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Marcu-Antone Orsoni

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THÈSE

PRÉSENTÉE POUR OBTENIR LE GRADE DE

Docteur de l'Université de Bordeaux

ÉCOLE DOCTORALE MATHÉMATIQUES ET INFORMATIQUE

Spécialité Mathématiques Pures

par **Marcu-Antone Orsoni**

Spaces of analytic functions and Reachable space of the heat equation

Sous la direction de : **Andreas Hartmann**

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Titre : Espaces de fonctions holomorphes et espace atteignable de l'équation de la chaleur.

Résumé : Cette thèse est consacrée à la description de l'espace atteignable de l'équation de la chaleur à l'aide de méthodes de l'analyse complexe moderne. Ce problème central de la théorie du contrôle est vieux de 50 ans et a captivé de nombreuses recherches depuis les travaux pionniers de Fattorini et Russell en 1971. Dans ce travail, on s'intéresse à l'équation de la chaleur 1-D sur un segment avec contrôle de Dirichlet au bord. Dans une première partie, on démontre à l'aide d'un théorème de type Paley-Wiener que l'espace atteignable est égal à la somme de deux espaces de Bergman, puis qu'il contient un espace de Smirnov-Zygmund en étudiant la régularité de la transformée de Cauchy. Dans une deuxième partie, en utilisant des méthodes de noyaux reproduisants et de $\bar{\partial}$, on résout le problème de séparation de singularités (problème de type Cousin) pour l'espace de Bergman dans plusieurs configurations. On en déduit ainsi une caractérisation définitive de l'espace atteignable comme espace de Bergman sur un carré. Enfin, la dernière partie est consacrée à l'équation de chaleur avec un potentiel quadratique et à son espace atteignable.

Mots-clés : Espaces de fonctions holomorphes, espace atteignable, équation de la chaleur, noyaux reproduisants, analyse complexe, théorie du contrôle.

Title: Spaces of analytic functions and reachable space of the heat equation.

Abstract: This thesis is devoted to the description of the reachable space of the heat equation using methods of modern complex analysis. This central problem in control theory is about 50 years old and has captivated a lot of research efforts since the pioneering work of Fattorini and Russell in 1971. In this work, we are interested in the 1-D heat equation on a segment with Dirichlet boundary control. In the first part, using a Paley-Wiener theorem we prove that the reachable space is the sum of two Bergman spaces, then that it contains a Smirnov-Zygmund space by studying the regularity of the Cauchy transform. In a second part, using reproducing kernels and $\bar{\partial}$ -methods, we solve the problem of separation of singularities (Cousin-type problem) for the Bergman space in several configurations. This enables us to deduce a definitive characterization of the reachable space as the Bergman space on a square. Finally, the last part is devoted to the heat equation with a quadratic potential and its reachable space.

Keywords: Spaces of analytic functions, reachable space, heat equation, reproducing kernels, complex analysis, control theory.

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Introduction (Français)

Introduite par le mathématicien Joseph Fourier au début du XIX^e siècle dans le but de modéliser la propagation de la chaleur dans un milieu donné, l'équation de la chaleur est l'une des équations différentielles les plus célèbres des mathématiques. Dans le discours introductif de son célèbre ouvrage *Théorie analytique de la chaleur* paru en 1822, Fourier écrira d'elle : « Cette théorie formera désormais une des branches les plus importantes de la physique générale ». Presque deux siècles plus tard, l'équation de la chaleur a largement dépassé les frontières de la physique et intervient dans de nombreux autres domaines comme la biologie, la chimie, la mécanique des fluides, les probabilités ou la géométrie différentielle. Son importance est due à la fois à son omniprésence en tant que prototype de l'équation parabolique, et à la multitude d'outils très puissants que son étude systématique a permis de développer, comme l'analyse de Fourier ou plus généralement l'analyse harmonique. L'objectif de cette thèse est de poursuivre modestement cette étude en essayant de l'enrichir de nouvelles approches.

La théorie du contrôle est le domaine des mathématiques qui étudie la possibilité d'agir sur un système dynamique au moyen d'une commande, aussi appelée contrôle. Un bon exemple d'un tel système est donné par l'équation de la chaleur unidimensionnelle sur un segment avec un contrôle de type Dirichlet au bord.

$$\begin{cases} \frac{\partial w}{\partial t}(t, x) - \frac{\partial^2 w}{\partial x^2}(t, x) = 0 & t > 0, x \in]0, \pi[, \\ w(t, 0) = u_0(t), \quad w(t, \pi) = u_\pi(t) & t > 0, \\ w(0, x) = f(x) & x \in]0, \pi[, \end{cases}$$

Cette équation modélise l'évolution au cours du temps t de la température w sur une barre que l'on chauffe aux deux extrémités. Ici, on agit sur le système au moyen de la fonction $u := (u_0, u_\pi)$ qui décrit notre façon de chauffer les extrémités de la barre à chaque instant t . Si pour toute température initiale f et pour toute température finale g , il est possible de trouver une manière de chauffer les extrémités

(c'est-à-dire un contrôle u) qui amène la barre de la température f (au temps $t = 0$) à la température g (au temps $t = \tau$), on dit que le système est (exactement) contrôlable. Malheureusement, cette propriété de contrôlabilité n'est pas vérifiée par notre équation de la chaleur ci-dessus. C'est une conséquence de l'extrême régularité de la température en la variable spatiale x . Plus précisément, il est bien connu dans le milieu des équations aux dérivées partielles que les états atteignables g sont des fonctions qui se prolongent de manière holomorphe sur le carré D dont l'une des diagonales est le segment $[0, \pi]$. Cela conduit à la question naturelle suivante : quels sont exactement les états finaux que l'on peut atteindre ? L'ensemble de ces états finaux atteignables est appelé espace atteignable de l'équation de la chaleur et sa description est une question centrale en théorie du contrôle qui remonte aux travaux pionniers de Fattorini et Russell il y a 50 ans. Le but de cette thèse est de fournir une caractérisation complète et définitive de cet espace atteignable.

L'analyse complexe moderne est la partie de l'analyse fonctionnelle qui s'occupe des espaces de fonctions holomorphes (appelées aussi fonctions analytiques). Ces fonctions, qui comme on vient de le voir sont liées au problème de caractérisation de l'espace atteignable de l'équation de la chaleur, vérifient de belles et surprenantes propriétés. Plusieurs espaces classiques de fonctions holomorphes comme les espaces de Hardy ou de Bergman sont des cas particuliers d'espaces de Hilbert à noyaux reproduisants (RKHS), dans lesquels on connaît la valeur d'une fonction en la testant contre un noyau. Cela procure des outils théoriques de calcul très puissants pour, par exemple, déterminer l'image d'une transformée intégrale de type Laplace. Cette méthode a déjà été utilisée par Aikawa, Hayashi et Saitoh pour décrire l'espace atteignable de l'équation de la chaleur sur une demi-droite cette fois, et nous fournit une nouvelle approche pour notre équation de la chaleur sur un segment. En effet, nous verrons que la description des états atteignables de l'équation de la chaleur sur un segment se réduit à une analyse fine d'une certaine transformation de type Fourier-Laplace. Cela nous permettra de déplacer le problème dans le champ de l'analyse complexe et d'utiliser plusieurs techniques spécifiques à ce domaine comme la séparation de singularités. Cette dernière technique permet d'écrire une fonction holomorphe sur l'intersection $\Omega_1 \cap \Omega_2$ de deux domaines Ω_1 et Ω_2 du plan complexe comme somme de deux fonctions holomorphes respectivement sur Ω_1 et Ω_2 . En allant plus loin, nous pouvons nous demander si ce résultat se généralise à des espaces de Banach ou de Fréchet de fonctions analytiques. Autrement dit, si $\mathcal{B}(\Omega)$ désigne un espace de Banach de fonctions analytiques sur un domaine Ω , l'égalité $\mathcal{B}(\Omega_1 \cap \Omega_2) = \mathcal{B}(\Omega_1) + \mathcal{B}(\Omega_2)$ est-elle vraie ? Ce problème est connu sous le nom de problème de séparation de singularités. Comme nous le verrons plus tard, bien que très naturel en analyse complexe, il joue un rôle prédominant dans la description de l'espace

atteignable qui se révèle être la somme de deux espaces de Bergman définis sur des secteurs s'intersectant.

Dans la première partie de ce manuscrit, nous donnons le matériel de base nécessaire pour mener à bien nos travaux, aussi bien en analyse complexe qu'en théorie du contrôle. Du côté de l'analyse complexe, nous rappelons la définition et les propriétés classiques de quelques espaces de fonctions holomorphes classiques comme ceux mentionnés plus haut. Nous introduisons également la théorie des noyaux reproductifs et plusieurs théorèmes de type Paley-Wiener. Ces derniers nous seront très utiles pour déterminer l'image de certains opérateurs de type Laplace qui sont au coeur du problème de contrôle considéré dans le deuxième paragraphe ci-dessus. Finalement, nous présentons le problème de séparation de singularités pour les fonctions holomorphes et ses généralisations pour des espaces de Banach (ou de Fréchet) de fonctions analytiques. Du côté de la théorie du contrôle, nous donnons quelques bases de la théorie et nous construisons précautionneusement le cadre théorique des systèmes de contrôle au bord bien posés dont nous avons besoin pour définir correctement notre problème. Finalement, nous introduisons le concept d'espace atteignable, objet central de cette thèse, et nous passons en revue les résultats précédents le concernant.

Dans une seconde partie, nous donnons nos résultats originaux sur ce problème obtenus durant cette thèse. Ceux contenus dans les chapitres 3 et 4 ont fait l'objet de deux articles acceptés pour publication dans des journaux, tandis que le contenu du chapitre 5 est une partie d'un travail encore en préparation. Dans le chapitre 3, nous fournissons une caractérisation exacte de l'espace atteignable de l'équation de la chaleur unidimensionnelle sur un segment avec contrôle de type Dirichlet au bord, comme somme de deux espaces de Bergman sur des secteurs dont l'intersection est le carré D défini plus haut. La première idée, développée par Hartmann, Kellay et Tucsnak est d'écrire la solution de l'équation comme une série dont les deux termes principaux sont les solutions respectives du problème de contrôle sur les demi-droites $]0, +\infty[$ et $] - \infty, \pi[$ et les termes restant sont traités comme des perturbations. Enfin le résultat est obtenu en appliquant un certain théorème de Paley-Wiener pour les espaces de Bergman à une transformée de type Laplace. Dans le chapitre 4, nous commençons pas résoudre le problème de séparation de singularités pour les espaces de Bergman sur des secteurs, des polygones ou des configurations plus générales d'intersections de parties convexes. L'idée repose sur un résultat clé d'égalité entre noyaux reproductifs pour le quart de plan, sur des estimées L^p de type Hörmander pour la solution du $\bar{\partial}$ et d'un découpage astucieux des polygones. Puis nous appliquons nos résultats à la somme d'espaces de Bergman mentionnée ci-dessus décrivant l'espace atteignable. Cela donne une solution définitive à notre

problème initial : l'espace atteignable de l'équation de la chaleur est l'espace de Bergman du carré D . Nous en déduisons également une caractérisation exacte des espaces atteignables pour d'autres types de contrôles au bord, comme le contrôle de type Neumann ou le contrôle unilatéral (de Dirichlet ou de Neumann). Dans le chapitre 5, nous appliquons les méthodes développées dans les précédents chapitres à un autre genre d'équation parabolique appelée équation de la chaleur de Hermite. Cette équation consiste à remplacer le Laplacien par l'oscillateur harmonique, toujours en une dimension d'espace sur un segment avec contrôle de Dirichlet au bord. Nous montrons que l'espace atteignable de cette équation est encore contenu dans l'espace de Bergman du carré D . Au passage, nous obtenons une caractérisation exacte de l'espace atteignable pour la demi-droite $]0, +\infty[$.

Finalement, dans l'appendice B, nous résumons brièvement un autre travail effectué durant ma thèse qui traite des ensembles dominants sur des espaces de Bergman. Il consiste en une estimation optimale de la constante d'échantillonnage. Dans un cadre étendu, ce sujet est aussi lié à la théorie du contrôle. Ces résultats ont aussi donné lieu à un article accepté pour publication.

Une célèbre citation du mathématicien Paul Painlevé, issue de son *Analyse des travaux scientifiques* paru en 1900, affirme : « Entre deux vérités réelles, le chemin le plus facile et le plus court passe souvent par le domaine complexe ». Dans un certain sens, par ce travail, l'auteur espère donner, s'il en était encore besoin, une autre preuve de la véracité de cette phrase.

Introduction (English)

Introduced by the mathematician Joseph Fourier at the beginning of the 19th century to model the heat propagation through a given region, the heat equation is one of the most famous differential equations of mathematics. In his famous volume *Théorie analytique de la chaleur* published in 1822, Fourier wrote about it : “This theory will henceforth form one of the most important branches of general physics”. Almost two centuries later, the heat equation has gone far beyond the boundaries of physics and is involved in many other fields such as biology, chemistry, fluid mechanics, probability and differential geometry. Its central importance is due both to its omnipresence as a prototypical parabolic equation, and to the multiplicity of powerful tools that its systematic study has given birth to, like Fourier analysis or more generally harmonic analysis. This thesis intends modestly to continue this study trying to develop new approaches.

Control theory studies the possibility of acting on a dynamical system by means of a command, also called control. A good example of such a system is given by the one-dimensional heat equation on a segment with Dirichlet boundary control

$$\begin{cases} \frac{\partial w}{\partial t}(t, x) - \frac{\partial^2 w}{\partial x^2}(t, x) = 0 & t > 0, x \in (0, \pi), \\ w(t, 0) = u_0(t), \quad w(t, \pi) = u_\pi(t) & t > 0, \\ w(0, x) = f(x) & x \in (0, \pi), \end{cases}$$

It models the evolution of the temperature w on a rod depending on the time t when we heat the rod at both ends. Here we act on the system by means of the boundary heating function $u := (u_0, u_\pi)$ at each instant t . If for any initial state f and any final target g we can find a way to heat the rod at both ends in order to steer the temperature from the initial state f (at time $t = 0$) to the final target g (at time $t = \tau$), then we say that the system is (exactly) controllable. Unfortunately, this controllability property is not fulfilled by our heat system. This is a consequence of the extreme regularity in the space variable x of the temperature. Actually, it is a folklore result of PDEs that the reachable states g are holomorphic on the square

D one diagonal of which is the interval $(0, \pi)$. This leads to the natural question: which are exactly the targets we can reach? The set of all the reachable final states is called reachable space of the heat equation and its description is a central question in control theory which goes back to the pioneering work of Fattorini and Russell 50 years ago. The aim of the present work is to provide a definitive exact description of this reachable space.

Modern complex analysis is the part of functional analysis which deals with holomorphic function spaces. These functions enjoy surprising and beautiful properties. Several classical holomorphic function spaces like Hardy or Bergman spaces are particular cases of so-called Reproducing Kernel Hilbert Spaces, on which we know a function from testing on a kernel. This provides powerful computational tools to, for example, determine the range of an integral Laplace-type transform. This method has been applied by Aikawa, Hayashi and Saitoh to the description of the reachable space of the heat equation on a half-line and provides a new approach for the heat equation on a rod. Indeed, the description of the reachable states of the heat equation on a rod reduces to a fine analysis of a certain Fourier-Laplace type transform. This allows us to move the problem into the field of complex analysis and use several specific technics as separation of singularities. This last technique allows us to write an analytic function on the intersection $\Omega_1 \cap \Omega_2$ of two domains Ω_1 and Ω_2 of the complex plane as the sum of two holomorphic functions respectively on Ω_1 and Ω_2 . Going further, we can ask if this result generalizes to Banach or Fréchet spaces of analytical functions. In other words, if $\mathcal{B}(\Omega)$ denotes a Banach space of analytical functions over a domain Ω , is the equality $\mathcal{B}(\Omega_1 \cap \Omega_2) = \mathcal{B}(\Omega_1) + \mathcal{B}(\Omega_2)$ true? This problem is known as the separation of singularities problem. As we will see later, this very natural problem in complex analysis plays a central role in the control of the heat equation, where it appears in the setting of the sum of two Bergman spaces defined on intersecting sectors.

In a first part of this manuscript, we shall give the basic material needed to carry out our work, both in complex analysis and control theory. On the complex analysis side, we remind the definition and the basic properties of some classical spaces of analytic functions as those mentioned above. We also introduce the theory of reproducing kernels and several Paley-Wiener type theorems. They will be useful to determine the range of Laplace type operators which will play a prominent rôle in the control problem we consider. Finally we present the separation of singularities problem for holomorphic functions and its counter parts for Banach spaces. On the control side, we give some background of the theory and we construct carefully the framework of well-posed boundary control systems that our problem needs to be well-defined. Finally, we introduce the concept of reachable space and we survey the

previous results on this topic.

In a second part, we give our original results on the problem. Those contained in Chapters 3 and 4 have been accepted for publication, the contents of Chapter 5 is a part of a work still in preparation. In Chapter 3, we provide an exact characterization of the reachable space of the 1-D heat equation on a finite interval with Dirichlet boundary control as sum of two Bergman spaces on sectors whose intersection is the square D . The first idea, developed by Hartmann, Kellay and Tucsnak is to write the solution of the heat equation as a series with two main terms given by the solutions on the half-lines $(0, \infty)$ and $(-\infty, \pi)$ and to treat the remaining terms as perturbation. The result is achieved applying a certain Paley-Wiener theorem for Bergman spaces to a Laplace type transform. In Chapter 4, we start solving the separation of singularities problem for Bergman spaces on sectors, polygons and in more general settings of intersections of convex sets. The proof is based on a key equality between reproducing kernels for the quarter plane, Hörmander type L^p -estimates for the solutions of the $\bar{\partial}$ -equation, and a clever cutting of the polygons. Then we apply our results to the sum of Bergman spaces mentioned above. This gives a definitive solution of our initial problem: the reachable space of the heat equation is the Bergman space on the square D . We also deduce an exact characterization of the reachable spaces for other types of boundary controls, like Neumann type or unilateral (Dirichlet or Neumann) boundary control. In Chapter 5, we apply the methods developed in the preceding chapters to another kind of parabolic equation, namely the Hermite heat equation. It consists in replacing the Laplacian by the harmonic oscillator, again in the one-dimensional case on a segment with Dirichlet boundary control. It follows that the reachable space of this equation is still contained in the Bergman space of the square D . Actually, we obtain also an exact characterization of the reachable space of this equation on the half-line $(0, \infty)$.

Finally, in Appendix B, we resume another work carried out during my PhD and dealing with dominating sets on weighted Bergman spaces. It consists in giving an optimal estimate for the sampling constant. In some extended area, this topic is also related to control theory. These results have also been accepted for publication.

A famous quote of the mathematician Paul Painlevé, from his *Analyse des travaux scientifiques* published in 1900, claims: "The easiest and shortest path between two real truths quite often passes through the complex domain". In a certain sense, by this work, we hope to give another proof of the veracity of this sentence.

Part I
Preliminaries

Chapter 1

Some spaces of analytic functions

1.1 Zoology

In this manuscript, we denote by $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ the unit disc and by $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \text{Im}z > 0\}$ the upper-half plane. If $\Omega \subset \mathbb{C}$ is an open set, we write $\text{Hol}(\Omega)$ for the algebra of holomorphic functions on Ω .

1.1.1 Reproducing Kernel Hilbert Spaces

Basic properties.

In this subsection we give a brief introduction to the theory of Reproducing Kernel Hilbert Spaces. It has been mainly developed by S. Bergman and N. Aronszajn. The material we introduce here is classical and can be found in [PR16], [Aro50].

From now on, Ω will be an arbitrary set and we will denote by $\mathcal{F}(\Omega, \mathbb{C})$ the vector space of complex valued functions defined on Ω .

Definition 1.1.1. Let Ω be a set and let $\mathcal{H} \subset \mathcal{F}(\Omega, \mathbb{C})$ be a Hilbert space. We say that \mathcal{H} is a *Reproducing Kernel Hilbert Space* (RKHS) if for every $\lambda \in \Omega$ the linear evaluation functional $E_\lambda : \mathcal{H} \rightarrow \mathbb{C}$ given by

$$E_\lambda(f) = f(\lambda), \quad \forall f \in \mathcal{H}$$

is bounded.

By the Riesz-Fréchet representation theorem this is equivalent to the following condition:

$$\forall \lambda \in \Omega, \exists! k_\lambda \in \mathcal{H}, \forall f \in \mathcal{H}, f(\lambda) = \langle f, k_\lambda \rangle.$$

The function k_λ is called the *reproducing kernel* at λ . The function $K : \Omega \times \Omega \rightarrow \mathbb{C}$ defined by $K(z, \lambda) := k_\lambda(z)$ is called the *reproducing kernel* of \mathcal{H} . Important cases appear when Ω is an open subset of \mathbb{C} and $\mathcal{H} \subset \text{Hol}(\Omega)$. We shall see a few examples of RKHS of holomorphic functions in the next subsections.

As immediate consequences of the definition, we obtain for every $\lambda, z \in \Omega$,

$$\|k_\lambda\|^2 = k_\lambda(\lambda),$$

and

$$k_\lambda(z) = \overline{k_z(\lambda)}.$$

It is also direct that the family $(k_\lambda)_{\lambda \in \Omega}$ spans a dense subspace of \mathcal{H} . Indeed, if f belongs to $\{k_\lambda, \lambda \in \Omega\}^\perp$, then $f(\lambda) = \langle f, k_\lambda \rangle = 0$ for every $\lambda \in \Omega$, so $f = 0$. In addition, since the evaluation functionals are bounded, it follows that the convergence in \mathcal{H} implies the pointwise convergence. Actually, when the map $\lambda \mapsto \|k_\lambda\|$ is continuous (as in the next subsections), we obtain with the Cauchy-Schwarz's inequality that the embedding $\mathcal{H} \subset \text{Hol}(\Omega)$ is bounded. This gives us an effective way to compute the reproducing kernel from an orthonormal basis $(e_n)_{n \in \mathbb{N}}$ of \mathcal{H} . Decomposing on (e_n) , we have

$$k_\lambda(z) = \sum_{n \in \mathbb{N}} \langle k_\lambda, e_n \rangle e_n(z) = \sum_{n \in \mathbb{N}} \overline{e_n(\lambda)} e_n(z), \quad (1.1)$$

where the series converge in \mathcal{H} and hence pointwisely. Assume now that \mathcal{H} is a sub-Hilbert space of a Hilbert space \mathcal{G} and let P denote the orthogonal projection from \mathcal{G} onto \mathcal{H} . Then for every $g \in \mathcal{G}$, we have

$$Pg(z) = \langle Pg, k_z \rangle = \langle g, Pk_z \rangle = \langle g, k_z \rangle, \quad \forall z \in \Omega. \quad (1.2)$$

Conversely, if \mathbf{H} is a Hilbert subspace of \mathcal{H} , it is also obviously a RKHS. We denote by $k_\lambda^{\mathbf{H}}$ its reproducing kernel and we write $P_{\mathcal{H} \rightarrow \mathbf{H}}$ for the orthogonal projection. Therefore, for every $f \in \mathbf{H}$

$$f(\lambda) = \langle f, k_\lambda \rangle = \langle P_{\mathcal{H} \rightarrow \mathbf{H}} f, k_\lambda \rangle = \langle f, P_{\mathcal{H} \rightarrow \mathbf{H}} k_\lambda \rangle.$$

Thus, by unicity of the reproducing kernel, we obtain $k_\lambda^{\mathbf{H}} = P_{\mathcal{H} \rightarrow \mathbf{H}} k_\lambda$. Nevertheless, this equality does not permit to compute effectively the reproducing kernel in general. Finally, the reproducing kernel characterizes its RKHS.

Proposition 1.1.2. *[see Proposition 2.3 in [PR16]] Let H_1 and H_2 be two RKHS on Ω with reproducing kernel K_1 and K_2 respectively. If $K_1 = K_2$ then $H_1 = H_2$.*

Proof. Write $K(z, \lambda) := K_1(z, \lambda) = K_2(z, \lambda)$. We have seen in the discussion above that $X := \text{Span}\{k_\lambda, \lambda \in \Omega\}$ is dense in H_i . Moreover, for $f := \sum_{i=1}^n \alpha_i k_{\lambda_i} \in X$, we have $\|f\|_2^2 = \sum_{i,j=1}^n \alpha_i \bar{\alpha}_j \langle k_{\lambda_i}, k_{\lambda_j} \rangle_1 = \sum_{i,j=1}^n \alpha_i \bar{\alpha}_j K(\lambda_j, \lambda_i)$ and identically $\|f\|_2^2 = \sum_{i,j=1}^n \alpha_i \alpha_j K(\lambda_j, \lambda_i)$. Hence,

$$\|f\|_1 = \|f\|_2, \quad \forall f \in X. \quad (1.3)$$

Let us prove now that $H_1 \subset H_2$. Pick $f \in H_1$. By density, there exists a sequence $(f_n)_{n \in \mathbb{N}}$ in X which converges to f in H_1 (and so pointwisely). Therefore, (f_n) is Cauchy in H_1 and then in H_2 by (1.3). Hence, (f_n) converges in H_2 (and so pointwisely) to a function g . This implies $f(\lambda) = \lim_{n \rightarrow \infty} f_n(\lambda) = g(\lambda)$, i.e f belongs to H_2 . By the same arguments we obtain also $H_2 \subset H_1$. Thus $H_1 = H_2$.

Finally, the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are equal on X which is dense, so they are equal everywhere. \square

Kernel functions.

Hence, if we know *a priori* that K is the reproducing kernel of a Hilbert space on Ω , this Hilbert space is necessarily unique. But, how can we recognize a reproducing kernel? More precisely, given a map $K : \Omega \times \Omega \rightarrow \mathbb{C}$, is there a necessary and sufficient condition for K to be a reproducing kernel of an Hilbert space?

Definition 1.1.3. We say that a map $K : \Omega \times \Omega \rightarrow \mathbb{C}$ is a *kernel function*, and we will write $K \geq 0$, if for every finite sequence $(\lambda_i)_{1 \leq i \leq n}$ of Ω and every $\alpha_1, \dots, \alpha_n \in \mathbb{C}$, we have

$$\sum_{i,j=1}^n \bar{\alpha}_i \alpha_j K(\lambda_i, \lambda_j) \geq 0,$$

i.e. the matrix $(K(\lambda_i, \lambda_j))_{1 \leq i,j \leq n}$ is positive.

A reproducing kernel is a kernel function. Indeed,

$$\sum_{i,j=1}^n \bar{\alpha}_i \alpha_j K(\lambda_i, \lambda_j) = \sum_{i,j=1}^n \bar{\alpha}_i \alpha_j \langle k_{\lambda_j}, k_{\lambda_i} \rangle = \left\| \sum_{i=1}^n \alpha_i k_{\lambda_i} \right\|^2 \geq 0.$$

Conversely, it turns out that every kernel function is a reproducing kernel for some RKHS.

Theorem 1.1.4 (Moore). *[see Theorem 2.14 in [PR16]] Let Ω be a set and $K : \Omega \times \Omega \rightarrow \mathbb{C}$ be a function. If K is a kernel function, there exists a unique RKHS \mathcal{H} on Ω such that K is the reproducing kernel of \mathcal{H} .*

The idea of the proof is quite simple. Roughly speaking the RKHS associated with K is the completion \mathcal{H} of the linear span $W := \text{span}\{(k_\lambda)_{\lambda \in \Omega}\}$ with respect to the inner product given by $B(\sum_i \alpha_i k_{\lambda_i}, \sum_j \beta_j k_{\lambda_j}) := \sum_{i,j} \beta_j \alpha_i K(\lambda_j, \lambda_i)$. The big part of the proof consists of proving that this application defines actually an inner product on W . It remains to identify the Hilbert space \mathcal{H} with the Hilbert space of functions $\hat{\mathcal{H}} := \{\hat{h}, h \in \mathcal{H}\}$ where \hat{h} is defined by $\hat{h}(\lambda) = \langle h, k_\lambda \rangle_{\mathcal{H}}$ for any $\lambda \in \Omega$. The uniqueness comes from Proposition 1.1.2. The interested reader can find a complete proof with all the necessary precautions in [PR16].

From now on, given a kernel function K , we will denote by \mathcal{H}_K the RKHS associated with.

Application to linear transforms.

Let Ω be a (non-empty) set and \mathcal{H} be a Hilbert space (not necessarily a RKHS). For a function $h : \Omega \rightarrow \mathcal{H}$, we denote by $L : \mathcal{H} \rightarrow \mathcal{F}(\Omega, \mathbb{C})$ the linear mapping given by

$$\forall f \in \mathcal{H}, (Lf)(\lambda) = \langle f, h(\lambda) \rangle_{\mathcal{H}}, \quad (\lambda \in \Omega). \quad (1.4)$$

An important problem is to determine the range $\text{Ran}(L)$ of the mapping L . The following theorem gives an answer.

Theorem 1.1.5. *[see Theorem 1 p.21 [Sai97]] Let L denote the operator defined by (1.4). Endowed with the Sarason norm*

$$\|g\|_{\text{Ran}(L)} := \inf\{\|f\|_{\mathcal{H}}, f \in \mathcal{H} \text{ and } Lf = g\},$$

the space $\text{Ran}(L)$ is a RKHS with reproducing kernel

$$K(z, \lambda) = \langle h(\lambda), h(z) \rangle_{\mathcal{H}}, \quad (z, \lambda \in \Omega).$$

Furthermore, the linear mapping L is an isometric isomorphism from \mathcal{H} onto $\text{Ran}(L)$ if and only if the family $(h(\lambda))_{\lambda \in \Omega}$ is complete in \mathcal{H} .

Proof. Note that $\|Lf\|_{\text{Ran}(L)} = \inf\{\|f - \tilde{f}\|_{\mathcal{H}}, \tilde{f} \in \text{Ker}(L)\}$, so that in particular L is bounded. Hence, it is clear that $L : \text{Ker}(L)^\perp \rightarrow \text{Ran}(L)$ is an isometric isomorphism. Since $\text{Ker}(L)^\perp$ is a closed subspace of \mathcal{H} , it is a Hilbert space and therefore $\text{Ran}(L)$ is also a Hilbert space. Indeed, its norm verifies the parallelogram identity by isometry (i.e. it is induced by an inner product), and again by isometry the completeness is conserved. Finally, we have also

$$\langle Lf, Lg \rangle_{\text{Ran}(L)} = \langle f, g \rangle_{\mathcal{H}}, \quad \forall f, g \in \text{Ker}(L)^\perp.$$

Obviously, $k_\lambda := K(\cdot, \lambda) = \langle h(\lambda), h(\cdot) \rangle_{\mathcal{H}} = L[h(\lambda)] \in \text{Ran}(L)$. Moreover, remark that $\text{Ker}(L)^\perp$ is exactly the closure of the linear span of $\{h(\lambda), \lambda \in \Omega\}$. So we have for every $f \in \mathcal{H}$,

$$\begin{aligned} \langle Lf, k_\lambda \rangle_{\text{Ran}(L)} &= \left\langle L \left[P_{\text{Ker}(L)^\perp} f \right], L[h(\lambda)] \right\rangle_{\text{Ran}(L)} = \left\langle P_{\text{Ker}(L)^\perp} f, h(\lambda) \right\rangle_{\mathcal{H}} \\ &= L \left[P_{\text{Ker}(L)^\perp} f \right] (\lambda) \\ &= Lf(\lambda). \end{aligned}$$

Hence, $\text{Ran}(L)$ is a RKHS with reproducing kernel K .

Finally, if $(h(\lambda))_{\lambda \in \Omega}$ is complete in \mathcal{H} , then $\text{Ker}(L)^\perp = \overline{\text{span}\{h(\lambda), \lambda \in \Omega\}} = \mathcal{H}$ and L is an isometry as seen above. \square

If we are able to identify the reproducing kernel of a well-known RKHS, the previous theorem will permit to determine exactly the range of linear integral transformations. We shall see in Section 1.2 how to use it in order to obtain several Paley-Wiener theorems. This will also be the main tool of Section 2.6 where it will be powerful in characterizing reachable spaces.

Operations on reproducing kernels.

Now, we will see how the operations on the kernel modify the RKHS. We will just state the results. The proofs can be found in [PR16, Chapter 5].

Given two kernel functions K_1 and K_2 on Ω , we write $K_1 \leq K_2$ to mean that $K_2 - K_1 \geq 0$ i.e. $K_2 - K_1$ is a kernel function. Then we have

Theorem 1.1.6 (Inclusion). *For $i = 1, 2$, let $K_i : \Omega \times \Omega \rightarrow \mathbb{C}$ be kernel functions. Then $\mathcal{H}_{K_1} \subset \mathcal{H}_{K_2}$ if and only if there exists a constant $c > 0$ such that $K_1 \leq c^2 K_2$. Moreover, $\|f\|_2 \leq c \|f\|_1$ for all $f \in \mathcal{H}_{K_1}$.*

If K_1 and K_2 are kernel functions on a set Ω , then so is the function $K = K_1 + K_2$. The next result describes the relationship between the three RKHSs.

Theorem 1.1.7 (Sum of reproducing kernels). *For $i = 1, 2$, let $K_i : \Omega \times \Omega \rightarrow \mathbb{C}$ be kernel functions. If $K = K_1 + K_2$, then*

$$\mathcal{H}_K = \mathcal{H}_{K_1} + \mathcal{H}_{K_2}$$

as set equality, and the norm of \mathcal{H}_K is given by

$$\|f\|^2 = \inf\{\|f_1\|_1^2 + \|f_2\|_2^2, \quad f_i \in \mathcal{H}_{K_i} \text{ for } i = 1, 2 \text{ and } f = f_1 + f_2\}.$$

Let Ω_0 be a set and let $\varphi : \Omega_0 \rightarrow \Omega$ be a function. Given a kernel function K , we define the pull-back $\varphi^*K : \Omega_0 \times \Omega_0 \rightarrow \mathbb{C}$ of K by φ as

$$\varphi^*K(z, \lambda) = K(\varphi(z), \varphi(\lambda)), \quad \forall z, \lambda \in \Omega_0.$$

Theorem 1.1.8 (Pull-back of a RKHS). *Let Ω_0 be a set, $\varphi : \Omega_0 \rightarrow \Omega$ be a function and K denote a kernel function on Ω . Then φ^*K is a kernel function and*

$$\mathcal{H}_{\varphi^*K} = \{f \circ \varphi, f \in \mathcal{H}_K\}$$

with norm given by

$$\forall g \in \mathcal{H}_{\varphi^*K}, \quad \|g\| = \min\{\|f\|, f \in \mathcal{H}_K \text{ and } f \circ \varphi = g\}.$$

Obviously, if the linear mapping $f \mapsto f \circ \varphi$ is one-to-one, then it is also an isometric isomorphism from \mathcal{H}_K onto \mathcal{H}_{φ^*K} . Indeed, in that case the norm of a function $g \in \mathcal{H}_{\varphi^*K}$ is given by $\|g\|_{\mathcal{H}_{\varphi^*K}} = \|f\|_{\mathcal{H}_K}$ where f is the unique element of \mathcal{H}_K such that $f \circ \varphi = g$.

When Ω_0 is a subset of Ω and $\varphi : \Omega_0 \rightarrow \Omega$ denotes the inclusion, we obtain the following result.

Corollary 1.1.9 (Restriction of a RKHS). *Let $\Omega_0 \subset \Omega$ and let K denote a kernel function on Ω . If we write $K|_{\Omega_0}$ the restriction of K to $\Omega_0 \times \Omega_0$, then*

$$\mathcal{H}_{K|_{\Omega_0}} = \{f|_{\Omega_0}, f \in \mathcal{H}_K\}$$

with norm given by

$$\forall \tilde{f} \in \mathcal{H}_{K|_{\Omega_0}}, \quad \|\tilde{f}\| = \min\{\|f\|, f \in \mathcal{H}_K \text{ and } f|_{\Omega_0} = \tilde{f}\}.$$

Again, if the restriction is one-to-one in the previous theorem, i.e. Ω_0 is a set of uniqueness for \mathcal{H}_K , then it is also an isometric isomorphism.

Finally, we will see how the reproducing kernel changes when we multiply the RKHS by a function.

Definition 1.1.10. Let \mathcal{H}_1 and \mathcal{H}_2 be two RKHS on Ω . A function $f : \Omega \rightarrow \mathbb{C}$ is called a *multiplier* of \mathcal{H}_1 into \mathcal{H}_2 if $f\mathcal{H}_1 := \{fg, g \in \mathcal{H}_1\} \subset \mathcal{H}_2$. We denote by $\mathcal{M}(\mathcal{H}_1, \mathcal{H}_2)$ the vector space of all multipliers of \mathcal{H}_1 into \mathcal{H}_2 . When $\mathcal{H}_1 = \mathcal{H}_2 := \mathcal{H}$, we simply write $\mathcal{M}(\mathcal{H}) := \mathcal{M}(\mathcal{H}, \mathcal{H})$.

Proposition 1.1.11 (Multiplication of a RKHS by a function). *Let \mathcal{H} be a RKHS on Ω with kernel K and let $f : \Omega \rightarrow \mathbb{C}$ be a function. Then the space $\mathcal{H}_f := f\mathcal{H}$ is a RKHS on Ω when it is endowed with the norm*

$$\forall g \in \mathcal{H}_f, \quad \|g\| := \inf\{\|h\|_{\mathcal{H}}, h \in \mathcal{H} \text{ and } fh = g\}.$$

Moreover, its reproducing kernel is given by $K_f(z, \lambda) = f(z)K(z, \lambda)\overline{f(\lambda)}$.

1.1.2 Hardy and Smirnov spaces

The Hardy space is probably one of the most famous spaces of analytic functions. We give here the basic material which will be needed in this thesis. To go further, one can read the numerous and complementary textbooks on this topic [Nik02], [Gar07], [Dur70], [Koo98], [Nik19].

Let $0 < p < +\infty$. Let Ω be a simply connected domain in the complex plane with at least two boundary points. We denote by H^∞ the space of bounded analytic functions on Ω . It is a Banach space under the norm $\|f\|_\infty = \sup_{z \in \Omega} |f(z)|$. We say that $f \in \text{Hol}(\Omega)$ belongs to the Hardy space $H^p(\Omega)$ if the subharmonic function $|f|^p$ admits a harmonic majorant on Ω . We say that $f \in \text{Hol}(\Omega)$ belongs to the Smirnov space $E^p(\Omega)$ if there exists a sequence $(\gamma_n)_{n \in \mathbb{N}}$ of rectifiable Jordan curves eventually surrounding each compact subdomain of Ω , such that

$$\|f\|_{E^p}^p = \sup_{n \in \mathbb{N}} \int_{\gamma_n} |f(z)|^p |dz| < \infty.$$

For $\Omega = \mathbb{D}$, it is well-known that these two last spaces coincide, and we will simply denote them by H^p (see [Dur70, Theorem 2.12]). For an arbitrary conformal mapping $\varphi : \mathbb{D} \rightarrow \Omega$, $f \in H^p(\Omega)$ if and only if $f \circ \varphi \in H^p$, and $f \in E^p(\Omega)$ if and only if $(f \circ \varphi)\varphi^{1/p} \in H^p$. It is clear that the two spaces coincide if there exists $a, b > 0$ such that $a \leq |\varphi'| \leq b$. Actually, this condition is also necessary (see [Dur70, Theorem 10.2]). When $1 \leq p < +\infty$, $E^p(\Omega)$ is a Banach space endowed with the norm $\|\cdot\|_{E^p}$. When $\Omega = \mathbb{C}^+$, the space $E^p(\Omega)$ consists of all the functions holomorphic on \mathbb{C}^+ such that

$$\|f\|_p^p = \sup_{y>0} \int_{\mathbb{R}} |f(x+iy)|^p dx < \infty.$$

This space is often called the Hardy space of the upper-half plane ([Gar07], [Nik02], [Lev96], [Koo98]), and denoted by $H^p(\mathbb{C}^+)$. Assume now that Ω is a domain bounded by a rectifiable Jordan curve γ . In this case, each $f \in E^p(\Omega)$ ($1 \leq p < \infty$) admits a non-tangential limit almost everywhere on γ (denoted again by f) which belongs to $L^p(\partial\Omega)$, and satisfies the Cauchy formula

$$\forall z \in \Omega, f(z) = \frac{1}{2i\pi} \int_{\gamma} \frac{f(u)}{u-z} du. \quad (1.5)$$

We mention also the Smirnov-Zygmund space $E_{L \log^+ L}(\Omega)$ which, at least for the domains we consider here (so-called Smirnov domains), can be defined as the space of functions $f \in \text{Hol}(\Omega)$ such that $f \in E^1(\Omega)$ and the non-tangential limit of f on γ

belongs to $L \log^+ L(\partial\Omega)$, that is

$$\int_{\gamma} |f(z)| \log^+(|f(z)|) |dz| < +\infty. \quad (1.6)$$

Therefore, the following inclusions are clear

$$\forall 1 < p < +\infty, \quad E^p(\Omega) \subsetneq E_{L \log^+ L}(\Omega) \subsetneq E^1(\Omega).$$

For more details on the Hardy space and the Smirnov space, we refer to [Dur70, Chap. 10 and 11]. For the cases of the disc and the upper-half plane, see also [Rud87],[Gar07], [Nik02], [Lev96], [Koo98].

