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Heythem Farhat

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# Application de la théorie des contrats à la régulation des marchés d'énergie, et étude des lois jointes d'une martingale et son maximum courant

Thèse de doctorat de l'Institut Polytechnique de Paris  
préparée à l'École polytechnique

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Spécialité de doctorat : Mathématiques appliquées

Thèse présentée et soutenue à Palaiseau (visioconférence), le 04/02/2021, par

**HEYTHEM FARHAT**

Composition du Jury :

Caroline Hillairet Professeur, ENSAE ParisTech	Pésidente du jury
Aurélien Alfonsi Professeur, École Nationale des Ponts et Chaussées	Rapporteur
Saïd Hamadène Professeur, Le Mans Université	Rapporteur
René Aïd Professeur, Université Paris-Dauphine	Examineur
Mathieu Rosenbaum Professeur, École polytechnique	Examineur
Nizar Touzi Professeur, École polytechnique	Directeur de thèse
Clémence Alasseur Ingénieur de recherche, EDF R&D	Co-encadrante de thèse



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# Résumé

Cette thèse est composée de deux parties indépendantes. La première partie se focalise sur la résolution de problématiques de régulation en lien avec les marchés d'énergie en utilisant le modèle du Principal-Agent des jeux différentiels stochastiques à somme non nulle (c.f. [Cvitanić & Zhang \(2013\)](#) et [Cvitanić et al. \(2018\)](#)).

Nous nous intéressons dans un premier temps au marché des capacités électriques, et en particulier aux mécanismes de rémunération de capacité. Étant donné la part croissante des énergies renouvelables dans la production d'électricité, les centrales de production "classiques" (à gaz ou à charbon par exemple) sont de moins en moins sollicitées, ce qui les rends peu rentables et non viables économiquement. Cependant, leur fermeture définitive exposerait les consommateurs à un risque de Blackout en cas de pic de demande d'électricité puisque celle-ci ne peut pas être stockée. Ainsi, la capacité de production doit être toujours maintenue à un niveau au-dessus de la demande, ce qui nécessite un "mécanisme de rémunération de capacité" pour rémunérer les centrales rarement sollicitées, ce qui peut être compris comme une assurance à payer contre les Blackout électriques. Nous modélisons cette situation avec un Principal qui représente l'agrégation des consommateurs (ou le gestionnaire du réseau, RTE dans le cas de la France), et qui cherche à se protéger du risque de Blackout tout en satisfaisant sa demande électrique et minimisant sa facture. L'Agent est le fournisseur d'électricité qui prend les décisions d'investissement dans les centrales électriques en fonction de la rémunération (ou contrat) promis par le Principal (les consommateurs). Nous résolvons le problème pour les deux parties, et nous donnons des résultats numériques qui justifient la nécessité d'un tel mécanisme, et compare notre modèle avec le marché de capacité actuel.

D'une façon assez similaire, nous traitons la problématique des incitations à la décarbonation. L'objectif est désormais de proposer un modèle d'instrument qui puisse être utilisé par un agent public (l'état) en vue d'inciter les différents secteurs à baisser leurs émissions de carbone dans un contexte de risque moral (où l'état n'observe pas l'action des acteurs et ne peut donc pas savoir si une baisse des émissions provient d'une baisse de production et de consommation ou d'un effort de gestion et d'investissement en recherche et développement); ce qui fournit une alternative à la taxe carbone qui nécessite une symétrie d'information, i.e., que l'état observe les efforts des différents agents. Dans notre modèle, le Principal est l'état qui propose une compensation aux différents acteurs (l'Agent) pour les inciter à utiliser des modes de production plus chers mais moins polluants (en investissant dans la recherche et le développement) tout en satisfaisant la demande d'un certain bien. Dans un cadre Linéaire-Quadratique, nous résolvons ce problème explicitement en terme d'équations de Riccati matricielles. Ceci permet de pallier à l'une des difficultés d'implémentation numérique dans un cadre plus général. En effet, la résolution consiste désormais à une résolution d'une équation de Riccati matricielle plutôt qu'une équation aux dérivées partielles semilinéaire, beaucoup plus simple surtout en grandes dimensions.

La deuxième partie de cette thèse traite d'un sujet complètement indépendant, et présente un résultat sur les lois



jointes entre les marginales d'une martingale et son maximum courant. Nous cherchons à établir une extension au théorème de Kellerer qui à une famille de probabilités sur  $\mathbb{R}$  donnée,  $(\mu_t)_{0 \leq t \leq T}$ , donne une condition nécessaire et suffisante pour qu'il existe une martingale  $(X_t)_{t \in [0, T]}$  telle que  $X_t \stackrel{\text{loi}}{=} \mu_t$ , pour  $t \in [0, T]$ .

Nous considérons une famille de probabilités  $(\nu_t)_{t \in [0, T]}$  sur  $\mathbb{R}^2$  et nous cherchons des conditions nécessaires et suffisantes assurant l'existence d'une martingale  $(X_t)_{t \in [0, T]}$  telle que  $(X_t, M_t) \stackrel{\text{loi}}{=} \nu_t$  pour  $t \in [0, T]$ , avec  $M_t := M_0 \vee \sup_{u \leq t} X_u$ . Nous suivons la méthodologie de [Hirsch & Roynette \(2012\)](#) où ils construisent la martingale en question en se basant sur l'unicité de l'EDP de Fokker-Planck vérifiée par les lois marginales de cette martingale sous des hypothèses de régularité, puis dans un cadre général avec une régularisation et un passage à la limite.

Pour caractériser les lois jointes  $(X_t, M_t)_{t \in [0, T]}$  d'une martingale continue  $(X_t)_{t \in [0, T]}$ , un résultat de projection markovienne de [Brunick & Shreve \(2013\)](#) nous montre qu'il suffit d'étudier les martingales solutions de l'EDS

$$d\hat{X}_t = \sigma(t, \hat{X}_t, \hat{M}_t) dW_t \text{ avec } (\hat{X}_0, \hat{M}_0) = (x_0, m_0) \text{ et } x_0 \leq m_0 \text{ dans } \mathbb{R}^2, \quad (1)$$

car sous quelques hypothèses d'intégrabilité, toute martingale  $X$  peut être "représentée" par une solution (faible) de l'EDS (1) telle que  $(X_t, M_t) \stackrel{\text{loi}}{=} (\hat{X}_t, \hat{M}_t)$  pour  $t \in [0, T]$ . Nous étudions donc l'EDP de Fokker-Planck en lien avec (1). L'existence (au sens le plus large) découle de celle de l'existence faible de l'EDS (1) et nous démontrons l'unicité mais sous des hypothèses supplémentaires sur  $\sigma$  et de régularité sur les lois marginales. Sans les hypothèses de régularités, nous donnons des conditions suffisantes assurant l'existence d'une famille de martingales continues dont les lois marginales jointes  $\mathcal{V}^\epsilon$  convergent vers  $\mathcal{V}$  et dont on peut extraire une limite grâce à un critère de tension. Le résultat final (dans le cas régulier) est sous la forme : "On suppose qu'il existe une fonction  $\sigma$  telle que la famille  $(\nu_t)_{t \in [0, T]}$  satisfait une EDP de Fokker-Planck qui dépend de  $\sigma$ , alors il existe une martingale telle que  $(X_t, M_t) \stackrel{\text{loi}}{=} \nu_t$  pour  $t \in [0, T]$ ". Cette martingale peut être construite comme solution (faible) de l'EDS (1). L'égalité en loi des marginales est assurée dans ce cas par l'unicité de l'EDP de Fokker-Planck.

**Mots-Clés.** Principal-Agent, risque moral, marché des capacités électrique, décarbonation, martingale et maximum courant.

# Abstract

This dissertation treats two independent topics. The first one is the application of stochastic differential games with non zero sum; the Principal-Agent models (c.f. [Cvitanic & Zhang \(2013\)](#) and [Cvitanic et al. \(2018\)](#)) to solve some contemporary challenges of energy market regulation.

The first work is about Capacity Remuneration Mechanisms (CRM) in the electricity market. Given the growing share of renewable energies in the production of electricity, "conventional" power plants (gas or coal-fired for instance) are less used, which makes them not viable economically. However, shutting down these power plants would expose consumers to a risk of shortage or blackout in the event of an electricity demand peak. This is due to the fact that electricity can hardly be stored, and so the production capacity should always be maintained at a level above demand. This explains the necessity of a "Capacity Remuneration Mechanism" (CRM) to pay for rarely used power plants, which can be understood as an insurance against electricity shortages and blackouts.

We model this situation with a Principal that represents the aggregation of consumers (or an equivalent entity like the transmission system operator; RTE in the case of France), and that seeks a protection from the risk of shortage while satisfying the electricity demand and minimizing the corresponding costs. The Agent is the electricity supplier who makes the decisions of investment policies in power plants based on the remuneration or contract promised by the Principal (the consumers). We solve the problem for both parties, and provide numerical results justifying the need for such a mechanism, and a comparison between our model and the current capacity market.

In a very similar way, we address the issue of the incentives for decarbonation. The goal is to propose a model of instrument that can be used by a public entity (the state) in order to incentivize different sectors of the economy to reduce their carbon emissions in a context of moral hazard; where the state does not observe the actions, and is unable to distinguish between reduction in carbon emissions coming from a reduced production and consumption, or from a management effort towards a less polluting production (investment in research and development, for example). This provides an alternative to the carbon tax, and does not require perfect information as for the latter. In this case, the Principal is the state, which proposes a compensation to the various actors (the Agent) to encourage them to use more expensive but less polluting modes of production (for example, by investing in research and development) while satisfying the demand of a certain good. We provide an explicit resolution in a Linear-Quadratic framework in terms of matrix Riccati equations. This allows to overcome one of the difficulties of numerical implementation in a more general setting. Indeed, the resolution now consists of a numerical resolution of a matrix Riccati equation rather than a semilinear partial differential equation, much simpler, especially in large dimensions.

The second part of this thesis is about an independent subject, and provides a result concerning the joint laws between the marginals of a martingale and its running maximum. We seek to establish an extension of Kellerer's theorem which to a given family of probabilities  $(\mu_t)_{t \in [0, T]}$  on  $\mathbb{R}$ , provides a necessary and sufficient condition for

the existence of a martingale  $(X_t)_{t \in [0, T]}$  such that  $X_t \stackrel{law}{=} \mu_t$ , for  $t \in [0, T]$ . We consider a family of probabilities  $(\nu_t)_{t \in [0, T]}$  on  $\mathbb{R}^2$  and we provide necessary and sufficient conditions for the existence of a martingale  $(X_t)_{t \geq 0}$  such that  $(X_t, M_t) \stackrel{law}{=} \nu_t$  for  $t \in [0, T]$ , with  $M_t := M_0 \vee \sup_{u \leq t} X_u$ . We follow the methodology of [Hirsch & Roynette \(2012\)](#) where they construct a martingale based on the uniqueness of Fokker-Planck's PDE satisfied by the marginal laws of this martingale under smoothness assumptions, then using a regularization in the general case. To characterize the joined laws  $(X_t, M_t)_{t \in [0, T]}$  of a continuous martingale  $(X_t)_{t \in [0, T]}$  a Markovian projection result of [Brunick & Shreve \(2013\)](#) proves that it is sufficient to characterize martingales which are solutions of the SDE

$$d\hat{X}_t = \sigma(t, \hat{X}_t, \hat{M}_t) dW_t \text{ with } (\hat{X}_0, \hat{M}_0) = (x_0, m_0) \text{ and } x_0 \leq m_0 \text{ in } \mathbb{R}^2, \quad (1)$$

since under some integrability assumptions, any martingale  $X$  can be "represented" by a (weak) solution of the SDE (1) such that  $(X_t, M_t) \stackrel{law}{=} (\hat{X}_t, \hat{M}_t)$  for  $t \in [0, T]$ . Thus, we study the Fokker-Planck's PDE in connection with (1). The existence (in the large sense) derives from the weak existence of the SDE (1), and the uniqueness is proved under additional assumptions on  $\sigma$  and smoothness of the marginal laws. Without the regularity assumptions, we provide sufficient conditions insuring the existence of a family of continuous martingales with joint marginal laws  $\mathcal{V}^\epsilon$  converging to  $\mathcal{V}$ , and we extract some converging subsequence of  $\mathcal{V}^\epsilon$  using a tightness criteria. The final result (in the smooth case) has the following form: "Assume that there exists some function  $\sigma$  such that the family  $(\nu_t)_{t \in [0, T]}$  satisfies a certain Fokker-Planck EDP that depends on  $\sigma$ , then there is exists a martingale such that  $(X_t, M_t) \stackrel{law}{=} \nu_t$  for  $t \in [0, T]$ . This martingale can be constructed as a (weak) solution of the SDE (1). The equality in law of the marginals is ensured in this case by the uniqueness of the Fokker-Planck's PDE.

**Key words.** Principal-Agent, moral hazard, Capacity Remuneration Mechanism, Decarbonation, martingale and its running maximum.





# Chapter 1

## Introduction

Cette thèse est constituée de deux parties indépendantes. La première (chapitres 2 et 3) se focalise sur la modélisation de deux problématiques de régulation dans le marché d'énergie. La deuxième (chapitre 4) porte sur la description des lois jointes entre une martingale continue et son maximum courant. Les chapitres peuvent être lus séparément, et cette introduction résume la motivation derrière chacun d'entre eux, et donne un aperçu sur les modèles et outils utilisés, ainsi que les résultats obtenus.

### 1.1 Partie I : Théorie des contrats et application au marché d'énergie

Nous commençons par un rappel sur le cadre général du problème du Principal-Agent inspiré de la présentation plus détaillée dans Cvitanic & Zhang (2013) ou Cvitanic et al. (2018), en donnant un bref aperçu de la littérature de plus en plus abondante sur ce problème, dont nous nous servons que de quelques résultats à des fins de modélisation de deux problématiques; celles des marchés de capacités et des incitations à la décarbonation. En passant, nous mettons en avant une résolution explicite de ce problème en terme d'équations de Riccati matricielles dans un cadre linéaire quadratique avec des utilités exponentielles.

#### 1.1.1 Formulation du problème du Principal-Agent en temps continu

La théorie des contrats, ou problème du Principal-Agent est un problème classique en microéconomie qui étudie le choix et l'impact des incitations de la part d'une entité (le Principal) sur les actions ou les efforts d'une autre (l'Agent) dans le but de maximiser un certain critère. Ce choix (ainsi que l'utilité qu'il permet d'atteindre) dépend d'une façon cruciale des fonctions objectifs des deux entités et du degré d'asymétrie d'information, étant donné que dans la plupart des situations, le principal n'observe pas les actions de l'agent, ce qui crée un risque moral.

Plus concrètement, la formulation la plus simple du problème du principal-agent nécessite un processus contrôlé  $X$ , et deux entités; le Principal et l'Agent. Par exemple, le processus contrôlé peut représenter la valeur d'une firme et est désigné comme étant "l'output". Le Principal détient la firme, et délègue sa gestion à l'Agent, i.e., l'Agent se charge du contrôle du processus  $X$  avec un effort  $\alpha$ , en modifiant sa loi  $\mathbb{P}^\alpha$ . Le Principal embauche donc l'Agent à l'instant  $t = 0$  pour la période  $[0, T]$ , en échange d'un paiement terminal (un contrat)  $\xi$  payé à maturité  $T$ , basé sur l'évolution du processus output pendant l'intervalle de temps  $[0, T]$ . En d'autres termes,  $\xi$  est une variable aléatoire  $\mathcal{F}_T$ -mesurable, (une fonction des incertitudes réalisées sur  $[0, T]$  par l'output  $X$ ), et donc  $\xi$  peut être une fonction de la valeur de la firme  $X$  (un pourcentage du gain final par exemple, ou une autre fonction de la trajectoire

entière, etc..). Cependant, l'effort de l'Agent  $\alpha$  n'est pas toujours observable pour le principal, et donc  $\xi$  ne peut pas dépendre de cet effort, d'où le risque moral.

Chacun des deux acteurs cherche à maximiser sa fonction objectif. L'Agent agit sur le processus output  $X$  via  $\alpha$  ou la loi induite  $\mathbb{P}^\alpha$  (sa décision de gestion) et subit le coût  $c^\alpha(\alpha)$  en fonction de son effort contre un paiement  $\xi$  à la date  $T$  de la part du Principal. Pour bien poser le problème, on impose que l'Agent ait une utilité de réservation (ou contrainte de participation) sous la forme  $U_A(\mathcal{R})$ , avec un équivalent certain monétaire  $\mathcal{R}$ : l'Agent n'accepte un contrat  $\xi$  que lorsque ce dernier satisfait  $V^A(\xi) \geq U_A(\mathcal{R})$ , avec pour  $\xi \in \mathcal{C}$ ,

$$V^A(\xi) := \sup_{\mathbb{P}^\alpha \in \mathcal{P}} J^A(\xi, \mathbb{P}^\alpha) = \sup_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_A \left( \xi - \int_0^T c^\alpha(\alpha_t) dt \right) \right], \text{ pour } \xi \in \mathcal{C}, \quad (1.1.1)$$

où  $\mathcal{P}$  est l'ensemble des contrôles admissibles; typiquement ceux assurant que le coût soit fini (ou intégrable), et  $\mathcal{C}$  désigne l'ensemble des contrats (les variables aléatoires  $\mathcal{F}_T$  mesurables suffisamment intégrables). Pour un contrat donné  $\xi$ , on définit  $\mathcal{P}^*(\xi)$  comme étant l'ensemble de contrôles  $\mathbb{P}^\alpha \in \mathcal{P}$  permettant d'atteindre le maximum dans (1.1.1), i.e.,

$$\mathcal{P}^*(\xi) := \{ \mathbb{P}^\alpha \in \mathcal{P}, \text{ avec } V^A(\xi) = J^A(\xi, \mathbb{P}^\alpha) \}. \quad (1.1.2)$$

Dans le cas où l'Agent accepte le contrat, le problème peut être formulé comme suit; le Principal bénéficie du processus output  $X$ , et essaie d'inciter l'Agent à faire un effort  $\alpha$  via un contrat  $\xi$ . Le Principal cherche à trouver l'incitation optimale  $\xi$ , tout en respectant la contrainte de participation de l'Agent. Comme déjà mentionné, le problème du principal dépend de l'information qui lui est disponible. Nous écrivons d'abord sa fonction objectif, puis détaillons les différentes situations

$$J^P(\xi, \mathbb{P}^\alpha) := \mathbb{E}^{\mathbb{P}^\alpha} [U_P(-\xi + X_T^\alpha)], \text{ pour } (\xi, \mathbb{P}^\alpha) \in \mathcal{C} \times \mathcal{P}. \quad (1.1.3)$$

### Le statu quo ou “business as usual”

Cette situation correspond au cas où le Principal choisit de ne pas offrir de contrat, i.e.,  $\xi = 0$ . Il s'agit bien évidemment d'un choix qui n'est pas forcément optimal, mais reste intéressant à étudier, puisqu'il fournit un bon estimateur de la contrainte de participation de l'Agent (ou son équivalent monétaire  $\mathcal{R}$ ). En effet, dans ce cas le problème de l'Agent s'écrit

$$U_A(\mathcal{R}) := V^A(0) = \sup_{\mathbb{P}^\alpha \in \mathcal{P}} J^A(0, \mathbb{P}^\alpha). \quad (1.1.4)$$

Il s'agit d'un problème de contrôle stochastique markovien qui peut être résolu par programmation dynamique (Fleming & Soner (2006), Touzi (2012)). Quand la borne sup est atteinte dans (1.1.4) pour un certain contrôle admissible, on dénote ce contrôle par  $\mathbb{P}^{\alpha^{\mathcal{R}}}$ . Du point de vue du Principal, la fonction objectif est comme suit

$$V_{\text{BU}}^P := J^P(0, \mathbb{P}^{\alpha^{\mathcal{R}}}), \quad (1.1.5)$$

et peut être interprétée comme une référence pour la fonction valeur du Principal, par analogie avec la contrainte de participation de l'Agent (1.1.4).

### Le partage du risque ou “first best”

La situation de partage du risque ou “first best” correspond au cas où le Principal a accès à toute l’information, en particulier les incertitudes ainsi que l’effort de l’agent. Dans ce cas le Principal choisit l’effort de l’Agent et le paiement terminal. Sa fonction objectif s’écrit

$$V_{\text{FB}}^{\text{P}} := \sup_{\xi \in \mathcal{C}} \sup_{\mathbb{P}^\alpha \in \mathcal{P}} J^{\text{P}}(\xi, \mathbb{P}^\alpha) \text{ tel que } J^{\text{A}}(\xi, \mathbb{P}^\alpha) \geq U_{\text{A}}(\mathcal{R}). \quad (1.1.6)$$

La résolution de ce problème peut être réalisée par une linéarisation de la contrainte de participation, possible grâce à la linéarité de l’espérance (Borch (1992)), ce qui réduit (1.1.6) à un problème de contrôle stochastique markovien pouvant être résolu par programmation dynamique et vérification. Cette approche sera celle utilisée dans ce qui suit. Notons qu’un autre travail récent permet de résoudre ce problème par comparaison de différents contrats à l’aide des inégalités de Hölder (Martin & Réveillac (2019)).

### Le risque moral ou “second best”

Le risque moral ou situation “second best” pour le Principal correspond au cas où il n’observe plus l’effort de l’Agent, mais seulement le processus “output”. La fonction valeur du Principal s’écrit

$$V_{\text{sb}}^{\text{P}} := \sup_{\substack{\xi \in \mathcal{C} \\ V^{\text{A}}(\xi) \geq U_{\text{A}}(\mathcal{R})}} \sup_{\mathbb{P}^\alpha \in \mathcal{P}^*(\xi)} J^{\text{P}}(\xi, \mathbb{P}^\alpha). \quad (1.1.7)$$

Notons que le contrat  $\xi$  satisfait la contrainte de participation  $V^{\text{A}}(\xi) \geq U_{\text{A}}(\mathcal{R})$ . La première maximisation (sur l’ensemble  $\mathcal{P}^*(\xi)$ ) dépend du choix du contrat  $\xi$  et exprime la réponse de l’Agent à ce contrat puisqu’elle appartient à l’ensemble de ses réponses optimales  $\mathcal{P}^*(\xi)$ . Dans le cas où  $\mathcal{P}^*(\xi)$  n’est pas réduit à un singleton, on supposera que l’agent est coopératif et choisit le contrôle qui maximise le critère du principal.

Cette situation est plus intéressante et réaliste que les deux précédentes, et permet de modéliser une multitude de problématiques. Cependant, de part sa définition, le problème (1.1.7) est non markovien, et sa dépendance par rapport à la variable  $\xi$  est non linéaire; sa résolution n’est donc pas directe.

Nous nous appuyons sur le principe de programmation dynamique pour la résolution de ce problème pour les systèmes non dégénérés, initiée par Sannikov (2008), puis formalisée dans Cvitanic et al. (2018) et généralisée dans Lin et al. (2020). Elle consiste à résoudre le problème dans un sous ensemble de contrats révélateurs “Incentive Compatible”; sous lesquels la réponse de l’agent peut être anticipée, puis démontrer que cet ensemble de contrats révélateurs est assez riche pour ne pas perdre en généralité.

L’idée sous-jacente est de considérer le contrat comme un indice de performance (en fonction de l’utilité de continuation de l’Agent dans un contexte dynamique) comme une variable d’état et de considérer le problème de contrôle résultant; désormais markovien et avec une variable d’état supplémentaire, et qui peut donc être résolu par les méthodes de contrôle stochastique markovien. La richesse du sous ensemble de contrats révélateurs est exprimée via un résultat de représentation (d’Equations aux Dérivées Stochastiques Rétrogrades).



## Sélection adverse ou “third best”

La sélection adverse représente un cas encore plus général, où il existe différents types d’Agents avec différentes caractéristiques, et que le Principal ne connaît pas le type de l’Agent, en plus de ne pas observer son action. Nous présentons brièvement ce problème comme dans [Cvitanic & Zhang \(2013\)](#)[chapitre 1].

Dans notre exemple où le Principal délègue la gestion de la firme à l’Agent, il est raisonnable de supposer que l’Agent est caractérisé par un niveau de compétence  $\theta$  (un type) qui impacte ses efforts et ses coûts, et sa réponse aux incitations. Pour résoudre ce problème le Principal offre cette fois un menu de contrats bien choisis “révélateurs de types”, i.e., le type de l’agent ( $\theta$ ) est révélé par son choix du contrat  $\xi(\theta)$  au Principal.

Le problème de l’Agent dépend désormais de son type et s’écrit, lorsqu’il est de type  $\theta$  et que le Principal lui propose un contrat  $\xi(\theta')$ :

$$V^A(\xi(\theta'), \theta) := \sup_{\mathbb{P}^\alpha \in \mathcal{P}} J^A(\xi(\theta'), \mathbb{P}^\alpha, \theta). \quad (1.1.8)$$

On note  $\Theta$  l’ensemble des types d’Agents, et on suppose que le Principal possède un à priori sur la distribution de  $\Theta$  exprimé par la fonction  $F$ . On note aussi  $\mathcal{T}$  l’ensemble des menus de contrats révélateurs, i.e., qui s’écrivent  $(\xi(\theta'))_{\theta' \in \Theta}$  et tels que l’Agent choisit un certain  $\xi(\theta)$  du menu seulement si il est de type  $\theta$ . On suppose aussi que l’optimum dans le problème de l’agent de type  $\theta$  est atteint par un unique contrôle  $\mathbb{P}^\alpha(\theta)$ . Le problème du Principal s’écrit donc

$$V_{\text{TB}}^P := \sup_{\xi \in \mathcal{T}} \int_{\theta \in \Theta} (J^P(\xi(\theta), \mathbb{P}^\alpha(\theta)) + \lambda(\theta) J^A(\xi(\theta), \mathbb{P}^\alpha(\theta), \theta)) dF(\theta), \quad (1.1.9)$$

où  $\lambda(\theta)$  est un multiplicateur de Lagrange associé à la contrainte de participation de l’Agent. Sous certaines hypothèses ce problème est traité dans [Cvitanic & Zhang \(2013\)](#), mais ne fera pas l’objet de cette thèse.

### 1.1.2 Application aux mécanismes de rémunération de capacité

Notre première application est sur la modélisation des mécanismes de rémunération de capacité électrique–CRM pour Capacity remuneration mechanism–. Ces derniers représentent une rémunération complémentaire aux prix spot, et servent à garantir la sécurité d’approvisionnement pour les consommateurs, et la rentabilité des centrales électrique.

Pour mieux expliquer les CRM, nous nous focalisons sur l’exemple Français, où un mécanisme de capacité est instauré depuis janvier 2017. Rappelons d’abord que l’électricité ne peut pas être stockée en grande quantité, et que la demande est inélastique (dépend très peu des prix, sauf en cas d’incitations (par exemple [Aïd et al. \(2018\)](#)), ce qui est plus l’exception que la règle). Ainsi la production électrique doit être supérieure ou égale à la demande en permanence, sous peine de délestages dans le système, ou d’un blackout total en cas extrêmes, avec des impacts économiques désastreux.

D’autre part, la demande est très volatile (en plus d’être saisonnière), avec une moyenne stagnante ces dernières années (autour de 55 GW) et des pics bien supérieurs à cette moyenne (plus que 100 GW en 2012 en heure de pointe hivernale).

Ceci implique qu'une partie des centrales de production n'est sollicitée que rarement (aux heures de pointe), et devient incapable de couvrir ses coûts avec seulement la rémunération de l'énergie électrique produite, et non viable économiquement, mais quand même nécessaire; la fermeture définitive de ces centrales exposerait les consommateurs aux délestages lors d'un pic de demande.

Ce phénomène est accentué par la part croissante des énergies renouvelables dans le système qui ont des coûts de production quasi-nuls et sont toujours sollicitées en premier, diminuant encore plus la part de marché (et la rentabilité) des centrales classiques.

Pour toutes ces raisons, le régulateur impose un objectif de nombre d'heures de délestage annuel à ne pas dépasser (LoLe, pour Loss of Load Expectation), fixé à trois heures par an pour la plupart des pays [Newbery \(2016\)](#). Dans le cas Français, pour atteindre cet objectif, le Réseau de Transport d'Électricité (RTE) impose aux fournisseurs d'électricité au début de chaque année de détenir un certain nombre de certificats ou garanties de capacité de production en fonction du nombre de leur clients (consommateurs).

Ces garanties de capacités sont achetées auprès des producteurs d'électricité à un prix fixé par enchères sur le marchés des capacités. Les revenus générés par ces certificats représentent une rémunération pour la disponibilité des centrales de production, indépendamment de la production qu'ils fourniront au futur, et permet d'améliorer leur rentabilité tout en garantissant la sécurité d'approvisionnement; ce qui constitue le CRM Français.

Une étude extensive des mécanismes de rémunération de capacité est menée dans [Léautier \(2016\)](#), [Bhagwat et al. \(2017\)](#), ainsi que [Scoufflaire \(2019\)](#) et [Bublitz et al. \(2019\)](#), et montre qu'il n'y a pas encore de consensus sur un design optimal d'un tel mécanisme.

L'objectif du chapitre 2 est de proposer un modèle de mécanisme de rémunération de capacité avec une approche Principal-Agent, qui permet de prendre en compte l'asymétrie d'information et les incertitudes externes.

L'Agent est le producteur d'électricité, qui choisit l'investissement (ou l'effacement) optimal dans les centrales de production, définissant ainsi une capacité de production instantanée  $X^C$ , pour satisfaire une demande en électricité  $X^D$ , contre une rémunération spot et une compensation de capacité.

Le Principal est l'agrégation des consommateurs, ou une entité représentative (le réseau de transport d'électricité par exemple), et son but est de choisir la compensation de capacité optimale (sous forme de contrat) qui permet de trouver un bon compromis entre coûts élevés ou risque de délestage.

Nous supposons que les deux acteurs sont risque-averse, avec des fonctions d'utilité exponentielles, et nous résolvons le problème et explicitons le paiement optimal ainsi que la politique d'investissement qu'il induit en terme de solution d'EDP semi-linéaires, que nous solvons numériquement.

Le paiement de capacité obtenu est défini ex-ante et payé ex-post, i.e., son montant exact n'est pas connu à la date initiale, mais seulement sa forme en fonction des incertitudes à venir, ce qui permet de transférer une partie du

risque financier du producteur au consommateur. En outre on obtient une décomposition sous la forme

$$\text{Rémunération de capacité} + \text{Rémunération spot} = \mathcal{R} + \text{Coûts du producteur} + \text{Risque partagé} + \text{compensation de risque}, \quad (1.1.10)$$

avec  $\mathcal{R}$  l'équivalent monétaire de la contrainte de participation du producteur. Ainsi, on voit que dans notre modèle le producteur est payé en fonction de ses coûts et des aléa, mais au même temps compensé pour cet aléa en fonction de son aversion au risque.

Enfin nous étudions différents scénarios sous cette politique et calculons différentes sensibilités, et nous comparons nos résultats avec le marché de capacité actuel (Français).

### 1.1.3 Incitations à la décarbonation dans un cadre linéaire-Quadratique

Dans cette partie, nous nous intéressons à la réduction des émissions des gaz à effets de serre (GES) pour lutter contre le réchauffement climatique.

En ce qui concerne la France, la stratégie nationale bas-carbone ([SNBC \(2020\)](#)) a pour objectif de réduire les émissions des Français et d'atteindre la neutralité carbone en 2050, i.e., un équilibre entre les émissions et l'absorption des GES par l'écosystème.

L'un des moyens identifiés pour arriver à cet objectif se concentre sur l'économie et l'industrie et consiste à accompagner et soutenir les entreprises dans une transition vers des systèmes de production bas carbone dans le but de diminuer leurs émissions de 35% d'ici 2030, et de 81% pour 2050. La question qui se pose alors est sur la forme et l'intensité optimale des subventions que l'État devrait employer pour arriver à cet objectif, étant donné la multitude de possibilité; plus de 700 formes d'interventions publique identifiées pour un montant total de plus 100 milliards d'euros dans l'Union Européenne en 2012 ([Alberici et al. \(2014\)](#)).

L'un des principaux défis dans ce choix est celui du risque moral. En effet, en choisissant une subvention pour l'économie ou pour un certain secteur d'activité, l'État (ou un observateur externe quelconque) ne peut pas distinguer entre une diminution des émissions de GES due à une diminution de consommation et de production, ou due à un effort d'amélioration des moyens de production vers d'autres moins polluants, sans perturber le niveau de production.

L'objectif du chapitre 3 est de définir une subvention optimale sous la forme d'un contrat dynamique qui permet de pallier à cette difficulté avec un modèle de Principal-Agent.

Dans ce cas l'Agent est le secteur d'activité (ou l'économie), dont le but est de déterminer un certain niveau d'investissement en capital  $K$  pour satisfaire une demande  $D$  (en termes monétaires). En l'absence de toute forme d'incitations, l'objectif de l'agent est de maximiser la richesse totale, équivalente à la somme des surplus du consommateur et du producteur.

Une formulation alternative (et équivalente) du problème plus simple à interpréter consiste à utiliser les mêmes variables d'états  $K$  et  $D$ , mais comme étant cette fois respectivement le produit intérieur brut (PIB) pour  $K$  et le produit intérieur brut ciblé pour  $D$ . Le problème de l'agent est encore de choisir le niveau d'investissement optimal,

afin de minimiser l'écart entre PIB ciblé et PIB réalisé.

En partant du constat empirique que les émissions de carbone sont procycliques i.e., augmentent en période d'expansion économique et diminuent en récession [Khan et al. \(2019\)](#), nous supposons que le capital (ou le PIB)  $K$  est une source d'émissions des GES dont le taux est le processus  $E$ .

Le principal est l'État, et son objectif est de proposer une compensation à l'agent pour l'inciter à faire des efforts pour diminuer les émissions  $E$ , tout en continuant à investir dans  $K$ .

Pour simplifier la modélisation et obtenir des résultats explicites, nous supposons une dynamique linéaire (contrôlée) pour les variables d'états avec des fonctions objectifs et des coûts quadratiques et des fonctions d'utilités exponentielles pour chacun des deux acteurs.

Ces hypothèses sur le modèle nous permettent de réduire le problème à la résolution d'équation de Riccati Matricielles dans les trois différents cas; statu quo, partage de risque, et risque moral.

Ces équations de Riccati présentent l'avantage d'être faciles à implémenter numériquement, même en grande dimension, et nous permettent de résoudre le problème et de comparer différentes politiques d'incitation et de voir les sensibilités des acteurs par rapport aux coûts et aux incertitudes.

## 1.2 Partie II : Étude des lois jointes d'une martingale et de son maximum courant

La deuxième partie de cette thèse présente un résultat sur les lois jointes entre les marginales d'une martingale et son maximum courant. Nous commençons par introduire notre problème et poser le cadre mathématique tout en expliquant ses motivations, et en donnant un aperçu sur la littérature sous-jacente. Puis, dans la même lignée, nous décrivons brièvement la méthode de résolution et indiquons notre résultat principal.

### 1.2.1 Problématique et motivations

Nous nous intéressons au problème suivant:

$\mathcal{Q}_1$ : “Étant donné  $\mathcal{V} := (\nu_t, t \in \mathbb{T})$  une famille de probabilités sur  $\mathbb{R}^2$ , existe-t-il une martingale  $(X_t)_{t \geq 0}$  avec un maximum courant  $M_t := M_0 \vee \sup_{u \leq t} X_u$  telle que  $\nu_t \stackrel{\text{loi}}{=} (X_t, M_t)$ , pour  $t \in \mathbb{T}$  ?” où  $\mathbb{T}$  peut être un intervalle, un ensemble (fini ou dénombrable) ou un singleton.

Pour donner plus de perspective aux motivations et implications de  $\mathcal{Q}_1$ , nous rappelons un problème similaire plus simple et bien connu dans la littérature:

$\mathcal{Q}_2$ : “Étant donné  $\mu := (\mu_t, t \in \mathbb{T})$  une famille de probabilités sur  $\mathbb{R}$ , existe-t-il une martingale  $(X_t)_{t \geq 0}$  telle que  $\mu_t \stackrel{\text{loi}}{=} X_t$ , pour  $t \in \mathbb{T}$  ?”.

Le problème  $\mathcal{Q}_2$  fut résolu successivement par [Strassen \(1965\)](#) pour un ensemble  $\mathbb{T}$  fini ou dénombrable, et [Kellerer \(1972\)](#) pour  $\mathbb{T}$  un intervalle (pouvant être infini) en introduisant la notion de Peacock, terminologie dérivée

de PCOC (Processus Croissant dans l'Ordre Convexe). On définit  $\mathcal{P}_1(\mathbb{R})$  comme l'ensemble des mesures de probabilités intégrables sur  $\mathbb{R}$ , et pour une famille  $\boldsymbol{\mu}$  de probabilités, on définit  $\mathcal{M}(\boldsymbol{\mu})$  l'ensemble des martingales satisfaisant  $\mu_t \stackrel{\text{loi}}{=} X_t$ , pour  $t \in \mathbb{T}$ . Les deux résultats sont donnés par ce théorème

**Theorem 1.2.1** (Strassen'64 et Kellerer'72).  $\mathcal{M}(\boldsymbol{\mu}) \neq \emptyset$  si et seulement si la famille  $(\mu_t, t \in \mathbb{T})$  est un Peacock, i.e., satisfait

(i)  $\forall t \geq 0, \mu_t \in \mathcal{P}_1(\mathbb{R})$ ,

(ii) pour toute fonction convexe  $f : \mathbb{R} \rightarrow \mathbb{R}$ , la fonction  $t \mapsto \int_{\mathbb{R}} f(x) \mu_t(dx)$  est croissante.

Plus récemment, le théorème de Kellerer à été redémontré avec une preuve constructive [Hirsch & Roynette \(2012\)](#) et [Hirsch et al. \(2015\)](#).

Un résultat de ce type présente un intérêt double en finance de marché, en terme de gestion des risques de modèle, et d'arbitrage.

En effet, rappelons que l'hypothèse fondamentale en valuation et couverture de produits dérivés en finance de marché est l'absence d'opportunités d'arbitrages. Cette hypothèse peut être reformulé par dualité ([Kreps \(1981\)](#), [Harrison & Pliska \(1981\)](#) et [Delbaen & Schachermayer \(1994\)](#)) comme l'existence d'au moins une probabilité équivalente "risque neutre" sous laquelle tout actif échangé sur le marché suit une martingale. Par ailleurs, pour un actif financier  $X$  (ayant assez de liquidité) les prix des options européennes cotés sur le marché permettent d'estimer les lois marginales de  $X$  ([Breedon & Litzenberger \(1978\)](#) et [Dupire \(1994\)](#)). En prenant comme exemple les options de vente (Put) ayant une maturité  $t \in \mathbb{R}_+$  et un prix d'exercice (strike)  $x$ , et en négligeant le taux d'intérêts sans risques et les dividendes, le prix de cette option est donné comme l'espérance de son payoff sous la probabilité risque neutre (sous laquelle  $(X_t)_{t \geq 0}$  est une martingale), i.e.,  $P_t(x) := \mathbb{E}[(x - X_t)^+]$ . On voit donc qu'une dérivation (sous réserve de régularité ou dans un sens faible) de ce prix par rapport au strike  $x$  nous donne  $\mu_t = \partial_{xx} P_t$ , et toute la famille  $(\mu_t, t \geq 0)$  peut être estimée d'une façon similaire.

Ainsi, on obtient deux cas distincts en fonction de la réponse à  $\mathcal{Q}_2$ ; si  $\mathcal{M}(\boldsymbol{\mu}) = \emptyset$ , alors il existe une opportunité d'arbitrage sur les prix cotés, et il est possible (en négligeant toute friction ou coût de transaction) de construire un arbitrage, i.e., un portefeuille auto-finançant, fournissant un gain positif. Sinon, si  $\mathcal{M}(\boldsymbol{\mu}) \neq \emptyset$ , alors  $\mathcal{M}(\boldsymbol{\mu})$  constitue l'ensemble des modèles pour le processus  $X$  qui sont cohérents avec les données du marchés, i.e., sous-lesquels les prix des produits dérivés (européens) sont les mêmes que ceux observés (et qui ne dépendent que de  $\boldsymbol{\mu}$ ). La question qui se pose alors est celle de la calibration; qui consiste à trouver un élément  $\mathcal{M}(\boldsymbol{\mu})$  qui soit assez explicite, l'exemple le plus connu étant celui de volatilité locale ([Dupire \(1994\)](#)). Par ailleurs,  $\mathcal{M}(\boldsymbol{\mu})$  peut être aussi interprété comme l'ensemble des solutions admissible (non vide dans ce cas) du problème de transport optimal sous contrainte martingale; qui s'écrit pour une variable aléatoire  $\xi$  comme suit

$$V(\boldsymbol{\mu}) := \sup_{\mathbb{P} \in \mathcal{M}(\boldsymbol{\mu})} \mathbb{E}^{\mathbb{P}}[\xi(X)], \text{ avec } \mathbb{E}^{\mathbb{P}}[|\xi(X)|] < \infty, \text{ pour } \mathbb{P} \in \mathcal{M}(\boldsymbol{\mu}). \quad (1.2.1)$$

Ce problème de transport optimal avec contrainte martingale est une variante du transport optimal classique avec une contrainte supplémentaire, et admet une formulation duale (de Kantorovich) qui s'interprète comme la couverture robuste (par rapport au modèle) des produits dérivés ([Galichon et al. \(2014\)](#)); en l'occurrence celui ayant un payoff  $\xi$  dans la formulation (1.2.1).

Dans la pratique, l'ensemble des produits financiers cotés sur le marché pour un certain actif peut être bien plus riche que les payoffs de type européens (les options de vente et d'achat), et il est légitime de s'attendre à observer une structure d'information plus riche que celle des lois marginales. En particulier, les événements conditionnés au dépassement d'un certain seuil (maximal ou minimal) sont importants et souvent présents dans les payoffs, car ils peuvent représenter une condition de défaut ou de faillite. On trouve donc souvent sur le marché des options de type barrière qui s'activent ou se désactivent en cas d'atteinte ou de dépassement d'un certain niveau par l'actif sous-jacent. Par exemple, une option de vente (put) barrière offre à son détenteur le droit de vendre un actif à un prix (strike)  $x$  et une maturité  $T$  fixée, mais seulement si le prix de l'actif a dépassé un seuil  $m$  pendant la période  $[0, T]$ . Le prix de ce produit peut être calculé comme l'espérance de son payoff sous la probabilité risque neutre, et s'écrit comme suit

$$P_t(x, m) := \mathbb{E} \left[ (x - X_t)^+ \mathbf{1}_{M_t \geq m} \right], \text{ avec } M_t := M_0 \vee \sup_{0 \leq u \leq t} X_u. \quad (1.2.2)$$

Les prix de ces produits sont cotés sur les marchés, et en considérant une collection de strikes, de seuils et de maturités et en les dérivant par rapport à  $x$  et  $m$  (sous réserve de régularité ou dans un sens faible) on obtient pour une maturité fixée  $\nu_t := -\partial_{x m} P_t$ , la probabilité jointe de  $(X_t, M_t)$ , et les données observés sur le marché ne sont plus restreint à la famille des lois marginales  $\mu$ , mais la famille  $\mathcal{V} = (\nu_t, t \in \mathbb{T})$ , où  $\nu_t \stackrel{\text{loi}}{=} (X_t, M_t)$ , pour  $t \in \mathbb{T}$ .

Cette observation introduit naturellement le problème  $\mathcal{Q}_1$  comme extension à  $\mathcal{Q}_2$ , et par analogie servirait à détecter l'existence d'arbitrages sur les options barrières dans le cas  $\mathcal{M}(\mathcal{V}) = \emptyset$ , et dans le cas contraire,  $\mathcal{M}(\mathcal{V}) \neq \emptyset$  servirait à décrire les modèles cohérents avec  $\mathcal{V}$ , plus précis que  $\mu$  dans le sens où  $\mathcal{M}(\mathcal{V}) \subset \mathcal{M}(\mu)$ , où servir de première étape pour poser une version du transport optimal sous contrainte martingale ainsi qu'une contrainte supplémentaire; celle de la loi du maximum.

## 1.2.2 Méthodologie et résultat

La majorité des résultats dans la littérature concernant ce problème sont des résultats de calibration, i.e., où on suppose déjà que les la famille des lois  $\mathcal{V}$  est issue d'une martingale et de son maximum courant, et on a pour but de décrire cette martingale à l'aide d'une EDS (Hambly et al. (2016), ou Henry-Labordere (2017), ou Forde (2014)).

A la connaissance de l'auteur, les résultats assurant l'existence de cette martingale sont dans un cas statique; où  $\mathbb{T}$  est un singleton. Pour une loi donnée  $\mu$  sur  $\mathbb{R}^2$ , plusieurs auteurs se sont intéressés à des conditions nécessaires pour que  $\mathcal{M}(\{\mu\}) \neq \emptyset$ , i.e., il existe une martingale  $X_{t \geq 0}$  telle que  $\mu$  soit la loi jointe de  $(X_\infty, X_\infty^*)$ , nous en citons Blackwell & Dubins (1963) et Azéma & Yor (1979) qui donnent une condition nécessaire sur la loi du maximum en terme de sa transformation de Hardy, ainsi que Vallois (1994) qui caractérise l'ensemble des lois du maximum dans le cas continue, et Rogers (1993) qui donne une condition nécessaire et suffisante assurant que  $\mathcal{M}(\{\mu\}) \neq \emptyset$  et/ou  $\mathcal{M}^0(\{\mu\}) \neq \emptyset$ , avec  $\mathcal{M}^0(\{\mu\})$  le sous ensemble de martingales continues de  $\mathcal{M}(\{\mu\})$ .

L'objectif du chapitre 4 est d'étendre ce résultat pour  $\mathbb{T} = [0, T]$ , i.e., de montrer que  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$ . La restriction qu'on fait sur les trajectoires (en imposant la continuité) est justifiée par son implication sur le maximum courant. En effet, dans ce cas la mesure engendrée par le maximum  $(M_t)_{t \in [0, T]}$  est concentrée sur la diagonale  $(M_t = X_t)$  puisque  $M_t$  ne varie que lorsque il est égal à  $X_t$ . Ainsi  $\int_0^T (X_t - M_t) dM_t = 0$ , ce qui simplifie les calculs.

Nous suivons la méthodologie dans Hirsch & Roynette (2012) (voir également (Hirsch et al. 2011, Chapitre 6)),

et notre résultat est constructif. Il peut donc aussi servir pour la calibration et rejoint les travaux de [Hambly et al. \(2016\)](#) et [Henry-Labordere \(2017\)](#), en relaxant les hypothèses de régularité.

Rappelons brièvement la méthodologie de [Hirsch & Roynette \(2012\)](#). L'idée est de supposer que  $\mathcal{M}^0(\mu) \neq \emptyset$ , puis d'identifier un élément de  $\mathcal{M}^0(\mu)$  sous la forme d'une solution d'une EDS avec un coefficient de diffusion  $\sigma$  déterministe

$$dX_t = \sigma(t, X_t)dW_t. \quad (1.2.3)$$

Ceci est en général possible (sous quelques hypothèses techniques) en vertu du résultat de [Gyöngy \(1986\)](#), et donne lieu dans ce cas au modèle de volatilité locale. Ensuite, les lois marginales de l'EDS (1.2.3) sont caractérisées comme unique solution de l'EDP de Fokker-Planck associée ([Hirsch et al. 2011](#), Chapitre 6.1). Ainsi, la famille des lois marginales de l'EDS (1.2.3) correspond bien à  $\mu$  si et seulement si  $\sigma$  est défini comme une certaine fonction de  $\mu$ , qui dans ce cas est donnée par les prix des options de ventes comme suit (dans le cas régulier)

$$\frac{1}{2}\sigma^2(t, x) = \frac{\partial_t P(t, x)}{\mu(t, x)}. \quad (1.2.4)$$

L'égalité (1.2.4) suggère que  $\sigma$  encode toute l'information contenue dans  $\mu$ , et que la seule condition suffisante (en plus d'être nécessaire par l'inégalité de Jensen) pour que  $\mathcal{M}(\mu) \neq \emptyset$  est que  $P$  soit croissant par rapport au temps, ce qui est équivalent à la croissance dans l'ordre convexe de la famille  $\mu$  ([Hirsch et al. 2011](#), Exercice 1.7). Ceci est enfin prouvé par construction dans le cas régulier, en définissant d'abord  $\sigma$  comme (1.2.4), puis l'EDS (1.2.3), et de montrer que cette dernière est un élément de  $\mathcal{M}(\mu)$  en utilisant le résultat d'unicité pour l'équation de Fokker-Planck associée à (1.2.3). Le cas général est enfin traité par régularisation et passage à la limite.

Dans notre cas, en supposant que  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$ , l'identification d'un élément de  $\mathcal{M}^0(\mathcal{V})$  qui soit une EDS est possible grâce à un résultat récent de projection markovienne de [Brunick & Shreve \(2013\)](#), qui généralise le résultat de [Gyöngy \(1986\)](#) pour les lois des fonctionnelles dépendantes de la trajectoire en plus des lois marginales; en particulier le maximum courant. En effet, sous quelques hypothèses d'intégrabilité, pour toute martingale continue  $X$ , il existe une fonction  $\sigma$  déterministe, et une solution faible de l'EDS

$$d\hat{X}_t = \sigma(t, \hat{X}_t, \hat{M}_t)dW_t \text{ avec } (\hat{X}_0, \hat{M}_0) = (x_0, m_0) \text{ et } x_0 \leq m_0 \text{ dans } \mathbb{R}^2, \quad (1.2.5)$$

telle que  $(X_t, M_t) \stackrel{\text{loi}}{=} (\hat{X}_t, \hat{M}_t)$  pour  $t \in [0, T]$ . Ainsi, si  $X \in \mathcal{M}^0(\mathcal{V})$  alors  $\hat{X} \in \mathcal{M}^0(\mathcal{V})$ . Pour  $\mathcal{V}$  donné et en supposant que  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$ , la calibration de  $\sigma$  a été faite dans [Hambly et al. \(2016\)](#) et [Henry-Labordere \(2017\)](#) ainsi que [Forde \(2014\)](#).

Pour caractériser les lois marginales  $(\hat{X}_t, \hat{M}_t)_{t \in [0, T]}$ , nous étudions l'EDP de Fokker-Planck associée à (1.2.5) et démontrons qu'elle est bien posée dans un certain cadre; l'existence découle de celle de l'EDS (1.2.5), et l'unicité peut être prouvée sous des hypothèses supplémentaires sur  $\sigma$  et de régularité. Une étude de cette EDP de Fokker-Planck (similaire) est faite dans [Coutin & Pontier \(2017\)](#) dans un cas d'une diffusion homogène par rapport au temps.

Cependant, pour reconstruire la diffusion à partir de l'EDP de Fokker-Planck (comme dans [Hirsch & Roynette \(2012\)](#)), une difficulté apparaît; les conditions de bord de l'EDP de Fokker-Planck (qui sont des conditions nécessaires pour que  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$ ) sont difficiles à expliquer ou transposer en termes probabilistes. On reformule donc le



problème à l'aide des options barrières pour donner une expression explicite à ces conditions et au calcul de la fonction  $\sigma$ . Nous obtenons deux types de conditions; une condition statique assurant que chacune des lois fixées est une probabilité jointe d'une martingale et de son maximum à une date fixe, et une condition dynamique qui décrit l'évolution de ces lois au cours du temps. On remarque donc une différence notable entre notre construction et celle de [Hirsch & Roynette \(2012\)](#), qui réside dans le fait que  $\sigma$  encode l'information contenue dans  $\mathcal{V}$ , mais ne permet pas de conclure quant au fait que  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$ , puisque on doit vérifier la condition statique à part.

Une fois ces conditions identifiées, on peut établir le résultat par une construction de l'EDS (1.2.5) dans un cas régulier, puis on généralise cette construction par une régularisation et un passage à la limite.

Sans les hypothèses de régularités, nous donnons des conditions suffisantes assurant l'existence d'une famille de martingales continues dont les lois marginales jointes  $\mathcal{V}^\epsilon$  convergent vers  $\mathcal{V}$ , et dont on peut extraire une limite grâce à un critère de tension. Les difficultés principales dans cette partie sont la régularisation des lois  $(\nu_t)_{t \geq 0}$  observées, étant donné qu'une simple convolution nous ferait perdre la structure état-maximum courant de ces lois; ainsi que la démonstration de l'unicité le l'EDP de Fokker-Planck qui s'avère plus délicate qu'en dimension 1, et nécessite une hypothèse technique supplémentaire.





## **Part I**

# **Théorie des contrats et application au marché d'énergie**



## Chapter 2

# A Principal–Agent approach to Capacity Remuneration Mechanisms

Joint work with Clémence Alasseur and Marcelo Saguan, published in the International Journal of Theoretical and Applied Finance (Alasseur et al. (2020)).

### Abstract

We propose to study electricity capacity remuneration mechanism design through a principal-agent approach. The principal represents the aggregation of electricity consumers (or a representative entity), subject to the physical risk of shortage, and the agent represents the electricity capacity owners, who invest in capacity and produce electricity to satisfy consumers' demand, and are subject to financial risks. Following the methodology of Cvitanić et al. (2017), we propose an optimal contract, from consumers' perspective, which complements the revenue capacity owners achieved from the spot energy market, and incentivizes both parties to perform an optimal level of investment while sharing the physical and financial risks. Numerical results provide insights on the necessity of a capacity remuneration mechanism and also show how this is especially true when the level of uncertainties on demand or production side increases.

**Key words.** Capacities market, capacity remuneration mechanism, principal-agent problem, contract theory.

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## 2.1 Introduction

Electricity market is characterized by the constraint that production must be equal to the consumption at any time. In case of non respect of this constraint, the system can incur a power outage whose consequences might be highly problematic. For example, the total economic cost of the August 2003 blackout in the USA was estimated to be between seven and ten billion dollars (Council (2004)). This blackout resulted in the loss of around 62 GW of electric load that served more than 50 million people at the USA-Canada border. Besides, it took 2 days for major affected areas to have the power restored, while some regions had to wait up to a full week.

As electricity can hardly be stored; hydro storage is limited in size, and developing a large fleet of batteries is still highly costly, the power production capacity must be high enough to cope with major peak load events which can reach extreme levels compared to the average load. In France for example the average load was around 55GW in 2017<sup>1</sup>, whereas the peak of electricity consumption record was above 100 GW in February 2012. Indeed, electrical load is characterized by a high variability implied by meteorological variations and economic conditions on different time scales. Again, in France for example, the difference between peak load in 2012 and 2014 is around 20 GW, which corresponds to an equivalent capacity of around 40 combined cycle gas turbines (CCGT) of 500 MW<sup>2</sup>. To ensure security of supply, most electricity systems specify the *Loss of Load Expectation* (LoLE) which is a reliability target for the electricity system, and has been fixed in some countries at three hours at most per year Newbery (2016).

The consequence of such constraints on the production system is that some power plants are used rarely (only during extreme peak load) but remain necessary for the system security, and insuring their economical viability with energy markets only is not guaranteed. This question has already motivated a great amount of economic literature under the name of “missing money” Joskow (2006). This lack of revenue can occur because of energy market imperfections such as price caps or out-of-market actions made by the transmission system operator as well as reliability targets going beyond reliability outcome provided by the market.

The “missing money” issue might be further increased when the share of renewable energies increases in the system Newbery (2016). Indeed, renewable energies have low variable costs so their introduction has made electricity prices lower. See Brown (2018) or Levin & Botterud (2015) for a proof that subsidized renewable capacity pushes downward energy prices. This could result in the withdrawal of most expensive power plants, jeopardizing the security of electricity system and the lack of incentives to invest in new capacities.

In addition, even without the “missing money” problem, electricity markets are highly volatile for the reasons already stated (i.e. low level of storage, high uncertainties on load and production levels induced by outages and meteorological conditions impacting the production capacities) and suffer from “missing markets” issues Newbery (2016) such as the horizon shortness of contracts proposed by electricity markets compared to the lifetime of power-plants. For all these reasons, the financial risk is particularly high for investors in electricity capacities and may lead to high hurdle rates, see Hobbs et al. (2007) for a model which studies how capacity markets variations can lower capital costs for generators by reducing risks).

For all the reasons cited above, several regions of the world have decided to put in place a capacity remuneration

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<sup>1</sup>See the French TSO website <http://bilan-electrique-2017.rte-france.com>.

<sup>2</sup>See <https://www.ecologique-solidaire.gouv.fr/securite-dapprovisionnement-en-electricite>.

mechanism (CRM), in addition to energy markets. This kind of markets aims at insuring a payment for electricity generating assets for the capacities they provide, regardless of their actual production. This market can be thought of as a payment for an insurance provided by the power plant against shortage and blackout risk. However, no consensus on the design of such CRM arose so far, see for example [Bublitz et al. \(2019\)](#) for a review of theoretical studies and implementations of CRM or [Bhagwat et al. \(2017\)](#) for a survey of different capacity markets implemented in the USA.

In [Scouflaire \(2018\)](#), the author argues that CRM do improve security of supply, in exchange of a significant impact on consumer’s bill in the USA, as opposed (surprisingly) to the EU where the impact on the end users price is not significant. An analysis of the impact of capacity on welfare under a price-capped electricity market is made on the Texas market (ERCOT) in [Bajo-Buenestado \(2017\)](#) showing that capacity markets have several effects: an increase of the wholesale electricity price and reliability and a reduction of price volatility. Several mechanisms and their corresponding conditions for achieving efficiency are studied in [Léautier \(2016\)](#). In [Briggs & Kleit \(2013\)](#), distortion of capacity markets implied by subsidies of base load capacities are pointed out and correction mechanisms such as the minimal offer price rule (MOPR) tested in the PJM markets are studied. Currently in Europe, several designs of CRM have been adopted such as a market capacity for example in the United Kingdom, France, Italy, Belgium and Ireland<sup>3</sup>(under construction); a capacity payment as in Spain or Portugal; strategic reserves in Sweden or in Germany.

In the literature, several papers study different CRM designs with distinct modeling approaches. For example in [Hary et al. \(2016\)](#), the authors compare the benefits of capacity markets or strategic reserves versus energy-only design in terms of security of supply, investment and generation costs in a dynamic model of investing. This is the same approach developed by [Hach et al. \(2016\)](#) and applied to the UK market. [Höschle et al. \(2017\)](#) analyse the impact of capacity mechanism on energy markets and on the remuneration of flexibility and emission-neutral renewable capacities. In [Bhagwat et al. \(2017\)](#), the authors implement the UK capacity markets in an Agent-based model where Agents have a limited vision of the future. In [Hermon et al. \(2007\)](#), the authors model two CRMs—in particular under information asymmetry—using agency theory. They model capacity payments as a menu of contracts and strategic reserves as a retention rule of a bilateral contract between the TSO and a producer and then compare these CRMs. The information asymmetry is mainly on the “type” of the generator, namely its access to the capital market which impacts its efficiency.

In this work, we propose a principal-agent framework to shed light on the design of CRM in a context of information asymmetry and external uncertainties (in production and demand). Using the recent developments of contract theory [Cvitanić et al. \(2018\)](#), we model and solve the problem in a continuous time setting, which allows us to dive deeper into the incentive mechanism, and provide a recommended policy for investment in electric power plants, with an optimal dynamic capacity payment allowing for an efficient (financial and physical) risk sharing between consumers and producers.

In the scope of the paper, producers and retailers are fully separated and exchange electricity through spot markets, and the electricity demand is considered to be inelastic. The relationship between consumers and producers is modeled by a principal-agent problem, with the principal being the aggregation of power consumers (or an equivalent entity representing them), and the agent the collection of producers. Note that the transmission system operator

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<sup>3</sup>See <https://www.sem-o.com/markets/capacity-market-overview> and [https://ec.europa.eu/commission/presscorner/detail/en/IP\\_17\\_4944](https://ec.europa.eu/commission/presscorner/detail/en/IP_17_4944).

(TSO) which operates in real-time in many electricity markets could be considered as a representative entity of the aggregation of consumers.

Our model accounts for the information asymmetry on the actions of the agent –moral hazard–, i.e., consumers do not observe producers’ actions but only the results of their actions. In fact, consumers want to incentivize producers in an optimal way to provide electricity when needed, but at the same time they have no information on the commitment of the latter (producers) to build or maintain power plants, as they only observe the volume of electricity produced, not the effort of capacity owners to install new power plants or to keep the existing ones in good operation conditions. The model developed enables us to specify an optimal contract which incentivizes the producers to make the right level of investment to achieve a certain level of security for the system.

This proposed contract remunerates the capacity owners depending on realized uncertainties on the demand and available capacities while sharing the financial risks between consumers and producers. It is also shown that the more uncertainties on the system, such as the increase share of variable renewable capacities, the more a capacity remuneration is needed to ensure correct levels of investment and maintenance.

Finally, we provide a numerical illustration of the optimal capacity payment obtained with our proposed optimal contract, compared with the payment supplied by the spot market. This numerical illustration is inspired by the French electricity system.

The second section of this paper is devoted to the presentation of the model and the objective functions of both the agent and the principal, with a brief summary of the resolution methodology (the details of which are left to the Appendix). In the third section, we present our case study; the French electricity system and provide some numerical interpretations of the optimal capacity payment. Mathematical proofs and details are included in the Appendix for the sake of clarity.

## 2.2 The model

In order to study CRM in a context of information asymmetry, we propose a non-zero sum Stackelberg game with a principal-agent formulation, i.e., the gain of one party does not come necessarily from the loss of the other. In this setting, the aggregation of consumers or an entity representing them such as the TSO, proposes to producers a capacity payment which optimally complements the revenue they (producers) obtain on the spot energy market. This payment incentivizes them to invest optimally in power plants management (construction, maintenance, etc..) to ensure an acceptable level (for consumers) of shortage occurrences. The proposed payment is a way to correct the information asymmetry faced by consumers (as they cannot observe directly producers decisions concerning the capacities of the production mix, thus the need for incentive), and to share the risks coming from demand and available capacities uncertainties between the two parties. Moreover, the proposed payment limits producers’ potential abuse of market power. Indeed, without capacity payment, producers may decide to under-invest in order to obtain high remunerations from a spot market with more shortages and price spikes.

### 2.2.1 Principal-Agent Problem: a brief review

Contract theory, or principal–agent problem, is a classical moral hazard problem in microeconomics. The simplest formulation involves a controlled process  $X$  and two parties; the principal and the agent. The controlled process is

called output process and represents the value of the firm for example. Principal owns the firm, and delegates its management to agent, i.e., the control of the output process  $X$ . So principal hires agent at time  $t = 0$  for the period  $[0, T]$ , in exchange for a terminal payment (a contract)  $\xi$  paid at time  $T$ , based upon the evolution of the output process during the contracting time period. In other words,  $\xi$  is an  $\mathcal{F}_T$ -measurable random variable, (a function of the realized uncertainties on  $X$  up to time  $T$ ), and thus can be a function of the firm value (a percentage of the final gain for example, or a function of the whole trajectory, etc.). However, agent's effort is not observable and/or not contractible for principal, which means that  $\xi$  cannot depend on the effort (work) of agent, hence the moral hazard.

Each of the parties aims at maximizing a utility function. The agent acts on the output process  $X$  via some control  $\alpha$  (his management decision) and has to pay a cost  $c^A(\alpha)$  as a function of the efforts (the management decision  $\alpha$ ), and expects a payment  $\xi$  from principal at time  $T$ . Agent also has a reservation utility  $U_A(\mathcal{R})$ , to be thought of as a participation constraint, with  $\mathcal{R}$  the cash equivalent of this constraint: agent accepts the contract  $\xi$  only if  $\xi$  satisfies  $V^A(\xi) \geq U_A(\mathcal{R})$ , otherwise he will refuse it. In the case where agent accepts the contract, we can formulate his problem as follow:

$$V^A(\xi) = \sup_{\alpha} \mathbb{E} \left[ U_A \left( \xi - \int_0^T c^A(\alpha_t) dt \right) \right]. \quad (2.2.1)$$

The principal benefits from the output process  $X$ , and so incites agent to put the effort  $\alpha$  (to work hard) via the contract  $\xi$ . Principal tries to find the optimal incentive ( $\xi$ ), while respecting agent's participation constraint. Principal's problem is written therefore as:

$$V^P = \sup_{\substack{\xi \\ V^A(\xi) \geq U_A(\mathcal{R})}} \sup_{\alpha^*(\xi)} \mathbb{E} \left[ U_P \left( -\xi + X_T^{\alpha^*(\xi)} \right) \right], \quad (2.2.2)$$

where the contract  $\xi$  satisfies the participation constraint  $V^A(\xi) \geq U_A(\mathcal{R})$  and  $\alpha^*(\xi)$  denotes agent's optimal effort (response) given the contract  $\xi$ , i.e., the solution to (2.2.1). The first supremum (over  $\alpha^*(\xi)$ ) expresses the fact that given a contract  $\xi$ , agent solves his problem (we will see later that the existence of at least one solution is guaranteed). Then in case of existence of multiple solutions to (2.2.1), we assume that agent would choose the one which maximizes principal's value function (once his utility maximized, agent is cooperative with principal). We refer the interested reader to [Cvitanic & Zhang \(2013\)](#), and [Cvitanic et al. \(2018\)](#) for a more detailed exposition of contract theory and principal-agent problem.

As mentioned before, in our model, principal is the aggregation of consumers or an equivalent entity representing them, and agent is the collection of producers. In the sequel we will ease the presentation by referring to these two parties by simply saying "the consumer" and "the producer".

So agent is the producer who exerts an effort (a process which we will denote  $\alpha$ ), to build or invest in the maintenance of peak power plants, to increase the total capacity of the fleet. Agent is compensated an amount  $\xi$  by principal (the consumer) for the utility received; the satisfaction of consumption and the insurance against shortage risk. We also account for the moral-hazard (second best in principal-agent terminology), in the sense that effort performed by the agent is not observable by principal. Therefore, principal does not observe  $\alpha$ , and is not able



to know if the available capacity is the result of decisions of maintenance and investments made by agent, or if it is due to market conditions not controlled by the producer, such as unanticipated failures, good or bad weather conditions for renewable energy sources of production. In mathematical words, the capacity compensation  $\xi$  given to the producer, cannot be a function of the effort  $\alpha$ .

### 2.2.2 Model, state variables and control

We fix a maturity  $T \in (0, +\infty)$ , and describe the system with two continuous processes  $X^C$  and  $X^D$ , denoting respectively the electricity generation capacity available at each time  $t \in [0, T]$  and the instantaneous electricity demand, both in GigaWatt (GW).  $X^C$  represents the aggregation of all production capacities, regardless of the corresponding production technology. The uncertainty of  $X^C$  represents the power outages of conventional power plants and the variability of the availability factor of renewable productions. We denote the state variable  $X := (X^C, X^D)^\top$ , a stochastic process valued in  $\mathbb{R}^2$ .

Agent (electricity producers) controls the generation capacity  $X^C$  via an  $\mathbb{F}$ -predictable process  $(\alpha_t)_{t \in [0, T]}$ ; at each time  $t \in [0, T]$  the control is only based on information prior to  $t$  without knowledge of the future. The control  $\alpha$  is expressed as a yield in  $[\text{Year}]^{-1}$  and represents the decision at each time  $t$  to change the generation capacity by building or dismantling peak power plants; gas turbines for instance. This restriction in the choice of only one technology for the control (maintaining/building or destroying) simplifies parameters calibration and allows for the use of a continuous time setup. Numerically, this is approximated by a small-step discretization as peak power plants are quite rapidly adjustable. The control is only on the average value of the available capacity and not the volatility, which spares us a lot of technicalities. Indeed, one could expect that investing in wind power or solar panels would increase the uncertainty of available generation capacity (volatility), as opposed to thermal plants which have a more controllable production.

The instantaneous available capacity process  $X^C$  is driven by a controlled geometric Brownian motion, which has the property of staying positive consistent with available capacity.  $X^C$  starts from  $x_0^C$  and has the infinitesimal increments over  $dt$ :

$$dX_t^C = \alpha_t X_t^C dt + \sigma^C X_t^C dW_t^{C, \alpha}, \text{ for } t \text{ in } [0, T], \quad (2.2.3)$$

where  $\alpha_t X_t^C dt$  is the variation on average capacity implied by the effort  $\alpha$  and  $\sigma^C X_t^C dW_t^{C, \alpha}$  is the stochastic part in the available capacities due to uncertainties, with  $\sigma^C > 0$  the volatility parameter. Remark that we overlook ageing and deterioration in our model, which is justified by taking a short maturity  $T$  compared to the average lifetime of power plants.

The demand  $X^D$  is modeled as the exponential of a mean reverting process to ensure that  $X_t^D > 0$ , for  $t$  in  $[0, T]$ , and that the demand oscillates around some average level. The initial condition is fixed as  $X_0^D = x_0^D$ , and the infinitesimal variation over  $dt$  is modeled by:

$$d \log (X_t^D) = \mu^D (m^D - \log (X_t^D)) dt + \sigma^D dW_t^D, \text{ for } t \text{ in } [0, T]. \quad (2.2.4)$$

The term  $\sigma^D dW_t^D$  is the random part in the variation with  $\sigma^D > 0$ , and  $m^D \in \mathbb{R}$  the long term average (of  $\log (X_t^D)$ ) and  $\mu^D > 0$  the speed of mean reversion.

We denote by  $\mathcal{U}$  the set of admissible control processes  $\alpha$ , defined as the  $\mathbb{F}$ -predictable processes valued in the compact interval  $[\alpha_{min}, \alpha_{max}]$ , with  $\alpha_{min} < 0$  and  $\alpha_{max} > 0$ , which implies that construction and dismantling are both allowed. The restriction  $\alpha \in [\alpha_{min}, \alpha_{max}]$  although reasonable from an economical perspective, will only be used to simplify the rigorous mathematical resolution of the problem, and so  $|\alpha_{min}|$  and  $|\alpha_{max}|$  are set arbitrarily large.

We write in a more compact form the dynamic of the state variables

$$X_t = x_0 + \int_0^t \mu(X_s, \alpha_s) ds + \int_0^t \sigma(X_s) dW_s^\alpha, \quad \forall t \in [0, T], \quad (2.2.5)$$

with  $\alpha$  the control process,  $x_0 \in \mathbb{R}_+^2$  a fixed initial condition and

$$\mu(x, \alpha) := \tilde{\mu}(x) + \begin{pmatrix} \alpha x^C \\ 0 \end{pmatrix}, \quad \text{and } \sigma(x) := \begin{pmatrix} \sigma^C x^C & 0 \\ 0 & \sigma^D x^D \end{pmatrix}, \quad (2.2.6)$$

with

$$\tilde{\mu}(x) := \begin{pmatrix} 0 \\ \left( \mu^D (m^D - \log(x^D)) + \frac{\sigma_D^2}{2} \right) x^D \end{pmatrix}, \quad (2.2.7)$$

or equivalently

$$\begin{pmatrix} X_t^C \\ \log(X_t^D) \end{pmatrix} = \begin{pmatrix} x_0^C \\ \log(x_0^D) \end{pmatrix} + \int_0^t \begin{pmatrix} \alpha_s X_s^C \\ \mu^D (m^D - \log(X_s^D)) \end{pmatrix} ds + \int_0^t \begin{pmatrix} \sigma^C X_s^C & 0 \\ 0 & \sigma^D \end{pmatrix} dW_t^\alpha. \quad (2.2.8)$$

**Remark 2.2.1.** In reality, the processes  $X^C$  and  $X^D$  exhibit a strong seasonal behavior (annual, weekly and daily patterns). These seasonalities are explained by patterns of electricity consumption in day to day life and weather conditions (heating in winter, solar production in the day ...). For expository purposes, we consider a deseasonalized version of state variables. The aim of this simplification is to focus the analysis on random consumption peaks, and how they should be dealt with, as opposed to seasonal variations which can be anticipated.

**Remark 2.2.2.** Power demand is considered to be inelastic with respect to electricity prices.

**Remark 2.2.3.** Even though we work on the space of processes valued in  $\mathbb{R}^2$ , our model ensures that  $X$  takes only positive values, in  $\mathbb{R}_+^2$ , since the capacity process  $X^C$  follows a log-normal distribution, and  $X^D$  is defined as the exponential of an Ornstein Uhlenbeck process. One could simplify the model and define these two variables as the canonical processes on the space of exponentials of continuous functions. This would indeed simplify the description of the dynamics. Adjustements of the cost functions would also be required. However, the numerical calibration on real data of these adjusted costs is not possible, so we chose to keep on with the current model.

**Remark 2.2.4.** As mentioned above, the capacity and demand processes defined by the controlled SDE (2.2.5) are both non negative. However, they are unbounded from above and have poor integrability properties which could cause technical issues especially since we will be using exponential utility functions.

To avoid this problem we define an arbitrarily large constant  $x_\infty \in \mathbb{R}_+$ , and the function  $x \mapsto \underline{x} := x \wedge x_\infty$ . So whenever necessary we will use  $\underline{X}^C = X^C \wedge x_\infty$  and  $\underline{X}^D = X^D \wedge x_\infty$  or the vector version defined componentwise,

i.e.,  $\underline{X} = (\underline{X}^C, \underline{X}^D)^\top$ . Remark that since  $x_\infty$  can be set arbitrarily large, it neither impacts the results nor represents a restriction on the capacity or the demand.

### 2.2.3 Spot energy payment

Without capacity payment, the only transaction between consumers and producers is the reward for energy production. This reward corresponds to the spot price of electricity. Therefore, for a time interval  $[0, T]$ , the consumer pays for his consumption the amount

$$S_T := \int_0^T s(X_t) dt, \quad (2.2.9)$$

and the producer receives  $S_T$ , where  $s : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  is the reward per unit of time, defined as

$$s(x) := P(\underline{x}) \underline{x}^C \wedge \underline{x}^D, \quad (2.2.10)$$

and  $P : \mathbb{R}^2 \rightarrow \mathbb{R}_+$  is the spot price function which we define as

$$P(x) := \beta_0 e^{-\beta_1(x^C - x^D)}, \text{ with } \beta_0, \beta_1 > 0. \quad (2.2.11)$$

Note that the spot energy payment only accounts for delivered energy, i.e., the minimum between the demand and available capacity; the requested power  $X^D$  in standard situations, and just the available capacity  $X^C$  in the case of a shortage. Different choices of electricity spot price functions can be found in [Aïd \(2015\)](#) or [Aïd et al. \(2009\)](#) and our model is directly inspired by them. In particular, the function  $P$  in (2.2.11) captures a key feature in our problem: the relationship between the spot price and the residual capacity ( $X^C - X^D$ ). In the sequel, we will call  $S_T$  the spot payment. Remark that  $S_T$  represents a cost for the Principal and a reward for the Agent.

### 2.2.4 Producer's problem

The agent is the electricity producer, and provides consumer with electricity, for a terminal payment  $\xi + S_T$ . The producer is in charge of choosing the investment policy in power plants via the process  $\alpha$ , and is subject to its costs.

We model producer's instantaneous costs as a quadratic function of state variables

$$c^A(x, \alpha) := \tilde{c}^A(x) + \kappa_1(\alpha \underline{x}^C) + \kappa_2 \frac{(\alpha \underline{x}^C)^2}{2}, \text{ for } (x, \alpha) \in \mathbb{R}_+^2 \times [\alpha_{min}, \alpha_{max}], \text{ and } \kappa_1, \kappa_2 > 0. \quad (2.2.12)$$

The term  $\kappa_1(\alpha \underline{x}^C) + \kappa_2 \frac{(\alpha \underline{x}^C)^2}{2}$  is the cost of building or dismantling peak power plants, where  $\kappa_1$  is the cost per unit, and  $\kappa_2$  is a penalization adjustment term as the quadratic cost of construction, since the marginal cost of building at a given time step is increasing. We define then

$$\tilde{c}^A(x) := a \underline{x}^C + b(\underline{x}^C \wedge \underline{x}^D), \text{ with } a, b > 0, \quad (2.2.13)$$

where the first term is the cost of maintenance and the second models the variable cost of production. These variable costs of production are proportional to the available generation capacity  $x^C$  and the minimum between this capacity and demand  $x^C \wedge x^D$ .

During the time period  $[0, T]$ , producer provides electricity to consumer, and receives the payment  $\xi + S_T$  at time  $T$ . The amount  $\xi$  represents the payment producer receives for the availability of capacity, in addition to the spot payment  $S_T$ .

To include the moral hazard in our problem, we use the weak formulation and so we introduce  $\mathbb{P}^\alpha$ , the law of the process  $X$ , weak solution of the stochastic differential equation (2.2.5) with a control process  $\alpha$ , and  $\mathcal{P}$  the set of probability measures  $\mathbb{P}^\alpha$ .

The producer's objective function or his average perceived utility is defined as

$$J_0^A(\xi, \mathbb{P}^\alpha) := \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_A \left( \xi + S_T - \int_0^T c^A(X_t, \alpha_t) dt \right) \right], \text{ for } (\xi, \mathbb{P}^\alpha) \in \Xi \times \mathcal{P}, \quad (2.2.14)$$

for a given contract  $\xi \in \Xi$  and a choice of  $\alpha \in \mathcal{U}$  to which we associate the probability measure  $\mathbb{P}^\alpha \in \mathcal{P}$ , and where  $U_A$  is a utility function expressing the risk aversion; increasing and concave. For tractability, we choose an exponential utility function,  $U_A(x) := -\exp(-\eta_A x)$  with  $\eta_A > 0$ , the agent's risk aversion. A rigorous definition of  $\mathcal{P}$  and  $\Xi$  is provided in Appendix 2.5.1, along with the weak formulation of the problem.

The producer is encouraged to provide enough capacity, otherwise the consumer would reduce his payment  $\xi$ . The moral hazard is modeled by adding the restriction that the payment  $\xi$  is a function only of  $X$ , not  $\alpha$  or  $\mathbb{P}^\alpha$ . So  $\xi$  is  $\mathcal{F}_T$ -measurable, where  $\mathcal{F}_T$  by definition models the information gathered from the observation of the process  $X$  up to time  $T$ , and  $\xi$  is a function of this information. That is to say the consumer only observes the state variables  $X$  as stochastic processes, and has no access to the control  $\alpha$  or  $\mathbb{P}^\alpha$  and cannot see if the randomness of  $X$  is coming from external uncertainties or from producer's actions. In other words, the producer controls the law  $\mathbb{P}^\alpha$  of the process  $X$ , i.e., the probability of having some trajectories rather than others, and the consumer observes the realized trajectory and fixes the payment  $\xi$  as a function of  $X$ . We stress here that an important feature of our model is that the structure of the contract is defined *ex-ante* while its exact value is provided only *ex-post* depending on the realized uncertainties.

In addition, the producer has a participation constraint  $U_A(\mathcal{R}) \in \mathbb{R}$ , with  $\mathcal{R} \geq 0$  its cash equivalent. Thus, producer will accept the contract  $\xi$  only if he can expect to retrieve from  $\xi$  a utility above the level  $U_A(\mathcal{R})$ . Indeed, the producer has no obligation to accept the contract and is free to refuse it before the start of the time period  $[0, T]$ .

Whenever the producer accepts a given contract  $\xi$ , he wants to make the optimal investment by choosing an appropriate control  $\mathbb{P}^\alpha$ . Producer (agent) solves the problem

$$V_0^A(\xi) := \sup_{\mathbb{P}^\alpha \in \mathcal{P}} J_0^A(\xi, \mathbb{P}^\alpha). \quad (2.2.15)$$

An agent's control  $\mathbb{P}^{\alpha^*} \in \mathcal{P}(\xi)$  (or equivalently  $\alpha^*(\xi)$ ) is said to be optimal if it satisfies

$$V_0^A(\xi) = J_0^A(\xi, \mathbb{P}^{\alpha^*}). \quad (2.2.16)$$

We denote by  $\mathcal{P}^*(\xi)$  the set of agent's optimal controls for some admissible contract  $\xi$ .

### 2.2.5 Consumer’s problem

The consumer buys and consumes electricity from the producer during the time period  $[0, T]$  and pays for the energy consumed at the spot price  $S_T = \int_0^T s(X_t)dt$ , and a capacity remuneration given by the contract  $\xi$ . Consumer gets an instantaneous utility from electricity consumption, and a disutility in case of shortage. We therefore model consumer (principal) overall instantaneous utility as

$$c^P(x) := \theta (\underline{x}^C \wedge \underline{x}^D) - k (\underline{x}^D - \underline{x}^C)^+ \quad \text{with } \theta, k > 0. \quad (2.2.17)$$

The first term represents consumers’ reservation value or their willingness to pay for effective consumption (which is  $\min(x^C, x^D)$ ). The larger  $\theta$ , the more valuable consumption to the consumers. In the literature,  $\theta$  is often set using the Value of Lost Load (VOLL) [Fabra \(2018\)](#).

The second term can be thought of as consumer’s disutility induced by the risk of total or partial blackout. Indeed, in critical situations where some shortage (whenever available capacity is less than the total demand), the system operator’s ability to keep the system running decreases and total (all the system) or partial (large geographical zones) blackout may occur. The coefficient  $k$  is defined to represent this disutility, which is represented in the model with a function proportional to the level of shortage (the higher the shortage level, the higher the risk of blackout). This term plays a role of “punishment”–via the contract– for producer whenever there is a failure to provide sufficient generation capacity to cover the instantaneous demand.

