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Basic sets and decomposition matrices of finite groups of Lie type in small characteristic

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Thèse de doctorat de mathématiques

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Titre : Ensembles basiques et matrices de décomposition de groupes finis de type de Lie en petite caractéristique

Résumé : Cette thèse se concentre sur les aspects modulaires de la théorie des représentations. Plus précisément, nous nous intéressons aux ensembles basiques des blocs unipotents des groupes finis de type de Lie qui vérifient une propriété d' « unitriangularité ». Dans la première partie de cette thèse , en nous inspirant des travaux de Lusztig sur le paramétrage des représentations unipotentes en caractéristique 0, nous introduisons une méthode pour compter les représentations modulaires irréductibles contenues dans les blocs unipotents. Nous conjecturons que cette méthode est valable pour tout les groupes finis de type de Lie définis sur un corps dont la caractéristique est bonne et nous montrons que la conjecture est vraie dans un certain nombre de cas. La seconde partie de cette thèse a consisté à généraliser les résultats de Geck sur l'existence d'ensemble basiques unitriangulaire pour les 2-blocs unipotents des groupes classiques au cas ou le centre est non connexe. Le dernier aspect de cette thèse porte sur les matrices de décomposition des groupes finis de type de Lie dans le cas de mauvais nombres premier. Nous obtenons des résultats pour le groupe le groupe $\mathrm{Sp}_4(q)$ et le groupe exceptionnel $G_2(q)$.

Mots clefs : Deligne-Lusztig, Représentations, ensembles basiques, matrices de décompositions, Représentations modulaires, groupes réductifs, groupes réductifs finis, groupes finis de type de Lie.

Title : Basic sets and decomposition matrices of finite groups of Lie type in small characteristic.

Abstract : This thesis is focused on the modular aspect of representation theory. More precisely, we are interisted in basic sets for unipotent blocks of finite groups of Lie typ which are « unitriangular ». In the first part of the thesis, following Lusztig's work on the parametrisation of unipotent representations in characeristic , we introduce a method to count irreducible modular representations lying in unipotent blocks. We conjecture that our method holds for every finite groups of Lie type defined over a field of good characteristic and we verify our conjecture in many cases. The second part of the thesis consists to generalize results of Geck on the existence of unitriangular basic sets for unipotent 2-blocks of classical groups to the case where the center is disconnected. The last aspect of the thesis is the computation of decomposition matrices of finite groups of Lie type for bad primes. We got results for $\mathrm{Sp}_4(q)$ and $G_2(q)$.

Keywords : Deligne-Lusztig, Representations, basic sets, decomposition matrices, modular representations, reductive groups, finite reductive groups, finite groups of Lie type.

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Introduction

Representations of finite groups

Let G be a finite group and K be a field. A *representation* of G over K is a finite-dimensional K-vector space on which G acts by linear automorphisms. Let KG be the K-algebra whose basis consists of elements of G and whose multiplication law is induced by the group law of G. Then, representations of G can be viewed as finitely generated KG-modules. Since a representation is a vector space V on which G acts, it is natural to look at the non-zero subvector spaces which are stable by G. These are the *subrepresentations* of G. The representations without non-zero proper subrepresentation are said to be *irreducible*.

When the field has characteristic zero (*e.g.* $K = \mathbb{C}$), the representation theory of G is much simpler. In that case, by Maschke's theorem, any representation of V over K can be written as a sum

$$V = \bigoplus_{i=1}^{n} V_i,$$

where $V_1, \ldots, V_r$ are irreducible subrepresentations of V. In other words, the ordinary (characteristic zero) representation theory of G is governed by the irreducible representations. These can be studied via the Grothendieck group $R_K(G)$ of G which is the abelian group where the :

- Generators are the finitely generated KG-modules. We denote by $[V]$ the image of a KG-module V in $R_K(G)$.
- Relations are $[V] = [V_1] + [V_2]$ whenever we have a short exact sequence $0 \to V_1 \to V \to V_2 \to 0$.

Note that we have $R_K(G) = \bigoplus_{V \in \mathrm{Irr}_K(G)} \mathbb{Z}[V]$ where $\mathrm{Irr}_K(G)$ is a set of representatives of isomorphism classes of irreducible representations of G over K.

Maschke's theorem states that, when K has characteristic zero, the isomorphism class of a finitely generated KG-module is completely determined by its image in $R_K(G)$.

For representations over a field of positive characteristic (*modular* representations), the situation is much more complicated. From now on, we assume that K has characteristic 0, and we denote by k a field of characteristic ℓ. We will assume that both K and k are algebraically closed. When ℓ divides the order of G, the image of a representation in $R_k(G)$ determines only its composition factors. However, we can use connections between representations of G over K and k to deduce information on representations over k from what we know on representations over K. The first step toward this goal is the decomposition matrix. To a representation V over K (an *ordinary* representation), we can associate some modular representation $\bar{V}$ which is constructed as a "reduction mod ℓ" of V. The *ℓ-decomposition map* of G is the map $d : R_K(G) \to R_k(G)$ defined by $d([V]) = [\bar{V}]$. Let $\{V_1, \ldots, V_n\}$ (resp. $\{W_1, \ldots, W_r\}$) be a set of representatives of isomorphism classes of irreducible representations of G over K (resp. over k). For $j = 1, \ldots, n$, let us write $d([V_j]) = \sum_{i=1}^{r} d_{i,j}[W_i]$. Then the matrix $(d_{i,j})$ is called the *ℓ-decomposition matrix* of G. Let $\mathcal{B} = \{E_1, \ldots, E_r\}$ be a set of r distinct irreducible representations of G over K. Let us write the decomposition matrix as follows :

$$(d_{i,j}) = \begin{array}{c} \\ E_1 \\ \vdots \\ E_r \\ \hline \vdots \end{array} \begin{array}{c} \begin{array}{ccc} W_1 & \ldots & W_r \end{array} \\ \left[\begin{array}{c} \\ D \\ \\ \hline * \end{array}\right] \end{array}.$$

We say that $\mathcal{B}$ is a *basic set* of G if the matrix D is invertible. If in addition D is lower unitriangular, we say that D is a *unitriangular basic set* of G. If $\mathcal{B}$ is a unitriangular basic set, D has the following form.

$$D = \begin{array}{c} \\ E_1 \\ \\ \vdots \\ \\ E_r \end{array} \begin{array}{c} \begin{array}{ccc} W_1 & \ldots & W_r \end{array} \\ \left[\begin{array}{ccc} 1 & & 0 \\ & \ddots & \\ * & & 1 \end{array}\right] \end{array}.$$

Then we obtain a natural parametrisation of $\mathrm{Irr}_k(G)$ by $\mathcal{B}$ given by $W_i \leftrightarrow E_i$. It is an open problem whether any group admits a (unitriangular) basic set of ordinary representations.

Finite groups of Lie type

Finite groups of Lie type are finite groups arising as rational points of a connected reductive group defined over $\mathbb{F}_q$ where q is a power of a prime number p. That construction allows us to transfer the geometric structure of connected reductive groups into finite groups of Lie type : each element has a Jordan decomposition, each finite group of Lie type has a BN-pair, finite groups of Lie type are classified in terms of root data and graph automorphisms. Let us set the notation for finite groups of Lie type. If $\mathbf{G}$ is a connected reductive group defined over $\mathbb{F}_q$ and $F : \mathbf{G} \to \mathbf{G}$ is the corresponding Frobenius map then we denote by $G := \mathbf{G}^F := \{g \in \mathbf{G} \mid F(g) = g\}$ the corresponding finite group of Lie type. For example, if we set $\mathbf{G} := \mathrm{GL}_n(\bar{\mathbb{F}}_q)$ and

$$F : \mathbf{G} \to \mathbf{G}, \quad (a_{i,j}) \mapsto (a_{i,j}^q),$$

then $G := \mathrm{GL}_n(q)$ is a finite group of Lie type. More generally, most finite classical groups over finite fields are finite groups of Lie type (the special linear groups, the special orthogonal groups, the symplectic groups, the spin groups, ...). In 1976, by constructing varieties associated to finite reductive group and by studying the ℓ-adic cohomology attached to those varieties, Deligne and Lusztig constructed virtual representations which we now refer to as *Deligne–Lusztig representations*. In 1984, Lusztig went further and provided a parametrisation of $\mathrm{Irr}_K(G)$. This was done as follows.

- Lusztig provided a partition of $\mathrm{Irr}_K(G)$ parametrised by semi-simple elements of a dual group. More precisely, to a reductive group $\mathbf{G}$ defined over $\mathbb{F}_q$ we can associate another reductive group $\mathbf{G}^*$ defined over $\mathbb{F}_q$, said to be *dual* to $\mathbf{G}$ which has the dual root datum. We denote by G^* the finite group of Lie type associated to $\mathbf{G}^*$. For example $\mathrm{GL}_n(q)$ is self-dual, the dual of $\mathrm{SL}_n(q)$ is $\mathrm{PGL}_n(q)$. We have a partition.

  $$\mathrm{Irr}_K(G) = \bigsqcup \mathcal{E}(G, s)$$

  where s runs over representatives of conjugacy classes of semi-simple elements of G^* (for instance, if $G = \mathrm{GL}_n(q)$, semi-simple elements are diagonalisable matrices). The set $\mathcal{E}(G, s)$ is the *rational Lusztig series* associated to s. Elements of $\mathcal{E}(G, 1)$ are called *unipotent* representations.

- In [41] Lusztig classified unipotent characters of finite groups of Lie type in terms of some unipotent classes of $\mathbf{G}$ (we recall that unipotent classes are conjugacy classes containing upper unitriangular matrices). Using the Springer correspondence, Lusztig defined the notion of special unipotent classes. To each special unipotent class $(u)_{\mathbf{G}}$ he associated a set $\mathcal{M}(u)$ and showed that $\mathcal{E}(G,1)$ is parametrised by pairs $((u)_{\mathbf{G}}, x)$ where $(u)_{\mathbf{G}}$ is an F-stable special unipotent conjugacy class of $\mathbf{G}$ and $x \in \mathcal{M}(u)$.
- Finally, in the case $Z(\mathbf{G})$ is connected, Lusztig showed that the study of irreducible characters can be reduced to the study of unipotent characters. More precisely, he constructed a bijection

$$\mathcal{E}(G,s) \simeq \mathcal{E}(C_{G^*}(s),1).$$

To summarise, when $Z(\mathbf{G})$ is connected, irreducible characters can be parametrised by triples $((s),(u),x)$ where (s) is a semi-simple class of the finite group G^*, (u) is an F-stable special unipotent class of $C_{\mathbf{G}^*}(s)$ and $x \in \mathcal{M}(u)$.

Unipotent blocks and basic sets

From now on, we work in *non-defining characteristic*, *i.e.* we assume that ℓ does not divide p. There exists a partition of both $\mathrm{Irr}_K(G)$ and $\mathrm{Irr}_k(G)$ into ℓ-blocks. The decomposition map preserves the block partition. Therefore, we can study the decomposition matrices and basic sets of G blockwise. Let s be a semi-simple element of G^* of order prime to ℓ. Let

$$B_s := \bigcup \mathcal{E}(G,st)$$

where t runs over semi-simple elements of $C_{G^*}(s)$ whose order is a power of ℓ. Then Broué–Michel showed in [6] that B_s is a union of ℓ-blocks. Blocks contained in B_1 are called unipotent ℓ-blocks.

In the past 30 years, the study of decomposition matrices and basic sets for finite groups of Lie type has been an active research area. In order to summarise the known results on the subject, we need to introduce the notion of *good prime* and *bad prime* for $\mathbf{G}$. The prime number ℓ is good for $\mathbf{G}$ if it does not divide the order of its fundamental group. The prime number ℓ is bad otherwise. The behavior of the basic sets and the decomposition matrices of G depends heavily on whether ℓ is good or bad for $\mathbf{G}$. In [17], Dipper showed the existence of a unitriangular basic set for G when $G = \mathrm{GL}_n(q)$. Then, Geck generalised the work of Dipper for $\mathrm{GU}_n(q)$ by an entirely different method using generalised Gelfand–Graev representations introduced

by Kawanaka [34]. He even showed that $\mathcal{E}(G,s)$ is a unitriangular basic set for B_s. Geck–Hiss and Geck generalised partially those results in [27, 21] to most cases by showing that $\mathcal{E}(G,s)$ gives a basic set of B_s whenever ℓ is good for $\mathbf{G}$ and does not divide the order of $(Z(\mathbf{G})/Z(\mathbf{G})^\circ)^F$ where $Z(\mathbf{G})$ is the center of $\mathbf{G}$. Furthermore, they conjectured in [28] that when ℓ is good, $\mathcal{E}(G,1)$ forms a unitriangular basic set for unipotent blocks.

If ℓ is bad, the situation is different. For example, if $G = \mathrm{SL}_2(3)$ and $\ell = 2$, there are not enough unipotent characters to form a basic set for B_1. In [36], using Clifford Theory, Kleshchev–Tiep showed the existence of a basic set for $\mathrm{SL}_n(q)$. Denoncin in [13] generalised this result to $\mathrm{SU}_n(q)$. In [22], using generalised Gelfand–Graev representations and results of Lusztig on character sheaves, Geck showed the existence of a basic set for unipotent blocks of classical groups whose center is connected when $\ell = 2$. We generalise this result for groups with disconnected center.

Theorem

Let $\mathbf{G}$ be a reductive group, defined over $\mathbb{F}_q$ where q is odd. Assume that $\mathbf{G}$ is simple and of type B, C or D except spin of half-spin. Then there exists a unitriangular basic set for unipotent 2-blocks of G.

The proof, following Geck's ideas can be summarised into three steps.

- We show that the number of unipotent modular representations equals the number of unipotent classes of G.
- Let $\Gamma_1, \ldots, \Gamma_r$ be the generalised Gelfand–Graev representations of G. There exist $\chi_1, \ldots, \chi_r \in \mathrm{Irr}_K(G)$ such that $(\langle \chi_i, \Gamma_j \rangle)_{i,j}$ is lower unitriangular.
- By using the fact that $\chi_1, \ldots, \chi_r$ lie in Lusztig series associated to 2-elements, we note that they belong to unipotent blocks. Then, by a classical result of modular representation theory, the scalar product property above allows us to conclude that $\chi_1, \ldots, \chi_r$ is a unitriangular basic set for B_1.

Counting unipotent modular representations

Since unipotent characters form a basic set of B_1 when ℓ is good and does not divide $|(Z(\mathbf{G})/Z(\mathbf{G})^\circ)^F|$, Lusztig's parametrisation of unipotent representations can be used to count unipotent modular representations. If $\ell = 2$ and G is classical, unipotent representations can be counted by unipotent

classes of the finite group G. We introduce a method to count unipotent modular representations generalising previous cases. Then, we conjecture that our method is effective for every finite group of Lie type. Let us give more details about this method. We first generalise the definition of special classes. In [41], using the Springer correspondence and j-induction of representations of Weyl groups, Lusztig defined a surjective map Φ from the set of special conjugacy classes of $\mathbf{G}^*$ onto the set of unipotent classes of $\mathbf{G}$. Let $(u)_{\mathbf{G}}$ be an F-stable unipotent class of $\mathbf{G}$. The class $(u)_{\mathbf{G}}$ is *ℓ-special* if there exists $g \in \mathbf{G}^*$ with Jordan decomposition $g = sv$ such that

- $\Phi((g)_{\mathbf{G}^*}) = (u)_{\mathbf{G}}$,
- s is an isolated ℓ-element of $\mathbf{G}^*$ (*i.e.* the order of s is a power of ℓ and $C_{\mathbf{G}^*}(s)^\circ$ is not contained in any proper Levi subgroup of $\mathbf{G}^*$),
- v is special in $C_{\mathbf{G}^*}(s)^\circ$.

In particular, special classes are ℓ-special since the neutral element is an isolated ℓ-element for all prime ℓ. If ℓ is good and does not divides $|(Z(\mathbf{G})/Z(\mathbf{G})^\circ)^F|$, ℓ-special classes are the special ones. As Lusztig did with special classes, to each ℓ-special class $(u)_{\mathbf{G}}$, we associate a set $\mathcal{M}_\ell(u)$ which we conjecture to parametrise a family of unipotent representations over k.

Conjecture

The number of irreducible unipotent modular representations of G is equal to the number of pairs $((u)_{\mathbf{G}}, x)$ where $(u)_{\mathbf{G}}$ is an F-stable unipotent ℓ-special class of $\mathbf{G}$ and $x \in \mathcal{M}_\ell(u)$.

The conjecture holds when ℓ is good and does not divide the order of $(Z(\mathbf{G})/Z(\mathbf{G})^\circ)^F$ since there exists a bijection between $\mathcal{M}_\ell(u)$ and $\mathcal{M}(u)$. We show that the conjecture holds in the following additional cases.

Theorem

Let $\mathbf{G}$ be a connected reductive group defined over $\mathbb{F}_q$ where q is the power of a good prime number p. Assume we are in one of the following cases.

- $\mathbf{G}$ is $\mathrm{SL}_n(\overline{\mathbb{F}}_p)$ (hence, G is either $\mathrm{SL}_n(q)$ or $\mathrm{SU}_n(q)$) for any $\ell \leq n$.
- $\mathbf{G}$ is simple of type B, C or D, for $\ell = 2$.
- $\mathbf{G}$ is a simple exceptional group of adjoint type for any bad prime number ℓ.

Then, the above conjecture holds.

Application to groups of small rank

The last part of the thesis has two goals :

- Use the result above to exhibit unitriangular basic sets.
- Use work of Dudas from [19], to get small upper bounds on decomposition numbers.

The results of Dudas are especially effective when we have a unitriangular decomposition matrix with a few missing coefficients. We focus on two cases : $G = \mathrm{Sp}_4(q)$ and $G_2(q)$. We assume that q is a power of a good prime for G.

Case $G = \mathrm{Sp}_4(q)$ and $\ell = 2$. For irreducible representations of G over K, we use the notation from Srinivasan's character table of G [61]. The above theorem on basic sets for classical groups provides a basic set for unipotent blocks of G. Moreover, we are able to detect in which Lusztig series and families each representation of this basic set lies. Using White's 2-decomposition matrix of G [65] and Dudas's result we are able to complete the decomposition matrix.

Theorem. *Assume $\ell = 2$ and q is odd. Then, $\mathcal{B} = \{1_G, \theta_3, \theta_4, \theta_9, \theta_{10}, \Phi_3, \Phi_4\}$ is a unitriangular basic set for B_1. The decomposition matrix of B_1 with respect to $\mathcal{B}$ is*

	P_1	P_2	P_3	P_4	P_5	P_6	P_7
1_G	1						
θ_3	1	1					
θ_4	1		1				
θ_9	2	1	1	1			
θ_{10}					1		
Φ_3		1			x	1	
Φ_4			1		x		1

where $0 \leq x \leq 1$ *and* $x = 1$ *if* $q \equiv 3 \mod 4$.

Case $G = G_2(q)$ and $\ell = 2, 3$. We do not have a theorem providing a unitriangular basic set for G but our conjecture allows us to detect which irreducible representations we should pick to hypothetically form a basic set for G. Using the 2 and 3-decomposition matrices computed by Hiss–Shamash in [32] and [31], and using the same results of Dudas we get the following result. We use the notation of [8, §13] for irreducible characters of G.

Theorem.

1. *Assume* $\ell = 2$. *Then,*

$$\mathcal{B}_2 = \{\phi_{1,0}, \phi_{1,6}, \phi'_{1,3}, \phi''_{1,3}, G_2[1], G_2[-1], G_2[\theta], G_2[\theta^2], \chi_{s,(2,2)}\}$$

is a unitriangular basic set for B_1. *Here,* $\chi_{s,(2,2)}$ *denotes a character from a Lusztig series associated to a non-trivial isolated semi-simple 2-element* s. *The characters* $G_2[\theta]$ *and* $G_2[\theta^2]$ *lie in blocks of defect zero and the decomposition matrix of the principal block with respect to* $\mathcal{B}_2$ *is*

	P_1	P_2	P_3	P_4	P_5	P_6	P_7
$\phi_{1,0}$	1						
$\chi_{s,(2,2)}$	1	1					
$G_2[-1]$			1				
$G_2[1]$				1			
$\{\phi'_{1,3}, \phi''_{1,3}\}$	1				1		
	1					1	
$\phi_{1,6}$	1		α	β	1	1	1

where

- $0 \leq \alpha \leq 2$ *and* $0 \leq \beta \leq (q+2)/3$ *if* $q \equiv 1 \mod 4$,

- $1 \leq \alpha \leq 2$ *and* $1 \leq \beta \leq (q+2)/3$ *if* $q \equiv -1 \mod 4$.

2. *Assume* $\ell = 3$. *Then,*

$$\mathcal{B}_3 = \{\phi_{1,0}, \phi_{1,6}, \phi'_{1,3}, \phi''_{1,3}, \phi_{2,2}, G_2[-1], G_2[1], \chi_{t,3}\}$$

is a unitriangular basic set for B_1. *Here,* $\chi_{t,3}$ *denotes a character lying in a Lusztig series associated to a non-trivial isolated semi-simple 3-element* t. *The decomposition matrix of* B_1 *with respect to* $\mathcal{B}_3$ *depends on the value of* $q \mod 3$.

- *Assume* $q \equiv 1 \mod 3$. *Then* $G_2[1]$ *lies in a block of defect zero and the decomposition matrix of the principal block is*

	P_1	P_2	P_3	P_4	P_5	P_6	P_7
$\phi_{1,0}$	1						
$\chi_{t,3}$	1	1					
$G_2[1]$			1				
$\{\phi'_{1,3}, \phi''_{1,3}\}$		1		1			
			α		1		
$\phi_{2,2}$	1					1	
$\phi_{1,6}$	1		β	γ		1	1

where $1 \leq \alpha \leq 2$, $0 \leq \beta \leq q-2$ *and* $1 \leq \gamma \leq 2$.

- *Assume* $q \equiv -1 \mod 3$. *Then* $\phi_{2,2}$ *lies in a block of defect zero and the decomposition matrix of the principal block is*

	P_1	P_2	P_3	P_4	P_5	P_6	P_7
$\phi_{1,0}$	1						
$\chi_{t,3}$		1					
$G_2[1]$			1				
$\{\phi'_{1,3}, \phi''_{1,3}\}$	1	1		1			
	1		1		1		
$G_2[-1]$						1	
$\phi_{1,6}$	1	1	α	β	1	γ	1

where $1 \leq \alpha \leq q+1$ *and* $(\beta, \gamma) \in \{(1,1), (1,2), (2,1)\}$.

Outline of the Thesis

This thesis is divided into four chapters. In Chapter 1, we gather useful definitions and results on representations of finite groups of Lie type. We first review general results on representation theory of finite groups,

including decomposition matrices, basic sets, blocks and Clifford Theory. Then, we introduce connected reductive algebraic groups and we recall the main results on their structure and their classification. That allows us to define the $\mathbb{F}_q$-structure of a connected reductive algebraic group and define finite groups of Lie type and their main properties. We move on to Deligne–Lusztig varieties and their ℓ-adic cohomology from which we can define Deligne–Lusztig virtual representations. After listing their principal properties, we define the Lusztig rational series and give the Jordan decomposition theorem. Finally, we state two results explaining how Deligne–Lusztig theory behaves with respect to modular representation theory. The first one is a theorem by Broué–Michel which allows us to define the union of blocks B_s. The second one is a theorem of Bonnafé–Rouquier giving, under some restriction on s, a Morita equivalence between B_s and unipotent blocks of some Levi subgroup of G. It can be viewed as a modular version of the Jordan decomposition.

The second chapter is devoted to our conjecture. After introducing families of representations of Weyl group and families of unipotent representations, we give Lusztig's classification theorem in term of families. Then, we introduce the Springer correspondence between unipotent classes of $\mathbf{G}$ and irreducible characters of the Weyl group W of $\mathbf{G}$. The Springer correspondence allows us to define the special unipotent classes of $\mathbf{G}$ and to interpret Lusztig's classification theorem in terms of special unipotent classes. Then we introduce the ℓ-special unipotent classes of $\mathbf{G}$ and we conjecture that unipotent modular representations can be counted in terms of these ℓ-special classes. We show that our conjecture holds for $\mathrm{SL}_n(q)$ and $\mathrm{SU}_n(q)$ for any ℓ using results of Kleshchev–Tiep and Denoncin. Then we focus on simple classical groups, *i.e.* simple groups of type B, C or D when $\ell = 2$. We show that proving our conjecture is equivalent to proving that the number of unipotent modular representations of G is equal to the number of unipotent classes of G. The latter is done by generalising a result of Geck in [22]. Finally, we show, by a case-by-case analysis that the conjecture holds for simple adjoint exceptional groups.

The aim of Chapter 3 is to show the existence of a unitriangular basic set for unipotent blocks of classical groups when $\ell = 2$. We show that it is enough to prove the existence of some projectives and irreducible characters such that their scalar product respects the unitriangularity condition mentioned previously. We explain the construction of generalised

Gelfand–Graev representations (GGGRs) and we review the properties we will need from them. The GGGRs will give the projective characters we need. Then, we introduce character sheaves of a connected reductive group and local systems on unipotent classes. We give a brief review of Geck–Hézard results [26] on character sheaves which will allow us to show the existence of a basic set for groups with connected center. The next section explains how Taylor generalised Hézard's results to groups with disconnected center using regular embeddings. Finally, in the last section we show the existence of a unitriangular basic set for every classical group with the exception of spin and half-spin groups.

In the last chapter, we start by introducing Dudas's result on computation of decomposition numbers. We first recall Rickard's and Bonnafé–Rouquier's results [51, 4] on cohomology complexes of Deligne–Lusztig varieties. In [18], to each element w of the Weyl group, Dudas attached a virtual projective representation P_w from which we can extract information on the decomposition numbers. Then, we focus on the case $G = \mathrm{Sp}_4(q)$ and $\ell = 2$. After giving some details on the combinatorics of representations of the Weyl group of G and on the Jordan decomposition of some Lusztig series, we state our result on basic sets and decomposition matrices of unipotent blocks of G. We use Dudas's method to get a better bound on the remaining unknown coefficient of the decomposition matrix. Finally, we consider the case where $G = \mathrm{G}_2(q)$. We review information on isolated elements and irreducible characters of G. Then, as we did for $\mathrm{Sp}_4(q)$, we give results on basic sets and decomposition matrices of G for $\ell = 2$ and 3.

1

Finite groups of Lie type and their representations

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We introduce in this chapter concepts and notation that will be used in the rest of this thesis. This chapter consists of three sections, that we will overview. In the first section, we begin by focusing on the positive characteristic case, we define the decomposition map d and its transpose e and we introduce the notion of blocks. Then, we finish with results on Clifford Theory. In particular, Proposition 1.1.5 will be useful for the Jordan decomposition for groups with disconnected center.

In the second section, we define finite groups of Lie type and give the main results on their structure. We first give general definitions about reductive groups and give their classification in terms of root data (Theorem 1.2.3). Then we define Frobenius maps of an algebraic group which leads to the definition of finite groups of Lie type. We state the Lang-Steinberg Theorem 1.2.6 and several applications. We explain how finite groups of Lie type can be classified by their root datum and the action of the Frobenius map on the roots. Then we state in the last part of this section the Theorem 1.2.12 on the classification of semi-simple isolated classes for adjoint simple groups.

The last section focuses on representations of finite groups of Lie type from the point of view of Deligne–Lusztig theory. We first give generalities on ℓ-adic cohomology and Deligne–Lusztig varieties that allow us to define Deligne–Lusztig induction. We give the main properties of Deligne–Lusztig characters, in particular the scalar product formula in Theorem 1.3.10 and the fact that every irreducible character lies in a Deligne–Lusztig character in Theorem 1.3.12. Then, we introduce the Lusztig's rational series and Theorem 1.3.20 on the Jordan decomposition of characters and we explain that the Jordan decomposition can be generalised to groups with disconnected center. In the last part, we explain how the Deligne-Lusztig theory matches with modular representation theory, in particular we give in Theorem 1.3.28 a result of Broué–Michel which explains how we can get unions of blocks by taking unions of rational series.

1.1 Representations of finite groups

1.1.1 Modular representations of finite groups

Decomposition map. Let G be a finite group and ℓ be a prime number. We fix an ℓ-modular system $(K, \mathcal{O}, k)$, that is :

- a complete discrete valuation ring $\mathcal{O}$ with unique maximal ideal $\mathfrak{m}$,
- the fraction field $K := \mathrm{Frac}(\mathcal{O})$, which we assume to have characteristic 0 and to be big enough for G (*i.e.* K contains all $|G|$-roots of unity),

- the residue field $k := \mathcal{O}/\mathfrak{m}$, of characteristic $\ell > 0$.

Let Λ be either K or k. Throughout this thesis, we will only consider finitely generated left ΛG-modules. We denote by $\mathrm{Irr}_\Lambda(G)$ the set of isomorphism classes of irreducible ΛG-modules. As in the introduction, we write $R_\Lambda(G)$ for the Grothendieck group of the category of finitely generated ΛG-modules. In the case of characteristic zero, we can identify $R_K(G)$ with the abelian subgroup generated by irreducible characters of the space $CF_K(G)$ of K-valued central functions of G. We will often identify representations over K with their character.

Let E be a KG-module, let E' be a finitely generated $\mathcal{O}G$-submodule of E such that $E' \otimes_{\mathcal{O}} K = E$ (for instance E' can be a $\mathcal{O}G$-module generated by a basis of E over K). The reduction mod ℓ, $E' \otimes_{\mathcal{O}} k$ is a kG-module whose isomorphism class may depend on the choice of E' but according to Theorem 32 of [53], the image of $E' \otimes_{\mathcal{O}} k$ in $R_k(G)$ does not depend on E'. Therefore, we can define a map

$$d : R_K(G) \to R_k(G), \quad [E] \mapsto [E' \otimes_{\mathcal{O}} k]$$

which we call the *decomposition map* of G. By convention, we will call the *decomposition matrix* of G the transpose of the matrix of d with respect to the bases $\mathrm{Irr}_K(G)$ and $\mathrm{Irr}_k(G)$. We recall that a kG-module is *projective* if it is a direct summand of a free kG-module (in particular kG is projective). We define the Grothendieck group $P_k(G)$ of finitely generated projective kG-modules as follows.

- The generators are the finitely generated projective kG-modules. We denote by $[P]$ the image of a finitely generated kG-module P.
- Relations are $[P] = [P_1] + [P_2]$ whenever we have an isomorphism of kG-modules $P \simeq P_1 \oplus P_2$.

Let V be a kG-module. According to Proposition 41 of [53], there exists (up to isomorphism) a unique projective kG-module P_V with a surjective map $p_V : P_V \twoheadrightarrow V$ such that $p_V(M) \neq V$ for any proper submodule M of P_V. The module P_V is the *projective cover* of V. Let $\{P_1, \ldots, P_r\}$ be a set of representatives of indecomposable projective kG-modules. Every projective kG-module P is the direct sum of indecomposable projective kG-modules :

$$P \simeq \bigoplus_{i=1}^{r} P_i^{n_i}$$

where the multiplicities n_i are uniquely determined. Moreover, each indecomposable projective kG-module is the projective cover of an irreducible

kG-module. Therefore, the set $\{[P_V] \mid V \in \mathrm{Irr}_k(G)\}$ is a $\mathbb{Z}$-basis of $P_k(G)$. We have a bilinear form

$$\langle\, .,. \,\rangle_{KG} : R_K(G) \times R_K(G) \to \mathbb{Z}$$

such that

$$\langle [V], [W] \rangle_{KG} = \dim_K(\mathrm{Hom}_{KG}(V, W))$$

for two KG-modules V and W. Note that this form corresponds to the scalar product of characters viewed as K-valued class functions. Similarly, we have a form

$$\langle\, .,. \,\rangle_{kG} : P_k(G) \times R_k(G) \to \mathbb{Z}$$

such that

$$\langle P, W \rangle_{kG} = \dim_k(\mathrm{Hom}_{kG}(P_V, W))$$

for any projective kG-module P and any kG-module W. According to Proposition 42 of [53], every projective kG-module lifts to a unique (up to isomorphism) projective $\mathcal{O}G$-module P' such that $P' \otimes_{\mathcal{O}} k = P$. Then, we can define a map

$$e : P_k(G) \to R_K(G), \quad [P] \mapsto [P' \otimes_{\mathcal{O}} K],$$

which is adjoint to d in the following sense. If $P \in P_k(G)$ and $V \in R_K(G)$, then

$$\langle P, d(V) \rangle_{kG} = \langle e(P), V \rangle_{KG}.$$

In particular, the *decomposition matrix* of G defined above is the matrix of e with respect to the basis of indecomposable projectives kG-modules and irreducible KG-modules.

Example 1.1.1.

1. Assume that ℓ does not divide the order of G. Then every irreducible kG-module is projective. Indeed, for any irreducible kG-module V and for any free kG-module L such that V is a quotient L/M of L, consider a k-linear projector p of L onto M. Since $|G|$ is invertible in k, we can define a kG-linear projector $p_G := \frac{1}{|G|} \sum_{g \in G} gpg^{-1}$ of L onto M. Therefore, V is isomorphic to a direct factor of the free kG-module L and is projective. Now let P be the projective $\mathcal{O}G$-module such that $P \otimes_{\mathcal{O}} k = V$. Then $P \otimes_{\mathcal{O}} K$ is irreducible (if $P \otimes_{\mathcal{O}} K$ is not irreducible, neither is P), so the decomposition matrix is the identity matrix. In particular, any irreducible KG-module remains irreducible under ℓ-reduction.

2. If $|G|$ is a power of ℓ then the unique irreducible kG-module is the trivial one (see [53, §8.3]). Hence, the decomposition map sends a KG-module V to $\dim V$ copies of the trivial kG-module. Let P be the unique indecomposable projective kG-module. Then by definition of the decomposition matrix, $e([P]) = KG$ so $P = kG$.
3. Assume $G = \mathfrak{S}_3$ and $\ell = 2$. There are three simple KG-modules : the trivial representation 1_G, the sign representation ε_G and the reflection representation that we denote by r. The reduction mod 2 of 1_G and ε_G affords the trivial kG-module $\bar{1}_G$. The reduction mod 2 of r remains irreducible hence the decomposition matrix of G is

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Blocks. We recall that $\mathcal{O}G$ has a unique decomposition into indecomposable two-sided ideals :

$$\mathcal{O}G = B_1 \oplus \cdots \oplus B_r.$$

The ideals B_i are called the *ℓ-blocks* of G. They have a structure of an $\mathcal{O}$-algebra and the representation theory of $\mathcal{O}G$ is controlled by the representation theory of each of its ℓ-blocks. We will use the term blocks if there is no ambiguity on the prime number ℓ. For $\Lambda = K$ or k, the decomposition into blocks of $\mathcal{O}G$ induces a decomposition of ΛG as follows :

$$\Lambda G = (B_1 \otimes_{\mathcal{O}} \Lambda) \oplus \cdots \oplus (B_r \otimes_{\mathcal{O}} \Lambda).$$

Given a ΛG-module V and $i \in \{1, \ldots, r\}$, we say that V belongs to the blocks B_i if

$$\begin{cases} (B_i \otimes_{\mathcal{O}} \Lambda).V = V, \\ (B_j \otimes_{\mathcal{O}} \Lambda).V = 0 \text{ if } i \neq j. \end{cases}$$

The *principal block* is the unique block which contains the trivial representation. Assume that V is irreducible. There is a unique $i \in \{1, \ldots, r\}$ such that V belongs to B_i. Let us denote by $\mathrm{Irr}_\Lambda(B_i)$ the set of irreducible ΛG-modules lying in B_i. We have a partition :

$$\mathrm{Irr}_\Lambda(G) = \bigsqcup_{i=1}^{r} \mathrm{Irr}_\Lambda(B_i).$$

We will use the term "block" indistinctly for B_i, $\mathrm{Irr}_K(B_i)$ or $\mathrm{Irr}_k(B_i)$.

Example 1.1.2.

1. Let $G = \mathfrak{S}_3$ and $\ell = 2$ and let us write $s_1 = (12)$ and $s_2 = (23)$. There are two blocks of G : the principal block B_0 containing the trivial and the sign representations and the block B_1 containing the reflection representation r. The blocks can be determined as follows. According to [10, §56B], each block B has the form $\mathcal{O}Gb$ where b is a central primitive idempotent of $\mathcal{O}G$. Moreover, according to [10, 56.25], b has the following form :
$$b := \frac{1}{|G|} \sum_{\chi \in \mathrm{Irr}_K(B), g \in G} \chi(1)\chi(g^{-1})g.$$
Hence, $B_0 = \mathcal{O}G.b_0$ where $b_0 = \frac{1}{3}(1 + s_1s_2 + s_2s_1)$ and $B_1 = \mathcal{O}G.b_1$ where $b_1 = \frac{1}{3}(2 - s_1s_2 - s_2s_1)$.
2. Suppose that G is an ℓ'-group. Then each irreducible character lies in a single block. Let $\chi_1, \ldots, \chi_r$ be the irreducible characters of G. Let $b_{\chi_i} = \frac{1}{|G|} \sum_{g \in G} \chi_i(1)\chi_i(g^{-1})g$ and $B_i = \mathcal{O}Gb_{\chi_i} \simeq \mathrm{M}_{\chi_i(1)}(\mathcal{O})$. Then
$$\mathcal{O}G := B_1 \oplus \cdots \oplus B_r.$$
3. If G is an ℓ-group, kG is indecomposable so there is a unique block.

If $I \subset \{1, \ldots, r\}$ and $B = \bigoplus_{i \in I} B_i$, then we may refer to B as a union of blocks. We denote by $\mathrm{Irr}_\Lambda(B)$ the set $\bigsqcup_{i \in I} \mathrm{Irr}_\Lambda(B_i)$ and by $R_\Lambda(B)$ the Grothendieck group of the category of ΛG-modules lying in B, it is the subgroup of $R_\Lambda(G)$ generated by $\mathrm{Irr}_\Lambda(B)$. Note that the decomposition map d induces a map
$$R_K(B) \to R_k(B).$$

Definition 1.1.3. A *basic set* of B is a subset $\mathcal{B} \subset \mathrm{Irr}_K(B)$ such that $d(\mathcal{B})$ is a $\mathbb{Z}$-basis of $R_k(B)$. Let D be the part of the decomposition matrix of B with rows labelled by $\mathcal{B}$. We say that $\mathcal{B}$ is *unitriangular* if, up to permutations of rows and columns, D has lower unitriangular shape.

Proving the existence of a (unitriangular) basic set for a given finite group is an open problem. However, it is expected that unipotent blocks of finite reductive groups have a unitriangular basic set in non-defining characteristic (see Chapter 3). This is the main problem that we look at in this thesis.

1.1.2 Clifford theory

Let $\Lambda = K$ or k, H be a subgroup of G and V be a ΛH-module. The *induction* operation allows us to associate to V a ΛG-module $\mathrm{Ind}_H^G V$ as

follows :

$$\mathrm{Ind}_H^G V = \Lambda G \otimes_{\Lambda H} V.$$

Moreover, we have naturally an adjoint functor : any ΛG-module can be viewed as a KH-module. Let V be a ΛG-module, V viewed as a ΛH-module will be denoted by $\mathrm{Res}_H^G V$, it is the *restriction* of V.

Remark 1.1.4. Assume that H is an ℓ'-subgroup of G. According to 1.1.1, every kH-module is projective. Since a projective kH-module V is a direct factor of a free kH module, $\mathrm{Ind}_H^G V = V \otimes_{kH} kG$ is a direct factor of a free kG-module and is projective. Then, the induction from any ℓ'-subgroup provides projective kG-modules.

From now on, we assume that H is a normal subgroup of G. We are interested in Clifford theory, that is the connection between representations of G and representations of H when H is a normal subgroup of G. For a KH-module W with underlying representation ρ and $g \in G$, we denote by ${}^gW$ the KH-module where

- the underlying K-vector space of ${}^gW$ is W,
- the underlying representation of ${}^gW$ is the map $H \to \mathrm{GL}_K(W), \quad h \mapsto \rho(ghg^{-1})$.

If χ is the character of W, we denote by ${}^g\chi$ the character of ${}^gW$. Remark that ${}^gW$ depends only on the class of g modulo H. By remarking that if V is irreducible, ${}^gV$ remains irreducible, we see that there is a natural action of G/H on $\mathrm{Irr}_K(H)$. Let χ be an irreducible character of G and ϕ be an irreducible constituent of $\mathrm{Res}_H^G\chi$. Then

$$\mathrm{Res}_H^G\chi = \sum_{g\in G/H} ({}^g\phi)^m$$

where m is the multiplicity of ϕ in $\mathrm{Res}_H^G\chi$.

Now, assume that G/H is abelian. Then $\mathrm{Irr}_K(G/H)$ is an abelian group under tensor product. Moreover, by inflation, representations of (G/H) can be viewed as representations of G. Then the tensor product induces an action of $\mathrm{Irr}_K(G/H)$ on $\mathrm{Irr}_K(G)$. We have a natural duality

$$G/H \times \mathrm{Irr}_K(G/H) \to K \quad (\bar{g}, \theta) \mapsto \theta(\bar{g}).$$

Proposition 1.1.5 ([47, 9a]). *Assume that G/H is abelian and that any irreducible character of G restricts to H without multiplicity. We have a bijection*

$$(G/H)\text{-}orbits\ on\ \mathrm{Irr}_K(H) \longleftrightarrow (\mathrm{Irr}_K(G/H))\text{-}orbits\ on\ \mathrm{Irr}_K(G)$$

with the following properties : Let $(G/H).\phi$ *be a* (G/H)*-orbit of* $\mathrm{Irr}_K(H)$ *and* $(\mathrm{Irr}_K(G/H)).\chi$ *be the* $(\mathrm{Irr}_K(G/H))$*-orbit of* $\mathrm{Irr}_K(G)$ *given by the bijection. Then*

$$\mathrm{Ind}_H^G\phi = \sum_{\theta\in\mathrm{Irr}_K(G/H)} \theta\otimes\chi, \text{ and}$$

$$\mathrm{Res}_H^G\chi = \sum_{g\in G/H} {}^g\phi.$$

Moreover, the stabiliser of ϕ *in* (G/H) *and the stabiliser of* χ *in* $\mathrm{Irr}_K(G/H)$ *are orthogonal to each other.*

1.2 Finite groups of Lie type

1.2.1 Reductive algebraic groups

We recall some elementary notions related to algebraic groups. The reader can find more details on most of the results mentioned below in [8], [14] and [24].

