suppose that the following conditions on datum (i)-(ii) are satisfied :

(C1) For all $A, B \in \mathcal{N}$, there is a map $E_{A,B} : \mathrm{Ext}^1_{\mathcal{T}}(\pi(A), \pi(B)) \to \mathcal{N}$, such that $\pi(E_{A,B}(\varepsilon)) = \mathbf{mt}(\varepsilon)$ and

$$E_{g+A,g'+B}(\varepsilon) = g + g' + E_{A,B}(\varepsilon), \quad \forall g, g' \in G;$$

(C2) For all $A, B, C \in \mathcal{N}$, for all $\varepsilon \in \mathrm{Ext}^1_{\mathcal{T}}(\pi(A), \pi(B))$, we have the following identities :

$$\frac{\phi(A+B, C)}{\phi(E_{A,B}(\varepsilon), C)} = |\,\mathrm{Ker}(k_\varepsilon^C)|; \tag{3.1}$$

$$\frac{\phi(C, A+B)}{\phi(C, E_{A,B}(\varepsilon))} = |\,\mathrm{Ker}(l_\varepsilon^C)|. \tag{3.2}$$

For $N \in \mathcal{N}$, let us denote by $\mathrm{Ext}^1_{\mathcal{T}}(\pi(A), \pi(B))_N$ the subset $E^{-1}_{A,B}(N) \subset \mathrm{Ext}^1_{\mathcal{T}}(\pi(A), \pi(B))$. We have a decomposition

$$\mathrm{Ext}^1(\underline{A}, \underline{B}) = \coprod_N \mathrm{Ext}^1(\underline{A}, \underline{B})_N.$$

Let $V(\mathcal{T}, \mathcal{N}, \phi)$ be a $\mathbb{Q}-$vector space with a basis $\{[N]\}$ parameterized by elements $N \in \mathcal{N}$.

We define the *Hall algebra* $\mathcal{H}(\mathcal{T}, \mathcal{N}, \phi)$ to be the vector space V endowed with the multiplication defined by the formula

$$[A] \diamond [B] = \phi(A, B) \sum_{\varepsilon \in \mathrm{Ext}^1_{\mathcal{T}}(\pi(A), \pi(B))} [E_{A,B}(\varepsilon)] =$$

$$= \phi(A, B) \sum_{N \in \mathcal{N}} \mathrm{Ext}^1_{\mathcal{T}}(\pi(A), \pi(B))_N [N].$$

Remark 3.1.1. Since $K_0^{add}(\mathcal{T})$ is a free abelian group, the short exact sequence from (i) splits. However, we need more information than just the direct sum $G \oplus K_0^{add}(\mathcal{T})$, so we describe our construction this way.

Note that if one of the factors belongs to (the image of) the group G, the multiplication is given by the formulas :

$$[g] \diamond [B] = \phi(g, B)[g + B], \quad [B] \diamond [g] = \phi(B, g)[g + B]. \tag{3.3}$$

Theorem 3.1.2. *$\mathcal{H}(\mathcal{T}, \mathcal{N}, \phi)$ is an associative unital algebra.*

Proof. Our proof is an easy modification of the proof of the associtiativity of derived Hall algebras ; we follow [79][80].

Assume that we have a homotopy cartesian square in $\mathcal{T}$

$$\begin{tikzcd} \underline{F} \arrow[r, "f'"] \arrow[d, "m'"'] & \underline{H} \arrow[d, "m"'] \\ \underline{B} \arrow[r, "f"] & \underline{E}, \end{tikzcd} \tag{3.4}$$

By the octahedral axiom, it can be completed to a commutative diagram of the following form :

$$\begin{tikzcd} \underline{C} \arrow[r, equal] \arrow[d] & \underline{C} \arrow[d] & & & \\ \underline{F} \arrow[r, "f'"] \arrow[d, "m'"'] & \underline{H} \arrow[r] \arrow[d, "m"'] & \underline{A} \arrow[r] \arrow[d, equal] & \underline{F}[1] \arrow[d] \\ \underline{B} \arrow[r, "f"] \arrow[d, "\nu"] & \underline{E} \arrow[r] \arrow[d, "\mu"] & \underline{A} \arrow[r, "\varepsilon"] & \underline{B}[1] \\ \underline{C}[1] \arrow[r, equal] & \underline{C}[1], & & \end{tikzcd} \tag{3.5}$$

for some $\underline{A}, \underline{C} \in \mathcal{T}$, and with rows and columns being distinguished triangles. Fix lifts $A, B, C \in \mathcal{N}$ of $\underline{A}, \underline{B}$ and $\underline{C}$. Let us denote $E_{A,B}(\varepsilon)$ by E, $E_{B,C}(\nu)$ by F. Then we have

$$\underline{E} = \pi(E), \underline{F} = \pi(E).$$

This determines a lift $H \in \mathcal{N}$ of $\underline{H}$. Namely, $H = E_{E,F}(\alpha) - B$, where α is the class of the extension corresponding to the homotopy cartesian square 3.4. Then $\pi(H) = \underline{H}$. The diagram (3.5) can be rewritten as follows :

$$\begin{tikzcd} \pi(C) \arrow[r, equal] \arrow[d] & \pi(C) \arrow[d] & & & \\ \pi(F) \arrow[r, "f'"] \arrow[d, "m'"'] & \pi(H) \arrow[r] \arrow[d, "m"'] & \pi(A) \arrow[r, "\delta"] \arrow[d, equal] & \pi(F)[1] \arrow[d] \\ \pi(B) \arrow[r, "f"] \arrow[d] & \pi(E) \arrow[r] \arrow[d, "\mu"] & \pi(A) \arrow[r, "\varepsilon"] & \pi(B)[1] \\ \pi(C)[1] \arrow[r, equal] & \pi(C)[1]. & & \end{tikzcd} \tag{3.6}$$

Fix A, B, C, E, F and H. Slightly modifying notation of [80], denote by $\mathrm{Ext}^1(\underline{A}, \underline{B})_E^F$ the subgroup of $\mathrm{Ext}^1(\underline{A}, \underline{B})_E$ given by those morphisms that appear in (the third row of) diagrams of the form (3.5), for some f, f', m, m'. We have a decomposition

$$\mathrm{Ext}^1(\underline{A}, \underline{B})_E = \coprod_F \mathrm{Ext}^1(\underline{A}, \underline{B})_E^F.$$

For each $\varepsilon \in \mathrm{Ext}^1(\underline{A}, \underline{B})_E^F$, we define $\mathrm{Ext}^1(\underline{E}, \underline{C})_H^{\varepsilon,F}$ to be the subgroup of $\mathrm{Ext}^1(\underline{E}, \underline{C})_G$ given by those morphisms μ that appear in (the second column of) diagrams of the form (3.5) with ε being the morphism $A \to B[1]$ in the third row, for some f, f', m, m'. In other words, $\mathrm{Ext}^1(\underline{E}, \underline{C})_H^{\varepsilon,F}$ consists of morphisms μ that appear in (the second column of) diagrams of the form (3.5) with ε being the morphism $A \to B[1]$ in the third row for some f, f', m, m'. We define $\mathrm{Ext}^1(\underline{B}, \underline{C})_F^E$ and $\mathrm{Ext}^1(\underline{A}, \underline{F})_H^{\nu,E}$ in a similar way. The same proof as in [80] provides the following result.

Lemma 3.1.3. *[80, cf. The symmetry-II]*

(i) Fix $\varepsilon \in \mathrm{Ext}^1(\pi(A), \pi(B))_E^F$, then there exists a surjective map

$$f_* : \mathrm{Ext}^1(\pi(E), \pi(C))_G^{\varepsilon,F} \to \mathrm{Ext}^1(\pi(B), \pi(C))_F^E$$

such that the cardinality of any fibre is

$$|(f_*)^{-1}| = |\,\mathrm{Ext}^1(\pi(A), \pi(C))| \times |\,\mathrm{Im}(f_\varepsilon^C)| = |\,\mathrm{Hom}(\pi(A), \pi(C)[1])| \frac{\phi(A+B, C)}{\phi(E_{A,B}(\varepsilon), C)}.$$

(ii) Fix $\nu \in \mathrm{Ext}^1(\pi(B), \pi(C))_{\pi(F)}^{\pi(E)}$, then there exists a surjective map

$$m'_* : \mathrm{Ext}^1(\pi(A), \pi(F))_{\pi(H)}^{\nu,\pi(E)} \to \mathrm{Ext}^1(\pi(A), \pi(B))_{\pi(E)}^{\pi(F)}$$

such that the cardinality of any fibre is

$$|(m'_*)^{-1}| = |\,\mathrm{Ext}^1(\pi(A), \pi(C))| \times |\,\mathrm{Im}(h_\varepsilon^C)| = |\,\mathrm{Hom}(\pi(A), \pi(C)[1])| \frac{\phi(A, B+C)}{\phi(A, E_{B,C}(\mu))};$$

To be more precise, in [80] the authors proved the first identities in (i) and (ii) for $\mathrm{Ext}^1(\pi(A), \pi(B))_{\pi(E)}^{\pi(F)}$. The argument applies for our case as well, because the maps we consider are restrictions of surjective maps with fibers of fixed cardianlity given in [80]. The second identities follow from the condition (C2) on the form ϕ.

To prove the associativity of the algebra $\mathcal{H}(\mathcal{T}, \mathcal{N}, \phi)$, we need to show that, for all $A, B, C \in \mathcal{N}$, we have the following identity :

$$([A] \diamond [B]) \diamond [C] = [A] \diamond ([B] \diamond [C]). \tag{3.7}$$

The left hand side can be rewritten as follows :

$$([A]\diamond[B])\diamond[C]=\phi(A,B)\sum_{\varepsilon\in\mathrm{Ext}^1_{\mathcal{T}}(\pi(A),\pi(B))}([E_{A,B}(\varepsilon)]\diamond[C])=$$
$$=\phi(A,B)\sum_{E\in\mathcal{N}}(\sum_{\varepsilon\in\mathrm{Ext}^1_{\mathcal{T}}(\underline{A},\underline{B})_E}([E]\diamond[C]))=\phi(A,B)\sum_{E,F\in\mathcal{N}}(\sum_{\varepsilon\in\mathrm{Ext}^1_{\mathcal{T}}(\underline{A},\underline{B})^F_E}([E]\diamond[C]))=$$
$$=\phi(A,B)\sum_{E,F,H\in\mathcal{N}}(\sum_{\varepsilon\in\mathrm{Ext}^1_{\mathcal{T}}(\underline{A},\underline{B})^F_E}\phi(E,C)\sum_{\mu\in\mathrm{Ext}^1_{\mathcal{T}}(\underline{E},\underline{C})^{\varepsilon,F}_H}[H])=$$
$$=\phi(A,B)\sum_{E,F,H\in\mathcal{N}}(\sum_{\varepsilon\in\mathrm{Ext}^1_{\mathcal{T}}(\underline{A},\underline{B})^F_E}\phi(E,C)|\mathrm{Ext}^1_{\mathcal{T}}(\underline{E},\underline{C})^{\varepsilon,F}_H|[H])$$

By part (i) of Lemma 3.1.3, this can be further rewritten as follows :

$$\sum_{E,F,H\in\mathcal{N}}(\phi(A,B)\phi(E,C)|\mathrm{Ext}^1_{\mathcal{T}}(\underline{A},\underline{B})^F_E||\mathrm{Ext}^1_{\mathcal{T}}(\underline{B},\underline{C})^E_F||\mathrm{Ext}^1_{\mathcal{T}}(\underline{A},\underline{C})|\tfrac{\phi(A+B,C)}{\phi(E,C)}[H])=$$
$$=\sum_{E,F,H\in\mathcal{N}}(\phi(A,B)\phi(A,C)\phi(B,C)|\mathrm{Ext}^1_{\mathcal{T}}(\underline{A},\underline{B})^{\underline{E}}_{\underline{E}}||\mathrm{Ext}^1_{\mathcal{T}}(\underline{B},\underline{C})^{\underline{E}}_{\underline{F}}||\mathrm{Ext}^1_{\mathcal{T}}(\underline{A},\underline{C})|[H]). \tag{3.8}$$

Using part (ii) of Lemma 3.1.3, we can rewrite the right hand side of (3.7) in a similar way and obtain the same result (3.8). This proves the associativity of $\mathcal{H}(\mathcal{T},\mathcal{N},\phi)$.

It is clear that the class of $0\in\mathcal{N}$ is the unit of $\mathcal{H}(\mathcal{T},\mathcal{N},\phi)$. $\square$

Remark 3.1.4. Note that G is a subgroup of the group of units of the monoid $\mathcal{N}$. It seems that we can slightly generalize our construction while keeping this observation in mind. Namely, we can take $\mathcal{N}$ to be only unit-cancellative and modify the datum as follows :

(i') A unit-cancellative commutative monoid $\mathcal{N}$ with the group of units $G=\mathcal{N}^\times$ and an isomorphism $\mathcal{N}/G\xrightarrow{\sim}K^{add}_{0,+}(\mathcal{T})$;

(ii') A multiplicative bilinear form $\phi:\widetilde{\mathcal{N}}\times\widetilde{\mathcal{N}}\to\mathbb{Q}^\times$, for $\widetilde{\mathcal{N}}$ the group completion of $\mathcal{N}$.

Our proof requires the cancellativity of the monoid $\mathcal{N}$ (we use it to determine the lift of $\underline{H}$ inside $\mathcal{N}$). However, it seems that the result might stay true if we use (i')-(i") instead of (i)-(ii). In particular, in the case of algebras coming from exact categories with weak equivalences, in the construction using replacements, the (alternative) proof uses different approach and does not refer to the short exact sequence from (i) at all. The same is true for the semi-derived Hall algebras from Chapter 2. Since this was the definition that appeared the first, we decided to keep it unchanged in this thesis and use Grothendieck monoids instead of their cancellative quotients. By abuse of notation, we denote all the algebras associated to data of type (i')-(i") in the same manner as the algebras associated to data of type (i)-(ii) as above.

Remark 3.1.5. As shown below, starting from a Frobenius category $\mathcal{F}$, we can recover the semi-derived Hall algebra from [24] via this construction. Since the non-localized algebra $\mathcal{H}(\mathcal{F})$ is still an associative unital algebra, it appears that at least some of the properties of these twisted extended Hall algebras are preserved if we consider a monoid M instead of a group G in the

definition. However, formulating precise statements concerning exact sequences of monoids is a very tricky subject. Since in the most important examples we do have a suitable group, we decided not to go deep into technical issues that appear if one wants to cover the case of monoids.

From the definition, we observe that the algebras thus defined admit natural gradings :

Proposition 3.1.6. *$\mathcal{H}(\mathcal{T},\mathcal{N},\phi)$ is naturally graded by the quotient monoid $\mathcal{N}/\sim_{\mathcal{T}}$, where the congruence $\mathcal{T}$ is generated by the relations*

$$A+B\sim_{\mathcal{T}} E_{A,B,\varepsilon},\quad \forall\quad A,B\in\mathcal{N},\varepsilon\in \mathrm{Ext}^1_{\mathcal{T}}(\pi(A),\pi(B)).$$

It is also naturally graded by the quotient group $\widetilde{\mathcal{N}}/\sim_{\mathcal{T}}$.

The group $\widetilde{\mathcal{N}}/\sim_{\mathcal{T}}$ is an abelian extension of the Grothendieck group $K_0(\mathcal{T})$ by a certain quotient of the group G. Contrary to $K_0^{add}(\mathcal{T})$, $K_0(\mathcal{T})$ might not be free, and this extension is not necessarily split, cf. Remark 3.1.1.

Clearly, we recover the usual derived Hall algebras :

Proposition 3.1.7. *If $\mathcal{T}$ is left locally homologically finite, then $\mathcal{H}(\mathcal{T},M_0^{add}(\mathcal{T}),\langle\cdot,\cdot\rangle_{\leq 0})$ is just the derived Hall algebra $\mathcal{DH}(\mathcal{T})$.*

Assume that $\mathcal{T}$ is left locally homologically finite. Let $\mathcal{H}(\mathcal{T},\mathcal{N},\phi)_{tw}$ be the twist of $\mathcal{H}(\mathcal{T},\mathcal{N},\phi)_{tw}$ by the form $\frac{1}{\phi(\cdot,\cdot)<\cdot,\cdot>_{\leq 0}}$.

Theorem 3.1.8. *(i) $\mathcal{H}(\mathcal{T},\mathcal{N},\phi)$ is a free module over the group algebra $\mathbb{Q}G$ twisted by the form ϕ.*

(ii) If $\mathcal{T}$ is left locally homologically finite, then for any $\mathcal{N},\phi$ satisfying conditions (C1) and (C2), $\mathcal{H}(\mathcal{T},\mathcal{N},\phi)_{tw}$ is a flat $G-$ deformation of $\mathcal{DH}(\mathcal{T})$.

Proof. The proof of part (i) is similar to the proof of Theorem 2.1.7. The classes in $M_0^{add}(\mathcal{T})$ generate $\mathcal{H}(\mathcal{T},\mathcal{N},\phi)$ over the twisted group algebra. As a module, $\mathcal{H}(\mathcal{T},\mathcal{N},\phi)$ is naturally graded by the monoid $\mathcal{N}$. Then the embedding $G\hookrightarrow N$ demonstrates the freeness.

The proof of part (ii) is similar to the proof of Theorem 2.1.11. Because of part (i), it is enough to show that the multiplication in $\mathcal{H}(\mathcal{T},\mathcal{N},\phi)_{tw}/I(G)$ coincides with the one in $\mathcal{DH}(\mathcal{T})$. This follows directly from the definitions of these two algebras. $\square$

By comparing the definitions, we immediately check that we recover all the algebras from Chapter 2 as well.

Theorem 3.1.9. *If $\mathcal{F}$ is a Frobenius category satisfying conditions (C1) and (C2) from Chapter 2, the semi-derived Hall algebra $\mathcal{SDH}(\mathcal{F},\mathcal{P}(\mathcal{F}))$ is nothing but*

$$\mathcal{H}(\underline{\mathcal{F}},M_0^{add}(\mathcal{F})[[P]^{-1}|P\in\mathcal{P}(\mathcal{F})],\langle\cdot,\cdot\rangle_{\mathcal{F}}).$$

In Section 3.3, we show that, more generally, exact categories $\mathcal{E}$ with weak equivalences W whose localized categores $\mathcal{E}[\mathcal{W}^{-1}] =: \mathcal{T}$ are triangulated naturally give rise to twisted extended Hall algebras of $\mathcal{T}$.

The algebras we defined behave naturally under the change of initial data.

Theorem 3.1.10. *Assume that we have a commutative diagram*

$$\begin{array}{ccccc} G_1 & \rightarrowtail & \widetilde{\mathcal{N}}_1 & \twoheadrightarrow & K_0^{add}(\mathcal{T}) \\ \downarrow & & \downarrow & & \| \\ G_2 & \rightarrowtail & \widetilde{\mathcal{N}}_1 & \twoheadrightarrow & K_0^{add}(\mathcal{T}) \end{array} \tag{3.9}$$

and bilinear forms ϕ_1, ϕ_2 on $\widetilde{\mathcal{N}}_1$, resp. on $\widetilde{\mathcal{N}}_1$. Assume that both of these data satisfy conditions (C1) and (C2) so that the Hall algebras are defined. Than the diagram above induces a natural algebra homomorphism

$$\mathcal{H}(\mathcal{T}, \mathcal{N}_1, \phi_1)_{tw} \to \mathcal{H}(\mathcal{T}, \mathcal{N}_1, \phi_1),$$

where the twist is given by the quotient $\frac{\phi_2}{\phi_1}$. It is injective if the vertical arrows are monic and surjective if they are epic.

The theorem follows from the definitions as in Chapter 2.

3.2 Idempotented forms

It follows from condition (C2) that for any $A, B \in \mathcal{N}, g \in G, \varepsilon \in \mathrm{Ext}^1_{\mathcal{T}}(\pi(A), \pi(B))$, we have the identities

$$\phi(A+B, g) = \phi(E_{A,B,\varepsilon}, g), \qquad \phi(g, A+B) = \phi(g, E_{A,B,\varepsilon}). \tag{3.10}$$

In other words, the bilinear form ϕ descends to a form on $G \times (\widetilde{\mathcal{N}}/\sim_{\mathcal{T}})$ and on $(\widetilde{\mathcal{N}}/\sim_{\mathcal{T}}) \times G$. Assume now that the following condition is satisfied.

(C3) The restriction of ϕ on $G \times G$ is trivial.

When (C3) holds, the value of form $\phi(A, B)$ clearly depends only on $\pi(A)$ and $\pi(B)$. By identities (3.10), ϕ descends to a form $G \times K_0(\mathcal{T})$ and on $K_0(\mathcal{T}) \times G$.

Consider an arbitrary embedding $i_h : K_0(\mathcal{T}) \to H$, for some group H, such that the form ϕ extends to a bilinear form on $G \times H \to \mathbb{Q}^\times$. Let ψ be the anti-symmetrization of ϕ considered as a form on $G \times H$:

$$\psi : G \times H \to \mathbb{Q}^\times, \quad \psi(g, \lambda) = \frac{\phi(g, \lambda)}{\phi(\lambda, g)}. \tag{3.11}$$

Assume that ψ is non-degenerate. By abuse of notation, for $g \in G, B \in \mathcal{N}$, we will denote $\psi(g, i_h \circ \pi_{\mathcal{T}} \circ \pi(B))$ simply by $\psi(g, B)$.

In this setting, we can define an idempotented form of the algebra $\mathcal{H}(\mathcal{T},\mathcal{N},\phi)$. To avoid overlong notations, until the end of this section, we will write simply $\mathcal{H}$ for $\mathcal{H}(\mathcal{T},\mathcal{N},\phi)$. For $\lambda',\lambda''\in H$, we set

$$ {}_{\lambda'}\mathcal{H}_{\lambda''} = \mathcal{H}\Big/\big\langle [g]\diamond[B]=\psi(g,\lambda')[B],[B]\diamond[g]=\psi(\lambda',g)[B] \quad | \quad g\in G, B\in\mathcal{N}\big\rangle\,. \tag{3.12}$$

Let $\pi_{\lambda',\lambda''}$ be the canonical projection. By formulas (3.3), the relations are equivalent to setting simultaneously

$$[g+B]=\frac{\psi(g,\lambda')}{\phi(g,B)}[B]$$

and

$$[g+B]=\frac{\psi(g,\lambda'')}{\phi(B,g)}[B].$$

By rewriting, we find that $[B]$ and $[g+B]$ can be non-zero in the quotient if and only if $\psi(g,\lambda'-\lambda'')=\psi(g,B)$. Let $\mathcal{N}(\nu)$ be the subset of all $B\in\mathcal{N}$ whose image in $K_0(\mathcal{T})$ under the composition

$$\widetilde{\mathcal{N}} \overset{\pi}{\twoheadrightarrow} K_0^{add}(\mathcal{T}) \overset{\pi_{\mathcal{T}}}{\twoheadrightarrow} K_0(\mathcal{T})$$

is ν. Consider the direct sum decomposition

$$\mathcal{H}=\bigoplus_{\nu\in K_0(\mathcal{T})}\mathcal{H}(\nu),$$

where $\mathcal{H}(\nu)$ is a subspace with the basis $\{[B]|B\in\mathcal{N}(\nu)\}$. Since ψ is assumed to be non-degenerate, the image of $[B]\in\mathcal{H}(\nu)$ in ${}_{\lambda'}\dot{\mathcal{H}}_{\lambda''}$ is non-zero if and only if $\nu=\lambda'-\lambda''$. When this is satisfied, we have

$$\frac{\psi(g,\lambda')}{\phi(g,B)}=\frac{\psi(g,\lambda'')}{\phi(B,g)}=\frac{\phi(g,\lambda'')}{\phi(\lambda',g)}.$$

We get that

$$\begin{aligned} {}_{\lambda'}\mathcal{H}_{\lambda''} &= {}_{\lambda'}\mathcal{H}(\nu)_{\lambda''} = \\ &= \mathcal{H}(\nu)\Big/\left\langle [g+B]=\frac{\phi(g,\lambda'')}{\phi(\lambda',g)}[B] \quad | \quad g\in G, B\in\mathcal{N}(\nu)\right\rangle, \end{aligned}$$

for $\nu=\lambda'-\lambda''$. We define $\dot{\mathcal{H}}$ as the direct sum

$$\dot{\mathcal{H}}=\bigoplus_{\lambda',\lambda'',\nu}{}_{\lambda'}\mathcal{H}(\nu)_{\lambda''}.$$

It has a natural associative algebra structure inherited from that of $\mathcal{H}$. It does not generally

have the unit, but it has a family of orthogonal idempotents. Namely, for $\lambda \in H$, we set

$$1_\lambda = \pi_{\lambda,\lambda}([0]).$$

Then we have

$$1_\lambda 1_{\lambda'} = \delta_\lambda^{\lambda'} 1_\lambda$$

and

$${}_{\lambda'}\mathcal{H}_{\lambda''} = 1_{\lambda'}\dot{\mathcal{H}}1_{\lambda''}.$$

Thus, we have a decomposition

$$\dot{\mathcal{H}} = \bigoplus_{\lambda',\lambda'',\nu} 1_{\lambda'}\dot{\mathcal{H}}(\nu)1_{\lambda''}.$$

Recall that the summand $1_{\lambda'}\dot{\mathcal{H}}(\nu)1_{\lambda''}$ is zero, unless $\lambda' - \lambda'' = \nu$.

3.3 Hall algebras of exact categories with weak equivalences

From now on and until the end of the chapter, we assume that the following is satisfied :

(EW1) $\mathcal{E}$ is an essentially small, Hom −finite, idempotent complete, $\mathbb{F}_q$−linear category.

(EW2) $(\mathcal{E}, \mathcal{W})$ is an exact category with weak equivalences; either a Frobenius pair with a larger exact structure, or a hereditary exact model category. Let $\mathcal{T} := \mathcal{E}[\mathcal{W}^{-1}]$ be the localized triangulated category.

(EW3) For each $A, B \in \mathcal{E}$, there exists $N \in \mathbb{N}$ such that

$$\mathrm{Ext}^i_{\mathcal{E}}(A, B) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{T}}(A, B), \qquad \forall \quad i > N,$$

$$\dim \mathrm{Ext}^j_{\mathcal{E}}(A, B) < \infty, \dim \mathrm{Ext}^j_{\mathcal{T}}(A, B) < \infty, \qquad \forall \quad 0 \geq j \geq N.$$

3.3.1 Relative Grothendieck monoids and groups

In this section, we define some variants of monoids $\mathcal{N}$ appearing in the construction of twisted extended Hall algebras from exact categories with weak equivalences. In some sense, we would like to understand what is "the difference" between two weak equivalent objects in $\mathcal{E}$. Ideally, this would be an element of the Grothendieck group of $\mathcal{W}$, but, unfortunately, the author does not know if one can do it in the full generality. In general, we will define it as an element of a certain quotient of $K_0(\mathcal{W})$; and under certain conditions we will show that this quotient is the entire group $K_0(\mathcal{W})$. This quotient will play the role of a group G needed to define a twisted extended Hall algebra as in Section 3.1.

Recall that for an exact category we defined its Grothendieck monoid and group, and we also have their additive counterparts. Define the *localized additive Grothendieck monoid* $M_0^{add}(\mathcal{E}, \mathcal{W})$

to be the localization of $M_0^{add}(\mathcal{E})$ at classes of objects in $\mathcal{W}$. Naturally, we have an embedding

$$K_0^{add}(\mathcal{W}) \hookrightarrow M_0^{add}(\mathcal{E}, \mathcal{W}).$$

The group completion of $M_0^{add}(\mathcal{E}, \mathcal{W})$ is just $K_0^{add}(\mathcal{E})$.

Similarly, we define the *localized Grothendieck monoid* $M_0(\mathcal{E}, \mathcal{W})$ to be the localization of $M_0(\mathcal{E})$ at classes of objects in $\mathcal{W}$, its group completion is $K_0(\mathcal{E})$. This time, the monoid was not necessarily free before localization, and the homomorphism $K_0(\mathcal{W}) \to M_0(\mathcal{E}, \mathcal{W})$ is not necessarily injective.

The most important monoids for us sit somewhere between $M_0^{add}(\mathcal{E}, \mathcal{W})$ and $M_0(\mathcal{E}, \mathcal{W})$. Let us explain this in detail.

First, we define some equivalence relations in the localized additive Grothendieck monoid $M_0^{add}(\mathcal{E}, \mathcal{W})$:

(i)

$$\sim_1 \quad : \quad \langle [B] \sim [A] + [C] \quad | \quad \exists A \rightarrowtail B \twoheadrightarrow C, A \in \mathcal{W} \rangle \, ;$$

(ii)

$$\sim_2 \quad : \quad \langle [A] + [D] \sim [B] + [C] \rangle \, ,$$

whenever there exists a bicartesian commutative square

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

where all morphisms are $\mathcal{W}-$admissible and objects in one of the rows belong to $\mathcal{W}$;

(iii)

$$\sim_3 \quad : \quad \langle [B] \sim [A] + [C] \quad | \quad \exists A \rightarrowtail B \twoheadrightarrow C, C \in \mathcal{W} \rangle \, .$$

Lemma 3.3.1. *These equivalence relations coincide and give a congruence on* $M_0^{add}(\mathcal{E}, \mathcal{W})$.

The lemma is easy to proof using [7, Corollary 2.14].

We define the *relative Grothendieck monoid* $M_0'(\mathcal{E}, \mathcal{W})$ as the quotient

$$M_0^{add}(\mathcal{E}, \mathcal{W}) / \sim_1 = M_0^{add}(\mathcal{E}, \mathcal{W}) / \sim_2 = M_0^{add}(\mathcal{E}, \mathcal{W}) / \sim_3 \, .$$

We define the *relative Grothendieck group* $K_0'(\mathcal{E}, \mathcal{W})$ as its group completion. It can also be understood as the quotient of $K_0^{add}(\mathcal{E}, \mathcal{W})$ by the same congruence. We have a canonical homomorphism

$$M_0'(\mathcal{E}, \mathcal{W}) \to K_0^{add}(\mathcal{E}, \mathcal{W})$$

and we define by $K_{0,+}'(\mathcal{E}, \mathcal{W})$ its image. It is the maximal cancellative quotient of $M_0'(\mathcal{E}, \mathcal{W})$.

We do not know in which generality $M_0'(\mathcal{E},\mathcal{W})$ is cancellative itself and thus is isomorphic to $K_{0,+}'(\mathcal{E},\mathcal{W})$.

The ideal $I_{\mathcal{T}}$ in $M_0^{add}(\mathcal{T})$ of relations of the form $[G]=[F]+[\Sigma E]$, for each distinguished triangle $E\to F\to G\to \Sigma E$ in $\mathcal{T}$, has the following natural lift to $M_0'(\mathcal{E},\mathcal{W})$. We define a congruence $\widetilde{I}_{\mathcal{T}}$ of $M_0'(\mathcal{E},\mathcal{W})$, generated by relations of the form $[C]\sim_{\mathcal{T}}[F]+[H]$ for each commutative diagram of the form

$$\begin{tikzcd}
E \arrow[r, tail] \arrow[d, tail] & F \arrow[d, tail] \\
W \arrow[r, tail] \arrow[d, two heads] & C \arrow[d, two heads] \\
H \arrow[r, equal] & H
\end{tikzcd}$$

where the top square is bicartesian and $W\in\mathcal{W}$. We can also consider $\widetilde{I}_{\mathcal{T}}$ as an ideal of relations in $K_0^{add}(\mathcal{E},\mathcal{W})$.