Altogether, consumer’s objective function or expected utility is defined as

$$J_0^P(\xi, \mathbb{P}^\alpha) := \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_P \left( -\xi - S_T + \int_0^T c^P(X_t) dt \right) \right], \quad \text{for } (\xi, \mathbb{P}^\alpha) \in \Xi \times \mathcal{P}, \quad (2.2.18)$$

with  $U_P$  denoting principal’s utility function, similar to agent’s utility function with a risk aversion  $\eta_P$ ;

$$U_P(x) := -\exp(-\eta_P x) \quad \text{with } \eta_P > 0. \quad (2.2.19)$$

Principal’s goal is to choose the optimal incentive (payment) for the agent to make an optimal effort. Principal’s problem is written

$$V_0^P := \sup_{\xi \in \Xi} \sup_{\mathbb{P}^\alpha \in \mathcal{P}^*(\xi)} J_0^P(\xi, \mathbb{P}^\alpha), \quad (2.2.20)$$

i.e., given the optimal response of the agent to the compensation scheme, the Principal chooses the best contract which maximizes  $\sup_{\mathbb{P}^\alpha \in \mathcal{P}^*(\xi)} J_0^P(\xi, \mathbb{P}^\alpha)$ . Furthermore, (as stated earlier) we assume that when given different optimal controls, agent will choose the one that maximizes principal’s objective function, which is a standard assumption in contract theory [Cvitanic & Zhang \(2013\)](#), [Holmström & Milgrom \(1987\)](#), [Sannikov \(2008\)](#).

### 2.2.6 Optimal contract and capacity payment

To find producer’s and consumer’s optimal policy, we follow the approach presented in [Cvitanic et al. \(2018\)](#) for principal-agent problems. We start by considering a special class of contracts; the “revealing contracts” as capacity payments. These contracts satisfy the incentive compatibility property, which means that consumer provides them

with a recommended policy (or effort) for producer, and producer’s optimal response to these contracts corresponds to the recommended effort. Therefore, the consumer can maximize his utility over the set of revealing contracts by identifying first producer’s response and choosing the best trade-off between the payment of the contract and utility induced by the corresponding response.

We next use a representation result to prove that any contract can be represented as “revealing”, and thus there is no loss of generality or utility for consumers in optimizing only over such contracts.

In mathematical words we solve the problem for contracts which can be written as a terminal value of a (special) controlled forward stochastic differential equation (SDE) designed to make agent’s response “predictable”. Then we prove that we can associate to any admissible contract such a controlled SDE, obtained by solving an appropriate backward SDE.

The revealing contracts are introduced via an appropriate parametrization of contracts; principal considers the contract as a terminal value of a controlled diffusion process, and controls its initial level and the increments linear in the state variable. The class of *revealing contracts*  $\mathcal{Z}$  is then defined as

$$\mathcal{Z} := \left\{ Y_T^{Y_0, Z} \text{ for some } (Y_0, Z) \in \mathbb{R} \times \mathcal{V} \text{ with } Y_T^{Y_0, Z} \in \Xi \right\}, \quad (2.2.21)$$

where  $\mathcal{V}$  is the set of  $\mathbb{F}$ -predictable processes  $Z$  valued in  $\mathbb{R}^2$  satisfying some integrability conditions (rigorously defined in Appendix 2.5.2.1), and

$$Y_t^{Y_0, Z} := Y_0 + \int_0^t Z_s \cdot dX_s - \int_0^t H(X_s, Z_s) ds, \text{ for all } t \in [0, T], \quad (2.2.22)$$

with  $H$  corresponding to producer’s Hamiltonian, defined by

$$H(x, z) := \sup_{\alpha \in [\alpha_{min}, \alpha_{max}]} h(x, z, \alpha), \text{ for } (x, z) \in \mathbb{R}^2 \times \mathbb{R}^2, \quad (2.2.23)$$

and  $h : \mathbb{R}^2 \times \mathbb{R}^2 \times [\alpha_{min}, \alpha_{max}] \rightarrow \mathbb{R}$  defined as

$$h(x, z, \alpha) := z \cdot \mu(x, \alpha) + s(x) - c^A(x, \alpha) - \frac{\eta^A}{2} |\sigma(x)z|^2, \text{ for } (x, z, \alpha) \in \mathbb{R}^2 \times \mathbb{R}^2 \times [\alpha_{min}, \alpha_{max}]. \quad (2.2.24)$$

For completeness, we provide a derivation of the class of revealing contracts in Appendix 2.5.2.2.

We denote by  $\hat{\alpha} : \mathbb{R}^2 \rightarrow [\alpha_{min}, \alpha_{max}]$  the maximizer of  $h$  which can be easily computed

$$\hat{\alpha}(x^C, z^C) := \alpha_{min} \vee \left( \frac{z^C x^C - \kappa_1 \underline{x}^C}{(\underline{x}^C)^2 \kappa_2} \right) \wedge \alpha_{max}, \text{ for } (x^C, z^C) \in \mathbb{R}^2 \text{ with } x^C > 0, \quad (2.2.25)$$

where  $x^C$  (respectively  $z^C$ ) denotes the first component of  $x$  (respectively  $z$ ). The function  $\hat{\alpha}$  will be referred to as the “recommended effort”–Sannikov (2008)–, and can be reasonably approximated when  $|\alpha_{min}|, \alpha_{max}, x_\infty \rightarrow +\infty$  as

$$\hat{\alpha}(x^C, z^C) \approx \frac{z^C - \kappa_1}{x^C \kappa_2} \text{ for } (x^C, z^C) \in \mathbb{R}^2 \text{ with } x^C > 0. \quad (2.2.26)$$

We will stick to the expression (2.2.25) for the mathematical proofs, and use the approximation (2.2.26) for the interpretations.

Remark that the process  $(Y_t^{Y_0, Z})_{t \in [0, T]}$  depends only on observations of  $X$  (consumption and available capacities) which are observable by Principal, as opposed to the effort  $\alpha$  and the Brownian motion  $W^\alpha$ . This is consistent with the moral-hazard of this problem.

The revealing contracts class  $\mathcal{Z}$  plays a central role in principal-agent problems. Not only does it allow principal to predict agent's optimal control, but also to overcome the main difficulty; the *non-Markovianity* of  $\xi$ , i.e., the dependence of the payment on the whole paths of the demand  $X^D$  and the capacity  $X^C$ .

We can interpret the revealing contract as a performance index, closely related to agent's continuation value, which comes with a recommended effort  $\hat{\alpha}$  defined in (2.2.25); recall that the actual effort provided by agent is neither observable, nor contractible, and therefore principal can only propose  $\hat{\alpha}$  as a recommendation and not an obligation.

Nevertheless, Proposition 2.2.5 proves that whenever agent (producer) is rational -which is a reasonable assumption- he will follow the recommended effort since it maximizes his expected utility, and so principal (consumer) can predict agent's (producer's) effort. Furthermore, Proposition 2.2.6, identifies  $\mathcal{Z}$  to  $\Xi$ , meaning that any admissible contract can be represented as a revealing one. We present these two results and provide their proofs in the Appendix.

**Proposition 2.2.5.** *For every contract  $Y_T^{Y_0, Z}$  in the class  $\mathcal{Z}$ , producer's value function is characterized as*

$$V_0^A \left( Y_T^{Y_0, Z} \right) = U_A \left( Y_0 \right), \quad (2.2.27)$$

and his optimal control is given by consumer's recommended effort  $\left( \hat{\alpha} \left( X_t^C, Z_t^C \right) \right)_{t \in [0, T]}$ .

The proof is reported in Appendix 2.5.3.

**Proposition 2.2.6.** *Let  $\xi \in \Xi$ . Then there exists a pair  $(Y_0, Z) \in \mathcal{Z}$  such that*

$$\begin{cases} Y_T^{Y_0, Z} = \xi, \\ dY_t^{Y_0, Z} = Z_t \cdot dX_t - H(X_t, Z_t) dt. \end{cases} \quad (2.2.28)$$

Furthermore,

$$\mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ e^{(\eta_A \vee \eta_P)(1+\delta) \sup_{t \in [0, T]} |Y_t|} \right] < +\infty. \quad (2.2.29)$$

In particular,  $\mathcal{Z} = \Xi$ , and therefore

$$V_0^P = \sup_{Y_0 \geq \mathcal{R}} \sup_{Z \in \mathcal{V}} J_0^P \left( Y_T^{Y_0, Z}, \mathbb{P}^{\hat{\alpha}} \right). \quad (2.2.30)$$

The proof is reported in Appendix 2.5.4.

The main conclusion is that there is no loss of generality in restricting consumer's problem to contracts in  $\mathcal{Z}$ , which are general enough (since  $\mathcal{Z} = \Xi$ ), and properly parameterized to make producer's response predictable (by

proposition 2.2.5). So consumer only needs to solve the reduced problem (2.2.30), i.e., to maximize his objective function over the set  $\mathcal{Z}$  (with the two new control variables  $Y_0$  and  $Z$ ). This corresponds to a Markovian stochastic control problem which can be solved by standard techniques, and is the object of Proposition 2.5.6 reported in Appendix 2.5.5.

By virtue of Proposition 2.5.6, we can provide a straightforward decomposition of Principal's optimal control in different parts as in the following Corollary 2.2.7.

**Corollary 2.2.7.** *Under the assumptions of Proposition 2.5.6, Principal's optimal contract  $\xi^*$  can be written as  $\xi^* := Y_T^{\mathcal{R}, Z^*}$  with the following decomposition:*

$$Y_T^{\mathcal{R}, Z^*} = \mathcal{R} + \int_0^T Z_t^* \cdot dX_t - \int_0^T H(X_t, Z_t^*) dt, \quad (2.2.31)$$

or equivalently,

$$Y_T^{\mathcal{R}, Z^*} + S_T = \mathcal{R} + \int_0^T c^A(X_t, \hat{\alpha}(X_t^C, Z_t^{*,C})) dt + \int_0^T Z_t^* \cdot \sigma(X_t) dW_t^{\mathbb{P}^{\hat{\alpha}}} + \frac{\eta_A}{2} \int_0^T |\sigma(X_t) Z_t^*|^2 dt. \quad (2.2.32)$$

with  $(Z_t^*)_{t \in [0, T]}$  defined in (2.5.72) and  $\hat{\alpha}$  the recommended effort function defined in (2.2.25).

This optimal contract consists in a terminal payment to the producer of the random amount  $Y_T^{\mathcal{R}, Z^*}$ , which incites him to follow the recommended effort. Note that any different effort (from the producer) would be sub-optimal in terms of his utility by Proposition 2.2.5.

**Remark 2.2.8.** We can make some observations on the remuneration of the producers and the optimal recommended effort:

(i) Under smoothness assumptions on consumer's certainty equivalent  $u$  (the solution of PDE (2.5.60)), the recommended effort  $(\hat{\alpha}(X_t^C, Z_t^{*,C}))_{t \in [0, T]}$  is a feedback control as a function of  $X_t^C$  and  $\partial_{x^C} u(t, X_t^C, X_t^D)$ . Recall

$$\hat{\alpha}(X_t^C, Z_t^{*,C}) = \alpha_{\min} \vee \left( \frac{Z_t^{*,C} X_t^C - \kappa_1 X_t^C}{(X_t^C)^2 \kappa_2} \right) \wedge \alpha_{\max}, \quad (2.2.25)$$

$$\approx \frac{Z_t^{*,C} - \kappa_1}{X_t^C \kappa_2} \text{ for } |\alpha_{\min}|, \alpha_{\max}, x_\infty \rightarrow +\infty, \quad (2.2.26)$$

and from the definitions (2.5.72) and (2.5.66), and the approximation (2.5.68) when  $|\alpha_{\min}|, \alpha_{\max}, x_\infty \rightarrow +\infty$ ,

$$\begin{aligned} Z_t^{*,C} &= \frac{\eta_P (\sigma^C x^C)^2 + \frac{1}{\kappa_2} \left( \frac{X_t^C}{X_t^C} \right)^2}{(\eta_A + \eta_P) (\sigma^C X_t^C)^2 + \frac{1}{\kappa_2} \left( \frac{X_t^C}{X_t^C} \right)^2} \partial_{x^C} u(t, X_t^C, X_t^D) + \frac{\kappa_1}{\kappa_2} \frac{\left( \frac{X_t^C}{X_t^C} \right) - \left( \frac{X_t^C}{X_t^C} \right)^2}{(\eta_A + \eta_P) (\sigma^C X_t^C)^2 + \frac{1}{\kappa_2} \left( \frac{X_t^C}{X_t^C} \right)^2}, \\ &\approx \frac{\eta_P (\sigma^C X_t^C)^2 + \frac{1}{\kappa_2}}{(\eta_A + \eta_P) (\sigma^C X_t^C)^2 + \frac{1}{\kappa_2}} \partial_{x^C} u(t, X_t^C, X_t^D). \end{aligned} \quad (2.2.33)$$

Therefore, for  $t \in [0, T]$  and a given position  $(X_t^C, X_t^D)$  the rate of investment  $\hat{\alpha}$  recommended by consumer is a function of the capacity  $X_t^C$  and  $\partial_{x^C} u(t, X_t^C, X_t^D)$ ; the sensitivity of his certainty equivalent with respect to



capacity at time  $t$ . The latter term depends on the triplet  $(t, X_t^C, X_t^D)$ , and so the recommended effort depends obviously on  $X^C$ , and implicitly on  $t$  and  $X^D$  through the sensitivity of consumer  $\partial_{x^C} u(t, X_t^C, X_t^D)$ . This point will be further highlighted with numerical experiments.

(ii) Since the effort  $(\hat{\alpha}(X_t^C, Z_t^{*,C}))_{t \in [0, T]}$  is defined as a rate (in  $[\text{Year}]^{-1}$ ), it can be better understood by observing the process  $(X_t^C \hat{\alpha}(X_t^C, Z_t^{*,C}))_{t \in [0, T]}$  which corresponds to the actual construction or dismantling of power plants (in  $[\text{GW}/\text{Year}]$ ) which is given from the previous expressions as

$$\begin{aligned} X_t^C \hat{\alpha}(X_t^C, Z_t^{*,C}) &\approx \frac{1}{\kappa_2} \frac{\eta_P (\sigma^C X_t^C)^2 + \frac{1}{\kappa_2}}{(\eta_A + \eta_P) (\sigma^C X_t^C)^2 + \frac{1}{\kappa_2}} \partial_{x^C} u(t, X_t^C, X_t^D) - \frac{\kappa_1}{\kappa_2}, \\ &\approx \frac{w(X_t^C) \partial_{x^C} u(t, X_t^C, X_t^D) - \kappa_1}{\kappa_2}, \end{aligned} \quad (2.2.34)$$

with  $w(X_t^C) \in (0, 1)$  a positive weight function related to the risk aversion of producer (because  $\eta_A > 0$ ). In the extreme case  $\eta_A \rightarrow 0$ ,  $w(X_t^C) = 1$  and the recommended effort becomes clear; the (linear) marginal cost of construction is  $\kappa_1$ , and so the optimal control is to construct if  $\partial_{x^C} u(t, X_t^C, X_t^D) > \kappa_1$  and dismantle power plants if  $\partial_{x^C} u(t, X_t^C, X_t^D) < \kappa_1$ . In particular,  $\hat{\alpha}$  depends only on the certainty equivalent of consumer (and  $X^C$ ), not producer, since the latter is compensated by the contract instead. The general case  $\eta_A > 0$ , with  $0 < w(X_t^C) < 1$  needs further considerations in terms of the magnitude of the sensitivity  $\partial_{x^C} u(t, X_t^C, X_t^D)$  to offset the weight  $w(X_t^C)$ , but leads to similar results.

(iii)  $Y_T^{\mathcal{R}, Z^*}$  covers all the costs the producer has to pay to follow the recommended capacity policy (2.2.25) and to produce electricity to match the demand. Therefore, the optimal contract compensates those costs taking into account what the producer is earning on the spot market. We recall below producer's costs (we omit the truncation function for exposition clarity)

$$\begin{aligned} \underbrace{\int_0^T c^A(X_t, \hat{\alpha}(X_t^C, Z_t^{*,C})) dt}_{\text{Producer's costs}} &= \underbrace{\int_0^T \kappa_1 \hat{\alpha}(X_t^C, Z_t^{*,C}) X_t^C + \kappa_2 \frac{(\hat{\alpha}(X_t^C, Z_t^{*,C}) X_t^C)^2}{2} dt}_{\text{Construction costs}} \\ &+ \underbrace{\int_0^T a X_t^C dt}_{\text{Maintenance costs}} + \underbrace{\int_0^T b(X_t^C \wedge X_t^D) dt}_{\text{Production costs}}, \end{aligned} \quad (2.2.35)$$

(vi)  $Y_T^{\mathcal{R}, Z^*}$  shares the risk (realized uncertainties on demand and capacity) between producers and consumers, by transferring part of the randomness to the agent, while providing him with a risk compensation at the same time, to overcome his risk-aversion:

$$\text{Risk part} = \underbrace{\int_0^T Z_t^* \cdot \sigma(X_t) dW_t^{\hat{\alpha}}}_{\text{Risk shared}} + \underbrace{\frac{\eta_A}{2} \int_0^T |\sigma(X_t) Z_t^*|^2 dt}_{\text{Risk compensation}}, \quad (2.2.36)$$

and the risk shared can be interpreted as a “reward for good luck” and a “punishment for bad luck” as in Hoffmann & Pfeil (2010), for both of the external noises  $W^C$  and  $W^D$ , which is possible since the contract is defined *ex-ante* and paid *ex-post*. In particular, our model accounts for the risk on the uncontrolled demand  $X^D$  in two ways; through the recommended effort (as explained in (i) and (ii)), and the optimal contract via the “Risk shared” term.

(v)  $Y_T^{\mathcal{R}, Z^*}$  is a random variable which depends on the scenario. In particular, its value changes as the uncertainties change, and might even become negative. This means that agent might earn less or more than his total costs, depending on the outcome of uncertainties (for example very sunny or windy years might lead to low spot prices and therefore to a higher capacity remuneration). Nevertheless, in expectation, agent (producer) is guaranteed to earn  $\mathcal{R}$ ; the cash equivalent of the reservation utility.

(vi) We can rewrite the decomposition (2.2.32) as follow:

$$\text{Capacity remuneration} + \text{Spot compensation} = \mathcal{R} + \text{Producer's costs} + \text{Risk shared} + \text{Risk compensation.} \quad (2.2.37)$$

## 2.2.7 Producer's participation constraint: the problem without capacity payment

In absence of a capacity payment, producer's only income is the spot compensation and therefore his problem is a standard Markovian stochastic control problem:

$$\hat{V}_0^A := V_0^A(0) = \sup_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_A \left( S_T - \int_0^T c^A(X_t, \alpha_t) dt \right) \right], \text{ for } \mathbb{P}^\alpha \in \mathcal{P}. \quad (2.2.38)$$

In this case, the consumer has no bargaining power and no control on investment decisions in capacity, and is then subject to shortage risk. The producer does not care anymore about consumer's value function, and there is no guarantee that a criteria such as the LoLE constraint is satisfied (recall that the Loss of Load Expectation (LoLE) is the targeted maximum number of hours of shortage per year, set at 3 hours per year for most European countries).

Because of the structure of the spot function (decreasing in  $x^C - x^D$ ), the producer makes a compromise between having few installed capacities (less than in the case with a capacity payment) to save maintenance costs and increase the spot prices, and enough capacities to satisfy (part of) the demand, to earn more on the spot market (since only sold energy generates a cash flow). Remark that this kind of arbitrage can be seen in practice even among producers in perfect competition.

The resolution of problem (2.2.38) is the object of Proposition 2.5.8 reported in Appendix 2.5.6. In the absence of a contract, producer's value function  $\hat{V}_0^A$ , i.e., the solution to problem (2.2.38) given in Proposition 2.5.8 provides a good proxy for the participation constraint, which we define as follow :

$$\mathcal{R} := U_A^{-1} \left( \hat{V}_0^A \right) \vee 0. \quad (2.2.39)$$

The maximum is taken in the previous equation between 0 and  $U_A^{-1} \left( \hat{V}_0^A \right)$  as the producer has two choices: to operate the power plants if  $U_A^{-1} \left( \hat{V}_0^A \right) \geq 0$  and earn the spot price which provides a utility  $\hat{V}_0^A$ , or (if  $U_A^{-1} \left( \hat{V}_0^A \right) < 0$ ), to stop all activities which would lead to 0 earnings (assuming that we neglect any agency costs related to bankruptcy).

## 2.3 Numerical results and interpretations

In this section our model is numerically solved for a stylized system, based on the French electricity power system. We implement the optimal capacity contract and optimal policy, by numerically solving the PDE (2.5.60) describing consumer's value function, with parameters calibrated on the French power system. Then we observe multiple

scenarios and the evolution of state variables under this policy. A more precise description of the numerical resolution procedure is provided in Appendix 2.5.7.

### 2.3.1 Case study: the French power system

We consider a time horizon  $T = 5$  [Years], and we discretize it with a time step  $\Delta t = \frac{1}{400}$  [Years] for the diffusion of state variables, roughly speaking, over one time step per day.

The state variables  $X^C$  and  $X^D$  are expressed in GW, and the contract and costs (quantities inside of the utility function) are in  $10^6\text{€}$ , ([M€]). As stated earlier,  $X^C$  is the instantaneous overall available capacity, and the control is only on the peak power plants which are assumed to be gas turbine.

#### 2.3.1.1 Capacity and demand

We use the generation capacity, demand and spot prices available online<sup>4</sup> for the time period 29/06/2009-15/12/2014. Remark that we stop at 2014 because the latest available capacity data is provided in that year, as the French TSO stopped publishing available capacity records on an aggregated basis per technology. Nevertheless, the French production mix did not change a lot in the past period, and we can reasonably assume that the uncertainties on capacity generation remain unchanged too.

We start by calibrating the parameters of SDE (2.2.5) modeling the dynamics of  $X^C$  and  $X^D$ . As mentioned in Remark 2.2.1, we only consider deseasonalized state variables in our model. Therefore, to deseasonalize the input data, we use a locally weighted scatterplot smoothing algorithm implemented in the software R; the function “STL” which decomposes time series into three components: a trend, a seasonal component and a residual noise. This algorithm extracts the trend by averaging locally, then computes the seasonality on residuals by averaging across a given frequency. Once the seasonal component is computed, it is subtracted from the original time series to get the deseasonalized data. We apply this procedure twice; once for the annual seasonality and another time for the weekly seasonality.

The demand  $X^D$  modeled as the exponential of an Orstein–Uhlenbeck process is calibrated by linear regression of the returns of daily data fixed at 7 p.m., the hour of the day with the highest demand.

As for the capacity  $X^C$ , we take the daily sum of the different generation technology capacities: nuclear, gas, coal, fuel, hydro-power (reservoir and run-of-the-river) and then  $\sigma^C$  is calibrated as to have simulated trajectories with similar behaviour with historical (observed) capacity data.

Table 2.1 summarizes our estimated parameters for capacity and demand processes, and we can see in figure 2.1 a comparison between historical (deseasonalized) data with generated scenarios of demand and capacity with our calibrated parameters.

<sup>4</sup>The French TSO RTE website for capacity and demand <https://clients.rte-france.com>, and the EPEX SPOT website <https://www.epexspot.com> for spot prices.

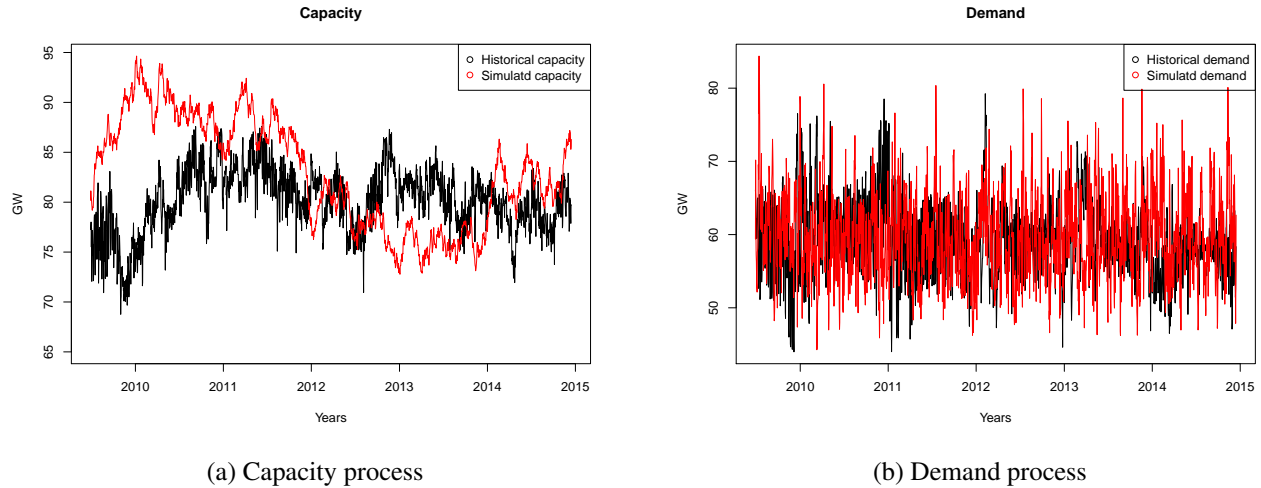


Figure 2.1: Comparison between historical data and simulated processes. The black line represents the evolution of the historical data (generation capacity (GW) in the left figure and demand (GW) in the right figure), plotted against a simulated trajectory (in red) with our model and the calibrated parameters of table 2.1.

	Parameter	Value	Unit
Available generation capacity	$x_0^C$	90	[GW]
	$\sigma^C$	0.1	[Year] <sup>-1/2</sup>
Demand	$x_0^D$	60	[GW]
	$\mu^D$	61.92	[Year] <sup>-1</sup>
	$\exp(m^D)$	60	[GW]
	$\sigma^D$	0.86	[Year] <sup>-1/2</sup>

Table 2.1: Calibrated parameters for capacity and demand. This table provides the set of parameters for which our model fits the deseasonalized data of the French generation capacity and demand over the time period 29/06/2009 – 15/12/2014.

### 2.3.1.2 Spot price function

We calibrate our spot price function  $P$  defined in (2.2.11) using historical data, and taking one price per day, at 7 p.m.; the same as for demand data (the hour of daily demand peak). The calibration is simply done by taking the log of the time series, and then applying a linear regression. Remark that the spot price function  $P$  is completely characterized by the capacity and demand, so the seasonality is naturally accounted for. Table 2.2 summarizes our choice of the spot price function  $P$  with its calibrated parameters, and the figure 2.2 represents a comparison between historical and simulated spot prices.

Remark that our model reproduces quite well the behaviour of spot prices but without the largest peaks. This is coherent with a market with a low price cap (a price cap which can be seen for example in many European countries).

	Model: $\frac{\text{M€}}{\text{GW} \cdot \text{Year}}$	Parameter	Value	Unit
Spot price function	$P(x) = \beta_0 e^{-\beta_1(x^C - x^D)}$	$\beta_0$	102.8	[€/MWh]
		$\beta_1$	$335.3 \times 10^{-4}$	[GW] <sup>-1</sup>

Table 2.2: Spot price model and its calibrated parameters. The first column recalls the model of electricity spot price as a function of the capacity margin ( $x^C - x^D$ ), and the last three columns provide the parameters of this model calibrated on the French electricity spot prices.

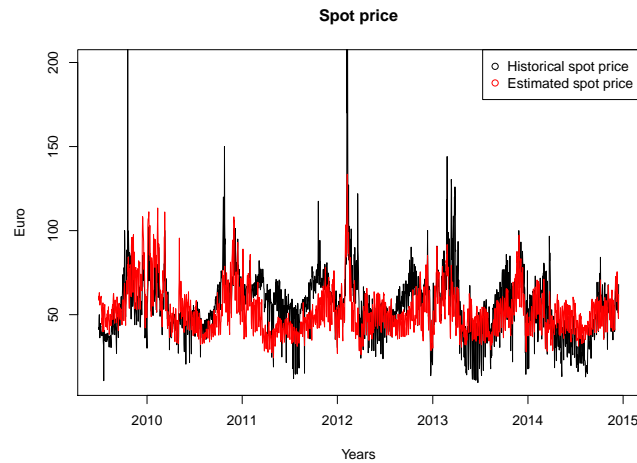


Figure 2.2: Historical and estimated spot prices in euros. In black the historical spot prices, and in red the reconstruction of the spot price using the function  $P$  from table 2.2 and the historical realizations of demand and capacity generation.

Technology	Installed capacities	Percentage of production
Nuclear	48 %	72 %
Coal	2 %	1 %
Gas and Fuel	8 %	4 %
Wind turbines and Photovoltaic	18 %	7 %
Hydropower	19 %	12 %
Bioenergy	6 %	4 %

Table 2.3: Different technologies and their weights. The first column lists the technologies present in the French mix. The second column represents the percentages in terms of installed capacities for each technology. The third column represents the percentages in terms of energy produced.

### 2.3.1.3 Costs

Electricity supplier has to take into account the construction, maintenance and production costs of different power plants. These costs are provided by the French TSO and WEO 2018<sup>5</sup>. The cost of maintenance  $a$  and the cost of production  $b$  in [€/MWh] are estimated as weighted averages between the different costs of technologies, where the weights used for maintenance are the same as for the installed capacities, and those for production costs are taken as the proportions of production; cf Table 2.3.

The cost of construction per unit  $\kappa_1$  in [€/MWh] is taken as the equivalent annual cost of a gas turbine power plant instead of a weighted average since only peak power plants are used for the control (construction or dismantling). The equivalent annual cost of a gas turbine power plant with a total cost of investment  $C_{\text{Total cost}} = 550$  [€/kW]– which should not be confused with the levelized cost of energy, and is computed by dividing the annual cost of investment by the total number of hours per year (8760 hours)–, a lifetime  $T_{\text{Gas Turbine}} = 30$  [Years], and a discount rate  $r = 8\%$  is computed as  $\kappa_1 = \frac{nrC_{\text{Total cost}}}{1-(1+r)^{T_{\text{Gas Turbine}}}} = 122.13$  [€/kW], where  $n$  is the number of upcoming annuities approximated by  $n \simeq 2.5$ . A more precise computation requires to take  $n$  as the number of annuities left to pay during the contract time (between 1 and 5 years), but we chose to simplify and take an average value  $n = 2.5$ . The adjustment coefficient  $\kappa_2$  is taken as  $\kappa_2 = 2 \times \kappa_1$  where  $\kappa_2$  is in [€/(MWh×MW)]. Different sensitivities with respect to this parameter are then performed and ensure that results are stable within a reasonable range of parameters.

A summary of our calibrated parameters for producer’s costs can be found in table 2.4.

### 2.3.1.4 Utility, disutility and risk aversion

Risk aversions and utility preferences,  $\theta, k, \eta_A, \eta_P$  and the participation constraint  $\mathcal{R}$ , are parameters less straightforward to calibrate.

The participation constraint (or cash equivalent of the reservation utility)  $\mathcal{R}$  is defined as a function of the solution to producer’s problem in absence of capacity payment, and given by 2.2.39.  $\mathcal{R}$  is computed by numerically solving the PDE (2.5.107).

<sup>5</sup>See the “WEO 2018 report” and “Impact assessment of the French Capacity Market, 2018.” by the French TSO RTE.

	Parameter	Value	Unit
Producer's costs	$\kappa_1$	122.13	[€/kW]
	$\kappa_2$	$31.8 \times 10^{-4}$	[€/(MWh×MW)]
	$a$	75.35	[€/(kW×year)]
	$b$	17.6	[€/MWh]

Table 2.4: Calibrated parameters for construction, maintenance and production costs.  $\kappa_1$  and  $\kappa_2$  are set to the equivalent annual cost of a gas turbine, and  $a$  and  $b$  are weighted averages (with weights from table 2.3) of the costs of maintenance and production of all technologies.

The parameter  $\theta$  is expressed in [€/MWh], and reflects consumers' satisfaction in consumption per GW over time, or how much they are willing to pay for the electricity. This parameter is calibrated as the *Value of Lost Load* (VoLL)<sup>6</sup>.

We assume that producer should be more risk averse than consumer. The reason of this assumption is that consumer's risk aversion embeds the aversion to shortage represented by the term in  $k$ . Roughly speaking, as soon as there is a shortage, consumer's utility starts to decrease because of the term  $\eta_P \times k(X^D - X^C)^+$ . We can interpret it by saying that consumer is willing to accept more financial risk than producer, in exchange of offsetting the risk of having a shortage. We choose therefore  $\eta_A > \eta_P$  and we take values inspired by the calibration in Aïd et al. (2018). A sensitivity analysis is then performed on these parameters and we find that our results are not affected if producer is more risk averse than the consumer or the contrary.

Regarding the parameter  $k$ , we use a further constraint which is that the average number of shortage hours per year should be reasonable. Indeed, recall that for most European electricity systems, the targeted maximum number of hours of shortage (LoLE), is 3 hours Newbery (2016). Therefore we calibrate these parameters by an iterative procedure, i.e., by repeatedly solving the problem, diffusing the state variables, computing the total period of shortage and adjusting the parameters until we attain a reasonable number of shortage hours per year. A possible set of consumer's preferences and risk aversions of both parties is given in the table 2.5. Of course, one could always argue that this set of parameters is not unique because of the degrees of liberties compared to the number of constraints, but this set seems quite reasonable and produces stable numerical results, which we present in the next section along with a sensitivity analysis.

### 2.3.2 Numerical results : Comparison between the system with and without a CRM

Once the parameters fixed, we simulate  $N = 5000$  scenarios and compare three different cases; one “without a CRM”, where producer adjusts capacities to maximize his utility, another “with a CRM”; using the optimal policy for both consumer (optimal compensation (2.2.32)) and producer (recommended effort (2.2.25)), and a third one with no capacity adjustment (“No adjustment”), i.e., no building or dismantling of capacities; leaving them subject to external uncertainties. Table 2.6 summarizes the results of our simulations.

<sup>6</sup>See again “Impact assessment of the French Capacity Market, 2018.” by RTE.

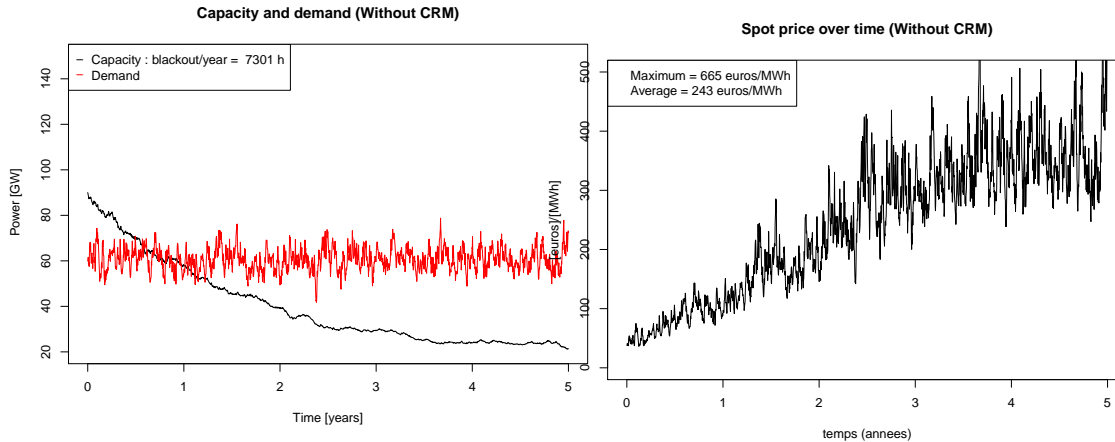
	Parameter	Value	Unit
Consumer's preferences	$\theta$	20000	[€/MWh]
	$k$	200000	[€/MWh]
Risk aversions	$\eta_A$	$0.852 \times 10^{-4}$	[M€] <sup>-1</sup>
	$\eta_P$	$0.8094 \times 10^{-5}$	[M€] <sup>-1</sup>
Participation constraint	$\mathcal{R}$	2.8	[€/MWh]

Table 2.5: Choice of risk aversions, and calibrated preferences. This table provides estimates for risk aversions  $\eta_A$  and  $\eta_P$ , and consumers' utility for consumption  $\theta$  (the VoLL), together with their aversion to shortage  $k$ , and the minimal payment required by producer  $\mathcal{R}$ . This set of parameters generates scenarios with a reasonable number of shortage hours.

	Without CRM		With a CRM		No adjustment	
	mean	sd	mean	sd	mean	sd
Shortage hours per year [Hours]	6165.4	1459.1	2.2	7.4	178.4	594.3
Average Spot price [euro/MWh]	146.5	30	37.6	9.6	43	16.4
Average Margin [GW]	-6	8.2	32.4	9.0	28.1	11.4
Spot revenues [euro/MWh]	134.6	25.6	37.8	9.6	43.1	16.2
Capacity payment [euro/MWh]	NA	NA	12.3	13.2	NA	NA
Spot + Capacity payment [euro/MWh]	134.6	25.6	50.1	10.9	43.1	16.2
Participation constraint [euro/MWh]	NA	NA	2.8	0.0	NA	NA
Risk shared [euro/MWh]	NA	NA	-0.2	11.7	NA	NA
Risk compensation [euro/MWh]	NA	NA	14.9	3.9	NA	NA
Total costs [euro/MWh]	77.4	9.3	32.7	3.2	30.5	1.6
Construction and dismantling [euro/MWh]	50.3	9.7	1.8	3.7	0	0
Maintenance [euro/MWh]	9.5	0.7	13.2	1.3	12.9	1.6
Production [euro/MWh]	17.6	0	17.6	0.0	17.6	0

Table 2.6: Comparison between different policies. This table regroups the average and standard deviation of the most relevant quantities over  $N = 5000$  simulations. The first two columns represent the scenarios generated without a CRM, i.e., where producer controls the capacity and his only income is the spot revenue. The next two columns represent our proposed CRM, with scenarios generated following the recommended effort and a capacity payment. The last columns provide the results for scenarios generated without any control on capacity and without capacity payment.





(a) Capacity and demand without a CRM. In red the demand with a constant average, and in black the capacity generation decreasing over time and suggesting consistent dismantling of powerplants and an increasing number of shortage hours per year.

(b) Spot price without a CRM. The evolution of spot price is the reverse of the margin (capacity-demand), increasing over time and reaching a maximum of 665 euros/MWh near maturity and an average of 243 euros/MWh on the whole period.

Figure 2.3: Evolution over time of the state variables without a CRM. The capacity and demand (GW) and the corresponding spot price (Euros/MWh), for a scenario where producer controls the generation capacity, and his only income is the spot revenue.

### 2.3.2.1 The system evolution without a capacity payment

We start by analyzing the system without a capacity payment. As mentioned before, the producer has market power and no incentive to satisfy the LoLE constraint, since his goal is to maximize his utility function instead of just offsetting his marginal costs. On the contrary, as his only compensation is from the spot market, his optimal strategy consists in finding the equilibrium between high enough spot prices (corresponding to low or even negative capacity margins) and high enough available capacity as spot compensation is  $\int_0^T P(X_t) X_t^C \wedge X_t^D dt$ . We can see from table 2.6 that producer settles for a -6 GW average, which corresponds to an average spot price of 146 euros/MWh. It follows from this negative equilibrium average margin that the system is in shortage situation most of the period  $[0, T]$ , which is confirmed by the numerical results (6165 shortage hours per year).

In figure 2.3a, we can observe one scenario without a CRM. As stated before, producer's optimal strategy is to decrease the capacity level which leads to the spot prices increasing, see figure 2.3b. This decrease in capacity continues even after reaching the average demand level (about 60 GW) and attains an equilibrium (around 20 GW in the scenario which is more severe than average). This explains the high construction and dismantling costs mainly due to dismantling actions.

The figures 2.3a and 2.3b illustrate that without a CRM, producer will be better off with low capacity and high spot revenues, and therefore if we aim at keeping a reasonable level of available capacity (which implies lower spot prices), it is necessary to provide him with a complementary compensation to replace his losses in spot revenues. We highlight the fact that the absence of a capacity payment would have much less drastic impacts in the real life than what we observe in our numerical simulations. This is due from one side to the regulation authorities which would

not allow for such levels of shortage to occur, and from the other side because of the presence of multiple producers in competition who might decide to invest more –breaking the market power–, or even new actors (producers) willing to invest and enter the market in such favorable conditions (i.e., with a spot price much higher than costs.)

The loss of spot revenues incurred by producer when keeping high capacity levels is partly captured by his participation constraint, since  $\mathcal{R}$  also accounts for the change in construction, maintenance and production costs. In fact, the more profitable the system without a CRM to the producer, the higher  $\mathcal{R}$ , and the more inciting the contract (CRM) needs to be. In our setting  $\mathcal{R} = 2.8\text{€/MWh}$ , and is decreasing in  $\sigma^C$  and increasing in  $\sigma^D$ . When the volatility of capacity is high, the efforts of producer have less and less impact on  $X^C$  -and therefore on the system- and his utility (or its cash equivalent  $\mathcal{R}$ ) is lower. On the other side, whenever the volatility of demand is high, the probability of shortage increases and when capacity margin becomes low this drives the spot prices up in an amplified manner which gives the producer more utility.

Similar to the volatility of capacity  $\sigma^C$ , the parameters  $\kappa_2$  and  $x_0^C$  have the same impact on  $\mathcal{R}$ ; the cost of control (in this case, the cost of shutting down powerplants) becomes higher with  $\kappa_2$  which lowers producer’s utility, while a positive change in the initial value  $x_0^C$  increases the capacity margin and decreases the spot prices and  $\mathcal{R}$  as a result. Finally, and obviously, higher spot prices (because of higher spot levels  $\beta_0$ ) increase  $\mathcal{R}$ .

Note that it is not possible to infer from the first column of table 2.6 the participation constraint  $\mathcal{R} = 2.8\text{€/MWh}$ , since the utility function is concave we have  $\mathcal{R} = U_A^{-1}(\mathbb{E}[U_A(\text{Spot} + \text{Capacity payment} - \text{Total Costs})])$  which is lower than what can be read from the table and which corresponds to  $\mathbb{E}[\text{Spot}] + \mathbb{E}[\text{Capacity payment}] - \mathbb{E}[\text{Total Costs}]$ .

### 2.3.2.2 Analysis of the system evolution under the optimal policy

Coming back to table 2.6, we can see that introducing the CRM drastically improves the security of the system; (an average of 2 hours shortage per year–*respecting the LoLE constraint*– compared to 178 hours per year when there is no capacity adjustment and 6165 hours per year when producer has market power.) Remark that this also reduces consumer’s payments: it is less costly for the consumer to pay for capacity and the spot prices (which is in average rather low because the system margin is high) than paying only the spot prices “without CRM” where spot prices are very high.

Observe also that when comparing the system with a CRM and without capacity adjustment, we see that the average margin is positive in both cases and quite high (32 GW and 28 GW) so one would expect that these two settings would be quite similar. However, we see that we obtain a substantial gain in the average number of shortage hours per year when following the dictated policy (with 32 GW capacity margin), going from 178 hours per year to only 2 hours per year. This owes to the design of the contract in the CRM taking into account the magnitude of uncertainties and other characteristics of the system.

In order to better interpret producer’s optimal policy, we select and analyze two of the 5000 simulated scenarios; a severe scenario –the one with the highest number of shortage hours over the period of simulation, selected a posteriori– and a Favorable scenario. Table 2.7 provides the outcomes of these scenarios compared with the average scenario.

We plot first the evolution of state variables in figure 2.4. We can see in red the evolution of demand, and in black the available capacity. The demand process is by construction a mean-reverting process. However, the capacity is a geometric Brownian motion. So the capacity has a priori no reason to exhibit a mean-reverting behavior which is

	Average scenario	Favorable scenario	Severe scenario
Shortage hours per year [Hours]	2.2	0.0	113.9
Average Spot price [euro/MWh]	37.6	31.9	59.4
Average Margin [GW]	32.4	36.0	17.3
Spot revenues [euro/MWh]	37.8	32.1	59.7
Capacity payment [euro/MWh]	12.3	57.9	21.7
Spot + Capacity payment [euro/MWh]	50.1	90.0	81.4
Participation constraint [euro/MWh]	2.8	2.7	2.7
Risk shared [euro/MWh]	-0.2	43.4	-40.7
Risk compensation [euro/MWh]	14.9	12.7	52.1
Total costs [euro/MWh]	32.7	31.2	67.3
Construction and dismantling [euro/MWh]	1.8	0.0	38.6
Maintenance [euro/MWh]	13.2	13.6	11.1
Production [euro/MWh]	17.6	17.6	17.6

Table 2.7: Comparison between different scenarios with CRM. This table regroups the results of the diffusion of  $N = 5000$  scenarios with the CRM and following the recommended effort to assess the extreme outcomes that might occur. The first column represents the average over all scenarios for reference, the second column provides the results for one “favorable” scenario with 0 shortage hours, and the third column provides the results of a “severe” scenario; the one with the maximal number of shortage hours in our simulations.

nevertheless observed in 2.4 on the right figure (b). This mean-reversion can be explained by the effort rate  $\hat{\alpha}$  which readjusts the capacity depending on the randomness, and the level of security fixed by consumers’ preferences. This readjustment can also be seen in the difference between construction costs in table 2.7; the severe scenario having the highest cost suggesting a policy with intensive construction.

Figure 2.5 provides an interpretation of producer’s optimal control, by comparing two policies. In black, we see the capacity evolution obtained with producer’s optimal policy, and in red the capacity without adjustment, i.e.,  $\alpha_t = 0$  for  $t \in [0, T]$ . This is interpreted as a comparison between our model and a the “No adjustment” policy model in which producer sets an initial capacity margin (30 GW in this case) and the system is then only impacted by the uncertainties.

The figure shows that the Favorable scenario is a scenario where the capacities naturally experience favorable outcomes. For example, the inputs of hydro-powerplants, the load factor of wind and photovoltaic should have been very high, or no major failure of power plant should have been observed. On the contrary, the severe scenario is a scenario where capacities uncertainties are very unfavorable (strong drop of capacity after year 2, which can be seen in the “no capacity adjustment policy”). We can see that optimal policy absorbs this shock, at least partially because of the costs of construction preventing producer from restoring a higher capacity margin.

A heuristic observation of consumer’s value function suggests that whenever the capacity margin is tight (when  $x^C - x^D$  is small), there is a high risk of shortage and a low satisfaction from consumption. We can guess then that  $\partial_{x^C} u \gg 0$  as a higher capacity level will make consumer’s situation better. It makes sense then for consumer to recommend a positive control  $\hat{\alpha}$  as suggested from (2.2.34), which is confirmed numerically as  $\hat{\alpha}$  takes higher values, and pushes the capacity process up. This represents a typical situation where we can see the implicit dependence of  $\hat{\alpha}$  on  $x^D$  (in particular on  $x^C - x^D$ ) as mentioned in Remark 2.2.8 ((i) and (ii)). The same pattern is observed at year 2 and 2.5 in the severe scenario where the producer invests to counteract a negative shock in capacity. However, between year 3 and 4.5, as it becomes very expensive to keep a positive capacity margin, the optimal control does not follow the shock and a serie of shortages occurs. This implies that

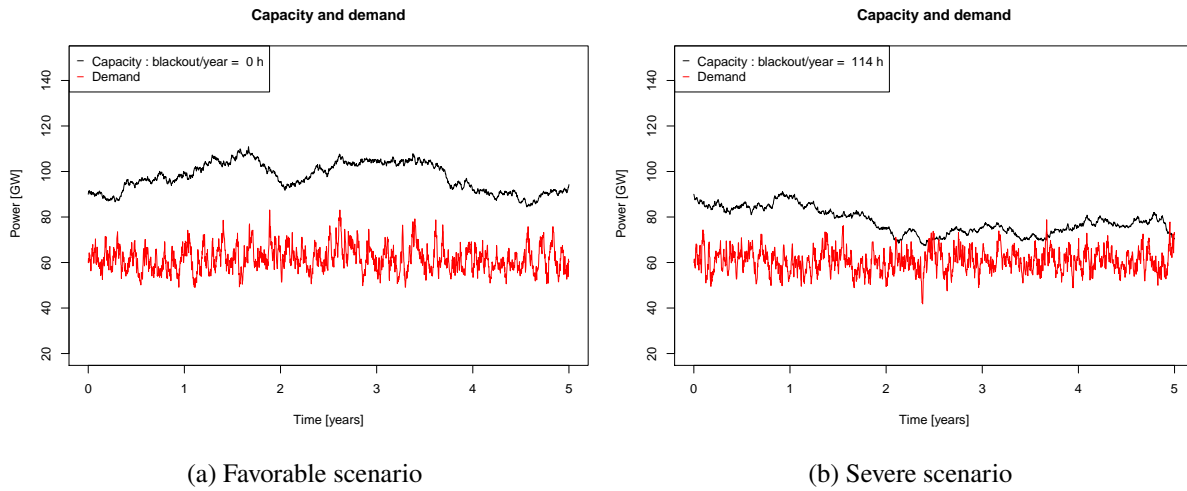


Figure 2.4: Evolution of capacity and demand under the recommended effort  $\hat{\alpha}$ . We compare the evolution of state variables for the “favorable” and “severe” scenarios. In the favorable scenario (left), the capacity (black) remains above the demand (red) with a high margin, as opposed to the severe scenario (right) where capacity is drawn down from external uncertainties and goes beyond the demand process a few times corresponding to the occurrences of shortage.

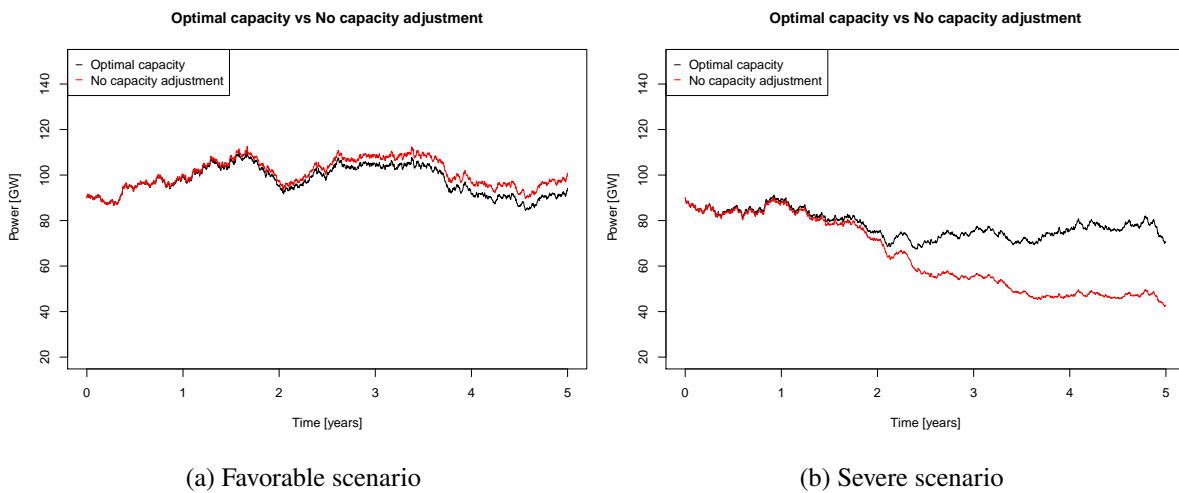
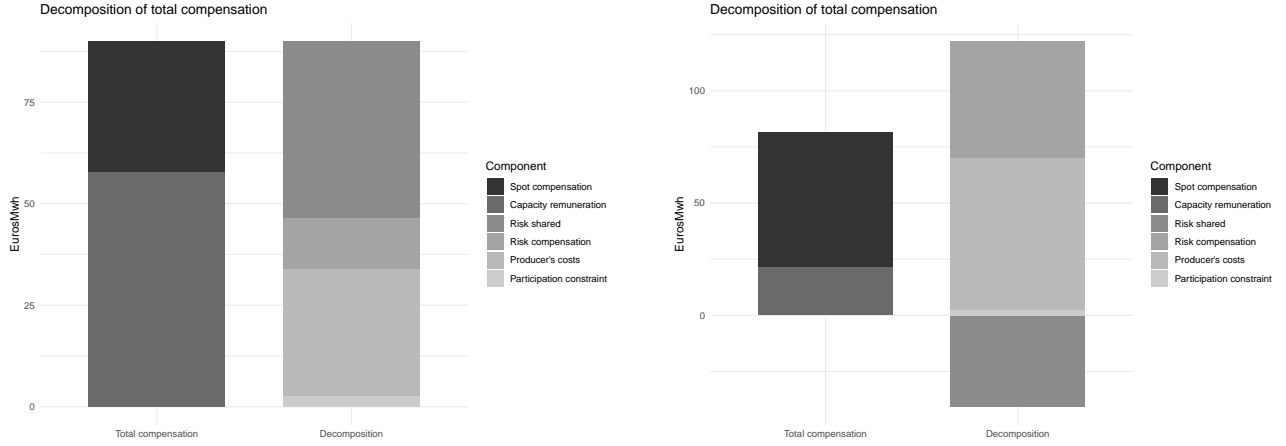


Figure 2.5: Comparison between producer’s optimal policy (the recommended effort) and a “no adjustment” policy for two different scenarios. The black line represents the controlled capacity for the favorable (left) and severe (right) scenarios. The externalities in these scenarios are made explicit with the uncontrolled capacity processes (red), and the impact of the control can be seen from the difference between the black line and the red line. As expected, the favorable scenario has positive externalities and the control is slightly negative, while the severe scenario has negative externalities and a positive control.



(a) Favorable scenario. The left bar aggregates the total compensation : (spot revenues + capacity remuneration), and is equal to the right bar (participation constraint + producer’s costs + risk shared + risk compensation) where we can see that all these terms are positive for the favorable scenario.

(b) Severe scenario. The left bar is the sum (spot revenues + capacity remuneration). The right bar has a part above zero representing the sum (participation constraint + producer’s costs+risk compensation), and a negative “risk shared” component because of the unfavorable uncertainties.

Figure 2.6: Decomposition of  $\xi^*$  for the favorable and severe scenarios highlighting the equality (spot + capacity) = (participation constraint + producer’s costs + risk shared + risk compensation).

starting from some threshold, consumers are willing to accept a shortage instead of paying a very high price to avoid it.

Finally, remark that from consumer’s perspective, the total payment (capacity remuneration + spot) is higher than average in the extreme scenarios, whether favorable or severe. Indeed, when the scenario is severe, the spot prices are high, and the capacity compensation is also high because of construction costs, and so consumer has to pay a lot for both. On the other hand, when the scenario is favorable, the spot prices are low but the consumer still needs to incentivize the producer with a high capacity compensation and share the positive risks (the “reward for good luck”).

### 2.3.3 Analysis of the optimal contract

#### 2.3.3.1 Decomposition of capacity payment

To understand how the contract is designed, we use the decomposition (2.2.37) suggested in section 2.2.6, recalled below:

$$\text{Capacity remuneration } (\xi^*) + \text{Spot compensation} = \mathcal{R} + \text{Producer's costs} + \text{Risk shared} + \text{Risk compensation.} \quad (2.2.37)$$

This decomposition is represented for the favorable and severe scenarios in figures 2.6a and 2.6b. In each of these figures, the first bar represents the total compensation (the left hand side terms of equality (2.2.37)), and the second bar corresponds to the right hand side. Remark that in the favorable scenario all the components are positive, and add up to the total compensation, while in the severe scenario, there is a negative component which is the risk shared. Nevertheless in this case also the algebraic sum of the components is equal to the total compensation.

In the severe scenario –figure 2.6b– the realized randomness is very unprofitable to the system. Our contract

	Without CRM	With a CRM	No adjustment
Spot revenues/Total revenues [%]	100	70	100
Scenarios with missing money [%]	2	28	26
Scenarios with negative capacity remuneration [%]	NA	18	NA
Scenarios with negative total compensation [%]	NA	0	NA
Scenarios with negative net revenues [%]	2	5	26

Table 2.8: Comparison between policies outcomes for producer with and without CRM. Each column of this table provides the average ratio (Spot revenues/Total revenues) on a sample of  $N = 5000$  simulated scenarios, together with the percentages of scenarios with missing money (total costs > spot revenues), the scenarios with (capacity remuneration < 0) and the scenarios with (spot revenue + capacity remuneration) < 0, and those with negative net revenues (total compensation - total costs) < 0. The first column represents the scenarios without a CRM, where producer controls the capacity and his only income is the spot revenue. The second column represents our CRM, with scenarios following the recommended effort and a capacity payment. The third column provides the results for scenarios generated without any control on capacity and without capacity payment.

automatically shares this negative randomness with the producer (negative “risk shared”). On the contrary, in the favorable scenario –figure 2.6a– where capacity outcomes are naturally high and profitable, this positive randomness is also shared with the producer but positively.

In addition, the contract accounts for the costs needed to implement the optimal policy, the remuneration from spot and the risk compensation. Remark that, as expected, the risk compensation is positive in both cases, even when the shared randomness is positive. This helps to offset the impact of the risk shared, for example in the severe scenario the negative risk shared (-40 euros/MWh) is completely canceled by the risk compensation (52 euros/MWh), see Table 2.7.

In the severe scenario the compensation for costs (the difference between blue and green) is quite high (this is mainly linked to the high costs needed to follow the optimal policy, consisting in investing a lot), whereas under the favorable scenario, this part is much limited (investment to be made are small). However, the remuneration obtained from the spot market in the favorable scenario (with a high capacity margin) is much less than in the severe scenario (with a low capacity margin). This leads to a low capacity payment under the severe scenario compared to the favorable scenario.

### 2.3.3.2 Link between capacity payment and spot compensation

Table 2.8 provides another comparison between the system without a CRM, with a CRM and with no adjustment, but this time with regard to the occurrences of missing money (when total costs are more than spot revenues) and the scenarios with negative net revenues for producer, along with the role played by spot in total compensation in the case of a CRM, and the percentage of scenarios with a negative capacity remuneration.

Regarding the system with a CRM, we obtain an average number of shortage hours per year less than three hours, which satisfies the LoLE constraint. The revenue provided by the capacity payment is about 30% of the total compensation (compared to 70% for spot market).

In our simulations  $\mathbb{E}^{\mathbb{P}^{\alpha^*}} \left[ \frac{S_T + \xi^*}{S_T} \right] \in [1.42, 1.43]$  with a 95% confidence level and a standard deviation of 0.53.

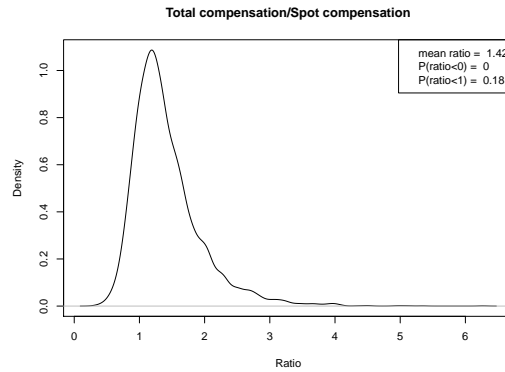


Figure 2.7: Distribution of  $\frac{S_T + \xi^*}{S_T}$  with a CRM. The ratio (total compensation/spot revenues) is observed on  $N = 5000$  scenarios, with an average of 1.42, and a positive probability (0.18) of being lower than 1, i.e., of having (total compensation < spot revenues).

This corresponds roughly to a partition of total revenues into 70% from spot market and 30% from capacity compensation. However, as we can see in figure 2.7, the distribution of  $\frac{S_T + \xi^*}{S_T}$  can take values less than 1 (even negative theoretically, but not observed in practice (cf. Table 2.8)) when  $\xi^*$  is negative. Whenever they occur, the negative capacity prices could be interpreted as a reimbursement from producers when their revenues from the spot market are high, similar to reliability options used in Italy and Ireland, and which by definition require such a money transfer when the spot price exceeds a certain level [Bhagwat & Meeus \(2019\)](#).

We investigate more this ratio in figure 2.8 by decomposing the total compensation into spot revenues and capacity remuneration for the favorable and severe scenarios, together with another extreme scenario.

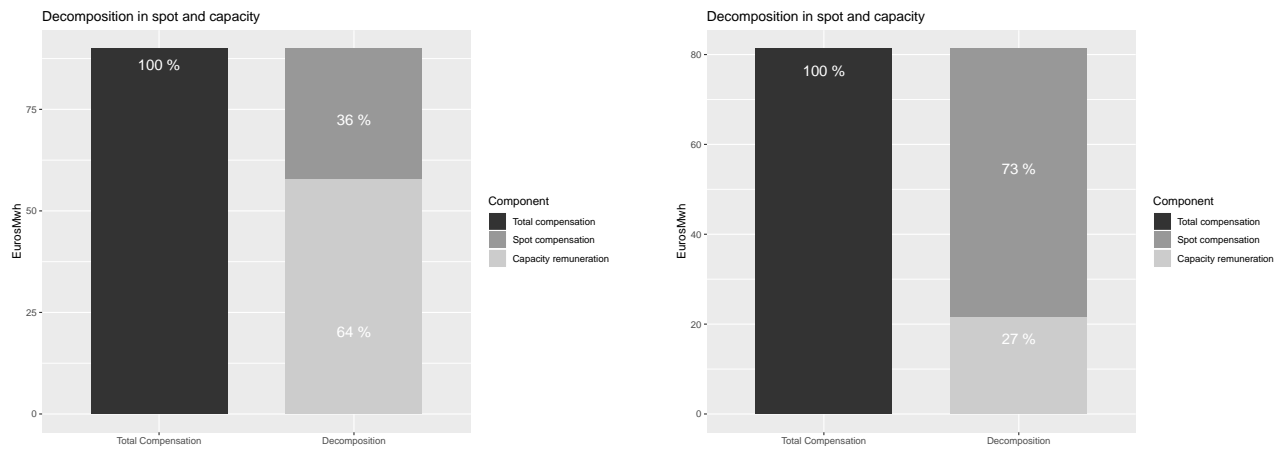
In the favorable scenario previously analyzed, figure 2.8a shows that the capacity payment is going to be more active than in the severe scenario 2.8b to complement spot remuneration. For the favorable scenario, the capacity payment represents 64% of the total remuneration of the producers, compared to 27% in the severe scenario. This is explained by the capacity margin, which dictates the level of spot prices and the need for a complementary compensation. Figure 2.8c illustrates a scenario where the capacity margin is very little during all the period leading to high spot prices, accompanied by a negative capacity remuneration.

Observe finally that the probability of getting negative net revenues drops from 26% in the case of no adjustment (very uncertain for producer) to 5% in the case with a CRM. It remains however slightly more than in the case without a CRM (2%) which is natural since in this scenario the producer’s only goal is to maximize his utility.

### 2.3.3.3 When is there missing money or a negative capacity payment

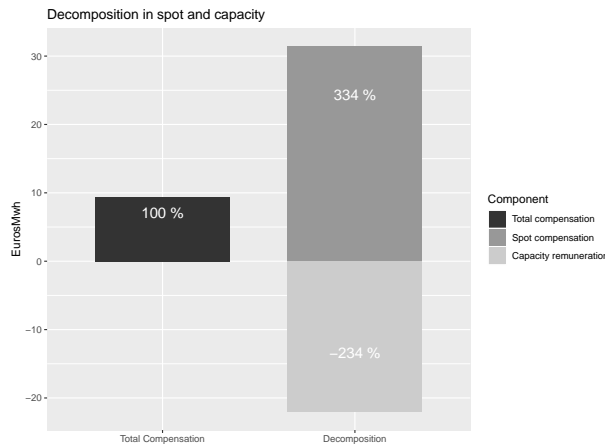
We investigate in this section “unfavorable” scenarios. Over the 5000 scenarios simulated earlier with a CRM, we select the ones where there is a missing money (MM) (where the spot remuneration is less than producer’s total costs (28% of scenarios)), and the scenarios where the capacity remuneration is negative (NCR); (18% of scenarios), and the ones where there is a missing money and the capacity remuneration is negative (MM and NCR); (1% of scenarios).

We compute the same indicators as before for two groups of the selected scenarios (the ones with missing money “MM” and the ones with a negative capacity remuneration “NCR”), and we discard the third group (with both “MM



(a) Favorable scenario. This scenario corresponds to high capacity levels, with low spot prices, so the capacity remuneration revenues have the biggest share in the total compensation.

(b) Severe scenario. This scenario is characterized by low capacities, and tight generation margins, which increase the spot revenues, whereas the capacity remuneration is low because of the (negative) risk shared.



(c) Scenario with the lowest  $\frac{S_T + \xi^*}{S_T}$  ratio. This particular extreme scenario shows the “punishment for bad luck” where the capacity remuneration is negative (because of the risk shared term), which considerably decreases the total compensation compared to the spot compensation.

Figure 2.8: Decomposition of  $\xi^*$  in terms of spot and capacity payment for different scenarios. In the three figures, the left bar represents the total compensation, and the right bar represents the decomposition total compensation = (spot compensation + capacity remuneration).



	MM	NCR	MM and NCR
Percentage [%]	28	18	1

Table 2.9: Missing money and negative capacity payment with a CRM. This table focuses on our model for CRM, and presents the percentages of “pathological” scenarios over  $N = 5000$ ; the scenarios with missing money (spot compensation < total costs), those with a negative capacity remuneration, and those with both missing money and negative capacity remuneration.

	Reference		MM		NCR	
	mean	sd	mean	sd	mean	sd
Shortage hours per year [Hours]	2.2	7.4	0.4	5.1	3.5	8.7
Average Spot price [euro/MWh]	37.6	9.6	25.8	4.9	44.6	6.5
Average Margin [GW]	32.4	9.0	43.9	6.9	26.2	4.7
Spot revenues [euro/MWh]	37.8	9.6	26.0	4.9	44.8	6.5
Capacity payment [euro/MWh]	12.3	13.2	22.7	11.6	-6.4	5.3
Spot + Capacity payment [euro/MWh]	50.1	10.9	48.7	10.4	38.4	7.7
Participation constraint [euro/MWh]	2.8	0.0	2.8	0.0	2.8	0.0
Risk shared [euro/MWh]	-0.2	11.7	0.1	11.2	-14.6	8.2
Risk compensation [euro/MWh]	14.9	3.9	13.1	2.8	16.7	4.7
Total costs [euro/MWh]	32.7	3.2	32.8	2.7	33.5	4.1
Construction and dismantling [euro/MWh]	1.8	3.7	0.3	2.8	3.5	4.5
Maintenance [euro/MWh]	13.2	1.3	14.9	1.0	12.4	0.7
Production [euro/MWh]	17.6	0.0	17.6	0.0	17.6	0.0

Table 2.10: Missing money and negative capacity remuneration with a CRM. This table regroups the average and sd of the relevant quantities for a simulation of  $N = 5000$  scenarios with the recommended policy. The first two columns provide the averages and sds over all the scenarios for reference. The second two columns provide the average and sd of the same quantities but only on the scenarios with missing money (spot revenues < total costs), and the last two columns provide the averages and sds over the scenarios where the capacity payment is negative.

and NCR”) since it concerns only 1% of the scenarios (average quantities are indeed meaningless in that case as the number of scenarios are very low). We summarize the results in table 2.10. Note that these indicators are computed with a Monte-Carlo method using different sizes of samples, and so do not have the same accuracy because of the different confidence intervals.

The major effect which explains missing money and negative capacity remuneration is the margin of the system. Missing money often comes with high margin while a negative capacity remuneration happens mostly with a low margin.

A close look at table 2.10 shows that there is missing money whenever the average margin is high (43 GW) in average, and so spot revenues are low (26 euros/MWh), and are not enough to cover total costs especially since maintenance costs are higher than average (14.9 euros/MWh compared to 13.2 euros/MWh) which overrules the fact that construction costs are close to zero. The CRM completes producer’s earnings on spot market since it compensates him systematically for his costs (whether high or low), and this can be seen in table 2.10 by the fact that

average total compensation is above costs.

The scenarios with negative capacity remuneration are more subtle to understand. In fact, they occur because our CRM is designed to take into account the spot compensation and complement it only when needed. In other words, if we recall again the decomposition of total compensation (2.2.37):

$$\text{Capacity remuneration} + \text{Spot compensation} = \mathcal{R} + \text{Producer's costs} + \text{Risk shared} + \text{Risk compensation}, \quad (2.2.37)$$

we can see that negative capacity remuneration is equivalent to

$$\text{Spot compensation} > \mathcal{R} + \text{Producer's costs} + \text{Risk shared} + \text{Risk compensation}. \quad (2.3.1)$$

This happens whenever the risk shared is negative (as confirmed by table 2.10 : -14.6 euros/MWh), suggesting unfavorable uncertainties and low capacity margins (26 GW compared to 32 GW in average), which increases spot revenues (44.8 euros/MWh) and consolidates the inequality (2.3.1). A typical scenario with a negative capacity remuneration would be one with a consistent low demand and high capacity without much realized volatility. This would keep the risk compensation low, with negative risk shared and high spot compensation. Nevertheless, observe that even in such conditions (unfavorable uncertainties), producer manages to keep an acceptable number of shortage hours per year (3.5 hours per year).

### 2.3.4 The optimal capacity payment in other setups

We test our system under different conditions, in the case where there are more renewable energies; and thus more uncertainties in the system (associated with higher  $\sigma^C$  (50% higher)), or when there is a demand response program applied on consumers, i.e., assuming demand volatility  $\sigma^D$  is lower (50% lower).

We also test different risk aversions; with more risk averse consumers or more risk averse producers (multiplying by 10 the risk aversion parameter each time).

#### 2.3.4.1 More renewable energies or a demand response program ( $\sigma^C$ and $\sigma^D$ )

A brief summary of the numerical results with perturbations of volatilities can be found in table 2.11.

We can see that introducing more renewable energies in the system yields a higher percentage of scenarios with missing money (74% as opposed to 28%), and to positive net revenues all the time. In this context, the capacity compensation is never negative (in the observed scenarios), and plays a much more important role in complementing producer revenues (59% of total revenues instead of only 30%). This is similar to what we observed in section 2.3.3.2, when comparing a severe and favorable scenario.

By looking further into table 2.12, we can explain the missing money by the high costs of construction and dismantling (7.6 euros/MWh compared to 1.8 euros/MWh), and the role of capacity payment by the considerable amount of risk compensation (34.5 euros/MWh compared to 14.9 euros/MWh).

So in summary the more uncertainties in production, the more the need for a capacity mechanism in order to cope with the random electricity demand and the random available capacity, since otherwise the number of shortage hours would increase. The capacity mechanism results in increasing the capacity margin of the system. As the

	Reference	More renewables	Demand response program
Spot revenues/Total revenues [%]	70	41	98
Scenarios with missing money [%]	28	74	25
Scenarios with negative capacity remuneration [%]	18	0	60
Scenarios with negative total compensation [%]	0	0	0
Scenarios with negative net revenues [%]	5	0	13

Table 2.11: Impacts of a shock in volatility on the CRM. This table presents the variation of total compensation composition and the percentages of scenarios with missing money, negative capacity remuneration, negative total compensation and negative net revenues. The first column recalls the reference results with our calibrated parameters. The second column provides the results when the volatility of capacity is 50% higher, interpreted as an increase of the proportion of renewable energies (which are more variable). The third column provides the results when the volatility of demand is 50% lower, interpreted as the introduction of some demand-response program driving consumers to have less variable demand.

system is longer in terms of capacity, spot prices are lower. Therefore, in average, consumers have to pay a higher capacity remuneration when the capacity volatility increases, and producers receive a higher total compensation, and higher earnings even though the spot compensation decreases.

The third column of tables 2.11 and 2.12 summarizes numerical results when the demand volatility is lower, which is a model for a demand response program; i.e., we assume that consumer is somehow incentivized to behave in a more predictable manner, so that the uncertainty on demand fluctuations ( $\sigma^D$ ) is lower.

In this case, the security of the system can be ensured with a lower capacity margin (29.9 GW instead of 32.4 GW) since there are less uncertainties. Therefore the average spot price and the spot revenues are higher (40.9 euros/MWh instead of 37.6 euros/MWh), so we are likely to obtain the inequality (2.3.1) which we recall

$$\text{Spot compensation} > \mathcal{R} + \text{Producer's costs} + \text{Risk shared} + \text{Risk compensation}, \quad (2.3.1)$$

which is equivalent to having a negative capacity remuneration especially since none of the terms on the right hand side should be high. This explains the high percentage of scenarios with negative capacity remuneration (60%).

These conditions suggest an auto-regulated market and less need for a CRM, since spot revenues represent 98% of total compensation, even though there is missing money 25% percent of the time, and often a negative capacity remuneration (60% of the time). We can guess that taking away the capacity remuneration would lead to an equilibrium situation with more uncertainties for producer: he incurs losses when there is a missing money, but these losses are balanced by the scenarios where the spot price is high, and he does not have to “pay” for capacity remuneration which is now a cost for him rather than a revenue, making his net revenues negative in 13% of the scenarios. Nevertheless, the total remuneration of producer (spot + capacity payment) is still above in average the total costs: in average with the CRM, the producer is going to earn money.

### 2.3.4.2 More risk aversion producer or consumer ( $\eta_A$ and $\eta_P$ )

We analyze the impacts of the CRM when producer or consumer is more risk averse. The numerical results are summarized in tables 2.13 and 2.14.

The first observation is that a more risk averse producer has a zero participation constraint, which means that the

	Reference	More renewables	Demand response program
Shortage hours per year [Hours]	2.2	2.8	0.7
Average Spot price [euro/MWh]	37.6	32.5	40.9
Average Margin [GW]	32.4	39.1	29.9
Spot revenues [euro/MWh]	37.8	32.6	40.7
Capacity payment [euro/MWh]	12.3	41.0	-3.2
Spot + Capacity payment [euro/MWh]	50.1	73.6	37.4
Participation constraint [euro/MWh]	2.8	0.0	2.2
Risk shared [euro/MWh]	-0.2	-0.4	-0.1
Risk compensation [euro/MWh]	14.9	34.5	4.0
Total costs [euro/MWh]	32.7	39.4	31.3
Construction and dismantling [euro/MWh]	1.8	7.6	0.8
Maintenance [euro/MWh]	13.2	14.2	12.9
Production [euro/MWh]	17.6	17.6	17.6

Table 2.12: Impacts of a shock in volatility on the number of shortage hours, and the costs and revenues of the CRM. The first column recalls the reference results with our calibrated parameters. The second column provides the results when the volatility of capacity is 50% higher, interpreted as an increase of the proportion of renewable energies (which are more variable). The third column provides the results when the volatility of demand is 50% lower, interpreted as the introduction of some demand-response incentive driving consumers to have less variable demand.

	Reference	Risk averse producer	Risk averse consumer
Shortage hours per year [Hours]	2.2	0.8	0.5
Average Spot price [euro/MWh]	37.6	32.8	34.4
Average Margin [GW]	32.4	36.6	34.9
Spot revenues [euro/MWh]	37.8	33.0	34.6
Capacity payment [euro/MWh]	12.3	51.2	380.6
Spot + Capacity payment [euro/MWh]	50.1	84.2	415.3
Participation constraint [euro/MWh]	2.8	0.0	2.7
Risk shared [euro/MWh]	-0.2	-0.1	-0.3
Risk compensation [euro/MWh]	14.9	49.7	378.7
Total costs [euro/MWh]	32.7	34.6	34.1
Construction and dismantling [euro/MWh]	1.8	3.2	2.9
Maintenance [euro/MWh]	13.2	13.8	13.6
Production [euro/MWh]	17.6	17.6	17.6

Table 2.13: Impacts of a shock in risk aversion parameters on the number of shortage hours, and the costs and revenues of the CRM. The first column recalls the reference results with our calibrated parameters. The second column provides the results when producer's risk aversion is higher ( $10 \times \eta_A$ ). The third column provides the results when consumer's risk aversion is higher ( $10 \times \eta_P$ ).

	Reference	Risk averse producer	Risk averse consumer
Spot revenues/Total revenues [%]	70	39	8
Scenarios with missing money [%]	28	53	41
Scenarios with negative capacity remuneration [%]	18	0	0
Scenarios with negative total compensation [%]	0	0	0
Scenarios with negative net revenues [%]	5	0	0

Table 2.14: Impacts of a shock in risk aversion parameters on the CRM. This table presents the variation of total compensation composition and the percentages of scenarios with missing money, negative capacity remuneration, negative total compensation and negative net revenues. The first column recalls the reference results with our calibrated parameters. The second column provides the results when producer’s risk aversion is higher ( $10 \times \eta_A$ ). The third column provides the results when consumer’s risk aversion is higher ( $10 \times \eta_P$ ).

utility gained from a system without a CRM is negative.

Apart from this observation, we have in both cases the same impacts but with different magnitudes. We can see that the costs of construction are slightly higher, with lower spot revenues. This leads to an increase in occurrences of missing money scenarios, and a larger part of total revenues coming from capacity compensation (always positive, and representing 41% and 92% from total compared to 30%). This capacity compensation comes mainly from the risk compensation (51.2 euros/MWh and 380 euros/MWh compared to 15 euros/MWh in the reference case).

To summarize, whenever one of the two parties is more risk averse, it becomes very costly for consumer to pay for capacity remuneration. A risk averse producer would require more risk compensation (the risk compensation is proportional to producer’s risk aversion by definition). From the other hand, a risk averse consumer would be ready to spend a lot to avoid potential shortage or blackout. The consequence is that the producer gets positive total compensation and positive net revenues 100% of the time.

## 2.4 Conclusion

In this paper, we provide some insights on how electricity producers and consumers could share the financial and physical risks (and uncertainties) to ensure the security of the system. We propose a CRM based on contract theory, which incentivizes producers to perform an optimal level of effort to maintain and develop new power-plants. It takes the form of a contract and a recommended effort, with payment adjusted to the uncertainty of outcomes (weather, outages...) ensuring to producers the right level of average earnings and financial risks while accounting for the spot revenues, as long as they follow the recommended effort.

This means that the CRM does not disrupt the spot market operations, which remains therefore able to ensure the short-term optimal economical dispatch. Given a predefined level of security for the system, the capacity mechanism we propose provides the right level of investment needed to insure it, and gives us insights to challenge real implementations of capacity markets.

One of our main results is that we point out the necessity of a CRM. This is reinforced with the level of randomness of the capacity and demand. Higher share of random renewable production in the electrical system means that a higher fraction of the compensation of the producers needs to come from a capacity remuneration system. As a matter of fact, in that case the volume of installed capacities should be more important to ensure the

security of the system. Meanwhile, the spot price decreases (the spot price is decreasing with respect to installed capacities, which is consistent with the increase of the supply curve), and so it is essential to support installed capacities.

In the meantime, the higher the volatility of consumption or supply, the higher the volatility of the capacity market. The mechanism we proposed also enables to study how risks should be share between the producers and the consumers. The increase of the financial risk is principally supported by the producers who are then compensated by a higher average revenue. However, even when there is a capacity payment, the number of hours of shortage increases when consumption or production become more volatile. In fact, if consumers do not modify the virtual value they associate to shortage, it is economically optimal for them to accept more hours of shortage instead of increasing suppliers' compensation. The capacity payment enables to share the financial risk between the producers and consumers depending on their risk aversion. The producers accept to take more financial risk if it comes with an increase of their average revenue. This is also what happens when the consumers want to reduce the physical risk.

Finally, we propose some variants for further research. It would be interesting to challenge our results by using other spot functions, especially ones that could reach higher peaks than the function (2.2.11). Furthermore, we assumed that the demand process is completely exogenous and uncontrolled. We explored this aspect of the model to a certain extent in section 2.3.4 by studying the sensitivities of our results with respect to the volatility of demand. Adding a direct control on the demand process would be relevant but would also change drastically the resolution methodology and is therefore an open research question. The same can be said about the assumption that dismantling power plants has the same cost as building new ones.

In addition, representing several technologies to produce electricity would also make a lot of sense. This could be the starting point for a future work by considering for example the possibility to control the volatility of the capacity process or to have several producers with different technologies instead of one. We also used a continuous time setting, which is convenient for modeling and computations, but requires using small time steps to discretize and approximate the optimal control (in our simulations we used a daily time step). This is equivalent to ignoring the delay needed to build new power plants which is not completely realistic, but remains quite common in such models because decision-making delay is hard to capture. Finally, our model only represents one design of CRM (a kind of sophisticated capacity payment). Further work may include other possible market designs.

## 2.5 Appendix

### 2.5.1 Mathematical framework and weak formulation

This section is devoted to the mathematical formulation of the problem. For  $T \in (0, +\infty)$  a fixed maturity, we denote by  $\Omega := \mathcal{C}([0, T], \mathbb{R}^2)$  the space of continuous functions from  $[0, T]$  to  $\mathbb{R}^2$ . The system is described by the state variable  $X := (X^C, X^D)^T$  which is the canonical process on  $\Omega$ . Finally we endow  $\Omega$  with its Borel  $\sigma$ -algebra  $\mathcal{F}_T$  and define the completed filtration  $\mathbb{F}$  generated by the process  $X$ .