Definitions and properties. Let p be a prime number, $q = p^a$ be a power of p. A *linear algebraic group* over $\bar{\mathbb{F}}_q$ is an affine algebraic variety over $\bar{\mathbb{F}}_q$ equipped with a group structure such that the multiplication and the inverse map are morphisms of algebraic varieties. From now on, every algebraic group that we consider will be linear and over $\bar{\mathbb{F}}_q$. A morphism of algebraic groups is a group morphism between algebraic groups which is also a morphism of varieties.

Example 1.2.1.

1. The group $(\bar{\mathbb{F}}_q, +)$ is called the *additive group* and is denoted by $\mathbf{G}_a = \mathbf{G}_a(\bar{\mathbb{F}}_q)$.
2. The group $(\bar{\mathbb{F}}_q, \cdot)$ is called the *multiplicative group* and is denoted by $\mathbf{G}_m = \mathbf{G}_m(\bar{\mathbb{F}}_q)$.
3. The group $\mathrm{GL}_n := \mathrm{GL}_n(\bar{\mathbb{F}}_q)$ is a linear algebraic group. Moreover, all classical groups SL_n, SO_n, Sp_{2n} are linear algebraic groups.

Let $\mathbf{G}$ be a linear algebraic group. The connected component of $\mathbf{G}$ containing the identity is denoted be $\mathbf{G}^\circ$. An element of $\mathbf{G}$ is *semi-simple* (resp. *unipotent)* if its image in some embedding of $\mathbf{G}$ in GL_n is semi-simple (resp. unipotent). That property does not depend on the embedding. Let $g \in \mathbf{G}$, g has a unique decomposition $g = su$ where s is semi-simple, u is unipotent

and $su = us = g$, this is the *Jordan decomposition* of g. There is a unique maximal closed connected solvable normal subgroup of $\mathbf{G}$, the *radical* of $\mathbf{G}$, denoted by $R(\mathbf{G})$. The set $R_u(\mathbf{G})$ of unipotent elements of $R(\mathbf{G})$ is a normal closed connected subgroup of $R(\mathbf{G})$ called the *unipotent radical* of $\mathbf{G}$. We say that $\mathbf{G}$ is *reductive* if $R_u(\mathbf{G}) = \{1\}$. We say that $\mathbf{G}$ is *semi-simple* if $\mathbf{G}$ is connected and $R(\mathbf{G}) = \{1\}$. Semi-simple groups are reductive. The linear algebraic group $\mathbf{G}$ is *simple* if $\mathbf{G}$ is connected but has no proper closed connected normal non-trivial subgroup.

Example 1.2.2.

1. $\mathbf{G} = \mathrm{GL}_n$ is reductive but not semi-simple. Indeed $R(\mathbf{G}) = Z(\mathbf{G})$ is the subgroup of scalar matrices.
2. $\mathbf{G} = \mathrm{SL}_n$ is semi-simple.

A *torus* of $\mathbf{G}$ is a closed subgroup of $\mathbf{G}$ isomorphic to a product of copies of $\mathbf{G}_m$. A torus of $\mathbf{G}$ is *maximal* if it is not contained in any larger torus, it can be shown that all maximal tori are conjugate under $\mathbf{G}$. For any maximal torus $\mathbf{T}$, we define its *Weyl group* $W := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$. It is a finite group which does not depend on the choice of $\mathbf{T}$. A *Borel subgroup* of $\mathbf{G}$ is a maximal closed connected solvable subgroup of $\mathbf{G}$. Every maximal torus $\mathbf{T}$ is contained in a Borel subgroup $\mathbf{B}$ and all pairs $(\mathbf{T}, \mathbf{B})$ where $\mathbf{T}$ is a maximal torus in a Borel subgroup $\mathbf{B}$ are conjugate under $\mathbf{G}$.

Let us fix a pair $(\mathbf{T}, \mathbf{B})$ where $\mathbf{T}$ is a maximal torus and $\mathbf{B}$ is a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$ and let $\mathbf{U} := R_u(\mathbf{B})$. Then $\mathbf{U}$ is normal in $\mathbf{B}$ and $\mathbf{B}$ is the semidirect product of $\mathbf{U}$ by $\mathbf{T}$.

Root subgroups and root data From now on, unless otherwise stated, we assume that $\mathbf{G}$ is connected and reductive. Let $X := X(\mathbf{T}) := \mathrm{Hom}(\mathbf{T}, \mathbf{G}_m)$ be the group of *characters* of $\mathbf{T}$ and $Y := Y(\mathbf{T}) := \mathrm{Hom}(\mathbf{G}_m, \mathbf{T})$ be the group of *cocharacters of* $\mathbf{T}$. Those groups are isomorphic to $\mathbb{Z}^n$ where $n = \dim\mathbf{T}$. For $\chi \in X$ and $\gamma \in Y$, there exists $m \in \mathbb{Z}$ such that $\chi \circ \gamma(t) = t^m$ for any $t \in \mathbf{G}_m$. Then we can define a $\mathbb{Z}$-bilinear form $X(\mathbf{T}) \times Y(\mathbf{T}) \to \mathbb{Z}$ by setting $(\chi, \gamma) = m$. This form provides a duality between X and Y. The Weyl group acts on X and Y as follows :

$$^{w}\chi(t) := \chi(^{w}t) \quad \forall w \in W, \chi \in X, t \in \mathbf{T},$$

$$^{w}\gamma(\lambda) :=^{w} (\gamma(\lambda)) \quad \forall w \in W, \gamma \in Y, \lambda \in \mathbf{G}_m.$$

We consider the minimal connected unipotent subgroups of $\mathbf{G}$ normalised by $\mathbf{T}$ (see [14, 0.31]). Those groups are isomorphic to $\mathbf{G}_a$. Since $\mathrm{Aut}(\mathbf{G}_a) \simeq$

$\mathbf{G}_m$, the action of $\mathbf{T}$ on one of those groups induces a character α. Distinct subgroups will provide distinct characters. The group corresponding to α will be denoted by $\mathbf{U}_\alpha$. We denote by Φ the set of characters obtained this way. These are the *roots* of $\mathbf{G}$ with respect to $\mathbf{T}$ and $\mathbf{U}_\alpha$ is the *root subgroup* of $\mathbf{G}$ corresponding to α. Let $\mathbf{B}$ be a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$ and $\mathbf{U} = R_u(\mathbf{B})$. Then there exists a subset Φ^+ of Φ such that $\mathbf{U} = \prod_{\alpha\in\Phi^+} \mathbf{U}_\alpha$, Φ^+ is the set of *positive roots* corresponding to $\mathbf{B}$. We have $\Phi = \Phi^+ \sqcup -\Phi^+$.

For $\alpha \in \Phi$, there exists a surjective morphism $\phi : \mathrm{SL}_2(\bar{\mathbb{F}}_q) \to \langle \mathbf{U}_\alpha, \mathbf{U}_{-\alpha}\rangle$ sending upper (resp. lower) triangular matrices to $\mathbf{U}_\alpha$ (resp. $\mathbf{U}_{-\alpha}$) and diagonal matrices into $\mathbf{T}$. The cocharacter sending λ to

$$\phi\left(\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}\right),$$

denoted by $\alpha^\vee$, is the *coroot* of α. We denote by $\Phi^\vee := \{\alpha^\vee \mid \alpha \in \Phi\}$ the set of coroots. The quadruple $(X, \Phi, Y, \Phi^\vee)$ has the following properties :

- X and Y are free $\mathbb{Z}$-modules in duality.
- There exists a bijection $\Phi \to \Phi^\vee,\ \alpha \to \alpha^\vee$ such that $(\alpha, \alpha^\vee) = 2$ for each $\alpha \in \Phi$
- The maps $s_\alpha : X \to X,\ \chi \mapsto \chi - (\chi, \alpha^\vee)\alpha$ and $s_{\alpha^\vee} : Y \to Y,\ \gamma \to \gamma - (\alpha, \gamma)\alpha^\vee$ satisfy $s_\alpha(\Phi) = \Phi$ and $s_{\alpha^\vee}(\Phi^\vee) = \Phi^\vee$.

We say that the quadruple $(X, \Phi, Y, \Phi^\vee)$ is a *root datum.* Let $(X', \Phi', Y', \Phi'^\vee)$ be another root datum. A isomorphism is a group isomorphism $\phi : X \to X'$ such that $\phi(\Phi) = \Phi'$ and such that $\phi^*(\Phi'^\vee) = \Phi^\vee$ where ϕ^* is the adjoint map of ϕ.

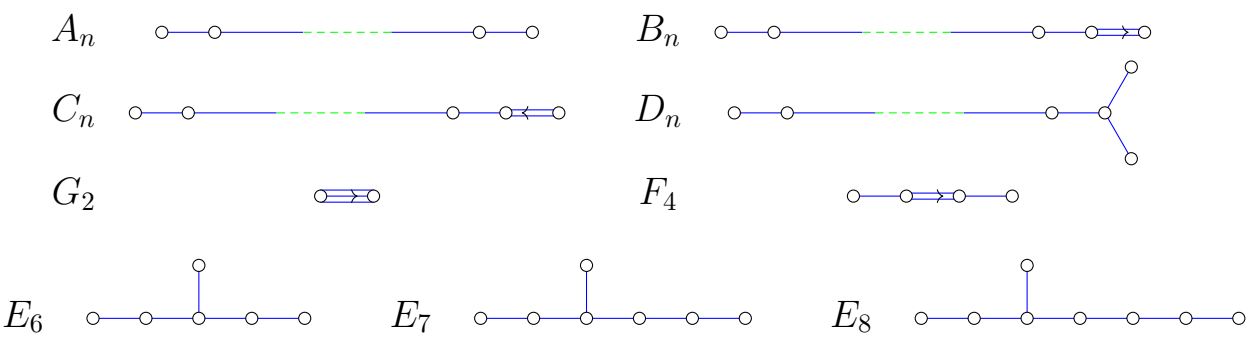
Theorem 1.2.3 ([14, 0.45])**.** *Two connected reductive algebraic groups are isomorphic if and only if their root data are isomorphic. Every root datum is the root datum of some reductive group.*

Let $V := \mathbb{Z}\Phi \otimes_{\mathbb{Z}} \mathbb{R}$, we identify V^* with $\mathbb{Z}\Phi^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. For $v \in V$ and $v^* \in V^*$ we set $(v, v^*) := v^*(v)$. Then (V, Φ) is a *root system.*. The set of positive roots Φ^+ gives a *basis* Δ of Φ. The basis Δ is a subset of Φ^+ such that each root can be written uniquely as a linear combination of elements of Δ with integer coefficients all of the same sign. The roots of Δ are said *simple.* There is a faithul action of W on Φ which induces a injective morphism $W \to \mathrm{GL}(V)$ whose image is generated by the $s'_\alpha s$ for $\alpha \in \Phi$. For $\alpha \in \Phi$, we will still denote by s_α the element of W whose image in $\mathrm{GL}(V)$ is s_α. The

group W is defined by generators and relations as follows. Let $\alpha_1, \ldots, \alpha_n$ be the simple roots and m_{ij} be the order of $s_{\alpha_i} s_{\alpha_j}$, then

$$W = \langle s_{\alpha_i}, \ldots, s_{\alpha_n} \mid s_{\alpha_i}^2 = 1,\ (s_{\alpha_i} s_{\alpha_j})^{m_{ij}} = 1 \rangle.$$

A root system is *irreducible* if there is no partition of Φ into orthogonal subsets Φ_1 and Φ_2 such that Φ_1 and Φ_2 are root systems of the spaces they generate. To the root system Φ, we can associate a graph called the *Dynkin diagram* of Φ. The Dynkin diagram has n vertices corresponding to simple roots $\alpha_1, \ldots, \alpha_n$. Vertices i and j are connected by n_{ij} edges where $n_{ij} := (\alpha_i, \alpha_j^\vee)(\alpha_j, \alpha_i^\vee)$. If $n_{ij} = 2$ or 3, we attach an arrow from i to j when $(\alpha_j, \alpha_i^\vee) \neq -1$. The root system Φ is irreducible if and only if its Dynkin diagram is connected. We describe below all Dynkin diagrams of irreducible root systems.



We say that two reductive groups are *isogenous* it they have same dynkin diagram.

Fundamental group. Assume that $\mathbf{G}$ is semi-simple and let $\Omega := \mathrm{Hom}(\mathbb{Z}\Phi^\vee, \mathbb{Z})$. Then X can naturally be viewed as a subgroup of Ω. The groups Ω/X is the *fundamental group* of $\mathbf{G}$. We say that $\mathbf{G}$ is *adjoint* if $X = \mathbb{Z}\Phi$, in that case the fundamental group is maximal and $Z(\mathbf{G}) = \{1\}$. We say that $\mathbf{G}$ is *simply connected* if the fundamental group is trivial. We denote by $\mathbf{G}_{ad}$ (resp. $\mathbf{G}_{sc}$) an algebraic group with root system Φ which is adjoint (resp. simply connected). For each semi-simple group $\mathbf{G}$ with root system Φ, there exist morphisms whith central finite kernel

$$\mathbf{G}_{sc} \twoheadrightarrow \mathbf{G} \twoheadrightarrow \mathbf{G}_{ad}.$$

Moreover, the kernel of the second morphism if $Z(\mathbf{G})$. We list below the adjoint and simply connected groups for each Dynkin diagram of classical type (see [8, 1.11] for the complete list).

- Type A_n : $\Omega/\mathbb{Z}\Phi$ is a cyclic group of order $n+1$. Moreover, $\mathbf{G}_{ad}$ is isomorphic to PGL_{n+1} and $\mathbf{G}_{sc}$ is isomorphic to SL_{n+1}.
- Type B_n : $\Omega/\mathbb{Z}\Phi$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ so the only possibilities are the adjoint and simply connected groups. We have $\mathbf{G}_{ad} \simeq \mathrm{SO}_{2n+1}$ and $\mathbf{G}_{sc} \simeq \mathrm{Spin}_{2n+1}$.
- Type C_n : $\Omega/\mathbb{Z}\Phi$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ so the only possibilities are the adjoint and simply connected groups. We have $\mathbf{G}_{ad} \simeq \mathrm{PCSp}_{2n}$ and $\mathbf{G}_{sc} \simeq \mathrm{Sp}_{2n}$.
- Type D_n :
$$\Omega/\mathbb{Z}\Phi \simeq \begin{cases} \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \text{ if } n \text{ is even,} \\ \mathbb{Z}/4\mathbb{Z} \text{ if } n \text{ is odd.} \end{cases}$$
Moreover, $\mathbf{G}_{ad} \simeq \mathrm{P}(\mathrm{CO}^{\circ}_{2n})$ and $\mathbf{G}_{sc} \simeq \mathrm{Spin}_{2n}$.

Parabolic and Levi subgroups. Assume that $\mathbf{G}$ is connected and reductive. We are now interested in a particular class of subgroups of $\mathbf{G}$. Let I be a subset of $\{1, \ldots, n\}$, the *parabolic subgroup* of W corresponding to I is the group :
$$W_I = \langle (s_{\alpha_i})_{i \in I} \rangle.$$
Let $N_I(\mathbf{T})$ be the subgroup of $N_{\mathbf{G}}(\mathbf{T})$ containing $\mathbf{T}$ such that $N_I(\mathbf{T})/\mathbf{T} = W_I$. Then the *standard parabolic subgroups* of $\mathbf{G}$ are the groups of the form $\mathbf{P}_I = \mathbf{B}N_I(\mathbf{T})\mathbf{B}$. A parabolic subgroup of $\mathbf{G}$ is a subgroup of $\mathbf{G}$ conjugate to a standard parabolic subgroup. It can be shown that parabolic subgroups of $\mathbf{G}$ are exactly the closed subgroups of $\mathbf{G}$ containing Borel subgroups. Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$. Then $\mathbf{P}$ has a decomposition into semi-direct product $\mathbf{P} = \mathbf{L}.R_u(\mathbf{P})$. Such a subgroup $\mathbf{L}$ is called a *Levi subgroup* of $\mathbf{P}$. It is a connected reductive group whose Weyl group is W_I. Set $\Delta_I := \{\alpha_i \mid i \in I\}$. Let V_I be the subspace of V generated by Δ_I and Φ_I be the set of roots contained in V_I. Then (V_I, Φ_I) is a root system with Weyl group W_I.

1.2.2 Finite groups of Lie type

Frobenius endomorphism. Let $X \subset \bar{\mathbb{F}}_q^n$ be an affine variety over $\bar{\mathbb{F}}_q$. Let $I \subset \bar{\mathbb{F}}_q[T_1, \ldots, T_n]$ be the ideal of polynomials vanishing on X and $A := \bar{\mathbb{F}}_q[T_1, \ldots, T_n]/I$ be the algebra of X.

Definition 1.2.4. Let $F : X \to X$ be a morphism of variety and $F^* : A \to A$ be the corresponding algebra homomorphism. We say that X is *defined over* $\mathbb{F}_q$ (or has an $\mathbb{F}_q$-structure) if the following conditions hold. There exists an F as above such that :

- F^* is injective,
- $F^*(A) = \{a^q \mid a \in A\}$,
- $\forall a \in A$, there exists $m \geq 1$ such that $(F^*)^m(a) = a^{q^m}$.

We say that F is the *Frobenius map* corresponding to the $\mathbb{F}_q$-structure of X.

Let X, Y be two affine varieties defined over $\mathbb{F}_q$, F_X, F_Y be their Frobenius maps and $\varphi : X \to Y$ be a morphism of varieties. We say that φ is defined over $\mathbb{F}_q$ (or rational) if $\varphi \circ F_X = F_Y \circ \varphi$. Let $\mathbf{G}$ be a linear algebraic group. We say that $\mathbf{G}$ is defined over $\mathbb{F}_q$ if the affine variety $\mathbf{G}$ is defined over $\mathbb{F}_q$ and the Frobenius map is a group morphism. Let F be a Frobenius map of $\mathbf{G}$, we denote by $G := \mathbf{G}^F$ the group of fixed points of $\mathbf{G}$ under F. The finite groups of the form $G = \mathbf{G}^F$ where $\mathbf{G}$ is a connected reductive group and F is a Frobenius map of $\mathbf{G}$ are called *finite groups of Lie type* or *finite reductive groups.*

Example 1.2.5.

- Let $\mathbf{G} = \mathbf{G}_m$. The Frobenius endomorphisms of $\mathbf{G}$ are of the form $x \mapsto x^{\varepsilon q}$ where $\varepsilon = \pm 1$. If $\varepsilon = 1$, then $G = \mathbb{F}_q^\times$. If $\varepsilon = -1$, G is the subgroup μ_{q+1} of $(q+1)$-th roots of unity of $\mathbf{G}_m$.
- Let $\mathbf{G} = \mathbf{G}_a$, the only Frobenius endomorphisms of $\mathbf{G}$ are of the form $x \mapsto x^q$. Then G is the additive group $(\mathbb{F}_q, +)$.
- Let $\mathbf{G} = \mathrm{GL}_n$ and

$$F_q : \mathbf{G} \to \mathbf{G},\ (a_{i,j}) \mapsto (a_{i,j}^q).$$

  Then $G = \mathrm{GL}_n(q)$ is the group of invertible n by n matrices with coefficients in $\mathbb{F}_q$. More generally, classical matrix groups are usually stable by F_q and provide finite groups of Lie type. For example, the groups $\mathrm{SL}_n(q)$, $\mathrm{SO}_{2n+1}(q)$, $\mathrm{Sp}_{2n}(q)$ can be constructed this way.
- We keep $\mathbf{G} = \mathrm{GL}_n$ but instead of considering the above Frobenius map F_q, we will consider the following morphism

$$F : \mathbf{G} \to \mathbf{G}, \quad A \to F_q(A^{-1})^{tr}.$$

  The morphism F is the composition of F_q with the map sending a matrix to the transpose of its inverse. Then F is a Frobenius map. The finite group $G \leq \mathrm{GL}_n(q)$ denoted by $\mathrm{GU}_n(q)$ is the *general unitary group* over $\mathbb{F}_{q^2}$. If we consider $\mathbf{G} := \mathrm{SL}_n$ instead of GL_n, then G, denoted by $\mathrm{SU}_n(q)$, is the *special unitary group.*

Lang-Steinberg theorem and applications.

Theorem 1.2.6 (Lang-Steinberg, [14, 3.10])**.** *Let $\mathbf{G}$ be a connected reductive group and F be a Frobenius map of $\mathbf{G}$. Then the map $\mathcal{L} : \mathbf{G} \to \mathbf{G}$, $g \mapsto g^{-1}F(g)$, is surjective.*

Corollary 1.2.7 ([14, 3.12,3.21])**.** *Let $\mathbf{G}$ be a connected reductive group defined over $\mathbb{F}_q$, F be a Frobenius map and X be a set on which $\mathbf{G}$ and F act. Assume that $F(g.x) = g.F(x)$ for all $g \in \mathbf{G}$ and $x \in X$. Let $\mathcal{O}$ be a non-empty F-stable $\mathbf{G}$-orbit in X. Then*

1. *The set of fixed points $\mathcal{O}^F$ is non-empty.*
2. *There is a well-defined map sending the G-orbit of $g.x \in \mathcal{O}^F$ to the F-conjugacy class of the image of $\mathcal{L}(g)$ in $\mathrm{Stab}_{\mathbf{G}}(x)/\mathrm{Stab}_{\mathbf{G}}(x)^\circ$ and this is a one-to-one correspondence.*

Example 1.2.8.

1. Let $\mathcal{B}$ be the set of Borel subgroups of $\mathbf{G}$. Then according to Corollary 1.2.7 1., $\mathcal{B}^F$ is non-empty so there exists an F-stable Borel subgroup of $\mathbf{G}$.
2. Let $\mathbf{B}$ be an F-stable Borel subgroup of $\mathbf{G}$, if we apply Corollary 1.2.7 to the set of maximal tori lying in $\mathbf{B}$, we see that $\mathbf{B}$ contains an F-stable maximal torus. Such a torus is said to be *maximally split*.
3. Let $\mathbf{P}$ be an F-stable parabolic subgroup of $\mathbf{G}$. Then $\mathbf{P}$ contains an F-stable Levi subgroup.
4. Let $\mathcal{O} = (g)_{\mathbf{G}}$ be an F-stable conjugacy class of $\mathbf{G}$. Then $\mathcal{O}^F$ is non-empty. Using Corollary 1.2.7 2., we can parametrise the G-conjugacy classes of $\mathcal{O}^F$ by F-conjugacy classes of $C_{\mathbf{G}}(g)/C_{\mathbf{G}}(g)^\circ$. In particular, when $C_{\mathbf{G}}(g)$ is connected all the elements of $\mathcal{O}^F$ are conjugate under G.
5. Let us fix an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ and let $W := N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ be the associated Weyl group. The G-conjugacy classes of F-stable maximal tori are parametrised by the F-conjugacy classes of W. Then an F-stable maximal torus $\mathbf{T}_w$ corresponding to the F-class of $w \in W$ has the form ${}^g\mathbf{T}$ where $g^{-1}F(g) = n_w$ and n_w is a representative of w in $N_{\mathbf{G}}(\mathbf{T})$. Moreover, $T_w := \mathbf{T}_w^F$ is isomorphic as a finite group to
$$\mathbf{T}^{wF} := \{t \in \mathbf{T} \mid n_w F(t) n_w^{-1} = t\}.$$
6. Let $\mathbf{H}$ be an F-stable closed and connected subgroup of $\mathbf{G}$. Then, $(\mathbf{G}/\mathbf{H})^F \simeq G/H$. Indeed, the natural map $G/H \to (\mathbf{G}/\mathbf{H})^F$ is injective since if $g_1, g_2 \in G$ have the same image, $g_1 g_2^{-1} \in H$. By Corollary

1.2.7 1., any left $\mathbf{H}$-coset contains an F-stable element so the map is surjective.

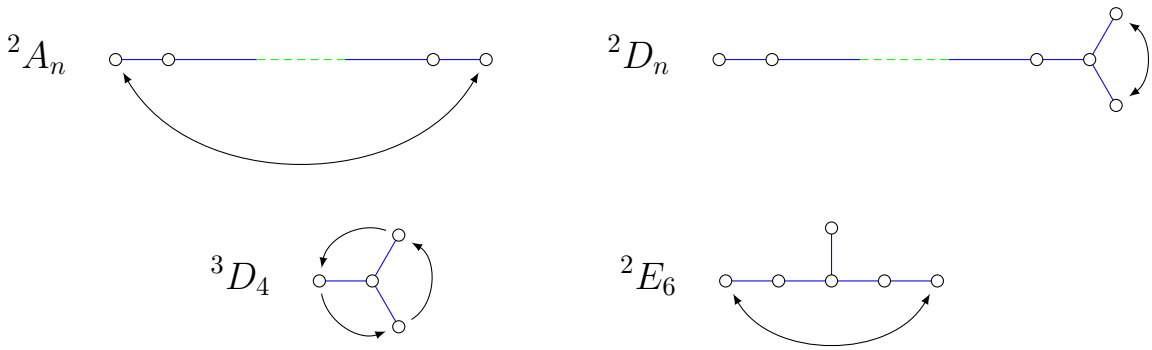
Classification of finite groups of Lie type. Let $\mathbf{G}$ be a connected reductive group, F be a Frobenius morphism on $\mathbf{G}$ and $(\mathbf{T}, \mathbf{B})$ be an F-stable maximal torus contained in an F-stable Borel subgroup of $\mathbf{G}$, Φ be the corresponding root system and Δ the basis of Φ induced by $\mathbf{B}$. Let $\mathbf{U} := R_u(\mathbf{B}) = \prod_{\alpha \in \Phi^+} \mathbf{U}_\alpha$. According to [49, 22.2], the unipotent group $\mathbf{U}$ is F-stable and F induces a permutation of the groups $\mathbf{U}_\alpha$ with the following property : there exists a permutation ρ of Φ^+ such that $F(\mathbf{U}_\alpha) = \mathbf{U}_{\rho(\alpha)}$ if $\alpha \in \Phi^+$. Then ρ induces a graph automorphism of the Dynkin diagram of Φ and F acts on $X = X(\mathbf{T})$ and $Y = Y(\mathbf{T})$ as follows :

$$^F\chi(t) = \chi(F(t)) \quad \forall \chi \in X, t \in \mathbf{T},$$
$$^F\gamma(\lambda) = F(\gamma(\lambda)) \quad \forall \gamma \in Y, \lambda \in \mathbf{G}_m.$$

We denote by ρ_X the automorphism of X induced by ρ. Then, we have $F = q\rho_X^{-1}$.

Theorem 1.2.9 ([49, 22.5])**.** *Let $\mathbf{G}$ be a simple algebraic group, F be a Frobenius map of $\mathbf{G}$ and ρ be as above. Then G is determined, up to isomorphism, by ρ and q.*

We list below the non-trivial graph automorphisms induced by Frobenius maps for connected Dynkin diagrams.



1.2.3 Isolated classes

Definition 1.2.10. Let $\mathbf{G}$ be a connected reductive group, let $g \in \mathbf{G}$ and $g = su$ be its Jordan decomposition. We say that g is *quasi-isolated* if $C_{\mathbf{G}}(s)$ is not contained in a Levi subgroup of a proper parabolic subgroup of $\mathbf{G}$. The element g is *isolated* if $C_{\mathbf{G}}(s)^\circ$ is not contained in a Levi subgroup of a proper parabolic subgroup of $\mathbf{G}$.

In [2], Bonnafé gave a complete description of the conjugacy classes of quasi-isolated semi-simple elements in terms of the root datum of $\mathbf{G}$. To simplify, we only consider the case where $\mathbf{G}$ is adjoint and simple. Let $\iota : (\mathbb{Q}/\mathbb{Z})_{p'} \xrightarrow{\sim} \bar{\mathbb{F}}_q^\times$ be an isomorphism and $\tilde{\iota}$ be the composition

$$\mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow (\mathbb{Q}/\mathbb{Z})_{p'} \xrightarrow{\iota} \bar{\mathbb{F}}_q^\times.$$

We define

$$\tilde{\iota}_{\mathbf{T}} : Y(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbf{T},\ x \otimes_{\mathbb{Z}} \lambda \mapsto \lambda(\tilde{\iota}(x)).$$

According to Proposition 25 of [5, VI], there exists a unique root $\tilde{\alpha}$, the *highest root* of Φ, which we decompose as

$$\tilde{\alpha} = \sum_{\alpha \in \Delta} n_\alpha \alpha, \quad n_\alpha \in \mathbb{N}$$

such that $\sum_{\alpha \in \Delta} n_\alpha$ is maximal. Let $\alpha_0 := -\tilde{\alpha}$ and $\tilde{\Delta} = \Delta \cup \{\alpha_0\}$. Let $(w_\alpha^\vee)_{\alpha \in \Delta} \subset Y(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the dual basis of $\Delta \subset X(\mathbf{T}) \otimes_{\mathbb{Z}} \mathbb{Q}$. The element $w_\alpha^\vee$ is the *fundamental coweight* associated to α. By convention, we set $\omega_{\alpha_0}^\vee = 0$ and $n_{\alpha_0} = 1$. We denote by $N_W(\tilde{\Delta})$ the subgroup of W normalising $\tilde{\Delta}$, *i.e.* :

$$N_W(\tilde{\Delta}) := \{w \in W \mid w(\tilde{\Delta}) = \tilde{\Delta}\}.$$

Let $\mathcal{Q}$ be the set of subsets $\Omega \subseteq \tilde{\Delta}$ such that the stabiliser of Ω in $N_W(\tilde{\Delta})$ acts transitively on Ω and such that p does not divide $|\Omega|$. Given $\Omega \in \mathcal{Q}$, we define

$$t_\Omega := \tilde{\iota}_{\mathbf{T}} \left(\sum_{\alpha \in \Omega} \frac{1}{n_\alpha |\Omega|} \omega_\alpha^\vee \right).$$

In order to state the classification theorem in a more convenient form we need the following definition.

Definition 1.2.11. We say that p is *bad* for $\mathbf{G}$ if there exists $\alpha \in \Delta$ such that p divides n_α. We say that p is *good* if it is not bad.

The following table lists the bad primes for simple algebraic groups.

$\mathbf{G}$	A_n	B_n	C_n	D_n	G_2	F_4	E_6	E_7	E_8
bad primes		2	2	2	$2,3$	$2,3$	$2,3$	$2,3$	$2,3,5$

Theorem 1.2.12 ([2, 5.1])**.** *Assume that* $\mathbf{G}$ *is adjoint and simple,* p *is good and does not divides* $|N_W(\tilde{\Delta})|$*. Then the following holds :*

- *The map $\mathcal{Q} \to \mathbf{T}, \Omega \mapsto t_\Omega$ induces a bijection between the set of $N_W(\tilde{\Delta})$-orbits of $\mathcal{Q}$ and the set of conjugacy classes of quasi-isolated semi-simple elements in $\mathbf{G}$.*
- *For $\Omega \in \mathcal{Q}$, we have*
  1. *The Weyl group of $C_{\mathbf{G}}(t_\Omega)^\circ$ is generated by the reflections s_α for $\alpha \in \tilde{\Delta} \setminus \Omega$.*
  2. *n_α is constant for $\alpha \in \Omega$ and the order of t_Ω is $n_\alpha|\Omega|$.*
  3. *t_Ω is isolated if and only if $|\Omega| = 1$.*

We reproduce [2], Table 2, describing quasi-isolated elements in simple adjoint classical groups. The second column describes the elements Ω of $\mathcal{Q}$, the third one gives the order $o(t_\Omega)$ of t_Ω. We follow the notation of [2] : we write $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. For $\Omega := \{\alpha_{n/2}\}$ in type D_n, the order of t_Ω in the original table of Bonnafé is 4. However, in [62, 4.1] Taylor remarked that this element is actually of order 2.

$\mathbf{G}$	Ω	$o(t_\Omega)$	$C_G(t_\Omega)^\circ$	Isolated ?
A_n	$\{\alpha_{j(n+1)/d} \mid 0 \leq j \leq d-1\}$ for $d \mid n+1$	d	$(A_{(n+1)/d} - 1)^d$	yes iff $d = 1$
B_n	$\{\alpha_0\}$	1	B_n	yes
	$\{\alpha_0, \alpha_1\}$	2	B_{n-1}	no
	$\{\alpha_d\}, 2 \leq d \leq n$	2	$D_d \times B_{n-d}$	yes
C_n	$\{\alpha_0\}$	1	C_n	yes
	$\{\alpha_d\}, 1 \leq d < n/2$	2	$C_d \times C_{n-d}$	yes
	$\{\alpha_{n/2}\}$ (n even)	2	$C_{n/2} \times C_{n/2}$	yes
	$\{\alpha_0, \alpha_n\}$	2	A_{n-1}	no
	$\{\alpha_d, \alpha_{n-d}\}, 1 \leq d < n/2$	4	$(C_d)^2 \times A_{n-2d-1}$	no
D_n	$\{\alpha_0\}$	1	D_n	yes
	$\{\alpha_d\}, 1 \leq d < n/2$	2	$D_d \times D_{n-d}$	yes
	$\{\alpha_{n/2}\}$ (n even)	2	$D_{n/2} \times D_{n/2}$	yes
	$\{\alpha_d, \alpha_{n-d}\}, 1 \leq d < n/2$	4	$(D_d)^2 \times A_{n-2d-1}$	no
	$\{\alpha_0, \alpha_1, \alpha_{n-1}, \alpha_n\}$	4	A_{n-3}	no
	$\{\alpha_0, \alpha_1\}$	2	A_{n-1}	no
	$\{\alpha_0, \alpha_{n-1}\}$ (n even)	2	A_{n-1}	no
	$\{\alpha_0, \alpha_n\}$ (n even)	2	A_{n-1}	no

Remark 1.2.13. Let $\mathbf{G}$ be simple, $\mathbf{G}_{ad}$ be the adjoint quotient of $\mathbf{G}$ and $\pi : \mathbf{G} \to \mathbf{G}_{ad}$ the canonical morphism. By [2, 2.3], if g is quasi-isolated then $\pi(g)$ is quasi-isolated. Moreover, g is isolated if and only if $\pi(g)$ is isolated. Thus, even if the table above concerns $\mathbf{G}_{ad}$, we can deduce information on quasi-isolated elements of $\mathbf{G}$.

1.3 Deligne–Lusztig theory

1.3.1 ℓ-adic cohomology

Let $\mathbf{G}$ be a connected algebraic group defined over $\mathbb{F}_q$ and G the associated finite group of Lie type. We fix a prime number ℓ different from p and we consider an ℓ-modular system $(K, \mathcal{O}, k)$ such that K is a finite extension of $\mathbb{Q}_\ell$. Let X be a quasi-projective algebraic variety defined over $\bar{\mathbb{F}}_q$ on which G acts by automorphisms. There exist two bounded complexes of $\mathcal{O}G$-modules $R\Gamma(X, \mathcal{O})$ and $R\Gamma_c(X, \mathcal{O})$ defined up to quasi-isomorphism. The complex $R\Gamma(X, \mathcal{O})$ is the *cohomology complex* of X and $R\Gamma_c(X, \mathcal{O})$ is the *cohomology complex with compact support* of X.

We set Λ to be either K or k. By extension of scalars, we have complexes :

$$R\Gamma(X, \Lambda) := R\Gamma(X, \mathcal{O}) \otimes^L_{\mathcal{O}} \Lambda,$$
$$R\Gamma_c(X, \Lambda) := R\Gamma_c(X, \mathcal{O}) \otimes^L_{\mathcal{O}} \Lambda,$$

where $\otimes^L_{\mathcal{O}}$ denotes the left-derived functor of the tensor product. In particular, we have $R\Gamma_c(X, K) \simeq R\Gamma_c(X, \mathcal{O}) \otimes_{\mathcal{O}} K$. The group $H^i(X, \Lambda) := H^i(R\Gamma(X, \Lambda))$ is the *i-th cohomology group* of X with coefficients in Λ, $H^i_c(X, \Lambda) := H^i(R\Gamma_c(X, \Lambda))$ is the *i-th cohomology group with compact support* of X with coefficients in Λ. These constructions are functorial : if Y is another quasi-projective variety on which G acts and if $f : Y \to X$ is a G-equivariant morphism of algebraic varieties then it induces a morphism in the derived category of finitely generated ΛG-modules $f^* : R\Gamma(X, \Lambda) \to R\Gamma(Y, \Lambda)$. Moreover, if f is a finite morphism it also induces a morphism between cohomology complexes with compact support $f^* : R\Gamma_c(X, \Lambda) \to R\Gamma_c(Y, \Lambda)$. The following theorem introduces results from [11]. See also [52] (or [14, §10] when $\Lambda = K$).

Theorem 1.3.1. *Let $d = \dim X$.*

1. *$H^i(X, \Lambda)$ and $H^i_c(X, \Lambda)$ are finitely generated over Λ for all i.*

2. *$H^i(X,\Lambda)$ and $H^i_c(X,\Lambda)$ vanish if $i<0$ or $i>2d$. In particular $R\Gamma(X,\Lambda)$ and $R\Gamma_c(X,\Lambda)$ are quasi-isomorphic to complexes of finitely generated ΛG-modules with terms in degree $0,\dots,2d$.*
3. *Let I be the set of irreducible components of X, G acts naturally on I. Then $H^{2d}_c(X,\Lambda)$ is isomorphic to the permutation module generated by I.*
4. *$H^i_c(X/G,K)\simeq H^i_c(X,K)^G$.*

Example 1.3.2.

1. Assume that X is an affine variety of pure dimension d. Then $H^i(X,\Lambda)=0$ if $i>d$ and $H^i_c(X,\Lambda)=0$ if $i<d$.
2. Let $\mathbf{A}_n$ be the affine space of dimension n. Then $R\Gamma_c(\mathbf{A}_n,\Lambda)\simeq\Lambda[-2n]$.
3. Let $\mathbf{P}^1:=\mathbf{P}^1(\bar{\mathbb{F}}_q)$ be the projective line over $\bar{\mathbb{F}}_q$. Then
$$H^i_c(\mathbf{P}^1,\Lambda)=\begin{cases}\Lambda \text{ if } i=0,2,\\ 0 \text{ otherwise.}\end{cases}$$
4. Let $\mathbf{G}_m$ be the multiplicative group. We have
$$H^i_c(\mathbf{G}_m,\Lambda)=\begin{cases}\Lambda \text{ if } i=1,2,\\ 0 \text{ otherwise.}\end{cases}$$

1.3.2 Deligne–Lusztig induction

Bruhat decomposition. From now on, we assume that $\mathbf{G}$ is a connected reductive group. We fix an F-stable pair $(\mathbf{T},\mathbf{B})$ where $\mathbf{B}$ is a Borel subgroup of $\mathbf{G}$ and $\mathbf{T}$ is a maximal torus of $\mathbf{B}$. Let $S=\{s_1,\dots,s_n\}$ be the set of simple reflections of W. For $w\in W$, we define its *length* $\ell(w)$ to be the smallest non-negative integer such that w is a product of $\ell(w)$ simple reflections. An expression of w as the product of $\ell(w)$ simple reflections is said to be *reduced*.

Theorem 1.3.3 ([14, 1.2,1.4])**.**

1. *We have a partition*
$$\mathbf{G}=\bigsqcup_{w\in W}\mathbf{B}w\mathbf{B}.$$
The spaces $\mathbf{B}w\mathbf{B}$ are called the Bruhat cells.
2. *Let $s\in S$ and $w\in W$. Then*
$$\mathbf{B}s\mathbf{B}w\mathbf{B}=\begin{cases}\mathbf{B}sw\mathbf{B} \text{ if } \ell(sw)>\ell(w),\\ \mathbf{B}w\mathbf{B}\sqcup\mathbf{B}sw\mathbf{B} \text{ otherwise.}\end{cases}$$
3. *$\mathbf{B}w\mathbf{B}/\mathbf{B}$ is an affine space of dimension $\ell(w)$ called a* Schubert cell.

Deligne–Lusztig varieties. Let $\mathbf{U} = R_u(\mathbf{B})$, let us fix a set $\{n_w\}_{w\in W}$ of representatives of w in $N_{\mathbf{G}}(\mathbf{T})$. Then the Deligne–Lusztig varieties attached to w are

$$Y_{\mathbf{G}}(w) := Y_w := \{g\mathbf{U} \in \mathbf{G}/\mathbf{U} \mid g^{-1}F(g) \in \mathbf{U}n_w\mathbf{U}\},$$

$$X_{\mathbf{G}}(w) := X_w := \{g\mathbf{B} \in \mathbf{G}/\mathbf{B} \mid g^{-1}F(g) \in \mathbf{B}n_w\mathbf{B}\}.$$

The Deligne–Lusztig varieties X_w and Y_w are smooth, projective and of pure dimension $\ell(w)$. The finite group G acts on X_w and Y_w by left multiplication. Recall that

$$\mathbf{T}^{wF} = \{t \in \mathbf{T} \mid n_wF(t)n_w^{-1} = t\}.$$

Then $\mathbf{T}^{wF}$ acts on Y_w by right multiplication. Indeed, let $g\mathbf{U} \in Y_w$ and $t \in \mathbf{T}^{wF}$. Then

$$(tg)^{-1}F(tg) = t^{-1}g^{-1}F(g)F(t) \in t^{-1}\mathbf{U}n_w\mathbf{U}F(t).$$

Since $\mathbf{T}$ normalises $\mathbf{U}$ we have $t^{-1}\mathbf{U}n_w\mathbf{U}F(t) = \mathbf{U}t^{-1}n_wF(t)\mathbf{U}$. Now since $t \in \mathbf{T}^{wF}$, $t^{-1}n_wF(t) = t$ so $tg\mathbf{U} \in Y_w$. The restriction of $\pi : \mathbf{G}/\mathbf{U} \to \mathbf{G}/\mathbf{B}$ to Y_w induces a G-equivariant surjective morphism $\pi_w : Y_w \to X_w$. The fibers of π_w are exactly the orbits of the action of $\mathbf{T}^{wF}$ on Y_w, hence we have a G-equivariant isomorphism of varieties $Y_w/\mathbf{T}^{wF} \simeq X_w$. Hence, by Theorem 1.3.1 4. we have that $H_c^i(Y_w, K) \simeq H_c^i(X_w, K)^{T_w}$.