Proposition 3.3.2. *The quotient $M_0'(\mathcal{E},\mathcal{W})/\widetilde{I}_{\mathcal{T}}$ is isomorphic to $M_0(\mathcal{E},\mathcal{W})$. Similarly, we have an isomorphism $K_0'(\mathcal{E},\mathcal{W})/\widetilde{I}_{\mathcal{T}}=K_0(\mathcal{E})$.*

Proof. Consider the quotient $M_0^{add}(\mathcal{E},\mathcal{W})/\widetilde{I}_{\mathcal{T}}$. For each conflation $E\rightarrowtail W\twoheadrightarrow H$, with $W\in\mathcal{E}^{\mathcal{W}}$, we have the diagram

$$\begin{tikzcd}
E \arrow[r, equal] \arrow[d, tail] & F \arrow[d, tail] \\
W \arrow[r, equal] \arrow[d, two heads] & W \arrow[d, two heads] \\
H \arrow[r, equal] & H
\end{tikzcd}$$

Therefore, in the quotient, we have $[W]\sim_{\mathcal{T}}[E]+[H]$. Let $E\rightarrowtail F\twoheadrightarrow G$ be a conflation in $\mathcal{E}$. It fits into a commutative diagram of the form

$$\begin{tikzcd}
E \arrow[r, tail] \arrow[d, tail] & F \arrow[r, two heads] \arrow[d, tail] & G \arrow[d, equal] \\
W \arrow[r, tail] \arrow[d, two heads] & C \arrow[r, two heads] \arrow[d, two heads] & G \\
H \arrow[r, equal] & H & \\
\end{tikzcd}$$

We have the congruences

$$\begin{aligned}[F]+[W]\sim_{\mathcal{T}}[F]+([E]+[H])\sim_{\mathcal{T}}([F]+[H])+[E]\sim_{\mathcal{T}}[C]+[E]\sim_{\mathcal{T}}\\ \sim_{\mathcal{T}}([W]+[G])+[E]\sim_{\mathcal{T}}([E]+[G])+[W]\Leftrightarrow\\ [F]\sim_{\mathcal{T}}[E]+[G].\end{aligned}\tag{3.13}$$

Thus, the ideal $I_{\mathcal{E}}$ is generated by relations and relations in $\widetilde{I}_{\mathcal{T}}$. By definition, all the relations in I_1 and in $\widetilde{I}_{\mathcal{T}}$ belong to $I_{\mathcal{E}}$. This implies the desired isomorphism of quotients of $M_0^{add}(\mathcal{E},\mathcal{W})$ and of $K_0(\mathcal{E})$. $\square$

Proposition 3.3.3. *We have an exact sequence*

$$0\to\ker(i_w)\to K_0(\mathcal{W})\overset{i_w}{\to}K_0'(\mathcal{E},\mathcal{W})\overset{\pi_w}{\to}K_0^{add}(\mathcal{T})\to 0.\tag{3.14}$$

Proof. The homomorphism i_w is induced by the inclusion of the subcategory $\mathcal{W}$ into $\mathcal{E}$. The homomorpism $K_0'(\mathcal{E},\mathcal{W})\overset{\pi_w}{\to}K_0^{add}(\mathcal{T})$ is also natural, and its kernel clearly coincides with the image of i_w. $\square$

Denote by $K_0'(\mathcal{W})$ the image of i_w. Since $K_0^{add}(\mathcal{T})$ is a free abelian group, we actually get a direct sum decomposition :

Corollary 3.3.4.

$$K_0'(\mathcal{E},\mathcal{W})\cong K_0'(\mathcal{W})\oplus K_0^{add}(\mathcal{T}).\tag{3.15}$$

Indeed, any set-theoretic section of the projection π_w is a group homomorphism.

3.3.2 Euler forms and numerical quotients

By assumption (EW3), we can define the following bilinear form $\langle\cdot,\cdot\rangle_{\mathcal{E}}^{\mathcal{T}_{\geq 0}}$ on objects in $\mathcal{E}$: for$A,B\in\mathcal{E}$, we define it to be

$$\langle A,B\rangle_{\mathcal{E}}^{\mathcal{T}_{\geq 0}}=\prod_{i=0}^{\infty}\left(\frac{|\operatorname{Ext}_{\mathcal{T}}^i(E_1,F)|}{|\operatorname{Ext}_{\mathcal{E}}^i(E_1,F)|}\right)^{(-1)^i}.$$

The product has only finitely many terms different from 1.

This form clearly descends on the relative Grothendieck group $K_0'(\mathcal{E},\mathcal{W})$, but it does not descend on $K_0(\mathcal{E})$. When one of the terms is weakly trivial, the form is simply the Euler form of the category $\mathcal{E}$, defined on $K_0(\mathcal{W})\times K_0'(\mathcal{E},\mathcal{W})$ and on $K_0'(\mathcal{E},\mathcal{W})\times K_0(\mathcal{W})$.

Let $K_0^{num}(\mathcal{W})$ be the quotient by left and right kernels of the form on $K_0(\mathcal{W})\times K_0(\mathcal{W})$. By using the calculus of fractions, it is easy to check that it the same as the quotient by left, resp. right kernels of the form on $K_0(\mathcal{W})\times K_0'(\mathcal{E},\mathcal{W})$ and on $K_0'(\mathcal{E},\mathcal{W})\times K_0(\mathcal{W})$.

Let $K_0^{num}(\mathcal{E},\mathcal{W})$ be the quotient of $K_0'(\mathcal{E},\mathcal{W})$ by the following congruence : $[A] \sim_{num} [B]$ if $\pi(A) = \pi(B)$ and

$$\langle A, ?\rangle_{\mathcal{E}}^{\mathcal{T}_{\geq 0}} = \langle B, ?\rangle_{\mathcal{E}}^{\mathcal{T}_{\geq 0}}, \langle ?, A\rangle_{\mathcal{E}}^{\mathcal{T}_{\geq 0}} = \langle ?, B\rangle_{\mathcal{E}}^{\mathcal{T}_{\geq 0}}.$$

Monoids $M_0^{num}(\mathcal{E},\mathcal{W})$ and $K_{0,+}^{num}(\mathcal{E},\mathcal{W})$ are defined similarly.

Proposition 3.3.5. *The Euler form $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ vanishes on $\ker(i_w) \times K_0'(\mathcal{E},\mathcal{W})$ and on $K_0'(\mathcal{E},\mathcal{W}) \times \ker(i_w)$.*

Proof. It is enough to show that for any $C, D, E \in \mathcal{W}$, if there exist conflations in $\mathcal{E}$ $A \hookrightarrow B \twoheadrightarrow C$ and $A \hookrightarrow B \twoheadrightarrow D$, for some $A, B \in \mathcal{E}$, then $\langle C, E\rangle_{\mathcal{E}} = \langle D, E\rangle_{\mathcal{E}}$ and $\langle E, C\rangle_{\mathcal{E}} = \langle E, D\rangle_{\mathcal{E}}$. Let us prove the first of these identities. By our assumption, there exist N, such that for any $n \geq N$, we have $\mathrm{Ext}^n_{\mathcal{E}}(C, E) = \mathrm{Ext}^n_{\mathcal{D}}(C, E) = 0$ and $\mathrm{Ext}^n_{\mathcal{E}}(D, E) = \mathrm{Ext}^n_{\mathcal{D}}(D, E) = 0$. By taking long exact sequences, we observe that this means that for any $n > N$, we have $\mathrm{Ext}^n_{\mathcal{D}}(A, E) = \mathrm{Ext}^n_{\mathcal{E}}(B, E)$. Therefore, the Euler forms both equal the following alternate product :

$$\langle C, E\rangle_{\mathcal{E}} = \prod_{0\leq i\leq N} \left(\frac{|\mathrm{Ext}^i_{\mathcal{E}}(B, E)|}{|\mathrm{Ext}^i_{\mathcal{E}}(A, E)|}\right)^{(-1)^i} = \langle D, E\rangle_{\mathcal{E}}. \tag{3.16}$$

The second identity is proved in a similar way. □

Corollary 3.3.6. *We have a commutative diagram*

$$\begin{tikzcd} K_0'(\mathcal{W}) \arrow[r, tail] \arrow[d, two heads] & K_0'(\mathcal{E},\mathcal{W}) \arrow[r, two heads] \arrow[d, two heads] & K_0^{add}(\mathcal{T}) \arrow[d, equal] \\ K_0^{num}(\mathcal{W}) \arrow[r, tail] & K_0^{num}(\mathcal{E},\mathcal{W}) \arrow[r, two heads] & K_0^{add}(\mathcal{T}) \end{tikzcd} \tag{3.17}$$

3.3.3 Middle terms

Let us fix $A, B \in \mathrm{Iso}(\mathcal{E})$. An element ε of the group $\mathrm{Ext}^1_{\mathcal{T}}(A, B)$ is an equivalence class of diagrams

$$\begin{tikzcd} B \arrow[r, two heads] & E' \arrow[r, two heads] & A' \arrow[d, two heads] \\ & & A, \end{tikzcd} \tag{3.18}$$

see Chapter 1. In the case of a Frobenius pair with larger exact structure, we can find a diagram where the horisontal row is a conflation in $\mathcal{F}$; the vertical morphism is a weak equivalence and a deflation in $\mathcal{F}$.

Let K' be the kernel of the vertical deflation. Denote by $E_{A,B,\varepsilon}$ the element $[E'] - [K']$ of the (effective) relative Grothendieck monoid.

Lemma 3.3.7. *The element $E_{A,B,\varepsilon}$ of $K'_{0,+}(\mathcal{E}, \mathcal{W})$ is well-defined. In other words, it does not depend on the choice of the diagram (4.18) in its equivalence class.*

Proof. Assume that we have two diagrams representing the same element $E_{A,B,\varepsilon}$: (4.18) and

$$\begin{tikzcd} & & K'' \arrow[d, two heads] \\ B \arrow[r, tail] & E'' \arrow[r, two heads] & A'' \arrow[d, two heads] \\ & & A. \end{tikzcd} \tag{3.19}$$

The calculus of fractions implies that we have another diagram representing $E_{A,B,\varepsilon}$

$$\begin{tikzcd} & & \widetilde{K} \arrow[d, two heads] \\ B \arrow[r, tail] & \widetilde{E} \arrow[r, two heads] & \widetilde{A} \arrow[d, two heads] \\ & & A \end{tikzcd} \tag{3.20}$$

that fits into the following diagrams :

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] & N' \arrow[r, equal] \arrow[d, tail] & N' \arrow[d, two heads] \\ B \arrow[r, tail] \arrow[d, equal] & \widetilde{E} \arrow[r, two heads] \arrow[d, two heads] & \widetilde{A} \arrow[d, two heads] \\ B \arrow[r, tail] & E' \arrow[r, two heads] & A'; \end{tikzcd} \tag{3.21}$$

$$\begin{tikzcd} 0 \arrow[r, tail] \arrow[d, tail] & N'' \arrow[r, equal] \arrow[d, tail] & N'' \arrow[d, two heads] \\ B \arrow[r, tail] \arrow[d, equal] & \widetilde{E} \arrow[r, two heads] \arrow[d, two heads] & \widetilde{A} \arrow[d, two heads] \\ B \arrow[r, tail] & E'' \arrow[r, two heads] & A''. \end{tikzcd} \tag{3.22}$$

Then, in the relative Grothendieck group, we have

$$[E'] - [K'] = [\widetilde{E}] - [N'] - [K'] = [\widetilde{E}] - ([\widetilde{A}] - [A']) + ([A'] - [A]) =$$

$$= [\widetilde{E}] - [\widetilde{A}] + [A] = [\widetilde{E}] - ([\widetilde{A}] - [A'']) + ([A''] - [A]) = [\widetilde{E}] - [N''] - [K''] = [E''] - [K''].$$

□

By using the calculus of fractions and the dual diagrams (with weak equivalences in B instead of A), we can naturally generalize this result :

Lemma 3.3.8. *Let $B' \rightarrowtail E' \twoheadrightarrow A'$ be any conflation in $\mathcal{E}$ representing ε. Then*

$$E_{A,B,\varepsilon} = E' + (A - A') + (B - B') \tag{3.23}$$

in $K'_{0,+}(\mathcal{E}, \mathcal{W})$.

We can alternatively take the right hand side of the identity3.23 as the definition of $E_{A,B,\varepsilon}$.

The subgroup $\mathrm{Ext}^1_{\mathcal{T}}(A, B)_\alpha$ consists of all the equivalence classes of diagrams 4.18 with E belonging to the weak equivalence class α. Denote by $E_{A,B,\alpha}$ the image of $E_{A,B,\varepsilon}$, for $\varepsilon \in \mathrm{Ext}^1_{\mathcal{T}}(A, B)_\alpha$, in the (numerical) effective relative Grothendieck monoid $K'_{0,+}(\mathcal{E}, \mathcal{W})$ and in $K^{num}_{0,+}(\mathcal{E}, \mathcal{W})$.

Lemma 3.3.9. *If the natural map $K_0(\mathcal{W}) \to K_0(\mathcal{E})$ is injective, the element $E_{A,B,\alpha}$ is well-defined. In other words, it does not depend on the class $\varepsilon \in \mathrm{Ext}^1_{\mathcal{T}}(A, B)_\alpha$.*

Proof. Let $\varepsilon_1, \varepsilon_2 \in \mathrm{Ext}^1_{\mathcal{T}}(A, B)_\alpha$. By using the calculus of fractions if necessary, we can represent them by the conflations

$$B' \rightarrowtail E_1 \twoheadrightarrow A' \tag{3.24}$$

and

$$B' \rightarrowtail E_2 \twoheadrightarrow A' \tag{3.25}$$

with the same end objects. The classes of E_1 and E_2 in $K_0(\mathcal{E})$ coincide, thus, their difference vanishes in $K_0(\mathcal{W})$. Then it vanishes in $K^{num}_0(\mathcal{W})$ as well and so $[E_1] = [E_2]$ both in $K'_{0,+}(\mathcal{E}, \mathcal{W})$ and in $K^{num}_{0,+}(\mathcal{E}, \mathcal{W})$. □

For $C \in K'_{0,+}(\mathcal{E}, \mathcal{W})$, denote by $\mathrm{Ext}^1_{\mathcal{T}}(A, B)_C$ the subset of $\mathrm{Ext}^1_{\mathcal{T}}(A, B)$ consisting of all the extensions ε with $E_{A,B,\varepsilon} = C$. For $D \in K^{num}_{0,+}(\mathcal{E}, \mathcal{W})$, denote by $\mathrm{Ext}^1_{\mathcal{T}}(A, B)_D$ the subgroup of

$\mathrm{Ext}^1_{\mathcal{T}}(A,B)$ consisting of all the extensions ε with the image $E_{A,B,\varepsilon}$ in $K^{num}_{0,+}(\mathcal{E},\mathcal{W})$ being equal to D.

3.3.4 Hall algebras $\mathcal{H}(\mathcal{E},\mathcal{W})$ and $\mathcal{H}^{num}(\mathcal{E},\mathcal{W})$

Define the quantum torus $\mathbb{T}'(\mathcal{W})$ of the exact category $\mathcal{W}$ as the $\mathbb{Q}-$group algebra of $K'_0(\mathcal{W})$, with the multiplication twisted by the inverse of the Euler form $\langle\cdot,\cdot\rangle_{\mathcal{E}}$. Consider the $\mathbb{Q}-$vector space with basis parametrized by the elements of $K'_{0,+}(\mathcal{E},\mathcal{W})$. Define on this space a structure of a bimodule $\mathcal{M}(\mathcal{E},\mathcal{W})$ over $\mathbb{T}'(\mathcal{W})$ by the rule

$$[K]\diamond[M] := \frac{1}{\langle K,M\rangle}[K\oplus M], \qquad [M]\diamond[K] := \frac{1}{\langle M,K\rangle}[M\oplus K]$$

for $K\in\mathcal{W}, M\in\mathcal{E}$.

Definition 3.3.10. We endow $\mathcal{M}(\mathcal{E},\mathcal{W})$ with the following multiplication : the product of the classes of two objects $A,B\in\mathrm{Iso}(\mathcal{E})$ is defined as

$$\begin{aligned}[A]\diamond[B] &= \sum_{\varepsilon\in\mathrm{Ext}^1_{\mathcal{T}}(A,B)}\left(\frac{1}{|\mathrm{Hom}_{\mathcal{E}}(A,B)|}\prod_{i=1}^{\infty}\left(\frac{|\mathrm{Ext}^i_{\mathcal{T}}(A,B)|}{|\mathrm{Ext}^i_{\mathcal{E}}(A,B)|}\right)^{(-1)^i}\right)[E_{A,B,\varepsilon}]\\ &= \sum_{C\in K'_{0,+}(\mathcal{E},\mathcal{W})}\left(\frac{|\mathrm{Ext}^1_{\mathcal{T}}(A,B)_C|}{|\mathrm{Hom}_{\mathcal{E}}(A,B)|}\prod_{i=1}^{\infty}\left(\frac{|\mathrm{Ext}^i_{\mathcal{T}}(A,B)|}{|\mathrm{Ext}^i_{\mathcal{E}}(A,B)|}\right)^{(-1)^i}\right)[C].\end{aligned} \tag{3.26}$$

We call the resulting algebra the Hall algebra of $(\mathcal{E},\mathcal{W})$ and denote it by $\mathcal{H}(\mathcal{E},\mathcal{W})$.

Theorem 3.3.11. *The algebra $\mathcal{H}(\mathcal{E},\mathcal{W})$ is a twisted extended Hall algebra of the category $\mathcal{E}[\mathcal{W}^{-1}]$.*

Proof. The triple $(\mathcal{T},K'_{0,+}(\mathcal{E},\mathcal{W}),\langle,\rangle^{\mathcal{T}_{\geq 0}}_{\mathcal{E}})$ satisfies the conditions from Section 3.1. Then $\mathcal{H}(\mathcal{E},\mathcal{W})$ is nothing but the algebra $\mathcal{H}(\mathcal{E}[\mathcal{W}^{-1},K'_{0,+}(\mathcal{E},\mathcal{W}),\langle,\rangle^{\mathcal{T}_{\geq 0}}_{\mathcal{E}})$. $\square$

Corollary 3.3.12. *(1) $\mathcal{M}(\mathcal{E},\mathcal{W})$ is free as a left and as a right module. Each choice of representative in $K'_{0,+}(\mathcal{E},\mathcal{W})$ of the isomorphism classes of objects in the category $\mathcal{T}$ yields a basis of this module.*

(2) The algebra $\mathcal{H}(\mathcal{E},\mathcal{W})$ is unitary and associative.

(3) $\mathcal{H}(\mathcal{E},\mathcal{W})$ with an appropriately twisted multiplication is a flat deformation of the derived Hall algebra of $\mathcal{E}[\mathcal{W}^{-1}]$ over the group algebra of $K'_0(\mathcal{W})$, when the derived Hall algebra is well-defined.

The twist is given by the form $\langle,\rangle^{\mathcal{E}}_{\mathcal{T}} := \frac{\langle,\rangle_{\mathcal{T}_{<0}}}{\langle,\rangle^{\mathcal{T}_{\geq 0}}_{\mathcal{E}}}$.

Proposition 3.3.13. *The algebra $\mathcal{H}(\mathcal{E},\mathcal{W})$ is naturally graded by the group $K_0(\mathcal{E})$ and by the monoid $K_{0,+}(\mathcal{E},\mathcal{W})$.*

Proof. The formula for the multiplication implies that the algebra is graded by the quotient of $K_0^{add}(\mathcal{E})$ by the relations of the form $[A] - [E_{A,B,\varepsilon}] + [B]$, for $\varepsilon \in \operatorname{Ext}^1_{\mathcal{T}}(A, B)$. By the definition of $E_{A,B,\varepsilon}$, these are precisely the generating relations of the ideal $\widetilde{I}_{\mathcal{T}}$. By Proposition 3.3.2, $K_0^{add}(\mathcal{E})/\widetilde{I}_{\mathcal{T}} = K_0(\mathcal{E})$. $\square$

We can define the numerical quotient of $\mathcal{H}(\mathcal{E}, \mathcal{W})$. Let $\mathbb{T}^{num}(\mathcal{W})$ be the group algebra of $K_{(}^{num}\mathcal{W})$ twisted by the inverse of the Euler form, $\mathcal{M}^{num}(\mathcal{E}, \mathcal{W})$ be the bimodule over $\mathbb{T}^{num}(\mathcal{W})$ with the underlying vector space having a basis parametrized by the elements of $K_{0,+}^{num}(\mathcal{E}, \mathcal{W})$ and the action defined by the same formulas as above. We define the numerical Hall algebra $\mathcal{H}^{num}(\mathcal{E}, \mathcal{W})$ as this vector space equipped with the multiplication defined by the following rule :

$$[A] \diamond [B] = \sum_{C \in K_{0,+}^{num}(\mathcal{E},\mathcal{W})} \left(\frac{|\operatorname{Ext}^1_{\mathcal{T}}(A, B)_C|}{|\operatorname{Hom}_{\mathcal{E}}(A, B)|} \prod_{i=1}^{\infty} \left(\frac{|\operatorname{Ext}^i_{\mathcal{T}}(A, B)|}{|\operatorname{Ext}^i_{\mathcal{E}}(A, B)|} \right)^{(-1)^i} \right) [C] \tag{3.27}$$

Clearly, this is a quotient of $\mathcal{H}(\mathcal{E}, \mathcal{W})$ and the statements of Theorem 3.3.11 and Corollary 3.3.12 hold for this algebra, up to replacement of $K'_{0,+}(\mathcal{E}, \mathcal{W})$ by $K_{0,+}^{num}(\mathcal{E}, \mathcal{W})$.

If the natural inclusion $K_0(\mathcal{W}) \to K_0(\mathcal{E})$ is injective, we can rewrite the multiplication as follows :

$$[A] \diamond [B] = \sum_{\alpha \in \operatorname{Iso} \mathcal{T}} \left(\frac{|\operatorname{Ext}^1_{\mathcal{T}}(A, B)_\alpha|}{|\operatorname{Hom}_{\mathcal{E}}(A, B)|} \prod_{i=1}^{\infty} \left(\frac{|\operatorname{Ext}^i_{\mathcal{T}}(A, B)|}{|\operatorname{Ext}^i_{\mathcal{E}}(A, B)|} \right)^{(-1)^i} \right) [E_{A,B,\alpha}]. \tag{3.28}$$

3.3.5 Replacements and relation to $\mathcal{H}(\mathcal{E})$

The following follows directly from the multiplication formula in $\mathcal{H}(\mathcal{E})$.

Proposition 3.3.14. *Let $A, B \in \mathcal{E}$ be such that the natural map $\operatorname{Ext}^i_{\mathcal{E}}(A, B) \to \operatorname{Ext}^i_{\mathcal{T}}(A, B)$ is surjective for $i = 1$ and an isomorphism for all $i > 1$. Then their product in $\mathcal{H}(\mathcal{E}, \mathcal{W})$ is given by the same formula as the usual Hall product in $\mathcal{H}(\mathcal{E})$:*

$$[A] \diamond [B] = \sum_{C \in K'_{0,+}(\mathcal{E},\mathcal{W})} \frac{\operatorname{Ext}^1_{\mathcal{E}}(A, B)_C|}{|\operatorname{Hom}_{\mathcal{E}}(A, B)|} [C]. \tag{3.29}$$

Corollary 3.3.15. *Assume that for all $A, B \in \mathcal{E}$, the natural map $\operatorname{Ext}^i_{\mathcal{E}}(A, B) \to \operatorname{Ext}^i_{\mathcal{T}}(A, B)$ is surjective for $i = 1$ and an isomorphism for all $i > 1$. Then the algebra $\mathcal{H}(\mathcal{E}, \mathcal{W})$ is isomorpic the quotient of the localization of the Hall algebra of the exact category $\mathcal{E}$:*

$$I \rightarrowtail \mathcal{H}(\mathcal{E})[[W]^{-1} \quad | \quad W \in \mathcal{W}] \overset{p}{\twoheadrightarrow} \mathcal{H}(\mathcal{E}, \mathcal{W}), \tag{3.30}$$

where I is the ideal defined by either of the congruences $\sim_1, \sim_2, \sim_3$ from Section 3.3.1.

Corollary 3.3.15 follows immediately from Propostion 3.3.14. The only thing one needs to

check is that the localization is well-defined in the sense that the set $\{[W] \quad | \quad W \in \mathcal{W}\}$ satisfies the Ore conditions. This follows from the multiplication formula.

The map p can be understood as a relative variant of the integration map defined by Reineke [63]. Namely, he considers a map from the Hall algebra of an abelian category $\mathcal{A}$ to the Grothendieck group $K_0(\mathcal{A})$ (actually, to the Grothendieck monoid) that can be understood as a composition of 2 maps. The first is an automorphism of $\mathcal{H}(\mathcal{A})$ that sends the algebra to the dual of the coalgebra discussed in Chapter 1. This can be understood as forgetting the automorphism groups of objects. The second sends the class of an object to its class in the Grothendieck group. For hereditary abelian categories, the second map, and thus the composition, is an algebra homomorphism. The map p is a version of the second map, sending a module over $\mathcal{H}(\mathcal{W})$ to a module over $\mathbb{Q}K_0'(\mathcal{W})$. Under assumption of Corollary 3.3.15, the category $\mathcal{W}$ is hereditary and the corollary means that the naturally defined map from the left hand side to the right hand side is an algebra homomorphism.

Proposition 3.3.14 can be generalized as follows.

Proposition 3.3.16. *Assume that we have decompositions* $[A] = [\widetilde{A}] + [A^w], [B] = [\widetilde{B}] + [B^w]$ *in* $K_0(\mathcal{E}, \mathcal{W})$, *where* $[A^w], [B^w] \in K_0'(\mathcal{W})$ *and the natural map* $\mathrm{Ext}^i_{\mathcal{E}}(\widetilde{A}, \widetilde{B}) \to \mathrm{Ext}^i_{\mathcal{T}}(\widetilde{A}, \widetilde{B})$ *is surjective for* $i = 1$ *and an isomorphism for all* $i > 1$. *Then we have the following :*

$$[A] \diamond [B] = \frac{\left\langle A^w, \widetilde{A} \right\rangle_{\mathcal{E}} \left\langle B^w, \widetilde{B} \right\rangle_{\mathcal{E}} \left\langle B^w, \widetilde{A} \right\rangle_{\mathcal{E}}}{\left\langle \widetilde{A}, B^w \right\rangle_{\mathcal{E}} \langle A^w, B^w \rangle_{\mathcal{E}}} [A^w + B^w] \sum_{\widetilde{C} \in K_{0,+}'(\mathcal{E}, \mathcal{W})} \frac{\mathrm{Ext}^1_{\mathcal{E}}(\widetilde{A}, \widetilde{B})_C|}{|\mathrm{Hom}_{\mathcal{E}}(\widetilde{A}, \widetilde{B})|} [\widetilde{C}] = \tag{3.31}$$

$$= \frac{1}{\left\langle \widetilde{A}, \widetilde{B} \right\rangle_{\mathcal{E}} \left\langle \widetilde{A}, B \right\rangle_{\mathcal{E}} \left\langle A, \widetilde{B} \right\rangle_{\mathcal{E}}} \sum_{\widetilde{C} \in K_{0,+}'(\mathcal{E}, \mathcal{W})} \frac{\mathrm{Ext}^1_{\mathcal{E}}(\widetilde{A}, \widetilde{B})_C|}{|\mathrm{Hom}_{\mathcal{E}}(\widetilde{A}, \widetilde{B})|} [A^w + B^w + \widetilde{C}] \tag{3.32}$$

Assume that each pair of objects $A, B \in \mathcal{E}$ admits such a pair of decompositions. Then we can define the $\mathcal{H}(\mathcal{E}, \mathcal{W})$ in an alternative way. First, we endow the underlying vector space with the structure of a $\mathbb{Q}K_0'(\mathcal{W})$-bimodule. As before, it will be free both as the right and the left module. Then we define the product $\widetilde{A} \diamond \widetilde{B}$ for pairs $(\widetilde{A}, \widetilde{B})$ satisfying conditions in Proposition 3.3.16 to be given by the same formula as in $\mathcal{H}(\mathcal{E})$. Then we define $[A] \diamond [B]$ as the right hand side of the equation 3.32.

In this case, we have an alternative proof of the associativity of $\mathcal{H}(\mathcal{E}, \mathcal{W})$. It follows from the associativity of $\mathcal{H}(\mathcal{E})$. Recall that the proof presented earleir was a modification of the proof of associativity of the derived Hall algebra of $\mathcal{T}$ instead. We do not give this alternative proof in the full generality here. In the case of categories of complexes with quasi-isomorphisms as the weak equivalences, with only certain decompositions (of the first factor) allowed, this alternatitive construction was given in [23] and we present it Chapter 4.

3.3.6 Hereditary exact model categories

When the localization comes from a hereditary exact model category, we can take bifibrant replacements of A and B as $\widetilde{A}, \widetilde{B}$ in Proposition 3.3.16.

Let $\mathcal{M}$ be a hereditary exact model structure on an exact category $\mathcal{E}$ with $\mathcal{W}$ the full subcategory of weakly trivial objects. Let $\mathcal{T} := \mathrm{Ho}(\mathcal{M})$ be the homotopy category.

Proposition 3.3.17. *The homomorphism $K_0(\mathcal{W}) \xrightarrow{i_w} K'_{0,+}(\mathcal{E}, \mathcal{W})$ is injective.*

We do not give the proof here; instead we present the proof in the case of a special projective model structure from [23] in Chapter 4. It generalizes naturally by using Becker's resolution lemma [3].

It follows that $K'_0(\mathcal{W})$ in this case is simply $K_0(\mathcal{W})$. By using bifibrant replacements, we can relate the Hall algebra of the model category to the semi-derived Hall algebra of the (Frobenius) category of bifibrant objects defined in Chapter 2.

Proposition 3.3.18. *If the inclusion $K_0(\mathcal{C} \cap \mathcal{W} \cap \mathcal{F}) \rightarrowtail K_0(\mathcal{E})$ is an isomorphism, the fully faithful embedding of categories $\mathcal{C} \cap \mathcal{F} \to \mathcal{E}$ induces an isomorphism*

$$\mathcal{SDH}(\mathcal{C} \cap \mathcal{F}, \mathcal{C} \cap \mathcal{W} \cap \mathcal{F}) \xrightarrow{\sim} \mathcal{H}(\mathcal{T}, K'_{0,+}(\mathcal{E}, \mathcal{W}), \langle -, - \rangle_{\mathcal{E}}^{\mathcal{T}_{\geq 0}}). \tag{3.33}$$

This is a direct corollary of Proposition 3.3.16.

3.3.7 Functoriality and heart property

As in Chapter 2, the algebras we defined are functorial. Namely, we have the following corollaries of Theorem 3.1.10.

Corollary 3.3.19. *Let $\mathcal{E}_1 \to \mathcal{E}_2$ be an exact functor, preserving weak equivalences and inducing an embedding of the localized categories. Then we have an embedding*

$$\mathcal{H}(\mathcal{T}_1, K'_{0,+}(\mathcal{E}_1, \mathcal{W}_1), \langle -, - \rangle_{\mathcal{E}_1}^{\mathcal{T}_{1,\geq 0}})_{tw} \hookrightarrow \mathcal{H}(\mathcal{T}, K'_{0,+}(\mathcal{E}_2, \mathcal{W}_2), \langle -, - \rangle_{\mathcal{E}_2}^{\mathcal{T}_{2,\geq 0}}),$$

where the twist is given by the quotient of the forms.

The following is a version of the heart property of derived Hall algebras in our setting.

Corollary 3.3.20. *Let $\mathcal{A}$ be a heart of a bounded $t-$structure on a category $\mathcal{E}[\mathcal{W}^{-1}] =: \mathcal{T}$. Let $\widetilde{\mathcal{A}}$ be the full exact subcategory of $\mathcal{E}$ consisting of all objects whose image under the localization belongs to $\mathcal{A}$. Then we have an embedding*

$$\mathcal{H}(\widetilde{\mathcal{A}}) \hookrightarrow \mathcal{H}(\mathcal{E}, \mathcal{W}),$$

and the algebra $\mathcal{H}_{tw}(\widetilde{\mathcal{A}})$ is a flat deformation of $\mathcal{H}(\mathcal{A})$ over $\mathbb{Q}K_0^{num}(\mathcal{W})$, where the twist is given by the form

$$\frac{\langle -,- \rangle_{\widetilde{\mathcal{A}}}}{\langle -,- \rangle_{\mathcal{A}}} = \langle -,- \rangle_{\mathcal{T}}^{\mathcal{E}}$$

More generally, we have the following result.

Corollary 3.3.21. *Let $\mathcal{E}'$ be a fully exact subcategory of $\mathcal{E}$ such that $\mathcal{E}'[\mathcal{W}^{-1}]$ is an extension closed subcategory of $\mathcal{E}[\mathcal{W}^{-1}]$. Then we have an embedding*

$$\mathcal{H}(\mathcal{E}', \mathcal{W}) \hookrightarrow \mathcal{H}(\mathcal{E}, \mathcal{W}),$$

and the algebra $\mathcal{H}_{tw}(\mathcal{E}', \mathcal{W})$ is a flat deformation of $\mathcal{H}(\mathcal{E}'[\mathcal{W}^{-1}])$ over $(\mathbb{Q}K_0^{num}(\mathcal{E}' \cap \mathcal{W})$, where the twist is given by the form

$$\langle -,- \rangle_{\mathcal{E}'}^{\mathcal{E}'[\mathcal{W}^{-1}]} = \langle -,- \rangle_{\mathcal{E}}^{\mathcal{T}} .$$

Chapitre 4

Bounded and periodic complexes

In this chapter, we discuss twisted extended Hall algebras of bounded and periodic derived categories. We begin by establishing some properties of periodic complexes in Section 4.1.

In Section 4.2, we define the semi-derived Hall algebras of exact categories and study their properties. These are twisted extended Hall algebras of bounded derived categories. The multiplication is defined via replacements of the first factor, see Section 3.3.16. The proof of the associativity thus uses the associativity of the (exact) categories of bounded complexes and not the associativity of the (triangulated) derived categories.