We define the reference probability measure as the weak solution of the controlled equation (2.2.5) with a constant control set to zero, i.e.,  $\alpha_t = 0$  for  $t \in [0, T]$ , on the space  $(\Omega, \mathcal{F}_T)$  and we denote it  $\mathbb{P}^0$ . It is characterized as the unique probability measure such that  $\mathbb{P}^0 \circ (X_0)^{-1} = \delta_{x_0}$  for some  $x_0 \in \mathbb{R}_+^2$  and the processes  $\left( X_t - \int_0^t \mu(X_s, 0) ds \right)_{t \in [0, T]}$  is a  $(\mathbb{P}^0, \mathbb{F})$ -martingale with  $\langle X \rangle_t = \int_0^t \sigma(X_s) \sigma^\top(X_s) ds$  for  $t \in [0, T]$ . Note that

existence and uniqueness of this measure are insured by the existence of a unique strong solution to the corresponding SDE. Therefore there exists a 2–dimensional  $\mathbb{P}^0$ –Brownian motion  $W^{\mathbb{P}^0}$  such that

$$X_t = X_0 + \int_0^t \mu(X_s, 0) ds + \int_0^t \sigma(X_s) dW_s^{\mathbb{P}^0}, \text{ for } t \in [0, T]. \quad (2.5.1)$$

Notice that under  $\mathbb{P}^0$  the component  $X^C$  is a martingale (the drift part of  $X^C$  is zero when the control is constantly equal to 0).

**Definition 2.5.1** (Admissible controls). *Recall from section 2.2.2 that  $\mathcal{U}$  is the set of  $\mathbb{F}$ –predictable processes valued in  $[\alpha_{min}, \alpha_{max}]$ . For  $\alpha \in \mathcal{U}$ , the Novikov’s criterion is satisfied (from the boundedness of  $\alpha$ ):*

$$\mathbb{E}^{\mathbb{P}^0} \left[ e^{\frac{1}{2} \int_0^T \left( \frac{\alpha_t}{\sigma^C} \right)^2 dt} \right] < +\infty, \quad (2.5.2)$$

and so the process

$$\mathcal{E} \left( \int_0^t \frac{\alpha_s}{\sigma^C} dW_s^{C, \mathbb{P}^0} \right)_{t \in [0, T]} \text{ is a } (\mathbb{P}^0, \mathbb{F})\text{-UI martingale,} \quad (2.5.3)$$

and we can define the induced probability measure  $\mathbb{P}^\alpha$  on  $(\Omega, \mathcal{F}_T)$  as the equivalent measure to  $\mathbb{P}^0$  with its Radon–Nikodym derivative

$$\begin{cases} \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} := \mathcal{E} \left( \int_0^T \frac{\alpha_s}{\sigma^C} dW_s^{C, \mathbb{P}^0} \right), \\ L_t^\alpha := \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} \Big|_{\mathcal{F}_t}, \end{cases} \quad (2.5.4)$$

where  $W^{C, \mathbb{P}^0}$  is the capacity component of the Brownian motion. We denote by  $\mathcal{P}$  the collection of probability measures  $\mathbb{P}^\alpha$  induced by the set of controls  $\alpha \in \mathcal{U}$ .

By applying Girsanov’s theorem, we can see that for an admissible control  $\alpha$ , the process  $X$  has the law of SDE (2.2.5) under  $\mathbb{P}^\alpha$ .

Remark that the demand component is not affected by this change of probability, i.e.,  $\mathbb{P}^0 \circ (X^D)^{-1} = \mathbb{P}^\alpha \circ (X^D)^{-1}$ , for every admissible  $\alpha$ , which is consistent with our model: the control is only on (the drift) of the capacity.

**Definition 2.5.2.** *The set  $\Xi$  of admissible contracts is defined as the collection of  $\mathcal{F}_T$ –measurable random variables  $\xi$  satisfying*

$$\begin{cases} V_0^A(\xi) \geq U_A(\mathcal{R}), \\ \mathcal{P}^*(\xi) \neq \emptyset, \\ \sup_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}^{\mathbb{P}^\alpha} \left[ e^{(\eta_A \vee \eta_P)(1+\delta)|\xi|} \right] < \infty, \text{ for some } \delta > 0. \end{cases} \quad (2.5.5)$$

The definition of admissible contracts imposes the existence of an optimal control for agent which satisfies the participation constraint, and non-degeneracy conditions for both principal and agent problems.

## 2.5.2 Definition and derivation of the class of revealing contracts $\mathcal{Z}$

### 2.5.2.1 Definition

Let  $\mathcal{V}$  be the set of  $\mathbb{F}$ -predictable processes  $Z$  valued in  $\mathbb{R}^2$  satisfying

$$\begin{cases} \mathbb{E}^{\mathbb{P}^\alpha} \left[ \mathcal{E} \left( -\eta_A \int_0^T Z_t \cdot \sigma(X_t) dW_t^\alpha \right) \right] = 1, \text{ for } \mathbb{P}^\alpha \in \mathcal{P}, \\ \sup_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}^{\mathbb{P}^\alpha} \left[ e^{(\eta_A \vee \eta_P)(1+\delta) |Y_T^{0,Z}|} \right] < \infty, \end{cases} \quad (2.5.6)$$

where we recall

$$Y_t^{Y_0,Z} = Y_0 + \int_0^t Z_s \cdot dX_s - \int_0^t H(X_s, Z_s) ds, \text{ for } t \in [0, T], \quad (2.2.22)$$

and  $H$  defined as in (2.2.23). The class of revealing contracts  $\mathcal{Z}$  is then the subset of  $\Xi$  containing the terminal values of  $(Y_t^{Y_0,Z})_{t \in [0,T]}$  parameterized by the two control variables  $(Y_0, Z) \in (\mathbb{R} \times \mathcal{V})$ ;

$$\mathcal{Z} = \left\{ Y_T^{Y_0,Z} \text{ for some } (Y_0, Z) \in \mathbb{R} \times \mathcal{V} \text{ with } Y_T^{Y_0,Z} \in \Xi \right\}. \quad (2.2.21)$$

### 2.5.2.2 Derivation of the class $\mathcal{Z}$

The aim of this subsection is to explain the intuition behind the class of revealing contracts  $\mathcal{Z}$  in the spirit of [Sannikov \(2008\)](#), which is a key ingredient in the resolution of our problem. Rather than providing a rigorous treatment of this question (for which we refer the reader to [Cvitanic et al. \(2018\)](#)), we only present here the main ideas behind it. The main goal of introducing the class  $\mathcal{Z}$  which—we recall—plays the role of a performance index, is to overcome the non-markovianity of the principal-agent problem, and to make agent’s response “predictable” by principal. This is achieved by the martingale optimality principle.

We start by recalling agent’s value function:

$$V_0^A(\xi) = \sup_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_A \left( \xi + \int_0^T s(X_t) dt - \int_0^T c^A(X_t, \alpha_t) dt \right) \right]. \quad (2.5.7)$$

We first restrict the contracts to the ones which are terminal values of some diffusion process, i.e., of the form  $\xi = Y_T^{Y_0,Z}$  where  $Y_0$  and  $Z$  are new control variables, and  $Y_T^{Y_0,Z} = Y_0 + \int_0^T Z_t \cdot dX_t + \int_0^T g(t, X_t, Y_t, Z_t) dt$ , where  $g$  is a deterministic function to be determined.

Roughly speaking, this allows us to reduce to the markovian case by recapturing principal’s “missing” information and plugging it into the new process  $Y_T^{Y_0,Z}$ . This is done through the change of control variables from  $\xi$  to  $Y_0$  and  $Z$ . We stress here that this not a mathematical proof but just an explanation, since  $Y_T^{Y_0,Z}$  is a solution to some SDE and has a priori no reason to exist (so far we didn’t impose any restrictions on the function  $g$ ).

Now that we are in a markovian setting, we want to get rid of the moral-hazard, i.e., to have a predicable response from the agent. This is possible by a careful choice of the function  $g$ . In fact, by the structure of the exponential utility  $U_A(x) = -e^{-\eta_A x}$ , we have by a simple application of Itô formula that the process defined by  $U_A \left( Y_t^{Y_0,Z} + \int_0^t (s(X_r) - c^A(X_r, \alpha_r)) dr \right)_{t \in [0,T]}$ , will be a  $(\mathbb{P}^\alpha, \mathbb{F})$ -supermartingale and a martingale only for



some  $\hat{\alpha}$  if we choose  $g$  such that

$$g(t, x, y, z) = -H(x, z) \text{ for } (t, x, y, z) \in [0, T] \times \mathbb{R}_+^2 \times \mathbb{R} \times \mathbb{R}^2, \quad (2.5.8)$$

with  $H$  the hamiltonian defined in (2.2.23). This gives us first an upper bound on agent's optimal control  $V_0^A(Y_0^{Y_0, Z}) \leq U_A(Y_0)$  by the supermartingale property, and we have that this bound is attained for the control induced by  $\mathbb{P}^{\hat{\alpha}}$  by construction.

Therefore, since agent is rational and aims at maximizing his utility, he chooses the control  $\hat{\alpha}$ , which is a deterministic function of the pair  $(X_t, Z_t)_{t \in [0, T]}$ , both observable by principal.

### 2.5.3 Solving Producer's problem: proof of Proposition 2.2.5

Let  $Y_T^{Y_0, Z} \in \mathcal{Z}$  and  $\mathbb{P}^\alpha \in \mathcal{P}$ . By definition

$$\begin{aligned} J_0^A(Y_T^{Y_0, Z}, \mathbb{P}^\alpha) &= \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_A \left( Y_T^{Y_0, Z} + \int_0^T (s(X_t) - c^A(X_t, \alpha_t)) dt \right) \right], \\ &= \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_A \left( Y_0 + \int_0^T Z_t \cdot dX_t - \int_0^T H(X_t, Z_t) dt + \int_0^T (s(X_t) - c^A(X_t, \alpha_t)) dt \right) \right]. \end{aligned} \quad (2.5.9)$$

Using (2.2.24) :

$$\begin{aligned} \int_0^T Z_s \cdot dX_s &= \int_0^T h(X_t, Z_t, \alpha_t) dt + \int_0^T \sigma(X_t) Z_t \cdot dW_t^\alpha \\ &\quad - \int_0^T (s(X_t) - c^A(X_t, \alpha_t)) dt + \frac{\eta_A}{2} \int_0^T |\sigma(X_t) Z_t|^2 dt, \end{aligned} \quad (2.5.10)$$

and therefore replacing  $U_A$  with its expression and injecting (2.5.10) we get

$$\begin{aligned} J_0^A(Y_T^{Y_0, Z}, \mathbb{P}^\alpha) &= \mathbb{E}^{\mathbb{P}^\alpha} \left[ -e^{-\eta_A \left( Y_0 + \int_0^T h(X_t, Z_t, \alpha_t) - H(X_t, Z_t) dt + \int_0^T \sigma(X_t) Z_t \cdot dW_t^\alpha + \frac{\eta_A}{2} \int_0^T |\sigma(X_t) Z_t|^2 dt \right)} \right] \\ &= U_A(Y_0) \mathbb{E}^{\mathbb{P}^\alpha} \left[ e^{\eta_A \int_0^T \{H(X_t, Z_t) - h(X_t, Z_t, \alpha_t)\} dt} \mathcal{E} \left( -\eta_A \int_0^T Z_t \cdot \sigma(X_t) dW_t^\alpha \right) \right]. \end{aligned} \quad (2.5.11)$$

Since  $Z \in \mathcal{V}$  we have that

$$\mathbb{E}^{\mathbb{P}^\alpha} \left[ \mathcal{E} \left( -\eta_A \int_0^T Z_t \cdot \sigma(X_t) dW_t^\alpha \right) \right] = 1, \quad (2.5.12)$$

and we can define the probability measure  $\widetilde{\mathbb{P}}^\alpha$  equivalent to  $\mathbb{P}^\alpha$  with the Radon-Nikodym derivative

$$\frac{d\widetilde{\mathbb{P}}^\alpha}{d\mathbb{P}^\alpha} := \mathcal{E} \left( -\eta_A \int_0^T Z_t \cdot \sigma(X_t) dW_t^\alpha \right), \quad (2.5.13)$$

and so

$$J_0^A \left( Y_T^{Y_0, Z}, \mathbb{P}^\alpha \right) = U_A \left( Y_0 \right) \mathbb{E}^{\widetilde{\mathbb{P}}^\alpha} \left[ e^{\eta_A \int_0^T \{H(X_t, Z_t) - h(X_t, Z_t, \alpha_t)\} dt} \right]. \quad (2.5.14)$$

Recalling that  $U_A \left( Y_0 \right) < 0$ , and  $H \left( X_t, Z_t \right) - h \left( X_t, Z_t, \alpha_t \right) \geq 0$  with equality if and only if  $\alpha_t = \hat{\alpha} \left( X_t^C, Z_t^C \right)$  for all  $t$  in  $[0, T]$ , we obtain the upper bound

$$J_0^A \left( Y_T^{Y_0, Z}, \mathbb{P}^\alpha \right) \leq U_A \left( Y_0 \right) \text{ for all } \mathbb{P}^\alpha \in \mathcal{P}, \quad (2.5.15)$$

which is attained for  $\left( \hat{\alpha} \left( X_t^C, Z_t^C \right) \right)_{t \in [0, T]}$  since it is an admissible control (as it is a progressively measurable process valued in  $[\alpha_{min}, \alpha_{max}]$ ) and so

$$J_0^A \left( Y_T^{Y_0, Z}, \mathbb{P}^{\hat{\alpha}} \right) = U_A \left( Y_0 \right), \quad (2.5.16)$$

which yields

$$V_0^A \left( Y_T^{Y_0, Z} \right) = U_A \left( Y_0 \right), \quad (2.5.17)$$

and  $\left( \hat{\alpha} \left( X_t^C, Z_t^C \right) \right)_{t \in [0, T]}$  is producer's optimal response given the contract  $Y_T^{Y_0, Z}$ .  $\square$

## 2.5.4 Proof of Proposition 2.2.6

For readers familiar with BSDE theory, Proposition 2.2.6 can be seen as an existence result for BSDEs with a quadratic generator. The following proof is largely inspired by [Elie & Possamaï \(2019\)](#) and [El Euch et al. \(2018\)](#), is classical in the non-Markovian stochastic control theory and relies on the *Agent's continuation utility* as a natural candidate for the solution of the BSDE.

We start by defining Agent's continuation utility, and prove that it satisfies the Dynamic Programming Principle. Then we use the assumptions on the set of admissible contracts and the properties of Agent's continuation utility to conclude the proof of Proposition 2.2.6.

**Definition 2.5.3.** *Let  $\tau$  be a stopping time valued in  $[t, T]$ . We denote by  $\mathcal{U}_\tau$  the restriction of (Agent's) controls to  $[\tau, T]$ . We define the dynamic version of Agent's objective function for a given  $\xi \in \Xi$  as*

$$J_\tau^A \left( \xi, \mathbb{P}^\alpha \right) := \mathbb{E}_\tau^{\mathbb{P}^\alpha} \left[ U_A \left( \xi + \int_\tau^T s(X_r) dr - \int_\tau^T c^A \left( X_r, \alpha_r \right) dr \right) \right] \text{ and } \mathcal{J}_\tau^A \left( \xi \right) := \left( J_\tau^A \left( \xi, \mathbb{P}^\alpha \right) \right)_{\alpha \in \mathcal{U}_\tau}, \quad (2.5.18)$$

and his continuation utility

$$V_\tau^A \left( \xi \right) := \text{esssup}_{\alpha \in \mathcal{U}_\tau} J_\tau^A \left( \xi, \mathbb{P}^\alpha \right). \quad (2.5.19)$$

Remark that for any  $\mathbb{P}^\alpha \in \mathcal{P}$ , the conditional expectation  $\mathbb{E}_\tau^{\mathbb{P}^\alpha}$  depends only on the restriction of  $\alpha$  on  $[\tau, T]$ . It is then defined without ambiguity for  $\alpha \in \mathcal{U}_\tau$ .

**Lemma 2.5.4.** For  $\xi \in \Xi$ ,  $t \in [0, T]$ , and  $\tau$  an  $\mathbb{F}$ -stopping time in  $[t, T]$ , we have that

(i) The family  $J_\tau^A(\xi)$  satisfies the lattice property, therefore the limiting sequence approaching  $V_\tau^A(\xi)$  can be chosen to be non-decreasing, i.e., there exists a sequence of  $(\mathbb{P}^{\alpha^n})_{n \geq 0}$  such that

$$V_\tau^A(\xi) = \lim_{n \rightarrow +\infty} \uparrow J_\tau^A(\xi, \mathbb{P}^{\alpha^n}). \quad (2.5.20)$$

(ii) The dynamic programming principle for Agent's value function holds, i.e. for  $\tau_1$  and  $\tau_2$  two stopping times such that  $0 \leq \tau_1 \leq \tau_2 \leq T$ :

$$V_{\tau_1}^A(\xi) = \operatorname{esssup}_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}_{\tau_1}^{\mathbb{P}^\alpha} \left[ V_{\tau_2}^A(\xi) e^{\eta_A \int_{\tau_1}^{\tau_2} (c^A(X_t, \alpha_t) - s(X_t)) dt} \right]. \quad (2.5.21)$$

**Proof.** (i) We consider two controls  $\alpha$  and  $\alpha'$  in  $\mathcal{U}_\tau$ . We define then

$$\tilde{\alpha} := \alpha \mathbb{1}_{\{J_\tau^A(\xi, \mathbb{P}^\alpha) \geq J_\tau^A(\xi, \mathbb{P}^{\alpha'})\}} + \alpha' \mathbb{1}_{\{J_\tau^A(\xi, \mathbb{P}^\alpha) < J_\tau^A(\xi, \mathbb{P}^{\alpha'})\}} \quad (2.5.22)$$

Then  $\tilde{\alpha} \in \mathcal{U}_\tau$  and from the definition of  $\tilde{\alpha}$  we have the inequality

$$J_\tau^A(\xi, \mathbb{P}^{\tilde{\alpha}}) \geq \max \left( J_\tau^A(\xi, \mathbb{P}^\alpha), J_\tau^A(\xi, \mathbb{P}^{\alpha'}) \right), \quad (2.5.23)$$

which proves the lattice property, implying (i) (Neveu 1972, Proposition VI.I.I, p121).

(ii) The proof of this part is similar to the one in (Cvitanic & Karatzas 1993, Proposition 6.2). We proceed in two steps proving each of the two inequalities. The first inequality is a direct consequence of the tower property. In fact, for  $0 \leq \tau_1 \leq \tau_2 \leq T$ , we have by definition

$$\begin{aligned} V_{\tau_1}^A(\xi) &= \operatorname{esssup}_{\alpha \in \mathcal{U}_{\tau_1}} \mathbb{E}_{\tau_1}^{\mathbb{P}^\alpha} \left[ -e^{-\eta_A \left( \xi - \int_{\tau_1}^T (c^A(X_r, \alpha_r) - s(X_r)) dr \right)} \right], \\ &= \operatorname{esssup}_{\alpha \in \mathcal{U}_{\tau_1}} \mathbb{E}_{\tau_1}^{\mathbb{P}^\alpha} \left[ -e^{-\eta_A \left( \xi - \int_{\tau_1}^{\tau_2} (c^A(X_r, \alpha_r) - s(X_r)) dr - \int_{\tau_2}^T (c^A(X_r, \alpha_r) - s(X_r)) dr \right)} \right]. \end{aligned} \quad (2.5.24)$$

By the tower property of the expectation we write

$$V_{\tau_1}^A(\xi) = \operatorname{esssup}_{\alpha \in \mathcal{U}_{\tau_1}} \mathbb{E}_{\tau_1}^{\mathbb{P}^\alpha} \left[ e^{\eta_A \int_{\tau_1}^{\tau_2} (c^A(X_r, \alpha_r) - s(X_r)) dr} \mathbb{E}_{\tau_2}^{\mathbb{P}^\alpha} \left[ -e^{-\eta_A \left( \xi - \int_{\tau_2}^T (c^A(X_r, \alpha_r) - s(X_r)) dr \right)} \right] \right]. \quad (2.5.25)$$

Using Bayes rule and remarking that  $\mathbb{E}_{\tau_2}^{\mathbb{P}^\alpha} \left[ -e^{-\eta_A \left( \xi - \int_{\tau_2}^T (c^A(X_r, \alpha_r) - s(X_r)) dr \right)} \right]$  depends only on values of  $\alpha$  after

$\tau_2$ , we have that for an arbitrary  $\alpha \in \mathcal{U}$

$$\begin{aligned} \mathbb{E}_{\tau_2}^{\mathbb{P}^\alpha} \left[ -e^{-\eta_A \left( \xi - \int_{\tau_2}^T (c^A(X_r, \alpha_r) - s(X_r)) dr \right)} \right] &\leq \text{esssup}_{\alpha \in \mathcal{U}} \mathbb{E}_{\tau_2}^{\mathbb{P}^\alpha} \left[ U_A \left( \xi - \int_{\tau_2}^T (c^A(X_r, \alpha_r) - s(X_r)) dr \right) \right], \\ &= \text{esssup}_{\alpha \in \mathcal{U}_{\tau_2}} \mathbb{E}_{\tau_2}^{\mathbb{P}^\alpha} \left[ U_A \left( \xi - \int_{\tau_2}^T (c^A(X_r, \alpha_r) - s(X_r)) dr \right) \right], \\ &= V_{\tau_2}^A(\xi), \end{aligned} \quad (2.5.26)$$

and then

$$V_{\tau_1}^A(\xi) \leq \text{esssup}_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}_{\tau_1}^{\mathbb{P}^\alpha} \left[ V_{\tau_2}^A(\xi) e^{\eta_A \int_{\tau_1}^{\tau_2} (c^A(X_r, \alpha_r) - s(X_r)) dr} \right]. \quad (2.5.27)$$

We proceed next to prove the second inequality. Consider  $\alpha \in \mathcal{U}$  and  $\nu \in \mathcal{U}_{\tau_2}$ . Define then the concatenation of the two controls for  $0 \leq t \leq T$  as  $(\alpha \otimes_{\tau_2} \nu)_t := \alpha_t \mathbb{1}_{0 \leq t < \tau_2} + \nu_t \mathbb{1}_{\tau_2 \leq t \leq T}$ , where  $\tau_2$  is an  $\mathbb{F}$ -stopping time.

We have then  $(\alpha \otimes_{\tau_2} \nu) \in \mathcal{U}$  and by definition of the essential supremum (where we denote  $\mathbb{E}_{\tau_1}^{\alpha \otimes_{\tau_2} \nu}$  instead of  $\mathbb{E}_{\tau_1}^{\mathbb{P}^{\alpha \otimes_{\tau_2} \nu}}$ ):

$$\begin{aligned} V_{\tau_1}^A(\xi) &\geq \mathbb{E}_{\tau_1}^{\alpha \otimes_{\tau_2} \nu} \left[ -e^{-\eta_A \left( \xi - \int_{\tau_1}^T (c^A(X_r, (\alpha \otimes_{\tau_2} \nu)_r) - s(X_r)) dr \right)} \right], \\ &= \mathbb{E}_{\tau_1}^{\alpha \otimes_{\tau_2} \nu} \left[ -e^{-\eta_A \left( -\int_{\tau_1}^{\tau_2} (c^A(X_r, \alpha_r) - s(X_r)) dr - \int_{\tau_2}^T (c^A(X_r, \nu_r) - s(X_r)) dr \right)} e^{-\eta_A \xi} \right], \\ &= \mathbb{E}_{\tau_1}^{\alpha \otimes_{\tau_2} \nu} \left[ e^{\eta_A \int_{\tau_1}^{\tau_2} (c^A(X_r, \alpha_r) - s(X_r)) dr} \mathbb{E}_{\tau_2}^{\alpha \otimes_{\tau_2} \nu} \left[ -e^{-\eta_A \left( \xi - \int_{\tau_2}^T (c^A(X_r, \nu_r) - s(X_r)) dr \right)} \right] \right]. \end{aligned} \quad (2.5.28)$$

Using again Bayes formula on the conditional expectation w.r.t  $\mathcal{F}_{\tau_2}$ , we have that

$$\mathbb{E}_{\tau_2}^{\alpha \otimes_{\tau_2} \nu} \left[ -e^{-\eta_A \left( \xi - \int_{\tau_2}^T (c^A(X_r, \nu_r) - s(X_r)) dr \right)} \right] = \mathbb{E}_{\tau_2}^0 \left[ -\frac{L_T^{\alpha \otimes_{\tau_2} \nu}}{L_{\tau_2}^{\alpha \otimes_{\tau_2} \nu}} e^{-\eta_A \left( \xi - \int_{\tau_2}^T (c^A(X_r, \nu_r) - s(X_r)) dr \right)} \right]. \quad (2.5.29)$$

Now notice that  $\frac{L_T^{\alpha \otimes_{\tau_2} \nu}}{L_{\tau_2}^{\alpha \otimes_{\tau_2} \nu}} = \frac{L_T^\nu}{L_{\tau_2}^\nu}$  (as stated earlier the change of measure applied to the conditional expectation depends only on the control after  $\tau_2$ ). We have therefore

$$\begin{aligned} \mathbb{E}_{\tau_2}^{\alpha \otimes_{\tau_2} \nu} \left[ -e^{-\eta_A \left( \xi - \int_{\tau_2}^T (c^A(X_r, \nu_r) - s(X_r)) dr \right)} \right] &= \mathbb{E}_{\tau_2}^0 \left[ -\frac{L_T^\nu}{L_{\tau_2}^\nu} e^{-\eta_A \left( \xi - \int_{\tau_2}^T (c^A(X_r, \nu_r) - s(X_r)) dr \right)} \right], \\ &= J_{\tau_2}^A(\xi, \mathbb{P}^\nu). \end{aligned} \quad (2.5.30)$$

Thus we obtain the following inequality

$$V_{\tau_1}^A(\xi) \geq \mathbb{E}_{\tau_1}^{\alpha \otimes_{\tau_2} \nu} \left[ e^{\eta_A \int_{\tau_1}^{\tau_2} (c^A(X_r, \alpha_r) - s(X_r)) dr} J_{\tau_2}^A(\xi, \mathbb{P}^\nu) \right]. \quad (2.5.31)$$

We use again Bayes Formula for the change of measure and the tower property of conditional expectation leading to

$$\begin{aligned}
V_{\tau_1}^A(\xi) &\geq \mathbb{E}_{\tau_1}^0 \left[ \frac{L_T^{\alpha \otimes \tau_2 \nu}}{L_{\tau_1}^{\alpha \otimes \tau_2 \nu}} e^{\eta_A \int_{\tau_1}^{\tau_2} (c^A(X_r, \alpha_r) - s(X_r)) dr} J_{\tau_2}^A(\xi, \mathbb{P}^\nu) \right], \\
&= \mathbb{E}_{\tau_1}^0 \left[ \mathbb{E}_{\tau_2}^0 \left[ \frac{L_T^{\alpha \otimes \tau_2 \nu}}{L_{\tau_1}^{\alpha \otimes \tau_2 \nu}} \frac{L_{\tau_2}^{\alpha \otimes \tau_2 \nu}}{L_{\tau_1}^{\alpha \otimes \tau_2 \nu}} e^{\eta_A \int_{\tau_1}^{\tau_2} (c^A(X_r, \alpha_r) - s(X_r)) dr} J_{\tau_2}^A(\xi, \mathbb{P}^\nu) \right] \right], \\
&= \mathbb{E}_{\tau_1}^0 \left[ \mathbb{E}_{\tau_2}^0 \left[ \frac{L_T^{\alpha \otimes \tau_2 \nu}}{L_{\tau_2}^{\alpha \otimes \tau_2 \nu}} \right] \frac{L_{\tau_2}^{\alpha \otimes \tau_2 \nu}}{L_{\tau_1}^{\alpha \otimes \tau_2 \nu}} e^{\eta_A \int_{\tau_1}^{\tau_2} (c^A(X_r, \alpha_r) - s(X_r)) dr} J_{\tau_2}^A(\xi, \mathbb{P}^\nu) \right], \\
&= \mathbb{E}_{\tau_1}^0 \left[ \frac{L_{\tau_2}^{\alpha \otimes \tau_2 \nu}}{L_{\tau_1}^{\alpha \otimes \tau_2 \nu}} e^{\eta_A \int_{\tau_1}^{\tau_2} (c^A(X_r, \alpha_r) - s(X_r)) dr} J_{\tau_2}^A(\xi, \mathbb{P}^\nu) \right].
\end{aligned} \tag{2.5.32}$$

Now recall that for  $0 \leq t \leq \tau_2$  we have by definition  $(\alpha \otimes_{\tau_2} \nu)_t = \alpha_t$ , and therefore  $\frac{L_{\tau_2}^{\alpha \otimes \tau_2 \nu}}{L_{\tau_1}^{\alpha \otimes \tau_2 \nu}} = \frac{L_{\tau_2}^\alpha}{L_{\tau_1}^\alpha}$  leading to

$$\begin{aligned}
V_{\tau_1}^A(\xi) &\geq \mathbb{E}_{\tau_1}^0 \left[ \frac{L_{\tau_2}^\alpha}{L_{\tau_1}^\alpha} e^{\eta_A \int_{\tau_1}^{\tau_2} (c^A(X_r, \alpha_r) - s(X_r)) dr} J_{\tau_2}^A(\xi, \mathbb{P}^\nu) \right], \\
&= \mathbb{E}_{\tau_1}^\alpha \left[ e^{\eta_A \int_{\tau_1}^{\tau_2} (c^A(X_r, \alpha_r) - s(X_r)) dr} J_{\tau_2}^A(\xi, \mathbb{P}^\nu) \right].
\end{aligned} \tag{2.5.33}$$

The inequality (2.5.33) holds for  $\alpha \in \mathcal{U}$  and  $\nu \in \mathcal{U}_{\tau_2}$ , we can then by virtue of (i) choose a sequence  $(\nu^n)_{n \in \mathbb{N}}$  of controls in  $\mathcal{U}_{\tau_2}$  such that

$$V_{\tau_2}^A(\xi) = \lim_{n \rightarrow +\infty} \uparrow J_{\tau_2}^A(\xi, \mathbb{P}^{\nu^n}), \tag{2.5.34}$$

then we have by the monotone convergence theorem that for  $\alpha \in \mathcal{U}$

$$\begin{aligned}
V_{\tau_1}^A(\xi) &\geq \lim_{n \rightarrow +\infty} \uparrow \mathbb{E}_{\tau_1}^\alpha \left[ e^{\eta_A \int_{\tau_1}^{\tau_2} (c^A(X_r, \alpha_r) - s(X_r)) dr} J_{\tau_2}^A(\xi, \mathbb{P}^{\nu^n}) \right] \\
&= \mathbb{E}_{\tau_1}^\alpha \left[ e^{\eta_A \int_{\tau_1}^{\tau_2} (c^A(X_r, \alpha_r) - s(X_r)) dr} \lim_{n \rightarrow +\infty} \uparrow J_{\tau_2}^A(\xi, \mathbb{P}^{\nu^n}) \right] \\
&= \mathbb{E}_{\tau_1}^\alpha \left[ e^{\eta_A \int_{\tau_1}^{\tau_2} (c^A(X_r, \alpha_r) - s(X_r)) dr} V_{\tau_2}^A(\xi) \right],
\end{aligned} \tag{2.5.35}$$

concluding the proof of Lemma 2.5.4.  $\square$

### Proof of Proposition 2.2.6

Now that we proved the Dynamic Programming Principle, we move to the existence of the BSDE. We have by definition  $\mathcal{Z} \subset \Xi$ . To prove the second inclusion, we fix some  $\xi \in \Xi$ , and define agent's continuation utility as in (2.5.19).

By virtue of Lemma 2.5.4, the family  $\left( V_\tau^A(\xi) e^{\eta_A \int_0^\tau (c^A(X_r, \alpha_r) - s(X_r)) dr} \right)_{\tau \in \mathcal{T}_{0,T}}$  is a  $(\mathbb{P}^\alpha, \mathbb{F})$ -supermartingale system. Therefore, by the results of [Lenglart & Dellacherie \(1981\)](#), it can be aggregated by a unique  $\mathbb{F}$ -optional process up to indistinguishability, which coincides with  $\left( V_t^A(\xi) e^{\eta_A \int_0^t (c^A(X_r, \alpha_r) - s(X_r)) dr} \right)_{t \in [0,T]}$  and remains a  $(\mathbb{P}^\alpha, \mathbb{F})$ -supermartingale, which then admits a càd-làg modification since the filtration considered satisfies the usual conditions.

Then, from the admissibility constraint of the contract  $\xi$ ; that is  $\mathcal{P}^*(\xi) \neq \emptyset$ , there exists some probability measure  $\mathbb{P}^{\alpha^*}(\xi)$  (referred to as  $\mathbb{P}^{\alpha^*}$  to ease notations) such that  $V_t^A(\xi) = J_t^A(\xi, \mathbb{P}^{\alpha^*})$ , for  $t \in [0, T]$ , and so the process  $\left( J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{\eta_A \int_0^t (c^A(X_r, \alpha_r) - s(X_r)) dr} \right)_{t \in [0, T]}$  is a  $(\mathbb{P}^{\alpha^*}, \mathbb{F})$ -supermartingale, for  $\mathbb{P}^{\alpha^*} \in \mathcal{P}$ , while the processes  $\left( J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{\eta_A \int_0^t (c^A(X_r, \alpha_r^*) - s(X_r)) dr} \right)_{t \in [0, T]}$  is a  $(\mathbb{P}^{\alpha^*}, \mathbb{F})$ -UI martingale.

In fact the integrability is guaranteed by the ceiling function in  $c^A$  and  $s$ , together with the admissibility condition on the contract  $\xi$ ; for  $t \in [0, T]$

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^{\alpha^*}} \left[ \left| J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{\eta_A \int_0^t (c^A(X_r, \alpha_r) - s(X_r)) dr} \right| \right] &= \mathbb{E}^{\mathbb{P}^{\alpha^*}} \left[ \left| \mathbb{E}_t^{\mathbb{P}^{\alpha^*}} \left[ e^{-\eta_A \left( \xi + \int_0^T (s(X_r) - c^A(X_r, (\alpha \otimes_t \alpha^*)_r)) dr \right)} \right] \right| \right], \\ &\leq K \mathbb{E}^{\mathbb{P}^{\alpha^*}} \left[ \mathbb{E}_t^{\mathbb{P}^{\alpha^*}} \left[ e^{-\eta_A \xi} \right] \right] = K \mathbb{E}^{\mathbb{P}^{\alpha^*}} \left[ e^{-\eta_A \xi} \right] < +\infty, \end{aligned} \quad (2.5.36)$$

where we used again the ceiling function (and thus the boundedness of the exponential term), together with the admissibility condition on  $\xi$ . On the other hand, by the super-martingale inequality and the tower property of conditional expectations, we have for every  $t_1 \leq t_2 \in [0, T]$ :

$$\begin{aligned} J_{t_1}^A(\xi, \mathbb{P}^{\alpha^*}) e^{\eta_A \int_0^{t_1} (c^A(X_r, \alpha_r^*) - s(X_r)) dr} &\geq \mathbb{E}_{t_1}^{\mathbb{P}^{\alpha^*}} \left[ J_{t_2}^A(\xi, \mathbb{P}^{\alpha^*}) e^{\eta_A \int_0^{t_2} (c^A(X_r, \alpha_r^*) - s(X_r)) dr} \right], \\ &= \mathbb{E}_{t_1}^{\mathbb{P}^{\alpha^*}} \left[ \mathbb{E}_{t_2}^{\mathbb{P}^{\alpha^*}} \left[ U_A \left( \xi - \int_0^T (c^A(X_r, \alpha_r^*) - s(X_r)) dr \right) \right] \right], \\ &= \mathbb{E}_{t_1}^{\mathbb{P}^{\alpha^*}} \left[ U_A \left( \xi - \int_0^T (c^A(X_r, \alpha_r^*) - s(X_r)) dr \right) \right], \\ &= J_{t_1}^A(\xi, \mathbb{P}^{\alpha^*}) e^{\eta_A \int_0^{t_1} (c^A(X_r, \alpha_r^*) - s(X_r)) dr}. \end{aligned} \quad (2.5.37)$$

Therefore all the previous terms are equal a.s., in particular, for  $t \in [0, T]$

$$J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{\eta_A \int_0^t (c^A(X_r, \alpha_r^*) - s(X_r)) dr} = \mathbb{E}_t^{\mathbb{P}^{\alpha^*}} \left[ U_A \left( \xi + \int_0^T (s(X_r) - c^A(X_r, \alpha_r^*)) dr \right) \right], \quad (2.5.38)$$

which proves that  $\left( J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{\eta_A \int_0^t (c^A(X_r, \alpha_r^*) - s(X_r)) dr} \right)_{t \in [0, T]}$  is a  $\mathbb{P}^{\alpha^*}$ -closed martingale, with a terminal value at  $T$  given by

$$J_T^A(\xi, \mathbb{P}^{\alpha^*}) e^{\eta_A \int_0^T (c^A(X_r, \alpha_r^*) - s(X_r)) dr} = -e^{-\eta_A \left( \xi - \int_0^T (c^A(X_r, \alpha_r^*) - s(X_r)) dr \right)}. \quad (2.5.39)$$

Then, by the martingale representation theorem, there exists a predictable process  $\tilde{Z} \in \mathbb{H}_{loc}^2$  valued in  $\mathbb{R}^2$  such that

$$\begin{aligned} J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{\eta_A \int_0^t (c^A(X_r, \alpha_r^*) - s(X_r)) dr} &= J_0^A(\xi, \mathbb{P}^{\alpha^*}) + \int_0^t \tilde{Z}_s dW_s^{\alpha^*}, \\ &= J_0^A(\xi, \mathbb{P}^{\alpha^*}) \mathcal{E} \left( -\eta_A \int_0^t Z_r \cdot \sigma(X_r) dW_r^{\alpha^*} \right), \end{aligned} \quad (2.5.40)$$

where

$$Z_t := -\frac{\sigma^{-1}(X_t) \tilde{Z}_t}{\eta_A J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{\eta_A \int_0^t (c^A(X_r, \alpha_r^*) - s(X_r)) dr}}, \quad (2.5.41)$$

and

$$\mathcal{E} \left( -\eta_A \int_0^t Z_r \cdot \sigma(X_r) dW_r^{\alpha^*} \right)_{t \in [0, T]} \text{ is a } (\mathbb{P}^{\alpha^*}, \mathbb{F})\text{-UI martingale.} \quad (2.5.42)$$

We define

$$Y_t = U_A^{-1} \left( V_t^A(\xi) \right), \text{ for } t \in [0, T], \quad (2.5.43)$$

and our goal is to prove that the pair  $(Y, Z)$  is a solution to (2.2.28), and that  $Z \in \mathcal{V}$ .

Recall that for an arbitrary  $\alpha \in \mathcal{U}$ , the process  $\tilde{Y}_t^\alpha := J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{\eta_A \int_0^t (c^A(X_r, \alpha_r) - s(X_r)) dr}$  is a  $(\mathbb{P}^\alpha, \mathbb{F})$ -supermartingale. Replacing  $J_t^A(\xi, \mathbb{P}^{\alpha^*})$  by its representation, we obtain

$$\frac{\tilde{Y}_t^\alpha}{\eta_A} = \frac{1}{\eta_A} J_0^A(\xi, \mathbb{P}^{\alpha^*}) \mathcal{E} \left( -\eta_A \int_0^t Z_r \cdot \sigma(X_r) dW_r^{\alpha^*} \right) e^{\eta_A \int_0^t (c^A(X_r, \alpha_r) - c^A(X_r, \alpha_r^*)) dr}. \quad (2.5.44)$$

We apply then Itô formula and Girsanov Theorem, therefore

$$\begin{aligned} \frac{d\tilde{Y}_t^\alpha}{\eta_A \tilde{Y}_t^\alpha} &= \left( c^A(X_t, \alpha_t) - c^A(X_t, \alpha_t^*) \right) dt - Z_t \cdot \sigma(X_t) dW_t^{\alpha^*}, \\ &= -Z_t \cdot \sigma(X_t) dW_t^\alpha - \left\{ \left( Z_t \cdot \mu(X_t, \alpha_t) - c^A(X_t, \alpha_t) \right) - \left( Z_t \cdot \mu(X_t, \alpha_t^*) - c^A(X_t, \alpha_t^*) \right) \right\} dt, \end{aligned} \quad (2.5.45)$$

and by the supermartingale property and the sign of  $\tilde{Y}_t^\alpha$  we conclude that

$$\alpha^* \in \operatorname{argmax} \left( Z_t \cdot \mu(X_t, \alpha_t) - c^A(X_t, \alpha_t) \right). \quad (2.5.46)$$

Finally, applying Itô Formula

$$\begin{aligned} Y_t &= U_A^{-1} \left( J_0^A(\xi, \mathbb{P}^{\alpha^*}) \right) + \int_0^t Z_r \cdot \sigma(X_r) dW_r^{\alpha^*} + \int_0^t (c^A(X_r, \alpha_r^*) - s(X_r)) dr, \\ &= U_A^{-1} \left( J_0^A(\xi, \mathbb{P}^{\alpha^*}) \right) + \int_0^t Z_r \cdot dX_r - \int_0^t \left( Z_r \cdot \mu(X_r, \alpha_r^*) + s(X_r) - c^A(X_r, \alpha_r^*) - \eta_A |\sigma(X_r) Z_r|^2 \right) dr, \end{aligned} \quad (2.5.47)$$

and so the pair  $(Y, Z)$ , satisfy (2.2.28). Furthermore, following the line of proof of Briand & Hu (2008)[corollary 4]

and using the integrability assumption on admissible contracts, we obtain

$$\mathbb{E}^{\mathbb{P}^{\alpha^*}} \left[ e^{(\eta_A \vee \eta_P)(1+\delta) \sup_{t \in [0, T]} |Y_t|} \right] < +\infty, \quad (2.5.48)$$

with  $\mathbb{P}^{\alpha^*} = \mathbb{P}^{\hat{\alpha}}$  from proposition 2.2.5, which concludes the proof.  $\square$

## 2.5.5 Solving consumer's problem : optimal contract and capacity payment

From the Proposition 2.2.6, consumer's problem is reduced to

$$V_0^P = \sup_{Y_0 \geq \mathcal{R}} \sup_{Z \in \mathcal{V}} J_0^P \left( Y_T^{Y_0, Z}, \mathbb{P}^{\hat{\alpha}} \right), \quad (2.2.30)$$

so the optimal contract we are looking for is of the form  $\xi^* = Y_T^{Y_0^*, Z^*}$ , with a pair  $(Y_0^*, Z^*) \in [\mathcal{R}, +\infty[ \times \mathcal{V}$ . Using the identity  $Y^{Y_0, Z} = Y_0 + Y^{0, Z}$ , we have

$$\begin{aligned} V_0^P &= \sup_{Y_0 \geq \mathcal{R}} \sup_{Z \in \mathcal{V}} \left[ -e^{-\eta_P \left( -Y_T^{Y_0, Z} - S_T + \int_0^T c^P(X_t) dt \right)} \right], \\ &= \sup_{Y_0 \geq \mathcal{R}} e^{\eta_P Y_0} \sup_{Z \in \mathcal{V}} \left[ -e^{-\eta_P \left( -Y_T^{0, Z} - S_T + \int_0^T c^P(X_t) dt \right)} \right], \\ &= e^{\eta_P \mathcal{R}} \sup_{Z \in \mathcal{V}} \left[ -e^{-\eta_P \left( -Y_T^{0, Z} - S_T + \int_0^T c^P(X_t) dt \right)} \right], \end{aligned} \quad (2.5.49)$$

and therefore  $Y_0^* = \mathcal{R}$ . We can rewrite (2.2.30) as

$$V_0^P = \sup_{Z \in \mathcal{V}} \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ U_P \left( -Y_T^{\mathcal{R}, Z} - \int_0^T s(X_t) dt + \int_0^T c^P(X_t) dt \right) \right], \quad (2.5.50)$$

with the state variables following agent's optimal response (which we recall is the same as principal's recommendation) i.e.,  $\alpha_t^* = \hat{\alpha} \left( X_t^C, Z_t^C \right)$ , for  $t \in [0, T]$ :

$$\begin{cases} X_t &= \begin{pmatrix} x_0^C \\ x_0^D \end{pmatrix} + \int_0^t \mu \left( X_r, \hat{\alpha} \left( X_r^C, Z_r^C \right) \right) dr + \int_0^t \sigma \left( X_r \right) dW_r^{\hat{\alpha}}, \\ Y_t &= \mathcal{R} + \int_0^t \left( c^A \left( X_r, \hat{\alpha} \left( X_r^C, Z_r^C \right) \right) + \frac{\eta_A}{2} |\sigma \left( X_r \right) Z_r|^2 - s \left( X_r \right) \right) dr + \int_0^t Z_r \cdot \sigma \left( X_r \right) dW_r^{\hat{\alpha}}. \end{cases} \quad (2.5.51)$$

Define then the continuation utility as

$$V^P(t, x, y) := \sup_{Z \in \mathcal{V}_t} \mathbb{E}_{t, x, y}^{\mathbb{P}^{\hat{\alpha}}} \left[ -e^{-\eta_P \left( -Y_T - \int_t^T s(X_r) dr + \int_t^T c^P(X_r) dr \right)} \right], \quad (2.5.52)$$



Observe that  $Z = 0$  is an admissible control and therefore

$$\begin{aligned} V^P(t, x, y) &\geq \mathbb{E}_{t,x,y}^{\mathbb{P}^{\hat{\alpha}(0)}} \left[ -e^{\eta_P Y_T + \eta_P \int_t^T s(X_r) dr - \eta_P \int_t^T c^P(X_r) dr} \right], \\ &= -e^{\eta_P y} \mathbb{E}_{t,x,y}^{\mathbb{P}^{\hat{\alpha}(0)}} \left[ e^{\eta_P \int_t^T (c^A(X_r, \hat{\alpha}(X_r^C, 0)) - c^P(X_r)) dr} \right], \\ &\geq -e^{\eta_P y + C(T-t)} \text{ for some constant } C, \end{aligned} \quad (2.5.53)$$

where the last inequality follows from the bound

$$-kx_\infty - c^A(x_\infty, \alpha_{max}) \leq c^P(x) - c^A(x, \alpha) \leq x_\infty(\theta - \kappa_1 \alpha_{min}), \text{ for } (x, \alpha) \in \mathbb{R}_+^2 \times [\alpha_{min}, \alpha_{max}]. \quad (2.5.54)$$

On the other hand, we have

$$\begin{aligned} \mathbb{E}_{t,x,y}^{\mathbb{P}^{\hat{\alpha}}} \left[ -e^{\eta_P (Y_T \int_t^T s(X_r) dr - \int_t^T c^P(X_r) dr)} \right] &= -e^{\eta_P y} \\ &\quad \times \mathbb{E}_{t,x,y}^{\mathbb{P}^{\hat{\alpha}}} \left[ e^{\eta_P \int_t^T (c^A(X_r, \hat{\alpha}(X_r^C, Z_r^C)) - c^P(X_r) + \frac{\eta_A}{2} |\sigma(X_r) Z_r|^2) dr + \eta_P \int_t^T Z_r \cdot \sigma(X_r) dW_r^{\hat{\alpha}}} \right], \\ &\leq -e^{\eta_P y + x_\infty \eta_P (\kappa_1 \alpha_{min} - \theta)(T-t)} \mathbb{E}_{t,x,y}^{\mathbb{P}^{\hat{\alpha}}} \left[ e^{\eta_P \int_t^T Z_r \cdot \sigma(X_r) dW_r^{\hat{\alpha}}} \right], \\ &\leq -e^{\eta_P y + x_\infty \eta_P (\kappa_1 \alpha_{min} - \theta)(T-t)}, \end{aligned} \quad (2.5.55)$$

where the last inequality follows from the Jensen inequality and the concavity of  $x \mapsto -e^x$ . Therefore, taking the supremum over  $Z \in \mathcal{V}_t$ , we obtain

$$V^P(t, x, y) \leq -e^{\eta_P y + x_\infty \eta_P (\kappa_1 \alpha_{min} - \theta)(T-t)}. \quad (2.5.56)$$

We can see then that  $|V^P(t, x, y)| \leq e^{\eta_P y + C(T-t)}$  for some constant  $C$ . By standard stochastic control theory (Touzi (2012)),  $V^P$  is characterized as the unique viscosity solution of the HJB equation :

$$\begin{cases} -\partial_t V^P - G(x, V^P, DV^P, D^2V^P) = 0, \text{ for } (t, x, y) \in [0, T) \times \mathbb{R}^3, \\ V^P(T, x, y) = -\exp(\eta_P y), \text{ for } (x, y) \in \mathbb{R}^3, \end{cases} \quad (2.5.57)$$

with growth  $|V^P(t, x, y)| \leq e^{\eta_P y + C(T-t)}$  for some constant  $C$ , and  $G : \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^3 \times \mathcal{M}_3(\mathbb{R}) \rightarrow \mathbb{R}$  the hamiltonian defined as

$$G(x, q, p, \gamma) := \sup_{z \in \mathbb{R}^2} g(x, q, p, \gamma, z), \quad (2.5.58)$$

with

$$\begin{aligned} g(x, q, p, \gamma, z) &:= \left\{ \mu(x, \hat{\alpha}(x^C, z^C)) \cdot p_x + \left( c^A(x, \hat{\alpha}(x, z)) + \frac{\eta_A}{2} z^\top \sigma \sigma^\top(x) z - s(x) \right) p_y + \eta_P q (s(x) - c^P(x)) \right. \\ &\quad \left. + \frac{1}{2} \sigma \sigma^\top(x) : \gamma_{xx} + \frac{1}{2} z^\top \sigma \sigma^\top(x) z \gamma_{yy} + z^\top \sigma \sigma^\top(x) \gamma_{xy} \right\}. \end{aligned} \quad (2.5.59)$$

Using the change of variable  $V^P(t, x, y) = -e^{\eta_P(y-u(t,x))}$ , we can simplify the PDE (2.5.57), and express consumer's value function with a 2-dimensional state variable equation instead of 3, that is  $u : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ , which is the unique bounded viscosity solution to

$$\begin{cases} -\partial_t u - \bar{G}(x, Du, D^2u) = 0, & \text{for } (t, x) \in [0, T] \times \mathbb{R}^2, \\ u(T, x) = 0, & \text{for } x \in \mathbb{R}^2, \end{cases} \quad (2.5.60)$$

where the boundedness is obtained from the growth condition of  $V^P$ , and with  $\bar{G} : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{M}_2(\mathbb{R}) \rightarrow \mathbb{R}$  defined as

$$\bar{G}(x, p, \gamma) := \sup_{z \in \mathbb{R}^2} \bar{g}(x, p, \gamma, z), \quad (2.5.61)$$

and

$$\begin{aligned} \bar{g}(x, p, \gamma, z) := & \mu(x, \hat{\alpha}(x^C, z^C)) \cdot p + c^P(x) - c^A(x, \hat{\alpha}(x^C, z^C)) + \frac{1}{2} \sigma \sigma^\top(x) : (\gamma - \eta_P p p^\top) \\ & - \frac{\eta_A + \eta_P}{2} z^\top \sigma \sigma^\top(x) z + \eta_P z^\top \sigma \sigma^\top(x) p, \end{aligned} \quad (2.5.62)$$

which is strictly concave (and separable) in the control variables  $z^C$  and  $z^D$

$$\begin{aligned} \bar{g}(x, p, \gamma, z) = & \tilde{\mu}(x) p + c^P(x) - \tilde{c}^A(x) + \frac{1}{2} \sigma \sigma^\top(x) : (\gamma - \eta_P p p^\top), \\ & - \frac{\kappa_2 (\underline{x}^C)^2}{2} (\hat{\alpha}(x^C, z^C))^2 + \hat{\alpha}(x^C, z^C) (x^C p_{x^C} - \kappa_1 \underline{x}^C), \\ & - \frac{\eta_A + \eta_P}{2} (z^C)^2 (\sigma^C x^C)^2 + z^C \eta_P (\sigma^C x^C)^2 p_{x^C}, \\ & - \frac{\eta_A + \eta_P}{2} (z^D)^2 (\sigma^D x^D)^2 + z^D \eta_P (\sigma^D x^D)^2 p_{x^D}, \end{aligned} \quad (2.5.63)$$

and so

$$\bar{G}(x, p, \gamma) = \bar{g}(x, p, \gamma, \hat{z}(x, p)), \quad (2.5.64)$$

with the maximizer  $\hat{z}$ , that can be expressed explicitly as

$$\hat{z}(x, p) := \begin{pmatrix} \hat{z}^C(x, p) \\ \hat{z}^D(x, p) \end{pmatrix}, \quad (2.5.65)$$

where for sufficiently large  $|\alpha_{max}|$  and  $|\alpha_{min}|$ ,

$$\hat{z}^C(x, p) := \frac{\eta_P (\sigma^C x^C)^2 + \frac{1}{\kappa_2} \left(\frac{x^C}{\underline{x}^C}\right)^2}{(\eta_A + \eta_P) (\sigma^C x^C)^2 + \frac{1}{\kappa_2} \left(\frac{x^C}{\underline{x}^C}\right)^2} p_{x^C} + \frac{\kappa_1}{\kappa_2} \frac{\left(\frac{x^C}{\underline{x}^C}\right) - \left(\frac{x^C}{\underline{x}^C}\right)^2}{(\eta_A + \eta_P) (\sigma^C x^C)^2 + \frac{1}{\kappa_2} \left(\frac{x^C}{\underline{x}^C}\right)^2}, \quad (2.5.66)$$

and

$$\hat{z}^D(x, p) := \frac{\eta_P}{(\eta_A + \eta_P)} p_{x^D}. \quad (2.5.67)$$

**Remark 2.5.5.** Whenever the truncation coefficients go to infinity  $x_\infty, \alpha_{max} \rightarrow +\infty$  and  $\alpha_{min} \rightarrow -\infty$ , we obtain

$$\hat{z}(x, p) \approx \begin{pmatrix} \frac{\eta_P (\sigma^C x^C)^2 + \frac{1}{\kappa_2}}{(\eta_A + \eta_P) (\sigma^C x^C)^2 + \frac{1}{\kappa_2}} & 0 \\ 0 & \frac{\eta_P}{(\eta_A + \eta_P)} \end{pmatrix} p, \quad (2.5.68)$$

which can be used as an approximation of (2.5.65). Similarly, the PDE (2.5.60) can be approximated by the solution to (2.5.69)

$$\begin{cases} \partial_t u + \left( \tilde{\mu}(x) - \begin{pmatrix} \frac{\kappa_1}{\kappa_2} \\ 0 \end{pmatrix} \right) \cdot Du + \frac{1}{2} \sigma \sigma^\top(x) : D^2 u + f(x) - \frac{\kappa_2}{2\kappa_2} + \frac{1}{2} \rho(x) \cdot \begin{pmatrix} (\partial_{x^C} u)^2 \\ (\partial_{x^D} u)^2 \end{pmatrix} = 0 \\ u(T, x) = 0 \end{cases}, \text{ for } (t, x) \in [0, T] \times \mathbb{R}^2, \quad (2.5.69)$$

where  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is defined as  $f := \tilde{c}^A - c^P$ , and

$$\rho(x) := \begin{pmatrix} \frac{\frac{1}{\kappa_2} + \frac{1}{\kappa_2} \eta_P (\sigma^C x^C)^2 - \eta_A \eta_P ((\sigma^C x^C)^2)^2}{(\eta_A + \eta_P) (\sigma^C x^C)^2 + \frac{1}{\kappa_2}} \\ \frac{-\eta_A \eta_P (\sigma^D x^D)^2}{(\eta_A + \eta_P)} \end{pmatrix}. \quad (2.5.70)$$

We next proceed by verification to solve the problem

**Proposition 2.5.6** (Verification). (i) Assume that (2.5.60) has a bounded  $C^{1,2}([0, T], \mathbb{R}^2)$  solution  $u$ . Then there exists a  $C^{1,2}([0, T], \mathbb{R}^3)$  solution to the PDE (2.5.57) denoted  $v$  with growth  $|v(t, x, y)| \leq e^{\eta_P y + C(T-t)}$  for some constant  $C$ , which satisfies

$$V_0^P \leq v(0, x_0, \mathcal{R}). \quad (2.5.71)$$

(ii) Define

$$Z_t^* := \hat{z}(X_t, Du(t, X_t)) \text{ for } t \in [0, T], \quad (2.5.72)$$

and assume that  $(Z_t^*)_{t \in [0, T]} \in \mathcal{V}$ . Then

$$V_0^P = v(0, x_0, \mathcal{R}), \quad (2.5.73)$$

and

$$\xi^* := \mathcal{R} + \int_0^T Z_t^* \cdot dX_t - \int_0^T H(X_t, Z_t^*) dt \quad (2.5.74)$$

is an optimal contract.

**Proof.** (i) Let  $u$  be a bounded  $C^{1,2}([0, T], \mathbb{R}^2)$  solution to the PDE (2.5.60). Then defining  $v(t, x, y) := -e^{\eta_P(y-u(t,x))}$ , we can see that  $v \in C^{1,2}([0, T], \mathbb{R}^3)$  as a composition of smooth functions, and that  $v(T, x, y) = -e^{\eta_P y}$  and  $v$  satisfies the growth condition  $|v(t, x, y)| \leq e^{\eta_P y + C(T-t)}$  for some constant  $C$  from the boundedness of  $u$ . By straightforward differentiation, we obtain that  $v$  is a solution to (2.5.57). Define then

$$\beta_{t_1, t_2} := e^{\eta_P \int_{t_1}^{t_2} (s(X_r) - c^P(X_r)) dr} \text{ for } 0 \leq t_1 \leq t_2 \leq T, \quad (2.5.75)$$

and the sequence of stopping times

$$T_n := T \wedge \inf \left\{ t \geq 0, \int_0^t (\beta_{0,r})^2 \left( |\sigma(X_r) D_x v(r, X_r, Y_r)|^2 + (D_y v(r, X_r, Y_r))^2 |\sigma(X_r) z_r|^2 \right) dr \geq n \right\} \text{ for } n \geq 1. \quad (2.5.76)$$

For an arbitrary control  $z \in \mathcal{V}$ , we apply Itô formula and take the expectation under  $\mathbb{P}^{\hat{\alpha}}$  to obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} [\beta_{t,T_n} v(T_n, X_{T_n}, Y_{T_n})] &= v(t, x, y) + \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ \int_t^{T_n} \beta_{t,r} (\partial_t v + g(X_r, v(r, X_r, Y_r), Dv(r, X_r, Y_r), D^2V(r, X_r, Y_r), z_r)) dr \right] \\ &+ \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ \int_t^{T_n} \beta_{t,r} D_x v(r, X_r, Y_r) \cdot \sigma(X_r) dW_r^{\hat{\alpha}} \right] + \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ \int_t^{T_n} \beta_{t,r} D_y v(r, X_r, Y_r) z_r \cdot \sigma(X_r) dW_r^{\hat{\alpha}} \right]. \end{aligned} \quad (2.5.77)$$

From the definition of the localizing sequence  $(T_n)_{n \geq 1}$ , we obtain that

$$0 = \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ \int_t^{T_n} \beta_{t,r} D_x v(r, X_r, Y_r) \cdot \sigma(X_r) dW_r^{\hat{\alpha}} \right] = \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ \int_t^{T_n} \beta_{t,r} D_y v(r, X_r, Y_r) z_r \cdot \sigma(X_r) dW_r^{\hat{\alpha}} \right]. \quad (2.5.78)$$

Furthermore, since  $v$  is a solution to the PDE (2.5.57), and by the linearity of the expectation

$$\mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ \int_t^{T_n} \beta_{t,r} (\partial_t v + g(X_r, v(r, X_r, Y_r), Dv(r, X_r, Y_r), D^2V(r, X_r, Y_r), z_r)) dr \right] \leq 0, \quad (2.5.79)$$

and so

$$\mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} [\beta_{t,T_n} v(T_n, X_{T_n}, Y_{T_n})] \leq v(t, x, y). \quad (2.5.80)$$

When  $n \rightarrow +\infty$ , the following a.s convergence holds

$$\beta_{t,T_n} v(T_n, X_{T_n}, Y_{T_n}) \rightarrow U_P \left( -Y_T - \int_t^T s(X_r) dr + \int_t^T c^P(X_r) dr \right), \quad (2.5.81)$$

and for  $n \geq 1$  and the growth condition of  $v$  we have

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} [|\beta_{t,T_n} v(T_n, X_{T_n}, Y_{T_n})|] &\leq \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ \beta_{t,T_n} e^{\eta_P Y_{T_n} + C(T-T_n)} \right], \\ &= \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ e^{\eta_P \int_t^{T_n} (s(X_r) - c^P(X_r)) dr} e^{\eta_P Y_{T_n} + C(T-T_n)} \right], \\ &\leq \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ e^{C(T-T_n) + \eta_P (T_n-t) x_{\infty}(P(0, x_{\infty}) + k_1)} e^{\eta_P Y_{T_n}} \right], \\ &\leq \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ e^{C(T-T_n) + \eta_P (T_n-t) x_{\infty}(P(0, x_{\infty}) + k_1)} e^{\eta_P \sup_{t \in [0, T]} |Y_t|} \right] < +\infty \text{ for } n \geq 1, \end{aligned} \quad (2.5.82)$$

where we used the bound  $(s(x) - c^P(x)) \leq x_{\infty}(P(0, x_{\infty}) + k_1)$  for  $x \in \mathbb{R}_+^2$  and the estimate on  $Y$  from Proposition 2.2.6. Therefore, by the dominated convergence theorem,

$$\mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ U_P \left( -Y_T - \int_t^T s(X_r) dr + \int_t^T c^P(X_r) dr \right) \right] \leq v(t, x, y), \quad (2.5.83)$$

and

$$\sup_{z \in \mathcal{V}} \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ U_P \left( -Y_T - \int_t^T s(X_r) dr + \int_t^T c^P(X_r) dr \right) \right] \leq v(t, x, y). \quad (2.5.84)$$

In particular for  $t = 0$ , we have an upper bound for the value function  $V_0^P$

$$V_0^P \leq v(0, x_0, \mathcal{R}). \quad (2.5.85)$$

(ii) Assuming that  $(Z_t^*)_{t \in [0, T]} \in \mathcal{V}$ , we can go over the same steps as in (i) with  $Z^*$  instead of an arbitrary control from  $\mathcal{V}$ . Therefore the inequalities (2.5.79) and (2.5.80) and (2.5.85) become equalities with

$$V_0^P = v(0, x_0, \mathcal{R}), \quad (2.5.86)$$

so that the upper bound is reached for the control  $Z^*$  which is then an optimal feedback control to the problem (2.2.30), and therefore the optimal contract corresponds to terminal value of the controlled state variable  $(Y_t^{\mathcal{R}, Z^*})_{t \in [0, T]}$  which we denote  $\xi^* := Y_T^{\mathcal{R}, Z^*}$ , and which satisfies  $\xi^* \in \Xi$  since  $Y_0 = \mathcal{R}$  and  $Z^* \in \mathcal{V}$ , concluding the proof.  $\square$

**Remark 2.5.7.** The HJB-PDE (2.5.60) (the same remark can be said about the approximating PDE(2.5.69)) is a semi-linear parabolic PDE of second order; with a non-linearity in the gradient term. This PDE has a-priori no solution in the classical sense, i.e., a smooth function (in  $C^{1,2}([0, T], \mathbb{R}^2)$ ) with a clear definition of  $\partial_t u$ ,  $Du$ , and  $D^2u$  solution to (2.5.60) because of the non-linearity. However, the existence can be proved in a weaker sense, by taking the candidate  $u(t, x) := y - U_p^{-1}(V^P(t, x, y))$  defined in (2.5.52), which can be proved to be a viscosity solution to (2.5.60) (not necessarily smooth). In this case one cannot define  $\partial_t u$ , and  $Du$ , and  $D^2u$ , and a more technical approach is required. We make the assumption  $u \in C^{1,2}([0, T], \mathbb{R}^2)$  to simplify the exposition. Nevertheless, (2.5.60) might satisfy some conditions (unknown to the authors) which insure the regularity of the viscosity solution defined above.

## 2.5.6 Producer's participation constraint: the problem without capacity payment

The problem (2.2.38) is a markovian stochastic control problem, and can be solved by classical techniques. We define producer's continuation utility function  $\hat{V}^A : [0, T] \times \mathbb{R}_+^2 \rightarrow \mathbb{R}$  as

$$\hat{V}^A(t, x) := \sup_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ -e^{-\eta_A \int_t^T (s(X_r) - c^A(X_r, \alpha_r)) dr} \right]. \quad (2.5.87)$$

Recall from the definition (2.2.10) and (2.2.12) of  $s$  and  $c^A$  that for  $(x, \alpha) \in \mathbb{R}_+^2 \times [\alpha_{min}, \alpha_{max}]$  we have the following bounds

$$0 \leq s(x) \leq P(0, x_\infty)x_\infty, \text{ and } \alpha_{min}\kappa_1 x_\infty \leq c^A(x, \alpha) \leq c^A(x_\infty, \alpha_{max}), \text{ and } 0 \leq c^A(x, 0) \leq (a+b)x_\infty, \quad (2.5.88)$$

so

$$\hat{V}^A(t, x) \leq -e^{-\eta_A(T-t)(P(0, x_\infty)x_\infty - \alpha_{min}\kappa_1 x_\infty)}, \quad (2.5.89)$$

and since  $\alpha = 0$  is an admissible control, then

$$\hat{V}^A(t, x) \geq \mathbb{E}_{t,x}^{\mathbb{P}^0} \left[ -e^{-\eta_A \int_t^T (s(X_r) - c^A(X_r, 0)) dr} \right] \geq -e^{-\eta_A(T-t)(a+b)x_\infty}, \quad (2.5.90)$$

so

$$\left| \hat{V}^A(t, x) \right| \leq e^{C(T-t)} \text{ for } (t, x) \in [0, T] \times \mathbb{R}_+^2, \text{ and some constant } C. \quad (2.5.91)$$

We identify then a candidate for the solution of the problem with the unique viscosity solution to the HJB equation :

$$\begin{cases} -\partial_t \hat{V}^A - \hat{H}(x, \hat{V}^A, D\hat{V}^A, D^2\hat{V}^A) = 0, & \text{in } [0, T] \times \mathbb{R}^2, \\ \hat{V}^A(T, \cdot) = -1, \end{cases} \quad (2.5.92)$$

with growth controlled by  $|\hat{V}^A(t, x)| \leq e^{C(T-t)}$  for some constant  $C$ , (in particular  $\hat{V}^A$  is bounded), where  $\hat{H} : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathcal{M}_2(\mathbb{R}) \rightarrow \mathbb{R}$  is the Hamiltonian of the producer acting on his own defined as

$$\hat{H}(x, y, z, \gamma) := \sup_{\alpha \in [\alpha_{min}, \alpha_{max}]} \hat{h}(x, y, z, \gamma, \alpha), \quad (2.5.93)$$

with

$$\hat{h}(x, y, z, \gamma, \alpha) := \mu(x, \alpha) \cdot z + \frac{1}{2} \sigma \sigma^\top : \gamma + \eta_A (c^A(x, \alpha) - s(x)) y. \quad (2.5.94)$$

Remark that the maximum is attained for

$$\alpha_{pc}^*(x, y, z) := \alpha_{min} \vee \left( -\frac{z^C x^C + \eta_A y \kappa_1 \underline{x}^C}{\eta_A y (\underline{x}^C)^2 \kappa_2} \right) \wedge \alpha_{max}, \text{ for } (x, y, z) \in \mathbb{R}_+^2 \times \mathbb{R}_+ \times \mathbb{R}^2 \text{ and } x^C, y \neq 0, \quad (2.5.95)$$

(where “pc” stands for participation constraint) and the PDE (2.5.92) can be written as

$$\begin{cases} -\partial_t \hat{V}^A - \hat{h}(x, \hat{V}^A, D\hat{V}^A, D^2\hat{V}^A, \alpha_{pc}^*(x, \hat{V}^A, D\hat{V}^A)) = 0 & \text{in } [0, T] \times \mathbb{R}_+^2, \\ \hat{V}^A(T, \cdot) = -1. \end{cases} \quad (2.5.96)$$

We next proceed by verification.

**Proposition 2.5.8.** *Assume that the PDE (2.5.92) has a  $C^{1,2}([0, T], \mathbb{R}^2)$  solution  $v$  with growth controlled by  $|v(t, x)| \leq e^{C(T-t)}$  for some constant  $C$ , then*

$$\hat{V}_0^A = v(0, x_0), \quad (2.5.97)$$

and  $\left( \alpha_{pc}^*(X_t, v(t, X_t), Dv(t, X_t)) \right)_{t \in [0, T]}$  with  $\alpha_{pc}^*$  defined in (2.5.95) is an optimal feedback control to the problem (2.2.38).

**Proof.** Let  $v$  be a  $C^{1,2}([0, T], \mathbb{R}^2)$  solution to the PDE (2.5.92), such that  $|v(t, x)| \leq e^{C(T-t)}$  for some constant  $C$  and define the sequence of stopping times

$$T_n := T \wedge \inf \left\{ t \geq 0, \int_0^t (\beta_{0,r}^\alpha)^2 |\sigma(X_r) Dv(r, X_r)|^2 dr \geq n \right\} \text{ for } n \geq 1. \quad (2.5.98)$$

For an arbitrary control  $\alpha \in \mathcal{U}$ , we denote

$$\beta_{t_1, t_2}^\alpha := e^{-\eta_A \int_{t_1}^{t_2} (s(X_r) - c^A(X_r, \alpha_r)) dr} \text{ for } 0 \leq t_1 \leq t_2 \leq T, \quad (2.5.99)$$

and we have by applying Itô formula and taking the expectation under  $\mathbb{P}^\alpha$

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^\alpha} \left[ \beta_{t,T_n}^\alpha v(T_n, X_n) \right] &= v(t, x) + \mathbb{E}^{\mathbb{P}^\alpha} \left[ \int_t^{T_n} \beta_{t,r}^\alpha \left( \partial_t v(r, X_r) + \hat{h} \left( X_r, v(r, X_r), Dv(r, X_r), D^2v(r, X_r), \alpha \right) \right) dr \right], \\ &+ \mathbb{E}^{\mathbb{P}^\alpha} \left[ \int_t^{T_n} \beta_{t,r}^\alpha Dv(r, X_r) \cdot \sigma(X_r) dW_r^\alpha \right]. \end{aligned} \quad (2.5.100)$$

By definition of the stopping sequence  $T_n$ , we have that  $\mathbb{E}^{\mathbb{P}^\alpha} \left[ \int_t^{T_n} \beta_{t,r}^\alpha Dv(r, X_r) \cdot \sigma(X_r) dW_r^\alpha \right] = 0$ , and since  $v$  is a solution to the PDE (2.5.92), then

$$\mathbb{E}^{\mathbb{P}^\alpha} \left[ \int_t^{T_n} \beta_{t,r}^\alpha \left( \partial_t v(r, X_r) + \hat{h} \left( X_r, v(r, X_r), Dv(r, X_r), D^2v(r, X_r), \alpha \right) \right) dr \right] \leq 0, \text{ for } \alpha \in \mathcal{U}. \quad (2.5.101)$$

Therefore

$$\mathbb{E}^{\mathbb{P}^\alpha} \left[ \beta_{t,T_n}^\alpha v(T_n, X_n) \right] \leq v(t, x) \text{ for } \alpha \in \mathcal{U}, \quad (2.5.102)$$

Remark then that the following almost sure convergence holds as  $n \rightarrow +\infty$

$$\beta_{t,T_n}^\alpha v(T_n, X_n) \rightarrow -e^{-\eta_A \int_t^T (s(X_r) - c^A(X_r, \alpha_r)) dr}. \quad (2.5.103)$$

Furthermore, we have from the growth of  $v$  and the bounds in (2.5.88)

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^\alpha} \left[ \beta_{t,T_n}^\alpha v(T_n, X_n) \right] &\leq \mathbb{E}^{\mathbb{P}^\alpha} \left[ \beta_{t,T_n}^\alpha e^{C(T-T_n)} \right], \\ &= \mathbb{E}^{\mathbb{P}^\alpha} \left[ e^{-\eta_A \int_t^{T_n} (s(X_r) - c^A(X_r, \alpha_r)) dr} e^{C(T-T_n)} \right], \\ &\leq \mathbb{E}^{\mathbb{P}^\alpha} \left[ e^{\eta_A c^A(x_\infty, \alpha_{max})(T_n-t)} e^{C(T-T_n)} \right] < +\infty \text{ for } n \geq 1. \end{aligned} \quad (2.5.104)$$

So by dominated convergence, we obtain that

$$\mathbb{E}^{\mathbb{P}^\alpha} \left[ -e^{-\eta_A \int_t^T (s(X_r) - c^A(X_r, \alpha_r)) dr} \right] \leq v(t, x) \text{ for } \alpha \in \mathcal{U}, \quad (2.5.105)$$

and in particular

$$\hat{V}_0^A = \sup_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_A \left( \int_0^T \left( s(X_t) - c^A(X_t, \alpha_t) \right) dt \right) \right] \leq v(0, x_0) \quad (2.5.106)$$

So  $v(0, x_0)$  is an upper bound for the maximization problem (2.2.38). Furthermore, for the feedback control  $(\alpha_{pc}^*(X_t, v(t, X_t), Dv(t, X_t)))_{t \in [0, T]}$  and going through the same steps as previously, the inequalities (2.5.101) and (2.5.102) become equalities, and so the upper bound is reached for  $\alpha_{pc}$  which is then an optimal control since it is admissible (as a progressively measurable process w.r.t  $\mathbb{F}$  valued in  $[\alpha_{min}, \alpha_{max}]$ ) which concludes the proof.  $\square$

Remark that the PDE (2.5.92) can be characterized in terms of certainty equivalent by a straightforward change of

variable  $\hat{V}^A(t, x) = -e^{-\eta_A \hat{u}^A(t, x)}$ , which leads to  $\hat{u}^A$  being the unique bounded viscosity solution to the PDE

$$\begin{cases} -\partial_t \hat{u}^A - \hat{h}\left(x, -\frac{1}{\eta_A}, D\hat{u}^A, D^2\hat{u}^A - \eta_A D\hat{u}^A (D\hat{u}^A)^\top, \alpha_{pc}^*\left(x, -\frac{1}{\eta_A}, D\hat{u}^A\right)\right) = 0 \text{ in } [0, T) \times \mathbb{R}_+^2, \\ \hat{u}^A(T, \cdot) = 0, \end{cases} \quad (2.5.107)$$

and the optimal feedback control is then written (under further smoothness assumptions) as

$$\alpha_{pc}^*\left(x, -\frac{1}{\eta_A}, D\hat{u}^A(t, x)\right) = \alpha_{min} \vee \frac{\partial_{x^C} \hat{u}^A(t, x) x^C - \kappa_1 \underline{x}^C}{\underline{x}^C \kappa_2} \wedge \alpha_{max}. \quad (2.5.108)$$

## 2.5.7 Description of the numerical procedure

To implement the optimal capacity contract and optimal policy, the first step is to numerically solve the PDE (2.5.60) describing consumer's value function, which we recall

$$\begin{cases} -\partial_t u - \bar{G}(x, Du, D^2u) = 0, \text{ for } (t, x) \in [0, T) \times \mathbb{R}^2, \\ u(T, x) = 0, \text{ for } x \in \mathbb{R}_+^2. \end{cases} \quad (2.5.60)$$

We use a finite differences method with an explicit Euler scheme to solve this PDE. We discretize the time horizon  $[0, T]$  by defining for some  $n_T \in \mathbb{N}$  the vector  $(t_0, t_2, \dots, t_{n_T})$  with  $t_k = k \frac{T}{n_T}$  for  $k \in \{0, \dots, n\}$ . We recall then that the state variables are non negative and so we define the boundaries of the space using positive constants  $0 < x_{c, min} < x_{c, max}$  and  $0 < x_{d, min} < x_{d, max}$ . The space grid is of size  $(n_c + 1) \times (n_d + 1)$  with  $n_c, n_d \in \mathbb{N}$ , and for  $0 \leq i \leq n_c$  and  $0 \leq j \leq n_d$ , a function  $M : \mathbb{R}_+^2 \mapsto \mathbb{R}$  is approximated by a matrix  $\mathbb{M}$  such that  $\mathbb{M}_{ij}$  represents  $M(x_{c, min} + i \frac{x_{c, max}}{n_c}, x_{d, min} + j \frac{x_{d, max}}{n_d})$ .