Remark 1.3.4. Consider an F-stable maximal torus $\mathbf{T}_w$ of type w. Since $T_w \simeq \mathbf{T}^{wF}$ we have a right action of T_w on Y_w.

Example 1.3.5. Assume $w = 1$. Then $Y_w = (\mathbf{G}/\mathbf{U})^F \simeq G/U$ is a finite set. Similarly, $X_w \simeq G/B$.

Deligne–Lusztig induction. Let $\mathbf{T}_w$ be a torus of type w. Recall that $\Lambda = K$ or k. The action of G and T_w on Y_w gives to the cohomology groups $H_c^i(Y_w, \Lambda)$ a structure of $(\Lambda G, \Lambda T_w)$-bimodule. Let $H_c^*(Y_w, \Lambda) = \sum(-1)^i[H_c^i(Y_w, \Lambda)] \in R_\Lambda(G)$. We consider the following functions between Grothendieck groups of T_w and G :

$$R_{T_w}^G : R_K(T_w) \to R_K(G), \quad \theta \mapsto H_c^*(Y_w, K) \otimes_{KT_w} \theta,$$

$$\bar{R}_{T_w}^G : R_k(T_w) \to R_k(G), \quad \theta \mapsto H_c^*(Y_w, k) \otimes_{kT_w} \theta.$$

Those maps are known as the *Deligne–Lusztig induction maps*. If $\theta \in \mathrm{Irr}_K(T_w)$, then $R_{T_w}^G(\theta)$ is a virtual character of G. Virtual characters

constructed this way are called the *Deligne–Lusztig characters* of G. Actually, Deligne–Lusztig induction can be defined for any F-stable Levi subgroup of $\mathbf{G}$, see [14, §11] for more details.

Let $\mathbf{T}_1$, $\mathbf{T}_2$ be F-stable tori, $\theta_1 \in \mathrm{Irr}_K(T_1)$ and $\theta_2 \in \mathrm{Irr}_K(T_2)$. If there exists $g \in G$ such that $\mathbf{T}_2 =^g \mathbf{T}_1$ and $\theta_2 =^g \theta_1$ then $R_{T_1}^G(\theta_1) = R_{T_2}^G(\theta_2)$. In particular, if $\mathbf{T}_v$ and $\mathbf{T}_w$ are F-stable tori of type v and w, the virtual characters $R_{T_w}^G(1)$ and $R_{T_v}^G(1)$ are equal when v and w are F-conjugate in W.

Properties of Deligne–Lusztig characters.

Proposition 1.3.6 ([14, 4.6])**.** *Let $(\mathbf{T}, \mathbf{B})$ be an F-stable maximal torus $\mathbf{T}$ contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. If $\theta \in \mathrm{Irr}_K(T)$, we denote by $\tilde{\theta}$ the inflation of θ to B. Then $R_T^G(\theta) = \mathrm{Ind}_B^G(\tilde{\theta})$.*

The map $\alpha \mapsto \mathrm{Ind}_B^G(\tilde{\alpha})$ is known as the *Harish–Chandra induction*. This proposition shows that the Deligne–Lusztig induction generalises the Harish–Chandra induction.

Example 1.3.7. According to Proposition 1.3.6, $R_T^G(1) = \mathrm{Ind}_B^G(1_B)$. The theory of Hecke algebras provides a bijection :

$$\begin{aligned} \mathrm{Irr}_K(W)^F &\to \{\chi \in \mathrm{Irr}_K(G) \mid \langle \chi, R_T^G(1)\rangle_{KG} \neq 0\}, \\ \phi &\mapsto \phi_q, \end{aligned}$$

such that the multiplicity of ϕ_q in $R_T^G(1)$ is the degree of ϕ. The *principal series* of G is the set of irreducible constituents of $R_T^G(1)$.

The image of 1_W by the bijection above is the trivial character of G, which appears with multiplicity 1 in $R_T^G(1)$. Let $\varepsilon : w \mapsto (-1)^{\ell(w)}$ be the sign representation of W, the image ε_q of ε by the above bijection is called the *Steinberg character* of G and denoted St, according to the previous example it appears with multiplicity 1 in $R_T^G(1)$. We list some results on $1_{\mathbf{G}}$ and St.

Proposition 1.3.8 ([14])**.**

Let $\mathbf{T}_w$ be an F-stable maximal torus of type w.

1. $\langle R_{T_w}^G(\theta), 1_G\rangle_{KG} = \begin{cases} 1 & \text{if } \theta = 1, \\ 0 & \text{otherwise.} \end{cases}$
2. *St is a character of degree $|U| = q^{|\Phi^+|}$ which is 0 on unipotent elements of G. Moreover,*

$$\langle R_{T_w}^G(\theta), St\rangle_{KG} = \begin{cases} (-1)^{\ell(w)} & \text{if } \theta = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The properties of the Steinberg character can be used to compute the degrees of Deligne–Lusztig characters, see [14, 12.9] for more details.

Proposition 1.3.9. *Let $\mathbf{T}_w$ be an F-stable maximal torus of type w and $\theta \in \mathrm{Irr}(T_w)$. We have*

$$R_{T_w}^G(\theta)(1) = (-1)^{\ell(w)}|G|_{p'}|T_w|^{-1}.$$

Theorem 1.3.10 ([14, 11.14])**.** *Let $\mathbf{T}_v$, $\mathbf{T}_w$ be F-stable maximal tori, respectively parametrised by $v, w \in W$. Let $\theta \in \mathrm{Irr}_K(T_v)$ and $\theta' \in \mathrm{Irr}_K(T_w)$. We have*

$$\langle R_{T_v}^G(\theta), R_{T_w}^G(\theta')\rangle_{KG} = |T_v|^{-1}|\{n \in G \mid \quad n\mathbf{T}_v n^{-1} = \mathbf{T}_w \text{ and } {}^n\theta = \theta'\}|.$$

Let $\mathbf{T}_v, \mathbf{T}_w$ be as in Theorem 1.3.10. Then the set

$$\{g \in G \mid g\mathbf{T}_v g^{-1} = \mathbf{T}_w\}$$

is empty if w and v are not F-conjugate (1.2.8). Assume that $w = v$, then $N_{\mathbf{G}}(\mathbf{T}_w)/\mathbf{T}_w = W(\mathbf{T}_w)$ is isomorphic to W. We have the commutative diagram :

$$\begin{tikzcd} W \arrow[r, "\sim"] \arrow[d, "{}^wF"'] & W(\mathbf{T}_w) \arrow[d, "F"] \\ W \arrow[r, "\sim"] & W(\mathbf{T}_w) \end{tikzcd}$$

where ${}^wF(x) = wF(x)w^{-1}$. Hence $W(\mathbf{T}_w)^F \simeq W^{{}^wF}$ and we get the following result.

Corollary 1.3.11.

$$\langle R_{T_v}^G(1), R_{T_w}^G(1)\rangle_{KG} = \begin{cases} |W^{{}^wF}| & \text{if } v \text{ and } w \text{ are } F\text{-conjugate,} \\ 0 & \text{otherwise.} \end{cases}$$

A class function on G is said to be *uniform* if it is a linear combination of Deligne–Lusztig characters. In general, class functions are not necessarly uniform but the following result implies that all irreducible characters of G are irreducible constituents of some Deligne–Lusztig character.

Theorem 1.3.12 ([14, 12.14])**.** *The character of the regular representation of G is uniform. The character χ_{reg} of KG is given by*

$$\chi_{reg} = \frac{1}{|W|}\sum_{w\in W}\sum_{\theta\in\mathrm{Irr}_K(T_w)} R_{T_w}^G(\theta)(1)R_{T_w}^G(\theta).$$

1.3.3 Rational series

Recall that $\mathbf{T}$ is an F-stable maximal torus contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$, $X := X(\mathbf{T})$ is the character group and $Y := Y(\mathbf{T})$ is the cocharacter group. We denote by Φ and $\Phi^\vee$ the set of roots and coroots. The isomorphism class of $\mathbf{G}$ is completely determined by the root datum $(X, \Phi, Y, \Phi^\vee)$. Note that the quadruple $(Y, \Phi^\vee, X, \Phi)$ is also a root datum. That leads to the following definition :

Definition 1.3.13. Let $\mathbf{G}^*$ be a connected reductive group and $\mathbf{T}^*$ be a maximal torus of $\mathbf{G}^*$. We say that $(\mathbf{G}^*, \mathbf{T}^*)$ is *dual* to $(\mathbf{G}, \mathbf{T})$ if the root datum of $\mathbf{G}^*$ with respect to $\mathbf{T}^*$ is isomorphic $(Y, \Phi^\vee, X, \Phi)$.

According to Theorem 1.2.3, $\mathbf{G}$ has a unique dual group up to isomorphism. Let $\mathbf{G}^*$ and $\mathbf{T}^*$ be as in the definition and $(X^*, \Phi^*, Y^*, \Phi^{*\vee})$ be the root datum of $\mathbf{G}^*$ with respect to $\mathbf{T}^*$. The group $\mathbf{G}^*$ is dual to $\mathbf{G}$ if and only if there is an isomorphism φ from X to Y^* sending Φ to $\Phi^{*\vee}$. From now on, for any reductive group $\mathbf{G}$ we will denote by $\mathbf{G}^*$ a reductive group which is dual to $\mathbf{G}$.

Example 1.3.14. If $\mathbf{G}$ is simple and $\mathbf{G} = \mathbf{G}_{ad}$, then $\mathbf{G}^* = G_{sc}$ excepted for types B and C. If $\mathbf{G}$ is adjoint of type B (resp. C) then $\mathbf{G}^*$ is the simple simply connected group of type C (resp. B)

Let F and F^* be Frobenius maps of $\mathbf{G}$ and $\mathbf{G}^*$, such that $\mathbf{T}$ and $\mathbf{T}^*$ are stable by F and F^*. We say that the pair $(\mathbf{G}^*, F^*)$ is dual to $(\mathbf{G}, F)$ if the following diagram is commutative.

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[d, "F"'] & Y^* \arrow[d, "F^*"] \\ X \arrow[r, "\varphi"] & Y^* \end{tikzcd}$$

In that case we denote $\mathbf{G}^{*F^*}$ by G^* and we say that the finite groups G and G^* are in duality. For convenience, we may denote F^* by F when there is no ambiguity.

Proposition 1.3.15 ([14, 13.11,13.13])**.** *Assume that* $(\mathbf{G}, F)$ *and* $(\mathbf{G}^*, F^*)$ *are in duality as above. Then we have an isomorphism* $T^* \simeq \mathrm{Irr}_K(T)$ *which induces a bijection between :*

1. *the set of* G*-conjugacy classes of pairs* $(\mathbf{S}, \theta)$ *where* $\mathbf{S}$ *is an* F*-stable maximal torus of* $\mathbf{G}$ *and* $\theta \in \mathrm{Irr}_K(S)$ *and*

2. *the set of G^*-conjugacy classes of pairs $(\mathbf{S}^*, s)$ where $\mathbf{S}^*$ is an F^*-stable maximal torus of $\mathbf{G}^*$ and $s \in S^*$.*

Moreover, if $s \in Z(G^*)$, then the image of s via the isomorphism extends to a linear character $\hat{s} \in \mathrm{Irr}_K(G)$.

Let $(\mathbf{S}, \theta)$ be in duality with $(\mathbf{S}^*, s)$ as in Proposition 1.3.15. Since $R_S^G(\theta)$ depends only on the G-conjugacy class of $(\mathbf{S}, \theta)$, we can set $R_{S^*}^G(s) := R_S^G(\theta)$ without ambiguity. We can now state this fundamental result on Deligne–Lusztig characters.

Theorem 1.3.16 ([14, 14.41])**.** *The virtual characters $R_{T_1^*}^G(s_1)$ and $R_{T_2^*}^G(s_2)$ have no irreducible constituent in common if s_1 and s_2 are not G^*-conjugate.*

Using Theorems 1.3.12 and 1.3.16, we can define a partition of $\mathrm{Irr}_K(G)$:

$$\mathrm{Irr}_K(G) = \bigsqcup_{(s)} \mathcal{E}(G, s)$$

where (s) runs over G^*-conjugacy classes of semi-simple elements of G^* and $\mathcal{E}(G, s)$ is the set of irreducible constituents of the virtual characters $R_{S^*}^G(s)$ for F^*-stable maximal tori $\mathbf{S}^*$ of $\mathbf{G}^*$ containing s. The set $\mathcal{E}(G, s)$ is a *rational Lusztig series.*

Definition 1.3.17. The character $\chi \in \mathrm{Irr}_K(G)$ is called *unipotent* if $\chi \in \mathcal{E}(G, 1)$. We will sometimes denote the set of unipotent characters by $\mathcal{U}(G)$.

Remark 1.3.18. The character χ is unipotent if and only if it lies in a Deligne–Lusztig character $R_S^G(1)$ for some F-stable torus $\mathbf{S}$.

The following result shows that unipotent characters depend only on the root system of the group.

Proposition 1.3.19 ([14, 13.20])**.** *Let $\mathbf{G}$ and $\mathbf{G}'$ be two connected reductive groups defined over $\mathbb{F}_q$. Let $f : \mathbf{G} \to \mathbf{G}'$ be a morphism of algebraic groups defined over $\mathbb{F}_q$ such that $\ker f \subset Z(\mathbf{G})$ and $[\mathbf{G}', \mathbf{G}'] \subset f(\mathbf{G})$. Then we have a bijection :*

$$\mathcal{E}(G', 1) \to \mathcal{E}(G, 1),$$
$$\chi \mapsto \chi \circ f.$$

1.3.4 Jordan decomposition

Connected center case. One of the main achievement of Lusztig's classification of irreducible characters of G is to reduce the problem to unipotent

characters. To do so, for groups with connected center, he provided a bijection between the series $\mathcal{E}(G,s)$ and the set of unipotent characters $\mathcal{U}(C_{G^*}(s))$ of $C_{G^*}(s)$. Note that by [14, 2.3, 13.15], whenever $Z(\mathbf{G})$ is connected, $C_{\mathbf{G}^*}(s)$ is a connected reductive group for any semi-simple element s of $\mathbf{G}^*$. We can now state one of the main results of Lusztig.

Theorem 1.3.20 ([41, 4.23])**.** *Assume that $Z(\mathbf{G})$ is connected, let $s \in G^*$ be semi-simple. There exists a bijection*

$$\psi_s : \mathcal{E}(G,s) \to \mathcal{U}(C_{G^*}(s)).$$

Moreover, there exists a sign ε such that if we extend ψ_s by linearity to virtual characters we have $\psi_s(R_{S^*}^{C_{G^*}}(s)) = \varepsilon R_{S^*}^{G^*}(1_{T^*})$ for any F-stable maximal torus $\mathbf{S}^*$ of $C_{\mathbf{G}^*}(s)$. Moreover, if $\chi \in \mathcal{E}(G,s)$,

$$\chi(1) = \frac{|G|_{p'}}{|C_{G^*}(s)|_{p'}}\psi_s(\chi)(1).$$

Disconnected center case. Using Clifford theory, Lusztig generalised this result to groups with disconnected center. The following definition will allow us to work "as in the connected center case".

Definition 1.3.21. Let $\mathbf{G}$ be a connected reductive group defined over $\mathbb{F}_q$, a *regular embedding* of $\mathbf{G}$ is a connected reductive group $\tilde{\mathbf{G}}$ with connected center with a rational morphism $\iota : \mathbf{G} \to \tilde{\mathbf{G}}$ such that

- ι is an isomorphism from $\mathbf{G}$ onto its image,
- $\iota(\mathbf{G})$ and $\tilde{\mathbf{G}}$ have the same derived subgroup.

Any connected reductive group has a regular embedding, for example we can take

$$\tilde{\mathbf{G}} = (\mathbf{G} \times \mathbf{T})/\{(z, z^{-1}) \mid z \in Z(\mathbf{G})\}.$$

Let $\mathbf{G}$ be a connected reductive group and $\tilde{\mathbf{G}}$ be a regular embedding of $\mathbf{G}$, we can view $\mathbf{G}$ as a closed subgroup of $\tilde{\mathbf{G}}$. In that case, we have $\tilde{\mathbf{G}} = \mathbf{G}Z(\tilde{\mathbf{G}})$ since $\mathbf{G}$ contains the derived subgroup of $\tilde{\mathbf{G}}$. Let us a fix an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ and an F-stablemaximal torus $\tilde{\mathbf{T}}$ of $\tilde{\mathbf{G}}$ containing $\mathbf{T}$. Let $(\mathbf{G}^*, \mathbf{T}^*, F)$ (resp. $(\tilde{\mathbf{G}}^*, \tilde{\mathbf{T}}^*, F)$) be in duality with $(\mathbf{G}, \mathbf{T}, F)$ (resp. $(\tilde{\mathbf{G}}, \tilde{\mathbf{T}}, F)$).

Lemma 1.3.22 ([62, 1.71])**.** *We have a rational surjective morphism $\iota^* : \tilde{\mathbf{G}}^* \to \mathbf{G}^*$ such that $\iota^*(\tilde{\mathbf{T}}^*) = \mathbf{T}^*$ which is unique up to conjugation by an element of $\tilde{\mathbf{T}}^*$. Moreover, $\ker(\iota^*) \subset Z(\tilde{\mathbf{G}}^*)$.*

Example 1.3.23. Assume that $\mathbf{G} = \mathrm{SL}_n$, then we can set $\tilde{\mathbf{G}} = \mathrm{GL}_n$ since SL_n is the derived subgroup of GL_n. We have $\mathbf{G}^* = \mathrm{PGL}_n$ and $\tilde{\mathbf{G}}^* = \mathrm{GL}_n$, $\iota^* : \mathrm{GL}_n \to \mathrm{PGL}_n$ is the natural morphism.

Theorem 1.3.24 (Lusztig, [7, 15.5,15.15])**.** *Let $\tilde{s} \in \tilde{G}^*$ be semi-simple and $s := \iota^*(\tilde{s}) \in G^*$. Let $\chi \in \mathcal{E}(\tilde{G}, \tilde{s})$. Then $\mathrm{Res}_G^{\tilde{G}}\chi$ is multiplicity-free and its irreducible constituents belong to $\mathcal{E}(G, s)$.*

Let $s \in G^*$ be a semi-simple element. Let $C_{G^*}(s)^\circ := C_{\mathbf{G}^*}(s)^{\circ F}$ and let us denote by $A_{G^*}(s)$ the group $C_{G^*}(s)/C_{G^*}(s)^\circ \simeq (C_{\mathbf{G}^*}(s)/C_{\mathbf{G}^*}(s)^\circ)^F$. The natural action of $A_{G^*}(s)$ on $\mathrm{Irr}_K(C_{G^*}(s))$ preserves unipotent characters. Similarly, $\tilde{G}/G$ acts naturally on $\mathrm{Irr}_K(G)$ and preserves Lusztig series.

Theorem 1.3.25 ([47, 5.1])**.** *There is a surjective map*

$$\psi_s : \mathcal{E}(G, s) \to \{A_{G^*}(s)\text{-orbits on } \mathcal{U}(C_{G^*}(s)^\circ)\}$$

such that the fibres of ψ_s are the orbits of the action of $\tilde{G}/G$ on $\mathcal{E}(G, s)$. Let Θ be an $A_{G^*}(s)$-orbit on $\mathcal{U}(C_{G^*}(s)^\circ)$ and $\Gamma \subset A_{G^*}(s)$ be the stabiliser of an element in Θ, then $|\Gamma| = |\psi_s^{-1}(\Theta)|$. Let $\rho \in \psi_s^{-1}(\Theta)$ and $\mathbf{S}^*$ be an F-stable torus containing s, then

$$\langle \rho, R_{S^*}^G(s)\rangle_{KG} = \varepsilon \sum_{\chi \in \Theta} \langle \chi, R_{S^*}^{C_{\mathbf{G}^*}(s)}(1_{S^*})\rangle_{KC_{G^*}(s)^\circ}.$$

Let us explain how Lusztig constructed ψ_s. We fix a semi-simple element $\tilde{s} \in \tilde{G}^*$ such that $\iota^*(\tilde{s}) = s$. Let $I := (\ker \iota^*)^F$, the isomorphism of Proposition 1.3.15 induces an isomorphism between I and $\mathrm{Irr}_K(\tilde{G}/G)$. Hence, we have an action of I on $\mathrm{Irr}_K(\tilde{G})$ and for any $x \in I$, the action of x induces a bijection $\mathcal{E}(\tilde{G}, \tilde{s}) \simeq \mathcal{E}(\tilde{G}, \tilde{s}x)$. The restriction of irreducible characters of $\tilde{G}$ to G have no multiplicity, so by Proposition 1.1.5 and Theorem 1.3.24 we have a bijection between $\tilde{G}/G$-orbits of $\mathcal{E}(G, s)$ and I-orbits of $\bigcup_{x \in I} \mathcal{E}(\tilde{G}, \tilde{s}x)$. Lusztig showed that $A_{G^*}(s)$ is isomorphic to the stabiliser of $\mathcal{E}(\tilde{G}, \tilde{s})$ in I. That isomorphism provides an action of $A_{G^*}(s)$ on $\mathcal{E}(\tilde{G}, \tilde{s})$ and we have a bijection between $\tilde{G}/G$-orbits of $\mathcal{E}(G, s)$ and $A_{G^*}(s)$-orbits of $\mathcal{E}(\tilde{G}, \tilde{s})$. Moreover we have a succession of bijections :

$$\mathcal{E}(\tilde{G}, \tilde{s}) \longleftrightarrow \mathcal{U}(C_{\tilde{G}^*}(s)) \longleftrightarrow \mathcal{U}(C_{G^*}(s)^\circ).$$

The first bijection is the Jordan decomposition for groups with connected center. The second bijection comes from Proposition 1.3.19 by remarking that $\iota^*(C_{\tilde{\mathbf{G}}^*}(\tilde{s})) = C_{\mathbf{G}^*}(s)^\circ$. Lusztig showed that the composition of those

bijections is compatible with the action of $A_{G^*}(s)$. Finally, ψ_s is the composition of the maps

$$\mathcal{E}(G,s) \to \{A_{G^*}(s)\text{-orbits on } \mathcal{E}(\tilde{G},\tilde{s})\} \longleftrightarrow \{A_{G^*}(s)\text{-orbits on } \mathcal{U}(C_{G^*}(s)^\circ)\}.$$

To get a statement similar to Theorem 1.3.20, we need to define unipotent characters and Deligne–Lusztig characters for non-connected groups.

Definition 1.3.26. Let $\mathbf{G}$ be a non-connected reductive group defined over $\mathbb{F}_q$.

1. A unipotent character of G is an irreducible character of G such that the irreducible constituents of its restriction to G° are unipotent. We denote by $\mathcal{U}(G)$ the set of unipotent characters of G.
2. Let $\mathbf{S}$ be an F-stable maximal torus of $\mathbf{G}$ and $\theta \in \mathrm{Irr}_K(S)$, we define the Deligne–Lusztig character of G as $R_S^G(\theta) := \mathrm{Ind}_{G^\circ}^G R_S^{G^\circ}(\theta)$.

Corollary 1.3.27. *Let $\mathbf{G}$ be a connected reductive group defined over $\mathbb{F}_q$ and $s \in G$ be semi-simple. We have a bijection*

$$\mathcal{E}(G,s) \longleftrightarrow \mathcal{U}(C_{G^*}(s)).$$

Moreover, for any F-stable maximal torus $\mathbf{S}^*$ of $C_{\mathbf{G}^*}(s)$, there is a sign ε such that the bijection sends $R_{S^*}^G(s)$ to $\varepsilon R_{S^*}^G(1_{S^*})$.

Proof. We can reduce the proof to the case where s is quasi-isolated. Indeed, assume that s is not quasi-isolated. Let $\mathbf{L}^*$ be a minimal proper F-stable Levi subgroup of $\mathbf{G}^*$ containing $C_{\mathbf{G}^*}(s)$, then s is quasi-isolated in $\mathbf{L}^*$ and $C_{\mathbf{L}^*}(s) = C_{\mathbf{G}^*}(s)$. Let $\mathbf{L}$ be a Levi subgroup $\mathbf{G}$ in duality with $\mathbf{L}^*$. According to [15, 11.37], there exists a sign ε such that the function εR_L^G induces a bijection between $\mathcal{E}(L,s)$ and $\mathcal{E}(G,s)$.

Now assume that s is quasi-isolated. According to [15, 11.51], the restriction of characters from $\mathcal{E}(C_{G^*}(s),1)$ to $C_{G^*}(s)^\circ$ has no multiplicity. Hence, we have the following sequence of bijections

$$\begin{gathered}\{(\tilde{G}/G)\text{-orbits on } \mathcal{E}(G,s)\}\\ \updownarrow \\ \{A_{G^*}(s)\text{-orbits on } \mathcal{U}(C_{G^*}(s)^\circ)\}\\ \updownarrow \\ \{\mathrm{Irr}_K(A_{G^*}(s))\text{-orbits on } \mathcal{U}(C_{G^*}(s))\}.\end{gathered}$$

The first bijection comes from Theorem 1.3.25 and the second from Proposition 1.1.5. Let $\Theta := (\tilde{G}/G) \cdot \chi$ be the orbit of χ under the action of $\tilde{G}/G$. Let $\Phi := A_{G^*}(s) \cdot \rho$ and $\Phi' := \mathrm{Irr}_K(A_{G^*}(s)) \cdot \rho'$ be its image by the above bijections. It is enough to show that $|\Theta| = |\Phi'|$. Let Γ be the stabiliser of ρ in $A_{G^*s}(s)$ and Γ' be the stabiliser of ρ' in $\mathrm{Irr}_K(A_{G^*s}(s))$. By Theorem, 1.3.25 $|\Theta| = |\Gamma|$. Moreover, by Proposition 1.1.5, Γ is orthogonal to Γ'. Hence, $|\Phi'| = \frac{|A_{G^*}(s)|}{|\Gamma'|} = |\Gamma| = |\Theta|$ and we are done. ■

1.3.5 Blocks and rational series

Let $\mathbf{G}$ be a connected reductive group defined over $\mathbb{F}_q$ where q is a power of a prime number p. Let $\ell \neq p$ be a prime number and $(K, \mathcal{O}, k)$ be an ℓ-modular system. Broué and Michel proved that the partition of $\mathrm{Irr}_K(G)$ into Lusztig series behaves particularly well with respect to the partition of $\mathrm{Irr}_K(G)$ into ℓ-blocks.

Theorem 1.3.28 (Broué–Michel [6]). *Let $s \in G^*$ be a semi-simple element of order prime to ℓ and set*

$$B_s(G) := \bigcup_t \mathcal{E}(G, st)$$

where t runs over semi-simple ℓ-elements of $C_{G^*}(s)$. Then $B_s(G)$ is a union of blocks.

When there is no ambiguity on the underlying group, we will denote $B_s(G)$ by B_s. Blocks in B_1 will be called *unipotent blocks*, elements of $\mathrm{Irr}_k(B_1)$ are *unipotent modular representations.*

Theorem 1.3.29 (Bonnafé–Rouquier [4, 11.8]). *Let $s \in G^*$ be a semi-simple element of order prime to ℓ, $\mathbf{L}^*$ be an F-stable Levi subgroup of $\mathbf{G}^*$ containing $C_{\mathbf{G}^*}(s)$. Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ dual to $\mathbf{L}^*$. The functor R_L^G induces a Morita equivalence between $B_s(L)$ and $B_s(G)$.*

Remark 1.3.30. This theorem induces a "Jordan decomposition" for union of blocks B_s. More precisely, assume $\mathbf{L}_s^* := C_{\mathbf{G}^*}(s)$ is a Levi subgroup of $\mathbf{G}^*$, and let $\mathbf{L}_s$ be a Levi subgroup of $\mathbf{G}$ in duality with $\mathbf{L}_s^*$. Let $\hat{s}$ be the linear character of L_s corresponding to s. Since s is an ℓ'-element, $\hat{s}$ can be viewed as a character of an $\mathcal{O}G$-module of $\mathcal{O}$-rank 1. Then $R_{L_s}^G(\hat{s} \otimes_{\mathcal{O}} .)$ induces a Morita equivalence between $B_1(L_s)$ and $B_s(G)$.

2

Counting unipotent modular representations

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The aim of this chapter is to introduce our new method for counting the number of irreducible modular representations lying in unipotent blocks. If ℓ is good for $\mathbf{G}$ and does not divide $|(Z(\mathbf{G})/Z(\mathbf{G})^\circ)^F|$, Geck–Hiss and Geck have shown that unipotent characters form a basic set for unipotent blocks [27, 22]. Moreover, unipotent characters have been classified in terms of special unipotent classes. Hence, using Geck–Hiss results on basic sets and Lusztig's classification, we can count unipotent modular representations in term of special classes of $\mathbf{G}$. However, when ℓ is bad or divides $|(Z(\mathbf{G})/Z(\mathbf{G})^\circ)^F|$ we cannot use the result above anymore, see Example 2.3.3. Actually, in that case, there are not enough special classes and we need to introduce a set of unipotent classes larger than the set of special classes : the *ℓ-special classes*. We conjecture in 2.3.14 that ℓ-special classes are the good objects to count unipotent modular representations. Then, we prove the conjecture for $\mathrm{SL}_n(q)$ and $\mathrm{SU}_n(q)$, for simple classical groups and for simple adjoint exceptional groups.

In the first section, we recall some notions on representations of Weyl groups. In particular, we introduce the notions of *families* and *special representations* and we state Lusztig's classification theorem for unipotent characters in terms of families. Then, we introduce the Springer correspondence between unipotent classes of $\mathbf{G}$ and representations of the Weyl group of $\mathbf{G}$. It allows us to define *special unipotent classes* and to state Lusztig's classification in terms of those special unipotent classes.

In the second section, we introduce the notion of ℓ-special unipotent classes which are associated to special classes of centralisers of isolated ℓ-elements. Then, with the idea of adapting Lusztig's approach to the modular case, we associate to each ℓ-special class $(u)_{\mathbf{G}}$ a group Ω_u^ℓ that we call the *ℓ-canonical quotient*. Together with Ω_u^ℓ, we associate a set $\bar{\mathcal{M}}_\ell(\Omega_u^\ell)$. Then, we conjecture that we can use those objects to count unipotent modular representations.

The three following sections are devoted to showing that our conjecture holds for most simple groups.

- In the third section, interpreting results of Kleshchev–Tiep and Denoncin, we show that the conjecture is true for $\mathrm{SL}_n(q)$ and $\mathrm{SU}_n(q)$.
- In Section 4, we are interested in groups of type B, C or D. In that case, the conjecture is equivalent to showing that the number of unipotent modular representations is equal to the number of unipotent classes of G. Geck proved that result for groups with connected center and we generalise it to the case of groups with disconnected center.

- In the last section, we prove that the conjecture holds for simple adjoint exceptional groups. To this end, we present an algorithm that counts the objects attached to the ℓ-special classes introduced in the second section. By implementing it in GAP3 using the package CHEVIE [50], and by a case-by-case analysis, we check that the numerical results we get coincide with the number of the unipotent modular representations of these groups computed by Geck and Hiss.

2.1 Classification of unipotent characters

2.1.1 On representations of W

Throughout this section, $\mathbf{G}$ will be a connected reductive algebraic group over $\bar{\mathbb{F}}_q$, $F : \mathbf{G} \to \mathbf{G}$ a Frobenius map corresponding to a $\mathbb{F}_q$-structure and $G := \mathbf{G}^F$ the corresponding finite reductive group. We will assume that q is a power of a good prime number. Let $\mathbf{T}$ be an F-stable maximal torus contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. We denote by W the Weyl group of $\mathbf{G}$.

Fake degrees and generic degrees. Let V be the reflection representation of W *i.e.* the $\mathbb{R}$-vector space on which W acts naturally, and let us denote $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$. Let $P := P(V_{\mathbb{C}})$ be the algebra of polynomial functions on $V_{\mathbb{C}}$. It is a graded $\mathbb{C}$-algebra whose n-th component consists of the homogeneous polynomials of degree n. The Weyl group W acts naturally on P via $w \cdot f(v) = f(w^{-1}v)$ for $w \in W$, $f \in P$ and $v \in V_{\mathbb{C}}$. Let I be the homogeneous ideal of P generated by W-invariant polynomials vanishing at 0. Let $A := P/I$. This is a graded $\mathbb{C}W$-module isomorphic to the regular representation of W [8, 2.4.6]. Let $\phi \in \mathrm{Irr}_{\mathbb{C}}(W)$ be an irreducible character of W over $\mathbb{C}$ and n_i be the multiplicity of ϕ in the i-th component of A. Then the *fake degree* of ϕ is the polynomial

$$P_\phi(X) := \sum_{i \geq 0} n_i X^i.$$

The theory of Hecke algebras provides a family of unipotent characters of G, namely the *principal series* which is in bijection with $\mathrm{Irr}_{\mathbb{C}}(W)$ and the degrees of those unipotent characters provide other polynomials. More precisely, if we assume that F acts trivially on W, recall that there is a bijection

$$\begin{aligned} \mathrm{Irr}_{\mathbb{C}}(W) &\to \{\chi \in \mathrm{Irr}_{\mathbb{C}}(G) \mid \langle \chi, \mathrm{Ind}_B^G 1_B \rangle_{\mathbb{C}G} \neq 0\}, \\ \phi &\mapsto \phi_q. \end{aligned}$$

Then, the degree of ϕ_q is a polynomial in q. In [41, 3.3], Lusztig generalised the construction of ϕ_q to the case where the Frobenius map does not act trivially on the Weyl group for $\phi \in \mathrm{Irr}_{\mathbb{C}}(W)^F$.

Definition 2.1.1. Let $\phi \in \mathrm{Irr}_{\mathbb{C}}(W)^F$. The *generic degree* of ϕ, denoted by $\widetilde{P}_\phi$, is the polynomial $\phi_q(1)$.

Special representations and families. Consider $\phi \in \mathrm{Irr}_{\mathbb{C}}(W)$ and let us write the fake and generic degree associated to ϕ :

$$\begin{aligned} P_\phi(X) &= \gamma_\phi X^{a_\phi} + \cdots + \delta_\phi X^{b_\phi}, \quad a_\phi \leq b_\phi, \\ \widetilde{P}_\phi(X) &= \widetilde{\gamma}_\phi X^{\widetilde{a}_\phi} + \cdots + \widetilde{\delta}_\phi X^{\widetilde{b}_\phi}, \quad \widetilde{a}_\phi \leq \widetilde{b}_\phi, \end{aligned}$$

where $\gamma_\phi, \widetilde{\gamma}_\phi \neq 0$. We always have $\widetilde{a}_\phi \leq a_\phi$ [8, 11.3.4]. We are interested in the irreducible representations for which we have an equality.

Definition 2.1.2. A character $\phi \in \mathrm{Irr}_{\mathbb{C}}(W)$ is called *special* if $a_\phi = \widetilde{a}_\phi$.

In [41], Lusztig defined a partition of $\mathrm{Irr}_{\mathbb{C}}(W)$, into *families* in a recursive way :

- If W is the trivial group, there is a unique family, containing the trivial representation.
- Assume that families have been defined for proper standard parabolic subgroups of W. Let us denote by ε_W the sign representation of W. For $\phi_1, \phi_2 \in \mathrm{Irr}_{\mathbb{C}}(W)$, we say that $\phi_1 \simeq \phi_2$ if there exists a standard parabolic subgroup $W' \subsetneq W$ and $\varphi_1, \varphi_2 \in \mathrm{Irr}_{\mathbb{C}}(W')$ in the same family such that one of the following conditions is satisfied :

$$\begin{array}{rlll} \langle \varphi_i, \mathrm{Res}^W_{W'} \phi_i \rangle_{\mathbb{C}W'} & \neq 0 \text{ and } & a_{\phi_i} & = a_{\varphi_i}, \quad i = 1, 2, \text{ or} \\ \langle \varphi_i, \mathrm{Res}^W_{W'} \phi_i \otimes \varepsilon_{W'} \rangle_{\mathbb{C}W'} & \neq 0 \text{ and } & a_{\phi_i \otimes \varepsilon_W} & = a_{\varphi_i}, \quad i = 1, 2. \end{array}$$

  Then we define families of W as equivalence classes for the equivalence relation generated by $\simeq$.

By a case-by-case verification, Lusztig noted a remarkable property of families.

Proposition 2.1.3 ([41, 4.14.2])**.** *Each family of W contains exactly one special character.*

Let us look at how irreducible characters fall into families in the following examples.

- Assume W is of type A_n *i.e.* W is the symmetric group $\mathfrak{S}_{n+1}$. Its irreducible characters are parametrised by partitions of $n+1$. If $\alpha = (\alpha_1, \ldots, \alpha_r)$ (where $\alpha_1 \leq \cdots \leq \alpha_r$) is such a partition, we denote by χ_α the corresponding character of W which is uniquely determined by the following properties :
  - the restriction of χ_α to $\prod_{i=1}^{r} \mathfrak{S}_{\alpha_i}$ contains the trivial character,

- let α^* be the dual partition of α. Then, the restriction of χ_α to $\prod_{i=1}^{r} \mathfrak{S}_{\alpha_i^*}$ contains the sign character.

All irreducible characters of W are special and each family has one element.

- Assume W is of type B_n or C_n. Irreducible characters of W are parametrised by bipartitions of n, *i.e.* pairs of partitions (α, β) such that $|\alpha| + |\beta| = n$. We denote by $\phi_{\alpha,\beta}$ the corresponding character. Adding zeros to α or β if necessary, we can assume that α has one more part than β. To (α, β) we associate its symbol Λ defined as the array :

$$\Lambda = \begin{pmatrix} \alpha_1 & & \alpha_2 + 1 & & \dots & & \alpha_m + (m-1) & & \alpha_{m+1} + m \\ & \beta_1 & & \beta_2 + 1 & & \dots & & \beta_m + (m-1) & \end{pmatrix}$$

Since the number of parts can be increased at our will by adding zeros to the partitions, we define an equivalence relation on the set of symbols. Let

$$\Lambda' = \begin{pmatrix} 0 & \alpha_1 + 1 & & \alpha_2 + 2 & & \dots & & \alpha_m + m & & \alpha_{m+1} + m + \\ & 0 & \beta_1 + 1 & & \beta_2 + 2 & & \dots & & \beta_m + m & \end{pmatrix}$$

The relation $\Lambda \simeq \Lambda'$ generates an equivalence relation and we denote by $[\Lambda]$ the equivalence class of Λ. Each bipartition defines a unique equivalence class of symbols under this relation. Moreover, if

$$\begin{pmatrix} \lambda_1 & & \lambda_2 & & \dots & & \lambda_m & & \lambda_{m+1} \\ & \mu_1 & & \mu_2 & & \dots & & \mu_m & \end{pmatrix}$$

is a symbol, the character attached to this symbol is special if and only if

$$\lambda_1 \leq \mu_1 \leq \lambda_2 \leq \mu_2 \leq \cdots \leq \mu_m \leq \lambda_{m+1}.$$

If Λ_1 and Λ_2 are two symbols, then their corresponding characters lie in the same family if and only if the unordered sets of entries of those symbols are the same.

Example 2.1.4. Let W be the Coxeter group of type B_2. The following table lists the bipartitions of 2 and their corresponding symbols.

(α, β)	$(2.-)$	$(1^2.-)$	(1.1)	$(-.1^2)$	$(-.2)$
Symbol	$\begin{pmatrix} 2 \\ - \end{pmatrix}$	$\begin{pmatrix} 1 & & 2 \\ & 0 & \end{pmatrix}$	$\begin{pmatrix} 0 & & 2 \\ & 1 & \end{pmatrix}$	$\begin{pmatrix} 0 & & 1 & & 2 \\ & 1 & & 2 & \end{pmatrix}$	$\begin{pmatrix} 0 & & 1 \\ & 2 & \end{pmatrix}$

There are 3 families : $\{\phi_{2.-}\}$, $\{\phi_{-.1^2}\}$ and $\{\phi_{1^2.-}, \phi_{1.1}, \phi_{-.2}\}$. The special characters are $\phi_{2.-}$, $\phi_{-.1^2}$ and $\phi_{1.1}$.

- Assume that W is a Coxeter group of type G_2. Then, W has 6 irreducible characters denoted by $\phi_{1,0}, \phi_{2,1}, \phi_{2,2}, \phi'_{1,3}, \phi''_{1,3}, \phi_{1,6}$ in [8, §13.2]. Here, a character denoted by $\phi_{d,e}$ (or $\phi'_{d,e}, \phi''_{d,e}$) is a character of degree d such that its fake degree is divisible by X^e but not by X^{e+1} (*i.e.* $e = a_{\phi_{e,d}}$). The families of W are $\{\phi_{1,0}\}, \{\phi_{2,1}, \phi_{2,2}, \phi'_{1,3}, \phi''_{1,3}\}$ and $\{\phi_{1,6}\}$. The special characters are $\phi_{1,0}, \phi_{2,1}$ and $\phi_{1,6}$.