In Section 4.3, we describe the similar construction for 2-periodic derived categories. It is defined for exact categories $\mathcal{E}$ where every object has a finite projective resolution. We find a subcategory of $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})$ admitting a hereditary projective model structure whose homotopy category is $\mathcal{D}_{\mathbb{Z}/2}(\mathcal{E})$. Then we construct an algebra by methods of Section 3.3.6 and show that we recover the algebra defined by Bridgeland in order to categorify entire quantum group. The material of Sections 4.2 and 4.3 follows [23]. This is the original approach of the author to the problem of the derived invariance of Bridgeland's algebras. It can be generalized to the case of $m-$periodic categories in a straightforward way.

In Section 4.4, we come back to the approach of Section 3.2 and give a more general construction without using replacements and model structures. By the results of Chapter 3, it has the same nice properties. It also generalizes the modified Hall algebras of Lu and Peng [49].

4.1 Periodic complexes

In this section, we discuss in details *2-periodic*, or $\mathbb{Z}/2-$*graded complexes*. All of the results generalize word for word to the $m-$periodic complexes, for any $m \geq 2$. Since the $2-$periodic case is the most important in the framework of quantum groups and Drinfeld doubles, we write down explicit formulas only in this case.

4.1.1 $\mathbb{Z}/2-$graded complexes

Let $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})$ be the exact category of $\mathbb{Z}/2$-graded complexes over $\mathcal{E}$. Namely, an object M of this category is a diagram with objects and morphisms in $\mathcal{E}$:

$$M^0 \underset{d^1}{\overset{d^0}{\rightleftarrows}} M^1 \ , \quad d^1 \circ d^0 = d^0 \circ d^1 = 0.$$

All indices of components of $\mathbb{Z}/2-$graded objects will be understood modulo 2. A *morphism* $s : M \to N$ is a diagram

$$\begin{array}{ccc} M^0 & \underset{d^1}{\overset{d^0}{\rightleftarrows}} & M^1 \\ {\scriptstyle s^0}\downarrow & & \downarrow{\scriptstyle s^1} \\ N^0 & \underset{d'^1}{\overset{d'^0}{\rightleftarrows}} & N^1 \end{array}$$

with $s^{i+1} \circ d^i = d'^i \circ s^i$. Two morphisms $s, t : M \to N$ are *homotopic* if there are morphisms $h^i : M^i \to N^{i+1}$ such that

$$t^i - s^i = d'^{i+1} \circ h^i + h^{i+1} \circ d^i.$$

Denote by $\mathcal{K}_{\mathbb{Z}/2}(\mathcal{E})$ the category obtained from $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})$ by identifying homotopic morphisms. Let us also denote by $\mathcal{D}_{\mathbb{Z}/2}(\mathcal{E})$ the $\mathbb{Z}/2-$graded derived category, i.e. the localization of $\mathcal{K}_{\mathbb{Z}/2}(\mathcal{E})$ with respect to all quasi-isomorphisms.

The shift functor Σ of complexes induces involutions

$$\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E}) \overset{*}{\longleftrightarrow} \mathcal{C}_{\mathbb{Z}/2}(\mathcal{E}), \quad \mathcal{K}_{\mathbb{Z}/2}(\mathcal{E}) \overset{*}{\longleftrightarrow} \mathcal{K}_{\mathbb{Z}/2}(\mathcal{E}), \quad \mathcal{D}_{\mathbb{Z}/2}(\mathcal{E}) \overset{*}{\longleftrightarrow} \mathcal{D}_{\mathbb{Z}/2}(\mathcal{E}).$$

These involutions shift the grading and change the sign of the differential as follows :

$$M^0 \underset{d^1}{\overset{d^0}{\rightleftarrows}} M^1 \ \overset{*}{\longleftrightarrow} \ M^1 \underset{-d^0}{\overset{-d^1}{\rightleftarrows}} M^0 \ .$$

We have an exact functor

$$\pi : \mathcal{C}^b(\mathcal{E}) \to \mathcal{C}_{\mathbb{Z}/2}(\mathcal{E}),$$

sending a complex $(M^i)_{i\in\mathbb{Z}}$ to the $\mathbb{Z}/2-$graded complex

$$\bigoplus_{i\in\mathbb{Z}} M^{2i} \rightleftarrows \bigoplus_{i\in\mathbb{Z}} M^{2i+1} \tag{4.1}$$

with the naturally defined differentials. It is easy to check that

$$\mathrm{Hom}_{\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})}(\pi(A), \pi(B)) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}^b(\mathcal{E})}(A, \Sigma^{2i}B).$$

Note that we actually have a whole family of isomorphisms

$$\mathrm{Ext}^p_{\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})}(\pi(A),\pi(B)) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(A,\Sigma^{2i}B), \quad \forall p\geq 0. \tag{4.2}$$

Similarly, we have exact functors

$$\mathcal{K}^b(\mathcal{E}) \to \mathcal{K}_{\mathbb{Z}/2}(\mathcal{E}), \quad \mathcal{D}^b(\mathcal{E}) \to \mathcal{D}_{\mathbb{Z}/2}(\mathcal{E}),$$

satisfying analogous isomorphisms. By abuse of notation, we will also denote them by π. Assume that M, N are stalk complexes in $\mathcal{C}^b(\mathcal{E}) : M = M^j[-j], N = N^i[-i]$. Then it is well-known that

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{E})}(A,B) = \begin{cases} 0, & j<i, \\ \mathrm{Ext}^{j-i}_{\mathcal{E}}(A^j,B^i), & j\geq i. \end{cases} \tag{4.3}$$

Here and everywhere below we write $\mathrm{Ext}^0(-,-)$ for $\mathrm{Hom}(-,-)$. Now, by simple calculations, we get the following result.

Lemma 4.1.1. *For any $A, B \in \mathcal{E}$, for any $i \geq 0$, we have*

$$\mathrm{Ext}^{2i}_{\mathcal{D}_2(\mathcal{E})}(\ 0 \rightleftarrows A\ ,\ 0 \rightleftarrows B\ ) = \bigoplus_{k\geq 0} \mathrm{Ext}^{2k}_{\mathcal{E}}(A,B);$$

$$\mathrm{Ext}^{2i}_{\mathcal{C}_2(\mathcal{E})}(\ 0 \rightleftarrows A\ ,\ 0 \rightleftarrows B\ ) = \bigoplus_{0\leq k\leq i} \mathrm{Ext}^{2k}_{\mathcal{E}}(A,B);$$

$$\mathrm{Ext}^{2i+1}_{\mathcal{D}_2(\mathcal{E})}(\ 0 \rightleftarrows A\ ,\ 0 \rightleftarrows B\ ) = \bigoplus_{k\geq 0} \mathrm{Ext}^{2k+1}_{\mathcal{E}}(A,B);$$

$$\mathrm{Ext}^{2i+1}_{\mathcal{C}_2(\mathcal{E})}(\ 0 \rightleftarrows A\ ,\ 0 \rightleftarrows B\ ) = \bigoplus_{0\leq k\leq i} \mathrm{Ext}^{2k+1}_{\mathcal{E}}(A,B);$$

and similarly, we have

$$\mathrm{Ext}^{2i}_{\mathcal{D}_2(\mathcal{E})}(\ A \rightleftarrows 0\ ,\ 0 \rightleftarrows B\ ) = \bigoplus_{k\geq 0} \mathrm{Ext}^{2k+1}_{\mathcal{E}}(A,B);$$

$$\mathrm{Ext}^{2i}_{\mathcal{C}_2(\mathcal{E})}(\ A \rightleftarrows 0\ ,\ 0 \rightleftarrows B\ ) = \bigoplus_{0\leq k< i} \mathrm{Ext}^{2k+1}_{\mathcal{E}}(A,B);$$

$$\mathrm{Ext}^{2i+1}_{\mathcal{D}_2(\mathcal{E})}(\ A \rightleftarrows 0\ ,\ 0 \rightleftarrows B\ ) = \bigoplus_{k\geq 0} \mathrm{Ext}^{2k}_{\mathcal{E}}(A,B);$$

$$\mathrm{Ext}^{2i+1}_{\mathcal{C}_2(\mathcal{E})}(\ A \rightleftarrows 0\ ,\ 0 \rightleftarrows B\ ) = \bigoplus_{0\leq k\leq i} \mathrm{Ext}^{2k}_{\mathcal{E}}(A,B).$$

Proposition 4.1.2. *Assume that gldim$(\mathcal{E}) = n$. Then for any $A, B \in \mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})$ both being iterated extensions of stalk complexes, the natural map $\mathrm{Ext}^i_{\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})}(A,B) \to \mathrm{Ext}^i_{\mathcal{D}_{\mathbb{Z}/2}(\mathcal{E})}(A,B)$ is an*

epimorphism for $i = n$ and an isomorphism for all $i > n$.

Proof. By Lemma 4.1.1, for A, B stalk complexes, we have

$$\mathrm{Ext}^i_{\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})}(A, B) \xrightarrow{\sim} \mathrm{Ext}^i_{\mathcal{D}_{\mathbb{Z}/2}(\mathcal{E})}(A, B), \forall i \geq n.$$

Indeed, for $i \geq n$, the non-zero terms in the direct sum decompositions of $\mathrm{Ext}^i_{\mathcal{D}_{\mathbb{Z}/2}(\mathcal{E})}(A, B)$ given in the Lemma are precisely those extensions of A by B that appear in the decompositions of $\mathrm{Ext}^i_{\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})}(A, B)$.

Then the statement can be proved by an appropriate application of both of the four-lemmas to the maps between the long exact sequences of extensions in $\mathcal{C}_{\mathbb{Z}/2}$ and in $\mathcal{D}_{\mathbb{Z}/2}$. $\square$

Corollary 4.1.3. *Assume that gldim$(\mathcal{E}) = n$. If $A, B \in \mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})$ both are iterated extensions of stalk complexes and A or B is acyclic, then $\mathrm{Ext}^i_{\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})}(A, B)$ vanishes for all $i > n$.*

To each object $M \in \mathcal{E}$, we attach a pair of acyclic (in fact, even contractible) complexes

$$K_M := \; M \underset{0}{\overset{1}{\rightleftarrows}} M \;, \qquad K^*_M := \; M \underset{1}{\overset{0}{\rightleftarrows}} M \;.$$

Let $\mathcal{P}$ be the full subcategory of projective objects in $\mathcal{E}$. The following fact was shown in [5].

Lemma 4.1.4 ([5, Lemma 3.2])**.** *Suppose that each object in $\mathcal{E}$ has a finite projective resolution. Then for any acyclic complex of projectives $M \in \mathcal{C}_{\mathbb{Z}/2}(\mathcal{P})$, there are objects $P, Q \in \mathcal{P}$, unique up to isomorphism, such that $M \cong K_P \oplus {K_Q}^*$.*

Remark 4.1.5. In [5] this Lemma was proved only for $\mathcal{E}$ abelian and of finite global dimension; nonetheless, the same proof works in the generality stated above.

Assume that $\mathcal{A}$ is hereditary and abelian and has enough projectives. Under these conditions, every object $A \in \mathcal{A}$ has a projective resolution

$$0 \to P \xrightarrow{f} Q \xrightarrow{g} A \to 0, \tag{4.4}$$

and decomposing P and Q into finite direct sums of indecomposables $P = \oplus_i P_i$, $Q = \oplus_j Q_j$, one may write $f = (f_{ij})$ in matrix form with $f_{ij} : P_i \to Q_j$. The resolution (4.4) is said to be *minimal* if none of the morphisms f_{ij} is an isomorphism. Here we use that $\mathcal{P}$ is Krull-Schmidt as an additive subcategory of the Krull-Schmidt category $\mathcal{A}$.

Lemma 4.1.6 ([5, Lemma 4.1])**.** *Any resolution (4.4) is isomorphic to a resolution of the form*

$$0 \to R \oplus P' \xrightarrow{1 \oplus f'} R \oplus Q' \xrightarrow{(0, g')} A \to 0,$$

with some object $R \in \mathcal{P}$ and some minimal projective resolution

$$0 \to P' \xrightarrow{f'} Q' \xrightarrow{g'} A \to 0.$$

Given an object $A \in \mathcal{A}$, take a minimal projective resolution

$$0 \to P_A \xrightarrow{f_A} Q_A \xrightarrow{g} A \to 0.$$

We define a $\mathbb{Z}/2$-graded complex

$$C_A := \; P_A \underset{0}{\overset{f_A}{\rightleftarrows}} Q_A \; \in \mathcal{C}(\mathcal{P}).$$

By Lemma 4.1.6, any two minimal projective resolutions of A are isomorphic, hence the complex C_A is well-defined up to isomorphism.

Lemma 4.1.7 ([5, Lemma 4.2])**.** *Every object $M \in C_{\mathbb{Z}/2}(\mathcal{P})$ has a direct sum decomposition*

$$M = C_A \oplus C_B^* \oplus K_P \oplus K_Q^*.$$

Moreover, the objects $A, B \in \mathcal{A}$ and $P, Q \in \mathcal{P}$ are unique up to isomorphism.

Remark 4.1.8. Note that each of these four direct summands can be obtained as an extension of stalk complexes. Indeed, in $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{P})$ we have conflations

$$(0 \underset{0}{\overset{0}{\rightleftarrows}} Q_A) \rightarrowtail C_A \twoheadrightarrow (P_A \underset{0}{\overset{0}{\rightleftarrows}} 0),$$

$$(0 \underset{0}{\overset{0}{\rightleftarrows}} P) \rightarrowtail K_P \twoheadrightarrow (P \underset{0}{\overset{0}{\rightleftarrows}} 0),$$

and similarly for C_B^* and K_Q^*.

Proposition 4.1.9. *Assume that either :*

— *$\mathcal{E}$ is an abelian category or*
— *$\mathcal{E}$ is the full exact subcategory of projective objects in a hereditary abelian category.*

Then any object in $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})$ is an iterated extension of stalk complexes.

Proof. The second statement is an immediate consequence of Lemma 4.1.7 an Remark 4.1.8. Let us prove the first one. Consider any complex $M^0 \underset{d^1}{\overset{d^0}{\rightleftarrows}} M^1$ in $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})$, where $\mathcal{E}$ is abelian. Let Z^1 be the kernel of d^1. Then we have a natural short exact sequence of $\mathbb{Z}/2-$graded complexes :

$$\begin{tikzcd}
M^0 \arrow[r, shift left, "d^0"] \arrow[d, equal] & Z^1 \arrow[l, shift left, "0"] \arrow[d, hook] \\
M^0 \arrow[r, shift left, "d^0"] \arrow[d, two heads] & M^1 \arrow[l, shift left, "d^1"] \arrow[d, two heads] \\
0 \arrow[r, shift left, "0"] & M^1/Z^1. \arrow[l, shift left, "0"]
\end{tikzcd} \tag{4.5}$$

The bottom complex is a stalk complex, the top one is an extension of $0 \underset{0}{\overset{0}{\rightleftarrows}} Z^1$ by $M^0 \underset{0}{\overset{0}{\rightleftarrows}} 0$. $\square$

4.1.2 Grothendieck groups of $\mathbb{Z}/2-$graded complexes

Define the *left relative Grothendieck monoid* $M_0'(\widetilde{\mathcal{E}})$ as the free monoid generated by the set $\mathrm{Iso}(\widetilde{\mathcal{E}})$, divided by the following set of relations :

$$\left\langle [L] = [K \oplus M] | K \rightarrowtail L \twoheadrightarrow M \quad \text{is a conflation}, K \in \mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}) \right\rangle .$$

Similarly, let *left relative Grothendieck group* $K_0'(\widetilde{\mathcal{E}})$ be the free group generated by the set $\mathrm{Iso}(\widetilde{\mathcal{E}})$, divided by the following set of relations :

$$\left\langle [K] - [L] + [M] = 0 | K \rightarrowtail L \twoheadrightarrow M \quad \text{is a conflation}, K \in \mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}) \right\rangle .$$

Let $\mathcal{A}$ be an abelian category.

Lemma 4.1.10. *We have an isomorphism*

$$K_0(\mathcal{C}_{\mathbb{Z}/2}(\mathcal{A})) \xrightarrow{\sim} K_0(\mathcal{A})^{\oplus 2}; \quad [A^\bullet] \mapsto ([A^0], [A^1]).$$

Proof. This is an immediate consequence of the existence of the "brutal truncations" constructed in the proof of Proposition 4.1.9. $\square$

Lemma 4.1.11. *Any short exact sequence*

$$A^\bullet \overset{f^\bullet}{\rightarrowtail} B^\bullet \overset{g^\bullet}{\twoheadrightarrow} C^\bullet \tag{4.6}$$

of complexes in $\mathcal{C}$, with $A^\bullet$ being acyclic, is CE-exact.

Proof. We want to show that at any degree i, the sequence $Z^i(A^\bullet) \xrightarrow{f^i} Z^i(B^\bullet) \xrightarrow{g^i} Z^i(C^\bullet)$ is short exact. The exactness in the first and the second term follows from the snake lemma. Let

us prove the exactness in the third term by diagram chasing for module categories; the result will follow then from the Mitchell embedding theorem. Assume $c \in Z^i(C^\bullet)$. Then there exists $b \in B^i$, s.t. $c = g^i(b)$. Also, $d^i_C(c) = 0$. Then $g^i(d^i_B(b)) = d^i_C(g^i(b)) = d^i_C(c) = 0$. Therefore, $d^i_B(b) \in \mathrm{Ker}(g^i)$. By the exactness of the sequence 4.6, $d^i_B(b) \in \mathrm{Im}(f^{i+1})$, i.e. there exists $a \in A^{i+1}$, such that $f^{i+1}(a) = d^i_B(b)$. Then $f^{i+2}d^{i+1}_A(a) = d^{i+1}_B(d^i_B(b)) = 0$. Since f^{i+2} is a monomorphism, it follows that $d^{i+1}_A(a) = 0$, hence $a \in \mathrm{Ker}(d^{i+1}_A)$. But $A^\bullet$ is acyclic; therefore, $\mathrm{Ker}(d^{i+1}_A) = \mathrm{Im}(d^i_A)$. This means that there exists $\widetilde{a} \in A^i$, such that $d^i_A(\widetilde{a}) = a$. Then $d^i_B(f^i(\widetilde{a})) = f^{i+1}(d^i_A(\widetilde{a})) = f_{i+1}(a) = d^i_B(b)$. We see that $b - f^i(\widetilde{a}) \in \mathrm{Ker}(d^i_B) = Z^i(B^\bullet)$ and $c = g^i(b) = g^i(b) - g^i(f^i(\widetilde{a})) = g^i(b - f^i(\widetilde{a}))$. This proves the exactness in the third term. □

Corollary 4.1.12. *The natural inclusion $i : K_0(\mathcal{A}) \oplus K_0(\mathcal{A}) \to K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{A}))$ is an embedding of a direct summand. The same holds for the inclusion $K_0(\mathcal{A}) \oplus K_0(\mathcal{A}) \to K'_0(\mathcal{C}_{\mathbb{Z}/2}(\mathcal{A}))$.*

Proof. By Lemma 4.1.11, the map

$$A^\bullet \to ([\mathrm{Im}\, d^1(A^\bullet], [\mathrm{Im}\, d^0(A^\bullet])$$

defines sections for both of the inclusions, and these sections are group homomorphisms. □

The following lemma is well-known in the case of bounded complexes; nonetheless, we present a proof here, as we will use later the explicit quasi-isomorphisms.

Lemma 4.1.13. *If $\mathcal{A}$ is hereditary, then every complex in $\mathcal{C}_{\mathbb{Z}/m}(\mathcal{A})$ is quasi-isomorphic to the direct sum of its cohomologies.*

Proof. Consider a complex $A^\bullet = (A^i, d^i)_{i\in\mathbb{Z}/m}$ and its cohomology complex $H^\bullet = (H^i, 0) = \oplus H^i[i]$. We have families of short exact sequences of the form $\mathrm{Ker}\, d^{i-1} \rightarrowtail A^{i-1} \twoheadrightarrow \mathrm{Im}\, d^{i-1}$ and of the form $\mathrm{Im}\, d^{i-1} \rightarrowtail \mathrm{Ker}\, d^i \twoheadrightarrow H^i$. The following arguments hold for all $i \in \mathbb{Z}/m$. Since $\mathcal{A}$ is hereditary, we have $\mathrm{Ext}^2(H^i, \mathrm{Ker}\, d^{i-1}) = 0$. Therefore, the natural map $\mathrm{Ext}^1(H^i, A^{i-1}) \to \mathrm{Ext}^1(H^i, \mathrm{Im}\, d^{i-1})$ is a surjection, and we can construct the following commutative diagram, with $S^i \in \mathcal{A}$.

$$\begin{tikzcd}
\mathrm{Ker}\, d^{i-1} \arrow[r, equal] \arrow[d, tail] & \mathrm{Ker}\, d^{i-1} \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\
A^{i-1} \arrow[r, tail, "f^{i-1}"] \arrow[d, two heads] & S^i \arrow[r, two heads] \arrow[d, two heads] & H^i \arrow[d, equal] \\
\mathrm{Im}\, d^{i-1} \arrow[r, tail] & \mathrm{Ker}\, d^i \arrow[r, two heads] & H^i.
\end{tikzcd} \tag{4.7}$$

We have natural maps $g^i : S^i \twoheadrightarrow \mathrm{Ker}\, d^i \rightarrowtail A^i$, and it is easy to check that the following morphism

of complexes is an epimorhic quasi-isomorphism.

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \oplus S^{i-1} \arrow[r, "\begin{pmatrix} 0 & f^{i-1} \\ 0 & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & g^i \end{pmatrix}"'] & A^i \oplus S^i \arrow[r, "\begin{pmatrix} 0 & f^i \\ 0 & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & g^i \end{pmatrix}"'] & A^{i+1} \oplus S^{i+1} \arrow[r, "\begin{pmatrix} 0 & f^{i+1} \\ 0 & 0 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 1 & g^{i+1} \end{pmatrix}"'] & \cdots \\
\cdots \arrow[r] & A^{i-1} \arrow[r, "d^{i-1}"] & A^i \arrow[r, "d^i"] & A^{i+1} \arrow[r] & \cdots
\end{tikzcd} \tag{4.8}$$

□

Explicitly, we have 2 conflations where the deflations are quasi-isomorphisms :

$$K_1^\bullet \rightarrowtail E^\bullet \twoheadrightarrow A^\bullet \tag{4.9}$$

and

$$K_2^\bullet \rightarrowtail E^\bullet \twoheadrightarrow H^\bullet, \tag{4.10}$$

with

$$H^\bullet = \bigoplus_{i=0}^{m} H^i;$$

$$K_1^\bullet = \bigoplus_{i=0}^{m} (Z^i(A^\bullet) \xrightarrow{1} Z^i(A^\bullet));$$

$$K_2^\bullet = \bigoplus_{i=0}^{m} (A^i \xrightarrow{1} A^i);$$

$$E^\bullet = \bigoplus_{i=0}^{m} A^i \overset{f^i}{\rightarrowtail} S^{i+1};$$

where the term with index i is in degree i.

Lemma 4.1.14. *If $\mathcal{A}$ is hereditary, then for any $m \geq 1$, we have an isomorphism*

$$K_0(\mathcal{C}_{\mathbb{Z}/m,ac}(\mathcal{A})) \xrightarrow{\sim} K_0(\mathcal{A})^{\oplus m}; \quad A^\bullet \mapsto ([\operatorname{Im} d^0], [\operatorname{Im} d^1], \ldots, [\operatorname{Im} d^{(m-1)}]).$$

Proof. This map is an epimorphism by Lemma 4.1.11. From the existence of conflations (4.9) and (4.10) for $A^\bullet$ acyclic, it follows that it is a monomorphism. □

Lemma 4.1.15. *If $\mathcal{A}$ is hereditary, the natural inclusion $i : K_0(\mathcal{C}_{Z/2,ac}(\mathcal{A})) \to K_0'(\mathcal{C}_{Z/2,ac}(\mathcal{A}))$ is an injection.*

Proof. This is a byproduct of Lemma 4.1.14 and Corollary 4.1.12. □

If the Euler form on $K_0(\mathcal{C}_{Z/2,ac}(\mathcal{A})) \times K_0(\mathcal{C}_{Z/2,ac}(\mathcal{A}))$ is well-defined, we have a natural counterpart of these lemmas for the numerical Grothendieck groups.

Lemma 4.1.16. *(i) The natural inclusion $i : K_0^{num}(\mathcal{A}) \oplus K_0^{num}(\mathcal{A}) \to K_0^{num}(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{A}))$ is an embedding of a direct summand. The same holds for the inclusion $K_0^{num}(\mathcal{A}) \oplus K_0^{num}(\mathcal{A}) \to K_0'^{num}(\mathcal{C}_{\mathbb{Z}/2}(\mathcal{A}))$.*

(ii) If $\mathcal{A}$ is hereditary, then for any $m \geq 1$, we have an isomorphism

$$K_0^{num}(\mathcal{C}_{\mathbb{Z}/m,ac}(\mathcal{A})) \xrightarrow{\sim} K_0^{num}(\mathcal{A})^{\oplus m}; \quad A^\bullet \mapsto ([\operatorname{Im} d^0], [\operatorname{Im} d^1], \ldots, [\operatorname{Im} d^{(m-1)}]).$$

(iii) If $\mathcal{A}$ is hereditary, the natural inclusion $i : K_0^{num}(\mathcal{C}_{Z/2,ac}(\mathcal{A})) \to K_0'^{num}(\mathcal{C}_{Z/2,ac}(\mathcal{A}))$ is an injection.

4.2 Semi-derived Hall algebras

4.2.1 Finiteness conditions

In this section, we will discuss the various conditions we impose on the exact category $\mathcal{E}$. We shall always assume that

(C1) $\mathcal{E}$ is essentially small, idempotent complete and linear over some ground field k;

(C2) For each pair of objects $A, B \in \mathcal{E}$ and for each $p > 0$, we have

$$|\operatorname{Ext}^p(A, B)| < \infty; \qquad |\operatorname{Hom}(A, B)| < \infty;$$

(C3) For each pair of objects $A, B \in \mathcal{E}$, there exists $N > 0$ such that for all $p > N$, we have

$$\operatorname{Ext}^p(A, B) = 0.$$

(C4) For each pair of complexes $A, B \in \mathcal{C}^b(\mathcal{E})$, there exists a quasi-isomorphism deflation $A' \to A$ such that for all $i > 0$, the natural map

$$\operatorname{Ext}^i_{\mathcal{C}^b(\mathcal{E})}(A, B) \to \operatorname{Ext}^i_{\mathcal{D}^b(\mathcal{E})}(A, B)$$

is an isomorphism.

The first part of the assumption (C1) and the finiteness of the $\operatorname{Hom}-$ and Ext^1-spaces are important to our naive approach to Hall algebras involving counting isomorphism classes. Assumption (C1) and $\operatorname{Hom}-$finiteness ensure that $\mathcal{E}$ is Krull-Schmidt. Moreover, it implies that all contractible complexes are acyclic, cf. [38], [7]. Finiteness of $\operatorname{Ext}^p(A, B)$ and the assumption (C3) are crucial for the multiplicative version of the Euler form that we use.

Proposition 4.2.1. *Suppose that for two bounded complexes $M, N \in \mathcal{C}^b(\mathcal{E})$, all extensions between their components are trivial :*

$$\operatorname{Ext}^p_{\mathcal{E}}(M^i, N^j) = 0, \quad \forall p > 0, \forall i, j \in \mathbb{Z}. \tag{4.11}$$

Then the canonical map :

$$\mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(M,N) \to \mathrm{Ext}^p_{\mathcal{D}^b(\mathcal{E})}(M,N)$$

is bijective for all $p > 0$.

Proof of Proposition 4.2.1. It is a classical fact that we have a family of conflations

$$\Sigma^{-i}M \rightarrowtail C(1_{\Sigma^{-i}M}) \twoheadrightarrow \Sigma^{-(i-1)}M, \qquad i > 0,$$

where $C(f)$ is a mapping cone and Σ is the shift functor. Thus, we have a complex

$$R(M)^\bullet := \ldots \to C(1_{\Sigma^{-3}M}) \to C(1_{\Sigma^{-2}M}) \to C(1_{\Sigma^{-1}M}) \to 0, \tag{4.12}$$

with a quasi-isomorphism $R(M)^\bullet \overset{\text{qis}}{\to} M$ in $\mathcal{C}(\mathcal{C}^b(M))$. Consider the category of bounded graded objects $\mathbf{gr^b}(\mathcal{E})$, whose objects are the $\mathbb{Z}-$graded families of objects of $\mathcal{E}$ and morphisms are given component-wise. There exists a natural forgetful functor

$$\mathcal{C}^b(\mathcal{E}) \overset{\text{forget}}{\to} \mathbf{gr^b}(\mathcal{E}),$$

that is known to be exact and admits an exact left adjoint forget_λ :

$$\mathcal{C}^b(\mathcal{E}) \overset{\mathrm{forget}_\lambda}{\leftarrow} \mathbf{gr^b}(\mathcal{E}).$$

It is known as well that

$$\mathrm{forget}_\lambda \circ \mathrm{forget}(X) \overset{\sim}{\to} C(1_X),$$

hence we have the following isomorphisms :

$$\mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(C(1_X),Y) \overset{\sim}{\to} \mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(\mathrm{forget}_\lambda \circ \mathrm{forget}(X),Y) \overset{\sim}{\to} \mathrm{Ext}^p_{\mathbf{gr^b}(\mathcal{E})}(\mathrm{forget}(X),\mathrm{forget}(Y)). \tag{4.13}$$

Clearly, by (4.11),

$$\mathrm{Ext}^p_{\mathbf{gr^b}(\mathcal{E})}(\mathrm{forget}(\Sigma^{-i}M),\mathrm{forget}(N)) = 0, \qquad \forall i,$$

hence

$$\mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(C(1_{\mathrm{forget}(\Sigma^{-i}M)},N) = 0, \qquad \forall i.$$

Therefore, the complex $R(M)^\bullet$ is actually a $\mathrm{Hom}(?,N)-$acyclic resolution of the complex M in $\mathcal{C}^b(\mathcal{E})$ and can be used to compute $\mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(M,N)$. Namely, these extensions are exactly the homologies of the complex $\mathrm{Hom}_{\mathcal{C}^b(\mathcal{E})}(R(M)^\bullet,N)$, i.e. quotients of the set of morphisms $\mathrm{Hom}_{\mathcal{C}^b(\mathcal{E})}(\Sigma^{-(p)}M,N)$ by the subset of morphisms which factor through $C(1_{\Sigma^{-p}M})$. This last subset is known to be the subset of null-homotopic morphisms. Therefore, we obtain that exten-

sions are exactly the morphisms in the homotopy category of bounded complexes $\mathbf{K}^b(\mathcal{E})$:

$$\mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(M,N) = \mathrm{Hom}_{\mathbf{K}^b(\mathcal{E})}(\Sigma^{-(p)}M, N).$$

Lemma 4.2.2. *Under the condition (4.11), we have a canonical bijection :*

$$\mathrm{Hom}_{\mathbf{K}^b(\mathcal{E})}(M,N) = \mathrm{Hom}_{\mathcal{D}^b(\mathcal{E})}(M,N).$$

The lemma uses nothing but the dévissage. The statement of the lemma holds, of course, for shifted complexes as well; thus, we get

$$\mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(M,N) = \mathrm{Hom}_{\mathcal{D}^b(\mathcal{E})}(\Sigma^{-(p)}M, N).$$

The right-hand side is nothing but $\mathrm{Hom}_{\mathcal{D}^b(\mathcal{E})}(M, \Sigma^p N)$, that is equal to $\mathrm{Ext}^p_{\mathcal{D}^b(\mathcal{E})}(M,N)$. □

Recall that $\mathcal{P}$ denotes the full subcategory of the projective objects of $\mathcal{E}$.

Corollary 4.2.3. *For all $P \in \mathcal{C}^b(\mathcal{P})$ and all $M \in \mathcal{C}^b(\mathcal{E})$, we have a canonical bijection :*

$$\mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(P,M) = \mathrm{Ext}^p_{\mathcal{D}^b(\mathcal{E})}(P,M), \quad \forall p > 0.$$

In particular, bounded acyclic complexes of projectives are projective objects in the category $\mathcal{C}^b(\mathcal{P})$.

Lemma 4.2.4 ([37, 4.1, Lemma, b)])**.** *Assume that $\mathcal{E}$ has enough projectives, and each object of $\mathcal{E}$ has a finite projective resolution. Then for each $A \in \mathcal{C}^b(\mathcal{E})$, there exists a deflation*

$$P \overset{\mathrm{qis}}{\twoheadrightarrow} A$$

that is a quasi-isomorphism, with $P \in \mathcal{C}^b(\mathcal{P})$.

Theorem 4.2.5. *The condition (C4) holds if $\mathcal{E}$ is of one of the following two types :*

1) $\mathcal{E}$ has enough projectives, and each object has a finite projective resolution ;

2) $\mathcal{E} = Coh(X)$ is the category of coherent sheaves on a smooth projective variety X.

Proof. Let us return to the proof of Theorem 4.2.5. For part 1), by lemma 4.2.4, in condition (C4) we can take a deflation quasi-isomorphism with $A'\mathcal{C}^b(\mathcal{P})$.

