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Fine properties of solutions for quasi-linear elliptic and parabolic equations with non-local and non-standard growth

Rakesh Arora

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Fine properties of solutions for quasi-linear elliptic and parabolic equations with non-local and non-standard growth

THESIS SUBMITTED TO

Université de Pau et des Pays de l'Adour, France

FOR THE AWARD OF THE DEGREE OF

DOCTOR OF PHILOSOPHY
IN MATHEMATICS

BY

Rakesh Arora

Thesis defence at Pau on 4 December, 2020
in front of the juries

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DECLARATION

This thesis is a presentation of my original research work. Wherever contributions of others are involved, every effort is made to indicate this clearly, with due reference to the literature and acknowledgement of collaborative research and discussions.

The work was completed under the guidance of Prof. Jacques Giacomoni and Prof. Guillaume Warnault at the University of Pau, France.

Rakesh Arora.

In my capacity as supervisor of the candidate's thesis, I certify that the above statements are true to the best of my knowledge.

Prof. Jacques Giacomoni

Prof. Guillaume Warnault

Date: 30/10/2020.

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Dedicated to all my teachers.

The thesis is divided into two volumes.

In volume I, we present the comprehensive state of the art and main results with a glimpse of the proof including framework of function spaces, new mathematical tools and techniques.

In volume II, we explained the proof of our main results with complete details and estimates.

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Volume I

**Introduction and main results with
a glimpse of proof**

Parabolic problems with non-standard growth

In the recent years, there has been an increasing interest in the study of equations with non-standard growth conditions. On one hand, the interest in such equations is motivated by their applications in the mathematical modelling of various real-world processes, such as the flows of electrorheological or thermorheological fluids [24, 25, 151, 230, 230], the problem of thermistor [264], processing of digital images [86], filtration process in complex media, stratigraphy problems [129] and heterogeneous biological interactions [62]. On the other hand, their theoretical study is very interesting and challenging from a purely mathematical point of view.

The theme of this chapter is to study the qualitative properties of a class of parabolic problems with non-standard growth conditions. The main purpose of this chapter is threefold.

Firstly, we study the strong solution of the evolution equations with $p(x, t)$ -Laplacian operator. For this, we establish new higher integrability interpolations and trace-interpolation inequalities. Using Galerkin method, we find the sufficient conditions on the initial data for existence and uniqueness of strong solution. The global higher integrability and second order regularity of the strong solution are the byproduct of interpolations inequalities and uniform estimates of Galerkin's approximations.

Secondly, we study the double phase parabolic problem with variable growth and nonlinear source term. Using the method of Galerkin and establishing new weighted trace-interpolation inequalities, we prove the existence of strong solution with better integrability and regularity properties promoted by the energy equality.

Thirdly, we study the doubly nonlinear diffusion parabolic equations (D.N.E. for short) involving $p(x)$ -homogeneous operator with nonlinear time derivative and sub-homogeneous non-monotone forcing terms. For this, we develop a new version of Picone identity for $p(x)$ -homogeneous operators and as an application of this identity, we extend the well-known Díaz-Saá inequality for the non-standard growth operators. This inequality enables us to es-

establish several new results on the uniqueness of solution and comparison principles for some anisotropic quasilinear elliptic equations. Using semi-discretization in time and approximations methods, we prove the existence, uniqueness and regularity results for the weak solution of D.N.E. . In addition, we prove the continuous and monotone dependency of the solution with the respect to the initial data and potentials or coefficients in the forcing term. We also study the stabilization property of the weak solution using semigroup theory.

This chapter includes the results of the following research articles:

- (i) R. Arora, J. Giacomoni, G. Warnault, *A Picone identity for variable exponent operators and applications*, Adv. Nonlinear Anal. 9 (2020), no. 1, 327-360.
- (ii) R. Arora, J. Giacomoni, G. Warnault, *Doubly nonlinear equation involving $p(x)$ - homogeneous operators: local existence, uniqueness and global behaviour*, J. Math. Anal. Appl. 487 (2020), no. 2, 124009, 27 pp.
- (iii) R. Arora, S. Shmarev, *Strong solutions of evolution equations with $p(x, t)$ -Laplacian: existence, global higher integrability of the gradients and second-order regularity*, J. Math. Anal. Appl., 493 (2021), 124506.
- (iv) R. Arora, S. Shmarev, *Double phase parabolic problems with variable growth*, submitted.

Turning to the layout of this chapter: In section 1.1, we discuss two physical process as a source of origin and motivation for non-standard growth operators, double phase functionals and doubly nonlinear equations. In section 1.2, we present our main problems. In section 1.3, we introduce the functions spaces and the comprehensive state of the art for evolution equations involving $p(x, t)$ -Laplacian operator, Double phase parabolic equations, Doubly nonlinear equations involving constant and variable exponent operators, and Picone identity. In section 1.4, we develop the main tools and techniques, and state main results of this chapter with a glimpse of the proof.

1.1 Physical motivation

In this section, we discuss the origin of the interest to study the evolution equations with non-standard growth conditions and also with double source of nonlinearity.