2.1.2 Lusztig's theorem

In this subsection, we introduce a theorem of Lusztig on the classification of unipotent characters of G in terms of families of representations of the Weyl group of G.

Families of unipotent characters. Recall that $\mathcal{U}(G)$ denotes the set of unipotent characters of G. We can derive a partition of $\mathcal{U}(G)$ from the one of $\mathrm{Irr}_{\mathbb{C}}(W)$ into families as follows. Let γ be the generator of an infinite cyclic group and $\tilde{W}$ be the semi-direct product of W with the infinite cyclic group generated by γ such that $\gamma \cdot w \cdot \gamma^{-1} = F(w)$ for $w \in W$. The following result of Lusztig tells us how F-stable characters of W extends to $\tilde{W}$.

Proposition 2.1.5 ([41, 3.2])**.** *Let $\phi \in (\mathrm{Irr}_{\mathbb{C}}(W))^F$. There exist exactly two irreducible characters of $\tilde{W}$ which factor through a finite group, can be realised over $\mathbb{Q}$ and whose restriction to W is ϕ.*

Let $\phi \in (\mathrm{Irr}_{\mathbb{C}}(W))^F$ and let $\tilde{\phi}$ be such an extension of ϕ. For each $w \in W$, we fix an F-stable maximal torus $\mathbf{T}_w$ of type w as in §1.3.2. We define the class function

$$R_{\tilde{\phi}} := \frac{1}{|W|} \sum_{w \in W} \tilde{\phi}(\gamma \cdot w) R_{T_w}^G(1).$$

Theorem 2.1.6 ([41, 4.23])**.** *Let $\phi_1, \phi_2 \in \mathrm{Irr}_{\mathbb{C}}(W)$ and $\tilde{\phi}_1, \tilde{\phi}_2$ be extensions to $\tilde{W}$ as above. Then $R_{\tilde{\phi}_1}$ and $R_{\tilde{\phi}_2}$ have a common irreducible constituent if and only if ϕ_1 and ϕ_2 lie in the same family of $\mathrm{Irr}_{\mathbb{C}}(W)$.*

Let $\mathcal{F}$ be an F-stable family of W and let us denote by $\mathcal{F}(G)$ the set of irreducible constituents of $R_{\tilde{\phi}}$ where $\tilde{\phi}$ is an extension of some $\phi \in \mathcal{F}$. Then we have a partition

$$\mathcal{U}(G) = \bigsqcup_{\mathcal{F}} \mathcal{F}(G)$$

where $\mathcal{F}$ runs over F-stable families of W. We will call $\mathcal{F}(G)$ a *family of unipotent characters* and we will use the notation $\mathcal{F}$ if there is no ambiguity.

Example 2.1.7.

- Assume that $\mathbf{G} = \mathrm{SL}_2(\bar{\mathbb{F}}_q)$ where q is odd and $F = F_q$. Let us denote by $W = \{e, s\}$ the Weyl group of $\mathbf{G}$. Its irreducible representations are 1 and ε. We have : $R_1 = \frac{1}{2}(R_{T_e} + R_{T_s}) = \chi_1$ and $R_\varepsilon = \frac{1}{2}(R_{T_e} - R_{T_s}) = \chi_{1^2}$. Therefore, the two unipotents characters lie in distinct families.
- Assume that $\mathbf{G} = \mathrm{Sp}_4(\bar{\mathbb{F}}_q)$ where q is odd. The table below give for each $\phi \in \mathrm{Irr}_{\mathbb{C}}(W)$, the decomposition of $R_{\tilde{\phi}}$ into irreducible characters.

ϕ	R_ϕ
$\phi_{2.-}$	$\chi_{2.-}$
$\phi_{1.1}$	$\frac{1}{2}(\chi_{1^2.-} + \chi_{1.1} + \chi_{-.2} + \theta)$
$\phi_{1^2.-}$	$\frac{1}{2}(\chi_{1^2.-} + \chi_{1.1} - \chi_{-.2} - \theta)$
$\phi_{-.2}$	$\frac{1}{2}(-\chi_{1^2.-} + \chi_{1.1} + \chi_{-.2} - \theta)$
$\phi_{-.1^2}$	$\chi_{-.1^2}$

  Here, for a bipartition (α, β), $\chi_{\alpha,\beta}$ is the character of the principal series corresponding to $\phi_{\alpha,\beta}$ and θ is the cuspidal unipotent character of G. The table gives us the partition of $\mathcal{U}(G)$ into families of unipotent characters of G : the family containing the trivial character $\chi_{2.-}$, the family containing the Steinberg character $\chi_{-.1^2}$ and the family containing the 4 remaining characters.

Finite groups associated to families. To each family $\mathcal{F}$ of W, Lusztig associated in [41, §4] a group $\Omega_{\mathcal{F}}$. This is done case-by-case, starting with irreducible Weyl groups and then generalising to any Weyl group. For each family $\mathcal{F}$, $\Omega_{\mathcal{F}}$ is a product of the groups $\mathbb{Z}/2\mathbb{Z}$, $\mathfrak{S}_3$, $\mathfrak{S}_4$ and $\mathfrak{S}_5$. The construction of those groups is an important step toward the classification of unipotent characters.

Definition 2.1.8. Let Ω be a finite group. Let us write $\mathcal{M}(\Omega)$ for the set of Ω-conjugacy classes of pairs (x, σ) where $x \in \Omega$ and $\sigma \in \mathrm{Irr}_{\mathbb{C}}(C_\Omega(x))$. Moreover, we define a pairing $\{\,.\,,\,.\,\} : \mathcal{M}(\Omega) \times \mathcal{M}(\Omega) \to \mathbb{C}$ by

$$\{(x, \sigma), (y, \tau)\} := \sum_g \frac{1}{|C_\Omega(x)||C_\Omega(y)|} \tau(g^{-1}x^{-1}g)\sigma(gyg^{-1})$$

where g runs over the set of elements of Ω such that gyg^{-1} commutes with x.

The set $\mathcal{M}(\Omega_{\mathcal{F}})$ plays an important role for the classification of unipotent characters of G but is not suited for the case where F does not act trivially on W, that is why we need to "twist" this set. Let Ω be a finite group on which F acts, as above, we denote by $\tilde{\Omega}$ the semidirect product of Ω with the infinite cyclic group generated by γ such that for $x \in \Omega$, $\gamma \cdot x \cdot \gamma^{-1} = F(x)$. We will define two sets and a pairing between those sets.

Definition 2.1.9. To Ω we can associate the following sets :

- $\tilde{\mathcal{M}}(\Omega)$ is the set of $\tilde{\Omega}$-conjugacy class of pairs (x, ϕ) where x is an element of Ω such that $C_{\tilde{\Omega}}(x) \cap \Omega.\gamma \neq \emptyset$ and ϕ is an irreducible representation of $C_{\tilde{\Omega}}(x)$ which factors through a finite quotient and whose restriction to $C_{\Omega}(x)$ remains irreducible.
- $\bar{\mathcal{M}}(\Omega)$ is the set of $\tilde{\Omega}$-conjugacy class of pairs (x, ϕ) where x is an element of $\Omega \cdot \gamma$ and $\phi \in \mathrm{Irr}_{\mathbb{C}}(C_{\Omega}(x))$.
- We can define a pairing $\bar{\mathcal{M}}(\Omega) \times \tilde{\mathcal{M}}(\Omega) \to \mathbb{C}$ by the same formula as in 2.1.8.

Remark 2.1.10.

1. If F acts trivially on Ω, then $\tilde{\Omega}$ is the direct product of Ω with the infinite cyclic group generated by γ and $\bar{\mathcal{M}}(\Omega)$ is in bijection with $\mathcal{M}(\Omega)$.
2. If $x = a.\gamma \in \tilde{\Omega}.\gamma$, then the $\tilde{\Omega}$-conjugates of x are the elements of the form $b.\gamma$ where $b \in \Omega$ is F-conjugate to a.

Example 2.1.11.

- Assume that W is a Coxeter group of type A_n. Then, for each family $\mathcal{F}$, the group $\Omega_{\mathcal{F}}$ is trivial and the set $\bar{\mathcal{M}}(\Omega_{\mathcal{F}})$ has only one element.
- Assume that W is a Coxeter group of type B_n or C_n. Let $\mathcal{F}$ be a family of W and let ϕ be a character of $\mathcal{F}$ whose class symbol is $[\Lambda]$ where

$$\Lambda = \begin{pmatrix} \lambda_1 & & \lambda_2 & & \dots & & \lambda_m & & \lambda_{m+1} \\ & \mu_1 & & \mu_2 & & \dots & & \mu_m & \end{pmatrix}$$

  is chosen such that m is as small as possible. Let Z be the set of entries of Λ which appear a single time. Note that Z depends only on $\mathcal{F}$ and has odd cardinality. Let us write $|Z| = 2d + 1$ and let V be the set of subsets of Z of even cardinality. For $M_1, M_2 \in V$, we define the operation $M_1 + M_2 = (M_1 \cup M_2) \setminus (M_1 \cap M_2)$. This operation endows

V with a structure of vector space over $\mathbb{F}_2$ of dimension $2d$ whose zero element is the empty set. The function $\langle M_1|M_2\rangle = |M_1 \cap M_2|$ mod 2 defines a symplectic form on V. Then $\Omega_{\mathcal{F}}$ is defined as a maximal isotropic subspace of V. In particular, $\Omega_{\mathcal{F}} \simeq (\mathbb{Z}/2\mathbb{Z})^d$.

Lusztig's theorem. Let $\mathcal{F}$ be an F-stable family of W. Then there is an action of F on $\Omega_{\mathcal{F}}$ and the constructions above can be done. Let us denote $\mathcal{M}(\Omega_{\mathcal{F}}), \tilde{\mathcal{M}}(\Omega_{\mathcal{F}})$ and $\bar{\mathcal{M}}(\Omega_{\mathcal{F}})$ by $\mathcal{M}_{\mathcal{F}}, \tilde{\mathcal{M}}_{\mathcal{F}}$ and $\bar{\mathcal{M}}_{\mathcal{F}}$ respectively. Let $\tilde{\mathcal{F}}$ be the set of irreducible characters of $\tilde{W}$ whose restriction to W is irreducible and belongs to $\mathcal{F}$.

Theorem 2.1.12 ([41, 4.23])**.** *For any F-stable family $\mathcal{F}$ of $\mathrm{Irr}_{\mathbb{C}}(W)$, there is a bijection*

$$\begin{aligned} \bar{\mathcal{M}}_{\mathcal{F}} &\to \mathcal{U}_{\mathcal{F}} \\ x &\mapsto \chi_x \end{aligned}$$

and an embedding

$$\begin{aligned} &\tilde{\mathcal{F}} \hookrightarrow \bar{\mathcal{M}}_{\mathcal{F}} : \\ &\tilde{\phi} \mapsto x_{\tilde{\phi}} \end{aligned}$$

such that given $\mathcal{F}$, $\mathcal{F}'$ two F-stable families, $\phi \in \mathcal{F}$ and $x \in \bar{\mathcal{M}}_{\mathcal{F}'}$, then :

$$\langle R_{\tilde{\phi}}, \chi_x\rangle_{\mathbb{C}G} = \begin{cases} \varepsilon_x\{x, x_\phi\} & \text{if} \quad \mathcal{F} = \mathcal{F}', \\ 0 & \text{otherwise.} \end{cases}$$

where $\varepsilon_x \in \{-1, 1\}$ depends on x.

Remark 2.1.13.

1. $\varepsilon_x = 1$ except for the exceptional families of the exceptional groups E_7 and E_8.
2. By Remark 2.1.10, if F acts trivially on W we can replace $\bar{\mathcal{M}}_{\mathcal{F}}$ by $\mathcal{M}_{\mathcal{F}}$ in the theorem.

Example 2.1.14. Assume that $G = \mathrm{Sp}_4(q)$. Let us describe how we can parametrise unipotent characters of G using Lusztig's theorem. If (α, β) is a bipartition of 2, we denote by $\chi_{\alpha.\beta}$ the character of the principal series corresponding to $\phi_{\alpha.\beta} \in \mathrm{Irr}_{\mathbb{C}}(W)$. We denote by θ the cuspidal unipotent character of G. The trivial and the Steinberg character $\chi_{2.-}$ and $\chi_{-.1^2}$ form families of one element corresponding respectively to the special characters $\phi_{2.-}$ and $\phi_{-.1^2}$ of W. Let $\mathcal{F}$ be the family such that $\mathcal{F}(G) = \{\chi_{1^2.-}, \chi_{1.1}, \chi_{-.2}, \theta\}$. Then $\Omega_{\mathcal{F}}$ is a group of order 2 which we identify with $\mathfrak{S}_2 = \{e, s_1\}$. According to [41, §4.5], $\chi_{1.1}$, $\chi_{-.2}$ and $\chi_{1^2.-}$ correspond respectively to $(e, 1)$, (e, ε)

and $(s_1, 1)$ and the cuspidal character θ corresponds to (s_1, ε). We give below the corresponding Fourier matrix of $\mathcal{F}$, *i.e.* the matrix of the form $\{.,.\}$.

$$\begin{array}{c} \\ (e,1) \\ (e,\varepsilon) \\ (s_1,1) \\ (s_1,\varepsilon) \end{array} \begin{array}{c} \begin{array}{cccc} (e,1) & (e,\varepsilon) & (s_1,1) & (s_1,\varepsilon) \end{array} \\ \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \end{array}.$$

Non unipotent characters. Assume that the center of $\mathbf{G}$ is connected. Using the Jordan decomposition, we can apply theorem 2.1.12 to parametrise all irreducible character of G. Recall that $\mathrm{Irr}_{\mathbb{C}}(G)$ is partitioned into rational Lusztig series $\mathcal{E}(G, s)$, where $(s)_{G^*}$ is a semi-simple conjugacy class of G^*. Jordan decomposition Theorem 1.3.20 provides a bijection between $\mathcal{E}(G, s)$ and $\mathcal{E}(C_{G^*}(s), 1)$. Theorem 2.1.12 gives a parametrisation of $\mathcal{E}(C_{G^*}(s), 1)$ by pairs $(\mathcal{F}, x)$ where $\mathcal{F}$ is an F-stable family of the Weyl group W_s of $C_{\mathbf{G}^*}(s)$ and $x \in \bar{\mathcal{M}}_{\mathcal{F}}$. Combining this parametrisation with the Jordan decomposition, we get a parametrisation of $\mathcal{E}(G, s)$, *i.e.* we have :

$$\mathcal{E}(G, s) = \bigsqcup_{\mathcal{F}} \mathcal{E}(G, s)_{\mathcal{F}}$$

where $\mathcal{F}$ runs over families of W_s and $\mathcal{E}(G, s)_{\mathcal{F}}$ is the subset of $\mathcal{E}(G, s)$ in bijection with $\mathcal{F}(C_{G^*}(s))$.

2.2 Springer correspondence

2.2.1 Springer's construction of representations of W

Springer in [60] introduced a geometric construction of all irreducible representations of W using unipotent classes of $\mathbf{G}$. The construction was initially done when $\mathbf{G}$ is a simple adjoint groups over $\mathbb{C}$, but Lusztig generalised it to fields of any characteristic and for any connected reductive algebraic group. Let $u \in G$ be a unipotent element and let $\mathcal{B}_u$ be the variety of Borel subgroups of $\mathbf{G}$ containing u. Then $\mathcal{B}_u$ is a projective variety. Let $H_u := H^{2\dim \mathcal{B}_u}(\mathcal{B}_u, \bar{\mathbb{Q}}_\ell)$ be the top non-vanishing ℓ-adic cohomology group of $\mathcal{B}_u$. According to Spaltenstein [58], all irreducible components have the

same dimension hence, by 1.3.1, the dimension of H_u as a $\bar{\mathbb{Q}}_\ell$-vector space is the number of irreducible components of $\mathcal{B}_u$.

Moreover, $C_{\mathbf{G}}(u)$ acts on $\mathcal{B}_u$ by conjugation and $C_{\mathbf{G}}(u)^\circ$, the identity component of $C_{\mathbf{G}}(u)$, fixes each irreducible component of $\mathcal{B}_u$. Thus, $A_{\mathbf{G}}(u) := C_{\mathbf{G}}(u)/C_{\mathbf{G}}(u)^\circ$ acts on the set of irreducible components of $\mathcal{B}_u$ so $A_{\mathbf{G}}(u)$ acts on H_u by permutation of the basis. In [60], Springer defined an action of W on H_u when the characteristic p is large enough. A generalisation of this action in arbitrary characteristic is given in [40]. This action of W commutes with the action of $A_{\mathbf{G}}(u)$ so we have an action of $W \times A_{\mathbf{G}}(u)$ on H_u. For an irreducible character ϕ of $A_{\mathbf{G}}(u)$, we denote by $H_{u,\phi}$ the sum of the $A_{\mathbf{G}}(u)$-submodules of H_u affording ϕ. Because this space is stable by the action of W, it can be viewed as a $\bar{\mathbb{Q}}_\ell W$-module. Then Springer proved that $H_{u,\phi}$ is either 0 or a direct sum of isomorphic copies of an irreducible $\bar{\mathbb{Q}}_\ell W$-module that we denote by $E_{u,\phi}$. By convention, we set $E_{u,\phi} := 0$ when $H_{u,\phi} = 0$. In addition, Springer showed that :

- $E_{u,\phi} \sim E_{v,\psi}$ if and only if u is conjugate to v and $\phi = \psi$.
- Every irreducible representation of W can be obtained by the construction above.

This construction provides a parametrisation of $\mathrm{Irr}_{\mathbb{C}}(W)$ by pairs $\{\mathcal{C}, \phi\}$ where $\mathcal{C}$ is a unipotent class of $\mathbf{G}$, $u \in \mathcal{C}$ and $\phi \in \mathrm{Irr}_{\mathbb{C}}(A_{\mathbf{G}}(u))$. This parametrisation, known as the *Springer correspondence*, was explicitly described when p is good for $\mathbf{G}$ by Springer for type A_n and G_2 [59], by Alvis–Lusztig for groups of type E_n [1]. The correspondence for F_4 and the classical groups was described by Shoji in [56] and [55].

Example 2.2.1.

- Assume $\mathbf{G} = \mathrm{GL}_n(\bar{\mathbb{F}}_q)$. If $u \in \mathbf{G}$ is unipotent, the $\mathbf{G}$-conjugacy class of u is determined by its Jordan normal form, hence by a partition of n. Therefore, unipotent classes of $\mathbf{G}$ are parametrised by partitions of n. If α is a partition of n, we denote by $\mathcal{C}_\alpha$ the corresponding unipotent class. For every class $\mathcal{C}_\alpha$ and $u_\alpha \in \mathcal{C}_\alpha$, $A_{\mathbf{G}}(u_\alpha)$ is trivial and $E_{u_\alpha,1} = \phi_\alpha \in \mathrm{Irr}_{\mathbb{C}}(\mathfrak{S}_n)$.
- Assume $\mathbf{G} = \mathrm{Sp}_{2n}(\bar{\mathbb{F}}_q)$, q odd. Unipotent classes of $\mathbf{G}$ are parametrised by partitions of $2n$ such that any odd part occurs an even number of times. Moreover, according to [42, 10.4], if $\lambda = (\lambda_1, \ldots, \lambda_r)$ is such a partition and $u_\lambda \in \mathbf{G}$ is a representative of the corresponding class then $A_{\mathbf{G}}(u_\lambda) = (\mathbb{Z}/2\mathbb{Z})^a$. The integer a is the number of integers appearing an even number of times in λ. We will follow the process in [29, 2B] to

describe $E_{u_\lambda,1}$. By adding zeros if necessary, we can suppose that $\lambda_1 = 0$ and that r is odd. We can partition $(\lambda_1, \dots, \lambda_r)$ into blocks of length 1 or 2 such that all even parts lie in blocks of length 1 and odd parts lie in blocks of length 2. Let us define

$$\begin{cases} c_i := \lambda_i/2 + (i-1) & \text{if } \{\lambda_i\} \text{ is a block,} \\ c_i := c_{i+1} := (\lambda_i + 1)/2 + i - 1 & \text{if } \{\lambda_i, \lambda_{i+1}\} \text{ is a block.} \end{cases}$$

Now let us define the symbol

$$\Lambda_\lambda := \begin{pmatrix} c_1 & & c_3 - 2 & \dots & c_r - 2r \\ & c_2 - 1 & \dots & & c_{r-1} - (2r-1) \end{pmatrix}.$$

Let (α, β) be the bipartition of n such that the parts of α are the non-zeros entries of the first row of Λ_λ and the part of β are the non-zeros entries of the second row of Λ_λ. Then the Springer correspondence is given by $E_{u_\lambda,1} = \phi_{\alpha,\beta}$.

Remark 2.2.2. This construction is a particular case of a more general construction on partitions. The bipartitions (α, β) constructed this way are 2-quotients of partitions λ which have trivial 2-core (see [33, 2.7] for more details).

Let us look at the case where $\mathbf{G} = \mathrm{Sp}_4(\bar{\mathbb{F}}_q)$ and q is odd. In the following table for each partition λ of 4 corresponding to a unipotent class $\mathcal{C}_\lambda$, we describe the blocks of the partitions, Λ_λ and the Springer correspondent $E_{u_\lambda,1}$ for $u_\lambda \in \mathcal{C}_\lambda$.

λ	blocks	Λ_λ	$E_{u_\lambda,1}$
(4)	$\{0\}, \{0\}, \{4\}$	$\begin{pmatrix} 0 & & 4 \\ & 1 & \end{pmatrix}$	$\phi_{2.-}$
(2^2)	$\{0\}, \{2\}, \{2\}$	$\begin{pmatrix} 1 & & 1 \\ & 1 & \end{pmatrix}$	$\phi_{1.1}$
$(1^2 2)$	$\{0\}, \{0\}, \{1,1\}, \{2\}$	$\begin{pmatrix} 0 & & 1 & & 1 \\ & 0 & & 0 & \end{pmatrix}$	$\phi_{1^2.-}$
(1^4)	$\{0\}, \{1,1\}, \{1,1\}$	$\begin{pmatrix} 0 & & 0 & & 0 \\ & 1 & & 1 & \end{pmatrix}$	$\phi_{-.1^2}$

- Assume $\mathbf{G} = G_2(\bar{\mathbb{F}}_q)$ and p is good for $\mathbf{G}$. Then $\mathbf{G}$ has 5 unipotent classes denoted as follows : 1 (the trivial class), A_1, $\tilde{A}_1$, $G_2(a_1)$ and G_2. The notation correspond to some Levi subgroup of a parabolic subgroup

attached to the class, see [8, 5.9.6] for more details. The following table taken from [8, §13.3] gives for each unipotent class $(u)_{\mathbf{G}}$ of $\mathbf{G}$, the group $A_{\mathbf{G}}(u)$ and the Springer correspondent $E_{u,\phi}$ for each irreducible character ϕ of $A_{\mathbf{G}}(u)$. The blank entry in the last column means that $E_{u,\phi} = 0$.

$(u)_{\mathbf{G}}$	$A_{\mathbf{G}}(u)$	$\mathrm{Irr}_{\mathbb{C}}(A_{\mathbf{G}}(u))$	$\mathrm{Irr}_{\mathbb{C}}(W)$
1	1	1	$\phi_{1,6}$
A_1	1	1	$\phi_{1,3''}$
$\tilde{A}_1$	1	1	$\phi_{2,2}$
$G_2(a_1)$	$\mathfrak{S}_3$	$\phi_3(=1)$	$\phi_{2,1}$
		ϕ_{12}	$\phi_{1,3'}$
		$\phi_{1^3}(=\varepsilon)$	
G_2	1	1	$\phi_{1,0}$

By a case-by-case verification, Lusztig noted that special representations of W have a specific form under the Springer correspondence.

Proposition 2.2.3 ([41, 13.1.1])**.** *Any special irreducible representation of W is of the form $E_{u,1}$ where u is a unipotent element of $\mathbf{G}$.*

This proposition and the facts that the converse is not true leads to the following definition :

Definition 2.2.4. Let u be a unipotent element of $\mathbf{G}$. We say that u is *special* if $E_{u,1}$ is a special irreducible representation of W. Since $E_{u,1}$ only depends on the $\mathbf{G}$-conjugacy class of u, we can say that the class $(u)_{\mathbf{G}}$ is special.

Example 2.2.5.

- Since every character is special in type A_n, every unipotent class is special.
- If $\mathbf{G} = \mathrm{Sp}_4(\bar{\mathbb{F}}_q)$, special characters of W are $\phi_{2.-}, \phi_{-.1^2}$ and $\phi_{1.1}$. So, according to Table 2.2.1, special unipotent classes of $\mathbf{G}$ are the classes corresponding to the partitions (4), (2^2) and (1^4).
- Amongst the five unipotent classes of $\mathbf{G} = G_2(\bar{\mathbb{F}}_q)$, three are special : 1, $G_2(a_1)$ and G_2.

2.2.2 Canonical quotient

Because each family contains exactly one special character, the Springer correspondence induces a bijection between special unipotent classes of $\mathbf{G}$ and families of $\mathrm{Irr}_{\mathbb{C}}(W)$. In [41], Lusztig attached to each special class a finite group closely related to the group $\Omega_{\mathcal{F}}$.

Definition 2.2.6. Let $u \in \mathbf{G}$ be a special unipotent element, and $b_u := \dim \mathcal{B}_u$ where $\mathcal{B}_u$ is the variety of Borel subgroups of $\mathbf{G}$ containing u. Let $\mathcal{S} \subset \mathrm{Irr}_{\mathbb{C}}(A_{\mathbf{G}}(u))$ be defined by

$$\mathcal{S} := \{\psi \in \mathrm{Irr}_{\mathbb{C}}(A_{\mathbf{G}}(u)) \mid \quad a_{E_{u,\psi}} = b_u\}.$$

The *canonical quotient* Ω_u is the largest quotient of $A_{\mathbf{G}}(u)$ through which every element of $\mathcal{S}$ factors. In other words,

$$\Omega_u := A_{\mathbf{G}}(u) / \bigcap_{\psi \in \mathcal{S}} \ker \psi.$$

Remark 2.2.7.

- We may use the notation $\Omega_u = \Omega_u(\mathbf{G})$ if there is an ambiguity on the underlying group.
- For simple groups, the groups defined above can be trivial, a product of copies of $\mathbb{Z}/2\mathbb{Z}$ or the symmetric groups $\mathfrak{S}_3, \mathfrak{S}_4$ and $\mathfrak{S}_5$.
- If $\mathcal{C}$ is an F-stable unipotent class of $\mathbf{G}$ and $u \in \mathcal{C}^F$, F acts naturally on $A_{\mathbf{G}}(u)$ so we can define $\bar{\mathcal{M}}(\Omega_u)$. According to [41, 13.1.3], $\bar{\mathcal{M}}(\Omega_u)$ only depends on $\mathcal{C}$.

Lusztig showed in [41, 13.1.3] that canonical quotients correspond to the finite groups which appear in the classification of unipotent characters.

Proposition 2.2.8. *Let u be a special unipotent element of $\mathbf{G}$, and $\mathcal{F}$ be the family of $\mathrm{Irr}_{\mathbb{C}}(W)$ containing the character $\phi_{u,1} \otimes \varepsilon$. Then, the triple $\Omega_u \subset \tilde{\Omega}_u \supset \Omega_u \cdot \gamma$ is isomorphic to $\Omega_{\mathcal{F}} \subset \tilde{\Omega}_{\mathcal{F}} \supset \Omega_{\mathcal{F}} \cdot \gamma$.*

Hence, there is a parametrisation of unipotent characters in terms of special unipotent classes.

Theorem 2.2.9. *Unipotent characters of G are parametrised by pairs (u, x) where $u \in G$ is a special unipotent element up to $\mathbf{G}$-conjugacy and $x \in \bar{\mathcal{M}}(\Omega_u)$.*

Example 2.2.10. Let $G = \mathrm{Sp}_4(q)$. The following table describes the parametrisation of unipotent characters of G in terms of special classes :

λ	$E_{u_\lambda,1} \otimes \varepsilon$	Ω_{u_λ}	(x, σ)	Unipotent character
1^4	$\phi_{2.-}$	1	$(1,1)$	$\chi_{2.-}$
2^2	$\phi_{1.1}$	$\mathfrak{S}_2$	$(1,1)$	$\chi_{1.1}$
			$(1,\varepsilon)$	$\chi_{-.2}$
			$(s_1,1)$	$\chi_{1^2.-}$
			(s_1,ε)	θ
4	$\phi_{-.1^2}$	1	$(1,1)$	$\chi_{-.1^2}$

2.2.3 Parametrisation of $\mathrm{Irr}_{\mathbb{C}}(G)$ by special classes

Definition 2.2.11. We say that $g \in \mathbf{G}$ is special if g has semi-simple part s, unipotent part u and u is special as an element of $C_{\mathbf{G}}(s)^\circ$.

Assume that $\mathbf{G}$ has connected center. Using 2.1.2 we can state a parametrisation of $\mathrm{Irr}_{\mathbb{C}}(G)$ in terms of special classes of $\mathbf{G}^*$. Recall from §1.3.3 that we have a partition :

$$\mathrm{Irr}_{\mathbb{C}}(G) = \bigsqcup \mathcal{E}(G, s)$$

where s runs over representatives of semi-simple conjugacy classes of G^*. Moreover, by the subsection 2.1.2 and the previous results, we have a partition of $\mathcal{E}(G,s)$ depending on families of the Weyl group W_s of $C_{\mathbf{G}^*}(s)$. Let g be a special element of G^* with Jordan decomposition $g = su$ and let $\mathcal{F}$ be the family of W_s corresponding to $u \in C_{\mathbf{G}^*}(s)$. Then we denote $\mathrm{Irr}_{\mathbb{C}}(G)_g := \mathcal{E}(G,s)_{\mathcal{F}}$. By [14, 2.3, 13.15], since $Z(\mathbf{G})$ is connected, $C_{\mathbf{G}^*}(s)$ is a connected reductive group for any semi-simple element s of $\mathbf{G}^*$. So by 1.2.8 4., semi-simple elements of G^* conjugate under $\mathbf{G}^*$ are conjugate under G^*. Hence, we have a partition :

$$\mathrm{Irr}_{\mathbb{C}}(G) = \bigsqcup_g \mathrm{Irr}_{\mathbb{C}}(G)_g$$

where g runs over representatives of F-stable special classes of $\mathbf{G}^*$. For each special element g of $\mathbf{G}^*$ with Jordan decomposition $g = su$, let us denote by Ω_g the canonical quotient of $A_{C_{\mathbf{G}^*}(s)}(u)$. Then, $\mathrm{Irr}_{\mathbb{C}}(G)_g$ is parametrised by $\bar{\mathcal{M}}(\Omega_g)$.

2.3 ℓ-special classes and ℓ-canonical quotient

As in §1.1.1, let ℓ be a prime number different from p, and let $(K, \mathcal{O}, k)$ be an ℓ-modular system such that K and k are big enough for G. We recall that

$$B_1 := \bigcup_t \mathcal{E}(G,t),$$

where t runs over semi-simple ℓ-elements of G^*, is a union of blocks. The irreducible modular representations lying in B_1 will be called *unipotent*. In this section, we will stick to this problem : trying to use Lusztig's parametrisation theorem 2.2.9 to count unipotent modular representations. We start from the following observation : the cardinal of $\mathrm{Irr}_k(B_1)$ equals the cardinal of any basic set for B_1. Moreover, Geck and Hiss proved that, under some conditions on ℓ, unipotent characters form a basic set for B_1. Let us give more details on this result. We first need to define what conditions on ℓ are needed.

Definition 2.3.1. Let ℓ be a prime number.

- We say that ℓ is *very good* for G if ℓ is good for $\mathbf{G}$ and does not divides $|(Z(\mathbf{G})/Z(\mathbf{G})^\circ)^F|$.

When ℓ is very good for G, Geck and Hiss have shown that unipotent characters form a basic set for unipotent blocks. More generally :

Theorem 2.3.2 ([27, 21]). *Assume that ℓ is very good for G and $\ell \neq p$. Let s be a semi-simple ℓ'-element of G^*. Then $\mathcal{E}(G, s)$ is a basic set for B_s.*

This theorem implies that Lusztig's parametrisation can be used to count unipotent modular representations. However when ℓ is not very good for G, there are cases where the number of unipotent modular representations differs from the number of unipotent characters. The following examples show that unipotent characters are not enough to provide a basic set or that there can be more unipotent characters than unipotent modular representations.

Example 2.3.3.

- Assume $G = \mathrm{SL}_2(q)$ with q odd and $\ell = 2$ (ℓ is good but not very good for G). According to [3, 9.4.9], G has 2 unipotent characters but there are 3 unipotent modular representations.
- Assume $G = G_2(q)$ and $\ell = 2$ (ℓ is bad for G). The decomposition matrix of G was computed in [32]. In that case unipotent characters form a generating set but do not form a basic set : there are actually 10 unipotent characters but 9 unipotent modular representations.

With the goal of adapting Lusztig's results to the positive characteristic framework, we will introduce "modular versions" of the notions of special classes and canonical quotient. We conjecture that these are the good objects to count unipotent modular representations.

Truncated induction. Recall that for $\phi \in \mathrm{Irr}_{\mathbb{C}}(W)$, a_ϕ is the valuation of the fake degree of ϕ.

Definition 2.3.4. Let W' be a subgroup of W generated by reflections and $\phi \in \mathrm{Irr}(W')$. We define the *truncated induction* (or *j-induction*) $j_{W'}^{W}(\phi)$ of ϕ as the unique irreducible summand ψ of $\mathrm{Ind}_{W'}^{W}(\phi)$ such that $a_\psi = a_\phi$.

For the existence and the uniqueness see [8, 11.2.1,11.2.5]. This construction comes originally from Macdonald and has been generalised by Lusztig and Spaltenstein. A more natural construction of the truncated induction can be found in [8, §11.3]. This operation can also be used to provide an alternative construction of the families of $\mathrm{Irr}_{\mathbb{C}}(W)$, see [8, §12.4] for example. The j-induction can be used to construct special representations of W from special representations of parabolic subgroups, more precisely :

Theorem 2.3.5 ([39]). *Let W' be a parabolic subgroup of W and let ψ be a special character of W'. Then $\phi := j_{W'}^{W}(\psi)$ is a special irreducible character*

of W and $a_\phi = a_\psi$. Conversely, if ϕ is a special character of W then either ϕ or $\phi \otimes \varepsilon_W$ is of the form $j_{W'}^W(\psi)$ for some parabolic subgroup W' of W and some special character ψ of W'.

Example 2.3.6.

- Suppose $W = \mathfrak{S}_{n+1}$ is of type A_n. Let $\alpha = (\alpha_1, \dots, \alpha_r)$ be a partition of $n+1$ and $\phi_\alpha \in \mathrm{Irr}_{\mathbb{C}}(W)$. Let $\alpha^* = (\alpha_1^*, \dots, \alpha_s^*)$ be the dual partition of α and let
$$W' = \mathfrak{S}_{\alpha_1^*} \times \cdots \times \mathfrak{S}_{\alpha_s^*}.$$
Then W' can be naturally viewed as a parabolic subgroup of W. Then $\phi_\alpha = j_{W'}^W(\varepsilon_{W'})$ where $\varepsilon_{W'}$ is the sign character of W'.
- Suppose W is of type G_2 and W' a parabolic subgroup of type A_1. Then $\phi_{1,0} = j_{W'}^W 1$, and $\phi_{2,1} = j_{W'}^W \varepsilon_{W'}$.

ℓ-special classes. Following our strategy of mimicking Lusztig's parametrisation in the context of modular representations, we will introduce a new set of unipotent classes depending on ℓ. In [41, §13.1], Lusztig defined when p is good a map between special classes of $\mathbf{G}^*$ and unipotent classes of $\mathbf{G}$ as follows : let $g = su$ be a special element of $\mathbf{G}^*$, we denote by W_s° the Weyl group of $C_{\mathbf{G}^*}(s)^\circ$. The Springer correspondence for $C_{\mathbf{G}^*}(s)$ affords a special irreducible character $E_{u,1} \in \mathrm{Irr}_{\mathbb{C}}(W_s^\circ)$. By applying the truncated induction to $E_{u,1}$, we get a character $\chi := j_{W_s^\circ}^{W^*} E_{u,1}$ of W^* which can be seen as a character of W. According to [41, §13.3], there is a well-defined unipotent element v of $\mathbf{G}$ such that $\chi = E_{v,1}$. Then, the image of g is v. We summarise the process in the following definition :

Definition 2.3.7. We denote by Φ the map defined as follows :

$$\begin{tikzcd}
\text{special } \mathbf{G}^*\text{-classes} \arrow[rr, "\Phi"] \arrow[d, "Springer"] & & \text{unipotent } \mathbf{G}\text{-classes} \\
\mathrm{Irr}_{\mathbb{C}}(W_s^\circ) \arrow[r, "j_{W_s^\circ}^{W^*}"] & \mathrm{Irr}_{\mathbb{C}}(W^*) \arrow[r, "\sim"] & \mathrm{Irr}_{\mathbb{C}}(W) \arrow[u, "Springer"']
\end{tikzcd}.$$

We can now introduce the "modular version" of the special unipotent classes :

Definition 2.3.8. A unipotent element u of $\mathbf{G}$ is *ℓ-special* if there exists a special element $g \in \mathbf{G}^*$ meeting the following conditions :

- $\Phi((g)_{\mathbf{G}^*}) = (u)_{\mathbf{G}}$,

- the semi-simple part of g is an isolated ℓ-element of $\mathbf{G}^*$.

Remark 2.3.9. From the definition, we can deduce the following information :

- Any special class is ℓ-special. Indeed, let $(u)_{\mathbf{G}}$ be a special unipotent class, by definition $E_{u,1}$ is a special character of W. Seen as a character of W^*, $E_{u,1}$ is still special and is the Springer correspondent of a special unipotent class $(v)_{\mathbf{G}^*}$ of $\mathbf{G}^*$. Hence, $\Phi((v)_{\mathbf{G}^*}) = (u)_{\mathbf{G}}$ and any special unipotent class is ℓ-special.
- If ℓ is good for $\mathbf{G}$ there are no non-trivial isolated semi-simple ℓ-element so ℓ-special unipotent classes are exactly the special ones.

Image of the Springer correspondence. Let S_W be the set of special characters of W and let $\bar{S}_W$ be the image of the injective map $(u)_{\mathbf{G}} \mapsto E_{u,1}$ from the set of unipotent classes of $\mathbf{G}$ to $\mathrm{Irr}_{\mathbb{C}}(W)$, we have $S_W \subset \bar{S}_W$ by Proposition 2.2.3. We can describe the set $\bar{S}_W$ in terms of j-induction of characters of maximal subgroups of W. More precisely, let us fix a set of roots Φ and a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. For $\alpha \in \Phi$, let $s_\alpha \in W$ be the corresponding reflection. We have $W = \langle s_{\alpha_i} \mid 1 \leq i \leq n \rangle$. Let $\alpha_0 = \sum n_{\alpha_i} \alpha_i$ be the highest short root. For $i \in \{0, \ldots, n\}$, let

$$W_i = \langle s_{\alpha_j} \mid 0 \leq j \leq n,\ j \neq i \rangle.$$

Assume that p is good for $\mathbf{G}$. It was shown by Shoji [54, 55], Springer [59] and Alvis–Lusztig [1] that $\bar{S}_W$ is exactly the set of irreducible characters of W coming from j-induction of special characters of the groups W_i. Moreover, the classification of quasi-isolated semi-simple elements by Bonnafé in [2] shows that every group of the form W_i is the Weyl group of the centraliser of an isolated semi-simple element s_i of $\mathbf{G}^*$ whose order is equal to n_{α_i}. Those facts motivate the following definition.

Definition 2.3.10. Let ℓ be a prime number, $E \in \mathrm{Irr}_{\mathbb{C}}(W)$. We say that E is *ℓ-special* if there exists $i \in \{1, \ldots, n\}$ such that :

- E is the j-induction of a special character of W_i,
- n_{α_i} is a power of ℓ.

The following proposition comes from the definition

Proposition 2.3.11. *Let $(u)_{\mathbf{G}}$ be a unipotent class of $\mathbf{G}$. The character $E_{u,1}$ is ℓ-special if and only if $(u)_{\mathbf{G}}$ is ℓ-special.*

ℓ-canonical quotient. We construct an analogue of the group Ω_u in characteristic ℓ. The definition involves characters of projective indecomposables $kA_{\mathbf{G}}(u)$-modules instead of complex irreducible characters of $A_{\mathbf{G}}(u)$.

Definition 2.3.12.

1) Let u be an ℓ-special unipotent element of $\mathbf{G}$. Let P be a projective indecomposable $kA_{\mathbf{G}}(u)$-module and Ψ be its character. We define a_Ψ as :
$$a_\Psi := \min a_{E_{u,\psi}}$$
where ψ runs over the ordinary irreducible constituents of Ψ such that $E_{u,\psi} \neq 0$. Let $\mathcal{S}_\ell$ be the set of indecomposable projectives Ψ such that a_Ψ is maximal. We define the *ℓ-canonical quotient* Ω_u^ℓ as the largest quotient of $A_{\mathbf{G}}(u)$ such that every $\Psi \in \mathcal{S}_\ell$ factors through Ω_u^ℓ.
2) Let Ω be a finite group on which F acts. Recall that $\tilde{\Omega}$ is the semidirect product of Ω with the infinite cyclic group generated by γ such that $\gamma x \gamma^{-1} = F(x)$ if $x \in \Omega$. We denote by $\bar{\mathcal{M}}_\ell(\Omega)$ the set of $\tilde{\Omega}$-conjugacy class of pairs (x, ϕ) where x is an element of $\Omega.\gamma$ and $\phi \in \mathrm{Irr}_k(C_\Omega(x))$.