Now, for part 2) we recall the next known facts, the first of which easily follows from Serre's theorem which connects the category $Coh(X)$ and the category of finitely generated modules over the homogeneous coordinate algebra of X :

Lemma 4.2.6. *(a) For any $F \in Coh(X), N \in \mathbb{N}$, there exists a deflation $E \twoheadrightarrow F$, where E is of the form*

$$E = \bigoplus_{i=1}^{n} \mathcal{O}(-a_i), \qquad a_i > N, \forall i = 1, \ldots, n,$$

for some n.

(b) [?, Theorem III.5.2.b)] For every $F \in Coh(X)$, there exists a nonnegative integer $n_0(F)$, such that for each $n > n_0(F)$, we have :

$$\mathrm{Ext}^p_{Coh(X)}(\mathcal{O}(-n), F) = H^p(X, F(n)) = 0, \qquad \forall p > 0.$$

Objects of the above form E generate an additive category, namely the category $\mathcal{V}_N(X)$ of vector bundles over X which are sums of line bundles $\mathcal{O}(-j), j \geq N$.

Let us return to the proof of part 2) of Theorem 4.2.5. Similarly to Lemma 4.2.4 (cf. also [7, Theorem 12.7]), we can show that for each $N > 0$ and for each $A \in \mathcal{C}^b(Coh(X))$, there exists a deflation quasi-isomorphism $A' \overset{\text{qis}}{\twoheadrightarrow} A$, with $A' \in \mathcal{C}^b(\mathcal{V}_N(X))$. Now we take N to be the maximum among the numbers $n_0(B^j)$, where B^j are among the (finitely many !) non-zero components of B, and find $A' \in \mathcal{V}_{A'}(X)$. As in part 1), for such A', all the conditions (i)-(iii) hold, by Proposition 4.2.1. □

4.2.2 Euler form and quantum tori

Let $\mathcal{C}^b_{ac}(\mathcal{E})$ denote the category of bounded acyclic complexes over $\mathcal{E}$.

Definition 4.2.7. The *multiplicative Euler form*

$$\langle \cdot, \cdot \rangle : K_0(\mathcal{C}^b_{ac}(\mathcal{E})) \times K_0(\mathcal{C}^b(\mathcal{E})) \to \mathbb{Q}^*$$

is given by the alternating product :

$$\langle K, A \rangle := \prod_{p=0}^{+\infty} |\,\mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(K, A)|^{(-1)^p}.$$

Using a similar product we define

$$\langle \cdot, \cdot \rangle : K_0(\mathcal{C}^b(\mathcal{E})) \times K_0(\mathcal{C}^b_{ac}(\mathcal{E})) \to \mathbb{Q}^*.$$

These two forms clearly coincide on $K_0(\mathcal{C}^b_{ac}(\mathcal{E})) \times K_0(\mathcal{C}^b_{ac}(\mathcal{E}))$, so it is harmless to denote them by the same symbol.

By the following lemma, this alternating product is well-defined, i.e. all but a finite number of factors equal 1. Therefore, by the five-lemma, it is bilinear. In fact, in our framework, one can define this form on $K_0(\mathcal{C}^b(\mathcal{E})) \times K_0(\mathcal{C}^b(\mathcal{E}))$; but in the $\mathbb{Z}/2-$graded case, this will not be true any more.

Lemma 4.2.8. *For each pair of objects $A, B \in \mathcal{C}^b(\mathcal{E})$, the property of local homological finiteness holds, i.e. there exists $n_0 > 0$ such that for all $p > n_0$, we have*

$$\mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(A, B) = 0.$$

Proof. Using the five-lemma and the stupid truncations (dévissage) for both arguments, we observe that it is enough to prove the local homological finiteness only for stalk complexes A and B. For each A, we have a conflation

$$\Sigma^{-1}A \rightarrowtail C(1_{\Sigma^{-1}A}) \twoheadrightarrow A.$$

By condition (C3), the desired local homological finiteness holds for the graded objects. Thus, by (4.13), it holds for $C(1_{\Sigma^{-1}A})$ and B, hence it holds for A, B if only if it holds for $\Sigma^{-1}A, B$. By induction, we get that it is enough to prove the statement for stalk complexes A, B both concentrated in the same degree. For those, the extensions are extensions in $\mathcal{E}$ of their non-zero components, and the condition (C3) implies the desired statement. $\square$

Consider the set $\mathrm{Iso}(\mathcal{C}^b_{ac}(\mathcal{E}))$ of isomorphism classes $[K]$ of bounded acyclic complexes and its quotient by the following set of relations :

$$\langle [K_2] = [K_1 \oplus K_3] | K_1 \rightarrowtail K_2 \twoheadrightarrow K_3 \quad \text{is a conflation} \rangle .$$

If we endow $\mathrm{Iso}(\mathcal{C}^b_{ac}(\mathcal{E}))$ with the addition given by direct sums, this quotient gives the Grothendieck monoid $M_0(\mathcal{C}^b_{ac}(\mathcal{E}))$ of the exact category $\mathcal{C}^b_{ac}(\mathcal{E})$. We define the *quantum affine space of acyclic complexes* $\mathbb{A}_{ac}(\mathcal{E})$ as the $\mathbb{Q}-$vector space generated by elements of $M_0(\mathcal{C}^b_{ac}(\mathcal{E}))$. We endow it with the bilinear multiplication defined below.

Definition 4.2.9. For $K_1, K_2 \in \mathcal{C}^b_{ac}(\mathcal{E})$, we define their product as

$$[K_1] \diamond [K_2] := \frac{1}{\langle K_1, K_2 \rangle} [K_1 \oplus K_2].$$

By Lemma 4.2.8, this product is well-defined; moreover, it is clearly associative. We see that this ring has the class of the zero complex $[0]$ as the unit. Moreover, it is clear that the set of all elements of the form $[K]$ satisfies the Ore conditions. This means that we can make all of them invertible and consider the *quantum torus of acyclic complexes* $\mathbb{T}_{ac}(\mathcal{E})$. It is generated by classes $[K]$ and their inverses $[K]^{-1}$. Here are simple relations concerning the product.

Lemma 4.2.10. *For $K_1, K_2 \in \mathcal{C}^b_{ac}(\mathcal{E})$, we have*

$$[K_1]^{-1} \diamond [K_2]^{-1} = \langle K_2, K_1 \rangle \, [K_1 \oplus K_2]^{-1}; \quad [K_1]^{-1} \diamond [K_2] = \frac{\langle K_1, K_2 \rangle}{\langle K_2, K_1 \rangle} [K_2] \diamond [K_1]^{-1}.$$

Proof. The first identity is trivial, let us prove the second one.

$$[K_1] \diamond [K_2] = \frac{1}{\langle K_1, K_2 \rangle} [K_1 \oplus K_2] \Rightarrow [K_2] = \frac{1}{\langle K_1, K_2 \rangle} [K_1]^{-1} \diamond [K_1 \oplus K_2] \Rightarrow$$

$$\Rightarrow [K_2] \diamond [K_1]^{-1} = \frac{1}{\langle K_1, K_2 \rangle} [K_1]^{-1} \diamond [K_1 \oplus K_2] \diamond [K_1]^{-1}.$$

Similarly, we find that

$$[K_1]^{-1} \diamond [K_2] = \frac{1}{\langle K_2, K_1 \rangle} [K_1]^{-1} \diamond [K_1 \oplus K_2] \diamond [K_1]^{-1},$$

and the desired equality follows. $\square$

Another way to define $\mathbb{T}_{ac}(\mathcal{E})$ is provided by the following simple statement.

Lemma 4.2.11. *The quantum torus of acyclic complexes is isomorphic to the quantum torus of the Grothendieck group $K_0(\mathcal{C}^b_{ac}(\mathcal{E}))$, twisted by the inverse of the Euler form :*

$$\mathbb{T}_{ac}(\mathcal{E}) \xrightarrow{\sim} \mathbb{T}(K_0(\mathcal{C}^b_{ac}(\mathcal{E})), \langle \cdot, \cdot \rangle^{-1}).$$

In other words, it is the $\mathbb{Q}-$group algebra of $K_0(\mathcal{C}^b_{ac}(\mathcal{E}))$, with the multiplication twisted by the inverse of the Euler form.

Proof. By the construction, $\mathbb{A}_{ac}(\mathcal{E})$ is the monoid algebra of $M_0(\mathcal{C}^b_{ac}(\mathcal{E}))$, with the multiplication, twisted by the Euler form. Therefore, for the proof it is enough to verify that the universal properties of the monoid algebra $\mathbb{Q}[M]$, localized at the elements of the monoid, and of the group algebra $\mathbb{Q}[G]$ of the group G, corresponding to this monoid M by the Grothendieck construction, are the same. Indeed, for each $\mathbb{Q}-$algebra A, we have

$$\mathrm{Hom}_{\mathbb{Q}-\mathbf{Alg}}(\mathbb{Q}[G], A) = \mathrm{Hom}_{\mathbf{Grp}}(G, A^\times) = \mathrm{Hom}_{\mathbf{Mon}}(M, A^\times);$$

$$\mathrm{Hom}_{\mathbb{Q}-\mathbf{Alg}}(\mathbb{Q}[M][[m]^{-1}, m \in M], A) = \left\{ f \in \mathrm{Hom}_{\mathbb{Q}-\mathbf{Alg}}(\mathbb{Q}[M], A) | f(M) \subset A^\times \right\} =$$

$$= \mathrm{Hom}_{\mathbf{Mon}}(M, A^\times). \ \square$$

4.2.3 Module structure over the quantum torus

Let us consider the vector space $\mathcal{M}_1(\mathcal{E})$ over $\mathbb{Q}$ whose basis is formed by the isomorphism classes $[M]$, where $M \in \mathcal{C}^b(\mathcal{E})$. On $\mathcal{M}_1(\mathcal{E})$, let us define a multiplication by classes of acyclic complexes, generalizing the one from the previous section. Namely, for K acyclic, M arbitrary, we define their products as follows :

$$[K] \diamond [M] = \frac{1}{\langle K, M \rangle} [K \oplus M]; \quad [M] \diamond [K] = \frac{1}{\langle M, K \rangle} [K \oplus M].$$

We get a bimodule over $\mathbb{A}_{ac}(\mathcal{E})$, let us call it $\mathcal{M}_1'(\mathcal{E})$.

We quotient $M_1(\mathcal{E})$ by the set of relations

$$\left\langle [L] = [K \oplus M] | K \rightarrowtail L \twoheadrightarrow M \quad \text{is a conflation}, K \in \mathcal{C}^b_{ac}(\mathcal{E}) \right\rangle, \tag{4.14}$$

to obtain the space $\mathcal{M}_2(\mathcal{E})$. We will denote classes after the factorization by the same symbols $[M]$. Of course, one gets the same vector space if one starts with the quotient of $\mathrm{Iso}(\mathcal{C}^b(\mathcal{E}))$ by this set of relations and then considers the vector space on this basis. It is easy to check that these relations respect the bimodule structure. Thus, $\mathcal{M}_2(\mathcal{E})$ has the induced bimodule structure, let us denote it by $\mathcal{M}_2'(\mathcal{E})$. By taking the tensor product with the quantum torus, we get a bimodule $\mathcal{M}(\mathcal{E}) := \mathbb{T}_{ac}(\mathcal{E}) \otimes_{\mathbb{A}_{ac}(\mathcal{E})} \mathcal{M}_2'(\mathcal{E}) \cong \mathcal{M}_2'(\mathcal{E}) \otimes_{\mathbb{A}_{ac}(\mathcal{E})} \mathbb{T}_{ac}(\mathcal{E})$ over $\mathbb{T}_{ac}(\mathcal{E})$.

4.2.4 Freeness over the quantum torus

Theorem 4.2.12. *$\mathcal{M}(\mathcal{E})$ is free as a right module over $\mathbb{T}_{ac}(\mathcal{E})$. Each choice of representatives of the quasi-isomorphism classes in $\mathcal{C}^b(\mathcal{E})$ yields a basis.*

Proof. Assume that two complexes M and M' are quasi-isomorphic to each other. This means that there is a sequence of objects $M_0 = M, M_1, M_2, \ldots, M_n = M'$, such that for each $i = 1, 2, \ldots, n$ there is either a conflation

$$K \rightarrowtail M_{i-1} \overset{\text{qis}}{\twoheadrightarrow} M_i,$$

or a conflation

$$K \rightarrowtail M_i \overset{\text{qis}}{\twoheadrightarrow} M_{i-1},$$

with K acyclic. Therefore, we either have

$$[M_i] = [K \oplus M_{i-1}] = \langle K, M_{i-1} \rangle \, [K] \diamond [M_{i-1}],$$

or

$$[M_{i-1}] = [K \oplus M_i] = \langle K, M_i \rangle \, [K] \diamond [M_i] \Rightarrow [M_i] = \frac{1}{\langle K, M_i \rangle} [K]^{-1} \diamond [M_{i-1}].$$

It follows that $[M'] \in \mathbb{T}_{ac}(\mathcal{E}) \diamond [M]$. Therefore, the quasi-isomorphism classes of complexes generate $\mathcal{M}(\mathcal{E})$ over $\mathbb{T}_{ac}(\mathcal{E})$. It remains to prove that they are independent over this quantum torus.

Since each relation in $\mathcal{M}_1(\mathcal{E})$ from the definition of the underlying vector space of $\mathcal{M}(\mathcal{E})$ identifies two elements in the same quasi-isomorphism class, one can decompose $\mathcal{M}(\mathcal{E})$ into the direct sum

$$\mathcal{M}(\mathcal{E}) = \bigoplus_{\alpha \in \mathrm{Iso}(\mathcal{D}^b(\mathcal{E}))} \mathcal{M}_\alpha(\mathcal{E}),$$

where $\mathcal{M}_\alpha(\mathcal{E})$ is the component containing the classes of all objects whose isomorphism class in $\mathcal{D}^b(\mathcal{E})$ is α. We claim that for each α, the $\mathbb{T}_{ac}-$submodule $\mathcal{M}_\alpha(\mathcal{E})$ is free of rank one. Let M be an object of $\mathcal{C}^b(\mathcal{E})$. By the above argument, the map

$$\mathbb{T}_{ac}(\mathcal{E}) \to \mathcal{M}_{[M]}(\mathcal{E}), \quad [K] \mapsto [K] \diamond [M]$$

is surjective. Since $\mathbb{T}_{ac}(\mathcal{E})$ is the (twisted) group algebra of $K_0(\mathcal{C}^b_{ac}(\mathcal{E}))$, the following Lemma

4.2.13 shows that its composition with the natural map

$$\mathcal{M}_{[M]}(\mathcal{E}) \to \mathcal{M}(\mathcal{E}) \to \mathbb{Q}[K_0(\mathcal{C}^b(\mathcal{E}))]$$

is injective. Therefore, it is bijective. This completes the proof. □

Lemma 4.2.13. *The natural map*

$$i : K_0(C^b_{ac}(\mathcal{E})) \to K_0(C^b(\mathcal{E})), [M] \mapsto [M]$$

is injective.

Proof.

Now to prove injectivity of i, it is enough to prove the injectivity of the following composed map :

$$\coprod_{\mathbb{Z}} K_0(\mathcal{E}) \overset{h}{\to} K_0(\mathcal{C}^b_{ac}(\mathcal{E})) \overset{i}{\to} K_0(\mathcal{C}^b(\mathcal{E})) \overset{f}{\to} \coprod_{\mathbb{Z}} K_0(\mathcal{E}).$$

We will do it by contradiction. Assume that the element

$$\mathbf{X} = (\ldots, \sum_{i=1}^{k_n} x_i[X_i], \ldots)$$

of $\coprod_{\mathbb{Z}} K_0(\mathcal{E})$ (the described term sits in $n-$th component), is nonzero and belongs to the kernel of $f \circ i \circ h$. Consider the maximal m such that the mth component of $\mathbf{X}$ is nonzero. Following the definitions of the homomorphisms f, i, h, one observes that then the mth components of $\mathbf{X}$ and of $f(i(h(\mathbf{X})))$ coincide, thus the latter is also non-zero. Contradiction. □

4.2.5 Multiplication

Consider a pair of complexes $A, B \in \mathcal{C}^b(\mathcal{E})$. For each class ε in the extension group $\mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(A, B)$ represented by a conflation $B \rightarrowtail E \twoheadrightarrow A$, we denote by $\mathbf{mt}(\varepsilon)$ the isomorphism class of E in $\mathcal{C}^b(\mathcal{E})$. It is well-defined, i.e. it does not depend on the choice of the representative of ε. We also consider its class $[\mathbf{mt}(\varepsilon)]$ in $\mathcal{M}(\mathcal{E})$.

Definition 4.2.14. We define a $\mathbb{Q}-$bilinear map

$$\diamond : \mathcal{M}_1(\mathcal{E}) \times \mathcal{M}_1(\mathcal{E}) \to \mathcal{M}(\mathcal{E})$$

by the following rule :

$$[L] \diamond [M] = \frac{1}{\langle K, L\rangle}[K]^{-1} \diamond \sum_{E \in \mathrm{Iso}(\mathcal{C}^b(\mathcal{E}))} (\frac{|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L', M)_E|}{|\operatorname{Hom}(L', M)|}[E]), \tag{4.15}$$

or, equivalently, as

$$[L] \diamond [M] = \frac{1}{\langle K, L\rangle}[K]^{-1} \diamond \sum_{\varepsilon' \in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L', M)} \frac{[\mathbf{mt}(\varepsilon')]}{|\operatorname{Hom}(L', M)|}, \tag{4.16}$$

where $K \rightarrowtail L' \overset{\text{qis}}{\twoheadrightarrow} L$ is a conflation such that L' satisfies

$$\mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(L', M) = \mathrm{Ext}^p_{\mathcal{D}^b(\mathcal{E})}(L', M), \quad \forall p > 0. \tag{4.17}$$

By part (i) of assumption (C4), at least one such L' always exists. We want to prove that the definition is correct and compatible with the bimodule structure over $\mathbb{A}_{ac}(\mathcal{E})$ and that it descends to $\mathcal{M}(\mathcal{E})$.

Proposition 4.2.15. *The map $\diamond$ is well-defined, i.e. it does not depend on the choice of a conflation $K \rightarrowtail L' \overset{\text{qis}}{\twoheadrightarrow} L$ such that the isomorphism (4.17) holds.*

Proof. Suppose we have two conflations whose deflations are quasi-isomorphisms

$$K_1 \rightarrowtail L'_1 \overset{\text{qis}}{\twoheadrightarrow} L, K \rightarrowtail L' \overset{\text{qis}}{\twoheadrightarrow} L,$$

and isomorphism (4.17) holds for both L' and L'_1. Then for the pullback L'_2 of the two deflations arising in these conflations, we also have deflations which are quasi-isomorphisms $L'_2 \overset{\text{qis}}{\twoheadrightarrow} L', L'_2 \overset{\text{qis}}{\twoheadrightarrow} L'_1$. By assumption (C4), there exists a deflation quasi-isomorphism $L'' \overset{\text{qis}}{\twoheadrightarrow} L'_2$, with L'' satisfying the condition (4.17) for the extensions by M. Since the set of deflations is closed under composition, we have a deflation $L'' \overset{\text{qis}}{\twoheadrightarrow} L$. Completing all these deflations to conflations and using the axiom of the exact category concerning push-outs, we can obtain the following commutative diagram with H, N, K acyclic :

$$\begin{tikzcd}
H \arrow[r, equal] \arrow[d, tail] & H \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\
N \arrow[r, tail] \arrow[d, two heads] & L'' \arrow[r, two heads] \arrow[d, two heads] & L \arrow[d, equal] \\
K \arrow[r, tail] & L' \arrow[r, two heads] & L
\end{tikzcd} \tag{4.18}$$

Therefore, it is sufficient to prove that for an arbitrary pair of conflations

$$K \rightarrowtail L' \overset{\text{qis}}{\twoheadrightarrow} L, N \rightarrowtail L'' \overset{\text{qis}}{\twoheadrightarrow} L$$

satisfying the condition (4.17) and such that there exists a commutative diagram (4.18), we have

the identity

$$\frac{1}{\langle K,L\rangle}[K]^{-1}\diamond \sum_{\varepsilon'\in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L',M)} \frac{[\mathbf{mt}(\varepsilon')]}{|\operatorname{Hom}(L',M)|} = \frac{1}{\langle N,L\rangle}[N]^{-1}\diamond \sum_{\varepsilon''\in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L'',M)} \frac{[\mathbf{mt}(\varepsilon'')]}{|\operatorname{Hom}(L'',M)|}. \tag{4.19}$$

Using the conflation $H \rightarrowtail N \twoheadrightarrow K$, we find out that

$$[N] = [K\oplus H] = \langle H,K\rangle\,[H]\diamond[K] \quad \Leftrightarrow \quad [N]\diamond[K]^{-1} = \langle H,K\rangle\,[H] \quad \Leftrightarrow$$

$$\Leftrightarrow \quad [K]^{-1} = \langle H,K\rangle\,[N]^{-1}\diamond[H].$$

Therefore, the right hand side of (4.19) is equal to

$$\frac{\langle H,K\rangle}{\langle K,L\rangle}[N]^{-1}\diamond[H]\diamond \sum_{\varepsilon'\in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L',M)} \frac{[\mathbf{mt}(\varepsilon')]}{|\operatorname{Hom}(L',M)|} =$$

$$= \frac{\langle H,L'\rangle}{\langle K,L\rangle\,\langle H,L\rangle}[N]^{-1}\diamond[H]\diamond \sum_{\varepsilon'\in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L',M)} \frac{[\mathbf{mt}(\varepsilon')]}{|\operatorname{Hom}(L',M)|} =$$

$$= \frac{\langle H,L'\rangle}{\langle N,L\rangle}[N]^{-1}\diamond[H]\diamond \sum_{\varepsilon'\in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L',M)} \frac{[\mathbf{mt}(\varepsilon')]}{|\operatorname{Hom}(L',M)|}. \tag{4.20}$$

From condition (4.17) for $i=1$ for L' and L'' which are quasi-isomorphic to each other, we see that the morphism $\mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L',M)\to \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L'',M)$ arising from the long exact sequence of extensions of elements of the conflation $N \rightarrowtail L'' \overset{\text{qis}}{\twoheadrightarrow} L'$ by M is in fact an isomorphism. That means that for each class ε' in $\mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L',M)$ represented by a conflation

$$M \rightarrowtail E' \twoheadrightarrow L',$$

there is a unique class ε'' in $\mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L',M)$ represented by a conflation

$$M \rightarrowtail E'' \twoheadrightarrow L''$$

such that the following diagram commutes :

$$\begin{tikzcd}
0 \arrow[r, tail] \arrow[d, tail] & H \arrow[r, equal] \arrow[d, tail] & H \arrow[d, tail] \\
M \arrow[r, tail] \arrow[d, equal] & E'' \arrow[r, two heads] \arrow[d, two heads] & L'' \arrow[d, two heads] \\
M \arrow[r, tail] & E' \arrow[r, two heads] & L'.
\end{tikzcd} \tag{4.21}$$

Using the conflation

$$H \rightarrowtail E' \twoheadrightarrow E'',$$

we see that formula (4.20) can be rewritten as follows :

$$\frac{\langle H, L'\rangle}{\langle N, L\rangle}[N]^{-1} \diamond \sum_{\varepsilon'' \in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L'', M)} \frac{[\mathbf{mt}(\varepsilon'')]}{|\operatorname{Hom}(L', M)| \langle H, L'\rangle \langle H, M\rangle} =$$

$$= \frac{1}{\langle N, L\rangle \langle H, M\rangle}[N]^{-1} \diamond \sum_{\varepsilon'' \in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L'', M)} \frac{[\mathbf{mt}(\varepsilon'')]}{|\operatorname{Hom}(L', M)|} =$$

$$= \frac{1}{\langle N, L\rangle}[N]^{-1} \diamond \sum_{\varepsilon'' \in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L'', M)} \frac{[\mathbf{mt}(\varepsilon'')]}{|\operatorname{Hom}(L'', M)|},$$

as desired. In the last equation, we use nothing more than

$$\langle H, M\rangle = \frac{|\operatorname{Hom}(L'', M)|}{|\operatorname{Hom}(L', M)|}.$$

This identity holds, since, by assumption, in the long exact sequence of extensions of elements of the conflation $H \rightarrowtail L'' \overset{\text{qis}}{\twoheadrightarrow} L'$ by M all morphisms

$$\mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(L', M) \to \mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(L'', M)$$

are isomorphisms. This completes the proof. □

Proposition 4.2.16. *The products $K \diamond L$ and $L \diamond K$, given in this section coincide with those given in section 5.1, for an acyclic complex K and an arbitrary L.*

Proof. Denote the map defined in this section by $\diamond_1$, the one used in the definition of $\mathbb{T}_{ac}(\mathcal{E})-$action by $\diamond_2$. To compute the $K \diamond_1 L$, we consider some conflation $K'' \rightarrowtail K' \overset{\text{qis}}{\twoheadrightarrow} K$ such that

$$\mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(K', L) \cong \mathrm{Ext}^p_{\mathcal{D}^b(\mathcal{E})}(K', L), \forall p > 0.$$

Since K is acyclic, so is K'. Thus, we find that $\mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(K', L) = 0, \forall p > 0$. Hence

$$K \diamond_1 L = \frac{1}{\langle K'', K\rangle}[K'']^{-1} \diamond_2 \frac{1}{|\operatorname{Hom}(K', L)|}[K' \oplus L] =$$

$$= \frac{1}{\langle K'', K\rangle \langle K', L\rangle}[K'']^{-1} \diamond_2 [K' \oplus L] =$$

$$= \frac{1}{\langle K'', K\rangle \langle K'', L\rangle \langle K, L\rangle}[K'']^{-1} \diamond_2 [K' \oplus L]. \tag{4.22}$$

Using the conflation $K'' \rightarrowtail K' \oplus L \overset{\text{qis}}{\twoheadrightarrow} K \oplus L$, we get

$$[K''] \diamond_2 [K \oplus L] = \frac{1}{\langle K'', K \oplus L\rangle}[K' \oplus L].$$

Using this, we rewrite the expression (4.22) as $\frac{1}{\langle K,L\rangle}[K \oplus L]$, which equals $K \diamond_2 L$ by definition. For $L \diamond K$, the proof is similar. $\square$

Proposition 4.2.17. *The map $\diamond$ is compatible with the bimodule structure on $\mathcal{M}_1'(\mathcal{E})$ and with relations (4.14).*

Proof. It is easy to check that $\diamond$ is simultaneously compatible with the bimodule structure and with the relations if and only if for any conflations $K \rightarrowtail L \twoheadrightarrow M, N \rightarrowtail P \twoheadrightarrow Q$ with $K, Q \in \mathcal{C}^b_{ac}(\mathcal{E})$, we have

$$\frac{1}{\langle K, M\rangle}[L] \diamond [N] = [K] \diamond ([M] \diamond [N]); \frac{1}{\langle N, Q\rangle}[M] \diamond [P] = ([M] \diamond [N]) \diamond [Q]. \tag{4.23}$$

Thus, it is enough to check the identities (4.23). Using the pull-back as in Proposition (4.2.15), we can construct the following diagram :

$$\begin{tikzcd}
K'' \arrow[r, tail] \arrow[d, tail] & L'' \arrow[r, equal] \arrow[d, tail] & M'' \arrow[d, tail] \\
K' \arrow[r, tail] \arrow[d, two heads] & L' \arrow[r, two heads] \arrow[d, two heads] & M' \arrow[d, two heads] \\
K \arrow[r, tail] & L \arrow[r, two heads] & M,
\end{tikzcd} \tag{4.24}$$

where all elements in the first row and the first column are acyclic, and the last two columns can be used to define $L \diamond N$ and $M \diamond N$. We have

$$\frac{1}{\langle K, M\rangle}[L] \diamond [N] = \frac{1}{\langle K, M\rangle \langle L'', L\rangle}[L'']^{-1} \diamond \sum_{\varepsilon_L \in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L', N)} \frac{[\mathbf{mt}(\varepsilon_L)]}{|\operatorname{Hom}(L', N)|}. \tag{4.25}$$

Using the same arguments as in the proof of Proposition 4.2.15 and the diagram similar to (4.21), we can rewrite this as follows :

$$\frac{\langle K', N\rangle \langle K', M'\rangle}{\langle K, M\rangle \langle L'', L\rangle}[L'']^{-1} \diamond [K'] \diamond \sum_{\varepsilon_M \in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(M', N)} \frac{[\mathbf{mt}(\varepsilon_M)]}{|\operatorname{Hom}(L', N)|} =$$

$$= \frac{\langle K', N\rangle \langle K, M''\rangle \langle K'', M''\rangle}{\langle K'', K\rangle \langle M'', K\rangle \langle M'', M\rangle}[L'']^{-1} \diamond [K'] \diamond \sum_{\varepsilon_M \in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(M', N)} \frac{[\mathbf{mt}(\varepsilon_M)]}{|\operatorname{Hom}(L', N)|} =$$

$$= \frac{\langle K', N\rangle \langle K, M''\rangle}{\langle M'', K\rangle \langle M'', M\rangle}[M'']^{-1} \diamond [K] \diamond \sum_{\varepsilon_M \in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(M', N)} \frac{[\mathbf{mt}(\varepsilon_M)]}{|\operatorname{Hom}(L', N)|} =$$

$$= \frac{\langle K', N\rangle}{\langle M'', M\rangle}[K] \diamond [M'']^{-1} \diamond \sum_{\varepsilon_M \in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(M', N)} \frac{[\mathbf{mt}(\varepsilon_M)]}{|\operatorname{Hom}(L', N)|} =$$

$$= [K] \diamond \left(\frac{1}{\langle M'', M\rangle}[M'']^{-1} \diamond \sum_{\varepsilon_M \in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(M', N)} \frac{[\mathbf{mt}(\varepsilon_M)]}{|\operatorname{Hom}(L', N)|} \right) = [K] \diamond ([M] \diamond [N]).$$

In the last but one equation, we use that $\langle K', M\rangle = \frac{|\operatorname{Hom}(L', M)|}{|\operatorname{Hom}(M', M)|}$, cf. the end of the proof of Proposition 4.2.15. We proved the first of identities (4.23) ; the proof of the second one is dual. □

Since the homomorphism $\diamond$ is compatible with the bimodule structure, it can be considered as a module homomorphism

$$\mathcal{M}_1'(\mathcal{E}) \times \mathcal{M}_1'(\mathcal{E}) \to \mathcal{M}(\mathcal{E}).$$

Since it is compatible with relations (4.14), it descends on $\mathcal{M}_2'(\mathcal{E})$ and, moreover, defines a multiplication

$$\diamond : \mathcal{M}(\mathcal{E}) \times \mathcal{M}(\mathcal{E}) \to \mathcal{M}(\mathcal{E}).$$

Definition 4.2.18. We define the *semi-derived Hall algebra* $\mathcal{SDH}(\mathcal{E})$ as $\mathcal{M}(\mathcal{E})$ with the multiplication $\diamond$.

Remark 4.2.19. It is easy to check that the product $\diamond$ given by formulae (4.15)–(4.16) is the unique $\mathbb{T}_{ac}(\mathcal{E})-$bilinear multiplication on $\mathcal{M}(\mathcal{E})$ which coincides with that of the classical Hall algebra $\mathcal{H}(\mathcal{C}^b(\mathcal{E})$ on the pairs (L', M) satisfying (4.17).

We can give an alternative definition of the multiplication, where the summation is taken over the set of the isomorphism classes in the bounded derived category. It turns out that this gives us the structure constants.

Definition 4.2.20. Given a complex $M \in \mathcal{C}^b(\mathcal{E})$, denote by $\overline{M}$ its isomorphism class in the bounded derived category $\mathcal{D}^b(\mathcal{E})$. For any $\alpha \in \operatorname{Iso}(\mathcal{D}^b(\mathcal{E}))$ and any $L, M \in \mathcal{C}^b(\mathcal{E})$, define the subset

$$\mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(A, B)_\alpha \subset \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(A, B)$$

as the set of all extensions of L by M whose middle term belongs to α.

Lemma 4.2.21. *Assume that for $E, E' \in \mathcal{C}^b(\mathcal{E})$, we have :*

1) $\overline{E} = \overline{E'} = \alpha$ and

2) there exist $A, B \in \mathcal{C}^b(\mathcal{E})$, such that

$$\mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(A, B)_E \neq 0, \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(A, B)_{E'} \neq 0.$$

Then in $\mathcal{M}(\mathcal{E})$ these two complexes determine the same element : $[E] = [E']$.