We define then a collection  $(U^k)_{\{0 \leq k \leq n_T\}}$  of  $(n_T + 1)$  matrices of size  $(n_c + 1) \times (n_d + 1)$  to stock the (approximation) of the values of  $u$ . Our goal is to have

$$U_{ij}^k = u(k\Delta t, x_{c, min} + i\Delta x_c, x_{d, min} + j\Delta x_d) \text{ for } k \in \{0, \dots, n_T\}, i \in \{0, \dots, n_c\}, j \in \{0, \dots, n_d\}. \quad (2.5.109)$$

where we defined  $\Delta t := \frac{T}{n_T}$ ,  $\Delta x_c := \frac{x_{c, max}}{n_c}$  and  $\Delta x_d := \frac{x_{d, max}}{n_d}$ , and recall the following approximations based on taylor expansion

$$\begin{aligned} \partial_t u(t, x) &= \frac{u(t + \Delta t, x) - u(t, x)}{\Delta t} + o(\Delta t), \\ &\approx \frac{u(t + \Delta t, x) - u(t, x)}{\Delta t}, \end{aligned} \quad (2.5.110)$$

and

$$\begin{aligned} Du(t, x) &= \left( \begin{array}{c} \frac{u(t, x_c + \Delta x_c, x_d) - u(t, x_c - \Delta x_c, x_d)}{2\Delta x_c} + o(\Delta x_c) \\ \frac{u(t, x_c, x_d + \Delta x_d) - u(t, x_c, x_d - \Delta x_d)}{2\Delta x_d} + o(\Delta x_d) \end{array} \right), \\ &\approx \left( \begin{array}{c} \frac{u(t, x_c + \Delta x_c, x_d) - u(t, x_c - \Delta x_c, x_d)}{2\Delta x_c} \\ \frac{u(t, x_c, x_d + \Delta x_d) - u(t, x_c, x_d - \Delta x_d)}{2\Delta x_d} \end{array} \right), \end{aligned} \quad (2.5.111)$$



and

$$D^2u(t, x) = \begin{pmatrix} \partial_{x^C}^2 u(t, x) & \partial_{x^C x^D}^2 u(t, x) \\ \partial_{x^C x^D}^2 u(t, x) & \partial_{x^D}^2 u(t, x) \end{pmatrix}, \quad (2.5.112)$$

with

$$\begin{cases} \partial_{x^C}^2 u(t, x) \approx \frac{u(t, x_c + \Delta x_c, x_d) + u(t, x_c - \Delta x_c, x_d) - 2u(t, x_c, x_d)}{(\Delta x_c)^2}, \\ \partial_{x^D}^2 u(t, x) \approx \frac{u(t, x_c, x_d + \Delta x_d) + u(t, x_c, x_d - \Delta x_d) - 2u(t, x_c, x_d)}{(\Delta x_d)^2}, \\ \partial_{x^C x^D}^2 u(t, x) \approx \frac{u(t, x_c + \Delta x_c, x_d + \Delta x_d) + u(t, x_c - \Delta x_c, x_d - \Delta x_d) - u(t, x_c + \Delta x_c, x_d - \Delta x_d) - u(t, x_c - \Delta x_c, x_d + \Delta x_d)}{4(\Delta x_d)(\Delta x_c)}. \end{cases} \quad (2.5.113)$$

Inspired by the previous expressions, we define for a matrix  $\mathbb{M}$  of size  $(n_c + 1) \times (n_d + 1)$  the following matrices also of size  $(n_c + 1) \times (n_d + 1)$  (corresponding to the gradient and the Hessian components)

$$\text{diff}_{x^C}(\mathbb{M}), \text{ and } \text{diff}_{x^D}(\mathbb{M}) \text{ and } \text{diff}2_{x^C}(\mathbb{M}) \text{ and } \text{diff}2_{x^D}(\mathbb{M}) \text{ and } \text{diff}_{x^C x^D}(\mathbb{M}), \quad (2.5.114)$$

such that

$$\begin{cases} \text{diff}_{x^C}(\mathbb{M})_{ij} := \frac{\mathbb{M}_{i+1,j} - \mathbb{M}_{i-1,j}}{2\Delta x_c}, \\ \text{diff}_{x^D}(\mathbb{M})_{ij} := \frac{\mathbb{M}_{i,j+1} - \mathbb{M}_{i,j-1}}{2\Delta x_d}, \\ \text{diff}_{x^C x^D}(\mathbb{M})_{ij} := \frac{\mathbb{M}_{i+1,j+1} + \mathbb{M}_{i-1,j-1} - \mathbb{M}_{i+1,j-1} - \mathbb{M}_{i-1,j+1}}{4(\Delta x_d)(\Delta x_c)}, \quad \text{for } i \in \{0, \dots, n_c\}, j \in \{0, \dots, n_d\}, \\ \text{diff}2_{x^C}(\mathbb{M})_{ij} := \frac{\mathbb{M}_{i+1,j} + \mathbb{M}_{i-1,j} - 2\mathbb{M}_{i,j}}{(\Delta x_c)^2}, \\ \text{diff}2_{x^D}(\mathbb{M})_{ij} := \frac{\mathbb{M}_{i,j+1} + \mathbb{M}_{i,j-1} - 2\mathbb{M}_{i,j}}{(\Delta x_d)^2}, \end{cases} \quad (2.5.115)$$

with slight changes near the boundaries to avoid out of range indices, i.e., by defining for  $i \in \{0, \dots, n_c\}$  and  $j \in \{0, \dots, n_d\}$

$$\mathbb{M}_{(-1)j} := \mathbb{M}_{0j} \text{ and } \mathbb{M}_{(n_c+1)j} := \mathbb{M}_{n_c j} \text{ and } \mathbb{M}_{i(-1)} := \mathbb{M}_{i0} \text{ and } \mathbb{M}_{i(n_d+1)} := \mathbb{M}_{i n_d}, \quad (2.5.116)$$

used in the definition (2.5.115). We define then the equivalent of the hamiltonian of PDE (2.5.60) in the space grid as a matrix valued function  $\mathbb{G}$  (of size  $(n_c + 1) \times (n_d + 1)$ ) such that for  $0 \leq i \leq n_c$ , and  $0 \leq j \leq n_d$ ,

$$\mathbb{G}(\mathbb{M})_{ij} := \bar{G} \left( \begin{pmatrix} x_{c,min} + i\Delta x_c \\ x_{d,min} + j\Delta x_d \end{pmatrix}, \begin{pmatrix} \text{diff}_{x^C}(\mathbb{M})_{ij} \\ \text{diff}_{x^D}(\mathbb{M})_{ij} \end{pmatrix}, \begin{pmatrix} \text{diff}2_{x^C}(\mathbb{M})_{ij} & \text{diff}_{x^C x^D}(\mathbb{M})_{ij} \\ \text{diff}_{x^C x^D}(\mathbb{M})_{ij} & \text{diff}2_{x^D}(\mathbb{M})_{ij} \end{pmatrix} \right). \quad (2.5.117)$$

The numerical approximation of (2.5.60) becomes straightforward, and consists in initializing the terminal value  $U_{ij}^{n_T} = 0$  for  $0 \leq i \leq n_c$  and  $0 \leq j \leq n_d$ . Then computing by backward induction

$$U^{k-1} := U^k + \Delta t \times \mathbb{G}(U^k), \text{ for } k \in \{n_T, n_T - 1, \dots, 1\}. \quad (2.5.118)$$

Recall then the definition of the optimal feedback control

$$Z_t^* = \begin{pmatrix} Z_t^{C,*} \\ Z_t^{D,*} \end{pmatrix} := \hat{z}(X_t, Du(t, X_t)) \text{ for } t \in [0, T]. \quad (2.5.72)$$

We approximate then  $\left(Z_t^{C,*}\right)_{t \in [0, T]}$  and  $\left(Z_t^{D,*}\right)_{t \in [0, T]}$  with the collection of matrices  $(Z_c^k)_{\{0 \leq k \leq n_T\}}$  and  $(Z_d^k)_{\{0 \leq k \leq n_T\}}$  of sizes  $(n_c + 1) \times (n_d + 1)$  defined as

$$\begin{pmatrix} (Z_c)_{ij}^k \\ (Z_d)_{ij}^k \end{pmatrix} := \hat{z} \left( \begin{pmatrix} x_{c, \min} + i \Delta x_c \\ x_{d, \min} + j \Delta x_d \end{pmatrix}, \begin{pmatrix} \text{diff}_{x_c}(U^k)_{ij} \\ \text{diff}_{x_d}(U^k)_{ij} \end{pmatrix} \right) \text{ for } k \in \{0, \dots, n_T\}, i \in \{0, \dots, n_c\}, j \in \{0, \dots, n_d\}. \quad (2.5.119)$$

Similarly, the recommended effort  $\left(\hat{\alpha}(X_t^C, Z_t^{*,C})\right)_{t \in [0, T]}$  is approximated with matrices  $(\hat{A}^k)_{\{0 \leq k \leq n_T\}}$  of sizes  $(n_c + 1) \times (n_d + 1)$  defined as

$$\hat{A}_{ij}^k := \hat{\alpha}(x_{c, \min} + i \Delta x_c, (Z_c)_{ij}^k) \text{ for } k \in \{0, \dots, n_T\}, i \in \{0, \dots, n_c\}, j \in \{0, \dots, n_d\}. \quad (2.5.120)$$

**Remark 2.5.9.** For a fixed  $k \in \{0, \dots, n_T\}$ , the matrices  $U^k$ ,  $(Z_c)^k$ ,  $(Z_d)^k$  and  $\hat{A}^k$  represent approximations on the whole space  $\mathbb{R}_+^2$ . So for a given  $x = (x^C, x^D) \in \mathbb{R}_+^2$ , one needs first to find the indices  $(i, j)$  such that  $(x_{c, \min} + i \Delta x_c, x_{d, \min} + j \Delta x_d) \approx x$ ; for example by taking  $(i, j) = \left( \left\lfloor \frac{x^C - x_{c, \min}}{\Delta x_c} \right\rfloor, \left\lfloor \frac{x^D - x_{d, \min}}{\Delta x_d} \right\rfloor \right)$  to estimate the required quantities, for example  $u(k \Delta t, x) \approx U_{ij}^k$ .

The second step in the simulation is the forward diffusion of state variables using the optimal controls for a number of scenarios  $N$ , to use the Monte-carlo for the estimations. We recall the dynamics of the controlled state variables

$$\begin{cases} X_t &= \begin{pmatrix} x_0^C \\ x_0^D \end{pmatrix} + \int_0^t \mu \left( X_r, \hat{\alpha} \left( X_r^C, Z_r^{C,*} \right) \right) dr + \int_0^t \sigma \left( X_r \right) dW_r^{\hat{\alpha}}, \\ Y_t &= \mathcal{R} + \int_0^t \left( c^A \left( X_r, \hat{\alpha} \left( X_r^C, Z_r^{C,*} \right) \right) + \frac{\eta_A}{2} |\sigma \left( X_r \right) Z_r^*|^2 - s \left( X_r \right) \right) dr + \int_0^t Z_r^* \cdot \sigma \left( X_r \right) dW_r^{\hat{\alpha}}. \end{cases} \quad (2.5.121)$$

This is also done through an explicit Euler scheme, which we provide for a single scenario. We define the matrix  $\hat{X}$  of size  $2 \times (n_T + 1)$  and the vector  $\hat{Y}$  of size  $1 \times (n_T + 1)$  which provides  $\hat{X}^k \approx X_{k \Delta t}$  and  $\hat{Y}^k \approx Y_{k \Delta t}$ . We start by initializing  $\hat{X}^0 := \begin{pmatrix} x_0^C \\ x_0^D \end{pmatrix}$  and  $\hat{Y}^0 := \mathcal{R}$ . Then we compute by induction for  $k \in \{1, \dots, n_T\}$ :

$$\begin{cases} (i, j) := \left( \left\lfloor \frac{(x_{c, \min} \vee \hat{X}_1^{k-1} \wedge x_{c, \max}) - x_{c, \min}}{\Delta x_c} \right\rfloor, \left\lfloor \frac{(x_{d, \min} \vee \hat{X}_2^{k-1} \wedge x_{d, \max}) - x_{d, \min}}{\Delta x_d} \right\rfloor \right), \\ \text{Generate a (normalized) 2 dimensional Brownian increment } (\Delta W)^k \text{ with a law } \mathcal{N}(0_2, I_2), \\ \hat{X}^k := \hat{X}^{k-1} + \Delta t \mu \left( \hat{X}^{k-1}, \hat{A}_{ij}^{k-1} \right) + \sqrt{\Delta t} \sigma \left( \hat{X}^{k-1} \right) (\Delta W)^k, \\ \hat{Y}^k := \hat{Y}^{k-1} + \Delta t \left( c^A \left( \hat{X}^{k-1}, \hat{A}_{ij}^{k-1} \right) + \frac{\eta_A}{2} \left| \sigma \left( \hat{X}^{k-1} \right) \begin{pmatrix} (Z_c)_{ij}^{k-1} \\ (Z_d)_{ij}^{k-1} \end{pmatrix} \right|^2 - s \left( \hat{X}^{k-1} \right) \right) + \sqrt{\Delta t} \begin{pmatrix} (Z_c)_{ij}^{k-1} \\ (Z_d)_{ij}^{k-1} \end{pmatrix} \cdot \sigma \left( \hat{X}^{k-1} \right) (\Delta W)^k. \end{cases} \quad (2.5.122)$$

Remark that it is fundamental that for each time step  $k$ , the Brownian increment used to compute  $\hat{X}^k$  is the same as the one to compute  $\hat{Y}^k$ .

The resulting matrices correspond to a realization for a scenario of the capacity and demand  $(\hat{X}_1^k$  and  $\hat{X}_2^k)$  for  $k \in \{0, \dots, n_T\}$ , and  $\hat{Y}^{n_T}$  the the contract for this scenario. Prior values of  $\hat{Y}$ , i.e.,  $\hat{Y}^k$  for  $k < n_T$  could be used to understand the composition of the contract, but are only informative and less important than  $\hat{Y}^{n_T}$  which represents the actual amount paid, since the contracting is in a lump-sum payment form.



## Chapter 3

# Linear-Quadratic Principal-Agent problem with application to the decarbonation of economies

Joint work with René Aïd and Nizar Touzi.

### Abstract

The goal of this paper is to provide a class of tractable Principal-Agent problems. We follow the resolution methodology of Cvitanić et al. (2018) in a Linear Quadratic setting with exponential utilities. We tackle then the incentives for decarbonation as an example of application.

**Key words.** Principal-Agent problem, Contract Theory, Linear-Quadratic stochastic control.

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### 3.1 Introduction

We study the principal-agent problem in a linear-quadratic setting with exponential utilities, and we provide explicit solutions in terms of matrix Riccati equations in three different situations : the case without contracting -or business as usual-, the first best when principal observes both the effort and the state variable, and the second best of moral hazard when principal only observes the state variable. We illustrate then the results through an application to the optimal incentives for the decarbonation of economies. Section 3.2 describes the framework and formulates the problem, which is solved in the sections 3.3, 3.4 and 3.5. The section 3.6 provides an application through a model of decarbonation. The appendix 3.7 contains the technical proofs.

#### Notations

We denote for a normed space  $(\mathbb{M}, |\cdot|)$ ;  $\mathbb{M}$  is typically  $\mathbb{R}^n$  or  $\mathbb{R}^{n \times m}$  for some  $n, m \in \mathbb{N}$ :

$$\mathbb{L}^\infty([0, T], \mathbb{M}) := \left\{ f : [0, T] \mapsto \mathbb{M} \text{ such that } f \text{ is measurable and } \sup_{[0, T]} |f(t)| < \infty \right\}, \quad (3.1.1)$$

and by  $\mathcal{S}^n$  the set of symmetric matrices (of  $\mathbb{R}^{n \times n}$ ),  $\mathcal{S}_+^n$  symmetric and non negative, and  $\mathcal{S}_{++}^n$  symmetric and positive. For a square matrix  $A \in \mathbb{R}^{n \times n}$ , we recall that  $e^A \in \mathbb{R}^{n \times n}$  is well defined as  $e^A := \sum_{k=0}^{+\infty} \frac{A^k}{k!}$ , in particular, for  $t \in [0, +\infty)$  we have  $\frac{d}{dt} e^{At} = e^{At} A$ . We denote by  $\lambda_{max}(A)$  and  $\lambda_{min}(A)$  the maximal and the minimal eigenvalue of  $A$  whenever they are real (for instance if  $A$  is symmetric).

### 3.2 Finite horizon Linear Quadratic Principal-Agent

#### 3.2.1 State variable and controlled equation

Let  $T > 0$  be a fixed finite time horizon, and  $\Omega := \mathcal{C}([0, T], \mathbb{R}^d)$  the space of continuous functions from  $[0, T]$  to  $\mathbb{R}^d$ , endowed with its Borel  $\sigma$ -algebra  $\mathcal{B}(\Omega)$ , and define the (completed) filtration  $\mathbb{F}$  generated by the canonical process  $X$ . We define the reference probability measure  $\mathbb{P}^0$  on  $\Omega$  as the one under which the canonical process  $X$  satisfies  $\mathbb{P}^0 \circ (X_0)^{-1} = \delta_{x_0}$  for some fixed  $x_0 \in \mathbb{R}^d$ , and there exists a  $d$ -dimensional  $(\mathbb{P}^0, \mathbb{F})$ -Brownian motion  $W^0$  such that

$$X_t - x_0 = \int_0^t (A_0 + A_1 X_s) ds + \sigma W_t^0, \text{ for } t \in [0, T], \quad (3.2.1)$$

where  $A_0 \in \mathbb{R}^d$ ,  $A_1 \in \mathbb{R}^{d \times d}$  and  $\sigma \in \mathcal{S}_{++}^n$ . The reference probability measure  $\mathbb{P}^0$  is well defined as the law of the unique strong solution of the linear SDE (3.2.1).

For an  $\mathbb{F}$ -predictable processes  $\alpha$  valued in  $\mathbb{R}^k$  with  $k \leq d$  satisfying

$$\mathbb{E}^{\mathbb{P}^0} \left[ \mathcal{E} \left( \int_0^T \sigma^{-1} B_1 \alpha_t \cdot dW_t^0 \right) \right] = 1, \quad (3.2.2)$$

with  $B_1 \in \mathbb{R}^{d \times k}$ , we can define the probability measure  $\mathbb{P}^\alpha$  equivalent to  $\mathbb{P}^0$  via the Radon-Nikodym derivative

$$\begin{cases} \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} := \mathcal{E} \left( \int_0^T \sigma^{-1} B_1 \alpha_t \cdot dW_t^0 \right), \\ L_t^\alpha := \left. \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} \right|_{\mathcal{F}_t} \text{ for } t \in [0, T]. \end{cases} \quad (3.2.3)$$

Therefore, the process

$$W_t^\alpha := W_t^0 - \int_0^t \sigma^{-1} B_1 \alpha_s ds, \text{ for } t \in [0, T], \quad (3.2.4)$$

is a  $(\mathbb{P}^\alpha, \mathbb{F})$ -Brownian motion, and so

$$X_t - x_0 = \int_0^t b(\alpha_s, X_s) ds + \sigma W_t^\alpha, \text{ for } t \in [0, T], \quad (3.2.5)$$

with

$$b(\alpha, x) := A_0 + A_1 x + B_1 \alpha. \quad (3.2.6)$$

Equivalently, the state variable can be written as

$$X_t = e^{A_1 t} x_0 + \int_0^t e^{A_1(t-s)} (A_0 + B_1 \alpha_s) ds + \int_0^t e^{A_1(t-s)} \sigma dW_s^\alpha, \text{ for } t \in [0, T]. \quad (3.2.7)$$

### 3.2.2 Agent's problem

Agent acts on the output process  $X$  via some admissible control  $\alpha \in \mathcal{U}$  (or equivalently  $\mathbb{P}^\alpha \in \mathcal{P}$ ), for some cost  $c^\alpha$  in exchange of an (admissible) contract (or a terminal payment)  $\xi \in \mathcal{C}$  from Principal. The sets of admissible controls and admissible contracts  $\mathcal{U}$  (or  $\mathcal{P}$ ) and  $\mathcal{C}$  will be defined soon. Agent's objective function is written

$$J^\alpha(\xi, \mathbb{P}^\alpha) := \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_A \left( \xi - \int_0^T c^\alpha(\alpha_s, X_s) ds \right) \right] \text{ for } (\xi, \mathbb{P}^\alpha) \in \mathcal{C} \times \mathcal{P}, \quad (3.2.8)$$

where  $c^\alpha(\alpha, x) := \frac{1}{2} C_2 \alpha \cdot \alpha + C_1 \cdot \alpha + \frac{1}{2} M_2 x \cdot x + M_1 \cdot x$ , and  $U_A(x) := -e^{-rx}$ , with  $r > 0$ , and  $C_2 \in \mathcal{S}_{++}^k$ ,  $M_2 \in \mathcal{S}_+^d$  and  $C_1 \in \mathbb{R}^k$ ,  $M_1 \in \mathbb{R}^d$ . Agent's value function is defined for a given  $\xi \in \mathcal{C}$  by

$$V^\alpha(\xi) := \sup_{\mathbb{P}^\alpha \in \mathcal{P}} J^\alpha(\xi, \mathbb{P}^\alpha). \quad (3.2.9)$$

To make sure that agent's problem is non-degenerate, we discard controls with infinite cost, and we impose some integrability conditions on both controls and contracts.

**Definition 3.2.1** (Admissible controls). *Let  $p_1, p_2 > 1$ , with  $\frac{1}{p_1} + \frac{1}{p_2} < 1$ . We define the set of admissible controls  $\mathcal{U}$  as the set of  $\mathbb{F}$ -predictable processes  $\alpha$  valued in  $\mathbb{R}^k$  such that*

$$\mathcal{E} \left( \int_0^t \sigma^{-1} B_1 \alpha_s \cdot dW_s^0 \right)_{t \in [0, T]} \text{ is a } (\mathbb{P}^0, \mathbb{F})\text{-martingale in } \mathbb{L}^{p_1}, \quad (3.2.10)$$

and

$$\mathbb{E}^0 \left[ e^{r p_2 \int_0^T c^A(\alpha_s, X_s) ds} \right] < +\infty. \quad (3.2.11)$$

The set of admissible probabilities  $\mathcal{P}$  is therefore the collection of probability measures induced by the processes in  $\mathcal{U}$  via the Radon Nikodym derivative (3.2.3), which is well defined by virtue of (3.2.10).

Agent accepts the contract only if it satisfies his participation constraint, i.e., if he expects to receive an utility higher than a certain threshold  $V^A(\xi) \geq U_A(\mathcal{R})$ , for some fixed  $\mathcal{R} \in \mathbb{R}$ . For a given contract  $\xi$  we define the set of optimal controls

$$\mathcal{P}^*(\xi) := \{\mathbb{P}^\alpha \in \mathcal{P} \text{ such that } V^A(\xi) = J^A(\xi, \mathbb{P}^\alpha)\}. \quad (3.2.12)$$

### 3.2.3 Principal's problem

Principal's problem consists in choosing the appropriate contract  $\xi$  to incentivize the Agent to put the effort  $\alpha$ . She wants to maximize her liquidation function and some running criteria of the form

$$J^P(\xi, \mathbb{P}^\alpha) := \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_P \left( -\xi + \int_0^T \ell(X_s) ds + g(X_T) \right) \right] \text{ for } (\xi, \mathbb{P}^\alpha) \in \mathcal{C} \times \mathcal{P}, \quad (3.2.13)$$

with  $\ell(x) := L_1 \cdot x - \frac{1}{2} L_2 x \cdot x$ , and  $g(x) := G_1 \cdot x$  and  $U_P(x) := -e^{-\rho x}$ , where  $\rho > 0$ , and  $L_1 \in \mathbb{R}^d$  and  $G_1 \in \mathbb{R}^d$  and  $L_2 \in \mathcal{S}_+^d \setminus \{0\}$ .

We distinguish three different situations; the business as usual case, the first best, and the second best.

#### The business as usual case

When principal chooses not to contract in the first place i.e., when  $\xi = 0$ , a good proxy for the participation constraint  $\mathcal{R}$  can be given by agent's value function (or more accurately his certainty equivalent) which is a solution to the Markovian stochastic control

$$U_A(\mathcal{R}) := V^A(0) = \sup_{\mathbb{P}^\alpha \in \mathcal{P}} J^A(0, \mathbb{P}^\alpha). \quad (3.2.14)$$

Whenever the supremum is attained for some admissible control, we will denote it  $\mathbb{P}^{\alpha^{\mathcal{R}}}$ , and principal's value function is then written

$$V_{\text{BU}}^P := J^P(0, \mathbb{P}^{\alpha^{\mathcal{R}}}), \quad (3.2.15)$$

which is in some sense a reference for principal's value function, by analogy to agent's participation constraint (3.2.14).

### The first best

The first best contract corresponds to the case where principal observes everything, and chooses both the effort for agent and the contract. Principal's value function is then written

$$V_{\text{FB}}^{\text{P}} := \sup_{\xi \in \mathcal{C}} \sup_{\mathbb{P}^\alpha \in \mathcal{P}} J^{\text{P}}(\xi, \mathbb{P}^\alpha) \text{ such that } J^{\text{A}}(\xi, \mathbb{P}^\alpha) \geq U_{\text{A}}(\mathcal{R}). \quad (3.2.16)$$

### The second best

The second best contract, which corresponds to the case where principal does not observe agent's effort, but only the output process. principal's value function is then written

$$V_{\text{SB}}^{\text{P}} := \sup_{\xi \in \Xi} \sup_{\mathbb{P}^\alpha \in \mathcal{P}^*(\xi)} J^{\text{P}}(\xi, \mathbb{P}^\alpha). \quad (3.2.17)$$

We now define the sets  $\mathcal{C}$  and  $\Xi$  of admissible contracts;

**Definition 3.2.2** (Admissible contracts). *We define the set of pre-admissible contracts  $\mathcal{C}$  as*

$$\mathcal{C} := \left\{ \xi \in \mathcal{F}_T, \text{ such that, } \mathbb{E}^0 \left[ e^{\lambda p_3 |\xi|} \right] < +\infty, \text{ for some } \lambda > r \vee \rho \right\}, \text{ with } \frac{1}{p_3} = 1 - \frac{1}{p_1} - \frac{1}{p_2}, \quad (3.2.18)$$

and the admissible contracts as the subset  $\Xi \subset \mathcal{C}$  :

$$\Xi := \{ \xi \in \mathcal{C} \text{ such that } V^{\text{A}}(\xi) \geq U_{\text{A}}(\mathcal{R}) \text{ and } \mathcal{P}^*(\xi) \neq \emptyset \}. \quad (3.2.19)$$

Remark that the condition  $\mathcal{P}^*(\xi) \neq \emptyset$  can be omitted in the previous definition, since the Principal would not choose a contract such that  $\mathcal{P}^*(\xi) = \emptyset$ , because then  $\sup_{\mathbb{P}^\alpha \in \emptyset} J^{\text{P}}(\xi, \mathbb{P}^\alpha) = -\infty$ . However, we keep it to simplify notations. Remark also that we have the following inequality

$$V_{\text{BU}}^{\text{P}} \leq V_{\text{SB}}^{\text{P}} \leq V_{\text{FB}}^{\text{P}}. \quad (3.2.20)$$

We introduce some further notations which will be useful in the sequel:

$$\begin{aligned} \bar{A}_0 &:= A_0 - B_1 C_2^{-1} C_1, \text{ and } C_0 := \frac{1}{2} C_1^{\text{T}} C_2^{-1} C_1, \\ \varrho &:= \frac{r\rho}{r + \rho}, \text{ and } B_0 := B_1 C_2^{-1} B_1^{\text{T}}, \end{aligned}$$

$$Q_{\text{BU}} := B_0 - r\sigma^2, \text{ and } Q_{\text{FB}} := B_0 - \varrho\sigma^2, \text{ and } Q_{\text{SB}} := \left( B_0 + \rho\sigma^2 \right) \left( B_0 + (\rho + r)\sigma^2 \right)^{-1} \left( B_0 + \rho\sigma^2 \right) - \rho\sigma^2. \quad (3.2.21)$$

### 3.2.4 Non degeneracy of Principal and Agent problems

In this section, we prove that Agent's objective function is finite, and provide a sufficient condition for Principal's objective function to be finite. The next lemma deals with Agent's problem.

**Lemma 3.2.3.** *For  $(\xi, \mathbb{P}^\alpha) \in \mathcal{C} \times \mathcal{P}$ , we have*

$$|J^{\text{A}}(\xi, \mathbb{P}^\alpha)| < +\infty. \quad (3.2.22)$$



The proof is reported in Appendix 3.7.2.

Principal's objective function involves the exponential of a quadratic term in the state variable, so we need to make some assumptions on the coefficients of the problem and the maturity  $T$  to prove that it is finite. We define

$$\begin{aligned} \lambda_{A_1} &:= (\lambda_{\max}(A_1 + A_1^\top))^+, \text{ and } \lambda_\sigma := \sqrt{\lambda_{\max}(\sigma\sigma^\top)}, \\ \varepsilon &:= \frac{(\rho p_2) \wedge (\varrho p_3)}{2} \lambda_{\max}(L_2), \text{ and } \kappa := \sqrt{|2\varepsilon\lambda_\sigma^2 - \lambda_{A_1}^2|}, \end{aligned} \quad (3.2.23)$$

Remark that  $\lambda_{A_1} \leq \|A_1 + A_1^\top\|$ , where  $\|A_1 + A_1^\top\|$  is the operator norm of matrices of  $(A_1 + A_1^\top)$ . We make the following assumption.

**Assumption 3.2.4.** *Assume that*

$$2\varepsilon\lambda_\sigma^2 - \lambda_{A_1}^2 > 0, \text{ and } T < \frac{1}{\kappa} \left( \frac{\pi}{2} + \arctan\left(\frac{-\lambda_{A_1}}{\kappa}\right) \right), \quad (3.2.24)$$

so that  $\kappa = \sqrt{2\varepsilon\lambda_\sigma^2 - \lambda_{A_1}^2}$  (without the absolute value).

The following lemma states that Principal's objective function is finite under the Assumption 3.2.4.

**Lemma 3.2.5.** *Under assumption 3.2.4, and for  $(\xi, \mathbb{P}^\alpha) \in \mathcal{C} \times \mathcal{P}$  we have that*

$$|J^p(\xi, \mathbb{P}^\alpha)| < +\infty. \quad (3.2.25)$$

The proof of this result is reported in Appendix 3.7.3, and the rest of the paper will be set under Assumption 3.2.4.

### 3.3 Solving the business as usual case

We start by computing agent's participation constraint, solution of his problem in the absence of contract (3.2.14). We solve first the related HJB-PDE, with a matrix Riccati ODE, and then we proceed by verification.

#### 3.3.1 Derivation of a candidate for the solution

Recall agent's value function whenever  $\xi = 0$ ;

$$V^A(0) = \sup_{\mathbb{P}^\alpha \in \mathcal{P}} J^A(0, \mathbb{P}^\alpha) = \sup_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}^{\mathbb{P}^\alpha} \left[ -e^r \int_0^T c^A(\alpha_s, X_s) ds \right]. \quad (3.2.14)$$

Agent's continuation utility is defined as

$$V^A(t, x) := \sup_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ -e^r \int_t^T c^A(\alpha_s, X_s^{t,x}) ds \right], \text{ for } (t, x) \in [0, T] \times \mathbb{R}^d, \quad (3.3.1)$$

and is related to the HJB-PDE

$$\begin{cases} V(T, \cdot) = -1, \\ -\partial_t V - \mathbf{H}_{\text{BU}}(x, V, DV, D^2V) = 0, \text{ in } [0, T] \times \mathbb{R}^d, \end{cases} \quad (3.3.2)$$

with

$$\mathbf{H}_{\text{BU}}(x, y, z, \gamma) := \sup_{\alpha \in \mathbb{R}^k} h_{\text{BU}}(x, y, z, \gamma, \alpha), \quad (3.3.3)$$

where

$$h_{\text{BU}}(x, y, z, \gamma, \alpha) := b(\alpha, x) \cdot z + \frac{1}{2} \sigma \sigma^\top : \gamma + r y c^\wedge(\alpha, x), \quad (3.3.4)$$

and for  $y \neq 0$ ,

$$\begin{aligned} h_{\text{BU}}(x, y, z, \gamma, \alpha) &:= b(0, x) \cdot z + \frac{1}{2} \sigma \sigma^\top : \gamma + r y c^\wedge(0, x) + \frac{r y}{2} \left\{ C_2 \alpha \cdot \alpha - 2 C_2^{-\frac{1}{2}} \left( B_1^\top \frac{-z}{r y} - C_1 \right) \cdot C_2^{\frac{1}{2}} \alpha \right\}, \\ &= -r y \left( (A_0 + A_1 x) \cdot \frac{-z}{r y} + \frac{1}{2} \sigma \sigma^\top : \frac{-\gamma}{r y} - \frac{1}{2} M_2 x \cdot x - M_1 \cdot x \right) \\ &\quad - \frac{r y}{2} C_2^{-1} \left( B_1^\top \frac{-z}{r y} - C_1 \right)^2 + \frac{r y}{2} \left( C_2^{\frac{1}{2}} \alpha - C_2^{-\frac{1}{2}} \left( B_1^\top \frac{-z}{r y} - C_1 \right) \right)^2, \end{aligned} \quad (3.3.5)$$

and so, for  $\alpha_{\text{BU}}^* \left( \frac{-z}{r y} \right) := C_2^{-1} \left( B_1^\top \frac{-z}{r y} - C_1 \right)$ , we have

$$\mathbf{H}_{\text{BU}}(x, y, z, \gamma) = -r y \left( (A_0 + A_1 x) \cdot \frac{-z}{r y} + \frac{1}{2} \sigma \sigma^\top : \frac{-\gamma}{r y} - \frac{1}{2} M_2 x \cdot x - M_1 \cdot x + \frac{1}{2} C_2^{-1} \left( B_1^\top \frac{-z}{r y} - C_1 \right)^2 \right). \quad (3.3.6)$$

The following Lemma provides an explicit solution to the PDE (3.3.2) in terms of a matrix Riccati equation

**Lemma 3.3.1.** *Assume that there exists a solution to the following matrix Riccati equation:*

$$\begin{cases} \Gamma_T = 0, \\ \dot{\Gamma}_t + A_1^\top \Gamma_t + \Gamma_t A_1 + \Gamma_t Q_{\text{BU}} \Gamma_t - M_2 = 0, \text{ for } t \in [0, T], \end{cases} \quad (3.3.7)$$

Then the function  $v(t, x) := -e^{-r \left( \frac{1}{2} \Gamma_t x \cdot x + P_t x + R_t \right)}$  is a classical solution to the PDE (3.3.2), where  $P$  is the solution to the following linear ODE

$$\begin{cases} P_T = 0, \\ \dot{P}_t + \Gamma_t \bar{A}_0 + A_1^\top P_t - M_1 + \Gamma_t Q_{\text{BU}} P_t = 0, \text{ for } t \in [0, T], \end{cases} \quad (3.3.8)$$

and  $R$  is defined as

$$R_t := \int_t^T \left( \bar{A}_0 \cdot P_s + \frac{1}{2} \sigma^2 : \Gamma_s + \frac{1}{2} Q_{\text{BU}} P_s \cdot P_s + C_0 \right) ds, \text{ for } t \in [0, T], \quad (3.3.9)$$

and the maximizer of the Hamiltonian (3.3.3) is given by

$$\alpha_{\text{BU}}^*(t, x) = C_2^{-1} \left( B_1^\top (\Gamma_t x + P_t) - C_1 \right), \text{ for } (t, x) \in [0, T] \times \mathbb{R}^d. \quad (3.3.10)$$

The proof is reported in Appendix 3.7.4

### 3.3.2 Verification

The following verification argument provides a solution to the problem (3.2.14), i.e., agent's optimal policy in absence of a contract and his participation constraint.

**Proposition 3.3.2.** (i) *Assume that  $Q_{BU} > 0$ . Then the Riccati system (3.3.7) and (3.3.8) and (3.3.9) admits a unique solution  $(\Gamma, P, R)$ , and we have*

$$V^A(0) \leq -e^{-r(\frac{1}{2}\Gamma_0 x_0 \cdot x_0 + P_0 \cdot x_0 + R_0)}. \quad (3.3.11)$$

(ii) *Assume further that the induced feedback control  $\alpha_{BU}^*(t, x) := C_2^{-1}(B_1^\top(\Gamma_t x + P_t) - C_1)$  is admissible, i.e.,  $(\alpha_{BU}^*(t, X_t))_{t \in [0, T]} \in \mathcal{U}$ , then*

$$V^A(0) = -e^{-r(\frac{1}{2}\Gamma_0 x_0 \cdot x_0 + P_0 \cdot x_0 + R_0)}, \quad (3.3.12)$$

i.e.,

$$\mathcal{R} = \frac{1}{2}\Gamma_0 x_0 \cdot x_0 + P_0 \cdot x_0 + R_0, \quad (3.3.13)$$

and the maximizer of the problem (3.2.14) is the probability measure  $\mathbb{P}^{\alpha^{\mathcal{R}}}$  induced by the feedback control  $\alpha_{BU}^*$ .

The proof is reported in Appendix 3.7.5.

The following lemma provides a sufficient condition for the feedback control  $\alpha_{BU}^*$  defined in Proposition 3.3.2 to be admissible. Let

$$\varepsilon_{BU} := \frac{\lambda_{max}(B_0)}{2} \left( \frac{q_1}{(\sqrt{q_1} - 1)^2} \frac{\lambda_{max}(B_0) \sup_{t \in [0, T]} (\lambda_{max}(\Gamma_t^2))}{\lambda_{min}(\sigma^2)} \right) \vee \left( rp_2 \left( \sup_{t \in [0, T]} (\lambda_{max}(\Gamma_t^2)) + \lambda_{max}(M_2) \right) \right), \quad (3.3.14)$$

where  $q_1$  is such that  $\frac{1}{q_1} = 1 - \frac{1}{p_1}$ , (or equivalently  $\frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{p_3}$ , and  $\Gamma$  the solution to the matrix Riccati equation (3.3.7). Define also

$$\kappa_{BU} := \sqrt{|2\varepsilon_{BU}\lambda_\sigma^2 - \lambda_{A_1}^2|}. \quad (3.3.15)$$

**Lemma 3.3.3.** *Assume that*

$$2\varepsilon_{BU}\lambda_\sigma^2 - \lambda_{A_1}^2 > 0, \text{ and } T < \frac{1}{\kappa_{BU}} \left( \frac{\pi}{2} + \arctan \left( \frac{-\lambda_{A_1}}{\kappa_{BU}} \right) \right), \quad (3.3.16)$$

Then  $(\alpha_{BU}^*(t, X_t))_{t \in [0, T]} \in \mathcal{U}$ .

The proof is reported in Appendix 3.7.6.

**Remark 3.3.4.** (i) *The participation constraint  $\mathcal{R}$  depends on the (spatial) initial condition  $x_0$ .*

(ii) *Agent's control in the absence of incentive is characterized by the probability measure  $\mathbb{P}^{\alpha^{\mathcal{R}}}$  under which*

$$\begin{aligned} X_t - x_0 &= \int_0^t \left( A_0 - B_1 C_2^{-1} C_1 + A_1 X_s + C_2^{-1} B_1^\top (\Gamma_s X_s + P_t) \right) ds + \sigma W_t^{\alpha^{\mathcal{R}}}, \\ &= \int_0^t \left( \bar{A}_0 + C_2^{-1} B_1^\top P_s + (A_1 + C_2^{-1} B_1^\top \Gamma_s) X_s \right) ds + \sigma W_t^{\alpha^{\mathcal{R}}}, \text{ for } t \in [0, T], \end{aligned} \quad (3.3.17)$$

with the  $(\mathbb{P}^{\alpha^{\mathcal{R}}}, \mathbb{F})$ -Brownian motion  $W^{\alpha^{\mathcal{R}}}$  defined as

$$W_t^{\alpha^{\mathcal{R}}} := W_t^0 - \int_0^t \sigma^{-1} B_1 \alpha_{BU}^*(s, X_s) ds, \text{ for } t \in [0, T]. \quad (3.3.18)$$

(iii) From Principal's perspective, we have

$$\begin{aligned} V_{BU}^P &= J^P(0, \mathbb{P}^{\alpha^{\mathcal{R}}}), \\ &= \mathbb{E}^{\mathbb{P}^{\alpha^{\mathcal{R}}}} \left[ -e^{-\rho \left( \int_0^T \ell(X_s) ds + g(X_T) \right)} \right]. \end{aligned} \quad (3.3.19)$$

with  $|V_{BU}^P| < +\infty$ , as a direct consequence of Lemma 3.2.5.

### 3.4 Solving the first best problem

We start by identifying a candidate for the solution of the first best by reducing the problem to a standard Markovian setting, then we use a verification argument to conclude.

#### 3.4.1 Derivation of a candidate for the solution

The first best problem is solved by introducing a Lagrange multiplier and then using the linearity of the expectation

$$\begin{aligned} V_{FB}^P &= \sup_{\xi \in \mathcal{C}} \sup_{\mathbb{P}^{\alpha} \in \mathcal{P}} J^P(\xi, \mathbb{P}^{\alpha}) \text{ such that } J^A(\xi, \mathbb{P}^{\alpha}) \geq U_A(\mathcal{R}), \\ &= \inf_{\lambda \geq 0} \sup_{\xi \in \mathcal{C}} \sup_{\mathbb{P}^{\alpha} \in \mathcal{P}} \left( J^P(\xi, \mathbb{P}^{\alpha}) + \lambda \left( J^A(\xi, \mathbb{P}^{\alpha}) - U_A(\mathcal{R}) \right) \right), \\ &= \inf_{\lambda \geq 0} \left( -\lambda U_A(\mathcal{R}) + \sup_{\xi \in \mathcal{C}} \sup_{\mathbb{P}^{\alpha} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}^{\alpha}} \left[ U_P \left( -\xi + \int_0^T \ell(X_s) ds + g(X_T) \right) + \lambda U_A \left( \xi - \int_0^T c^A(\alpha_s, X_s) ds \right) \right] \right). \end{aligned} \quad (3.4.1)$$

For a fixed  $\lambda$ , we compute the first order condition with respect to the contract, corresponding to

$$-U_P' \left( -\xi + \int_0^T \ell(X_s) ds + g(X_T) \right) + \lambda U_A' \left( \xi - \int_0^T c^A(\alpha_s, X_s) ds \right) = 0, \quad (3.4.2)$$

which yields

$$\xi_\lambda = \frac{1}{\rho + r} \log \left( \frac{r\lambda}{\rho} \right) + \frac{r}{\rho + r} \int_0^T c^A(\alpha_s, X_s) ds + \frac{\rho}{\rho + r} \int_0^T \ell(X_s) ds + \frac{\rho}{\rho + r} g(X_T), \quad (3.4.3)$$

and so the problem (in the first best case) reduces to

$$V_{FB}^P = \inf_{\lambda \geq 0} \left( -\lambda U_A(\mathcal{R}) + \lambda \frac{\rho}{\rho+r} \left( \frac{r}{\rho} \right)^{\frac{-r}{\rho+r}} \left( 1 + \frac{r}{\rho} \right) \bar{V}_0 \right), \quad (3.4.4)$$

where

$$\bar{V}_0 := \sup_{\mathbb{P}^{\alpha} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}^{\alpha}} \left[ -e^{-\rho \left( \int_0^T (\ell(X_s) - c^A(\alpha_s, X_s)) ds + g(X_T) \right)} \right], \quad (3.4.5)$$

and so after some computations, we obtain the optimal Lagrange multiplier  $\lambda^*$  and  $V_{\text{FB}}^p$  as

$$\lambda^* := \frac{\rho}{r} \left( \frac{\bar{V}_0}{U_A(\mathcal{R})} \right)^{1+\frac{\rho}{r}} \quad \text{and} \quad V_{\text{FB}}^p = U_A(\mathcal{R}) \left( \frac{\bar{V}_0}{U_A(\mathcal{R})} \right)^{1+\frac{\rho}{r}}, \quad (3.4.6)$$

with the state variable  $X$  following the controlled SDE (3.2.5).

### 3.4.2 Solving the Markovian stochastic control problem

We solve now the Markovian stochastic control problem  $\bar{V}_0$  using standard stochastic control arguments. We define the function  $\bar{V} : [0, T] \times \mathbb{R}^d$  as

$$\bar{V}(t, x) := \sup_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ -e^{-\rho \left( \int_t^T (\ell(X_s) - c^A(\alpha_s, X_s)) ds + g(X_T) \right)} \right] \quad \text{for } (t, x) \in [0, T] \times \mathbb{R}^d, \quad (3.4.7)$$

and  $\bar{V}$  is characterized by the HJB-PDE

$$\begin{cases} \bar{V}(T, \cdot) = -e^{-\rho g(\cdot)}, \\ -\partial_t \bar{V}(t, x) - \mathbf{H}_{\text{FB}}(x, \bar{V}, D\bar{V}, D^2\bar{V}) = 0 \text{ in } [0, T] \times \mathbb{R}^d, \end{cases} \quad (3.4.8)$$

where

$$\mathbf{H}_{\text{FB}}(x, y, z, \gamma) := \sup_{\alpha \in \mathbb{R}^k} h_{\text{FB}}(x, y, z, \gamma, \alpha), \quad (3.4.9)$$

and

$$\begin{aligned} h_{\text{FB}}(x, y, z, \gamma, \alpha) := & (A_0 + A_1 x) \cdot z + \frac{1}{2} \sigma^2 : \gamma - y \rho \left( (L_1 - M_1) \cdot x - \frac{1}{2} (L_2 + M_2) x \cdot x \right) \\ & - y \rho \frac{1}{2} C_2^{-1} \left( B_1^\top \frac{z}{y \rho} + C_1 \right) \cdot \left( B_1^\top \frac{z}{y \rho} + C_1 \right) \\ & + y \rho \frac{1}{2} \left( C_2^{-\frac{1}{2}} \left( B_1^\top \frac{z}{y \rho} + C_1 \right) + C_2^{\frac{1}{2}} \alpha \right)^2. \end{aligned} \quad (3.4.10)$$

Since  $y$  represents the continuation value which is negative by definition of the utility function, we have that the maximum is attained in

$$\operatorname{argmax}_{\alpha \in \mathbb{R}^k} h_{\text{FB}}(x, y, z, \gamma, \alpha) = -C_2^{-1} \left( B_1^\top \frac{z}{y \rho} + C_1 \right). \quad (3.4.11)$$

and

$$\begin{aligned} \mathbf{H}_{\text{FB}}(x, y, z, \gamma) = & (A_0 + A_1 x) \cdot z + \frac{1}{2} \sigma^2 : \gamma - y \rho \left( (L_1 - M_1) \cdot x - \frac{1}{2} (L_2 + M_2) x \cdot x \right) \\ & - y \rho \frac{1}{2} C_2^{-1} \left( B_1^\top \frac{z}{y \rho} + C_1 \right) \cdot \left( B_1^\top \frac{z}{y \rho} + C_1 \right). \end{aligned} \quad (3.4.12)$$

The next lemma uses the quadratic structure of the problem to construct a solution to (3.4.8) as a function of some Riccati system.

**Lemma 3.4.1.** *Assume that there exists a solution to the following matrix Riccati equation:*

$$\begin{cases} \Gamma_T = 0, \\ \dot{\Gamma}_t + A_1^\top \Gamma_t + \Gamma_t A_1 + \Gamma_t Q_{FB} \Gamma_t - (L_2 + M_2) = 0, \text{ for } t \in [0, T], \end{cases} \quad (3.4.13)$$

Then the function  $v(t, x) := -e^{-\varrho(\frac{1}{2}\Gamma_t x \cdot x + P_t x + R_t)}$  is a classical solution to the PDE (3.4.8), where  $P$  is the solution to the following linear ODE

$$\begin{cases} P_T = G_1, \\ \dot{P}_t + \Gamma_t \bar{A}_0 + A_1^\top P_t + (L_1 - M_1) + \Gamma_t Q_{FB} P_t = 0, \text{ for } t \in [0, T], \end{cases} \quad (3.4.14)$$

and  $R$  is defined as

$$R_t := \int_t^T \left( \bar{A}_0 \cdot P_s + \frac{1}{2} \sigma^2 : \Gamma_s + \frac{1}{2} Q_{FB} P_s \cdot P_s + C_0 \right) ds, \text{ for } t \in [0, T], \quad (3.4.15)$$

and the maximizer of the Hamiltonian (3.4.9) is given by

$$\alpha_{FB}^*(t, x) = C_2^{-1} (B_1^\top (\Gamma_t x + P_t) - C_1), \text{ for } (t, x) \in [0, T] \times \mathbb{R}^d. \quad (3.4.16)$$

The proof is reported in Appendix 3.7.7.

### 3.4.3 Verification

**Proposition 3.4.2.** (i) *Assume that  $Q_{FB} > 0$ . Then the Riccati equation (3.4.13) is wellposed, with a unique solution  $(\Gamma, P, R)$ , and we have*

$$\bar{V}_0 \leq -e^{-\varrho(\frac{1}{2}\Gamma_0 x \cdot x + P_0 x + R_0)}. \quad (3.4.17)$$

(ii) *Assume also that the induced feedback control  $\alpha_{FB}^*(t, x) := C_2^{-1} (B_1^\top (\Gamma_t x + P_t) - C_1)$  is admissible, i.e.,  $(\alpha_{FB}^*(t, X_t))_{t \in [0, T]} \in \mathcal{U}$ , then*

$$\bar{V}_0 = -e^{-\varrho(\frac{1}{2}\Gamma_0 x \cdot x + P_0 x + R_0)}. \quad (3.4.18)$$

(iii) *Assume further that the induced optimal contract is admissible, i.e.,  $\xi_{FB}^* \in \mathcal{C}$ , with*

$$\begin{aligned} \xi_{FB}^* = & \mathcal{R} - \frac{\rho}{\rho + r} \left( \frac{1}{2} \Gamma_0 x_0 \cdot x_0 + P_0 \cdot x_0 + R_0 \right) \\ & + \frac{\rho}{\rho + r} \left( g(X_T) + \int_0^T \ell(X_s) ds + \frac{r}{\rho} \int_0^T c^A(\alpha_{FB}^*(s, X_s), X_s) ds \right) \end{aligned} \quad (3.4.19)$$

then

$$V_{FB}^P = U_P \left( -\mathcal{R} + \frac{1}{2} \Gamma_0 x_0 \cdot x_0 + P_0 \cdot x_0 + R_0 \right), \text{ and } \alpha_{FB}^*(t, x) = C_2^{-1} (B_1^\top D u_{FB}(t, x) - C_1), \quad (3.4.20)$$

where  $u_{FB}(t, x)$  is the certainty equivalent dynamic function defined as

$$u_{FB}(t, x) := -\mathcal{R} + \frac{1}{2}\Gamma_t x \cdot x + P_t \cdot x + R_t, \text{ for } (t, x) \in [0, T] \times \mathbb{R}^d. \quad (3.4.21)$$

The proof is reported in Appendix 3.7.8.

The following lemma provides a sufficient condition for the feedback control and the optimal contract defined in Proposition 3.4.2 to be admissible. Define  $q_1$  such that  $\frac{1}{q_1} = 1 - \frac{1}{p_1}$ , (or equivalently  $\frac{1}{q_1} = \frac{1}{p_2} + \frac{1}{p_3}$ ), and

$$\varepsilon_{FB} := \max \left\{ \frac{1}{2} \left( \lambda_{max}(L_2) \frac{\lambda p_3 \rho}{\rho + r} \right), \left( \left( \lambda_{max}(M_2) + \sup_{t \in [0, T]} \lambda_{max}(\Gamma_t^2) \times \lambda_{max}(B_0) \right) \times \left( \frac{\lambda p_3 r}{\rho + r} \vee r p_2 \vee \frac{q_1 \lambda_{max}(B_0)}{\lambda_{min}(\sigma \sigma^\top)(\sqrt{q_1} - 1)^2} \right) \right) \right\}, \quad (3.4.22)$$

where  $\Gamma$  is the solution to the matrix Riccati equation (3.4.13), and define also

$$\kappa_{FB} := \sqrt{|2\varepsilon_{FB} \lambda_\sigma^2 - \lambda_{A_1}^2|}. \quad (3.4.23)$$

**Lemma 3.4.3.** *Assume that*

$$2\varepsilon_{FB} \lambda_\sigma^2 - \lambda_{A_1}^2 > 0, \text{ and } T < \frac{1}{\kappa_{FB}} \left( \frac{\pi}{2} + \arctan \left( \frac{-\lambda_{A_1}}{\kappa_{FB}} \right) \right), \quad (3.4.24)$$

then  $(\alpha_{FB}^*(t, X_t))_{t \in [0, T]} \in \mathcal{U}$  and  $\xi_{FB}^* \in \mathcal{C}$ .

The proof is reported in Appendix 3.7.9.

### 3.5 Solving the second best problem

We follow the methodology of Cvitanić et al. (2018), by introducing the set of revealing contracts, which induce tractable agent responses and make principal's problem easy to solve. Then we prove that there is no loss of generality in considering only such contracts.

#### 3.5.1 Revealing contracts

Let  $H : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$  be the Hamiltonian of agent's problem defined as

$$H(x, z) := \sup_{\alpha \in \mathbb{R}^k} h(x, z, \alpha), \quad (3.5.1)$$

with

$$\begin{aligned} h(x, z, \alpha) &:= b(\alpha, x) \cdot z - c^\wedge(\alpha, x), \\ &= b(0, x) \cdot z + B_1 \alpha \cdot z - \frac{1}{2} C_2 \alpha \cdot \alpha - C_1 \cdot \alpha - \frac{1}{2} M_2 x \cdot x - M_1 \cdot x. \end{aligned} \quad (3.5.2)$$

We define also the recommended policy  $\hat{\alpha} : \mathbb{R}^d \rightarrow \mathbb{R}^k$  as the function

$$\hat{\alpha}(z) := C_2^{-1} (B_1^\top z - C_1), \quad (3.5.3)$$

which satisfies for  $(x, z) \in \mathbb{R}^d \times \mathbb{R}^d$

$$\begin{aligned} H(x, z) &= h(x, z, \hat{\alpha}(z)), \\ &= (A_0 + A_1 x) \cdot z + \frac{1}{2} C_2^{-1} (B_1^\top z - C_1) \cdot (B_1^\top z - C_1) - \frac{1}{2} M_2 x \cdot x - M_1 \cdot x, \\ &= (\bar{A}_0 + A_1 x) \cdot z + \frac{1}{2} B_0 z \cdot z + C_0 - \frac{1}{2} M_2 x \cdot x - M_1 \cdot x. \end{aligned} \quad (3.5.4)$$

We introduce the set of revealing contracts  $\mathcal{Z}$  as the processes controlled by a pair  $(Y_0, Z) \in \mathbb{R} \times \mathcal{V}$  defined as

$$\mathcal{Z} := \left\{ Y_T^{Y_0, Z} \text{ for some } (Y_0, Z) \in \mathbb{R} \times \mathcal{V} \text{ with } Y_T^{Y_0, Z} \in \Xi \right\}, \quad (3.5.5)$$

where

$$Y_t^{Y_0, Z} := Y_0 + \int_0^t Z_s \cdot dX_s - \int_0^t H(X_s, Z_s) ds + \frac{r}{2} \int_0^t |\sigma Z_s|^2 ds, \text{ for } t \in [0, T], \quad (3.5.6)$$

and  $\mathcal{V}$  is the set of  $\mathbb{F}$ -predictable processes  $Z$  valued in  $\mathbb{R}^d$  satisfying :

$$\left\{ \begin{array}{l} \mathbb{E}^{\mathbb{P}^\alpha} \left[ \mathcal{E} \left( -r \int_0^T \sigma Z_s \cdot dW_s^\alpha \right) \right] = 1, \text{ for all } \mathbb{P}^\alpha \in \mathcal{P}, \\ \text{and } (\hat{\alpha}(Z_t))_{0 \leq t \leq T} \in \mathcal{U}. \end{array} \right. \quad (3.5.7)$$

### 3.5.2 Representation theorem

The following proposition provides a representation result which proves that  $\mathcal{Z} = \Xi$ , i.e., that there is no loss of generality for Principal in considering only contracts from  $\mathcal{Z}$  (which are designed to easily characterize Agent's response and value function).

**Proposition 3.5.1.** *We have that  $\mathcal{Z} = \Xi$ , i.e., for  $\xi \in \Xi$  there exists a pair  $(Y_0, Z) \in \mathbb{R} \times \mathcal{V}$  such that*

$$\xi = Y_0 + \int_0^T Z_t \cdot dX_t - \int_0^T H(X_t, Z_t) dt + \frac{r}{2} \int_0^T |\sigma Z_t|^2 dt. \quad (3.5.8)$$

Furthermore, we have the estimate on  $Y$

$$\mathbb{E}^0 \left[ e^{\lambda p_3 \sup_{t \leq r \leq T} |Y_r|} \right] < +\infty. \quad (3.5.9)$$

The proof is reported in Appendix 3.7.10.

### 3.5.3 Agent response to revealing contracts

Using a simple verification argument, we have the following lemma which characterizes Agent's value function and reaction given a revealing contract from  $\mathcal{Z}$ :

**Lemma 3.5.2.** *For  $Y_T^{Y_0, Z} \in \mathcal{Z}$ ,  $V^A(Y_T^{Y_0, Z}) = U_A(Y_0)$ , and Agent's optimal control corresponds to the recommended*



policy  $(\hat{\alpha}(Z_t))_{t \in [0, T]}$ , so the state variable has the following dynamic

$$\begin{aligned} X_t - x_0 &= \int_0^t b(\hat{\alpha}(Z_s), X_s) ds + \sigma W_t^{\hat{\alpha}}, \\ &= \int_0^t (\bar{A}_0 + A_1 X_s + B_0 Z_s) ds + \sigma W_t^{\hat{\alpha}}, \text{ for } t \in [0, T]. \end{aligned} \quad (3.5.10)$$

The proof is reported in Appendix 3.7.12.

### 3.5.4 Equivalent markovian stochastic control problem

As we have seen from Lemma 3.5.2, Principal is able to predict Agent's effort whenever she gives him a contract from the class  $\mathcal{Z}$ . Therefore, her problem restricted to the class  $\mathcal{Z}$  is easy to solve. Furthermore, Proposition 3.5.1 asserts that  $\mathcal{Z} = \Xi$  so there is no loss of generality in considering only such contracts.

In order to write properly the equivalent (markovian) problem, we start by defining the subset  $\mathcal{V}_0$  of  $\mathcal{V}$  as

$$\mathcal{V}_0 := \left\{ Z \in \mathcal{V} \text{ such that } \mathbb{E}^0 \left[ e^{\lambda p_3 |Y_T^{0,Z}|} \right] < +\infty \right\}, \quad (3.5.11)$$

i.e.,  $\mathcal{V}_0$  is the set of  $\mathbb{F}$ -predictable processes  $Z$  valued in  $\mathbb{R}^d$  satisfying:

$$\begin{cases} \mathbb{E}^{\mathbb{P}^\alpha} \left[ \mathcal{E} \left( -r \int_0^T \sigma Z_s \cdot dW_s^\alpha \right) \right] = 1, \text{ for all } \mathbb{P}^\alpha \in \mathcal{P}, \\ \text{and } (\hat{\alpha}(Z_t))_{0 \leq t \leq T} \in \mathcal{U}, \\ \text{and } \mathbb{E}^0 \left[ e^{\lambda p_3 |Y_T^{0,Z}|} \right] < +\infty. \end{cases} \quad (3.5.12)$$

and we can see from Lemma 3.5.2 and the definition of  $\Xi$  and  $\mathcal{V}_0$  that

$$\begin{aligned} \mathcal{Z} &= \left\{ Y_T^{Y_0, Z} \text{ for some } (Y_0, Z) \in \mathbb{R} \times \mathcal{V} \text{ with } Y_T^{Y_0, Z} \in \Xi \right\}, \\ &= \left\{ Y_T^{Y_0, Z} \text{ for some } (Y_0, Z) \in [\mathcal{R}, +\infty[ \times \mathcal{V}_0 \right\}. \end{aligned} \quad (3.5.13)$$

From Proposition 3.5.1,  $\mathcal{Z} = \Xi$ , and so we can write Principal's problem as follow

$$\begin{aligned} V_{\text{sb}}^p &= \sup_{Y_0 \geq \mathcal{R}} \sup_{Z \in \mathcal{V}_0} \mathbb{E}^{\mathbb{P}^\alpha} \left[ -e^{-\rho \left( -Y_T^{Y_0, Z} + \int_0^T \ell(X_s) ds + g(X_T) \right)} \right], \\ &= \sup_{Y_0 \geq \mathcal{R}} e^{\rho Y_0} \sup_{Z \in \mathcal{V}_0} \mathbb{E}^{\mathbb{P}^\alpha} \left[ -e^{-\rho \left( -Y_T^{0, Z} + \int_0^T \ell(X_s) ds + g(X_T) \right)} \right], \\ &= \sup_{Y_0 \geq \mathcal{R}} e^{\rho Y_0} V_0, \end{aligned} \quad (3.5.14)$$

where we used the identity  $Y_T^{Y_0, Z} - Y_0 = Y_T^{0, Z}$ , and we introduced

$$V_0 := \sup_{Z \in \mathcal{V}_0} \mathbb{E}^{\mathbb{P}^\alpha} \left[ -e^{-\rho \left( -Y_T^{0, Z} + \int_0^T \ell(X_s) ds + g(X_T) \right)} \right]. \quad (3.5.15)$$

Since  $V_0 \leq 0$  we obtain that  $Y_0^* = \mathcal{R}$ , i.e.,

$$V_{\text{SB}}^{\text{P}} = e^{\rho \mathcal{R}} V_0, \quad (3.5.16)$$

and the second best problem is reduced to the resolution of

$$V_0 = \sup_{Z \in \mathcal{V}_0} \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ -e^{-\rho \left( -Y_T + \int_0^T \ell(X_s) ds + g(X_T) \right)} \right], \quad (3.5.17)$$

which is a Markovian stochastic control problem with the dynamics

$$\begin{cases} dX_t = b(\hat{\alpha}(Z_t), X_t) dt + \sigma dW_t^{\hat{\alpha}}, \\ dY_t = \left( c^{\wedge}(\hat{\alpha}(Z_t), X_t) + \frac{1}{2} r \sigma^2 Z_t \cdot Z_t \right) dt + Z_t \cdot \sigma dW_t^{\hat{\alpha}}. \end{cases} \quad (3.5.18)$$

We introduce the continuation utility

$$V(t, x, y) := \sup_{Z \in \mathcal{V}_t} \mathbb{E}_{t,x,y}^{\mathbb{P}^{\hat{\alpha}}} \left[ -e^{-\rho \left( -Y_T + \int_t^T \ell(X_s) ds + g(X_T) \right)} \right], \quad (3.5.19)$$

which can be characterized by standard stochastic control arguments, using the HJB PDE

$$\begin{cases} V(T, x, y) = -e^{-\rho(-y+g(x))}, \\ -\partial_t V - \mathbf{H}_{\text{SB}}(\cdot, V, DV, D^2V) = 0. \end{cases} \quad (3.5.20)$$

where

$$\mathbf{H}_{\text{SB}}(x, y, r, p, \gamma) := \sup_{z \in \mathbb{R}^d} h_{\text{SB}}(x, y, r, p, \gamma, z), \quad (3.5.21)$$

and

$$\begin{aligned} h_{\text{SB}}(x, y, r, p, \gamma, z) := & (\bar{A}_0 + A_1 x) \cdot p_x + \frac{1}{2} \sigma^2 : \gamma_{xx} - \rho r \ell(x) - p_y (C_0 - \frac{1}{2} M_2 x \cdot x - M_1 \cdot x) \\ & + \frac{1}{2} (p_y B_0 + (\gamma_{yy} + r p_y) \sigma^2) z \cdot z + (B_0 p_x + \sigma^2 \gamma_{xy}) \cdot z. \end{aligned} \quad (3.5.22)$$

As in the first best case, the following lemma provides a solution to (3.5.20) as a function of a solution to a matrix Riccati equation.

**Lemma 3.5.3.** *Assume that there exists a solution to the following matrix Riccati equation:*

$$\begin{cases} \Gamma_T = 0, \\ \dot{\Gamma}_t + A_1^{\top} \Gamma_t + \Gamma_t A_1 + \Gamma_t Q_{\text{SB}} \Gamma_t - (L_2 + M_2) = 0, \text{ for } t \in [0, T), \end{cases} \quad (3.5.23)$$

where we recall that

$$Q_{\text{SB}} := (B_0 + \rho \sigma^2) (B_0 + (\rho + r) \sigma^2)^{-1} (B_0 + \rho \sigma^2) - \rho \sigma^2. \quad (3.5.24)$$

Then the function  $v(t, x, y) := -e^{-\rho(-y + \frac{1}{2} \Gamma_t x \cdot x + P_t x + R_t)}$  is a classical solution to the PDE (3.5.20), where  $P$  is the

solution of the following linear ODE

$$\begin{cases} P_T = G_1, \\ \dot{P}_t + \Gamma_t \bar{A}_0 + A_1^\top P_t + (L_1 - M_1) + \Gamma_t Q_{SB} P_t = 0, \text{ for } t \in [0, T], \end{cases} \quad (3.5.25)$$

and  $R$  is defined as

$$R_t := \int_t^T \left( \bar{A}_0 \cdot P_s + \frac{1}{2} \sigma^2 : \Gamma_s + \frac{1}{2} Q_{SB} P_s \cdot P_s + C_0 \right) ds, \text{ for } t \in [0, T], \quad (3.5.26)$$

and the maximizer of the Hamiltonian (3.5.21) is given by

$$z^*(t, x) = \left( B_0 + (\rho + r) \sigma^2 \right)^{-1} \left( B_0 + \rho \sigma^2 \right) (\Gamma_t x + P_t). \quad (3.5.27)$$

The proof is reported in Appendix 3.7.13.

### 3.5.5 Verification

We proceed to the verification argument.

**Proposition 3.5.4.** (Verification)

(i) Assume that  $Q_{SB} > 0$ . Then the matrix Riccati system (3.5.23) and (3.5.25) and (3.5.26) is wellposed with a unique solution  $(\Gamma, P, R)$ , and we have

$$V_{SB}^P \leq U_P \left( -\mathcal{R} + \frac{1}{2} \Gamma_0 x \cdot x + P_0 \cdot x + R_0 \right), \quad (3.5.28)$$

(ii) Assume further that the induced feedback control is admissible, i.e.,  $(z^*(t, X_t))_{t \in [0, T]} \in \mathcal{V}_0$ , where

$$z^*(t, x) := \left( B_0 + (\rho + r) \sigma^2 \right)^{-1} \left( B_0 + \rho \sigma^2 \right) (\Gamma_t x + P_t) \quad (3.5.29)$$

then

$$V_{SB}^P = U_P \left( -\mathcal{R} + \frac{1}{2} \Gamma_0 x \cdot x + P_0 \cdot x + R_0 \right), \quad (3.5.30)$$

and the optimal contract  $\xi_{SB}^*$  is given by

$$\xi_{SB}^* := \mathcal{R} + \int_0^T z^*(s, X_s) \cdot dX_s - \int_0^T H(X_s, z^*(s, X_s)) ds + \frac{r}{2} \int_0^T |\sigma z^*(s, X_s)|^2 ds, \quad (3.5.31)$$

or equivalently (under  $\mathbb{P}^{\hat{\alpha}}$ ) as

$$\xi_{SB}^* = \mathcal{R} + \int_0^T z^*(s, X_s) \cdot \sigma dW_s^{\hat{\alpha}} + \int_0^T c^A(\hat{\alpha}(z^*(s, X_s)), X_s) ds + \frac{r}{2} \int_0^T |\sigma z^*(s, X_s)|^2 ds, \quad (3.5.32)$$

while Agent's response is given by

$$\alpha_{SB}^*(t, X_t) := \hat{\alpha}(z^*(t, X_t)) = C_2^{-1} (B_1^\top z^*(t, X_t) - C_1). \quad (3.5.33)$$

or equivalently

$$z^*(t, x) = \left( B_0 + (\rho + r) \sigma^2 \right)^{-1} \left( B_0 + \rho \sigma^2 \right) D u_{SB}(t, x), \quad (3.5.34)$$

where the certainty equivalent dynamic function  $u_{SB}$  is defined as

$$u_{SB}(t, x) := -\mathcal{R} + \frac{1}{2} \Gamma_t x \cdot x + P_t \cdot x + R_t. \quad (3.5.35)$$

The proof is reported in Appendix 3.7.14.

We provide in the next Lemma sufficient conditions for the optimal contract and the feedback control defined in Proposition 3.5.4 to be admissible. Recall that  $q_1$  is such that  $\frac{1}{q_1} = 1 - \frac{1}{p_1}$ , and define

$$\begin{aligned} \varepsilon_{SB} &:= \lambda_{\max}(\sigma \sigma^\top) \left( \frac{q_1 r^2}{2} \vee 8(\lambda p_3)^2 \vee 2\lambda p_3 r \right) \vee \lambda_{\max}(B_0) \left( \lambda p_3 \vee \frac{q_1 \lambda_{\max}(B_0)}{2(\sqrt{q_1} - 1)^2 \lambda_{\min}(\sigma \sigma^\top)} \vee r p_2 \right), \\ \iota_{SB} &:= \mathbf{1}_{\{M_2 \neq 0\}} \lambda_{\max}(M_2) \times (r p_2) \vee (2\lambda p_3) + \mathbf{1}_{\{M_2 = 0\}} \varepsilon_{SB} \lambda_\sigma^2 \lambda_{\max}(D^2) \sup_{0 \leq t \leq T} \lambda_{\max}(\Gamma_t^2) \end{aligned} \quad (3.5.36)$$

and

$$\kappa_{SB} := \sqrt{\left| 2 \left( \iota_{SB} \wedge \varepsilon_{SB} \lambda_\sigma^2 \lambda_{\max}(D^2) \sup_{0 \leq t \leq T} \lambda_{\max}(\Gamma_t^2) \right) - \lambda_{A_1}^2 \right|}, \quad (3.5.37)$$

where  $\Gamma$  is the solution to (3.5.23), and  $D = (B_0 + (\rho + r) \sigma^2)^{-1} (B_0 + \rho \sigma^2)$ .

**Lemma 3.5.5.** *Assume that*

$$2 \left( \iota_{SB} \wedge \varepsilon_{SB} \lambda_\sigma^2 \lambda_{\max}(D^2) \sup_{0 \leq t \leq T} \lambda_{\max}(\Gamma_t^2) \right) - \lambda_{A_1}^2 > 0, \text{ and } T < \frac{1}{\kappa_{SB}} \left( \frac{\pi}{2} + \arctan \left( \frac{-\lambda_{A_1}}{\kappa_{SB}} \right) \right). \quad (3.5.38)$$

Then  $(z^*(t, X_t))_{t \in [0, T]} \in \mathcal{V}_0$ .

The proof is reported to Appendix 3.7.15.

### 3.5.6 Special case 1: The problem in one dimension

The case  $k = d = 1$  is a special case in the previous analysis, and all the results remain true. Furthermore, the optimal contract can be written in a more explicit way. Recall that under  $\mathbb{P}^{\hat{\alpha}}$  we have for  $t \in [0, T]$

$$\begin{aligned} X_t - x_0 &= \int_0^t b(\hat{\alpha}(z^*(s, X_s)), X_s) ds + \sigma W_t^{\hat{\alpha}}, \\ &= \int_0^t \left( A_0 + A_1 X_s + B_1 C_2^{-1} (B_1^\top z^*(s, X_s) - C_1) \right) ds + \sigma W_t^{\hat{\alpha}} \\ &= \int_0^t \left( \bar{A}_0 + A_1 X_s + \frac{B}{C} B_0 \partial_x u_{SB}(s, X_s) \right) ds + \sigma W_t^{\hat{\alpha}} \end{aligned} \quad (3.5.39)$$

and

$$\begin{aligned}\xi_{\text{SB}}^* &= \mathcal{R} + \int_0^T z^*(s, X_s) \sigma dW_s^{\hat{\alpha}} + \int_0^T c^\Lambda(\hat{\alpha}(z^*(s, X_s)), X_s) ds + \frac{r}{2} \int_0^T |\sigma z^*(s, X_s)|^2 ds, \\ &= \mathcal{R} + \frac{\mathbf{B}}{\mathbf{C}} \int_0^T \partial_x u_{\text{SB}}(s, X_s) \sigma dW_s^{\hat{\alpha}} + \int_0^T c^\Lambda(\hat{\alpha}(z^*(s, X_s)), X_s) ds + \frac{r}{2} \left(\frac{\mathbf{B}}{\mathbf{C}}\right)^2 \int_0^T |\sigma \partial_x u_{\text{SB}}(s, X_s)|^2 ds,\end{aligned}\tag{3.5.32}$$

where  $\mathbf{C} := (B_0 + (\rho + r)\sigma^2)$  and  $\mathbf{B} := (B_0 + \rho\sigma^2)$ . Then recall that the function  $u_{\text{SB}}$  defined in (3.5.35) satisfies the PDE (3.7.156) up to some constant, i.e.,  $u_{\text{SB}}(T, x) = G_1 x - \mathcal{R}$  and

$$\begin{aligned}0 &= \partial_t u_{\text{SB}} + \frac{1}{2} \sigma^2 \partial_{xx} u_{\text{SB}} + (\bar{A}_0 + A_1 x) \partial_x u_{\text{SB}} + \ell(x) + C_0 \\ &\quad - \frac{1}{2} M_2 x \cdot x - M_1 \cdot x + \frac{1}{2} Q_{\text{SB}} (\partial_x u_{\text{SB}})^2, \quad \text{for } (t, x) \in [0, T) \times \mathbb{R},\end{aligned}\tag{3.5.40}$$

or equivalently, since

$$Q_{\text{SB}} = \frac{\mathbf{B}^2}{\mathbf{C}} - \rho\sigma^2, \quad \text{and } c^\Lambda(\hat{\alpha}(z^*(t, x)), x) = \frac{1}{2} B_0 \left(\frac{\mathbf{B}}{\mathbf{C}}\right)^2 (\partial_x u_{\text{SB}}(t, x))^2 - C_0 + \frac{1}{2} M_2 x \cdot x + M_1 \cdot x,\tag{3.5.41}$$

then  $u_{\text{SB}}(T, x) = G_1 x - \mathcal{R}$  and

$$\begin{aligned}0 &= \partial_t u_{\text{SB}} + \frac{1}{2} \sigma^2 \partial_{xx} u_{\text{SB}} + b(\hat{\alpha}(z^*(t, x)), x) \partial_x u_{\text{SB}} + \ell(x) \\ &\quad - c^\Lambda(\hat{\alpha}(z^*(t, x)), x) - \frac{1}{2} \rho\sigma^2 (\partial_x u_{\text{SB}})^2 + \frac{\mathbf{B}}{2\mathbf{C}} \left( B_0 \left(\frac{-r\sigma^2}{\mathbf{C}}\right) + \rho\sigma^2 \right) (\partial_x u_{\text{SB}})^2, \quad \text{for } (t, x) \in [0, T) \times \mathbb{R},\end{aligned}\tag{3.5.42}$$

and so by applying Itô formula

$$\begin{aligned}u_{\text{SB}}(T, X_T) - u_{\text{SB}}(0, x_0) &= \int_0^T \left( \partial_t u_{\text{SB}}(t, X_t) + \partial_x u_{\text{SB}}(t, X_t) b(\hat{\alpha}(z^*(s, X_t)), X_t) + \frac{1}{2} \sigma^2 \partial_{xx} u_{\text{SB}}(t, X_t) \right) dt, \\ &\quad + \int_0^T \partial_x u_{\text{SB}}(t, X_t) \sigma dW_t^{\hat{\alpha}}, \\ &= - \int_0^T (\ell(X_t) - c^\Lambda(\hat{\alpha}(z^*(t, X_t)), X_t)) dt, \\ &\quad - \int_0^T \left( -\frac{1}{2} \rho\sigma^2 (\partial_x u_{\text{SB}})^2 + \frac{\mathbf{B}}{2\mathbf{C}} \left( B_0 \left(\frac{-r\sigma^2}{\mathbf{C}}\right) + \rho\sigma^2 \right) (\partial_x u_{\text{SB}})^2 \right) dt, \\ &\quad + \int_0^T \partial_x u_{\text{SB}}(t, X_t) \sigma dW_t^{\hat{\alpha}},\end{aligned}\tag{3.5.43}$$

We can then multiply the previous equality by  $\frac{\mathbf{B}}{\mathbf{C}}$ , and replace the stochastic integral term in the expression of  $\xi_{\text{SB}}^*$ , and after some simplifications, we obtain

$$\begin{aligned} \xi_{\text{SB}}^* = & \mathcal{R} \left( 1 - \frac{\mathbf{B}}{\mathbf{C}} \right) - \left( \frac{\mathbf{B}}{\mathbf{C}} \right) u_{\text{SB}}(0, x_0) + \left( \frac{\mathbf{B}}{\mathbf{C}} \right) G_1 X_T + \left( \frac{\mathbf{B}}{\mathbf{C}} \right) \int_0^T \ell(X_s) ds \\ & + \left( 1 - \frac{\mathbf{B}}{\mathbf{C}} \right) \int_0^T c^\Lambda(\hat{\alpha}(z^*(s, X_s)), X_s) ds + \frac{1}{2} \frac{B_0 r^2 \sigma^2}{\mathbf{C}^2} \int_0^T \left( \frac{\mathbf{B}}{\mathbf{C}} \right) (\sigma \partial_x u_{\text{SB}})^2 dt \end{aligned} \quad (3.5.44)$$

Remark that the assumption  $k = d = 1$  is essential to obtain the previous expression, since the optimal control involves a stochastic integral with a scalar product in a multidimensional setting preventing us from the factorization by  $\mathbf{C}^{-1}\mathbf{B}$ .

### 3.5.7 Special case 2: Risk neutral principal and agent

The risk neutral case, i.e., with  $U_A(x) = x$  and  $U_P(x) = x$ , is a special case in the previous analysis, and provides an explicit expression of the optimal contract under some controllability assumptions. We provide in this section a sketch of the derivation of this expression. Recall Agent's and principal's objective functions

$$J^A(\xi, \mathbb{P}^\alpha) := \mathbb{E}^{\mathbb{P}^\alpha} \left[ \xi - \int_0^T c^\Lambda(\alpha_t, X_t) dt \right], \text{ and } J^P(\xi, \mathbb{P}^\alpha) := \mathbb{E}^{\mathbb{P}^\alpha} \left[ -\xi + \int_0^T \ell(X_t) dt \right], \quad (3.5.45)$$

and we can see that the integrability assumptions on the set of admissible contracts can be relaxed. We assume that the cost function is of the form

$$c^\Lambda(\alpha, x) = \frac{1}{2} C_2 \alpha \cdot \alpha + c^\Lambda(0, x) \text{ for some } C_2 \in \mathcal{S}_{++}^k, \quad (3.5.46)$$

and

$$b(\alpha, x) = B_1 \alpha + b(0, x) \text{ for some } B_1 \in \mathbb{R}^{d \times k}. \quad (3.5.47)$$

For a given admissible contract  $\xi$ , the corresponding revealing contract  $Y$  is given by the continuation utility of agent, and can be derived by standard BSDE theory as

$$dY_t = Z_t dX_t - H(X_t, Z_t) dt \quad (3.5.48)$$

where

$$\begin{aligned} H(x, z) &:= \sup_{\alpha \in \mathbb{R}^k} \{ b(\alpha, x) \cdot z - c^\Lambda(\alpha, x) \}, \\ &= b(0, x) \cdot z + c^\Lambda(0, x) + B_1 \alpha \cdot \hat{\alpha}(z) - \frac{1}{2} C_2 \hat{\alpha}(z) \cdot \hat{\alpha}(z), \end{aligned} \quad (3.5.49)$$

and

$$\hat{\alpha}(z) := C_2^{-1} B_1 z. \quad (3.5.50)$$

Principal's problem is then written

$$V^P = \sup_{\xi \in \Xi} \sup_{\mathbb{P}^\alpha \in \mathcal{P}^*(\xi)} \mathbb{E}^{\mathbb{P}^\alpha} \left[ -\xi + \int_0^T \ell(X_t) dt \right], \quad (3.5.51)$$

and using the representation theorem, we can express  $V^P$  as follow

$$V^P = -y_0 + \sup_{Z \in \mathcal{V}} \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ \int_0^T (\ell(X_t) - c^\wedge(\hat{\alpha}(X_t, Z_t), X_t)) dt \right], \quad (3.5.52)$$

where we used the integrability of the class of revealing contracts. The controlled state variable is written

$$dX_t = b(\hat{\alpha}(Z_s), X_s) dt + \sigma dW_t^\alpha, \quad (3.5.53)$$

and the continuation utility for principal is defined as

$$V^P(t, x) = \sup_{Z \in \mathcal{V}_t} \mathbb{E}_{t,x}^{\mathbb{P}^{\hat{\alpha}}} \left[ \int_t^T (\ell(X_t) - c^\wedge(\hat{\alpha}(X_t, Z_t), X_t)) dt \right], \quad (3.5.54)$$

and characterized (with the appropriate growth conditions) as the unique viscosity solution to

$$\begin{cases} V^P(T, x) = 0, \\ -\partial_t V^P - G(x, DV^P, D^2V^P) = 0, \end{cases} \quad (3.5.55)$$

with

$$\begin{aligned} G(x, p, \gamma) &:= \sup_{z \in \mathbb{R}^d} \left\{ b(\hat{\alpha}(z), x) \cdot p + \frac{1}{2} \sigma \sigma^\top : \gamma + \ell(x) - c^\wedge(\hat{\alpha}(z), x) \right\}, \\ &= b(0, x) \cdot p + \frac{1}{2} \sigma \sigma^\top : \gamma + \ell(x) - c^\wedge(0, x) + \sup_{z \in \mathbb{R}^d} \left\{ B_1 \hat{\alpha}(z) \cdot p - \frac{1}{2} C_2 \hat{\alpha}(z) \cdot \hat{\alpha}(z) \right\}, \\ &= b(0, x) \cdot p + \frac{1}{2} \sigma \sigma^\top : \gamma + \ell(x) - c^\wedge(0, x) + \sup_{z \in \mathbb{R}^d} \left\{ B_0 z \cdot p - \frac{1}{2} B_0 z \cdot z \right\}. \end{aligned} \quad (3.5.56)$$

where we recall that  $B_0 = B_1^\top C_2^{-1} B_1$ . Now assume that  $k = d$ , and  $B_0$  is invertible, then the optimal feedback control  $z^*(DV) = DV$ , and the verification is written (up to a localization argument, and the admissibility of the optimal control),

$$\begin{aligned} V^P(T, X_T) = 0 &= V^P(0, x_0) + \int_0^T \partial_t V^P dt + \int_0^T DV^P \cdot dX_t + \frac{1}{2} \int_0^T \sigma \sigma^\top : D^2V dt, \\ &= V^P(0, x_0) + \int_0^T DV^P \cdot \sigma dW_t + \int_0^T \left( c^\wedge(DV^P(t, X_t), X_t) - \ell(X_t) \right) dt. \end{aligned} \quad (3.5.57)$$

Recall then the expression of the optimal contract

$$\begin{aligned}\xi_{\text{SB}}^* &= Y_T = \mathcal{R} + \int_0^T Z_t^* \cdot dX_t - H(X_t, Z_t^*)dt \\ &= \mathcal{R} + \int_0^T DV^P(t, X_t) \cdot \sigma dW_t + \int_0^T c^\Lambda(DV^P(t, X_t), X_t)dt.\end{aligned}\tag{3.5.58}$$

where we used the fact that  $Z_t^* = DV^P(t, X_t)$ , so we can replace the stochastic integral term with the one in (3.5.57),

$$\begin{aligned}\xi_{\text{SB}}^* &= \mathcal{R} - V^P(0, x_0) - \left( \int_0^T \left( c^\Lambda(DV^P(t, X_t), X_t) - \ell(X_t) \right) \right) + \int_0^T c^\Lambda(DV^P(t, X_t), X_t)dt, \\ &= \mathcal{R} - V^P(0, x_0) + \int_0^T \ell(X_t)dt.\end{aligned}\tag{3.5.59}$$

A second verification on agent's value function provides

$$\begin{aligned}V^A(\xi_{\text{SB}}^*) &= V^A\left(\mathcal{R} - V^P(0, x_0) + \int_0^T \ell(X_t)dt\right) = \mathcal{R} - V^P(0, x_0) + \sup_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}^\alpha \left[ \int_0^T \ell(X_t)dt - \int_0^T c^\Lambda(\alpha_t, X_t)dt \right], \\ &= \mathcal{R} - V^P(0, x_0) + V^P(0, x_0) = \mathcal{R}.\end{aligned}\tag{3.5.60}$$

with an optimal control  $\hat{\alpha}$ . Remark that the previous representation holds under the assumption that  $k = d$ , i.e., all the state variables are controlled, and both principal and agent are risk neutral.

## 3.6 Application : A decarbonation model

### 3.6.1 Presentation of the problem

Carbon emission reduction is considered at the level of a sector or the whole economy and the goal is to determine the relative intensity of the subsidy that should be given to optimally reduce carbon emissions.

Assume that the economy produces a good to satisfy a certain demand. By considering the economy as a single entity we can take the aggregated goods produced and their monetary equivalent  $K$ , and  $D$  the (monetary) demand for capital (or for goods). We model  $D$  and  $K$  as follow

$$\begin{aligned}dD_t &= \mu D_t dt + \sigma^D dW_t^D, \\ dK_t &= (i_t - \delta K_t)dt + \beta dW_t^K,\end{aligned}\tag{3.6.1}$$

where  $W^D$  and  $W^K$  are two independent Brownian motions, and  $\mu > 0$  is the growth rate of demand, and  $i_t$  is a control on the investment (the intensity of production for example), and  $\delta > 0$  the depreciation rate of capital (for example, the deterioration impacting the means of production used), and  $\sigma^D, \beta > 0$  the levels of uncertainty.