Remark 2.3.13. If F acts trivially on Ω, $\tilde{\Omega}$ is the direct product of Ω with the infinite cyclic group generated by γ and $\bar{\mathcal{M}}_\ell(\Omega)$ is in bijection with the set $\mathcal{M}_\ell(\Omega)$ consisting of Ω-conjugacy classes of pairs (x, ϕ) where $x \in \Omega$ and $\phi \in \mathrm{Irr}_k(C_\Omega(x))$.

If $(u)_{\mathbf{G}}$ is an ℓ-special F-stable unipotent class, we denote by $\alpha_{\ell,u}$ the cardinal of $\bar{\mathcal{M}}_\ell(\Omega_u^\ell)$ and by α_ℓ the number of pairs $((u)_{\mathbf{G}}, x)$ where $(u)_{\mathbf{G}}$ is an F-stable ℓ-special unipotent class of G and $x \in \bar{\mathcal{M}}_\ell(\Omega_u^\ell)$ (we will use the notation $\alpha_\ell(G)$ if there is an ambiguity on the underlying group). We conjecture the following :

Conjecture 2.3.14. *Assume that p is good for $\mathbf{G}$. Then, the number of irreducible unipotent ℓ-modular representations of G equals α_ℓ.*

Proposition 2.3.15. *The conjecture holds whenever ℓ is very good for G.*

Proof. It is enough to show that for any unipotent class $(u)_{\mathbf{G}}$, $A_{\mathbf{G}}(u)$ is an ℓ'-group. Indeed, in that case $\mathcal{S}_\ell = \mathcal{S}$ so $\Omega_u^\ell = \Omega_u$ and $|\bar{\mathcal{M}}_\ell(\Omega_u^\ell)| = |\tilde{\mathcal{M}}(\Omega_u)|$ so we can conclude by 2.3.2 and 2.2.9. Let us begin by the following remark. Let $\pi : \mathbf{G}_1 \to \mathbf{G}_2$ be a surjective morphism between two algebraic groups with central kernel. Let $u_1 \in \mathbf{G}$ be unipotent and $u_2 = \pi(u_1)$. Then, since u_1 is unipotent, the restriction of π to $C_{\mathbf{G}_1}(u_1)$ is a surjective morphism $C_{\mathbf{G}_1}(u_1) \twoheadrightarrow C_{\mathbf{G}_2}(u_2)$. More precisely, if $\pi(g) \in C_{\mathbf{G}_2}(u_2)$ then $gu_1g = u_1z$

where $z \in \ker \pi$. The element z is central, then semi-simple so has to be trivial. Then, π induces a surjective morphism $A_{\mathbf{G}_1}(u_1) \twoheadrightarrow A_{\mathbf{G}_2}(u_2)$.

- Assume first that $\mathbf{G}$ is simple and has type A, B, C or D. Let $\pi_{sc} : \mathbf{G}_{sc} \to \mathbf{G}$ be a simply connected covering of $\mathbf{G}$, let u_{sc} be a unipotent element of $\mathbf{G}_{sc}$ whose image in $\mathbf{G}$ is u. Since we have a surjective morphism $A_{\mathbf{G}_{sc}}(u_{sc}) \twoheadrightarrow A_{\mathbf{G}}(u)$, we can assume that $\mathbf{G}$ is simply connected. If $\mathbf{G}$ has type A_n, $|A_{\mathbf{G}}(u)|$ divides $|Z(\mathbf{G})/Z(\mathbf{G})^\circ|$ according to [42, 10.3] so $\ell \nmid |A_{\mathbf{G}}(u)|$. If $\mathbf{G}$ is of type B, C or D, 2 is the only prime number which does not satisfies the assumptions and $A_{\mathbf{G}}(u)$ is a 2-group (see [37, 3.1, 3.3.5]).
- Assume that $\mathbf{G}$ is a simple exceptional group. The case of exceptional adjoint groups can be checked case-by-case in [37, §22]. Let $\mathbf{G}_{ad}$ be the adjoint quotient of $\mathbf{G}$ and u_{ad} be the image of u in $\mathbf{G}$. The kernel of the surjective morphism $A_{\mathbf{G}}(u) \to A_{\mathbf{G}_{ad}}(u_{ad})$ is $Z(\mathbf{G})/(Z(\mathbf{G}) \cap C_{\mathbf{G}}(u)^\circ)$ and $Z(\mathbf{G})$ is an ℓ'-group so $A_{\mathbf{G}}(u)$ is an ℓ'-group.
- Let us consider the general case. The restriction of the morphism $\pi : \mathbf{G} \to \mathbf{G}/Z(\mathbf{G})^\circ$ to $C_{\mathbf{G}}(u)$ induces an isomorphism $A_{\mathbf{G}}(u) \simeq A_{\mathbf{G}/Z(\mathbf{G})^\circ}(\pi(u))$. Indeed, we know that the morphism is surjective by the remark above and the kernel of the morphism is trivial since $Z(\mathbf{G})^\circ \subset C_{\mathbf{G}}(u)^\circ$. Therefore, replacing $\mathbf{G}$ by $\mathbf{G}/Z(\mathbf{G})^\circ$ if necessary, we can assume that $\mathbf{G}$ is semi-simple. In particular, $Z(\mathbf{G})$ is a finite ℓ'-group. Using the same arguments as above, it is enough to show that $A_{\mathbf{G}_{ad}}(u_{ad})$ is an ℓ'-group. But since $\mathbf{G}_{ad}$ is a direct product of simple groups, $A_{\mathbf{G}_{ad}}(u_{ad})$ is a direct product of ℓ'-groups so is an ℓ'-group and we are done.

■

The following theorem summarise the results that will be shown in the rest of this chapter.

Theorem 2.3.16. *Let $\mathbf{G}$ be a connected reductive group defined over $\mathbb{F}_q$ where q is the power of a good prime number p. Assume we are in one of the following cases.*

- *$\mathbf{G}$ is $\mathrm{SL}_n(\overline{\mathbb{F}}_q)$ (hence, G is either $\mathrm{SL}_n(q)$ or $\mathrm{SU}_n(q)$) for any prime number $\ell \leq n$.*
- *$\mathbf{G}$ is simple of type B, C or D, for $\ell = 2$.*
- *$\mathbf{G}$ is a simple exceptional group of adjoint type for any bad prime number ℓ.*

Then, Conjecture 2.3.14 holds for $\mathbf{G}$.

Remark 2.3.17. Our Theorem includes the twisted groups ${}^2D_n$, ${}^3D_4$ for $\ell = 2$ and ${}^2E_6$ for $\ell = 2, 3$. Suzuki and Ree groups are not included. This is due to the fact that we do not consider Frobenius roots (or Steinberg morphisms) *i.e.* morphisms of algebraic groups whose power is a Frobenius morphism.

- Section 2.4 contains the proof of Theorem 2.3.16 for $\mathrm{SL}_n(q)$ and $\mathrm{SU}_n(q)$ for any prime number $\ell \leq n$.
- Section 2.5 contains the proof of Theorem 2.3.16 for simple groups of type B, C or D when $\ell = 2$. The proof is valid for ${}^2D_n$ and ${}^3D_4$.
- Section 2.6 contains the proof of Theorem 2.3.16 for adjoint exceptional groups and any prime number ℓ. The proof is valid for ${}^2E_6$.

2.4 Type A_n

Throughout this section, $\tilde{\mathbf{G}}$ will denote $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$, F will be a Frobenius endomorphism of $\tilde{\mathbf{G}}$ and, depending on F, $\tilde{G} := \tilde{\mathbf{G}}^F$ will be either $\mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q)$. Similarly, $\mathbf{G}$ will denote $\mathrm{SL}_n(\overline{\mathbb{F}}_q)$ and G will be either $\mathrm{SL}_n(q)$ or $\mathrm{SU}_n(q)$. Note that according to Theorem 2.3.15, the conjecture holds for $\tilde{G}$. Actually, Dipper and Geck have shown a stronger result.

Theorem 2.4.1 ([16, 20])**.** *The set $\mathcal{U}(\tilde{G})$ is a unitriangular basic set for unipotent blocks of $\tilde{G}$.*

Restriction to G. Kleshchev–Tiep [36] and Denoncin [13] showed the existence of a unitriangular basic set for unipotent blocks of G. We will briefly explain how they proceeded. Since G is of type A, any unipotent class is special and F-stable so we can directly work with unipotent classes of $\mathbf{G}$ which are parametrised by partitions of n via the Jordan normal form. For any partition λ, we denote by u_λ a representative of the corresponding unipotent class in $\mathbf{G}$ and by ρ_λ the unipotent character corresponding to $(u_\lambda)_{\tilde{\mathbf{G}}}$.

Example 2.4.2.

1. The class $(u_{1^n})_{\tilde{\mathbf{G}}}$ is the class containing the unit element and ρ_{1^n} is the Steinberg representation.
2. The class $(u_n)_{\tilde{\mathbf{G}}}$ is the regular class, *i.e.* the unique class of maximal dimension of $\tilde{\mathbf{G}}$. The representation ρ_n is the trivial one.

Even if unipotent characters form a unitriangular basic set for the union of unipotent blocks $B_1(\tilde{G})$ of $\tilde{G}$, irreducible constituents of their restriction to G do not form a basic set for $B_1(G)$. That is why, starting from $\mathcal{U}(G)$, we need to construct another basic set for $B_1(\tilde{G})$. Let $\lambda := (\lambda_1, \ldots, \lambda_r)$ be a partition of n, $m_\lambda := \gcd(\lambda_1, \ldots, \lambda_r)_{p'}$ and $m_{\lambda,F} := \gcd(|Z(\tilde{G})|, \lambda_1, \ldots, \lambda_r)_{p'}$. If the number of irreducible constituents of $\mathrm{Res}_G^{\tilde{G}}(\rho_\lambda)$ is different from $(m_{\lambda,F})_\ell$, we replace ρ_λ by another character as follows. Let us choose a semi-simple ℓ-element s as in [13, 4.4]. More precisely, if ω is a primitive $(m_{\lambda,F})_\ell$-root of unity, we choose s such that the set of its eigenvalues is $\{\omega^i \mid 0 \leq i \leq (m_{\lambda,F})_\ell - 1\}$, each eigenvalue having multiplicity $d_\lambda := \frac{n}{(m_{\lambda,F})_\ell}$. Using the Jordan decomposition and the fact that $C_{\tilde{\mathbf{G}}}(s)$ is isomorphic to $(m_{\lambda,F})_\ell$ copies of $\mathrm{GL}_{d_\lambda}(\bar{\mathbb{F}}_q)$, we can parametrise $\mathcal{E}(\tilde{G}, s)$ by collections $\{\delta^i \mid 0 \leq i \leq (m_{\lambda,F})_\ell - 1\}$ of partitions of d_λ. If δ is such a collection we denote by $\rho_{s,\delta}$ the corresponding character. Let us denote by $\frac{\lambda}{(m_{\lambda,F})_\ell}$ the partition $(\frac{\lambda_1}{(m_{\lambda,F})_\ell}, \ldots, \frac{\lambda_r}{(m_{\lambda,F})_\ell})$ of $\frac{n}{(m_{\lambda,F})_\ell}$. Then we replace ρ_λ, by $\rho_{s,\delta}$ where $\delta^i := \frac{\lambda}{(m_{\lambda,F})_\ell}, 0 \leq i \leq n$.

Applying the above procedure for each element $\mathcal{U}(G)$, we obtain a set $\mathcal{B}_{\tilde{G}}$ of irreducible characters of $\tilde{G}$ which is a unitriangular basic set for $B_1(\tilde{G})$. Let $\mathcal{B}_G$ be the set consisting of irreducible constituents of the restrictions of characters of $\mathcal{B}_{\tilde{G}}$ to G. Using Clifford Theory, Kleshchev–Tiep and Denoncin showed that $\mathcal{B}_G$ is a unitriangular basic set for $B_1(G)$. Moreover, for any partition λ of n, the restriction to G of the character of $\mathcal{B}_{\tilde{G}}$ corresponding to λ has $(m_{\lambda,F})_\ell$ irreducible constituents. Hence, to show that Conjecture 2.3.14 holds for G, it is enough to show that for any partition λ of n, $\alpha_{\ell,u_\lambda}(G) = (m_{\lambda,F})_\ell$.

ℓ-canonical quotient for G. If λ is a partition of n, $A_{\mathbf{G}}(u_\lambda)$ is a cyclic group of order m_λ [42, 10.3]. Let us now compute $\Omega^\ell_{u_\lambda}$. With the notation of 2.3.12, recall that $\Omega^\ell_{u_\lambda}$ is the smallest quotient of $A_{\mathbf{G}}(u_\lambda)$ through which every projective character in $\mathcal{S}_\ell$ factors. The only representation of $A_{\mathbf{G}}(u_\lambda)$ whose Springer correspondent is non-zero is the trivial representation. Using the fact that the decomposition of the projective indecomposable into irreducible is given by the transpose of the decomposition map, projective characters in $\mathcal{S}_\ell$ are sum of the irreducible characters whose image by the decomposition map are trivial. Since $A_{\mathbf{G}}(u_\lambda) \simeq \mathbb{Z}/(m_\lambda)_\ell\mathbb{Z} \times \mathbb{Z}/(m_\lambda)_{\ell'}\mathbb{Z}$, the only projective representation in $\mathcal{S}_\ell$ is $k(\mathbb{Z}/(m_\lambda)_\ell\mathbb{Z}) \otimes_k 1_{\mathbb{Z}/(m_\lambda)_{\ell'}\mathbb{Z}}$ whose kernel is the cyclic subgroup of $A_{\mathbf{G}}(u_\lambda)$ of order $(m_\lambda)_{\ell'}$. Hence, $\Omega^\ell_{u_\lambda}$ is a cyclic group of order

$(m_\lambda)_\ell$, in particular it is an ℓ-group so for any $x \in \tilde{\Omega}_u^\ell$, $|\mathrm{Irr}_k(C_{\Omega_{u_\lambda}^\ell}(x))| = 1$. Hence, $|\bar{\mathcal{M}}_\ell(\Omega_{u_\lambda}^\ell)|$ is the number of F-classes of $\Omega_{u_\lambda}^\ell$, that is $(m_\lambda)_\ell$. Indeed, F acts on $\Omega_{u_\lambda}^\ell$ by multiplication by εq where $\varepsilon = 1$ if $\tilde{G} = \mathrm{GL}_n(q)$ and $\varepsilon = -1$ if $\tilde{G} = \mathrm{GU}_n(q)$. The F-conjugacy classes of $\Omega_{u_\lambda}^\ell$ are of the form $x + (q+\varepsilon).\Omega_{u_\lambda}^\ell$ where $x \in \Omega_{u_\lambda}^\ell$. Hence, since $q + \varepsilon = |Z(\tilde{G})|$, the number of F-conjugacy classes of is $\frac{|\Omega_{u_\lambda}^\ell|}{|(q+\varepsilon).\Omega_{u_\lambda}^\ell|} = (m_{\lambda,F})_\ell$. Finally, as conjectured, the number of unipotent modular representations of G is $\alpha_\ell(G)$.

Example 2.4.3. Assume $G = \mathrm{SL}_2(q)$, $\tilde{G} = \mathrm{GL}_2(q)$ and $\ell = 2$. Let $\tilde{T}$ (resp. T) be the maximal torus consisting of diagonal matrices of $\tilde{G}$ (resp. G). Since $\tilde{\chi}_2$ and $\tilde{\chi}_{1^2}$ are respectively the trivial and the Steinberg characters, their restriction to G remain irreducible. But $m_2 = \gcd(2, q-1)_2 = 2$ so we should replace the Steinberg character by an irreducible character lying in $\mathcal{E}(\tilde{G}, s)$ where

$$s = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Since $C_{\tilde{G}}(s) = \tilde{T}$, $\mathcal{E}(\tilde{G}, s)$ contains a unique character that we will denote by $\tilde{\chi}_s$. Moreover, $\tilde{\chi}_s = R_{\tilde{T}}^{\tilde{G}}(\tilde{\theta})$ where $\tilde{\theta}$ is a character of order 2 of $\tilde{T}$ whose restriction θ to T is not trivial. According to [7, 15.15], $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}_s) = R_T^G(\theta)$. Moreover, by [14, §15.9], $R_T^G(\theta)$ has two irreducible constituents that we will denote by χ_s^+ and χ_s^-. Hence, $\{\chi_{1^2}, \chi_s^+, \chi_s^-\}$ form a unitriangular basic set for the unipotent blocks of G and we can view χ_{1^2} as "attached" to u_{1^2} and $\{\chi_s^+, \chi_s^-\}$ as "attached" to u_2.

2.5 Classical types

In this section, unless otherwise stated, $\mathbf{G}$ will be a connected reductive group defined over $\mathbb{F}_q$ with q odd whose simple components are of type B, C or D. In that case $Z(\mathbf{G})/Z(\mathbf{G})^\circ$ is a 2-group so the only prime which is not very good for G is 2. Hence, we suppose $\ell = 2$.

A remark on the groups Ω_u^2. Since 2 is the only bad prime of G, every unipotent class of $\mathbf{G}$ is 2-special (see Section 1.2.3). Let $\mathcal{C}$ be an F-stable unipotent class of $\mathbf{G}$ and $u \in \mathcal{C}^F$. The following lemma gathers information about $A_{\mathbf{G}}(u)$.

Lemma 2.5.1. *The group $A_{\mathbf{G}}(u)$ is a 2-group. Moreover if $\mathbf{G}$ is a simple group which is neither spin or half-spin, $A_{\mathbf{G}}(u)$ is isomorphic to a product of copies of $\mathbb{Z}/2\mathbb{Z}$.*

Proof. The fact that $A_{\mathbf{G}}(u)$ is a 2-group has been shown in the proof of 2.3.15 so we just have to show the second assumption. The results in [37, 3.1, 3.3.5] show that $A_{\mathbf{G}}(u)$ is of the form $(\mathbb{Z}/2\mathbb{Z})^k$ if $\mathbf{G}$ is a simple orthogonal or a symplectic group. The only groups left are the adjoint groups of type C and D. If $\mathbf{G}$ is a simple adjoint group of type C_n (resp. D_n), then $\mathbf{G}$ is a quotient of a symplectic group (resp. simple orthogonal group). Therefore, $A_{\mathbf{G}}(u)$ is the quotient of a group of the form $(\mathbb{Z}/2\mathbb{Z})^k$ and the conclusion follows. ■

The group $A_{\mathbf{G}}(u)$ being a 2-group, the only indecomposable projective $kA_{\mathbf{G}}(u)$-module (resp. irreducible $kA_{\mathbf{G}}(u)$-module) is the regular representation (resp. the trivial representation). This shows that $\Omega_u^2 = A_{\mathbf{G}}(u)$ so $\bar{\mathcal{M}}_2(\Omega_u^2)$ corresponds bijectively to the set of F-conjugacy classes of $A_{\mathbf{G}}(u)$. By 1.2.8, the F-conjugacy classes of $A_{\mathbf{G}}(u)$ parametrise the G-orbits in $(u)_{\mathbf{G}}^F$. Summing over all unipotent $\mathbf{G}$-conjugacy classes, we have that α_2 is the number of unipotent classes of G. Hence, showing that the conjecture holds for G is equivalent to showing the following result.

Theorem 2.5.2. *Assume that $\mathbf{G}/Z(\mathbf{G})^\circ$ is a simple group of type B, C or D. The number of unipotent modular representations of G for $\ell = 2$ equals the number of unipotent classes of G.*

If $\mathbf{G}$ has connected center, that proposition was shown by Geck in [22]. The remainder of this section will be devoted to showing that we can generalise this result to simple classical groups with non-connected center.

Preliminary results. From now on, for a semi-simple element of odd order of G^*, we will denote by m_s the number of irreducible modular representations lying in B_s (or $m_s(G)$ if we need to specify the underlying group G). In [22], Geck showed that for $\mathbf{G}$ with connected center and simple components of classical type, $m_1(G)$ is equal to the number of unipotent classes of G. We first recall that the centraliser in $\mathbf{G}$ of a semi-simple element of odd order is a rational Levi subgroup of $\mathbf{G}$.

Lemma 2.5.3. *Let $\mathbf{G}$ be a connected reductive group such that $Z(\mathbf{G})/Z(\mathbf{G})^\circ$ is a 2-group.*

(1) *For any Levi subgroup* $\mathbf{L}$ *of* $\mathbf{G}$, $Z(\mathbf{L})/Z(\mathbf{L})^\circ$ *is a* 2*-group.*

(2) *If* s *is a semi-simple element of odd order of* $\mathbf{G}^*$, *then* $C_{\mathbf{G}^*}(s)$ *is connected. Moreover, if the order of* s *is divisible by good primes only then* $C_{\mathbf{G}^*}(s)$ *is a rational Levi subgroup of* $\mathbf{G}$.

Proof. Let X be the group of characters of a rational torus $\mathbf{T}$ contained in $\mathbf{L}$, Φ be the corresponding set of roots of $\mathbf{G}$ and $\Phi_{\mathbf{L}} \subset \Phi$ be the set of roots of $\mathbf{L}$. Given A a subgroup of X and $\mathbf{S}$ a subtorus of $\mathbf{T}$ we define :

$$A^\perp := \{t \in \mathbf{T} \mid \chi(t) = 0 \ \forall \chi \in A\},$$
$$\mathbf{S}^\perp := \{\chi \in X \mid \chi(t) = 0 \ \forall t \in \mathbf{S}\}.$$

Note that $A \subset A^{\perp\perp}$ but there is no equality in general. The following properties can be found for example in [14, 0.24, 13.14, 13.15].

(a) $Z(\mathbf{G})/Z(\mathbf{G})^\circ$ is isomorphic to the torsion group of $X/\mathbb{Z}\Phi^{\perp\perp}$.

(b) $A^{\perp\perp}/A$ is the p-torsion subgroup of X/A.

(c) If $(\mathbf{G}^*, \mathbf{T}^*)$ is dual to $(\mathbf{G}, \mathbf{T})$, then for any $s \in \mathbf{T}^*$ the group $C_{\mathbf{G}^*}(s)/C_{\mathbf{G}^*}(s)^\circ$ is isomorphic to a subgroup of $Z(\mathbf{G})/Z(\mathbf{G})^\circ$.

(d) The exponent of $C_{\mathbf{G}^*}(s)/C_{\mathbf{G}^*}(s)^\circ$ divides the order of s.

By (a), the group $Z(\mathbf{G})/Z(\mathbf{G})^\circ$ is a 2-group if and only if $X/\mathbb{Z}\Phi^{\perp\perp}$ has only 2-torsion. This is equivalent to $X/\mathbb{Z}\Phi$ having only 2 and p-torsion by observing that

$$X/Z\Phi^{\perp\perp} \simeq (X/\mathbb{Z}\Phi)/(\mathbb{Z}\Phi^{\perp\perp}/\mathbb{Z}\Phi)$$

and then applying (b) with $A = \mathbb{Z}\Phi$. Moreover, $\mathbb{Z}\Phi_{\mathbf{L}}$ is a direct summand of $\mathbb{Z}\Phi$. Indeed, we can assume that we are in the setting of §1.2.1 so that Δ is a basis of Φ and $\mathbb{Z}\Phi_L$ is generated by a subset Δ_I of Δ. Therefore, $X/\mathbb{Z}\Phi_{\mathbf{L}}$ has only 2-torsion and p-torsion. Indeed, suppose $X/\mathbb{Z}\Phi_{\mathbf{L}}$ has m-torsion where m is an integer prime to 2 and p. Then there exists an element $\chi \in X$ such that $m\chi \in \mathbb{Z}\Phi_{\mathbf{L}}$, but that implies that $\chi \in \mathbb{Z}\Phi$ because $X/\mathbb{Z}\Phi$ has only 2 and p-torsion. The fact that $\chi \in \mathbb{Z}\Phi$ and $m\chi \in \mathbb{Z}\Phi_{\mathbf{L}}$ forces $\chi \in \mathbb{Z}\Phi_L$ since $\mathbb{Z}\Phi_{\mathbf{L}}$ is a direct summand of the free $\mathbb{Z}$-module $\mathbb{Z}\Phi$. Therefore $X/\mathbb{Z}\Phi_{\mathbf{L}}$ has no m-torsion. By (b), this is equivalent to $Z(\mathbf{L})/Z(\mathbf{L})^\circ$ being a 2-group. That proves (1).

By (c), $C_{\mathbf{G}^*}(s)/C_{\mathbf{G}^*}(s)^\circ$ is isomorphic to a subgroup of $Z(\mathbf{G})/Z(\mathbf{G})^\circ$, hence is a 2-group. But the exponent of this group divides the order of s by (d) so $C_{\mathbf{G}^*}(s)/C_{\mathbf{G}^*}(s)^\circ$ is trivial and $C_{\mathbf{G}^*}(s)$ is connected. Finally, the connectedness of $C_{\mathbf{G}^*}(s)$ and Proposition 2.1 of [27] implies that $C_{\mathbf{G}^*}(s)$ is a rational Levi subgroup of $\mathbf{G}^*$ which gives (2). ■

Corollary 2.5.4. *If $\mathbf{G}$ is a reductive group with simple components of type B, C or D and s is a semi-simple element of odd order of G, then $C_{\mathbf{G}}(s)$ is a rational Levi subgroup of $\mathbf{G}$.*

Let $S_{2'}(G)$ be a set of representatives of G-conjugacy classes of semi-simple elements of odd order of G. The result in [27, 4.2] gives, under some conditions, a bijection between $S_{2'}(G)$ and $S_{2'}(G^*)$ which is one of the main ingredients to prove Theorem 2.5.2. We need a slight modification of the statement to be able to get the same bijection when the center is not connected. If $s \in S_{2'}(G)$, we denote by $C_{\mathbf{G}}(s)^*$ a Levi subgroup of $\mathbf{G}^*$ dual to $C_{\mathbf{G}}(s)$ (we recall that by the lemma above and under the assumptions on $\mathbf{G}$, $C_{\mathbf{G}}(s)$ is a rational Levi subgroup of $\mathbf{G}$).

Proposition 2.5.5 ([7, 14.1])**.** *Let ℓ be a prime number dividing neither q, $|(Z(\mathbf{G})/Z(\mathbf{G})^\circ)^F|$ or $|(Z(\mathbf{G}^*)/Z(\mathbf{G}^*)^\circ)^F|$. There is a one-to-one map from the set of G-conjugacy classes of ℓ-element of G onto the set of G^*-conjugacy classes of ℓ-elements of G^*. If the class of $x \in G_\ell$ maps to the class of $y \in G^*_\ell$, then there is a isomorphism defined over $\mathbb{F}_q$ between $C_{\mathbf{G}}(x)^{\circ *}$ and $C_{\mathbf{G}^*}(y)^\circ$.*

Applying this result for classical groups, we get :

Corollary 2.5.6. *Assume that $\mathbf{G}$ is a reductive group with simple components of type B, C or D. There is a bijection*

$$S_{2'}(G) \to S_{2'}(G^*) \quad t \mapsto t'$$

such that there is an isomorphism $C_{\mathbf{G}}(t)^* \simeq C_{\mathbf{G}^*}(t')$ defined over $\mathbb{F}_q$. The results still holds for any F-stable Levi subgroup of $\mathbf{G}$.

Proof. Let ℓ be an odd prime number. Remark that if ℓ divides q, the only semi-simple ℓ-element if trivial, so we can assume that ℓ does not divides q. The group $\mathbf{G}$ being of type B, C or D, $(Z(\mathbf{G})/Z(\mathbf{G})^\circ)^F$ and $(Z(\mathbf{G}^*)/Z(\mathbf{G}^*)^\circ)^F$ are 2-groups. Moreover, the centralisers of semi-simple elements of odd order of $\mathbf{G}^*$ are connected by 2.5.4 (2), so we can apply 2.5.5 to $\mathbf{G}$ for any odd prime number to get the result. Using 2.5.4 (1) for $\mathbf{G}$ and $\mathbf{G}^*$, we can use the same arguments for any F-stable Levi subgroup of $\mathbf{G}$. ■

Proof of Theorem 2.5.2. We will proceed by induction on the dimension of $\mathbf{G}$: let $s \in S_{2'}(G^*)$ and $\mathbf{L}_s$ be a rational Levi subgroup of $\mathbf{G}$ dual to the

Levi subgroup $C_{\mathbf{G}^*}(s)$ of $\mathbf{G}^*$. We denote by $u(L_s)$ the number of unipotent classes of L_s. If s in non-central, then $\dim \mathbf{L}_s < \dim \mathbf{G}$, so

$$u(L_s) = m_1(L_s) = m_s(G).$$

The first equality comes from the induction hypothesis, the second one from the Morita equivalence between $B_1(L_s)$ and $B_s(G)$ (see Theorem 1.3.29). Let a be the number of 2-regular classes of G. Using the fact that a is also the number of elements in $\mathrm{Irr}_k(G)$, we have

$$\begin{aligned} a &= \sum_{s \in S_{2'}(G^*)} m_s(G) \\ &= \sum_{\substack{s \in S_{2'}(G^*) \\ s \notin Z(G^*)}} u(L_s) + \sum_{s \in Z(G^*)_{2'}} m_s(G) \\ &= \sum_{\substack{s \in S_{2'}(G^*) \\ s \notin Z(G^*)}} u(L_s) + |Z(G^*)_{2'}| \cdot m_1(G). \end{aligned}$$

The last equality comes from the fact that for any central element s of odd order of G^*, tensoring by the linear character of $C_{\mathbf{G}^*}(s)$ attached to s as in [14, 13.30] provides a natural isomorphism between $B_1(G)$ and $B_s(G)$, so $m_1(G) = m_s(G)$. By using the fact that a is the number of 2-regular classes of G we also have :

$$\begin{aligned} a &= \sum_{\substack{s \in S_{2'}(G) \\ s \notin Z(G)}} u(C_G(s)) + |Z(G)_{2'}| \cdot u(G) \\ &= \sum_{\substack{s \in S_{2'}(G) \\ s \notin Z(G)}} u(L_{s'}) + |Z(G)_{2'}| \cdot u(G) \end{aligned}$$

where the second equality comes from the fact that the isomorphism $C_{\mathbf{G}}(s)^* \simeq C_{\mathbf{G}^*}(s')$ is defined over $\mathbb{F}_q$.

To conclude it remains to show that $|Z(G)_{2'}| = |Z(G^*)_{2'}|$. This comes from the fact that the bijection $S_{2'}(G^*) \simeq S_{2'}(G)$ of Corollary 2.5.6 induces a bijection between central elements of odd order of G and G^*.

■

2.6 Exceptional groups

For simple exceptional groups of adjoint type, the number of unipotent modular representation was determined by Geck–Hiss (see [28, 6.6]).

Type	$\ell = 2$	$\ell = 3$	$\ell = 5$	ℓ good
G_2	8	9		10
F_4	28	35		37
$E_6, {}^2E_6$	27	28		30
E_7	64	72		76
E_8	131	150	162	166

Using this table, we will show that Conjecture 2.3.14 holds for any exceptional adjoint group G and any prime number ℓ. The following section introduces a method for computing α_ℓ.

Remark 2.6.1. By [63, 2.4], for each F-stable unipotent class $\mathcal{C}$, there exists $u \in \mathcal{C}$ such that F acts trivially on $A_{\mathbf{G}}(u)$. Hence, we can replace the set $\bar{\mathcal{M}}_\ell(\Omega_u^\ell)$ of Conjecture 2.3.14 by the set $\mathcal{M}_\ell(\Omega_u^\ell)$ according to Remark 2.3.13.

Computing α_ℓ. Let $(\mathbf{T}, \mathbf{B})$ be an F-stable pair where $\mathbf{T}$ is a maximal torus contained in a Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. Let Φ be the root system of $\mathbf{G}$ associated to $\mathbf{T}$ and $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots induced by $\mathbf{B}$. Let α_0 be the highest short root. Since we work with with adjoint exceptional groups in good characteristic, we can use Theorem 1.2.12 (see [2, 4.7]) and the groups

$$W_i = \langle s_{\alpha_j} \mid 0 \leq j \neq i \leq n \rangle$$

are isomorphic to Weyl groups of neutral components of centralisers of isolated semi-simple elements of $\mathbf{G}^*$. Since $\mathbf{G}$ is adjoint, Proposition 2.3.11 states that a unipotent class is ℓ-special if and only if its Springer correspondent is ℓ-special. Using this property, we can sketch a strategy to compute α_ℓ.

First step. Finding all ℓ-special unipotent classes :

1. Determine all subgroups W_i^* of the Weyl group W^* of $\mathbf{G}^*$ for $i \in \{0, \ldots, n\}$.
2. Using j-induction tables, detect which irreducible characters of W^* are ℓ-special.
3. For every ℓ-special character E, use the Springer correspondence to get the unipotent class $(u)_{\mathbf{G}}$ of $\mathbf{G}$ such that $E = E_{u,1}$ where E is viewed as a representation of W via the natural isomorphism $W \simeq W^*$.

Second step. Now that we have every ℓ-special class, compute α_ℓ :

1. For each F-stable ℓ-special class C_u, compute the ℓ-special quotient Ω_u^ℓ of $A_{\mathbf{G}}(u)$.

2. For each conjugacy class (x) of Ω_u^ℓ, compute $|\mathrm{Irr}_k(C_{\Omega_u^\ell}(x))|$ by counting the number of ℓ'-classes. Let $\alpha_{\ell,u}$ be the sum over conjugacy classes of the numbers obtained this way.
3. Then α_ℓ is equal to $\sum_u \alpha_{\ell,u}$ where u runs over the set of ℓ-special classes.

Example 2.6.2. Assume that $\mathbf{G} = G_2$ and $\ell = 2$, we can identify $\mathbf{G}$ with $\mathbf{G}^*$. Then $\mathbf{G}$ has 3 special classes : 1, G_2 and $G_2(a_1)$ and the class $\tilde{A}_1$ is 2-special. Indeed, $\mathbf{G}$ has a unique class of non-trivial isolated semi-simple 2-elements $(s)_\mathbf{G}$ and the Weyl group W_s of $C_\mathbf{G}(s)$ is of type $A_1 \times A_1$. Therefore, irreducible characters of W_s are special. If $u \in \tilde{A}_1$, $E_{u,1} = \phi_{2,2} = j_{W_s}^W(\phi_{1^2} \otimes \phi_{1^2})$ is 2-special so is $\tilde{A}_1$.

For each of those classes, $A_\mathbf{G}(u)$ is trivial except for $(u)_\mathbf{G} = G_2(a_1)$ where $A_\mathbf{G}(u) = \mathfrak{S}_3$, let us compute Ω_u^2. Then $\mathfrak{S}_3$ has two indecomposable projective characters : $\Phi_1 = \phi_3 + \phi_{1^3}$ and $\Phi_2 = \phi_{12}$. The Springer correspondence for G_2 described in the Table 2.2.1 gives us $E_{u,\phi_3} = \phi_{2,1}$, $E_{u,\phi_{1^3}} = 0$ and $E_{u,\phi_{12}} = \phi_{1,3'}$. According to 2.1.1, $a_{\phi_{12}} = a_{\phi_{1,3'}} = 1$, so $a_{\Phi_1} = a_{\Phi_2} = 1$ and $\mathcal{S}_2 = \{\Phi_1, \Phi_2\}$. Hence, $\Omega_u^2 = \mathfrak{S}_3$ and $\alpha_{2,u} = 5$, adding 1 for each other 2-special classes we get $\alpha_2 = 8$.

Computations were made for adjoint exceptional groups using CHEVIE [50]. For each group of exceptional type $\mathbf{G}$ and each bad prime ℓ, we provide in the appendix tables listing ℓ-special classes of $\mathbf{G}$ (we use the same labelling as in CHEVIE) and for each special class the groups $A_\mathbf{G}(u)$, Ω_u^ℓ and the number $\alpha_{\ell,u}$. Finally, the proof of Theorem 2.3.16 is complete.

3

Basic sets for unipotent blocks of classical groups

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Assume that $\mathbf{G}$ is a connected reductive group defined over $\mathbb{F}_q$ where q is a power of a prime number p and let F be the corresponding Frobenius map. Let us denote by $G = \mathbf{G}^F$ the finite group of fixed points of $\mathbf{G}$ under F. Throughout this chapter, we assume that p is good for $\mathbf{G}$. Let ℓ be a prime number different from p. We are interested in finding (unitriangular) basic sets for unipotent ℓ-blocks of G. It is to be expected that one can find a natural basic set of characters for unipotent blocks of finite reductive groups such that the decomposition matrix has unitriangular shape in this basic set. The behaviour of such basic sets depends on whether ℓ is a good or a bad prime number for $\mathbf{G}$.

When ℓ is good and does not divide the order of the component group of the center of $\mathbf{G}$, the unipotent characters form a basic set for the unipotent blocks. This was first proved for $\mathrm{GL}_n(q)$ by Dipper [17] and $\mathrm{GU}_n(q)$ by Geck [20] using the so-called generalised Gelfand–Graev representations (GGGRs). In both cases the basic set was shown to be unitriangular. The case of a general finite reductive group was settled by Geck–Hiss [27] and Geck [21], but the question whether the decomposition matrix has unitriangular shape remains open in general.

When ℓ is a bad prime number, much less is known : for classical groups with connected center and $\ell = 2$ the existence of a unitriangular basic set for the unipotent blocks was shown in [22], using again GGGRs. More recently, Kleshchev–Tiep found a unitriangular basic set for $\mathrm{SL}_n(q)$ [36] by studying the restrictions of characters of $\mathrm{GL}_n(q)$. Using this method and the results of Geck for $\mathrm{GU}_n(q)$, Denoncin generalised this result to $\mathrm{SU}_n(q)$ [13].

The aim of this chapter is to show the existence of such a basic set for classical groups by generalising results of Geck obtained in [22]. This chapter is organised as follows : in the first section we recall results on basic sets for finite reductive groups and state our main result (see 3.1.1). The second section introduces GGGRs and their properties, the third section presents results of Geck–Hézard and Taylor that we will need. The last section is devoted to the proof of the main theorem of this chapter. Throughout this chapter, we assume $\mathbf{G}$ has simple components of type B, C or D.

3.1 Introduction to the main theorem

3.1.1 Statement of the theorem

Let $\mathbf{G}$ be a connected reductive group defined over $\mathbb{F}_q$ where q is the power of a good prime number p. As in §1.1.1, we fix a prime number $\ell \neq p$ and an ℓ-modular system $(K, \mathcal{O}, k)$ for G. If B is a union of ℓ-blocks, we recall that a basic set for B is a subset $\mathcal{B} \subset \mathrm{Irr}_K(B)$ such that $d(\mathcal{B})$ is a $\mathbb{Z}$-basis of $R_k(B)$. We recall that according to Theorem 1.3.28

$$B_s = B_s(G) = \bigcup_t \mathcal{E}(G^F, st)$$

where t runs over semi-simple ℓ-elements of $C_{G^*}(s)$ is a union of blocks. Under some restrictions on ℓ, Geck and Hiss proved that there is a natural basic set for B_s. By Theorem 2.3.2, $\mathcal{E}(G, s)$ is a basic set for B_s when ℓ is very good for G but we don't know whether the basic set is unitriangular. It was conjectured by Geck to be the case, at least for unipotent blocks.

Conjecture 3.1.1 (Geck [28, 3.4])**.** *Assume that ℓ is good for $\mathbf{G}$. Then the unipotent characters of G form a unitriangular ℓ-basic set of B_1.*

Our strategy for constructing a unitriangular basic set consists in counting the number of irreducible unipotent modular representations and then finding enough projective modules which will satisfy some unitriangularity condition. Under the condition that the center of $\mathbf{G}$ is connected, Geck achieved this for classical groups in [22]. Following the same strategy, we generalise those results to cases where the center is disconnected :

Theorem. *Assume that p is an odd prime number, $\ell = 2$ and let $\mathbf{G}$ be a simple group of type B, C or D which is neither spin or half-spin. Then there exists a unitriangular basic set for the unipotent blocks of G.*

3.1.2 Strategy of proof

The following result is valid for arbitrary finite groups and has been used in [22] to prove that the decomposition matrix of the unipotent blocks of G is unitriangular when the center of $\mathbf{G}$ is connected.

Proposition 3.1.2. *Let B be a union of blocks and $n := |\mathrm{Irr}_k(B)|$. Assume that there exist ordinary characters $\chi_1, \ldots, \chi_n$ and projective kG-modules $P_1, \ldots, P_n$ in B such that the matrix of scalar products $\big(\langle \chi_i, e(P_j)\rangle_{KG}\big)_{1 \leq i,j \leq n}$ has lower unitriangular shape. Then the characters $\chi_1, \ldots, \chi_n$ form a unitriangular basic set for B.*

Proof. Let $d : R_K(G) \longrightarrow R_k(G)$ be the decomposition map. Since e and d are in duality, we have the relation $\langle \chi_i, e(P_j)\rangle_{KG} = \langle d(\chi_i), P_j\rangle_{kG}$. Let us denote by $\varphi_1, \ldots, \varphi_n$ the irreducible modular representations lying in B. For $1 \leq i \leq n$, we can write $d(\chi_i) = \sum_{l=1}^{n} d_{\chi_i,\varphi_l}\varphi_l$. So we have

$$\langle \chi_i, e(P_j)\rangle_{KG} = \sum_{l=1}^{n} d_{\chi_i,\varphi_l}\langle \varphi_l, P_j\rangle_{kG}.$$

In terms of matrices, this equality is :

$$(d_{\chi_i,\varphi_j})_{1\leq i,j\leq n} \times \big(\langle \varphi_i, P_j\rangle_{kG}\big)_{1\leq i,j\leq n} = \big(\langle \chi_i, e(P_j)\rangle_{KG}\big)_{1\leq i,j\leq n}.$$

We want to show that (d_{χ_i,φ_j}) is lower unitriangular, using the fact that the matrix on the right side of the equality is lower unitriangular. The result follows from the lemma below (you can find an alternative proof in [12]). ■

Lemma 3.1.3. *Let A, B and C be three square matrices of size $n \times n$ with non-negative integer coefficients such that $AB = C$ with C lower unitriangular. Then, up to permutations of columns, A is unitriangular as well.*

Proof. Let us write $A = (a_{i,j})_{1\leq i,j\leq n}$, $B = (b_{i,j})_{1\leq i,j\leq n}$ and $C = (c_{i,j})_{1\leq i,j\leq n}$. The matrix C is lower unitriangular so

$$0 = c_{i,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j} \quad \forall 1 \leq i < j \leq n.$$

$$1 = c_{i,i} = \sum_{k=1}^{n} a_{i,k}b_{k,i} \quad \forall 1 \leq i \leq n.$$

All the coefficients are non-negative so we have

1. $a_{i,k}b_{k,j} = 0$ for all $1 \leq k \leq n$ and $1 \leq i < j \leq n$,
2. Given $1 \leq i \leq n$, there is a unique integer k_i with $1 \leq k_i \leq n$ such that $a_{i,k_i} = b_{k_i,i} = 1$. If $k \neq k_i$, then $a_{i,k}b_{k,i} = 0$.