First we discuss two physical processes of image recovery and non-Newtonian fluids whose mathematical modelling leads to the equations involving non-standard growth conditions. Let u be the true image and v be the input image defined on the domain $\Omega \subset \mathbb{R}^N$, a result of a linear transformation A on the true image and corrupted by a random noise $v = Au + \eta$ where η is a random variable with zero mean and u, v represent shades of gray. The effect of noise can be eliminated by smoothing the input which corresponds to minimizing the following energy functional

$$\mathcal{I}(u) := \int_{\Omega} |\nabla u|^2 + \frac{\lambda}{2} |Au - v|^2.$$

with a given Lagrange multiplier $\lambda = \text{const}$. This smoothing eliminates the noise effect, but unfortunately it destroys small details of the true image. A better approach, total variation smoothing method corresponds to minimizing the new energy functional

$$\mathcal{J}(u) := \int_{\Omega} |\nabla u| + \frac{\lambda}{2} |Au - v|^2.$$

This method preserves edges when $|\nabla u|$ is high. However, the drawback of the method is that it may also create edges due to the presence of the random noise (called staircase effect). A combination of the two methods consists in minimizing the following energy

$$\mathcal{V}(u) := \int_{\Omega} |\nabla u|^{p(x)} + \frac{\lambda}{2} |Au - v|^2$$

where the exponent $p(x)$ close to 2 where there are likely no edges, and close to 1 where the edges are expected. The approximate location of the edges can be determined by smoothing the input and looking for the zones where $|\nabla u|$ is high. The minimizer of the functional \mathcal{V} is a solution of the $p(x)$ -Laplace equation. A detailed discussion with more complicated models in the image restoration problems can be found in [86, 187].

A second process is the modeling of electrorheological fluids where the perturbations of the variable exponents operators appear in a natural way. This kind of fluids is characterized by their ability to drastically change the mechanical properties under the influence of an external electromagnetic field. For example, many electrorheological fluids are suspensions consisting of solid particles and a carrier oil. These suspensions change their material properties radically if they are exposed to an electric field (see [229]). The mathematical model for the motion of an electrorheological fluid is given by

$$u_t + \operatorname{div} P(u) + (u \cdot \nabla)u + \nabla \pi = f,$$

where $u : \mathbb{R}^3 \times \mathbb{R} \rightarrow \mathbb{R}^3$ is the velocity of the fluid at a point in space-time, $\nabla = (\partial_1, \partial_2, \partial_3)$ is the gradient operator, $\pi : \mathbb{R}^3 \rightarrow \mathbb{R}$ is the pressure, $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ represents external forces, and the stress tensor $P : W_{loc}^{1,1} \rightarrow \mathbb{R}^{3 \times 3}$ is of the form

$$P(u)(x) = \mu(x)(\nu + |Du(x)|^2)^{\frac{p(x)-2}{2}} Du(x)$$

where Du is the symmetric part of the gradient of u . The above model for $p = 2$, reduces to the usual non-dimensionalized Navier-Stokes equation. The case $\nu = 0$ and $\mu = 1$ corresponds to the $p(x)$ -Laplacian operator. For more details, we refer to [229, 230].

The study of the double phase problems started in the late 80th by the works of V. Zhikov [266, 269] where the models of strongly anisotropic materials were considered in the context of homogenization. Later on, the double phase functionals

$$u \rightarrow \int_{\Omega} (|\nabla u|^p + a(x)|\nabla u|^q) dx$$

attracted attention of many researchers. The significant case occurs when the modulating coefficient $a(\cdot)$ vanishes on a set of non-zero measure and $p < q$. The double phase functional changes the ellipticity/growth depending upon the support of the modulating coefficient $a(\cdot)$. Indeed, the energy density of functional exhibits a growth/ellipticity in the gradient of order q in those points x where $a(x)$ is positive and of order p on the points x where $a(x) = 0$. Such double phase functionals provide an elementary model for describing the behaviour of strongly anisotropic materials whose hardening properties linked to the exponent of the gradient variables. The modulating coefficient $a(\cdot)$ serves to regulate the mixture between two different materials, with p and q hardening respectively. On the one hand, the study of these functionals is a challenging mathematical problem due to its most dramatic phase-transition and on the other hand, the double phase functionals appear in a variety of physical models. We refer here to [42, 268] for applications in the elasticity theory, [40] for transonic flows, [48] for quantum physics and [91] for reaction-diffusion systems.