Boundedness of the Riesz projection.

In this paragraph, we restrict our attention to $\Omega = \mathbb{D}$ or $\Omega = \mathbb{C}^+$. As written above, if $\Omega = \mathbb{D}$, we can identify the Hardy space $H^p(\mathbb{D})$ ($1 \leq p < \infty$) with the subspace of $L^p(\mathbb{T})$ formed by all the non-tangential limits of its elements. Moreover, the norm of an element $f \in H^p(\mathbb{D})$ can be computed using the L^p -norm of its non-tangential limit: $\|f\|_{H^p} = \|f\|_{L^p(\mathbb{T})}$. Hence it is closed subspace of $L^p(\mathbb{T})$. Since the Hardy space of the half plane $H^p(\mathbb{C}^+)$ is isometrically isomorphic to the Hardy space of the disc, this result also holds for $H^p(\mathbb{C}^+)$ (see [Dur70, Section 11.2]). When $p = 2$, $H^2(\Omega)$ is a RKHS and if we write k_λ the reproducing kernel in $\lambda \in \mathbb{D}$, the orthogonal projection P_+ from $L^2(\partial\Omega)$ to $H^2(\partial\Omega)$ is given by

$$\forall f \in L^2(\partial\Omega), \quad P_+ f(z) = \langle P_+ f, k_z \rangle = \langle f, P_+ k_z \rangle = \langle f, k_z \rangle,$$

where we have used that the orthogonal projection is self-adjoint. A famous theorem of M. Riesz says that the orthogonal projection extends to a bounded operator from $L^p(\partial\Omega)$ to H^p for $1 < p < \infty$ (see [CMR06, Theorem 3.2.1]). However, it is not bounded neither from L^∞ into H^∞ nor from L^1 into H^1 . Actually, a theorem of D. Newman shows that there does not exist any bounded projection from L^p onto H^p for $p = 1, \infty$ (see [Nik02, Exercise 5.7.3.e]).

In the disc case, a Zygmund result (see [CMR06, Theorem 3.2.7]) states that if we permit a little more regularity, the projection is bounded into $H^1(\mathbb{T})$: P_+ is bounded from $L \log^+ L$ to $H^1(\mathbb{T})$.

On the upper half plane, the reproducing kernel at $\lambda \in \mathbb{C}^+$ is given by

$$k_\lambda(z) = \frac{1}{i\pi(z - \bar{\lambda})}, \quad \forall z \in \mathbb{C}^+$$

and thus the projection P_+ coincides with the Cauchy Transform (or Borel Transform) \mathcal{C} defined by

$$\forall f \in L^2(\mathbb{R}), (\mathcal{C}f)(z) = \frac{1}{i\pi} \int_{\mathbb{R}} \frac{f(t)}{t-z} dt, \quad \forall z \in \mathbb{C}^+.$$

For more details on this operator, we refer to [CMR06, Section 3.8]. Unfortunately, Zygmund's theorem is no longer true, when the domain is unbounded. Nevertheless, the result holds locally for compactly supported functions as stated in the following result.

Proposition 1.1.12. *Let $f \in L \log^+ L(\mathbb{R})$ have compact support. Let Ω be a square in the upper half plane, one side of which is a segment $I \subset \mathbb{R}$. Let $L > 0$ such that $I \subset (-\frac{L}{2}, \frac{L}{2})$ and $\text{supp } f \subset (-\frac{L}{2}, \frac{L}{2})$. Denote by $\Omega_L := (-L, L) \times (0, 2L)$ the square contained in \mathbb{C}^+ . Then the Cauchy transform $\mathcal{C}f$ belongs to $E^1(\Omega_L)$.*

This result is certainly known to the experts in harmonic analysis. Since we found no reference and since we use it in Section 3.4, we include its proof for convenience of the reader.

We start with a first intermediate result.

Lemma 1.1.13. *Let $f \in L \log^+ L(\mathbb{R})$ having compact support. Then the Cauchy Transform $\mathcal{C}f$ satisfies:*

$$\sup_{y>0} \int_{-L}^L |\mathcal{C}f(x+iy)| dx < +\infty$$

This result is essentially based on the following theorem by Calderón and Zygmund [CZ52, Thm 2 p.100]. Let $\tilde{f}_\lambda(x) = \int_{|x-y|>\lambda} f(y)/(x-y) dy$. Note that $\lim_{\lambda \rightarrow 0} \tilde{f}_\lambda(x)$ corresponds to the Hilbert transform of f .

Theorem 1.1.14 (Calderón-Zygmund). *If $|f|(1 + \log_+ |f|)$ is integrable over \mathbb{R} , then \tilde{f}_λ is integrable over every set S of finite measure. Moreover,*

$$\int_S |\tilde{f}_\lambda| dx \leq A_S \int_{\mathbb{R}} |f|(1 + \log_+ |f|) dx + B_S,$$

where A_S and B_S are constants depending only on S , but neither on f nor on λ .

Proof of Lemma 1.1.13. For $y > 0$, let $P_y(x) = \frac{y}{\pi(x^2+y^2)}$ and $Q_y(x) = \frac{x}{\pi(x^2+y^2)}$ be the Poisson and the conjugate Poisson kernels (Q_0 corresponds to the kernel of the Hilbert transform). Then we have

$$\forall z \in \mathbb{C}^+, \mathcal{C}f(z) = \mathcal{P}f(z) + i\mathcal{Q}f(z)$$

where we have written $\mathcal{P}f(x + iy) = (P_y * f)(x)$ and $\mathcal{Q}f(x + iy) = (Q_y * f)(x)$. So it suffices to show

$$\sup_{y>0} \int_{-L}^L |\mathcal{P}f(x + iy)| dx < +\infty \quad \text{and} \quad \sup_{y>0} \int_{-L}^L |\mathcal{Q}f(x + iy)| dx < +\infty.$$

The first inequality is clear from classical properties of the Poisson kernel (for this it is even enough that $f \in L^1(\mathbb{R})$, see [Gar07, Thm 3.1]). Indeed, for $f \in L^1(\mathbb{R})$ we have

$$\begin{aligned} \forall y > 0, \|\mathcal{P}f(\cdot + iy)\|_{L^1(-R,R)} &\leq \|\mathcal{P}f(\cdot + iy)\|_{L^1(\mathbb{R})} \\ &= \|P_y * f\|_{L^1(\mathbb{R})} \\ &\leq \|P_y\|_{L^1(\mathbb{R})} \|f\|_{L^1(\mathbb{R})} \\ &= \|f\|_{L^1(\mathbb{R})} < +\infty. \end{aligned}$$

Consider the second inequality. Recall the following estimate (see for example [Gar07, p. 105])

$$\forall y > 0, \forall x \in \mathbb{R}, \left| \mathcal{Q}_y f(x + iy) - \tilde{f}_y(x) \right| \leq CMf(x)$$

where Mf is the Hardy-Littlewood Maximal function. This, together with the classical result on the regularity of Mf (see [Gar07, p. 23])

$$f \in L \log^+ L([-L, L]) \iff Mf \in L^1([-L, L])$$

and Theorem 1.1.14 above yields the desired result. \square

We are now in a position to prove the proposition.

Proof of Proposition 1.1.12. In order to prove $\mathcal{C}f \in E^1(\Omega_L)$, pick $(\omega_\varepsilon)_{0 < \varepsilon < \varepsilon_0}$, $\varepsilon_0 < L/2$, a sequence of rectifiable Jordan curves given by the sides of the squares contained in Ω_L one side of which is $\omega_{\varepsilon,0} = [-L + \varepsilon, L - \varepsilon] + i\varepsilon$ and $\omega_{\varepsilon,1}$ corresponds to the remaining three sides of the square (see Figure 1). Let $\omega_\varepsilon = \omega_{\varepsilon,0} \vee \omega_{\varepsilon,1}$ (concatenation of the two Jordan curves, orientated counterclockwise). Then for any $0 < \varepsilon \leq \varepsilon_0$, we have $d(\omega_{\varepsilon,1}, \text{supp } f) > 0$. Thus, from the very definition of the Cauchy transform and triangular inequality,

$$\sup_{0 < \varepsilon < \varepsilon_0} \int_{\omega_{\varepsilon,1}} |\mathcal{C}f| |dz| < +\infty.$$

It remains to show that

$$\sup_{0 < \varepsilon < \varepsilon_0} \int_{\omega_{\varepsilon,0}} |\mathcal{C}f| |dz| = \sup_{0 < \varepsilon < \varepsilon_0} \int_{-L+\varepsilon}^{L-\varepsilon} |\mathcal{C}f(t + i\varepsilon)| dt < +\infty.$$

Using Lemma 1.1.13, we conclude that $\mathcal{C}f \in E^1(\Omega_L)$ □

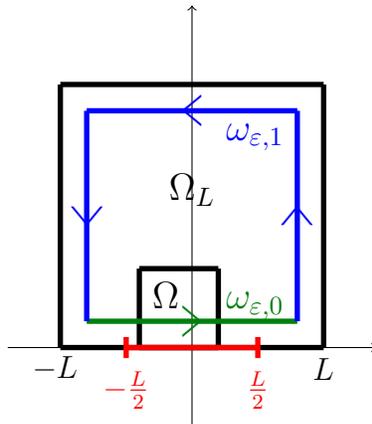


Figure 1: The squares Ω et Ω_L , and the path ω_ε .

1.1.3 Bergman space.

In this subsection, we focus on another classical space, namely the Bergman space. The material of this subsection can be found in the two main textbooks on this topic [HKZ00], [DS04] and in the lecture notes [BBG⁺04]. For the interested readers, a collection of open problems is proposed in [AHR05].

Let Ω be an open subset of \mathbb{C} . Let $1 \leq p < \infty$. The weighted Bergman space $A^p(\Omega, \omega)$, where ω is a non negative measurable function on Ω , consists of all functions $f \in \text{Hol}(\Omega)$ such that

$$\|f\|_{A^p(\Omega, \omega)}^p = \int_{\Omega} |f(x + iy)|^p \omega(x + iy) dx dy < +\infty.$$

Endowed with the norm $\|\cdot\|_{A^p(\Omega, \omega)}$, it is a Banach space. When $\omega = 1$, $A^p(\Omega, \omega)$ is the classical Bergman space which we simply denote $A^p(\Omega)$. When $p = 2$, $A^2(\Omega)$ is a RKHS and we denote by k_λ^Ω its reproducing kernel, i.e. for any $f \in A^2(\Omega)$ and any

$\lambda \in \Omega$, $f(\lambda) = \langle f, k_\lambda^\Omega \rangle_{L^2(\Omega)}$. Clearly, $A^2(\Omega)$ is a closed subspace of $L^2(\Omega)$, and the corresponding orthogonal projection, called *Bergman projection*, is given by

$$(P_\Omega f)(\lambda) = \langle f, k_\lambda^\Omega \rangle_{L^2(\Omega)}, \quad \lambda \in \Omega. \quad (1.7)$$

Also, if Ω_1 and Ω_2 are two open subsets of \mathbb{C} , and $\varphi : \Omega_1 \rightarrow \Omega_2$ is a conformal mapping, we have the following conformal invariance property (see [QQ17, Chapter VIII, Theorem 4.9, p.280] or [DS04, Chapter 1, § 1.3, Theorem 3])

$$k_\lambda^{\Omega_1}(z) = k_{\varphi(\lambda)}^{\Omega_2}(\varphi(z))\varphi'(z)\overline{\varphi'(\lambda)} \quad (1.8)$$

Observe that in the last formula, $A^2(\Omega_1)$ is not exactly the pull-back of $A^2(\Omega_2)$ as introduced in 1.1.8 since we have to take into account also the Jacobian. The most prominent case is when Ω is the unit disk in the complex plane $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\omega = 1$. Then for $\lambda \in \mathbb{D}$ (see e.g. [DS04, Section 1.2]),

$$k_\lambda^{\mathbb{D}}(z) = \frac{1}{\pi(1 - \bar{\lambda}z)^2}, \quad z \in \mathbb{D}. \quad (1.9)$$

Regularity of the Bergman projection and density of $A^2 \cap A^p$ in A^p .

By definition, the Bergman projection P_Ω is bounded on $L^2(\Omega)$. As for the Riesz projection, we can ask whether it extends to a bounded operator on $L^p(\Omega)$. This question is closely related to the geometry of the domain Ω and can be used as a way to classify the domains. For simply connected domains, it is linked in a sense to the regularity of the Riemann mapping on the closure of the domain. We refer to the survey [Zey20] for more details. When $\Omega = \mathbb{D}$ or \mathbb{C}_+ , the boundedness can be obtained *via* Schur's lemma. In both cases, P_Ω is bounded on L^p if and only if $1 < p < \infty$.

The boundedness of the Bergman projection is also related to the density of $A^2(\Omega) \cap A^p(\Omega)$ in $A^p(\Omega)$. Indeed, we have seen in 1.7 that the Bergman projection P_Ω can be written as an integral operator. It follows that the reproducing property

$$f(\lambda) = \int_\Omega f(z)\overline{k_\lambda^\Omega(z)}dA(z), \quad \lambda \in \Omega$$

holds for $f \in L^2(\Omega) \cap L^p(\Omega)$, providing that k_λ^Ω belongs to $L^{p'}$. Hence assuming the boundedness of P_Ω on $L^p(\Omega)$ and density of $A^2(\Omega) \cap A^p(\Omega)$ in $A^p(\Omega)$, this equality extends to all the functions in $L^p(\Omega)$. In Chapter 4, these two properties will allow us to obtain another kind of reproducing property. Although the density of $A^2(\Omega) \cap A^p(\Omega)$ in $A^p(\Omega)$ seems to be a difficult problem in general (see [Hed02, Proposition 2.2]), it is well-known in the particular case $\Omega = \mathbb{C}_+$.

Proposition 1.1.15. [see [BBG⁺04, Proposition 1.17]] For all $1 \leq p < \infty$, the set $A^2(\mathbb{C}_+) \cap A^p(\mathbb{C}_+)$ is dense in $A^p(\mathbb{C}_+)$.

We will give an adaptation of this proposition for $\Omega = \mathbb{C}^{++} := \{z \in \mathbb{C}; \operatorname{Re}(z) > 0, \text{ and } \operatorname{Im}(z) > 0\}$ in Lemma 4.2.1.

A Hardy-Littlewood Theorem. Finally we finish this subsection with the following interesting theorem which is due to Hardy and Littlewood [HL32, thm31] (see also [Vuk03] or [QQ17, Thm4.11, p. 282] for a more elementary proof).

Theorem 1.1.16 (Hardy-Littlewood). *The Hardy space $H^1(\mathbb{D})$ embeds continuously into $A^2(\mathbb{D})$.*

1.2 Paley-Wiener type theorems

In this section, we give an overview of the main Paley-Wiener type theorems. They are theorems which describe the range of the Fourier (or Laplace) transform as a space of holomorphic functions, depending on the space on which it is taken. They show in the extreme case of analyticity, the well-known links between decay properties of a function and regularity of its Fourier transform, and conversely. They are named in reference of R. Paley and N. Wiener who gave the first two theorems in this way ([PW87, Theorem V and Theorem X], see also [Rud87, Theorems 19.2 and 19.3]) which are classical today. Paley-Wiener theorem will turn out to be powerful tools for describing the reachable spaces. Since the Laplace Transform is a particular case of the linear transform defined in (1.4), we will do an intensive use of Theorem 1.1.5 in this section, revisiting the original proofs.

1.2.1 On a segment

Let \mathcal{F} denote the Fourier transform formally given by $(\mathcal{F}f)(z) = \frac{1}{\sqrt{2A}} \int_{\mathbb{R}} e^{-ixz} f(x) dx$. Remind that for $A > 0$ the Paley-Wiener space PW_A consists of all entire functions f of exponential type A , i.e. $|f(z)| \leq A_\varepsilon e^{(A+\varepsilon)|z|}$ for every $\varepsilon > 0$ and every $z \in \mathbb{C}$ (as it turns out, in this context $e^{A|z|}$ can be safely replaced by $e^{A|\operatorname{Im}z|}$), which satisfy $\|f\|_{PW}^2 := \int_{\mathbb{R}} |f(x)|^2 dx < \infty$. It is a RKHS with reproducing kernel

$$k_\lambda(z) = \operatorname{sinc}(A(\bar{\lambda} - z)), \quad (\lambda, z \in \mathbb{C})$$

where $\operatorname{sinc}(z) := \frac{\sin(z)}{z}$.

Theorem 1.2.1 (Paley-Wiener). [see Theorem X in [PW87]] $\mathcal{F} : L^2(-A, A) \rightarrow PW_A$ is an isometric isomorphism.

We can now appreciate the power of Theorem 1.1.5 proving very easily this theorem.

Proof. Writing $h(z) = \frac{1}{\sqrt{2A}}e^{i\bar{z}}$, we have

$$(\mathcal{F}f)(z) = \langle f, h(z) \rangle_{L^2(-A, A)}.$$

Therefore, using Theorem 1.1.5, we obtain that the range of the Fourier transform is a RKHS with reproducing kernel

$$k_\lambda(z) = \langle h(\lambda), h(z) \rangle_{L^2(-A, A)} = \frac{1}{2A} \int_{-A}^A e^{i(\bar{\lambda}-z)t} dt = \frac{e^{i(\bar{\lambda}-z)A} - e^{-i(\bar{\lambda}-z)A}}{2Ai(\bar{\lambda}-z)} = \text{sinc}(A(\bar{\lambda}-z))$$

which is the result expected. Finally, since the family $(e^{iz})_{z \in \mathbb{C}}$ is complete in $L^2(-A, A)$ (it contains the Fourier basis), the Fourier transform is isometric. \square

Let \mathcal{L} be the Laplace transform defined formally by $(\mathcal{L}f)(z) = \int_0^\infty e^{-zt} f(t) dt$. For θ an inner function of the right half plane, i.e. a function uniformly bounded by one having non tangential boundary values of modulus one a.e. on $i\mathbb{R}$, we denote by K_θ the associated model space, i.e. $K_\theta = H^2(\mathbb{C}_+) \ominus \theta H^2(\mathbb{C}_+)$. It is a RKHS of analytic functions on \mathbb{C}_+ with kernel

$$k_\lambda^\theta(z) = \frac{1 - \theta(z)\overline{\theta(\bar{\lambda})}}{2\pi(z + \bar{\lambda})}.$$

Proposition 1.2.2. Let θ_A be the inner function defined by $\theta_A(z) = e^{-zA}$. $\mathcal{L} : L^2(0, A) \rightarrow K_{\theta_A}$ is an isomorphism. Moreover, we have $\|\mathcal{L}f\|_{K_{\theta_A}} = \sqrt{2\pi}\|f\|_{L^2}$.

In this particular case, we can show that K_θ is a space of entire functions. Obviously, the result can be deduced from Theorem 1.2.1 by a translation and a dilation, but we use only Theorem 1.1.5.

Proof. The proof is similar as the one above. We give only the computation of the reproducing kernel :

$$k_\lambda(z) = \langle \overline{e^{-\lambda}}, \overline{e^{-z}} \rangle_{L^2(0, A)} = \int_0^A e^{-(z+\bar{\lambda})t} dt = \frac{1 - e^{-(z+\bar{\lambda})A}}{z + \bar{\lambda}},$$

which is precisely, up to the constant 2π , the reproducing kernel of K_{θ_A} as claimed. \square

In Theorem 1.2.1, if we ask for more regularity on the starting function, we have to add a decay condition on the Fourier transform as expected. This leads to the following Paley-Wiener theorem for test functions.

Theorem 1.2.3. [see Theorem 8.1, p. 123, [Zui02]] Let $A > 0$. The following conditions are equivalent :

1. $F = \mathcal{F}f$ with $f \in C_c^\infty(\mathbb{R})$ and $\text{supp}f \subset [-A, A]$,
2. F is entire and satisfies

$$\forall N \in \mathbb{N}, \exists C_N > 0, |F(z)| \leq C_N(1 + |z|)^{-N} e^{A|\text{Im}z|}, \quad \forall z \in \mathbb{C}.$$

Let us give a little modification of this theorem, which can be useful for describing the reachable space.

Theorem 1.2.4. Let $A > 0$. The following conditions are equivalent :

1. $F = \mathcal{L}f$ with $f \in C_c^\infty(0, \infty)$ and $\text{supp}f \subset [0, A]$,
2. F is entire and satisfies

$$\forall N \in \mathbb{N}, \exists C_N > 0, |F(z)| \leq C_N(1 + |z|)^{-N} e^{A \max(0, -\text{Re}z)}, \quad \forall z \in \mathbb{C}.$$

L. Schwartz gave an extension of the Paley-Wiener theorem to the space of compactly supported distributions \mathcal{E}' . We recall that a distribution $T \in \mathcal{D}'(\mathbb{R})$ is said of order $N \in \mathbb{N}$ if for every compact set K there exists $C_K > 0$ such that

$$|\langle T, \varphi \rangle| \leq C_K \sum_{k=0}^N \sup_{x \in K} |\varphi^{(k)}(x)|, \quad \forall \varphi \in C_c^\infty(K).$$

In other words, the integer $N \in \mathbb{N}$ does not depend on the compact set K . If the previous inequality holds for a $N \in \mathbb{N}$, T is said to be of finite order. It is well-known that compactly supported distributions are of finite order. The Paley-Wiener-Schwartz theorem is the following.

Theorem 1.2.5 (Schwartz). [see Theorem 8.3, p. 123, [Zui02]] Let $A > 0$. The following conditions are equivalent :

1. $F = \mathcal{F}T$ with $T \in \mathcal{E}'$, T is of order $N_0 \in \mathbb{N}$ and $\text{supp}T \subset [-A, A]$,
2. F is entire and satisfies

$$\exists C > 0, |F(z)| \leq C(1 + |z|)^{N_0} e^{A|\text{Im}z|}, \quad \forall z \in \mathbb{C}.$$

Remark 1.2.6. For Paley-Wiener type theorems on more general compact subset of \mathbb{C} , the reader can read [Lin02] and the references given there.

1.2.2 On the half-line

We will now give the same type of results for functions which are supported on the half-line $(0, \infty)$. Let us start with the second classical theorem of Paley and Wiener. Let $H^2(\mathbb{C}_+)$ be the Hardy space of the right half-plane.

Theorem 1.2.7 (Paley-Wiener). *[see Theorem V in [PW87]]* $\mathcal{L} : L^2(0, \infty) \rightarrow H^2(\mathbb{C}_+)$ is an isomorphism. Moreover, we have $\|\mathcal{L}f\|_{H^2(\mathbb{C}_+)}^2 = 2\pi\|f\|_{L^2}^2$ for every $f \in L^2(0, \infty)$.

Proof. Again, the result follows essentially from Theorem 1.1.5. We compute the reproducing kernel:

$$k_\lambda(z) = \int_0^\infty e^{-(z+\bar{\lambda})t} dt = \frac{1}{z + \bar{\lambda}},$$

which is the reproducing kernel of the Hardy space $H^2(\mathbb{C}_+)$ up to the constant $\frac{1}{2\pi}$. Finally, we have an isometry if and only if the family $(e^{-\lambda \cdot})_{\lambda \in \mathbb{C}_+}$ is complete in $L^2(0, \infty)$, which is equivalent (by the change of variable $s = e^{-t}$) to the fact that the family $(t^\lambda)_{\lambda \in \mathbb{C}_+}$ is complete in $L^2[0, 1]$. This follows from the classical Müntz theorem (see [Rud87, Theorem p. 313]). \square

Remark 1.2.8. In the previous proof, we have used the Müntz theorem to prove the Paley-Wiener theorem. Conversely, the Paley-Wiener theorem can be used as a bridge between the Müntz spaces (closure of the span of a family $(t^\lambda)_{\lambda \in \Lambda}$) and the Hardy (or Model) space of the right half plane. The interested reader can see the paper [FL19] for more information.

Let $\tilde{\nu}$ be a positive regular measure on $[0, \infty)$ which is doubling at the origin, i.e

$$\exists C_{\tilde{\nu}}, \forall x > 0, \tilde{\nu}([0, 2x]) \leq C_{\tilde{\nu}}\tilde{\nu}([0, x]) < \infty.$$

Let λ be the Lebesgue measure on \mathbb{R} . We denote by ν the positive regular measure on $\overline{\mathbb{C}_+} := [0, \infty) \times \mathbb{R}$ given by $\nu := \tilde{\nu} \otimes \lambda$. The Zen space $A_{\tilde{\nu}}^2$ consists of all functions f analytic on \mathbb{C}_+ which satisfy

$$\|f\|_{A_{\tilde{\nu}}^2}^2 := \sup_{\epsilon > 0} \int_{\overline{\mathbb{C}_+}} |f(z + \epsilon)|^2 d\nu(z) < \infty.$$

These spaces have been constructed by Zen Harper in [Har09] and [Har10]. They generalize the Hardy space (which is obtained for $\tilde{\nu} = \delta_0$) and the (weighted) Bergman space (obtained for $d\tilde{\nu}(x) = x^\alpha dx$ and $\alpha > -1$). The next proposition generalizes the Paley-Wiener theorem above for weighted L^2 spaces. It appeared in [JPP13, Proposition 2.3], a big part being already known in [Har09, Section 2] and [Har10, Section 2]. Notice that it gives only an isometry (and not an isomorphism).

Proposition 1.2.9. [see Proposition 2.3 in [JPP13]] Let A_ν^2 be a Zen space, and let $w : (0, \infty) \rightarrow \mathbb{R}_+$ be given by

$$w(t) := 2\pi \int_0^\infty e^{-2rt} d\tilde{\nu}, \quad \forall t > 0.$$

Then $\mathcal{L} : L^2(0, \infty; w) \rightarrow A_\nu^2(\mathbb{C}_+)$ defines an isometric map.

The proof of this proposition is straightforward, using Plancherel's and Fubini's theorems.

Proof. Note first that, since $\tilde{\nu}$ is a doubling measure, the integral defining w converges for all $t > 0$. Indeed, this can be seen using a dyadic decomposition of the interval $[1, +\infty)$. Let $\mathcal{F}f(y) = \int_{\mathbb{R}} e^{-iyt} f(t) dt$. For every $f \in L^2(0, \infty; w)$, we have

$$\begin{aligned} \sup_{\varepsilon > 0} \int_{\mathbb{C}_+} |(\mathcal{L}f)(z + \varepsilon)|^2 d\nu(z) &= \sup_{\varepsilon > 0} \int_{[0, \infty)} \int_{-\infty}^\infty |(\mathcal{L}f)(x + iy + \varepsilon)|^2 d\lambda(y) d\tilde{\nu}(x) \\ &= \sup_{\varepsilon > 0} \int_{[0, \infty)} \|(\mathcal{L}f)(x + i \cdot + \varepsilon)\|_{L^2(\mathbb{R})}^2 d\tilde{\nu}(x) \\ &= \sup_{\varepsilon > 0} \int_{[0, \infty)} \|(\mathcal{F}[e^{-(x+\varepsilon)\cdot} f])\|_{L^2(\mathbb{R})}^2 d\tilde{\nu}(x) \\ &= \sup_{\varepsilon > 0} \int_{[0, \infty)} 2\pi \|e^{-(x+\varepsilon)\cdot} f\|_{L^2(0, \infty)}^2 d\tilde{\nu}(x) \\ &= \sup_{\varepsilon > 0} \int_0^\infty |f(y)|^2 2\pi \int_{[0, \infty)} e^{-2(x+\varepsilon)y} d\tilde{\nu}(x) dy \\ &= \int_0^\infty |f(y)|^2 w(y) dy. \end{aligned}$$

□

Let $R_w(\mathbb{C}_+) := \mathcal{L}[L^2(0, \infty; w)]$ denote the range of the Laplace transform. It is a closed subspace of $A_\nu^2(\mathbb{C}_+)$ and hence a RKHS endowed with the norm of $A_\nu^2(\mathbb{C}_+)$. The following proposition gives an exact characterization of the range $R_w(\mathbb{C}_+)$ through of its reproducing kernel.

Theorem 1.2.10. [see Theorem 1.2.4, [Kuc17]] The reproducing kernel of $R_w(\mathbb{C}_+)$ is given by

$$k_\lambda^{\mathbb{R}_w(\mathbb{C}_+)}(z) := \int_0^\infty \frac{e^{-t(z+\bar{\lambda})}}{w(t)} dt, \quad \forall z, \lambda \in \mathbb{C}_+.$$

As done in [Kuc17], it suffices to prove that the given function $k_\lambda^{\mathbf{R}_w(\mathbb{C}_+)}$ belongs to $\mathbf{R}_w(\mathbb{C}_+)$ and satisfies the reproducing property. But, once again we will just use Theorem 1.1.5.

Proof. We have for every $z \in \mathbb{C}_+$,

$$\forall f \in L^2(0, \infty; w), \quad \mathcal{L}f(z) = \int_0^\infty f(t) \frac{e^{-zt}}{w(t)} w(t) dt = \left\langle f, \frac{e^{-\bar{z}\cdot}}{w(t)} \right\rangle_{L^2(0, \infty; w)}.$$

Hence by Theorem 1.1.5, the reproducing kernel of $\mathbf{R}_w(\mathbb{C}_+)$ is given by

$$k_\lambda^{\mathbf{R}_w(\mathbb{C}_+)}(z) = \left\langle \frac{e^{-\bar{\lambda}\cdot}}{w(t)}, \frac{e^{-\bar{z}\cdot}}{w(t)} \right\rangle_{L^2(0, \infty; w)} = \int_0^\infty \frac{e^{-t(z+\bar{\lambda})}}{w(t)} dt.$$

□

As said above, we have $\mathbf{R}_w(\mathbb{C}_+) \neq A_\nu^2(\mathbb{C}_+)$ in general. Harper gives several counterexamples in [Har09, Section 2]. In [Har10, Theorem 2.1], he gives a sufficient condition for the equality to hold when $d\tilde{\nu}(x) = v(x)dx$: if for every $0 < \alpha < \beta < \infty$ there exists $\epsilon(\alpha, \beta) > 0$ such that $\int_\alpha^\beta v(x)^{-\epsilon(\alpha, \beta)} dx < \infty$, then $\mathbf{R}_w(\mathbb{C}_+) = A_\nu^2(\mathbb{C}_+)$. There are few simple known cases in which this condition holds. Some of them are resumed in the following folk theorem which is proved in [DGGMR07]. Let us introduce some notations. For $\alpha > 0$, let $L_\alpha^2(\mathbb{R}_+)$ denote the space of complex-valued measurable functions on \mathbb{R}_+ for which

$$\|f\|_{L_\alpha^2(\mathbb{R}_+)}^2 := \int_0^\infty |f(t)|^2 t^{-\alpha} dt < \infty.$$

For $\beta > -1$ and $d\tilde{\nu}(x) = x^\beta dx$, we will denote as for the disc $A_\beta^2(\mathbb{C}_+)$ the weighted Bergman space on the right half plane endowed with the norm

$$\|F\|_{A_\beta^2(\mathbb{C}_+)}^2 := \int_{\mathbb{C}_+} |F(z)|^2 \operatorname{Re}(z)^\beta dA(z).$$

Its reproducing kernel is given by

$$k_\lambda^{A_\beta^2} = [\mathcal{L}(t^{\beta+1})](z + \bar{\lambda}) \left(\frac{\Gamma(\beta + 2)}{(z + \bar{\lambda})^{\beta+2}} \right) \quad \lambda \in \mathbb{C}_+.$$

The theorem states as follows.

Theorem 1.2.11. [see [DGGMR07, Theorem 1]] Let $\alpha > 0$. $\mathcal{L} : L_\alpha^2(\mathbb{R}_+) \rightarrow A_{\alpha-1}^2(\mathbb{C}_+)$ is an isomorphism. Moreover, we have $\|\mathcal{L}f\|_{A_{\alpha-1}^2(\mathbb{C}_+)}^2 = \frac{2\pi\Gamma(\alpha)}{2^\alpha} \|f\|_{L_\alpha^2(\mathbb{R}_+)}^2$ for every $f \in L_\alpha^2(\mathbb{R}_+)$.

Remark 1.2.12. It seems that this theorem appears for the first time in [Rot60, Theorem 2.7] for the (unweighted) Bergman space on tubes over self-dual cones (e.g. the right half plane is a tube over the half line) and in [Kor62, Theorem1] for the Bergman space of general tubes (unlike the first paper, the result is stated explicitly). In these two papers, the theorem is used in order to obtain the expression of the Bergman kernel. In our case, the one-dimensional setting allows us to get the Bergman kernel from its expression on the disc using a conformal mapping. The weighted case appears in [BBG⁺04, Theorem 1.22] for the upper half plane and in [BBG⁺04, Theorem 3.11] for tubes over symmetric convex cones. The authors get it as a consequence of the Paley-Wiener theorem for the Hardy space.

In our context, the proof is just a direct corollary of the Theorem 1.2.10 above.

Proof. By Theorem 1.2.10, the reproducing kernel of the range $R_\alpha(\mathbb{C}_+)$ is given by

$$k_\lambda^{\mathbb{R}_\alpha(\mathbb{C}_+)}(z) := \int_0^\infty e^{-t(z+\bar{\lambda})} t^\alpha dt = [\mathcal{L}(t^\alpha)](z + \bar{\lambda}) = \frac{\Gamma(\alpha + 1)}{(z + \bar{\lambda})^{\alpha+1}}$$

which is, up to a constant the reproducing kernel of $A_{\alpha-1}^2(\mathbb{C}_+)$. In the computation above, the last inequality is true (by changing the variable) on $(0, \infty)$ and thus holds on \mathbb{C}_+ by the identity principle. \square

In the proof given in [DGGMR07], the main idea is to show that there is an orthonormal basis of $A_{\alpha-1}^2(\mathbb{C}_+)$ which is in the range of the Laplace transform. For this, remark that, if φ is a conformal mapping from \mathbb{C}_+ to \mathbb{D} , the operator $f \mapsto (f \circ \varphi)\varphi'$ is an isometric isomorphism from $A_\alpha^2(\mathbb{D})$ onto $A_\alpha^2(\mathbb{C}_+)$, where $A_\alpha^2(\mathbb{D})$ consists of all functions holomorphic on \mathbb{D} which satisfy

$$\int_{\mathbb{D}} |f(z)|^2 (1 - |z|^2)^\alpha dA(z) < \infty.$$

Hence, we can carry the orthonormal basis $(z^n)_{n \in \mathbb{N}}$ of $A_\alpha^2(\mathbb{D})$ to its corresponding basis in $A_\alpha^2(\mathbb{C}_+)$. Finally, it suffices roughly speaking to show that the Fourier transform of each element of this new basis belongs to $L_\alpha^2(\mathbb{R}_+)$ and use the Fourier inversion theorem. It is interesting to note that the Fourier transforms of these elements are related to with Laguerre polynomials.

All the previous methods (including Theorem 1.1.5) used to obtain Paley-Wiener type theorems cannot get out of the frame of the Hilbertian case. There are also L^p -versions of the Theorems 1.2.7 and 1.2.11. They treat only the boundedness of the Laplace transform (no isometry, no isomorphism), and are given by Rydhe in [Ryd19]. They state as follows.

Theorem 1.2.13. [see Theorem 1.6, [Ryd19]] *If $2 < p < \infty$, then*

$$\mathcal{L} : L^p_\alpha(\mathbb{R}_+, x^{p-2}dx) \rightarrow H^p(\mathbb{C}_+)$$

is bounded.

Theorem 1.2.14. [see [Ryd19, Theorem 1.5]] *If $2 < p \leq q < \infty$ and $\alpha < p/q' - 1$, then*

$$\mathcal{L} : L^p_\alpha(\mathbb{R}_+, x^\alpha dx) \rightarrow A^q_{q/p'-2-\alpha q/p}(\mathbb{C}_+)$$

is bounded.

Remark 1.2.15. For more generalizations on the spaces, the reader can take a look at the thesis manuscript of Kucik [Kuc17, Chapter 1].

1.2.3 Discrete case

For $\delta > 0$, let H_δ denote the space of complex-valued functions which are continuous on \mathbb{R} , 2π -periodic and extend analytically to the strip $S_\delta := \{z \in \mathbb{C}, |\text{Im}z| < \delta\}$. Write

$$c_n(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x)e^{-inx} dx, \quad n \in \mathbb{Z}$$

the (complex) Fourier coefficients of a function f . The following theorem is a classical discrete (i.e. for Fourier series) Paley-Wiener theorem and can be found in [QZ13, Chap.IV, sect.V, Theorem V.1 vi), p. 98].

Theorem 1.2.16. *If $f \in H_\delta$, then $|c_n(f)| = O(e^{-\frac{\delta}{2e}|n|})$ when $|n| \rightarrow \infty$. Conversely, if there exists $\epsilon > 0$ such that $|c_n(f)| = O(e^{-\epsilon|n|})$ when $|n| \rightarrow \infty$, then f belongs to H_ϵ .*

In our context, we will be more interested in sine Fourier series. So, we give a little modification of the previous theorem. For $\delta > 0$, denote by H_δ^0 the space of continuous functions which are π -periodic on \mathbb{R} , which extend holomorphically on the strip S_δ and which vanish with all its derivatives of even orders in 0 and in π . We remind that the family $(\sin(n\pi \cdot))_{\mathbb{N}^*}$ is an orthonormal basis of $L^2(0, \pi)$ and we write $(a_n(f))_{n \in \mathbb{N}^*}$ the sequence of sine-Fourier coefficients of a function f of $L^2(0, \pi)$.

Proposition 1.2.17. *If $f \in H_\delta^0$ with $\delta > 0$, then $|a_n(f)| = O(e^{-\frac{\delta}{2e}n})$ when $n \rightarrow \infty$. Conversely, if there exists $\delta > 0$ such that $|a_n(f)| = O(e^{-\delta n})$ when $n \rightarrow \infty$, then f belongs to H_δ^0 .*

We give a proof for the sake of completeness.

Proof. Let $\delta > 0$ and pick $f \in H_\delta^0$. Using Cauchy's inequalities on the compact $K = [-\delta/2, \pi + \delta/2] \times [-\delta/2, \delta/2]$, we obtain

$$\forall x \in [0, \pi], |f^{(k)}(x)| \leq \left(\frac{2}{\delta}\right)^k k! \sup_{\partial K} |f(z)| \leq C \left(\frac{2k}{\delta}\right)^k.$$

Pick $n \in \mathbb{N}^*$. By multiple integrations by part and since the derivative of even orders vanish, we obtain

$$|a_n(f)| = |n^{-2k}| |a_n(f^{(2k)})| \leq |n^{-2k}| \sup_{[0, \pi]} |f^{(2k)}|, \quad k \in \mathbb{N}.$$

Combining the two previous estimates, we get

$$|a_n(f)| \leq C \left(\frac{4k}{|n|\delta}\right)^{2k} \leq \exp\left(2k \log\left(\frac{4k}{|n|\delta}\right)\right).$$

We optimize the right hand side taking $k = \lfloor \frac{|n|\delta}{4e} \rfloor$. Then

$$2k \log\left(\frac{4k}{|n|\delta}\right) \leq 2k \log(1/e) = -2k \leq 2 - \frac{|n|\delta}{2e}$$

and so $|a_n(f)| \lesssim e^{-\frac{\delta}{2e}|n|}$.