Proof. By condition 1), both $[E]$ and $[E']$ belong to the same component $\mathcal{M}(\mathcal{E})_\alpha$ of $\mathcal{M}(\mathcal{E})$. By Theorem 4.2.12, there exists a unique element t of the quantum torus $\mathbb{T}_{ac}(\mathcal{E})$, such that $[E'] = t \diamond [E]$. By condition 2), the classes of $[E]$ and $[E']$ in $K_0(\mathcal{C}^b(\mathcal{E}))$ coincide, hence the class of t in $K_0(\mathcal{C}^b(\mathcal{E}))$ equals zero. By Lemma 4.2.13, this implies that the class of t in $K_0(\mathcal{C}^b_{ac}(\mathcal{E}))$ equals zero. Therefore, we have $t = [0]$ in $\mathbb{T}_{ac}(\mathcal{E}))$. We finally obtain that $[E'] = [0] \diamond [E] = [E]$, q.e.d. □

Corollary 4.2.22. *For all objects A, B of $\mathcal{C}^b(\mathcal{E})$, we have the following identity :*

$$\sum_{E \in \mathrm{Iso}(\mathcal{C}^b(\mathcal{E}))} (|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(A,B)_E|[E]) = \sum_{\alpha \in \mathrm{Iso}(\mathcal{D}^b(\mathcal{E}))} (|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(A,B)_\alpha|[E_{\alpha,A,B}]), \tag{4.26}$$

where for each α, the complex $E_{\alpha,A,B}$ is the middle term of any extension belonging to $\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(A,B)_\alpha$.

Proof. Lemma 4.2.21 implies that the right-hand side is well-defined. Now, for any $\alpha \in \mathrm{Iso}(\mathcal{D}^b(\mathcal{E}))$, we have, just by definition,

$$\sum_{E \in \mathrm{Iso}(\mathcal{C}^b(\mathcal{E})), \overline{E} = \alpha} |\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(A,B)_E| = |\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(A,B)_\alpha|$$

and if this number is non-zero, then for each E from the left-hand side, we have $[E] = [E_{\alpha,A,B}]$ in $\mathcal{M}(\mathcal{E})$. □

Corollary 4.2.23. *For each pair of bounded complexes $L, M \in \mathcal{C}^b(\mathcal{E})$, the product $[L] \diamond [M]$ in the semi-derived Hall algebra is equal to the following sum :*

$$[L] \diamond [M] = \frac{1}{\langle K, L \rangle}[K]^{-1} \diamond \sum_{\alpha \in \mathrm{Iso}(\mathcal{D}^b(\mathcal{E}))} \left(\frac{|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L',M)_\alpha|}{|\operatorname{Hom}(L',M)|}[E_{\alpha,L',M}]\right), \tag{4.27}$$

where

$$K \rightarrowtail L' \overset{\text{qis}}{\twoheadrightarrow} L$$

is a conflation such that L' satisfies

$$\operatorname{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(L',M) = \operatorname{Ext}^p_{\mathcal{D}^b(\mathcal{E})}(L',M), \quad \forall p > 0.$$

4.2.6 Associativity

Theorem 4.2.24. *For each triple of bounded complexes $A, B, C \in \mathcal{C}^b(\mathcal{E})$, we have*

$$([A] \diamond [B]) \diamond [C] = [A] \diamond ([B] \diamond [C]).$$

Proof. We have

$$([A]\diamond[B])\diamond[C] = (\frac{1}{\langle K_1, A\rangle}[K_1]^{-1}\diamond \sum_{\varepsilon\in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L_1,B)} (\frac{[\mathbf{mt}(\varepsilon)]}{|\operatorname{Hom}(L_1,B)|}))\diamond[C] =$$

$$= \frac{1}{\langle K_1, A\rangle}[K_1]^{-1}\diamond \sum_{\varepsilon\in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L_1,B)} (\frac{1}{|\operatorname{Hom}(L_1,B)|}([\mathbf{mt}(\varepsilon)]\diamond[C])).$$

Using part (ii) of condition (C4), we obtain that for each extension ε of L_1 by B represented by a conflation $B \rightarrowtail X \twoheadrightarrow L_1$, there exists a unique extension ε' of L_1 by L_2 represented by $L_2 \rightarrowtail Y \twoheadrightarrow L_1$, such that the following diagram commutes :

$$\begin{tikzcd}
K_2 \arrow[r, equal] \arrow[d, tail] & K_2 \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\
L_2 \arrow[r, tail] \arrow[d, two heads] & Y \arrow[r, two heads] \arrow[d, two heads] & L_1 \arrow[d, equal] \\
B \arrow[r, tail] & X \arrow[r, two heads] & L_1.
\end{tikzcd} \tag{4.28}$$

We have

$$[X] = \frac{1}{\langle K_2, X\rangle}[K_2]^{-1}\diamond[Y] = \frac{1}{\langle K_2, B\rangle\langle K_2, L_1\rangle}[K_2]^{-1}\diamond[Y],$$

i.e.

$$[\mathbf{mt}(\varepsilon)] = \frac{1}{\langle K_2, B\rangle\langle K_2, L_1\rangle}[K_2]^{-1}\diamond[\mathbf{mt}(\varepsilon')].$$

Then, by part (iii) of condition (C4), the expression above can be rewritten as follows :

$$\frac{1}{\langle K_1, A\rangle\langle K_2, B\rangle\langle K_2, L_1\rangle}([K_1]^{-1}\diamond[K_2]^{-1})\diamond\frac{|\operatorname{Hom}(L_1,L_2)|}{|\operatorname{Hom}(L_1,B)|}\times$$

$$\times(\sum_{\varepsilon'\in \mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L_1,L_2)} \frac{1}{|\operatorname{Hom}(L_1,L_2)|}[\mathbf{mt}(\varepsilon')]\diamond[C]) =$$

$$= \frac{|\operatorname{Hom}(L_1,L_2)|}{\langle K_1, A\rangle\langle K_2, B\rangle\langle K_2, L_1\rangle|\operatorname{Hom}(L_1,B)|}([K_1]^{-1}\diamond[K_2]^{-1})\diamond(([L_1]\bullet[L_2])\diamond[C]),$$

where $\bullet$ is the usual Hall product in the exact category $\mathcal{C}^b(\mathcal{E})$. Thus, by part (iii) of condition (C4), both multiplications in the formula $(([L_1]\diamond[L_2])\diamond[C])$ are the usual Hall multiplications. Therefore, we can use the associativity property for this expression and rewrite the last formula in the above calculations as follows :

$$([A]\diamond[B])\diamond[C] = \frac{|\operatorname{Hom}(L_1,L_2)|}{\langle K_1, A\rangle\langle K_2, B\rangle\langle K_2, L_1\rangle|\operatorname{Hom}(L_1,B)|}([K_1]^{-1}\diamond[K_2]^{-1})\diamond([L_1]\bullet([L_2]\bullet[C])).$$

Using the associativity of the product with $[K]^{-1}$ and the commutation rule from Lemma 4.2.10 for $[L_1]$ and $[K_2]^{-1}$, we obtain that this is equal to :

$$\frac{|\operatorname{Hom}(L_1,L_2)|}{\langle K_1,A\rangle\langle K_2,B\rangle\langle K_2,L_1\rangle|\operatorname{Hom}(L_1,B)|}[K_1]^{-1}\diamond\frac{1}{|\operatorname{Hom}(L_1,K_2)|}\langle K_2,L_1\rangle[L_1]\bullet([K_2]^{-1}\diamond([L_2]\bullet[C])).$$

Just by definition (and since the two $\langle K_2,L_1\rangle$ cancel each other), it equals

$$\frac{|\operatorname{Hom}(L_1,L_2)|}{\langle K_1,A\rangle|\operatorname{Hom}(L_1,B)||\operatorname{Hom}(L_1,K_2)|}[K_1]^{-1}\diamond[L_1]\bullet([B]\diamond[C])=$$

$$=\frac{|\operatorname{Hom}(L_1,L_2)|}{|\operatorname{Hom}(L_1,B)||\operatorname{Hom}(L_1,K_2)|}[A]\diamond([B]\diamond[C]).$$

It is easy to check that part (ii) of condition (C4) implies that

$$\frac{|\operatorname{Hom}(L_1,L_2)|}{|\operatorname{Hom}(L_1,B)||\operatorname{Hom}(L_1,K_2)|}=1;$$

therefore, we have $([A]\diamond[B])\diamond[C]=[A]\diamond([B]\diamond[C])$. $\square$

4.2.7 Derived invariance

In this section, we prove that our algebra is invariant under some (quite big) class of derived equivalences :

Proposition 4.2.25. *An equivalence of bounded derived categories of exact categories*

$$F:\mathcal{D}^b(\mathcal{E}')\xrightarrow{\sim}\mathcal{D}^b(\mathcal{E})$$

induces the equivalence of bounded derived categories of their categories of bounded complexes :

$$F:\mathcal{D}^b(\mathcal{C}^b(\mathcal{E}'))\xrightarrow{\sim}\mathcal{D}^b(\mathcal{C}^b(\mathcal{E})).$$

Proof. Denote by $\mathcal{C}^{[-N,N]}(\mathcal{E})$ the category of complexes over $\mathcal{E}$, concentrated in degrees $-N,-N+1,\dots,N$. Since we have

$$\mathcal{D}^b(\mathcal{C}^b(\mathcal{E}'))=\bigcup_{N=0}^{\infty}\mathcal{D}^b(\mathcal{C}^{[-N,N]}(\mathcal{E}')),$$

and similarly for $\mathcal{E}$, it is enough to prove that F induces an equivalence

$$\mathcal{D}^b(\mathcal{C}^{[-N,N]}(\mathcal{E}'))\overset{F}{\xrightarrow{\sim}}\mathcal{D}^b(\mathcal{C}^{[-N,N]}(\mathcal{E})),$$

for all $N\in\mathbb{Z}_{\geq 0}$. For this we use the same “dévissage” trick as in the proof of Proposition 4.2.8.

We start by recalling the family of evaluation functors

$$ev_k : \mathcal{C}^{[-N,N]}(\mathcal{E}) \to \mathcal{E}, k = -N, -N+1, \dots, N$$

$$(\dots \to 0 \to A^{-N} \to \dots \to A^k \to \dots \to A^N \to 0 \to \dots) \quad \overset{ev_k}{\mapsto} \quad A^k.$$

Each of these functors is exact and admits an exact left adjoint $ev_k^\lambda : \mathcal{E} \to \mathcal{C}^{[-N,N]}(\mathcal{E})$:

$$A \quad \overset{ev_{-N}^\lambda}{\mapsto} \quad (\dots \to 0 \to A \to 0 \to \dots \to 0 \to \dots \to 0 \to 0 \to \dots),$$

$$A \quad \overset{ev_k^\lambda}{\mapsto} \quad (\dots \to 0 \to 0 \to 0 \to \dots \to A = A \to \dots \to 0 \to 0 \to \dots),$$

for $-N+1 \leq k \leq N$. Here A sits in the degree $-N$ and in degrees $k-1$ and k, respectively. Since ev_k and ev_k^λ are exact adjoint, they induce a pair of adjoint functors between $\mathcal{D}^b(\mathcal{C}^{[-N,N]}(\mathcal{E}))$ and $\mathcal{D}^b(\mathcal{E})$, which we will denote by the same symbols ; this holds for each $k = -N, -N+1, \dots, N$. We consider similar functors ev'_k and ${ev'_k}^\lambda$ for $\mathcal{E}'$. Then for each $X, Y \in \mathcal{D}^b(\mathcal{E}'), k, l \in [-N, N]$, we have isomorphisms

$$\mathrm{Hom}_{\mathcal{D}^b(\mathcal{C}^{[-N,N]}(\mathcal{E}'))}(ev_k^\lambda X, ev_l^\lambda Y) = \mathrm{Hom}_{\mathcal{D}^b(\mathcal{E}')}(X, ev_k \circ ev_l^\lambda Y) =$$

$$= \mathrm{Hom}_{\mathcal{D}^b(\mathcal{E})}(FX, F(ev_k \circ ev_l^\lambda Y)) = \mathrm{Hom}_{\mathcal{D}^b(\mathcal{E})}(FX, ev'_k \circ {ev'_l}^\lambda(FY)) =$$

$$= \mathrm{Hom}_{\mathcal{D}^b(\mathcal{C}^{[-N,N]}(\mathcal{E}'))}({ev'_k}^\lambda FX, {ev'_l}^\lambda FY).$$

It follows that it remains to show that the images of $\mathcal{D}^b(\mathcal{E})$ under $ev_k^\lambda, k = -N, -N+1, \dots, N$ generate $\mathcal{D}^b(\mathcal{C}^{[-N,N]}(\mathcal{E}))$ by triangles, and similarly for $\mathcal{E}'$. Let us prove that these images generate the subcategories $\mathcal{D}^b(\mathcal{C}_k^b(\mathcal{E}'))$ and $\mathcal{D}^b(\mathcal{C}_k^b(\mathcal{E}))$ generated by the complexes concentrated in degree k, for $k \in [-N, N]$. We give the proof by induction on k, and write it only for $\mathcal{E}$, since for $\mathcal{E}'$ everything is the same. For $k = -N$ the statement is clear by the definition of ev_{-N}^λ. Assume that we have proved the statement for $k \leq l$. Consider an arbitrary stalk complex

$$A^\bullet := \dots \to 0 \to \dots \to 0 \to A \to 0 \to \dots \to 0 \to \dots,$$

with A sitting in degree l. Then there is a conflation of complexes $\Sigma^{-1}A^\bullet \rightarrowtail C(1_{\Sigma^{-1}A^\bullet}) \twoheadrightarrow A^\bullet$ inducing a triangle in $\mathcal{D}^b(\mathcal{C}^{[-N,N]}(\mathcal{E}))$. But $\Sigma^{-1}A^\bullet \in \mathcal{D}^b(\mathcal{C}_{l-1}^b(\mathcal{E}))$, hence it is in the subcategory of $\mathcal{D}^b(\mathcal{C}^{[-N,N]}(\mathcal{E}))$ generated by the images $C(1_{\Sigma^{-1}A^\bullet}) = ev_l^\lambda(A)$. Therefore all stalk complexes in $\mathcal{D}^b(\mathcal{C}_l^b(\mathcal{E}))$ are in the subcategory generated by the images of ev_k^λ, but the stalk complexes generate by triangles the whole $\mathcal{D}^b(\mathcal{C}_l^b(\mathcal{E}))$.

Now we verify easily that the closure under extensions of the $\mathcal{D}^b(\mathcal{C}_l^b(\mathcal{E}))$, for $l = -N, -N+1, \dots, N$, is $\mathcal{D}^b(\mathcal{C}^{[-N,N]}(\mathcal{E}))$, and this completes the proof. $\square$

Proposition 4.2.26. *Under the assumptions of Proposition 4.2.25, we have an equivalence of*

derived categories of the categories of acyclic bounded complexes :

$$\mathcal{D}^b(\mathcal{C}^b_{ac}(\mathcal{E}')) \overset{F}{\xrightarrow{\sim}} \mathcal{D}^b(\mathcal{C}^b_{ac}(\mathcal{E}))$$

which induces an isomorphism of the Grothendieck groups

$$\mathbf{K_0}(\mathcal{C}^b_{ac}(\mathcal{E}')) \xrightarrow{\sim} \mathbf{K_0}(\mathcal{C}^b_{ac}(\mathcal{E}))$$

and respects the Euler form $\langle \cdot, \cdot \rangle$ *.*

Proof. By the equivalence of $\mathcal{D}^b(\mathcal{C}^b(\mathcal{E}'))$ and $\mathcal{D}^b(\mathcal{C}^b(\mathcal{E}'))$, it is sufficient to prove that the canonical functors

$$\mathcal{D}^b(\mathcal{C}^b_{ac}(\mathcal{E}')) \hookrightarrow \mathcal{D}^b(\mathcal{C}^b(\mathcal{E}')), \mathcal{D}^b(\mathcal{C}^b_{ac}(\mathcal{E})) \hookrightarrow \mathcal{D}^b(\mathcal{C}^b(\mathcal{E})) \tag{4.29}$$

are fully faithful and that an object A' of $\mathcal{D}^b(\mathcal{C}^b(\mathcal{E}'))$ lies in the essential image of $\mathcal{D}^b(\mathcal{C}^b_{ac}(\mathcal{E}'))$ iff $F(A')$ lies in the essential image of $\mathcal{D}^b(\mathcal{C}^b_{ac}(\mathcal{E}))$. The functors (4.29) are fully faithful by [38, Theorem 12.1] applied to $\mathcal{B} = \mathcal{C}^b_{ac}(\mathcal{E}), \mathcal{A} = \mathcal{C}^b(\mathcal{E})$. The followin lemma provides a functorial characterization of complexes of complexes, isomorphic in $\mathcal{D}^b(\mathcal{C}^b(\mathcal{E}))$ to complexes of acyclics, that gives us the desired condition on essential images and, therefore, completes the proof.

Lemma 4.2.27. *A complex* $X \in \mathcal{C}^b(\mathcal{C}^b(\mathcal{E}))$ *is isomorphic in* $\mathcal{D}^b(\mathcal{C}^b(\mathcal{E}))$ *to an object of* $\mathcal{D}^b(\mathcal{C}^b_{ac}(\mathcal{E}))$ *if and only if its total complex* $\mathrm{Tot}(X)$ *is acyclic, i.e. the functor*

$$\mathrm{Tot} : \mathcal{D}^b(\mathcal{C}^b(\mathcal{E})) \to \mathcal{D}^b(\mathcal{E})$$

sends X *to 0.*

Proof. We start by an argument similar to Lemma 4.2.4 : for each complex of complexes $X \in \mathcal{C}^b(\mathcal{C}^b(\mathcal{E}))$ there exists a deflation quasi-isomorphism $X' \overset{\text{qis}}{\twoheadrightarrow} X$, where all but one of the components of X' are acyclic. Indeed, for each complex $X^p \in \mathcal{C}^b(\mathcal{E})$ there exists a deflation from an acyclic complex $C(1_{\Sigma^{-1}X})$ to X^p. Without loss of generality, we can assume that X is of form

$$X : \ldots \to 0 \to X^0 \to \ldots \to X^n \to 0 \to \ldots, \quad X^p \in \mathcal{C}^b(\mathcal{E}).$$

We will construct the components of X' step by step. Take a deflation $A^n \twoheadrightarrow X^n$ with A^n acyclic. Then take the pullback of a morphism $X^{n-1} \to X^n$ and this deflation, and define A^{n-1} (with a

morphism $A^{n-1} \to A_n$ and a deflation $A^{n-1} \twoheadrightarrow X^{n-1}$) by an acyclic deflation to this pullback :

$$\begin{tikzcd}
A^{n-1} \arrow[dr, two heads] \arrow[drr, dotted, two heads] \arrow[ddr, dotted, two heads] & & & \\
& \bullet \arrow[r] \arrow[d, two heads] & A^n \arrow[r] \arrow[d, two heads] & 0 \arrow[d, equal] \\
& X^{n-1} \arrow[r] & X^n \arrow[r] & 0
\end{tikzcd} \tag{4.30}$$

In similar manner we construct $A^{n-2}, \dots, A^1, A^0$. At the last step, we do not consider a deflation from an acyclic complex, just a pullback B given by the diagram

$$\begin{tikzcd}
B \arrow[r] \arrow[d, two heads] & A^0 \arrow[d, two heads] \\
0 \arrow[r] & X^0
\end{tikzcd} \tag{4.31}$$

By construction, we get a deflation quasi-isomorphism $X' \twoheadrightarrow X$, where

$$X' = \dots \to 0 \to B \to A^0 \to \dots \to A^n \to 0 \to \dots.$$

Now we see that X is isomorphic in $\mathcal{D}^b(\mathcal{C}^b(\mathcal{E}))$ to a complex from $\mathcal{D}^b(\mathcal{C}^b_{ac}(\mathcal{E}))$ if and only if X' is. But all A^i are acyclic, hence this last condition holds if and if B is acyclic. Clearly, in this and only this case we have $\mathrm{Tot}(X') = 0$. But Tot is a functor from the derived category $\mathcal{D}^b(\mathcal{C}^b(\mathcal{E}))$, which gives us the desired criterion. $\square$

Corollary 4.2.28. *Under the assumptions of Proposition 4.2.25, we have an isomorphism of quantum tori of acyclic complexes :*

$$\mathbb{T}_{ac}(\mathcal{E}') \xrightarrow{\sim} \mathbb{T}_{ac}(\mathcal{E}).$$

Theorem 4.2.29. *Under the assumptions of Proposition 4.2.25, we have an isomorphism of algebras :*

$$\mathcal{SDH}(\mathcal{E}') \xrightarrow{\sim} \mathcal{SDH}(\mathcal{E}).$$

Proof. By the Corollary 4.2.28, we have an isomorphism $\mathcal{M}(\mathcal{E}') \xrightarrow{\sim} \mathcal{M}(\mathcal{E})$ of free modules over isomorphic quantum tori with bases which are in bijection by the derived equivalence. Since we can consider all bounded complexes as stalk complexes in $\mathcal{D}^b(\mathcal{C}^b(\mathcal{E}'))$ and $\mathcal{D}^b(\mathcal{C}^b(\mathcal{E}))$ respectively, all extensions in the categories of bounded complexes are just morphisms in these derived categories, hence they are preserved under the functor $\mathcal{D}^b(\mathcal{C}^b(\mathcal{E}')) \to \mathcal{D}^b(\mathcal{C}^b(\mathcal{E}))$ induced by F. It follows

that the multiplication of the semi-derived Hall algebra is preserved as well, which completes the proof. □

4.2.8 Case with enough projectives

Throughout this section, we assume that our category $\mathcal{E}$ satisfies the following conditions :

1) $\mathcal{E}$ has enough projectives ;
2) each object in $\mathcal{E}$ has a finite projective resolution.

Here we record some simple facts concerning extensions in the category $\mathcal{C}^b(\mathcal{E})$.

Lemma 4.2.30. *For each complex $K \in \mathcal{C}^b_{ac}(\mathcal{E})$, there exists an acyclic complex of complexes*

$$0 \to P_d \to P_{d-1} \to \ldots \to P_1 \to P_0 \to K \to 0, \tag{4.32}$$

where d is the maximum of the projective dimensions of the components of K; and the P_i are bounded acyclic complexes of projectives.

Proof. By Lemma 4.2.4, there exists a conflation whose deflation is a quasi-isomorphism

$$K_0 \rightarrowtail P_0 \overset{\text{qis}}{\twoheadrightarrow} K,$$

with $P_0 \in \mathcal{C}^b(\mathcal{P})$. Then P_0 is acyclic, and from the long exact sequences of extensions of components by arbitrary objects of $\mathcal{E}$, it follows that the components of K_1 are of projective dimension at most $d-1$. Now we prove the statement by induction on the maximal projective dimension of the components of our acyclic complex. □

Corollary 4.2.31. *For each complex $M \in \mathcal{C}^b(\mathcal{E})$, there exists an acyclic complex of complexes*

$$0 \to P_d \to P_{d-1} \to \ldots \to P_1 \to P_0 \overset{\text{qis}}{\twoheadrightarrow} M \to 0, \tag{4.33}$$

where d is the maximum of the projective dimensions of the components of M and the P_i are bounded complexes of projectives which are acyclic for $i \geq 1$.

Proof. By Lemma 4.2.4 there exists a conflation whose deflation is a quasi-isomorphism

$$K_0 \rightarrowtail P_0 \overset{\text{qis}}{\twoheadrightarrow} M,$$

with $P_0 \in \mathcal{C}^b(\mathcal{P})$. Now apply Lemma 4.2.30 to K_0, whose components have projective dimension at most $d-1$. □

Corollary 4.2.32. *For each acyclic complex $K \in \mathcal{C}^b_{ac}(\mathcal{E})$ and each $L \in \mathcal{C}^b(\mathcal{E})$ the following holds :*

$$\mathrm{Ext}^p(K, L) = 0, \quad \forall p > d,$$

where d is the maximum of projective dimensions of the components of K.

Proof. By Proposition 4.2.3, the complex (4.32) is a projective resolution of K in $\mathcal{C}^b(\mathcal{E})$. $\square$

Corollary 4.2.33. *For each pair of complexes $L, M \in \mathcal{C}^b(\mathcal{E})$ the following holds :*

$$\mathrm{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(M, L) = \mathrm{Ext}^p_{\mathcal{D}^b(\mathcal{E})}(M, L), \quad \forall p > d,$$

where d is the maximum of projective dimensions of the components of M.

Proof. Write down the long exact sequences of extensions in $\mathcal{C}^b(\mathcal{E})$ and in $\mathcal{D}^b(\mathcal{E})$ induced by morphisms from elements of a conflation $K_0 \rightarrowtail P_0 \overset{\text{qis}}{\twoheadrightarrow} M$, with $P_0 \in \mathcal{C}^b(\mathcal{P})$, to L; and apply Corollary 4.2.33 to K_0 and L and the five-lemma to these two sequences. Note that the maximum of projective dimensions of K_0 is $d-1$. $\square$

With the conditions 1) and 2) on $\mathcal{E}$, Proposition 4.2.4 establishes that the inclusion $\mathcal{P} \hookrightarrow \mathcal{E}$ induces an equivalence of derived categories $\mathcal{D}^b(\mathcal{P}) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{E})$. Then Theorem 4.2.29 provides an isomorphism

$$I : \mathcal{SDH}(\mathcal{P}) \xrightarrow{\sim} \mathcal{SDH}(\mathcal{E}), \quad [P] \mapsto [P]. \tag{4.34}$$

Proposition 4.2.34. *The map*

$$F : [M] \mapsto \frac{1}{|\mathrm{Hom}(\bigoplus_{k\in\mathbb{Z}_{\geq 0}} P_{2k+1}, M)|}[\bigoplus_{k\in\mathbb{Z}_{\geq 0}} P_{2k+1}]^{-1} \diamond [\bigoplus_{k\in\mathbb{Z}_{\geq 0}} P_{2k}];$$

$$[K]^{-1} \mapsto [F([K])]^{-1}, \quad K \in \mathcal{C}^b_{ac}(\mathcal{E})$$

provides an isomorphism

$$\mathcal{SDH}(\mathcal{E}) \xrightarrow{\sim} \mathcal{SDH}(\mathcal{P}),$$

inverse to the isomorphism (4.34).

It is easy to check that F is inverse to I as a map. I being an isomorphism, so is F.

Note that all extensions in $\mathcal{C}^b(\mathcal{P})$ are isomorphic to those in $\mathcal{D}^b(\mathcal{P})$, therefore the multiplication in $\mathcal{SDH}(\mathcal{P})$ is just the usual Hall multiplication in the exact category $\mathcal{C}^b(\mathcal{P})$. It means that $\mathcal{SDH}(\mathcal{P})$ is the usual Hall algebra of $\mathcal{C}^b(\mathcal{P})$, localized at the classes of acyclic complexes (all of which are contractible) :

Proposition 4.2.35. $\mathcal{SDH}(\mathcal{P}) = \mathcal{H}(\mathcal{C}^b(\mathcal{P}))[[K]^{-1} | K \in \mathcal{C}^b(\mathcal{P})]$.
If $\mathcal{E}$ has enough projectives and any object in $\mathcal{E}$ has finite projective resolution, then

$$\mathcal{SDH}(\mathcal{E}) = \mathcal{H}(\mathcal{C}^b(\mathcal{P}))[[K]^{-1} | K \in \mathcal{C}^b(\mathcal{P})].$$

4.2.9 Hereditary case

It turns out that if the category $\mathcal{E}$ is hereditary, i.e of global dimension 1, and has enough projectives, then the multiplication in our algebra is given by the same formula as the usual Ringel-Hall multiplication.

Theorem 4.2.36. *Assume that $\mathcal{E}$ is hereditary and has enough projectives. Then for $L, M \in \mathcal{C}^b(\mathcal{E})$, the product $[L] \diamond [M]$ in the semi-derived Hall algebra is equal to the following sum :*

$$[L] \diamond [M] = \sum_{X \in \mathrm{Iso}(\mathcal{C}^b(\mathcal{E}))} \frac{|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L, M)_X|}{|\operatorname{Hom}(L, M)|}[X],$$

Proof. We can choose a conflation $K \rightarrowtail L' \overset{\text{qis}}{\twoheadrightarrow} L$ with $K \in \mathcal{C}^b_{ac}(\mathcal{P}), L' \in \mathcal{C}^b(\mathcal{P})$ to write down the product $[L] \diamond [M]$ (if L' has projective components, then, by the heredity assumption, K has projective components as well). By Proposition 4.2.3, we have

$$\operatorname{Ext}^p_{\mathcal{C}^b(\mathcal{E})}(K, M) = \operatorname{Ext}^p_{\mathcal{D}^b(\mathcal{E})}(K, M) = 0, \quad \forall p > 0,$$

hence $\langle K, M \rangle = |\operatorname{Hom}(K, M)|$. Consider the part of the long exact sequence of extensions in the category $\mathcal{C}^b(\mathcal{E})$:

$$\begin{aligned} 0 \to \operatorname{Hom}(L, M) \to \operatorname{Hom}(L', M) \to \operatorname{Hom}(K, M) \to \\ \to \operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L, M) \to \operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L', M) \to \operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(K, M) = 0. \end{aligned} \tag{4.35}$$

We find out that $\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L, M)$ surjects onto $\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L', M) = \operatorname{Ext}^1_{\mathcal{D}^b(\mathcal{E})}(L, M)$. That means that for each extension $\varepsilon \in \operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L, M)$ represented by

$$M \rightarrowtail X \twoheadrightarrow L$$

there exists a unique extension $\varepsilon' \in \operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L', M)$ represented by

$$M \rightarrowtail E \twoheadrightarrow L'$$

such that the following diagram commutes :

$$\begin{tikzcd} 0 \arrow[r, equal] \arrow[d, tail] & K \arrow[r, equal] \arrow[d, tail] & K \arrow[d, tail] \\ M \arrow[r, tail] \arrow[d, equal] & E \arrow[r, two heads] \arrow[d, two heads] & L' \arrow[d, two heads] \\ M \arrow[r, tail] & X \arrow[r, two heads] & L \end{tikzcd} \tag{4.36}$$

and for each ε' there are exactly $\frac{|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L,M)|}{|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L',M)|}$ of ε corresponding to ε'. We see that for each such pair $(\varepsilon, \varepsilon')$, we have

$$[\mathbf{mt}(\varepsilon')] = \langle K, M\rangle \langle K, L\rangle [K] \diamond [\mathbf{mt}(\varepsilon)],$$

cf. proofs of Proposition 4.2.15 and Theorem 4.2.24. We get

$$[L] \diamond [M] = \frac{1}{\langle K, L\rangle}[K]^{-1} \diamond \sum_{\varepsilon' \in \operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L',M)} \frac{[\mathbf{mt}(\varepsilon')]}{|\operatorname{Hom}(L', M)|} =$$

$$= \frac{1}{\langle K, L\rangle}[K]^{-1} \frac{|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L', M)|}{|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L, M)|} \diamond \sum_{\varepsilon \in \operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L,M)} \frac{\langle K, M\rangle \langle K, L\rangle [K] \diamond [\mathbf{mt}(\varepsilon)]}{|\operatorname{Hom}(L', M)|} =$$

$$= (|\operatorname{Hom}(K, M)| \frac{|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L', M)|}{|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L, M)|} \frac{|\operatorname{Hom}(L, M)|}{|\operatorname{Hom}(L', M)|}) \sum_{\varepsilon \in \operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L,M)} \frac{[\mathbf{mt}(\varepsilon)]}{|\operatorname{Hom}(L, M)|} =$$

$$= (|\operatorname{Hom}(K, M)| \frac{|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L', M)|}{|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L, M)|} \frac{|\operatorname{Hom}(L, M)|}{|\operatorname{Hom}(L', M)|}) \sum_{X \in \operatorname{Iso}(\mathcal{C}^b(\mathcal{E}))} \frac{|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L, M)_X|}{|\operatorname{Hom}(L, M)|}[X].$$

It follows from the long exact sequence (4.35) that

$$|\operatorname{Hom}(K, M)| \frac{|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L', M)|}{|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{E})}(L, M)|} \frac{|\operatorname{Hom}(L, M)|}{|\operatorname{Hom}(L', M)|} = 1,$$

and we find out the desired formula. □

Corollary 4.2.37. *Assume that $\mathcal{E}$ is hereditary and has enough projectives. Then there exists an algebra homomorphism*

$$p : \mathcal{H}(\mathcal{C}^b(\mathcal{E})) \to \mathcal{SDH}(\mathcal{E}).$$

The homomorphism p induces an algebra isomorphism

$$(\mathcal{H}(\mathcal{C}^b(\mathcal{E}))/I)[S^{-1}] \xrightarrow{\sim} \mathcal{SDH}(\mathcal{E}), \tag{4.37}$$

where I is the two-sided ideal generated by all differences $[L] - [K \oplus M]$, where $K \rightarrowtail L \twoheadrightarrow M$ is a conflation in $\mathcal{C}^b(\mathcal{E})$ with acyclic K, and S is the set of all classes $[K]$ of acyclic complexes.