As observed in [Khan et al. \(2019\)](#), the greenhouse gas emissions are procyclical (move in the same direction as the overall economy). We can assume then that  $K$  induces a carbon emission rate  $E$  with the dynamics

$$dE_t = \eta dK_t + \lambda E_t dt - a_t dt + \varphi dW_t^E,\tag{3.6.2}$$

with  $W^E$  a Brownian motion independent of  $W^D$  and  $W^K$ , and  $\eta > 0$  some multiplicative factor, and  $\lambda E_t$  an



exponential increase in the emissions rate, and  $(-a_t)$  a control (an effort) to reduce them, and  $\varphi > 0$  the level of uncertainty. Remark that in the absence of control ( $i_t = 0$  and  $a_t = 0$ ), the evolution of the emissions rates  $E$  is not straightforward and depends on the parameters, especially the difference between the decay in emissions caused by the deterioration of  $K$  and the exponential growth of the emissions (in  $\lambda$ ).

The principal is the social planner, and the agent is the economy considered as the aggregation of its entities. It makes sense to state that the baseline emission of a sector is difficult to identify: it is difficult to know if a reduction in emissions is caused by a decrease in production and consumption or by appropriate efforts made by the sector to reduce the emissions without impacting the production level, so the social planner's problem is one of moral hazard.

We present hereafter two different possibilities to model principal and agent criterias; even though we consider only one of them (section 3.6.1.2) for the numerical resolution.

### 3.6.1.1 Maximization of social welfare measured by total surplus

In disregard of carbon emissions, we consider that agent's problem (the economy) is to minimize the costs and maximize the social welfare measured by the total surplus; the sum of consumer's surplus and producer's surplus, which is achieved at the equilibrium. Recall that consumers surplus is the utility they gain from acquiring the good at a price lower than their willingness to pay, while producers surplus is their benefit when the market price is higher than their willingness to sell, which is the same as the profit in the case where producers do not sell at a loss. The figure 3.1 provides a representation of these quantities for affine prices.

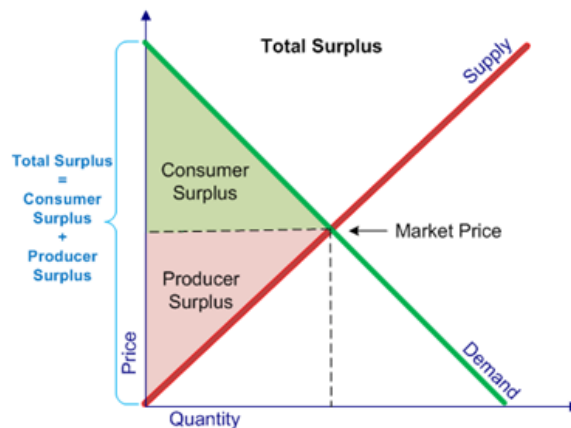


Figure 3.1: Total surplus represented as the sum of the green area (consumers' surplus) and the red area (producers' surplus).

If we assume that the willingness to pay a certain demand level is expressed in terms of the inverse demand function  $p(z) := \theta_0 - \theta z$ , with  $\theta_0, \theta > 0$ , we can compute consumers' surplus when production capacity is roughly

speaking, proportional or equal to the total capital  $K_t$ , namely

$$\begin{aligned} CS_t &:= \int_0^{K_t} p(D_t - q) dq - p_t K_t, \\ &= \frac{\theta}{2} (K_t - D_t)^2 - \frac{\theta}{2} D_t^2 + (\theta_0 - p_t) K_t, \end{aligned} \quad (3.6.3)$$

where  $p_t$  is the actual price. On the other hand, the profit (or equivalently producers' surplus) is

$$\Pi_t := (p_t - c) K_t - f, \quad (3.6.4)$$

where  $c$  is a proportional cost and  $f$  is a fixed cost. Thus we have:

$$\begin{aligned} CS_t + \Pi_t &= \frac{\theta}{2} (K_t - D_t)^2 - \frac{\theta}{2} D_t^2 + (\theta_0 - c) K_t - f, \\ &\approx \frac{\theta}{2} (K_t - D_t)^2 - \frac{\theta}{2} D_t^2 - f, \text{ for } \theta_0 - c \approx 0. \end{aligned} \quad (3.6.5)$$

This justifies a simplified criterion for the agent (the economy), to be maximized, in the business as usual situation of the form

$$\tilde{J}^A(0, \mathbb{P}^\alpha) := \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_A \left( \int_0^T \left( \frac{\theta}{2} (K_t - D_t)^2 - \frac{\theta}{2} D_t^2 - g(i_t) - c(a_t) \right) dt \right) \right], \quad (3.6.6)$$

for some utility function  $U_A$ , and two cost functions  $g$  and  $c$  for the efforts  $(i_t)_{t \in [0, T]}$  and  $(a_t)_{t \in [0, T]}$ . Remark here that there is no incentive to make an effort towards carbon reduction and we expect to have  $a_t = 0$  and  $c(a_t) = 0$ , for  $t \in [0, T]$ .

The social planner (the principal) aims at finding the optimal subsidy to incentivize the agent to minimize the induced emissions, which is a trade-off between total welfare and carbon emissions.

For an admissible contract  $\xi$  (a subsidy), agent's objective function becomes

$$\tilde{J}^A(\xi, \mathbb{P}^\alpha) := \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_A \left( \xi + \int_0^T \left( \frac{\theta}{2} (K_t - D_t)^2 - \frac{\theta}{2} D_t^2 - g(i_t) - c(a_t) \right) dt \right) \right], \quad (3.6.7)$$

and the social planner's objective function is the following

$$J^P(\xi, \mathbb{P}^\alpha) := \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_P \left( -\xi + \int_0^T \ell(E_t) dt \right) \right], \quad (3.6.8)$$

where  $\ell(E_t)$  captures a constraint on the emissions (a target) or damages. Using (3.6.7) and (3.6.8), and an appropriate set of parameters, we can study the different cases introduced in section 3.2.3; the business as usual and the participation constraint, then the first best and second best.

Remark that the incentive to pollute less is through a penalty on the observed rate of emissions rather than the total emissions which can be computed a-posteriori. Remark also that the demand process  $D$  is exogenous (uncontrolled)

and can be expressed explicitly as

$$D_t = d_0 e^{\mu t} + \int_0^t e^{\mu(t-s)} \sigma dW_s^D, \quad (3.6.9)$$

and so for  $t \geq 0$ ,  $D_t \stackrel{law}{=} \mathcal{N} \left( d_0 e^{\mu t}, \frac{\sigma^2}{2\mu} [e^{2\mu t} - 1] \right)$ , i.e., the demand is expected to increase in average, and to have a higher volatility.

### 3.6.1.2 Minimizing the difference between targeted and realised GDP

The problem can be restated slightly differently by interpreting  $(D_t, K_t, E_t)_{t \in [0, T]}$  as the targeted GDP, the actual realised GDP, and the emissions rate. This has the advantage of both simplifying the calibration and the computations. This approach does not have the same economical interpretation as in 3.6.1 and considers the economy as a whole. In this case we replace the objective function  $\tilde{J}^\Lambda$  defined in (3.6.1) by the following  $J^\Lambda$

$$J^\Lambda(\xi, \mathbb{P}^\alpha) := \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_\Lambda \left( \xi + \int_0^T \left( -\frac{\theta}{2} (K_t - D_t)^2 - g(i_t) - c(a_t) \right) dt \right) \right], \quad (3.6.10)$$

where the term in  $\theta$  is now interpreted as a penalty for deviating from the targeted GDP. We will focus on this presentation in the following sections.

## 3.6.2 Formulation of the problem

### 3.6.2.1 State variable and controlled equation

As mentioned in section 3.6.1, we fixe a finite time horizon  $T > 0$ , and assume that the subsidy (the contract) is a lump sum payment  $\xi$  (instead of a continuous payment), and without constraints on the state variables, so that we can use the results of sections 3.3, 3.4 and 3.5.

Using the same notations as in section 3.2, we have  $d = 3$ , so  $\Omega := \mathcal{C}([0, T], \mathbb{R}^3)$ , and  $\mathbb{P}^0$  the reference probability measure under which the canonical process  $X$  satisfies  $\mathbb{P}^0 \circ (X_0)^{-1} = \delta_{x_0}$  for some  $x_0 \in \mathbb{R}^3$  and there exists a 3-dimensional  $(\mathbb{P}^0, \mathbb{F})$ -Brownian motion  $W^{\mathbb{P}^0}$  such that

$$X_t - X_0 = \int_0^t A_1 X_s ds + \int_0^t \sigma dW_s^{\mathbb{P}^0}, \text{ for } t \in [0, T], \quad (3.6.11)$$

with the notations  $X_t := \begin{pmatrix} D_t \\ K_t \\ E_t \end{pmatrix}$  and  $W_t^{\mathbb{P}^0} := \begin{pmatrix} W_t^{D, \mathbb{P}^0} \\ W_t^{K, \mathbb{P}^0} \\ W_t^{E, \mathbb{P}^0} \end{pmatrix}$ ,  $A_1 := \begin{pmatrix} \mu & 0 & 0 \\ 0 & -\delta & 0 \\ 0 & -\eta\delta & \lambda \end{pmatrix}$ , and  $\sigma := \begin{pmatrix} \sigma^D & 0 & 0 \\ 0 & \beta & 0 \\ 0 & \eta\beta & \varphi \end{pmatrix}$ .

The existence (and uniqueness) of the reference probability measure  $\mathbb{P}^0$  holds since the coefficients of SDE (3.6.11) are linear which implies the existence of a unique strong solution. The set of controls  $\mathcal{U}$  is defined with the same

conditions as in Definition 3.2.1; a subset of the  $\mathbb{F}$ -predictable  $\mathbb{R}^2$  valued processes  $\alpha_t := \begin{pmatrix} a_t \\ i_t \end{pmatrix}$  satisfying

$$\mathbb{E}^{\mathbb{P}^0} \left[ \mathcal{E} \left( \int_0^T \sigma^{-1} B_1 \alpha_s \cdot dW_s^{\mathbb{P}^0} \right) \right] = 1, \quad (3.6.12)$$

where  $B_1 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ -1 & \eta \end{pmatrix}$  a constant matrix. Therefore  $\frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} := \mathcal{E} \left( \int_0^T \sigma^{-1} B_1 \alpha_s \cdot dW_s^{\mathbb{P}^0} \right)$  is well defined and we have under  $\mathbb{P}^\alpha$

$$X_t - X_0 = \int_0^t (A_1 X_u + B_1 \alpha_u) du + \sigma W_t^\alpha, \text{ for } t \geq 0. \quad (3.6.13)$$

with

$$W_t^\alpha = W_t^0 - \int_0^t \sigma^{-1} B_1 \alpha_s ds, \text{ for } t \in [0, T], \quad (3.6.14)$$

is a  $(\mathbb{P}^\alpha, \mathbb{F})$ -Brownian motion.

### 3.6.2.2 Principal and agent objective functions

Recall that agent is the economy whose objective is maximize the total welfare. For some given contract  $\xi \in \mathcal{C}$ , a subset of the  $\mathcal{F}_T$  measurable random variables, and  $\mathbb{P}^\alpha \in \mathcal{P}$  a probability measure induced by some admissible control (in  $\mathcal{U}$ ), agent's objective function is defined as in Section 3.6.1.2

$$\begin{aligned} J^A(\xi, \mathbb{P}^\alpha) &:= \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_A \left( \xi - \frac{1}{2} \int_0^T \left( \theta (K_t - D_t)^2 + C_2 \alpha_t \cdot \alpha_t \right) dt \right) \right], \\ &= \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_A \left( \xi - \int_0^T c^A(\alpha_t, X_t) dt \right) \right], \end{aligned} \quad \text{for } (\xi, \mathbb{P}^\alpha) \in \mathcal{C} \times \mathcal{P}, \quad (3.6.10)$$

where  $c^A(\alpha, x) := \frac{1}{2} C_2 \alpha \cdot \alpha + \frac{1}{2} M_2 x \cdot x$ , and  $C_2 := \begin{pmatrix} \frac{1}{g} & 0 \\ 0 & \frac{1}{c} \end{pmatrix}$  with  $g, c > 0$ ; positive costs for both production (investment) and deviation from the target  $\frac{1}{c}$ , or reduction of emissions rate  $\frac{1}{g}$ . On the other hand

$$M_2 := \theta \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (3.6.15)$$

so that  $M_2 x \cdot x = \theta(k - d)^2$ . Agent's value function is then defined as

$$V^A(\xi) := \sup_{\mathbb{P}^\alpha \in \mathcal{P}} J^A(\xi, \mathbb{P}^\alpha), \quad (3.6.16)$$

and he accepts a contract only if it satisfies  $V^A(\xi) \geq U_A(\mathcal{R})$ , with  $\mathcal{R}$  the certainty equivalent of the participation constraint, given by the solution of the Business as usual case.

The principal optimizes over the set of contracts to find a trade-off between total welfare and carbon emissions. For an admissible contract  $\xi$  (a subsidy), her objective function is defined for  $(\xi, \mathbb{P}^\alpha) \in \mathcal{C} \times \mathcal{P}$  as in Section 3.6.1

$$\begin{aligned} J^P(\xi, \mathbb{P}^\alpha) &:= \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_P \left( -\xi + \int_0^T \ell(E_t) dt \right) \right], \\ &= \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_P \left( -\xi - \int_0^T \frac{1}{2} L_2 X_t \cdot X_t dt \right) \right], \end{aligned} \quad (3.6.8)$$

with

$$L_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & l_2 \end{pmatrix} \text{ and } l_2 > 0. \quad (3.6.17)$$

where  $l_2$  is a penalty term to lower the emissions rate.

We can see that our problem fits into the Linear-Quadratic framework, and we fix then the sets of admissible controls and admissible contracts  $\mathcal{U}$  (or equivalently  $\mathcal{P}$ ) and  $\mathcal{C}$  and  $\Xi$  as in definitions 3.2.1 and 3.2.2. Furthermore, we make the assumption 3.2.4. This allows us to use the results established in sections 3.3, 3.4 and 3.5 to find the participation constraint from the business as usual scenario

$$U_A(\mathcal{R}) := V^A(0) = \sup_{\mathbb{P}^\alpha \in \mathcal{P}} J^A(0, \mathbb{P}^\alpha), \quad (3.6.18)$$

and to solve the first best contracting which we recall

$$V_{\text{FB}}^P := \sup_{\xi \in \mathcal{C}} \sup_{\mathbb{P}^\alpha \in \mathcal{P}} J^P(\xi, \mathbb{P}^\alpha) \text{ such that } J^A(\xi, \mathbb{P}^\alpha) \geq U_A(\mathcal{R}), \quad (3.6.19)$$

and the second best :

$$V_{\text{SB}}^P := \sup_{\xi \in \Xi} \sup_{\mathbb{P}^\alpha \in \mathcal{P}^*(\xi)} J^P(\xi, \mathbb{P}^\alpha). \quad (3.6.20)$$

We present in the sequel the results in this problem as direct applications of the previous propositions, before moving to the numerical resolution and the interpretations.

**Remark 3.6.1.** *As mentioned earlier, the demand is assumed to be exogenous, and remains the same under any policy. It can be expressed explicitly as*

$$D_t = d_0 e^{\mu t} + \int_0^t e^{\mu(t-s)} \sigma dW_s^D, \quad (3.6.9)$$

and for  $t \geq 0$ ,  $D_t \stackrel{\text{law}}{=} \mathcal{N} \left( d_0 e^{\mu t}, \frac{\sigma^2}{2\mu} [e^{2\mu t} - 1] \right)$ .

### 3.6.2.3 The Business as usual case

Following the same methodology as in section 3.3, we solve the problem of the economy with disregard to environmental issues. The problem reduces then to the maximization of total welfare, and has one less state variable.

We recall agent's value function whenever  $\xi = 0$ ; to see the reduction in dimension which follows for the the pair of controls  $\alpha = (a, i)$  from the positivity of the cost  $\frac{1}{2g}(a_s)^2$  leading to  $a_s = 0$  for  $s \in [0, T]$ , and the fact that  $M_2 x \cdot x = \theta(k - d)^2$ , independent of the emissions rate  $E$ .

$$\begin{aligned}
V^A(0) &= \sup_{\mathbb{P}^\alpha \in \mathcal{P}} J^A(0, \mathbb{P}^\alpha), \\
&= \sup_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}^{\mathbb{P}^\alpha} \left[ -e^r \int_0^T c^A(\alpha_s, X_s) ds \right], \\
&= \sup_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}^{\mathbb{P}^\alpha} \left[ -e^r \int_0^T \left( \frac{1}{2} C_2 \alpha_s \cdot \alpha_s + \frac{1}{2} M_2 X_s \cdot X_s \right) ds \right], \\
&= \sup_{\mathbb{P}^{(0,i)} \in \mathcal{P}} \mathbb{E}^{\mathbb{P}^{(0,i)}} \left[ -e^r \int_0^T \left( \frac{1}{2c} i_s \cdot i_s + \frac{\theta}{2} (K_s - D_s)^2 \right) ds \right]
\end{aligned} \tag{3.2.14}$$

From the previous computations, the problem can be reduced to a matrix Riccati equation of the form

$$\begin{cases} \Gamma_T = 0 \text{ and } R_T = 0, \\ \dot{\Gamma}_t + A_1^T \Gamma_t + \Gamma_t A_1 + \Gamma_t Q_{BU} \Gamma_t - M_2 = 0, \\ \dot{R}_t + \frac{1}{2} \sigma^2 : \Gamma_t = 0, \end{cases} \tag{3.6.21}$$

where we recall

$$Q_{BU} := B_0 - r\sigma^2. \tag{3.6.22}$$

We use then the verification theorem 3.3.2 to characterize agent's optimal control in the business as usual  $\mathbb{P}^{\alpha^{\mathcal{R}}}$ , together with the participation constraint  $U_A(\mathcal{R})$ .

**Proposition 3.6.2.** (i) *Assume that the Riccati equation (3.6.21) is wellposed, then we have*

$$V^A(0) \leq -e^{-r(\frac{1}{2}\Gamma_0 x_0 \cdot x_0 + R_0)}. \tag{3.6.23}$$

(ii) *Assume further that the induced feedback control  $\alpha_{BU}^*(t, x) := C_2^{-1} B_1^T \Gamma_t x$  is admissible, i.e.,  $(\alpha_{BU}^*(t, X_t))_{t \in [0, T]} \in \mathcal{U}$ , then*

$$V^A(0) = -e^{-r(\frac{1}{2}\Gamma_0 x_0 \cdot x_0 + R_0)}, \tag{3.6.24}$$

i.e.,

$$\mathcal{R} = \frac{1}{2} \Gamma_0 x_0 \cdot x_0 + R_0, \tag{3.6.25}$$

and the maximizer of the problem (3.6.18) is the probability measure  $\mathbb{P}^{\alpha^{\mathcal{R}}}$  induced by the feedback control  $\alpha_{BU}^*$ , i.e., Agent's control in the absence of incentive is characterized by the probability measure  $\mathbb{P}^{\alpha^{\mathcal{R}}}$  under which

$$X_t - x_0 = \int_0^t \left( A_1 + C_2^{-1} B_1^T \Gamma_s \right) X_s ds + \sigma W_t^{\alpha^{\mathcal{R}}}. \tag{3.6.26}$$

As mentioned in Remark 3.3.4, the participation constraint  $\mathcal{R}$  depends on the initial condition  $x_0$ , and from

Principal perspective, we have

$$V_{\text{BU}}^{\text{P}} = J^{\text{P}}(0, \mathbb{P}^{\alpha^{\mathcal{R}}}) = \mathbb{E}^{\mathbb{P}^{\alpha^{\mathcal{R}}}} \left[ -e^{-\rho \int_0^T \ell(X_s) ds} \right]. \quad (3.6.27)$$

### 3.6.2.4 Solving the first best case

We follow the steps of section 3.4, and we derive the matrix Riccati system

$$\begin{cases} \Gamma_T = 0 \text{ and } R_T = 0, \\ \dot{\Gamma}_t + A_1^{\top} \Gamma_t + \Gamma_t A_1 + \Gamma_t Q_{\text{FB}} \Gamma_t - (L_2 + M_2) = 0, \\ \dot{R}_t + \frac{1}{2} \sigma^2 : \Gamma_t = 0. \end{cases} \quad (3.6.28)$$

Before moving to the verification, we recall that  $Q_{\text{FB}} := B_0 - \rho \sigma^2$ , with

$$B_0 := B_1 C_2^{-1} B_1^{\top} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & c & \eta c \\ 0 & \eta c & g + (\eta)^2 c \end{pmatrix},$$

so  $Q_{\text{FB}}$  does not satisfy the condition of Proposition 3.4.2 so there is (a priori) no guarantee for the existence of a solution to the Riccati system (3.6.28). We apply Proposition 3.4.2 to exhibit the optimal contract and the optimal control.

**Proposition 3.6.3.** (i) *Assume that the Riccati equation (3.6.28) is wellposed, then we have*

$$\bar{V}_0 \leq -e^{-\rho(\frac{1}{2}\Gamma_0 x \cdot x + R_0)}. \quad (3.6.29)$$

(ii) *Assume further that the induced feedback control is admissible, i.e.,*

$$(\alpha_{\text{FB}}^*(t, X_t))_{t \in [0, T]} := (C_2^{-1} B_1^{\top} \Gamma_t X_t)_{t \in [0, T]} \in \mathcal{U}, \quad (3.6.30)$$

then

$$\bar{V}_0 = -e^{-\rho(\frac{1}{2}\Gamma_0 x \cdot x + R_0)}, \quad (3.6.31)$$

(iii) *Assume further that the induced optimal contract is admissible;  $\xi_{\text{FB}}^* \in \mathcal{C}$ , with*

$$\xi_{\text{FB}}^* = \mathcal{R} - \frac{\rho}{\rho + r} \left( \frac{1}{2} \Gamma_0 x_0 \cdot x_0 + R_0 \right) + \frac{\rho}{\rho + r} \int_0^T \left( \ell(X_t) + \frac{r}{\rho} c^A (\alpha_{\text{FB}}^*(t, X_t), X_t) \right) dt, \quad (3.6.32)$$

then

$$V_{\text{FB}}^{\text{P}} = U_{\text{P}} \left( -\mathcal{R} + \frac{1}{2} \Gamma_0 x_0 \cdot x_0 + R_0 \right),$$

and  $\alpha_{\text{FB}}^*(t, x) = C_2^{-1} B_1^{\top} D u_{\text{FB}}(t, x)$ , where  $u_{\text{FB}}(t, x)$  is the certainty equivalent dynamic function

$$u_{\text{FB}}(t, x) := -\mathcal{R} + \frac{1}{2} \Gamma_t x \cdot x + R_t.$$

### 3.6.2.5 Solving the second best case

Now we follow the methodology in section 3.5 to derive the matrix Riccati equation

$$\begin{cases} \Gamma_T = 0 \text{ and } R_T = 0, \\ \dot{\Gamma}_t + A_1^T \Gamma_t + \Gamma_t A_1 + \Gamma_t Q_{SB} \Gamma_t - (L_2 + M_2) = 0, \\ \dot{R}_t + \frac{1}{2} \sigma^2 : \Gamma_t = 0, \end{cases} \quad (3.6.33)$$

where we recall

$$Q_{SB} := (B_0 + \rho \sigma^2) (B_0 + (\rho + r) \sigma^2)^{-1} (B_0 + \rho \sigma^2) - \rho \sigma^2. \quad (3.6.34)$$

We use then the verification theorem 3.5.4 to obtain the optimal contract and value function, and the corresponding optimal response.

**Proposition 3.6.4.** (i) *Assume that the Riccati equation (3.6.33) is wellposed, then we have*

$$V_{SB}^P \leq U_P \left( -\mathcal{R} + \frac{1}{2} \Gamma_0 x \cdot x + R_0 \right), \quad (3.6.35)$$

(ii) *Assume further that the induced feedback control (the maximizer of the Hamiltonian) is admissible, i.e.,  $(z^*(t, X_t))_{t \in [0, T]} \in \mathcal{V}$ , with*

$$z^*(t, x) := (B_0 + (\rho + r) \sigma^2)^{-1} (B_0 + \rho \sigma^2) \Gamma_t x, \quad (3.6.36)$$

then

$$V_{SB}^P = U_P \left( -\mathcal{R} + \frac{1}{2} \Gamma_0 x \cdot x + P_0 \cdot x + R_0 \right), \quad (3.6.37)$$

and the optimal contract  $\xi_{SB}^*$  is given by

$$\xi_{SB}^* := \mathcal{R} + \int_0^T z^*(s, X_s) \cdot dX_s - \int_0^T H(X_s, z^*(s, X_s)) ds + \frac{r}{2} \int_0^T |\sigma z^*(s, X_s)|^2 ds, \quad (3.6.38)$$

or equivalently (under  $\mathbb{P}^{\hat{\alpha}}$ ) as

$$\xi_{SB}^* = \mathcal{R} + \int_0^T z^*(s, X_s) \cdot \sigma dW_s^{\hat{\alpha}} + \int_0^T c^A(\hat{\alpha}(z^*(s, X_s)), X_s) ds + \frac{r}{2} \int_0^T |\sigma z^*(s, X_s)|^2 ds, \quad (3.6.39)$$

while Agent's response is given by

$$\alpha_{SB}^*(t, X_t) := \hat{\alpha}(z^*(t, X_t)) = C_2^{-1} B_1^T z^*(t, X_t). \quad (3.6.40)$$

or equivalently

$$\alpha_{SB}^*(t, x) = C_2^{-1} B_1^T (B_0 + (\rho + r) \sigma^2)^{-1} (B_0 + \rho \sigma^2) Du_{SB}(t, x). \quad (3.6.41)$$



and

$$z^*(t, x) = \left( B_0 + (\rho + r) \sigma^2 \right)^{-1} \left( B_0 + \rho \sigma^2 \right) D u_{SB}(t, x), \quad (3.6.42)$$

where the certainty equivalent dynamic function  $u_{SB}$  is defined as

$$u_{SB}(t, x) := -\mathcal{R} + \frac{1}{2} \Gamma_t x \cdot x + R_t. \quad (3.6.43)$$

### 3.6.3 Numerical experiments and possible interpretations

Before diving into the numerical resolution, we would like to mention that the choice of the title of this section has the purpose of warning the reader that up to the day of redaction of these notes, the uncertainties in the choice of our model and some of its parameters might lead to results hard to explain economically (notably the order of magnitude of the optimal first and second best contracts).

#### 3.6.3.1 Calibration

We take the French economy as an example in our calibration. Our data<sup>1</sup> is the French GDP per capita (per person) in  $[10^6\$]$  and Greenhouse gas emissions per capita in  $[10^3\text{Tons of CO}_2][\text{Year}]^{-1}$ . We list below the parameters and the choice or calibration methodology for each of them, and we provide then a summary in tables 3.6.3.1, 3.6.3.1, 3.6.3.1 and 3.6.3.1

The time horizon (or maturity of the subsidy) is fixed to  $T = 5[\text{Years}]$ , and we take the risk aversion parameters  $r = \rho = \frac{1}{2}[10^6\$]^{-1}$ , scaled to match the unit of the monetary values, and the uncertainties (the volatilities) are fixed at  $\sigma^D = \varphi = \beta = 0.1$ .

	Value	Unit
T	5	[Year]
r	0.5	$[10^6\$]^{-1}$
$\rho$	0.5	$[10^6\$]^{-1}$
$\sigma^D$	0.0043	$[10^6\$][\text{Year}]^{-2}$
$\beta$	0.0043	$[10^6\$][\text{Year}]^{-2}$
$\varphi$	0.000457	$[10^3\text{Tons of CO}_2][\text{Year}]^{-3}$

**The capital and demand** (i) The demand  $D_t$  is calibrated as the French GDP per capita starting from  $d_0 = 0.043 \times [10^6\$]$ , and growing in expectation by an (optimistic) factor of  $\mu = 1.2\%$  per year.

(ii) The capital  $K_t$  starts at  $k_0 = d_0 = 0.043[10^6\$]$ , and its depreciation (the loss of capital) per year is given by  $\delta = 3.0\%$ . The control in investment  $i$  (to be computed) is expressed as a yield, in  $[10^6\$][\text{Year}]^{-2}$ .

<sup>1</sup>From [www.worldbank.org](http://www.worldbank.org), yearly for the time period 1960 – 2014.

	Value	Unit
$d_0$	0.043	$[10^6\$]$
$k_0$	0.043	$[10^6\$]$
$\mu$	0.012	$[\text{Year}]^{-1}$
$\delta$	0.03	$[\text{Year}]^{-1}$
$i$	control	$[10^6\$][\text{Year}]^{-1}$

**The emissions rate** (iii) The emissions rate  $E_t$  starts from  $0.00457[10^3\text{Tons of CO}_2][\text{Year}]^{-1}$ , and the emissions per capital is calibrated as  $\eta \approx \frac{\Delta E_t}{\Delta D_t} \approx 0.53$ , i.e., 53% of investments are non green. We assume that in the absence of intervention  $a = 0$ , the emissions rate increases by  $\lambda = 0.5\%$ . The control to reduce carbon emissions  $a$  is expressed in  $[10^3\text{Tons of CO}_2][\text{Year}]^{-2}$ .

	Value	Unit
$e_0$	0.00457	$[10^3\text{Tons of CO}_2][\text{Year}]^{-1}$
$\eta$	0.530	$[10^3\text{Tons of CO}_2][\text{Year}]^{-1}[10^6\$]^{-1}$
$\lambda$	0.005	$[\text{Year}]^{-1}$
$a$	control	$[10^3\text{Tons of CO}_2][\text{Year}]^{-2}$

**Costs** (iv) The economy (the Agent) has a cost for investment  $\frac{1}{c}$  in and a cost for carbon reduction  $\frac{1}{g}$ . He also has a term in  $\theta$ ; which is a deviation from plan penalty.

(v) The social planner (the Principal) has another penalty parameter  $l_2$  for the excess in carbon emissions.

	Value	Unit
$g$	5	$[10^6\$]^{-1}[10^3\text{Tons of CO}_2]^2[\text{Year}]^{-3}$
$c$	10	$[10^6\$][\text{Year}]^{-1}$
$\theta$	2	$[10^6\$]^{-1}[\text{Year}]^{-1}$
$l_2$	1	$[10^6\$][\text{Year}][10^3\text{Tons of CO}_2]^{-2}$

### 3.6.3.2 Comparison between different setups

**State variables under different scenarios** We solve the problem for each of the three situations (Business as usual, First best, and second best). A first sanity check is to see whether the demand process has the same dynamics under the different scenarios, which is confirmed by the superposition of all the lines in Figure 3.2 (the legend might even seem superfluous, but we keep it for clarity).

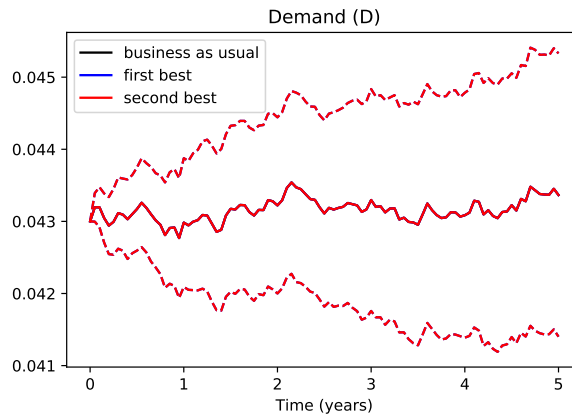


Figure 3.2: Evolution of the demand process

We compare then the evolution of the controlled variables; the GDP  $K$  and the emissions rate  $E$  under each policy in Figure 3.3.

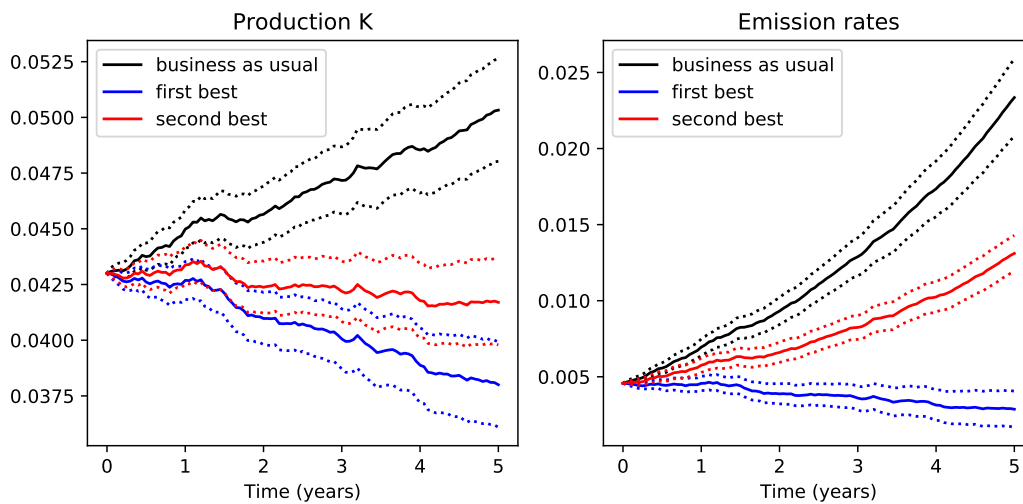


Figure 3.3: Evolution of controlled state variables with 95% confidence interval

**Empirical distributions of first and second best contracts** The figure 3.4 illustrates the distributions of the first best and second best contracts.

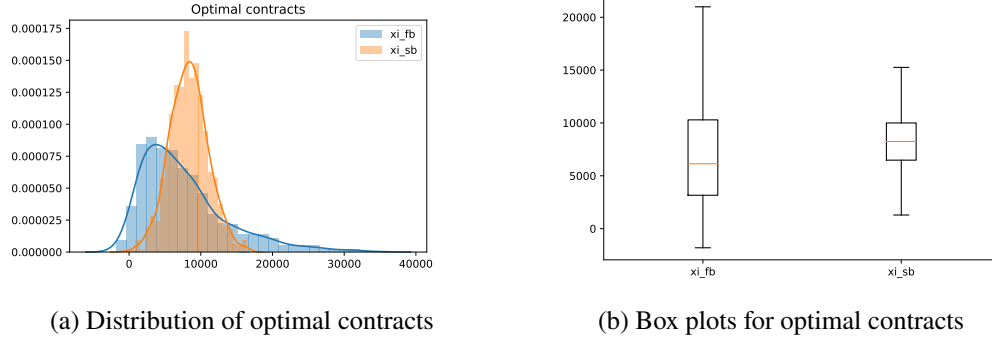


Figure 3.4: Optimal contracts

### 3.6.3.3 Analysis of the optimal control and the optimal contract : First best

It is convenient to use the certainty equivalent function  $u_{FB}$  which satisfies  $u_{FB}(t, x) = U_p^{-1} \circ V_{FB}^P(t, x)$  for the interpretations of the optimal control and contract, instead of the abstract terms  $\Gamma$  and  $R$ , solutions of the matrix Riccati equation. From Proposition 3.6.3, we can write the controlled equation under  $\mathbb{P}^{\alpha_{FB}^*}$  as

$$dX_t = (A_1 X_t + B_1 \alpha_{FB}^*(t, X_t)) dt + \sigma dW_t^{\alpha_{FB}^*},$$

where

$$\alpha_{FB}^*(t, x) := C_2^{-1} B_1^\top D u_{FB}^P(t, x) = \begin{pmatrix} 0 & 0 & -g \\ 0 & c & \eta c \end{pmatrix} D u_{FB}^P(t, x) = \begin{pmatrix} -g \partial_E u_{FB}^P(t, x) \\ c(\partial_K u_{FB}^P(t, x) + \eta \partial_E u_{FB}^P(t, x)) \end{pmatrix} = \begin{pmatrix} a_{FB}^*(t, x) \\ i_{FB}^*(t, x) \end{pmatrix}.$$

Recall that  $\alpha_{FB}$  is the optimal feedback effort (in the first best) to reduce the emissions rates. It is inversely proportional to its cost  $\frac{1}{g}$ , and proportional to  $\partial_E u_{FB}^P$  which should be negative since an increase in  $E$  implies an increase in the actual emissions which are negative in Principal's objective function. As for  $i_{FB}^*$ , it represents the construction (or investment effort) and is inversely proportional to its cost  $\frac{1}{g}$  and proportional to  $(\partial_K u_{FB}^P + \eta \partial_E u_{FB}^P)$ . i.e., we construct when  $\partial_K u_{FB}^P + \eta \partial_E u_{FB}^P > 0$  and destroy otherwise, and since as explained earlier  $\partial_E u_{FB}^P$  it can be interpreted as a constraint; The construction is done not only if  $\partial_K u_{FB}^P > 0$  (we need more power), but also under the condition that the need for this power is higher than its environmental impact  $\partial_K u_{FB}^P > -\eta \partial_E u_{FB}^P$  (where the right hand side of the previous inequality should be positive since as explained above we want the emissions to be low). Thus the appearance of the amplifying factor  $\eta$ ; the higher  $\eta$ , the more constraining this term becomes.

To give a more detailed account on the evolution of state variables under the optimal policy, we recall that the demand process  $D$  is exogenous, and is a gaussian process with increasing average and volatility. So it suffices to interpret the behavior of the production  $K$  and the emission rates  $E$  under the optimal policy.

Recall that

$$\begin{aligned} dK_t &= (i_t^* - \delta K_t) dt + \beta dB_t^1, \\ &= \left( c \partial_K u_{FB}^P(t, x) + c \eta \partial_E u_{FB}^P(t, X_t) - \delta K_t \right) dt + \beta dB_t^1, \\ &= \left( \underbrace{(c \partial_K + c \eta \partial_E) u_{FB}^P(t, X_t)}_{:= \mathcal{L}_K} - \delta K_t \right) dt + \beta dB_t^1, \end{aligned}$$

and

$$\begin{aligned}
dE_t &= \eta dK_t + \lambda E_t dt - a_t dt + \varphi dB_t^2, \\
&= \eta \left( i_t^* - \frac{a_t^*}{\eta} \right) dt + (\lambda E_t - \eta \delta K_t) dt + \varphi dB_t^2 + \eta \beta dB_t^1, \\
&= \eta \left( (c \partial_K + c \eta \partial_E) u_{FB}^P(t, X_t) + \frac{g}{\eta} \partial_E u_{FB}^P(t, X_t) \right) dt + (\lambda E_t - \eta \delta K_t) dt + \eta \beta dB_t^1 + \varphi dB_t^2, \\
&= \left( \underbrace{(\eta c \partial_K + (c \eta^2 + g) \partial_E)}_{:= \mathcal{L}_E} u_{FB}^P(t, X_t) \right) dt + (\lambda E_t - \eta \delta K_t) dt + \eta \beta dB_t^1 + \varphi dB_t^2
\end{aligned}$$

So by defining the change of variable

$$\hat{X}_t := \left( B_0 + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) X_t$$

we obtain that  $\mathcal{L}_K u_{FB}^P(t, X_t) = \partial_{\hat{K}} u_{FB}^P(t, \hat{X}_t)$  and  $\mathcal{L}_E u_{FB}^P(t, X_t) = \partial_{\hat{E}} u_{FB}^P(t, \hat{X}_t)$ , so that the controls can be expressed as directional derivatives with respect to the new variables  $\hat{K}$  and  $\hat{E}$

$$\begin{aligned}
dK_t &= \left( \partial_{\hat{K}} u_{FB}^P(t, \hat{X}_t) - \delta K_t \right) dt + \beta dB_t^1, \\
dE_t &= \left( \partial_{\hat{E}} u_{FB}^P(t, \hat{X}_t) + \lambda E_t - \eta \delta K_t \right) dt + \eta \beta dB_t^1 + \varphi dB_t^2,
\end{aligned}$$

where as defined from the change of variable

$$\begin{pmatrix} \hat{K}_t \\ \hat{E}_t \end{pmatrix} = \begin{pmatrix} cK_t + c\eta E_t \\ c\eta K_t + (c\eta^2 + g) E_t \end{pmatrix} \text{ or equivalently } \begin{pmatrix} K_t \\ E_t \end{pmatrix} = \begin{pmatrix} \left( \frac{1}{c} + \frac{\eta^2}{g} \right) \hat{K}_t - \frac{\eta}{g} \hat{E}_t \\ -\frac{\eta}{g} \hat{K}_t + \frac{1}{g} \hat{E}_t \end{pmatrix},$$

which seem to express better the evolution of the system than  $(K, E)$ , but remain ambiguous in terms of economical interpretations.

The optimal contract is given as the sum of multiple terms

$$\xi_{FB}^* = \mathcal{R} - \frac{\rho}{\rho + r} \left( \frac{1}{2} \Gamma_0 x_0 \cdot x_0 + R_0 \right) + \frac{\rho}{\rho + r} \int_0^T \left( \ell(X_t) + \frac{r}{\rho} c^\Lambda (\alpha_{FB}^*(t, X_t), X_t) \right) dt,$$

This contract provides a constant corresponding to the participation constraint, and a term which captures the evolution of the certainty equivalent over time, then compensates Agent for his effort but penalizes him for the costs incurred by Principal.

### 3.6.3.4 Analysis of the optimal control and the optimal contract : Second best

From 3.6.4, we can write the controlled equation under the optimal (recommended) effort  $\mathbb{P}^{\hat{\alpha}}$

$$dX_t = (A_1 X_t + B_1 \alpha_{\text{SB}}^*(t, X_t)) dt + \sigma dW_t^{\alpha_{\text{SB}}^*},$$

with

$$\alpha_{\text{SB}}^*(t, x) = C_2^{-1} B_1^\top \left( B_0 + (\rho + r) \sigma^2 \right)^{-1} \left( B_0 + \rho \sigma^2 \right) Du_{\text{SB}}(t, x).$$

and

$$z^*(t, x) = \left( B_0 + (\rho + r) \sigma^2 \right)^{-1} \left( B_0 + \rho \sigma^2 \right) Du_{\text{SB}}(t, x),$$

as Principal cannot observe the effort this time, she controls it via a proxy; the function  $z^*(t, x)$ . In fact, this function could be interpreted as perturbation of the certainty equivalent by Agent actions (which are unobservable). This is highlighted in the limiting case since when  $r \rightarrow 0$ , we have that  $(B_0 + (\rho + r) \sigma^2)^{-1} (B_0 + \rho \sigma^2) \rightarrow I_4$  and  $Q_{\text{SB}} \rightarrow Q_{\text{FB}}$ . So in the second best, Principal's incentives (the function  $z^*$ ) plays the same role as  $\alpha_{\text{SB}}^*$  in the first best. As for the optimal contract, a complete decomposition provides

$$\xi_{\text{SB}}^* = \mathcal{R} + \int_0^T c^A (\alpha_{\text{SB}}^*(s, X_s)) ds + \int_0^T z^*(s, X_s) \cdot \sigma dW_s^{\alpha_{\text{SB}}^*} + \frac{r}{2} \int_0^T |\sigma z^*(s, X_s)|^2 ds,$$

which corresponds to a constant payment (the participation constraint) plus a compensation for the effort, and a term of risk sharing (the stochastic integral) together with a risk compensation, which is proportional to the risk shared and to Agent's risk aversion. Remark that contrary to the first best case, the optimal contract does not penalize Agent (explicitly) for the costs incurred by Principal, but implicitly through the incentive.

## 3.7 Appendix

### 3.7.1 Wellposedness of matrix Riccati ODEs

The following result is elementary, and we provide it here for completeness.

**Lemma 3.7.1.** *Let  $A \in \mathbb{R}^{d \times d}$ ,  $Q \in \mathcal{S}_{++}^d$  and  $L \in \mathcal{S}_+^d$ . Then the matrix Riccati equation*

$$\begin{cases} \Gamma_T = 0, \\ \dot{\Gamma}_t + A^\top \Gamma_t + \Gamma_t A + \Gamma_t Q \Gamma_t - L = 0, \text{ for } t \in [0, T], \end{cases} \quad (3.7.1)$$

*admits a unique solution  $(\Gamma_t)_{t \in [0, T]}$ , which is symmetric for  $t \in [0, T]$ .*

**Proof.** To prove the existence we recall that by the Radon Lemma (which can be found in Reid (1972)[Chapter 2]) the matrix Riccati equation (3.7.1) is related to the following linear matrix differential system

$$\begin{pmatrix} \dot{x}_t \\ \dot{y}_t \end{pmatrix} = \begin{pmatrix} A & Q \\ L & -A^\top \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} \text{ with } \begin{pmatrix} x_T \\ y_T \end{pmatrix} = \begin{pmatrix} I_d \\ 0 \end{pmatrix}. \quad (3.7.2)$$

The statement of the Radon Lemma is that whenever the unique solution to the linear ODE (3.7.2) is such that  $x_t$  is invertible for  $t \in [0, T]$ , then  $\Gamma_t := y_t x_t^{-1}$  is a solution to the matrix Riccati equation. In fact, we can check in that case the terminal condition  $\Gamma_T = 0$  since  $y_T = 0$ , and by differentiating we have

$$\begin{aligned}\dot{\Gamma}_t &= \dot{y}_t x_t^{-1} + y_t \left( \dot{x}_t^{-1} \right), \\ &= \dot{y}_t x_t^{-1} - y_t x_t^{-1} \dot{x}_t x_t^{-1}, \\ &= (L x_t - A^\top y_t) x_t^{-1} - y_t x_t^{-1} (A x_t + Q y_t) x_t^{-1}, \\ &= L - A^\top \Gamma_t - \Gamma_t A - \Gamma_t Q \Gamma_t,\end{aligned}\tag{3.7.3}$$

which is exactly the Riccati equation (3.7.1). So in order to prove existence we now check that for  $Q \in \mathcal{S}_{++}^d$  and  $L \in \mathcal{S}_+^d$ ,  $x_t$  is invertible for  $t \in [0, T]$ . The proof is by contradiction; remark first that for  $t \in [0, T]$  and  $v \in \mathbb{R}^d$  we have

$$v^\top y_T^\top x_T v - v^\top y_t^\top x_t v = \int_t^T (v^\top x_s^\top L x_s v + v^\top y_s^\top Q y_s v) ds.\tag{3.7.4}$$

Now assume that  $x_t v = 0$  for some  $\mathbb{R}^d \ni v \neq 0$ . Therefore

$$0 = \int_t^T (v^\top x_s^\top L x_s v + v^\top y_s^\top Q y_s v) ds,\tag{3.7.5}$$

and  $y_s v = 0$  for  $s \in [t, T]$  since  $L \geq 0$  and  $Q > 0$ . From the ODE (3.7.2) we have

$$x_T = e^{A(T-t)} x_t + \int_t^T e^{A(T-s)} Q y_s ds,\tag{3.7.6}$$

and so

$$x_T v = e^{A(T-t)} x_t v + \int_t^T e^{A(T-s)} Q y_s v ds = 0,\tag{3.7.7}$$

which contradicts the terminal condition  $x_T = I_d$  in (3.7.2), and we conclude that  $x_t$  is invertible in  $[0, T]$ , and therefore we have existence of at least a solution to (3.7.1).

To prove the uniqueness, we take  $\Gamma$  a solution to (3.7.1) and we compare it to  $y x^{-1}$ , where  $(x, y)^\top$  is the solution to the ODE (3.7.2). We define  $\hat{y}_t := \Gamma_t x_t$  for  $t \in [0, T]$ . Since  $x_t$  is invertible in  $t \in [0, T]$ , it suffices to prove that  $\hat{y}_t = y_t$  for  $t \in [0, T]$  to obtain  $\Gamma = y x^{-1}$  and conclude. By differentiating and using the definition of the ODE (3.7.2), we have that

$$\begin{aligned}\dot{\Gamma}_t x_t &= \left( \dot{\hat{y}}_t x_t^{-1} - \hat{y}_t x_t^{-1} \dot{x}_t x_t^{-1} \right) x_t, \\ &= \dot{\hat{y}}_t - \hat{y}_t x_t^{-1} \dot{x}_t, \\ &= \dot{\hat{y}}_t - \hat{y}_t x_t^{-1} (A x_t + Q y_t), \\ &= \dot{\hat{y}}_t - \Gamma_t (A x_t + Q y_t).\end{aligned}\tag{3.7.8}$$

Then, since  $\Gamma$  is a solution to (3.7.1), we have that  $\hat{y}_T = \Gamma_T x_T = 0$  and

$$\begin{aligned}\dot{\Gamma}_t x_t &= -(A_t^\top \Gamma_t + \Gamma_t A_t + \Gamma_t Q \Gamma_t - L) x_t, \\ &= -A_t^\top \Gamma_t x_t - \Gamma_t A_t x_t - \Gamma_t Q \Gamma_t x_t + L x_t, \\ &= -A_t^\top \hat{y}_t - \Gamma_t A_t x_t - \Gamma_t Q \hat{y}_t + L x_t.\end{aligned}\tag{3.7.9}$$

From (3.7.8) and (3.7.9), we have

$$\dot{\hat{y}}_t - \Gamma_t Q y_t = -A_t^\top \hat{y}_t - \Gamma_t Q \hat{y}_t + L x_t,\tag{3.7.10}$$

Then, from ODE (3.7.2),  $L x_t = \dot{y}_t + A_t^\top y_t$  and so

$$\dot{\hat{y}}_t - \Gamma_t Q y_t = -A_t^\top \hat{y}_t - \Gamma_t Q \hat{y}_t + \dot{y}_t + A_t^\top y_t,\tag{3.7.11}$$

and therefore

$$\dot{\hat{y}}_t - \dot{y}_t = -(A_t^\top + \Gamma_t Q) (\hat{y}_t - y_t).\tag{3.7.12}$$

We can see then that  $\hat{y} - y$  satisfies a linear ODE, with terminal condition  $\hat{y}_T - y_T = 0$ , and so  $\hat{y}_t - y_t = 0$  for  $t \in [0, T]$ , which provides uniqueness.

Finally, remark that for a solution  $(\Gamma_t)_{t \in [0, T]}$  to (3.7.1), by transposition and symmetry, we can see that  $(\Gamma_t^\top)_{t \in [0, T]}$  is also a solution to (3.7.1), and the uniqueness provides  $\Gamma_t = \Gamma_t^\top$  for  $t \in [0, T]$ . Thus the unique solution to (3.7.1) is symmetric, which concludes the proof.  $\square$

### 3.7.2 Proof of Lemma 3.2.3: non degeneracy of agent's problem

The proof is a direct application of Hölder inequality

$$\begin{aligned}|J^A(\xi, \mathbb{P}^\alpha)| &= \mathbb{E}^{\mathbb{P}^\alpha} \left[ \left| U_A \left( \xi - \int_0^T c^A(\alpha_s, X_s) ds \right) \right| \right], \\ &= \mathbb{E}^{\mathbb{P}^0} \left[ \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} e^{-r\xi} e^r \int_0^T c^A(\alpha_s, X_s) ds \right], \\ &\leq \mathbb{E}^{\mathbb{P}^0} \left[ \left( \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} \right)^{p_1} \right]^{\frac{1}{p_1}} \mathbb{E}^{\mathbb{P}^0} \left[ e^{rp_2} \int_0^T c^A(\alpha_s, X_s) ds \right]^{\frac{1}{p_2}} \mathbb{E}^{\mathbb{P}^0} \left[ e^{-rp_3 \xi} \right]^{\frac{1}{p_3}}, \\ &\leq \mathbb{E}^{\mathbb{P}^0} \left[ \left( \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} \right)^{p_1} \right]^{\frac{1}{p_1}} \mathbb{E}^{\mathbb{P}^0} \left[ e^{rp_2} \int_0^T c^A(\alpha_s, X_s) ds \right]^{\frac{1}{p_2}} \mathbb{E}^{\mathbb{P}^0} \left[ e^{\lambda p_3 |\xi|} \right]^{\frac{1}{p_3}} < +\infty,\end{aligned}\tag{3.7.13}$$

since  $\lambda \geq r$ , and from the definition of the sets  $\mathcal{P}$  and  $\mathcal{C}$ .  $\square$

### 3.7.3 Proof of Lemma 3.2.5: non degeneracy of principal's problem

The proof is in two steps, we prove first that

$$\mathbb{E}^0 \left[ e^{(rp_2) \wedge (rp_3) \left( \int_0^T \ell(X_s) ds + g(X_T) \right)} \right] < +\infty,\tag{3.7.14}$$



then we prove that for  $(\xi, \mathbb{P}^\alpha) \in (\mathcal{C} \times \mathcal{P})$ ,

$$|\mathcal{J}^\mathbb{P}(\xi, \mathbb{P}^\alpha)| < +\infty. \quad (3.7.15)$$

### Step 1

Recall the dynamics of the SDE (3.2.1) under  $\mathbb{P}^0$

$$X_t - x_0 = \int_0^t (A_0 + A_1 X_s) ds + \sigma W_t^0, \text{ for } t \in [0, T], \quad (3.7.16)$$

and that

$$\begin{aligned} \mathbb{E}^0 \left[ e^{(\rho p_2) \wedge (\varrho p_3) \left( \int_0^T \ell(X_s) ds + g(X_T) \right)} \right] &= \mathbb{E}^0 \left[ e^{- (\rho p_2) \wedge (\varrho p_3) \left( \int_0^T (L_1 \cdot X_s - \frac{1}{2} L_2 X_s \cdot X_s) ds + G_1 \cdot X_T \right)} \right], \\ &\mathbb{E}^0 \left[ e^{- (\rho p_2) \wedge (\varrho p_3) \left( \int_0^T L_1 \cdot X_s ds + G_1 \cdot X_T \right)} e^{(\rho p_2) \wedge (\varrho p_3) \left( \int_0^T \frac{1}{2} L_2 X_s \cdot X_s ds \right)} \right]. \end{aligned} \quad (3.7.17)$$

The random variable  $\left( \int_0^T L_1 \cdot X_s ds + G_1 \cdot X_T \right)$  is Gaussian, so it has exponential moments of all orders, and therefore

$$\mathbb{E}^0 \left[ e^{\frac{(\rho p_2) \wedge (\varrho p_3)}{2} \int_0^T L_2 X_s \cdot X_s ds} \right] < +\infty \implies \mathbb{E}^0 \left[ e^{(\rho p_2) \wedge (\varrho p_3) \left( \int_0^T \ell(X_s) ds + g(X_T) \right)} \right] < +\infty. \quad (3.7.18)$$

Since  $L_2 \geq 0$ , we have that

$$\mathbb{E}^0 \left[ e^{\frac{(\rho p_2) \wedge (\varrho p_3)}{2} \int_0^T L_2 X_s \cdot X_s ds} \right] \leq \mathbb{E}^0 \left[ e^{\varepsilon \int_0^T |X_s|^2 ds} \right], \quad (3.7.19)$$

where we recall that  $\varepsilon = \frac{(\rho p_2) \wedge (\varrho p_3)}{2} \lambda_{\max}(L_2)$ . We want to prove now that

$$\mathbb{E}^0 \left[ e^{\varepsilon \int_0^T |X_s|^2 ds} \right] < +\infty. \quad (3.7.20)$$

Remark that there is no loss of generality in taking  $x_0 = 0$  and  $A_0 = 0$  to prove the integrability condition, and we will do so to ease notations.

We define then the following PDE in  $[0, T] \times \mathbb{R}_+$  (closely related to the conditional expectation) which will be crucial to prove the integrability result:

$$\begin{cases} V(T, \cdot) = 1, \\ \partial_t V + (2\lambda_{A_1} y + \lambda_\sigma^2 d) DV + 2y\lambda_\sigma^2 D^2 V + \varepsilon y V = 0 \text{ in } [0, T] \times \mathbb{R}_+, \end{cases} \quad (3.7.21)$$

where we recall that  $\lambda_{A_1} = (\lambda_{\max}(A_1 + A_1^\top))^+$  and  $\lambda_\sigma = \sqrt{\lambda_{\max}(\sigma\sigma^\top)}$  and  $d$  is the dimension of the space variable  $X$ .

The following lemma provides an explicit solution to the PDE (3.7.21).

**Lemma 3.7.2.** *Under the assumption 3.2.4, the PDE (3.7.21) admits at least a  $C^{1,2}([0, T] \times \mathbb{R}_+, \mathbb{R})$  solution  $v$  satisfying*

$$v > 0 \text{ and } Dv \geq 0 \text{ and } D^2v \geq 0, \quad (3.7.22)$$

given by

$$v(t, y) := e^{\varphi(t) + \psi(t)y} \text{ for } (t, y) \in [0, T] \times \mathbb{R}_+, \quad (3.7.23)$$

where for  $t \in [0, T]$

$$\varphi(t) := \frac{\lambda_{A_1} \times d}{2}(t - T) - \frac{d}{2} \log \left( \frac{\cos \left( \arctan \left( \frac{-\lambda_{A_1}}{\kappa} \right) - \kappa(T - t) \right)}{\cos \left( \arctan \left( \frac{-\lambda_{A_1}}{\kappa} \right) \right)} \right), \quad (3.7.24)$$

and

$$\psi(t) := -\frac{\lambda_{A_1}}{2\lambda_\sigma^2} - \frac{\kappa}{2\lambda_\sigma^2} \tan \left( \arctan \left( \frac{-\lambda_{A_1}}{\kappa} \right) - \kappa(T - t) \right). \quad (3.7.25)$$

where we recall that  $\kappa = \sqrt{2\varepsilon\lambda_\sigma^2 - \lambda_{A_1}^2}$ .

**Proof.** By straightforward verification, we can see that  $v$  is  $C^{1,2}([0, T] \times \mathbb{R}_+, \mathbb{R})$  (it has even more regularity as composition of smooth functions), satisfies the terminal condition, and is a solution to (3.7.21) by differentiating. Furthermore,  $v > 0$  by definition of the exponential function. From assumption 3.2.4, and the fact that  $\lambda_{A_1} \geq 0$ , we have for  $t \in [0, T]$

$$-\frac{\pi}{2} < \arctan \left( \frac{-\lambda_{A_1}}{\kappa} \right) - \kappa(T - t) \leq \arctan \left( \frac{-\lambda_{A_1}}{\kappa} \right), \quad (3.7.26)$$

and so applying the tangent function (which is increasing)

$$-\infty < \frac{\kappa}{2\lambda_\sigma^2} \tan \left( \arctan \left( \frac{-\lambda_{A_1}}{\kappa} \right) - \kappa(T - t) \right) \leq \frac{-\lambda_{A_1}}{2\lambda_\sigma^2}, \quad (3.7.27)$$

therefore

$$\frac{\lambda_{A_1}}{2\lambda_\sigma^2} \leq -\frac{\kappa}{2\lambda_\sigma^2} \tan \left( \arctan \left( \frac{-\lambda_{A_1}}{\kappa} \right) - \kappa(T - t) \right) < +\infty, \quad (3.7.28)$$

and so  $0 \leq \psi(t) < +\infty$ , providing that  $Dv = v\psi \geq 0$  and  $D^2v = v\psi^2 \geq 0$ , which is enough to conclude the proof. Nevertheless, we present hereafter a more detailed derivation of the solution mainly to justify the choice of assumption 3.2.4.

By looking for a solution of the form  $V(t, y) = e^{\varphi(t) + \psi(t)y}$ , and from the terminal condition we can see that  $\varphi(T) = \psi(T) = 0$ , and by differentiating

$$\begin{cases} \varphi' + d\lambda_\sigma^2\psi = 0, \\ \psi' + 2\lambda_{A_1}\psi + 2\lambda_\sigma^2\psi^2 + \varepsilon = 0. \end{cases} \quad (3.7.29)$$

the function  $\psi$  satisfies an autonomous (Riccati) ODE

$$\psi' + 2\lambda_{A_1}\psi + 2\lambda_\sigma^2\psi^2 + \varepsilon = 0, \text{ and } \psi(T) = 0, \quad (3.7.30)$$

with  $\varepsilon > 0$  and  $2\lambda_\sigma^2 > 0$ , so we cannot solve it by the same method as in Lemma, ?? (for any time horizon), and we use further assumptions instead on the coefficients and the maturity  $T$ . After completion of squares we obtain

$$\psi' + 2\left(\lambda_\sigma\psi + \frac{\lambda_{A_1}}{2\lambda_\sigma}\right)^2 + \varepsilon - \frac{1}{2}\left(\frac{\lambda_{A_1}}{\lambda_\sigma}\right)^2 = 0 \text{ and } \psi(T) = 0. \quad (3.7.31)$$

Assumption 3.2.4 guarantees that  $\varepsilon - \frac{1}{2}\left(\frac{\lambda_{A_1}}{\lambda_\sigma}\right)^2 > 0$ , so  $\psi$  satisfies the same ODE as a the tangent function up to some factor, and by identification we obtain the announced solution

$$\psi(t) = -\frac{\lambda_{A_1}}{2\lambda_\sigma^2} - \frac{\kappa}{2\lambda_\sigma^2} \tan\left(\arctan\left(\frac{-\lambda_{A_1}}{\kappa}\right) - \kappa(T-t)\right). \quad (3.7.32)$$

as soon as

$$T < \frac{1}{\kappa} \left( \frac{\pi}{2} + \arctan\left(\frac{-\lambda_{A_1}}{\kappa}\right) \right). \quad (3.7.33)$$

In fact, for  $f : t \mapsto \left(\arctan\left(\frac{-\lambda_{A_1}}{\kappa}\right) - \kappa(T-t)\right)$  defined for  $t \in [0, T]$  we need to have  $f([0, T]) \subset ]-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi[$  for some  $k \in \mathbb{Z}$  so that  $\psi$  remains finite (otherwise by continuity of  $f$ , the tangent function would explode). Observing that  $f(T) = \arctan\left(\frac{-\lambda_{A_1}}{\kappa}\right) \in ]-\frac{\pi}{2}, 0[$ , we obtain  $k = 0$ , and since  $f$  is increasing, we need to impose that  $f(0) > -\frac{\pi}{2}$  for  $\psi$  to be finite, which explains the condition on  $T$ .

We compute then the function  $\varphi$  defined as

$$\varphi' + d\lambda_\sigma^2\psi = 0, \text{ and } \varphi(T) = 0, \quad (3.7.34)$$

by integration and we obtain the explicit announced solution

$$\varphi(t) = \frac{\lambda_{A_1} \times d}{2}(t-T) - \frac{d}{2} \log\left(\frac{\cos\left(\arctan\left(\frac{-\lambda_{A_1}}{\kappa}\right) - \kappa(T-t)\right)}{\cos\left(\arctan\left(\frac{-\lambda_{A_1}}{\kappa}\right)\right)}\right) \quad (3.7.35)$$

which is well defined and finite on  $[0, T]$  since by assumption  $\left(\arctan\left(\frac{-\lambda_{A_1}}{\kappa}\right) - \kappa(T-t)\right) \in ]-\frac{\pi}{2}, 0[$ , so the cosine is strictly positive.  $\square$

Now let  $v$  be the solution of PDE (3.7.21) given by Lemma 3.7.2, and  $(T_n)_{n \geq 1}$  a localizing sequence defined by

$$T_n := T \wedge \inf\left\{t \geq 0, \int_0^t e^{2\varepsilon \int_0^s |X_u|^2 du} |\sigma^\top X_s|^2 ds \geq n\right\} \text{ for } n \geq 1. \quad (3.7.36)$$

By Itô formula and taking the expectation we obtain that

$$\begin{aligned}
\mathbb{E}^0 \left[ e^{\varepsilon \int_0^{T_n} |X_t^2| dt} v(T_n, X_{T_n}^2) \right] &= v(0, X_0^2) + \mathbb{E}^0 \left[ \int_0^{T_n} 2e^{\varepsilon \int_0^t |X_s^2| ds} \sigma^\top X_t dW_t^0 \right] \\
&\quad + \mathbb{E}^0 \left[ \int_0^{T_n} e^{\varepsilon \int_0^t |X_s^2| ds} \left( \varepsilon |X_t^2| v(t, X_t^2) + \partial_t v(t, X_t^2) \right) dt \right] \\
&\quad + \mathbb{E}^0 \left[ \int_0^{T_n} e^{\varepsilon \int_0^t |X_s^2| ds} \left( Dv(t, X_t^2) \left( (A + A^\top) X_t \cdot X_t + \text{tr}(\sigma \sigma^\top) \right) + 2 |\sigma^\top X_t|^2 D^2 v(t, X_t^2) \right) dt \right]. \quad (3.7.37)
\end{aligned}$$

From the definition of the localizing sequence, we have  $\mathbb{E}^0 \left[ \int_0^{T_n} 2e^{\varepsilon \int_0^t |X_s^2| ds} \sigma^\top X_t dW_t^0 \right] = 0$  for  $n \geq 1$ . Furthermore, from Lemma 3.7.2, we have  $Dv \geq 0$ , and  $D^2v \geq 0$ , and the following estimate holds:

$$\begin{aligned}
\mathbb{E}^0 \left[ \int_0^{T_n} e^{\varepsilon \int_0^t |X_s^2| ds} \left( Dv(t, X_t^2) \left( (A + A^\top) X_t \cdot X_t + \text{tr}(\sigma \sigma^\top) \right) + 2 |\sigma^\top X_t|^2 D^2 v(t, X_t^2) \right) dt \right] \\
\leq \mathbb{E}^0 \left[ \int_0^{T_n} e^{\varepsilon \int_0^t |X_s^2| ds} \left( Dv(t, X_t^2) \left( \lambda_{A_1} |X_t|^2 + \lambda_\sigma^2 d \right) + 2 \lambda_\sigma^2 |X_t|^2 D^2 v(t, X_t^2) \right) dt \right], \quad (3.7.38)
\end{aligned}$$

so

$$\begin{aligned}
\mathbb{E}^0 \left[ e^{\varepsilon \int_0^{T_n} |X_t^2| dt} v(T_n, X_{T_n}^2) \right] &\leq v(0, X_0^2) + \mathbb{E}^0 \left[ \int_0^{T_n} e^{\varepsilon \int_0^t |X_s^2| ds} \left( \varepsilon |X_t^2| v(t, X_t^2) + \partial_t v(t, X_t^2) \right) dt \right] \\
&\quad + \mathbb{E}^0 \left[ \int_0^{T_n} e^{\varepsilon \int_0^t |X_s^2| ds} \left( Dv(t, X_t^2) \left( \lambda_{A_1} |X_t|^2 + \lambda_\sigma^2 d \right) + 2 \lambda_\sigma^2 |X_t|^2 D^2 v(t, X_t^2) \right) dt \right], \\
&= v(0, X_0^2), \quad (3.7.39)
\end{aligned}$$

where we used that  $v$  is a solution to the PDE (3.7.21). Finally, recall that  $v(T, \cdot) = 1$ , and observe that we have the almost sure convergence

$$e^{\varepsilon \int_0^{T_n} |X_t^2| dt} v(T_n, X_{T_n}^2) \rightarrow e^{\varepsilon \int_0^T |X_t^2| dt}, \quad (3.7.40)$$

and we conclude by Fatou lemma (since  $v \geq 0$ ) that

$$\mathbb{E}^0 \left[ \liminf_{n \rightarrow +\infty} e^{\varepsilon \int_0^{T_n} |X_t^2| dt} v(T_n, X_{T_n}^2) \right] \leq v(0, X_0^2). \quad (3.7.41)$$

Therefore

$$\mathbb{E}^0 \left[ e^{\varepsilon \int_0^T |X_t^2| dt} \right] \leq v(0, X_0^2) < +\infty, \quad (3.7.42)$$

and so

$$\mathbb{E}^0 \left[ e^{\frac{(\rho p_2) \wedge (\rho p_3)}{2} \int_0^T L_2 X_s \cdot X_s ds} \right] \leq \mathbb{E}^0 \left[ e^{\varepsilon \int_0^T |X_s|^2 ds} \right] < +\infty, \quad (3.7.43)$$

and

$$\mathbb{E}^0 \left[ e^{(\rho p_2) \wedge (\rho p_3)} \left( \int_0^T \ell(X_s) ds + g(X_T) \right) \right] < +\infty. \quad (3.7.44)$$

## Step 2

Now we prove that Principal's objective function is finite, by a direct application of Hölder inequality. For  $(\xi, \mathbb{P}^\alpha) \in \mathcal{C} \times \mathcal{P}$ , we have

$$\begin{aligned} |J^p(\xi, \mathbb{P}^\alpha)| &= \mathbb{E}^{\mathbb{P}^\alpha} \left[ \left| U_p \left( -\xi + \int_0^T \ell(X_s) ds + g(X_T) \right) \right| \right], \\ &= \mathbb{E}^{\mathbb{P}^0} \left[ \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} e^{\rho \xi} e^{-\rho \left( \int_0^T \ell(X_s) ds + g(X_T) \right)} \right], \\ &\leq \mathbb{E}^{\mathbb{P}^0} \left[ \left( \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} \right)^{p_1} \right]^{\frac{1}{p_1}} \mathbb{E}^{\mathbb{P}^0} \left[ e^{-\rho p_2 \left( \int_0^T \ell(X_s) ds + g(X_T) \right)} \right]^{\frac{1}{p_2}} \mathbb{E}^{\mathbb{P}^0} \left[ e^{\rho p_3 \xi} \right]^{\frac{1}{p_3}}, \\ &\leq \mathbb{E}^{\mathbb{P}^0} \left[ \left( \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} \right)^{p_1} \right]^{\frac{1}{p_1}} \mathbb{E}^{\mathbb{P}^0} \left[ e^{\rho p_2 \left( \int_0^T \ell(X_s) ds + g(X_T) \right)} \right]^{\frac{1}{p_2}} \mathbb{E}^{\mathbb{P}^0} \left[ e^{\lambda p_3 |\xi|} \right]^{\frac{1}{p_3}} < +\infty, \end{aligned} \quad (3.7.45)$$

where we used the definitions of  $\mathcal{P}$  and  $\mathcal{C}$ , and the result of step 1.  $\square$

### 3.7.4 Proof of Lemma 3.3.1: Riccati equation for the Business as usual case

The function  $v(t, x) := -e^{-r \left( \frac{1}{2} \Gamma_t x \cdot x + P_t x + R_t \right)}$  is smooth as a composition of smooth functions. In particular, it is in  $C^{1,2} \left( [0, T], \mathbb{R}^d \right)$ . It satisfies the boundary condition (at  $t = T$ ) of (3.3.2), since the Riccati system has the appropriate terminal conditions. Furthermore, we define  $u(t, x) := \frac{1}{2} \Gamma_t x \cdot x + P_t \cdot x + R_t$  to ease notations and we write first the PDE in terms of  $u$

$$\begin{aligned} \frac{-1}{rv} \left( \partial_t v + \mathbf{H}_{\text{BU}}(x, v, Dv, D^2v) \right) &= \partial_t u + (A_0 + A_1 x) \cdot Du + \frac{1}{2} \sigma \sigma^\top : (D^2 u - r Du Du^\top) \\ &\quad - \frac{1}{2} M_2 x \cdot x - M_1 \cdot x + \frac{1}{2} C_2^{-1} (B_1^\top Du - C_1)^2, \\ &= \partial_t u + \frac{1}{2} \sigma \sigma^\top : D^2 u + (\bar{A}_0 + A_1 x) \cdot Du(t, x) \\ &\quad - M_1 \cdot x - \frac{1}{2} M_2 x \cdot x + \frac{1}{2} Q_{\text{BU}} Du \cdot Du + C_0, \end{aligned} \quad (3.7.46)$$

then, replacing  $u$  by its definition we obtain

$$\begin{aligned}
\frac{-1}{rv} \left( \partial_t v + \mathbf{H}_{\text{BU}} \left( x, v, Dv, D^2v \right) \right) &= \frac{1}{2} \dot{\Gamma}_t x \cdot x + \dot{P}_t \cdot x + \dot{R}_t + \frac{1}{2} \sigma^2 : \Gamma_t + (\bar{A}_0 + A_1 x) \cdot (\Gamma_t x + P_t) \\
&\quad - M_1 \cdot x - \frac{1}{2} M_2 x \cdot x + \frac{1}{2} Q_{\text{BU}} (\Gamma_t x + P_t) \cdot (\Gamma_t x + P_t) + C_0, \\
&= \frac{1}{2} \left( \dot{\Gamma}_t + A_1^\top \Gamma_t + \Gamma_t A_1 + \Gamma_t Q_{\text{BU}} \Gamma_t - M_2 \right) x \cdot x \\
&\quad + \left( \dot{P}_t + \Gamma_t \bar{A}_0 + A_1^\top P_t - M_1 + \Gamma_t Q_{\text{BU}} P_t \right) \cdot x \\
&\quad + \left( \dot{R}_t + \bar{A}_0 \cdot P_t + \frac{1}{2} \sigma^2 : \Gamma_t + \frac{1}{2} Q_{\text{BU}} P_t \cdot P_t + C_0 \right), \\
&= 0,
\end{aligned} \tag{3.7.47}$$

where we used the Riccati system in the last equality to cancel each of the terms, which leads to

$$-\partial_t v - \mathbf{H}_{\text{BU}} \left( x, v, Dv, D^2v \right) = 0 \text{ in } [0, T] \times \mathbb{R}^d, \tag{3.7.48}$$

in particular, the maximizer of the Hamiltonian (3.3.3) can be expressed as

$$\alpha_{\text{BU}}^* \left( \frac{-Dv(t, x)}{rv(t, x)} \right) = C_2^{-1} (B_1^\top (\Gamma_t x + P_t) - C_1), \text{ for } (t, x) \in [0, T] \times \mathbb{R}^d, \tag{3.7.49}$$

concluding the proof.  $\square$

### 3.7.5 Proof of Proposition 3.3.2: Solving the business as usual case by verification

(i) Since  $Q_{\text{BU}} > 0$  and  $M_2 \geq 0$ , the matrix Riccati system (3.3.7) and (3.3.8) and (3.3.9) is wellposed from Lemma 3.7.1 with a unique solution denoted  $(\Gamma, P, R)$ . We introduce the  $C^{1,2}([0, T], \mathbb{R}^d)$  function

$$v(t, x) := -e^{-r \left( \frac{1}{2} \Gamma_t x \cdot x + P_t \cdot x + R_t \right)}, \tag{3.7.50}$$

which is a classical solution to the PDE (3.3.2) from lemma 3.3.1. We introduce

$$\beta_{t,T}^\alpha := e^r \int_t^T c^\alpha(\alpha_s, X_s) ds, \tag{3.7.51}$$

and the sequence of stopping times

$$T_n := T \wedge \inf \left\{ s > t, \int_t^s \left( \beta_{t,r}^\alpha \right)^2 |\sigma Dv(r, X_r)|^2 dr \geq n \right\}. \tag{3.7.52}$$

For an arbitrary control  $\alpha \in \mathcal{U}$ , we apply Itô formula and take the expectation to obtain

$$\begin{aligned}
\mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \beta_{t,T_n}^\alpha v(T_n, X_{T_n}) \right] &= v(t, x) + \mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \int_t^{T_n} \beta_{t,s}^\alpha Dv(s, X_s) \cdot \sigma dW_s^\alpha \right] \\
&\quad + \mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \int_t^{T_n} \beta_{t,s}^\alpha \left( \partial_t v(s, X_s) + h_{\text{BU}}(X_s, v(s, X_s), Dv(s, X_s), D^2v(s, X_s), \alpha_s) \right) ds \right],
\end{aligned} \tag{3.7.53}$$

From the definition of the localizing sequence  $T_n$ , we have

$$\mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \int_t^{T_n} \beta_{t,s}^\alpha Dv(s, X_s) \cdot \sigma dW_s^\alpha \right] = 0, \quad (3.7.54)$$

and so

$$\begin{aligned} \mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \beta_{t,T_n}^\alpha v(T_n, X_{T_n}) \right] &= v(t, x) \\ &+ \mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \int_t^{T_n} \beta_{t,s}^\alpha \left( \partial_t v(s, X_s) + h_{\text{BU}}(X_s, v(s, X_s), Dv(s, X_s), D^2v(s, X_s), \alpha_s) \right) ds \right]. \end{aligned} \quad (3.7.55)$$

Then, since  $v$  is a solution to the PDE (3.3.2), we can use the monotonicity of the expectation and we obtain

$$\mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \int_t^{T_n} \beta_{t,s}^\alpha \left( \partial_t v(s, X_s) + h_{\text{BU}}(X_s, v(s, X_s), Dv(s, X_s), D^2v(s, X_s), \alpha_s) \right) ds \right] \leq 0, \quad (3.7.56)$$

and therefore

$$\mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \beta_{t,T_n}^\alpha v(T_n, X_{T_n}) \right] \leq v(t, x). \quad (3.7.57)$$

Since we have the convergence a.s

$$\beta_{t,T_n}^\alpha v(T_n, X_{T_n}) \rightarrow -e^r \int_t^T c^A(\alpha_s, X_s) ds \quad (3.7.58)$$

and for some  $\mathbb{P}^\alpha \in \mathcal{P}$ , by Hölder inequality (for  $q_1$  such that  $\frac{1}{p_1} + \frac{1}{q_1} = 1$ ),

$$\begin{aligned} \mathbb{E}^\alpha \left[ \left| \beta_{0,T}^\alpha v(T, X_T) \right| \right] &= \mathbb{E}^{\mathbb{P}^\alpha} \left[ e^r \left( \int_0^T c^A(\alpha_s, X_s) ds \right) \right], \\ &= \mathbb{E}^{\mathbb{P}^0} \left[ \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} e^r \int_0^T c^A(\alpha_s, X_s) ds \right], \\ &\leq \mathbb{E}^{\mathbb{P}^0} \left[ \left( \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} \right)^{p_1} \right]^{\frac{1}{p_1}} \mathbb{E}^{\mathbb{P}^0} \left[ e^{rq_1 \int_0^T c^A(\alpha_s, X_s) ds} \right]^{\frac{1}{q_1}} < +\infty \end{aligned} \quad (3.7.59)$$

from the admissibility of  $\mathbb{P}^\alpha$  (and the fact that  $q_1 < p_2$ ). In particular, by localizing and moving the time origin, we obtain

$$\mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \left| \beta_{t,T_n}^\alpha v(T_n, X_{T_n}) \right| \right] < +\infty, \text{ for } n \geq 1. \quad (3.7.60)$$

So by dominated convergence, we have for an arbitrary admissible control

$$\mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ -e^r \int_t^T c^A(\alpha_s, X_s) ds \right] \leq v(t, x), \quad (3.7.61)$$

and by taking  $t = 0$  and the supremum over  $\mathbb{P}^\alpha \in \mathcal{P}$ , we obtain

$$V^A(0) \leq v(0, x_0) = -e^{-r(\frac{1}{2}\Gamma_0 x_0 \cdot x_0 + P_0 \cdot x_0 + R_0)}. \quad (3.7.62)$$

(ii) Assuming that the induced feedback control  $\alpha_{\text{BU}}^*(t, x) := C_2^{-1}(B_1^\top(\Gamma_t x + P_t) - C_1)$  is admissible, i.e., the process  $(\alpha_{\text{BU}}^*(t, X_t))_{t \in [0, T]}$  is in  $\mathcal{U}$  as defined in 3.2.1, then we can go over the same steps in (i) by applying Itô formula and taking  $\alpha_{\text{BU}}^*$  to obtain equality instead of inequality in (3.7.56), and so the upper bound  $v(0, x_0)$  in the maximization problem is attained for the probability measure  $\mathbb{P}^{\alpha^{\mathcal{R}}}$  induced by the admissible control  $\alpha_{\text{BU}}^*$ , and we have

$$V^\Lambda(0) = v(0, x_0) = -e^{-r(\frac{1}{2}\Gamma_0 x_0 \cdot x_0 + P_0 \cdot x_0 + R_0)}, \quad (3.7.63)$$

in particular, by applying  $U_A^{-1}$  to the equality (3.7.63), we obtain the participation constraint

$$\mathcal{R} = \frac{1}{2}\Gamma_0 x_0 \cdot x_0 + P_0 \cdot x_0 + R_0, \quad (3.7.64)$$

which concludes the proof.  $\square$

### 3.7.6 Proof of Lemma 3.3.3 : Admissibility of the optimal control in the business as usual case

Observe first that following the same line of proof as in Lemma 3.2.5, and by assumption (3.3.16), we have that

$$\mathbb{E}^0 \left[ e^{\varepsilon_{\text{BU}} \int_0^T |X_s|^2 ds} \right] < +\infty. \quad (3.7.65)$$