Conversely, if $|a_n(f)| \lesssim e^{-\delta n}$, then the series $\sum a_n(f) \sin(n\pi \cdot)$ converges normally on each compact subset of S_δ . So, the function defined by

$$F(z) = \sum_{n=1}^{\infty} a_n(f) \sin(n\pi z),$$

is an analytic continuation of f on S_δ . Moreover, it is clear that the function f and all its derivatives of even orders vanish in 0 and in π . \square

1.3 Separation of singularities

For $\Omega \subset \mathbb{C}$, an open set in the complex plane, we denote by $\text{Hol}(\Omega)$ the space of holomorphic functions on Ω . Given Ω_1 and Ω_2 two open subsets of \mathbb{C} with non empty intersection, a natural question is to know whether every function $f \in \text{Hol}(\Omega_1 \cap \Omega_2)$ can be written as a sum of two functions $f_1 \in \text{Hol}(\Omega_1)$ and $f_2 \in \text{Hol}(\Omega_2)$, i.e. does the equality $\text{Hol}(\Omega_1 \cap \Omega_2) = \text{Hol}(\Omega_1) + \text{Hol}(\Omega_2)$ hold? This problem is known as the separation of singularities problem for holomorphic functions and has a quite long history. A simple example is given by $\Omega_2 = \{z \in \mathbb{C} \mid |z| < r_2\}$ and $\Omega_1 = \{z \in \mathbb{C} \mid |z| > r_1\}$ with $0 < r_1 < r_2$. Then $\Omega_1 \cap \Omega_2$ is a ring and the problem can be solved affirmatively using Laurent series. Poincaré [Poi92, V, Ch. 3, § 21] discussed the solution in the particular case when $\Omega_1 = \mathbb{C} \setminus [-1, 1]$ and $\Omega_2 = \mathbb{C} \setminus ((-\infty, -1] \cup [1, +\infty))$, and Aronszajn [Aro35] gave a positive answer for any pair (Ω_1, Ω_2) of open sets in \mathbb{C} .

Theorem 1.3.1 (Aronszajn). *[see [Aro35]] Let Ω_1 and Ω_2 be two open subsets of \mathbb{C} with non empty intersection. For every function $f \in \text{Hol}(\Omega_1 \cap \Omega_2)$, there exist two functions $f_1 \in \text{Hol}(\Omega_1)$ and $f_2 \in \text{Hol}(\Omega_2)$, such that*

$$f = f_1 + f_2, \quad \text{on } \Omega_1 \cap \Omega_2.$$

Let us give a modern proof of this theorem. For that, we introduce the standard Wirtinger derivative

$$\bar{\partial} := \frac{\partial}{\partial \bar{z}} := \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

We recall also a classical result of complex analysis about solution of the $\bar{\partial}$ -equation (non-homogeneous Cauchy-Riemann equation).

Theorem 1.3.2. *[see [AM04, Theorem 9.3.1]] Let Ω be an open set of \mathbb{C} and $v \in C^\infty(\Omega)$. Then there exists a function $u \in C^\infty(\Omega)$ such that $\bar{\partial}u = v$.*

Modern proof of Aronszajn's theorem. Pick $f \in \text{Hol}(\Omega_1 \cap \Omega_2)$. Take χ a bounded C^∞ -function on $\Omega_1 \cup \Omega_2$ such that $\chi = 1$ on $\Omega_1 \setminus \Omega_2$ and $\chi = 0$ on $\Omega_2 \setminus \Omega_1$. So we can define $h_1 = f(1 - \chi)$ on Ω_1 and $h_2 = f\chi$ on Ω_2 . Using the analyticity of f , we have $\bar{\partial}h_1 = -f\bar{\partial}\chi = -\bar{\partial}h_2$ on $\Omega_1 \cap \Omega_2$, which implies the existence of a C^∞ -continuation v on $\Omega_1 \cup \Omega_2$ such that $v = \bar{\partial}h_1$ on Ω_1 and $v = -\bar{\partial}h_2$ on Ω_2 . By the previous theorem, there exists $u \in C^\infty(\Omega_1 \cup \Omega_2)$ such that $\bar{\partial}u = v$. Finally, defining $f_1 = h_1 - u$ on Ω_1 and $f_2 = h_2 + u$ on Ω_2 , we obtain $f = f_1 + f_2$ on $\Omega_1 \cap \Omega_2$ and $f_i \in \text{Hol}(\Omega_i)$ by definition of u . The proof is complete. \square

The separation of singularities problem is a special case of the First Cousin Problem which reduces the problem to solving a $\bar{\partial}$ -equation and can be reformulated in sheaf cohomology terms (see [Hör90, Thm 1.4.5 and Thm 5.5.1]). The First Cousin problem has been solved a few years after Aronszajn, first by Oka [Oka39] on domains of holomorphy and then in the Cartan seminar [Car52] on Stein manifolds. Today, it is well-known that the First Cousin Problem on \mathbb{C} is equivalent to the Mittag-Leffler theorem (see [Hör90, pp. 11-14] and [AM04, section 9.4]).

We would also like to mention two other simple proofs of the separation of singularities problem. The first one, given by Havin [Hav58] (see also [Aiz14] or [MK71]), is based on a very beautiful duality argument. We reproduce the proof here, with additional details. See also [BKN18, p.16] for a concentrated proof.

Havin's proof of the Aronszajn's theorem. Let us start with a theoretic lemma.

Lemma 1.3.3. *Let X_0, X_1, X_2 be locally convex topological vector spaces (LCTVS) such that $X_0 \subset X_1$ and $X_0 \subset X_2$ (in the sense that X_0 is algebraically isomorphic to a subset of X_i , $i = 1, 2$). Let X_0 be endowed with the topology generated by the topologies induced on X_0 by X_1 and X_2 . If T_0 is a bounded linear form on X_0 , then there exist two linear forms T_1 and T_2 bounded respectively on X_1 and X_2 such that*

$$T_0(x) = T_1(x) + T_2(x), \forall x \in X_0.$$

Proof. Let $\widetilde{X}_0 := \{(x, x) \mid x \in X_0\}$ be endowed with the topology induced by $X_1 \times X_2$. Then the linear form defined by $\widetilde{T}_0(x, x) = T_0(x)$ is bounded on \widetilde{X}_0 . By the Hahn-Banach theorem, \widetilde{T}_0 extends on $X_1 \times X_2$ as a bounded linear form. Finally, the result is obtained setting $T_1(x_1) = \widetilde{T}_0(x_1, 0)$ for $x_1 \in X_1$ and $T_2(x_2) = \widetilde{T}_0(0, x_2)$ for $x_2 \in X_2$. \square

Let $F_1 := \mathbb{C} \setminus \Omega_1$, $F_2 := \mathbb{C} \setminus \Omega_2$ and $F_0 := F_1 \cup F_2 = \mathbb{C} \setminus (\Omega_1 \cap \Omega_2)$. For $i \in \{1, 2\}$, let $(O_n^i)_{n \in \mathbb{N}}$ be a decreasing sequence of open sets which are bounded by a finite number of rectifiable Jordan curves and such that $\bigcap_{n \in \mathbb{N}} O_n^i = F_i$. For every $n \in \mathbb{N}$, set $O_n^0 = O_n^1 \cup O_n^2$. We write $H^\infty(O_n^i)$ the classical Banach space of bounded holomorphic functions on O_n^i endowed with the norm

$$\|f\|_{H^\infty(O_n^i)} = \sup_{z \in O_n^i} |f(z)|.$$

Finally, let $X_{F_i} := \bigcup_{n \in \mathbb{N}} H^\infty(O_n^i)$ be endowed with the topology of inductive limit (of the topologies of $H^\infty(O_n^i)$). Some reminders on this topology can be found at Appendix A. With these notations in mind, it is an exercise to verify that the spaces $X_{F_0}, X_{F_1}, X_{F_2}$ satisfy the hypotheses of Lemma 1.3.3. We write it in the following lemma which is proved in the Appendix A.

Lemma 1.3.4. *The topology on X_0 is the topology generated by the topologies induced on X_{F_0} by X_{F_1} and X_{F_2} .*

Let us now prove the theorem. Pick $f \in \text{Hol}(\Omega_1 \cap \Omega_2)$. We define the operator T_0 on X_{F_0} by

$$\forall n \in \mathbb{N}, \forall g \in H^\infty(O_n^1 \cup O_n^2), T_0 g = \int_{\gamma_n} g(z) f(z) dz,$$

where γ_n denotes a finite union of rectifiable pairwise-disjoint Jordan curves which forms the boundary of a domain containing F_0 and such that $O_n^1 \cup O_n^2$ includes the closure of that domain (see Figure 2).

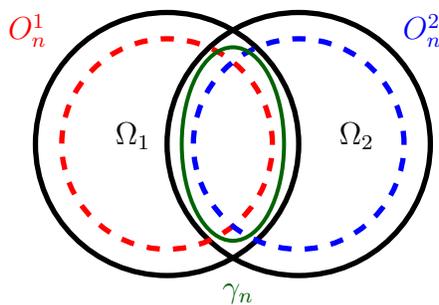


Figure 2: The Jordan curve γ_n .

Note that γ_n is contained in $\Omega_1 \cap \Omega_2$, so f is well-defined on γ_n for every $n \in \mathbb{N}$. It turns out that T_0 is a bounded linear form on X_{F_0} . Indeed, for all $n \in \mathbb{N}$ and for all $g \in H^\infty(O_n^1 \cup O_n^2)$ we have

$$|T_0 g| \leq \int_{\gamma_n} |g(z) f(z)| |dz| \leq \|g\|_{H^\infty(O_n^1 \cup O_n^2)} \sup_{z \in \gamma_n} |f(z)| |\gamma_n|$$

where $|\gamma|$ means the length of γ . Hence, combining Lemma 1.3.3 and Lemma 1.3.4 together, there exist two bounded linear forms T_1 and T_2 respectively on X_{F_1} and X_{F_2} such that $T_0 g = T_1 g + T_2 g$ for every $g \in X_{F_0}$. But by Cauchy formula, we have $f(w) = T_0 g_w$ where $g_w(z) = \frac{1}{z-w}$. Thus $f(w) = T_1 g_w + T_2 g_w := f_1(w) + f_2(w)$. By continuity of T_1 and T_2 , the functions f_1 and f_2 are holomorphic respectively on Ω_1 and Ω_2 . This completes the proof. \square

The second one, given by Müller and Wengenroth [MW98, Theorem 1], uses the open mapping theorem, Roth's fusion lemma and Runge theorem. It establishes a link between the separation of singularities problem and approximation theory.

In the n -dimensional setting, Aronszajn's theorem is no longer true. One can read [Aiz67] and [Yuz72] for more details.

A related question is to know whether there exists a bounded linear operator $T : \text{Hol}(\Omega_1 \cap \Omega_2) \rightarrow \text{Hol}(\Omega_1) \times \text{Hol}(\Omega_2)$, $f \mapsto (f_1, f_2)$, such that $f = f_1 + f_2$. Mityagin and Khenkin proved in [MK71, Proposition 5.1] that such an operator does not always exist.

The problem has attracted a lot of interest in particular in Banach spaces of analytic functions. A challenging situation is the separation of singularities problem in the space H^∞ of bounded analytic functions, which arises naturally in connection with interpolation problems [Pol83, PK90]. Havin, Nersessian [HN01], Havin [Hav05], and Havin, Nersessian, Ortega-Cerdá [HNOC07] solved it in several general configurations. Unlike the classical problem, they proved also that the problem has not a positive solution for arbitrary pairs of open sets, giving a lot of instructive counterexamples. The authors used an explicit Cauchy integral approach in the first two papers cited above and a reduction to the $\bar{\partial}$ -equation (as in the modern proof of the Aronszajn theorem) in the last paper. They constructed bounded linear separation operators explicitly in both cases.

Another interesting situation previously studied concerns Smirnov spaces. Aizenberg solved the problem for the Smirnov space E^p ($1 < p < +\infty$) in the case of the intersection of k bounded domains with regular boundaries (Ahlfors-regularity). Bounded domains with Ahlfors-regular boundaries are particular cases of Smirnov domains (see [Zin85, Corollaire 1]). These last domains are exactly the domains on which the space E^p (identified with its set of non-tangential limits on the boundary) coincides with the L^p closure of the polynomials (see [Dur70, Section 10.3]).

Theorem 1.3.5 (Aizenberg). *[see [Aiz14, Theorem 2]] Let $k \in \mathbb{N}^*$ and let $\Omega_1, \dots, \Omega_k$ be bounded domains with Ahlfors-regular boundaries such that $\Omega := \bigcap_{i=1}^k \Omega_i$ is non-empty. If $f \in E^p(\Omega)$ with $1 < p < \infty$, then there exist f_1, \dots, f_k such that $f_i \in E^p(\Omega_i)$ for any $i = 1, \dots, k$ and $f = \sum_{i=1}^k f_i$ on Ω .*

The proof relies heavily on a strong result by David [Dav84, Theorem 1], who studies the boundedness of the Hilbert transform on such regular curves. Since it is a very short proof, we reproduce it here.

Proof. Let $1 < p < \infty$ and pick $f \in E^p(\Omega)$. We recall that such a function satisfies the Cauchy formula (1.5). Now, we decompose the boundary $\partial\Omega$ into k parts $\Gamma_1, \dots, \Gamma_k$ such that $\Gamma_i \subset \partial\Omega_i$ for any $i = 1, \dots, k$. Thus we can write

$$f(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(u)}{u-z} du = \frac{1}{2\pi i} \sum_{i=1}^k \int_{\partial\Omega_i} \frac{F_i(u)}{u-z} du,$$

where the functions F_i are defined on $\partial\Omega_i$ by

$$F_i(z) = \begin{cases} f(z) & \text{if } z \in \Gamma_i \\ 0 & \text{if } z \in \partial\Omega_i \setminus \Gamma_i. \end{cases}$$

Finally, the curves $\partial\Omega_i$ are also Ahlfors-regular, and as said above, the result of G. David implies that, on such curves, the Cauchy transform is bounded from $L^p(\partial\Omega_i)$ onto $E^p(\Omega_i)$. Hence $F_i \in E^p(\Omega_i)$ as required. \square

It can easily be generalized to finitely multiply connected domains using the same argument (see [Dur70, p. 182] for the definition of Smirnov space on a finitely connected domain). L. Aizenberg also gave the same kind of result (with more regularity hypotheses) for the Hardy space in several complex variables (see [Aiz14, Theorem 9] for the definition of this space and the theorem).

Müller and Wengenroth [MW98, Theorem 3] proved that solving the problem on the space \mathcal{A} of holomorphic functions which are continuous up to the boundary is equivalent to prove a Roth's fusion type lemma. In the same vein, Kaufman [HN94, section 16.18] asked for a solution to the problem on the space $\mathcal{A}^{(n)}$ of functions which have their first n derivatives in \mathcal{A} . He mentioned that a positive answer for any $n \geq 1$ would provide information on the triviality of $\mathcal{A}^{(n)}(\Omega_1 \cap \Omega_2)$, where triviality means $\mathcal{A}^{(n)}(\Omega_1 \cap \Omega_2) = \text{Hol}(\mathbb{C})|_{\Omega_1 \cap \Omega_2}$.

One of the two aims of this thesis is to discuss the problem of separation of singularities for the Bergman space. We shall do this in Chapter 4. Then we will apply the separation of singularities to the other goal of this thesis, namely the description of the reachable space of the heat equation.

Chapter 2

Some background in Control theory

In this chapter, we give the basic definitions and properties we need from control theory, and we introduce the main object of this thesis, namely the reachable set of a Linear Time Invariant (LTI) system. This chapter is essentially based on the textbook [TW09].

2.1 Operator theory

Dual with respect to a pivot space. By the Riesz representation theorem, we can identify a Hilbert space with its dual, but sometimes it can be more useful not to do so. Especially in the theory of PDE's, the dual with respect to a pivot space is often a better choice of realization of the dual space. We explain shortly this concept in this paragraph.

Let V and H be two Hilbert spaces such that $V \subset H$ densely and continuously. Let V' and H' their topological dual spaces. Then H' embeds densely and continuously into V' . The Riesz representation theorem states that the operator $J : H \ni z \mapsto \langle \cdot, z \rangle_H \in H'$ is an antilinear isometric isomorphism. Now, since $H' \subset V'$ our goal is to enlarge the space H in order to obtain an isometric isomorphism onto V' . For this purpose, we denote by V^* the completion of H with respect to the norm $\| \cdot \|_*$ given by

$$\|z\|_* = \sup_{\substack{v \in V \\ \|v\|_V \leq 1}} |\langle z, v \rangle_H|, \quad \forall z \in H.$$

Since H is dense and continuously embedded into V^* by construction, the operator

$J : H \rightarrow V'$ extends as a bounded linear operator from V^* into V' . Moreover, this extension is an isometric isomorphism from V^* onto V' . From now on, we identify (through J) the spaces V^* and V' , i.e. we do not distinguish z and Jz for $z \in V^*$. Also, for $z \in V^*$ and $v \in V$, we write $\langle z, v \rangle_{V',V}$ instead of $\langle Jz, v \rangle_{V',V}$. Note that for $z \in H$, we have $\langle Jz, v \rangle_{V',V} = \langle z, v \rangle_H$ by construction. Finally, we have

$$V \subset H \subset V'$$

densely and continuously. Under this identification, we say that V' is *the dual of V with respect to the pivot space H* , and $V \subset H \subset V'$ is called a *Gelfand triple*.

Remark 2.1.1. Denote $\mathcal{D}(\Omega)$ the space of functions which are infinitely differentiable on Ω and compactly supported. Assume that the embeddings $\mathcal{D}(\Omega) \subset V \subset H$ are dense and continuous. Using H as pivot space, it is clear that we have $\mathcal{D}(\Omega) \subset V \subset H \subset V' \subset \mathcal{D}'(\Omega)$. Thus, the dual space V' can be viewed as a subset of the distribution space.

Example 2.1.2. Let $d \in \mathbb{N}^*$ and let Ω be an open space in \mathbb{R}^d . Let $W_0^{1,2}(\Omega)$ denote the classical Sobolev space which consists of all L^2 -functions with all their first order derivatives in $L^2(\Omega)$, and such that their Dirichlet trace is zero. If we take $L^2(\Omega)$ as pivot space, then the dual of $W_0^{1,2}(\Omega)$ is the Sobolev space of negative order $W^{-1,2}(\Omega)$ defined by

$$W^{-1,2}(\Omega) = \left\{ f_0 + \sum_{k=1}^d \partial_k f_k \in \mathcal{D}'(\Omega) \mid \forall 0 \leq k \leq d, f_k \in L^2(\Omega) \right\}.$$

Extension of operators. In order to define the notion of mild solution in the next section, we need to extend an unbounded operator and its semigroup to bigger spaces. We explain now how to do that. We insist on the fact that the following construction can be made fairly general (see [TW09, Section 2.10]) but we focus here on selfadjoint operators.

Let $\mathcal{L}(E, F)$ denote the space of bounded linear operators from a Banach space E to another Banach space F . We write $\mathcal{L}(E)$ instead of $\mathcal{L}(E, E)$.

Let H be a Hilbert space. Let $A : D(A) \subset H \rightarrow H$ be an unbounded self-adjoint linear operator which generates a strongly continuous semigroup of operators $(\mathbb{T}_t)_{t>0}$ (see [TW09, Definitions 2.1.1 and 2.1.4]). Let H_1 denote the Hilbert space $D(A)$ when it is endowed with one of the equivalent norms

$$\|x\|_{H_1} := \|(\beta I - A)x\|_H$$

where β belongs to the resolvent set $\rho(A)$ of A . Note that this norm is equivalent to the graph norm

$$\|x\|_{\text{graph}}^2 := \|x\|_H^2 + \|Ax\|_H^2.$$

With one of these norms, H_1 embeds densely and continuously into H (just write $x = (\beta I - A)^{-1}(\beta I - A)x$). Let H_{-1} denote the dual space of H_1 with respect to the pivot space H . It can be defined alternatively as the completion of H with respect to the norm $\|x\|_{H_{-1}} := \|(\beta I - A)^{-1}x\|_H$. Hence, as explained in the previous paragraph, we have

$$H_1 \subset H \subset H_{-1}$$

densely and with continuous embeddings.

Proposition 2.1.3. *[see [TW09, Propositions 2.10.3 and 2.10.4]] The operator A has a unique bounded extension $\tilde{A} : H \rightarrow H_{-1}$ and for every $t \geq 0$, \mathbb{T}_t has a unique bounded extension $\tilde{\mathbb{T}}_t : H_{-1} \rightarrow H_{-1}$. Moreover, $(\tilde{\mathbb{T}}_t)_{t \geq 0}$ is a strongly continuous semigroup on H_{-1} with generator \tilde{A} .*

Proof. Let $\beta \in \rho(A)$. By definition of the norm $\|\cdot\|_{H_1}$, it is clear that $\beta I - A$ is a unitary operator from H_1 onto H . Since H_1 embeds continuously into H , we have $\beta I - (\beta I - A) = A \in \mathcal{L}(H_1, H)$. Let us define $\tilde{A} \in \mathcal{L}(H, H_{-1})$ by $\langle \tilde{A}x, x_1 \rangle_{H_{-1}, H_1} := \langle x, Ax_1 \rangle_H$ for all $x \in H$ and $x_1 \in H_1$ (i.e. \tilde{A} is the Banach-adjoint $A^t : H' \ni \varphi \mapsto \varphi \circ A \in H_{-1}$ under the identification $H \simeq H'$). Hence, using that A is selfadjoint, we have for every $x \in H_1$,

$$\langle \tilde{A}x, x_1 \rangle_{H_{-1}, H_1} := \langle x, Ax_1 \rangle_H = \langle Ax, x_1 \rangle_H, \quad \forall x_1 \in H_1.$$

Therefore \tilde{A} is an extension of A , and the uniqueness follows from the density of H_1 in H .

Let us show that $(\beta I - \tilde{A})^{-1}$ exists (and so belongs to $\mathcal{L}(H_{-1}, H)$ by the Banach isomorphism theorem) and is the unique bounded extension of $(\beta I - A)^{-1}$. In the same way as above, the unitary operator $R := (\beta I - A)^{-1} \in \mathcal{L}(H, H_1)$ has a unique extension $\tilde{R} \in \mathcal{L}(H_{-1}, H)$ given by

$$\langle \tilde{R}x_{-1}, x \rangle_H := \langle x_{-1}, Rx \rangle_{H_{-1}, H_1}, \quad \forall x_{-1} \in H_{-1}, \forall x \in H.$$

Moreover, since $\|z\|_{H_{-1}} = \|Rz\|_H$ for every $z \in H_{-1}$ and using the density of H in H_{-1} , \tilde{R} is an isometry. Finally, the equalities

$$(\beta I - \tilde{A})\tilde{R}x = x = \tilde{R}(\beta I - \tilde{A})x$$

holds for every $x \in H_1$. Thus by density of H_1 in H_{-1} (for the left-hand-side equality) and in H (for the right-hand-side equality), we obtain $\tilde{R} = (\beta I - \tilde{A})^{-1}$. This means that $(\beta I - \tilde{A})^{-1}$ exists and is a unitary operator which extends $(\beta I - A)^{-1}$. Now, for every $t > 0$, we define $\tilde{\mathbb{T}}_t \in \mathcal{L}(H_{-1})$ by

$$\tilde{\mathbb{T}}_t z = (\beta I - \tilde{A}) \mathbb{T}_t (\beta I - \tilde{A})^{-1} z, \quad \forall z \in H_{-1}.$$

It is clear that $\tilde{\mathbb{T}}_t$ extends \mathbb{T}_t (\mathbb{T}_t commutes with A on H_1) and it is bounded on H_{-1} . Finally, direct computations show that $(\tilde{\mathbb{T}}_t)_{t \geq 0}$ is a strongly continuous semigroup whose generator $(\beta I - \tilde{A}) A (\beta I - \tilde{A})^{-1} \in \mathcal{L}(H, H_{-1})$ is a bounded extension of A , i.e. \tilde{A} generates $(\tilde{\mathbb{T}}_t)_{t \geq 0}$. \square

Strictly positive operators and their extensions. For the boundary control case, we will need to extend our operator (which will be the Laplacian or Schrödinger operator) to a bigger space. This will be done using the positivity. We refer to [TW09, Sections 3.3 and 3.4] for this paragraph. Here, H is a Hilbert space and A is a selfadjoint operator on H .

Definition 2.1.4. We say that A is *positive* and we write $A \geq 0$ if for all $z \in D(A)$ we have $\langle Az, z \rangle \geq 0$. We say that A is *strictly positive* and we write $A > 0$ if there exists $\lambda > 0$ such that $A - \lambda I$ is positive.

We also write $A \geq \lambda I$ to mean $A - \lambda I \geq 0$. As for the bounded case, we have $A \geq \lambda I$ for $\lambda \in \mathbb{R}$, if and only if the spectrum $\sigma(A)$ of A is included in $[\lambda, +\infty[$.

From now on, we assume that A is a strictly positive operator. So we can define its square root $A^{\frac{1}{2}}$ i.e. the unique positive operator $S : D(S) \rightarrow H$ satisfying $S^2 = A$. Moreover, $A^{\frac{1}{2}} > 0$.

Let H_1 and H_{-1} be defined as in the previous paragraph. In the same way, we denote by $H_{\frac{1}{2}}$ the domain $D(A^{\frac{1}{2}})$ with the norm $\|z\|_{\frac{1}{2}} := \|A^{\frac{1}{2}} z\|$. Since $A^{\frac{1}{2}}$ is selfadjoint, we have $\|z\|_{\frac{1}{2}} = \sqrt{\langle Az, z \rangle_H}$ for $z \in D(A)$ and $H_{\frac{1}{2}}$ can be defined equivalently as the completion of $D(A)$ with respect to that norm. We define also $H_{-\frac{1}{2}}$ the dual of $H_{\frac{1}{2}}$ with respect to the pivot space H , or equivalently the completion of H with respect to the norm $\|z\|_{-\frac{1}{2}} := \|A^{-\frac{1}{2}} z\| = \sqrt{\langle A^{-1} z, z \rangle_H}$. Thus, we have the following dense and continuous embeddings

$$H_1 \subset H_{\frac{1}{2}} \subset H \subset H_{-\frac{1}{2}} \subset H_{-1}.$$

Proposition 2.1.5. [see [TW09, Proposition 3.4.5 and Corollary 3.4.6]] *If A is a strictly positive selfadjoint operator which generates a strongly continuous semigroup*

$(\mathbb{T}_t)_{t \geq 0}$, then A has a unique strictly positive densely defined extension \tilde{A} on $H_{-\frac{1}{2}}$ with domain $H_{\frac{1}{2}}$, and for every $t \geq 0$, \mathbb{T}_t has a unique bounded extension $\tilde{\mathbb{T}}_t \in \mathcal{L}(H_{-\frac{1}{2}})$. Moreover, $(\tilde{\mathbb{T}}_t)_{t \geq 0}$ is a strongly continuous semigroup on $H_{-\frac{1}{2}}$ with generator \tilde{A} .

Proof. By Proposition 2.1.3, $A^{\frac{1}{2}}$ has a unique bounded extension $\tilde{A}^{\frac{1}{2}} \in \mathcal{L}(H, H_{-\frac{1}{2}})$. Hence, writing

$$\tilde{A} = \tilde{A}^{\frac{1}{2}} A (\tilde{A}^{\frac{1}{2}})^{-1}$$

we obtain the expected extension. Furthermore, noting that $A^{\frac{1}{2}} \in \mathcal{L}(H_1, H_{\frac{1}{2}})$ is a unitary operator, we obtain the domain of \tilde{A} .

In a similar way, writing

$$\tilde{\mathbb{T}}_t = \tilde{A}^{\frac{1}{2}} \mathbb{T}_t (\tilde{A}^{\frac{1}{2}})^{-1}, \quad \forall t \geq 0$$

the claim on the semigroup follows as in 2.1.3. \square

Remark 2.1.6. If we write $\tilde{H} = H_{-\frac{1}{2}}$, the previous proposition implies that \tilde{A} is a strictly positive operator on \tilde{H} , and we have

$$\tilde{H}_1 := D(\tilde{A}) = H_{\frac{1}{2}}, \quad \tilde{H}_{\frac{1}{2}} := D(\tilde{A}^{\frac{1}{2}}) = H, \quad \tilde{H}_{-\frac{1}{2}} = H_{-1}$$

with equal norms. Again, \tilde{A} is a strictly positive (densely defined) operator on \tilde{H} . So, by Proposition 2.1.5 it has a unique strictly positive extension to $\tilde{H}_{-\frac{1}{2}} = H_{-1}$ with domain $\tilde{H}_{\frac{1}{2}} = H$.

Finally, for a selfadjoint bounded perturbation, if the sum is strictly positive then the spaces involved do not change.

Proposition 2.1.7. [see [TW09, Proposition 3.4.5 and Corollary 3.4.6]] Let $A > 0$ be selfadjoint and $Q \in \mathcal{L}(H)$ be selfadjoint such that $B = A + Q > 0$. Then the space H_{-1} , $H_{-\frac{1}{2}}$, $H_{\frac{1}{2}}$, H_1 induced by A and B are the same.

Proof. It suffices to prove that the norms defined by $\|z\|'_1 := \|Bz\|$ and $\|z\|'_{\frac{1}{2}} := \|B^{\frac{1}{2}}z\|$ are respectively equivalent to the norm $\|\cdot\|_1$ and $\|\cdot\|_{\frac{1}{2}}$. For that, let $\lambda > 0$ be such that $A \geq \lambda I$. Then for all $z \in D(A)$,

$$\|z\|'_1 = \|(A + Q)z\| \leq \|Az\| + \|Q\|\|z\| \leq (1 + \frac{\|Q\|}{\lambda})\|Az\| = (1 + \frac{\|Q\|}{\lambda})\|z\|_1.$$

The converse inequality can be proved in a similar way. So the norms $\|\cdot\|_1$ and $\|\cdot\|'_1$, are equivalent.

For the remainder norms, note that

$$Q \leq \|Q\|I \leq \frac{\|Q\|}{\lambda} \lambda I \leq kA$$

where we have denoted $k = \frac{\|Q\|}{\lambda}$. Therefore we have for all $z \in D(A)$,

$$(\|z\|'_{\frac{1}{2}})^2 = \langle Bz, z \rangle \leq \langle Az, z \rangle + \langle Qz, z \rangle \leq (1+k) \langle Az, z \rangle = (1+k) \|z\|_{\frac{1}{2}}^2.$$

Again, the converse inequality can be proved in a similar way. So $\|\cdot\|'_{\frac{1}{2}}$ and $\|\cdot\|_{\frac{1}{2}}$ are equivalent on $D(A)$. \square

From now on, the extension will be denoted as the initial operator.

Reminders on the Laplacian. For the convenience of the reader, we recall quickly some well known facts about the Dirichlet Laplacian and its perturbations, both on bounded and unbounded domains.

Let first $\Omega \subset \mathbb{R}^n$ be a bounded domain with boundary of class C^2 . Then $A_0 = -\Delta := -\sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$ is a strictly positive selfadjoint operator on $H := L^2(\Omega)$ with domain $H_1 := D(A_0) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. Since Ω is bounded, the embedding $D(A_0) \subset L^2(\Omega)$ is compact and thus A_0 has a compact resolvent. Hence, A_0 is diagonalizable with an orthonormal basis of $L^2(\Omega)$ formed by eigenfunctions $(\varphi_n)_{n \in \mathbb{N}}$ and the corresponding sequence of eigenvalues $(\lambda_n)_{n \in \mathbb{N}}$ satisfies $\lambda_n > 0$ for every $n \in \mathbb{N}$ and $\lambda_n \rightarrow \infty$. This implies that $\Delta = -A_0$ generates a strongly continuous semigroup given by $\mathbb{T}_t f = \sum_{n \in \mathbb{N}} e^{-\lambda_n t} \langle f, \varphi_n \rangle \varphi_n$ for every $t \geq 0$.

Following Proposition 2.1.5, A_0 admits a unique strictly positive extension on $H_{-\frac{1}{2}}$ with domain $H_{\frac{1}{2}}$.

Proposition 2.1.8. *We have $H_{\frac{1}{2}} = W_0^{1,2}(\Omega)$ and $H_{-\frac{1}{2}} = W^{-1,2}(\Omega)$.*

Proof. We remind that the Sobolev spaces $W_0^{1,2}(\Omega)$ and $W^{-1,2}(\Omega)$ have been introduced in the first paragraph of this section. The space $H_{\frac{1}{2}}$ is the completion of $H_1 = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ with respect to the norm

$$\|f\|_{\frac{1}{2}}^2 = \langle A_0 f, f \rangle_{L^2(\Omega)} = \|\nabla f\|_{L^2(\Omega)}^2.$$

Since $W_0^{1,2}(\Omega)$ is the completion of $\mathcal{D}(\Omega)$ with respect to the previous norm and $\mathcal{D}(\Omega) \subset H_1$, we have $H_{\frac{1}{2}} = W_0^{1,2}(\Omega)$. Finally, the assertion on $H_{-\frac{1}{2}}$ is given in Example 2.1.2 2. \square

Depending on the context, it should be convenient to consider the operator A_0 on $L^2(\Omega)$ or on $W^{-1,2}(\Omega)$. On both spaces, the operator $-A_0 = \Delta$ is called the *Dirichlet Laplacian*. Moreover, we have seen that the semigroup $(\mathbb{T}_t)_{t \geq 0}$ extends boundedly on $H_{-\frac{1}{2}}$. It is called *Dirichlet Laplacian semigroup*, or *Dirichlet heat semigroup* on both spaces.

Remark 2.1.9. Note that on $W_0^{1,2}(\Omega)$, the Dirichlet Laplacian $-A_0$ coincides with the operator Δ defined in the distributional sense. This follows from the fact that $\mathcal{D}(\Omega)$ is dense in $W_0^{1,2}(\Omega)$. On the contrary, they are different on $\tilde{H}_{\frac{1}{2}} = H = L^2(\Omega)$ (see Remark 2.1.6 for an extension of $-A_0$ on this space). This is because, if $f \in \tilde{H}_{\frac{1}{2}} = L^2(\Omega)$, $-A_0 f$ is now in $\tilde{H}_{-\frac{1}{2}} = H_{-1}$ which is the dual of $H_1 = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$ with respect to the dual space $H = L^2(\Omega)$. But $\mathcal{D}(\Omega)$ is not dense in $W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$.

An other way to see that they do not coincide on L^2 is the following: if f is a non-zero constant, then $\Delta f = 0$ but $-A_0 f$ cannot be zero since $A_0 > 0$.

Now, we restrict our attention to the one dimensional unbounded case. If $\Omega = \mathbb{R}$ or $(0, \infty)$, then $-\Delta$ is still selfadjoint and generates a strongly continuous semigroup which can be obtained using the Fourier transform (resp. the Laplace transform) for $\Omega = \mathbb{R}$ (resp. for $\Omega = (0, \infty)$). But in these cases, $-\Delta$ is not strictly positive any more, as $\sigma(-\Delta) = [0, \infty)$. Nevertheless, note that the operator $-\Delta + I$ is strictly positive and now the same argument as above gives $D((-\Delta + I)^{\frac{1}{2}}) = W^{1,2}(\Omega)$. Since for such Ω we have $W_0^{1,2}(\Omega) = W^{1,2}(\Omega)$, $-\Delta + I$ extends as a strictly positive operator on $W^{-1,2}(\Omega)$ with domain $W^{1,2}(\Omega)$, and so $-\Delta$ also does.

Finally, let q be a positive and bounded real-valued function on Ω . Then, the operator $-\Delta + q$ is still strictly positive and generates a strongly continuous semigroup (see [TW09, Theorem 2.11.2]). The previous results hold with same spaces by Proposition 2.1.7.

2.2 Well-posed Linear Time Invariant Control Systems

The notion of Linear Time Invariant (LTI) control systems provides the abstract general framework unifying the concepts of control theory. As we shall see, it permits also to extend rigorously the basic concepts to the case of boundary control systems. Here, X will denote a complex Hilbert space and $(\mathbb{T}_t)_{t \geq 0}$ a strongly continuous semigroup on X with generator $A : D(A) \rightarrow X$. We write \dot{z} for the time derivative of a function z .

Solution of a Linear Time Invariant system. We introduce now the notion of solutions we will use here.

Definition 2.2.1. Let $f \in L^1_{\text{loc}}([0, \infty); X_{-1})$. Let $z_0 \in X$.

We say that $z \in L^1_{\text{loc}}([0, \infty); X) \cap C([0, \infty); X_{-1})$ is a (weak) solution in X_{-1} of the Cauchy problem

$$\begin{cases} \dot{z}(t) = Az(t) + f(t), \\ z(0) = z_0 \end{cases} \quad (2.1)$$

if it satisfies for every $\varphi \in D(A^*)$ and every $t \geq 0$,

$$\langle z(t) - z_0, \varphi \rangle_{X_{-1}, D(A^*)} = \int_0^t \langle z(s), A^* \varphi \rangle_X + \langle f(s), \varphi \rangle_{X_{-1}, D(A^*)} ds.$$

Proposition 2.2.2. [see [TW09, Prop 4.1.4]] Let $f \in L^1_{\text{loc}}([0, \infty); X_{-1})$. Let $z_0 \in X$. If z is a solution of (2.1), then z is given by

$$z(t) = \mathbb{T}_t z_0 + \int_0^t \mathbb{T}_{t-s} f(s) ds, \quad \forall t \geq 0. \quad (2.2)$$

Remark that the semigroup involved in the integral is in fact the bounded extension of $(\mathbb{T}_t)_{t \geq 0}$ to X_{-1} given in Proposition 2.1.3. The previous proposition implies that there exists at most one solution in X_{-1} of (2.1). The function z defined by (2.2) is called the *mild solution* of (2.1). Obviously, the mild solution is not necessarily a solution.

Admissible control operator. Let U (the input or control space) be a Hilbert space. Let $B \in \mathcal{L}(U, X_{-1})$ (the control operator). We are now interested in equations of the form $\dot{z}(t) = Az(t) + Bu(t)$ where $u \in L^2([0, \infty); U)$. We shall see later that a large class of partial differential equations with control on the boundary of a domain can be written in this way. For every $\tau \geq 0$, we call *input-to-state map* (or *controllability operator*) the bounded operator $\Phi_\tau : L^2([0, \infty); U) \rightarrow X_{-1}$ defined by

$$\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-s} B u(s) ds. \quad (2.3)$$

Hence, when it exists, the solution of the Cauchy problem

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), & t \geq 0 \\ z(0) = z_0 \end{cases} \quad (2.4)$$

for $z_0 \in X$ and $u \in L^2([0, \infty); U)$ is given by

$$z(\tau) = \mathbb{T}_\tau z_0 + \Phi_\tau u, \quad \forall \tau \geq 0. \quad (2.5)$$

As we shall see in the next section, the operator Φ_τ will play a central rôle in the rest of this thesis. We denote by $\text{Ran}\Phi_\tau$ its range.

Definition 2.2.3. The operator $B \in \mathcal{L}(U, X_{-1})$ is called an *admissible control operator* for the semigroup $(\mathbb{T}_t)_{t \geq 0}$ if there exists $\tau > 0$ such that $\text{Ran}\Phi_\tau \subset X$.

In this case, we say that (A, B) define a *well-posed control LTI system*.

If B is admissible, i.e. $\text{Ran}\Phi_\tau \subset X$ for some $\tau > 0$, then for every $t \geq 0$, we have $\Phi_t \in \mathcal{L}(L^2([0, \infty); U), X)$. Obviously, if $B \in \mathcal{L}(U, X)$ then B is admissible.

If I is a open interval of \mathbb{R} , we denote by $W^{k,p}(I)$ ($1 \leq p \leq \infty$ and $k \in \mathbb{N}$) the Sobolev space which consists of all the $L^p(I)$ -functions which have all their (distributional) derivatives of order at most k in $L^p(I)$.