The following known result is useful for us to give a set of generators of $\mathcal{SDH}(\mathcal{A})$:

Lemma 4.2.38. *Suppose that $\mathcal{A}$ is a hereditary abelian category. Then each object in $\mathcal{C}^b(\mathcal{A})$ is quasi-isomorphic to the direct sum of its homology objects, where H^i is concentrated in the degree i. Equivalently, each object in $\mathcal{D}^b(\mathcal{A})$ is isomorphic to a direct sum of indecomposable stalk complexes.*

The proof can be found, e.g., in [35]. Let us denote by $u_{A,m}$ a stalk complex with non-zero component A, concentrated in the degree m. Let us also define the complexes $v_{A,m}$, $A \in \mathcal{A}$, by

$$v_{A,m} := \ldots \to 0 \to 0 \to A \xrightarrow{1_A} A \to 0 \to 0 \to \ldots,$$

where A sits in degrees m and $(m+1)$. For any class $\alpha \in K_0(\mathcal{A})$ and for any integer m, let us define the class $v_{\alpha,m}$ as the product

$$[v_{A,m}] \diamond [v_{B,m}]^{-1},$$

for any $A, B \in \mathcal{A}$ such that $\alpha = A - B$ in the Grothendieck group $K_0(\mathcal{A})$.

From now on up to the end of the section, we suppose that $\mathcal{A}$ is a hereditary abelian category with enough projectives, satisfying all the conditions (C1)-(C3).

Proposition 4.2.39. *$\mathcal{SDH}(\mathcal{A})$ is generated by the classes of complexes $u_{A,m}$ and the classes $v_{\alpha,m}$.*

Proof. From Theorem 4.2.12, it follows that $\mathcal{SDH}(\mathcal{A})$ has a basis over $\mathbb{Q}$, each of whose elements can be presented as the product of an element of the quantum torus by the class of an object in $\mathcal{D}^b(\mathcal{A})$. It is easy to verify that for any acyclic bounded complex M and any integer i, we have

$$[M] = a_i[\tau_{\geq i}M] \diamond [\tau_{<i}M],$$

for a certain $a_i \in \mathbb{Q}$. Recall that $\tau_{\geq i}M, \tau_{<i}M$ are the intelligent truncations of M. By induction, we prove that $\mathbb{A}_{ac}(\mathcal{A})$ is generated by the classes of the complexes $v_{A,m}$. Then it is clear that $\mathbb{T}_{ac}(\mathcal{A})$ is generated by the $v_{\alpha,m}$. Similarly, we prove that the class of any direct sum $\bigoplus\limits_{n_1 \leq i \leq n_2} A_i$, where A_i is a stalk complex concentrated in the degree i, equals, up to a rational multiplicative coefficient, the product

$$[A_{n_1}] \diamond [A_{n_1+1}] \diamond \cdots \diamond [A_{n_2}],$$

i.e. a product of classes of $u_{A_i,i}$. By Lemma 4.2.38, any quasi-isomorphism class in $\mathcal{C}^b(\mathcal{A})$ contains such a direct sum. $\square$

Lemma 4.2.40. *There is a family of injective ring homomorphisms*

$$I_m : \mathcal{H}(\mathcal{A}) \hookrightarrow \mathcal{SDH}(\mathcal{A}), \quad [A] \mapsto [u_{A,m}].$$

Proof. It is enough to prove that, for any $A, B \in \mathcal{A}$, we have

$$\sum_{\mathcal{C}} \frac{|\operatorname{Ext}^1_{\mathcal{A}}(A,B)_C|}{|\operatorname{Hom}_{\mathcal{A}}(A,B)|} u_{C,m} = \sum_{\mathcal{X}} \frac{|\operatorname{Ext}^1_{\mathcal{C}^b(\mathcal{A})}(u_{A,m}, u_{B,m})_X|}{|\operatorname{Hom}_{\mathcal{C}^b(\mathcal{A})}(u_{A,m}, u_{B,m})|} X,$$

which is easy to check. $\square$

Let us write down some simple relations on generators in $\mathcal{SDH}(\mathcal{A})$.

Lemma 4.2.41. *Assume that for $A, B \in \mathcal{A}$, we have $\mathrm{Hom}(A, B) = 0$. Then in $\mathcal{SDH}(\mathcal{A})$ we have*

$$[u_{A,m}] \diamond [u_{B,n}] = [u_{A,m} \oplus u_{B,n}],$$

for any $m \neq n \in \mathbb{Z}$.

Lemma 4.2.42. *Assume that for $A \in \mathcal{A}$, we have $\dim \mathrm{Hom}(A, A) = 1$. Assume also that $\mathcal{A}$ is linear over a finite field $k = \mathbb{F}_q$. Then in $\mathcal{SDH}(\mathcal{A})$ we have*

$$[u_{A,m}] \diamond [u_{A,n}] = [u_{A,m} \oplus u_{A,n}] + (q-1)\delta_m^{n-1}[v_{A,n}],$$

for any $m \neq n \in \mathbb{Z}$. As a corollary, we have

$$[v_{A,n}] = (q-1)^{-1}([u_{A,n}] \diamond [u_{A,(n+1)}] - [u_{A,(n+1)}] \diamond [u_{A,n}]), \quad \forall n \in \mathbb{Z}.$$

To prove both of these lemmas, one should use the form of the product given in Theorem 4.2.36. Note that

$$\mathrm{Hom}_{\mathcal{C}^b(\mathcal{A})}(u_{A,m}, u_{B,n}) = 0, \quad \forall A, B \in \mathcal{A}, \forall m \neq n \in \mathbb{Z},$$

and $\mathrm{Ext}^1_{\mathcal{C}^b(\mathcal{A})}(u_{A,m}, u_{B,n})$ can be easily computed.

We can also calculate the value of the Euler form on pairs of generators.

Lemma 4.2.43. *For any $A, B \in \mathcal{A}$ and any $m, n \in \mathbb{Z}$, we have*

$$\langle v_{\alpha,m}, u_{B,n} \rangle = \delta_m^n \langle \alpha, B \rangle_{\mathcal{A}}; \quad \langle v_{\alpha,m}, v_{\beta,n} \rangle = (\delta_m^n + \delta_m^{n+1}) \langle \alpha, \beta \rangle_{\mathcal{A}};$$

$$\langle u_{B,n}, v_{\alpha,m} \rangle = \delta_m^{n-1} \langle B, \alpha \rangle_{\mathcal{A}}.$$

Corollary 4.2.44. *For any $A, B \in \mathcal{A}$ and any $n > m \in \mathbb{Z}$, we have*

$$\langle u_{A,m}, u_{B,n} \rangle = (\langle A, B \rangle_{\mathcal{A}})^{(-1)^{(n-m)}}; \quad \langle u_{A,m}, u_{B,m} \rangle = \langle A, B \rangle_{\mathcal{A}};$$

$$\langle u_{B,n}, u_{A,m} \rangle = 1.$$

One may twist the multiplication in $\mathcal{SDH}(\mathcal{A})$ in different ways. Since the classes of the stalk complexes yield the full set of generators, we can define the twist of their products and then extend it by bilinearity. Here are some natural twists :

1) $[u_{A,m}] *_1 [u_{B,n}] := \langle u_{A,m}, u_{B,n} \rangle [u_{A,m}] \diamond [u_{B,n}]$. With this twist, the quantum torus becomes commutative;
2) $[u_{A,m}] *_2 [u_{B,n}] := \sqrt{\langle u_{A,m}, u_{B,n} \rangle} [u_{A,m}] \diamond [u_{B,n}]$. This twist is probably the most natural analogue of Ringel's twist;
3) $[u_{A,m}] *_3 [u_{B,n}] := (\sqrt{\langle A, B \rangle_{\mathcal{A}}})^{\delta_m^n} [u_{A,m}] \diamond [u_{B,n}]$. This twist is a $\mathbb{Z}-$graded analogue of the twist used in [5];

4) $[u_{A,m}] *_4 [u_{B,n}] := (\sqrt{\langle A, B\rangle_{\mathcal{A}}})^{(-1)^{(n-m)}} [u_{A,m}] \diamond [u_{B,n}]$. Note that here we take arbitrary $n, m \in \mathbb{Z}$, which need not satisfy $n > m$. This twist is similar to the one used in [31], [73].

In any of cases 2), 3) and 4), we get a family of embeddings of $\mathcal{H}_{tw}(\mathcal{A})$ into $\mathcal{SDH}_{tw}(\mathcal{A})$, with the twist thus defined. It is easy to show that all of these twisted algebras are invariant under the same class of derived equivalences as in Theorem 4.2.29, cf. Theorem 4.3.18 below.

4.2.10 Example : quiver representations

Let Q be a simply-laced Dynkin quiver on the vertices $1, \ldots, n$. Let $\mathcal{A}$ be the category of finite-dimensional representations of Q over the field $k = \mathbb{F}_q$. This abelian category satisfies all the assumptions (C1)-(C3) ; moreover, it is hereditary and has enough projectives. For each vertex i of Q, we denote the corresponding one-dimensional simple module by $S_i \in \mathcal{A}$. Two objects associated to a quiver Q, the Hall algebra $\mathcal{H}(\mathcal{A})$ and the quantum group $U_{\sqrt{q}}(\mathfrak{g})$ are related to each other by the following important result.

Theorem 4.2.45 (Ringel, [64, Sch])**.** *There are isomorphisms of algebras*

$$R : U_{\sqrt{q}}(\mathfrak{n}^+) \xrightarrow{\sim} \mathcal{H}_{tw}(\mathrm{rep}_{\mathbb{F}_q}(Q)), \quad R^e : U_{\sqrt{q}}(\mathfrak{b}^+) \xrightarrow{\sim} H^e_{tw}(\mathrm{rep}_{\mathbb{F}_q}(Q)),$$

defined on generators by

$$R(E_i) = R^e(E_i) = \frac{[S_i]}{(q-1)}, \quad R^e(K_i) = K_{S_i}.$$

In the algebra $\mathcal{SDH}(\mathcal{A})$, we choose a set of generators consisting of elements of two types, according to Proposition 4.2.39 : generators of the quantum torus $\mathbb{T}_{ac}(\mathcal{E})$ and classes of stalk complexes. Let $u_{i,m} := u_{S_i,m}$, for $i = 1, \ldots, n$ and $m \in \mathbb{Z}$.

Proposition 4.2.46. *The algebra $\mathcal{SDH}(\mathcal{A})$ is generated by the classes $[u_{i,m}], [v_{\alpha,m}]$, subject only to the following relations : for every $p > m+1 \in \mathbb{Z}$ and all $1 \leq i, j \leq n$ and $\alpha, \beta \in K_0(\mathcal{A})$, we have*

(U) the relations in $I_m(\mathcal{H}(\mathcal{A}))$ for the $u_{i,m}$, $i = 1, \ldots, n$;

$$u_{i,m}u_{j,m+1} = u_{j,m+1}u_{i,m} + \delta_i^j (q-1) v_{S_i,m}; \qquad u_{i,m}u_{j,p} = u_{j,p}u_{i,m};$$

(V)

$$v_{\alpha,m}v_{\beta,m} = \frac{\langle \beta, \alpha\rangle}{\langle \alpha, \beta\rangle} v_{\beta,m}v_{\alpha,m}; \qquad v_{\alpha,m}v_{\beta,m+1} = \langle \beta, \alpha\rangle v_{\beta,m+1}v_{\alpha,m}; \qquad v_{\alpha,m}v_{\beta,p} = v_{\beta,p}v_{\alpha,m};$$

(UV)

$$v_{\alpha,m}u_{j,m+1} = \langle S_j, \alpha\rangle u_{j,m+1}v_{\alpha,m}; \qquad u_{i,m}v_{\beta,m} = \langle \beta, S_i\rangle v_{\beta,m}u_{i,m};$$

$$u_{i,m}v_{\beta,m+1} = v_{\beta,m+1}u_{i,m}; \qquad u_{i,m}v_{\beta,p} = v_{\beta,p}u_{i,m}; \qquad v_{\alpha,m}u_{j,p} = u_{j,p}v_{\alpha,m}.$$

Proof. Let B be the algebra generated by the classes $[u_{i,m}], [v_{\alpha,m}]$, subject only to the above relations. We have a natural map

$$\phi : B \to \mathcal{SDH}(\mathcal{A}), [X] \mapsto [X].$$

Due to Theorem 4.2.45, we know that the classes $[u_{i,m}]$ give a basis of $I_m(\mathcal{H}(\mathcal{A}))$, for any integer m. Therefore, by Proposition 4.2.39, the classes $[u_{i,m}], [v_{\alpha,m}]$ generate $\mathcal{SDH}(\mathcal{A})$. Applying lemmas 4.2.40-4.2.43 and Theorem 4.2.45, one easily proves that all the relations in the statement are satisfied in the algebra $\mathcal{SDH}(\mathcal{A})$, i.e. the map ϕ is a surjective homomorphism. Let us prove its injectivity. Being the twisted group algebra of $K_0(\mathcal{C}^b(\mathcal{A}))$, the quantum torus has a basis consisting of the elements of this Grothendieck group. We know (cf. the proof of Lemma 4.2.13) that $K_0(\mathcal{C}^b_{ac}(\mathcal{E})) \cong \coprod_{\mathbb{Z}} K_0(\mathcal{E})$. From here it immediately follows that the elements of the form

$$[v_{\alpha_1,m_1}] \diamond [v_{\alpha_2,m_2}] \diamond \ldots \diamond [v_{\alpha_k,m_k}], m_1 > m_2 > \ldots > m_k,$$

form a basis of $\mathbb{T}_{ac}(\mathcal{A})$ over $\mathbb{F}_q$. The set formed by the elements

$$([u_{i_1,l_1}] \diamond [v_{i_2,l_2}] \diamond \ldots \diamond [v_{i_k,l_k}]) \diamond ([v_{\alpha_1,m_1}] \diamond [v_{\alpha_2,m_2}] \diamond \ldots \diamond [v_{\alpha_k,m_k}]),$$

where $l_1 > l_2 > \ldots > l_k, m_1 > m_2 > \ldots > m_k$, spans B. Thanks to the above argument and the freeness of the algebra $\mathcal{SDH}(\mathcal{A})$ over the quantum torus, this set is mapped to a basis of $\mathcal{SDH}(\mathcal{A})$. Thus, the homomorphism ϕ is injective, i.e. it is an isomorphism. $\square$

Note that with any of the twists 2), 3) and 4), the relations (U1) will be transformed into the quantum Serre relations.

4.2.11 Tilting objects and derived equivalences

We say that T is a *tilting* object in an exact category $\mathcal{E}$ satisfying conditions (C1) - (C4), if the following conditions hold :

(T1) the groups $\mathrm{Ext}^i(T, T)$ vanish for $i > 0$;

(T2) the full exact category $\mathrm{add}\, T$ containing all finite direct sums of indecomposable summands of T generates $\mathcal{D}^b(\mathcal{A})$;

(T3) the algebra $A = \mathrm{End}(T)$ is of finite global dimension.

Note that the exact structure on $\mathrm{add}\, T$ splits. Therefore, the inclusion $I_T : \mathrm{add}(T) \to \mathcal{E}$ induces an equivalence

$$\mathcal{D}^b(\mathrm{add}(T)) \xrightarrow{\sim} \mathcal{D}^b(\mathcal{E})$$

by conditions (T1) and (T2). Moreover, we have an exact fully faithful functor

$$\mathrm{Hom}(T, ?) : \mathrm{add}(T) \to \mathrm{mod}(A).$$

This functor induces an equivalence from $\mathrm{add}(T)$ onto the subcategory $\mathrm{proj}(A)$ of finitely generated projective A-modules. Therefore, by condition (T3), $\mathrm{Hom}(T, ?)$ also induces an equivalence

$$\mathcal{D}^b(\mathrm{add}(T)) \xrightarrow{\sim} \mathcal{D}^b(\mathrm{mod}(A)).$$

Theorem 4.2.47. *We have the following isomorphisms between the semi-derived Hall algebras :*

$$\mathcal{SDH}(\mathrm{mod}(\mathrm{End}(T))) \xleftarrow{\sim} \mathcal{SDH}(\mathrm{add}\, T) \overset{I_T}{\xrightarrow{\sim}} \mathcal{SDH}(\mathcal{E}).$$

We will consider a special case of these tilting equivalences - namely, BGP-reflections, - in Section 9.6.

4.3 $\mathbb{Z}/2$-graded version of $\mathcal{SDH}(\mathcal{E})$

In this section, we will develop a $\mathbb{Z}/2$-graded version of the construction from the previous section. We should point out once again that the whole concept of the semi-derived Hall algebras $\mathcal{SDH}(\mathcal{E})$ was inspired by the work [5] of Bridgeland where he considered $\mathbb{Z}/2$-graded complexes with projective components. When we try to generalize this construction to arbitrary exact categories with enough projectives we encounter a major obstacle : the stupid truncation functors $\sigma_{\geq n}$ are no longer defined. Recall that these truncations have been an important ingredient of our construction : we used them in the proofs of Propositions 4.2.8 and 4.2.26 and of Lemmas 4.2.2, Lemma 4.2.13 and Lemma 4.2.21. Another difficulty is that neither the $\mathbb{Z}/2$-graded analogue of Lemma 4.2.4, nor its analogue for vector bundles seem to work in full generality. These difficulties can be resolved and we can construct a $\mathbb{Z}/2$-graded analogue of $\mathcal{SDH}(\mathcal{E})$. The key point is to replace the category $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})$ with a suitable subcategory.

4.3.1 Choice of a subcategory

From now on, we assume that the exact category $\mathcal{E}$ satisfies conditions (C1) and (C2) from Section 3 and the following condition :

(C4') $\mathcal{E}$ has enough projectives, and each object has a finite projective resolution.

Note that (C4') implies condition (C3) from Section 3. We are interested in a subcategory $\widetilde{\mathcal{E}}$ of $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})$, where each object admits a deflation quasi-isomorphism from a complex P with projective components such that, moreover, for each complex $M \in \widetilde{\mathcal{E}}$, we have :

$$\mathrm{Ext}^p_{\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})}(P, M) = \mathrm{Ext}^p_{\mathcal{D}_{\mathbb{Z}/2}(\mathcal{E})}(P, M), \quad \forall p > 0. \tag{4.38}$$

Definition 4.3.1. For any exact category $\mathcal{E}$, denote by $\widetilde{\mathcal{E}}$ the closure with respect to extensions and quasi-isomorphism classes of the full subcategory of all stalk complexes inside $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})$.

We consider the category $\widetilde{\mathcal{P}}$. Note that it coincides with the intersection $\widetilde{\mathcal{E}} \cap \mathcal{C}_{\mathbb{Z}/2}(\mathcal{P})$. Indeed,

the inclusion $\widetilde{\mathcal{P}} \subset \widetilde{\mathcal{E}} \cap \mathcal{C}_{\mathbb{Z}/2}(\mathcal{P})$ follows from the definition. The inclusion in the other direction will follow from Proposition 4.3.5 below. The following lemmas will be very useful for us.

Lemma 4.3.2. *For any acyclic complex $K \in \mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E})$, there exists a deflation quasi-isomorphism $P_K \overset{\text{qis}}{\twoheadrightarrow} K$, with $P_K \in \widetilde{\mathcal{P}}$.*

Proof. Given an acyclic complex $K = K^0 \underset{f^1}{\overset{f^0}{\rightleftarrows}} K^1 \in \mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})$, we get two conflations

$$Z^0 \xrightarrow{i^0} K^0 \xrightarrow{\pi^1} Z^1, \qquad\qquad Z^1 \xrightarrow{i^1} K^1 \xrightarrow{\pi^0} Z^0.$$

By (C4'), we have deflations $P^0 \overset{p^0}{\twoheadrightarrow} Z^0$, $P^1 \overset{p^1}{\twoheadrightarrow} Z^1$, with $P^0, P^1 \in \mathcal{P}$. Now we lift p^0 to a morphism $g^0 : P^0 \to K^1$, such that $i^0 \circ g^0 = p^0$, and similarly we get $g^1 : P^1 \to K^0$. This determines two commutative diagrams :

$$\begin{tikzcd} P^1 \arrow[r, tail] \arrow[d, two heads] & P^0 \oplus P^1 \arrow[r, two heads] \arrow[d] & P^0 \arrow[d, two heads] \\ Z^1 \arrow[r, tail] & K^0 \arrow[r, two heads] & Z^0 \end{tikzcd} \qquad \begin{tikzcd} P^0 \arrow[r, tail] \arrow[d, two heads] & P^0 \oplus P^1 \arrow[r, two heads] \arrow[d] & P^1 \arrow[d, two heads] \\ Z^0 \arrow[r, tail] & K^1 \arrow[r, two heads] & Z^1 \end{tikzcd} \tag{4.39}$$

By the five-lemma, the two middle vertical arrows are deflations. Also, we get another two commutative diagrams :

$$\begin{tikzcd} P^1 \arrow[r, equal] \arrow[d, two heads] & P^1 \arrow[r, two heads] \arrow[d] & 0 \arrow[d, two heads] \\ K^1 \arrow[r, tail] & K^0 \arrow[r, two heads] & K^1 \end{tikzcd} \qquad \begin{tikzcd} P^0 \arrow[r, equal] \arrow[d, two heads] & P^0 \arrow[r, two heads] \arrow[d] & 0 \arrow[d, two heads] \\ K^0 \arrow[r, tail] & K^1 \arrow[r, two heads] & K^0 \end{tikzcd} \tag{4.40}$$

Consider the complex

$$P_K = P^0 \underset{0}{\overset{1}{\rightleftarrows}} P^0 \oplus P^1 \underset{1}{\overset{0}{\rightleftarrows}} P^1 .$$

Since diagrams (4.40) are commutative, we get a deflation $P_K \overset{p'}{\to} K$ that is clearly a quasi-isomorphism. By Lemma 4.1.4, all acyclic complexes with projective components lie in $\widetilde{\mathcal{P}}$. $\square$

Lemma 4.3.3. *Any $\mathbb{Z}/2-$graded complex in the image $\pi(\mathcal{C}^b(\mathcal{E}))$ can be obtained by a finite sequence of extensions of stalk complexes. In particular, $\pi(\mathcal{C}^b(\mathcal{E}))$ is contained in $\widetilde{\mathcal{E}}$.*

Proof. The statement follows from the principle of finite dévissage for bounded complexes and from the exactness of the functor π. $\square$

Now we can verify that $\widetilde{\mathcal{P}}$ and $\widetilde{\mathcal{E}}$ have the desired properties.

Lemma 4.3.4. *For any $P \in \widetilde{\mathcal{P}}, M \in \widetilde{\mathcal{E}}$, the isomorphism (4.38) holds.*

Proof. All the arguments of the proof of Proposition 4.2.1, except for Lemma 4.2.2, work for $P \in \mathcal{C}_{\mathbb{Z}/2}(\mathcal{P}), M \in \mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})$. It remains to prove that, for any $P \in \widetilde{\mathcal{P}}, M \in \widetilde{\mathcal{E}}$, we have

$$\mathrm{Hom}_{\mathbf{K}_{\mathbb{Z}/2}(\mathcal{E})}(P, M) = \mathrm{Hom}_{\mathcal{D}_{\mathbb{Z}/2}(\mathcal{E})}(P, M). \tag{4.41}$$

It is easy to check that for P, M stalk complexes, concentrated in degrees i and j respectively, both sides of (4.41) are isomorphic to $\mathrm{Hom}_{\mathcal{E}}(P^i, M^j)$ if $i = j$ and vanish if $i \neq j$. By Lemma 4.1.4, any $P \in \mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{P})$ is an iterated extension of stalk complexes with projective components, thus, by the five-lemma, the identity (4.41) holds for any such P and stalk M. In all the following steps we will apply the five-lemma in an appropriate way. Firstly, we prove that the identity (4.41) holds for stalk complexes M and any P from the closure with respect to extensions and quasi-isomorphisms of the full subcategory of stalk complexes in $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{P})$, i.e. for any $P \in \widetilde{\mathcal{P}}$ and stalk complex M. Then, using once again Lemma 4.1.4, we prove it for any $P \in \widetilde{\mathcal{P}}, M \in \mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{P})$. By Lemma 4.3.2 and induction, we prove the identity for any $P \in \widetilde{\mathcal{P}}, M \in \mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E})$. Applying the five-lemma once more, we prove it in the desired generality, i.e. for any $P \in \widetilde{\mathcal{P}}, M \in \widetilde{\mathcal{E}}$. $\square$

Proposition 4.3.5. *For any object $A \in \widetilde{\mathcal{E}}$, there exists a deflation quasi-isomorphism $P \overset{\mathrm{qis}}{\twoheadrightarrow} A$, with $P \in \widetilde{\mathcal{P}}$.*

Proof. To prove the statement, it is enough to show the following :

(i) for any $\mathbb{Z}/2-$graded stalk complex there exists such a deflation ;

(ii) the condition of the existence of such a deflation is closed under extensions ;

(iii) this condition is stable under quasi-isomorphism.

We first prove (i). Without loss of generality, one can consider a stalk complex of the form $M^\bullet = M \underset{0}{\overset{0}{\rightleftarrows}} 0$. Then $\pi(P^\bullet) \overset{\mathrm{qis}}{\twoheadrightarrow} M^\bullet$ is a desired deflation quasi-isomorphism, for

$$P^\bullet := \ldots \to 0 \to P^{-d} \to P^{-(d-1)} \to \ldots \to P^0 \to 0 \to \ldots,$$

where $0 \to P^{-d} \to P^{-(d-1)} \to \ldots \to P^0 \to M \to 0$ is a finite projective resolution of M in $\mathcal{E}$. Point (iii) is an easy consequence of (ii). Thus, it remains to prove point (ii).

Lemma 4.3.6. *Assume given a conflation $L \rightarrowtail M \twoheadrightarrow N$ in $\widetilde{\mathcal{E}}$, whose end terms L, N admit deflation quasi-isomorphisms*

$$P_L \overset{\mathrm{qis}}{\twoheadrightarrow} L, \qquad P_N \overset{\mathrm{qis}}{\twoheadrightarrow} N,$$

with $P_L, P_N \in \widetilde{\mathcal{P}}$. Then the middle term M also admits a deflation quasi-isomorphism $P(M) \overset{\mathrm{qis}}{\twoheadrightarrow} M$, with $P(M) \in \widetilde{\mathcal{P}}$.

Proof. Since $\Sigma = \Sigma^{-1}$, we have a conflation $\Sigma P_N \rightarrowtail C(1_{\Sigma P_N}) \overset{\mathrm{qis}}{\twoheadrightarrow} P_N$. Since $C(1_{\Sigma P_N})$ is contractible with projective components, it is projective in $\widetilde{\mathcal{P}}$. In particular, the composition $C(1_{\Sigma P_N}) \rightarrowtail P_N \overset{\mathrm{qis}}{\twoheadrightarrow} N$ lifts, with respect to the deflation $M \twoheadrightarrow N$, to a morphism

$C(1_{\Sigma P_N}) \to M$. We have an induced map of kernels $\Sigma P_N \xrightarrow{f} L$, completing the commutative diagram

$$\begin{tikzcd} \Sigma P_N \arrow[r, tail] \arrow[d] & C(1_{\Sigma P_N}) \arrow[r, two heads] \arrow[d] & P_N \arrow[d, two heads] \\ L \arrow[r, tail] & M \arrow[r, two heads] & N. \end{tikzcd} \tag{4.42}$$

Consider the conflation $A_L \rightarrowtail P_L \overset{\text{qis}}{\twoheadrightarrow} L$, where A_L is acyclic. Since P_N belongs to $\widetilde{\mathcal{P}}$, so does ΣP_N. Thus,

$$\mathrm{Ext}^1_{\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})}(\Sigma P_N, A_L) = \mathrm{Ext}^1_{\mathcal{D}_{\mathbb{Z}/2}(\mathcal{E})}(\Sigma P_N, A_L) = 0,$$

and f can be lifted to a morphism $f' : \Sigma P_N \to P_L$. The cone $C(f')$ of the morphism f' coincides with the push-out of the diagram $P_L \xleftarrow{f'} \Sigma P_N \rightarrowtail C(1_{\Sigma P_N})$. Thus, we get morphisms $C(f') \to P_N$ and $C(f') \to M$ such that the following diagram commutes :

$$\begin{tikzcd} \Sigma P_N \arrow[r, tail] \arrow[d] & C(1_{\Sigma P_N}) \arrow[r, two heads] \arrow[d] & P_N \arrow[d, equal] \\ P_L \arrow[r, tail] \arrow[d, two heads] & C(f') \arrow[r] \arrow[d] & P_N \arrow[d, two heads] \\ L \arrow[r, tail] & M \arrow[r, two heads] & N. \end{tikzcd} \tag{4.43}$$

The morphism $C(f') \to M$ is a deflation, by the five-lemma applied to the last two rows. The second row is the conflation induced from the first row by the morphism f'. The category $\widetilde{\mathcal{P}}$ being closed under extensions, $C(f')$ lies in $\widetilde{\mathcal{P}}$. Since $C(1_{\Sigma P_N})$ is acyclic, we find that the deflation $C(f') \twoheadrightarrow M$ is a quasi-isomorphism. This completes the proof of the lemma and of the proposition. $\square$

Now we can prove the inclusion $\widetilde{\mathcal{E}} \cap \mathcal{C}_{\mathbb{Z}/2}(\mathcal{P}) \subset \widetilde{\mathcal{P}}$, mentioned above. We have just shown that for any $P \in \widetilde{\mathcal{E}}$, in $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})$, there exists a conflation $K \rightarrowtail P' \overset{\text{qis}}{\twoheadrightarrow} P$, with $P' \in \widetilde{\mathcal{P}}$. If P has projective components, so does K. Thus, this conflation belongs to $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{P})$ and induces a quasi-isomorphism in this category. But $\widetilde{\mathcal{P}}$ is closed with respect to such quasi-isomorphisms, therefore, P also lies in $\widetilde{\mathcal{P}}$. We proved the desired inclusion. Thus, $\widetilde{\mathcal{E}} \cap \mathcal{C}_{\mathbb{Z}/2}(\mathcal{P}) = \widetilde{\mathcal{P}}$.

In fact, in the two most motivating cases, the category $\widetilde{\mathcal{E}}$ coincides with the entire category $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})$.

Proposition 4.3.7. *Assume that either :*

— *$\mathcal{E}$ is an abelian category or*

— *$\mathcal{E}$ is the full exact subcategory of projective objects in a hereditary abelian category.*

Then the category $\widetilde{\mathcal{E}}$ equals $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})$.

Proof. This is a rephrasing of Proposition 4.1.9. □

Proposition 4.3.8. *The triple* $(\widetilde{\mathcal{P}}, \mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}), \widetilde{\mathcal{E}})$ *endows the category* $\mathcal{E}$ *with a hereditary projective exact model stucture.*

Proof. By Lemma 4.3.4, $((\widetilde{\mathcal{P}}, \mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E})$ and $(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{P})), \widetilde{\mathcal{E}})$ are cotorsion pairs. By the existence of deflation quasi-isomorphsims from Proposion 4.3.5and their mapping cones, we find that the first pair is complete. Completeness of the second pair is proved similarly in the usual way. By recalling that $\widetilde{\mathcal{P}} = \widetilde{\mathcal{E}} \cap \mathcal{C}_{\mathbb{Z}/2}(\mathcal{P})$, we easily show that both of the pairs are hereditary. □

4.3.2 Grothendieck monoids and groups

Recall that the *left relative Grothendieck monoid* $M_0'(\widetilde{\mathcal{E}})$ was defined as the free monoid generated by the set $\mathrm{Iso}(\widetilde{\mathcal{E}})$, divided by the following set of relations :

$$\left\langle [L] = [K \oplus M] | K \rightarrowtail L \twoheadrightarrow M \quad \text{is a conflation}, K \in \mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}) \right\rangle .$$

Similarly, the *left relative Grothendieck group* $K_0'(\widetilde{\mathcal{E}})$ is the free group generated by the set $\mathrm{Iso}(\widetilde{\mathcal{E}})$, divided by the following set of relations :

$$\left\langle [K] - [L] + [M] = 0 | K \rightarrowtail L \twoheadrightarrow M \quad \text{is a conflation}, K \in \mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}) \right\rangle .$$

Lemma 4.3.9. *The natural map*

$$i : K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E})) \to K_0'(\widetilde{\mathcal{E}}), [M] \mapsto [M]$$

is injective.