(i) Recall that  $\alpha_{\text{BU}}^*(t, x) := C_2^{-1}(B_1^\top(\Gamma_t x + P_t) - C_1)$  with  $(\Gamma, P, R)$  the solution to the Riccati system (3.3.7) and (3.3.8) and (3.3.9), for  $(t, x) \in [0, T] \times \mathbb{R}^d$ . Without loss of generality, we can consider that the constant term in the control is null, i.e.,  $C_1 = 0$ . From (3.7.65) we have

$$\begin{aligned} \mathbb{E}^0 \left[ e^{\frac{1}{2} \int_0^T \left| \frac{\sqrt{q_1}}{\sqrt{q_1-1}} \sigma^{-1} B_0 \Gamma_s X_s \right|^2 ds} \right] &\leq \mathbb{E}^0 \left[ e^{\left( \frac{\sqrt{q_1}}{\sqrt{q_1-1}} \right)^2 \frac{\lambda_{\max}(B_0)^2 \sup_{t \in [0, T]} (\lambda_{\max}(\Gamma_t^2))}{2\lambda_{\min}(\sigma^2)} \int_0^T |X_s|^2 ds} \right], \\ &\leq \mathbb{E}^0 \left[ e^{\varepsilon_{\text{BU}} \int_0^T |X_s|^2 ds} \right] < +\infty, \end{aligned} \quad (3.7.66)$$

and so

$$\mathbb{E}^0 \left[ e^{\frac{1}{2} \int_0^T \left| \frac{\sqrt{q_1}}{\sqrt{q_1-1}} \sigma^{-1} B_1 \alpha_{\text{BU}}^*(s, X_s) \right|^2 ds} \right] = \mathbb{E}^0 \left[ e^{\frac{1}{2} \int_0^T \left| \frac{\sqrt{q_1}}{\sqrt{q_1-1}} \sigma^{-1} B_0 (\Gamma_s X_s + P_s) \right|^2 ds} \right] < +\infty, \quad (3.7.67)$$



Then, by [Kazamaki \(2006\)](#)[Theorem 1.5], for  $a := \frac{\sqrt{q_1}+1}{\sqrt{q_1}-1}$  and a stopping time  $\tau \in \mathcal{T}_{0,T}$ , we have that

$$\begin{aligned}
\mathbb{E}^0 \left[ \mathcal{E} \left( \int_0^\tau \sigma^{-1} B_1 \alpha_{\text{BU}}^*(s, X_s) \cdot dW_s^0 \right)^{p_1} \right] &\leq \mathbb{E}^0 \left[ \mathcal{E} \left( \sqrt{ap_1} \int_0^\tau \sigma^{-1} B_1 \alpha_{\text{BU}}^*(s, X_s) \cdot dW_s^0 \right) \right]^{\frac{1}{a}}, \\
&\quad \times \mathbb{E}^0 \left[ \exp \left( \frac{\sqrt{q_1}}{2(\sqrt{q_1}-1)} \int_0^\tau \sigma^{-1} B_1 \alpha_{\text{BU}}^*(s, X_s) \cdot dW_s^0 \right) \right]^{\frac{2}{\sqrt{q_1}+1}}, \\
&\leq \mathbb{E}^0 \left[ \exp \left( \frac{\sqrt{q_1}}{2(\sqrt{q_1}-1)} \int_0^\tau \sigma^{-1} B_1 \alpha_{\text{BU}}^*(s, X_s) \cdot dW_s^0 \right) \right]^{\frac{2}{\sqrt{q_1}+1}}, \\
&\leq \mathbb{E}^0 \left[ \exp \left( \frac{1}{2} \int_0^\tau \left| \frac{\sqrt{q_1}}{\sqrt{q_1}-1} \sigma^{-1} B_1 \alpha_{\text{BU}}^*(s, X_s) \right|^2 ds \right) \right]^{\frac{1}{\sqrt{q_1}+1}},
\end{aligned} \tag{3.7.68}$$

and so

$$\begin{aligned}
\sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}^0 \left[ \mathcal{E} \left( \int_0^\tau \sigma^{-1} B_1 \alpha_{\text{BU}}^*(s, X_s) \cdot dW_s^0 \right)^{p_1} \right] &\leq \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}^0 \left[ e^{\frac{1}{2} \int_0^\tau \left| \frac{\sqrt{q_1}}{\sqrt{q_1}-1} \sigma^{-1} B_1 \alpha_{\text{BU}}^*(s, X_s) \right|^2 ds} \right]^{\frac{1}{\sqrt{q_1}+1}}, \\
&\leq \mathbb{E}^0 \left[ e^{\frac{1}{2} \int_0^T \left| \frac{\sqrt{q_1}}{\sqrt{q_1}-1} \sigma^{-1} B_1 \alpha_{\text{BU}}^*(s, X_s) \right|^2 ds} \right]^{\frac{1}{\sqrt{q_1}+1}} < +\infty.
\end{aligned} \tag{3.7.69}$$

Therefore

$$\mathcal{E} \left( \int_0^t \sigma^{-1} B_1 \alpha_{\text{BU}}^*(s, X_s) \cdot dW_s^0 \right)_{t \in [0,T]} \text{ is a } (\mathbb{P}^0, \mathbb{F})\text{-martingale in } \mathbb{L}^{p_1}. \tag{3.7.70}$$

(ii) We have that

$$\begin{aligned}
\mathbb{E}^0 \left[ e^{rp_2 \int_0^T c^A(\alpha_{\text{BU}}^*(s, X_s), X_s) ds} \right] &= \\
&\mathbb{E}^0 \left[ e^{rp_2 \int_0^T \left( \frac{1}{2} B_0(\Gamma_s X_s + P_s) \cdot (\Gamma_s X_s + P_s) + C_1^\top C_2^{-1} B_1^\top (\Gamma_s X_s + P_s) + \frac{1}{2} M_2 X_s \cdot X_s + M_1 \cdot X_s \right) ds} \right].
\end{aligned} \tag{3.7.71}$$

We discard the linear terms in  $X$  since it has exponential moments of all order, and we have then

$$\begin{aligned}
\mathbb{E}^0 \left[ e^{\frac{rp_2}{2} \int_0^T (\Gamma_s B_0 \Gamma_s + M_2) X_s \cdot X_s ds} \right] &\leq \mathbb{E}^0 \left[ e^{\frac{rp_2 \lambda_{\max}(B_0)}{2} \int_0^T \lambda_{\max}(\Gamma_s^2) X_s \cdot X_s ds + \frac{rp_2 \lambda_{\max}(M_2)}{2} \int_0^T X_s \cdot X_s ds} \right], \\
&\leq \mathbb{E}^0 \left[ e^{\varepsilon_{\text{BU}} \int_0^T |X_s|^2 ds} \right] < +\infty,
\end{aligned} \tag{3.7.72}$$

and so

$$\mathbb{E}^0 \left[ e^{rp_2 \int_0^T c^A(\alpha_{\text{BU}}^*(s, X_s)) ds} \right] < +\infty. \tag{3.7.73}$$

Thus  $\alpha_{\text{BU}}^*(t, X_t)_{t \in [0,T]} \in \mathcal{U}$ , concluding the proof.  $\square$

### 3.7.7 Proof of Lemma 3.4.1: Riccati equation for the first best

The Riccati equation (3.4.13) is a locally lipschitz ODE with constant coefficients, therefore its solution is differentiable with respect to time. Furthermore, once  $\Gamma$  defined (by assumption), the linear ODE (3.4.14) has a unique solution differentiable with respect to time, and so is  $R$  defined by (3.4.15). Therefore, the function  $v(t, x) := -e^{-\varrho(\frac{1}{2}\Gamma_t x \cdot x + P_t x + R_t)}$  is smooth as a composition of smooth functions. In particular, it is in  $C^{1,2}([0, T], \mathbb{R}^d)$ . It satisfies the boundary condition (at  $t = T$ ) of (3.4.8), since the Riccati system has the appropriate terminal condition, together with  $R_T = 0$ . Furthermore, we define  $u(t, x) := \frac{1}{2}\Gamma_t x \cdot x + P_t \cdot x + R_t$  to ease notations and we compute

$$\begin{aligned}
\frac{-1}{\varrho v} \left( \partial_t v(t, x) + \mathbf{H}_{\text{FB}}(x, v, Dv, D^2v) \right) &= \partial_t u(t, x) + (A_0 + A_1 x) \cdot Du(t, x) + \frac{1}{2} \sigma^2 : (D^2 u - \varrho Du Du^\top) \\
&\quad + \left( (L_1 - M_1) \cdot x - \frac{1}{2} (L_2 + M_2) x \cdot x \right) \\
&\quad + \frac{1}{2} C_2^{-1} (B_1^\top Du(t, x) - C_1) \cdot (B_1^\top Du(t, x) - C_1) \\
&= \partial_t u(t, x) + \frac{1}{2} \sigma^2 : D^2 u(t, x) + (\bar{A}_0 + A_1 x) \cdot Du(t, x) \\
&\quad + \left( (L_1 - M_1) \cdot x - \frac{1}{2} (L_2 + M_2) x \cdot x \right) \\
&\quad + \frac{1}{2} Q_{\text{FB}} Du(t, x) \cdot Du(t, x) + C_0,
\end{aligned} \tag{3.7.74}$$

we replace then  $u$  and its derivatives by its definition (and use the symmetry of  $\Gamma$ ) and so

$$\begin{aligned}
\frac{-1}{\varrho v} \left( \partial_t v(t, x) + \mathbf{H}_{\text{FB}}(x, v, Dv, D^2v) \right) &= \frac{1}{2} \dot{\Gamma}_t x \cdot x + \dot{P}_t \cdot x + \dot{R}_t + \frac{1}{2} \sigma^2 : \Gamma_t + (\bar{A}_0 + A_1 x) \cdot (\Gamma_t x + P_t) \\
&\quad + \left( (L_1 - M_1) \cdot x - \frac{1}{2} (L_2 + M_2) x \cdot x \right) \\
&\quad + \frac{1}{2} Q_{\text{FB}} (\Gamma_t x + P_t) \cdot (\Gamma_t x + P_t) + C_0 \\
&= \frac{1}{2} \left( \dot{\Gamma}_t + A_1^\top \Gamma_t + \Gamma_t A_1 + \Gamma_t Q_{\text{FB}} \Gamma_t - (L_2 + M_2) \right) x \cdot x \\
&\quad + \left( \dot{P}_t + \Gamma_t \bar{A}_0 + A_1^\top P_t + (L_1 - M_1) + \Gamma_t Q_{\text{FB}} P_t \right) \cdot x \\
&\quad + \left( \dot{R}_t + \bar{A}_0 \cdot P_t + \frac{1}{2} \sigma^2 : \Gamma_t + \frac{1}{2} Q_{\text{FB}} P_t \cdot P_t + C_0 \right) \\
&= 0,
\end{aligned} \tag{3.7.75}$$

where we used the system (3.4.13) and the definition of  $R$  in (3.4.15) in the last equality to cancel each of the terms, which leads to

$$\begin{aligned}
0 &= \partial_t u(t, x) + (A_0 + A_1 x) \cdot Du(t, x) + \frac{1}{2} \sigma^2 : (D^2 u - \rho Du Du^\top) \\
&\quad + \left( (L_1 - M_1) \cdot x - \frac{1}{2} (L_2 + M_2) x \cdot x \right) + \frac{1}{2} C_2^{-1} (B_1^\top Du(t, x) - C_1) \cdot (B_1^\top Du(t, x) - C_1), \\
&= \partial_t u(t, x) + \frac{1}{2} \sigma^2 : D^2 u(t, x) + (\bar{A}_0 + A_1 x) \cdot Du(t, x) + \left( L_1 \cdot x - \frac{1}{2} L_2 x \cdot x \right) \\
&\quad + \frac{1}{2} Q_{\text{FB}} Du(t, x) \cdot Du(t, x) + C_0,
\end{aligned} \tag{3.7.76}$$

and

$$\partial_t v + \mathbf{H}_{\text{FB}}(x, v, Dv, D^2 v) = 0 \text{ in } [0, T) \times \mathbb{R}^d, \tag{3.7.77}$$

concluding the proof.  $\square$

### 3.7.8 Proof of Proposition 3.4.2 : Solving the first best by verification

(i) Since  $Q_{\text{FB}} > 0$  and  $(L_2 + M_2) \geq 0$ , then the matrix Riccati system (3.4.13) and (3.4.14) and (3.4.15) is wellposed from Lemma 3.7.1 with a unique solution denoted  $(\Gamma, P, R)$ . We introduce the  $C^{1,2}([0, T], \mathbb{R}^d)$  function

$$v(t, x) := -e^{-\varrho \left( \frac{1}{2} \Gamma_t x \cdot x + P_t \cdot x + R_t \right)}, \tag{3.7.78}$$

which is a classical solution to the PDE (3.4.8) from lemma 3.4.1. We also introduce

$$\beta_{t,T}^\alpha := e^{-\varrho \int_t^T (\ell(X_s) - c^\alpha(\alpha_s, X_s)) ds}, \tag{3.7.79}$$

and we define the sequence of stopping times  $T_n := T \wedge \inf \left\{ s > t, \int_t^s \left( \beta_{t,r}^\alpha \right)^2 |\sigma Dv(r, X_r)|^2 dr \geq n \right\}$ . By applying Itô formula with an arbitrary control  $\alpha \in \mathcal{U}$  and taking the expectation, we obtain

$$\begin{aligned}
\mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \beta_{t,T_n}^\alpha v(T_n, X_{T_n}) \right] &= v(t, x) + \mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \int_t^{T_n} \beta_{t,s}^\alpha Dv(s, X_s) \cdot \sigma dW_s^\alpha \right] \\
&\quad + \mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \int_t^{T_n} \beta_{t,s}^\alpha \left( \partial_t v(s, X_s) + h_{\text{FB}}(X_s, v(s, X_s), Dv(s, X_s), D^2 v(s, X_s), \alpha_s) \right) ds \right],
\end{aligned} \tag{3.7.80}$$

From the definition of the localizing sequence  $T_n$ , the stochastic integral has zero expectation, and so

$$\begin{aligned}
\mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \beta_{t,T_n}^\alpha v(T_n, X_{T_n}) \right] &= v(t, x) \\
&\quad + \mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \int_t^{T_n} \beta_{t,s}^\alpha \left( \partial_t v(s, X_s) + h_{\text{FB}}(X_s, v(s, X_s), Dv(s, X_s), D^2 v(s, X_s), \alpha_s) \right) ds \right].
\end{aligned} \tag{3.7.81}$$

Then, since  $v$  is a solution to (3.4.8) and by monotonicity of the expectation we have

$$\mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \int_t^{T_n} \beta_{t,s}^\alpha \left( \partial_t v(s, X_s) + h_{\text{FB}}(X_s, v(s, X_s), Dv(s, X_s), D^2v(s, X_s), \alpha_s) \right) ds \right] \leq 0, \quad (3.7.82)$$

and therefore

$$\mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \beta_{t,T_n}^\alpha v(T_n, X_{T_n}) \right] \leq v(t, x). \quad (3.7.83)$$

Since we have the convergence a.s

$$\beta_{t,T_n}^\alpha v(T_n, X_{T_n}) \rightarrow -e^{-\varrho \left( \int_t^T (\ell(X_s) - c^A(\alpha_s, X_s)) ds + g(X_T) \right)}, \quad (3.7.84)$$

and for some  $\mathbb{P}^\alpha \in \mathcal{P}$ , by Hölder inequality

$$\begin{aligned} \mathbb{E}^\alpha \left[ \left| \beta_{0,T}^\alpha v(T, X_T) \right| \right] &= \mathbb{E}^{\mathbb{P}^\alpha} \left[ e^{-\varrho \left( \int_0^T (\ell(X_s) - c^A(\alpha_s, X_s)) ds + g(X_T) \right)} \right], \\ &= \mathbb{E}^{\mathbb{P}^0} \left[ \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} e^{\varrho \int_0^T c^A(\alpha_s, X_s) ds} e^{-\varrho \left( \int_0^T (\ell(X_s)) ds + g(X_T) \right)} \right], \\ &\leq \mathbb{E}^{\mathbb{P}^0} \left[ \left( \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} \right)^{p_1} \right]^{\frac{1}{p_1}} \mathbb{E}^{\mathbb{P}^0} \left[ e^{\varrho p_2 \int_0^T c^A(\alpha_s, X_s) ds} \right]^{\frac{1}{p_2}} \mathbb{E}^{\mathbb{P}^0} \left[ e^{\varrho p_3 \left( \int_0^T \ell(X_s) ds + g(X_T) \right)} \right]^{\frac{1}{p_3}}, \\ &< +\infty, \end{aligned} \quad (3.7.85)$$

from the admissibility of  $\mathbb{P}^\alpha$  (and the fact that  $\varrho < r$ ), and the assumption 3.2.4 on the coefficients. In particular, by localizing and moving the time origin, we obtain

$$\mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ \left| \beta_{t,T_n}^\alpha v(T_n, X_{T_n}) \right| \right] < +\infty, \text{ for } n \geq 1. \quad (3.7.86)$$

So by dominated convergence, we have for an arbitrary admissible control

$$\mathbb{E}_{t,x}^{\mathbb{P}^\alpha} \left[ -e^{-\varrho \left( \int_t^T (\ell(X_s) - c^A(\alpha_s, X_s)) ds + g(X_T) \right)} \right] \leq v(t, x), \quad (3.7.87)$$

and by taking  $t = 0$  and the supremum over  $\mathbb{P}^\alpha \in \mathcal{P}$ , we obtain

$$\bar{V}_0 \leq v(0, x). \quad (3.7.88)$$

(ii) Assuming that the induced feedback control  $\alpha_{\text{FB}}^*(t, x) := C_2^{-1}(B_1^\top(\Gamma_t x + P_t) - C_1)$  is admissible, i.e., the process  $(\alpha_{\text{FB}}^*(t, X_t))_{t \in [0, T]}$  is in  $\mathcal{U}$  as defined in 3.2.1, then we can go over the same steps in (i) by applying Itô formula and taking  $\alpha_{\text{FB}}^*$  to obtain equality instead of inequality in (3.7.82), and so

$$\bar{V}_0 = v(0, x). \quad (3.7.89)$$

(iii) Finally, we can inject  $\bar{V}_0$  in (3.4.6) to compute  $\lambda^*$  and in (3.4.3) to get the optimal contract whenever it is

admissible (in  $\mathcal{C}$ ), which leads to

$$\begin{aligned} \xi_{\text{FB}}^* &= \mathcal{R} - \frac{\rho}{\rho + r} \left( \frac{1}{2} \Gamma_0 x_0 \cdot x_0 + P_0 \cdot x_0 + R_0 \right), \\ &+ \frac{\rho}{\rho + r} \left( g(X_T) + \int_0^T \ell(X_s) ds + \frac{r}{\rho} \int_0^T c^\wedge(\alpha_{\text{FB}}^*(s, X_s), X_s) ds \right) \end{aligned} \quad (3.7.90)$$

and so

$$V_{\text{FB}}^p = U_p \left( -\mathcal{R} + \frac{1}{2} \Gamma_0 x_0 \cdot x_0 + P_0 \cdot x_0 + R_0 \right). \quad (3.7.91)$$

In that case we define the certainty equivalent dynamic value as

$$u_{\text{FB}}(t, x) := -\mathcal{R} + \frac{1}{2} \Gamma_t x \cdot x + P_t \cdot x + R_t, \quad (3.7.92)$$

which characterizes the control as  $\alpha_{\text{FB}}^*(t, x) = C_2^{-1}(B_1^\top Du_{\text{FB}}(t, x) - C_1)$ .  $\square$

### 3.7.9 Proof of Lemma 3.4.3 : Admissibility of the optimal First Best policy

Observe first that following the same line of proof as in Lemma 3.2.5, and by assumption (3.4.24), we have that

$$\mathbb{E}^0 \left[ e^{\varepsilon_{\text{FB}}} \int_0^T |X_s|^2 ds \right] < +\infty. \quad (3.7.93)$$

We prove first (in (i) and (ii)) that  $(\alpha_{\text{FB}}^*(t, X_t))_{t \in [0, T]} \in \mathcal{U}$  (with the same arguments as in Lemma 3.3.3), then we prove that  $\xi_{\text{FB}}^* \in \mathcal{C}$  in (iii).

(i) Recall that  $\alpha_{\text{FB}}^*(t, x) := C_2^{-1}(B_1^\top(\Gamma_t x + P_t) - C_1)$ , for  $(t, x) \in [0, T] \times \mathbb{R}^d$ . Without loss of generality, we can consider that the constant term in the control is null, i.e.,  $C_1 = 0$ . From (3.7.93) we have

$$\begin{aligned} \mathbb{E}^0 \left[ e^{\frac{1}{2} \int_0^T \left| \frac{\sqrt{q_1}}{\sqrt{q_1-1}} \sigma^{-1} B_0 \Gamma_s X_s \right|^2 ds} \right] &\leq \mathbb{E}^0 \left[ e^{\left( \frac{\sqrt{q_1}}{\sqrt{q_1-1}} \right)^2 \frac{\lambda_{\max}(B_0)^2 \sup_{t \in [0, T]} (\lambda_{\max}(\Gamma_t^2))}{2\lambda_{\min}(\sigma^2)} \int_0^T |X_s|^2 ds} \right] \\ &\leq \mathbb{E}^0 \left[ e^{\varepsilon_{\text{FB}}} \int_0^T |X_s|^2 ds \right] < +\infty, \end{aligned} \quad (3.7.94)$$

and so

$$\mathbb{E}^0 \left[ e^{\frac{1}{2} \int_0^T \left| \frac{\sqrt{q_1}}{\sqrt{q_1-1}} \sigma^{-1} B_1 \alpha_{\text{FB}}^*(s, X_s) \right|^2 ds} \right] = \mathbb{E}^0 \left[ e^{\frac{1}{2} \int_0^T \left| \frac{\sqrt{q_1}}{\sqrt{q_1-1}} \sigma^{-1} B_0 (\Gamma_s X_s + P_s) \right|^2 ds} \right] < +\infty, \quad (3.7.95)$$

Then, from Kazamaki (2006)[Theorem 1.5], for  $a := \frac{\sqrt{q_1}+1}{\sqrt{q_1}-1}$  and a stopping time  $\tau \in \mathcal{T}_{0,T}$ , we have that

$$\begin{aligned} \mathbb{E}^0 \left[ \mathcal{E} \left( \int_0^\tau \sigma^{-1} B_1 \alpha_{\text{FB}}^*(s, X_s) \cdot dW_s^0 \right)^{p_1} \right] &\leq \mathbb{E}^0 \left[ \mathcal{E} \left( \sqrt{ap_1} \int_0^\tau \sigma^{-1} B_1 \alpha_{\text{FB}}^*(s, X_s) \cdot dW_s^0 \right) \right]^{\frac{1}{a}} \\ &\quad \times \mathbb{E}^0 \left[ \exp \left( \frac{\sqrt{q_1}}{2(\sqrt{q_1}-1)} \int_0^\tau \sigma^{-1} B_1 \alpha_{\text{FB}}^*(s, X_s) \cdot dW_s^0 \right) \right]^{\frac{2}{\sqrt{q_1}+1}}, \\ &\leq \mathbb{E}^0 \left[ \exp \left( \frac{\sqrt{q_1}}{2(\sqrt{q_1}-1)} \int_0^\tau \sigma^{-1} B_1 \alpha_{\text{FB}}^*(s, X_s) \cdot dW_s^0 \right) \right]^{\frac{2}{\sqrt{q_1}+1}}, \\ &\leq \mathbb{E}^0 \left[ \exp \left( \frac{1}{2} \int_0^\tau \left| \frac{\sqrt{q_1}}{\sqrt{q_1}-1} \sigma^{-1} B_1 \alpha_{\text{FB}}^*(s, X_s) \right|^2 ds \right) \right]^{\frac{1}{\sqrt{q_1}+1}}, \end{aligned} \quad (3.7.96)$$

and so

$$\begin{aligned} \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}^0 \left[ \mathcal{E} \left( \int_0^\tau \sigma^{-1} B_1 \alpha_{\text{FB}}^*(s, X_s) \cdot dW_s^0 \right)^{p_1} \right] &\leq \sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}^0 \left[ e^{\frac{1}{2} \int_0^\tau \left| \frac{\sqrt{q_1}}{\sqrt{q_1}-1} \sigma^{-1} B_1 \alpha_{\text{FB}}^*(s, X_s) \right|^2 ds} \right]^{\frac{1}{\sqrt{q_1}+1}}, \\ &\leq \mathbb{E}^0 \left[ e^{\frac{1}{2} \int_0^T \left| \frac{\sqrt{q_1}}{\sqrt{q_1}-1} \sigma^{-1} B_1 \alpha_{\text{FB}}^*(s, X_s) \right|^2 ds} \right]^{\frac{1}{\sqrt{q_1}+1}} < +\infty. \end{aligned} \quad (3.7.97)$$

Therefore

$$\mathcal{E} \left( \int_0^t \sigma^{-1} B_1 \alpha_{\text{FB}}^*(s, X_s) \cdot dW_s^0 \right)_{t \in [0,T]} \text{ is a } (\mathbb{P}^0, \mathbb{F})\text{-martingale in } \mathbb{L}^{p_1}. \quad (3.7.98)$$

(ii) We have that

$$\begin{aligned} \mathbb{E}^0 \left[ e^{rp_2 \int_0^T c^A(\alpha_{\text{FB}}^*(s, X_s), X_s) ds} \right] &= \\ &\mathbb{E}^0 \left[ e^{rp_2 \int_0^T \left( \frac{1}{2} B_0(\Gamma_s X_s + P_s) \cdot (\Gamma_s X_s + P_s) + C_1^\top C_2^{-1} B_1^\top (\Gamma_s X_s + P_s) + \frac{1}{2} M_2 X_s \cdot X_s + M_1 \cdot X_s \right) ds} \right]. \end{aligned} \quad (3.7.99)$$

To prove the integrability, we can always discard the linear terms in  $X$ , and we have then

$$\begin{aligned} \mathbb{E}^0 \left[ e^{\frac{rp_2}{2} \int_0^T (\Gamma_s B_0 \Gamma_s + M_2) X_s \cdot X_s ds} \right] &\leq \mathbb{E}^0 \left[ e^{\frac{rp_2 \lambda_{\max}(B_0)}{2} \int_0^T \lambda_{\max}(\Gamma_s^2) X_s \cdot X_s ds + \frac{rp_2 \lambda_{\max}(M_2)}{2} \int_0^T X_s \cdot X_s ds} \right], \\ &\leq \mathbb{E}^0 \left[ e^{\varepsilon_{\text{FB}} \int_0^T |X_s|^2 ds} \right] < +\infty, \end{aligned} \quad (3.7.100)$$

and so

$$\mathbb{E}^0 \left[ e^{rp_2 \int_0^T c^A(\alpha_{\text{FB}}^*(s, X_s)) ds} \right] < +\infty. \quad (3.7.101)$$

Thus  $\alpha_{\text{FB}}^*(t, X_t)_{t \in [0,T]} \in \mathcal{U}$  from (i) and (ii).

(iii) Since for  $x \in \mathbb{R}$ , we have  $\cosh(x) \leq e^{|x|} \leq 2 \cosh(x)$ , we obtain that

$$\mathbb{E}^0 \left[ e^{\lambda p_3 |\xi_{\text{FB}}^*|} \right] < +\infty \text{ if and only if } \mathbb{E}^0 \left[ \frac{1}{2} \cosh(\lambda p_3 |\xi_{\text{FB}}^*|) \right] < +\infty. \quad (3.7.102)$$

We recall then the definition of  $\xi_{\text{FB}}^*$

$$\begin{aligned} \xi_{\text{FB}}^* &= \mathcal{R} - \frac{\rho}{\rho+r} \left( \frac{1}{2} \Gamma_0 x_0 \cdot x_0 + P_0 \cdot x_0 + R_0 \right) + \frac{\rho}{\rho+r} \left( g(X_T) + \int_0^T \left( \ell(X_s) + \frac{r}{\rho} c^\Lambda(\alpha_{\text{FB}}^*(s, X_s)) \right) ds \right), \\ &= \mathcal{R} - \frac{\rho}{\rho+r} \left( \frac{1}{2} \Gamma_0 x_0 \cdot x_0 + P_0 \cdot x_0 + R_0 \right), \\ &\quad + \frac{\rho}{\rho+r} \left( G \cdot X_T + \int_0^T \left( L_1 \cdot X_s + \frac{r}{\rho} C_1 \cdot C_2^{-1} B_1^\top (\Gamma_s X_s + P_s) \right) ds \right), \\ &\quad + \frac{\rho}{\rho+r} \left( \int_0^T \frac{r}{\rho} \left( \frac{1}{2} B_0 (2\Gamma_s X_s \cdot P_s + M_1 \cdot X_s + P_s \cdot P_s) \right) ds \right) \\ &\quad + \frac{1}{2} \frac{\rho}{\rho+r} \left( \int_0^T -L_2 X_s \cdot X_s ds + \frac{r}{\rho} \int_0^T (\Gamma_s B_0 \Gamma_s + M_2) X_s \cdot X_s ds \right), \end{aligned} \quad (3.7.103)$$

We discard again the linear terms in  $X$  with linear growth and remark that

$$\begin{aligned} \mathbb{E}^0 \left[ e^{\frac{\lambda p_3}{2} \frac{\rho}{\rho+r} \left( \int_0^T (L_2 - \frac{r}{\rho} (\Gamma_s B_0 \Gamma_s + M_2)) X_s \cdot X_s ds \right)} \right] &\leq \mathbb{E}^0 \left[ e^{\frac{\lambda p_3}{2} \frac{\rho}{\rho+r} \lambda_{\max}(L_2) \int_0^T |X_s|^2 ds} \right] \\ &\leq \mathbb{E}^0 \left[ e^{\varepsilon_{\text{FB}} \int_0^T |X_s|^2 ds} \right] < +\infty, \end{aligned} \quad (3.7.104)$$

and

$$\begin{aligned} \mathbb{E}^0 \left[ e^{\frac{\lambda p_3}{2} \frac{\rho}{\rho+r} \left( \int_0^T \left( \frac{r}{\rho} (\Gamma_s B_0 \Gamma_s + M_2) - L_2 \right) X_s \cdot X_s ds \right)} \right] &\leq \mathbb{E}^0 \left[ e^{\frac{\lambda p_3}{2} \frac{r}{\rho+r} (\lambda_{\max}(M_2) + \lambda_{\max}(B_0) \sup_{t \in [0, T]} (\lambda_{\max}(\Gamma_t^2))) \int_0^T |X_s|^2 ds} \right] \\ &\leq \mathbb{E}^0 \left[ e^{\varepsilon_{\text{FB}} \int_0^T |X_s|^2 ds} \right] < +\infty, \end{aligned} \quad (3.7.105)$$

and so  $\mathbb{E}^0 \left[ e^{\lambda p_3 |\xi_{\text{FB}}^*|} \right] < +\infty$  and  $\xi_{\text{FB}}^* \in \mathcal{C}$  concluding the proof.  $\square$

### 3.7.10 Proof of Proposition 3.5.1 : The representation $\mathcal{Z} = \Xi$

We already have the inclusion  $\mathcal{Z} \subset \Xi$  by definition of  $\mathcal{Z}$ , and so we need to prove the other inclusion  $\Xi \subset \mathcal{Z}$  which is a BSDE representation. The following proof is classical in the non-Markovian stochastic control theory and relies on the Agent's continuation utility as a natural candidate for the solution of the BSDE.

We start by defining Agent's continuation utility, and prove that it satisfies the Dynamic Programming Principle. Then we use the assumptions on the set of admissible contracts and the properties of Agent's continuation utility to conclude.

**Definition 3.7.3.** For  $t \in [0, T]$ , we denote by  $\mathcal{T}_{t, T}$  the set of stopping times valued in  $[t, T]$ . Let  $\tau \in \mathcal{T}_{t, T}$  an denote

by  $\mathcal{U}_\tau$  the restriction of (Agent's) controls to  $[\tau, T]$ . We define the dynamic version of Agent's objective function for a given  $\xi \in \Xi$  and  $\tau \in \mathcal{T}_{t,T}$  as the  $\mathcal{F}_\tau$  measurable random variable under some  $\mathbb{P}^\alpha$

$$J_\tau^A(\xi, \mathbb{P}^\alpha) := \mathbb{E}_\tau^{\mathbb{P}^\alpha} \left[ U_A \left( \xi - \int_\tau^T c^A(\alpha_s, X_s) ds \right) \right] \text{ and } \mathcal{J}_\tau^A(\xi) := \left( J_\tau^A(\xi, \mathbb{P}^\alpha) \right)_{\alpha \in \mathcal{U}_\tau}, \quad (3.7.106)$$

$\mathcal{J}_\tau^A$  being the collection of such random variables with different controls, and his continuation utility and certainty equivalent

$$V_\tau^A(\xi) := \text{esssup}_{\alpha \in \mathcal{U}_\tau} J_\tau^A(\xi, \mathbb{P}^\alpha) \text{ and } u_\tau^A(\xi) := -\frac{1}{r} \log \left( -V_\tau^A(\xi) \right). \quad (3.7.107)$$

Remark that for any  $\mathbb{P}^\alpha \in \mathcal{P}$ , the conditional expectation  $\mathbb{E}_\tau^{\mathbb{P}^\alpha}$  depends only on the restriction of  $\alpha$  on  $[\tau, T]$ . It is then defined without ambiguity for  $\alpha \in \mathcal{U}_\tau$ . We state now the Dynamic Programming Principle and report its proof to Appendix 3.7.11.

**Lemma 3.7.4** (Dynamic Programming Principle). *For  $\xi \in \Xi$ , the family  $(V_\tau^A(\xi), \tau \in \mathcal{T}_{0,T})$  can be aggregated by a unique optional process  $(V_t^A(\xi))_{t \in [0,T]}$  which we consider in its càd-làg modification, and which satisfies the dynamic programming principle:*

$$V_{\tau_1}^A(\xi) = \text{esssup}_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}_{\tau_1}^{\mathbb{P}^\alpha} \left[ V_{\tau_2}^A(\xi) e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} \right], \text{ for } \tau_1, \tau_2 \in \mathcal{T}_{0,T} \text{ with } \tau_1 \leq \tau_2. \quad (3.7.108)$$

or equivalently

$$-e^{-ru_{\tau_1}^A(\xi)} = \text{esssup}_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}_{\tau_1}^{\mathbb{P}^\alpha} \left[ -e^{-r \left( u_{\tau_2}^A(\xi) - \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds \right)} \right], \text{ for } \tau_1, \tau_2 \in \mathcal{T}_{0,T} \text{ with } \tau_1 \leq \tau_2. \quad (3.7.109)$$

Finally we prove the inclusion  $\Xi \subset \mathcal{Z}$ . We fix some  $\xi \in \Xi$ , and define Agent's continuation utility as in (3.7.107). Since  $\mathcal{P}^*(\xi) \neq \emptyset$  by definition of  $\Xi$ , there exists  $\mathbb{P}^{\alpha^*}$ <sup>2</sup> such that  $V_t^A(\xi) = J_t^A(\xi, \mathbb{P}^{\alpha^*})$ , for  $t \in [0, T]$ , and so the process  $\left( J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{r \int_0^t c^A(\alpha_s, X_s) ds} \right)_{t \in [0,T]}$  is a  $(\mathbb{P}^{\alpha^*}, \mathbb{F})$ -supermartingale, for  $\mathbb{P}^\alpha \in \mathcal{P}$ , and so is the process  $\left( -e^{-r \left( u_t^A(\xi) - \int_0^t c^A(\alpha_s, X_s) ds \right)} \right)_{t \in [0,T]}$ .

<sup>2</sup>Note here that we wrote  $\mathbb{P}^{\alpha^*}$  instead of  $\mathbb{P}^{\alpha^*(\xi)}$  to ease notations.



In fact, the integrability condition follows from the admissibility of controls and contracts; for instance, we have that

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^\alpha} \left[ \left| J_t^A(\xi, \mathbb{P}^{\alpha^*}) \right| e^{r \int_0^t c^A(\alpha_s, X_s) ds} \right] &= \mathbb{E}^{\mathbb{P}^\alpha} \left[ \left| \mathbb{E}_t^{\mathbb{P}^{\alpha^*}} \left[ U_A \left( \xi - \int_t^T c^A(\alpha_s^*, X_s) ds \right) \right] \right| e^{r \int_0^t c^A(\alpha_s, X_s) ds} \right], \\
&= \mathbb{E}^{\mathbb{P}^\alpha} \left[ \left| \mathbb{E}_t^{\mathbb{P}^{\alpha^*}} \left[ -e^{-r \left( \xi - \int_0^T c^A((\alpha \otimes_t \alpha^*)_s, X_s) ds \right)} \right] \right| \right], \\
&= \mathbb{E}^{\mathbb{P}^\alpha} \left[ \left| \mathbb{E}_t^{\mathbb{P}^{\alpha^*}} \left[ e^{-r \left( \xi - \int_0^T c^A((\alpha \otimes_t \alpha^*)_s, X_s) ds \right)} \right] \right| \right], \\
&= \mathbb{E}^{\mathbb{P}^{\alpha \otimes_t \alpha^*}} \left[ \left| \mathbb{E}_t^{\mathbb{P}^{\alpha \otimes_t \alpha^*}} \left[ e^{-r \left( \xi - \int_0^T c^A((\alpha \otimes_t \alpha^*)_s, X_s) ds \right)} \right] \right| \right], \\
&= \mathbb{E}^{\mathbb{P}^{\alpha \otimes_t \alpha^*}} \left[ e^{-r \left( \xi - \int_0^T c^A((\alpha \otimes_t \alpha^*)_s, X_s) ds \right)} \right], \\
&\leq \mathbb{E}^0 \left[ \left( L_T^{\alpha \otimes_t \alpha^*} \right)^{p_1} \right]^{\frac{1}{p_1}} \mathbb{E}^0 \left[ e^{rp_2 \int_0^T c^A((\alpha \otimes_t \alpha^*)_s, X_s) ds} \right]^{\frac{1}{p_2}} \mathbb{E}^0 \left[ e^{-rp_3 \xi} \right]^{\frac{1}{p_3}}, \\
&< +\infty.
\end{aligned} \tag{3.7.110}$$

Furthermore, the processes  $\left( J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{r \int_0^t c^A(\alpha_s^*, X_s) ds} \right)_{t \in [0, T]}$  is a  $(\mathbb{P}^{\alpha^*}, \mathbb{F})$ -UI martingale, since by the super-martingale inequality and the tower property of conditional expectations, we have for every  $t_1 \leq t_2 \in [0, T]$ :

$$\begin{aligned}
J_{t_1}^A(\xi, \mathbb{P}^{\alpha^*}) e^{r \int_0^{t_1} c^A(\alpha_s^*, X_s) ds} &\geq \mathbb{E}_{t_1}^{\mathbb{P}^{\alpha^*}} \left[ J_{t_2}^A(\xi, \mathbb{P}^{\alpha^*}) e^{r \int_0^{t_2} c^A(\alpha_s^*, X_s) ds} \right], \\
&= \mathbb{E}_{t_1}^{\mathbb{P}^{\alpha^*}} \left[ \mathbb{E}_{t_2}^{\mathbb{P}^{\alpha^*}} \left[ U_A \left( \xi - \int_0^T c^A(\alpha_s^*, X_s) ds \right) \right] \right], \\
&= \mathbb{E}_{t_1}^{\mathbb{P}^{\alpha^*}} \left[ U_A \left( \xi - \int_0^T c^A(\alpha_s^*, X_s) ds \right) \right], \\
&= J_{t_1}^A(\xi, \mathbb{P}^{\alpha^*}) e^{r \int_0^{t_1} c^A(\alpha_s^*, X_s) ds},
\end{aligned} \tag{3.7.111}$$

and therefore all the previous terms are equal a.s., in particular, for  $t \in [0, T]$

$$J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{r \int_0^t c^A(\alpha_s^*, X_s) ds} = \mathbb{E}_t^{\mathbb{P}^{\alpha^*}} \left[ U_A \left( \xi - \int_0^T c^A(\alpha_s^*, X_s) ds \right) \right], \tag{3.7.112}$$

which proves that  $\left( J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{r \int_0^t c^A(\alpha_s^*, X_s) ds} \right)_{t \in [0, T]}$  is a  $\mathbb{P}^{\alpha^*}$ -closed (and so UI) martingale. The same can be said using the certainty equivalent process  $\left( -e^{-r \left( u_t^A(\xi) - \int_0^t c^A(\alpha_s^*, X_s) ds \right)} \right)_{t \in [0, T]}$ :

$$e^{-r u_t^A(\xi)} = \mathbb{E}_t^{\mathbb{P}^{\alpha^*}} \left[ e^{-r \left( \xi - \int_t^T c^A(\alpha_s^*, X_s) ds \right)} \right]. \tag{3.7.113}$$

We can then apply the martingale representation theorem and Itô formula to prove that there exists a predictable process  $\tilde{Z} \in \mathbb{H}_{loc}^2$  valued in  $\mathbb{R}^d$  such that the following representation holds

$$\begin{aligned} J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{r \int_0^t c^A(\alpha_s^*, X_s) ds} &= J_0^A(\xi, \mathbb{P}^{\alpha^*}) + \int_0^t \tilde{Z}_s dW_s^{\alpha^*} \\ &= J_0^A(\xi, \mathbb{P}^{\alpha^*}) \mathcal{E} \left( -r \int_0^t Z_s \cdot \sigma dW_s^{\alpha^*} \right), \end{aligned} \quad (3.7.114)$$

where

$$Z_t := -\frac{\sigma^{-1} \tilde{Z}_t}{r J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{r \int_0^t c^A(\alpha_s^*, X_s) ds}}, \quad (3.7.115)$$

and

$$\mathcal{E} \left( -r \int_0^t Z_s \cdot \sigma dW_s^{\alpha^*} \right)_{t \in [0, T]} \text{ is a UI martingale,} \quad (3.7.116)$$

from the integrability of  $\left( J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{r \int_0^t c^A(\alpha_s^*, X_s) ds} \right)_{t \in [0, T]}$ . In particular, and under  $\mathbb{P}^{\alpha^*}$ , we have  $u_T^A(\xi) = \xi$ , and

$$-e^{-r \left( \xi - \int_t^T c^A(\alpha_s^*, X_s) ds - \int_t^T Z_s \cdot \sigma dW_s^{\alpha^*} - \frac{r}{2} \int_t^T |\sigma Z_s|^2 ds \right)} = -e^{-r u_t^A(\xi)}. \quad (3.7.117)$$

Now, for an arbitrary  $\alpha \in \mathcal{U}$  we recall that  $\tilde{Y}_t^\alpha := J_t^A(\xi, \mathbb{P}^{\alpha^*}) e^{r \int_0^t c^A(\alpha_s, X_s) ds}$  is a  $(\mathbb{P}^\alpha, \mathbb{F})$ -supermartingale and we compute

$$\frac{\tilde{Y}_t^\alpha}{r} = \frac{1}{r} J_0^A(\xi, \mathbb{P}^{\alpha^*}) \mathcal{E} \left( -r \int_0^t Z_s \cdot \sigma dW_s^{\alpha^*} \right) e^{r \int_0^t c^A(\alpha_s, X_s) - c^A(\alpha_s^*, X_s) ds}. \quad (3.7.118)$$

We apply then Itô formula and Girsanov Theorem, therefore

$$\begin{aligned} \frac{d\tilde{Y}_t^\alpha}{r\tilde{Y}_t^\alpha} &= (c^A(\alpha_t, X_t) - c^A(\alpha_t^*, X_t)) dt - Z_t \cdot \sigma dW_t^{\alpha^*}, \\ &= -\{((A_0 + A_1 X_t + B_1 \alpha_t) \cdot Z_t - c^A(\alpha_t, X_t)) - ((A_0 + A_1 X_t + B_1 \alpha_t^*) \cdot Z_t - c^A(\alpha_t^*, X_t))\} dt, \\ &\quad - Z_t \cdot \sigma dW_t^\alpha \end{aligned} \quad (3.7.119)$$

and by the supermartingale property and the sign of  $\tilde{Y}_t^\alpha$  we conclude that

$$\alpha_t^* \in \operatorname{argmax} \left( (A_0 + A_1 X_t + B_1 \alpha_t) \cdot Z_t - c^A(\alpha_t, X_t) \right), \text{ for } t \in [0, T], \quad (3.7.120)$$

and so  $\alpha_t^* = \hat{\alpha}(Z_t)$  for  $t \in [0, T]$ , and therefore the recommended effort is an admissible effort by assumption on the set of contracts.

Finally, we define the process  $Y_t := U_A^{-1} \left( J_t^A \left( \xi, \mathbb{P}^{\alpha^*} \right) \right)$  which, by Itô Formula has the following dynamics

$$\begin{aligned} Y_t &= U_A^{-1} \left( J_0^A \left( \xi, \mathbb{P}^{\alpha^*} \right) \right) + \int_0^t Z_s \cdot \sigma dW_s^{\alpha^*} + \int_0^t c^A \left( \alpha_s^*, X_s \right) ds + \frac{r}{2} \int_0^t |\sigma Z_s|^2 ds, \\ &= U_A^{-1} \left( J_0^A \left( \xi, \mathbb{P}^{\alpha^*} \right) \right) + \int_0^t Z_s \cdot dX_s - \int_0^t \left( (A_0 + A_1 X_s + B_1 \alpha_s^*) \cdot Z_s - c^A \left( \alpha_s^*, X_s \right) \right) ds + \frac{r}{2} \int_0^t |\sigma Z_s|^2 ds, \quad (3.7.121) \\ &= U_A^{-1} \left( J_0^A \left( \xi, \mathbb{P}^{\alpha^*} \right) \right) + \int_0^t Z_s \cdot dX_s - \int_0^t H(X_s, Z_s) ds + \frac{r}{2} \int_0^t |\sigma Z_s|^2 ds, \end{aligned}$$

which concludes the proof of the theorem; as we just proved the existence of a pair

$$\left( U_A^{-1} \left( J_0^A \left( \xi, \mathbb{P}^{\alpha^*} \right) \right), Z \right) \in \mathbb{R} \times \mathcal{V}, \quad (3.7.122)$$

satisfying  $Y_T^{U_A^{-1}(J_0^A(\xi, \mathbb{P}^{\alpha^*})), Z} = \xi$ , where we defined  $Y_t := U_A^{-1} \left( V_t^A(\xi) \right)$  and  $Z_t := -\frac{\sigma^{-1} \dot{Z}_t}{r V_t^A(\xi) e^{r \int_0^t c^A(\alpha_s^*, X_s) ds}}$ ,

such that  $\mathcal{E} \left( -r \int_0^t Z_t \cdot \sigma dW_t^\alpha \right)$  is a UI martingale. Finally by the estimates in [Briand & Hu \(2008\)](#)[Corollary 4] and the integrability assumption on admissible contracts, we obtain

$$\mathbb{E}^0 \left[ e^{\lambda p_3 \sup_{t \leq r \leq T} |Y_r|} \right] < +\infty. \quad (3.7.123)$$

□

### 3.7.11 Proof of Lemma 3.7.4 : The Dynamic Programming Principle

The proof is similar to the one in ([Cvitanic & Karatzas 1993](#), Proposition 6.2). We proceed in two steps proving each of the two inequalities. The first inequality is a direct consequence of the tower property. In fact, for  $\tau_1, \tau_2 \in \mathcal{T}_{0,T}$ , with  $\tau_1 \leq \tau_2$ , we have by definition

$$V_{\tau_1}^A(\xi) = \operatorname{esssup}_{\alpha \in \mathcal{U}_{\tau_1}} \mathbb{E}_{\tau_1}^{\mathbb{P}^{\alpha}} \left[ -e^{-r \left( \xi - \int_{\tau_1}^T c^A(\alpha_s, X_s) ds \right)} \right] = \operatorname{esssup}_{\alpha \in \mathcal{U}_{\tau_1}} \mathbb{E}_{\tau_1}^{\mathbb{P}^{\alpha}} \left[ -e^{-r \left( \xi - \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds - \int_{\tau_2}^T c^A(\alpha_s, X_s) ds \right)} \right]. \quad (3.7.124)$$

By the tower property of the expectation we write

$$V_{\tau_1}^A(\xi) = \operatorname{esssup}_{\alpha \in \mathcal{U}_{\tau_1}} \mathbb{E}_{\tau_1}^{\mathbb{P}^{\alpha}} \left[ e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} \mathbb{E}_{\tau_2}^{\mathbb{P}^{\alpha}} \left[ -e^{-r \left( \xi - \int_{\tau_2}^T c^A(\alpha_s, X_s) ds \right)} \right] \right]. \quad (3.7.125)$$

Using Bayes rule and remarking that  $\mathbb{E}_{\tau_2}^{\mathbb{P}^{\alpha}} \left[ -e^{-r \left( \xi - \int_{\tau_2}^T c^A(\alpha_s, X_s) ds \right)} \right]$  depends only on values of  $\alpha$  after  $\tau_2$ , we have that for an arbitrary  $\alpha \in \mathcal{U}$

$$\begin{aligned} \mathbb{E}_{\tau_2}^{\mathbb{P}^{\alpha}} \left[ -e^{-r \left( \xi - \int_{\tau_2}^T c^A(\alpha_s, X_s) ds \right)} \right] &\leq \operatorname{esssup}_{\alpha \in \mathcal{U}} \mathbb{E}_{\tau_2}^{\mathbb{P}^{\alpha}} \left[ U_A \left( \xi - \int_{\tau_2}^T c^A(\alpha_s, X_s) ds \right) \right], \\ &= \operatorname{esssup}_{\alpha \in \mathcal{U}_{\tau_2}} \mathbb{E}_{\tau_2}^{\mathbb{P}^{\alpha}} \left[ U_A \left( \xi - \int_{\tau_2}^T c^A(\alpha_s, X_s) ds \right) \right] = V_{\tau_2}^A(\xi), \end{aligned} \quad (3.7.126)$$

and so

$$V_{\tau_1}^A(\xi) \leq \operatorname{esssup}_{\mathbb{P}^\alpha \in \mathcal{P}} \mathbb{E}_{\tau_1}^{\mathbb{P}^\alpha} \left[ V_{\tau_2}^A(\xi) e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} \right]. \quad (3.7.127)$$

To prove the second inequality, we consider  $\alpha \in \mathcal{U}$  and  $\nu \in \mathcal{U}_{\tau_2}$ , and define then the concatenation of the two controls for  $0 \leq s \leq T$  as  $(\alpha \otimes_{\tau_2} \nu)_s := \alpha_s \mathbb{1}_{0 \leq s \leq \tau_2} + \nu_s \mathbb{1}_{\tau_2 \leq s \leq T}$ . We have then  $(\alpha \otimes_{\tau_2} \nu) \in \mathcal{U}$ , and by definition of the essential supremum (where we denote  $\mathbb{E}_{\tau_1}^{\alpha \otimes_{\tau_2} \nu}$  instead of  $\mathbb{E}_{\tau_1}^{\mathbb{P}^{\alpha \otimes_{\tau_2} \nu}}$ ):

$$\begin{aligned} V_{\tau_1}^A(\xi) &\geq \mathbb{E}_{\tau_1}^{\alpha \otimes_{\tau_2} \nu} \left[ -e^{-r \left( \xi - \int_{\tau_1}^T c^A((\alpha \otimes_{\tau_2} \nu)_s, X_s) ds \right)} \right], \\ &= \mathbb{E}_{\tau_1}^{\alpha \otimes_{\tau_2} \nu} \left[ -e^{-r \left( - \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds - \int_{\tau_2}^T c^A(\nu_s, X_s) ds \right)} e^{-r\xi} \right], \\ &= \mathbb{E}_{\tau_1}^{\alpha \otimes_{\tau_2} \nu} \left[ e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} \mathbb{E}_{\tau_2}^{\alpha \otimes_{\tau_2} \nu} \left[ -e^{-r \left( \xi - \int_{\tau_2}^T c^A(\nu_s, X_s) ds \right)} \right] \right]. \end{aligned} \quad (3.7.128)$$

Using again Bayes formula on the conditional expectation w.r.t  $\mathcal{F}_{\tau_2}$ , we have that

$$\mathbb{E}_{\tau_2}^{\alpha \otimes_{\tau_2} \nu} \left[ -e^{-r \left( \xi - \int_{\tau_2}^T c^A(\nu_s, X_s) ds \right)} \right] = \mathbb{E}_{\tau_2}^0 \left[ -\frac{L_T^{\alpha \otimes_{\tau_2} \nu}}{L_{\tau_2}^{\alpha \otimes_{\tau_2} \nu}} e^{-r \left( \xi - \int_{\tau_2}^T c^A(\nu_s, X_s) ds \right)} \right]. \quad (3.7.129)$$

Now notice that  $\frac{L_T^{\alpha \otimes_{\tau_2} \nu}}{L_{\tau_2}^{\alpha \otimes_{\tau_2} \nu}} = \frac{L_T^\nu}{L_{\tau_2}^\nu}$  (as stated earlier the change of measure applied to the conditional expectation depends only on the control after  $\tau_2$ ). We have therefore

$$\begin{aligned} \mathbb{E}_{\tau_2}^{\alpha \otimes_{\tau_2} \nu} \left[ -e^{-r \left( \xi - \int_{\tau_2}^T c^A(\nu_s, X_s) ds \right)} \right] &= \mathbb{E}_{\tau_2}^0 \left[ -\frac{L_T^\nu}{L_{\tau_2}^\nu} e^{-r \left( \xi - \int_{\tau_2}^T c^A(\nu_s, X_s) ds \right)} \right] \\ &= J_{\tau_2}^A(\xi, \mathbb{P}^\nu). \end{aligned} \quad (3.7.130)$$

Thus we obtain the following inequality

$$V_{\tau_1}^A(\xi) \geq \mathbb{E}_{\tau_1}^{\alpha \otimes_{\tau_2} \nu} \left[ e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} J_{\tau_2}^A(\xi, \mathbb{P}^\nu) \right]. \quad (3.7.131)$$

We use again Bayes Formula for the change of measure and the tower property of conditional expectation leading to

$$\begin{aligned} V_{\tau_1}^A(\xi) &\geq \mathbb{E}_{\tau_1}^0 \left[ \frac{L_T^{\alpha \otimes_{\tau_2} \nu}}{L_{\tau_1}^{\alpha \otimes_{\tau_2} \nu}} e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} J_{\tau_2}^A(\xi, \mathbb{P}^\nu) \right], \\ &= \mathbb{E}_{\tau_1}^0 \left[ \mathbb{E}_{\tau_2}^0 \left[ \frac{L_T^{\alpha \otimes_{\tau_2} \nu}}{L_{\tau_1}^{\alpha \otimes_{\tau_2} \nu}} \frac{L_{\tau_2}^{\alpha \otimes_{\tau_2} \nu}}{L_{\tau_2}^{\alpha \otimes_{\tau_2} \nu}} e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} J_{\tau_2}^A(\xi, \mathbb{P}^\nu) \right] \right], \\ &= \mathbb{E}_{\tau_1}^0 \left[ \mathbb{E}_{\tau_2}^0 \left[ \frac{L_T^{\alpha \otimes_{\tau_2} \nu}}{L_{\tau_2}^{\alpha \otimes_{\tau_2} \nu}} \right] \frac{L_{\tau_2}^{\alpha \otimes_{\tau_2} \nu}}{L_{\tau_1}^{\alpha \otimes_{\tau_2} \nu}} e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} J_{\tau_2}^A(\xi, \mathbb{P}^\nu) \right], \\ &= \mathbb{E}_{\tau_1}^0 \left[ \frac{L_T^{\alpha \otimes_{\tau_2} \nu}}{L_{\tau_1}^{\alpha \otimes_{\tau_2} \nu}} e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} J_{\tau_2}^A(\xi, \mathbb{P}^\nu) \right]. \end{aligned} \quad (3.7.132)$$

Now recall that for  $0 \leq s \leq \tau_2$  we have by definition  $(\alpha \otimes_{\tau_2} \nu)_s = \alpha_s$ , and therefore  $\frac{L_{\tau_2}^{\alpha \otimes_{\tau_2} \nu}}{L_{\tau_1}^{\alpha \otimes_{\tau_2} \nu}} = \frac{L_{\tau_2}^{\alpha}}{L_{\tau_1}^{\alpha}}$  leading to

$$V_{\tau_1}^A(\xi) \geq \mathbb{E}_{\tau_1}^0 \left[ \frac{L_{\tau_2}^{\alpha}}{L_{\tau_1}^{\alpha}} e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} J_{\tau_2}^A(\xi, \mathbb{P}^{\nu}) \right] = \mathbb{E}_{\tau_1}^{\alpha} \left[ e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} J_{\tau_2}^A(\xi, \mathbb{P}^{\nu}) \right]. \quad (3.7.133)$$

The inequality (3.7.133) holds for  $\alpha \in \mathcal{U}$  and  $\nu \in \mathcal{U}_{\tau_2}$ . Furthermore, the family  $\mathcal{J}_{\tau_2}^A(\xi)$  satisfies the lattice property, i.e., for every pair of controls  $\nu$  and  $\nu'$  in  $\mathcal{U}_{\tau_2}$ , there exists a control  $\tilde{\nu} \in \mathcal{U}_{\tau_2}$  such that

$$J_{\tau_2}^A(\xi, \mathbb{P}^{\tilde{\nu}}) \geq \max \left( J_{\tau_2}^A(\xi, \mathbb{P}^{\nu}), J_{\tau_2}^A(\xi, \mathbb{P}^{\nu'}) \right), \quad (3.7.134)$$

for instance  $\tilde{\nu}$  can be chosen as

$$\tilde{\nu} := \nu \mathbb{1}_{\{J_{\tau_2}^A(\xi, \mathbb{P}^{\nu}) \geq J_{\tau_2}^A(\xi, \mathbb{P}^{\nu'})\}} + \nu' \mathbb{1}_{\{J_{\tau_2}^A(\xi, \mathbb{P}^{\nu}) < J_{\tau_2}^A(\xi, \mathbb{P}^{\nu'})\}}. \quad (3.7.135)$$

So, by (Neveu 1972, Proposition VI.I.I, p121),  $V_{\tau_2}^A(\xi)$  can be approached by a non-decreasing sequence, i.e., there exists a sequence of  $(\mathbb{P}^{\nu^n})_{n \geq 0}$  such that

$$V_{\tau_2}^A(\xi) = \lim_{n \rightarrow +\infty} \uparrow J_{\tau_2}^A(\xi, \mathbb{P}^{\nu^n}). \quad (3.7.136)$$

We can then apply the monotone convergence theorem, so we have for  $\alpha \in \mathcal{U}$

$$\begin{aligned} V_{\tau_1}^A(\xi) &\geq \lim_{n \rightarrow +\infty} \uparrow \mathbb{E}_{\tau_1}^{\alpha} \left[ e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} J_{\tau_2}^A(\xi, \mathbb{P}^{\nu^n}) \right], \\ &= \mathbb{E}_{\tau_1}^{\alpha} \left[ e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} \lim_{n \rightarrow +\infty} \uparrow J_{\tau_2}^A(\xi, \mathbb{P}^{\nu^n}) \right], \\ &= \mathbb{E}_{\tau_1}^{\alpha} \left[ e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} V_{\tau_2}^A(\xi) \right], \end{aligned} \quad (3.7.137)$$

in particular

$$V_{\tau_1}^A(\xi) \geq \operatorname{esssup}_{\mathbb{P}^{\alpha} \in \mathcal{P}} \mathbb{E}_{\tau_1}^{\alpha} \left[ e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} V_{\tau_2}^A(\xi) \right]. \quad (3.7.138)$$

From the two inequalities (3.7.127) and (3.7.138), we have that

$$V_{\tau_1}^A(\xi) = \operatorname{esssup}_{\mathbb{P}^{\alpha} \in \mathcal{P}} \mathbb{E}_{\tau_1}^{\alpha} \left[ e^{r \int_{\tau_1}^{\tau_2} c^A(\alpha_s, X_s) ds} V_{\tau_2}^A(\xi) \right]. \quad (3.7.139)$$

Therefore, the family  $\left( V_{\tau}^A(\xi) e^{r \int_0^{\tau} c^A(\alpha_s, X_s) ds} \right)_{\tau \in \mathcal{T}_{0,T}}$  is a  $(\mathbb{P}^{\alpha}, \mathbb{F})$ -supermartingale system, and by the results of [Lenglart & Dellacherie \(1981\)](#), it can be aggregated by a unique  $\mathbb{F}$ -optional process up to indistinguishability, which coincides with  $\left( V_t^A(\xi) e^{r \int_0^t c^A(\alpha_s, X_s) ds} \right)_{t \in [0,T]}$ , and remains a  $(\mathbb{P}^{\alpha}, \mathbb{F})$ -supermartingale, which then admits a càd-làg modification since the filtration considered satisfies the usual conditions, concluding the proof of Lemma 3.7.4.  $\square$

### 3.7.12 Proof of Lemma 3.5.2: agent response to revealing contracts

For  $Y_T^{Y_0, Z} \in \mathcal{Z}$ , and for every  $\mathbb{P}^\alpha \in \mathcal{P}$ , we have that

$$J^A \left( Y_T^{Y_0, Z}, \mathbb{P}^\alpha \right) = \mathbb{E}^{\mathbb{P}^\alpha} \left[ U_A \left( Y_T^{Y_0, Z} - \int_0^T c^A(\alpha_s, X_s) ds \right) \right], \quad (3.7.140)$$

Using the definition of  $Y_T^{Y_0, Z}$ , we have

$$\begin{aligned} Y_T^{Y_0, Z} - \int_0^T c^A(\alpha_s, X_s) ds &= Y_0 + \int_0^T Z_s \cdot dX_s - \int_0^T H(s, X_s, Z_s) ds + \frac{r}{2} \int_0^T |\sigma Z_s|^2 ds - \int_0^T c^A(\alpha_s, X_s) ds, \\ &= Y_0 + \int_0^T (h(s, X_s, Z_s, \alpha_s) - H(s, X_s, Z_s)) ds + \int_0^T Z_s \cdot \sigma dW_s^\alpha + \frac{r}{2} \int_0^T |\sigma Z_s|^2 ds. \end{aligned} \quad (3.7.141)$$

therefore, applying the exponential utility, we obtain

$$U_A \left( Y_T^{Y_0, Z} - \int_0^T c^A(\alpha_s, X_s) ds \right) = U_A \left( Y_0 + \int_0^T (h(s, X_s, Z_s, \alpha_s) - H(s, X_s, Z_s)) ds \right) \mathcal{E} \left( -r \int_0^T \sigma Z_s \cdot dW_s^\alpha \right). \quad (3.7.142)$$

From the definition of the set  $\mathcal{V}$ , we have that  $Z$  satisfies

$$\mathbb{E}^{\mathbb{P}^\alpha} \left[ \mathcal{E} \left( -r \int_0^T \sigma Z_s \cdot dW_s^\alpha \right) \right] = 1, \quad (3.7.143)$$

and we can define the change of measure

$$\frac{d\widetilde{\mathbb{P}}^\alpha}{d\mathbb{P}^\alpha} := \mathcal{E} \left( -r \int_0^T \sigma Z_s \cdot dW_s^\alpha \right), \quad (3.7.144)$$

and so

$$J^A \left( Y_T^{Y_0, Z}, \mathbb{P}^\alpha \right) = \mathbb{E}^{\widetilde{\mathbb{P}}^\alpha} \left[ U_A \left( Y_0 + \int_0^T (h(s, X_s, Z_s, \alpha_s) - H(s, X_s, Z_s)) ds \right) \right], \quad (3.7.145)$$

and since  $H$  is defined as the supremum of  $h$ , we obtain an upper bound

$$V^A(Y_T^{Y_0, Z}) \leq U_A(Y_0), \quad (3.7.146)$$

with equality

$$V^A(Y_T^{Y_0, Z}) = U_A(Y_0), \quad (3.7.147)$$

for the recommended policy  $(\hat{\alpha}(Z_t))_{t \in [0, T]}$  which satisfies  $h(X_t, Z_t, \hat{\alpha}(Z_t)) = H(X_t, Z_t)$ , for  $t \in [0, T]$ , and is admissible from the definition of  $\mathcal{V}$ . Therefore, controlled state process has the following dynamic

$$\begin{aligned} X_t - x_0 &= \int_0^t (\bar{A}_0 + A_1 X_s + B_1 \hat{\alpha}(Z_s)) ds + \sigma W_t^{\hat{\alpha}}, \\ &= \int_0^t (\bar{A}_0 + A_1 X_s + B_0 Z_s) ds + \sigma W_t^{\hat{\alpha}}. \end{aligned} \quad (3.7.148)$$

□

### 3.7.13 Proof of Lemma 3.5.3: Riccati equation for second best

The proof is similar to the one of Lemma 3.4.1; The function  $v(t, x, y) := -e^{-\rho(-y + \frac{1}{2}\Gamma_t x \cdot x + P_t x + R_t)}$  is smooth as a composition of smooth functions and in particular in  $C^{1,2}([0, T], \mathbb{R}^{d+1})$ . It satisfies the boundary condition (at  $t = T$ ) of (3.5.20), since the Riccati equation has the appropriate terminal condition and  $R_T = 0$ . Furthermore, we define  $u(t, x) := \frac{1}{2}\Gamma_t x \cdot x + P_t \cdot x + R_t$ . We start by computing the maximizer of  $h_{\text{SB}}$ . Since

$$\begin{aligned} h_{\text{SB}}(x, y, v, Dv, D^2v, z) &= (\bar{A}_0 + A_1 x) \cdot D_x v + \frac{1}{2}\sigma^2 : D_{xx} v - \rho v \ell(x) - D_y v (C_0 - \frac{1}{2}M_2 x \cdot x - M_1 \cdot x) \\ &\quad + \frac{1}{2} \left( D_y v B_0 + (D_{yy} v + r D_y v) \sigma^2 \right) z \cdot z + \left( B_0 D_x v + \sigma^2 D_{xy}^2 v \right) \cdot z, \end{aligned} \quad (3.7.149)$$

it is equivalent to compute the maximizer of

$$\bar{h}(x, y, v, Dv, D^2v, z) := \frac{1}{2} \left( D_y v B_0 + (D_{yy} v + r D_y v) \sigma^2 \right) z \cdot z + \left( B_0 D_x v + \sigma^2 D_{xy}^2 v \right) \cdot z. \quad (3.7.150)$$

We can then use the guess on the form of the solution  $v$  which leads to

$$\begin{aligned} \bar{h}(x, y, v, Dv, D^2v, z) &= \frac{D_y v}{2} \left( B_0 + \left( \frac{D_{yy} v}{D_y v} + r \right) \sigma^2 \right) z \cdot z + \left( B_0 D_x v + \sigma^2 D_{xy}^2 v \right) \cdot z, \\ &= \frac{\rho v}{2} \left( (B_0 + (\rho + r) \sigma^2) z \cdot z - 2(B_0 + \sigma^2 \rho) Du \cdot z \right). \end{aligned} \quad (3.7.151)$$

where we used that  $D_y v = \rho v > 0$ , and  $\frac{D_{yy} v}{D_y v} = \rho$  together with  $D_{xy}^2 v = \rho D_x v$ , and  $D_x v = -\rho v Du$ . Since  $v < 0$  and  $(B_0 + (r + \rho)\sigma^2) \geq 0$ , we can compute the maximizer of  $\bar{h}$ , or equivalently of  $h_{\text{SB}}$  as a function of  $Du$ , which gives us

$$\begin{aligned} z^*(t, x) &= \left( B_0 + (\rho + r) \sigma^2 \right)^{-1} \left( B_0 + \rho \sigma^2 \right) Du(t, x), \\ &= \left( B_0 + (\rho + r) \sigma^2 \right)^{-1} \left( B_0 + \rho \sigma^2 \right) (\Gamma_t x + P_t), \end{aligned} \quad (3.7.152)$$

and we can compute

$$\begin{aligned} \frac{-1}{\rho v} \left( \partial_t v(t, x, y) + \mathbf{H}_{\text{SB}}(x, y, v, Dv, D^2v) \right) &= \partial_t u(t, x) + (\bar{A}_0 + A_1 x) \cdot Du(t, x) + \frac{1}{2}\sigma^2 : (D^2 u - \rho Du Du^\top) \\ &\quad + \ell(x) + C_0 - \frac{1}{2}M_2 x \cdot x - M_1 \cdot x, \\ &\quad + \frac{1}{2} \left( B_0 + \rho \sigma^2 \right) \left( B_0 + (\rho + r) \sigma^2 \right)^{-1} \left( B_0 + \rho \sigma^2 \right) Du(t, x) \cdot Du(t, x), \\ &= \partial_t u(t, x) + \frac{1}{2}\sigma^2 : D^2 u + (\bar{A}_0 + A_1 x) \cdot Du(t, x) \\ &\quad + \ell(x) + C_0 - \frac{1}{2}M_2 x \cdot x - M_1 \cdot x + \frac{1}{2}Q_{\text{SB}} Du(t, x) \cdot Du(t, x), \end{aligned} \quad (3.7.153)$$

we replace then  $u$  and its derivatives by its definition

$$\begin{aligned}
\frac{-1}{\rho v} \left( \partial_t v(t, x, y) + \mathbf{H}_{\text{SB}}(x, y, v, Dv, D^2v) \right) &= \frac{1}{2} \dot{\Gamma}_t x \cdot x + \dot{P}_t \cdot x + \dot{R}_t + \frac{1}{2} \sigma^2 : \Gamma_t + (\bar{A}_0 + A_1 x) \cdot (\Gamma_t x + P_t) \\
&\quad + \left( (L_1 - M_1) \cdot x - \frac{1}{2} (L_2 + M_2) x \cdot x \right) \\
&\quad + \frac{1}{2} Q_{\text{SB}} (\Gamma_t x + P_t) \cdot (\Gamma_t x + P_t) + C_0 \\
&= \frac{1}{2} \left( \dot{\Gamma}_t + A_1^T \Gamma_t + \Gamma_t A_1 + \Gamma_t Q_{\text{SB}} \Gamma_t - (L_2 + M_2) \right) x \cdot x \\
&\quad + \left( \dot{P}_t + \Gamma_t \bar{A}_0 + A_1^T P_t + (L_1 - M_1) + \Gamma_t Q_{\text{SB}} P_t \right) \cdot x \\
&\quad + \left( \dot{R}_t + \bar{A}_0 \cdot P_t + \frac{1}{2} \sigma^2 : \Gamma_t + \frac{1}{2} Q_{\text{SB}} P_t \cdot P_t + C_0 \right) \\
&= 0,
\end{aligned} \tag{3.7.154}$$

where we used the system (3.5.23) in the last equality to cancel each of the terms, which leads to

$$\partial_t v + \mathbf{H}_{\text{SB}}(x, y, v, Dv, D^2v) = 0 \text{ in } [0, T] \times \mathbb{R}^{d+1}. \tag{3.7.155}$$

and the function  $u$  satisfies

$$\begin{cases} u(T, x) = G_1 \cdot x, \\ \partial_t u + \frac{1}{2} \sigma^2 : D^2 u + (\bar{A}_0 + A_1 x) \cdot Du + \ell(x) + C_0 - \frac{1}{2} M_2 x \cdot x - M_1 \cdot x + \frac{1}{2} Q_{\text{SB}} Du \cdot Du = 0, \text{ in } [0, T] \times \mathbb{R}^d \end{cases} \tag{3.7.156}$$

□

### 3.7.14 Proof of Proposition 3.5.4 : Solving the second best by verification

(i) Since  $Q_{\text{SB}} > 0$  and  $(L_2 + M_2) \geq 0$ , we have from Lemma 3.7.1 that the matrix Riccati system (3.5.23) is wellposed and so is (3.5.25) and (3.5.25) with a unique solution denoted  $(\Gamma, P, R)$ . Define  $v \in C^{1,2}([0, T], \mathbb{R}^{d+1})$  the function

$$v(t, x, y) := -e^{-\rho(-y + \frac{1}{2} \Gamma_t x \cdot x + P_t \cdot x + R_t)}, \tag{3.7.157}$$

which is a classical solution to the PDE (3.5.20) from lemma 3.5.3. We also introduce

$$\beta_{t,T} := e^{-\rho \int_t^T \ell(X_s) ds}, \tag{3.7.158}$$

and we define the sequence of stopping times

$$T_n := T \wedge \inf \left\{ s > t, \int_t^s (\beta_{t,r})^2 \left( |\sigma D_x v(r, X_r, Y_r)|^2 + (D_y v(r, X_r, Y_r))^2 |\sigma Z_r|^2 \right) dr \geq n \right\}. \tag{3.7.159}$$

By applying Itô formula and taking the expectation, we obtain



$$\begin{aligned}
\mathbb{E}_{t,x,y}^{\mathbb{P}^{\hat{\alpha}}} [\beta_{t,T_n} v (T_n, X_{T_n}, Y_{T_n})] &= v(t, x, y) \\
&+ \mathbb{E}_{t,x,y}^{\mathbb{P}^{\hat{\alpha}}} \left[ \int_t^{T_n} \beta_{t,s} \left( \partial_t v(s, X_s, Y_s) + h_{\text{SB}}((s, X_s, Y_s), v, Dv, D^2v, Z_s) \right) ds \right] \\
&+ \mathbb{E}_{t,x,y}^{\mathbb{P}^{\hat{\alpha}}} \left[ \int_t^{T_n} \beta_{t,s} D_x v(s, X_s, Y_s) \cdot \sigma dW_s^{\hat{\alpha}} + \int_t^{T_n} \beta_{t,s} D_y v(s, X_s, Y_s) Z_s \cdot \sigma dW_s^{\hat{\alpha}} \right],
\end{aligned} \tag{3.7.160}$$

By construction of the family of stopping times  $T_n$ , the stochastic integral has zero expectation

$$\mathbb{E}_{t,x,y}^{\mathbb{P}^{\hat{\alpha}}} \left[ \int_t^{T_n} \beta_{t,s} D_x v(s, X_s, Y_s) \cdot \sigma dW_s^{\hat{\alpha}} + \int_t^{T_n} \beta_{t,s} D_y v(s, X_s, Y_s) Z_s \cdot \sigma dW_s^{\hat{\alpha}} \right] = 0, \tag{3.7.161}$$

and since  $v$  is a solution to the PDE (3.5.20), and from the monotonicity of the expectation, we have the inequality

$$\mathbb{E}_{t,x,y}^{\mathbb{P}^{\hat{\alpha}}} \left[ \int_t^{T_n} \beta_{t,s} \left( \partial_t v(s, X_s, Y_s) + h_{\text{SB}}(X_s, Y_s, v(s, X_s, Y_s), Dv(s, X_s, Y_s), D^2v(s, X_s, Y_s), Z_s) \right) ds \right] \leq 0, \tag{3.7.162}$$

which leaves us with

$$\mathbb{E}_{t,x}^{\mathbb{P}^{\hat{\alpha}}} [\beta_{t,T_n} v (T_n, X_{T_n}, Y_{T_n})] \leq v(t, x, y). \tag{3.7.163}$$

From the continuity of  $\beta$  and  $v$ , the a.s convergence holds

$$\beta_{t,T_n} v (T_n, X_{T_n}, Y_{T_n}) \rightarrow -e^{-\rho \left( -Y_T + \int_t^T \ell(X_s) ds + g(X_T) \right)}, \tag{3.7.164}$$

and for an admissible contract  $Y$  and the corresponding (admissible) response  $\hat{\alpha}$  we have

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} [|\beta_{0,T} v (T, X_T, Y_T)|] &= \mathbb{E}^{\mathbb{P}^{\hat{\alpha}}} \left[ e^{-\rho \left( -Y_T + \int_0^T \ell(X_s) ds + g(X_T) \right)} \right], \\
&\leq \mathbb{E}^0 \left[ \frac{d\mathbb{P}^{\hat{\alpha}}}{d\mathbb{P}^0} e^{\rho \left( \int_0^T \ell(X_s) ds + g(X_T) \right)} e^{\rho \sup_{0 \leq t \leq T} |Y_t|} \right], \\
&\leq \mathbb{E}^0 \left[ \left( \frac{d\mathbb{P}^{\hat{\alpha}}}{d\mathbb{P}^0} \right)^{p_1} \right]^{\frac{1}{p_1}} \mathbb{E}^0 \left[ e^{\rho p_2 \left( \int_0^T \ell(X_s) ds + g(X_T) \right)} \right]^{\frac{1}{p_2}} \mathbb{E}^0 \left[ e^{\rho p_3 \sup_{t \leq r \leq T} |Y_r|} \right]^{\frac{1}{p_3}}, \\
&< +\infty,
\end{aligned} \tag{3.7.165}$$

where we used Hölder inequality, and the admissibility of the control and the contract together with assumption 3.2.4. Therefore, by localizing and moving the time origin, we obtain

$$\mathbb{E}_{t,x,y}^{\mathbb{P}^{\hat{\alpha}}} [|\beta_{t,T_n} v (T_n, X_{T_n}, Y_{T_n})|] < +\infty, \text{ for } n \geq 1. \tag{3.7.166}$$

We can then use the dominated convergence theorem to obtain

$$\mathbb{E}_{t,x}^{\mathbb{P}^{\hat{\alpha}}} \left[ -e^{-\rho \left( -Y_T + \int_t^T \ell(X_s) ds + g(X_T) \right)} \right] \leq v(t, x, y). \quad (3.7.167)$$

for any admissible control  $Z \in \mathcal{V}_0$ , and so

$$V(t, x, y) \leq v(t, x, y), \quad (3.7.168)$$

in particular

$$V_0 \leq v(0, x_0, 0). \quad (3.7.169)$$

Therefore

$$V_{\text{SB}}^{\text{P}} \leq U_{\text{P}} \left( -\mathcal{R} + \frac{1}{2} \Gamma_0 x \cdot x + P_0 \cdot x + R_0 \right). \quad (3.7.170)$$

(ii) If we assume further that the induced feedback control (the maximizer of the Hamiltonian)  $Z_t^* := z^*(t, X_t)$ , for  $t \in [0, T]$  with

$$z^*(t, x) := \left( B_0 + (\rho + r) \sigma^2 \right)^{-1} \left( B_0 + \rho \sigma^2 \right) (\Gamma_t x + P_t) \quad (3.7.171)$$

is admissible, i.e.,  $Z^* \in \mathcal{V}_0$ :

$$\mathbb{E}^{\mathbb{P}^{\alpha}} \left[ \mathcal{E} \left( -r \int_0^T \sigma Z_s \cdot dW_s^{\alpha} \right) \right] = 1, \text{ for all } \mathbb{P}^{\alpha} \in \mathcal{P}, \quad (3.7.172)$$

and the recommended response is an admissible control, i.e.,  $(\hat{\alpha}(Z_t))_{0 \leq t \leq T} \in \mathcal{U}$ , then by applying Itô formula with  $Z^*$  in (i) we obtain an equality instead of an inequality in (3.7.162), and so using the same arguments

$$V_{\text{SB}}^{\text{P}} = U_{\text{P}} \left( -\mathcal{R} + \frac{1}{2} \Gamma_0 x \cdot x + P_0 \cdot x + R_0 \right), \quad (3.7.173)$$

and the optimal contract is  $\xi_{\text{SB}}^*$  is written

$$\xi_{\text{SB}}^* := \mathcal{R} + \int_0^T z^*(s, X_s) \cdot dX_s - \int_0^T H(X_s, z^*(s, X_s)) ds + \frac{r}{2} \int_0^T |\sigma z^*(s, X_s)|^2 ds, \quad (3.7.174)$$

and is admissible  $\xi_{\text{SB}}^* \in \Xi$  since  $Y_0 = \mathcal{R}$  and  $(z^*(t, X_t))_{t \in [0, T]} \in \mathcal{V}_0$ , and Agent's optimal response is given by

$$\alpha_{\text{SB}}^*(t, X_t) := \hat{\alpha}(z^*(t, X_t)) = C_2^{-1} (B_1^{\top} z^*(t, X_t) - C_1), \quad (3.7.175)$$

which in turn is admissible since  $(z^*(t, X_t))_{t \in [0, T]} \in \mathcal{V}_0$ . Then by defining the certainty equivalent dynamic function  $u_{\text{SB}}$  as

$$u_{\text{SB}}(t, x) := -\mathcal{R} + \frac{1}{2} \Gamma_t x \cdot x + P_t \cdot x + R_t, \quad (3.7.176)$$

we obtain that

$$z^*(t, x) = \left( B_0 + (\rho + r) \sigma^2 \right)^{-1} \left( B_0 + \rho \sigma^2 \right) Du_{\text{SB}}(t, x), \quad (3.7.177)$$

and

$$\alpha_{\text{SB}}^*(t, x) = C_2^{-1} \left( B_1^\top \left( B_0 + (\rho + r) \sigma^2 \right)^{-1} \left( B_0 + \rho \sigma^2 \right) Du_{\text{SB}}(t, x) - C_1 \right). \quad (3.7.178)$$

□

### 3.7.15 Proof of Lemma 3.5.5 : Admissibility of the optimal second Best policy

Our goal is to prove that  $(z^*(t, X_t))_{t \in [0, T]} \in \mathcal{V}_0$ , where we recall that  $z^*(t, x) = \mathbf{D}(\Gamma_t x + P_t)$ , where  $\mathbf{D} = (B_0 + (\rho + r) \sigma^2)^{-1} (B_0 + \rho \sigma^2)$ , with  $\Gamma$  and  $P$  solutions to (3.5.23) and (3.5.25). So we need to check that the following holds

$$\begin{cases} \mathbb{E}^{\mathbb{P}^\alpha} \left[ \mathcal{E} \left( -r \int_0^T \sigma z^*(s, X_s) \cdot dW_s^\alpha \right) \right] = 1, \text{ for all } \mathbb{P}^\alpha \in \mathcal{P}, \\ \hat{\alpha}(z^*(s, X_s))_{0 \leq t \leq T} \in \mathcal{U}, \\ \text{and } \mathbb{E}^0 \left[ e^{\lambda p_3 |\xi_{\text{SB}}^* - \mathcal{R}|} \right] < +\infty, \end{cases} \quad (3.7.179)$$

where we recall that  $\hat{\alpha}(z^*(t, x)) = C_2^{-1} (B_1^\top z^*(t, x) - C_1)$  and

$$\xi_{\text{SB}}^* = \mathcal{R} + \int_0^T z^*(s, X_s) \cdot dX_s - \int_0^T H(X_s, z^*(s, X_s)) ds + \frac{r}{2} \int_0^T |\sigma z^*(s, X_s)|^2 ds. \quad (3.7.180)$$

As in the proof of Lemma 3.7.9, remark that

$$\mathbb{E}^0 \left[ e^{\varepsilon_{\text{SB}}} \int_0^T |\mathbf{D} \Gamma_s X_s|^2 ds \right] \leq \mathbb{E}^0 \left[ e^{\varepsilon_{\text{SB}} \lambda_{\max}(\mathbf{D} \mathbf{D}^\top)} \sup_{0 \leq t \leq T} \lambda_{\max}(\Gamma_t^2) \int_0^T |X_s|^2 ds \right] < +\infty, \quad (3.7.181)$$

by assumption (3.5.38) (and the same argument as in proof of Lemma 3.2.5), and so

$$\mathbb{E}^0 \left[ e^{\varepsilon_{\text{SB}}} \int_0^T |z^*(s, X_s)|^2 ds \right] = \mathbb{E}^0 \left[ e^{\varepsilon_{\text{SB}}} \int_0^T |\mathbf{D}(\Gamma_s X_s + P_s)|^2 ds \right] < +\infty, \quad (3.7.182)$$

and

$$\mathbb{E}^0 \left[ e^{t_{\text{SB}}} \int_0^T |X_s|^2 ds \right] < +\infty, \quad (3.7.183)$$

which is the main ingredient for the whole proof.