Proposition 2.2.4. *[[TW09, Proposition 4.2.5]] If B is an admissible control operator for $(\mathbb{T}_t)_{t \geq 0}$, then for every $z_0 \in X$ and every $u \in L^2_{\text{loc}}([0, \infty); U)$, the Cauchy problem (2.4) has a unique solution z in X_{-1} . This solution is given by (2.5) and satisfies*

$$z \in C([0, \infty); X) \cap W^{1,2}_{\text{loc}}((0, \infty); X_{-1}).$$

This means that for every $z_0 \in X$ and every $u \in L^2_{\text{loc}}([0, \infty); U)$ there exists a unique $z \in C([0, \infty); X)$ such that for every $\psi \in D(A^*)$ and every $t \geq 0$,

$$\langle z(t) - z_0, \psi \rangle_X = \int_0^t \langle z(s), A^* \psi \rangle_X + \langle u(s), B^* \psi \rangle_U ds. \quad (2.6)$$

The last writing can be useful to determine the control operator B for PDE's with control at the boundary of a domain.

Finally, we mention that the admissibility of a control operator B is often tested using the duality notion of admissible observation operator. In fact, B is an admissible control operator for $(\mathbb{T}_t)_{t \geq 0}$ if and only if B^* is an admissible observation operator for $(\mathbb{T}_t^*)_{t \geq 0}$ (see [TW09, Definition 4.3.1 and Proposition 4.4.3]). This duality will also appear later to test the controllability of an admissible control operator B .

Example 2.2.5. Let Ω be an open bounded and connected set with boundary of class C^2 . We denote $X = L^2(\Omega)$ and we consider the Dirichlet Laplacian $A = \Delta : D(A) \rightarrow X$ where $D(A) = W^{2,2}(\Omega) \cap W_0^{1,2}(\Omega)$. We have already seen in the last section that this operator generates a strongly continuous semigroup $(\mathbb{T}_t)_{t \geq 0}$.

Let ω be an open subset of Ω and $U = L^2(\omega)$. We regard U as a closed subspace of $L^2(\Omega)$ by considering functions in U to vanish on $\Omega \setminus \omega$. Let $B : U \rightarrow X$ be the inclusion operator of U into X (i.e. $Bu = u$). Since we have obviously $B \in \mathcal{L}(U, X)$, the pair (A, B) defines a well-posed control LTI system.

In terms of PDEs, the system defined by (A, B) can be written

$$\begin{cases} \frac{\partial y}{\partial t}(t, x) - \Delta y(t, x) = \mathbb{1}_\omega(x)u(t, x) & t > 0, x \in \Omega, \\ y(t, x) = 0, & t > 0, x \in \partial\Omega, \\ y(0, x) = f(x) & x \in \Omega, \end{cases} \quad (2.7)$$

For $u \in L^2_{\text{loc}}([0, \infty); U)$ and $f \in X$, Proposition 2.2.4 implies that this equation admits a unique solution $y \in C([0, \infty); X)$.

Boundary control systems. A big part of the systems coming from PDE's, those with control at the boundary of a domain, appears in the following form:

$$\begin{cases} \dot{z}(t) = Lz(t) \\ Gz(t) = u(t) \\ z(0) = z_0 \end{cases} \quad (2.8)$$

where L is a differential operator and G a boundary trace operator. The aim of this paragraph is to give sufficient assumptions in order to translate such a system into the previous form (2.4).

Let Z (the *solution space*) be a Hilbert space such that $Z \subset X$ with continuous embedding.

Definition 2.2.6. A *boundary control system* on U , Z and X is a pair of operators (L, G) where

$$L \in \mathcal{L}(Z, X), \quad G \in \mathcal{L}(Z, U),$$

if there exists $\beta \in \mathbb{C}$ such that the following properties hold:

- (i) G is onto,
- (ii) $\text{Ker}(G)$ is dense in X ,
- (iii) $\beta I - L$ restricted to $\text{Ker}(G)$ is onto,
- (iv) $\text{Ker}(\beta I - L) \cap \text{Ker}(G) = \{0\}$.

Given a boundary control system (L, G) we can define the operator A by

$$X_1 = \text{Ker}(G), \quad A = L|_{X_1}.$$

Then X_1 is a Hilbert space (as a closed subspace of Z) and $A \in \mathcal{L}(X_1, X)$. Conditions (iii) and (iv) mean that $\beta I - A$ is an isomorphism of X_1 onto X , i.e. $(\beta I - L)^{-1} \in \mathcal{L}(X, X_1)$. Hence, the norm on X_1 is equivalent to the norm

$$\|x\|_{X_1} := \|(\beta I - A)x\|_X$$

which we have already seen in the previous section. Again, X_{-1} is the completion of X with respect to the norm $\|x\|_{X_{-1}} := \|(\beta I - A)^{-1}x\|_X$ and A admits an extension $\tilde{A} \in \mathcal{L}(X, X_{-1})$.

Proposition 2.2.7. *[[TW09, Proposition 10.1.2]] Let (L, G) be a boundary control system on U, Z and X . Let $\tilde{A} \in \mathcal{L}(X, X_{-1})$ be defined as above. There exists a unique operator $B \in \mathcal{L}(U, X_{-1})$ such that*

$$L = \tilde{A} + BG.$$

Proof. Since G is onto, it has a bounded right inverse $H \in \mathcal{L}(U, Z)$. Set

$$B = (L - \tilde{A})H. \tag{2.9}$$

By definition of H , we have $G(I - HG) = G - GHG = G - G = 0$. So the range of $I - HG$ is contained in $\text{Ker}(G) = X_1$. Therefore $(L - \tilde{A})(I - HG) = 0$ and then we obtain $BG = (L - \tilde{A})HG = L - \tilde{A}$ as required. The unicity is a direct consequence of the surjectivity of G . \square

This proposition means that the system (2.8) can be rewritten in the form (2.4). When A generates a strongly continuous semigroup and B is admissible, we say that (L, G) is a *well-posed* boundary control system. The control operator B is given by (2.9), but this characterization is not very useful in the applications. Often, it is more convenient to determine B^* rather than B . We get it with the formula

$$\langle Lz, \psi \rangle = \langle z, A^*\psi \rangle + \langle Gz, B^*\psi \rangle, \quad \forall z \in Z, \quad \forall \psi \in D(A^*) \tag{2.10}$$

which follows from the proposition. Indeed, the expression $\langle Lz, \psi \rangle - \langle z, A^*\psi \rangle$ can often be computed using integration by parts. Otherwise, when the solution of the boundary system is well-known and explicit, the operator B can be determined from this solution.

The heat equation with Dirichlet boundary control. We end this section with a translation of the previous concepts for the Dirichlet Laplacian. We consider here the Dirichlet Laplacian $-A_0$ extended to $X = W^{-1,2}(\Omega)$ where Ω is an open bounded and connected set with boundary of class C^2 . It is the good framework in order to write the heat equation with boundary control as a well-posed boundary control system.

First, let us focus on the case $\Omega = (0, \pi)$. Let $D : \mathbb{C}^2 \rightarrow W^{1,2}(0, \pi)$ be the *Dirichlet map* defined by

$$D(v_1, v_2)(x) = \frac{x}{\pi}v_2 + \frac{\pi - x}{\pi}v_1, \quad \forall x \in (0, \pi). \quad (2.11)$$

Consider the *Dirichlet trace* γ_0 defined on $C^\infty([0, \pi])$ by $\gamma_0 z = (z(0), z(\pi))$. Using the density of $C^\infty([0, \pi])$ in $W^{1,2}(0, \pi)$, γ_0 extends uniquely as a bounded operator from $W^{1,2}(0, \pi)$ into \mathbb{C}^2 , with kernel $W_0^{1,2}(0, \pi)$ (as it is the closure of $C_0^\infty(0, \pi)$ in $W^{1,2}(0, \pi)$). Note that for every $v \in \mathbb{C}^2$, we have

$$\gamma_0 Dv = v. \quad (2.12)$$

We consider the following boundary value problem.

$$\begin{cases} \frac{\partial y}{\partial t}(t, x) - \frac{\partial^2 y}{\partial x^2}(t, x) = 0 & t > 0, x \in (0, \pi), \\ y(t, 0) = u_0(t), \quad y(t, \pi) = u_\pi(t) & t > 0, \\ y(0, x) = f(x) & x \in (0, \pi), \end{cases} \quad (\text{HE})$$

To formulate this equation as a boundary control system, take $X = W^{-1,2}(0, \pi)$, $U = \mathbb{C}^2$ and $Z = W_0^{1,2}(0, \pi) + DU$ and define the operators $L \in \mathcal{L}(Z, X)$ and $G \in \mathcal{L}(Z, U)$ by

$$Lz = \frac{d^2 z}{dx^2} \text{ (in the distributional sense),} \quad Gz = \gamma_0 z. \quad (2.13)$$

It is clear that L and G are well-defined and bounded since $Z \subset W^{1,2}(\Omega)$.

We have the following result.

Theorem 2.2.8. [see [TW09, Proposition 10.7.1]] *The pair (L, G) defined by (2.13) is a well-posed boundary control system. Its generator A is the Dirichlet Laplacian $-A_0$ and its control operator is $B = A_0 D$.*

In the last equality of this theorem, A_0 means the unique extension of this operator given by Proposition 2.1.5 which belongs to $\mathcal{L}(X_{\frac{1}{2}}, X_{-\frac{1}{2}})$.

Proof. Let us prove that the pair (L, G) defines a boundary control system. Condition (i) in Definition 2.2.6 is satisfied in view of equation (2.12), and condition (ii) is the density of $W_0^{1,2}(0, \pi)$ in $W_0^{-1,2}(0, \pi)$. As explained in Remark 2.1.9, L coincides with the Dirichlet Laplacian $-A_0$ on $W_0^{1,2}(0, \pi) = \text{Ker}(G)$. Hence, $A := L|_{\text{Ker}(G)} = -A_0$, and conditions (iii) – (iv) are fulfilled for $\beta = 0$ since $A_0 > 0$.

By definition, the control operator B is determined (see Proposition 2.2.7) by the equality

$$L - \tilde{A} = BG$$

where $\tilde{A} \in \mathcal{L}(X, X_{-1})$ coincides on $X_{\frac{1}{2}}$ with the bounded extension of $-A_0$ mentioned before the proof. Therefore, for every $v \in U$, we have $LDv - \tilde{A}Dv = BGDv$. Since $LDv = 0$ and $GDv = v$, we finally obtain $Bv = -\tilde{A}Dv = A_0Dv$, which is the wanted result.

It remains to show that this boundary control LTI system is well-posed. We have already seen that the Dirichlet Laplacian generates a strongly continuous semigroup. We refer to [TW09] for the admissibility of B . \square

In the one-dimensional case, the operator B (or B^*) is easy to compute.

Lemma 2.2.9. *For $g \in X_{\frac{1}{2}} = L^2(0, \pi)$, we have*

$$B^*g = \begin{pmatrix} \frac{d(A_0^{-1}g)}{dx}(0) \\ \frac{d(A_0^{-1}g)}{dx}(\pi) \end{pmatrix}.$$

Proof. For $v \in U$ and $g \in X_{\frac{1}{2}}$, we have

$$\langle Bv, g \rangle_{X_{-\frac{1}{2}}, X_{\frac{1}{2}}} = \langle A_0Dv, g \rangle_{X_{-\frac{1}{2}}, X_{\frac{1}{2}}} = \left\langle A_0^{\frac{1}{2}}Dv, A_0^{\frac{1}{2}}g \right\rangle_X.$$

Since $A_0^{\frac{1}{2}}$ is a unitary operator from $D(A_0^{\frac{1}{2}}) := X_{\frac{1}{2}}$ to X and $X_{\frac{1}{2}} = L^2(0, \pi)$, we obtain

$$\langle Bv, g \rangle_{X_{-\frac{1}{2}}, X_{\frac{1}{2}}} = \langle Dv, g \rangle_{L^2(0, \pi)}. \quad (2.14)$$

Now, writing $g = A_0A_0^{-1}g = LA_0^{-1}g$ and using integration by parts, this leads to

$$\begin{aligned} \langle Bv, g \rangle_{X_{-\frac{1}{2}}, X_{\frac{1}{2}}} &= \langle Dv, LA_0^{-1}g \rangle_{L^2(0, \pi)} \\ &= (Dv)(0) \overline{\frac{d(A_0^{-1}g)}{dx}(0)} - (Dv)(\pi) \overline{\frac{d(A_0^{-1}g)}{dx}(\pi)} \end{aligned}$$

where we have used that $LDv = 0$ and $A_0^{-1}g \in W^{2,2}(0, \pi) \cap W_0^{1,2}(0, \pi)$. Hence, by definition of D , it follows

$$\langle Bv, g \rangle_{X_{-\frac{1}{2}}, X_{\frac{1}{2}}} = \left\langle v, \left(\begin{array}{c} \frac{d(A_0^{-1}g)}{dx}(0) \\ \frac{d(A_0^{-1}g)}{dx}(\pi) \end{array} \right) \right\rangle_{\mathbb{C}}.$$

□

Another way to see B is to replace g by the elements of the orthonormal basis $(\varphi_n) = (\sqrt{\frac{\pi}{2}} \sin(n \cdot))_{n \in \mathbb{N}^*}$ of $L^2(0, \pi)$ in (2.14). This leads to

$$\langle Bv, \varphi_n \rangle_{X_{-\frac{1}{2}}, X_{\frac{1}{2}}} = \int_0^\pi \left(\frac{x}{\pi} v_2 + \frac{\pi - x}{\pi} v_1 \right) \sqrt{\frac{\pi}{2}} \sin(nx) dx = \sqrt{\frac{\pi}{2}} \left(\frac{v_1}{n} + (-1)^n \frac{v_2}{n} \right).$$

Finally, since by the previous proposition B is admissible, the heat equation (HE) admits a unique solution $y \in C([0, \infty), X)$ i.e. y satisfies (2.6). We give now an equivalent formulation which is well-known by specialists of PDEs and which is easier to use.

Proposition 2.2.10. *[see [TW09, Proposition 10.7.3]] A function $y \in C([0, \infty), X)$ is a solution of the equation (HE) if and only if for every $t \geq 0$, and every $\varphi \in W^{2,2}(0, \pi) \cap W_0^{1,2}(0, \pi)$ such that $\frac{d^2\varphi}{dx^2} \in W_0^{1,2}(0, \pi)$, it satisfies*

$$\begin{aligned} \langle y(t), \varphi \rangle_{-1,1} &= \langle f, \varphi \rangle_{-1,1} + \int_0^t \left\langle y(s), \frac{d^2\varphi}{dx^2} \right\rangle_{-1,1} ds \\ &\quad + \int_0^t u_0(s) \overline{\frac{d\varphi}{dx}(0)} ds - \int_0^t u_\pi(s) \overline{\frac{d\varphi}{dx}(\pi)} ds \end{aligned} \quad (2.15)$$

where $\langle \cdot, \cdot \rangle_{-1,1}$ stands for the duality $W^{-1,2}(0, \pi) - W_0^{1,2}(0, \pi)$.

Proof. Let us write equation (2.6) in our setting. It becomes

$$\langle y(t) - f, \psi \rangle_X = - \int_0^t \langle y(s), A_0 \psi \rangle_X + \langle u(s), B^* \psi \rangle_{\mathbb{C}} ds$$

for every $\psi \in D(A_0) = W_0^{1,2}(0, \pi)$ and $t \geq 0$.

Using that $A_0^{\frac{1}{2}}$ is a unitary operator from $D(A_0^{\frac{1}{2}}) := X_{\frac{1}{2}}$ to X in the first two inner products above, and lemma 2.2.9, we obtain for every $\tilde{t} \geq 0$,

$$\langle y(t) - f, A_0^{-1} \psi \rangle_{-1,1} = - \int_0^t \langle y(s), \psi \rangle_{-1,1} + \left\langle u(s), \left(\begin{array}{c} \frac{d(A_0^{-1}g)}{dx}(0) \\ \frac{d(A_0^{-1}g)}{dx}(\pi) \end{array} \right) \right\rangle_{\mathbb{C}} ds, \quad \forall \psi \in D(A).$$

The results follows setting $\varphi = A_0^{-1} \psi$. □

It is worth mentioning that the previous results extend to the n -dimensional case with some additional technical efforts. We refer to [TW09, Sections 10.6 and 10.7] for more details.

The Hermite heat equation with Dirichlet boundary control. We give now quickly the results for a perturbation of the heat equation. More precisely, we set $q(x) = x^2$ and we consider the so called Hermite heat equation.

$$\begin{cases} \frac{\partial w}{\partial t}(t, x) - \frac{\partial^2 w}{\partial x^2}(t, x) + x^2 w(t, x) = 0 & t > 0, x \in (0, \pi), \\ w(t, 0) = u_0(t), \quad w(t, \pi) = u_\pi(t) & t > 0, \\ w(0, x) = f(x) & x \in (0, \pi), \end{cases} \quad (\text{HHE})$$

As in the previous paragraph, we set

$$U = \mathbb{C}^2, \quad Z = W_0^{1,2}(0, \pi) + DU, \quad X = W^{-1,2}(0, \pi)$$

where D is the Dirichlet map defined in (2.11). We define the operators $L \in \mathcal{L}(Z, X)$ and $G \in \mathcal{L}(Z, U)$ by

$$Lz = \frac{d^2 z}{dx^2} + x^2 z, \quad Gz = \gamma_0 z.$$

where $\frac{d^2 z}{dx^2}$ is the double derivative in the sense of distributions and γ_0 is the Dirichlet trace given in the previous paragraph. Then the result is the same as for the heat equation.

Theorem 2.2.11. *[see [TW09, Theorem 10.8.3]] The pair (L, G) defines a well-posed boundary control system on the spaces U, Z and X .*

This theorem implies that for every control $u \in L^2((0, \infty); \mathbb{C}^2)$ and every initial condition $f \in X$, the Hermite heat equation (HHE) admits a unique solution $w \in C([0, \infty); X)$. As for the heat equation, we can reformulate this fact saying that there exists a unique fonction $w \in C([0, \infty); X)$ which satisfies

$$\begin{aligned} \langle w(t), \psi \rangle_{-1,1} &= \int_0^t \left\langle w(s), \frac{d^2 \psi}{dx^2} - x^2 \psi \right\rangle_{-1,1} ds \\ &\quad + \int_0^t u_0(s) \overline{\frac{d\psi}{dx}(0)} ds - \int_0^t u_\pi(s) \overline{\frac{d\psi}{dx}(\pi)} ds \end{aligned}$$

for every $\psi \in W^{2,2}(0, \pi) \cap W_0^{1,2}(0, \pi)$ such that $\frac{d^2 \psi}{dx^2} \in W_0^{1,2}(0, \pi)$ and every $t \geq 0$.

2.3 Controllability and Reachability

In this section, we give the different notions of controllability and the definition of the reachable set of a well-posed control LTI system. Throughout the section, U , X and Y are complex Hilbert spaces. $(\mathbb{T}_t)_{t \geq 0}$ is a strongly continuous semigroup on X with generator $A : D(A) \rightarrow X$. $B \in \mathcal{L}(U, X_{-1})$ is an admissible control operator for $(\mathbb{T}_t)_{t \geq 0}$. Hence, (A, B) defines a well-posed control LTI system.

Controllability. We now give three classical notions of controllability. We consider the system defined by (A, B) which is

$$\begin{cases} \dot{z}(t) = Az(t) + Bu(t), & t \geq 0 \\ z(0) = z_0 \end{cases} \quad (2.16)$$

and we recall that for $z_0 \in X$ and $u \in L^2_{\text{loc}}((0, \infty); U)$ the unique solution is given by the mild solution

$$z(\tau) = \mathbb{T}_\tau z_0 + \Phi_\tau u, \quad \forall \tau \geq 0. \quad (2.17)$$

where $\Phi_\tau \in \mathcal{L}(L^2((0, \infty); U); X)$ is given by $\Phi_\tau u = \int_0^\tau \mathbb{T}_{\tau-s} B u(s) ds$ for $\tau \geq 0$.

Definition 2.3.1. Let $\tau > 0$.

- (i) The pair (A, B) is *exactly controllable in time τ* if for any $z_0, z_1 \in X$ there exists a control $u \in L^2((0, \tau); U)$ such that the solution of (2.16) satisfies $z(\tau) = z_1$.
- (ii) The pair (A, B) is *approximately controllable in time τ* if for any $z_0, z_1 \in X$ and any $\varepsilon > 0$, there exists a control $u \in L^2((0, \tau); U)$ such that the solution of (2.16) satisfies $\|z(\tau) - z_1\|_X \leq \varepsilon$.
- (iii) The pair (A, B) is *null-controllable in time τ* if for any $z_0 \in X$, there exists a control $u \in L^2((0, \tau); U)$ such that the solution of (2.16) satisfies $z(\tau) = 0$.

We say that (A, B) is exactly controllable (resp. approximately controllable, resp. null-controllable) if it is exactly controllable (resp. approximately controllable, resp. null-controllable) for some $\tau > 0$.

It is clear that (i) implies (ii) and (iii). These three notions of controllability can be reformulated in terms of the range of Φ_τ . The following proposition is an immediate consequence of equality (2.17).

Proposition 2.3.2. Let $\tau > 0$.

- (a) The pair (A, B) is exactly controllable in time τ if and only if $\text{Ran } \Phi_\tau = X$.
- (b) The pair (A, B) is approximately controllable in time τ if and only if $\text{Ran } \Phi_\tau$ is dense in X .
- (c) The pair (A, B) is null-controllable in time τ if and only if $\text{Ran } \mathbb{T}_\tau \subset \text{Ran } \Phi_\tau$.

These controllability concepts are often proved by duality using the so called Hilbert Uniqueness Method (HUM): they are equivalent to observability notions for the adjoint pair (A^*, B^*) which often consists in proving a good inequality. Since we will not prove controllability properties, we let this part aside.

When the pair (A, B) is not exactly controllable, it is of interest to know which states can be attained by the solution. This leads to the following definition.

Definition 2.3.3. For $f \in X$ and $\tau > 0$, we will say that $g \in X$ is reachable from f in time τ if there exists a boundary control $u \in L^2((0, \tau), \mathbb{C}^2)$ such that the solution of (2.16) satisfies $z(\tau) = g$. We call *reachable set* and we denote \mathcal{R}_τ^f the set of all reachable functions from f in time τ .

When $f \equiv 0$, the set \mathcal{R}_τ^0 is called null-reachable set and $g \in \mathcal{R}_\tau^0$ is said null-reachable in time τ .

This set is the central object of this thesis. We remind its basic properties in the next section.

Two examples. We give now two examples of controllability results for PDEs. We will see in the next section that they are related to the reachable set. It is standard to say that a PDE is controllable when the pair (A, B) associated with is.

Let Ω be an open bounded and connected set with boundary of class C^2 and ω be an open subset of Ω . We start with the heat equation with internal control that is

$$\begin{cases} \frac{\partial y}{\partial t}(t, x) - \Delta y(t, x) = \mathbf{1}_\omega(x)u(t, x) & t > 0, x \in \Omega, \\ y(t, x) = 0, & t > 0, x \in \partial\Omega, \\ y(0, x) = f(x) & x \in \Omega, \end{cases} \quad (2.18)$$

We have seen in Example 2.2.5 that it defines a well-posed control LTI system for

$$X = L^2(\Omega), \quad U = \mathbb{C}^2, \quad A = -\Delta$$

and B equals to the injection of $L^2(\omega)$ into $L^2(\Omega)$. This obviously still holds if we replace the Dirichlet Laplacian $A = -\Delta$ by the harmonic oscillator $A = -\Delta + |x|^2$.

Because of the smoothing effect of the heat equation, it is clear that the solution of (2.18) cannot reach arbitrary functions g in $L^2(\Omega)$, whatever the initial condition $f \in L^2(\Omega)$. This implies in particular that this equation is not exactly controllable. However we have the following result.

Theorem 2.3.4. *[see [TW09, Proposition 11.5.1] The heat equation with internal control (2.18) is null-controllable in any time $\tau > 0$.*

This theorem has been first proved in dimension one by [Ego63] and [FR71] using the moment method. For the n -dimensional case, it has been proved by [LR95] and [FI96] using the Hilbert Uniqueness Method and Carleman estimates. Finally, it follows from an argument of analyticity of the solutions that this equation is approximatively controllable in any time $\tau > 0$. The previous results hold for the Hermite heat equation.

Now, we consider the one-dimensional heat equation and Hermite heat equation with Dirichlet boundary control which are given respectively by

$$\begin{cases} \frac{\partial y}{\partial t}(t, x) - \frac{\partial^2 y}{\partial x^2}(t, x) = 0 & t > 0, x \in (0, \pi), \\ y(t, 0) = u_0(t), \quad y(t, \pi) = u_\pi(t) & t > 0, \\ y(0, x) = f(x) & x \in (0, \pi), \end{cases} \quad (\text{HE})$$

and

$$\begin{cases} \frac{\partial w}{\partial t}(t, x) - \frac{\partial^2 w}{\partial x^2}(t, x) + x^2 w(t, x) = 0 & t > 0, x \in (0, \pi), \\ w(t, 0) = u_0(t), \quad w(t, \pi) = u_\pi(t) & t > 0, \\ w(0, x) = f(x) & x \in (0, \pi), \end{cases} \quad (\text{HHE})$$

We have seen in the last section that they define well-posed boundary control systems. For the same reason as for the equations with internal control, they are not exactly controllable. Nevertheless, we can obtain, as a consequence of the null-controllability of the corresponding equations with internal control, that they are null-controllable (and approximatively controllable) in any time $\tau > 0$ (see [TW09, Proposition 11.5.4]). (See also [Jon77] and [Lit78] for other important contributions on the null-controllability with Dirichlet boundary control). As we will see in the next section, this null-controllability in any time will have a heavy impact on the structure of the reachable space. All results still hold in the n -dimensional setting.

Finally, if we consider the domain $(0, \infty)$ instead of $(0, \pi)$ in the last two equations with a control at the boundary $x = 0$, the null-controllability does not hold. In fact,

except the null function, there is no initial condition which can be steered to zero. However, the approximate controllability still holds. We refer to [DE19] and the references therein for more details.

2.4 Properties of the reachable set

We keep the same notation as in the previous section. In this last section, we have given the definition of the reachable set \mathcal{R}_τ^f of a well-posed control LTI system. Here we will focus on the general properties of the reachable set. This approach is developed in [KNT19, Section 3]. It follows that the null-controllability in any time of the pair (A, B) implies strong invariance properties.

Before giving these properties, let us make an obvious remark: it follows from the definition of the reachable set and from (2.17) the equality

$$\forall f \in X, \forall \tau > 0, \mathcal{R}_\tau^f = \mathbb{T}_\tau f + \text{Ran } \Phi_\tau. \quad (2.19)$$

The first invariance property is straightforward.

Proposition 2.4.1. *If (A, B) is null-controllable in time $\tau > 0$, the reachable set \mathcal{R}_τ^f does not depend on the initial condition $f \in X$. Hence, it is equal to the linear subspace $\text{Ran } \Phi_\tau$ of X .*

Proof. Let $f \in X$. Pick $g \in \mathcal{R}_\tau^f$. This means that there exists $u \in L^2((0, \tau); U)$ such that $g = \mathbb{T}_\tau f + \Phi_\tau u$. Since (A, B) is null-controllable in time $\tau > 0$, we have $\text{Ran } \mathbb{T}_\tau \subset \text{Ran } \Phi_\tau$. Hence there exists $v \in L^2((0, \tau); U)$ such that $g = \Phi_\tau v + \Phi_\tau u = \Phi_\tau(u+v)$, i.e. $g \in \mathcal{R}_\tau^0$. Conversely, if $g = \Phi_\tau w \in \mathcal{R}_\tau^0$, then $g = \Phi_\tau v + \Phi_\tau(w-v) = \mathbb{T}_\tau f + \Phi_\tau(w-v)$ i.e. $g \in \mathcal{R}_\tau^f$. The last assertion is a direct consequence of (2.19). \square

Hence, when the null-controllability holds, this set is $\text{Ran } \Phi_\tau$ (we will use this notation from now on) and we call it *reachable space*.

The second invariant property concerns the time. It has been first proved by Fattorini for the heat equation (HE) in [Fat78], and then by Seidman in [Sei79] in all generality. Before stating the result we need to introduce a notation. For $u \in L^2((0, \tau); U)$ and $v \in L^2((0, t); U)$, we define their concatenation $u \diamond_\tau v$ by

$$u \diamond_\tau v(s) = \begin{cases} u(s) & \text{if } 0 \leq s < \tau \\ v(s - \tau) & \text{if } \tau \leq s \leq t + \tau. \end{cases}$$

With this notation, we obtain easily

$$\Phi_{\tau+t}(u \diamond_\tau v) = \mathbb{T}_t \Phi_\tau u + \Phi_t v.$$

Proposition 2.4.2. *If the well-posed control LTI system (A, B) is null-controllable in any time $\tau > 0$, then the reachable space $\text{Ran } \Phi_\tau$ does not depend on time $\tau > 0$.*

Proof. We follow Seidman's proof [Sei79].

Let $0 < \tau < t$. If $u \in L^2((0, \tau); U)$ then $0 \underset{t-\tau}{\diamond} u \in L^2((0, t); U)$ and we have

$$\Phi_t(0 \underset{t-\tau}{\diamond} u) = \mathbb{T}_\tau \Phi_{t-\tau} 0 + \Phi_\tau u = \Phi_\tau u.$$

So $\text{Ran } \Phi_\tau \subset \text{Ran } \Phi_t$.

Let us prove the converse inclusion. Take $g \in \text{Ran } \Phi_t$ and $v \in L^2((0, t); U)$ such that $g = \Phi_t v$. Remark that if

$$\tilde{u}(s) = u(s + t - \tau), \quad \forall s \in (0, \tau)$$

then we have $u = u \underset{t-\tau}{\diamond} \tilde{u}$. Therefore with the concatenation formula we obtain

$$g = \Phi_t u = \Phi_{(t-\tau)+\tau}(u \underset{t-\tau}{\diamond} \tilde{u}) = \mathbb{T}_\tau \Phi_{t-\tau} u + \Phi_\tau \tilde{u}.$$

Finally we have $\text{Ran } \mathbb{T}_\tau \subset \Phi_\tau$ by hypothesis, so the last equality implies that g belongs to $\text{Ran } \Phi_\tau$. The proof is complete. \square

The last general result we will present has been found very recently in [KNT19, Proposition 3.2]. It means that we can ask the control to be very flat in zero without changing the reachable space.

Proposition 2.4.3. *Let (A, B) be a well-posed control LTI system which is null-controllable in any positive time. Then for every $\alpha > 0$ and $\tau > 0$, we have*

$$\text{Ran } \Phi_\tau = \Phi_\tau (L^2((0, \tau); U; t^{-\alpha} dt)).$$

Proof. It suffices to prove that $\text{Ran } \Phi_\tau \subset \Phi_\tau (L^2((0, \tau); U; t^{-\alpha} dt))$. Let $g \in \Phi_\tau$. By Proposition 2.4.2, there exists $u \in L^2((0, \tau/2); U)$ such that $\Phi_{\tau/2} u = g$. Then setting $\tilde{u} = 0 \underset{\tau/2}{\diamond} u$, it follows

$$\Phi_\tau \tilde{u} = \Phi_\tau(0 \underset{\tau/2}{\diamond} u) = \mathbb{T}_{\tau/2} \Phi_{\tau/2} 0 + \Phi_{\tau/2} u = g.$$

Obviously, $\tilde{u} \in (L^2((0, \tau); U; t^{-\alpha} dt))$, so $g \in \Phi_\tau (L^2((0, \tau); U; t^{-\alpha} dt))$. This ends the proof. \square

2.5 Overview on the reachable space of the heat equation on a segment

We focus now on the main object of this thesis, namely the reachable space of the heat equation on $(0, \pi)$ with Dirichlet boundary control. The description of this space is a central question in control theory which goes back to the work of Fattorini and Russell, 50 years ago and has gained quite some renewed interest in recent years. It turns out that this reachable space can be identified with a space of holomorphic functions and this thesis is devoted to obtaining a complete characterization of it.

In the first paragraph, we recall the problem and resume the properties viewed in the previous sections which are related to this specific equation. In the second paragraph, we present a survey of the many results achieved over the years, from the pioneer work of Fattorini and Russell to this thesis.

Problem setting. In this paragraph we present the central problem of this thesis and we regroup the results of control theory obtained in the previous sections which are related to it.

Let $X = W^{-1,2}(0, \pi)$ and $U = \mathbb{C}^2$. We consider the following boundary control problem of the heat equation.

$$\begin{cases} \frac{\partial w}{\partial t}(t, x) - \frac{\partial^2 w}{\partial x^2}(t, x) = 0 & t > 0, x \in (0, \pi), \\ w(t, 0) = u_0(t), \quad w(t, \pi) = u_\pi(t) & t > 0, \\ w(0, x) = f(x) & x \in (0, \pi), \end{cases} \quad (\text{HE})$$

We have seen in Section 2.2 that this equation determine a well-posed control LTI system. This implies that for any control (or input) $u := (u_0, u_\pi) \in L^2_{\text{loc}}((0, +\infty), U)$ and any initial condition $f \in X$, this equation admits a unique solution $w \in C([0, +\infty), X)$ defined by

$$\forall t > 0, w(t, \cdot) = \mathbb{T}_t f + \Phi_t u \quad (2.20)$$

where $(\mathbb{T}_t)_{t \geq 0}$ is the Dirichlet Laplacian semigroup and $\Phi_t \in \mathcal{L}(L^2([0, \tau], \mathbb{C}^2), X)$ is the input-to-state map (or controllability operator) defined in (2.3). As explained in Section 2.4, we can thus define the reachable set of the equation (HE).

We have seen in Section 2.3 that this equation is null- controllable in any time $\tau > 0$. By Propositions 2.4.1 and 2.4.2, this implies that the reachable set does not depend neither on the initial condition $f \in X$ nor time $\tau > 0$ and it is equal to the linear space $\text{Ran } \Phi_\tau$. In the rest of this thesis we will denote the reachable space by $\text{Ran } \Phi_\tau$.

Because of the smoothing effect of the heat kernel, it is clear that for arbitrary control $u \in L^2((0, \tau), \mathbb{C}^2)$, we cannot reach any non-regular functions $g \in X$. So, $\text{Ran } \Phi_\tau \subsetneq X$.

With all these properties in mind, it is thus natural to seek for more precise information on $\text{Ran } \Phi_\tau$. This question has been the source of an effervescent research. An historical account of related results is given below.

2.5.1 Known results

Trying to be exhaustive, we give now an overview of the results obtained over the years and which led to the starting point of this work thesis.

It seems to be known for a long time that the solutions of the heat equation (HE) extends analytically (in the space variable) to the square (see Figure 1)

$$D = \left\{ z = x + iy \in \mathbb{C} \mid \left| x - \frac{\pi}{2} \right| + |y| < \frac{\pi}{2} \right\}. \quad (2.21)$$

Although this result is considered as a folk result by specialists of PDEs, it seems that it appears for the first time in the literature in the work [MRR16, Theorem 2.1] of Martin, Rosier and Rouchon. This result is independent of the type of control considered (e.g. u can be chosen L^1 instead of L^2) and generalizes a result of Gevrey [Gev13] in which the control is assumed to be continuous. In our setting, identifying as usual the function and the holomorphic extension, it reads as follows.

Proposition 2.5.1. *We have $\text{Ran } \Phi_\tau \subset \text{Hol}(D)$.*

We give a proof¹ for the convenience of the reader.

Proof. Let K denote the heat kernel on the whole line, given by

$$K(t, x, y) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}}, \quad \forall t > 0, \forall x, y \in \mathbb{R}.$$

Pick $g \in \text{Ran } \Phi_\tau$. Then, there exists $u \in L^2((0, \tau); \mathbb{C}^2)$ such that the solution w of (HE) satisfies $w(\tau, \cdot) = g$ on $(0, \pi)$. Let $\varepsilon > 0$ and let $\eta \in C_0^\infty(0, \pi)$ be a cut-off such that $\eta = 1$ on $I_\varepsilon := \{x \in (0, \pi) \mid \min(x, \pi - x) > \varepsilon\}$. Then $v = \eta w$ can be extended on \mathbb{R} and satisfies

$$\begin{cases} \partial_t v - \partial_x^2 v = f, & t > 0, x \in \mathbb{R} \\ v(0, x) = 0, & x \in \mathbb{R} \\ v(\tau, x) = g(x), & x \in I_\varepsilon \end{cases}$$

¹I would like to thank Sylvain Ervedoza for bringing this proof to my attention.

with $f := -2\frac{d\eta}{dx}\partial_x w - (\Delta\eta)w$ supported in $(0, \pi)\setminus I_\varepsilon$. Hence, we can write

$$\begin{aligned} v(\tau, z) &= \int_0^\tau \int_{\mathbb{R}} K(\tau - t, z, y) f(t, y) dy dt \\ &= \int_0^\tau \int_{(0, \pi)\setminus I_\varepsilon} K(\tau - t, z, y) f(t, y) dy dt \\ &= \int_0^\tau \int_{(0, \pi)\setminus I_\varepsilon} \frac{1}{\sqrt{4\pi(\tau - t)}} e^{-\frac{(z-y)^2}{4(\tau-t)}} f(t, y) dy dt. \end{aligned}$$

This integral is holomorphic on $\{z \in \mathbb{C} \mid \forall y \in (0, \pi)\setminus I_\varepsilon, \operatorname{Re}((z - y)^2) > 0\}$. But for every $y \in (0, \pi)\setminus I_\varepsilon$, we have

$$\operatorname{Re}((z - y)^2) = (\operatorname{Re}(z) - y)^2 - \operatorname{Im}(z)^2 \geq (\min(|\operatorname{Re}(z)|, |\pi - \operatorname{Re}(z)|) - \varepsilon)^2 - \operatorname{Im}(z)^2.$$

Hence, $z \mapsto v(\tau, z)$ extends holomorphically to the domain

$$\{z \in \mathbb{C} \mid \operatorname{Re}(z) \in (0, \pi) \text{ and } (\min(|\operatorname{Re}(z)|, |\pi - \operatorname{Re}(z)|) - \varepsilon)^2 - \operatorname{Im}(z)^2 > 0\}.$$

Since $g = v(\tau, \cdot)$ on I_ε and $\varepsilon > 0$ is arbitrary, we obtain that g is holomorphic on

$$\{z \in \mathbb{C} \mid \operatorname{Re}(z) \in (0, \pi) \text{ and } \min(|\operatorname{Re}(z)|, |\pi - \operatorname{Re}(z)|)^2 - \operatorname{Im}(z)^2 > 0\} = D.$$

□

This proof can be easily generalized to the n -dimensional setting.

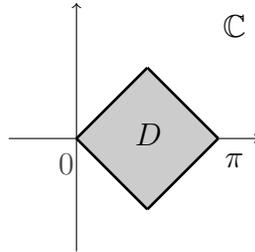


Figure 1: The square D .

Remark 2.5.2. The previous proof uses the analyticity of the heat kernel for the whole line. We refer to [Joh82, p.219] for another proof using this kernel. Make it appear seems to be determinant in order to obtain analytic properties or more

precisely to show a membership in a holomorphic function space. Although he uses the heat kernel of the segment instead of that of the whole line in [Can84, Section 10.5], Cannon writes the kernel in terms of the heat kernel on the whole line. This is the systematic strategy that we shall choose in this thesis to determine the reachable space.

It seems that the first actual work which considers explicitly the reachable space (or the reachable states) of the heat equation (HE) is in [FR71]. Using a *moment method*, Fattorini and Russell showed that if there exists $A > 0$ and $B > 0$ such that

$$\forall n \in \mathbb{N}^*, |a_n| \leq A \exp(-(\pi + B)n) \quad (2.22)$$

then the function defined by $g(x) = \sum_{n=1}^{+\infty} a_n \sin(nx)$ is reachable. We recall from Subsection 1.2.3 that, for $\delta > 0$, we denote H_δ^0 the space of continuous functions which are π -periodic on \mathbb{R} , which extend holomorphically on the strip $|\operatorname{Im}z| < \delta$ and which vanish with all the derivatives of even orders in 0 and in π . Using Proposition 1.2.17, we obtain from (2.22) that for δ large enough, H_δ is included in $\operatorname{Ran} \Phi_\tau$.