Proof. We will define a map $\Phi : K_0'(\widetilde{\mathcal{E}}) \to K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}))$, such that

$$\Phi \circ i = \mathrm{Id}_{K_0(C_{\mathbb{Z}/2,ac}(\mathcal{E}))}. \tag{4.44}$$

We start by constructing such a map for complexes with projective components, i.e. we construct a retraction $\Phi_{\mathcal{P}}$ for

$$i_{\mathcal{P}} : K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{P})) \to K_0'(\widetilde{\mathcal{P}}), [M] \mapsto [M].$$

Since acyclic complexes with projective components are projective objects in $\widetilde{\mathcal{P}}$, all relations in $K_0'(\widetilde{\mathcal{P}})$ come actually from direct sums. This yields a natural projection map

$$p : K_0'(\widetilde{\mathcal{P}}) \twoheadrightarrow K_0^{split}(\widetilde{\mathcal{P}}), [M] \mapsto [M].$$

Since $\mathcal{P}$ is Krull-Schmidt category, so is $\mathcal{C}_{\mathbb{Z}/2}(\mathcal{P})$ and, therefore, $\widetilde{\mathcal{P}}$. Therefore, one can define an "acyclic part" of a complex with projective components : each object $M \in \widetilde{\mathcal{P}}$ can be decomposed in a unique way (up to a permutation of factors) into a finite direct sum of indecomposable

complexes :

$$M = \bigoplus_{i=1}^{m(M)} M_i \oplus \bigoplus_{j=1}^{k(M)} M'_j,$$

where all M_i are acyclic while the M'_j are not. Then

$$\phi : K_0^{split}(\widetilde{\mathcal{P}}) \twoheadrightarrow K_0^{split}(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{P})) = K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{P})), [M] \mapsto \bigoplus_{i=1}^{m(M)} M_i$$

is a well-defined group epimorphism, and for $\Phi_{\mathcal{P}} := \phi \circ p$ we get

$$\Phi_{\mathcal{P}} \circ i_{\mathcal{P}} = \mathrm{Id}_{K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{P}))}.$$

Given any complex $M \in \widetilde{\mathcal{E}}$, by Proposition 4.3.5, we have at least one deflation quasi-isomorphism $P_M \overset{\text{qis}}{\twoheadrightarrow} M$, with $P_M \in \widetilde{\mathcal{P}}$; consider also the kernel of this deflation A_M. We define the left inverse to the inclusion i as

$$\Phi([M]) := [\Phi_{\mathcal{P}}(P_M)] - [A_M].$$

We should verify two things :

1) The element $\Phi(M)$ thus defined does not depend on the choice of a deflation quasi-isomorphism $P_M \overset{\text{qis}}{\twoheadrightarrow} M$;
2) The map Φ is additive on conflations $K \rightarrowtail L \twoheadrightarrow M$ of $\widetilde{E}$ with K acyclic, i.e. it is actually a well-defined group homomorphism.

We prove part 1) in the same manner as Proposition 4.2.15 concerning the well-definedness of the multiplication in $\mathcal{SDH}(\mathcal{E})$. Namely, using suitable pull-backs, we reduce the problem to the case where we have two resolutions of M linked by a deflation quasi-isomorphism $P_M \overset{\text{qis}}{\twoheadrightarrow} P'_M$. For these two resolutions, we construct a commutative $3 \times 3-$diagram, similar to diagram (4.18) :

$$\begin{tikzcd}
H \arrow[r, equal] \arrow[d, tail] & H \arrow[r, two heads] \arrow[d, tail] & 0 \arrow[d, tail] \\
N \arrow[r, tail] \arrow[d, two heads] & P_M \arrow[r, two heads] \arrow[d, two heads] & M \arrow[d, equal] \\
K \arrow[r, tail] & P'_M \arrow[r, two heads] & M.
\end{tikzcd} \tag{4.45}$$

Here H, N, K are acyclic complexes. By considering the second column we see that H has projective components. Since $\Phi_{\mathcal{P}}$ is well-defined on $K'_0(\mathcal{P})$, we have $\Phi_{\mathcal{P}}([P_M]) - \Phi_{\mathcal{P}}([P'_M]) =$

$\Phi_{\mathcal{P}}([H])$. But $\Phi_{\mathcal{P}}([H]) = [H] = [N] - [K]$, and it follows that

$$\Phi_{\mathcal{P}}([P_M)]) - [N] = \Phi_{\mathcal{P}}([P'_M]) - [K],$$

proving the desired independence.

Let us prove the additivity of the map Φ. We should show that for any conflation $K \rightarrowtail L \twoheadrightarrow M$, with K acyclic, we have $\Phi(K) - \Phi(L) + \Phi(M) = 0$. Take a pair of conflations $A_K \rightarrowtail P_K \overset{\text{qis}}{\twoheadrightarrow} K$ and $A_M \rightarrowtail P_M \overset{\text{qis}}{\twoheadrightarrow} M$. Since $\operatorname{Ext}^1(P_M, K) = 0$ (by Lemma 4.3.4), the morphism $P_M \twoheadrightarrow M$ can be lifted along the deflation $L \twoheadrightarrow M$ and we can find a deflation quasi-isomorphism $P_K \oplus P_M \overset{\text{qis}}{\twoheadrightarrow} L$ with the kernel A_L. With the induced conflation of kernels $A_K \rightarrowtail A_L \twoheadrightarrow A_M$, we get the commutative diagram

$$\begin{tikzcd}
A_K \arrow[r, tail] \arrow[d, tail] & A_L \arrow[r, two heads] \arrow[d, tail] & A_M \arrow[d, tail] \\
P_K \arrow[r, tail] \arrow[d, two heads] & P_K \oplus P_M \arrow[r, two heads] \arrow[d, two heads] & P_M \arrow[d, two heads] \\
K \arrow[r, tail] & L \arrow[r, two heads] & M.
\end{tikzcd} \tag{4.46}$$

Using the well-definedness of Φ and the additivity of $\Phi_{\mathcal{P}}$, it is easy to check now that $\Phi(K) - \Phi(L) + \Phi(M) = 0$.

It remains to check identity (4.44). Take an acyclic complex $M \in C_{\mathbb{Z}/2,ac}(\mathcal{E})$. By Lemma 4.3.2, there exists a deflation quasi-isomorphism $P_M \overset{\text{qis}}{\twoheadrightarrow} M$ with kernel A_M. Since M is acyclic, so is P_M. Therefore, $\Phi_{\mathcal{P}}(P_M) = [P_M]$ and

$$\Phi \circ i([M]) = \Phi(M) = [P_M] - [A_M] = [M].$$

By the additivity proved above, we get the desired identity of homomorphisms. $\square$

4.3.3 Euler forms

In order to construct $\mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E})$, we first define the Euler form

$$\langle \cdot, \cdot \rangle : K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}) \times K_0(\widetilde{\mathcal{E}}) \to \mathbb{Q}^\times$$

via

$$\langle [K], [A] \rangle := \prod_{p=0}^{+\infty} |\operatorname{Ext}^p_{\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})}(K, A)|^{(-1)^p}.$$

By the same rule, we define the form

$$\langle\cdot,\cdot\rangle : K_0(\widetilde{\mathcal{E}}) \times K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}) \to \mathbb{Q}^{\times}.$$

These two forms coincide on $K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}) \times K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E})$, thus we can denote them by the same symbol and call each of them the Euler form.

Proposition 4.3.10. *Each of the Euler forms $\langle\cdot,\cdot\rangle$ defined above is a well-defined group homomorphism.*

Proof. If such an Euler form is well-defined, it is clearly a group homomorphism, by the five-lemma; thus, we should verify only that in both of these two alternating products all but a finite number of factors vanish, for all K, A. This does hold by Corollary 4.1.3. $\square$

Warning 4.3.11. *The canonical map $i' : K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E})) \to K_0(\widetilde{\mathcal{E}})$ is not injective. E.g., we have*

$$i'([\, X \underset{0}{\overset{1}{\rightleftarrows}} X\,]) = [\, X \underset{0}{\overset{0}{\rightleftarrows}} 0\,] + [\, 0 \underset{0}{\overset{0}{\rightleftarrows}} X\,] = i'([\, X \underset{1}{\overset{0}{\rightleftarrows}} X\,]). \tag{4.47}$$

Cf. the proof of Theorem 4.3.16. Moreover, the Euler form cannot be defined as before on the whole product $K_0(\widetilde{\mathcal{E}}) \times K_0(\widetilde{\mathcal{E}})$, if $\mathcal{E}$ has non-trivial Euler form.

Indeed, assume that we have a group homomorphism

$$\langle\cdot,\cdot\rangle : K_0(\widetilde{\mathcal{E}}) \times K_0(\widetilde{\mathcal{E}}) \to \mathbb{Q}^{\times}, \quad \langle [K],[A]\rangle := \prod_{p=0}^{+\infty} |\operatorname{Ext}^p_{\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})}(K,A)|^{(-1)^p}.$$

If $\langle Z,Y\rangle_{\mathcal{E}} \neq 0$, for certain $Z, Y \in \mathcal{E}$, it is easy to check that

$$\left\langle [\, Z \underset{0}{\overset{1}{\rightleftarrows}} Z\,], [\, Y \underset{0}{\overset{1}{\rightleftarrows}} 0\,] \right\rangle = \langle Z,Y\rangle_{\mathcal{E}} \neq 0 = \left\langle [\, Z \underset{1}{\overset{0}{\rightleftarrows}} Z\,], [\, Y \underset{0}{\overset{1}{\rightleftarrows}} 0\,] \right\rangle,$$

but the classes of $Z \underset{0}{\overset{1}{\rightleftarrows}} Z$ and $Z \underset{1}{\overset{0}{\rightleftarrows}} Z$ in $K_0(\widetilde{\mathcal{E}})$ coincide, cf. (4.47). Here by $\langle\cdot,\cdot\rangle_{\mathcal{E}}$ we mean the Euler form of the category $\mathcal{E}$.

4.3.4 Construction of the algebra $\mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E})$

Define the quantum torus of $\mathbb{Z}/2-$graded acyclic complexes $\mathbb{T}_{\mathbb{Z}/2,ac}(\mathcal{E})$ as the $\mathbb{Q}-$group algebra of $K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}))$, with the multiplication twisted by the inverse of the Euler form, i.e. the product of classes of acyclic complexes $K_1, K_2 \in \mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E})$ is defined as follows :

$$[K_1] \diamond [K_2] := \frac{1}{\langle K_1, K_2\rangle}[K_1 \oplus K_2].$$

Define the *quantum affine space of* $\mathbb{Z}/2-$*graded acyclic complexes* $\mathbb{A}_{\mathbb{Z}/2,ac}(\mathcal{E})$ as the $\mathbb{Q}-$monoid algebra of the Grothendieck monoid $M_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}))$, with the multiplication twisted by the inverse of the Euler form.

Consider the $\mathbb{Q}-$vector space with basis parametrized by the elements of $M_0'(\widetilde{\mathcal{E}})$. Define on this space a structure of a bimodule $\mathcal{M}'_{\mathbb{Z}/2}(\mathcal{E})$ over $\mathbb{A}_{\mathbb{Z}/2,ac}(\mathcal{E})$ by the rule

$$[K] \diamond [M] := \frac{1}{\langle K, M\rangle}[K \oplus M], \qquad [M] \diamond [K] := \frac{1}{\langle M, K\rangle}[M \oplus K]$$

for $K \in \mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}), M \in \widetilde{\mathcal{E}}$. Then $\mathcal{M}_{\mathbb{Z}/2}(\mathcal{E}) := \mathbb{T}_{\mathbb{Z}/2,ac}(\mathcal{E}) \otimes_{\mathbb{A}_{\mathbb{Z}/2,ac}(\mathcal{E})} \mathcal{M}'_{\mathbb{Z}/2}(\mathcal{E}) \otimes_{\mathbb{A}_{\mathbb{Z}/2,ac}(\mathcal{E})} \mathbb{T}_{\mathbb{Z}/2,ac}(\mathcal{E})$ is a bimodule over the quantum torus $\mathbb{T}_{\mathbb{Z}/2,ac}(\mathcal{E})$.

Theorem 4.3.12. *$\mathcal{M}_{\mathbb{Z}/2}(\mathcal{E})$ is free as a right and as a left module over $\mathbb{T}_{\mathbb{Z}/2,ac}(\mathcal{E})$. Each choice of representatives of the quasi-isomorphism classes yields a basis.*

Proof. The proof of Theorem 4.2.12 would work if Lemma 4.2.13 did not fail : the natural map $i' : K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E})) \to K_0(\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E}))$ is not injective ; cf. the above warning. Note that nonetheless, this injectivity has been used only to prove that there is a grading on the whole module compatible with the grading of any quasi-isomorphism component $\mathcal{M}_\alpha(\mathcal{E})$ (if we fix a representative E such that $\overline{E} = \alpha$) by $K_0(\mathcal{C}^b_{ac}(\mathcal{E}))$. Note that the grading we have chosen in the $\mathbb{Z}-$graded case can be now refined : indeed, $\mathcal{M}_{\mathbb{Z}/2}(\mathcal{E})$ is naturally graded by $K_0'(\widetilde{\mathcal{E}})$. Lemma 4.3.9 provides the compatibility of the $K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}))-$grading of any quasi-isomorphism component with this one. With these two compatible gradings at hand, we can mimic the proof of Theorem 4.2.12, replacing everywhere bounded complexes by the $\mathbb{Z}/2-$graded and $K_0(\mathcal{C}^b(\mathcal{E}))$ by $K_0(\widetilde{\mathcal{E}})$. □

Definition 4.3.13. We endow $\mathcal{M}_{\mathbb{Z}/2}(\mathcal{E})$ with the following multiplication : the product of the classes of two complexes $L, M \in \widetilde{\mathcal{E}}$ is defined as

$$[L] \diamond [M] = \frac{1}{\langle A_L, L\rangle}[A_L]^{-1} \diamond \sum_{\varepsilon \in \mathrm{Ext}^1_{\widetilde{\mathcal{E}}}(P_L, M)} \frac{[\mathbf{mt}(\varepsilon)]}{|\operatorname{Hom}(P_L, M)|}, \tag{4.48}$$

or, equivalently, as

$$[L] \diamond [M] = \frac{1}{\langle A_L, L\rangle}[A_L]^{-1} \diamond \sum_{X \in \mathrm{Iso}(\widetilde{\mathcal{E}})} \left(\frac{|\operatorname{Ext}^1_{\widetilde{\mathcal{E}}}(P_L, M)_X|}{|\operatorname{Hom}(P_L, M)|}[X]\right), \tag{4.49}$$

where $A_L \rightarrowtail P_L \overset{\text{qis}}{\twoheadrightarrow} L$ is an arbitrary conflation with $A_L \in \mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}), P_L \in \widetilde{\mathcal{P}}$. We call the resulting algebra the $\mathbb{Z}/2-$*graded semi-derived Hall algebra* $\mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E})$.

Proposition 4.3.14. *The multiplication $\diamond$ of $\mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E})$ is well-defined and compatible with the module structure.*

Proof. The proof follows the lines of the proofs of Propositions 4.2.15 and 4.2.16. □

We do not know, whether the $\mathbb{Z}/2-$graded versions of Lemma 4.2.21, Corollary 4.2.22 and Corollary 4.2.23 do hold, since their proofs use Lemma 4.2.13.

In [5], Bridgeland uses a non-trivial twist. Let us construct a similarly twisted version of the algebra $\mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E})$. We define a bilinear form

$$\langle \cdot, \cdot \rangle_{cw} : \mathrm{Iso}(\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})) \times \mathrm{Iso}(\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})) \to \mathbb{Q}^{\times}, \quad \langle M, N \rangle_{cw} := \sqrt{\langle M^0, N^0 \rangle \cdot \langle M^1, N^1 \rangle},$$

where M has the form $M^0 \rightleftarrows M^1$, and similarly for N. This form descends to a bilinear form (denoted by the same symbol)

$$\langle \cdot, \cdot \rangle_{cw} : A \times B \to \mathbb{Q}^{\times},$$

where A and B are some of the Grothendieck groups considered above, i.e.

$$A, B \in \left\{ K_0(\mathcal{C}_{\mathbb{Z}/2}(\mathcal{E})), K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E})), K_0(\widetilde{\mathcal{E}}), K_0'(\widetilde{\mathcal{E}}) \right\}.$$

Lemma 4.3.15. *On the Grothendieck group of acyclic complexes, the bilinear form $\langle \cdot, \cdot \rangle_{cw}$ coincides with the usual Euler form :*

$$\langle \alpha, \beta \rangle_{cw} = \langle \alpha, \beta \rangle, \quad \forall (\alpha, \beta) \in K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E})) \times K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E})).$$

Proof. The statement is trivial for $\alpha, \beta \in \{[K_P], [K_P^*] | P \in \mathcal{P}\}$. We know (cf. the proof of Proposition 4.3.10) that these classes generate the whole Grothendieck group $K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}))$. $\square$

We define the *twisted $\mathbb{Z}/2-$graded semi-derived Hall algebra* $\mathcal{SDH}_{\mathbb{Z}/2,tw}(\mathcal{E})$ in the same way as the non-twisted version, but with replacing $\mathbb{A}_{\mathbb{Z}/2,ac}(\mathcal{E})$ and $\mathbb{T}_{\mathbb{Z}/2,ac}(\mathcal{E})$ by the monoid and the group algebras $\mathbb{Q}M_0'(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}))$ and $\mathbb{Q}K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}))$, respectively. The multiplication in $\mathcal{SDH}_{\mathbb{Z}/2,tw}(\mathcal{E})$ is given by the rule

$$[M_1] * [M_2] := \langle M_1, M_2 \rangle_{cw} [M_1] \diamond [M_2], \quad [M_1], [M_2] \in K_0'(\widetilde{\mathcal{E}}).$$

It is easy to check that this is a free bimodule over $\mathbb{Q}K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}))$, with the same type of bases as in theorem 4.3.12. We define the *reduced twisted* version of the $\mathbb{Z}/2-$graded semi-derived Hall algebra by setting $[K] = 1$ whenever K is an acyclic complex, invariant under the shift functor :

$$\mathcal{SDH}_{\mathbb{Z}/2,tw,red}(\mathcal{E}) := \mathcal{SDH}_{\mathbb{Z}/2,tw}(\mathcal{E}) / ([K] - 1 : K \in \mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}), K \cong K^*).$$

Due to isomorphism (4.51), this is the same as setting

$$K_\alpha * K_\alpha^* = 1, \quad \forall \alpha \in K_0(\mathcal{E}), \tag{4.50}$$

where

$$K_\alpha := [K_A] * [K_B]^{-1}, \quad K_\alpha^* := [{K_A}^*] * [{K_B}^*]^{-1},$$

for any $A, B \in \mathcal{E}$ such that $\alpha = A - B$. It is easy to check (cf. [5]) that these two elements are well-defined, i.e. they do not depend on choice of A, B.

4.3.5 Associativity and derived invariance

Theorem 4.3.16. *$\mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E})$ is an associative unital algebra.*

Proof. The class $[0]$ is clearly the unit of $\mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E})$. One can check that the proof of Theorem 4.2.24 concerning the associativity works in this $\mathbb{Z}/2-$graded case without any changes. $\square$

Theorem 4.3.17. *Suppose that $F : \mathcal{E}' \to \mathcal{E}$ is an exact functor between exact categories satisfying conditions (C1), (C2) and (C4'), inducing an equivalence of bounded derived categories*

$$F : \mathcal{D}^b(\mathcal{E}') \xrightarrow{\sim} \mathcal{D}^b(\mathcal{E})$$

Then F induces an isomorphism in the $\mathbb{Z}/2-$graded semi-derived Hall algebras :

$$F : \mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E}') \xrightarrow{\sim} \mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E}).$$

Proof. Since F induces an equivalence of the bounded derived categories, it induces an isomorphism of their Grothendieck groups and preserves the Euler form. But $K_0(\mathcal{D}^b(\mathcal{E})) = K_0(\mathcal{E})$, hence F induces an isomorphism $K_0(\mathcal{E}') \xrightarrow{\sim} K_0(\mathcal{E})$. It easily follows from the existence of resolutions (**??**), that $K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{P})) \xrightarrow{\sim} K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E}))$. By Lemma 4.1.4 and because the exact structure of $K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{P}))$ is split, we have a canonical isomorphism

$$K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{P})) \xrightarrow{\sim} K_0(\mathcal{P}) \oplus K_0(\mathcal{P}).$$

But $K_0(\mathcal{P}) \xrightarrow{\sim} K_0(\mathcal{E})$ by our assumptions. By composing all these isomorphisms, we finally find that $K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E})) \xrightarrow{\sim} K_0(\mathcal{E}) \oplus K_0(\mathcal{E})$. In other words, the natural homomorphism :

$$K_0(\mathcal{E}) \oplus K_0(\mathcal{E}) \to K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E})), \quad ([M],[N]) \mapsto [\, M \underset{0}{\overset{1}{\rightleftarrows}} M \,] + [\, N \underset{1}{\overset{0}{\rightleftarrows}} N \,] \tag{4.51}$$

is an isomorphism, and similarly for $\mathcal{E}'$. These two isomorphisms are compatible with F; therefore, F induces an isomorphism of the Grothendieck groups of acyclic $\mathbb{Z}/2-$graded complexes. It clearly preserves the Euler form. It follows that F induces an isomorphism

$$\mathbb{T}_{\mathbb{Z}/2,ac}(\mathcal{E}') \xrightarrow{\sim} \mathbb{T}_{\mathbb{Z}/2,ac}(\mathcal{E}).$$

Since F induces an equivalence $\mathcal{D}^b(\mathcal{E}') \xrightarrow{\sim} \mathcal{D}^b(\mathcal{E})$, it induces an equivalence between the images of the bounded derived categories under the natural map π into the $\mathbb{Z}/2-$graded derived categories and also an equivalence of the extension closures of these images. Let $\mathcal{D}'_{\mathbb{Z}/2}(\mathcal{E})$ denote this extension closure of $\pi(\mathcal{D}^b(\mathcal{E}))$, and similarly for $\mathcal{E}'$. Note that stalk complexes lie in the image

of π and, therefore, in $\mathcal{D}'_{\mathbb{Z}/2}(\mathcal{E}')$ and $\mathcal{D}'_{\mathbb{Z}/2}(\mathcal{E})$, respectively. We have natural fully faithful functors

$$\psi_{\mathcal{E}} : \widetilde{\mathcal{E}}[\mathrm{qis}^{-1}] \to \mathcal{D}'_{\mathbb{Z}/2}(\mathcal{E}), \quad \psi_{\mathcal{E}'} : \widetilde{\mathcal{E}}'[\mathrm{qis}^{-1}] \to \mathcal{D}'_{\mathbb{Z}/2}(\mathcal{E}'),$$

which are both the identity on objects (in fact, both of them are equivalences). Therefore, F induces a bijection between the sets of quasi-isomorphism classes of objects in $\widetilde{\mathcal{E}}$ and $\widetilde{\mathcal{E}}'$ i.e., by Theorem 4.3.12, between the bases of $\mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E}')$ and $\mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E})$ as modules over the (isomorphic to each other) quantum tori. By property (4.2) and Proposition 4.2.25, all homomorphism and extension spaces in $\widetilde{\mathcal{E}}$ are preserved under F. By the same reasons, the Euler form on $K_0(\mathcal{C}_{\mathbb{Z}/2,ac}(\mathcal{E})) \times K_0(\widetilde{\mathcal{E}})$ is preserved by F. Since $\psi_{\mathcal{E}'}$ and $\psi_{\mathcal{E}}$ are equivalences and F induces an equivalence between $\mathcal{D}'_{\mathbb{Z}/2}(\mathcal{E}')$ and $\mathcal{D}'_{\mathbb{Z}/2}(\mathcal{E})$, the multiplication (in particular, the action of the quantum torus) in $\mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E}')$ is preserved under F, hence $F : \mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E}') \xrightarrow{\sim} \mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E})$. $\square$

Theorem 4.3.18. *The algebras $\mathcal{SDH}_{\mathbb{Z}/2,tw}(\mathcal{E})$ and $\mathcal{SDH}_{\mathbb{Z}/2,tw,red}(\mathcal{E})$ are associative and unital. Each derived equivalence F from Theorem 4.3.17 induces an isomorphism of the twisted and reduced twisted $\mathbb{Z}/2-$graded semi-derived Hall algebras.*

Proof. The first part follows from Theorem 4.3.16 and the well-definedness of the form $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ as a homomorphism $K'_0(\widetilde{\mathcal{E}}) \times K'_0(\widetilde{\mathcal{E}}) \to \mathbb{Q}^{\times}$. To prove the invariance of $\mathcal{SDH}_{\mathbb{Z}/2,tw}(\mathcal{E})$, it is enough to prove that this form is preserved under the functor F. Since F is induced by an exact functor between exact categories, it sends the $i-$th component of a complex in $\widetilde{\mathcal{E}}'$ to the $i-$th component of its image in $\widetilde{\mathcal{E}}$, for $i = 0, 1$. The Euler form on the Grothendieck group $K_0(\mathcal{E}')$ is sent by F to the Euler form on $K_0(\mathcal{E})$, since all extension spaces are certain homomorphism spaces in (equivalent) derived categories. By these two arguments, the form $\langle \cdot, \cdot \rangle_{\mathcal{W}}$ is preserved under F. Isomorphism (4.51) being compatible with F, so is condition (4.50). $\square$

As in the $\mathbb{Z}-$graded case, we get the following corollaries of Theorems 4.3.17 and 4.3.18.

Corollary 4.3.19. *We have the following isomorphisms of algebras :*

$$\mathcal{H}(\widetilde{\mathcal{P}})[[K]^{-1}|H^{\bullet}(K) = 0] \cong \mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{P}) \overset{I}{\xrightarrow{\sim}} \mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E}); \tag{4.52}$$

where the isomorphism I is induced by the inclusion functor $\mathcal{P} \hookrightarrow \mathcal{A}$. These isomorphisms are compatible with the twist and the reduction considered above. The isomorphism F, inverse to I, is defined as in Proposition 4.2.34.

Corollary 4.3.20. *Let T be a tilting object in an exact category $\mathcal{E}$ satisfying conditions (C1), (C2) and (C4'). We have the following isomorphisms between the $\mathbb{Z}/2$-graded semi-derived Hall algebras :*

$$\mathcal{SDH}_{\mathbb{Z}/2}(\mathrm{mod}(\mathrm{End}(T))) \xleftarrow{\sim} \mathcal{SDH}_{\mathbb{Z}/2}(\mathrm{add}\, T) \overset{I_T}{\xrightarrow{\sim}} \mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E}),$$

where I_T is induced by the inclusion functor $I_T : \operatorname{add} T \hookrightarrow \mathcal{E}$. These isomorphisms are compatible with the twist and the reduction considered above.

4.3.6 Hereditary case : Bridgeland's construction, Drinfeld doubles and quantum groups

As in the $\mathbb{Z}-$graded case, we have an alternative formula for the product in $\mathcal{SDH}_{\mathbb{Z}/2}(\mathcal{E})$, if $\mathcal{E}$ is hereditary. The proof is the same as for Theorem 4.2.36.

Theorem 4.3.21. *Assume that $\mathcal{E}$ is hereditary and has enough projectives. Then for $L, M \in \widetilde{\mathcal{E}}$, the product $[L] \diamond [M]$ is equal to the following sum :*

$$[L] \diamond [M] = \sum_{X \in \mathrm{Iso}(\widetilde{\mathcal{E}})} \frac{|\operatorname{Ext}^1_{\widetilde{\mathcal{E}}}(L, M)_X|}{|\operatorname{Hom}(L, M)|}[X], \tag{4.53}$$

Corollary 4.3.22. *Assume that $\mathcal{E}$ is hereditary and has enough projectives. Then there exists an algebra homomorphism*

$$p : \mathcal{H}(\widetilde{\mathcal{E}}) \to \mathcal{SDH}_{\mathbb{Z}/\in}(\mathcal{E}).$$

The homomorphism p induces an algebra isomorphism

$$(\mathcal{H}(\widetilde{\mathcal{E}})/I_{\mathbb{Z}/2})[S^{-1}_{\mathbb{Z}/2}] \xrightarrow{\sim} \mathcal{SDH}_{\mathbb{Z}/\in}(\mathcal{E}), \tag{4.54}$$

where $I_{\mathbb{Z}/2}$ is the two-sided ideal generated by all differences $[L]-[K \oplus M]$, where $K \rightarrowtail L \twoheadrightarrow M$ is a conflation in $\widetilde{\mathcal{E}}$ with acyclic K, and $S_{\mathbb{Z}/2}$ is the set of all classes $[K]$ of acyclic $\mathbb{Z}/2-$graded complexes.

Corollary 4.3.23. *Assume that $\mathcal{E}$ is hereditary and has enough projectives. There is an embedding of algebras*

$$J^e_+ : \mathcal{H}^e_{tw}(\mathcal{E}) \hookrightarrow \mathcal{SDH}_{\mathbb{Z}/2,tw}(\mathcal{E})$$

defined by

$$[A] \longmapsto [\, 0 \rightleftarrows A \,], \quad K_\alpha \longmapsto K_\alpha,$$

where $A \in \mathcal{E}, \alpha \in K_0(\mathcal{E})$. By composing J^e_+ and the involution $*$, we also have an embedding

$$J^e_- : \mathcal{H}^e_{tw}(\mathcal{E}) \hookrightarrow \mathcal{SDH}_{\mathbb{Z}/2,tw}(\mathcal{E})$$

defined by

$$[A] \longmapsto [\, A \rightleftarrows 0 \,], \quad K_\alpha \longmapsto K^*_\alpha.$$

Let now $\mathcal{A}$ be a hereditary abelian category satisfying conditions (C1) and (C2) and having enough projectives. By Proposition 4.1.9, we have $\widetilde{\mathcal{A}} = \mathcal{C}_{\mathbb{Z}/2}(\mathcal{A})$; similarly, we have $\widetilde{\mathcal{P}} = \mathcal{C}_{\mathbb{Z}/2}(\mathcal{P})$.

By Corollary 4.3.19, we have an isomorphism

$$I : \mathcal{H}_{tw}(\mathcal{C}_{\mathbb{Z}/2}(\mathcal{P}))[[K]^{-1}|H^{\bullet}(K) = 0] \xrightarrow{\sim} \mathcal{SDH}_{\mathbb{Z}/2,tw}(\mathcal{A}). \tag{4.55}$$

The algebra on the left hand side is the algebra $\mathcal{DH}(\mathcal{A})$ from Bridgeland's work [5]. It has a set of generators $\{E_A, F_A, K_\alpha, K_\alpha^* | A \in \mathrm{Iso}(\mathcal{A}), \alpha \in K_0(\mathcal{A})\}$, cf. [5].

Proposition 4.3.24. *(i) The isomorphism*

$$F : \mathcal{SDH}_{\mathbb{Z}/2,tw}(\mathcal{A}) \xrightarrow{\sim} \mathcal{DH}(\mathcal{A}),$$

inverse to I, is defined on the generators in the following way :

$$[\, 0 \rightleftarrows A \,] \mapsto E_A, \quad [\, A \rightleftarrows 0 \,] \mapsto F_A, \quad [K_\alpha] \mapsto [K_\alpha], \quad [K_\alpha^*] \mapsto [K_\alpha^*]. \tag{4.56}$$

(ii) The multiplication map

$$m : a \otimes b \longmapsto J_+^e(a) * J_-^e(b)$$

defines an isomorphism of vector spaces

$$m : \mathcal{H}_{tw}^e(\mathcal{A}) \otimes_{\mathbb{C}} \mathcal{H}_{tw}^e(\mathcal{A}) \xrightarrow{\sim} \mathcal{SDH}_{\mathbb{Z}/2,tw}(\mathcal{A}).$$

Point (i) can be easily checked by hand (it is enough to show that F and I are inverse to each other as maps). Point (ii) follows from [5, Lemmas 4.6, 4.7].

Combining Yanagida's theorem [82, Theorem 1.26] with the isomorphism (4.55), we get another point of view on the semi-derived Hall algebras.

Theorem 4.3.25. *The algebra $\mathcal{SDH}_{\mathbb{Z}/2,tw}(\mathcal{A})$ is isomorphic to the Drinfeld double of the bialgebra $\mathcal{H}_{tw}^e(\mathcal{A})$.*

This fact combined with Theorem 4.3.18 yields a new proof of the following theorem of Cramer.

Theorem 4.3.26 ([8, Theorem 1])**.** *Suppose for two hereditary abelian categories $\mathcal{A}', \mathcal{A}$ satisfying conditions (C1) and (C2) and having enough projectives, an exact functor $F : \mathcal{A}' \to \mathcal{A}$ induces an equivalence of bounded derived categories*

$$F : \mathcal{D}^b(\mathcal{A}') \xrightarrow{\sim} \mathcal{D}^b(\mathcal{A}).$$

Then F induces an algebra isomorphism of the Drinfeld doubles of the bialgebras $\mathcal{H}_{tw}^e(\mathcal{A}')$ and $\mathcal{H}_{tw}^e(\mathcal{A})$.

In fact, Cramer proved a more general result : he did not assume that $\mathcal{A}'$ and $\mathcal{A}$ have enough projectives, and the derived equivalence did not have to be induced by an exact functor of abelian

categories. By Happel-Rickard-Schofield result [30, Theorem 1] and our Theorem 4.3.20 on tilting invariance, we recover many cases of Cramer's more general statement.

Now consider an acyclic quiver Q, a field $k = \mathbb{F}_q$ with q elements, and the corresponding quantum group $U_{\sqrt{q}}(\mathfrak{g})$. Thanks to Bridgeland's theorem [5, Theorem 4.9], we have the following corollary of Theorem 4.3.25.