Note that if $i < j$, then $k_i \neq k_j$ and the map $i \to k_i$ is a bijection. Indeed if $k_i = k_j$, then $a_{i,k_i}b_{k_j,j} = 0$ by 1 which is impossible. Let $j \in \{1, \ldots, n\}$ such that $j > i$. By 1, we have $a_{i,k_j}b_{k_j,j} = 0$. By 2, $b_{k_j,j} = 1$ so $a_{i,k_j} = 0$. Therefore, in the i-th row of A, there are at least $n - i$ zero coefficients. In particular, the first row of A has a unique non-zero coefficient whose value is 1. If $\sigma \in \mathfrak{S}_n$, we have

$$\sum_{k=1}^{n} a_{i,k}b_{k,j} = \sum_{k=1}^{n} a_{i,\sigma(k)}b_{\sigma(k),j}.$$

Then, we can apply the same permutation to the columns of A and to the rows of B without changing the equality $AB = C$. In particular, permuting the columns of A and the rows of B if necessary, we can assume that $a_{1,j} = \delta_{1,j}$. Then, by 1 and 2 we also have $b_{1,j} = \delta_{i,j}$. Now, we just have to show that the submatrix $(a_{i,j})_{2\leq i,j\leq n}$ is lower unitriangular up to permutations of columns. We can conclude by an inductive argument since we have

$$(a_{i,j})_{2\leq i,j\leq n} \times (b_{i,j})_{2\leq i,j\leq n} = (c_{i,j})_{2\leq i,j\leq n}.$$

■

Consequently, in order to find a unitriangular basic set of characters for unipotent blocks it is enough to :

- Compute the cardinal of $\mathrm{Irr}_k(B_1)$. We have seen in Theorem 2.5.2 that the cardinal of $\mathrm{Irr}_k(B_1)$ is the number of unipotent classes of G when $\mathbf{G}$ is a simple group of type B, C, D.
- Find projective and ordinary representations satisfying the assumptions of Proposition 3.1.2. This can be done using the theory of generalised Gelfand–Graev representations and results of Geck, Hézard and Taylor that we will introduce in Section 3.3.

3.2 Generalised Gelfand–Graev representations

We fix an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. We denote by Φ a root system associated to $(\mathbf{G}, \mathbf{T})$. The Borel subgroup $\mathbf{B}$ provides a set of simple roots $\Delta \subset \Phi$. To any unipotent element u of G, Kawanaka associated in [34] a projective representation Γ_u which depends only on the G-conjugacy class of u, called the generalised Gelfand–Graev representation of G associated to u (GGGR). Those representations will allow us to show the existence of a basic set for unipotent blocks of classical groups.

3.2.1 Weighted Dynkin diagrams

We will introduce a description of unipotent classes of $\mathbf{G}$: to a unipotent class one can attach a combinatorial object called the *weighted Dynkin diagram*.

Definition 3.2.1. A weighted Dynkin diagram is a map $d : \Phi \to \mathbb{Z}$ such that

- For all $\alpha \in \Delta$, $d(\alpha) \in \{0, 1, 2\}$,

- For all $\beta = \sum_{\alpha\in\Delta} n_\alpha \alpha \in \Phi$, $d(\beta) = \sum_{\alpha\in\Delta} n_\alpha d(\alpha)$.

We review the classification of unipotent classes by weighted Dynkin diagrams below :

Theorem 3.2.2 ([8, 5.6.7,5.6.8])**.** *To a unipotent class $\mathcal{C}$ of $\mathbf{G}$, we can attach a weighted Dynkin diagram $d_\mathcal{C}$. Moreover, if $\mathcal{C}'$ is another unipotent class, we have*

$$d_\mathcal{C} = d_{\mathcal{C}'} \Longleftrightarrow \mathcal{C} = \mathcal{C}'.$$

Remark 3.2.3.

- Even though the proofs in [8, 5.6] rely on the fact that p is large enough, the result still holds whenever p is good [35, 2.1.1].
- The map $\mathcal{C} \mapsto d_\mathcal{C}$ is not surjective.
- The weighted Dynkin diagrams for simple algebraic groups are listed in [8, §13.1].
- The unipotent class $\mathcal{C}$ is F-stable if and only if $d_\mathcal{C}$ is stable under the action of F on Φ, see for example [62, Lem. 2.40].

Let $\mathcal{C}$ be a unipotent class of $\mathbf{G}$. Without giving a proof of the above result, we will explain how we can describe $\mathcal{C}$ from $d_\mathcal{C}$ (we follow [35, §2.1]). For $\alpha \in \Phi$, let $\mathbf{U}_\alpha$ be the root subgroup of $\mathbf{G}$ attached to α. Recall that we have an isomorphism

$$\mathbf{G}_a \xrightarrow{\sim} \mathbf{U}_\alpha, \quad x \mapsto u_\alpha(x),$$

and that $\mathbf{U}_\alpha$ is the unique minimal closed, connected unipotent subgroup of $\mathbf{G}$ normalised by $\mathbf{T}$ such that that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for $x \in \mathbf{G}_a, t \in \mathbf{T}$. We define the following groups :

$$\begin{aligned}
\mathbf{L}_\mathcal{C} &:= \langle \mathbf{T}, \mathbf{U}_\alpha \mid \alpha \in \Phi, d_\mathcal{C}(\alpha) = 0\rangle, \\
\mathbf{U}_{\mathcal{C},i} &:= \prod_{\substack{\alpha\in\Phi^+ \\ d_\mathcal{C}(\alpha)\geq i}} \mathbf{U}_\alpha, \ i \in \mathbb{N}.
\end{aligned}$$

Then $\mathcal{C}$ is the unique unipotent class in $\mathbf{G}$ such that $\mathcal{C} \cap \mathbf{U}_{\mathcal{C},2}$ is dense in $\mathbf{U}_{\mathcal{C},2}$. Moreover, $\mathcal{C} \cap \mathbf{U}_{\mathcal{C},2}$ is a single conjugacy class of the parabolic group $\mathbf{U}_{\mathcal{C},1}\mathbf{L}_\mathcal{C}$.

Example 3.2.4.

1. Assume that $\mathcal{C}$ is the trivial class. Then $d_{\mathcal{C}}(\alpha) = 0$ for $\alpha \in \Phi$. Indeed, in that case $\mathbf{U}_{\mathcal{C},2} = \prod_{\substack{\alpha\in\Phi \\ d_{\mathcal{C}}(\alpha)\geq 2}} \mathbf{U}_\alpha = \{1\} = \mathcal{C}$.
2. Assume that $\mathcal{C}$ is the regular class, *i.e.* the unique unipotent class of $\mathbf{G}$ of maximal dimension. Then $d_{\mathcal{C}}(\alpha) = 2$ for all $\alpha \in \Delta$ (see [8, §13.1] for simple groups) so that $\mathbf{U}_{\mathcal{C},2} = \mathbf{U}$.

3.2.2 Generalised Gelfand–Graev representations

Although generalised Gelfand–Graev representations were introduced by Kawanaka in [34] and [35], we follow the recent definition given by Taylor in [64]. Let us introduce some further notation. Let $\mathfrak{g}$ be the Lie algebra of $\mathbf{G}$ over $\overline{\mathbb{F}}_q$, $\mathfrak{g}$ also has an $\mathbb{F}_q$-rational structure and is equipped with a Frobenius map $F : \mathfrak{g} \to \mathfrak{g}$. Let $\mathfrak{t}$ be the Lie algebra of $\mathbf{T}$, $\mathfrak{g}$ has a Cartan decomposition

$$\mathfrak{g} := \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$$

where $\mathfrak{t}$ is stable under F. The spaces $\mathfrak{g}_\alpha$ are called the *root subspaces* of $\mathfrak{g}$ and have dimension 1 so for each α we can choose $e_\alpha \in \mathfrak{g}_\alpha$ such that $\mathfrak{g}_\alpha = \overline{\mathbb{F}}_q e_\alpha$. Let ρ be the permutation of Φ induced by F (see 1.2.9). Then $F(\mathfrak{g}_\alpha) = \mathfrak{g}_{\rho(\alpha)}$. According to [64, 5.6], there exists a $\mathbf{G}$-invariant symmetric bilinear form defined over $\mathbb{F}_q$ denoted by $\kappa : \mathfrak{g} \times \mathfrak{g} \to \overline{\mathbb{F}}_q$ such that for $\alpha \in \Phi$ we have

$$\mathfrak{g}_\alpha^\perp = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi\setminus\{-\alpha\}} \mathfrak{g}_\alpha$$

where $\mathfrak{g}_\alpha^\perp := \{x \in \mathfrak{g} \mid \kappa(x, y) = 0\ \forall y \in \mathfrak{g}_\alpha\}$. Moreover, according to [64, 5.2], the map $\mathfrak{g} \to \mathfrak{g}$, $x \mapsto x^*$, defined by

- $t^* = -t$ if $t \in \mathfrak{t}$,
- $e_\alpha^* = -e_{-\alpha}$ for all $\alpha \in \Phi$,

is an $\mathbb{F}_q$-opposition automorphism of $\mathfrak{g}$, *i.e.* this is an automorphism such that $\mathfrak{t}^* = \mathfrak{t}$ and $e_\alpha^* \in \mathbb{F}_q e_{-\alpha}$ for all $\alpha \in \Phi$. The reductive group $\mathbf{G}$ acts on $\mathfrak{g}$ as follows. For $g \in \mathbf{G}$, let Int_g be the map defined by $\mathrm{Int}_g(x) := gxg^{-1}$ for $x \in \mathbf{G}$. Let $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ be the differential of Int_g. Then $g \cdot x := \mathrm{Ad}_g(x)$ for $g \in \mathbf{G}$ and $x \in \mathfrak{g}$. We need the following definition.

Definition 3.2.5 ([64, 2.10, 2.15])**.** Let $\mathbf{G}$ be a connected reductive group with root datum $(X, \Phi, Y, \Phi^\vee)$. We say that $\mathbf{G}$ is *proximate* if $Y/\mathbb{Z}\Phi^\vee$ has no p-torsion.

Let $\mathcal{U}_{ni} \subseteq \mathbf{G}$ be the variety of unipotent elements of $\mathbf{G}$ and let $\mathcal{N}_{il} \subset \mathfrak{g}$ be the nilpotent cone of $\mathfrak{g}$. According to [64, 3.4], if $\mathbf{G}$ is proximate there exists a $\mathbf{G}$-equivariant isomorphism of varieties $\phi : \mathcal{U}_{ni} \to \mathcal{N}_{il}$ which is compatible with the Frobenius endomorphisms on $\mathbf{G}$ and $\mathfrak{g}$. Moreover, according to [64, 2.16], for any connected reductive group $\mathbf{G}$, there exists a proximate algebraic group $\mathbf{G}'$ defined over $\mathbb{F}_q$ such that $G' \simeq G$. Hence, we can assume that $\mathbf{G}$ is proximate and that ϕ exists.

Let us fix a linear character $\chi_p : (\mathbb{F}_p, +) \to K^\times$. Let $\mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p} : \mathbb{F}_q \to \mathbb{F}_p$ be the field trace. Let $\chi_q := \chi_p \circ \mathrm{Tr}_{\mathbb{F}_q/\mathbb{F}_p}$. Let $\mathcal{C}$ be an F-stable unipotent class of $\mathbf{G}$ and let u be an element of $\mathcal{C} \cap U_{\mathcal{C},2}$. Note that u exists since $\mathcal{C} \cap \mathbf{U}_{\mathcal{C},2}$ is the orbit of some parabolic subgroup of $\mathbf{G}$. Let $\varphi_u : U_{\mathcal{C},2} \to K^\times$ be the map defined by

$$\varphi_u(x) = \chi_q(\kappa(\phi(u)^*, \phi(x))).$$

It is a linear character of $U_{\mathcal{C},2}$ by [64, 5.10]. Moreover, according to [64, 5.14, 5.15],

$$\gamma_u := |U_{\mathcal{C},1} : U_{\mathcal{C},2}|^{-1/2} \mathrm{Ind}_{U_{\mathcal{C},2}}^{G}(\varphi_u)$$

is a character of G and Γ_u. To γ_u, we associate a *generalised Gelfand–Graev representation*, that is a representation Γ_u of G whose character is γ_u. When F is split, *i.e.* if ρ acts trivially on Φ, Geck gave in [25] an alternate definition for φ_u. In that case :

- $\kappa(e_\alpha, e_\beta) \in \mathbb{F}_q$ for any $\alpha, \beta \in \Phi$.
- If $x \in U$ and if we write $x = \prod_{\alpha \in \Phi^+} u_\alpha(\mu_\alpha)$, then $\mu_\alpha \in \mathbb{F}_q$.

Let us write $u = \prod_{\substack{\alpha \in \Phi^+ \\ d_{\mathcal{C}}(\alpha) \geq 2}} u_\alpha(\lambda_\alpha)$ and let $x = \prod_{\substack{\alpha \in \Phi^+ \\ d_{\mathcal{C}}(\alpha) \geq 2}} u_\alpha(\mu_\alpha) \in U_{C,2}$ where $\lambda_\alpha, \mu_\alpha \in \mathbb{F}_q$. Then

$$\varphi_u(x) = \sum_{\substack{\alpha \in \Phi^+ \\ d_{\mathcal{C}}(\alpha) = 2}} \chi_q(\lambda_\alpha \mu_\alpha \kappa(e_\alpha^*, e_\alpha)).$$

Remark 3.2.6.

- By [35, 3.1.12], $|U_{\mathcal{C},1} : U_{\mathcal{C},2}|$ is an even power of q so it has a square root in $\mathbb{Z}_{\geq 0}$.
- In the original definition [35, 3.1.11], GGGRs are constructed by induction of a linear character of a p-group so GGGRs are projective kG-modules whenever $p \neq \ell$.
- The character γ_u is unipotently supported since it is the induction of a character of a unipotent subgroup of G.

- According to [34, 1.3.6], $\Gamma_u \simeq \Gamma_{u'}$ if u and u' are G-conjugate. Moreover, according to [25, 2.2], $\mathcal{C} \cap U_{\mathcal{C},2}$ contains a complete set of representatives of the G-conjugacy classes of $\mathcal{C}^F$. Hence, we can construct a GGGR from any unipotent class of G.

Example 3.2.7.

1. Assume that $\mathcal{C}$ is the trivial class. Then $\mathbf{U}_{\mathcal{C},2} = \{1\}$ and $\gamma_1 = \mathrm{Ind}_{\{1\}}^G 1$ is the character of the regular representation of G.
2. We assume that F is split for more convenience. Let $\mathcal{C}$ be the regular unipotent class and let $u = \prod_{\alpha \in \Phi^+} u_\alpha(\lambda_\alpha) \in \mathcal{C} \cap U_{\mathcal{C},2}$. Then $\mathbf{U}_{\mathcal{C},2} = \mathbf{U}$ since $d_{\mathcal{C}}(\alpha) = 2$ for all $\alpha \in \Delta$. Let $\mathbf{U}^* := \prod_{\alpha \in \Phi^+ \setminus \Delta} \mathbf{U}_\alpha = \mathbf{U}_{\mathcal{C},3}$. Then, φ_u is trivial on U^*. Indeed, if $x := \prod_{\alpha \in \Phi^+} u_\alpha(\mu_\alpha) \in \mathbf{U}_{\mathcal{C},3}$, then $\mu_\alpha = 0$ if $d_{\mathcal{C}}(\alpha) = 2$. Moreover, φ_u is clearly non-trivial on $\prod_{\alpha \in \Delta'} U_\alpha$ for any non-empty subset Δ' of Δ. Hence, φ_u is regular in the sense of [14, 14.27] and γ_u is a Gelfand–Graev character *i.e.* the induced character from U^* to G of a regular character.

The value of the character of a GGGR on a unipotent class is non-zero only under some conditions. For a subset X of $\mathbf{G}$, we denote by $\overline{X}$ its closure for the Zariski topology.

Proposition 3.2.8 ([48, 6.14])**.** *Let $\mathcal{C}, \mathcal{C}'$ be two F-stable unipotent classes of $\mathbf{G}$, $u \in \mathcal{C}^F$ and $v \in \mathcal{C}'^F$. If $\gamma_u(v) \neq 0$, then $\mathcal{C}' \subseteq \overline{\mathcal{C}}$.*

3.2.3 Unipotent support and wave front set

Following [48], we can associate to $\rho \in \mathrm{Irr}_K(G)$ a unique F-stable unipotent class $\mathcal{C}_\rho$ of $\mathbf{G}$ satisfying the two following conditions :

1. $\sum_{x \in \mathcal{C}_\rho^F} \rho(x) \neq 0$.
2. For any F-stable unipotent class $\mathcal{C}'$ of $\mathbf{G}$, $\sum_{x \in \mathcal{C}'^F} \rho(x) \neq 0$ implies $\dim \mathcal{C}' \leq \dim \mathcal{C}_\rho$.

We say that $\mathcal{C}_\rho$ is the *unipotent support* of ρ. Using GGGRs, we can also associate to ρ another F-stable unipotent class $\mathcal{C}_\rho^*$ of $\mathbf{G}$, the *wave front set* of ρ, which is the unique unipotent class satisfying the following conditions (see [48, 11.2]) :

1. $\langle \rho, \gamma_u \rangle_{KG} \neq 0$ for some $u \in \mathcal{C}_\rho^{*F}$.

2. For any unipotent element $v \in G$, $\langle \rho, \gamma_v \rangle_{KG} \neq 0$ implies $\dim (v)_{\mathbf{G}} \leq \dim \mathcal{C}_\rho^*$.

Unipotent support and wave front set are closely related. More precisely, let D_G be the Alvis–Curtis duality for representations of G. This is an operator on $R_K(G)$ with the following properties :

- $D_G \circ D_G = \mathrm{Id}$,
- $D_G(1_G) = \mathrm{St}_G$,
- For every $\chi \in \mathrm{Irr}_K(G)$, there exists a sign ε such that $\varepsilon D_G(\chi) \in \mathrm{Irr}_K(G)$.

See [14, §8] for more details.

Remark 3.2.9. Note that $D_G(\gamma_u)$ is unipotently supported. Indeed, D_G is constructed as a linear combination of composition of Harish-Chandra induction and restriction operators (see [14, 8.8]). Let $\mathbf{L}$ be an F-stable Levi subgroup of $\mathbf{G}$ contained in an F-stable parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. Let us denote by R_L^G (resp. ${}^*R_L^G$) the Harish-Chandra induction (resp. restriction). We have by Remark 8.9 of [14] that $R_L^G \circ {}^*R_L^G(\gamma_u)$ is the induction of a character of P which is zero on non-unipotent elements. Therefore, $R_L^G \circ {}^*R_L^G(\gamma_u)$ is unipotently supported and so is $D_G(\gamma_u)$.

The following result can be viewed as dual to Proposition 3.2.8.

Proposition 3.2.10 ([48, 8.6, 6.13])**.** *Let $\mathcal{C}, \mathcal{C}'$ be two F-stable unipotent classes, $u \in \mathcal{C}^F$ and $v \in \mathcal{C}'^F$. If $D_G(\gamma_u)(v) \neq 0$, then $\mathcal{C} \subseteq \overline{\mathcal{C}'}$.*

If $\rho \in \mathrm{Irr}_K(G)$, we denote by ρ^* the irreducible character of G such that $\rho^* = \pm D_G(\rho)$. We call ρ^* the dual character of ρ. The relation between unipotent support and wave front set comes from this duality :

$$\forall \rho \in \mathrm{Irr}_K(G) \quad \mathcal{C}_\rho^* = \mathcal{C}_{\rho^*},$$

see [64, 14.15]. Note that all these properties were first proved under the assumption that p and q were large enough. These were later generalised to the case where p is a good prime in [64]. Let us make an important remark about unipotent support and wave front set in regard of the classification of irreducible characters of G that we stated in §2.2.3 in the case $Z(\mathbf{G})$ is connected. Let $g \in G^*$ be a special element with Jordan decomposition $g = su$. Recall that $\mathrm{Irr}_K(G)_g$ is the subset of $\mathcal{E}(G, s)$ which, by the Jordan decomposition, corresponds bijectively to the family of unipotent characters of $C_{G^*}(s)$ parametrised by $(u)_{C_{\mathbf{G}^*}}(s)$ in Theorem 2.2.9. Let $\rho \in \mathrm{Irr}_K(G)_g$

and Φ be the map defined in 2.3.7 sending a special conjugacy class of $\mathbf{G}^*$ to a unipotent class of $\mathbf{G}$. According to [48, 11.1, 11.2, 10.5] and the above observations we have :

Proposition 3.2.11. *The unipotent class $\Phi((g)_{\mathbf{G}^*})$ is the unipotent support of ρ.*

3.2.4 GGGRs and regular embedding

We review some properties of GGGRs which will be useful later. Let $\tilde{\mathbf{G}} = \mathbf{G}\tilde{\mathbf{T}}$ be a regular embedding of $\mathbf{G}$ as in §1.3.4. If $g \in \tilde{G}$ and χ is a character of G, we denote by χ^g the central function of G defined by $\chi^g(x) = \chi(gxg^{-1})$ for $x \in G$. It is a character of G. Since G acts trivially on χ, for any $g \in \tilde{G}$ there exists $t \in \tilde{T}$ such that $\chi^g = \chi^t$.

Proposition 3.2.12. *Let $u \in G$ be unipotent and $\tilde{t} \in \tilde{T}$. Then $\gamma_u^{\tilde{t}} = \gamma_{\tilde{t}u\tilde{t}^{-1}}$.*

Proof. Since $\gamma_u = |U_{\mathcal{C},1} : U_{\mathcal{C},2}|^{-1/2}\mathrm{Ind}_{U_{\mathcal{C},2}}^G(\varphi_u)$ we just need to show that $\varphi_u^{\tilde{t}} = \varphi_{\tilde{t}u\tilde{t}^{-1}}$. Note that since $\mathbf{G}$ is a normal subgroup of $\tilde{\mathbf{G}}$, the adjoint action of $\mathbf{G}$ on $\mathcal{N}_{il}$ can be extended to $\tilde{\mathbf{G}}$. Let $\tilde{g} = gz \in \tilde{\mathbf{G}}$ where $g \in \mathbf{G}$ and $z \in Z(\tilde{\mathbf{G}})$. Then for any $x \in \mathcal{U}_{ni}$ (resp. $n \in \mathcal{N}_{il}$), $\tilde{g} \cdot x = g \cdot x$ (resp. $\tilde{g} \cdot n = g \cdot n$). Hence, ϕ is $\tilde{\mathbf{G}}$-equivariant. Let $x \in U_{\mathcal{C},2}$. We have :

$$\begin{aligned}
\varphi_{\tilde{t}u\tilde{t}^{-1}}(x) &= \chi_q(\kappa(\phi(\tilde{t}u\tilde{t}^{-1})^*, \phi(x))) \\
&= \chi_q(\kappa(\tilde{t}^{-1} \cdot \phi(u)^*, \phi(x))) \\
&= \chi_q(\kappa(\phi(u)^*, \tilde{t} \cdot \phi(x))) \\
&= \chi_q(\kappa(\phi(u)^*, \phi(\tilde{t}x\tilde{t}^{-1}))) \\
&= \varphi_u(\tilde{t}x\tilde{t}^{-1}) = \varphi_u^{\tilde{t}}(x).
\end{aligned}$$

The second equality comes from the fact that $(\tilde{t} \cdot n)^* = \tilde{t}^{-1} \cdot n^*$ for $n \in \mathcal{N}_{il}$ and $\tilde{t} \in \tilde{T}$. Indeed, without loss of generality, we can assume that $\tilde{t} = t \in \mathbf{T}$ and $n \in \mathfrak{g}_\alpha$ for some root α. Then, since $n^* \in \mathfrak{g}_{-\alpha}$, we have

$$(t \cdot n)^* = (\alpha(t)n)^* = \alpha(t)n^* = (-\alpha)(t^{-1})n^* = t^{-1} \cdot n^*.$$

Finally, we have the required result. ■

If $u \in \tilde{G}$ is unipotent, it also belongs to G so we can associate to u a GGGR of G and a GGGR of $\tilde{G}$. We denote by Γ_u the GGGR of G associated to u and by $\tilde{\Gamma}_u$ the GGGR of $\tilde{G}$ associated to u. Induction from G to $\tilde{G}$ provides a relation between GGGRs of $\tilde{G}$ and GGGRs of G.

Proposition 3.2.13. *Let $u \in \tilde{G}$ be unipotent, then*

$$\tilde{\Gamma}_u = \mathrm{Ind}_G^{\tilde{G}}(\Gamma_u).$$

Proof. That comes directly from the definition of Γ_u and the fact that $\mathrm{Ind}_{U_{\mathcal{C},2}}^{\tilde{G}}(\varphi_u) = \mathrm{Ind}_G^{\tilde{G}} \circ \mathrm{Ind}_{U_{\mathcal{C},2}}^{G}(\varphi_u)$. ■

3.3 Character Sheaves and GGGRs

We introduce in this section results of Geck–Hézard and Taylor on generalised Gelfand–Graev representations which will be essential to prove that GGGRs satisfies the assumption of Proposition 3.1.2. After introducing some generalities on character sheaves and local systems, we introduce a theorem of Geck–Hézard which was proved under the assumption that $\mathbf{G}$ has connected center. Finally, we briefly explain how Taylor generalised the results of Geck-Hézard for groups with non-connected center.

3.3.1 Character sheaves and local systems

We assume that the center of $\mathbf{G}$ is connected. Let $\hat{\mathbf{G}}$ be the set of character sheaves on $\mathbf{G}$. Characters sheaves, introduced by Lusztig in [43], are certain irreducible perverse sheaves which are equivariant for the action of $\mathbf{G}$ by conjugation. The structure of $\hat{\mathbf{G}}$ is in a certain sense very similar to $\mathrm{Irr}_K(G)$: Lusztig gave in [44, 11.2] a partition

$$\hat{\mathbf{G}} = \bigsqcup \hat{\mathbf{G}}_s$$

where s runs over a complete set of representatives of F-stable $\mathbf{G}^*$-conjugacy class of semi-simple elements of G^*. Moreover, by [45, 17.8.3], for each class $(s)_{\mathbf{G}^*}$, we have a partition

$$\hat{\mathbf{G}}_s = \bigsqcup \hat{\mathbf{G}}_g$$

where g runs over representatives of F-stable special classes of $\mathbf{G}^*$ whose semi-simple part is s.

We say that $A \in \hat{\mathbf{G}}$ is F-stable if it is isomorphic to its inverse image F^*A under the Frobenius map F. We denote by $\hat{G}$ the set of F-stable character sheaves. If $A \in \hat{G}$, any isomorphism $\phi : F^*A \to A$ induces a class function $\chi_A \in \mathrm{CF}(G)$. By [46, 25.1], we can choose ϕ such that $\langle \chi_A, \chi_A \rangle_{KG} = 1$. According to Theorem 25.2 of [46], the set obtained $\{\chi_A \mid A \in \hat{G}\}$ is an orthonormal basis of $\mathrm{CF}_K(G)$. In particular, there are as many F-stable

character sheaves as irreducible characters of G over K. The structure of $\hat{\mathbf{G}}$ can be transfered to $\hat{G}$, we have a partition :

$$\hat{G} = \bigsqcup \hat{G}_g$$

where g runs over a set of representatives of F-stable special $\mathbf{G}^*$-conjugacy classes. Moreover, according to [57, I 5.7, II 3.2] , if $A \in \hat{G}_g$ then χ_A is a linear combination of irreducible characters of $\mathrm{Irr}_K(G)_g$.

Let $\mathcal{C}$ be an F-stable unipotent class of $\mathbf{G}$, let $I_{\mathcal{C}}$ be the set of isomorphism classes of $\mathbf{G}$-equivariant irreducible local systems on $\mathcal{C}$. We denote by $I_{\mathcal{C}}^F$ the local systems $\mathcal{E} \in I_{\mathcal{C}}$ such that $\mathcal{E} \simeq F^*\mathcal{E}$. Let $\mathcal{E} \in I_{\mathcal{C}}^F$, and let us fix an isomorphism $\psi : F^*\mathcal{E} \to \mathcal{E}$. Let us define the following function $Y_{\mathcal{E}}$ on G :

$$Y_{\mathcal{E}}(g) = \begin{cases} \mathrm{Trace}(\psi, \mathcal{E}_g) & \text{if } g \in \mathcal{C}^F, \\ 0 & \text{otherwise.} \end{cases}$$

In the proof of [46, 24.2.7], Lusztig proved the following statement about the functions $Y_{\mathcal{E}}$.

Proposition 3.3.1. *The functions $Y_{\mathcal{E}}$, $\mathcal{E} \in I_{\mathcal{C}}^F$, form a basis for the vector space of G-invariant functions on $\mathcal{C}^F$.*

3.3.2 The unitriangularity condition

Let $\mathcal{C}$ be an F-stable unipotent class of $\mathbf{G}$, and g be a special element of G^* with Jordan decomposition $g = sv$. We say that g satisfies the property (P) with respect to $\mathcal{C}$ if :

- $\Phi((g)_{\mathbf{G}^*}) = \mathcal{C}$.
- $|\Omega_g| = |\Omega_u|$. Recall that Ω_u is the canonical quotion of $A_{\mathbf{G}}(u)$ and $\Omega_u(g)$ is the canonical quotient of $A_{C_{\mathbf{G}^*}(s)}(v)$.
- The image of s in the adjoint quotient of $\mathbf{G}^*$ is quasi-isolated.

By a case-by-case analysis, the existence of such a g for any F-stable unipotent class $\mathcal{C}$ has been proved by Geck–Hézard.

Proposition 3.3.2 ([26, 2.4])**.** *Let $\mathcal{C}$ be an F-stable unipotent class of $\mathbf{G}$. There exists $g \in G^*$ special satisfying (P) with respect to $\mathcal{C}$.*

Remark 3.3.3. In [26, 2.4], the semi-simple part s of g was mentioned to be isolated. But in [62, 2.28], Taylor noted that for $\mathbf{G}$ of type C_n, Hézard used in his thesis a special element $g \in G^*$ with semi-simple part s such that $C_{\mathbf{G}^*}(s)$ is a Levi subgroup of type B_{n-1} of $\mathbf{G}^*$.

This result has many consequences on the restriction of character sheaves to unipotent classes.

Theorem 3.3.4 ([26, 3.2]). *Let $\mathcal{C}$ be an F-stable unipotent class of $\mathbf{G}$. Assume that $g = sv \in G^*$ satisfies the property (P) with respect to $\mathcal{C}$. Then*

- *For any $A \in \hat{\mathbf{G}}_g$, $A_{|\mathcal{C}}$ is either zero or an irreducible $\mathbf{G}$-equivariant local system up to shift.*
- *There is a bijection*

$$\begin{array}{rcl} \{A \in \hat{\mathbf{G}}_g |\, A_{|\mathcal{C}} \neq 0\} & \to & I_{\mathcal{C}}, \\ A & \mapsto & A_{|\mathcal{C}}. \end{array}$$

Remark 3.3.5.

1. For any $\mathcal{E} \in I_{\mathcal{C}}^F$, the unique character sheaf $A \in \hat{\mathbf{G}}_g$ such that $A|_{\mathcal{C}} = \mathcal{E}$ is F-stable. Indeed, since $(s)_{\mathbf{G}^*}$ and $(v)_{C_{\mathbf{G}^*}(s)}$ are F-stable, $F^*A \in \hat{\mathbf{G}}_g$ by [57, 5.5]. Moreover, $F^*A|_{\mathcal{C}} = F^*\mathcal{E} \simeq \mathcal{E} = A|_{\mathcal{C}}$. Hence $F^*A \simeq A$ according to the theorem.
2. Let us keep the notation of the theorem. We can translate those results as follows : for any $A \in \hat{G}_g$ we have either $\chi_{A_{|\mathcal{C}}} = 0$ or we can choose the isomorphism $\phi : F^*A \to A$ such that $\chi_A(g) = Y_{\mathcal{E}}(g)$ for all $g \in \mathcal{C}^F$ where $\mathcal{E} = A_{|\mathcal{C}}$.

Although the following statement only involves irreducible characters of G, the proof relies on character sheaves and local systems. Recall that if $\rho \in \mathrm{Irr}_K(G)$, we denote by ρ^* the irreducible character of G such that $\rho^* = \pm D_G(\rho)$.

Proposition 3.3.6. *Let $\mathcal{C}$ be an F-stable unipotent class of $\mathbf{G}$, $u_1, \ldots, u_d$ be representatives of the G-conjugacy classes of $\mathcal{C}^F$ and $g = sv \in G^*$. Asssume that g satisfies (P) with respect to $\mathcal{C}$. Then there exists $\rho_1, \ldots, \rho_d \in \mathrm{Irr}_K(G)_g$ such that the matrix $(\rho_i(u_j))_{1 \leq i,j \leq d}$ is invertible.*

Proof. By Proposition 3.3.1, $I_{\mathcal{C}}^F$ has d elements $\mathcal{E}_1, \ldots, \mathcal{E}_d$. By Theorem 3.3.4 and Remark 3.3.5 1), there exist $A_1, \ldots, A_d \in \hat{G}_g$ such that $A_{i|\mathcal{C}} = \mathcal{E}_i$. According to Remark 3.3.5 2), we can assume that $\chi_{A_i} = Y_{\mathcal{E}_i}$ on $\mathcal{C}^F$ and Proposition 3.3.1 tells us that $(\chi_{A_i}(u_j))_{1 \leq i,j \leq d}$ is invertible. Since each χ_{A_i} is a linear combination of characters in $\mathrm{Irr}_K(G)_g$, there exist $\rho_1, \ldots, \rho_d \in \mathrm{Irr}_K(G)_g$ such that the matrix $(\rho_i(u_j))_{1 \leq i,j \leq d}$ is invertible. ■

We can now state the result which will provide the unitriangularity condition for groups with connected center.

Theorem 3.3.7 (Geck–Hézard, [26, 4.3]). *Assume that the center of* $\mathbf{G}$ *is connected and* $\mathbf{G}$ *has classical type. Let* $\mathcal{C}$ *be an* F*-stable unipotent class of* $\mathbf{G}$ *and* $u_1, \ldots, u_d$ *be representatives of* G*-conjugacy classes of* $\mathcal{C}^F$*. Let* $g = sv \in G^*$ *satisfying* (P) *with respect to* $\mathcal{C}$*. Then, there exist* $\chi_1, \ldots, \chi_d \in \mathrm{Irr}_K(G)_g$ *such that* $\langle \chi_i^*, \gamma_{u_j} \rangle_{KG} = \delta_{i,j}$ *for* $1 \leq i, j \leq d$*.*

Proof. According to Remark 3.8 of [29], for any $\rho \in \mathrm{Irr}_K(G)_g$,

$$\sum_{i=1}^{d} [A_{\mathbf{G}}(u_i) : A_{\mathbf{G}}(u_i)^F] \langle \rho^*, \gamma_{u_i} \rangle_{KG} = n_\rho^{-1} |A_{\mathbf{G}}(u_1)|$$

where n_ρ is the minimal integer such that $n_\rho \rho(1)$ is a polynomial in q with integer coefficients. More precisely, following the notation of Theorem 2.1.12, we denote by x_ρ the element of $\bar{\mathcal{M}}(\Omega_g)$ corresponding to ρ and by $x \in \bar{\mathcal{M}}(\Omega_g)$ the element corresponding to the special representation of W_s in the family corresponding to $(v)_{C_{\mathbf{G}^*}(s)}$ (*i.e.* $E_{v,1} \otimes \varepsilon$), then $n_\rho = \{x_\rho, x\}$. Since g has property (P), $|\Omega_g| = |A_{\mathbf{G}}(u_1)|$ and the fact that we are in classical type implies that Ω_g is abelian (see [41, 4.5,4.6]). So $n_\rho = |A_{\mathbf{G}}(u_1)|$ according to the definition of $\{.,.\}$ (see Definition 2.1.9). Hence,

$$\sum_{i=1}^{d} [A_{\mathbf{G}}(u_i) : A_{\mathbf{G}}(u_i)^F] \langle \rho^*, \gamma_{u_i} \rangle_{KG} = 1.$$

Since each term of the sum is non-negative, there is a unique $i \in \{1, \ldots, d\}$ such that $\langle \rho^*, \gamma_{u_i} \rangle_{KG} = 1$ and $\langle \rho^*, \gamma_{u_j} \rangle_{KG} = 0$ for $j \neq i$. For $i \in \{1, \ldots, d\}$, let I_i the subset of $\mathrm{Irr}_K(G)_g$ consisting of characters ρ such that $\langle \rho, \gamma_{u_i} \rangle_{KG} = 1$ and $\langle \rho, \gamma_{u_j} \rangle_{KG} = 0$ for $j \neq i$. Assume that $I_r = \emptyset$ for some $r \in \{1, \ldots, d\}$, *i.e.* assume that for all $\rho \in \mathrm{Irr}_K(G)_g$, $\langle \rho^*, \gamma_{u_r} \rangle_{KG} = 0$ or, equivalently, $\langle \rho, D_G(\gamma_{u_r}) \rangle_{KG} = 0$. Writing the definition of the scalar product for the last equality we have

$$|G|^{-1} \sum_{g \in G} \overline{\rho(g)} D_G(\gamma_{u_r})(g) = 0.$$

Recall that, by Remark 3.2.9, $D_G(\gamma_{u_r})$ is 0 on non-unipotent elements. According to Proposition 3.2.10, $D_G(\gamma_{u_r})$ is non-zero only on unipotent classes $\mathcal{C}'$ such that $\mathcal{C} \subseteq \overline{\mathcal{C}'}$. On the other hand, by Proposition 3.2.11, $\mathcal{C}$ is the unipotent support of ρ. Then, according to [48, 11.2], ρ is non-zero only on classes $\mathcal{C}'$ such that $\dim \mathcal{C}' < \dim \mathcal{C}$ or $\mathcal{C}' = \mathcal{C}$. Hence, we only need to sum over elements of $\mathcal{C}^F$ and we get

$$\sum_{j=1}^{d} |C_G(u_j)|^{-1}\overline{\rho(u_j)}D_G(\gamma_{u_r})(u_j) = 0.$$

Introducing characters $\rho_1, \ldots, \rho_d$ of Proposition 3.3.6 in that equation, we get a linear relation on columns of the invertible matrix $(\rho_i(u_j))_{1\leq i,j\leq d}$. Then, $D_G(\gamma_{u_r})(u_j) = 0$ for $j \in \{1, \ldots, d\}$. According to [23, 2.4], for any $\mathcal{E} \in I_{\mathcal{C}}^F$, $\langle D_G(\gamma_{u_r}), Y_{\mathcal{E}}\rangle_{KG}$ equals $\zeta q^b \overline{Y_{\mathcal{E}}(u_r)}$ where ζ is a 4-th root of unity and b is an integer depending on $\mathcal{E}$. Hence, $Y_{\mathcal{E}}(u_r) = 0$ for any $\mathcal{E} \in I_{\mathcal{C}}^F$ but this is in contradiction with Proposition 3.3.1. That contradiction implies that $I_i \neq \emptyset$ for $i \in \{1, \ldots, d\}$ and for each $i \in \{1, \ldots, d\}$ we choose $\chi_i \in I_i$ such that $\langle \chi_i^*, \gamma_{u_j}\rangle_{KG} = \delta_{i,j}$ for all $j \in \{1, \ldots, d\}$. ■

3.3.3 Unitriangularity for groups with disconnected center

In [63] Taylor generalised Theorem 3.3.7 for groups with disconnected center, we will briefly discuss this generalisation. Suppose that $\mathbf{G}$ is a simple classical group with disconnected center, let $\tilde{\mathbf{G}} = \mathbf{G}\tilde{\mathbf{T}}$ be a regular embedding of $\mathbf{G}$ and $\tilde{G} = \tilde{\mathbf{G}}^F$ the associated finite reductive group. This property on the regular embedding and the Alvis-Curtis duality will be needed in the proof of the next theorem.

Lemma 3.3.8 ([63, 5.3])**.** *Let $\tilde{\chi} \in \mathrm{Irr}_K(\tilde{G})$ and let us write*

$$\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}) = \chi_1 + \cdots + \chi_r$$

where $\chi_i \in \mathrm{Irr}_K(G)$. Then $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}^*) = \chi_1^* + \cdots + \chi_r^*$.

Recall that for a unipotent element $u \in \tilde{G}$, we denote by γ_u the character of the associated GGGR of G and by $\tilde{\gamma}_u$ the character of the associated GGGR of $\tilde{G}$. We have $\tilde{\gamma}_u = \mathrm{Ind}_G^{\tilde{G}}(\gamma_u)$ by Proposition 3.2.13. Let $\mathcal{C} = (u)_{\mathbf{G}}$ be an F-stable conjugacy class of $\mathbf{G}$ such that $A_{\mathbf{G}}(u)$ is abelian. Let us write the partition of $\mathcal{C}^F$ into G-conjugacy classes : $\mathcal{C}^F = \mathcal{C}_1 \sqcup \cdots \sqcup \mathcal{C}_d$. For $i \in \{1, \ldots, d\}$ let u_i be a representative of $\mathcal{C}_i$.