In a series of papers in the eighties [Sch80], [SS81], [Wec84a], [Wec84b], [Sch86], the authors tried successively to enlarge the class of reachable states (see [Sch86] for an overview). In particular they proved that it is not necessary to vanish at the boundary of the interval to be reachable, and they showed that the polynomials are reachable. In the last paper [Sch86], the author considers that "a characterization of reachable states remains elusive".

The investigation of $\operatorname{Ran} \Phi_\tau$ has gained quite some renewed interest with an acceleration in the last five years (2016-2020). Indeed, Martin, Rosier and Rouchon improved substantially the previous results in [MRR16, Theorem 1.1] using the *flatness approach* and showing that the holomorphic functions on the disk

$$B = \left\{ z \in \mathbb{C} \mid \left| z - \frac{\pi}{2} \right| < \frac{\pi}{2} e^{(2e)^{-1}} \right\}$$

are reachable. Dardé and Ervedoza improved again this latter result in [DE18, Theorem 1.1] showing that all the functions which are holomorphic on a neighborhood of D are reachable. Their method is based on a Carleman type estimate and on Cauchy's formula. This result combined with Proposition 2.5.1 means that $\operatorname{Ran} \Phi_\tau$ is a space of holomorphic functions on D .

Finally, before the works exposed in this thesis, the best known result on this problem to our knowledge was given in [HKT20], where the authors proved that the reachable space is sandwiched between two Hilbert spaces of holomorphic functions on the square D . More explicitly, it satisfies the inclusions

$$E^2(D) \subsetneq \operatorname{Ran} \Phi_\tau \subset A^2(D)$$

(see Section 1.1 for a definition of these spaces). The authors conjectured also (see [HKT20, Remark 1.4]) that $\text{Ran } \Phi_\tau$ should not be "too far from $A^2(D)$ ". Based on this observation, in what follows, we will call the following equality the HKT-conjecture:

$$\text{Ran } \Phi_\tau = A^2(D).$$

Key tools used in that paper include a unitary Laplace type integral operator studied by Aikawa, Hayashi and Saitoh [AHS90] (see Theorem 2.6.2), as well as a Riesz basis of exponentials in $E^2(D)$ discussed by Levin and Lyubarskii [LL75]. Writing the solution as a series involving the heat kernel of the half-line, they identified two main terms and treat the remaining terms as perturbation. The main terms correspond to the solution of the boundary control solutions on half-lines.

In Chapter 3, we will follow the strategy of [HKT20], examining precisely the integral transforms involved. We shall remind some of their arguments there. In that Chapter we shall prove that the reachable space is exactly the sum of two Bergman spaces. Finally, we shall see in Chapter 4 that this sum is equal to the Bergman space on D .

The last substantial results we would mention are in [KNT19]. The authors gave subsequently another proof of the result we obtain in Chapter 3. Using Proposition 2.4.3, they also proved that the reachable space equals the sum of two weighted Bergman spaces. We will come back on these results in Chapters 3 and 4.

2.6 Reachable space for the half-line

In this section, we recall the situation for the reachable space of the heat equation on the half-line $(0, \infty)$. This has been treated essentially in [AHS90] and [Sai91] (see also the textbook [Sai97] for a survey).

Let us remind the setting. We consider the following heat equation.

$$\begin{cases} \frac{\partial w}{\partial t}(t, x) - \frac{\partial^2 w}{\partial x^2}(t, x) = 0 & t > 0, x > 0, \\ w(t, 0) = u_0(t), & t > 0, \\ w(0, x) = f(x) & x > 0, \end{cases} \quad (2.23)$$

This equation defines a well-posed boundary control system for $X = W^{-1,2}(0, \infty)$ and $U = \mathbb{C}$. But as explained briefly at the end of Section 2.3, it is not null-controllable. Worse, it turns out that the lack of null-controllability is maximal: there is no non trivial initial condition $f \in X$ which can be steered to zero in whatever time. In other words, if $(\widehat{\mathbb{T}}_t)_{t \geq 0}$ denotes the heat Dirichlet semigroup on

the half-line and $\tilde{\Phi}_{\tau,0}$ the controllability operator of (2.23), then for any $0 \neq f \in X$ and any $\tau > 0$, we have $\widehat{\mathbb{T}}_{\tau} f \notin \text{Ran } \tilde{\Phi}_{\tau,0}$. This means that the reachable space of this equation, given by $\widehat{\mathcal{R}}_{\tau}^f = \widehat{\mathbb{T}}_{\tau} f + \text{Ran } \tilde{\Phi}_{\tau,0}$ is an affine space which is linear only for $f = 0$. Also, since the null-controllability is not fulfilled, the results of Section 2.4 are no longer true. The reachable space depends on the initial condition and on time.

From now on we consider only the null-reachable space $\text{Ran } \tilde{\Phi}_{\tau,0}$. It can be characterized by its reproducing kernel and turns out to be a strange sum of a Bergman space and a Hardy (Smirnov) space on the sector $\Delta = \{z \in \mathbb{C}, |\arg(z)| < \frac{\pi}{4}\}$.

Proposition 2.6.1. *[see [Sai91, Section 2] The (null-)reachable space $\text{Ran } \tilde{\Phi}_{\tau,0}$ is a RKHS of holomorphic functions on Δ with reproducing kernel*

$$k_{\lambda}(z) = z\bar{\lambda}e^{-\frac{z^2}{4t}}e^{-\frac{\bar{\lambda}^2}{4t}} \left[\frac{4}{\pi(z^2 + \bar{\lambda}^2)^2} + \frac{1}{\tau\pi(z^2 + \bar{\lambda}^2)} \right], \quad z, \lambda \in \Delta. \quad (2.24)$$

Proof. The unique solution of (2.23) in $C([0, \infty); X)$ is given by

$$w(t, x) = \int_0^{\tau} \frac{x e^{-\frac{x^2}{4(t-s)}}}{2\sqrt{\pi}(t-s)^{3/2}} u_0(s) ds.$$

Hence, the proposition follows from Theorem 1.1.5. Indeed, we have

$$\tilde{\Phi}_{\tau,0} u_0(z) = w(\tau, z) = \left\langle u_0, \frac{x e^{-\frac{x^2}{4(\tau-s)}}}{2\sqrt{\pi}(\tau-s)^{3/2}} \right\rangle_{L^2(0,\infty)}.$$

Therefore $\text{Ran } \tilde{\Phi}_{\tau,0}$ is a RKHS with reproducing kernel

$$\begin{aligned} k_{\lambda}(z) &= \left\langle \frac{\lambda e^{-\frac{\lambda^2}{4(\tau-s)}}}{2\sqrt{\pi}(\tau-s)^{3/2}}, \frac{z e^{-\frac{z^2}{4(\tau-s)}}}{2\sqrt{\pi}(\tau-s)^{3/2}} \right\rangle_{L^2(0,\infty)} \\ &= \frac{z\bar{\lambda}}{4\pi} \int_0^{\tau} \frac{e^{-\frac{z^2 + \bar{\lambda}^2}{4(\tau-s)}}}{(\tau-s)^3} ds \\ &= \frac{z\bar{\lambda}}{4\pi} \int_{1/\tau}^{\infty} \sigma e^{-\frac{z^2 + \bar{\lambda}^2}{4}\sigma} d\sigma \\ &= z\bar{\lambda}e^{-\frac{z^2}{4\tau}}e^{-\frac{\bar{\lambda}^2}{4\tau}} \left[\frac{4}{\pi(z^2 + \bar{\lambda}^2)^2} + \frac{1}{\tau\pi(z^2 + \bar{\lambda}^2)} \right]. \end{aligned}$$

It is clear that k_{λ} extends holomorphically to the sector Δ . □

Precisely, $\text{Ran } \tilde{\Phi}_{\tau,0}$ is the restriction to the half-line of a space of analytic functions on Δ . Now, note that the reproducing kernels of the Hardy space of the right half-plane $H^2(\mathbb{C}_+)$ and of the Bergman space of the right half-plane $A^2(\mathbb{C}_+)$ are given respectively by

$$k_\lambda^{H^2(\mathbb{C}_+)}(z) = \frac{1}{2\pi(z + \bar{\lambda})}, \quad \lambda, z \in \mathbb{C}_+$$

and

$$k_\lambda^{A^2(\mathbb{C}_+)}(z) = \frac{1}{\pi(z + \bar{\lambda})^2}, \quad \lambda, z \in \mathbb{C}_+.$$

Therefore, combining formula (1.8), Theorem 1.1.7 and Proposition 1.1.11, we obtain that g belongs to $\text{Ran } \tilde{\Phi}_{\tau,0}$ if and only if there exists $g_1 \in H^2(\Delta, \omega_0)$ and $g_2 \in A^2(\Delta, \omega_0)$ such that

$$g(z) = \sqrt{z}g_1(z) + g_2(z)$$

where we have denoted by ω_0 the weight $\omega_0(z) = e^{-\frac{\text{Re}(z^2)}{2\tau}}$. In view of this weight, it is clear now that the reachable space depends on time $\tau > 0$.

Moreover, if we restrict a little bit the class of controls u_0 picking them in $L^2\left((0, \tau), \frac{dt}{t}\right)$, we obtain a simpler description of the (modified) reachable space.

Theorem 2.6.2. *[see [AHS90, Section 2]] The operator $\tilde{\Phi}_{\tau,0} : (L^2((0, \tau), \frac{dt}{t})) \rightarrow A^2(\Delta, \omega_0)$ is an isometrical isomorphism.*

As for the previous proposition, the proof is a direct application of Theorem 1.1.5.

This result is one of the main tools of the paper [HKT20]. Indeed, we will see later that the controllability operator Φ_τ of the heat equation on $(0, \pi)$ can be written as

$$\Phi_\tau(u_0, u_\pi) = \tilde{\Phi}_{\tau,0}u_0 + \tilde{\Phi}_{\tau,\pi}u_\pi + R(u_0, u_\pi)$$

where $\tilde{\Phi}_{\tau,\pi}$ is the controllability operator for the half-line $(-\infty, \pi)$ and R is a perturbation term. This writing is fundamental in our work. It also allowed Kellay, Tucsnak and Normand to obtain in [KNT19] another exact characterization of the reachable space on $(0, \pi)$, using Proposition 2.4.3.

Finally, the last achievement of [AHS90] is to identify the trace of the Bergman space $A^2(\Delta)$ on the half-line with a space of C^∞ -functions whose norm is computed on the half-line.

Theorem 2.6.3. *[[AHS90, Theorem 4.1 and Corollary 4.1]] A function f belongs to $A^2(\Delta)$ if and only if f belongs to $C^\infty(0, \infty)$ and satisfies*

$$\|f\|_{C^\infty(0,\infty)}^2 := \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!} \int_0^\infty x^{2n+1} |f^{(n)}(x)|^2 dx < \infty.$$

Moreover, we have $\|f\|_{A^2(\Delta)}^2 = \|f\|_{C^\infty(0,\infty)}^2$.

This isometrical identity is derived from the heat equation and Theorem 2.6.2. It permits to identify the reachable states of the heat equation of the half-line with functions of class C^∞ on the half-line.

We mention also a generalization of these results to the n -dimensional half-space by Saitoh in [Sai94].

Part II

Main results

Chapter 3

The reachable space as sum of Bergman spaces

This chapter is based on the paper [Ors20]. Here, we shall prove our first result for the reachable space of the heat equation. It gives an exact characterization of the reachable space as a sum of two Bergman spaces on sectors the intersection of which is D . Using tools from classical harmonic analysis, we will also make a first step into the direction of a more precise information on this sum of Bergman spaces, proving that it contains the Smirnov-Zygmund space $E_{L \log^+ L}(D)$. This result will be the starting point of Chapter 4 where different tools lead to a complete characterization of this sum, even in more general geometric settings.

3.1 Main results

We remind that we consider the reachable space of the following heat equation

$$\begin{cases} \frac{\partial y}{\partial t}(t, x) - \frac{\partial^2 y}{\partial x^2}(t, x) = 0 & t > 0, x \in (0, \pi), \\ y(t, 0) = u_0(t), \quad y(t, \pi) = u_\pi(t) & t > 0, \\ y(0, x) = 0 & x \in (0, \pi). \end{cases} \quad (\text{HE})$$

It has already been discussed in Section 2.5. We recall the result of [HKT20]

$$E^2(D) \subsetneq \text{Ran } \Phi_\tau \subset A^2(D). \quad (3.1)$$

and the HKT-conjecture

$$\text{Ran } \Phi_\tau = A^2(D). \quad (3.2)$$

Let $\Delta = \{z \in \mathbb{C} \mid |\arg(z)| < \frac{\pi}{4}\}$. The first central result of this thesis is the following explicit characterization of the reachable space.

Theorem 3.1.1. *We have $\text{Ran } \Phi_\tau = A^2(\Delta) + A^2(\pi - \Delta)$.*

We mention another characterization which was very recently observed in [KNT19]. Denote by ω_0 and ω_π the weights defined by

$$\forall z \in \Delta, \omega_0(z) = \frac{e^{-\frac{\text{Re}(z^2)}{2\tau}}}{\tau} \quad \text{and} \quad \forall z \in \pi - \Delta, \omega_\pi(z) = \omega_0(\pi - z), \quad (3.3)$$

then

$$\text{Ran } \Phi_\tau = A^2(\Delta, \omega_0) + A^2(\pi - \Delta, \omega_\pi), \quad (3.4)$$

independently of $\tau > 0$. It is a direct consequence of Proposition 2.4.3, Aikawa-Hayashi-Saitoh's theorem 2.6.2 and the decomposition (3.6) that we shall see later. Note that the inclusion “ \supset ” in (3.4) was already known in [HKT20] and was used to prove the left inclusion in (3.1) (see the proof of Theorem 1.2 in [HKT20]). Since it is also used in the proof of Theorem 3.1.1, we will recall some ideas leading to this inclusion below, in Section 3.2. It should be mentioned that based on (3.4), the authors of [KNT19] were subsequently able to produce a new proof for the unweighed case.

To prove Theorem 3.1.1, the main idea is to write a certain integral operator as a Laplace type transform for which we have a Paley-Wiener type theorem. We will also use suitable analyticity properties of this transform and the fact that functions holomorphic in a neighborhood of D are reachable (as proved by Dardé and Ervedoza, see Section 2.5).

In this chapter, we will also discuss a first approach to the sum based on tools from harmonic analysis. Our starting result in this direction is the following.

Theorem 3.1.2.

$$E_{L \log^+ L}(D) \subset A^2(\Delta) + A^2(\pi - \Delta).$$

Note that in view of Theorem 3.1.1, this result improves the left inclusion (3.1) obtained in [HKT20] since $E^2(D) \subsetneq E_{L \log^+ L}(D)$. The methods involved in the proof of Theorem 3.1.2 are for the most part harmonic and complex analysis methods. More precisely, we use essentially the Cauchy formula for Smirnov functions, a local regularity result for the Cauchy Transform on the upper-half plane and the embedding $H^1(\mathbb{D}) \subset A^2(\mathbb{D})$ due to Hardy and Littlewood. The theorem is almost sharp in terms of growth of functions regarding the HKT-conjecture (3.2). Indeed, a function in $A^2(D)$ cannot grow faster than $\frac{1}{d(z, \partial D)}$, while the growth of a function

in $E_{L \log^+ L}(D)$ is bounded by $\frac{1}{d(z, \partial D) \log(\frac{1}{d(z, \partial D)})}$. This last estimate comes from the Cauchy formula and the Hölder inequality for Orlicz spaces.

This chapter is organized as follows. In Section 3.2, we recall some auxiliary results on representations of solutions of (HE), in Section 3.3 we prove Theorem 3.1.1. Finally, Section 3.4 is devoted to the proof of Theorem 3.1.2.

3.2 Reminders on solutions of (HE)

We recall several central facts for our discussion from Section 2.3 and [HKT20]. For $f \in W^{-1,2}(0, \pi)$ and $u := (u_0, u_\pi) \in L^2([0, \tau], \mathbb{C}^2)$, the unique solution $y \in C([0, \infty), W^{-1,2}(0, \pi))$ of (HE) satisfies for every $t \geq 0$, and every $\varphi \in W^{2,2}(0, \pi) \cap W_0^{1,2}(0, \pi)$ such that $\frac{d^2\varphi}{dx^2} \in W_0^{1,2}(0, \pi)$,

$$\begin{aligned} \langle y(t), \varphi \rangle_{-1,1} &= \int_0^t \left\langle y(s), \frac{d^2\varphi}{dx^2} \right\rangle_{-1,1} ds \\ &\quad + \int_0^t u_0(s) \overline{\frac{d\varphi}{dx}(0)} ds - \int_0^t u_\pi(s) \overline{\frac{d\varphi}{dx}(\pi)} ds \end{aligned} \quad (3.5)$$

where $\langle \cdot, \cdot \rangle_{-1,1}$ stands for the duality $W^{-1,2}(0, \pi) - W_0^{1,2}(0, \pi)$. The operator A_0 defined by

$$A_0\varphi = -\frac{d^2\varphi}{dx^2}$$

on $L^2(0, \pi)$ with domain $W^{2,2}(0, \pi) \cap W_0^{1,2}(0, \pi)$, admits an orthonormal basis of $L^2(0, \pi)$ composed of the eigenfunctions $\varphi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$, $n \in \mathbb{N}^*$ (associated with the eigenvalues n^2). Moreover, A_0 has a unique strictly positive extension to $W^{-1,2}(0, \pi)$ and the family of eigenfunctions $(\psi_n)_{n \in \mathbb{N}^*}$ given by $\psi_n = n\varphi_n$ is an orthonormal basis of $W^{-1,2}(0, \pi)$. Decomposing $y(t)$ on this orthonormal basis for each $t \geq 0$ and replacing in (3.5) with $\varphi = \varphi_n$, it is not difficult to check that the reachable states of the 1-D heat equation can be represented in the following way:

$$\begin{aligned} (\Phi_\tau u)(x) &= \frac{2}{\pi} \sum_{n \geq 1} n \left[\int_0^\tau e^{n^2(\sigma-\tau)} u_0(\sigma) d\sigma \right] \sin(nx) \\ &\quad + \frac{2}{\pi} \sum_{n \geq 1} n(-1)^{n+1} \left[\int_0^\tau e^{n^2(\sigma-\tau)} u_\pi(\sigma) d\sigma \right] \sin(nx), \quad \tau > 0, x \in (0, \pi) \end{aligned}$$

where the series converge in $W^{-1,2}(0, \pi)$. With the elementary formula $\cos u = \frac{e^{iu} + e^{-iu}}{2}$ in mind, the authors of [HKT20] showed that

$$(\Phi_\tau u)(x) = \int_0^\tau \frac{\partial K_0}{\partial x}(\tau - \sigma, x) u_0(\sigma) d\sigma + \int_0^\tau \frac{\partial K_\pi}{\partial x}(\tau - \sigma, x) u_\pi(\sigma) d\sigma,$$

where

$$\begin{aligned} K_0(\sigma, x) &= -\frac{2}{\pi} \left(\sum_{n \geq 1} e^{-n^2 \sigma} \cos(nx) + \frac{1}{2} \right) \\ &= -\frac{1}{\pi} \sum_{n \geq 1} e^{-n^2 \sigma} (e^{inx} + e^{-inx}) - \frac{1}{\pi} \\ &= -\frac{1}{\pi} \sum_{n \in \mathbb{Z}} e^{-n^2 \sigma} e^{inx}, \quad \sigma > 0, \quad x \in (0, \pi). \end{aligned}$$

and $K_\pi(\sigma, x) = -K_0(\sigma, \pi - x)$. And finally, using the Poisson summation formula applied to a suitable Gaussian function, they obtained

$$K_0(\sigma, x) = -\sqrt{\frac{1}{\pi\sigma}} \sum_{m \in \mathbb{Z}} e^{-\frac{(x+2m\pi)^2}{4\sigma}}, \quad \sigma > 0, \quad x \in (0, \pi).$$

Hence, setting $\widetilde{K}_0(\sigma, z) = -\sqrt{\frac{1}{\pi\sigma}} \sum_{m \in \mathbb{Z}^*} e^{-\frac{(z+2m\pi)^2}{4\sigma}}$, we can write (see [HKT20, equation (2.18)])

$$\Phi_\tau(u_0, u_\pi) = \widetilde{\Phi}_{0,\tau} u_0 + \widetilde{\Phi}_{\pi,\tau} u_\pi + R_{0,\tau} u_0 + R_{\pi,\tau} u_\pi \quad (3.6)$$

where

$$\left[\widetilde{\Phi}_{0,\tau} f \right](s) = \int_0^\tau \frac{s e^{-\frac{s^2}{4(\tau-\sigma)}}}{2\sqrt{\pi}(\tau-\sigma)^{\frac{3}{2}}} f(\sigma) d\sigma \quad \text{and} \quad \left[\widetilde{\Phi}_{\pi,\tau} f \right](s) = - \left[\widetilde{\Phi}_{0,\tau} f \right](\pi - s) \quad (3.7)$$

$$\left[R_{0,\tau} f \right](s) = \int_0^\tau \frac{\partial \widetilde{K}_0}{\partial s}(\tau - \sigma, s) f(\sigma) d\sigma \quad \text{and} \quad \left[R_{\pi,\tau} f \right](s) = - \left[R_{0,\tau} f \right](\pi - s).$$

Note that the operator $\widetilde{\Phi}_{0,\tau}$ is the controllability operator of the heat equation for the half-line introduced in Section 2.6.

Finally, since the inclusion

$$A^2(\Delta, \omega_0) + A^2(\pi - \Delta, \omega_\pi) \subset \text{Ran } \Phi_\tau \quad (3.8)$$

will be used later, we remind rapidly how the authors of [HKT20] proved it. Writing

$$\widetilde{K}_0(\sigma, z) = -\sqrt{\frac{1}{\pi\sigma}} \sum_{m \geq 1} e^{-\frac{(z+2m\pi)^2}{4\sigma}} - \sqrt{\frac{1}{\pi\sigma}} \sum_{m \leq -1} e^{-\frac{(z+2m\pi)^2}{4\sigma}} := A(\sigma, z) + B(\sigma, z),$$

we can decompose $R_{0,\tau}$ and $R_{\pi,\tau}$ as

$$[R_{0,\tau}f](s) = \int_0^\tau \frac{\partial A}{\partial s}(\tau-\sigma, s)f(\sigma)d\sigma + \int_0^\tau \frac{\partial B}{\partial s}(\tau-\sigma, s)f(\sigma)d\sigma := [R_{A,\tau}f](s) + [R_{B,\tau}f](s)$$

and

$$[R_{\pi,\tau}f](s) = -[R_{B,\tau}f](\pi-s) - [R_{A,\tau}f](\pi-s) := [R_{C,\tau}f](s) + [R_{D,\tau}f](s).$$

Let us start by assuming that for some small enough $\tau > 0$ the matricial operator M_τ defined by

$$M_\tau := \begin{pmatrix} \widetilde{\Phi}_{0,\tau} + R_{A,\tau} & R_{C,\tau} \\ R_{B,\tau} & \widetilde{\Phi}_{\pi,\tau} + R_{D,\tau} \end{pmatrix} = \begin{pmatrix} \widetilde{\Phi}_{0,\tau} & 0 \\ 0 & \widetilde{\Phi}_{\pi,\tau} \end{pmatrix} + \begin{pmatrix} R_{A,\tau} & R_{C,\tau} \\ R_{B,\tau} & R_{D,\tau} \end{pmatrix}$$

is bounded and invertible from $(L^2([0, \tau], \frac{dt}{t}))^2$ onto $A^2(\Delta, \omega_0) \times A^2(\pi - \Delta, \omega_\pi)$. Then for every couple $(\varphi_0, \varphi_\pi) \in A^2(\Delta, \omega_0) \times A^2(\pi - \Delta, \omega_\pi)$, there exists $u = (u_0, u_\pi) \in (L^2([0, \tau], \frac{dt}{t}))^2$ such that

$$\begin{pmatrix} \varphi_0 \\ \varphi_\pi \end{pmatrix} = M_\tau \begin{pmatrix} u_0 \\ u_\pi \end{pmatrix} = \begin{pmatrix} (\widetilde{\Phi}_{0,\tau} + R_{A,\tau})u_0 + R_{C,\tau}u_\pi \\ R_{B,\tau}u_0 + (\widetilde{\Phi}_{\pi,\tau} + R_{D,\tau})u_\pi \end{pmatrix}.$$

So, $\varphi_0 + \varphi_\pi = \Phi_\tau u$. In other words, we have

$$A^2(\Delta, \omega_0) + A^2(\pi - \Delta, \omega_\pi) \subset \Phi_\tau (L^2((0, \tau), dt/t; \mathbb{C}^2)) \subset \text{Ran } \Phi_\tau$$

(it is relevant to mention that the second inclusion is in fact an equality, as proved in [KNT19, Prop. 3.2]). Hence, it remains to prove the assertion above claiming that there exists $\tau > 0$ such that M_τ is boundedly invertible from $(L^2([0, \tau], \frac{dt}{t}))^2$ onto $A^2(\Delta, \omega_0) \times A^2(\pi - \Delta, \omega_\pi)$. For that, we can use that $\widetilde{\Phi}_{0,\tau}$ (resp. $\widetilde{\Phi}_{\pi,\tau}$) is an isometric isomorphism from $L^2([0, \tau], \frac{dt}{t})$ onto $A^2(\Delta, \omega_0)$ (resp. $A^2(\pi - \Delta, \omega_\pi)$) by a result of Aikawa, Hayashi and Saitoh [AHS90] (see also Remark 3.3.3 for another proof) and the Lemma 4.1 of [HKT20] which can be stated

$$\left\| \begin{pmatrix} R_{A,\tau} & R_{C,\tau} \\ R_{B,\tau} & R_{D,\tau} \end{pmatrix} \right\|_{\mathcal{L}((L^2([0, \tau], \frac{dt}{t}))^2, A^2(\Delta, \omega_0) \times A^2(\pi - \Delta, \omega_\pi))} \xrightarrow{\tau \rightarrow 0} 0.$$

The proof of this lemma is essentially the one which is recalled for Lemma 3.3.2 below since for $u \in L^2([0, \tau], \frac{dt}{t})$ we have $\|u\|_{L^2(0, \tau)} \leq \sqrt{\tau} \|u\|_{L^2([0, \tau], \frac{dt}{t})}$.

3.3 Sum of Bergman spaces

We are now in a position to prove our first main result.

Proof of Theorem 3.1.1. The key of the proof is that $\tilde{\Phi}_{0,\tau}$ is an isometry from $L^2(0, \tau)$ to $A^2(\Delta)$ and we can compute its range. Indeed, let \mathbb{C}_+ be the right half-plane (not to be confused with the notation of the upper half plane \mathbb{C}^+ introduced earlier). Denote by \mathcal{L} the normalized Laplace transform defined by $\mathcal{L}(f)(s) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-st} f(t) dt$ and $G : A^2(\mathbb{C}_+) \rightarrow A^2(\Delta)$ the unitary operator associated to the conformal mapping $z \mapsto z^2$ from Δ to \mathbb{C}_+ , defined by $(Gf)(z) = 2zf(z^2)$.

By the change of variables $t = \frac{1}{4(\tau-\sigma)}$, we obtain

$$\forall s \in \Delta, \left(\tilde{\Phi}_{0,\tau} f \right) (s) = \int_0^\tau \frac{se^{-\frac{s^2}{4(\tau-\sigma)}}}{2\sqrt{\pi}(\tau-\sigma)^{\frac{3}{2}}} f(\sigma) d\sigma = \frac{s}{\sqrt{\pi}} \int_{\frac{1}{4\tau}}^{+\infty} \frac{e^{-s^2 t}}{\sqrt{t}} f\left(\tau - \frac{1}{4t}\right) dt$$

Define for $f \in L^2(0, \tau)$,

$$(Tf)(t) = \begin{cases} \frac{f(\tau - \frac{1}{4t})}{2\sqrt{t}} & \text{if } t > \frac{1}{4\tau}, \\ 0 & \text{if } 0 < t \leq \frac{1}{4\tau}. \end{cases}$$

It is easily seen that the operator T is an isometry from $L^2(0, \tau)$ to $L^2(\mathbb{R}^+, \frac{dt}{t})$ with range $L^2\left(\left(\frac{1}{4\tau}, +\infty\right), \frac{dt}{t}\right)$. Hence

$$\left(\tilde{\Phi}_{0,\tau} f \right) (s) = 2s\mathcal{L}(Tf)(s^2) = (G\mathcal{L}Tf)(s)$$

where \mathcal{L} denotes the normalized Laplace transform

$$(\mathcal{L}f)(z) = \frac{1}{\sqrt{\pi}} \int_0^{+\infty} e^{-zt} f(t) dt$$

(note that this is not the normalization chosen in Section 1.2). The last step is the following Paley-Wiener type theorem 1.2.11 for Bergman spaces. With the normalization above of the Laplace transform, it states

Proposition 3.3.1. *The Laplace transform \mathcal{L} is a unitary operator of $L^2\left(\mathbb{R}_+, \frac{dt}{t}\right)$ onto $A^2(\mathbb{C}_+)$ where $\mathbb{C}_+ = \{z \mid \operatorname{Re} z > 0\}$.*

So, if \simeq means that the operator is unitary, we have the following diagram.

$$L^2(0, \tau) \xrightarrow[\simeq]{T} L^2\left(\left(\frac{1}{4\tau}, +\infty\right), \frac{dt}{t}\right) \subset L^2\left(\mathbb{R}_+, \frac{dt}{t}\right) \xrightarrow[\simeq]{\mathcal{L}} A^2(\mathbb{C}_+) \xrightarrow[\simeq]{G} A^2(\Delta)$$

Hence, by composition, $\tilde{\Phi}_{0,\tau}$ is isometric from $L^2(0, \tau)$ to $A^2(\Delta)$, and

$$\text{Ran } \tilde{\Phi}_{0,\tau} = G\mathcal{L} \left[L^2 \left(\left(\frac{1}{4\tau}, +\infty \right), \frac{dt}{t} \right) \right] \subset A^2(\Delta).$$

By (3.7), we have also $\text{Ran } \tilde{\Phi}_{\pi,\tau} \subset A^2(\pi - \Delta)$. In order to discuss the range of Φ_τ we thus have to investigate the remainder terms $R_{0,\tau}$ and $R_{\pi,\tau}$, which, morally speaking, are sums converging very quickly since they involve gaussians centered essentially at πn , $n \in \mathbb{Z}^*$. For these remainder terms we will use the lemma below which is a straightforward modification of Lemma 4.1 of [HKT20], the main difference being a square root in the integral operator, which does not change the boundedness.

Lemma 3.3.2. *Let ω_0 and ω_π be the weights defined in (3.3). Then $R_{0,\tau}$ and $R_{\pi,\tau}$ are bounded from $L^2(0, \tau)$ to $A^2(\Delta, \omega_0) + A^2(\pi - \Delta, \omega_\pi)$.*

For the sake of completeness, we recall the proof which corresponds essentially to the one given in [HKT20].

Proof. It is clear that the ranges of $R_{0,\tau}$ and $R_{\pi,\tau}$ are contained in $\text{Hol}(\Delta) + \text{Hol}(\pi - \Delta)$. It remains to prove the boundedness from $L^2(0, \tau)$ to $L^2(\Delta, \omega_0) + L^2(\pi - \Delta, \omega_\pi)$. This will be done for $R_{0,\tau} = R_{A,\tau} + R_{B,\tau}$ only, the boundedness of $R_{\pi,\tau}$ follows by symmetry. Pick $u \in L^2(0, \tau)$. Writing

$$\rho_{m,\tau}(s) = \int_0^\tau \frac{s + m\pi}{(\tau - \sigma)^{3/2}} e^{-\frac{(s+m\pi)^2}{4(\tau-\sigma)}} u(\sigma) d\sigma,$$

we have

$$[R_{A,\tau}u](s) = \frac{\sqrt{\pi}}{4} \sum_{m \geq 1} \rho_{2m,\tau}(s), \quad s \in \Delta \tag{3.9}$$

and analogously for $R_{B,\tau}$ where we replace $\rho_{2m,\tau}$ by $\rho_{-2m,\tau}$ and $s \in \pi - \Delta$. For $m \geq 1$, the norm of $\rho_{m,\tau}$ may be estimated via the triangular inequality and Cauchy-Schwarz inequality (see [HKT20] for details) to obtain:

$$\begin{aligned} \|\rho_{m,\tau}\|_{A^2(\Delta, \omega_0)}^2 &= \frac{1}{\tau} \int \int_{\Delta} e^{\frac{x^2-y^2}{2\tau}} \left| \int_0^\tau \frac{x + iy + m\pi}{(\tau - \sigma)^{3/2}} e^{-\frac{(x+iy+m\pi)^2}{4(\tau-\sigma)}} u(\sigma) d\sigma \right|^2 dx dy \\ &\leq \frac{1}{\tau} \|u\|_{L^2(0,\tau)}^2 \iint_{\Delta} e^{\frac{x^2-y^2}{2\tau}} (|x + m\pi|^2 + y^2) \int_0^\tau \frac{e^{-\frac{(x+m\pi)^2-y^2}{2\sigma}}}{\sigma^3} d\sigma dx dy \end{aligned}$$

With elementary computations, it easy to prove that

$$\iint_{\Delta} e^{\frac{x^2-y^2}{2\tau}} (|x+m\pi|^2+y^2) \int_0^\tau \frac{e^{-\frac{(x+m\pi)^2-y^2}{2\sigma}}}{\sigma^3} d\sigma dx dy \leq C\tau(1+\tau)^3 e^{-m^2\pi^2/2\tau}.$$

Hence for $m \geq 1$,

$$\|\rho_{m,\tau}\|_{A^2(\Delta,\omega_0)} \leq C'\|u\|_{L^2(0,\tau)}(1+\tau)^{3/2} e^{-m^2\pi^2/4\tau}.$$

Using (3.9), we obtain

$$\begin{aligned} \|R_{A,\tau}u\|_{A^2(\Delta,\omega_0)} &\leq \frac{\sqrt{\pi}}{4} C'\|u\|_{L^2(0,\tau)}(1+\tau)^{3/2} \sum_{m \geq 1} e^{-(2m)^2\pi^2/4\tau} \\ &\leq \frac{\sqrt{\pi}}{4} C'\|u\|_{L^2(0,\tau)}(1+\tau)^{3/2} \frac{e^{-\pi^2/\tau}}{1-e^{-\pi^2/\tau}}. \end{aligned}$$

$R_{B,\tau}$ is estimated similarly using $\|\rho_{m,\tau}\|_{A^2(\pi-\Delta,\omega_\pi)} = \|\rho_{|m|-1,\tau}\|_{A^2(\Delta,\omega_0)}$, for $m \leq -2$, and the result follows. \square

Since $A^2(\Delta, \omega_0) + A^2(\pi - \Delta, \omega_\pi) \subset A^2(\Delta) + A^2(\pi - \Delta)$, the inclusion $\text{Ran } \Phi_\tau \subset A^2(\Delta) + A^2(\pi - \Delta)$ is a direct consequence of the decomposition (3.6), the above discussion and Lemma 3.3.2.

For the converse inclusion, we will prove $A^2(\Delta) \subset \text{Ran } \Phi_\tau$ and $A^2(\pi - \Delta) \subset \text{Ran } \Phi_\tau$. Using that G and \mathcal{L} are unitary, we have the decomposition

$$\begin{aligned} A^2(\Delta) &= G\mathcal{L} \left[L^2 \left(\mathbb{R}_+, \frac{dt}{t} \right) \right] \\ &= G\mathcal{L} \left[L^2 \left(\left(0, \frac{1}{4\tau} \right), \frac{dt}{t} \right) \oplus L^2 \left(\left(\frac{1}{4\tau}, +\infty \right), \frac{dt}{t} \right) \right] \\ &= X_0 \oplus \text{Ran } \tilde{\Phi}_{0,\tau} \end{aligned}$$

where we wrote $X_0 := G\mathcal{L} \left[L^2 \left(\left(0, \frac{1}{4\tau} \right), \frac{dt}{t} \right) \right]$ and where, as usual \oplus means orthogonal sum. Similarly, we have $A^2(\pi - \Delta) = X_\pi \oplus \text{Ran } \tilde{\Phi}_{\pi,\tau}$, where X_π is the image of X_0 by the transformation $f \mapsto f(\pi - \cdot)$. Hence, it is enough to prove that $X_0, X_\pi, \text{Ran } \tilde{\Phi}_{0,\tau}$ and $\text{Ran } \tilde{\Phi}_{\pi,\tau}$ are contained in $\text{Ran } \Phi_\tau$. For this, note that for every $u_0 \in L^2(0, \tau)$, we have

$$\tilde{\Phi}_{0,\tau}u_0 = \Phi_\tau(u_0, 0) - R_{0,\tau}u_0.$$

Since $A^2(\Delta, \omega_0) + A^2(\pi - \Delta, \omega_\pi) \subset \text{Ran } \Phi_\tau$ (see 3.8 and discussion after), we get from Lemma 3.3.2 that $R_{0,\tau}u_0 \in \text{Ran } \Phi_\tau$. It follows that $\text{Ran } \tilde{\Phi}_{0,\tau} \subset \text{Ran } \Phi_\tau$. The case of $\text{Ran } \tilde{\Phi}_{\pi,\tau}$ is similar. Finally, note that $L^2\left(\left(0, \frac{1}{4\tau}\right), \frac{dt}{t}\right)$ is included in the space of functions in $L^1(\mathbb{R})$ with compact support. Therefore, up to the change of variable G which does not change the holomorphy, an element of X_0 is essentially a Fourier transform of a compactly supported integrable function, so it is entire by Theorem 1.2.5 (actually $\mathcal{L}\left[L^2\left(\left(0, \frac{1}{4\tau}\right), \frac{dt}{t}\right)\right]$ is a Model space by Proposition 1.2.2). Thus, X_0 is a space of entire functions and, as such, is contained in the reachable space. The same argument proves also that the reachable space includes X_π , and the proof is complete. \square

Remark 3.3.3. Note that the same Laplace integral method gives in a straightforward way the Aikawa-Hayashi-Saitoh Theorem 2.6.2.

3.4 Inclusion of the Smirnov-Zygmund space

We prove now Theorem 3.1.2.

Let $P(z) = z + 2i\pi$. It suffices to prove the following assertion.

$$\forall f \in E_{L \log^+ L}(D), \quad \frac{f}{P} \in A^2(\Delta) + A^2(\pi - \Delta) \quad (3.10)$$

Indeed, assume that (3.10) is true and let $g \in E_{L \log^+ L}(D)$. Since P is bounded analytic on \bar{D} , Pg belongs also to $E_{L \log^+ L}(D)$. Hence, by (3.10), $g = (Pg)/P$ belongs to $A^2(\Delta) + A^2(\pi - \Delta)$ and then to $\text{Ran } \Phi_\tau$ by Theorem 3.1.1, which proves the inclusion.

Remark 3.4.1. With a more refined argument, as used in [HKT20, corollary 3.6], we can prove that $E_{L \log^+ L}(D) \subset A^2(\Delta, \omega_0) + A^2(\pi - \Delta, \omega_\pi)$ where ω_0 and ω_π are defined in (3.3). This observation also follows from Thm 3.1.2 and (3.4) (see [KNT19]).

So, pick $f \in E_{L \log^+ L}(D)$ and let us prove (3.10).