(i) By the Hölder inequality, we have

$$\mathbb{E}^{\mathbb{P}^\alpha} \left[ e^{\frac{r^2}{2} \int_0^T |\sigma z^*(t, X_t)|^2 dt} \right] \leq \mathbb{E}^{\mathbb{P}^0} \left[ \left( \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} \right)^{p_1} \right]^{\frac{1}{p_1}} \mathbb{E}^{\mathbb{P}^0} \left[ e^{\frac{q_1 r^2}{2} \int_0^T |\sigma z^*(t, X_t)|^2 dt} \right]^{\frac{1}{q_1}} \text{ for } \mathbb{P}^\alpha \in \mathcal{P}. \quad (3.7.184)$$

From the definition of the set  $\mathcal{P}$ , we have

$$\mathbb{E}^{\mathbb{P}^0} \left[ \left( \frac{d\mathbb{P}^\alpha}{d\mathbb{P}^0} \right)^{p_1} \right] < +\infty \text{ for } \mathbb{P}^\alpha \in \mathcal{P}, \quad (3.7.185)$$

and the second term in the right hand side is finite since

$$\mathbb{E}^{\mathbb{P}^0} \left[ e^{\frac{q_1 r^2}{2} \int_0^T |\sigma z^*(t, X_t)|^2 dt} \right] \leq \mathbb{E}^{\mathbb{P}^0} \left[ e^{\frac{q_1 r^2 \lambda_{max}(\sigma \sigma^\top)}{2} \int_0^T |\sigma z^*(t, X_t)|^2 dt} \right] \leq \mathbb{E}^{\mathbb{P}^0} \left[ e^{\varepsilon_{SB} \int_0^T |z^*(t, X_t)|^2 dt} \right] < +\infty, \quad (3.7.186)$$

and so

$$\mathbb{E}^{\mathbb{P}^\alpha} \left[ e^{\frac{r^2}{2} \int_0^T |\sigma z^*(t, X_t)|^2 dt} \right] < +\infty \text{ for } \mathbb{P}^\alpha \in \mathcal{P}, \quad (3.7.187)$$

and it follows from Novikov's criterion that

$$\mathbb{E}^{\mathbb{P}^\alpha} \left[ \mathcal{E} \left( -r \int_0^T \sigma z^*(s, X_s) \cdot dW_s^\alpha \right) \right] = 1, \text{ for } \mathbb{P}^\alpha \in \mathcal{P}. \quad (3.7.188)$$

(ii) By the same argument as in the proof of Lemma 3.7.9 part (i), (while assuming again without loss of generality that the constant term is null i.e.,  $C_1 = 0$ ) we have that

$$\sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}^0 \left[ \mathcal{E} \left( \int_0^\tau \sigma^{-1} B_1 \hat{\alpha}(z^*(s, X_s)) \cdot dW_s^0 \right)^{p_1} \right] \leq \mathbb{E}^0 \left[ e^{\frac{1}{2} \int_0^T \left| \frac{\sqrt{q_1}}{\sqrt{q_1}-1} \sigma^{-1} B_1 \hat{\alpha}(z^*(s, X_s)) \right|^2 ds} \right]^{\frac{1}{\sqrt{q_1}+1}}, \quad (3.7.189)$$

but

$$\begin{aligned} \mathbb{E}^0 \left[ e^{\frac{1}{2} \int_0^T \left| \frac{\sqrt{q_1}}{\sqrt{q_1}-1} \sigma^{-1} B_1 \hat{\alpha}(z^*(s, X_s)) \right|^2 ds} \right] &\leq \mathbb{E}^0 \left[ e^{\frac{q_1 \lambda_{max}(B_0)^2}{2(\sqrt{q_1}-1)^2 \lambda_{min}(\sigma \sigma^\top)} \int_0^T |z^*(s, X_s)|^2 ds} \right], \\ &\leq \mathbb{E}^0 \left[ e^{\varepsilon_{SB} \int_0^T |z^*(s, X_s)|^2 ds} \right] < +\infty, \end{aligned} \quad (3.7.190)$$

since  $\mathbb{E}^0 \left[ \exp \left( \varepsilon_{SB} \int_0^T |z^*(s, X_s)|^2 ds \right) \right] < +\infty$ . So

$$\sup_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}^0 \left[ \mathcal{E} \left( \int_0^\tau \sigma^{-1} B_1 \hat{\alpha}(z^*(s, X_s)) \cdot dW_s^0 \right)^{p_1} \right] < +\infty, \quad (3.7.191)$$

and

$$\mathcal{E} \left( \int_0^t \sigma^{-1} B_1 \hat{\alpha}(z^*(s, X_s)) \cdot dW_s^0 \right)_{t \in [0, T]} \text{ is a } (\mathbb{P}^0, \mathbb{F})\text{-martingale in } \mathbb{L}^{p_1}. \quad (3.7.192)$$

Then recall

$$\begin{aligned} & \mathbb{E}^0 \left[ e^{rp_2 \int_0^T c^A(\hat{\alpha}(z^*(s, X_s))) ds} \right] \\ &= \mathbb{E}^0 \left[ e^{rp_2 \int_0^T \left( \frac{1}{2} C_2^{-1} (B_1^\top z^*(s, X_s) - C_1) \cdot (B_1^\top z^*(s, X_s) - C_1) + C_1 \cdot C_2^{-1} (B_1^\top z^*(s, X_s) - C_1) + \frac{1}{2} M_2 X_s \cdot X_s + M_1 \cdot X_s \right) ds} \right], \end{aligned} \quad (3.7.193)$$

we discard the linear terms in  $z^*$  (since they are linear in  $X$ ) and the one in  $M_1$ , and using Cauchy-Schwarz inequality

$$\begin{aligned} \mathbb{E}^0 \left[ e^{\frac{rp_2}{2} \int_0^T (B_0 z^*(s, X_s) \cdot z^*(s, X_s) + M_2 X_s \cdot X_s) ds} \right] &\leq \mathbb{E}^0 \left[ e^{\frac{rp_2}{2} \int_0^T (\lambda_{max}(B_0) |z^*(s, X_s)|^2 + \lambda_{max}(M_2) |X_s|^2) ds} \right] \\ &\leq \mathbb{E}^0 \left[ e^{rp_2 \lambda_{max}(B_0) \int_0^T |z^*(s, X_s)|^2 ds} \right]^{\frac{1}{2}} \mathbb{E}^0 \left[ e^{rp_2 \lambda_{max}(M_2) \int_0^T |X_s|^2 ds} \right]^{\frac{1}{2}} \\ &\leq \mathbb{E}^0 \left[ e^{\varepsilon_{SB} \int_0^T |z^*(s, X_s)|^2 ds} \right]^{\frac{1}{2}} \mathbb{E}^0 \left[ e^{\iota_{SB} \int_0^T |X_s|^2 ds} \right]^{\frac{1}{2}} < +\infty, \end{aligned} \quad (3.7.194)$$

and so

$$\mathbb{E}^0 \left[ e^{rp_2 \int_0^T c^A(\hat{\alpha}(z^*(s, X_s))) ds} \right] < +\infty. \quad (3.7.195)$$

Therefore  $\hat{\alpha}(z^*(s, X_s))_{0 \leq t \leq T} \in \mathcal{U}$ .

(iii) Finally we prove that

$$\mathbb{E}^0 \left[ e^{\lambda p_3 |\xi_{SB}^* - \mathcal{R}|} \right] < +\infty, \text{ or equivalently, } \mathbb{E}^0 \left[ e^{\lambda p_3 (\xi_{SB}^* - \mathcal{R})} + e^{-\lambda p_3 (\xi_{SB}^* - \mathcal{R})} \right] < +\infty. \quad (3.7.196)$$

Recall that

$$\xi_{SB}^* - \mathcal{R} = \int_0^T z^*(s, X_s) \cdot dX_s - \int_0^T H(X_s, z^*(s, X_s)) ds + \frac{r}{2} \int_0^T |\sigma z^*(s, X_s)|^2 ds, \quad (3.7.197)$$

so under the measure  $\mathbb{P}^0$ , we have (we denote  $\alpha^{\mathcal{R}} = -C_2^{-1} C_1$ )

$$\begin{aligned} \xi_{SB}^* - \mathcal{R} &= \int_0^T z^*(s, X_s) \cdot \sigma dW_s^0 + \frac{r}{2} \int_0^T |\sigma z^*(s, X_s)|^2 ds, \\ &\quad - \int_0^T \left( \frac{1}{2} B_0 z^*(s, X_s) \cdot z^*(s, X_s) + B_1 \alpha^{\mathcal{R}} \cdot z^*(s, X_s) + C_0 - \frac{1}{2} M_2 X_s \cdot X_s - M_1 \cdot X_s \right) ds. \end{aligned} \quad (3.7.198)$$

Using Cauchy-Schwarz inequality we obtain

$$\begin{aligned}
\mathbb{E}^0 \left[ e^{\lambda p_3 (\xi_{\text{SB}}^* - \mathcal{R})} \right] &\leq e^{-\lambda p_3 C_0} \mathbb{E}^0 \left[ e^{2\lambda p_3 \int_0^T \sigma^\top z^*(s, X_s) \cdot dW_s^0} \right]^{\frac{1}{2}} \\
&\quad \times \mathbb{E}^0 \left[ e^{-2\lambda p_3 \int_0^T \left( \frac{1}{2} (B_0 z^*(s, X_s) \cdot z^*(s, X_s) - M_2 X_s \cdot X_s) - M_1 \cdot X_s + B_1 \alpha^{\mathcal{R}} \cdot z^*(s, X_s) \right) ds + \lambda p_3 r \int_0^T |\sigma z^*(s, X_s)|^2 ds} \right]^{\frac{1}{2}}, \\
&\leq e^{-\lambda p_3 C_0} \mathbb{E}^0 \left[ e^{8(\lambda p_3)^2 \int_0^T |\sigma^\top z^*(s, X_s)|^2 ds} \right]^{\frac{1}{4}} \\
&\quad \times \mathbb{E}^0 \left[ e^{-2\lambda p_3 \int_0^T B_1 \alpha^{\mathcal{R}} \cdot z^*(s, X_s) ds + \lambda p_3 \int_0^T \left( \lambda_{\max}(M_2) |X_s|^2 + r |\sigma z^*(s, X_s)|^2 \right) ds} \right]^{\frac{1}{2}},
\end{aligned} \tag{3.7.199}$$

and similarly

$$\begin{aligned}
\mathbb{E}^0 \left[ e^{-\lambda p_3 (\xi_{\text{SB}}^* - \mathcal{R})} \right] &\leq e^{-\lambda p_3 C_0} \mathbb{E}^0 \left[ e^{-2\lambda p_3 \int_0^T \sigma^\top z^*(s, X_s) \cdot dW_s^0} \right]^{\frac{1}{2}} \\
&\quad \times \mathbb{E}^0 \left[ e^{2\lambda p_3 \int_0^T \left( \frac{1}{2} (B_0 z^*(s, X_s) \cdot z^*(s, X_s) - M_2 X_s \cdot X_s) - M_1 \cdot X_s + B_1 \alpha^{\mathcal{R}} \cdot z^*(s, X_s) \right) ds - \lambda p_3 r \int_0^T |\sigma z^*(s, X_s)|^2 ds} \right]^{\frac{1}{2}}, \\
&\leq e^{-\lambda p_3 C_0} \mathbb{E}^0 \left[ e^{8(\lambda p_3)^2 \int_0^T |\sigma^\top z^*(s, X_s)|^2 ds} \right]^{\frac{1}{4}} \\
&\quad \times \mathbb{E}^0 \left[ e^{2\lambda p_3 \int_0^T \left( \frac{1}{2} B_0 z^*(s, X_s) \cdot z^*(s, X_s) - M_1 \cdot X_s + B_1 \alpha^{\mathcal{R}} \cdot z^*(s, X_s) \right) ds} \right]^{\frac{1}{2}},
\end{aligned} \tag{3.7.200}$$

but

$$\mathbb{E}^0 \left[ e^{8(\lambda p_3)^2 \int_0^T |\sigma^\top z^*(s, X_s)|^2 ds} \right] \leq \mathbb{E}^0 \left[ e^{8(\lambda p_3)^2 \lambda_{\max}(\sigma \sigma^\top) \int_0^T |z^*(s, X_s)|^2 ds} \right] \leq \mathbb{E}^0 \left[ e^{\varepsilon_{\text{SB}} \int_0^T |z^*(s, X_s)|^2 ds} \right] < +\infty, \tag{3.7.201}$$

and using again Cauchy-Schwarz inequality

$$\begin{aligned}
\mathbb{E}^0 \left[ e^{\lambda p_3 \int_0^T \left( \lambda_{\max}(M_2) |X_s|^2 + r |\sigma z^*(s, X_s)|^2 \right) ds} \right] &\leq \mathbb{E}^0 \left[ e^{\lambda p_3 \lambda_{\max}(M_2) \int_0^T |X_s|^2 ds} e^{\lambda p_3 r \lambda_{\max}(\sigma \sigma^\top) \int_0^T |z^*(s, X_s)|^2 ds} \right], \\
&\leq \mathbb{E}^0 \left[ e^{2\lambda p_3 \lambda_{\max}(M_2) \int_0^T |X_s|^2 ds} \right]^{\frac{1}{2}} \mathbb{E}^0 \left[ e^{2\lambda p_3 r \lambda_{\max}(\sigma \sigma^\top) \int_0^T |z^*(s, X_s)|^2 ds} \right]^{\frac{1}{2}} \\
&\leq \mathbb{E}^0 \left[ e^{\varepsilon_{\text{SB}} \int_0^T |X_s|^2 ds} \right]^{\frac{1}{2}} \mathbb{E}^0 \left[ e^{\varepsilon_{\text{SB}} \int_0^T |z^*(s, X_s)|^2 ds} \right]^{\frac{1}{2}} < +\infty,
\end{aligned} \tag{3.7.202}$$

and

$$\mathbb{E}^0 \left[ e^{\lambda p_3 \int_0^T B_0 z^*(s, X_s) \cdot z^*(s, X_s) ds} \right] \leq \mathbb{E}^0 \left[ e^{\lambda p_3 \lambda_{\max}(B_0) \int_0^T |z^*(s, X_s)|^2 ds} \right] \leq \mathbb{E}^0 \left[ e^{\varepsilon_{\text{SB}} \int_0^T |z^*(s, X_s)|^2 ds} \right] < +\infty, \tag{3.7.203}$$

so

$$\mathbb{E}^0 \left[ e^{\lambda p_3 (\xi_{\text{SB}}^* - \mathcal{R})} + e^{-\lambda p_3 (\xi_{\text{SB}}^* - \mathcal{R})} \right] < +\infty, \text{ and } \mathbb{E}^0 \left[ e^{\lambda p_3 |\xi_{\text{SB}}^* - \mathcal{R}|} \right] < +\infty, \tag{3.7.204}$$

which concludes the proof.

□

## **Part II**

# **Étude des lois d'une martingale et de son maximum courant**





# Chapter 4

## An extension of Kellerer’s theorem with marginal joint laws of value and running maximum

Joint work with Nizar Touzi.

### Abstract

In this work, we prove a result similar to Kellerer’s theorem. Given a family of probability measures on  $\mathbb{R}^2$ , we provide sufficient conditions for the existence of a continuous martingale with these probabilities as the joint marginal laws of its state variable and running maximum.

**Key words.** Kellerer’s theorem, martingale, joint law of marginal and running maximum.

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## 4.1 Introduction

Our goal is to extend the following result by Kellerer

**Theorem 4.1.1** (Kellerer). *Let  $\mu := (\mu_t)_{t \geq 0}$  be a Peacock, i.e.,  $\forall t \geq 0$ ,  $\mu_t$  is a probability measure in  $\mathbb{R}$  with  $\int_{\mathbb{R}} |x| \mu_t(dx) < +\infty$ , and for every convex function  $f : \mathbb{R} \rightarrow \mathbb{R}$ , the mapping  $t \mapsto \int_{\mathbb{R}} f(x) \mu_t(dx)$  (valued in  $(-\infty, +\infty]$ ), is non-decreasing.*

*Then there exists a martingale  $(X_t)_{t \geq 0}$  valued in  $\mathbb{R}$  such that  $X_t \stackrel{\text{law}}{=} \mu_t, \forall t \geq 0$ .*

A constructive proof of this theorem was proposed recently in [Hirsch & Roynette \(2012\)](#), where they construct a martingale with given marginals as a solution to some SDE by relying on the Dupire equation, which plays a central role in financial mathematics, and a well-posedness result of the Fokker-Planck PDE related to the SDE used.

We extend [Theorem 4.1.1](#) in the sense that instead of considering a family of probability measures in  $\mathbb{R}$  and constructing a martingale with these probabilities as marginal laws, we take a family of probability measures in  $\mathbb{R}^2$ , and we provide sufficient conditions guaranteeing that it is possible to construct a martingale with these laws as marginal joint laws of the process and its corresponding running maximum.

Following the methodology of [Hirsch & Roynette \(2012\)](#), we will address two questions:

- (i) Given a weak solution to some SDE (with a diffusion coefficient involving the running maximum), can we characterize the marginal joint laws of the process and the running maximum using a Fokker-Planck PDE? Under which assumptions?
- (ii) Given a family of joint laws, can we construct a martingale (as a solution to an SDE) with these marginals joint laws? Under which conditions?

Section [4.2](#) is devoted to describing the setup of the problem, recalling some related existing results, and outlining the resolution methodology which relies on the use of the Fokker-Planck PDE as a tool to characterize the family of marginal probability measures. Section [4.3](#) focuses on this Fokker-Planck equation, and the proof of its well-posedness. Section [4.4](#) introduces barrier options, and restates the problem and the necessary conditions in terms of these functions. Section [4.5](#) states the main result, and provides a constructive proof in the case where the marginal joint laws satisfy some smoothness assumptions; and a regularization procedure in the general case. Finally, we regroup in the [Appendix 4.6](#) some technical proofs and useful tools for the analysis of the problem.

**Notations:** (i) Whenever  $(X_t)_{t \geq 0}$  is a real valued process, we denote by  $(M_t)_{t \geq 0}$  its running maximum, possibly starting from  $m_0 \geq x_0$  i.e.,  $M_t := m_0 \vee \sup_{0 \leq u \leq t} X_u$ , and with the convention  $m_0 = x_0 = 0$ , unless stated otherwise.

(ii) Let  $B$  be a Borel set in  $\mathbb{R}^2$ . We denote by  $\mathcal{P}(B)$  the set of probability measures on  $\mathbb{R}^2$  supported on  $B$ , equipped with the topology of weak convergence, and  $\mathcal{P}_1(B)$  is the subset of  $\mathcal{P}(B)$  of probability measures with finite first moment w.r.t the first variable i.e.,  $\mu \in \mathcal{P}(B)$  and satisfies  $\int_{\mathbb{R}^2} |x| \mu(dx, dm) < +\infty$ .

## 4.2 Setup and resolution methodology

Let  $T \in (0, +\infty)$  and  $\mathcal{V} := (\nu_t)_{t \in [0, T]}$  be a family of probability measures in  $\mathcal{P}_1(\mathbb{R}^2)$ , and  $\Omega := \mathcal{C}([0, T], \mathbb{R})$  the space of continuous functions from  $[0, T]$  to  $\mathbb{R}$ . Define then the measurable space  $(\Omega, \mathcal{B}(\Omega))$ , and its canonical

process denoted  $(X_t)_{t \in [0, T]}$  and given by  $X_t(\omega) = \omega(t)$  for  $t \in [0, T]$ . We define the (possibly enlarged) filtration  $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$  containing the canonical filtration  $\mathbb{F}^X := \sigma(X_t, t \in [0, T])$ ,

Consider the set

$$\mathcal{M}^0(\mathcal{V}) := \left\{ \begin{array}{l} \mathbb{P} \text{ is a probability measure on } (\Omega, \mathcal{F}_T) \text{ with } \mathbb{P} \circ (X_t, M_t)^{-1} = \nu_t, \text{ for } t \in [0, T] \\ \text{and } X \text{ is a } (\mathbb{P}, \mathbb{F})\text{-martingale.} \end{array} \right\}, \quad (4.2.1)$$

Our goal is to provide necessary and sufficient (up to some technical assumptions) conditions on the family  $\mathcal{V}$  to have  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$ . We assume first that  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$  and provide necessary conditions from the properties of its elements. Then we will identify which of these conditions are sufficient to construct a martingale consistent with the family  $\mathcal{V}$ . In the special case where the family  $\mathcal{V}$  is a singleton, i.e.,  $\mathcal{V} := \{\nu\}$ , we will use a similar notation

$$\mathcal{M}^0(\{\nu\}) := \left\{ \begin{array}{l} \text{There exists some } T \in \mathbb{R}_+ \text{ and } \mathbb{P} \text{ a probability measure on } (\Omega, \mathcal{F}_T) \text{ with } \mathbb{P} \circ (X_T, M_T)^{-1} = \nu, \\ \text{and } X \text{ is a } (\mathbb{P}, \mathbb{F})\text{-martingale.} \end{array} \right\}, \quad (4.2.2)$$

#### 4.2.1 First necessary conditions : The static case

We can establish a first necessary condition by considering separately each element of  $\mathcal{V}$ . In fact, we have the implication (and thus the necessary condition)

$$\mathcal{M}^0(\mathcal{V}) \neq \emptyset \Rightarrow \mathcal{M}^0(\{\nu_t\}) \neq \emptyset, \text{ for all } t \in [0, T], \quad (4.2.3)$$

as opposed to

$$\mathcal{M}^0(\{\nu_t\}) \neq \emptyset, \text{ for some } t \in [0, T] \not\Rightarrow \mathcal{M}^0(\mathcal{V}) \neq \emptyset, \quad (4.2.4)$$

since each set  $\mathcal{M}^0(\{\nu_t\})$  for  $t \in [0, T]$  might contain the law  $\mathbb{P}$  of some martingale  $X$  such that  $\mathbb{P} \circ (X_t, M_t)^{-1} = \nu_t$  and  $\mathbb{P} \circ (X_s, M_s)^{-1} \neq \nu_s$  for some  $s \neq t$  in  $[0, T]$ .

We recall a result by (Rogers 1993, Theorem 3.1) characterizing the set  $\mathcal{M}^0(\{\nu_t\})$  for  $\nu_t \in \mathcal{P}_1(\mathbb{R}^2)$ .

**Theorem 4.2.1** (Rogers 93'). *Let  $\mu \in \mathcal{P}_1(\mathbb{R}^2)$ , with  $(X, Y) \stackrel{\text{law}}{=} \mu$  and  $\mathbb{E}[X] = 0$ . Consider the function  $f_\mu : \mathbb{R} \rightarrow \mathbb{R}$*

$$f_\mu(B) := \mathbb{E}[X|Y > B] \mathbf{1}_{\{\mathbb{P}(Y > B) > 0\}} + B \mathbf{1}_{\{\mathbb{P}(Y > B) = 0\}}, \quad \forall B \in \mathbb{R}, \quad (4.2.5)$$

and define the set

$$\mathcal{S}_0 := \{(x, m) \in \mathbb{R}^2, x \vee 0 < m\}. \quad (4.2.6)$$

Then  $\mathcal{M}^0(\{\mu\}) \neq \emptyset$  if and only if

- (i)  $\text{supp}(\mu) \subset \text{cl}(\mathcal{S}_0)$ ,
- (ii)  $f_\mu(B) = B^+, \forall B \in \mathbb{R}$ .

Remark that there is a less restrictive result in (Rogers 1993, Theorem 2.2), guaranteeing the existence of a uniformly integrable martingale which is not necessarily continuous and with  $\mu$  as a terminal joint law of the process and its running maximum. We do not discuss this result since in this paper we only consider continuous processes.

### 4.2.2 Reminder of the Brunick & Shreve (2013) : The Markovian projection theorem

The main idea behind our characterization (similar to the one in Hirsch & Roynette (2012)), is that whenever the set  $\mathcal{M}^0(\mathcal{V})$  is non-empty, then it contains under rather general conditions a weak solution of some SDE; easier to describe than a general Itô process. In fact, assuming that there exists some probability measure  $\mathbb{P}^Y \in \mathcal{M}^0(\mathcal{V})$ , where  $\mathbb{P}^Y$  is the law of some diffusion process defined on some filtered space  $(\tilde{\Omega}, \tilde{\mathcal{F}}_T, \tilde{\mathbb{F}})$  as

$$Y_t = Y_0 + \int_0^t \tilde{\sigma}_u d\tilde{W}_u, \text{ for } t \in [0, T], \text{ and } \tilde{W} \text{ is a } (\mathbb{P}^Y, \tilde{\mathbb{F}})\text{-Brownian motion,} \quad (4.2.7)$$

Then we have the following Markovian projection result

**Proposition 4.2.2.** ((Brunick & Shreve 2013, Theorem 3.6 and Corollary 3.10))

Assume that  $\mathbb{E}^{\mathbb{P}^Y} \left[ \int_0^t \tilde{\sigma}_s^2 ds \right] < \infty$  for  $t \in [0, T]$ , i.e.,  $(\nu_t(x^2) - \nu_0(x^2)) < +\infty$  for  $t \in [0, T]$ . Then there exists at least an  $\mathbb{R}_+$ -valued measurable deterministic function  $\sigma$  defined on  $\mathbb{R}_+ \times \mathbb{R}^2$ , and a Lebesgue-null set  $N \subset [0, T]$ , satisfying<sup>1</sup>

$$\sigma^2(t, Y_t, Y_t^*) = \mathbb{E} \left[ \tilde{\sigma}_t^2 | Y_t, Y_t^* \right], \mathbb{P}^Y - a.s., t \in [0, T] \cap N^c, \quad (4.2.8)$$

with the notation  $Y_t^* := Y_0^* \vee \max_{0 \leq u \leq t} Y_u$ , and a probability measure  $\mathbb{P}^X$  defined on  $(\Omega, \mathcal{F}_T, \mathbb{F})$  and supporting a Brownian motion  $W$  such that  $\mathbb{P}^X$  a.s

$$X_t = X_0 + \int_0^t \sigma(u, X_u, X_u^*) dW_u, \text{ for } t \in [0, T], \quad (4.2.9)$$

with  $X_t^* := X_0^* \vee \max_{0 \leq u \leq t} X_u$ , and satisfying  $\mathbb{P}^X \circ (X_t, X_t^*)^{-1} = \mathbb{P}^Y \circ (Y_t, Y_t^*)^{-1}$ , for  $t \in [0, T]$ .

From the previous result, we have that  $\mathbb{P}^X \in \mathcal{M}^0(\mathcal{V})$ , where  $\mathbb{P}^X$  is some weak solution to the SDE (4.2.9), characterized by the function  $\sigma$  (instead of the family  $\mathcal{V}$ ).

This suggests that the deterministic function  $\sigma$  embeds all the information given by  $\mathcal{V}$ , which is a remark that has already been used in Hambly et al. (2016) and (Henry-Labordere 2017, Chapter 4.3) to design a model (a solution of an SDE) consistent with observed laws  $\mathbb{P}^Y \circ (Y_t, Y_t^*)^{-1}$ , for  $t \in [0, T]$ , by computing  $\sigma$  ( $\sigma^2$  to be more precise) as a function of the marginal distributions for an existing martingale.

In this paper we are interested in the case where we do not know whether the marginal laws that we are observing are coming from a martingale and its running maximum or not, and so we want to provide sufficient conditions under which this is the case. We will strongly rely on the SDE (4.2.9) and study its related Fokker-Planck PDE to obtain such conditions.

<sup>1</sup>Remark that  $\sigma$  is characterized outside of a set of measure 0 with respect to  $\lambda \otimes \mathbb{P}^Y \circ (Y_t, Y_t^*)^{-1}$ , where  $\lambda$  is the Lebesgue measure.

### 4.3 The Fokker-Planck equation

The goal of this section is to provide necessary conditions on the family  $\mathcal{V}$  when  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$ , and contains a solution to the following SDE

$$X_t = x_0 + \int_0^t \sigma(s, X_s, M_s) dW_s, \text{ for } t \in [0, T], \text{ with } (X_0, M_0) = (x_0, m_0), \quad (4.3.1)$$

with as usual  $M_t := m_0 \vee \max_{0 \leq u \leq t} X_u$ , for some  $x_0, m_0$  in  $\mathbb{R}$  with  $x_0 \leq m_0$ , and a continuous function  $\sigma$  (we will make further assumptions on  $\sigma$  in the next subsection).

We shall denote  $S_t := (X_t, M_t)$ , and

$$a(t, s) := \frac{1}{2} \sigma^2(t, s), \mathcal{S} := \{(x, m) \in \mathbb{R}^2, x \vee m_0 < m\}, \Delta := \partial \mathcal{S} \setminus \{(\cdot, m_0)\}, \text{ and } \bar{\mathcal{S}} := \text{cl}(\mathcal{S}). \quad (4.3.2)$$

Finally, define the open time-space subset  $U := ]0, T[ \times \mathcal{S}$ , and its closure  $\bar{U} := \text{cl}(U) = [0, T] \times \bar{\mathcal{S}}$ .

For any open subset  $\mathcal{O}$  of a finite dimensional space, we shall denote by  $C_c^\infty(\mathcal{O})$  the collection of all  $C^\infty$  compactly supported functions on  $\mathcal{O}$ . The marginal law  $p_t$  of the pair  $(X_t, M_t)$  is defined by  $p(t, ds) := \mathbb{P}[S_t \in ds]$ .

A straightforward observation is that

$$\text{supp}(p(t, \cdot)) \subset \bar{\mathcal{S}}, \text{ for } t \in [0, T]. \quad (4.3.3)$$

We will use the Fokker-Planck PDE related to SDE (4.3.1) to characterize the family  $(p_t)_{t \in [0, T]}$  of marginal laws, instead of the stronger characterization via the law of the corresponding process.

#### 4.3.1 Existence of a weak solution of SDE (4.3.1)

We denote by  $C_b^2(\mathbb{R})$  the set of twice continuously differentiable functions which are bounded together with their derivatives, and  $\mathcal{M}_2^{c,loc}$  the set of continuous locally square integrable martingales.

**Definition 4.3.1.** A solution of SDE (4.3.1) is a probability measure  $\mathbb{P}$  on  $(\Omega, \mathcal{B}(\Omega))$  such that  $\mathbb{P}$ -a.s

$$f(X_\cdot) - f(X_0) - \frac{1}{2} \int_0^\cdot \sigma^2(s, X_s, M_s) f''(X_s) ds \in \mathcal{M}_2^{c,loc} \text{ for all } f \in C_b^2(\mathbb{R}). \quad (4.3.4)$$

**Assumption 4.3.2.** Assume that  $\sigma$  is a function on  $\bar{U}$  satisfying

$$\begin{cases} \sigma \text{ is continuous on } \bar{U}, \\ \sigma(t, x, m) > 0 \text{ for every } (t, x, m) \in U. \end{cases} \quad (4.3.5)$$

**Proposition 4.3.3.** Under Assumption 4.3.2, the stochastic differential equation (4.3.1) admits at least a weak non-exploding solution.

**Proof.** We denote by  $E(\sigma)$  the SDE (4.3.1). The proof relies on the existence result in Ikeda & Watanabe (2014) in the case of a bounded  $\sigma$ . Since we do not assume boundedness, we start by proving the existence of at least a weak solution up to some explosion time, then we prove that this explosion time is infinite with probability one.

For each  $n \geq 1$ , we define the function  $\sigma_n$  on  $\bar{U}$ , with<sup>2</sup>  $\sigma_n(t, x, m) := \sigma(t, (x \vee -n) \wedge n, (m \wedge n))$ . From the continuity of  $\sigma$  on  $\bar{U}$ , it is bounded in every compact, and so for  $n \geq 1$ , we have that  $\sigma_n$  is continuous and bounded, and we can then apply the existence Theorem by [Ikeda & Watanabe \(2014\)](#) (more precisely Chapter IV, Theorem 2.2), to prove that  $E(\sigma_n)$  admits a weak solution denoted  $\mathbb{P}^{(X^{(n)})}$ .

We define then the family of non decreasing-stopping times  $\tau_n := T \wedge \inf \{t \geq 0; |X_t| = n\}$ , and for each solution  $\mathbb{P}^{(X^{(n)})}$ , we define  $\mathbb{P}^{(\widetilde{X}^{(n)})}$  as the law of the process  $X^{(n), \tau_n}$ , i.e., the process  $X^{(n)}$  stopped at  $\tau_n$ .

We remark then that the family  $\left(\mathbb{P}^{(\widetilde{X}^{(n)})}\right)_{n \geq 1}$  is consistent, in the sense that for a fixed  $n \geq 1$ , we have that  $\mathbb{P}^{(\widetilde{X}^{(n+1)})}$  is a solution to

$$X_t - X_0 = \int_0^t \sigma_n(s, X_s, M_s) dB_s, \text{ for } t \in [0, \tau_n], \quad (4.3.6)$$

as the initial solution  $\mathbb{P}^{(\widetilde{X}^{(n)})}$ . So we use this identity together with the monotonicity of  $(\tau_n)_{n \geq 1}$  to build recursively a solution on  $(\Omega, \mathcal{B}(\Omega))$  in  $[0, \tau_n)$  with  $\mathbb{P}^{(\widetilde{X}^{(n)})}$  and then gluing an increment in  $[\tau_n, \tau_{n+1})$  with  $\mathbb{P}^{(\widetilde{X}^{(n+1)})}$  and so on, up to  $e := \lim_{n \rightarrow \infty} \uparrow \tau_n$ , which is the explosion time defined as a stopping time (a limit of stopping times) in  $[0, T]$ .

We denote then the obtained solution  $\mathbb{P}^{(X)}$ . By definition of  $e$ , we have that for  $t < e$ , there exists some  $k \in \mathbb{N}$  such that  $t \leq \tau_k < e$ , and so from the continuity of  $\sigma$  and the boundedness of  $(s, X_s, M_s)$  for  $s \in [0, \tau_k]$  we have

$$\int_0^t \sigma^2(s, X_s, M_s) ds \leq \int_0^{\tau_k} \sigma^2(s, X_s, M_s) ds < +\infty, \text{ for } t \leq \tau_k < e, \mathbb{P}^{(X)} - a.s., \quad (4.3.7)$$

and therefore the stochastic integral

$$\int_0^t \sigma(s, X_s, M_s) dB_s, \text{ for } t \in [0, e), \quad (4.3.8)$$

is well defined and we have

$$X_t - X_0 = \int_0^t \sigma(s, X_s, M_s) dB_s, \text{ for } t \in [0, e), \mathbb{P}^{(X)} - a.s., \quad (4.3.9)$$

which is a solution of the SDE (4.3.1) stopped before  $e$ , and the process  $(X_t)_{0 \leq t < e}$  is a local martingale under  $\mathbb{P}^{(X)}$ . We prove now that  $\mathbb{P}^{(X)}(\{e < T\}) = 0$  by contradiction. In fact, in the event  $\{e < T\}$  we have by definition  $e = \lim_{n \rightarrow +\infty} \uparrow \tau_n < T$ , and so

$$\lim_{t \rightarrow e} |X_t| = +\infty, \quad (4.3.10)$$

Since the process  $(X_t)_{0 \leq t < e}$  is a local martingale, it is then a time changed Brownian motion by the Dambis, Dubins & Schwarz theorem, and there exists a Brownian motion  $W_{t \geq 0}$  such that

$$X_t = x_0 + W_{\int_0^t \sigma^2(s, X_s, M_s) ds} \text{ for } t < e, \quad (4.3.11)$$

<sup>2</sup>Remark that the SDE (4.3.1) is defined in a compact interval of time  $[0, T]$ , so there is no need for the truncation  $t \mapsto t \wedge n$  (as opposed to the case of an SDE defined with  $T = +\infty$ ).

and therefore, in the event  $\{e < T\}$  we have

$$\limsup_{t \rightarrow e} W \int_0^t \sigma^2(s, X_s, M_s) ds = +\infty, \text{ and } \liminf_{t \rightarrow e} W \int_0^t \sigma^2(s, X_s, M_s) ds = -\infty, \quad (4.3.12)$$

which is a contradiction with (4.3.10). Therefore  $\mathbb{P}^{(X)}(\{e < T\}) = 0$  for our solution  $\mathbb{P}^{(X)}$  of the SDE (4.3.1), and  $\mathbb{P}^{(X)}$  is a non-exploding solution of the SDE (4.3.1). In particular  $(\sigma(t, X_t, M_t))_{t \in [0, T]} \in \mathbb{H}_{loc}^2(\mathbb{P}^{(X)})$  where

$$\mathbb{H}_{loc}^2(\mathbb{P}^{(X)}) := \left\{ h : \text{measurable, } \mathbb{F} \text{ adapted processes with } \int_0^T h_s^2 ds < +\infty, \mathbb{P}^{(X)} - a.s. \right\}. \quad (4.3.13)$$

□

Recall that  $h \in \mathbb{H}_{loc}^2(\mathbb{P}^{(X)})$  if and only if  $h^n \in \mathbb{H}^2(\mathbb{P}^{(X)})$  for  $n \in \mathbb{N}$ , where we defined the process  $h^n := h \cdot \mathbf{1}_{\cdot \wedge \tau_n}$  with  $\tau_n := T \wedge \inf \left\{ t > 0, \int_0^t h_s^2 ds \geq n \right\}$ , and

$$\mathbb{H}^2(\mathbb{P}^{(X)}) := \left\{ h : \text{measurable, } \mathbb{F} \text{ adapted processes with } \mathbb{E}^{\mathbb{P}^{(X)}} \left[ \int_0^T h_s^2 ds \right] < +\infty \right\}. \quad (4.3.14)$$

Define the aggregated set of integrands over the family of weak solutions  $E(\sigma)$  of SDE (4.3.1),

$$\mathbb{H}_{loc}^2 := \cup_{\{\mathbb{P}^{(X)} \in E(\sigma)\}} \mathbb{H}_{loc}^2(\mathbb{P}^{(X)}), \text{ and } \mathbb{H}^2 := \cup_{\{\mathbb{P}^{(X)} \in E(\sigma)\}} \mathbb{H}^2(\mathbb{P}^{(X)}). \quad (4.3.15)$$

### 4.3.2 Derivation of the Fokker-Planck PDE

Let  $\mathbb{P}$  be one of the weak solutions of the SDE (4.3.1), and define the family of probability measures  $(\mathbb{P} \circ (X_t, M_t)^{-1})_{t \in [0, T]}$ , which we denote  $(p(t, dx, dm), t \in [0, T])$ . Our goal is to characterize this family via a Fokker-Planck PDE.

One of the difficulties of describing the family  $\mathcal{V}$  as a solution to some Fokker-Planck PDE, is to define properly the boundary condition on  $\Delta$ . This suggests imposing some regularity on the family  $(p(t, dx, dm), t \in [0, T])$ , either a Sobolev regularity (enough to define weak derivatives and the trace operator on  $\Delta$ ), or classical regularity (by assuming that the marginal probability measures have a smooth density w.r.t to the Lebesgue measure). The latter assumption was made in [Hambly et al. \(2016\)](#).

We derive the Fokker-Planck PDE with smoothness assumptions (similar to the one in [Hambly et al. \(2016\)](#)), and prove that it admits at most one solution (uniqueness). However, this leads to a gap between the existence and uniqueness result (we can always prove existence of a family of probability measures which is a solution to the PDE in a weak sense) and then we can prove the uniqueness but only under smoothness assumptions.

As we don't have a-priori estimates on the solution, we need a regularization procedure to overcome this problem. So we treat the smooth case (when  $\mathcal{V}$  is smooth in some sense), characterize it as a solution to a Fokker-Planck PDE, then we treat the general case by proving that we can always reduce to the smooth case, apply the characterization, and pass to the limit.

A similar work to ours can be found in [Coutin & Pontier \(2017\)](#), where the authors derive a Fokker-Planck PDE for a continuous diffusion and its running maximum. However, their result relies heavily on the homogeneity and the



uniform ellipticity of the diffusion term. In both papers cited above, the authors do not prove the uniqueness for the solution of the Fokker-Planck PDE, which is crucial to establish our result.

We start by some estimates, then we proceed to deriving the Fokker-Planck PDE by Itô formula. Recall that  $(\sigma(t, X_t, M_t))_{t \in [0, T]} \in \mathbb{H}_{loc}^2(\mathbb{P}^{(X)})$ , so  $\int_0^T \sigma(t, X_t, M_t)^2 dt < +\infty$ ,  $\mathbb{P}^{(X)}$  - a.s for some weak solution  $\mathbb{P}^{(X)}$  of the SDE (4.3.1). Then for some  $\varphi \in C_c^{2,1}(\mathbb{R}^2)$  we have that

$$\varphi(X_t, M_t) - \varphi(x_0, m_0) - \int_0^t a(u, X_u, M_u) \partial_{xx} \varphi(X_u, M_u) du - \int_0^t \partial_m \varphi(M_u, M_u) dM_u = \int_0^t \partial_x \varphi(X_u, M_u) dX_u, \quad (4.3.16)$$

and so the term on the right hand side is a martingale by the boundedness of  $\partial_x \varphi$  and  $\sigma$  on the compact support of  $\partial_x \varphi$  and we have that

$$\begin{aligned} \mathbb{E}[\varphi(X_t, M_t)] - \mathbb{E}[\varphi(x_0, m_0)] &= \mathbb{E} \left[ \int_0^t a(u, X_u, M_u) \partial_{xx} \varphi(X_u, M_u) du \right] + \mathbb{E} \left[ \int_0^t \partial_m \varphi(M_u, M_u) dM_u \right] \\ &= \mathbb{E} \left[ \int_0^t a(u, X_u, M_u) \partial_{xx} \varphi(X_u, M_u) du \right] + \mathbb{E} \left[ \int_{M_0}^{M_t} \partial_m \varphi(y, y) dy \right], \end{aligned} \quad (4.3.17)$$

where we got rid of the martingale term, and we used the fact that  $(X_u - M_u)dM_u = 0$ , i.e., the running maximum process increases only on the diagonal  $\{(X_t, M_t) \in \Delta\}$ . Notice that the last expression induced from Itô's formula only involves the joint marginal distributions of  $S_u = (X_u, M_u)$  for  $u \leq t$ . Then define  $G_t^*(m) := \mathbb{P}[M_t > m]$ , (a right continuous with left limits function); the survival cdf of  $M_t$ , integral of  $p_t^*(dm) := \int_{(-\infty, m]} p_u(dx, dm) = \mathbb{P}[M_t \in dm]$ , where  $p_t(ds)$  is the marginal distribution of the pair  $(X_t, M_t)$ . We can see that

$$\mathbb{E} \left[ \int_{M_0}^{M_t} \partial_m \varphi(y, y) dy \right] = \mathbb{E} \left[ \int_{\mathbb{R}} \partial_m \varphi(y, y) \mathbf{1}_{M_0 < y < M_t} dy \right] = \int_{\mathbb{R}} \varphi_m(x, x) (G_t^*(x) - G_0^*(x)) dx, \quad (4.3.18)$$

by Fubini's theorem. So

$$\int_{\mathcal{S}} \varphi(s) (p_t - p_0)(ds) = \int_0^t \int_{\mathcal{S}} \mathcal{A}_u \varphi(s) p_u(ds) du + \int_{\mathbb{R}} \varphi_m(x, x) (G_t^*(x) - G_0^*(x)) dx, \quad (4.3.19)$$

where  $\mathcal{A}_u \varphi(s) := a(u, s) \varphi_{xx}(s)$ .

**Definition 4.3.4.** We say that a family of probability measures  $(p(t, dx, dm), t \in [0, T])$  is variational-solution to the Fokker-Planck PDE if

- (i)  $\int \int_K \sigma^2(t, x, m) p(t, dx, dm) dt < +\infty$  for all compact set  $K \subset U$ ,
- (ii)  $(p(t, dx, dm), t \in [0, T])$  satisfies (4.3.19) for every  $\varphi \in C_c^{2,1}(\mathbb{R}^2)$ .

**Remark 4.3.5.** (i) Up to now, we didn't make any regularity assumption on the solution  $p$ . However, to further develop the equality (4.3.19), we need to integrate by parts the quadratic variation term, either by making regularity assumptions on the solution (as in Hambly et al. (2016)) or establishing a priori estimates (as in Coutin & Pontier (2017), but in an homogeneous framework).

(ii) By construction, there exists at least a solution to (4.3.19) given by the marginal laws of any weak solution to SDE (4.3.1).

### 4.3.2.1 Integration by parts with smoothness assumptions

We start by making regularity assumptions, to clarify the reasoning, then we will use the tools recalled in Appendix 4.6.1 to relax these assumptions.

For  $t > 0$ , we denote by  $ap_t$  the measure  $ap(t, \cdot)$  on  $\mathbb{R}^2$ . We denote by  $ap_t^\Delta$ , the image of  $ap_t$  by the trace operator on  $\Delta$  and by  $(ap)_x^\Delta$  and  $(ap)_m^\Delta$  the traces of the derivatives of  $ap_t$  on  $\Delta$  (c.f. Appendix 4.6.1).

**Proposition 4.3.6.** *Assume that  $ap_t \in H^2(\mathcal{S})$  for  $t > 0$ . Then  $p$  is a solution to (4.3.19) if and only if for every  $\varphi \in C_c^{2,1}(\mathbb{R}^2)$*

$$\begin{aligned} \int_{\mathcal{S}} \varphi(s) \left( p_t - p_0 - \int_0^t (ap_u)_{xx} du \right) (ds) &= -\frac{1}{\sqrt{2}} \int_0^t \int_{\Delta} \varphi(2(ap_u)_x + (ap_u)_m) (dm) du, \\ &+ \frac{1}{\sqrt{2}} \int_{\Delta} \varphi_m \left( G_t^* - G_0^* - \int_0^t (ap_u)^\Delta du \right) dm. \end{aligned} \quad (4.3.20)$$

**Proof.** The proof is simple and relies on integration by parts. For some  $\varphi \in C_c^{2,1}(\mathbb{R}^2)$ , assuming that  $ap_t \in H^2(\mathcal{S})$ , we can compute

$$\int_{\mathcal{S}} \varphi_{xx}(s) ap_t(ds) = - \int_{\mathcal{S}} \varphi_x(s) (ap_t)_x(ds) + \frac{1}{\sqrt{2}} \int_{\Delta} \varphi_x ap_t^\Delta, \quad (4.3.21)$$

Remark that we omit the argument in the boundary terms to stress the fact that  $ap_t^\Delta$  should be seen as a linear form and to avoid the confusion concerning the length element of the line  $\Delta$  as opposed to  $\mathbb{R}$  (see Appendix (4.6.5)).

We plug (4.3.21) in (4.3.19) to obtain

$$\begin{aligned} \int_{\mathcal{S}} \varphi(s) (p_t - p_0)(ds) &= - \int_{\mathcal{S}} \varphi_x \int_0^t (ap_u)_x du + \frac{1}{\sqrt{2}} \int_{\Delta} \varphi_x \int_0^t ap_u^\Delta du + \int_{\mathbb{R}} \varphi_m(m, m) (G_t^*(m) - G_0^*(m)) dm, \\ &= - \int_{\mathcal{S}} \varphi_x \int_0^t (ap_u)_x du + \frac{1}{\sqrt{2}} \int_{\Delta} \varphi_x \int_0^t ap_u^\Delta du + \frac{1}{\sqrt{2}} \int_{\Delta} \varphi_m (G_t^* - G_0^*) dm. \end{aligned} \quad (4.3.22)$$

where we added the term  $\frac{1}{\sqrt{2}}$  to compensate for the length element of the curve  $\Delta$  (which is  $\sqrt{2}$ ). Therefore we obtain

$$\int_{\mathcal{S}} \varphi(s) (p_t - p_0)(ds) = - \int_{\mathcal{S}} \varphi_x(s) \int_0^t (ap_u)_x(s) ds du + \frac{1}{\sqrt{2}} \int_{\Delta} \varphi_x(m, m) \int_0^t ap_u^\Delta du + \frac{1}{\sqrt{2}} \int_{\Delta} \varphi_m (G_t^* - G_0^*) dm. \quad (4.3.23)$$

We integrate by parts again (4.3.21)

$$\begin{aligned} \int_{\mathcal{S}} \varphi_{xx}(s) ap_t(ds) &= - \int_{\mathcal{S}} \varphi_x(s) (ap_t)_x(ds) + \frac{1}{\sqrt{2}} \int_{\Delta} \varphi_x ap_t^\Delta, \\ &= \int_{\mathcal{S}} \varphi(s) (ap_t)_{xx}(ds) + \frac{1}{\sqrt{2}} \int_{\Delta} \left( \varphi_x ap_t^\Delta - \varphi (ap_t^\Delta)_x \right). \end{aligned} \quad (4.3.24)$$

Then, from Lemma 4.6.1, we obtain

$$\int_{\mathcal{S}} \mathcal{A}_t \varphi(s) p_t(ds) = \int_{\mathcal{S}} \varphi(s) (ap_t)_{xx}(ds) - \frac{1}{\sqrt{2}} \int_{\Delta} \left( \varphi_m ap_t^\Delta + \varphi \left( 2(ap_t^\Delta)_x + (ap_t^\Delta)_m \right) \right), \quad (4.3.25)$$

therefore by defining  $\mathcal{A}^* p_t := \partial_{xx} a p(t)$ , we have that

$$\int_{\mathcal{S}} \mathcal{A}_t \varphi(s) p_t(ds) = \int_{\mathcal{S}} \varphi(s) \mathcal{A}^* p_t(ds) - \frac{1}{\sqrt{2}} \int_{\Delta} \varphi(2(ap_t^\Delta)_x + (ap_t^\Delta)_m) - \frac{1}{\sqrt{2}} \int_{\Delta} \varphi_m a p_t^\Delta. \quad (4.3.26)$$

We plug then (4.3.26) in (4.3.19) and we rescale by  $\frac{1}{\sqrt{2}}$  to integrate over  $\Delta$  instead of  $\mathbb{R}$  to obtain

$$\begin{aligned} \int_{\mathcal{S}} \varphi(s) \left( p_t - p_0 - \int_0^t (ap_u)_{xx} du \right) (ds) &= - \frac{1}{\sqrt{2}} \int_0^t \int_{\Delta} \varphi(2(ap_u)^\Delta_x + (ap_u)^\Delta_m) (dm) du \\ &+ \frac{1}{\sqrt{2}} \int_{\Delta} \varphi_m \left( G_t^* - G_0^* - \int_0^t (ap_u)^\Delta du \right) dm. \end{aligned} \quad (4.3.20)$$

which completes the proof (by using again the integration by parts "backwards" to prove the other sense).  $\square$

**Remark 4.3.7.** (i) *The regularity assumptions are needed to define the trace operator. In particular the equivalent definition of a Fokker-Planck solution (4.3.23) only requires  $ap_t \in H^1(\mathcal{S})$  for  $t > 0$ , whereas the main characterization (4.3.20) requires  $ap_t \in H^2(\mathcal{S})$ .*

(ii) *As opposed to the initial form of the PDE (4.3.19) where we had the existence of at least one solution given the marginal laws of the SDE (4.3.1), the expressions (4.3.23) and (4.3.20) are more involved, but since they require smoothness assumptions the existence of a solution is not guaranteed anymore.*

(iii) *Observe that there are two boundary expressions (on  $\Delta$ ) in both (4.3.23) and (4.3.20). This suggests a redundancy as will be seen in the classical case. In particular, we will see that  $G_t^* - G_0^* = \int_0^t ap_u^\Delta du$  in some appropriate sense.*

(iv) *Under the current assumptions ( $ap_t \in H^2(\mathcal{S})$ ), the trace  $ap_t^\Delta$  is a positive measure on  $\Delta$  absolutely continuous (with respect to the Lebesgue measure defined on  $\Delta$ , and its density is in  $L^2(\Delta)$ ).*

### 4.3.2.2 PDE in differential form and equivalent definitions

In this subsection, we push further the assumptions on the smoothness of the solution. This allows us to derive a differential form of the Fokker-Planck PDE and investigate the boundary condition.

We denote by  $AC(\overline{\mathcal{S}})$  the set of measures supported in  $\overline{\mathcal{S}}$  and absolutely continuous with respect to Lebesgue measure, and for a measure  $q \in AC(\overline{\mathcal{S}})$  we will abuse notations and denote also  $q$  its density.

**Proposition 4.3.8.** *Assume that for  $t > 0$ ,  $p_t \in AC(\overline{\mathcal{S}})$  and  $ap_t \in H^2(\mathcal{S})$ , and consider the following statements:*

- (i)  $p_0 = \delta_{x_0, m_0}$ ,
- (ii)  $t \mapsto p(t, s) ds$  is weakly continuous,
- (iii)  $\partial_t p_t - (ap_t)_{xx} = 0$  in  $\mathbb{R}_+^* \times \mathcal{S}$ ,
- (iv)  $\partial_t G_t^* = ap_t^\Delta$  in  $\mathbb{R}_+^* \times \Delta$ ,
- (iv')  $2(ap_t)^\Delta_x + (ap_t)^\Delta_m = 0$  in  $\mathbb{R}_+^* \times \Delta$ .

We have then

$$(i) \text{ and } (ii) \text{ and } (iii) \implies (iv) \text{ if and only if } (iv'), \quad (4.3.27)$$

and  $p$  is a variational solution to the Fokker-Planck PDE (4.3.19) if and only if  $p$  satisfies

$$\begin{cases} \text{(i) and (ii) and (iii) ,} \\ \text{(iv) or (iv') .} \end{cases} \quad (4.3.28)$$

**Proof.** Remark first that from the smoothness assumption on  $p$ , and Proposition 4.3.6, the formulation (4.3.19) of the Fokker-Planck PDE is equivalent to (4.3.20), and the trace operator is well defined in (iv) and (iv') . Assume that we have (i) and (ii) and (iii) and (iv) and let us prove (iv') . For a smooth test function  $\varphi \in C_c^{2,1}(\mathbb{R}^2)$  which does not depend on the variable  $x$ , using (i) and (ii) and (iii), we have that

$$\begin{aligned} \int_{\mathcal{S}} \varphi(m)(p_t - p_0)(ds) &= \int_{\mathcal{S}} \varphi(m) \int_0^t (ap_u)_{xx}(ds) du, \\ &= \int_{m_0}^{+\infty} \varphi(m) \int_0^t (ap_u)_x(m, m)(dm) du. \end{aligned} \quad (4.3.29)$$

On the other hand, if we integrate over the variable  $x$  to obtain the marginal law of the maximum, then using the boundary condition (iv);  $\partial_t G_t^* = ap_t^\Delta$ , and Fubini's theorem we obtain

$$\begin{aligned} \int_{\mathcal{S}} \varphi(m)(p_t - p_0)(ds) &= \int_{m_0}^{+\infty} \varphi(m)(p_t^* - p_0^*)(dm), \\ &= \int_{m_0}^{+\infty} \varphi(m) \frac{d}{dm} (G_t^* - G_0^*) dm, \\ &= \int_{m_0}^{+\infty} \varphi(m) \int_0^t \left( \frac{d}{dm} ap_u^\Delta \right) du. \end{aligned} \quad (4.3.30)$$

So we have

$$\int_{m_0}^{+\infty} \varphi(m) \left( \int_0^t (ap_u)_x^\Delta(m, m) du + \int_0^t \left( \frac{d}{dm} ap_u^\Delta \right) du \right) dm = 0, \quad (4.3.31)$$

which leads to (iv') .

On the other hand, assume that (i) and (ii) and (iii) and (iv') hold, and let us prove (iv), using a similar approach as in the previous step, a test function  $\varphi \in C_c^{2,1}(\mathbb{R}^2)$  which does not depend on the variable  $x$ . From (i) and (ii) and (iii) we have the equalities (4.3.29) and (4.3.30), so that

$$\int_{m_0}^{+\infty} \varphi(m) \int_0^t (ap_u)_x(m, m) dm du = \int_{m_0}^{+\infty} \varphi(m)(p_t^* - p_0^*) dm du, \quad (4.3.32)$$

then, using (iv');  $(ap_t)_x^\Delta = -\frac{d}{dm}(ap_t)^\Delta$  in  $\mathbb{R}_+^* \times \Delta$ , we obtain

$$\int_{m_0}^{+\infty} \varphi(m) \left( \int_0^t \frac{d}{dm} (ap_t)^\Delta du + (p_t^* - p_0^*) \right) dm = 0, \quad (4.3.33)$$

or equivalently, by integration by parts

$$\int_{m_0}^{+\infty} \varphi'(m) \left( \int_0^t (ap_t)^\Delta du - (G_t^* - G_0^*) \right) dm = 0. \quad (4.3.34)$$

which proves (iv) .

The rest of the proof is trivial. In fact, recall the formulation of the Fokker-Planck PDE

$$\begin{aligned} \int_{\mathcal{S}} \varphi(s) \left( p_t - p_0 - \int_0^t (ap_u)_{xx} du \right) (ds) &= -\frac{1}{\sqrt{2}} \int_0^t \int_{\Delta} \varphi(2(ap_u)_x^\Delta + (ap_u)_m^\Delta) (dm) du \\ &+ \frac{1}{\sqrt{2}} \int_{\Delta} \varphi_m \left( G_t^* - G_0^* - \int_0^t (ap_u)^\Delta du \right) dm. \end{aligned} \quad (4.3.20)$$

Assuming that  $p$  satisfies (4.3.28) then we have from (4.3.27) that all the integrals are zero in (4.3.20) and so  $p$  is a variational solution to the Fokker-Planck PDE.

On the other hand, if  $p$  is a variational solution to the Fokker-Planck PDE, then for  $\varphi \in C_c^{2,1}(\mathcal{S})$ , the right hand side of (4.3.20) becomes zero and so

$$p_t - p_0 - \int_0^t (ap_u)_{xx} du = 0 \text{ in } \mathcal{S}. \quad (4.3.35)$$

Then, for  $\varphi \in C_c^{2,1}(\mathbb{R}^2)$  with  $\varphi_m = 0$  on  $\Delta$ , the left hand side of (4.3.20) is zero by (4.3.35) and the remaining term is zero, i.e.,

$$\int_0^t \int_{\Delta} \varphi(2(ap_u)_x^\Delta + (ap_u)_m^\Delta) (dm) du = 0. \quad (4.3.36)$$

So

$$2(ap_t)_x^\Delta + (ap_t)_m^\Delta = 0 \text{ in } \mathbb{R}_+^* \times \Delta. \quad (4.3.37)$$

Finally, from (4.3.37) and (4.3.35), we have that the remaining term in (4.3.20) is zero which yields

$$G_t^*(m) - G_0^*(m) = \int_0^t ap_u(m, m)^\Delta du, \quad (4.3.38)$$

and we conclude by differentiating with respect to time, with the initial condition  $p_0 = \delta_{x_0, m_0}$  from the weak continuity of  $t \mapsto p(t, \cdot)$ , and so  $p$  satisfies (4.3.28) which concludes the proof.  $\square$

### 4.3.3 The uniqueness result for the Fokker-Planck PDE

Now that we derived the Fokker-Planck PDE satisfied by the probability measures induced by the solution SDE (4.3.1), under smoothness assumptions, we prove a uniqueness result for this PDE under further assumptions.

**Assumption 4.3.9.** Assume further that  $\sigma$  satisfies

$$\begin{cases} \sigma \text{ is differentiable w.r.t } t, \\ \log(\sigma) \text{ is lipschitz in } t \text{ uniformly in } s. \end{cases} \quad (4.3.39)$$

**Proposition 4.3.10.** Let  $(X_t)_{t \geq 0}$  be a weak solution of the SDE (4.3.1) in some probability space; (there exists at least one from proposition 4.3.3), and denote by  $(p(t, ds), t \geq 0)$  the family of probability measures induced by the laws of the pairs  $(X_t, M_t)_{t \geq 0}$ . Then

- (i) The family  $(p(t, ds), t \geq 0)$  is a solution to the variational Fokker-Planck PDE (4.3.19).
- (ii) Under the Assumption 4.3.9, there exists at most one family of probability measures  $(p(t, ds), t \geq 0)$  with  $C^2$

densities w.r.t to the Lebesgue measure such that  $ap(t) \in H^2(\mathcal{S})$  for  $t > 0$ , which is a solution to the Fokker-Planck PDE (4.3.19), (in this case equivalent to (4.3.28)).

In particular, every weak solution of (4.3.1) with marginal laws  $(X_t, M_t)_{t \geq 0}$  having  $C^{2,1}$  densities w.r.t to the Lebesgue measure, and appropriate integrability conditions ( $ap(t) \in H^2(\mathcal{S})$  for  $t > 0$ ) has the same marginal joint laws  $(X_t, M_t)_{t \geq 0}$ .

Before moving to the proof, remark that although the smoothness assumption seems restrictive and could possibly be improved, the uniqueness result that we obtain is enough for our application since we can always reduce to the "smooth case" via a regularization that will be introduced in section 4.5.

**Proof.** The existence result (i) follows from the weak existence of a solution to the SDE (4.3.1).

To prove the uniqueness, we define the function  $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}$  with

$$\Phi(t) := e^{-\lambda t} \int_{x_0}^{+\infty} \int_{-\infty}^m a(\delta p)^2(t, s) ds, \text{ for } t \geq 0$$

where  $\delta p$  is defined as the difference between two solutions of (4.3.19), and  $\lambda := \left\| \frac{\partial_t a}{a} \right\|_\infty$  which is well defined since  $\log(\sigma)$  is lipschitz in  $t$  uniformly in the space variables. Remark that we have  $\Phi(0) = 0$  since both solutions have the same initial condition, and  $\Phi(t) \geq 0$  for  $t \geq 0$ . We compute

$$\begin{aligned} \partial_t \Phi(t) &= e^{-\lambda t} \int_{m_0}^{+\infty} \int_{-\infty}^m a \partial_t (\delta p)^2(t, x, m) dx dm + e^{-\lambda t} \int_{m_0}^{+\infty} \int_{-\infty}^m \partial_t a (\delta p)^2(t, x, m) dx dm - \lambda \Phi(t), \\ &= e^{-\lambda t} \int_{m_0}^{+\infty} \int_{-\infty}^m a \partial_t (\delta p)^2(t, x, m) dx dm + e^{-\lambda t} \int_{m_0}^{+\infty} \int_{-\infty}^m \frac{\partial_t a}{a} a (\delta p)^2(t, x, m) dx dm - \lambda \Phi(t), \\ &\leq e^{-\lambda t} \int_{x_0}^{+\infty} \int_{-\infty}^m a \partial_t (\delta p)^2(t, x, m) dx dm + \underbrace{\left\| \frac{\partial_t a}{a} \right\|_\infty}_{=0} \Phi(t) - \lambda \Phi(t). \end{aligned} \tag{4.3.40}$$

Then, since  $\partial_t p = \partial_{xx} ap$  in  $\mathcal{S}$

$$\partial_t \Phi(t) \leq 2e^{-\lambda t} \int_{m_0}^{+\infty} \int_{-\infty}^m a(\delta p) \partial_t (\delta p)(t, x, m) dx dm = e^{-\lambda t} 2 \int_{m_0}^{+\infty} \int_{-\infty}^m a(\delta p) \partial_{xx} a(\delta p)(t, x, m) dx dm, \tag{4.3.41}$$

Integrating by parts and using the boundary condition  $(ap_t)_x^\Delta = -\frac{d}{dm}(ap_t)^\Delta$  in  $\mathbb{R}_+^* \times \Delta$ ,

$$\partial_t \Phi(t) \leq -2e^{-\lambda t} \int_{m_0}^{+\infty} a(\delta p) \frac{d}{dm} a(\delta p)(t, m, m) dm - 2e^{-\lambda t} \int_{m_0}^{+\infty} \int_{-\infty}^m (\partial_x a(\delta p))^2(t, x, m) dx dm, \tag{4.3.42}$$

$$\leq -e^{-\lambda t} \int_{m_0}^{+\infty} \frac{d}{dm} (a(\delta p)(t, m, m))^2 dm = 0, \tag{4.3.43}$$

where the last equality follows from the integrability condition on  $ap$  together with  $ap(t, m_0, m_0) = 0$  for  $t > 0$ . Therefore

$$\partial_t \Phi \leq 0, \text{ for } t \geq 0,$$

and  $\Phi(0) = 0$  from the equality of initial conditions, so  $\Phi$  is a non negative non increasing function such that  $\Phi(t) \leq 0$  for  $t \geq 0$ , i.e.,  $\Phi(t) = 0$  for  $t \geq 0$  which proves the uniqueness of the solution.

□

## 4.4 The problem in terms of barrier options

Now that we established the wellposedness of the Fokker-Planck PDE (with some regularity assumptions), we want to use this result as a necessary condition for our family  $\mathcal{V}$  to have  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$ . In fact, by the Proposition 4.3.10, we expect to have:

If  $\mathcal{V}$  is a smooth solution to PDE (4.3.19) for some function  $a$  then,  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$ ,

where smooth is in the sense that it has  $C^{2,1}$  densities w.r.t to the Lebesgue measure and  $a\nu(t, \cdot) \in H^2(\mathcal{S})$  for  $t > 0$  (as required by the uniqueness result of the Fokker-Planck PDE).

However there are two main problems with the previous assertion; the smoothness assumption on the family  $\mathcal{V}$ , and the fact that  $\sigma$  is unknown. We start therefore by making smoothness assumptions on  $\mathcal{V}$  and estimate  $\sigma$ , then we prove that we can always reduce to the smooth case using an appropriate regularization procedure.

As stated earlier we follow the methodology of [Hirsch & Roynette \(2012\)](#) where they use the call function and Dupire's formula to construct a martingale with given marginals and prove Kellerer's theorem. In fact, the presence of the unknown function  $\sigma$  (or equivalently  $a$ ) in the Fokker-Planck PDE (when  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$ ) makes the introduction of the call (or put) function necessary to estimate it. In our case, the prescribed laws are the joint probability measures of the process and its running maximum, and so by analogy, the construction (more precisely the estimation of  $\sigma$  as in [Hambly et al. \(2016\)](#) and [Henry-Labordere 2017, Chapter 4.3](#)) is based on barrier call and put functions.

**Definition 4.4.1** (Barrier call and put functions). *Let  $\nu \in \mathcal{P}_1(\mathbb{R}^2)$ . We define the corresponding barrier call function  $C_\nu : \mathbb{R}^2 \rightarrow \mathbb{R}$  as*

$$C_\nu(K, B) := \int_{\mathbb{R}^2} (x - K)^+ \mathbf{1}_{\{m > B\}} \nu(dx, dm), \quad (4.4.1)$$

and the put function  $P_\nu : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$P_\nu(K, B) := \int_{\mathbb{R}^2} (K - x)^+ \mathbf{1}_{\{m > B\}} \nu(dx, dm), \quad (4.4.2)$$

and  $\varphi_\nu : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\begin{aligned} \varphi_\nu(K, B) &:= \int_{\mathbb{R}^2} \left\{ (x - K)^+ - (B - K) \right\} \mathbf{1}_{\{m > B\}} \nu(dx, dm), \\ &= C_\nu(K, B) + (B - K) \partial_K C_\nu(-\infty, B). \end{aligned} \quad (4.4.3)$$

**Remark 4.4.2.** *The definition 4.4.1 might seem redundant since it introduces three different functions which are all characterized by the measure  $\nu$ . The most suitable function in our analysis is  $\varphi_\nu$ , and the reason we introduce  $C_\nu$*

and  $P_\nu$  is to exhibit a remarkable identity (the call put parity), which will have a very useful implication on  $\varphi_\nu$  when  $\nu$  is induced by a martingale and its running maximum (see Lemma 4.4.10 and Proposition 4.4.11).

**Remark 4.4.3.** (i) More specifically, definition 4.4.1 refers to up and in Barrier call (resp. put) function, which is a convention we use. All upcoming results can be transposed in terms of up and out Barrier call (or put) functions.  
(ii) In our definition, the measure  $\nu$  is not necessarily induced by a martingale and its running maximum. This will turn out to be useful later since we want to provide sufficient conditions for  $\mathcal{M}^0(\{\nu\})$  to be non-empty, and therefore we will work with measures which are a priori not induced by a martingale and its running maximum.  
(iii) In a financial context, the support of  $\nu$  is restricted to a subset of  $\mathbb{R}^+ \times \mathbb{R}^+$  since it represents the law of the asset price fluctuations (and its running maximum). We present our results in a more general framework by allowing the martingale to take negative values.

For a given family of probability measures  $\mathcal{V}$ , we add the time index as another variable in the barrier function  $\psi$ , where  $\psi \in \{C, P, \varphi\}$ , and therefore consider a single function instead of a family of functions,  $\psi_\mathcal{V} : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  with  $\psi_\mathcal{V}(t, \cdot) = \psi_{\nu_t}$ , for  $t \in [0, T]$ .

Now the observed data considered is a function  $\psi$  instead of a family of probability measures, and the necessary conditions previously established on  $\mathcal{V}$  need to be rephrased. These conditions can be organized as:

- (i) Static conditions: conditions insuring that for  $t \in [0, T]$ ,  $\psi(t, \cdot)$  characterizes a probability measure which is the joint law of a martingale and its running maximum at some time  $t \in [0, T]$ .
- (ii) Dynamic condition : time consistency condition insuring (up to some smoothness assumption) that there exists some function  $\sigma$  (or  $a$ ) such that the probability measures corresponding to  $(\psi(t, \cdot))_{t \in [0, T]}$  defines a solution to the Fokker-Planck PDE (4.3.19). For instance in the case of Kellerer's theorem, the dynamic condition is that the marginal laws are increasing in the convex order.

#### 4.4.1 Static conditions

In this section, we provide some properties of barrier functions introduced in definition 4.4.1; without the time variable. As in the one dimensional case where a probability measure  $\mu \in \mathcal{P}_1(\mathbb{R})$  is completely characterized by the corresponding call function (see for instance Hirsch & Roynette (2012)[Proposition 2.1]), we provide a similar characterization with the barrier call function for a measure  $\nu \in \mathcal{P}_1(\mathcal{S})$  such that  $\nu$  is the joint law of the terminal value and running maximum of a martingale.

We define first some “Monotonicity conditions”<sup>3</sup> which are necessary and sufficient for a function  $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$  to be a barrier function, i.e., induced by some measure  $\nu$  in  $\mathcal{P}_1(\mathbb{R}^2)$ . We refine then the result by imposing further conditions to obtain the “Static conditions” for the function  $\psi$  insuring that the corresponding probability measure  $\nu$  is the joint law of the maximum and terminal value of a martingale.

##### 4.4.1.1 Characterization of a probability measure in $\mathbb{R}^2$

We start by introducing the “Monotonicity conditions”. We define these conditions for  $C$ , since it is the easiest. Similar results can be established for  $P$  and  $\varphi$ .

<sup>3</sup>In particular the monotonicity and convexity allow us to define the derivatives in a weak sense.



**Definition 4.4.4** (Monotonicity conditions). *We say that a function  $C : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies the Monotonicity conditions if:*

- (i)  $\forall B \in \mathbb{R}$ , the function  $K \mapsto C(K, B)$  is convex. We can therefore define its left derivative everywhere, which we denote  $\partial_K C$  (instead of  $\partial_{K^-} C$ ) to ease notations,
- (ii)  $\lim_{K \rightarrow +\infty} C(K, B) = 0, \forall B \in \mathbb{R}$ ,
- (iii)  $C(0, -\infty) < +\infty$ , and  $\lim_{K \rightarrow -\infty} (C(K, -\infty) + K) = a$  for some  $a \in \mathbb{R}$ ,
- (iv)  $-\partial_K C$  is lower semi-continuous,
- (v)  $\lim_{K \rightarrow +\infty} \partial_K C(K, B) = 0, \forall B \in \mathbb{R}$ , and  $\lim_{B \rightarrow +\infty} \partial_K C(K, B) = 0, \forall K \in \mathbb{R}$ ,
- (vi)  $\lim_{\substack{K \rightarrow -\infty \\ B \rightarrow -\infty}} \partial_K C(K, B) = -1$ ,
- (vii)  $\forall (K, B) \in \mathbb{R}^2$ , and  $\forall (\varepsilon, \eta) \in \mathbb{R}_+^2$  we have

$$-\partial_K C(K, B) - \partial_K C(K + \varepsilon, B + \eta) + \partial_K C(K, B + \eta) + \partial_K C(K + \varepsilon, B) \geq 0. \quad (4.4.4)$$

Remark that (ii) implies the first part of (v), but we keep it for clarity. Remark also that we included the condition of the finite first moment in the Monotonicity conditions.

The following lemma establishes the one-to-one relationship between functions satisfying the Monotonicity conditions and probability measures on  $\mathbb{R}^2$ .

**Lemma 4.4.5.** *Let  $C : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Then  $C$  is a barrier call function if and only if it satisfies the Monotonicity conditions. We can then define the unique correspondent probability measure  $\nu \in \mathcal{P}_1(\mathbb{R}^2)$*

$$\begin{aligned} \nu([K, K + \varepsilon] \times [B, B + \eta]) := & -\partial_K C(K, B) - \partial_K C(K + \varepsilon, B + \eta) + \partial_K C(K, B + \eta) \\ & + \partial_K C(K + \varepsilon, B), \text{ for } (K, B, \varepsilon, \eta) \in \mathbb{R}^2 \times \mathbb{R}_+^2, \end{aligned} \quad (4.4.5)$$

and we write  $C_\nu$  instead of  $C$ .

The proof is reported in Appendix 4.6.2.1.

**Remark 4.4.6.** (i) *We have the two following identities (since  $-\partial_K C$  is the survival cumulative distribution function)*

$$\begin{cases} F_{\nu^*}(K, B) = 1 - \partial_K C(K, B) + \partial_K C(K, -\infty) + \partial_K C(-\infty, B), \\ -\partial_K C(K, B) = 1 + F_{\nu^*}(K, B) - F_{\nu^*}(+\infty, B) - F_{\nu^*}(K, +\infty), \end{cases} \text{ for } (K, B) \in \mathbb{R}^2. \quad (4.4.6)$$

where  $F_{\nu^*}$  denotes the lower semi-continuous envelope of  $F_\nu$ , the cumulative distribution function associated to  $\nu$  (c.f. Appendix, Theorem 4.6.3).

(ii) *Whenever the barrier call function  $C$  or more generally  $\psi \in \{C, P, \varphi\}$  is smooth, the measure  $\nu$  is absolutely continuous w.r.t Lebesgue measure and its density can be identified by  $\nu(ds, dm) = -\partial_{KKB}\psi(s, m) ds dm$ . Otherwise -when  $\psi$  is less smooth- we still have  $\nu = -\partial_{KKB}\psi$  but with the derivatives of  $\psi$  in the weak sense.*

#### 4.4.1.2 Characterization of the joint law of maximum and terminal value of a martingale

In this section, we refine the ‘‘Monotonicity conditions’’ to guarantee that  $C$  is induced by a probability measure which is the (terminal) joint law of a martingale and its running maximum. This is possible since the function  $f_\mu$

defined in Theorem 4.2.1 can be characterized in terms of  $C_\mu$  as

$$\begin{aligned} f_\mu(B) &= -\frac{\lim_{K \rightarrow -\infty} \{C_\mu(K, B) - K \partial_K C_\mu(-\infty, B)\}}{\partial_K C_\mu(-\infty, B)} \mathbf{1}_{\{\partial_K C_\mu(-\infty, B) < 0\}} + B^+ \mathbf{1}_{\{\partial_K C_\mu(-\infty, B) = 0\}}, \\ &= \left( -\lim_{K \rightarrow -\infty} \left\{ \frac{C_\mu(K, B)}{\partial_K C_\mu(-\infty, B)} - K \right\} \right) \mathbf{1}_{\{\partial_K C_\mu(-\infty, B) < 0\}} + B^+ \mathbf{1}_{\{\partial_K C_\mu(-\infty, B) = 0\}}, \end{aligned} \quad (4.4.7)$$

Therefore the following definition (in particular (v)) makes sense:

**Definition 4.4.7** (Static conditions). *We say that a function  $C : \mathbb{R}^2 \rightarrow \mathbb{R}$  satisfies the Static conditions if:*

- (i)  $C$  satisfies the Monotonicity conditions,
- (ii)  $\lim_{K \rightarrow -\infty} (C(K, -\infty) + K) = 0$ ,
- (iii)  $\forall B \in \mathbb{R}_+$ , and for  $\eta \geq 0$ ,  $\partial_K C(B, B) = \partial_K C(B, B - \eta)$ ,
- (iv)  $\forall K \in \mathbb{R}$ , and for  $\eta \geq 0$ ,  $\partial_K C(K, -\eta) = \partial_K C(K, 0)$ ,
- (v)  $f_\mu(B) = B^+$  for  $B \in \mathbb{R}$ .

The following Lemma provides an expression of Theorem 4.2.1 in terms of barrier options.

**Lemma 4.4.8.** *Let  $C : \mathbb{R}^2 \rightarrow \mathbb{R}$ . Then  $\mathcal{M}^0(-\partial_{KKB}C) \neq \emptyset$  if and only if  $C$  satisfies the Static conditions.*

*In that case (when  $\mathcal{M}^0(-\partial_{KKB}C) \neq \emptyset$ ), the function  $C$  is completely characterized by the probability measure  $-\partial_{KKB}C$ .*

The proof is reported in Appendix 4.6.2.2.

**Remark 4.4.9.** a) *Condition (ii) insures that the martingale is centered and thus can be relaxed up to some change of variable.*

b) *Properties (iii) and (iv) imply that whenever  $\mathcal{M}(-\partial_{KKB}C) \neq \emptyset$ , then  $C$  is completely characterized by its values in the set  $\mathcal{S}$  since we can always extend it by the boundary condition. Therefore it is sufficient to characterize  $C$  inside the set  $\mathcal{S}$ .*

The following Lemma provides a link between  $\varphi$  and  $P$  under the Static conditions

**Lemma 4.4.10.** *Assume that  $\nu$  is induced by a continuous martingale and its running maximum, i.e.,  $C$  satisfies the Static conditions, then we have for  $(B, K) \in \bar{\mathcal{S}}$*

$$\varphi(K, B) = P(K, B), \text{ and } \partial_B \varphi_\nu(B, B) = -G^*(B), \quad (4.4.8)$$

where  $G^*(B) := \int_B^{+\infty} \int_{(-\infty, m]} \nu_u(dx, dm)$ , i.e., for  $\nu \stackrel{\text{law}}{=} (X, M)$ ,  $G^*(B) := \mathbb{P}[M > B]$ .

**Proof.** We have for  $(B, K) \in \bar{\mathcal{S}}$ , and the fact that  $x^+ - (-x)^+ = x$ , and Theorem 4.2.1,

$$C(K, B) - P(K, B) = \int_{\mathcal{S}} (x - K) \mathbf{1}_{\{m > B\}} \nu(dx, dm), \quad (4.4.9)$$

$$= (B - K) \int_{\mathcal{S}} \mathbf{1}_{\{m > B\}} \nu(dx, dm) = -(B - K) \partial_K C(-\infty, B). \quad (4.4.10)$$

So

$$\varphi(K, B) = P(K, B), \quad (4.4.11)$$

and since  $\partial_K C(-\infty, B) = -G^*(B)$ , we obtain by differentiating with respect to  $B$ , then taking  $K = B$  that  $\partial_B \varphi_{\mathcal{V}}(B, B) = -G^*(B)$ , concluding the proof.  $\square$

#### 4.4.2 Dynamic conditions : time consistence and estimation of $\frac{1}{2}\sigma^2$

Now we consider the family of probability measures  $\mathcal{V}$  as a whole, i.e., the functions  $\psi_{\mathcal{V}} : [0, T] \times \mathbb{R}^2 \rightarrow \mathbb{R}$  with  $\psi_{\mathcal{V}}(t, \cdot) = \psi_{\nu_t}, \forall t \in [0, T]$ , and  $\psi \in \{C, P, \varphi\}$ .