Theorem 3.3.9 ([63, 5.5])**.** *Assume that $\mathcal{C} = (u)_{\mathbf{G}}$ with $A_{\mathbf{G}}(u)$ abelian. Then, there exists a semi-simple element $s \in G^*$ such that the image of s under an adjoint quotient of $\mathbf{G}^*$ is quasi-isolated and characters $\chi_1, \ldots, \chi_d \in \mathcal{E}(G, s)$ such that $\langle \chi_i^*, \gamma_{u_j}\rangle_{KG} = \delta_{i,j}$ for $1 \leq i, j \leq d$.*

Proof.

Let $\tilde{d}$ be the number of $\tilde{G}$-conjugacy classes of $\mathcal{C}^F$ and let $\mathcal{C}^F = \tilde{\mathcal{C}}_1 \sqcup \ldots \sqcup \tilde{\mathcal{C}}_{\tilde{d}}$ be the partition of $\mathcal{C}^F$ into $\tilde{G}$-conjugacy classes. For $i \in \{1, \ldots, \tilde{d}\}$, let us

decompose $\tilde{\mathcal{C}}_i \cap G = \mathcal{C}_{i,1} \sqcup \ldots \sqcup \mathcal{C}_{i,k_i}$ into G-conjugacy classes, where k_i depends on i. For each $i \in \{1, \ldots, \tilde{d}\}$ and $j \in \{1, \ldots, k_i\}$, let $u_{i,j}$ be a representative of $\mathcal{C}_{i,j}$. According to [63, 2.8], we can choose a representative $u \in \mathcal{C}^F$ such that F acts trivially on $A_{\tilde{\mathbf{G}}}(u)$ and

$$|A_{\mathbf{G}}(u)^F| = |Z_{\mathbf{G}}(u)^F||A_{\tilde{\mathbf{G}}}(u)| \tag{3.1}$$

where $Z_{\mathbf{G}}(u)$ is the image of $Z(\mathbf{G})$ in $A_{\mathbf{G}}(u)$. Because $\tilde{G}$-conjugacy classes of $\mathcal{C}^F$ are in bijection with F-classes of $A_{\tilde{\mathbf{G}}}(u)$ and $A_{\tilde{\mathbf{G}}}(u)$ is an abelian group on which F acts trivially, we have that $\tilde{d} = |A_{\tilde{\mathbf{G}}}(u)|$. According to [63, 3.2], we can chose $\tilde{g} = \tilde{s}\tilde{v} \in \tilde{G}^*$ such that

- the special element $\tilde{g}$ satisfies (P) with respect to $\mathcal{C}$,
- the restriction of any character of $\mathrm{Irr}_K(\tilde{G})_{\tilde{g}}$ to G has $|Z_{\mathbf{G}}(u)^F|$ irreducible constituents, without multiplicities.

Hence, we can use Theorem 3.3.7 to exhibit irreducible characters $\tilde{\chi}_1, \ldots, \tilde{\chi}_{\tilde{d}} \in \mathrm{Irr}_K(\tilde{G})_{\tilde{g}}$ such that $\langle \tilde{\chi}_i^*, \tilde{\gamma}_{u_{x,1}} \rangle_{K\tilde{G}} = \delta_{i,x}$ for $i, x \in \{1, \ldots, \tilde{d}\}$. Since the number of irreducible constituents of $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}_i^*)$ is $|Z_{\mathbf{G}}(u)^F|$, we can write by Lemma 3.3.8

$$\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}_i^*) = \chi_{i,1}^* + \cdots + \chi_{i,|Z_{\mathbf{G}}(u)^F|}^*$$

where the right side is a sum of distinct irreducible characters of G. In order to prove the theorem it is enough to prove that :

- we have $k_i = |Z_{\mathbf{G}}(u)^F|$ for every $i \in \{1, \ldots, \tilde{d}\}$, and
- $\langle \chi_{i,j}^*, \gamma_{u_{x,y}} \rangle_{KG} = \delta_{i,j}\delta_{x,y}$ for $1 \leq i, x \leq d$ and $1 \leq j, y \leq |Z_{\mathbf{G}}(u)^F|$.

Using Frobenius reciprocity and the fact that $\tilde{\gamma}_{u_{x,1}} = \mathrm{Ind}_G^{\tilde{G}}(\gamma_{u_{x,1}})$ we have that $\delta_{i,x} = \langle \mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}_i^*), \gamma_{u_{x,1}} \rangle_{KG}$. Then,

$$\delta_{i,x} = \sum_{j=1}^{|Z_{\mathbf{G}}(u)^F|} \langle \chi_{i,j}^*, \gamma_{u_{x,1}} \rangle_{KG}.$$

We can assume for more convenience that $\langle \chi_{i,j}^*, \gamma_{u_{i,1}} \rangle_{KG} = \delta_{j,1}$. Since the characters $\chi_{i,j}^*$ are irreducible constituents of $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}_i^*)$, we can choose $t_{i,1} = 1, \ldots, t_{i,|Z_{\mathbf{G}}(u)^F|} \in \tilde{T}$ such that $\chi_{i,j}^* = (\chi_{i,1}^*)^{t_{i,j}}$. Using 3.2.12, we have

$$1 = \langle \chi_{i,1}^*, \gamma_{u_{i,1}} \rangle_{KG} = \langle (\chi_{i,1}^*)^{t_{i,j}}, (\gamma_{u_{i,1}})^{t_{i,j}} \rangle_{KG} = \langle \chi_{i,j}^*, \gamma_{t_{i,j}u_{i,1}t_{i,j}^{-1}} \rangle_{KG}.$$

Moreover, if $j \neq k$, $t_{i,j}u_{i,1}t_{i,j}^{-1}$ and $t_{i,k}u_{i,1}t_{i,k}^{-1}$ lie in distinct G-conjugacy classes. Indeed, if $t_{i,j}u_{i,1}t_{i,j}^{-1}$ and $t_{i,k}u_{i,1}t_{i,k}^{-1}$ are in the same G-conjugacy class then $\gamma_{u_{i,1}} = \gamma_{t_{i,j}^{-1}t_{i,k}u_{i,1}t_{i,k}^{-1}t_{i,j}}$ and

$$1 = \langle \chi_{i,1}^*, \gamma_{t_{i,j}^{-1}t_{i,k}u_{i,1}t_{i,k}^{-1}t_{i,j}} \rangle_{KG} = \langle \chi_{i,j}^*, (\gamma_{u_{i,1}})^{t_{i,j}^{-1}t_{i,k}} \rangle_{KG} = \langle (\chi_{i,1}^*)^{t_{i,k}^{-1}t_{i,j}}, \gamma_{u_{i,1}} \rangle_{KG}.$$

The character $(\chi_{i,1}^*)^{t_{i,k}^{-1}t_{i,j}}$ is one of the characters $\chi_{i,1}^*, \ldots, \chi_{i,|Z_{\mathbf{G}}(u)^F|}^*$ but amongst those characters $\chi_{i,1}$ is the only one satisfying the above equality. Then, $\chi_{i,1}^* = (\chi_{i,1}^*)^{t_{i,j}t_{i,k}^{-1}}$ and $t_{i,j} = t_{i,k}$. Therefore, $k_i \leq |Z_{\mathbf{G}}(u)^F|$ and we can use the inequality below to deduce that $k_i = |Z_{\mathbf{G}}(u)^F|$:

$$|A_{\mathbf{G}}(u)^F| = |Z_{\mathbf{G}}(u)^F||A_{\tilde{\mathbf{G}}}(u)| = d|Z_{\mathbf{G}}(u)^F| \leq \sum_{i=1}^{\tilde{d}} k_i = |A_{\mathbf{G}}(u)^F|$$

where the first equality comes from equality (3.1). Now, by changing the numbering if necessary, we can assume that $u_{x,y} = t_{x,y}u_{x,1}t_{x,y}^{-1}$ for $x \in \{1, \ldots, \tilde{d}\}$ and $y \in \{1, \ldots, |Z_{\mathbf{G}}(u)^F|\}$. We have $\langle \chi_{i,j}^*, \gamma_{u_{x,y}} \rangle_{KG} = \langle (\chi_{i,j}^*)^{t_{x,y}^{-1}}, \gamma_{u_{x,1}} \rangle_{KG}$. Since $(\chi_{i,j}^*)^{t_{x,y}^{-1}}$ is an irreducible constituent of $\mathrm{Res}_G^{\tilde{G}}(\tilde{\chi}_{i,1}^*)$, we have that $\langle \chi_{i,j}^*, \gamma_{u_{x,y}} \rangle_{KG} = 0$ if $i \neq x$. We can prove similarly that $\langle \chi_{i,j}^*, \gamma_{u_{i,y}} \rangle_{KG} = 0$ if $j \neq y$. Hence, we can take the characters $\chi_{i,j}^*$ for $0 \leq i \leq \tilde{d}$ and $0 \leq j \leq |Z_{\mathbf{G}}(u)^F|$. Let s be the image of $\tilde{s}$ in G^*, then the characters $\chi_{i,j}^*$ belong to the rational series $\mathcal{E}(G, s)$ according to Theorem 1.3.24. Moreover, since s and $\tilde{s}$ have the same image under an adjoint quotient and $\tilde{g}$ satisfies the property (P), the image of s under an adjoint quotient is quasi-isolated. ■

3.4 Main result

Theorem 3.4.1. *Let* $\mathbf{G}$ *be a simple group of type* B*,* C *or* D *except spin of half-spin. Then there exists a unitriangular basic set for the unipotent* 2*-blocks of* G*.*

Proof. Let m be the number of unipotent modular representations of G. By Theorem 2.5.2, m is the number of unipotent classes of G so we can choose the GGGRs to be the projectives we need to use Proposition 3.1.2. Let us set some notation : let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be the F-stable unipotent classes of $\mathbf{G}$ ordered such that $i < j$ whenever $\dim \mathcal{C}_i < \dim \mathcal{C}_j$. For each class $\mathcal{C}_i$, let $u_{i,1}, \ldots, u_{i,r_i}$ be representatives of the G-conjugacy classes of $\mathcal{C}_i^F$. We denote by $\gamma_{u_{i,j}}$ the character of the GGGR corresponding to $u_{i,j}$ where $1 \leq i \leq r$ and

$1 \leq j \leq r_i$. According to Lemma 2.5.1, if $\mathcal{C}$ is a unipotent class of a simple group of type B, C, D which is not a spin group nor a half-spin group and $u \in \mathcal{C}$, then the group $A_{\mathbf{G}}(u)$ is abelian. Then, we can apply Theorem 3.3.9 and for each i there exists a semi-simple element $s_i \in G^*$ and irreducible characters $\chi_{i,j} \in \mathcal{E}(G, s_i)$ such that $\left(\langle \chi_{i,j}^*, \gamma_{u_{i,j'}} \rangle_{KG}\right)_{1 \leq j,j' \leq r_i}$ is the identity matrix. We will show that those characters form a basic set for unipotent blocks.

We first need to show that all the characters of the statement lie in a unipotent 2-block. By definition of B_1, it is enough to show that they all belong to rational Lusztig series associated to a semi-simple 2-element. The characters $\chi_{i,j}$ all belongs to $\mathcal{E}(G, s_i)$ and the image of s_i in the adjoint quotient of $\mathbf{G}^*$ is a quasi-isolated element. For adjoint groups of type B, C or D, all quasi-isolated elements are 2-elements. Therefore, there is an integer k such that $s_i^{2^k}$ belongs to $Z(\mathbf{G}^*)$ which is a 2-group, so s_i is a 2-element. We have m ordinary and projective characters in B_1 so, according to Proposition 3.1.2, we simply have to ensure that the matrix whose coefficients are $\left(\langle \chi_{i,j}^*, \gamma_{u_{i',j'}} \rangle_{KG}\right)_{(i,j),(i',j')}$ is lower unitriangular. Using the fact that for each class $\mathcal{C}_i$, $\left(\langle \chi_{i,j}^*, \gamma_{u_{i,j'}} \rangle_{KG}\right)_{1 \leq j,j' \leq r_i}$ is the identity matrix, we just have to check that $\langle \chi_{i,j}^*, \gamma_{u_{i',j'}} \rangle_{KG} = 0$ whenever $i < i'$.

Let $\tilde{\mathbf{G}}$ a regular embedding of $\mathbf{G}$, and let us fix i and i' such that $1 \leq i, i' \leq r$. Assume that there exists $(j, j') \in \{1, \ldots, r_i\} \times \{1, \ldots, r_{i'}\}$ such that $\langle \chi_{i,j}^*, \gamma_{u_{i',j'}} \rangle_{KG} \neq 0$. According to the proof of Theorem 3.3.9, there exists $\tilde{g}_i \in \tilde{\mathbf{G}}^*$ special such that $\Phi((\tilde{g}_i)_{\mathbf{G}^*}) = \mathcal{C}_i$ and $\tilde{\chi}_i \in \mathrm{Irr}_K(\tilde{G})_{\tilde{g}_i}$ such that $\chi_{i,j}$ is an irreducible constituent of $\mathrm{Res}_G^{\tilde{G}}(\chi_i)$. Moreover, by Proposition 3.2.11, $\tilde{\chi}_i$ has unipotent support $\mathcal{C}_i$ (so $\mathcal{C}_i$ is the wave front set of $\tilde{\chi}_i{}^*$). Since $\chi_{i,j}^*$ is an irreducible constituent of $\mathrm{Res}_G^{\tilde{G}} \tilde{\chi}_i^*$, we have $\langle \mathrm{Res}_G^{\tilde{G}} \tilde{\chi}_i^*, \gamma_{u_{i',j'}} \rangle_{KG} \neq 0$. Using Frobenius reciprocity, we get $\langle \tilde{\chi}_i^*, \tilde{\gamma}_{u_{i',j'}} \rangle_{K\tilde{G}} \neq 0$ so we have $i' \leq i$ since $\mathcal{C}_i$ is the wave front set of $\tilde{\chi}_i^*$. ■

4

Decomposition matrices and basic sets for groups of small rank

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This chapter focuses on basic sets and decomposition matrices for unipotent blocks of groups of small rank. Our first goal is to exhibit basic sets for unipotent blocks by using Theorem 2.3.16 and Theorem 3.4.1. The second goal is to apply the methods developed by Dudas in [18] to compute decomposition numbers.

In the first section, we begin by reviewing results on the complex of modules provided by the cohomology of Deligne–Lusztig varieties. Following the work of Rickard and Bonnafé–Rouquier, we can choose a representative consisting of a complex of projectives modules satisfying many properties (see 4.1.3). Then, going back to modular representation theory, we explain the consequences of this result on numerical conditions for decomposition numbers in Proposition 4.1.6.

In the second section, we apply the aforementioned results to unipotent 2-blocks of $G = \mathrm{Sp}_4(q)$ when q is odd. We first explain how the basic set mentioned in Theorem 3.4.1 can be constructed in the case of symplectic groups. For that, we introduce results of Hézard [30] on combinatorics of Weyl groups of type C. These results, combined with the Jordan decomposition of irreducible characters of $\mathrm{Sp}_4(q)$ allow us to determine which characters should be in the basic set of Theorem 3.4.1. Then, using the 2-decomposition matrix of $\mathrm{Sp}_4(q)$ computed by White in [65], we give the decomposition matrix for unipotent blocks with respect to this basic set. This matrix has an unknown coefficient $0 \leq x \leq (q-1)/2$. We use methods of the first sections to show that $x \in \{0, 1\}$.

In the last section, we are interested in ℓ-unipotent blocks of $G = G_2(q)$ when q is the power of a good prime number and $\ell = 2$ or 3. Theorem 2.3.16 provides a method to count unipotent modular representations but that method does not provide a parametrisation. However, since that theorem is based on Lusztig's parametrisation of unipotent characters, we can "guess" which characters should be chosen to get a basic set. After explaining how we proceed to select the characters, we use the decomposition matrix of G to check that the chosen set of characters indeed provides a basic set. The decomposition matrix of G was computed by Hiss–Shamash for $\ell = 2$ [32] and $\ell = 3$ [31]. Finally, we use again the result of Section 4.1 to get better bounds on decomposition numbers.

4.1 Cohomology complexes

Let $\mathbf{G}$ be a connected reductive group defined over $\mathbb{F}_q$ where q is a power of p, F be the corresponding Frobenius map and W be its Weyl group. Let $\mathbf{T}$

be an F-stable maximal torus of $\mathbf{G}$ contained in an F-stable Borel subgroup $\mathbf{B}$ of $\mathbf{G}$. As in Section 1.3, for $w \in W$, we fix a an F-stable torus of type w, which we denote by $\mathbf{T}_w$. Let ℓ be a prime number different from p, we fix an ℓ-modular system $(K, \mathcal{O}, k)$ where $\mathbb{Q}_\ell \subseteq K$ and such that K and k are big enough for G. We denote by d the decomposition map of G and by e its transpose (see Section 1.1.1). Let $\Lambda = K$ or k. We denote by

- $\mathrm{C}(\Lambda G)$ the category of complexes of ΛG-modules,
- $\mathrm{Ho}(\Lambda G)$ the homotopy category of ΛG-modules,
- $\mathrm{D}(\Lambda G)$ the derived category of ΛG-modules.

Decomposition map and Deligne–Lusztig Induction. For $w \in W$, we denote by X_w and Y_w the Deligne–Lusztig varieties associated to w (1.3.2). We recall that $G \times T_w^{op}$ acts on Y_w so that by §1.3.1, $R\Gamma_c(Y_w, \Lambda)$ is quasi-isomorphic to a bounded complex of $(\Lambda G, \Lambda T_w)$-bimodules. Let $H_c^*(Y_w, \Lambda) = \sum_i (-1)^i [H_c^i(Y_w, \Lambda)]$. We denote by $R_{T_w}^G$ and $\bar{R}_{T_w}^G$ the Deligne–Lusztig induction maps defined in Section 1.3. Let C be a bounded complex of $(\mathcal{O}G, \mathcal{O}T_w)$-bimodules representing $R\Gamma_c(Y_w, \mathcal{O})$ whose terms are free as $\mathcal{O}$-modules. In particular, $H_c^i(Y_w, \Lambda) = H_c^i(C \otimes_{\mathcal{O}} \Lambda)$. Let $\partial_i : C_i \to C_{i+1}$ be the differentiel map. By noting that $[C_i] = [H_c^i(Y_w, \Lambda)] + [\mathrm{Im}\, \partial_{i-1} \otimes_{\mathcal{O}} \Lambda] + [\mathrm{Im}\, \partial_i \otimes_{\mathcal{O}} \Lambda]$, we have $H_c^*(Y_w, \Lambda) = \sum_i (-1)^i [C_i \otimes_{\mathcal{O}} \Lambda]$. Let E be a KT_w-module and E' be a $\mathcal{O}T_w$-submodule such that $E = K \otimes_{\mathcal{O}} E'$. Then

$$\begin{aligned} H_c^*(Y_w, K) \otimes_{KT_w} [E] &= \left(\sum_i (-1)^i [C_i \otimes_{\mathcal{O}} K] \right) \otimes_{KT_w} [E] \\ &= \sum_i (-1)^i [C_i \otimes_{\mathcal{O}} K] \otimes_{KT_w} [E] \\ &= \sum_i (-1)^i [(C_i \otimes_{\mathcal{O}T_w} E') \otimes_{\mathcal{O}} K], \end{aligned}$$

and the following diagram commutes

$$\begin{tikzcd} R_K(T_w) \arrow[r, "R_{T_w}^G"] \arrow[d, "d"'] & R_K(G) \arrow[d, "d"] \\ R_k(T_w) \arrow[r, "\bar{R}_{T_w}^G"] & R_k(G). \end{tikzcd}$$

Complex of projective modules. We can represent $R\Gamma_c(Y_w, \Lambda)$ by a complex whose non-zero terms are in degrees $\ell(w)$ to $2\ell(w)$.

Theorem 4.1.1 ([38, 3.10c]). *$H^i_c(Y_w, \Lambda) = H^i_c(X_w, \Lambda) = 0$ if $i < \ell(w)$.*

Moreover, $R\Gamma_c(Y_w, \Lambda)$ can be represented by a complex whose termes are finitely generated projective ΛG-modules.

Theorem 4.1.2 ([51, 3.2]). *Let X be a quasi-projective variety defined over $\mathbb{F}_q$ on which G acts. If $|Stab_G(x)|$ is invertible in Λ for every $x \in X$, then $R\Gamma_c(X, \Lambda)$ is quasi-isomorphic to a bounded complex of finitely generated projective ΛG-modules.*

Corollary 4.1.3. *$R\Gamma_c(Y_w, \Lambda)$ can be represented by a complex of finitely generated projective ΛG-modules.*

Proof. Let $x = g\mathbf{U} \in Y_w$. The order of $\mathrm{Stab}_G(x) = (gUg^{-1})^F$ is a power of p so is invertible in Λ. Hence, we can apply Theorem 4.1.2 and represent $R\Gamma_c(Y_w, \Lambda)$ by a complex

$$0 \to P_{\ell(w)} \to P_{\ell(w)+1} \to \cdots \to P_{2\ell(w)} \to 0$$

where P_i is a finitely generated projective ΛG-module in degree i. ■

We denote by $R\mathrm{Hom}_{\Lambda G}$ the right-derived functor of the Hom-functor $\mathrm{Hom}_{\Lambda G}$. We recall that if C and D are two bounded complexes and if P is a projective resolution of C, then $R\mathrm{Hom}_{\Lambda G}(C, D)$ is quasi-isomorphic to the total Hom complex $\mathrm{Hom}^\bullet_{\Lambda G}(P, D)$.

The *Bruhat order* on W is a partial order defined as follows. For $v, w \in W$, we have $v \leq w$ if some substring of some reduced word for w is a reduced word for v.

Theorem 4.1.4 ([4, 8.10]). *Let $M \in \mathrm{Irr}_\Lambda(G)$ and $w \in W$ be minimal for the Bruhat order such that $R\mathrm{Hom}_{\Lambda G}(R\Gamma_c(Y_w, \Lambda), M) \neq 0$. Then, the cohomology of $R\mathrm{Hom}_{\Lambda G}(R\Gamma_c(Y_w, \Lambda), M)$ vanishes outside the degree $\ell(w)$.*

Corollary 4.1.5. *Let $M \in \mathrm{Irr}_\Lambda(G)$ and P_M be its projective cover. Let $w \in W$ be minimal for the Bruhat order such that $R\mathrm{Hom}_{\Lambda G}(R\Gamma_c(Y_w, \Lambda), M) \neq 0$. Then $R\Gamma_c(Y_w, \Lambda)$ can be represented by a complex*

$$0 \to P_{\ell(w)} \to \cdots \to P_{2\ell(w)} \to 0$$

such that

- *P_i is a finitely generated projective ΛG-module in degree i.*

- *P_M is a direct summand of P_i for $i = \ell(w)$ only.*

Proof. If M is an ΛG-module, we denote by $M[i]$ the complex of ΛG-modules where M is concentrated in degree i. Since $R\Gamma_c(Y_w, \Lambda)$ can be represented by a complex of finitely generated projective ΛG-modules we have, by definition of $R\mathrm{Hom}_{\Lambda G}$,

$$\begin{aligned} H^i(R\mathrm{Hom}_{\Lambda G})(R\Gamma_c(Y_w, \Lambda), M)) &\simeq \mathrm{Hom}_{\mathrm{D}(\Lambda G)}(R\Gamma_c(Y_w, \Lambda), M[i]) \\ &\simeq \mathrm{Hom}_{\mathrm{Ho}(\Lambda G)}(R\Gamma_c(Y_w, \Lambda), M[i]). \end{aligned}$$

Since the cohomology of $R\mathrm{Hom}_{\Lambda G}(R\Gamma(Y_w, \Lambda), M)$ vanishes outside the degree $\ell(w)$, we have $\mathrm{Hom}_{\mathrm{Ho}(\Lambda G)}(R\Gamma_c(Y_w, \Lambda), M[i]) = 0$ for $i \neq \ell(w)$.

Let $P := 0 \to P_{\ell(w)} \to \cdots \to P_{2\ell(w)} \to 0$ be a representative of $R\Gamma_c(Y_w, \Lambda)$ with finitely generated projective terms. By removing them if necessary, we can suppose that this complex has no null-homotopic direct summand. We have to show that P_M is a direct summand of P_i for $i = \ell(w)$ only. Let i be maximal such that P_M is a direct summand of P_i, in particular we have a surjective map $P_i \twoheadrightarrow P_M$. Assume that $i > \ell(w)$.We have two cases

- The composition $P_{i-1} \to P_i \twoheadrightarrow P_M$ is non-zero.
  Then the composition is surjective and the complex $\cdots 0 \to P_M \simeq P_M \to 0 \to \cdots$ (P_M is in degree $i-1$ and i) is a null-homotopic direct summand of P. Since we assumed that P has no null-homotopic direct summand, this is impossible.
- The composition $P_{i-1} \to P_i \twoheadrightarrow P_M$ is zero.
  Then, the map $P_i \twoheadrightarrow M$ induces a map in $\mathrm{Hom}_{\mathrm{Ho}(\Lambda G)}(R\Gamma_c(Y_w, \Lambda), M[i])$. Since $i > \ell(w)$, the map must be null-homotopic so it must factorise through P_{i+1}, which is impossible by the assumption on i.

Hence, we must have $i = \ell(w)$ and we are done. ■

Application to computation of decomposition numbers. We introduce a proposition giving numerical conditions on decomposition numbers which is a consequence of the previous results. We recall that $P_k(G)$ is the Grothendieck group of finitely generated projective kG-modules. Let $P_w := [H_c^*(Y_w, k)] = \bar{R}_{T_w}^G(kT_w) \in P_k(G)$. For $M \in \mathrm{Irr}_k(G)$, we denote by $W(M)$ the set of elements of W minimal for the Bruhat order such that $\langle P_w, M\rangle_{kG} \neq 0$ (*i.e.* such that $[P_M]$ occurs in the decomposition of P_w in the basis of projective indecomposable modules).

Proposition 4.1.6 ([18, 1.5]). *Let M be a simple kG-module and $w \in W(M)$. Then*

$$(-1)^{\ell(w)}\langle P_w, M\rangle_{kG} > 0.$$

Moreover, if N is a simple kG-module such that $(-1)^{\ell(w)}\langle P_w, N\rangle_{kG} < 0$, there exists $v < w$ such that $\langle P_v, N\rangle_{kG} \neq 0$.

Proof. Since $\langle P_w, M\rangle_{kG} \neq 0$, we have $R\mathrm{Hom}_{\Lambda G}(R\Gamma_c(Y_w, k), M) \neq 0$. Let $v \leq w$ be minimal for that property. Then, by Corollary 4.1.5, $R\Gamma_c(Y_v, k)$ can be represented by a complex of projective ΛG-modules $0 \to P_{\ell(v)} \to \cdots \to P_{2\ell(v)} \to 0$ such that P_M occurs only in $P_{\ell(v)}$. Hence, $\langle P_v, M\rangle_{kG} \neq 0$ and we must have $v = w$. We have

$$\langle (-1)^{\ell(w)} P_w, M\rangle_{kG} = \langle P_{\ell(w)}, M\rangle_{kG} > 0.$$

Let $N \in \mathrm{Irr}_k(G)$ such that $\langle (-1)^{\ell(w)} P_w, N\rangle_{kG} < 0$. Then, P_N can not only occur in $P_{2\ell(w)}$ so $w \notin W(N)$ according to the first part and there exists $v < w$ such that $\langle (-1)^{\ell(v)} P_v, N\rangle_{kG} \neq 0$. ■

We can use Proposition 4.1.6 to compute decomposition numbers when G has small rank and most of the decomposition numbers are known. Let B_1 be the union of unipotent ℓ-blocks of G and $\mathcal{B} := \{\chi_1, \ldots, \chi_r\}$ be a unitriangular basic set of B_1. Let $P_1, \ldots, P_r$ be the indecomposable projective kG-modules of B. We order χ_i and P_i such that the decomposition matrix is unitriangular (*i.e.* $e(P_j) = \chi_j + \sum_{i=j+1}^{r} d_{i,j}\chi_j$). Let b (resp. $\bar{b}$) be the idempotent of KG (resp. kG) associated to B_1. By definition of P_w and e, we have $e(\bar{b}P_w) = bR_w(KT_w)$. If we write $(-1)^{\ell(w)}\bar{b}P_w = \sum_{i=1}^{r} a_i P_i$, we have

$$(-1)^{\ell(w)} bR_w(KT_w) = \sum_{i=1}^{r} a_i e(P_i).$$

From the unitriangularity of the decomposition matrix and the decomposition of bP_w on the basic set $\mathcal{B}$ we can easily deduce the coefficients a_i from the decomposition numbers. Conversely, in the case where few decomposition numbers are missing, we can have bounds on them since we know that if w is minimal such that P_i appears in P_w, we have $a_i > 0$. That method will be used in the following section to compute decomposition numbers of $\mathrm{Sp}_4(q)$ (when q is odd and $\ell = 2$) and $G_2(q)$ (when $p > 3$ and $\ell = 2, 3$).

4.2 Basic set and decomposition matrix of $\mathrm{Sp}_4(q)$

4.2.1 Description of the basic set of $\mathbf{Sp}_{2n}(q)$

Let $\mathbf{G} = \mathrm{Sp}_{2n}(\bar{\mathbb{F}}_q)$ with q odd and let $\iota : \mathbf{G} \to \tilde{\mathbf{G}}$ be a regular embedding of $\mathbf{G}$ (we can take $\tilde{\mathbf{G}} = \mathrm{CSp}_{2n}(\bar{\mathbb{F}}_q)$). Let B_1 be the union of unipotent blocks of G. There exists, according to Theorem 3.3.9 a unitriangular basic set for B_1. We want to give enough details on this basic set to be able to exhibit it for $n = 2$. Let us denote by $\mathcal{B}$ a basic set satisfying the hypothesis of Theorem 3.4 and for each F-stable unipotent class $\mathcal{C} = (u)_{\mathbf{G}}$ of G, let us denote by d the number of G-conjugacy classes of $\mathcal{C}^F$. Then, $\mathcal{B}$ has the following property : there exists $\tilde{g} \in \tilde{G}^*$ special such that $\Phi((\tilde{g})_{\tilde{G}^*}) = \mathcal{C}$ and d characters $\chi_1, \ldots, \chi_d$ which are restrictions of characters of $\mathrm{Irr}_K(\tilde{G})_{\tilde{g}}$ such that $\chi_1^*, \ldots, \chi_d^* \in \mathcal{B}$. Each character of $\mathcal{B}$ can be constructed this way. We denote by $\tilde{s}$ the semi-simple part of $\tilde{g}$, by s the image of $\tilde{s}$ in $\mathbf{G}^*$. Let $A_{G^*}(s)$ be the group $C_{G^*}(s)/C_{G^*}(s)^\circ$.

Let us make a remark about the Jordan decomposition for $\mathcal{E}(G, s)$. We recall that the Jordan decomposition of characters for groups with disconnected center is given by the following bijection (see Theorem 1.3.25)

$$\{(\tilde{G}/G)\text{-orbits on } \mathcal{E}(G, s)\} \longleftrightarrow \{A_{G^*}(s)\text{-orbits on } \mathcal{E}(C_{G^*}(s)^\circ, 1)\}.$$

Assume that $\chi_1, \ldots, \chi_d$ are the irreducible constituents of $\mathrm{Res}_G^{\tilde{G}}\tilde{\chi} \in \mathcal{E}(\tilde{G}, \tilde{s})$ so they form a single $\tilde{G}/G$ orbit. Recall that, by Proposition 1.3.19, ι^* induces a bijection between $\mathcal{U}(C_{\tilde{G}^*}(\tilde{s}))$ and $\mathcal{U}(C_{G^*}(s)^\circ)$. Let $\tilde{\rho} \in \mathcal{U}(C_{\tilde{G}^*}(\tilde{s}))$ be the Jordan correspondent of $\tilde{\chi}$ and let ρ be the image of $\tilde{\rho}$ in $\mathcal{U}(C_{\mathbf{G}^*}(s))$. Then the Jordan correspondents of $\{\chi_1, \ldots, \chi_d\}$ are the characters of $\mathcal{U}(C_{G^*}(s))$ whose restriction to $C_{G^*}(s)^\circ$ lies in the $A_{G^*}(s)$-orbit of ρ. Let W_s° be the Weyl group of $C_{\mathbf{G}^*}(s)^\circ$ (it is isomorphic to the Weyl group of $C_{\tilde{\mathbf{G}}^*}(\tilde{s})$). We denote by $\mathcal{F}$ the family of W_s° corresponding to $\tilde{g}$. Then, if $\chi_1, \ldots, \chi_d$ are irreducible constituent of the restriction of a character of $\mathcal{E}(G, \tilde{g})$, their Jordan correspondents in $\mathcal{E}(C_{G^*}(s), 1)$ are irreducible constituent of the induction to $C_{G^*}(s)$ of a character in $\mathcal{U}(C_{G^*}(s)^\circ)_{\mathcal{F}}$. So to be able to describe $\chi_1, \ldots, \chi_d$ by the Jordan correspondence, we need to know :

$(\mathcal{Q})$
- What is the quasi-isolated class $(s)_{\mathbf{G}}^*$. Since $\mathbf{G}^*$ is adjoint, this is an isolated class.
- To which family of W_s° the special element $\tilde{g}$ corresponds.

Hézard answered those questions in his thesis [30, §4]. We will describe how he proceeded. We recall that unipotent classes of $\mathbf{G}$ are in bijection with partition of $2n$ such that any odd part occurs an even number of time. For

such a partition we denote by $\mathcal{C}_\lambda = (u_\lambda)_{\mathbf{G}}$ the corresponding class. Let δ be defined by

$$\delta(\mathcal{C}_\lambda) = \begin{cases} 1 & \text{if there exists } i \text{ even such that } \lambda \text{ has an odd number of parts equal to } i, \\ 0 & \text{otherwise.} \end{cases}$$

Let $c_1, \dots, c_r$ be the integers constructed from λ in Example 2.2.1. Let $m := (r-1)/2$ and let $[\Lambda]$ be the symbol

$$[\Lambda] = \begin{pmatrix} c_1 & & c_3 & \dots & c_r \\ & c_2 & \dots & c_{r-1} & \end{pmatrix} = \begin{pmatrix} a_1 & & a_2 & \dots & a_{m+1} \\ & b_1 & \dots & b_m & \end{pmatrix}.$$

We have two cases :

- $\delta(\mathcal{C}_\lambda) = 0$. Then the answer to $\mathcal{Q}$ is
  - $(s)_{\mathbf{G}^*}$ is the trivial class, so $\chi_1, \dots, \chi_d$ are unipotent characters.
  - The symbol Λ parametrise a special irreducible character of W. Let $\mathcal{F}$ be the family of $[\Lambda]$. Then $\mathcal{F}$ is the family corresponding to $\tilde{g}$ and $\chi_1, \dots, \chi_d \in \mathcal{U}(G)_{\mathcal{F}}$.
- $\delta(\mathcal{C}_\lambda) = 1$. Then W_s° is a group of type $D_a \times B_b$ with $a + b = n$. Let $s \in \mathbf{G}^* = \mathrm{SO}_{2n+1}$ be the following semi-simple element :

$$(s)_{\mathbf{G}^*} = (\mathrm{diag}(\overbrace{-1, \dots, -1}^{a \text{ times}}, \overbrace{1, \dots, 1}^{2b \text{ times}}, \overbrace{-1, \dots, -1}^{a \text{ times}}))_{\mathbf{G}^*}.$$

  We construct a symbol $[\Lambda_D]$ corresponding to an irreducible character of D_a.

$$\Lambda_D := \begin{pmatrix} \alpha_1 & \dots & \alpha_m \\ \beta_1 & \dots & \beta_m \end{pmatrix}$$

  where

$$\begin{cases} \alpha_i = a_{i+1} - \lfloor a_{i+1}/2 \rfloor - 1 \text{ for } i \in \{1, \dots, m\} \\ \beta_i = b_i - \lfloor b_i/2 \rfloor - 1 \text{ for } i \in \{1, \dots, m\} \end{cases}.$$

  Similarly we construct a symbol corresponding to an irreducible character of B_b.

$$\Lambda_B := \begin{pmatrix} \alpha'_1 & & \alpha'_2 & \dots & \alpha'_{m+1} \\ & \beta'_1 & \dots & \beta'_m & \end{pmatrix}$$

where

$$\begin{cases} \alpha'_i = \lfloor a_i/2 \rfloor \text{ for } i \in \{1, \dots, m+1\} \\ \beta'_i = \lfloor b_i/2 \rfloor \text{ for } i \in \{1, \dots, m\} \end{cases}$$

and $[\Lambda_D] \otimes [\Lambda_B]$ parametrises a special representation of W_s° corresponding to a family $\mathcal{F}$ of W_s. Finally, we can answer to $\mathcal{Q}$ as follows :

- We can choose the class $(s)_{\mathbf{G}^*}$ where s is the semi-simple element constructed above. This is an isolated class according to [30, 4.2.3].
- The family of W_s° corresponding to $\tilde{g}$ is the family $\mathcal{F}$.

4.2.2 Semi-simple elements of order 2 of $\mathbf{Sp}_4(q)$

We suppose from now on that $\mathbf{G} = \mathrm{Sp}_4(\bar{\mathbb{F}}_q)$ and

$$\mathbf{G}^* = \mathrm{SO}_5(\bar{\mathbb{F}}_q) := \{A \in \mathrm{SL}_n(k) \mid A^{\mathrm{tr}} Q A = Q\}$$

where

$$Q := \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We want to describe G^*-conjugacy classes of the semi-simple element of G^* involved in the previous part and their centralisers. It is enough to describe G^*-conjugacy classes of element of order 2. We need to fix some notation, we set

$$\mathbf{T}^* = \{\mathrm{diag}(\lambda_1, \lambda_2, 1, \lambda_2^{-1}, \lambda_1^{-1}), \quad \lambda_1, \lambda_2 \in \bar{\mathbb{F}}_q^\times\}.$$

The maximal torus $\mathbf{T}^*$ of $\mathbf{G}^*$ has 3 elements of order 2 :

$$r_1 := \mathrm{diag}(-1, -1, 1, -1, -1), \quad r_2 := \mathrm{diag}(-1, 1, 1, 1, -1), \quad r_3 := \mathrm{diag}(1, -1, 1, -1, 1)$$

Let us denote by a and b the generators of the Coxeter system W, we take the following representative n_a and n_b in $N_{\mathbf{G}^*}(\mathbf{T}^*)$:

$$n_a := \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad n_b := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The element r_1 is stable under the action of W. We have ${}^a r_2 = r_3$, ${}^b r_2 = r_2$, ${}^a r_3 = r_2$, ${}^b r_3 = r_3$. We now compute the centralisers of r_1, r_2 and r_3 in $\mathbf{G}^*$.

$$C_{\mathbf{G}^*}(r_1) = \left\{ \left(\begin{array}{c|c|c} A & 0 & B \\ \hline 0 & \varepsilon & 0 \\ \hline C & 0 & D \end{array} \right) \middle| \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \in \mathrm{O}_4(\bar{\mathbb{F}}_q), \quad \varepsilon = det \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \right\},$$

so $C_{\mathbf{G}^*}(r_1) \simeq \mathrm{O}_4(\bar{\mathbb{F}}_q)$. The centraliser of r_2 in $\mathbf{G}^*$ is

$$C_{\mathbf{G}^*}(r_2) = \left\{ \left(\begin{array}{c|c|c} \lambda & 0 & 0 \\ \hline 0 & A & 0 \\ \hline 0 & 0 & \lambda^{-1} \end{array} \right) \middle| A \in \mathrm{SO}_3(\bar{\mathbb{F}}_q), \quad \lambda \in \bar{\mathbb{F}}_q^\times \right\} . \langle Q \rangle$$

so $C_{\mathbf{G}^*}(r_1) \simeq (\mathrm{SO}_3 \times \bar{\mathbb{F}}_q^\times).2$. Let us fix $w \in W$ and $s \in T_w^*$ be an element of order 2. Let $g \in \mathbf{G}^*$ such that $\mathbf{T}_w^* = {}^g\mathbf{T}^*$. Conjugation by g induces an isomorphism $\mathbf{T}^{*wF} \simeq T_w$ where

$$\mathbf{T}^{*wF} = \{t \in \mathbf{T}^* \mid n_w F(t) n_w^{-1} = t\}$$

and we have $g^{-1}F(g) = n_w$. Let $r' \in \mathbf{T}^{*wF}$ such that $r' = {}^g r$. Then $C_{G^*}(r) \simeq C_{\mathbf{G}^*}(r')^{wF}$. Let us give more details about the sign ε occurring in Jordan decomposition Theorem 1.3.25 on Lusztig's series $\mathcal{E}(G, s)$. We have $\varepsilon = \varepsilon_G \varepsilon_{C_{G^*}(s)}$ where $\varepsilon_G = (-1)^{\mathbb{F}_q\text{-rank}(G)}$, see [14, 8.3] for the definition of the $\mathbb{F}_q$-rank. The following tables describe, for each r_i, if $r_i \in \mathbf{T}^{*wF}$, the group $C_{G^*}(r_i)^{wF}$ and $\varepsilon_{C_{G^*}(r_1)^{wF}}$. We denote by $\mu_{q+1} \subset \bar{\mathbb{F}}_q^\times$ the group of $(q+1)$-roots of unity.

w	$r_1 \in \mathbf{T}^{*wF}$?	$C_{G^*}(r_1)^{wF}$	$\varepsilon_{C_{G^*}(r_1)^{wF}}$
$\{e, a, bab, abab\}$	yes	$\mathrm{O}_4^+(q)$	1
$\{a, ab, ba, aba\}$	yes	$\mathrm{O}_4^-(q)$	-1

w	$r_2 \in \mathbf{T}^{*wF}$?	$C_{G^*}(r_2)^{wF}$	$\varepsilon_{C_{G^*}(r_2)^{wF}}$
$\{e, b\}$	yes	$(\mathrm{SO}_3(q) \times \mathbb{F}_q^\times).2$	1
$\{aba, abab\}$	yes	$(\mathrm{SO}_3(q) \times \mu_{q+1}).2$	-1
$\{a, ab, ba, bab\}$	no		

Since $r_3 = n_a r_2 n_a$ and n_a is F-stable, we have $C_{\mathbf{G}^*}(r_3)^{wF} \simeq C_{\mathbf{G}^*}(r_3)^{awaF}$ and the table for r_3 can be deduced from the table for r_2.

w	$r_3 \in \mathbf{T}^{*wF}$?	$C_{G^*}(r_3)^{wF}$	$\varepsilon_{C_{G^*}(r_3)^{wF}}$
$\{e, b\}$	yes	$(\mathrm{SO}_3(q) \times \mu_{q+1}).2$	-1
$\{aba, abab\}$	yes	$(\mathrm{SO}_3(q) \times \mathbb{F}_q^\times).2$	1
$\{a, ab, ba, bab\}$	no		

4.2.3 Unitriangular basic set and decomposition numbers for unipotent blocks of $\mathbf{Sp}_4(q)$

Recall that unipotent classes of $\mathbf{G}$ are parametrised by partitions of 4 such that any odd part occurs an even number of times. The following table gives, for each class $\mathcal{C}_\lambda$ corresponding to a partition λ, the eigenvalues of the corresponding semi-simple element s, the type of W_s°, the special character of W_s° constructed, the corresponding family in $\mathcal{E}(G,s)$ and the number of characters d in this family which lie in the basic set $\mathcal{B}$. We use notation of Srinivasan [61] for the irreducible characters of G and Table A1 of [65] for the Jordan decomposition.