Decomposition. Let γ be the boundary of D parameterized counterclockwise side by side as follows (see Figure 1).

$$\begin{array}{ll} \gamma_{1,+} : [0, 1] \longrightarrow \mathbb{C} & \gamma_{1,-} : [0, 1] \longrightarrow \mathbb{C} \\ t \longmapsto (1-t)\frac{\pi}{2}(1+i) & t \longmapsto (1-i)\frac{\pi}{2}t \\ \gamma_{2,+} : [0, 1] \longrightarrow \mathbb{C} & \gamma_{2,-} : [0, 1] \longrightarrow \mathbb{C} \\ t \longmapsto \pi(1-t) + t\left((1+i)\frac{\pi}{2}\right) & t \longmapsto (1-i)\frac{\pi}{2}(1-t) + t\pi \end{array}$$

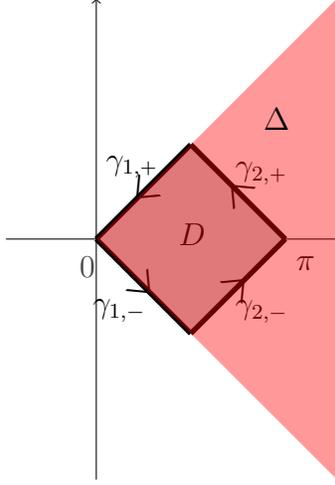


Figure 1: The square D , the path γ and the sector Δ .

The key idea is to decompose f via the Cauchy formula for functions in $E^1(D)$ (see (1.5)) :

$$\begin{aligned}
 \forall z \in D, f(z) &= \frac{1}{2i\pi} \int_{\gamma} \frac{f(u)}{u-z} du \\
 &= \frac{1}{2i\pi} \sum_{\substack{k \in \{1, 2\} \\ \varepsilon \in \{\pm\}}} \int_{\gamma_{k,\varepsilon}} \frac{f(u)}{u-z} du \\
 &= \frac{1}{2} \sum_{\substack{k \in \{1, 2\} \\ \varepsilon \in \{\pm\}}} f_{k,\varepsilon}(z)
 \end{aligned}$$

where we have written

$$f_{k,\varepsilon}(z) = \frac{1}{i\pi} \int_{\gamma_{k,\varepsilon}} \frac{f(u)}{u-z} du, \quad k \in \{1, 2\}, \quad \varepsilon \in \{\pm\}.$$

For the reader acquainted with Hardy spaces, the crucial observation here is that $f_{k,\varepsilon}$ can be seen — modulo rotation and translation — as a scalar product between a (compactly supported) function and a reproducing kernel of the Hardy space, which thus yields a (Riesz-) projection on the Hardy space. It is known that this projection is bounded when $f \in L^p$, $p > 1$, but not when $p = 1$. As it is explained in Subsection 1.1.2, on the real line, this boundedness remains valid when $f \in L \log^+ L$ and f is

compactly supported (see Proposition 1.1.12 and Theorem 1.1.16).

The remainder part of the section will be devoted to show that $f_{1,\varepsilon}/P \in A^2(\Delta)$ and $f_{2,\varepsilon}/P \in A^2(\pi - \Delta)$ for $\varepsilon \in \{\pm\}$. We cut each sector Δ and $\pi - \Delta$ in two disjoint parts, which will be treated separately. For that, given a fixed $a > 0$, denote by D_a the homothetic dilation of D with center 0 and obtained by adding length $a > 0$ to the sides of D (see Figure 2). We will consider the disjoint union $\Delta = D_a \cup \Delta \setminus D_a$ (and similarly for $\pi - \Delta$). The proof is composed of two steps.

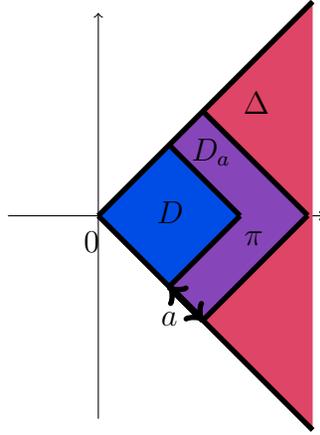


Figure 2: The squares D , D_a and the sector Δ .

- **Step 1** : In this step we prove the following claim:

$$f_{1,\varepsilon}/P \in A^2(\Delta \setminus D_a) \quad (3.11)$$

(the case $f_{2,\varepsilon}/P \in A^2((\pi - \Delta) \setminus (\pi - D_a))$ follows in a similar fashion).

To do so, remark that there exists a constant $C_a > 0$ such that for any $z \notin D_a$, $|z| + 1 \leq C_a d(z, \partial D)$. Using the triangular inequality, we have for any $k \in \{1, 2\}$ and $\varepsilon \in \{\pm\}$,

$$\begin{aligned} \forall z \notin D_a, |f_{k,\varepsilon}(z)| &\leq \frac{1}{\pi} \int_{\gamma_{k,\varepsilon}} \left| \frac{f(u)}{u - z} \right| |du| \\ &\leq \|f\|_{L^1(\partial D)} \frac{1}{\pi d(z, \partial D)} \\ &\leq \frac{C}{|z| + 1}, \end{aligned}$$

where we have used that $L \log^+ L \subset L^1$ on a segment.

So, since $-2i\pi \notin \overline{\Delta \setminus D_a}$, we obtain

$$\int_{\Delta \setminus D_a} \left| \frac{f_{1,\varepsilon}(z)}{P(z)} \right|^2 dA(z) \leq C \int_{\Delta \setminus D_a} \frac{dA(z)}{|2i\pi + z|^2 (1 + |z|)^2} < +\infty.$$

This proves claim (3.11).

• **Step 2 :** This step is more delicate and uses the Cauchy (or Hilbert) transform and the inclusion $H^1(\mathbb{D}) \subset A^2(\mathbb{D})$ (see Theorem 1.1.16).

We need to show the following claim

$$f_{1,\varepsilon}/P \in A^2(D_a) \tag{3.12}$$

(and $f_{2,\varepsilon}/P \in A^2(\pi - D_a)$). It is enough to treat the case $f_{1,+}$, the others follow in a similar way.

We recall that for $g \in L^1(\mathbb{R})$, its Cauchy Transform $\mathcal{C}g$ is defined by

$$(\mathcal{C}g)(z) = \frac{1}{i\pi} \int_{\mathbb{R}} \frac{g(t)}{t - z} dt, z \in \mathbb{C}^+ := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}.$$

See Section 1.1.2 for more details.

We first explain briefly how to translate $f_{1,+}$ to $\mathcal{C}g$ for some suitable g . It is essentially rotating and translating the line through $\gamma_{1,+}$ to \mathbb{R} . To be more explicite, let $\alpha_{1,+} : z \mapsto 1 + \frac{\sqrt{2}}{\pi} e^{i\frac{3\pi}{4}} z$. This is a direct similarity transformation which sends D_a to \mathbb{C}^+ and in particular $\gamma_{1,+}$ onto $[0, 1]$ (note that the orientation is preserved, e.g. the endpoint 0 of $\gamma_{1,+}$ is sent to 1). Let $f_{1,+}^\gamma(t) = \mathbf{1}_{[0,1]}(t) f(\gamma_{1,+}(t))$. For all $z \in D_a$, we have

$$\begin{aligned} f_{1,+}(z) &= \frac{1}{i\pi} \int_{\gamma_{1,+}} \frac{f(u)}{u - z} du = \frac{1}{i\pi} \int_0^1 \frac{f(\gamma_{1,+}(t))}{\gamma_{1,+}(t) - z} \gamma'_{1,+}(t) dt \\ &= \frac{1}{i\pi} \int_{\mathbb{R}} \frac{f_{1,+}^\gamma(t)}{(1-t)\frac{\pi}{2}(1+i) - z} \left(-\frac{\pi}{2}(1+i)\right) dt \\ &= \frac{1}{i\pi} \int_{\mathbb{R}} \frac{f_{1,+}^\gamma(t)}{t - \alpha_{1,+}(z)} dt \\ &= (\mathcal{C}f_{1,+}^\gamma)(\alpha_{1,+}(z)) \end{aligned}$$

So, since P does not vanish on $\overline{D_a}$, we obtain

$$\begin{aligned}
\int_{D_a} \left| \frac{f_{1,+}(z)}{P(z)} \right|^2 dA(z) &\leq C \int_{D_a} |f_{1,+}(z)|^2 dA(z) \\
&= C \int_{D_a} |(\mathcal{C}f_{1,+}^\gamma)(\alpha_{1,+}(z))|^2 dA(z) \\
&= C_2 \int_{\alpha_{1,+}(D_a)} |(\mathcal{C}f_{1,+}^\gamma)(z)|^2 dA(z), \tag{3.13}
\end{aligned}$$

where we have used in the last step that $\alpha_{1,+}$ is an affine change of variable with constant jacobian. As already written, $\alpha_{1,+}(D_a)$ is a square in the upper-half plane with a segment of the real line as one of its sides. We will next appeal to the following regularity result of the Cauchy transform which is essentially a combination of Proposition 1.1.12 (and so the Calderón-Zygmund theorem [CZ52, Thm 2, p.100]) and Theorem 1.1.16 of Hardy-Littlewood which gives the inclusion between H^1 and A^2 on the disk.

Proposition 3.4.2. *Let $f \in L \log^+ L(\mathbb{R})$ have compact support. Let Ω be a square in the upper-half plane one side of which is a segment $I \subset \mathbb{R}$. Then the Cauchy transform $\mathcal{C}f$ belongs to $A^2(\Omega)$.*

Remark 3.4.3. Note that in this proposition, we do not need to assume any link between the (compact) support of f and the segment I . However, we will apply later on the result for the case when the support of f is included in I (and $I = [0, 1]$).

Proof of Proposition 3.4.2. In view of Proposition 1.1.12, we already know that $\mathcal{C}f \in E^1(\Omega_L)$ where $\Omega \subset \Omega_L$. Now, if $\varphi : \mathbb{D} \rightarrow \Omega_L$ is a conformal mapping, then we will have $(\mathcal{C}f \circ \varphi) \varphi' \in H^1(\mathbb{D})$. From Theorem 1.1.16 we obtain $(\mathcal{C}f \circ \varphi) \varphi' \in A^2(\mathbb{D})$, or equivalently, by simple change of variable, $\mathcal{C}f \in A^2(\Omega_L)$. Since $\Omega \subset \Omega_L$, we obtain $\mathcal{C}f \in A^2(\Omega)$ which is what we want to prove. \square

From the preceding discussions we can now deduce the claim (3.12). Indeed, recall from (3.13) that

$$\int_{D_a} \left| \frac{f_{1,+}(z)}{P(z)} \right|^2 dA(z) \leq C \int_{\alpha_{1,+}(D_a)} |(\mathcal{C}f_{1,+}^\gamma)(z)|^2 dA(z).$$

Clearly, when $f \in L \log^+ L$ with compact support, the same will be true for $f_{1,+}^\gamma$ (which is essentially a truncation of f composed with a rotation/translation). From Proposition 3.4.2 (with $\Omega = \alpha_{1,+}(D_a)$ being a unit square in the upper half plane with base on the real line we deduce (3.12) (the argument is the same for $f_{1,-}$).

Proof of Theorem 3.1.2. By (3.10) it is enough to show that $f/P \in A^2(\Delta) + A^2(\pi - \Delta)$. The decomposition will be given by $F_1 = (f_{1,+} + f_{1,-})/P$ and $F_2 = (f_{2,+} + f_{2,-})/P$. By (3.11) we have $F_1 \in A^2(\Delta \setminus D_a)$, and (3.12) implies that $F_1 \in A^2(D_A)$. The case F_2 is treated in exactly the same way. \square

Chapter 4

Separation of singularities for the Bergman space and exact characterization of the reachable space

In Chapter 3 we have discussed a first approach to the description of the sum of two Bergman spaces defined on intersecting domains (more precisely on intersecting sectors). In particular we have seen that classical harmonic analysis allows to connect this separation of singularities problem with Hilbert transforms giving some inclusion relations. Pushing forward those tools led to the containment of the Smirnov-Zygmund space in the sum.

In this chapter, based on the paper [HO20], we solve completely the separation of singularities problem for Bergman spaces at least for certain geometries. More precisely, we show that if $\mathcal{P} \subset \mathbb{C}$ is a convex polygon which is the intersection of n half planes, then the Bergman space on \mathcal{P} decomposes into the sum of the Bergman spaces on these half planes. We obtain also a theorem for more general convex domains. This will allow us to give an affirmative answer to the HKT-conjecture (3.2) and solves definitively the problem.

4.1 Introduction

4.1.1 Results on separation of singularities in Bergman spaces

In Section 1.3, we presented the separation of singularities problem and its different counter parts for some Banach spaces of analytic functions and Theorem 3.1.1 showed its particular relevance in the setting of control problems. Following this way, we discuss here the separation of singularities problem for the Bergman space. Note that this problem is mentioned explicitly in [BKN18, p.17] without an exact reference. We refer to Section 1.1.3 for the basic properties of these spaces.

Let us now turn to our first set of results concerning separation of singularities in Bergman spaces.

The following weight will play a central role in our study: for $N \in \mathbb{N}$, we write

$$\omega_N(z) = (1 + |z|^{2p})^{-N}. \quad (4.1)$$

We start with a quasi separation of singularities theorem, in the sense that we have to add a weight with decay at infinity. It deals with general open sets Ω_1 and Ω_2 of \mathbb{C} such that $\Omega_1 \setminus \Omega_2$ and $\Omega_2 \setminus \Omega_1$ are far away. Note that this condition already appears in [HN01, Cor. 3.3] as an easy case for solving the separation of singularities problem in H^∞ .

Theorem 4.1.1. *Let $1 < p < \infty$. Let Ω_1 and Ω_2 be open sets of \mathbb{C} such that $\Omega_1 \cap \Omega_2 \neq \emptyset$. If $\text{dist}(\Omega_1 \setminus \Omega_2, \Omega_2 \setminus \Omega_1) > 0$, then we have $A^p(\Omega_1 \cap \Omega_2) \subset A^p(\Omega_1, \omega_1) + A^p(\Omega_2, \omega_1)$.*

The previous theorem is based on a reduction to the $\bar{\partial}$ -equation, as in the modern solution of Aronszajn's theorem (see Theorem 1.3.1), and on Hörmander type L^p -estimates for the $\bar{\partial}$ -equation. This method was already used in [Ors20, Thm 1.2] to prove another kind of weighted separation theorem (see Corollary 4.1.10 below for an improvement of this theorem). Using the fact that polynomials not vanishing on $\bar{\Omega}$ are invertible multipliers of the Bergman space on a bounded domain Ω , will allow us to show our first “real” (i.e. unweighted) separation result for bounded intersections.

Corollary 4.1.2. *Under the same hypotheses as in Theorem 4.1.1, if in addition $\Omega_1 \cap \Omega_2$ is bounded and $\overline{\Omega_1 \cup \Omega_2} \neq \mathbb{C}$, then $A^p(\Omega_1 \cap \Omega_2) = A^p(\Omega_1) + A^p(\Omega_2)$.*

The case when $\text{dist}(\Omega_1 \setminus \Omega_2, \Omega_2 \setminus \Omega_1) = 0$ is more intricate. Let us begin with the simplest configuration of interest for us: Ω_1 and Ω_2 are half planes which intersect perpendicularly. By rotation and translation we can of course reduce the situation to the upper and right half planes: $\mathbb{C}^+ = \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$ and $\mathbb{C}_+ = \{z \in \mathbb{C} \mid \text{Re}(z) > 0\}$. We write $\mathbb{C}^{++} = \mathbb{C}_+ \cap \mathbb{C}^+$ for the resulting quarter plane.

Theorem 4.1.3. *Let $1 < p < \infty$. Then $A^p(\mathbb{C}^+) + A^p(\mathbb{C}_+) = A^p(\mathbb{C}^{++})$.*

The proof of this theorem is strikingly simple when $p = 2$ where it uses only the explicit form of the reproducing kernels of the two half planes. Though the same idea does not apply to arbitrary sectors we can reduce that general situation to right angle sectors which leads to our next result.

Theorem 4.1.4. *Let $1 < p < \infty$. Let H_1, H_2 be two half planes such that $\Sigma := H_1 \cap H_2 \neq \emptyset$ is a sector. Then $A^p(\Sigma) = A^p(H_1) + A^p(H_2)$.*

The main result of this part of the chapter is the separation of singularities problem for n half planes, the intersection of which is a convex polygon.

Theorem 4.1.5. *Let $1 < p < \infty$. Let H_1, H_2, \dots, H_n be half planes such that $\mathcal{P} := \bigcap_{k=1}^n H_k \neq \emptyset$ is a convex polygon. Then $A^p(\mathcal{P}) = \sum_{k=1}^n A^p(H_k)$.*

It is worth mentioning that when Ω is a polygon it is known that Schwarz-Christoffel mappings allow to send the upper half plane conformally onto Ω , so that with (1.8) it is possible to determine the Bergman kernel for $A^2(\Omega)$. However, already for a square, the understanding of the corresponding reproducing kernel is a very non-trivial matter.

Let us consider a special case illustrating the above results:

$$\Omega = \{z = x + iy \in \mathbb{C} : 0 < x < 1, 0 < y < 1\}$$

i.e. Ω is the unit square with lower left corner 0. Let $\Omega_1 = \mathbb{C}^{++}$ and $\Omega_2 = (1+i) - \mathbb{C}^{++}$. Then $\Omega = \Omega_1 \cap \Omega_2$, and Theorems 4.1.4 and 4.1.5 yield the following immediate consequence which will resolve the conjecture on the reachable states of the 1-D heat equation on a finite rod with boundary controls as discussed in the next section.

Corollary 4.1.6. *We have $A^p(\Omega) = A^p(\mathbb{C}^{++}) + A^p((1+i) - \mathbb{C}^{++})$.*

It turns out that we can apply Theorem 4.1.5 to more general domains. More precisely, we will consider non-empty, bounded intersections of convex domains Ω_1 and Ω_2 . Then the boundaries $\partial\Omega_1$ and $\partial\Omega_2$ can meet in single points or along curves. We will assume that there are only finitely many single points and arcs, i.e. $\tilde{\partial}(\partial\Omega_1 \cap \partial\Omega_2)$ is finite (by $\partial\Omega$ we mean the boundary of a two dimensional manifold Ω , and by $\tilde{\partial}E$ the boundary of a one-dimensional manifold E).

Theorem 4.1.7. *Let $1 < p < \infty$. Let Ω_1 and Ω_2 be two open convex sets in \mathbb{C} such that*

(i) $\Omega_1 \cap \Omega_2$ is non-empty and bounded,

(ii) The set $\tilde{\partial}(\partial\Omega_1 \cap \partial\Omega_2)$ is finite.

Then $A^p(\Omega_1 \cap \Omega_2) = A^p(\Omega_1) + A^p(\Omega_2)$.

All these theorems have weighted versions with weights ω_l ($l \in \mathbb{N}$). Let us mention another direct consequence. Recall that the Dirichlet space $\mathcal{D}(\Omega)$ consists of all functions f holomorphic on Ω satisfying $f' \in A^2(\Omega)$ (since the formerly stated results for Bergman spaces work for $1 < p < \infty$, we can also consider the corresponding Dirichlet type spaces which are rather called Besov spaces). Applying the above decompositions to f' and taking anti-derivatives yields the corresponding decompositions in Dirichlet spaces. Note the following general results.

Proposition 4.1.8. *Let Ω_1 and Ω_2 be two simply connected domains in \mathbb{C} such that $\Omega_1 \cap \Omega_2 \neq \emptyset$. If $A^2(\Omega_1 \cap \Omega_2) = A^2(\Omega_1) + A^2(\Omega_2)$ then $\mathcal{D}(\Omega_1 \cap \Omega_2) = \mathcal{D}(\Omega_1) + \mathcal{D}(\Omega_2)$.*

An application of this observation solves the control problem of the heat equation with Neumann boundary control.

Finally, we emphasize that our proofs do not work for $p = 1$. This situation already occurs in Aizenberg's result for the Smirnov space [Aiz14]. While in his work it is the failure of boundedness of the Riesz projection which makes obstruction, here it is the Bergman projection which is not bounded on L^1 . This leads to the following open question.

Question: Is there a positive solution to the separation of singularities problem in E^1 and A^1 ?

4.1.2 The reachable states of the 1-D heat equation

Let us now explain how we apply the preceding results to give a definitive solution to the description of the reachable space of the heat equation.

The key result proved in Chapter 3 is the following.

Theorem 4.1.9. *Let $\Delta = \{z \in \mathbb{C} \mid |\arg(z)| < \pi/4\}$. We have $\text{Ran}\Phi_\tau = A^2(\Delta) + A^2(\pi - \Delta)$.*

By rotation and rescaling, Corollary 4.1.6 immediately yields

$$A^2(D) = A^2(\Delta) + A^2(\pi - \Delta), \quad (4.2)$$

which gives the following characterization of the reachable space.

Corollary 4.1.10. *We have $\text{Ran}\Phi_\tau = A^2(D)$.*

Hence, this proves the HKT-conjecture (3.2). This implies also obviously the following inclusion.

Corollary 4.1.11. *We have $A^2(D) \subset X := W^{-1,2}(0, \pi)$.*

The rest of this chapter is organized as follows. In Section 4.2 we prove the separation of singularities results, and in Section 4.3 we give a more transparent proof to Corollary 4.1.6 and apply the results to several related problems on reachable spaces of the heat equation.

4.2 Proof of theorems

Proof of Theorem 4.1.3. Obviously we only have to show the reverse inclusion. So, let us start with $f \in A^2(\mathbb{C}^{++})$. Using the conformal invariance property (1.8) applied to the kernel on \mathbb{D} introduced in (1.9) and using the conformal map $\varphi : \mathbb{C}^+ \rightarrow \mathbb{D}$,

$$\varphi(z) = \frac{z - i}{z + i},$$

we obtain first the reproducing kernel on \mathbb{C}^+ ,

$$k_\lambda^{\mathbb{C}^+}(z) = \frac{-1}{\pi(z - \bar{\lambda})^2}, \quad \lambda, z \in \mathbb{C}^+.$$

The kernel for \mathbb{C}_+ is deduced from this just by a suitable rotation

$$k_\lambda^{\mathbb{C}_+}(z) = \frac{1}{\pi(z + \bar{\lambda})^2}, \quad \lambda, z \in \mathbb{C}_+.$$

Finally, for the kernel on the quarter plane \mathbb{C}^{++} , use $\varphi : \mathbb{C}^{++} \rightarrow \mathbb{C}_+$, $\varphi(z) = z^2$ to get

$$k_\lambda^{\mathbb{C}^{++}}(z) = \frac{-4z\bar{\lambda}}{\pi(z^2 - \bar{\lambda}^2)^2}, \quad \lambda, z \in \mathbb{C}^{++}$$

An easy computation leads to the following key observation:

$$k_\lambda^{\mathbb{C}^+} + k_\lambda^{\mathbb{C}_+} = k_\lambda^{\mathbb{C}^{++}}, \quad \forall \lambda \in \mathbb{C}^{++}. \quad (4.3)$$

For a function f defined on \mathbb{C}^{++} , we write $S_{\mathbb{C}^+}f$ (resp. $S_{\mathbb{C}_+}f$) for the trivial extension of f by 0 on \mathbb{C}^+ (resp. \mathbb{C}_+) outside \mathbb{C}^{++} , i.e.

$$S_{\mathbb{C}^+}f(z) = \begin{cases} f(z) & \text{if } z \in \mathbb{C}^{++} \\ 0 & \text{if } z \in \mathbb{C}^+ \setminus \mathbb{C}^{++} \end{cases}$$

and correspondingly for $S_{\mathbb{C}_+}f$. Hence, since f was assumed in $A^2(\mathbb{C}^{++})$, for every $\lambda \in \mathbb{C}^{++}$, we have

$$\begin{aligned} f(\lambda) &= \left\langle f, k_\lambda^{\mathbb{C}^{++}} \right\rangle_{L^2(\mathbb{C}^{++})} = \int_{\mathbb{C}^{++}} f(z) \overline{k_\lambda^{\mathbb{C}^{++}}(z)} dA(z) \\ &= \int_{\mathbb{C}^{++}} f(z) \overline{(k_\lambda^{\mathbb{C}^+}(z) + k_\lambda^{\mathbb{C}_+}(z))} dA(z) \\ &= \int_{\mathbb{C}^+} S_{\mathbb{C}^+}f(z) \overline{k_\lambda^{\mathbb{C}^+}(z)} dA(z) + \int_{\mathbb{C}_+} S_{\mathbb{C}_+}f(z) \overline{k_\lambda^{\mathbb{C}_+}(z)} dA(z) \\ &= \left\langle S_{\mathbb{C}^+}f, k_\lambda^{\mathbb{C}^+} \right\rangle_{L^2(\mathbb{C}^+)} + \left\langle S_{\mathbb{C}_+}f, k_\lambda^{\mathbb{C}_+} \right\rangle_{L^2(\mathbb{C}_+)}. \end{aligned}$$

Finally, using the Bergman projection introduced in (1.7), we obtain on \mathbb{C}^{++}

$$f = P_{\mathbb{C}^+}(S_{\mathbb{C}^+}f) + P_{\mathbb{C}_+}(S_{\mathbb{C}_+}f) \in A^2(\mathbb{C}^+) + A^2(\mathbb{C}_+). \quad (4.4)$$

The result follows.

Consider now the case $p \neq 2$. We shall use the following density result.

Lemma 4.2.1. *Let $1 \leq p < \infty$. The set $A^2(\mathbb{C}^{++}) \cap A^p(\mathbb{C}^{++})$ is dense in $A^p(\mathbb{C}^{++})$.*

Proof. As explained in Subsection 1.1.3, for a general open set $\Omega \subset \mathbb{C}$, the density of $A^2(\Omega) \cap A^p(\Omega)$ in $A^p(\Omega)$ is a difficult problem (see [Hed02, Proposition 2.2]) but in our specific case it follows from Proposition 1.1.15 which gives the result for $\Omega = \mathbb{C}^+$. Indeed, moving back and forth between $A^p(\mathbb{C}^+)$ and $A^p(\mathbb{C}^{++})$ via the change of variables formula $T_p F(z) = z^{2/p} F(z^2)$, $F \in A^p(\mathbb{C}^+)$, $z \in \mathbb{C}^{++}$, will produce the desired result. More precisely, in [BBG⁺04, Proposition 2.2] the authors regularize the function F by shifting and multiplying with a suitable function: $F_{\varepsilon, \alpha}(z) = F(z + i\varepsilon)G_\alpha(\varepsilon z)$, where $G_\alpha(z) = (1 - iz)^{-(2+\alpha)}$, $\alpha \geq 0$ and $\varepsilon > 0$. Clearly $F_{\varepsilon, \alpha}(z) \rightarrow F(z)$, when $\varepsilon \rightarrow 0$, for every $z \in \mathbb{C}^+$. As observed in [BBG⁺04, Proposition 1.3], the function $F(z + i\varepsilon)$ is in the Hardy space of the upper half plane $H^p(\mathbb{C}^+)$ which allows an application of the dominated convergence theorem (actually to horizontal p -means of $F_{\varepsilon, \alpha}$) when $\varepsilon \rightarrow 0$. It remains to prove that

$$f_{\varepsilon, \alpha} := T_p F_{\varepsilon, \alpha} = z^{2/p} F(z^2 + i\varepsilon)G_\alpha(\varepsilon z^2) = \frac{z^{2/p-1}}{(1 - i\varepsilon z^2)^\alpha} T_2 F_{\varepsilon, 0}(z)$$

is in $A^2(\mathbb{C}^{++})$. Since $F_{\varepsilon,0} \leq C_\varepsilon/(1 + |z|)^2$, we have $F_{\varepsilon,0} \in A^2(\mathbb{C}^+)$, and so $f_{\varepsilon,\alpha} \in A^2(\mathbb{C}^{++})$ can now be reached by an appropriate choice of α depending on p (note that $T_2 F_{\varepsilon,0}$ is locally bounded at 0, so that only the case $1 < p < 2$ needs consideration of a suitable α). \square

Pick $f \in A^p(\mathbb{C}^{++})$. Since $k_\lambda^{\mathbb{C}^+}$ (resp. $k_\lambda^{\mathbb{C}_+}$) belongs to $L^q(\mathbb{C}^+)$ (resp. $L^q(\mathbb{C}_+)$) for $q > 1$, the right hand side is well-defined for $f \in L^p(\mathbb{C}^{++})$ if $1 \leq p < \infty$. In addition, it is well-known (see Subsection 1.1.3) that $P_{\mathbb{C}^+}$ (resp. $P_{\mathbb{C}_+}$) is bounded from $L^p(\mathbb{C}^+)$ (resp. $L^p(\mathbb{C}_+)$) onto $A^p(\mathbb{C}^+)$ (resp. $A^p(\mathbb{C}_+)$) if and only if $p > 1$. Finally, equality (4.4) holds for all $f \in A^2(\mathbb{C}^{++}) \cap A^p(\mathbb{C}^{++})$ and this last space is dense in $A^p(\mathbb{C}^{++})$ by Lemma 4.2.1, hence it holds also for every $f \in A^p(\mathbb{C}^{++})$ by continuity. The proof is complete. \square

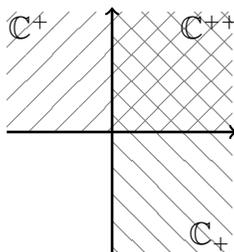


Figure 1: The half planes \mathbb{C}^+ and \mathbb{C}_+ , and their intersection, the quarter plane \mathbb{C}^{++} .

It should be pointed out that the above argument yields a linear bounded separation operator.

Remark 4.2.2. Obviously, the theorem holds for every half plane the intersection of which is a right-angle sector.

We will now move on to the proof of Theorem 4.1.4. For $-\pi \leq a < b \leq \pi$, we denote by Δ_a^b the angular sector $\Delta_a^b = \{z \in \mathbb{C} \mid a < \arg(z) < b\}$.

We can of course reduce the situation to the case $a = 0$, and consider $\Delta_0^b = \mathbb{C}^+ \cap H_1$ where H_1 is the half plane $\Delta_{b-\pi}^b$. While it is very tempting now to apply the same idea above to arbitrary sector Δ_0^b , the magic decomposition of the reproducing kernel on the right-angle sector breaks down. Indeed, with formula (1.8) in mind one can of course explicitly compute the kernel for H_1 which amounts essentially to multiply z and λ by a suitable unimodular constant α (more precisely $\alpha = e^{i(\pi-b)}$) in the expression of $k_\lambda^{\mathbb{C}^+}(z)$. However, the same transformation formula (1.8) applied

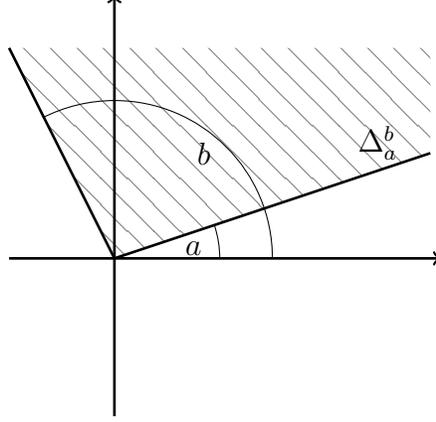


Figure 2: The sector Δ_a^b .

to transform the kernel of \mathbb{C}^+ to that of Δ_0^b involves a power function: $\varphi(z) = z^{\pi/b}$. A computation shows that the kernels of the half planes do not add up to the kernel of the sector.

Proof of Theorem 4.1.4. As already mentioned it is enough to prove the result for $\Sigma = \Delta_0^\theta$, $H_1 = \mathbb{C}^+$ and $H_2 = \mathbb{C}_\theta := \Delta_{-\pi+\theta}^\theta$, with $0 < \theta < \pi$:

$$A^p(\Delta_0^\theta) = A^p(\mathbb{C}^+) + A^p(\Delta_{-\pi+\theta}^\theta). \quad (4.5)$$

The heart of the proof is contained in the following lemma which shows in a way that we can double the opening of the sector.

Lemma 4.2.3. *Let a, b be real numbers such that $-\pi \leq a < b \leq \pi$. Then $A^p(\Delta_a^b) = A^p\left(\Delta_a^{\min(\pi, 2b-a)}\right) + A^p\left(\Delta_{\max(-\pi, 2a-b)}^b\right)$.*

Proof. Let $\varphi : \Delta_a^b \rightarrow \mathbb{C}^{++}$ be the conformal mapping given by $\varphi(z) = (e^{-ia}z)^{\frac{\pi}{2(b-a)}}$ where we have chosen the branch cut to be $(-\infty, 0]$. Let $T : A^p(\mathbb{C}^{++}) \rightarrow A^p(\Delta_a^b)$ be the isometric isomorphism associated with φ , i.e

$$\forall g \in A^p(\mathbb{C}^{++}), \quad Tg = (g \circ \varphi)(\varphi')^{2/p}.$$

Pick $f \in A^p(\Delta_a^b)$. Then $g := T^{-1}f$ belongs to $A^p(\mathbb{C}^{++})$. So, by Theorem 4.1.3, there exist $g_1 \in A^p(\mathbb{C}^+)$ and $g_2 \in A^p(\mathbb{C}_+)$ such that $g = g_1 + g_2$ on \mathbb{C}^{++} . Hence

$$f = Tg = Tg_1 + Tg_2 = (g_1 \circ \varphi)(\varphi')^{2/p} + (g_2 \circ \varphi)(\varphi')^{2/p}.$$

Remark now that the branch cut has been chosen such that φ continues analytically on $\Delta_a^{\min(\pi, 2b-a)}$ and $\Delta_{\max(-\pi, 2a-b)}^b$. Moreover, $\varphi(\Delta_a^{\min(\pi, 2b-a)}) \subset \mathbb{C}^+$ and $\varphi(\Delta_{\max(-\pi, 2a-b)}^b) \subset \mathbb{C}_+$, so that $g_1 \circ \varphi$ and $g_2 \circ \varphi$ are well defined holomorphic functions. Thus, $f \in A^p(\Delta_a^{\min(\pi, 2b-a)}) + A^p(\Delta_{\max(-\pi, 2a-b)}^b)$, which proves the lemma. \square

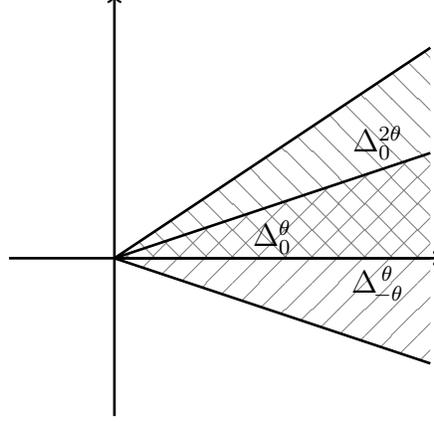


Figure 3: One step of the induction.

We are now in a position to prove the theorem. We start from $A^p(\Delta_0^\theta)$, where we assume for the moment that $\pi/2 < \theta < \pi$. Since $\min(\pi, 2\theta) = \pi$, the lemma yields

$$A^p(\Delta_0^\theta) = A^p(\Delta_0^\pi) + A^p(\Delta_{-\theta}^\theta)$$

so that $f \in A^p(\Delta_0^\theta)$ decomposes as $f = f_1 + f_2$ (considered on Δ_0^θ) with $f_1 \in A^p(\Delta_0^\pi) = A^p(\mathbb{C}^+)$ and $f_2 \in A^p(\Delta_{-\theta}^\theta)$. Since $\pi/2 < \theta$ we have $\Delta_{-\theta}^\theta \supset \Delta_{-\theta-\pi}^\theta$ which yields (4.5).

We will proceed by an inductive application of the lemma. In order to better understand this induction, let us also illustrate the case when $\pi/4 < \theta < \pi/2$. In order to not overcharge notation we will only mention the underlying sectors and not write out the Bergman spaces, see Figure 4. We stop the procedure when we have reached $H_1 = \mathbb{C}_+$ or $H_2 = \Delta_{-\pi}^\theta$.

It is clear from here that after n steps of applications of the lemma, there are 2^n sectors $\Delta_{-k\theta}^{(2^n-k)\theta}$, $k = 0, \dots, 2^n - 1$ (when $(2^n - k)\theta > \pi$ or $k\theta > \pi$ it should be replaced by π).

In the general case, let $N \in \mathbb{N}^*$ be the least natural number such that $2\pi < 2^N\theta$ (in the example above, where $\pi/4 < \theta < \pi/2$, we had to choose $N = 3$). Observe that when $0 \leq k \leq 2^{N-1}$ (which corresponds to one half of the possible k 's), then

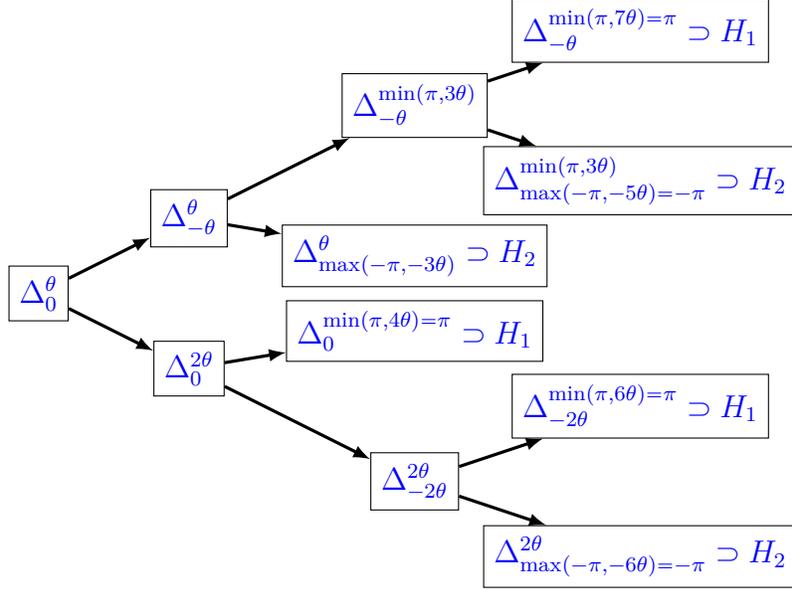


Figure 4: Iterative applications of Lemma 4.2.3 in the decomposition of the sector Δ_0^θ .

$(2^N - k)\theta \geq (2^N - 2^{N-1})\theta = 2^{N-1}\theta > \pi$, so that $\Delta_{-k\theta}^{(2^N - k)\theta} \supset \mathbb{C}^+$, while for $2^{N-1} < k \leq 2^N - 1$, we have $k\theta > \pi$ so that $\Delta_{-k\theta}^{(2^N - k)\theta} \supset \Delta_{-\pi}^\theta \supset \Delta_{\theta - \pi}^\theta$. (We mention again that as soon as $(2^N - k)\theta > \pi$ or $k\theta > \pi$ in the procedure, it should be replaced by π .)

Hence, any function $f \in A^p(\Delta_0^\theta)$ will be decomposed into 2^N functions, one half of which is in $A^p(\mathbb{C}^+)$ and the other one in $A^p(\Delta_{-\pi+\theta}^\theta)$. \square

The next result claims that Theorem 4.1.4 also holds for Bergman spaces with the same weight $\omega_N(z) = (1 + |z|^{2p})^{-N}$ (see (4.1)).

Corollary 4.2.4. *Let H_1, H_2 be two half planes such that $S_\theta := H_1 \cap H_2 \neq \emptyset$ is an angular sector. For any $N \in \mathbb{N}$, we have $A^p(S_\theta, \omega_N) = A^p(H_1, \omega_N) + A^p(H_2, \omega_N)$.*

Proof. The direct inclusion is obvious, let us prove the converse one. Denote by z_0 a complex number such that $\text{dist}(z_0, S_\theta) > 0$ and $P(z) = (z - z_0)^{2N}$. Pick $f \in A^p(S_\theta, \omega_N)$, then $\frac{f}{P}$ belongs to $A^p(S_\theta)$, and by Theorem 4.1.4, there exists $\tilde{f}_1 \in A^p(H_1)$ and $\tilde{f}_2 \in A^p(H_2)$ such that $\frac{f}{P} = \tilde{f}_1 + \tilde{f}_2$. Hence, $f = P\tilde{f}_1 + P\tilde{f}_2 := f_1 + f_2$ with $f_1 \in A^p(H_1, \omega_N)$ and $f_2 \in A^p(H_2, \omega_N)$. \square

We will now prove Theorem 4.1.1 which is an almost-separation of singularities in the simplest case. Denote by $\bar{\partial}$ the Cauchy-Riemann operator $\bar{\partial} = \frac{1}{2}(\frac{d}{dx} + i\frac{d}{dy})$. The main idea is to reduce the problem to a $\bar{\partial}$ -equation and to use Hörmander type L^p -estimates for the solution of the $\bar{\partial}$ -equation. The estimates are certainly well-known to experts but we include a proof for completeness.