**Proposition 4.4.11** (Necessary dynamic condition). *Let  $\mathcal{V} := (\nu_t)_{t \in [0, T]}$  a family of probability measures in  $\mathcal{P}(\mathcal{S})$ , such that  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$ . Then  $\varphi_{\mathcal{V}}$  satisfies (in the weak sense) :*

$$\partial_{tB} \varphi_{\mathcal{V}} \leq 0 \text{ in } [0, T] \times \mathcal{S}, \quad (4.4.12)$$

Furthermore

$$\varphi_{\mathcal{V}}(t, K, B) = P_{\mathcal{V}}(t, K, B), \text{ and } \partial_B \varphi_{\mathcal{V}}(t, B, B) = -G_t^*(B). \quad (4.4.13)$$

In particular,

$$\partial_t G_t^* = -\partial_{tB} \varphi_{\mathcal{V}} \text{ in } (0, +\infty) \times \Delta. \quad (4.4.14)$$

Assume further that  $\mathcal{M}^0(\mathcal{V})$  contains a weak solution of the SDE (4.3.1), and  $\varphi_{\mathcal{V}} \in C^{1,2,1}([0, T] \times \mathcal{S})$ . Then

$$\partial_{tB} \varphi_{\mathcal{V}} = \frac{1}{2} \sigma^2 \partial_{KK} \varphi_{\mathcal{V}} \text{ in } [0, T] \times \mathcal{S}. \quad (4.4.15)$$

**Proof.** By definition, we have

$$\varphi_{\mathcal{V}}(t, K, B) = \mathbb{E} \left[ \left\{ (X_t - K)^+ - (B - K) \right\} 1_{\{M_t > B\}} \right], \quad (4.4.16)$$

for some pair  $(X_t, M_t)_{t \in [0, T]}$  of a UI continuous martingale and its running maximum, since by assumption  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$ . We have therefore (from Theorem 4.2.1) that  $\mathbb{E} \left[ X_t 1_{\{M_t > B\}} \right] = B \mathbb{E} \left[ 1_{\{M_t > B\}} \right]$ , for  $B \geq 0$ , and  $t \in [0, T]$ . So we can write  $\varphi_{\mathcal{V}}$  as

$$\begin{aligned} \varphi_{\mathcal{V}}(t, K, B) &= \mathbb{E} \left[ \left\{ (X_t - K)^+ - (X_t - K) \right\} 1_{\{M_t > B\}} \right], \\ &= \mathbb{E} \left[ (K - X_t)^+ 1_{\{M_t > B\}} \right], \\ &= P_{\mathcal{V}}(t, K, B) \text{ for } t \in [0, T], \end{aligned} \quad (4.4.17)$$

and (4.4.14) follows after differentiating with respect to  $B$  and taking  $K = B$ .

Remark that by monotonicity of the expectation, we can see that  $\varphi_{\nu_t}$  is non-increasing w.r.t  $B$ , and therefore  $\partial_B \varphi_{\mathcal{V}}$  is well defined for almost all  $B$ . We apply Itô–Tanaka theorem to the payoff  $(K - X_{t+h})^+ 1_{\{M_{t+h} > B\}}$  :

$$\begin{aligned} (K - X_{t+h})^+ 1_{\{M_{t+h} > B\}} &= (K - X_t)^+ 1_{\{M_t > B\}} - \int_t^{t+h} 1_{\{K \geq X_u\}} 1_{\{M_u > B\}} dX_u, \\ &\quad + \frac{1}{2} \int_t^{t+h} 1_{\{M_u > B\}} dL_u^K(X) + \int_t^{t+h} (K - X_u)^+ d \left[ 1_{\{M_u > B\}} \right], \end{aligned} \quad (4.4.18)$$

where  $L_t^K(X)$  is the local time of  $X$  in  $K$ . Remark that  $1_{\{K \geq X_u\}}1_{\{M_u > B\}}$  is bounded and therefore

$$\mathbb{E} \left[ \int_t^{t+h} 1_{\{K \geq X_u\}} 1_{\{M_u > B\}} dX_u \right] = 0. \quad (4.4.19)$$

Remark also that  $(K - X_u)^+ d[1_{\{M_u > B\}}] = (K - B)^+ d[1_{\{M_u > B\}}]$  by the continuity of the process  $X$  (and of the process  $M$  by implication), i.e., the measure  $d[1_{\{M_u > B\}}]$  is supported on the set  $\{X_u = B\}$ . Furthermore, we have that  $(K, B) \in \mathcal{S}$ , which means that  $K \leq B$  leading to  $(K - B)^+ = 0$ , and so  $\int_t^{t+h} (K - X_u)^+ d[1_{\{M_u > B\}}] = 0$ . We can now rewrite (4.4.18), apply the expectation which leads to :

$$\varphi_{\mathcal{V}}(t+h, K, B) - \varphi_{\mathcal{V}}(t, K, B) = \frac{1}{2} \mathbb{E} \left[ \int_t^{t+h} 1_{\{M_u > B\}} dL_u^K(X) \right], \quad (4.4.20)$$

and for  $\varepsilon > 0$ ,

$$\varphi_{\mathcal{V}}(t+h, K, B+\varepsilon) - \varphi_{\mathcal{V}}(t, K, B+\varepsilon) = \frac{1}{2} \mathbb{E} \left[ \int_t^{t+h} 1_{\{M_u > B+\varepsilon\}} dL_u^K(X) \right], \quad (4.4.21)$$

so that

$$\varphi_{\mathcal{V}}(t+h, K, B+\varepsilon) + \varphi_{\mathcal{V}}(t, K, B) - \varphi_{\mathcal{V}}(t, K, B+\varepsilon) - \varphi_{\mathcal{V}}(t+h, K, B) = -\frac{1}{2} \mathbb{E} \left[ \int_t^{t+h} 1_{\{B+\varepsilon \geq M_u > B\}} dL_u^K(X) \right], \quad (4.4.22)$$

and we conclude then from the positivity of the measure  $dL_u^K(X)$ , as  $h \rightarrow 0$  and  $\varepsilon \rightarrow 0$  that  $\partial_{tB}\varphi_{\mathcal{V}} \leq 0$  in the weak sense.

Assume now that  $\mathcal{M}^0(\mathcal{V})$  contains a weak solution of the SDE (4.3.1), and  $\varphi_{\mathcal{V}} \in C^{1,2,1}([0, T] \times \mathcal{S})$ . We can then write the right hand side of (4.4.22) using the occupation time formula for the local time (see (Revuz & Yor 2013, VI, 1.6)), and we have from the smoothness of  $\varphi_{\mathcal{V}}$  that  $p$  (identified with  $-\partial_{KKB}\varphi_{\mathcal{V}}$ ) is absolutely continuous with respect to the Lebesgue measure (see (ii) in Remark 4.4.6), and so we have

$$\begin{aligned} -\frac{1}{2} \mathbb{E} \left[ \int_t^{t+h} 1_{\{B+\varepsilon \geq M_u > B\}} dL_u^K(X) \right] &= -\frac{1}{2} \int_t^{t+h} \int_B^{B+\varepsilon} \sigma^2 p(u, K, m) dm du, \\ &= \frac{1}{2} \int_t^{t+h} \int_B^{B+\varepsilon} \sigma^2 \partial_{KKB}\varphi_{\mathcal{V}}(u, K, m) dm du. \end{aligned} \quad (4.4.23)$$

On the other hand

$$\varphi_{\mathcal{V}}(K, B+\varepsilon, t+h) + \varphi_{\mathcal{V}}(K, B, t) - \varphi_{\mathcal{V}}(K, B+\varepsilon, t) - \varphi_{\mathcal{V}}(K, B, t+h) = \int_t^{t+h} \int_B^{B+\varepsilon} \partial_{tB}\varphi_{\mathcal{V}}(u, K, m) dm du, \quad (4.4.24)$$

and so, for  $(K, B) \in \mathcal{S}$ , we have

$$\int_t^{t+h} \int_B^{B+\varepsilon} \left( \partial_{tB}\varphi_{\mathcal{V}}(u, K, m) - \frac{1}{2} \sigma^2 \partial_{KKB}\varphi_{\mathcal{V}}(u, K, m) \right) dm du = 0, \quad (4.4.25)$$

which concludes the proof.  $\square$

We say that a family of probability measures  $\mathcal{V} = (\nu_t)_{t \in [0, T]}$  satisfies the Dynamic condition if the corresponding barrier call function satisfies (4.4.12).

Remark that (4.4.15) provides a formula for  $\frac{1}{2}\sigma^2$  similar to the one in Hambly et al. (2016) and (Henry-Labordere

2017, Chapter 4.3).

**Assumption 4.4.12.** *The family  $\mathcal{V}$  is smooth in the sense that for  $t > 0$ ,  $\nu_t$  has  $C^{2,1}$  density denoted  $p_t$  w.r.t Lebesgue measure.*

## 4.5 Main result

We are now ready to state our main result, which is a collection of sufficient conditions on  $\mathcal{V}$  that insures  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$ . We first provide a constructive proof under smoothness assumptions on  $\mathcal{V}$ . Then we prove that any family  $\mathcal{V}$  can be approximated by an appropriate collection of smooth families  $\mathcal{V}^\epsilon$ , thus relax the smoothness assumption provided that some relative compactness property (along with a technical condition) holds.

### 4.5.1 The smooth case

As mentioned in section 4.4, the assumption and resolution of the problem is performed by means of the function  $\varphi : [0, T] \times \mathcal{S} \rightarrow \mathbb{R}$ . We make the following assumption:

**Assumption 4.5.1.** *Assume that  $\varphi$  associated to  $\mathcal{V}$  satisfies*

- (i)  $\varphi$  satisfies the static conditions,
- (ii)  $\varphi$  satisfies the dynamic condition,
- (iii)  $\varphi \in C^\infty$ ,
- (iv)  $\partial_{KKB}\varphi(t, K, B) < 0$  for all  $(t, K, B) \in [0, T] \times \mathcal{S}$ ,
- (v)  $\partial_{tB}\varphi < 0$  for  $(t, K, B) \in [0, T] \times \mathcal{S}$ .

Under assumption 4.5.1, we can define

$$\hat{\sigma}(t, K, B) := \sqrt{\frac{\partial_{tB}\varphi(t, K, B)}{\frac{1}{2}\partial_{KKB}\varphi(t, K, B)}}, \text{ for } (t, K, B) \in [0, T] \times \mathcal{S}, \quad (4.5.1)$$

and provide a further assumption:

**Assumption 4.5.2.** *Assume that  $\hat{\sigma}$  defined by (4.5.1) satisfies the Assumption 4.3.9.*

Remark that in Assumption 4.5.1, the static conditions and dynamic conditions imply the two inequalities (iv) and (v) but only in the large sense. This point is addressed in the regularization. Remark also that by smoothness of  $\varphi$  and (iv) and (v),  $\hat{\sigma}$  satisfies Assumption 4.3.2.

We provide now the main result in the smooth case

**Proposition 4.5.3.** *Under Assumptions 4.5.1 and 4.5.2,  $\mathcal{M}^0(\mathcal{V})$  contains a weak solution of the SDE*

$$X_t = x_0 + \int_0^t \hat{\sigma}(u, X_u, M_u) dW_u, \quad (4.5.2)$$

with  $\hat{\sigma}$  defined by (4.5.1).

**Proof.** The proof is in two steps:

- (i) From the smoothness of  $\varphi$  and (iv) and (v), we have that  $\hat{\sigma}$  satisfies Assumption 4.3.2, which guarantees

the existence of at least a weak solution to the SDE (4.5.2). Furthermore, the assumption 4.5.2 guarantees the wellposedness of the associated Fokker-Planck PDE of the form (4.3.19) (or equivalently (4.3.28)). We denote by  $\hat{\mathcal{V}}$  the family of marginal laws induced by (4.5.2) and characterized by the Fokker-Planck PDE (4.3.28). Remark that by construction  $\mathcal{M}^0(\hat{\mathcal{V}}) \neq \emptyset$ .

(ii) From Assumption 4.5.1, the family  $\mathcal{V}$  is a solution to the Fokker-Planck PDE since it satisfies (4.3.28). In fact, we have that

$$\nu_t = -\partial_{KKB}\varphi_{\mathcal{V}}(t, \cdot) \text{ for } t \in [0, T], \quad (4.5.3)$$

and so, from the definition of  $\hat{\sigma}$  and the smoothness assumption on  $\varphi_{\mathcal{V}}$  we have in  $(0, T] \times \mathcal{S}$

$$\begin{aligned} \partial_t \nu_t - \partial_{KK} \left( \frac{1}{2} \hat{\sigma}^2 \nu_t \right) &= \partial_t (-\partial_{KKB}\varphi_{\mathcal{V}}(t, \cdot)) - \partial_{KK} \left( \frac{1}{2} \hat{\sigma}^2 (-\partial_{KKB}\varphi_{\mathcal{V}}(t, \cdot)) \right) \\ &= -(\partial_{KKtB}\varphi_{\mathcal{V}}(t, \cdot) - \partial_{KK}\partial_{tB}\varphi_{\mathcal{V}}(t, \cdot)) = 0. \end{aligned} \quad (4.5.4)$$

Furthermore, since  $\mathcal{V}$  satisfies the Static conditions, then

$$\partial_{tB}\varphi_{\mathcal{V}}(t, B, B) = -\partial_t G_t^*(B) \text{ in } \mathbb{R}_+^* \times \Delta, \quad (4.5.5)$$

so by replacing the left hand side term in (4.5.5) with the definition of  $\hat{\sigma}$  again, we obtain

$$\frac{1}{2} \hat{\sigma}^2 \partial_{KKB}\varphi_{\mathcal{V}}(t, B, B) = -\partial_t G_t^*(B) \text{ in } \mathbb{R}_+^* \times \Delta. \quad (4.5.6)$$

From (4.5.4) and (4.5.6), together with the initial condition and the smoothness with respect to time, we can see that  $\mathcal{V}$  satisfies (4.3.28) and so from Proposition 4.3.8,  $\mathcal{V}$  is a solution to the Fokker-Planck PDE (4.3.19).

From (i) and (ii), and the uniqueness result on the Fokker-Planck PDE (4.3.19) by assumption 4.5.2, we have that  $\mathcal{V} = \hat{\mathcal{V}}$ , and so  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$  since  $\mathcal{M}^0(\hat{\mathcal{V}}) \neq \emptyset$ .  $\square$

## 4.5.2 The general case

As stated earlier, we can relax the smoothness conditions for the family  $\mathcal{V}$ , i.e., (iii), and (iv) and (v) in Assumption 4.5.1 and regularize to reduce to the smooth case.

One difficulty arises which is that we need to preserve the structure of the martingale and its running maximum while regularizing, since it is crucial for the boundary condition in Fokker-Planck PDE and the uniqueness result. This implies constraints which are not linear with respect to the barrier functions, and are therefore not necessarily stable by direct convolution.

This is fundamentally linked to the fact that the probability measure representing the law of the sum of two independent random variables is characterized by the convolution of their two measures. For example, let's take a measure  $\mu \stackrel{\text{law}}{=} (X_t, M_t)$  for some UI continuous martingale  $X$  and its running maximum  $M_t := M_0 \vee \sup_{u \leq t} X_u$ . Consider then an independent pair  $(W_t, W_t^*)_{t \geq 0}$  of a Brownian motion and its running maximum with density

$\rho : \mathbb{R}_+ \times \mathcal{S} \rightarrow [0, 1]$  which we recall

$$\rho(t, s, m) := \frac{2(2m-s)}{\sqrt{2\pi}u^3} \exp\left(-\frac{(2m-s)^2}{2u}\right) 1_{\{m>s\}} 1_{\{m>0\}}. \quad (4.5.7)$$

Now consider a regularization of  $\mu$  of the form  $\mu^\varepsilon : \mathcal{S} \rightarrow \mathbb{R}$ , defined for  $\varepsilon \in (0, 1)$  as

$$\mu^\varepsilon(x, y) := \int_{\mathcal{S}} \rho(\varepsilon, x-s, y-m) \mu(ds, dm). \quad (4.5.8)$$

This regularization is clearly not convenient for our proof since  $\mu^\varepsilon$  is the joint law of the pair  $(X_t + W_\varepsilon, M_t + W_\varepsilon^*)$ , and since the sum of two martingales and their running maximums has no reason to be itself a martingale and its running maximum, we can rule out direct convolutions for our regularization (at least regarding space variables).

We propose therefore a more constructive approach for our regularization, in 4 steps, and a 5<sup>th</sup> step to prove the convergence of marginals. We start by regularizing the space state variables by appropriately approximating the martingale considered and its running maximum (step 1). We will then make sure that our approximation does not interfere with the dynamic consistence, i.e., that the dynamic condition is kept for the new process (step 2). Step 3 consists on regularizing with respect to the time variable, and finally, in step 4 we will make sure that we have the inequalities (iii) and (iv) of Assumption (4.5.1) in the strict sense to avoid dividing by zero when defining  $\hat{\sigma}$ .

#### 4.5.2.1 Reduction to the smooth case

The standing assumption on the family  $\mathcal{V}$  in the general case is the following

**Assumption 4.5.4.** Assume that  $\varphi$  associated to  $\mathcal{V}$  satisfies

- (i)  $\varphi$  satisfies the static conditions,
- (ii)  $\varphi$  satisfies the dynamic condition.

We present hereafter and prove the approximation result, which allows to reduce to the smooth case

**Proposition 4.5.5.** Let  $\mathcal{V} := (\nu_t)_{t \in [0, T]}$  a family of probability measures on  $\mathcal{S}$ , satisfying assumption 4.5.4. For  $\varepsilon \in (0, 1)$ , there exists a family  $\mathcal{V}^\varepsilon := (\nu_t^\varepsilon)_{t \in [0, T]}$  that satisfies assumption 4.5.1, and such that

$$\nu_t^\varepsilon \Rightarrow \nu_t, \text{ for all } t \in [0, T],$$

where  $\nu_t^\varepsilon \Rightarrow \nu_t$  denotes the convergence in the weak topology.

**Proof.**

*Step 1 : Regularization of space variables*

We start by fixing some  $t \in [0, T]$ . By assumption (Static conditions), the measure  $\nu_t = -\partial_{KKB}\varphi_{\mathcal{V}}(t, \cdot, \cdot)$  is well defined, as a probability measure induced by a continuous UI-martingale and its running maximum, which we shall denote  $(X_t, X_t^*)$ . As explained above, simply applying a convolution with a Gaussian kernel to  $\nu_t$  does not solve our problem. So instead, we consider a martingale  $\tilde{X}_{\{0 \leq u \leq t\}}$  whose terminal joint law with its maximum is  $(X_t, X_t^*)$ , and which exists by virtue of (Rogers 1993, Theorem 3.1). Then we take two independent Brownian motions (both independent of  $(\tilde{X}_u, \tilde{X}_u^*)_{u \in [0, t]}$  too) and their running maximums :  $(V_u, V_u^*)_{u \geq 0}$  and  $(W_u, W_u^*)_{u \geq 0}$ ,

and for  $\varepsilon \in (0, 1)$ , we construct by concatenation the process  $(Y_u)_{u \in [0, t+2\varepsilon]}$ , and  $(Z_u)_{u \in [0, t+\varepsilon]}$  :

$$Z := V \otimes_{\varepsilon} \tilde{X}, \text{ and } Y := V \otimes_{\varepsilon} \tilde{X} \otimes_{t+\varepsilon} W, \quad (4.5.9)$$

i.e.,

$$\begin{cases} Z_u := V_u 1_{\{u < \varepsilon\}} + (V_{\varepsilon} + \tilde{X}_{u-\varepsilon}) 1_{\{\varepsilon \leq u \leq t+\varepsilon\}}, \\ Y_u := Z_u 1_{\{u < t+\varepsilon\}} + (V_{\varepsilon} + \tilde{X}_t + W_{u-t}) 1_{\{t+\varepsilon \leq u\}}. \end{cases} \quad (4.5.10)$$

It is obvious that  $Y$  is –by construction– a continuous UI martingale, whose running maximum will be denoted  $Y^*$ . With a quick reasoning at times of connection ( $\varepsilon$ ,  $t + \varepsilon$  and at the terminal time  $t + 2\varepsilon$ ) we can express the joint laws of the pair  $(Y, Y^*)$  at those times in terms of  $(X_t, X_t^*)$  (since it is the same as  $(\tilde{X}_t, \tilde{X}_t^*)$  in time  $t$ ), and  $(V_{\varepsilon}, V_{\varepsilon}^*)$  and  $(W_{\varepsilon}, W_{\varepsilon}^*)$  whose laws we can compute by independence.

In fact, we have that  $(Y_{\varepsilon}, Y_{\varepsilon}^*) = (V_{\varepsilon}, V_{\varepsilon}^*)$ , and  $(Y_{t+\varepsilon}, Y_{t+\varepsilon}^*) = (V_{\varepsilon} + X_t, V_{\varepsilon}^* \vee (V_{\varepsilon} + X_t^*))$ . Therefore, at  $t + 2\varepsilon$  :

$$(Y_{t+2\varepsilon}, Y_{t+2\varepsilon}^*) = (V_{\varepsilon} + X_t + W_{\varepsilon}, V_{\varepsilon}^* \vee (V_{\varepsilon} + X_t^*) \vee (V_{\varepsilon} + X_t + W_{\varepsilon}^*)). \quad (4.5.11)$$

Finally, define  $(X_t^{\varepsilon}, X_t^{*\varepsilon}) := (Y_{t+2\varepsilon}, Y_{t+2\varepsilon}^*)$  for  $t$  fixed at the beginning, and repeat the procedure for  $t \in [0, T]$ , until we have a collection of space regularized joint measures  $(X_t^{\varepsilon}, X_t^{*\varepsilon})_{t \in [0, T]}$ . We stress the fact that an essential condition for our reasoning to hold is that the Brownian motions  $V$  and  $W$  are the same for every  $t \in [0, T]$ , and remark that  $(X_t^{\varepsilon}, X_t^{*\varepsilon})_{t \in [0, T]}$  is not a process but a collection of joint marginal laws, since for each iteration (each time  $t$ ), when concatenating with the Brownian motions  $V$  and  $W$ , the process  $\tilde{X}$  may differ because of the static nature of Rogers result (Rogers 1993, Theorem 3.1).

This remark will be crucial, especially when proving that the marginals of the regularized process satisfy the Fokker-Plank PDE. In fact, our space regularization is done for every maturity separately, by taking the joint law, connecting it to a Brownian motion  $V$ , and then connecting the resulting process to another Brownian motion  $W$ .

### Step 2 : Stability of static and dynamic conditions

With the notations of *step 1*, we introduce  $\tilde{\varphi}^{\varepsilon} : [0, T] \times \mathcal{S} \rightarrow \mathbb{R}$  as :

$$\tilde{\varphi}^{\varepsilon}(t, x, m) := \mathbb{E} \left[ (x - X_t^{\varepsilon})^+ 1_{\{X_t^{*\varepsilon} \geq m\}} \right], \text{ for } (t, x, m) \in [0, T] \times \mathcal{S}. \quad (4.5.12)$$

The function  $\tilde{\varphi}^{\varepsilon}$  naturally satisfies the Monotonicity conditions since it is defined as a Barrier put function, and by construction (step 1), for  $t \in [0, T]$ , the joint law  $(X_t^{\varepsilon}, X_t^{*\varepsilon})$  is induced by a UI-continuous martingale and its running maximum (in particular  $\tilde{\varphi}^{\varepsilon} = \tilde{P}^{\varepsilon}$ , which justifies our abuse of notation by directly defining  $\tilde{\varphi}^{\varepsilon}$  as a barrier put function, (see Lemma 4.4.10)). Furthermore, the smoothness of  $\tilde{\varphi}^{\varepsilon}$  with respect to space variables  $x$  and  $m$  is trivial. In fact, since we can write  $\tilde{\varphi}^{\varepsilon}(t, x, m) = \mathbb{E} [g(X_t^{\varepsilon} - x, X_t^{*\varepsilon} - m)]$  where  $g(x, y) := (-x)^+ 1_{\{y > 0\}}$ , and recalling the definition of  $(X_t^{\varepsilon}, X_t^{*\varepsilon})$ , we write

$$\begin{aligned} \tilde{\varphi}^{\varepsilon}(t, x, m) &= \mathbb{E} [g(X_t^{\varepsilon} - x, X_t^{*\varepsilon} - m)] \\ &= \mathbb{E} [g(V_{\varepsilon} + X_t + W_{\varepsilon} - x, (V_{\varepsilon}^* - m) \vee (V_{\varepsilon} + X_t^* - m) \vee (V_{\varepsilon} + X_t + W_{\varepsilon}^* - m))]. \end{aligned} \quad (4.5.13)$$

Recall that  $(V_{\varepsilon}, V_{\varepsilon}^*)$  and  $(W_{\varepsilon}, W_{\varepsilon}^*)$  have a density w.r.t Lebesgue measure which we will denote  $\rho_V$  and  $\rho_W$  to



differentiate them (even though its the same density), we write

$$\begin{aligned} & \mathbb{E} [g (V_\varepsilon + X_t + W_\varepsilon - x, (V_\varepsilon^* - m) \vee (V_\varepsilon + X_t^* - m) \vee (V_\varepsilon + X_t + W_\varepsilon^* - m))] \\ &= \int_{\mathcal{S}} \int_{\mathcal{S}} \mathbb{E} [g (v + X_t + w - x, (v^* - m) \vee (v + X_t^* - m) \vee (v + X_t + w^* - m))] \\ & \times \rho_V (\varepsilon, v, v^*) \rho_W (\varepsilon, w, w^*) dv dv^* dw dw^*, \end{aligned}$$

where we exhibited the expectations w.r.t Brownian motions. We now make the following change of variables :

$$\begin{cases} \tilde{v} := v - m, \\ \tilde{v}^* := v^* - m, \\ \tilde{w} := w + m - x, \end{cases} \quad (4.5.14)$$

we then write again

$$\begin{aligned} \tilde{\varphi}^\varepsilon (t, x, m) &= \int_{\mathcal{S}} \int_{\mathcal{S}} \mathbb{E} [g (\tilde{v} + X_t + \tilde{w}, \tilde{v}^* \vee (\tilde{v} + X_t^*) \vee (\tilde{v} + X_t + w^*))] \\ & \times \rho_V (\varepsilon, \tilde{v} + m, \tilde{v}^* + m) \rho_W (\varepsilon, \tilde{w} - (m - x), w^*) dv dv^* dw dw^*. \end{aligned} \quad (4.5.15)$$

We can see that we shifted the  $x$  and  $m$  variables to the  $C^\infty$  densities  $\rho_V$  and  $\rho_W$  which makes  $\tilde{\varphi}^\varepsilon$  smooth by application of Lebesgue Dominated convergence Theorem. Therefore  $\tilde{\varphi}^\varepsilon$  is  $C^\infty$  with respect to  $x$  and  $m$  (but not yet with respect to time  $T$ ).

The following Lemma 4.5.6 insures that the dynamic condition is satisfied for the constructed process.

**Lemma 4.5.6.** : *We have  $-\partial_{tm}\tilde{\varphi}^\varepsilon \geq 0$  (in a weak sense since we still did not regularize yet with respect to time).*

**Proof.** We define first

$$\bar{\varphi}^\varepsilon (t, x, m) := \mathbb{E} \left[ (x - Y_{t+\varepsilon})^+ 1_{\{Y_{t+\varepsilon}^* > m\}} \right] \text{ for } t \in [0, T]. \quad (4.5.16)$$

Since the pair  $(\bar{Y}_t, \bar{Y}_t^*)_{t \in [0, T]} := (Y_{t+\varepsilon}, Y_{t+\varepsilon}^*)_{t \in [0, T]}$  is a martingale and its running maximum (by construction) we get by a similar argument as in Proposition (4.4.11) that

$$\begin{aligned} \mathbb{E} \left[ (x - \bar{Y}_t)^+ 1_{\{\bar{Y}_t^* \geq m\}} \right] &= \mathbb{E} \left[ (x - \bar{Y}_0)^+ 1_{\{\bar{Y}_0^* \geq m\}} \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^t 1_{\{\bar{Y}_u^* \geq m\}} dL_u^x (\bar{Y}) \right], \\ &= \mathbb{E} \left[ (x - \bar{Y}_0)^+ 1_{\{\bar{Y}_0^* \geq m\}} \right] + \frac{1}{2} \mathbb{E} \left[ L_t^x (\bar{Y}) \right] - \frac{1}{2} \mathbb{E} \left[ \int_0^t 1_{\{\bar{Y}_u^* \leq m\}} dL_u^x (\bar{Y}) \right] \end{aligned}$$

then for a fixed  $x$ , we use the same reasoning as in (Billingsley 2007, Theorem 12.5, p.177) and define

$$\begin{aligned} -2\bar{\nu}_x ([m, m + \varepsilon] \times [t, t + \eta]) &:= \bar{\varphi}^\varepsilon (t, x, m) + \bar{\varphi}^\varepsilon (t + \eta, x, m + \varepsilon) - \bar{\varphi}^\varepsilon (t, x, m + \varepsilon) - \bar{\varphi}^\varepsilon (t + \eta, x, m) \\ &= \mathbb{E} \left[ \int_t^{t+\eta} 1_{\{m \leq \bar{Y}_u^* \leq m + \varepsilon\}} dL_u^x (\bar{Y}) \right] \geq 0, \text{ for } (x, m, t, \varepsilon, \eta) \in \mathcal{S} \times [0, T] \times \mathbb{R}_+^2, \end{aligned}$$

and therefore  $\partial_{tm}\bar{\varphi}^\varepsilon$  defines a negative measure.

Now for a fixed  $t \in [0, T]$ , we define another pair  $(\tilde{Y}_u, \tilde{Y}_u^*)_{u \in [0, \varepsilon]} := (Y_{t+\varepsilon+u}, Y_{t+\varepsilon+u}^*)_{u \in [0, \varepsilon]}$ . The pair  $(\tilde{Y}_u, \tilde{Y}_u^*)_{u \in [0, \varepsilon]}$  is a Brownian Motion and its running maximum starting from the point  $(Y_{t+\varepsilon}, Y_{t+\varepsilon}^*)$ , (in fact it has the same increments as  $W$  defined previously). We can reuse then the argument in Proposition (4.4.11) and obtain that

$$\mathbb{E} \left[ (x - \tilde{Y}_\varepsilon)^+ 1_{\{\tilde{Y}_\varepsilon^* > m\}} \right] = \mathbb{E} \left[ (x - \tilde{Y}_0)^+ 1_{\{\tilde{Y}_0^* > m\}} \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^\varepsilon 1_{\{\tilde{Y}_u^* > m\}} dL_u^x(\tilde{Y}) \right] \quad (4.5.17)$$

which can be written as

$$\tilde{\varphi}^\varepsilon(t, x, m) = \bar{\varphi}^\varepsilon(t, x, m) + \frac{1}{2} \mathbb{E} \left[ \int_0^\varepsilon 1_{\{\tilde{Y}_u^* > m\}} dL_u^x(\tilde{Y}) \right]$$

where the second right term does not depend on  $t$ , therefore  $\tilde{\varphi}^\varepsilon$  has the same monotonicity with respect to  $t$  as  $\bar{\varphi}^\varepsilon$ , in particular  $t \mapsto \partial_m \tilde{\varphi}^\varepsilon(t, K, B)$  is non-increasing, which concludes the proof of the Lemma 4.5.6.  $\square$

Remark that after the concatenation, we obtain, for each fixed time  $t$ , we obtain three different time intervals; the first one and the last one of length  $\varepsilon$  both corresponding to the Brownian increments of the regularization, and the middle time interval corresponding to the representation of  $(X_t, X_t^*)$ . This can be expressed as in Lemma 4.5.6 as

$$\begin{aligned} \tilde{\varphi}(t + 2\varepsilon, x, m) &= \tilde{\varphi}(0, x, m) + \frac{1}{2} \mathbb{E} \left[ \int_0^\varepsilon 1_{\{\tilde{V}_u^* > m\}} dL_u^x(V) \right] \\ &+ \frac{1}{2} \mathbb{E} \left[ \int_0^t 1_{\{\tilde{X}_u^* > m\}} dL_u^x(X) \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^\varepsilon 1_{\{W_u^* > m\}} dL_u^x(W) \right], \end{aligned} \quad (4.5.18)$$

### Step 3 : Regularization of time component

The only condition that involves the time component is the dynamic condition, which is linear with respect to time, and therefore stable by ordinary convolution. Consider  $\alpha$  a  $C^\infty$  density of a probability measure supported in a compact interval, and constant extrapolation of  $\tilde{\varphi}^\varepsilon$  outside of the interval  $[0, T]$ , i.e.,

$$\tilde{\varphi}^\varepsilon(t, x, m) := \tilde{\varphi}^\varepsilon(0 \vee t \wedge T, x, m), \text{ for } (t, x, m) \in \mathbb{R} \times \mathcal{S}. \quad (4.5.19)$$

Define then  $U$  as an independent random variable (of  $X$  and  $V$  and  $W$ ) with law  $\alpha$ . The regularization with respect to time is defined by  $\varphi_1^\varepsilon : [0, T] \times \mathcal{S} \rightarrow \mathbb{R}$  such that

$$\begin{aligned} \varphi_1^\varepsilon(t, x, m) &:= \frac{1}{\varepsilon} \int_{\mathbb{R}} \tilde{\varphi}^\varepsilon(t, x, m) \alpha\left(\frac{u}{\varepsilon}\right) du, \\ &= \int_{\mathbb{R}} \tilde{\varphi}^\varepsilon(t + \varepsilon u, x, m) \alpha(u) du, \end{aligned} \text{ for } (t, x, m) \in [0, T] \times \mathcal{S}, \quad (4.5.20)$$

where

$$\varphi_1^\varepsilon(t, x, m) = \mathbb{E} \left[ g(V_\varepsilon + X_{t+\varepsilon U} + W_\varepsilon - x, (V_\varepsilon^*) \vee (V_\varepsilon + X_{t+\varepsilon U}^*) \vee (V_\varepsilon + X_{t+\varepsilon U} + W_\varepsilon^*) - m) \right], \quad (4.5.21)$$

and so, from (4.5.18), we can write

$$\begin{aligned} \varphi_1^\varepsilon(t, x, m) = & \tilde{\varphi}(0, x, m) + \frac{1}{2} \mathbb{E} \left[ \int_0^\varepsilon 1_{\{\tilde{V}_u^* > m\}} dL_u^x(V) \right] + \frac{1}{2} \mathbb{E} \left[ \int_0^\varepsilon 1_{\{W_u^* > m\}} dL_u^x(W) \right] \\ & + \frac{1}{2} \int_{\mathbb{R}} \mathbb{E} \left[ \int_0^{t+\varepsilon u} 1_{\{\tilde{X}_u^* > m\}} dL_u^x(\tilde{X}) \right] \alpha(u) du, \text{ for } (t, x, m) \in [0, T] \times \mathcal{S}, \end{aligned} \quad (4.5.22)$$

and we can see that the first three terms of the right hand side are constant with respect to time.

*Step 4 : Insuring strict inequalities*

So far we regularized properly and obtained a  $\mathcal{C}^\infty$  function  $\varphi_1^\varepsilon$  such that

$$\begin{cases} -\partial_{xxm} \varphi_1^\varepsilon(t, x, m) \geq 0, \\ -\partial_{tm} \varphi_1^\varepsilon(t, x, m) \geq 0, \end{cases} \text{ for } (t, x, m) \in \bar{U}. \quad (4.5.23)$$

However we need these inequalities to be strict, so we define  $\varphi_2 : [0, T] \times \mathcal{S} \rightarrow \mathbb{R}$  as the barrier function induced by the Brownian motion and its running maximum (translated in time to avoid problems at  $t = 0$ ), which by definition satisfies  $\varphi_2 = P_2$

$$\varphi_2(t, K, B) := \int_{\mathbb{R}^2} (K - s)^+ 1_{\{m > B\}} \rho(t + 1, s, m) ds dm, \text{ for } (t, K, B) \in [0, T] \times \mathcal{S}. \quad (4.5.24)$$

By construction,  $\varphi_2$  satisfies Assumption 4.5.1. Now define  $\varphi^\varepsilon : [0, T] \times \mathcal{S} \rightarrow \mathbb{R}$  as

$$\varphi^\varepsilon(t, x, m) := (1 - \varepsilon) \varphi_1^\varepsilon(t, x, m) + \varepsilon \varphi_2(t, x, m), \text{ for } (t, x, m) \in [0, T] \times \mathcal{S}. \quad (4.5.25)$$

The inequalities (iv) and (v) of Assumption 4.5.1 are satisfied in the large sense for  $\varphi_1^\varepsilon$  and the strict sense for  $\varphi_2$ , so they are satisfied in the strict sense for  $\varphi^\varepsilon$ , and (i), (ii) and (iii) are stable by convex combination and thus satisfied by  $\varphi^\varepsilon$ . Therefore  $\varphi^\varepsilon$  satisfies all the conditions of Assumption 4.5.1.

*Step 5 : Convergence of the marginal laws*

**Lemma 4.5.7.** : For  $t \in [0, T]$ , we have  $\nu_t^\varepsilon \Rightarrow \nu_t$ .

**Proof.** Let  $f : \mathcal{S} \rightarrow \mathbb{R}$  be a continuous bounded function. We want to prove that for  $t \in [0, T]$ ,

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathcal{S}} f(x, m) \nu_t^\varepsilon(dx, dm) = \int_{\mathcal{S}} f(x, m) \nu_t(dx, dm). \quad (4.5.26)$$

We have by definition

$$\begin{aligned} \int_{\mathcal{S}} f(x, m) \nu_t^\varepsilon(dx, dm) = & (1 - \varepsilon) \int_{\mathcal{S}} \int_{\mathcal{S}} \int_{\mathcal{S}} \int_{\mathbb{R}} f(v + x + w, (v^*) \vee (v + m) \vee (v + x + w^*)) \nu_{t+\varepsilon u}(dx, dm) \\ & \times \rho_V(\varepsilon, v, v^*) \rho_W(\varepsilon, w, w^*) \alpha(u) dv dv^* dw dw^* du \\ & + \varepsilon \int_{\mathcal{S}} f(x, m) \rho(t + 1, x, m). \end{aligned} \quad (4.5.27)$$

From the boundedness of  $f$  we have that

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{\mathcal{S}} f(x, m) \rho(t+1, x, m) = 0. \quad (4.5.28)$$

As for the first term of the right hand side in (4.5.27), we have two integrations over  $\mathcal{S}$  to regularize with respect to space variables, and an integration over  $\mathbb{R}$  (the support of the kernel  $\alpha$ ) to regularize over time. So by classical analysis arguments, we have that

$$\begin{aligned} \int_{\mathcal{S}} f(x, m) \nu_t(dx, dm) &= \lim_{\varepsilon \rightarrow 0} (1 - \varepsilon) \int_{\mathcal{S}} \int_{\mathcal{S}} \int_{\mathcal{S}} \int_{\mathbb{R}} f(v + x + w, (v^* \vee (v + m) \vee (v + x + w^*))) \nu_{t+\varepsilon u}(dx, dm) \\ &\quad \times \rho_V(\varepsilon, v, v^*) \rho_W(\varepsilon, w, w^*) \alpha(u) dv dv^* dw dw^* du, \end{aligned} \quad (4.5.29)$$

which is the desired result.  $\square$

This completes the proof of Proposition 4.5.5.  $\square$

### 4.5.2.2 Main result

For  $\varepsilon \in (0, 1)$ , the function  $\varphi^\varepsilon$  defined above induces a family  $\mathcal{V}^\varepsilon$  of probability measures satisfying Assumption 4.5.1, which is the same as the one needed in the smooth case. We can then define  $\sigma^\varepsilon$  for  $\varepsilon \in (0, 1)$  as in the smooth case

$$a^\varepsilon := \frac{1}{2} \sigma^{\varepsilon 2} = \frac{\partial_{tm} \varphi^\varepsilon}{\partial_{xmx} \varphi^\varepsilon}. \quad (4.5.30)$$

Once  $\sigma^\varepsilon$  is defined, we can see that it is continuous from the smoothness of  $\varphi^\varepsilon$ , and satisfies  $\sigma^\varepsilon > 0$  on  $\bar{U}$ . Therefore, from Proposition 4.3.3, the SDE (4.5.2) (with  $\sigma^\varepsilon$ ) admits at least a weak non exploding solution which we denote  $\mathbb{P}^\varepsilon$ .

We still need the marginal laws of  $\mathbb{P}^\varepsilon$  to be uniquely characterized by the Fokker-Planck PDE, which requires the following technical assumption on  $\sigma^\varepsilon$ .

**Assumption 4.5.8.** *Assume that for  $\varepsilon \in (0, 1)$ , the function  $\log(\sigma^\varepsilon)$  is lipschitz in  $t$  uniformly in  $(x, m)$ .*

We are ready now to state our main theorem, which is a verification theorem under the previously announced Assumptions.

**Theorem 4.5.9** (Verification). *Under Assumptions 4.5.4 and 4.5.8, we have that  $\mathcal{M}^0(\mathcal{V}) \neq \emptyset$ .*

**Proof.** From Assumption 4.5.4, the function  $\varphi$  associated to  $\mathcal{V}$  satisfies the static and dynamic conditions. Therefore, using proposition 4.5.5 we get the existence of a family of laws  $\mathcal{V}_{\varepsilon \in (0, 1)}^\varepsilon$ , each of them satisfying assumption 4.5.1, and such that  $\mathcal{V}^\varepsilon \Rightarrow \mathcal{V}$  (in the sense of weak convergence for each marginal law).

Then, by the assumption 4.5.8 and the proposition 4.5.3 we obtain that  $\mathcal{M}^0(\mathcal{V}^\varepsilon) \neq \emptyset$  for  $\varepsilon \in (0, 1)$  and in particular  $\mathbb{P}^\varepsilon \in \mathcal{M}^0(\mathcal{V}^\varepsilon)$  where  $\mathbb{P}^\varepsilon$  is a weak solution of the SDE

$$X_t = x_0 + \int_0^t \sigma^\varepsilon(u, X_u, M_u) dW_u, \text{ for } t \in [0, T]. \quad (4.5.2)$$

We now prove that the family of measures  $(\mathbb{P}^\varepsilon)_{\varepsilon \in (0, 1)}$  defined on the space  $(\mathcal{C}([0, T], \mathbb{R}), \mathcal{B}(\mathcal{C}([0, T], \mathbb{R})))$  is relatively compact by Prokhorov's criterion (see Revuz & Yor (2013)[Chapter 0, Theorem 5.4]) since it is tight. In fact, for

$\varepsilon \in (0, 1)$ , we have the Doob martingale inequality

$$\mathbb{P}^\varepsilon \left[ \sup_{0 \leq t \leq T} |X_t| \geq n \right] \leq \frac{1}{n} \mathbb{E}^{\mathbb{P}^\varepsilon} [|X_T|], \text{ for } n \in \mathbb{N}. \quad (4.5.31)$$

Recall then that for a function  $f : \mathbb{R}^2 \mapsto \mathbb{R}$ , by definition of the regularization we have

$$\begin{aligned} \int_{\mathcal{S}} f(x, m) \nu_T^\varepsilon(dx, dm) &= (1 - \varepsilon) \int_{\mathcal{S}} \int_{\mathcal{S}} \int_{\mathcal{S}} \int_{\mathbb{R}} f(v + x + w, (v^*) \vee (v + m) \vee (v + x + w^*)) \nu_{T+\varepsilon u}(dx, dm) \\ &\quad \times \rho(\varepsilon, v, v^*) \rho(\varepsilon, w, w^*) \alpha(u) dv dv^* dw dw^* du \\ &\quad + \varepsilon \int_{\mathcal{S}} f(x, m) \rho(T + 1, x, m) dx dm. \end{aligned} \quad (4.5.27)$$

In particular, for  $f(x, m) := |x|$ , we denote by  $\nu_t^x$  the marginal law of  $\nu_t$  with respect to  $x$  and we abuse notations by taking  $\rho(t, x) := \int_{0 \vee x}^{+\infty} \rho(t, x, m) dm$ . We obtain for  $\varepsilon \in (0, 1)$

$$\begin{aligned} \int_{\mathbb{R}} |x| \nu_T^\varepsilon(dx) &= (1 - \varepsilon) \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |v + x + w| \nu_{T+\varepsilon u}^x(dx) \rho(\varepsilon, v) \rho(\varepsilon, w) \alpha(u) dv dw du + \varepsilon \int_{\mathbb{R}} |x| \rho(T + 1, x) dx, \\ &\leq (1 - \varepsilon) \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |x| \nu_{T+\varepsilon u}^x(dx) \rho(\varepsilon, v) \rho(\varepsilon, w) \alpha(u) dv dw du, \\ &\quad + (1 - \varepsilon) \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |v| \nu_{T+\varepsilon u}^x(dx) \rho(\varepsilon, v) \rho(\varepsilon, w) \alpha(u) dv dw du, \\ &\quad + (1 - \varepsilon) \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} |w| \nu_{T+\varepsilon u}^x(dx) \rho(\varepsilon, v) \rho(\varepsilon, w) \alpha(u) dv dw du, \\ &\quad + \varepsilon \int_{\mathbb{R}} |x| \rho(T + 1, x) dx, \\ &\leq (1 - \varepsilon) \int_{\mathbb{R}} \int_{\mathbb{R}} |x| \nu_{T+\varepsilon u}^x(dx) \alpha(u) du + 2(1 - \varepsilon) \int_{\mathbb{R}} |v| \rho(\varepsilon, v) dv + \varepsilon \int_{\mathbb{R}} |x| \rho(T + 1, x) dx. \end{aligned} \quad (4.5.32)$$

Since  $\nu_t^x = \nu_T^x$  for  $t \geq T$ , and  $(\nu_t^x)_{t \in [0, T]}$  is increasing in the convex order, and the function  $x \mapsto |x|$  is convex, we obtain

$$\begin{aligned} \int_{\mathbb{R}} |x| \nu_T^\varepsilon(dx) &\leq (1 - \varepsilon) \int_{\mathbb{R}} \int_{\mathbb{R}} |x| \nu_{T+\varepsilon u}^x(dx) \alpha(u) du + 2(1 - \varepsilon) \int_{\mathbb{R}} |v| \rho(\varepsilon, v) dv + \varepsilon \int_{\mathbb{R}} |x| \rho(T + 1, x) dx, \\ &\leq (1 - \varepsilon) \int_{\mathbb{R}} \int_{\mathbb{R}} |x| \nu_T^x(dx) \alpha(u) du + 2 \int_{\mathbb{R}} |v| \rho(1, v) dv + \varepsilon \int_{\mathbb{R}} |x| \rho(T + 1, x) dx, \\ &\leq \int_{\mathbb{R}} |x| \nu_T^x(dx) + 2 \int_{\mathbb{R}} |v| \rho(1, v) dv + \varepsilon \int_{\mathbb{R}} |x| \rho(T + 1, x) dx < +\infty, \end{aligned} \quad (4.5.33)$$

where we used that  $\alpha$  is the density of a probability measure, and the finite first moment of  $\nu_T^x$  which follows from the assumption of the static conditions (the integrability of the marginal laws). Therefore we have that

$$\lim_{n \rightarrow +\infty} \sup_{\varepsilon \in (0, 1)} \mathbb{P}^\varepsilon \left[ \sup_{0 \leq t \leq T} |X_t| \geq n \right] = 0, \quad (4.5.34)$$

concluding the tightness and the relative compactness of  $(\mathbb{P}^\varepsilon)_{\varepsilon \in (0, 1)}$ . So the family of measures  $(\mathbb{P}^\varepsilon)_{\varepsilon \in (0, 1)}$  on

$(\mathcal{C}([0, T], \mathbb{R}), \mathcal{B}(\mathcal{C}([0, T], \mathbb{R})))$  contains a weakly convergent subsequence  $(\mathbb{P}^{\varepsilon_j})_{j \in \mathbb{N}}$  to some  $\bar{\mathbb{P}}$ .

Since the family of marginal laws  $\mathcal{V}^{\varepsilon_j}$  of  $\mathbb{P}^{\varepsilon_j}$  converges to  $\mathcal{V}$  by Lemma 4.5.7, and  $\mathbb{P}^{\varepsilon_j} \rightarrow \bar{\mathbb{P}}$ , we can identify the marginal laws of  $\bar{\mathbb{P}}$  as  $\mathcal{V}$ . So it remains to prove that  $\bar{\mathbb{P}}$  is a martingale measure to obtain that  $\bar{\mathbb{P}} \in \mathcal{M}^0(\mathcal{V})$ .

By virtue of the Skorokhod representation theorem, there exists a probability space  $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$  and a family of random variables valued in  $\mathcal{C}([0, T], \mathbb{R})$  –processes–  $(X^j)_{j \in \mathbb{N}}$  and  $\bar{X}$  such that  $\mathbb{P}^{\varepsilon_j} = \tilde{\mathbb{P}} \circ (X^j)^{-1}$  for  $j \in \mathbb{N}$  and  $\bar{\mathbb{P}} = \tilde{\mathbb{P}} \circ (\bar{X})^{-1}$  and  $X^j \xrightarrow{\tilde{\mathbb{P}}-q.s.} \bar{X}$ .

Consider then  $n \in \mathbb{N}$  and a continuous bounded function  $g : \mathbb{R}^n \rightarrow \mathbb{R}$ , and  $0 \leq s_1 \leq \dots \leq s_n \leq s \leq t$  of  $[0, T]$ . We have that

$$\mathbb{E}^{\tilde{\mathbb{P}}} \left[ g \left( X_{s_1}^{\varepsilon_j}, \dots, X_{s_n}^{\varepsilon_j} \right) \left( X_t^{\varepsilon_j} - X_s^{\varepsilon_j} \right) \right] = 0 \text{ for } j \in \mathbb{N}, \quad (4.5.35)$$

and the  $\tilde{\mathbb{P}}$  almost-sure convergence  $j \rightarrow +\infty$

$$g \left( X_{s_1}^{\varepsilon_j}, \dots, X_{s_n}^{\varepsilon_j} \right) \left( X_t^{\varepsilon_j} - X_s^{\varepsilon_j} \right) \rightarrow g \left( \bar{X}_{s_1}, \dots, \bar{X}_{s_n} \right) \left( \bar{X}_t - \bar{X}_s \right). \quad (4.5.36)$$

On the other hand, for  $j \in \mathbb{N}$  and  $t \in [0, T]$  we have that

$$\mathbb{E}^{\tilde{\mathbb{P}}} \left[ \left| X_t^{\varepsilon_j} \right| \mathbf{1}_{\{|X_t^{\varepsilon_j}| > n\}} \right] = \int_{\mathbb{R}} |x| \mathbf{1}_{\{|x| > n\}} \nu_t^{\varepsilon_j}(dx). \quad (4.5.37)$$

Remark that for  $n \geq 0$  we have that

$$|x| \mathbf{1}_{\{|x| > n\}} \leq (2|x| - n)^+, \quad (4.5.38)$$

and the function  $x \mapsto (2|x| - n)^+$  is convex. Therefore, following the same argument as in (4.5.32) and we have that

$$\begin{aligned} \mathbb{E}^{\tilde{\mathbb{P}}} \left[ \left| X_t^{\varepsilon_j} \right| \mathbf{1}_{\{|X_t^{\varepsilon_j}| > n\}} \right] &= \int_{\mathbb{R}} |x| \mathbf{1}_{\{|x| > n\}} \nu_t^{\varepsilon_j}(dx) \leq \int_{\mathbb{R}} (2|x| - n)^+ \nu_t^{\varepsilon_j}(dx) \\ &\leq (1 - \varepsilon_j) \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} (2|v + x + w| - n)^+ \nu_{t+\varepsilon_j u}^x(dx) \rho(\varepsilon_j, v) \rho(\varepsilon_j, w) \alpha(u) dv dw du \\ &\quad + \varepsilon_j \int_{\mathbb{R}} (2|x| - n)^+ \rho(t + 1, x) dx. \end{aligned} \quad (4.5.39)$$

Recall then that  $(\nu_t^x)$  is increasing in the convex order, and so is  $\rho(t, \cdot)$ . From the convexity of  $x \mapsto (2|x| - n)^+$  and

since  $0 < \varepsilon_j < 1$  we have

$$\begin{aligned}
\mathbb{E}^{\tilde{\mathbb{P}}} \left[ \left| X_t^{\varepsilon_j} \right| \mathbf{1}_{\{|X_t^{\varepsilon_j}| > n\}} \right] &\leq (1 - \varepsilon_j) \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} (2|v+x+w| - n)^+ \nu_{t+\varepsilon_j}^x(dx) \rho(\varepsilon_j, v) \rho(\varepsilon_j, w) \alpha(u) dv dw du \\
&\quad + \varepsilon_j \int_{\mathbb{R}} (2|x| - n)^+ \rho(t+1, x) dx, \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} (2|v+x+w| - n)^+ \nu_T^x(dx) \rho(1, v) \rho(1, w) \alpha(u) dv dw du \\
&\quad + \int_{\mathbb{R}} (2|x| - n)^+ \rho(t+1, x) dx, \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} (2|v+x+w| - n)^+ \nu_T^x(dx) \rho(1, v) \rho(1, w) dv dw + \int_{\mathbb{R}} (2|x| - n)^+ \rho(t+1, x) dx.
\end{aligned} \tag{4.5.40}$$

From the finite first moment of  $\nu_T^x$  and  $\rho(1, \cdot)$  and  $\rho(t+1, \cdot)$  for  $t \in [0, T]$ , and the upper bound uniform in  $\varepsilon_j$ , we can see that by dominated convergence

$$\lim_{n \rightarrow +\infty} \sup_{j \in \mathbb{N}} \mathbb{E}^{\tilde{\mathbb{P}}} \left[ \left| X_t^{\varepsilon_j} \right| \mathbf{1}_{\{|X_t^{\varepsilon_j}| > n\}} \right] = 0. \tag{4.5.41}$$

Therefore  $(X_t^{\varepsilon_j})_{j \in \mathbb{N}}$  is uniformly integrable for  $t \in [0, T]$ , and so is  $(X_t^{\varepsilon_j} - X_s^{\varepsilon_j})_{j \in \mathbb{N}}$ . By the boundedness of the function  $g$  we obtain that  $(g(X_{s_1}^{\varepsilon_j}, \dots, X_{s_n}^{\varepsilon_j})(X_t^{\varepsilon_j} - X_s^{\varepsilon_j}))_{j \in \mathbb{N}}$  is uniformly integrable. Therefore we have the  $\mathbb{L}^1$  convergence, i.e.,

$$\lim_{j \rightarrow +\infty} \mathbb{E}^{\tilde{\mathbb{P}}} \left[ g(X_{s_1}^{\varepsilon_j}, \dots, X_{s_n}^{\varepsilon_j})(X_t^{\varepsilon_j} - X_s^{\varepsilon_j}) \right] = \mathbb{E}^{\tilde{\mathbb{P}}} \left[ g(\bar{X}_{s_1}, \dots, \bar{X}_{s_n})(\bar{X}_t - \bar{X}_s) \right], \tag{4.5.42}$$

and so

$$\mathbb{E}^{\tilde{\mathbb{P}}} \left[ g(\bar{X}_{s_1}, \dots, \bar{X}_{s_n})(\bar{X}_t - \bar{X}_s) \right] = 0, \tag{4.5.43}$$

or equivalently,  $\tilde{\mathbb{P}}$  is a martingale measure. Therefore  $\tilde{\mathbb{P}} \in \mathcal{M}^0(\mathcal{V})$ , concluding the proof.  $\square$

## 4.6 Appendix

### 4.6.1 Integration by parts

We provide hereafter a quick reminder on the tools needed for our computations and to properly define the boundary value conditions in the Fokker-Planck PDE: the Green formula and the trace operator.

Recall that  $H^1(\mathcal{S})$  is the Sobolev space defined as

$$H^1(\mathcal{S}) := \left\{ u \in L^2(\mathcal{S}) \text{ such that } |\partial_x u| + |\partial_m u| \in L^2(\mathcal{S}), \text{ for } i = x, m. \right\},$$

where the derivative is defined in distribution sense.  $H^1(\mathcal{S})$  equipped with the scalar product  $\langle u, v \rangle := \int_{\mathcal{S}} (uv + \nabla u \cdot \nabla v) dx$ , for  $u, v \in H^1(\mathcal{S})$  and the corresponding norm  $\|u\|_{H^1(\mathcal{S})} := \langle u, u \rangle^{\frac{1}{2}}$  is a Hilbert space and

can be characterized by a density argument as  $\overline{C_c^\infty(\overline{\mathcal{S}})}^{H^1(\mathcal{S})}$ , i.e., for every  $u \in H^1(\mathcal{S})$ , there exists a sequence  $(u_n)_{n \in \mathbb{N}} \in C_c^\infty(\overline{\mathcal{S}})$  such that  $\lim_{n \rightarrow \infty} \|u_n - u\|_{H^1(\mathcal{S})} = 0$ .

For  $u, v \in H^1(\mathcal{S})$  we have the Green formula

$$\int_{\mathcal{S}} uv_i = - \int_{\mathcal{S}} u_i v + \int_{\Delta} u v n_i, \text{ for } i = x \text{ or } m, \quad (4.6.1)$$

where  $\mathcal{S} := \{(x, m) \in \mathbb{R}^2, x - m < 0\}$ , and  $\vec{n} := (n_x, n_m) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)^T$  the outward-pointing unit normal vector. Since  $u$  and  $v$  are in  $H^1(\mathcal{S})$ , the term  $\int_{\Delta} u v n_i ds$  in the formula (4.6.1) is well defined using the trace operator. In fact we recall that the mapping

$$\begin{aligned} H^1(\mathcal{S}) \cap C(\overline{\mathcal{S}}) &\rightarrow L^2(\Delta) \cap C(\overline{\Delta}) \\ v &\mapsto v|_{\Delta}, \end{aligned}$$

can be extended by continuity to a linear operator (the trace operator) from  $H^1(\mathcal{S})$  to  $L^2(\Delta)$  by a density argument. This operator is continuous in the sense that  $\|v\|_{L^2(\Delta)} \leq C \|v\|_{H^1(\mathcal{S})}$  for some constant  $C \in \mathbb{R}$ . This is the sense in which we interpret the "restrictions" of  $u$  and  $v$  to  $\Delta$ . In the sequel, whenever there is an ambiguity, we will denote by  $u^{\Delta}$  the trace of some  $u \in H^1(\mathcal{S})$ .

Whenever we assume further regularity  $H^2(\mathcal{S})$  defined as

$$H^2(\mathcal{S}) := \left\{ u \in L^2(\mathcal{S}) \text{ such that } \partial_{\alpha} u \in L^2(\mathcal{S}), \text{ for } |\alpha| \leq 2 \right\},$$

and define instead the scalar product

$$\langle u, v \rangle_{H^2(\mathcal{S})} := \int_{\mathcal{S}} \sum_{|\alpha| \leq 2} \partial_{\alpha} u \partial_{\alpha} v,$$

then  $H^2(\mathcal{S})$  is a Hilbert space endowed with this scalar product and the corresponding norm. We also have the density result  $H^2(\mathcal{S}) = \overline{C_c^\infty(\overline{\mathcal{S}})}^{H^2(\mathcal{S})}$  and we can give a sense to the trace operator for first order derivatives of elements of  $H^2(\mathcal{S})$ . In fact the mapping

$$\begin{aligned} H^2(\mathcal{S}) \cap C(\overline{\mathcal{S}}) &\rightarrow L^2(\Delta) \cap C(\overline{\Delta}) \\ v &\mapsto \nabla v \cdot \vec{n}|_{\Delta}, \end{aligned}$$

and by the same reasoning as earlier we can extend it to a linear continuous operator from  $H^2(\mathcal{S})$  to  $L^2(\Delta)$  satisfying  $\|\nabla v \cdot \vec{n}\|_{L^2(\Delta)} \leq C \|v\|_{H^2(\mathcal{S})}$  for some constant  $C \in \mathbb{R}$ .

With the previous definition, we can give sense to another form of the Green formula, for  $u \in H^2(\mathcal{S})$  and  $v \in H^1(\mathcal{S})$  we have

$$\int_{\mathcal{S}} \Delta uv = - \int_{\mathcal{S}} \nabla u \cdot \nabla v + \int_{\Delta} \nabla u \cdot n v. \quad (4.6.2)$$



We use now the geometry of the domain  $\mathcal{S}$  to characterize the trace operator.

**Lemma 4.6.1.** *Let  $q \in H^1(\mathcal{S})$ , and denote by  $q^\Delta$  its trace on the boundary  $\Delta$ . Then we have for  $\varphi \in H^1(\mathcal{S})$*

$$q^\Delta(\varphi) = \sqrt{2}(q(\varphi_x) + q_x(\varphi)) = -\sqrt{2}(q(\varphi_m) + q_m(\varphi)), \quad (4.6.3)$$

where  $q^\Delta(\varphi)$  is a notation for  $\int_\Delta \varphi q^\Delta$  (not to be confused with the different quantity  $\int_{\mathbb{R}} \varphi q^\Delta$ ). Assume further that  $q \in H^2(\mathcal{S})$  then for  $\varphi \in H^2(\mathcal{S})$

$$(q^\Delta)'(\varphi) := -q^\Delta(\varphi_x + \varphi_m) = (q_x^\Delta + q_m^\Delta)(\varphi), \quad (4.6.4)$$

in the sense that

$$\int_\Delta (\varphi_x + \varphi_m) q^\Delta = - \int_\Delta \varphi (q_x^\Delta + q_m^\Delta).$$

**Remark 4.6.2.** *Whenever  $q, \varphi \in C_c^1(\overline{\mathcal{S}})$ , the trace operator reduces to the restriction of these functions to  $\Delta$  (so we can omit the notation  $q^\Delta$ ), and the relationship (4.6.4) reduces to the familiar integration by parts formula:*

$$\int_{\mathbb{R}} \frac{d}{dm} \varphi q(m, m) dm = - \int_{\mathbb{R}} \varphi \frac{d}{dm} q(m, m) dm,$$

where  $\frac{d}{dm} \varphi(m, m) = (\varphi_x + \varphi_m)(m, m)$ , and  $\frac{d}{dm} q(m, m) = (q_x + q_m)(m, m)$  for  $m \in \mathbb{R}$ .

**Proof.** The proof is simple and relies on a direct application of Green formula (4.6.1) for some  $\varphi \in H^1(\mathcal{S})$  which is possible since  $q \in H^1(\mathcal{S})$  too. We obtain

$$\int_{\mathcal{S}} q \varphi_x = - \int_{\mathcal{S}} q_x \varphi + \frac{1}{\sqrt{2}} \int_\Delta q^\Delta \varphi,$$

and

$$\int_{\mathcal{S}} q \varphi_m = - \int_{\mathcal{S}} q_m \varphi - \frac{1}{\sqrt{2}} \int_\Delta q^\Delta \varphi,$$

so we can write

$$q^\Delta(\varphi) = \sqrt{2}(q(\varphi_x) + q_x(\varphi)) = -\sqrt{2}(q(\varphi_m) + q_m(\varphi)).$$

Now, whenever  $q$  and  $\varphi$  are in  $H^2(\mathcal{S})$ ,  $q_x^\Delta$ ,  $q_m^\Delta$  are well defined as well as  $\varphi_x^\Delta$  and  $\varphi_m^\Delta$ , and using (4.6.3), we have

$$\begin{aligned} (q^\Delta)'(\varphi) &:= -q^\Delta(\varphi_x + \varphi_m) \\ &= -\sqrt{2}(q(\varphi_{xm}) + q_x(\varphi_m)) + \sqrt{2}(q(\varphi_{mx}) + q_m(\varphi_x)) \\ &= \sqrt{2}(q_m(\varphi_x) - q_x(\varphi_m)). \end{aligned}$$

We apply then again (4.6.1) and get

$$\int_{\mathcal{S}} q_m \varphi_x = - \int_{\mathcal{S}} q_{xm} \varphi + \int_\Delta q_m \varphi n_x,$$

and

$$\int_{\mathcal{S}} q_x \varphi_m = - \int_{\mathcal{S}} q_{mx} \varphi + \int_{\Delta} q_x \varphi^m,$$

so

$$q_m(\varphi_x) = -q_{xm}(\varphi) + \frac{1}{\sqrt{2}} q_m^\Delta(\varphi), \text{ and } q_x(\varphi_m) = -q_{xm}(\varphi) - \frac{1}{\sqrt{2}} q_x^\Delta(\varphi),$$

yielding to

$$(q^\Delta)'(\varphi) = \sqrt{2}(q_m(\varphi_x) - q_x(\varphi_m)) = q_m^\Delta(\varphi) + q_x^\Delta(\varphi),$$

which is the desired result. □

Remark that in the previous computations we only required  $\varphi \in H^1(\mathcal{S})$  or  $\varphi \in H^2(\mathcal{S})$ . In practice the function  $\varphi$  will have more regularity as we will use it to apply Itô formula.

Finally, remark that for  $q \in H^1(\mathcal{S})$  and  $\varphi \in H^1(\mathcal{S})$  for example, one needs to be careful when computing the boundary integral  $\int_{\Delta} q^\Delta \varphi^\Delta$ , as

$$\int_{\Delta} q^\Delta \varphi^\Delta \neq \int_{\mathbb{R}} q^\Delta \varphi^\Delta, \quad (4.6.5)$$

since  $\Delta$  is a parameterized curve defined as  $\Delta = \{(x, m) \in \mathbb{R}^2 \text{ with } m = f(x), x \in \mathbb{R}\}$  with  $f(x) := x$ , so its length element is equal to  $\sqrt{1 + f'(x)^2} = \sqrt{2}$  and we have that

$$\int_{\Delta} q^\Delta \varphi^\Delta = \sqrt{2} \int_{\mathbb{R}} q^\Delta \varphi^\Delta.$$

## 4.6.2 Technical proofs

### 4.6.2.1 Proof of Lemma 4.4.5

We start by recalling a basic result which can be found in (Billingsley 2007, Theorem 12.5), characterizing a probability measure in  $\nu \in \mathcal{P}(\mathbb{R}^2)$  (or  $\mathbb{R}^n$  in general) by its cumulative distribution function. The theorem actually characterizes any measure (non necessarily finite), but we will restrict to the case where the total mass equals 1.

**Theorem 4.6.3** (see Billingsley (2007)). *Let  $F : \mathbb{R}^2 \rightarrow [0, 1]$ . Then  $F$  is a cumulative distribution function, i.e., there exists a unique measure  $\nu \in \mathcal{P}(\mathbb{R}^2)$  such that  $F(K, B) = \int_{\mathbb{R}^2} 1_{\{s \leq K\}} 1_{\{m \leq B\}} \nu(ds, dm)$ , if and only if :*

- (i)  $F$  is upper semi-continuous,
- (ii)  $\lim_{K \rightarrow -\infty} F(K, B) = 0, \forall B \in \mathbb{R}$ , and  $\lim_{B \rightarrow -\infty} F(K, B) = 0, \forall K \in \mathbb{R}$ ,
- (iii)  $\lim_{\substack{K \rightarrow +\infty \\ B \rightarrow +\infty}} F(K, B) = 1$ ,
- (iv)  $\forall (K, B) \in \mathbb{R}^2$ , and  $\forall (\varepsilon, \eta) \in \mathbb{R}_+^2$  we have

$$F(K + \varepsilon, B + \eta) + F(K, B) - F(K, B + \eta) - F(K + \varepsilon, B) \geq 0. \quad (4.6.6)$$

In that case we write  $F_\nu$  instead of  $F$ , with  $\nu$  the unique probability measure in  $\mathbb{R}^2$  satisfying

$$\nu([K, K + \varepsilon] \times [B, B + \eta]) := F(K + \varepsilon, B + \eta) + F(K, B) - F(K, B + \eta) - F(K + \varepsilon, B). \quad (4.6.7)$$

We can see that the Monotonicity conditions are defined to guarantee that  $-\partial_K C_\nu$  (the left derivative of  $C_\nu$  w.r.t  $K$ ) is the survival cumulative distribution function of  $\nu$ , and so can be characterized by Theorem 4.6.3 with

$$F_\nu(K, B) = 1 - \partial_K C_{\nu^*}(K, B) + \partial_K C_{\nu^*}(K, -\infty) + \partial_K C_{\nu^*}(-\infty, B) \quad (4.6.8)$$

where  $\partial_K C_{\nu^*}$  is the lower semi-continuous envelope of  $\partial_K C_\nu$ .

**Proof of Lemma 4.4.5** "  $\implies$  " Let  $(X, Y)$  a pair of random variables valued in  $\mathbb{R}^2$ , with  $\nu \in \mathcal{P}_1(\mathbb{R}^2)$  as a joint law. We can write the barrier call function in probabilistic terms :

$$C_\nu(K, B) := \mathbb{E} \left[ (X - K)^+ 1_{\{Y \geq B\}} \right]$$

From the convexity of the function  $K \mapsto (X - K)^+ 1_{\{Y \geq B\}}$  and the monotonicity of the expectation we have (i) . Computing  $\lim_{K \rightarrow +\infty} (X - K)^+ 1_{\{Y \geq B\}} = 0$ , for  $B \in \mathbb{R}$ , we can apply the Lebesgue dominated convergence theorem since for  $K$  large enough  $\mathbb{E} \left[ (X - K)^+ 1_{\{Y \geq B\}} \right] \leq \mathbb{E} [|X|] < +\infty$  and therefore  $\lim_{K \rightarrow +\infty} C_\nu(K, B) = 0$  proving (ii) .

The condition (iii) follows from the finite first moment of  $\nu$ . In fact, using again the Lebesgue dominated convergence theorem we compute  $C_\nu(0, -\infty) = \mathbb{E} [X^+] \leq \mathbb{E} [|X|] < +\infty$  and

$$\begin{aligned} \lim_{K \rightarrow -\infty} (C(K, -\infty) + K) &= \lim_{K \rightarrow -\infty} (C(K, -\infty) + K), \\ &= \lim_{K \rightarrow -\infty} \mathbb{E} [X \vee K], \\ &= \mathbb{E} [X], \end{aligned}$$

and so  $a := \mathbb{E} [X] < +\infty$ . Finally, from the convexity of  $C_\nu$  w.r.t  $K$ , the left derivative function  $\partial_K C_\nu$  is defined everywhere and can be written as  $-\partial_K C_\nu(K, B) = \mathbb{E} \left[ 1_{\{X \geq K\}} 1_{\{Y \geq B\}} \right]$ . (iv) to (vii) are then straightforward as  $-\partial_K C_\nu$  is the survival cdf of  $\nu$ .

"  $\Leftarrow$  " From (i) we can define  $-\partial_K C$ , and then (iv) to (vii) characterize  $-\partial_K C$  as a survival cumulative distribution function and we can therefore define a unique probability measure  $\nu$  such that  $-\partial_K C(K, B) = \mathbb{E} \left[ 1_{\{X \geq K\}} 1_{\{Y \geq B\}} \right]$ . By integrating w.r.t  $K$  and applying Fubini theorem (the integrands are positive) we get

$$C(K, B) = \mathbb{E} \left[ (X - K)^+ 1_{\{Y \geq B\}} \right] + f(B),$$

where  $f$  is some integration constant (a function of  $B$ ), which we compute using (ii) obtaining  $f(B) = 0$ , for all  $B \in \mathbb{R}$ , proving that  $C$  is a barrier call function. Finally from (iii)  $C(0, -\infty) = \mathbb{E} [X^+] < \infty$  and  $\lim_{K \rightarrow -\infty} (C(K, -\infty) + K) = \mathbb{E} [X] < +\infty$ . Therefore  $\mathbb{E} [|X|] = 2\mathbb{E} [X^+] - \mathbb{E} [X] < +\infty$  and we conclude that  $\nu \in \mathcal{P}_1(\mathbb{R}^2)$ .  $\square$

### 4.6.2.2 Proof of Lemma 4.4.8

"  $\implies$  " By assumption there exists  $T \in \mathbb{R}_+^*$ , and a continuous uniformly integrable martingale on  $[0, T]$  vanishing at 0 such that  $C(K, B) = \mathbb{E} \left[ (X_T - K)^+ 1_{\{M_T \geq B\}} \right]$ ,  $\forall (K, B) \in \mathbb{R}^2$ , where  $M_T := \sup_{\{0 \leq u \leq T\}} X_u$ . Therefore (i) and (v) are naturally satisfied. As for (ii), it follows from the centeredness of  $(X_t)_{t \in [0, T]}$ , since  $\lim_{K \rightarrow -\infty} (C(K, -\infty) + K) = \mathbb{E}[X_T]$ .

Conditions (iii) and (iv) represent the requirements on the support. In fact, recall that  $-\partial_K C(K, B) = \mathbb{E} \left[ 1_{\{X_T \geq K\}} 1_{\{M_T \geq B\}} \right]$  and that by definition that  $M_T \geq X_T$ , and therefore, on the event  $\{X_T \geq B\}$ ,  $1_{\{M_T \geq B - \eta\}} = 1$ , for  $B \in \mathbb{R}_+$  and  $\eta \geq 0$ , and so  $\partial_K C(B, B) = \partial_K C(B, B - \eta) = -\mathbb{E} \left[ 1_{\{X_T \geq B\}} \right]$ . Condition (iii) is obtained by a similar reasoning combined to the centeredness of  $X_T$ . Since  $M_0 = X_0 = 0$  and  $(M_t)_{t \in [0, T]}$  is non decreasing then  $1_{\{M_T \geq -\eta\}} = 1_{\{M_T \geq 0\}} = 1$  for  $\eta \geq 0$  and so  $\partial_K C(K, -\eta) = \partial_K C(K, 0) = -\mathbb{E} \left[ 1_{\{X_T \geq K\}} \right]$ .

"  $\longleftarrow$  " From (i) we can define the unique probability measure  $\nu = -\partial_{KKB} C$  in  $\mathcal{P}_1(\mathbb{R}^2)$  such that

$$C(K, B) = \mathbb{E} \left[ (X - K)^+ 1_{\{Y \geq B\}} \right] \text{ for } (K, B) \in \mathbb{R}^2 \text{ and where } (X, Y) \stackrel{\text{law}}{=} \nu,$$

with  $\mathbb{E}[X] = 0$  from (ii). We compute  $-\partial_K C(K, B) = \mathbb{E} \left[ 1_{\{X \geq K\}} 1_{\{Y \geq B\}} \right]$ , and we define

$$\begin{aligned} F(K, B) &:= 1 - \partial_K C(K, B) + \partial_K C(K, -\infty) + \partial_K C(-\infty, B), \\ &= \mathbb{E} \left[ 1_{\{X < K\}} 1_{\{Y < B\}} \right]. \end{aligned}$$

From (iii), if  $B \leq 0$  then

$$\begin{aligned} F(K, B) &= 1 - \partial_K C(K, B) + \partial_K C(K, -\infty) + \partial_K C(-\infty, B) \\ &= 1 - \partial_K C(K, 0) + \partial_K C(K, 0) + \partial_K C(-\infty, B) \\ &= 1 + \partial_K C(-\infty, B), \\ &= 1 + \partial_K C(-\infty, 0). \end{aligned}$$

$F$  is constant for  $B \leq 0$  so by taking the limit when  $B \rightarrow -\infty$  we have that  $F(K, B) = 0$  for  $B < 0$  and  $\text{supp}(\nu) \subset \mathbb{R} \times \mathbb{R}_+$ . Finally, we use (iii) to compute for  $B \in \mathbb{R}^+$

$$\begin{aligned} \mathbb{E} \left[ 1_{\{X \geq B\}} 1_{\{Y < B\}} \right] &= \partial_K C(B, B) - \partial_K C(B, -\infty), \\ &= 0, \end{aligned}$$

implying that  $\text{supp}(\nu) \subset \mathcal{S}$ . We can finally use Theorem 4.2.1 since we have the integrability, the support condition and (v), which concludes the proof.  $\square$



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**Titre :** Application de la théorie des contrats à la régulation des marchés d'énergie, et étude des lois jointes d'une martingale et son maximum courant.

**Mots clés :** Principal-Agent, risque moral, marché des capacités électrique, décarbonation, martingale et maximum courant.

**Résumé :** Cette thèse est composée de deux parties indépendantes. La première partie se focalise sur l'application du problème du Principal-Agent (c.f. Cvitanic & Zhang (2013) et Cvitanic; et al. (2018)) pour la résolution de problématiques de modélisations sur les marchés d'énergie. La deuxième porte sur les lois jointes d'une martingale et de son maximum courant.

Nous nous intéressons dans un premier lieu au marché des capacités électriques, et en particulier les mécanismes de rémunération de capacité. Étant donné la part croissante des énergies renouvelables dans la production d'électricité, les centrales de production "classiques" (à gaz ou à charbon par exemple) sont de moins en moins sollicitées, ce qui les rends peu rentables et non viable économiquement. Cependant, leur fermeture exposerait les consommateurs à un risque de Blackout en cas de pic de demande d'électricité, puisque celle-ci ne peut pas être stockée. Ainsi, la capacité de production doit être toujours maintenue à un niveau au-dessus de la demande, ce qui nécessite un "mécanisme de rémunération de capacités" pour rémunérer les centrales rarement sollicitées, ce qui peut être compris comme une assurance à payer contre les Black-out électriques.

Nous traitons ensuite la problématique des incitations à la décarbonation. L'objectif est de proposer un modèle d'instrument qui puisse être utilisé par

un agent public (l'état) en vue d'inciter les différents secteurs à baisser leurs émissions de carbone dans un contexte de risque moral (où l'état n'observe pas l'effort des acteurs et ne peut donc pas savoir si une baisse des émissions provient d'une baisse de production et de consommation ou d'un effort de gestion; investissement en recherche et développement); ce qui fournit une alternative à la taxe carbone qui nécessite une information parfaite.

La deuxième partie (indépendante) est motivée par la calibration de modèles et l'arbitrage sur un marché financier avec des options barrière. Elle présente un résultat sur les lois jointes d'une martingale et son maximum courant. Nous considérons une famille de probabilités en dimension 2, et nous donnons des conditions nécessaires et suffisantes assurant l'existence d'une martingale telle que ses lois marginales couplées avec ceux de son maximum courant coïncident avec les probabilités données. Nous suivons la méthodologie de Hirsch et Roynette (2012) basée sur une construction de martingale par EDS associée à une EDP bien posée de Fokker-Planck vérifiée par les lois marginales données sous des hypothèses de régularité, puis dans un cadre général avec une régularisation et un passage à la limite.

**Title :** Application of contract theory to energy regulation problems, and study of the joint laws of a martingale and its running maximum.

**Keywords :** Principal-Agent, moral hazard, capacity market, decarbonation, martingale and its running maximum.

**Abstract :** This dissertation treats two independent topics. The first one is the application of stochastic differential games with non zero sum; the Principal-Agent models (c.f. Cvitanic & Zhang (2013) and Cvitanic et al.(2018)) to solve some contemporary challenges of energy markets regulation. The second concerns the study of the dynamics of the joint law of a continuous martingale and its running maximum.

The first work is about Capacity Remuneration Mechanisms (CRM) in the electricity market. Given the growing share of renewable energies in the production of electricity, "conventional" power plants (gas or coal-fired) are less and less used, which makes them not viable economically. However, shutting down these power plants would expose consumers to a risk of shortage or blackout in the event of a peak demand for electricity. This is due to the fact that electricity can hardly be stored, and so the production capacity should always be maintained at a level above demand. This explains the necessity of a "Capacity Remuneration Mechanism" (CRM) to pay for rarely used power plants, which can be understood as an insurance against electrical shortages and blackouts.

We address then the issue of the incentives for decarbonation. The goal is to propose a model of an instrument that can be used by a public agent (the

state) in order to incentivize different sectors to reduce their carbon emissions in a context of moral hazard (where the state does not observe the action of the actors and therefore cannot know whether a reduction in emissions comes from a reduction in production and consumption, or from a management effort towards a less polluting production (for example investment in research and development). This provides an alternative to the carbon tax, and does not require perfect information as the latter.

The second part of this thesis deals with a completely independent subject, motivated by model calibration and arbitrage in a financial market with barrier options. We provides a result on the joint laws between a martingale and its running maximum. We consider a family of 2 dimensional probabilities and we provide necessary and sufficient conditions for the existence of a martingale such that it's marginal laws coupled with those of its running maximum match the given probabilities. We follow the methodology of Hirsch & Roynette (2012) where they construct a martingale using an SDE corresponding to a wellposed Fokker-Planck PDE satisfied by the marginal laws of this martingale under smoothness assumptions, then using a regularization in the general case.