Proposition 4.3.27. *There is an injective homomorphism of algebras*

$$R' : U_{\sqrt{q}}(\mathfrak{g}) \hookrightarrow \mathcal{SDH}_{\mathbb{Z}/2,tw,red}(\mathrm{rep}_k\, Q),$$

defined on generators by

$$R'(E_i) = (q-1)^{-1} \cdot [\, 0 \rightleftarrows S_i \,], \quad R'(F_i) = (-\sqrt{q})(q-1)^{-1} \cdot [\, S_i \rightleftarrows 0 \,],$$

$$R'(K_i) = [K_{S_i}], \quad R'(K_i^{-1}) = [K^*_{S_i}].$$

The map R' is an isomorphism precisely when the graph underlying Q is a simply-laced Dynkin diagram.

4.3.7 Reflection functors and the braid group action on $U_{\sqrt{q}}(\mathfrak{g})$

A *source* of a quiver Q is a vertex without incoming arrows, a *sink* is a vertex without outgoing arrows. Let i be a sink and let Q' be the quiver obtained from Q by reversing all arrows starting in i. Let us denote by S_j and S'_j the simple objects concentrated at a vertex j in $\mathrm{rep}_k(Q)$ and $\mathrm{rep}_k(Q')$, respectively. Let $\tau^-(S_i)$ be the cokernel of the natural monomorphism

$$S_i = P_i \hookrightarrow \bigoplus_{j \to i} P_j.$$

Then

$$T := \bigoplus_{j \neq i} P_j \oplus \tau^-(S_i)$$

is a tilting object in $\mathrm{rep}_k(Q)$, and the indecomposable objects of the category $\mathrm{Fac}\, T$ of all quotients of finite direct sums of copies of T are all the indecomposables of $\mathrm{rep}_k(Q)$, except for S_i. In [72], this category is denoted by $\mathrm{rep}_{adm}(Q)$. Define $\mathcal{T}' = \mathrm{rep}_{adm}(Q')$ in a similar way, i.e. as the full exact subcategory containing all the indecomposables of $\mathrm{rep}_k(Q')$, except for S'_i. There is an equivalence of categories

$$s_i : \mathrm{rep}_{adm}(Q) \xrightarrow{\sim} \mathrm{rep}_{adm}(Q'),$$

cf. [72]. The functors s_i are called *reflection functors*, since s_i induces the action of a simple reflection in the Weyl group of Q on the lattice $\mathbb{Z}^n$ which can be identified with both Grothendieck groups $K_0(\mathrm{rep}_k(Q)$ and $K_0(\mathrm{rep}_k(Q'))$. For different i, these functors generate the action of the

whole Weyl group on the lattice. We have a diagram of functors

$$\mathrm{rep}_k(Q) \hookleftarrow \mathrm{Fac}\, T \xrightarrow{\sim} \mathcal{T}' \hookrightarrow \mathrm{rep}_k(Q'). \tag{4.57}$$

All of them induce derived equivalences, and we can finally define an equivalence of the bounded derived categories

$$s_i^d : \mathcal{D}^b(\mathrm{rep}_k(Q)) \xrightarrow{\sim} \mathcal{D}^b(\mathrm{rep}_k(Q'))$$

to be their composition (from left to right). These functors are the derived versions of the *Bernstein-Gelfand-Ponomarev reflection functors* [4]. By computing the effect of the functors (4.57) in the semi-derived Hall algebra we obtain the following theorem.

Theorem 4.3.28. *There is a unique algebra isomorphism*

$$t_i : \mathcal{SDH}_{\mathbb{Z}/2,tw,red}(\mathrm{rep}_k(Q)) \xrightarrow{\sim} \mathcal{SDH}_{\mathbb{Z}/2,tw,red}(\mathrm{rep}_k(Q')),$$

satisfying

$$\begin{aligned} * \circ t_i &= t_i \circ *; \\ t_i([\, S_i \rightleftarrows 0\,]) &= q^{-1/2} \cdot [\, 0 \rightleftarrows S_i'\,] * K^*_{S_i'}; \\ t_i([\, A \rightleftarrows 0\,]) &= [\, s_i(A) \rightleftarrows 0\,], \quad \forall A \in \mathrm{Fac}\, T; \\ t_i(K_\alpha) &= K_{s_i\alpha}. \end{aligned} \tag{4.58}$$

This isomorphism is induced by the derived equivalence s_i^d.

Proof. Only equation (4.58) is not straightforward. To prove this, note that we have a triangle

$$S_i \hookrightarrow \bigoplus_{j\to i} P_j \to \tau^-(S_i) \to \Sigma S_i$$

in the derived category $\mathcal{D}^b(\mathrm{rep}_k(Q)) \xrightarrow{\sim} \mathcal{D}^b(\mathrm{rep}_k(Q'))$. The first three objects here belong to the category $\mathrm{rep}_k(Q)$, while the images of the last three under s_i^d belong to $\mathrm{rep}_k(Q')$. We thus have conflations

$$(S_i \rightleftarrows 0) \;\rightarrowtail\; (\bigoplus_{j\to i} P_j \underset{0}{\rightleftarrows} \tau^-(S_i)) \;\twoheadrightarrow\; K_{\tau^-(S_i)}\,,$$

$$K_{s_i(\bigoplus_{j\to i} P_j)} \;\rightarrowtail\; (s_i(\bigoplus_{j\to i} P_j) \underset{0}{\rightleftarrows} s_i(\tau^-(S_i))) \;\twoheadrightarrow\; (0 \rightleftarrows S_i')$$

in $\mathrm{rep}_k(Q)$ and $\mathrm{rep}_k(Q')$, respectively. Formula (4.58) follows by simple calculations. □

Sevenhant-Van den Bergh [72] consider the *reduced Drinfeld double* $U(Q)$ of the algebra $\mathcal{H}^e_{tw}(\mathrm{rep}_k(Q))$, where reduction is given by imposing (4.50). By our results, the algebra $U(Q)$ is isomorphic to $\mathcal{SDH}_{\mathbb{Z}/2,tw,red}(\mathrm{rep}_k(Q))$. Therefore, Theorem 9.1 of [72] is a corollary of the

above theorem.

As shown in [72] and [81], by combining the isomorphisms t_i with an appropriate Fourier transform, one gets an induced action of the braid group of the quiver Q on the quantum group $U_{\sqrt{q}}(\mathfrak{g})$, identified with a subalgebra of $\mathcal{SDH}_{\mathbb{Z}/2,tw,red}(\mathrm{rep}_k(Q))$ by Proposition 4.3.27. This Fourier transform sends S_j to S'_j [81], and its action on the $\mathbb{Z}/2-$graded semi-derived Hall algebras is naturally induced. As shown on [72] and [81], this braid group action coincides with the one defined by Lusztig by different methods [53].

4.4 Twisted extended Hall algebras of derived categories

The approach of Chapter 3 allows us to generalize results of the previous 2 sections to the case when replacements of objects might not exist. The results here are direct applications of theorems from Chapter 3 and computations from Section 4.1, so we present them without detailed proofs.

Let $\mathcal{E}$ be an exact category such that the following conditions are satisfied :

(C1) $\mathcal{E}$ is essentially small, idempotent complete and linear over some ground field k;

(C2) For each pair of objects $A, B \in \mathcal{E}$ and for each $p > 0$, we have

$$|\operatorname{Ext}^p(A,B)| < \infty; \qquad |\operatorname{Hom}(A,B)| < \infty;$$

(C3) For each pair of objects $A, B \in \mathcal{E}$, there exists $N > 0$ such that for all $p > N$, we have

$$\operatorname{Ext}^p(A,B) = 0.$$

By Proposition 4.2.1, the form $\langle -,-\rangle_{\mathcal{C}^b(\mathcal{E})}^{\mathcal{D}^b(\mathcal{E})_{\geq 0}}$ is well-defined. Then we have the following.

Theorem 4.4.1. *(i) The algebra $\mathcal{H}(\mathcal{D}^b(\mathcal{E}), \mathrm{qis})$ is well-defined. It is a twisted extended Hall algebra of $\mathcal{D}^b(\mathcal{E})$. Therefore, it is a free module over the quantum torus of acyclics.*

(ii) Whenever $\mathcal{SDH}(\mathcal{C}^b(\mathcal{E}))$ is well-defined, we have an isomorphism

$$\mathcal{H}(\mathcal{D}^b(\mathcal{E}), \mathrm{qis}) \xrightarrow{\sim} \mathcal{SDH}(\mathcal{C}^b(\mathcal{E})).$$

(iii) $\mathcal{H}(\mathcal{D}^b(\mathcal{E}), \mathrm{qis})$ is functorial up to twist under exact functors preserving quasi-isomorphisms and tilting invariant.

(iv) If $\mathcal{E}$ is hereditary, we have an isomorphism

$$\mathcal{H}(\mathcal{D}^b(\mathcal{E}), \mathrm{qis}) \xrightarrow{\sim} (\mathcal{H}(\mathcal{E})[[K]^{-1}, K \in \mathcal{C}^b_{ac}(\mathcal{E})])/I.$$

(v) We have an embedding

$$\mathcal{H}(\mathcal{E}) \to \mathcal{H}(\mathcal{D}^b(\mathcal{E}), \mathrm{qis})$$

and an embedding of the extended algebra.

Similarly, the form $\langle -,-\rangle_{\mathcal{C}_m(\mathcal{E})}^{\mathcal{D}_m(\mathcal{E})_{\geq 0}}$ is well-defined, for each $m \geq 2$, and we have the following.

Theorem 4.4.2. *(i) The algebra $\mathcal{H}(\mathcal{D}_m(\mathcal{E}), \mathrm{qis})$ is well-defined. It is a twisted extended Hall algebra of $\mathcal{D}^b(\mathcal{E})$. Therefore, it is a free module over the quantum torus of acyclics.*

(ii) Whenever $\mathcal{SDH}_{\mathbb{Z}/m}(\mathcal{E})$ is well-defined, we have an isomorphism

$$\mathcal{H}(\mathcal{D}_m(\mathcal{E}), \mathrm{qis}) \xrightarrow{\sim} \mathcal{SDH}_{\mathbb{Z}/m}(\mathcal{E}).$$

(iii) $\mathcal{H}(\mathcal{D}_m(\mathcal{E}), \mathrm{qis})$, is functorial p to twist under exact functors preserving quasi-isomorphisms and tilting invariant.

(iv) If $\mathcal{E}$ is hereditary, we have an isomorphism

$$\mathcal{H}(\mathcal{D}_m(\mathcal{E}), \mathrm{qis}) \xrightarrow{\sim} (\mathcal{H}(\mathcal{E})[[K]^{-1}, K \in \mathcal{C}_{m,ac}(\mathcal{E})])\Big/ I\ .$$

The right hand side in (iv) was considered by Lu and Peng [49] under the name of *modified Ringel-Hall algebras*.

The periodic derived categories of abelian categories $\mathcal{D}_m(\mathcal{A})$ do not have $t-$structures, but we are still interested in the relation between the twisted extended Hall algebras of $\mathcal{D}_m(\mathcal{A})$ and of $\mathcal{A}$.

It follows from the computations in Lemma 4.1.1, that for an abelian category $\mathcal{A}$, the classes of $m-$periodic complexes quasi-isomorphic to stalk complexes concentrated in degree 0 form an extension-closed subcategory of $\mathcal{D}_m(\mathcal{A})$ if and only if $\mathrm{gl\,dim}(\mathcal{A}) \leq m$. They always form a fully exact subcategory of $\mathcal{C}_m(\mathcal{A})$ that we denote $\widetilde{\mathcal{A}}$. We have well-defined cohomologies $H^i, i = 0, \ldots, m-1$, with values in $\mathcal{A}$, and from the long exact sequence it follows that the complexes with cohomologies concentrated in degree 0 always form a fully exact subcategory $\mathcal{C}_m^0(\mathcal{A})$ of $\mathcal{C}_m(\mathcal{A})$ closed under quasi-isomorphisms and their classes form an extension-closed subcategory $\mathcal{D}_m^0(\mathcal{A})$ of $\mathcal{D}_m(\mathcal{A})$. Thus, we have the following result.

Theorem 4.4.3. *Let $\mathcal{A}$ be an $\mathbb{F}_q-$linear abelian category of finite global dimension with finite $\mathrm{Hom}-$ and Ext^i-space, for all $i > 0$. Then the Hall algebras $\mathcal{H}(\mathcal{A}), \mathcal{H}(\mathcal{C}_m(\mathcal{A}), \mathrm{qis}), \mathcal{H}(\mathcal{C}_m^0(\mathcal{A}), \mathrm{qis})$ are well-defined.*

(i) We have an embedding

$$\mathcal{H}(\widetilde{\mathcal{A}}) \hookrightarrow \mathcal{H}(\mathcal{C}_m(\mathcal{A}), qis),$$

and the algebra

$$\mathcal{H}_{tw}(\widetilde{\mathcal{A}})$$

is a flat deformation of $\mathcal{H}(\mathcal{A})$ over

$$\mathbb{Q}K_0^{num}(\mathcal{C}_{m,ac}(\mathcal{A})) \xrightarrow{\sim} \bigotimes_{i=1}^{m} K_0^{num}(\mathcal{A}),$$

if and only if $\mathrm{gl\,dim}(\mathcal{A}) \leq m$.

(ii) We have an embedding

$$\mathcal{H}(\mathcal{C}_m^0(\mathcal{A}), \mathrm{qis}) \hookrightarrow \mathcal{H}(\mathcal{C}_m(\mathcal{A}), qis)$$

together with isomorphisms

$$\mathcal{H}_{tw}(\mathcal{C}_m^0(\mathcal{A}), \mathrm{qis}) \xrightarrow{\sim} \mathcal{H}(\mathcal{D}_m^0(\mathcal{A})) \otimes \mathbb{Q}K_0^{num}(\mathcal{C}_{m,ac}(\mathcal{A})) \xrightarrow{\sim} \mathcal{H}(\mathcal{A}) \otimes \bigotimes_{i=1}^{m} K_0^{num}(\mathcal{A}),$$

for all $\mathcal{A}$.

Geng and Peng [21] proved the existence of the embedding and the isomorphisms in part (i) for $\mathcal{A}$ having enough projectives and of $\mathrm{gl\,dim} \leq 2$ and m being equal to 2. They used Bridgeland's construction of twisted extended Hall algebras of periodic derived categories.

Lu and Peng [49] proved that for hereditary abelian category $\mathcal{A}$, the algebra

$$((\mathcal{H}(\mathcal{C}_2(\mathcal{A})), qis))[[K]^{-1}, K \in \mathcal{C}_{2,ac}(\mathcal{A})])/I)_{cw}$$

is the Drinfeld double of the algebra $\mathcal{H}_{tw}^e(\mathcal{A})$, generalizing the rezult of Yanagida. The multiplication in $\mathcal{H}(\mathcal{C}_2^0(\mathcal{A}), \mathrm{qis})$ involves higher extensions in $\mathcal{A}$ and it might be interesting to check the relation between the multiplication and the comultiplication in it.

After the twist be the form $\langle -, - \rangle_{cw}$, we can take a reduction as we did in Section 4.3. This algebra satisfies conditions of Section 3.2 and we can take its idempotented form. When $\mathcal{A} = \mathrm{mod}\,\mathbb{F}_q\dot{Q}$, the natural group H is the root lattice, and we recover the subalgebra of Lusztig's algebra $\dot{U}$, where we take the root lattice instead of the weight lattice.

Chapitre 5

Reduction of exact structures and degeneration of Hall algebras

5.1 Exact structures and degenerations of Hall algebras

Let $\mathcal{A}$ be an additive category. As in the previous chapters, we assume that $\mathcal{A}$ is essentially smal, Hom −finite, idempotent complete and linear over some field k. We know (see Chapter 1) that $\mathcal{A}$ can be endowed with many different exact structures. Assume that $\mathcal{E}'$ and $\mathcal{E}$ are two different Ext^1 −finite exact structures. Then their Hall algebras $\mathcal{H}(\mathcal{E}')$ and $\mathcal{H}(\mathcal{E})$ are both well-defined. It is then natural to ask whether there is any sensible relation between these algebras. In this chapter, we find such a relation in the case when $\mathcal{E}$ is larger than $\mathcal{E}'$, i.e. it has more conflations. We denote it by $\mathcal{E}' < \mathcal{E}$. Note that in this case, $\mathcal{E}'$ has more projective objects than $\mathcal{E}$ and, in particular, more indecomposable projectives : $\mathrm{Ind}(\mathcal{P}(\mathcal{E})) \subset \mathrm{Ind}(\mathcal{P}(\mathcal{E}'))$, where $\mathrm{Ind}(-)$ denotes the set of indecomposable objects in a category. We denote by $\mathcal{E}\backslash\mathcal{E}'$ the class of conflations in $\mathcal{E}$ that are *not* conflations in $\mathcal{E}'$. Proceeding from $\mathcal{E}$ to $\mathcal{E}'$ is called the *reduction* of exact structures in [6], see the reference for motivation.

5.1.1 General result

Generalizing a definition from [14] from the category of quiver representations (considered as an abelian category), we introduce the following notions.

Definition 5.1.1. A function $w : \mathrm{Iso}(\mathcal{A}) \to \mathbb{N}$ is called

(i) *normalized* if $w(M) = 0$ only for $M = 0$,

(ii) *weakly $\mathcal{E}$−admissible* if $w(X) \leq w(M) + w(N)$ whenever there exists a conflation

$$N \rightarrowtail X \twoheadrightarrow M$$

in $\mathcal{E}$;

(iii) $\mathcal{E}-$*additive* if $w(X) = w(M) + w(N)$ whenever there exists a conflation

$$N \rightarrowtail X \twoheadrightarrow M$$

in $\mathcal{E}$;

(iv) $\mathcal{E}-$*admissible* if it is weakly $\mathcal{E}-$admissible and $\mathcal{E}^{add}-$additive, i.e. we have $w(M \oplus N) = w(M) + w(N)$ for all M and N;

(v) $\mathcal{E}$*-characteristic* if it is normalized, $\mathcal{E}-$additive and $w(X) < w(M) + w(N)$ whenever there exists a conflation

$$N \rightarrowtail X \twoheadrightarrow M$$

in $\mathcal{E}^{max} \backslash \mathcal{E}$.

Note that, for $\mathcal{E}' < \mathcal{E}$, any (weakly) $\mathcal{E}-$admissibe function is (weakly) $\mathcal{E}'-$admissible and any $\mathcal{E}'-$characteristic function is $\mathcal{E}-$admissible. $\mathcal{E}^{add}-$characteristic functions were called *strongly admissible* in [14].

Proposition 5.1.2. *(i) If w is a weakly $\mathcal{E}-$admissible function, there exists a filtration $\mathcal{F}_\bullet$ on $\mathcal{H}(\mathcal{E})$ where $\mathcal{F}_n$ is spanned by the $[M]$ such that $w(M) \leq n$, which is normalized if w is so.*

(ii) If w is $\mathcal{E}'-$characteristic, the associated graded algebra of $\mathcal{H}(\mathcal{E})$ with respect to this filtration is $\mathcal{H}(\mathcal{E}')$.

Proof. The proof of [14, Corollary 1] applies here word for word. It uses only [14, Lemma 3]. □

Corollary 5.1.3. *If w is $\mathcal{E}^{add}-$charateristic, the associated graded algebra of $\mathcal{H}(\mathcal{E})$ with respect to this filtration is $\mathcal{H}(\mathcal{E}^{add})$. It is a skew polynomial algebra in variables in* $\mathrm{Iso}(\mathcal{A})$, *namely, the monoid algebra of* $(\mathrm{Iso}(\mathcal{A}), \oplus)$, *twisted by the form* $\frac{1}{|\mathrm{Hom}_{\mathcal{A}}(-,-)|}$.

5.1.2 Categories of finite type

In this section, we assume that the additive category $\mathcal{A}$ is *of finite type*, that is, it has only finitely many indecomposable objects up to isomorphism. As before, we assume that $\mathcal{A}$ is $\mathrm{Hom}-$finite and linear over a field k. In this case, we can consider its *Auslander algebra* $\Gamma := \mathrm{End}_{\mathcal{A}}(M)$, where M is the direct sum of all non-isomorphic indecomposables in $\mathcal{A}$. It is finite-dimensional over k and thus artinian. Then all exact structures on $\mathcal{A}$ are *admissible* in the sense of Enomoto [17] and can be classified via Auslander-Reiten (AR) theory. We do not need an explicit definition of admissible exact structures in what follows, so we do not give it here.

Recall the basic notions of the AR theory. A morphism $g : X \to Z$ is called *right almost split* if it is not a retraction and any non-retraction $h : W \to Z$ factors through g. *Left almost split* are defined dually. A conflation $X \overset{f}{\rightarrowtail} Y \overset{g}{\twoheadrightarrow} Z$ in an exact category $\mathcal{E}$ is an AR conflation if f is left almost split and g is right almost split. We say that $\mathcal{E}$ has AR conflations if for every

indecomposable non-projective object Z there exists an AR conflation ending at Z, and for every indecomposable non-injective object X there exists an AR conflation starting at X.

Let $\mathrm{AR}(\mathcal{E})$ be the subgroup of $K_0^{add}(\mathcal{E})$ generated by

$$\{[X]-[Y]+[Z] \quad | \quad \text{there exists an AR conflation} X \rightarrowtail Y \twoheadrightarrow Z \quad \text{in} \mathcal{E}\},$$

$\mathrm{Ex}(\mathcal{E})$ the subgroup generated by

$$\{[X]-[Y]+[Z] \quad | \quad \text{there exists a conflation} X \rightarrowtail Y \twoheadrightarrow Z \quad \text{in} \mathcal{E}\}.$$

Let us summarize those results of Enomoto that we will need.

Theorem 5.1.4 (Enomoto). *Let $\mathcal{A}$ be a $\mathrm{Hom}-$finite idempotent complete $k-$linear additive category of finite type. Let $\mathcal{E}$ be an exact structure on $\mathcal{A}$. Then the following holds.*

(i) [17, Corollary 3.15] $\mathcal{E}$ has enough projectives and enough injectives.

(ii) [17, Corollary A.4] $\mathcal{E}$ has AR conflations.

(iii) [17, Proposition 3.8] $\mathcal{E}$ is Ext^1-finite.

(iv) [17, Corollary 3.10] There is a one-to-one correspondence between the exact structures on $\mathcal{A}$ and the subsets of the (finite) set $\mathrm{ind}(\mathcal{A})\setminus \mathrm{ind}(\mathcal{P}(\mathcal{E}^{max}))$. For an exact structure $\mathcal{E}$, the corresponding subset coincides with the set $\mathrm{ind}(\mathcal{P}(\mathcal{E}))\setminus \mathrm{ind}(\mathcal{P}(\mathcal{E}^{max}))$. In particular, $\mathrm{ind}(\mathcal{P}(\mathcal{E}^{add})) = \mathrm{ind}(\mathcal{A})$.

(v) [17, Corollary 3.18] $\mathrm{Ex}(\mathcal{E}) = \mathrm{AR}(\mathcal{E})$. Moreover, any element of $\mathrm{Ex}(\mathcal{E})$ is a non-negative linear combination of elements of $\mathrm{AR}(\mathcal{E})$.

The following result was proved in [14] for the categories of representations of Dynkin quivers, but the proof uses only the existence of AR conflations and the statement of Theorem 5.1.4(v), and thus applies in the general situation.

Theorem 5.1.5. *[14, Theorem 4 (1)] Let $\mathcal{A}$ be a $\mathrm{Hom}-$finite idempotent complete $k-$linear additive category of finite type. A function w is $\mathcal{E}^{max}-$admissible if and only if it is of the form $w(M) = \dim \mathrm{Hom}(V, M)$ for an object $V \in \mathcal{A}$.*

In the proof, it is first shown that any $\mathcal{E}^{add}-$additive function decomposes as $w = \sum_{X \in \mathrm{ind}\,\mathcal{A}} a_X w_X$, where $w_X(M) = \dim \mathrm{Hom}(X, M)$, for some $a_X \in \mathbb{Z}$. Then it is shown that for $X \in \mathrm{ind}(\mathcal{A})\setminus \mathrm{ind}(\mathcal{P}(\mathcal{E}^{max}))$, the coefficient a_X is given by the formula

$$a_X = w_X(X) - w_X(T) + w_x(\tau X),$$

where

$$\tau X \rightarrowtail T \twoheadrightarrow X$$

is the AR conflation in $\mathcal{E}^{max}$ ending at X. For this, [14] uses the categories of $\mathcal{A}-$modules, that is, additive $k-$linear functors from $\mathcal{A}$ to the category of finite-dimensional $k-$vector spaces.

Theorem 5.1.6. *Let $\mathcal{A}$ be a* Hom *−finite idempotent complete $k-$linear additive category of finite type. Let $\mathcal{E}$ be an exact structure on $\mathcal{A}$. Then the function*

$$w_{\mathcal{E}} : \mathrm{Iso}(\mathcal{A}) \to \mathbb{N}, \qquad w(M) = \dim \mathrm{Hom}(\bigoplus_{P \in \mathrm{ind}(\mathcal{P}(\mathcal{E}))} P, M) \tag{5.1}$$

is $\mathcal{E}-$characteristic.

Proof. By Theorem 5.1.5, the function $w_{\mathcal{E}}$ is $\mathcal{E}^{max}-$admissible. It is also clearly $\mathcal{E}-$additive. In its decomposition $w_{\mathcal{E}} = \sum_{X \in \mathrm{ind}\,\mathcal{A}} a_X w_X$, we have $a_S = 1$, for S simple projectives in $\mathcal{E}^{max}$. By the proof of [14, Theorem 4 (2)], this implies that $w_{\mathcal{E}}$ is normalized.

Let $X \rightarrowtail Y \twoheadrightarrow Z$ be a conflation in $\mathcal{E}^{max}$. By Theorem 5.1.4(v), we have a decomposition

$$[X] - [Y] + [Z] = \sum_{U \in \mathrm{ind}(\mathcal{A}) \setminus \mathrm{ind}(\mathcal{P}(\mathcal{E}^{max}))} ([\tau U] - [C] + [U]) b_U,$$

for some $b_U \in \mathbb{Z}_+$, where

$$\tau U \rightarrowtail C \twoheadrightarrow U$$

is the AR conflation in $\mathcal{E}^{max}$ ending at U.

Then we have

$$w_{\mathcal{E}}([X] - [Y] + [Z]) = \sum_{U \in \mathrm{ind}(\mathcal{A}) \setminus \mathrm{ind}(\mathcal{P}(\mathcal{E}^{max}))} w_{\mathcal{E}}([\tau U] - [C] + [U]) b_U =$$

$$= \sum_{\substack{U \in \mathrm{ind}(\mathcal{A}) \setminus \mathrm{ind}(\mathcal{P}(\mathcal{E}^{max})) \\ P \in \mathrm{ind}(\mathcal{P}(\mathcal{E}))) \setminus \mathrm{ind}(\mathcal{P}(\mathcal{E}^{max}))}} w_P([\tau U] - [C] + [U]) b_U =$$

$$= \sum_{\substack{U \in \mathrm{ind}(\mathcal{A}) \setminus \mathrm{ind}(\mathcal{P}(\mathcal{E}^{max})) \\ P \in \mathrm{ind}(\mathcal{P}(\mathcal{E}))) \setminus \mathrm{ind}(\mathcal{P}(\mathcal{E}^{max}))}} b_U \delta_P^U =$$

$$= \sum_{P \in \mathrm{ind}(\mathcal{P}(\mathcal{E}))) \setminus \mathrm{ind}(\mathcal{P}(\mathcal{E}^{max}))} b_P.$$

Since all $b_P \geq 0$, we have

$$w_{\mathcal{E}}([X] - [Y] + [Z]) = 0$$

if and only if $b_P = 0, \quad \forall P \in \mathrm{ind}(\mathcal{P}(\mathcal{E})))\setminus \mathrm{ind}(\mathcal{P}(\mathcal{E}^{max}))$. This happens if and only if we have a decomposition

$$[X] - [Y] + [Z] = \sum_{U \in \mathrm{ind}(\mathcal{A}) \setminus \mathrm{ind}(\mathcal{P}(\mathcal{E}))} ([\tau U] - [C] + [U]) b_U,$$

for some $b_U \in \mathbb{Z}_+$. Applying Theorem 5.1.4(v), we find that this happens if and only if the

conflation

$$X \rightarrowtail Y \twoheadrightarrow Z$$

is a conflation in $\mathcal{E}$. Then if this conflation is in $\mathcal{E}^{max} \backslash \mathcal{E}$ instead, we have

$$w_{\mathcal{E}}([X] - [Y] + [Z]) > 0,$$

or, equivalently,

$$w_{\mathcal{E}}(Y) < w_{\mathcal{E}}(X) + w_{\mathcal{E}}(Z).$$

This finishes the proof. □

Corollary 5.1.7. *For any $\mathcal{E} > \mathcal{E}'$, the function $w_{\mathcal{E}'}$ defines a filtration on $\mathcal{H}(\mathcal{E})$ whose associated graded is $\mathcal{H}(\mathcal{E}')$.*

Corollary 5.1.8. *Let $\mathcal{A}$ be a $\mathrm{Hom}-$finite idempotent complete $k-$linear additive category of finite type. The function w_V defined by $w_V(M) = \dim(V, M)$, where V is the direct sum of several pairwise different indecomposable objects and V contains each simple $\mathcal{E}^{max}-$projective, is a $\mathcal{E}^{max}-$admissible function on $\mathrm{Iso}(\mathcal{A})$. It induces a filtration on $\mathcal{H}(\mathcal{E}^{max})$ whose associated graded is $\mathcal{H}(\mathcal{E})$, where $\mathcal{E}$ is a unique exact structure whose set of projectives is the union of the set of the indecomposable summands of V and $\mathrm{ind}(\mathcal{P}(\mathcal{E}^{max}))$.*

Proposition 5.1.9. *For $\mathcal{A}$ being (the additive) category of representations of a Dynkin quiver Q over $\mathbb{F}_q$, all the linear degenerations defined in [14] are precisely the Hall algebras of all possible exact structures on $\mathcal{A}$. In particular, a strongly admissible function inducing the quantum PBW degeneration to $q-$symmetric polynomials corresponds to $\mathcal{E}^{add}$. The non-degenerate algebra (isomorphic to $U_{\sqrt{q}}(n)$) corresponds to the abelian exact structure.*

Remark 5.1.10. When $\mathcal{A}$ is the category of representations of a Dynkin quiver, $\mathcal{E}^{max}-$admissible functions bijectively correspond to the lattice points of the *quantum degree cone* considered in [1]. Results of Enomoto and Corollary 5.1.8 imply that the objects in the set $\mathrm{ind}(\mathcal{A}) \backslash \mathrm{ind}(\mathcal{P}(\mathcal{E}^{max}))$ bijectively correspond to the generators of this cone and the exact structures on $\mathcal{A}$ bijectively correspond to the faces.

5.2 2-periodic complexes and generalized quantum doubles

The following lemma is not difficult to prove by checking that the condition on the exact structure implies that all CE-projective objects must be projective in $\mathcal{E}$.

Lemma 5.2.1. *Let $\mathcal{A}$ be a $\mathrm{Hom}-$finite $k-$linear abelian category of finite type. Then the CE-exact structure on $\mathcal{C}_{\mathbb{Z}_2}(\mathcal{A})$ is the minimal exact structure $\mathcal{E}$ on the underlying additive category, such that all the abelian conflations in each degree are in $\mathcal{E}$ and all the conflations with acyclic first term are in $\mathcal{E}$. Its projective objects are the acyclic complexes with projective components and the stalk complexes with projective components.*

We know that the entire quantum group $U_{\sqrt{q}}(\mathfrak{g})$ is the quotient of the twisted localized Hall algebra of 2-periodic complexes over $\operatorname{mod} kQ$ with the abelian exact structure. On the other hand, it is the reduced Drinfeld double of the twisted extended Hall algebra of $\operatorname{mod} kQ$. Each Hopf pairing on this twisted extended Hall algebra defines a Hopf algebra structure on its tensor square. It is called a *generalized quantum double*, see, e.g. [36]. The most degenerate case provides the product Hopf algebra (on the tensor square).

Conjecture 5.2.2. *Hopf pairings of $\mathcal{H}^e_{tw}(\operatorname{mod} kQ)$ with itself bijectively correspond to exact stuctures on $\mathcal{C}_2(\operatorname{mod} kQ)$ sitting between the CE exact structure and the abelian exact structure. The associated quotient by I of the twisted localized Hall algebra for each such exact category is the corresponding generalized quantum double of $\mathcal{H}^e_{tw}(\operatorname{mod} kQ)$. CE exact category corresponds to the product Hopf algebra, the maximal exact structure gives the usual Drinfeld double.*

The quotient of the twisted localized Hall algebra $\mathcal{H}(\mathcal{C}_2(\operatorname{mod} kQ), CE)$ is isomorphic to the extended Hall algebra of the (hereditary) category $\operatorname{gr}_{\mathbb{Z}/2} \operatorname{mod} kQ$. The comultiplication on it is the usual Hall multiplication on the first factor and the opposite on the second. Since the comultiplication on all the generalized quantum doubles is the same, this also defines the comultiplication on the Drinfeld double $\mathcal{SDH}_{\mathbb{Z}/2}(\operatorname{mod} kQ)$.

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