Lemma 4.2.5. *Let $1 < p < \infty$. Let $\Omega \subset \mathbb{C}$ be an open connected set such that $\bar{\Omega} \neq \mathbb{C}$. If $f \in L^p(\Omega)$ then there exists a solution u of the equation $\bar{\partial}u = f$ on Ω such that $u \in L^p(\Omega, \omega_1)$.*

Note that we do not look for a solution in $L^p(\Omega)$ but we allow a weight to appear which makes the problem solvable in the setting under consideration. Still, this solution will be sufficient for our purpose.

Proof. The case $p = 2$ is a particular case of the famous Hörmander L^2 -estimates [Hör07, Thm 4.2.1]. From now on, let $1 < p \neq 2 < \infty$.

For bounded Ω the result can be found in [FS91, Sec. 2, p.134] and follows from Young's inequality and properties of the Cauchy kernel $\frac{1}{s-z}$. Indeed, the classical solution (i.e the solution with minimal weighted L^2 -norm) of $\bar{\partial}u = f$ on Ω is given by

$$u(z) = \frac{1}{\pi} \int_{\Omega} \frac{f(s)}{s-z} dA(s)$$

and satisfies

$$\|u\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} \left\| \frac{1}{z} \right\|_{L^1(\Omega-\Omega)} \leq C \|f\|_{L^p(\Omega)} \quad (4.6)$$

where C depends only the diameter of $\Omega - \Omega = \{u - v : u \in \Omega, v \in \Omega\}$.

For the general case, let $z_0 \in \mathbb{C}$ be such that $\text{dist}(z_0, \Omega) > 0$ and set $Q(z) = z - z_0$. Denote by g the function $g = \frac{f}{Q}$. Clearly, u_g satisfies $\bar{\partial}u_g = g$ on Ω if and only if $\bar{\partial}u_f = f$, where $u_f = Qu_g$. It thus suffices to prove the existence of a solution u_g to $\bar{\partial}u_g = g$ such that $u_g \in L^p(\Omega, (1 + |z|^p)^{-1})$. For this, we choose u_g to be the classical solution of $\bar{\partial}u_g = g$ on Ω defined by

$$u_g(z) = u_g^{\Omega}(z) = \frac{1}{\pi} \int_{\Omega} \frac{g(s)}{s-z} dA(s).$$

Let $\zeta \in \Omega$, then for every $z \in \Omega_{\zeta} := D(\zeta, 1) \cap \Omega$ we have

$$u_g(z) = \int_{\Omega \setminus D(\zeta, 2)} \frac{g(s)}{s-z} dA(s) + \int_{\Omega \cap D(\zeta, 2)} \frac{g(s)}{s-z} dA(s) := u_g^{\Omega \setminus D(\zeta, 2)}(z) + u_g^{\Omega \cap D(\zeta, 2)}(z).$$

Considering $u_g^{\Omega \cap D(\zeta, 2)}$ as the solution to the $\bar{\partial}$ -problem on the bounded domain $\Omega \cap D(\zeta, 2)$, we obtain by (4.6)

$$\begin{aligned} \|u_g^{\Omega \cap D(\zeta, 2)}\|_{L^p(\Omega_\zeta)} &\leq \|u_g^{\Omega \cap D(\zeta, 2)}\|_{L^p(\Omega \cap D(\zeta, 2))} \\ &\leq C \|g\|_{L^p(\Omega \cap D(\zeta, 2))} \\ &\leq \frac{C}{1 + |\zeta|} \|f\|_{L^p(\Omega \cap D(\zeta, 2))} \\ &\leq \frac{C}{1 + |\zeta|} \end{aligned}$$

where C is essentially given by $\|\frac{1}{z}\|_{L^1(D(\zeta, 2) - D(\zeta, 2))} \leq \|\frac{1}{z}\|_{L^1(D(0, 4))}$ and thus independent on ζ .

For $u_g^{\Omega \cap D(\zeta, 2)}$, Hölder's inequality gives for all $z \in \Omega_\zeta$,

$$|u_g^{\Omega \cap D(\zeta, 2)}(z)| \leq \|f\|_{L^p(\Omega \cap D(\zeta, 2))} \left\| \frac{1}{(z - \cdot)Q} \right\|_{L^{p'}(\Omega \cap D(\zeta, 2))} \leq \begin{cases} \frac{C}{1 + |\zeta|} & \text{if } p < 2 \\ C & \text{if } p > 2 \end{cases}$$

where p' is the conjugate exponent of p , and C is independent on ζ .

Putting the above estimates together, and observing that $1 + |\zeta| \simeq 1 + |z|$ for $z \in D(\zeta, 1)$ (the symbol “ \simeq ” means that both quantities are comparable up to multiplicative constants), we get for every $\zeta \in \Omega$,

$$\int_{\Omega_\zeta} |u_g(z)|^p dA(\zeta) \leq \frac{C^p}{(1 + |\zeta|)^p} + \pi \begin{cases} \frac{C^p}{(1 + |\zeta|)^p} & \text{if } p < 2 \\ C^p & \text{if } p > 2 \end{cases}$$

where the constants again do not depend on ζ . Hence, covering suitably Ω by Ω_ζ 's, we get

$$\int_{\Omega} \frac{|u_g(z)|^p}{1 + |z|^p} dA(z) \leq \begin{cases} \int_{\Omega} \frac{C'}{1 + |z|^{2p}} dA(z) & \text{if } p < 2 \\ \int_{\Omega} \frac{C'}{1 + |z|^p} dA(z) & \text{if } p > 2 \end{cases} < \infty.$$

□

Restating the previous lemma in the spirit of Hörmander's result is to say that the solution u of $\bar{\partial}u = f$ satisfies

$$\int_{\Omega} |u(z)|^p (1 + |z|^2)^p dA(z) \leq C_p \int_{\Omega} |f(z)|^p dA(z).$$

In particular, this is coherent in the power of $(1 + |z|^2)$ with Hörmander's result which yields exactly the above estimate for $p = 2$ (with a simpler proof). However,

our argument does not work for $p = 2$.

The proof below follows essentially the argument given in the proof of Theorem 1.3.1 combined with the L^p -estimates from the previous lemma.

Proof of Theorem 4.1.1. Pick $f \in A^p(\Omega_1 \cap \Omega_2)$. Take χ a bounded C^∞ -function such that $\chi = 1$ on $\Omega_1 \setminus \Omega_2$ and $\chi = 0$ on $\Omega_2 \setminus \Omega_1$. Since $\text{dist}(\Omega_1 \setminus \Omega_2, \Omega_2 \setminus \Omega_1) > 0$, we can assume that $\nabla \chi$ is uniformly bounded (and vanishes outside $\Omega_1 \cap \Omega_2$). So we can define $h_1 = f(1 - \chi)$ on Ω_1 and $h_2 = f\chi$ on Ω_2 . Using the analyticity of f , we have $\bar{\partial}h_1 = -f\bar{\partial}\chi = -\bar{\partial}h_2$ on $\Omega_1 \cap \Omega_2$, which implies the existence of a C^∞ -continuation v such that $v = \bar{\partial}h_1$ on Ω_1 and $v = -\bar{\partial}h_2$ on Ω_2 . Since $\bar{\partial}\chi$ is bounded, v belongs to $L^p(\Omega_1 \cup \Omega_2)$ and by Lemma 4.2.5, there exists $u \in L^p(\Omega_1 \cup \Omega_2, \omega_1)$ such that $\bar{\partial}u = v$. Finally, defining $f_1 = h_1 - u$ on Ω_1 and $f_2 = h_2 + u$ on Ω_2 , we obtain $f = f_1 + f_2$ on $\Omega_1 \cap \Omega_2$ and $f_i \in A^p(\Omega_i, \omega_1)$ by definition of u . The proof is complete. \square

Now, a multiplier argument gives us a general separation of singularities result.

Proof of Corollary 4.1.2. Let z_0 be such that $\text{dist}(z_0, \Omega_1 \cup \Omega_2) > 0$, and write $P(z) = (z - z_0)^2$. Pick $f \in A^p(\Omega_1 \cap \Omega_2)$, then $g := Pf$ belongs also to $A^p(\Omega_1 \cap \Omega_2)$ since P is bounded on $\Omega_1 \cap \Omega_2$. So, by Theorem 4.1.1, $g = g_1 + g_2$ with $g_1 \in A^p(\Omega_1, \omega_1)$ and $g_2 \in A^p(\Omega_2, \omega_1)$. Therefore $f = \frac{g}{P}$ belongs to $A^p(\Omega_1) + A^p(\Omega_2)$. \square

With the same argument used in Corollary 4.2.4, we obtain the following weighted version of Theorem 4.1.1.

Corollary 4.2.6. *Let Ω_1 and Ω_2 be open sets of \mathbb{C} such that $\Omega_1 \cap \Omega_2 \neq \emptyset$. If $\text{dist}(\Omega_1 \setminus \Omega_2, \Omega_2 \setminus \Omega_1) > 0$, then for any $l \in \mathbb{N}$, we have $A^p(\Omega_1 \cap \Omega_2, \omega_l) \subset A^p(\Omega_1, \omega_{l+1}) + A^p(\Omega_2, \omega_{l+1})$.*

Before proving Theorem 4.1.5 we need an auxiliary result on decompositions of Bergman spaces on unbounded intersections of half planes. This is provided by the following lemma.

Lemma 4.2.7. *Let $n \geq 2$ and H_1, \dots, H_n be half planes such that $\Omega := \bigcap_{k=1}^n H_k$ is non-empty, convex and unbounded. Then for any $l \in \mathbb{N}$, we have*

$$A^p(\Omega, \omega_l) \subset \sum_{k=1}^n A^p(H_k, \omega_{l+(n-1)}).$$

Proof. Pick $f \in A^p(\Omega, \omega_l)$. We can assume that Ω has exactly n sides, otherwise just add the zero function. Let us prove the result by induction. If $n = 2$, Ω is either a sector or a strip. In the first case, the result follows from Corollary 4.2.4. When Ω is a strip, then $\text{dist}(H_1 \setminus \Omega, H_2 \setminus \Omega) > 0$ and the result follows from Corollary 4.2.6. So the base case of the induction is established.

Assume now that the lemma is true for every $2 \leq n < N$. We shall prove it for N . Since Ω is convex and $N \geq 3$, the boundary $\partial\Omega$ is path connected. Without loss of generality we can assume that the half planes H_1, \dots, H_N are ordered such that the sides S_1, \dots, S_N of $\partial\Omega$ satisfy $S_i \subset \partial H_i$, $S_i \cap S_{i+1} \neq \emptyset$ ($1 \leq i \leq N-1$), and S_1 and S_N are the unbounded sides of $\partial\Omega$. Then, writing $\Omega = \left(\bigcap_{i=1}^{N-1} H_i\right) \cap$

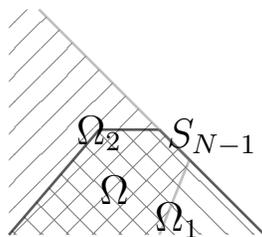


Figure 5: The decomposition of Ω as the intersection of Ω_1 and Ω_2 .

$(H_{N-1} \cap H_N) := \Omega_1 \cap \Omega_2$ (see Figure 5), we obtain $f = f_1 + f_2$ with $f_1 \in A^p(\Omega_1, \omega_{l+1})$ and $f_2 \in A^p(\Omega_2, \omega_{l+1})$. Indeed, this follows from Corollary 4.2.6 since Ω is convex, and $\text{dist}(\Omega_1 \setminus \Omega_2, \Omega_2 \setminus \Omega_1) = |S_{N-1}| > 0$. Here $|S_i|$ means the length of S_i . Finally, using the induction hypothesis on Ω_1 and Ω_2 , we conclude the inductive step, which proves the lemma. \square

Proof of Theorem 4.1.5. Again we will prove the result by induction. If $n = 3$, \mathcal{P} is a triangle and we make the decomposition $\mathcal{P} = \Sigma_3 \cap \Omega$ where Σ_3 is the angular sector $H_1 \cap H_2$ and Ω is an unbounded domain, as in Figure 6. We will again denote by S_i the sides of \mathcal{P} and $S_i \subset \partial H_i$. As previously, $\text{dist}(\Sigma_3 \setminus \Omega, \Omega \setminus \Sigma_3) > 0$, so by Corollary 4.1.2 we have

$$A^p(\mathcal{P}) = A^p(\Sigma_3) + A^p(\Omega). \quad (4.7)$$

(Observe that the corollary does not require any convexity assumption, and no weights appear here.) Hence, it remains to decompose Ω which will be done writing $\Omega = \Theta_1 \cap \Theta_2$, as in Figure 6. Again, $\text{dist}(\Theta_1 \setminus \Omega, \Theta_2 \setminus \Omega) > 0$, so by Theorem 4.1.4, we have

$$A^p(\Omega) \subset A^p(\Theta_1, \omega_1) + A^p(\Theta_2, \omega_1). \quad (4.8)$$

Define $\Sigma_2 = H_1 \cap H_3$ and $\Sigma_1 = H_2 \cap H_3$. Clearly $\Sigma_k \subset \Theta_k$, $k = 1, 2$, and so

$$A^p(\Theta_k, \omega_1) \subset A^p(\Sigma_k, \omega_1), \quad k = 1, 2. \quad (4.9)$$

From (4.7)-(4.9), we obtain $A^p(\mathcal{P}) \subset \sum_{k=1}^3 A^p(\Sigma_k, \omega_1)$. It can be checked that $\overline{\bigcup_{k=1}^3 \Sigma_k} \neq \mathbb{C}$, so that there exists z_0 with $\text{dist}(z_0, \overline{\bigcup_{k=1}^3 \Sigma_k}) > 0$. Define $P(z) = (z - z_0)^2$. Then, for $f \in A^p(\mathcal{P})$, the function $g = Pf$ is also in $A^p(\mathcal{P})$ (multiplication by P is actually a norm conserving operation on $A^p(\mathcal{P})$), which implies that g can be written as $g = g_1 + g_2 + g_3$ with $g_i \in A^p(\Sigma_i, \omega_1)$. Thus, $f = \frac{g}{P} = \sum_{k=1}^3 \frac{g_k}{P}$. Hence, setting $f_k = \frac{g_k}{P}$, and since $\|g_k\|_{A^p(\Sigma_k, \omega_1)}$ is comparable to $\|f_k\|_{A^p(\Sigma_k)} = \|g_k/P\|_{A^p(\Sigma_k)}$, we obtain that f belongs to $\sum_{k=1}^3 A^p(\Sigma_k)$. This means that $A^p(\mathcal{P}) = \sum_{k=1}^3 A^p(\Sigma_k)$ and using Theorem 4.1.4, we obtain the base case of the induction.

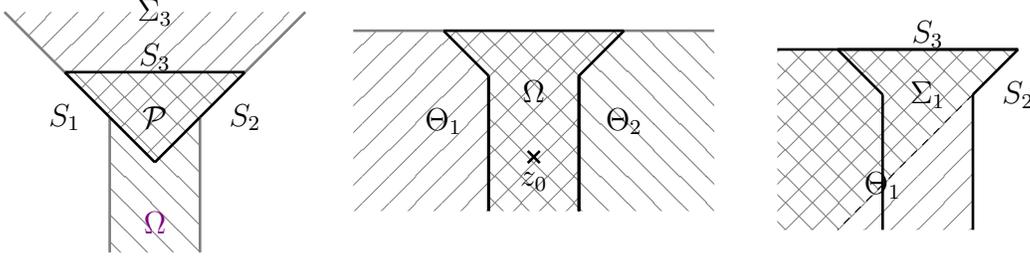


Figure 6: Decomposition of a triangle.

Now, assume that the result is true for every $3 \leq k < n$. We have to prove it for n . There are two cases. First suppose that \mathcal{P} has two non-consecutive non-parallel sides (see Figure 7). Denote by S_1, \dots, S_n its ordered sides ($S_i \cap S_{i+1} \neq \emptyset$, $S_n \cap S_1 \neq \emptyset$) and H_1, \dots, H_n the corresponding half planes. Let S_i and S_j be two non-parallel sides with $j \geq i + 2$. Write $\Omega_1 = \bigcap_{k=i}^j H_k$ and $\Omega_2 = \bigcap_{k \notin (i,j)} H_k$ (we denote $[i, j] = \{i, i+1, \dots, j\}$ and $(i, j) = \{i+1, \dots, j-1\}$). Observe that Ω_1 and Ω_2 have the sides S_i and S_j in common. It is clear that $\mathcal{P} = \Omega_1 \cap \Omega_2$ and $\text{dist}(\Omega_1 \setminus \Omega_2, \Omega_2 \setminus \Omega_1) = \min(|S_i|, |S_j|) > 0$. So, by Corollary 4.1.2, we have $A^p(\mathcal{P}) = A^p(\Omega_1) + A^p(\Omega_2)$. Since S_i and S_j are non-parallel, one of the sets Ω_i is a polygon and the other one is unbounded. Let us assume that Ω_1 is a polygon. Using Lemma 4.2.7 with $l = 0$, we obtain $A^p(\Omega_2) \subset \sum_{k \notin (i,j)} A^p(H_k, \omega_{n-1})$, and hence

$$A^p(\mathcal{P}) \subset A^p(\Omega_1) + \sum_{k \notin (i,j)} A^p(H_k, \omega_{n-1}),$$

where Ω_1 is a polygon with lower degree. By the induction hypothesis, $A^p(\Omega_1)$ -functions decompose in the required way into $A^p(H_k)$ -functions where $k \in [i, j]$.

In order to manage the second term, we need to get rid of the weight ω_{n-1} . This will again be done using the multiplication by an appropriate polynomial vanishing neither on H_k , $k \notin (i, j)$ nor on Ω_1 . For that, observe that Ω_2 is an unbounded convex domain, and S_i and S_j are non-parallel. We claim that there exists $z_0 \notin \overline{\bigcup_{k \notin (i, j)} H_k \cup \Omega_1}$. Indeed, for every $k \notin (i, j)$, $H_k \subset H_i \cup H_j$, and moreover $\Omega_1 \subset H_i \cup H_j$ (convexity comes into play here). Since S_i and S_j are non parallel, $H_i \cup H_j \neq \mathbb{C}$, and it is enough to pick $z_0 \notin H_i \cup H_j$ with $\text{dist}(z_0, H_i \cup H_j) > 0$. The polynomial we are looking for is $P(z) = (z - z_0)^{2(n-1)}$. Pick now $f \in A^p(\mathcal{P})$. As in the previous corollary, writing $g = Pf$, we have $g \in A^p(\mathcal{P})$ and so, by the reasoning above, $g = g_1 + \sum_{k \notin (i, j)} g_{2,k}$ with $g_1 \in A^p(\Omega_1)$ and $g_{2,k} \in A^p(H_k, \omega_{n-1})$, and hence $f_{2,k} = g_{2,k}/P \in A^p(H_k)$. Also, since multiplication (and division) by P is a norm conserving operation on $A^p(\Omega_1)$, we have $f_1 = g_1/P \in A^p(\Omega_1)$, and by the induction hypothesis (applied to Ω_1) f_1 decomposes into a sum of $A^p(H_k)$ -functions, $k \in [i, j]$:

$$f = f_1 + \sum_{k \notin (i, j)} \frac{g_{2,k}}{P} \in \sum_{k \in [i, j]} A^p(H_k) + \sum_{k \notin (i, j)} A^p(H_k).$$

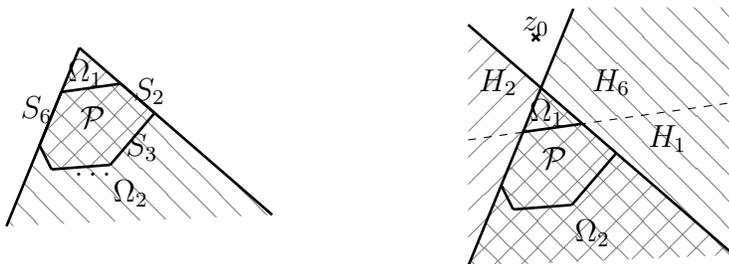


Figure 7: Decomposition of \mathcal{P} as $\Omega_1 \cap \Omega_2$ with a possible numbering of S_i , and the domain $H_i \cup H_j$. (S_2 and S_6 are two non-consecutive non-parallel sides.)

Secondly, suppose that *all* non-consecutive sides of \mathcal{P} are parallel. Then $n = 4$ and \mathcal{P} is a parallelogram. We treat this case directly. As in the first case, we denote by S_1, \dots, S_4 the consecutive sides of \mathcal{P} , H_1, \dots, H_4 the corresponding half planes and $\Sigma_k = H_k \cap H_{k+1}$ (with $H_5 = H_1$) the angular sectors. We make the same decomposition : $A^p(\mathcal{P}) = A^p(\Omega_1) + A^p(\Omega_2)$ where $\Omega_1 = H_1 \cap H_2 \cap H_3$ and $\Omega_2 = H_3 \cap H_4 \cap H_1$. This time Ω_1 and Ω_2 are both unbounded and $\text{dist}(\Omega_1 \setminus \Omega_2, \Omega_2 \setminus \Omega_1) > 0$. Using Theorem 4.1.1 we get $A^p(\mathcal{P}) \subset A^p(\Omega_1, \omega_1) + A^p(\Omega_2, \omega_1)$. Next we apply Lemma 4.2.7 to each of these spaces to get $A^p(\mathcal{P}) \subset \sum_{k=1}^4 A^p(H_k, \omega_3)$ (notice that the weight is given by $1 + (n - 1) = 3$ since we intersect 3 half planes). Observe that in this case $H_1 \cup H_2 \cup H_3 = \mathbb{C}$ (and similarly for H_3, H_4, H_1), so that at this step we cannot

find a z_0 allowing the multiplication trick by a polynomial. Instead, we use Corollary 4.2.4, to get sectors:

$$A^p(\mathcal{P}) \subset \sum_{k=1}^4 A^p(H_k, \omega_3) = A^p(\Sigma_1, \omega_3) + A^p(\Sigma_3, \omega_3),$$

where $\Sigma_1 = H_1 \cap H_2$ and $\Sigma_3 = H_3 \cap H_4$. Now, there exists a point z_0 which is not in $\overline{\Sigma_1 \cup \Sigma_3}$, so that we can use the multiplication trick as in the first case: $f \in A^p(\mathcal{P})$ implies $g = fP \in A^p(\mathcal{P})$ splits into $g = g_1 + g_2$ with $g_1 \in A^p(\Sigma_1, \omega_3)$, $g_2 \in A^p(\Sigma_3, \omega_3)$. With the suitable choice of P we compensate again the weight so that $f_1 = g_1/P \in A^p(\Sigma_1)$ and $f_2 = g_2/P \in A^p(\Sigma_3)$. Hence $A^p(\mathcal{P}) \subset A^p(\Sigma_1) + A^p(\Sigma_3)$, and we conclude using Theorem 4.1.4. \square

Let us indicate how the above results apply to more general separation of singularities problem in Bergman spaces, and not only on polygons.

Proof of Theorem 4.1.7. Assume that $\Omega_1 \not\subset \Omega_2$ and $\Omega_2 \not\subset \Omega_1$, otherwise the problem is trivial. Let us make some additional observations.

- By assumption, Ω_1 and Ω_2 are convex, so that $\Omega_1 \cap \Omega_2$ is also convex (and non-empty), hence $\partial(\Omega_1 \cap \Omega_2)$ is the image Γ of a Jordan curve γ .
- $\partial\Omega_1 \cap \partial\Omega_2$ includes at least two points and if we write $\tilde{\partial}(\partial\Omega_1 \cap \partial\Omega_2) := \{z_1, \dots, z_n\}$ ($n \geq 2$), we have

$$\partial(\Omega_1 \cap \Omega_2) = \Gamma = \bigcup_{k=1}^n \Gamma_{z_k, z_{k+1}}, \quad (z_{n+1} := z_1)$$

where $\Gamma_{z_k, z_{k+1}}$ is the path in $\partial(\Omega_1 \cap \Omega_2)$ connecting z_k to z_{k+1} . Moreover, $\Gamma_{z_k, z_{k+1}} \subset \partial\Omega_1$ or $\Gamma_{z_k, z_{k+1}} \subset \partial\Omega_2$. Note that it can happen that $\Gamma_{z_k, z_{k+1}} \subset \partial\Omega_1 \cap \partial\Omega_2$.

- By convexity, $[z_k, z_{k+1}] \subset \overline{\Omega_1 \cap \Omega_2}$.

Let us start assuming $n \geq 3$. Pick $f \in A^p(\Omega_1 \cap \Omega_2)$. Write

$$\mathcal{P}_{z_1, \dots, z_n} := \text{Int}(\text{Conv}(\{z_1, \dots, z_n\})) = \bigcap_{k=1}^n H_{z_k, z_{k+1}}$$

where $H_{z_k, z_{k+1}}$ is the half plane associated with the side $[z_k, z_{k+1}]$ of the polygon (see Figure 8), and note that by convexity $\mathcal{P}_{z_1, \dots, z_n} \subset \Omega_1 \cap \Omega_2$. So $f \in A^p(\mathcal{P}_{z_1, \dots, z_n})$ and by

Theorem 4.1.5 there exist f_1, \dots, f_n such that $f_k \in A^p(H_{z_k, z_{k+1}})$ and $f = \sum_{k=1}^n f_k$. It remains to prove that each f_k belongs either to $A^p(\Omega_1)$ or to $A^p(\Omega_2)$. Let $i_k \in \{1, 2\}$ be the index such that $\Gamma_{z_k, z_{k+1}} \subset \partial\Omega_{i_k}$ (when $\Gamma_{z_k, z_{k+1}}$ is in both boundaries, we can pick either of the values 1 or 2 for i_k). Two cases may occur. If $\Gamma_{z_k, z_{k+1}} = [z_k, z_{k+1}]$, then, by convexity, $\Omega_{i_k} \subset H_{z_k, z_{k+1}}$ and the result follows: $f_k \in A^p(H_{z_k, z_{k+1}}) \subset A^p(\Omega_{i_k})$. So, assume that $\Gamma_{z_k, z_{k+1}} \neq [z_k, z_{k+1}]$ and write

$$O_{z_k, z_{k+1}} := \bigcap_{j \neq k} H_{z_j, z_{j+1}} \cap \Omega_{i_k} \subset \Omega_1 \cap \Omega_2$$

(one side of the polygon has been replaced by the arc $\Gamma_{z_k, z_{k+1}}$). Since $f_k \in A^p(H_{z_k, z_{k+1}})$ it is obviously in $A^p(H_{z_k, z_{k+1}} \cap \Omega_{i_k})$. We claim that it extends to a function in $A^p(\Omega_{i_k})$. By definition $O_{z_k, z_{k+1}} \subset H_{z_j, z_{j+1}}$, $j \neq k$. Therefore, we have

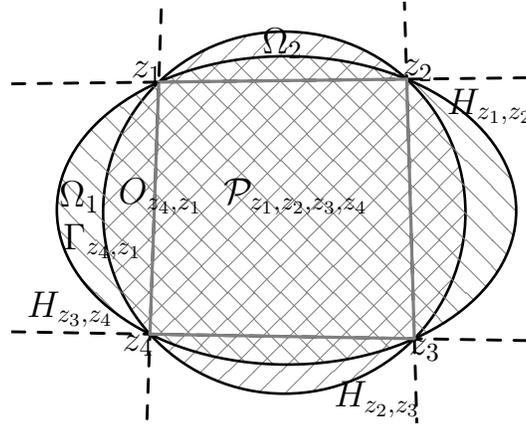


Figure 8: Intersection of two convex open sets.

$$f_k = \underbrace{f}_{\in A^p(\Omega_1 \cap \Omega_2) \subset A^p(O_{z_k, z_{k+1}})} - \sum_{j \neq k} \underbrace{f_j}_{\in A^p(H_{z_j, z_{j+1}}) \subset A^p(O_{z_k, z_{k+1}})} \in A^p(O_{z_k, z_{k+1}}).$$

Thus $f_k \in A^p(H_{z_k, z_{k+1}} \cup O_{z_k, z_{k+1}}) \subset A^p(\Omega_{i_k})$.

Finally, assume that $n = 2$. It is sufficient to add a point z_3 which belongs to $\partial(\Omega_1 \cap \Omega_2) \setminus \{z_1, z_2\}$, and construct $\mathcal{P}_{z_1, z_2, z_3}$ as before. The rest of the proof is dealt with as in the previous case. \square

It is worth mentioning that convex sets $\Omega = \Omega_1 \cap \Omega_2$ with $\tilde{\partial}(\partial\Omega_1 \cap \partial\Omega_2)$ infinite can be constructed easily.

4.3 Reachable states of the heat equation

As already discussed in the first section, the result of Chapter 3, which states that

$$\text{Ran}\Phi_\tau = A^2(\Delta) + A^2(\pi - \Delta) \quad (4.10)$$

together with our Corollary 4.1.6 yield the final characterization of the reachable states of the 1-D heat equation with L^2 -boundary controls

$$\text{Ran}\Phi_\tau = A^2(D), \quad (4.11)$$

as stated in Corollary 4.1.10.

In this section we would like to make some additional observations on this and related control problems. Also, the general arguments presented in Section 4.2 leading to Theorem 4.1.5 might hide the very simple ideas which are actually behind Corollary 4.1.6 and thus leading to (4.11). For this reason, we would like to present here a more direct proof of Corollary 4.1.6 based on Theorem 4.1.1, Corollary 4.1.2 and Theorem 4.1.3 (case $p = 2$).

4.3.1 A direct proof to Corollary 4.1.10

As already mentioned several times (see for instance Remark 4.2.2), the decomposition (4.10) is invariant by rotation and dilation. So, writing $\Sigma_2 := (1+i) - \mathbb{C}^{++}$ and denoting by D' the square $D' := \mathbb{C}^{++} \cap \Sigma_2$ (see Figure 9), it is enough to show that $A^2(D') = A^2(\mathbb{C}^{++}) + A^2(\Sigma_2)$. Let $z_0 \in \mathbb{C} \setminus (\overline{\mathbb{C}^{++} \cup \Sigma_2})$ and $P(z) = (z - z_0)^2$ which is bounded and non-vanishing on $\overline{D'}$, so that multiplication by P is an isomorphism on $A^p(D')$. In particular $f \in A^2(D')$ if and only if $g = Pf \in A^2(D')$.

The proof decomposes into 3 steps. We have already met some arguments in the proof of Theorem 4.1.5 when considering the case of a parallelogram \mathcal{P} .

Step 1: Let S_1 and S_2 be the half strips $S_1 = \{z = x + iy \in \mathbb{C} \mid y > 0, 0 < x < 1\}$ and $S_2 = (1+i) - S_1$. Note that $D' = S_1 \cap S_2$ (see Figure 10). Since D' is bounded and $\text{dist}(S_1 \setminus D', S_2 \setminus D') > 0$, by Corollary 4.1.2, there exist $g_1 \in A^2(S_1)$ and $g_2 \in A^2(S_2)$ such that $g = g_1 + g_2$ on D' . This step is complete.

Step 2: Denote by $Q_{1,1}$ and $Q_{1,2}$ (respectively $Q_{2,1} = \Sigma_2$ and $Q_{2,2}$) the left and right quarter planes the intersection of which is S_1 (resp. S_2) (see Figure 11). We can repeat the same argument as in the previous step (applying Theorem 4.1.1) and obtain $g_1 = g_{1,1} + g_{1,2}$ (resp. $g_2 = g_{2,1} + g_{2,2}$) with $g_{1,i} \in A^2(Q_{1,i}, \omega_1)$ (resp.

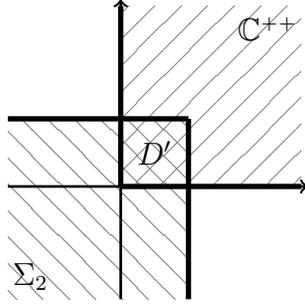


Figure 9: The decomposition of D' as the intersection of \mathbb{C}^{++} and Σ_2 .

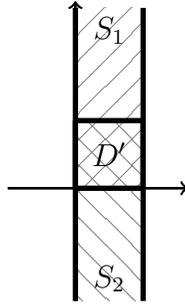


Figure 10: Step 1 — Decomposition of D' as the intersection of S_1 and S_2 .

$$g_{2,i} \in A^2(Q_{2,i}, \omega_1).$$

Step 3: Remark that $Q_{1,2} = \mathbb{C}^{++}$ and $Q_{2,1} = \Sigma_2$. So that we already have $g_{1,2} + g_{2,1} \in A^2(\mathbb{C}^{++}, \omega_1) + A^2(\Sigma_2, \omega_1)$, solving the problem for $g_{1,2}$ and $g_{2,1}$. Let us show the same for $g_{1,1}$ and $g_{2,2}$. Denote by $H_{1,1,1} = \mathbb{C}^+$ the upper half plane and by $H_{1,1,2} = 1 - \mathbb{C}_+$ the left half plane translated by 1 (resp. $H_{2,2,1}$ the lower half plane translated by 1 and $H_{2,2,2}$ the right half plane) the intersection of which is $Q_{1,1}$ (resp. $Q_{2,2}$), see Figure 12. By Corollary 4.2.4 with Remark 4.2.2, $g_{1,1}$ belongs to $A^2(H_{1,1,1}, \omega_1) + A^2(H_{1,1,2}, \omega_1)$ and $g_{2,2}$ belongs to $A^2(H_{2,2,1}, \omega_1) + A^2(H_{2,2,2}, \omega_1)$.

Now, observing that $Q_{1,2} = \mathbb{C}^{++} \subset H_{1,1,1}$, $\mathbb{C}^{++} \subset H_{2,2,2}$, $\Sigma_2 \subset H_{1,1,2}$ and $\Sigma_2 \subset H_{2,2,1}$, we obtain that $g_{1,1}$ belongs to $A^2(\mathbb{C}^{++}, \omega_1) + A^2(\Sigma_2, \omega_1)$ and the same is true for $g_{2,2}$. Thus $g \in A^2(\mathbb{C}^{++}, \omega_1) + A^2(\Sigma_2, \omega_1)$. Finally, by definition of P , $f = \frac{g}{P}$ belongs to $A^2(\mathbb{C}^{++}) + A^2(\Sigma_2)$, which concludes the proof. \square



Figure 11: Step 2 — The decompositions of S_1 and S_2 .



Figure 12: Step 3 — The decompositions of $Q_{1,1}$ and $Q_{2,2}$.

4.3.2 Remarks on related control problems

In this subsection we discuss some related control problems for the heat equation. The first one is mentioned in [KNT19] and the others in [HKT20].

Smooth boundary control.

In [KNT19], Kellay, Normand and Tucsnak gave a characterization of the reachable space when the control is smooth. Let $s \in \mathbb{N}$ and denote by $W_L^{s,2}((0, \tau), \mathbb{C}^2)$ the Sobolev type space given by

$$W_L^{s,2}((0, \tau), \mathbb{C}^2) := \left\{ v \in L^2((0, \tau), \mathbb{C}^2) \mid \begin{array}{l} \forall 1 \leq k \leq s, \frac{d^k v}{dt^k} \in L^2((0, \tau), \mathbb{C}^2) \\ \text{and } \forall 0 \leq k \leq s-1, \frac{d^k v}{dt^k}(0) = 0 \end{array} \right\}.$$

Write also

$$(A^2(\Delta) + A^2(\pi - \Delta))^{(s)} := \{ f \in A^2(\Delta) + A^2(\pi - \Delta) \mid \forall 1 \leq k \leq s, f^{(2k)} \in A^2(\Delta) + A^2(\pi - \Delta) \}$$

Then, combining their propositions 5.1 and 7.1, they proved that

$$\text{Ran} \left(\Phi_{\tau|W_L^{s,2}((0, \tau), \mathbb{C}^2)} \right) = (A^2(\Delta) + A^2(\pi - \Delta))^{(s)}. \quad (4.12)$$

Similarly as above, we let $(A^2)^{(s)}(D)$ be the space of functions $f \in A^2(D)$ such that $f^{(2k)} \in A^2(D)$ for all $1 \leq k \leq s$. Using Corollary 4.1.10, result (4.12) immediately leads to

Corollary 4.3.1. *We have $\text{Ran} \left(\Phi_{\tau|W_L^{s,2}((0,\tau),\mathbb{C}^2)} \right) = (A^2)^{(s)}(D)$.*

Neumann conditions.

It is also possible to ask for a description of the reachable space for other types of boundary conditions. For Neumann boundary conditions, the result follows directly from Corollary 4.1.10 and a trick used in [HKT20]. We remind that we are searching the reachable space of the equation

$$\begin{cases} \frac{\partial y}{\partial t}(t, x) - \frac{\partial^2 y}{\partial x^2}(t, x) = 0 & t > 0, x \in (0, \pi), \\ \frac{\partial y}{\partial x}(t, 0) = u_0(t), \quad \frac{\partial y}{\partial x}(t, \pi) = u_\pi(t) & t > 0, \\ y(0, x) = g(x) & x \in (0, \pi), \end{cases} \quad (4.13)$$

Again, for every initial condition $g \in L^2(0, \pi)$ and every control function $u = (u_0, u_\pi) \in L^2(\mathbb{R}_+, \mathbb{C})$, the previous equation (4.13) admits a unique solution $y \in C([0, +\infty), L^2(0, \pi))$. As for Dirichlet boundary control, the null-controllability in any time $\tau > 0$ holds in this case, and so the invariance of the reachable space with respect to the initial condition and to the time also. We denote by Φ_τ^{NN} the controllability map associated to this equation. Let $\mathcal{D}(D)$ be the Dirichlet space on D , which consists of all holomorphic functions F in D such that $F' \in A^2(D)$. As noted in [HKT20, Prop. 5.2], w is a solution of (4.13) if and only if $y = \frac{\partial w}{\partial x}$ is a solution of (HE) with initial condition $f = g'$. Thus the next result follows.

Corollary 4.3.2. *We have $\text{Ran} \Phi_\tau^{NN} = \mathcal{D}(D)$.*

Dirichlet condition at one end.

Now, we are looking for the reachable space for Dirichlet boundary condition at one end on the interval i.e. the reachable space of the equation

$$\begin{cases} \frac{\partial y}{\partial t}(t, x) - \frac{\partial^2 y}{\partial x^2}(t, x) = 0 & t > 0, x \in (0, \pi), \\ y(t, 0) = 0, \quad y(t, \pi) = u_\pi(t) & t > 0, \\ y(0, x) = g(x) & x \in (0, \pi). \end{cases} \quad (4.14)$$

Note that if y is a solution of (4.14), then its odd extension \tilde{y} to $[-\pi, \pi]$ is a solution of

$$\begin{cases} \frac{\partial \tilde{y}}{\partial t}(t, x) - \frac{\partial^2 \tilde{y}}{\partial x^2}(t, x) = 0 & t > 0, x \in (-\pi, \pi), \\ \tilde{y}(t, -\pi) = -u_\pi(t), \quad \tilde{y}(t, \pi) = u_\pi(t) & t > 0, \\ \tilde{y}(0, x) = \tilde{g}(x) & x \in (-\pi, \pi) \end{cases} \quad (4.15)$$

where \tilde{g} means the odd extension of g to $(-\pi, \pi)$. Denote by D_2 the square $D_2 = \{z = x + iy \in \mathbb{C} \mid |x| + |y| < \pi\}$. So, using Corollary 4.1.10 on $[-\pi, \pi]$ we obtain $\tilde{y} \in A^2(D_2)$ and $\tilde{y}(-z) = -\tilde{y}(z), \forall z \in D_2$. Conversely, if $\tilde{y} \in A^2(D_2)$ and $\forall z \in D_2, \tilde{y}(-z) = -\tilde{y}(z)$, then $\tilde{y}|_{[0, \pi]}$ is a solution of (4.14). Thus, we have proved

Corollary 4.3.3. *Denote by $\text{Ran}\Phi_\tau^{0D}$ the reachable space of (4.14).*

We have $\text{Ran}\Phi_\tau^{0D} = \{f \in A^2(D_2) \mid \forall z \in D_2, f(-z) = -f(z)\} := A_{\text{odd}}^2(D_2)$.