λ	$\delta(\mathcal{C}_\lambda)$	eigenvalues	W_s°	Char. of W_s°	Char. of $\mathcal{E}(G,s)_{\mathcal{F}\otimes\varepsilon}$	d
(1^4)	1	1	W	ε	$\{1_G\}$	1
(2^2)	1	1	W	$\phi_{1,1}$	$\{\theta_9,\theta_{10},\theta_{11},\theta_{12}\}$	2
(4)	1	$(-1,1,1,1,-1)$	$D_1\times B_1$	1	$\{\Phi_7,\Phi_8\}$ or $\{\Phi_3,\Phi_4\}$	2
$(1^2,2)$	1	$(-1-1,1,-1,-1)$	D_2	ε	$\{\theta_3,\theta_4\}$ or $\{\theta_7,\theta_8\}$	2

Hence, to construct a basic set for B_1 satisfying the hypothesis of Theorem 3.3.9, we should

- Take the trivial character.
- Take 2 characters amongst $\{\theta_9,\theta_{10},\theta_{11},\theta_{12}\}$.
- Take $\{\theta_3,\theta_4\}$ or $\{\theta_7,\theta_8\}$.
- Take $\{\Phi_3,\Phi_4\}$ or $\{\Phi_7,\Phi_8\}$.

Then, we set $\mathcal{B}$ to be

$$\mathcal{B} := \{1_G,\theta_3,\theta_4,\theta_9,\theta_{10},\Phi_3,\Phi_4\}.$$

By checking the decomposition matrix of G in [65], we can see to $\mathcal{B}$ is indeed a unitriangular basic set for the unipotent 2-blocks. Indeed, up to applying the permutation (26375) to the columns of the decomposition matrix of the principal block in [65, 3.1], we can check that the decomposition matrix of B_1 with respect to $\mathcal{B}$ has the following shape :

	P_1	P_2	P_3	P_4	P_5	P_6	P_7
1_G	1						
θ_3	1	1					
θ_4	1		1				
θ_9	2	1	1	1			
θ_{10}					1		
Φ_3		1			x	1	
Φ_4			1		x		1

where $0 \leq x \leq (q-1)/2$ and $x \geq 1$ if $q \equiv 3 \mod 4$. We will now use methods explained in 4.1 to determine x.

Proposition 4.2.1. *We have $0 \leq x \leq 1$ and $x = 1$ if $q \equiv 3 \mod 4$.*

Proof. We just have to show that $x \leq 1$. Let b be the idempotent of $\mathcal{O}G$ associated to B_1. In order to be able to use Proposition 4.1.6, we need to compute $(-1)^{\ell(w)} b R_w(KT_w)$ for $w \in W$. Characters of $\mathcal{B}$ are either unipotent or lie in a rational Lusztig series associated to an element of order 2. Hence, we only need to compute

$$R^G_{T^*_w}(1) + \sum_s R^G_{T^*_w}(s)$$

where T^*_w is a torus of G^* dual to T_w and s runs over semi-simple elements of order 2 of T_w. We can decompose the characters $R^G_{T^*_w}(s)$ into irreducibles using the Jordan decomposition of characters. We can apply Theorem 1.3.25 and the tables of [65, A1] to decompose $(-1)^{\ell(w)} b R^G_{T_w}(KT_w)$ into irreducibles. Using the unitriangularity of the decomposition matrix, we can deduce the decomposition of $(-1)^{\ell(w)} \bar{b} P_w$ into the indecomposable projectives $P_1, \ldots, P_7$. The following table gathers this information.

w	$(-1)^{\ell(w)} b R^G_{T_w}(KT_w)$	$(-1)^{\ell(w)} \bar{b} P_w$
$\{e\}$	$1 + 2\theta_9 + \theta_3 + \theta_4$	$[P_1]$
$\{s, tst\}$	$-1_G - \theta_3 - \theta_4$	$-[P_1] + 2[P_4]$
$\{t, sts\}$	$-1_G + \Phi_3 + \Phi_4$	$-[P_1] + [P_2] + [P_3]$
$\{st, tst\}$	$1_G - \theta_9 + \theta_{10}$	$[P_1] - [P_2] - [P_3] - [P_4] + [P_5] + (1-x)[P_6] + (1-x)[$

The projectives P_6 and P_7 do not appear in P_v when $v < st$. Hence, according to Proposition 4.1.6, we should have $(1-x) \geq 0$ which gives $x \leq 1$. ■

4.3 Basic sets and decomposition matrices for $G_2(q)$

4.3.1 Unitriangular basic set for $G_2(q)$

Isolated elements of $G_2(q)$

Let $\mathbf{G} := G_2(\bar{\mathbb{F}}_q)$ and assume that $q = p^\alpha$ where $p > 3$. We want to compute isolated elements of $\mathbf{G}$ and their centralisers. Let Φ be the root system associated to $\mathbf{T}$ and Δ be the basis of Φ associated to $\mathbf{B}$. All the following information about the root system comes from [2]. With $\Delta = \{\alpha_1, \alpha_2\}$ we have :

$$\Phi := \pm\{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}.$$

Let $w_{\alpha_1}^\vee, w_{\alpha_2}^\vee$ be, as in §1.2.3, the fundamental coweights associated to α_1 and α_2 respectively. Let $\tilde{\alpha} := n_{\alpha_1}\alpha_1 + n_{\alpha_2}\alpha_2$ be the highest root and $\alpha_0 = -\tilde{\alpha}$. We have $w_{\alpha_1}^\vee = 2\alpha_1^\vee + \alpha_2^\vee$, $w_{\alpha_2}^\vee = 3\alpha_1^\vee + 2\alpha_2^\vee$ and $\alpha_0 = -\tilde{\alpha} = -3\alpha_1 - 2\alpha_2$. Let $\tilde{\Delta}$ be as in Section 1.2.3. Since $\mathbf{G}$ is simple, both adjoint and simply connected and $p > 3$, Theorem 1.2.12 gives a bijection between $\tilde{\Delta}$ and the set of G-conjugacy classes of isolated elements of G. More precisely, let $\tilde{\iota} : \mathbb{Q} \to k$ be as in §1.2.3. Then the bijection is induced by the map

$$\begin{array}{ccc} \tilde{\Delta} & \to & \mathbf{T} \\ \alpha & \mapsto & t_\alpha := w_\alpha^\vee(\tilde{\iota}(\frac{1}{n_\alpha})) \end{array}.$$

Since $p > 3$ we have

$$\tilde{\Delta} = \{\alpha_1, \alpha_2, \alpha_0\}.$$

Let us describe the elements t_α for $\alpha \in \Delta$.

- t_{α_0} is the identity and its centraliser is $\mathbf{G}$.
- $C_{\mathbf{G}}(t_{\alpha_1})$ has type A_2 and its root system is $\Phi_1 := \pm\{\alpha_2, 3\alpha_1 + \alpha_2, 3\alpha_1 + 2\alpha_2\}$. The element t_{α_1} has order 3.
- $C_{\mathbf{G}}(t_{\alpha_2})$ has type $A_1 \times A_1$ and its root system is $\Phi_2 := \pm\{\alpha_1, 3\alpha_1 + 2\alpha_2\}$. The order of t_α is 2.

We will denote t_{α_1} by t and t_{α_2} by s.

Notation for irreducible characters of $G_2(q)$

Chang–Ree in [9] classified irreducible characters of $G_2(q)$, and Hiss–Shamash computed an approximation of the decomposition matrix of $G_2(q)$ for $\ell = 2, 3$ in [32] and [31] using Chang–Ree notation. Our aim is to find a basic set satisfying the following properties :

- This basic set should, in some sense, be parametrised by the objects of Conjecture 2.3.14.
- We should know in which Lusztig series lies each character of this basic set in order to be able to use Proposition 4.1.6 to compute decomposition numbers.

In order to do this, we want to make a correspondence between notation of Chang–Ree and notation of Chevie. The table below gives information about unipotent characters of $G_2(q)$. The first column give the Chang–Ree notation, the second the Chevie notation and the third the degree of the character. The fourth column gives the unipotent class $(u)_{\mathbf{G}}$ associated to the parametrisation of Theorem 2.2.9, the fifth the associated canonical quotient and the last column provides the pair (x, σ) of $\mathcal{M}(\Omega_u)$ parametrising the character. We know that the pair (X_{13}, X_{14}) corresponds to the pair $(\phi'_{1,3}, \phi''_{1,3})$ but we are not able to say which character corresponds to the other. That is why we put them indistinctly in the table. We denote by Φ_d the d-th cyclotomic polynomial.

Chang–Ree	Chevie	Degree	Unipotent class	Ω_u	Pair (x, σ)
X_{11}	$\phi_{1,0}$	1	G_2	1	$(1, 1)$
X_{12}	$\phi_{1,6}$	q^6	1	1	$(1, 1)$
$\{X_{13}, X_{14}\}$	$\{\phi'_{1,3}, \phi''_{1,3}\}$	$\frac{1}{3}q\Phi_3(q)\Phi_6(q)$	$G_2(a_1)$	$\mathfrak{S}_3$	$\{(1, r), (g_3, 1)\}$
X_{15}	$\phi_{2,2}$	$\frac{1}{2}q\Phi_2(q)^2\Phi_6(q)$	–	–	$(g_2, 1)$
X_{16}	$\phi_{2,1}$	$\frac{1}{2}q\Phi_2(q)^2\Phi_3(q)$	–	–	$(1, 1)$
X_{17}	$G_2[-1]$	$\frac{1}{2}q\Phi_1(q)^2\Phi_3(q)$	–	–	(g_2, ε)
X_{18}	$G_2[1]$	$\frac{1}{6}\Phi_1(q)^2\Phi_6(q)$	–	–	$(1, \varepsilon)$
$\{X_{19}, \bar{X}_{19}\}$	$\{G_2[\theta], G_2[\theta^2]\}$	$\frac{1}{3}\Phi_1(q)^2\Phi_2(q)^2$	–	–	$\{(g_3, \theta), (g_3, \theta^2)\}$

Since $\mathbf{G}$ is both adjoint and simply connected, $\mathbf{G}$ is isomorphic to its dual group $\mathbf{G}^*$ so we can parametrise the Lusztig series by semi-simple elements of G. The following tables describe the characters of the series $\mathcal{E}(G, s)$ and $\mathcal{E}(G, t)$. The Jordan decomposition provides a natural bijection between $\mathcal{E}(G, s)$ and $\mathcal{E}(C_G(s), 1)$. Since $C_G(s)$ is of type $A_1 \times A_1$, its unipotent characters are parametrised by pairs of partition of 2. We denote by $\chi_{2,2}$, $\chi_{11,2}$, $\chi_{2,11}$ and $\chi_{2,2}$ the unipotent characters of $C_G(s)$. Then $\chi_{2,2}$ is the trivial character and $\chi_{11,11}$ is the Steinberg character. The table below lists the

characters of $\mathcal{E}(G,s)$ according to the Chang–Ree notation and provides the Jordan correspondent for each of those characters.

Characters of $\mathcal{E}(G,s)$

Chang–Ree	Degree	Jordan correspondent
X_{21}	$q^2(q^4+q^2+1)$	$\chi_{11,11}$
X_{22}	q^4+q^2+1	$\chi_{2,2}$
$\{X_{23}, X_{24}\}$	$q(q^4+q^2+1)$	$\{\chi_{2,11}, \chi_{11,2}\}$

We provide the same table for $\mathcal{E}(G,t)$, $C_G(t)$ is of type A_2 and we have 3 unipotent characters parametrised by partitions of 3 : χ_3 (trivial), χ_{21} and χ_{111} (Steinberg). Let $\epsilon \equiv q \mod 3 \in \{\pm 1\}$.

Characters of $\mathcal{E}(G,t)$

Chang–Ree	Degree	Jordan correspondent
X_{31}	$q^3(q^3+\epsilon)$	χ_{111}
X_{32}	$q^3+\epsilon$	χ_3
X_{33}	$q(q+\epsilon)(q^3+\epsilon)$	χ_{21}

Basic set for unipotent blocks

We want to get a basic set for unipotent blocks of $G_2(q)$ which would be parameterised in the same way as unipotent Brauer characters are counted in Conjecture 2.3.14. Here is how we will proceed : let $(u)_{\mathbf{G}}$ be an ℓ-special class. Note that we always have $A_{\mathbf{G}}(u) = \Omega_u = \Omega_u^\ell$ (see Table 4.3.2).

- We construct a subset of $\mathcal{M}(\Omega_u)$ in bijection with $\mathcal{M}(\Omega_u^\ell)$ as follows : for each Ω_u-conjugacy class $(x)_{\Omega_u}$, we choose an ℓ-basic set $\{\sigma_{x,1}, \ldots, \sigma_{x,k}\}$ of $C_{\Omega_u}(x)$. We choose the unipotent characters in $\mathrm{Irr}_K(G)_u$ corresponding to this subset according to Lusztig's parametrisation.
- If u is ℓ-special but not special then $A_{\mathbf{G}}(u) = 1$ and there is a unique character to pick in $\mathcal{E}(G,s)$.

Basic set for $\ell = 2$. There are four 2-special classes : 1, G_2, $G_2(a_1)$ and $\tilde{A}_1$. The last one is not special. The groups corresponding to 1 and G_2 are trivial and their corresponding unipotent characters are respectively $\phi_{1,1}$ and $\phi_{1,6}$. We have, as explained above, to choose characters lying in the family corresponding to $G_2(a_1)$. If $(u)_{\mathbf{G}} = G_2(a_1)$, Ω_u is the symmetric group $\mathfrak{S}_3$ and we denote by e the trivial element, by g_2 the transposition (12) and by g_3 the 3-cycle (123).

- Since $C_{\Omega_u}(e) = \mathfrak{S}_3$, we should have two unipotent characters corresponding to $\{(1,\sigma_1),(1,\sigma_2)\}$ where σ_1,σ_2 is a basic set of $\mathfrak{S}_3$. We can have two such basic sets $\{1,r\}$ where r is the reflection representation and $\{\varepsilon,r\}$. So we can have either the pair of characters $\{\phi_{2,1},\phi'_{1,3}\}$ or the pair $\{G_2[1],\phi'_{1,3}\}$.
- Since $C_{\Omega_u}(g_2) \simeq \mathbb{Z}/2\mathbb{Z}$, we have to choose one character between $\phi_{2,2}$ and $G_2[-1]$.
- Since $C_{\Omega_u}(g_3) \simeq \mathbb{Z}/3\mathbb{Z}$, we have to take $\phi''_{1,3}$, $G_2[\theta]$ and $G_2[\theta^2]$.

It remains to look at the class $\tilde{A}_1$. Its component group is trivial so there is only one character from $\mathcal{E}(G,s)$ to add. The Jordan correspondent χ of that character should have the following property. Let $(v)_{C_{\mathbf{G}}(s)}$ be the unipotent class of $C_{\mathbf{G}}(s)$ corresponding to χ by the Lusztig parametrisation. Then the image of $(v)_{C_{\mathbf{G}}(s)}$ by Lusztig's map Φ should be $\tilde{A}_1$. Using CHEVIE [50], we can deduce that $(v)_{C_{\mathbf{G}}(s)}$ is the class parametrised by $(11,11)$. Hence, χ should correspond to the trivial representation, that is $\chi_{2,2}$. According to the table above, we should take the characters X_{22} that we will denote by $\chi_{s,(2,2)}$.

With the goal of having a basic set respecting the above requirements and giving the "nicest" decomposition matrix possible we chose the following 2-basic set for unipotent blocks :

$$\{\phi_{1,0},\phi_{1,6},\phi'_{1,3},\phi''_{1,3},G_2[1],G_2[-1],G_2[\theta],G_2[\theta^2],\chi_{s,(2,2)}\}.$$

Except for $G_2[\theta]$ and $G_2[\theta^2]$, all characters of the basic set lie in the principal block. The characters $G_2[\theta]$ and $G_2[\theta^2]$ lie in blocks containing a single modular representation so we just need to work on the principal block. The following table provides the decomposition matrix of the principal block in respect with our basic set, the first two columns give the ℓ-special class $(u)_{\mathbf{G}}$ and the corresponding pair as above. The decomposition matrix of Hiss–Shamash [32] uses the Chang–Ree notation so we are unable to distinguish the line corresponding to $\phi'_{1,3}$ from the line corresponding to $\phi''_{1,3}$.

2-special class	Pair (x,σ)	Character	P_1	P_2	P_3	P_4	P_5	P_6	P_7
G_2	$(1,1)$	$\phi_{1,0}$	1						
$\tilde{A}_1$	$(1,1)$	$\chi_{s,(2,2)}$	1	1					
$G_2[a_1]$	(g_2,ε)	$G_2[-1]$			1				
	$(1,\varepsilon)$	$G_2[1]$				1			
	$(1,r)$	$\{\phi'_{1,3},\phi''_{1,3}\}$	1				1		
	$(g_3,1)$		1					1	
G_2	$(1,1)$	$\phi_{1,6}$	1		α	β	1	1	1

We have upper and lower bounds on α and β depending on the value of q.

1. $0 \leq \alpha \leq q-1$ if $q \equiv 1 \mod 4$ and $1 \leq \alpha \leq q-1$ if $q \equiv -1 \mod 4$.
2. $0 \leq \beta \leq (q+2)/3$ if $q \equiv 1 \mod 4$ and $1 \leq \beta \leq (q+2)/3$ if $q \equiv -1 \mod 4$.

Basic set for $\ell = 3$. The 3-special classes are : 1, G_2, $G_2(a_1)$, and A_1, all of them are special except A_1. For all of those classes $(u)_{\mathbf{G}}$, Ω_u is trivial except for $G_2(a_1)$ where Ω_u is $\mathfrak{S}_3$.

- The characters $\phi_{1,1}$ and $\phi_{1,6}$ correspond respectively to G_2 and 1 by the Springer correspondence.
- Let $(u)_{\mathbf{G}} = G_2(a_1)$. The possible 3-basic sets of Ω_u are $(1,\epsilon)$, $(1,r)$ and (ϵ,r) so we have to choose 2 characters among $\phi_{2,1}, G_2[1], \phi'_{1,3}$. Then $C_{\Omega_u}(g_2) \simeq \mathbb{Z}/2\mathbb{Z}$ so we have to take $\phi_{2,2}$ and $G_2[-1]$. The group $C_{\Omega_u}(g_3)$ has order 3 so we have to take one character from $\phi''_{1,3}, G_2[\theta], G_2[\theta^2]$.
- It remains to look at the class A_1. Its component group is trivial so there is only one character from $\mathcal{E}(G,t)$ to add. This character should have the following property : let $\chi \in \mathcal{E}(C_G(t),1)$ be its Jordan correspondent. Since $C_{\mathbf{G}}(t)$ is of type A_2, each familly contains exactly one unipotent character. Let $(v)_{C_{\mathbf{G}}(t)}$ be the unipotent class of $C_{\mathbf{G}}(t)$ corresponding to the familly $\mathcal{F}$ of χ. Then the image of $(v)_{C_{\mathbf{G}}(t)}$ by Lusztig's map Φ should be $\tilde{A}_1$. By CHEVIE, $(v)_{C_{\mathbf{G}}(t)}$ is the class parameterised by (111) so χ should be in the familly corresponding to the trivial representation, that is χ_3. Then, we should take the characters X_{22} that we will denote by $\chi_{t,3}$.

Finally, using the 3-decomposition matrix computed in [31] by Hiss and Shamash we choose the following characters for our basic set :

$$\{\phi_{1,0}, \phi_{1,6}, \phi'_{1,3}, \phi''_{1,3}, \phi_{2,2}, G_2[-1], G_2[1], \chi_{t,3}\}.$$

For the decomposition matrix, we have to distinguish the cases $q \equiv 1 \mod 3$ and $q \equiv -1 \mod 3$:

- $q \equiv 1 \mod 3$

  The character $G_2[-1]$ lies in a block containing a unique modular representation, the other characters lie in the principal block. Here is the decomposition matrix of the principal block according to our basic set :

3-special class	Pair (x,σ)	Character	P_1	P_2	P_3	P_4	P_5	P_6	P_7
G_2	$(1,1)$	$\phi_{1,0}$	1						
A_1	$(1,1)$	$\chi_{t,3}$	1	1					
$G_2[a_1]$	$(1,\varepsilon)$	$G_2[1]$			1				
	$(1,r)$	$\{\phi'_{1,3},\phi''_{1,3}\}$		1		1			
	$(g_3,1)$				α		1		
	$(g_2,1)$	$\phi_{2,2}$	1					1	
G_2	$(1,1)$	$\phi_{1,6}$	1		β	γ		1	1

  We have $1 \leq \alpha \leq 2$, $0 \leq \beta \leq q-2$ and $1 \leq \gamma \leq q+1$.

- $q \equiv -1 \mod 3$

  The character $\phi_{2,2}$ lies in a block containing a unique irreducible modular representation. The other characters lie in the principal block. Here is the decomposition matrix of the principal block according to our basic set :

3-special class	Pair (x,σ)	Character	P_1	P_2	P_3	P_4	P_5	P_6	P_7
G_2	$(1,1)$	$\phi_{1,0}$	1						
A_1	$(1,1)$	$\chi_{t,3}$		1					
$G_2[a_1]$	$(1,\varepsilon)$	$G_2[1]$			1				
	$(1,r)$	$\{\phi'_{1,3},\phi''_{1,3}\}$	1	1		1			
	$(g_3,1)$		1		1		1		
	$(g_2,1)$	$G_2[-1]$						1	
G_2	$(1,1)$	$\phi_{1,6}$	1	1	α	β	1	γ	1

  We have $1 \leq \alpha \leq q+1$, $1 \leq \beta \leq q-1$ and $1 \leq \gamma \leq q/2$.

4.3.2 Decomposition numbers

Let a and b be the generators of the Weyl group W of $\mathbf{G}$. Let P_w be the projective modules defined as in 4.1. Since the methods of §4.1 are easier

to apply to unipotent characters, we will replace the basic sets of the previous section by basic sets consisting of unipotent characters for the principal blocks. Moreover, throughout this section we will use Chang-Ree notation. The following table provides the decomposition of $(-1)^{\ell(w)}bP_w$ in terms of the characters of the basic sets of the decomposition matrices below.

w	$(-1)^{\ell(w)}P_w$
1	$X_{11}+X_{12}+X_{13}+X_{14}+2X_{15}+2X_{16}+\dots$
a	$-X_{11}+X_{12}-X_{13}+X_{14}+\dots$
b	$-X_{11}+X_{12}+X_{13}-X_{14}+\dots$
(ab)	$X_{11}+X_{12}-X_{16}+X_{17}+X_{19}+\dots$
$(ab)^2$	$X_{11}+X_{12}-X_{15}+X_{18}-X_{19}+\dots$
$(ab)^3$	$X_{11}+X_{12}-X_{13}-X_{14}-2X_{17}-2X_{18}+\dots$

Case $\ell=2$. The decomposition matrix of the principal block in [32] provides us the following basic set of unipotent characters with the corresponding decomposition matrix :

	P_1	P_2	P_3	P_4	P_5	P_6	P_7
X_{11}	1						
X_{17}		1					
X_{18}			1				
X_{13}	1			1			
X_{14}	1				1		
X_{15}		1				1	
X_{12}	1	α	β	1	1		1

We have to decompose the projectives $(-1)^{\ell(w)}P_w$ in terms of the basis $(P_1,\dots,P_7)$ by using the unitriangularity of the decomposition matrix. We get $P_e=P_1+2P_6$, $-P_a=-P_1+2P_5$ and $-P_b=-P_1+2P_4$. Let us write

$$P_{ab}=\sum_{i=1}^{7}a_iP_i.$$

Since P_7 does not appear in P_w for $w<ab$, we must have $a_7\geq 0$ by Proposition 4.1.6. Using the unitriangularity of the decomposition matrix, and the decomposition of P_{ab} into the basic set we get $a_1=a_2=1, a_3=0, a_4=a_5=a_6=-1$. Studying the coefficients of X_{12} for each side of the equality we get $a_1+\alpha a_2+\beta a_3+a_4+a_5+a_7=1$, hence $\alpha=2-a_7$. Since $a_7\geq 0$, we have

$$\boxed{\alpha \leq 2}.$$

Case $\ell = 3$, $q \equiv 1 \mod 3$**.** We start from the following decomposition matrix of the principal block :

	P_1	P_2	P_3	P_4	P_5	P_6	P_7
X_{11}	1						
X_{18}		1					
X_{19}			1				
X_{14}		α		1			
X_{15}	1				1		
X_{16}		$\alpha - 1$		1		1	
X_{12}		β	γ		1		1

Using the unitriangularity of the decomposition matrix, we have $P_e = P_1 + P_4 + P_5 + P_6$, and $-P_a = -P_1 + P_4 + P_5 - P_6$ and $-P_b = -P_1 - P_4 + P_5 + P_6$. If we write

$$P_{ab} = \sum_{i=1}^{7} a_i P_i,$$

we have $a_1 = a_3 = 1$, $a_2 = a_4 = 0$, $a_5 = a_6 = -1$, hence we deduce $\gamma = 2 - a_7$. Using the same argument as above, we have $a_7 \geq 0$ and we deduce

$$\boxed{\gamma \leq 2}.$$

Case $\ell = 3$, $q \equiv -1 \mod 3$**.** We start from the following decomposition matrix of the principal block :

	P_1	P_2	P_3	P_4	P_5	P_6	P_7
X_{11}	1						
X_{18}		1					
X_{19}			1				
X_{14}	1	1		1			
X_{17}					1		
X_{16}	2			1		1	
X_{12}	1	α	β	1	γ	1	1

Using the unitriangularity of the decomposition matrix as above, we get

$P_e = -P_1$, $-P_a = -P_1 + 2P_4$ and $-P_b = -P_1 + 2P_6$. Writing

$$P_{ab} = \sum_{i=1}^{7} a_i P_i,$$

we have $a_1 = a_3 = a_5 = 1$, $a_2 = 0$, $a_4 = -1$ and $a_6 = -2$. Looking at the coefficients of X_{12} on each side of the equality we get $\beta + \gamma = 3 - a_7$. Since P_7 does not appear in any P_w for $w < ab$ we have $a_7 \geq 0$ and we deduce that

$$\boxed{\beta + \gamma \leq 3}.$$

Appendix

Appendix A : ℓ-special unipotent classes for exceptional groups (ℓ bad)

Here $A_{\mathbf{G}}(u)$ denote the component group of $C_{\mathbf{G}}(u)$, Ω_u^ℓ denote the ℓ-special quotient and α_ℓ, u the cardinal of $\bar{\mathcal{M}}_\ell(\Omega_u^\ell)$ (see Definition 2.3.12). Having blank entries for Ω_u^ℓ and $\alpha_{\ell,u}$ means that the corresponding class is not ℓ-special.

Type G_2

Classes	$A_{\mathbf{G}}(u)$	Ω_u^2	$\alpha_{2,u}$	Ω_u^3	$\alpha_{3,u}$
1	1	1	1	1	1
A_1	1			1	1
$\widetilde{A}_1$	1	1	1		
$G_2(a_1)$	$\mathfrak{S}_3$	$\mathfrak{S}_3$	6	$\mathfrak{S}_3$	5
G_2	1	1	1	1	1

Type F_4

Classes	$A_{\mathbf{G}}(u)$	Ω_u^2	$\alpha_{2,u}$	Ω_u^3	$\alpha_{3,u}$
1	1	1	1	1	1
A_1	1	1	1		
$\tilde{A}_1$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4
$A_1+\tilde{A}_1$	1	1	1	1	1
$\tilde{A}_2$	1	1	1	1	1
A_2	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	1	1
$A_2+\tilde{A}_1$	1	1	1		
$\tilde{A}_2+A_1$	1			1	1
B_2	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2		
$C_3(a_1)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2		
$F_4(a_3)$	$\mathfrak{S}_4$	$\mathfrak{S}_4$	8	$\mathfrak{S}_4$	18
C_3	1	1	1	1	1
B_3	1	1	1	1	1
$F_4(a_2)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	1	1
$F_4(a_1)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4
F_4	1	1	1	1	1

Type E_6

Classes	$A_{\mathbf{G}}(u)$	Ω_u^2	$\alpha_{2,u}$	Ω_u^3	$\alpha_{3,u}$
1	1	1	1	1	1
A_1	1	1	1	1	1
$2A_1$	1	1	1	1	1
$3A_1$	1	1	1		
A_2	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4
A_2+A_1	1	1	1	1	1
A_2+2A_1	1	1	1	1	1
$2A_2$	1	1	1	1	1
$2A_2+A_1$	1			1	1
A_3	1	1	1	1	1
A_3+A_1	1	1	1		
$D_4(a_1)$	$\mathfrak{S}_3$	$\mathfrak{S}_3$	6	$\mathfrak{S}_3$	5
A_4	1	1	1	1	1
D_4	1	1	1	1	1
A_4+A_1	1	1	1	1	1
$D_5(a_1)$	1	1	1	1	1
A_5	1	1	1		
$E_6(a_3)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4
D_5	1	1	1	1	1
$E_6(a_1)$	1	1	1	1	1
E_6	1	1	1	1	1

Type E_7

Classes	$A_{\mathbf{G}}(u)$	Ω_u^2	$\alpha_{2,u}$	Ω_u^3	$\alpha_{3,u}$
1	1	1	1	1	1
A_1	1	1	1	1	1
$2A_1$	1	1	1	1	1
$3A_1'$	1	1	1		
$3A_1''$	1	1	1	1	1
A_2	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4
$4A_1$	1	1	1		
A_2+A_1	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4
A_2+2A_1	1	1	1	1	1
A_2+3A_1	1	1	1	1	1
A_3	1	1	1	1	1
$2A_2$	1	1	1	1	1
$2A_2+A_1$	1			1	1
$(A_3+A_1)''$	1	1	1	1	1
$(A_3+A_1)'$	1	1	1		
A_3+2A1	1	1	1		
$D_4(a_1)$	$\mathfrak{S}_3$	$\mathfrak{S}_3$	6	$\mathfrak{S}_3$	5
$D_4(a_1)+A_1$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4
A_3+A_2	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	1	1
A_4	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4
$A_3+A_2+A_1$	1	1	1	1	1
D_4	1	1	1	1	1
A_4+A_1	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4
D_4+A_1	1	1	1		
$D_5(a_1)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4
A_5''	1	1	1	1	1
A_4+A_2	1	1	1	1	1
A_5+A_1	1			1	1
$D_5(a_1)+A_1$	1	1	1	1	1
A_5'	1	1	1		
$E_6(a_3)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4
$D_6(a_2)$	1	1	1		
D_5	1	1	1	1	1
$E_7(a_5)$	$\mathfrak{S}_3$	$\mathfrak{S}_3$	6	$\mathfrak{S}_3$	5
$D_6(a_1)$	1	1	1	1	1
D_5+A_1	1	1	1	1	1
A_6	1	1	1	1	1
$E_7(a_4)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	1	1
D_6	1	1	1		
$E_6(a_1)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4

Type E_8

Classes	$A_{\mathbf{G}}(u)$	Ω_u^2	$\alpha_{2,u}$	Ω_u^3	$\alpha_{3,u}$	Ω_u^5	$\alpha_{5,u}$
1	1	1	1	1	1	1	1
A_1	1	1	1	1	1	1	1
$2A_1$	1	1	1	1	1	1	1
$3A_1$	1	1	1				
A_2	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
$4A_1$	1	1	1				
A_2+A_1	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
A_2+2A_1	1	1	1	1	1	1	1
A_2+3A_1	1	1	1				
$2A_2$	2	2	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
A_3	1	1	1	1	1	1	1
$2A_2+A_1$	1			1	1		
A_3+A_1	1	1	1				
$2A_2+2A_1$	1			1	1		
$D_4(a_1)$	$\mathfrak{S}_3$	$\mathfrak{S}_3$	6	$\mathfrak{S}_3$	5	$\mathfrak{S}_3$	8
A_3+2A_1	1	1	1				
$D_4(a_1)+A_1$	$\mathfrak{S}_3$	$\mathfrak{S}_3$	6	$\mathfrak{S}_3$	5	$\mathfrak{S}_3$	8
A_3+A_2	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	1	1	1	1
$A_3+A_2+A_1$	1	1	1				
$D_4(a_1)+A_2$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
A_4	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
D_4	1	1	1	1	1	1	1
$2A_3$	1	1	1				
A_4+A_1	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
D_4+A_1	1	1	1				
A_4+2A_1	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
A_4+A_2	1	1	1	1	1	1	1
$D_5(a_1)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
$D_5(a_1)+A_1$	1	1	1	1	1	1	1
$A_4+A_2+A_1$	1	1	1	1	1	1	1
A_5	1	1	1				
D_4+A_2	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	1	1	1	1
A_4+A_3	1					1	1
$D_5(a_1)+A_2$	1	1	1				
A_5+A_1	1						
$E_6(a_3)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
$D_6(a_2)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2				
$E_6(a_3)+A_1$	$\mathfrak{S}_2$			$\mathfrak{S}_2$	4		
D_5	1	1	1	1	1	1	1
$E_7(a_5)$	$\mathfrak{S}_3$	$\mathfrak{S}_3$	6				

Type E_8 -continued

Classes	$A_{\mathbf{G}}(u)$	Ω_u^2	$\alpha_{2,u}$	Ω_u^3	$\alpha_{3,u}$	Ω_u^5	$\alpha_{5,u}$
$D_5{+}A_2$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	1	1	1	1
$E_6(a_1)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
$D_7(a_2)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
A_7	1	1	1				
$E_6(a_1){+}A_1$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
D_6	1	1	1				
$E_8(b_6)$	$\mathfrak{S}_3$	$\mathfrak{S}_3$	2	$\mathfrak{S}_3$	5	$\mathfrak{S}_2$	4
$E_7(a_3)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
E_6	1	1	1	1	1	1	1
$D_7(a_1)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	1	1	1	1
$E_6 + A_1$	1			1	1		
$E_8(a_6)$	$\mathfrak{S}_3$	$\mathfrak{S}_3$	6	$\mathfrak{S}_3$	5	$\mathfrak{S}_3$	8
$E_7(a_2)$	1	1	1				
D_7	1	1	1				
$E_8(b_5)$	$\mathfrak{S}_3$	$\mathfrak{S}_3$	6	$\mathfrak{S}_3$	5	$\mathfrak{S}_3$	8
$E_8(a_5)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
$E_7(a_1)$	1	1	1	1	1	1	1
$E_8(b_4)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	1	1	1	1
$E_8(a_4)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
E_7	1	1	1				
$E_8(a_3)$	$\mathfrak{S}_2$	$\mathfrak{S}_2$	2	$\mathfrak{S}_2$	4	$\mathfrak{S}_2$	4
$E_8(a_2)$	1	1	1	1	1	1	1
$E_8(a_1)$	1	1	1	1	1	1	1
E_8	1	1	1	1	1	1	1

Appendix B : Computation of ℓ-special classes and quotients

```
lspec:=function(W, l) #Give the list of l-special unipotent
   ↪ classes
    local C,spec,jind,M,lspec,u,sgmax,k,n;
    lspec:=[];
    n:=PositionProperty([1..Length(W.roots)],i ->Sum(W.roots[W
       ↪ .N])=Sum(W.coroots[i]));
#position of the longest coroot
```

```
    sgmax:=List([1..W.rank],
i->[ReflectionSubgroup(W,Concatenation([[1..i-1],[i+1..W.rank
   ↪ ],[HighestS
hortRoot(W)]])),W.coroots[n][i]]);#List maximal subgroups of
   ↪ the dual and give the order of the associated isolated
   ↪ element
    Add(sgmax,[W,1]);
    C:=List(sgmax,i -> ChevieCharInfo(i[1]));
    for k in [1..Length(sgmax)] do
        spec:=Filtered([1..Length(C[k].a)],i->C[k].a[i]=C[k].b
           ↪ [i]);
        jind:=jInductionTable(sgmax[k][1],W);
        M:=Transposed(Transposed(jind.scalar){spec});
        u:=Filtered([1..Length(M)],i->Sum(M[i])>0);
        Add(lspec, [sgmax[k][2],List(UnipotentClasses(W).
           ↪ springerSeries[1].locsys{u},i -> i[1])]);
    od;
    lspec:=Filtered(lspec,j -> ((j[1] mod l =0) and
       ↪ IsPrimePower(j[1])) or j[1]=1);
    return Set(Flat(List(lspec, j->j[2])));
end;

lprimeConj:=function(G,l)#Give the list of conjugacy classes
   ↪ of l'-elements.
    return Filtered(ConjugacyClasses(G), i -> Gcd(Order(G,
       ↪ Representative(i)),l)=1);
end;

lnumber:=function(G,l)#Count the number of l-modular
   ↪ representations of centralisers of l'-elements.
    return Sum(ConjugacyClasses(G),i -> Length(lprimeConj(
       ↪ Centraliser(G,Representative(i)),l)));
    end;


    Quotient:= function(G, L)#G is a group and L a list of
       ↪ representations parametrised as in CharTable. Retour
       ↪  the quotient of G by the intersection of kernels of
       ↪  representations in L.
```

```
    local e, C, T, K, H;
    if L=[] then
        return G;
    else
        e:=Elements(G);
        C:=ConjugacyClasses(G);
        T:=Transposed(CharTable(G).irreducibles);
        C:=C{Filtered([1..Length(C)], i -> T[i]{L}=T[1]{L})};
        H:=Flat(List(C, i->Elements(i)));
        return G/H;
    fi;
end;

lQuotient:= function(G, L, l)#Same as Quotient but take only
   ↪ the l'-part of the intersection of kernels.
    local e, C, T, K, H;
    if L=[] then
        return G;
    else
        e:=Elements(G);
        C:=ConjugacyClasses(G);
        T:=Transposed(CharTable(G).irreducibles);
        C:=C{Filtered([1..Length(C)], i -> T[i]{L}=T[1]{L})};
        H:=Flat(List(C, i->Elements(i)));
        H:=Filtered(H,i -> Gcd(Order(G,i),l)=1);
        return G/H;
    fi;
end;


    lAubar:=function(W,x,l)#Return the group Gamma_x^l
    local ucl,u,locsys,C, Au,Pos, i,j,a, D,Fil, Max,E,L;
    ucl:=UnipotentClasses(W);
    u:=ucl.classes[x];
    locsys:=ucl.springerSeries[1].locsys;
    C:=ChevieCharInfo(W);
    Au:=u.Au;
    Pos:=List([1..Length(ConjugacyClasses(Au))], i ->
       ↪ PositionProperty(locsys,j -> j[1]=x and j[2]=i));#
```

```
      ↪ Give the position of the Springer correspondent E_{x
      ↪ ,i}
    D:=Transposed(DecompositionMatrix(Au,l));
    a:=List([1..Length(D)], i -> -1);
    for i in [1..Length(D)] do
        Fil:=Filtered([1 .. Length(Pos)], j -> Pos[j] <> false
           ↪  and D[i][j]<>0);
        if Fil <> [] then
            a[i]:=Minimum(List(Pos{Fil}, j->C.a[j]));
        fi;
    od;
    Max:=Maximum([0,Maximum(a)]);
    E:=Transposed(D{Filtered([1..Length(D)], i -> a[i]=Max)});
    L:=List(E, i->Maximum(i));
    return Quotient(Au,PositionsProperty(L,i -> i<>0));

end;


  countublock:= function(W,l) # Return the number unipotent l
     ↪ modular representations of the simple adjoint
     ↪ exceptionnal group of type W.
    local i,ls;
    ls:=lspec(W,l);
    return Sum(ls, i -> lnumber(lAubar(W,i,l),l));
end;
```

Index of notation

$B_s(G)$ 45
$E_{u,\phi}$ 57
$P_k(G)$ 20
$R_K(G)$ 7
$Z_{\mathbf{G}}(u)$ 93
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