Neumann condition at one end.

Finally, let $\text{Ran}\Phi_\tau^{0N}$ be the reachable space for Neumann boundary condition at one end, i.e the reachable space of (4.13) with $u_0 = 0$. Using an even extension and with the same kind of arguments as in the previous case, we obtain

Corollary 4.3.4. *We have*

$$\begin{aligned} \text{Ran}\Phi_\tau^{0N} &= \{f \in \mathcal{D}(D_2) \mid \forall z \in D_2, f(-z) = f(z)\} := \mathcal{D}_{\text{even}}(D_2) \\ &= \{f \in \text{Hol}(D_2) \mid f' \in A_{\text{odd}}^2(D_2)\}. \end{aligned}$$

Chapter 5

Reachable space of the Hermite heat equation

Some of the tools developed in the preceding chapters apply to the case of the heat equation with the potential x^2 since in this case we know explicitly the kernel. This equation is called the Hermite heat equation. The aim of this chapter is to present some corresponding results for the inclusion of the reachable space in the Bergman space.

5.1 Introduction

For $u = (u_0, u_1) \in L^2((0, \tau), \mathbb{C}^2)$, we consider the following equation.

$$\begin{cases} \partial_t w - \partial_x^2 w + x^2 w = 0, & t > 0, x \in (0, \pi) \\ w(t, 0) = u_0(t), & w(t, \pi) = u_\pi(t), & t > 0 \\ w(0, x) = 0. \end{cases} \quad (5.1)$$

We have seen in Section 2.2 that this equation can be formulated as a well posed boundary control system for $X = W^{-1,2}(0, \pi)$ and $U = \mathbb{C}^2$. Therefore, for every $u \in L^2_{\text{loc}}((0, \infty); U)$, it admits a unique solution $w \in C([0, \infty), X)$. We recall that $w \in C([0, \infty), X)$ is called solution of (5.1) if it satisfies

$$\begin{aligned} \langle w(t), \psi \rangle_{-1,1} &= \int_0^t \left\langle w(s), \frac{d^2 \psi}{dx^2} - x^2 \psi \right\rangle_{-1,1} ds \\ &+ \int_0^t u_0(s) \overline{\frac{d\psi}{dx}(0)} ds - \int_0^t u_\pi(s) \overline{\frac{d\psi}{dx}(\pi)} ds \end{aligned} \quad (5.2)$$

for every $t \geq 0$ and every $\psi \in W^{2,2}(0, \pi) \cap W_0^{1,2}(0, \pi)$ such that $\frac{d^2\psi}{dx^2} \in W_0^{1,2}(0, \pi)$. Here $\langle \cdot, \cdot \rangle_{-1,1}$ denotes the duality $W^{-1,2}(0, \pi) - W_0^{1,2}(0, \pi)$.

As in the classical case, the Hermite heat equation (5.1) is null-controllable in any time. So, its reachable space does not depend on time. We denote by Φ_τ^H its controllability operator and $\text{Ran } \Phi_\tau^H$ its reachable space.

5.2 Computation of the solution

In this section, we compute the solution of equation (5.1). In Chapter 3, we followed the method of [HKT20] computing the solution of the heat equation with a decomposition on a Fourier basis and a use of the Poisson summation formula. This allowed the heat kernel of the line to appear. Here we use the method of images to avoid the Poisson formula step and obtain directly an expression involving the Mehler kernel of the line.

Mehler kernel. Let us recall some facts about the fundamental solution of the Hermite heat equation on \mathbb{R} , i.e. the solution of

$$\begin{cases} \partial_t K_1 - \partial_x^2 K_1 + x^2 K_1 = 0, & t > 0, x \in \mathbb{R} \\ K_1(0, x, \cdot) = \delta_x. \end{cases} \quad (5.3)$$

where δ_x stands for the Dirac delta distribution at x . It is given by the Mehler kernel

$$K_1(t, x, y) = \frac{1}{\sqrt{2\pi \sinh(2t)}} \exp\left(-\coth(2t) \frac{x^2 + y^2}{2} + \frac{xy}{\sinh(2t)}\right). \quad (5.4)$$

It can be obtained using the normalized Hermite functions defined by

$$h_k(x) = \frac{(-1)^k e^{-\frac{x^2}{2}}}{2^k k! \pi^{\frac{1}{2}}} \frac{d^k}{dx^k} e^{-x^2}, \quad x \in \mathbb{R}, k \in \mathbb{N}$$

which are known to form an orthonormal basis of $L^2(0, \pi)$ and to be the eigenfunctions of the operator $-\Delta + x^2$ on $L^2(\mathbb{R})$ associated with the eigenvalues $\lambda_k = 2k + 1$. Thus, the solution of (5.3) can be derived decomposing on this orthonormal basis. Finally, the above form of the Mehler kernel is obtained using the so called Mehler formula.

More general Mehler kernels can be found in [DE19, Subsection 2.1].

Computation of the solution using the method of images. For $\varphi \in \mathcal{D}(\mathbb{R}) := C_c^\infty(\mathbb{R})$, we denote by φ_{per} the function

$$\varphi_{\text{per}} = \sum_{k \in \mathbb{Z}} \varphi(\cdot + 2k\pi) - \varphi(2k\pi - \cdot).$$

Remark that the sum is finite and $\varphi_{\text{per}}, \frac{d^2 \varphi_{\text{per}}}{dx^2} \in W_0^{1,2}(0, \pi)$. Hence, for $T \in W^{-1,2}(0, \pi)$, we can define $T_{\text{per}} \in \mathcal{D}'(\mathbb{R})$ by duality

$$\forall \varphi \in \mathcal{D}(\mathbb{R}), \langle T_{\text{per}}, \varphi \rangle_{\mathcal{D}'(\mathbb{R}), \mathcal{D}(\mathbb{R})} := \langle T, \overline{\varphi_{\text{per}}} \rangle_{-1,1}.$$

It is easy to check that T_{per} is well defined as a distribution and $T_{\text{per}} \in \mathcal{S}'(\mathbb{R})$. Note that T_{per} is the continuation (in the sense of distributions) of T to \mathbb{R} , first by odd extension to $(-\pi, \pi)$ and then by periodic extension to \mathbb{R} with period 2π .

Let w be the unique solution of (5.1) in the sense of (5.2). Recall that $\mathcal{D}(\mathbb{R}_t) \otimes \mathcal{D}(\mathbb{R}_x)$ is dense in $\mathcal{D}(\mathbb{R}_t \times \mathbb{R}_x)$. Write W the tempered distribution defined on $\mathbb{R}_t \times \mathbb{R}_x$ by

$$\forall \zeta \in \mathcal{D}(\mathbb{R}_t), \forall \varphi \in \mathcal{D}(\mathbb{R}_x), \langle W, \zeta \otimes \varphi \rangle_{\mathcal{D}'(\mathbb{R}_t \times \mathbb{R}_x), \mathcal{D}(\mathbb{R}_t \times \mathbb{R}_x)} := \int_{\mathbb{R}_+} \langle w(t)_{\text{per}}, \varphi \rangle_{\mathcal{D}'(\mathbb{R}), \mathcal{D}(\mathbb{R})} \zeta(t) dt.$$

Note that

$$W|_{\mathbb{R}_+^* \times (0, \pi)} = w. \tag{5.5}$$

Lemma 5.2.1. *The distribution W is the unique solution in $\mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x)$ of the equation*

$$\begin{cases} \partial_t W - \partial_x^2 W + x^2 W = -2u_0 \mathbf{1}_{\mathbb{R}_+} \otimes \sum_{k \in \mathbb{Z}} \delta'_{2k\pi} + 2u_\pi \mathbf{1}_{\mathbb{R}_+} \otimes \sum_{k \in \mathbb{Z}} \delta'_{(2k+1)\pi} := F \\ \text{supp}(W) \subset \mathbb{R}_+ \times \mathbb{R}. \end{cases} \tag{5.6}$$

Proof. Since W and F belong to $\mathcal{S}'(\mathbb{R}_t \times \mathbb{R}_x)$ (recall that the $L^2 \subset \mathcal{S}'$), it suffices to prove the equality in $\mathcal{D}'(\mathbb{R}_t \times \mathbb{R}_x)$. For each $t \geq 0$ and for every $\varphi \in \mathcal{D}(\mathbb{R})$, we have

(using (5.2))

$$\begin{aligned}
& \langle w(t)_{\text{per}}, \varphi \rangle_{\mathcal{D}'(\mathbb{R}), \mathcal{D}(\mathbb{R})} = \langle w(t), \overline{\varphi_{\text{per}}} \rangle_{-1, 1} \\
& = \int_0^t \left\langle w(s), \frac{d^2 \overline{\varphi_{\text{per}}}}{dx^2} - x^2 \overline{\varphi_{\text{per}}} \right\rangle_{-1, 1} ds + \int_0^t u_0(s) \frac{d\varphi_{\text{per}}}{dx}(0) ds - \int_0^t u_\pi(s) \frac{d\varphi_{\text{per}}}{dx}(\pi) ds \\
& = \int_0^t \left\langle w(s), \overline{\left(\frac{d^2 \varphi}{dx^2} - x^2 \varphi \right)_{\text{per}}} \right\rangle_{-1, 1} ds \\
& \quad + \int_0^t 2u_0(s) \sum_{k \in \mathbb{Z}} \frac{d\varphi}{dx}(2k\pi) ds - \int_0^t 2u_\pi(s) \sum_{k \in \mathbb{Z}} \frac{d\varphi}{dx}(2(k+1)\pi) ds \\
& = \int_0^t \left\langle w(s)_{\text{per}}, \frac{d^2 \varphi}{dx^2} - x^2 \varphi \right\rangle_{\mathcal{D}', \mathcal{D}} ds \\
& \quad - 2 \int_0^t u_0(s) \left\langle \sum_{k \in \mathbb{Z}} \delta'_{2k\pi}, \varphi \right\rangle_{\mathcal{D}', \mathcal{D}} ds + 2 \int_0^t u_\pi(s) \left\langle \sum_{k \in \mathbb{Z}} \delta'_{2(k+1)\pi}, \varphi \right\rangle_{\mathcal{D}', \mathcal{D}} ds \quad (\star)
\end{aligned}$$

Hence, for every $\zeta \in \mathcal{D}(\mathbb{R}_t)$ and every $\varphi \in \mathcal{D}(\mathbb{R}_x)$,

$$\begin{aligned}
& \langle \partial_t W, \zeta \otimes \varphi \rangle_{\mathcal{D}'(\mathbb{R}_t \times \mathbb{R}_x), \mathcal{D}(\mathbb{R}_t \times \mathbb{R}_x)} = - \langle W, \partial_t \zeta \otimes \varphi \rangle_{\mathcal{D}'(\mathbb{R}_t \times \mathbb{R}_x), \mathcal{D}(\mathbb{R}_t \times \mathbb{R}_x)} \\
& = - \int_{\mathbb{R}_+} \langle w(t)_{\text{per}}, \varphi \rangle_{\mathcal{D}'(\mathbb{R}), \mathcal{D}(\mathbb{R})} \frac{d\zeta}{dt}(t) dt \\
& = \int_{\mathbb{R}_+} \frac{d}{dt} \left(\langle w(t)_{\text{per}}, \varphi \rangle_{\mathcal{D}'(\mathbb{R}), \mathcal{D}(\mathbb{R})} \right) \zeta(t) dt.
\end{aligned}$$

Finally using (\star) , we obtain

$$\begin{aligned}
& \langle \partial_t W, \zeta \otimes \varphi \rangle_{\mathcal{D}'(\mathbb{R}_t \times \mathbb{R}_x), \mathcal{D}(\mathbb{R}_t \times \mathbb{R}_x)} \\
&= \int_{\mathbb{R}_+} \left[\left\langle w(t)_{\text{per}}, \frac{d^2 \varphi}{dx^2} - x^2 \varphi \right\rangle_{\mathcal{D}', \mathcal{D}} - 2u_0(t) \left\langle \sum_{k \in \mathbb{Z}} \delta'_{2k\pi}, \varphi \right\rangle_{\mathcal{D}', \mathcal{D}} \right. \\
&\quad \left. + 2u_\pi(t) \left\langle \sum_{k \in \mathbb{Z}} \delta'_{2(k+1)\pi}, \varphi \right\rangle_{\mathcal{D}', \mathcal{D}} \right] \zeta(t) dt \\
&= \langle (\partial_x^2 - x^2)W, \zeta \otimes \varphi \rangle_{\mathcal{D}', \mathcal{D}} - 2 \left\langle u_0 \mathbf{1}_{\mathbb{R}_+} \otimes \sum_{k \in \mathbb{Z}} \delta'_{2k\pi}, \zeta \otimes \varphi \right\rangle_{\mathcal{D}', \mathcal{D}} \\
&\quad + 2 \left\langle u_\pi \mathbf{1}_{\mathbb{R}_+} \otimes \sum_{k \in \mathbb{Z}} \delta'_{2(k+1)\pi}, \zeta \otimes \varphi \right\rangle_{\mathcal{D}', \mathcal{D}}.
\end{aligned}$$

□

Proposition 5.2.2. *The solution of the Hermite heat equation (5.1) is given by*

$$\begin{aligned}
\forall t > 0, \forall x \in (0, \pi), \quad w(t, x) &= 2 \int_0^t \sum_{k \in \mathbb{Z}} \frac{\partial K_1}{\partial y}(t-s, x, 2k\pi) u_0(s) ds \\
&\quad - 2 \int_0^t \sum_{k \in \mathbb{Z}} \frac{\partial K_1}{\partial y}(t-s, x, (2k+1)\pi) u_\pi(s) ds.
\end{aligned}$$

Proof. By Lemma 5.2.1, W is the unique solution of equation (5.6) in \mathcal{S}' which is by Duhamel's formula

$$\begin{aligned}
W(t, x) &= \langle F, K_1(t - \cdot, x, \cdot) \rangle_{\mathcal{S}'(\mathbb{R}_s \times \mathbb{R}_y), \mathcal{S}(\mathbb{R}_s \times \mathbb{R}_y)} \\
&= -2 \int_0^t \left\langle \sum_{k \in \mathbb{Z}} \delta'_{2k\pi}, K_1(t-s, x, \cdot) \right\rangle_{\mathcal{S}', \mathcal{S}} u_0(s) ds \\
&\quad + 2 \int_0^t \left\langle \sum_{k \in \mathbb{Z}} \delta'_{(2k+1)\pi}, K_1(t-s, x, \cdot) \right\rangle_{\mathcal{S}', \mathcal{S}} u_\pi(s) ds \\
&= 2 \int_0^t \sum_{k \in \mathbb{Z}} \frac{\partial K_1}{\partial y}(t-s, x, 2k\pi) u_0(s) ds \\
&\quad - 2 \int_0^t \sum_{k \in \mathbb{Z}} \frac{\partial K_1}{\partial y}(t-s, x, (2k+1)\pi) u_\pi(s) ds
\end{aligned}$$

The results follows from (5.5).

□

5.3 Inclusion in the Bergman space of the square

In this section we prove that the reachable space of the Hermite heat equation (5.1) is contained in the Bergman space $A^2(D)$. We prove also that the (null)-reachable space of the Hermite heat equation on the half-line is included in the Bergman space of the sector Δ . Hence, the results are similar as those proved in Chapter 3.

In view of Proposition 5.2.2, we can write the solution of (5.1) as

$$w(\tau, x) := \Phi_\tau^H(u_0, u_\pi) = \Phi_{\tau,0}^H u_0(x) + \Phi_{\tau,\pi}^H u_\pi(x) + R_{\tau,0} u_0(x) + R_{\tau,\pi} u_\pi(x),$$

where $\Phi_{\tau,0}^H u_0$ and $\Phi_{\tau,\pi}^H u_\pi$ are respectively the zero-term of the first sum and the second sum. It can be proved easily that $R_{\tau,0} u_0, R_{\tau,\pi} u_\pi \in A^2(D)$ with the explicit structure of the Mehler kernel in mind (the proof is essentially the same as the one from [HKT20] given in Chapter 3).

In order to obtain information on the reachable space of the equation (5.1), it suffices to describe the range of $\Phi_{\tau,0}^H$ and $\Phi_{\tau,\pi}^H$.

Range of $\Phi_{\tau,0}^H$. Using a change of variable $x = \alpha(s) := \tanh(2(\tau - s))/2$ (which leads to $ds = -\frac{dx}{1-4x^2}$ and $\sinh(2(\tau - s)) = \frac{2x}{\sqrt{1-4x^2}}$), we obtain

$$\begin{aligned} \Phi_{\tau,0}^H u_0(z) &:= 2 \int_0^\tau \frac{\partial K_1}{\partial y}(\tau - s, z, 0) u_0(s) ds \\ &= \frac{2z}{\sqrt{2\pi}} \int_0^\tau \frac{e^{-\coth(2(\tau-s))z^2/2}}{\sinh(2(\tau-s))^{3/2}} u_0(s) ds \\ &= \frac{z}{2\sqrt{\pi}} \int_0^{\frac{\tanh(2\tau)}{2}} \frac{e^{-\frac{z^2}{4x}}}{x^{3/2}} (1-4x^2)^{-1/4} u_0(\alpha^{-1}(x)) dx. \end{aligned}$$

Now, setting $T = \tanh(2\tau)/2$ and the change of variable $x = T - t$ leads to

$$[\Phi_{\tau,0}^H u_0](z) = [\tilde{\Phi}_{0,T} \tilde{u}_0](z)$$

where $\tilde{\Phi}_{0,T}$ denotes the controllability operator of the heat equation for the half-line defined in (3.7), and \tilde{u}_0 is given by

$$\tilde{u}_0(t) = (1 - 4(T - t)^2)^{-1/4} u_0(\alpha^{-1}(T - t)).$$

It is clear that the operator $S : L^2((0, \tau); \mathbb{C}) \rightarrow L^2((0, T); \mathbb{C})$ defined by

$$S u_0 = \tilde{u}_0$$

is boundedly invertible and we have

$$\int_0^T |\tilde{u}_0(t)|^2 dt = \int_0^\tau \frac{1}{\cosh(2(\tau-s))} |u_0(s)|^2 ds \asymp \int_0^\tau |u_0(s)|^2 ds.$$

Hence, we have

$$\text{Ran } \Phi_{\tau,0}^H = \text{Ran } \tilde{\Phi}_{0,T} \subset A^2(D). \quad (5.7)$$

It is of interest to note that the function $y(t, x) = (\Phi_{t,0}^H u_0)(x)$ is the unique solution in $C((0, \infty), W^{-1,2}(0, \infty))$ of the Hermite heat equation on $(0, \infty)$ given by

$$\begin{cases} \partial_t y - \partial_x^2 y + x^2 y = 0, & t > 0, x \in (0, \infty) \\ y(t, 0) = u_0(t), & t > 0 \\ y(0, x) = 0, & x \in (0, \infty). \end{cases} \quad (5.8)$$

Hence, we have proved the following result.

Theorem 5.3.1. *For every $\tau > 0$, the (null-)reachable space $\text{Ran } \Phi_{\tau,0}^H$ of the equation (5.8) is included in $A^2(\Delta)$.*

Actually, in view of Section 2.6, equality (5.7) characterizes the reachable space $\text{Ran } \Phi_{\tau,0}^H$ by its reproducing kernel.

Remark 5.3.2. As for the heat equation on the half line, the Hermite heat equation on the half-line is not null-controllable. Therefore its reachable space $\mathcal{R}_\tau^{H,f}(\mathbb{R}_+^*)$ should depend on time τ and on the initial condition f . Actually, it can be proved that the null function is not contained in $\mathcal{R}_\tau^{H,f}(\mathbb{R}_+^*)$, except if $f = 0$ (see [DE19] for more details).

Range of $\Phi_{\tau,\pi}^H$.

Lemma 5.3.3. *We have $\text{Ran } \Phi_{\tau,\pi}^H \subset A^2(D)$.*

Proof. Let us write the expression of $\Phi_{\tau,\pi}^H u_\pi$.

$$\begin{aligned} [\Phi_{\tau,\pi}^H u_\pi](z) &:= -2 \int_0^\tau \frac{\partial K_1}{\partial y}(\tau-s, z, \pi) u_\pi(s) ds \\ &= -2 \int_0^\tau \left(-\coth(2(\tau-s)\pi) + \frac{z}{\sinh(2(\tau-s))} \right) \frac{e^{-\coth(2(\tau-s)) \frac{z^2 + \pi^2}{2} + \frac{\pi z}{\sinh(2(\tau-s))}}}{\sqrt{2\pi \sinh(2(\tau-s))}} u_\pi(s) ds. \\ &= -2 \int_0^\tau \left(-\coth(2(\tau-s)\pi) + \frac{z}{\sinh(2(\tau-s))} \right) \frac{e^{-\frac{\coth(2(\tau-s))}{2} (\pi-z)^2 + \frac{1-\cosh(2(\tau-s))}{\sinh(2(\tau-s))} \pi z}}{\sqrt{2\pi \sinh(2(\tau-s))}} u_\pi(s) ds. \end{aligned}$$

We remind that for the heat equation, the equality

$$(\tilde{\Phi}_{\tau, \pi} f)(z) = -(\tilde{\Phi}_{\tau, 0} f)(\pi - z) \quad (5.9)$$

holds. It is no longer true here. However, we can make $\Phi_{\tau, 0}^H$ appear in the above expression of $\Phi_{\tau, \pi}^H u_\pi$. For that, remark that the function ψ defined by $\psi(s) = \frac{1 - \cosh(2(\tau - s))}{\sinh(2(\tau - s))}$ is bounded on $[0, \tau]$. Hence writing $e^{\psi(s)\pi z} = \sum_{n=0}^{\infty} \frac{(\psi(s)\pi z)^n}{n!}$, we obtain

$$\begin{aligned} & [\Phi_{\tau, \pi}^H u_\pi](z) \\ &= -2 \int_0^\tau \sum_{n=0}^{\infty} \frac{(\psi(s)\pi z)^n}{n!} \left(-\coth(2(\tau - s)\pi + \frac{z}{\sinh(2(\tau - s))}) \right) \frac{e^{-\frac{\coth(2(\tau - s))}{2}(\pi - z)^2}}{\sqrt{2\pi \sinh(2(\tau - s))}} u_\pi(s) ds \\ &= -2 \sum_{n=0}^{\infty} \frac{(\pi z)^n}{n!} \int_0^\tau (\psi(s))^n \left(-\coth(2(\tau - s)\pi + \frac{z}{\sinh(2(\tau - s))}) \right) \frac{e^{-\frac{\coth(2(\tau - s))}{2}(\pi - z)^2}}{\sqrt{2\pi \sinh(2(\tau - s))}} u_\pi(s) ds. \end{aligned}$$

Now, since $-\coth(2(\tau - s)\pi + \frac{z}{\sinh(2(\tau - s))}) = \frac{z - \pi}{\sinh(2(\tau - s))} + \pi\psi(s)$, it follows

$$\begin{aligned} & [\Phi_{\tau, \pi}^H u_\pi](z) \\ &= \sum_{n=0}^{\infty} \frac{(\pi z)^n}{n!} \frac{2}{\sqrt{2\pi}} \int_0^\tau (\psi(s))^n \frac{\pi - z}{\sinh(2(\tau - s))^{\frac{3}{2}}} e^{-\frac{\coth(2(\tau - s))}{2}(\pi - z)^2} u_\pi(s) ds \\ &\quad - \sqrt{2\pi} \sum_{n=0}^{\infty} \frac{(\pi z)^n}{n!} \int_0^\tau (\psi(s))^{n+1} \frac{e^{-\frac{\coth(2(\tau - s))}{2}(\pi - z)^2}}{\sqrt{\sinh(2(\tau - s))}} u_\pi(s) ds \\ &= \sum_{n=0}^{\infty} \frac{(\pi z)^n}{n!} [\Phi_{\tau, 0}^H(\psi^n u_\pi)](\pi - z) \\ &\quad - \sqrt{2\pi} \sum_{n=0}^{\infty} \frac{(\pi z)^n}{n!} \int_0^\tau (\psi(s))^{n+1} \frac{e^{-\frac{\coth(2(\tau - s))}{2}(\pi - z)^2}}{\sqrt{\sinh(2(\tau - s))}} u_\pi(s) ds \\ &:= \sum_{n=0}^{\infty} \frac{(\pi z)^n}{n!} (A(z) - B(z)). \end{aligned}$$

We treat these two terms separately. On the first hand, we have an oscillatory

integral that we estimate as follows

$$\begin{aligned}
\|A\|_{A^2(D)} &= \left\| \left[\Phi_{\tau,0}^H(\psi^n u_\pi) \right] (\pi - \cdot) \right\|_{A^2(D)} \\
&\leq \left\| \left[\Phi_{\tau,0}^H(\psi^n u_\pi) \right] (\pi - \cdot) \right\|_{A^2(\pi - \Delta)} \\
&= \left\| \Phi_{\tau,0}^H(\psi^n u_\pi) \right\|_{A^2(\Delta)} \\
&\lesssim \|\psi^n u_\pi\|_{L^2(0,\tau)} \\
&\leq \|u_\pi\|_{L^2(0,\tau)} \|\psi\|_{L^\infty(0,\infty)}^n
\end{aligned}$$

On the other hand, we have an integral which converges easily. Indeed, using the Cauchy-Schwarz inequality

$$\|B\|_{A^2(D)}^2 \leq \|u_\pi\|_{L^2(0,\tau)}^2 \int_0^\tau \int_D |\psi(s)|^{2n+1} \left| \frac{e^{-\coth(2(\tau-s))(\pi-z)^2}}{\sinh(2(\tau-s))} \right| dA(z) ds.$$

So, using that $|e^{-\coth(2(\tau-s))(\pi-z)^2}| \leq 1$ on D , we obtain

$$\|B\|_{A^2(D)}^2 \lesssim \|u_\pi\|_{L^2(0,\tau)}^2 \int_0^\tau |\psi(s)|^{2n+1} \left| \frac{1}{\sinh(2(\tau-s))} \right| ds \leq \|u_\pi\|_{L^2(0,\tau)}^2 \|\psi\|_{L^\infty(0,\infty)}^{2n} \int_0^\tau \psi(s) \frac{1}{\sinh(2(\tau-s))} ds$$

Finally, we have

$$\begin{aligned}
\|\Phi_{\tau,\pi}^H u_\pi\|_{A^2(D)} &\leq \sum_{n=0}^{\infty} \frac{\pi^{2n}}{n!} (\|A\|_{A^2(D)} + \|B\|_{A^2(D)}) \\
&\leq \sum_{n=0}^{\infty} \frac{\pi^{2n}}{n!} (\|u_\pi\|_{L^2(0,\tau)} \|\psi\|_{L^\infty(0,\infty)}^n + C \|\psi\|_{L^\infty(0,\infty)}^n \|u_\pi\|_{L^2(0,\tau)}) \\
&= (1 + C) \|u_\pi\|_{L^2(0,\tau)} \sum_{n=0}^{\infty} \frac{\|\psi\|_{L^\infty(0,\infty)}^n \pi^{2n}}{n!}.
\end{aligned}$$

Since the last sum converges (to $e^{\|\psi\|_{L^\infty(0,\infty)} \pi^2}$), the proof is complete. \square

Remark 5.3.4. Since the potential x^2 is even, one could want to take a symmetric interval with respect to zero. In that setting, the symmetry (5.9) holds but the difficulty of describing the ranges of $\Phi_{\tau,0}^H$ and $\Phi_{\tau,\pi}^H$ does not change.

Combining the last results, we obtain the following theorem.

Corollary 5.3.5. *We have $\text{Ran } \Phi_\tau^H \subset A^2(D)$.*

The converse inclusion is more tricky if one wants to use the same kind of arguments as in the case of the heat equation. While $\Phi_{\tau,0}^H$ is still isomorphic, the situation for $\Phi_{\tau,\pi}^H$ is more complicated and awaits further investigation.

Chapter 6

Perspectives

In this chapter, we list few open questions that we plan to investigate in future researches.

Trace of the Bergman space on the real axis.

We have seen in Chapter 4 that the reachable states of the heat equation are exactly the functions which extend holomorphically to the square D and belong to the Bergman space of the square D . A natural and interesting question is whether we can get a testing condition on the segment $(0, \pi)$ to know whether a C^∞ function is reachable or not. In other words, can we characterize the trace of the Bergman functions on the segment $(0, \pi)$? We have seen in Theorem 2.6.3 that such a characterization has been established in [AHS90] for the half-line. For the heat equation on $(0, \pi)$, the question is still open.

Strongly continuous semigroup on the Bergman space.

The previous question is related to another important question that we expose now. We have seen in Section 2.1 that the (one-dimensional) Laplacian $\frac{d^2}{dx^2}$ generates a strongly continuous semigroup on the Sobolev space $X = W^{-1,2}(0, \pi)$ which is called Dirichlet heat semigroup. It can be viewed as an obvious consequence of our results that the range of this semigroup is contained in the Bergman space $A^2(D)$. We formulate now the question: can the Dirichlet heat semigroup be defined as a strongly continuous semigroup on $A^2(D)$? A positive answer should permit to make work fixed-point methods and obtain controllability results for non-linear parabolic equations. Such kind of results has been obtained in [LR95] for semilinear parabolic equations by another way.

Higher dimensional setting.

The question of describing the reachable space can be generalized to the n -dimensional setting considering a bounded convex open set Ω instead of $(0, \pi)$. If a general answer seems to be difficult to obtain for now, it seems accessible to prove results for some particular geometries. We would like to mention that very recently Strohmaier and Waters have made in [SW20] a big step forward giving an optimal result about the domain of holomorphy of the reachable states.

Separation of singularities.

The separation of singularities turned out to be very efficient in the above problem of control theory. It could also become a systematic tool in the theory of holomorphic functions. Hence, it seems important to develop this method on more general domains and in higher dimensions. This last point is obviously closely related to the reachable space in dimension n .

Internal control.

Although we mentioned it as an example in 2.2.5 or in Section 2.3, we did not treat the case of the reachable space for internal control in the present thesis. The problem seems trickier than for boundary control but it can be interesting to investigate this way.

More general inputs.

Finally the question of determining the reachable states has still a sense when we choose the control in L^p or in C_c^∞ . Partial results can be given using for example the Paley-Wiener type theorems 1.2.14 and 1.2.4 but the converse inclusion seems difficult to obtain without the Hilbertian setting. It would be more interesting to take the control in L^∞ . This would permit to obtain the result for intermediate spaces by interpolation. Finally, it would be also natural to take the control in $C^0[0, \tau]$ or in the Gevrey class $G^2[0, \tau]$ since inside of the square the solution of the heat equation is G^2 in time.

Other type of equations.

The reachable space problem holds also for other evolution equations which are not of parabolic type as the linear Korteweg- de Vries equation or the linear Zakharov-Kuznetsov equation. For these equations, some results can be found respectively in [MRRR19] and [CR20].

Appendix A

Proof of Havin's lemma

In this appendix, we give a proof of Lemma 1.3.4. Whilst it is a key result of the Havin proof of Aronszajn's theorem 1.3.1, this lemma is not proved in the original paper [Hav58] nor in the papers [Aiz14] and [MK71] which reproduce the Havin's proof.

First, we recall some topological generalities and introduce some notations. We conserve the definitions given in Section 1.3. We write

$$I_i : X_{F_0} \rightarrow X_{F_i}$$

for the canonical embedding given by : if $f \in H^\infty(O_n^1 \cup O_n^2)$ then $I_i(f) = f|_{O_n^i}$. We remind that the topology of inductive limit on X_{F_i} is the topology generated by the family \mathcal{P}_i of all the seminorms p such that for every $n \in \mathbb{N}$, $p : H^\infty(O_n^i) \rightarrow \mathbb{R}_+$ is continuous. Also, the topology generated by the topologies induced on X_{F_0} by X_{F_1} and X_{F_2} is the topology generated by the family of seminorms

$$\mathcal{P}_{1|X_{F_0}} \cup \mathcal{P}_{2|X_{F_0}}.$$

With this in mind we can start the proof.

Proof of Lemma 1.3.4. Let us start proving that the topology generated by the family of seminorms $\mathcal{P}_{1|X_{F_0}} \cup \mathcal{P}_{2|X_{F_0}}$ is included in the topology induced by the family \mathcal{P}_0 . For that, we will show that the seminorms $p \in \mathcal{P}_{1|X_{F_0}} \cup \mathcal{P}_{2|X_{F_0}}$ belong to \mathcal{P}_0 . Without loss of generalities, we can assume that $p \in \mathcal{P}_1$, the other case is similar. Let $n \in \mathbb{N}$. By definition of p , there exists $C_n > 0$ such that

$$p(f) \leq C_n \|f\|_{H^\infty(O_n^1)}, \quad \forall f \in H^\infty(O_n^1).$$

So for every $f \in H^\infty(O_n^1 \cup O_n^2)$, we have

$$p(f) \leq C_n \|f\|_{H^\infty(O_n^1)} \leq C_n \|f\|_{H^\infty(O_n^1 \cup O_n^2)},$$

i.e. p is continuous on $H^\infty(O_n^1 \cup O_n^2)$. Hence $p \in \mathcal{P}_0$.

Conversely, we shall prove that the topology generated by the family \mathcal{P}_0 is contained in the topology generated by the family $\mathcal{P}_{1|X_{F_0}} \cup \mathcal{P}_{2|X_{F_0}}$. For that, it is enough to show that every $p_0 \in \mathcal{P}_0$ is continuous from $(X_{F_0}, \mathcal{P}_{1|X_{F_0}} \cup \mathcal{P}_{2|X_{F_0}})$ to \mathbb{R}_+ . So, pick $p_0 \in \mathcal{P}_0$. By definition of p_0 , for every $n \in \mathbb{N}$ there exists $C'_n > 0$ such that

$$p_0(f) \leq C'_n \|f\|_{H^\infty(O_n^1 \cup O_n^2)}, \quad \forall f \in H^\infty(O_n^1 \cup O_n^2). \quad (\text{A.1})$$

We need now the following preliminary result: the seminorm

$$\rho : (X_{F_0}, \mathcal{P}_{1|X_{F_0}} \cup \mathcal{P}_{2|X_{F_0}}) \rightarrow \mathbb{R}_+$$

defined by $\rho(f) = C'_n \|f\|_{H^\infty(O_n^1 \cup O_n^2)}$ for $f \in H^\infty(O_n^1 \cup O_n^2)$, is continuous. To prove this result, it suffices to write $\rho = \max(p_1 \circ I_1, p_2 \circ I_2)$ where we have denoted by $p_i : X_{F_i} \rightarrow \mathbb{R}_+$ the seminorm given by $p_i(f) = C'_n \|f\|_{H^\infty(O_n^i)}$ for $f \in H^\infty(O_n^i)$. Since the seminorms p_i are clearly continuous, ρ also is. Therefore the inequality (A.1) becomes

$$p_0(f) \leq C'_n \|f\|_{H^\infty(O_n^1 \cup O_n^2)} = \rho(f), \quad \forall f \in H^\infty(O_n^1 \cup O_n^2),$$

which means that $p_0 \leq \rho$ on X_{F_0} . Hence p_0 is also continuous from $(X_{F_0}, \mathcal{P}_{1|X_{F_0}} \cup \mathcal{P}_{2|X_{F_0}})$ to \mathbb{R}_+ . □

Appendix B

Dominating sets and sampling constants

During my PhD, I have also worked on another kind of problems in collaboration with Andreas Hartmann, Dantouma Kamisso and Siaka Konate. This work constitutes a part of the PhD thesis of Siaka Konate. Here, I shall resume it briefly. For more details, the interested reader can refer to Siaka Konate's thesis and to the paper [HKKO20].

A challenging question for complex and harmonic analysts is whether we can recover the norm of an analytic function from its values on a sequence (sampling) or on a set (dominating sets). Both settings are particular cases of reverse Carleson measures (see [FHR17]) and they have wide applications in signal theory or control theory.

The most famous result concerning dominating sets is probably due to Logvinenko, Sereda [LS74] and Panejah [Pan66] and has been stated on the Paley-Wiener space (see Section 1.2 for a definition). In this setting, a set E is said dominating for the Paley-Wiener space if there exists $C > 0$ such that for every $f \in PW_\pi$ we have

$$\|f\|_{PW_\pi}^2 := \int_{\mathbb{R}} |f(x)|^2 dx \leq C \int_E |f(x)|^2 dx.$$

The Logvinenko-Sereda-Panejah result said that E is dominating for PW_π if and only if there exists $\gamma, r > 0$ such that for every $x \in \mathbb{R}$, $|E \cap [x - r, x + r]| \geq \gamma r$. In that case, we say that E is relatively dense.

The inequality of domination above can be viewed as a form of uncertainty principle and several variants of Logvinenko-Sereda-Panejah inequality have been proved useful in the very last years in control theory (see e.g. [LM19],[GJM19], [ES20], [EV20]).

The characterization of the dominating sets is known in other spaces of analytic functions like Fock spaces, Model spaces or Bergman spaces, and also relies to relative density type properties. Let us recall how the question is stated on Bergman spaces.

Let \mathbb{D} be the open unit disc of the complex plane. The weighted Bergman space $A^{p,\alpha}(\mathbb{D})$ consists of all the holomorphic functions on D which satisfy

$$\|f\|_{A^{p,\alpha}}^p := (\alpha + 1) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) < +\infty$$

where dA denotes the normalized planar Lebesgue measure. We say that a measurable set $E \subset \mathbb{D}$ is dominating for the (weighted) Bergman space $A^{p,\alpha}(\mathbb{D})$ if there exists $C > 0$ such that for every $f \in A^{p,\alpha}(\mathbb{D})$,

$$\|f\|_{A^{p,\alpha}}^p := (\alpha + 1) \int_{\mathbb{D}} |f(z)|^p (1 - |z|^2)^\alpha dA(z) \leq C^p (\alpha + 1) \int_E |f(z)|^p (1 - |z|^2)^\alpha dA(z).$$

Dominating sets in $A^p(\mathbb{D})$ have been completely characterized by Luecking in [Lue81] as the sets which are relatively dense. In this special geometric setting, this means that there exists $\gamma > 0$ and $r > 0$ such that

$$\forall z \in \mathbb{D}, \frac{|E \cap D_{phb}(z, r_0)|}{|D_{phb}(z, r_0)|} \geq \gamma$$

where

$$D_{phb}(z, r) = \{w \in \mathbb{D}; |\rho(z, w)| < r\}$$

is the pseudohyperbolic disc of center $z \in \mathbb{D}$ and radius $0 < r < 1$, and

$$\rho(z, w) = \left| \frac{z - w}{1 - \bar{z}w} \right|$$

is the pseudohyperbolic distance.

Having precise estimates on the sampling constants C is important in applications when one has to decide on the trade-off between the cost of the sampling and the accuracy of the estimates. It is also important that the constant is small enough to be useful in control theory.

In 2000, Kovrijkine [Kov01] considered the Paley-Wiener space and gave a precise and optimal estimate on C in function of the density γ . His method is based on Remez-type inequalities, and Bernstein's inequalities which hold in the Paley-Wiener space. Bernstein's inequalities are known are no longer true in Bergman spaces or Fock spaces.

In the work [HKKO20] in collaboration with Andreas Hartmann, Dantouma Kamissoko and Siaka Konate, we gave an estimate on C in function of γ and r in the case of the Bergman spaces $A^{p,\alpha}(\mathbb{D})$. Our main result is the following

Theorem B.0.1. *Let $1 \leq p < +\infty$. There exists L such that for every measurable set $E \subset \mathbb{D}$ which is (γ, r) -dense, we have*

$$\|f\|_{L^{p,\alpha}(E)} \geq \left(\frac{\gamma}{c}\right)^L \|f\|_{A^{p,\alpha}}$$

for every $f \in A^{p,\alpha}$.

The constants c and L depend on r . For L we can choose

$$L = c_1 \frac{1 + \alpha}{p} \frac{1}{(1 - r)^4} \ln \frac{1}{1 - r},$$

where c_1 is some universal constant.

To obtain this result, we used planar Remez-type inequalities established by Andrievskii and Ruscheweyh and we avoid Bernstein's inequalities defining "good" disks. Our method is quite general and can be used for other spaces.

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