The significant interest to study Doubly nonlinear equation (D.N.E. for short) comes from a wide spectrum of applications in real world phenomenons, for instance in fluid dynamics, soil science, combustion theory, reaction chemistry (see [37, 38, 44, 45, 47, 94, 154, 190, 225, 234]). In literature, there are a variety of evolution equations involving double nonlinearity depending upon the positioning of nonlinear exponents. One of the basic model for D.N.E. is given by

$$\partial_t u - \nabla \cdot (|\nabla(u^r)|^{p-2} \nabla(u^r)) = 0 \quad \text{in } \Omega \times (0, T). \quad (1.1.1)$$

For $p = 2$ and $r > 1$, (1.1.1) is well-known as the porous media equation. More generally, for $p > 1$ and $r > 0$, (1.1.1) is known as the Polytropic Filtration Equations (P.F.E.) (see [258]). The physical background of P.F.E. can be explained by considering the flow of compressible non-newtonian fluid in the homogeneous isotropic rigid medium which satisfies:

$$\begin{cases} \epsilon \partial_t u = -\nabla(u \vec{V}) & \text{Mass balance} \\ \mathcal{P} = \mathcal{P}_0 u^r & \text{State equation} \end{cases}$$

where u is the particle density of the fluid, \vec{V} is the momentum velocity, \mathcal{P} is the pressure, r is the polytropic constant and \mathcal{P}_0 is the reference pressure and ϵ is the porosity of the medium. Due to the influence of molecular and ion effects in non-newtonian fluids, the linear Darcy's law (\vec{V} is proportional to $\nabla \mathcal{P}$) is no longer valid. Instead, we have the nonlinear version of Darcy's law:

$$\mu \vec{V} = -\lambda |\nabla \mathcal{P}|^{p-2} \nabla \mathcal{P}$$

where μ is the viscosity of the fluid and λ is the permeability of the medium. By combining the two last equations, we obtain an analogous form of (1.1.1).

Another equivalent form of D.N.E. is given by

$$\partial_t u^m - \nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0 \quad \text{in } \Omega \times (0, T). \quad (1.1.2)$$

Depending upon the value of m and p , (1.1.2) is called as Slow Diffusion Equation (S.D.E.) if $p > 1 + m$ and Fast Diffusion Equation (F.D.E.) if $p < 1 + m$. A main difference between the two cases is the existence of solutions with compact support for the S.D.E whereas the occurrence of dead core type solutions can not occur for the F.D.E. due to the infinite speed of perturbations propagation (for more details see [60, 258]). In the framework of D.N.E., (1.1.2) is also referred in the literature (for instance see [60, 173]) as:

	$p \in (1, 2)$	$p > 2$
$m \in (0, 1)$	Singular-degenerate	Doubly degenerate
$m > 1$	Doubly singular	Degenerate-singular

1.2 Presentation of the problems

In this section, we present two different class of parabolic problems with variable nonlinearity depending upon time and space variable. For this, we start by introducing the suitable variable Lebesgue and Sobolev space for our study. We limit ourselves to collecting the most basic facts of the theory and refer to Chapter 4 and [112] for a detailed insight, see also [34, 111, 179, 223].

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a bounded domain with Lipschitz-continuous boundary $\partial\Omega$. Let $\mathcal{P}(\Omega)$ be the set of all measurable function $p : \Omega \rightarrow [1, \infty[$ in N -dimensional Lebesgue measure. The set

$$L^{p(\cdot)}(\Omega) = \{f : \Omega \rightarrow \mathbb{R} : f \text{ is measurable on } \Omega, A_{p(\cdot)}(f) = \int_{\Omega} |f(x)|^{p(x)} dx < \infty\}$$

equipped with the Luxemburg norm

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \lambda > 0 : A_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leq 1 \right\}$$

is a reflexive and separable Banach space. Throughout the chapter, we assume that

$$1 < p^- \stackrel{\text{def}}{=} \min_{\Omega} p(x) \leq p(x) \leq p^+ \stackrel{\text{def}}{=} \max_{\Omega} p(x) < \infty.$$

The variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the set of functions

$$W_0^{1,p(\cdot)}(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \mid u \in L^{p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega), |\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}} + \|\nabla u\|_{L^{p(\cdot)}}.$$

It is known that $C_c^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ and the Poincaré inequality holds if $p \in C_{\log}(\overline{\Omega})$, *i.e.*, the exponent p is continuous in $\overline{\Omega}$ with the logarithmic modulus of continuity:

$$|p(x_1) - p(x_2)| \leq \omega(|x_1 - x_2|),$$

where ω is a non-negative function satisfying the condition

$$\limsup_{\tau \rightarrow 0^+} \omega(\tau) \ln \left(\frac{1}{\tau} \right) = C < \infty.$$

For the study of parabolic problem with $p(x, t)$ -Laplacian and spaces of functions depending on $(x, t) \in Q_T$, we define the following spaces:

$$\begin{aligned} V_{p(\cdot, t)}(\Omega) &= \{u : \Omega \rightarrow \mathbb{R} \mid u \in L^2(\Omega) \cap W_0^{1,1}(\Omega), |\nabla u|^{p(\cdot, t)} \in L^1(\Omega)\}, \quad t \in (0, T), \\ W_{p(\cdot)}(Q_T) &= \{u : (0, T) \rightarrow V_{p(\cdot, t)}(\Omega) \mid u \in L^2(Q_T), |\nabla u|^{p(\cdot, \cdot)} \in L^1(Q_T)\}, \quad z = (x, t). \end{aligned}$$

The dual $W'(Q_T)$ of the space $W_{p(\cdot)}(Q_T)$ is defined as follows: $\Phi \in W'(Q_T)$ iff there exists $\Phi_0 \in L^2(Q_T)$, $\Phi_i \in L^{p'(\cdot)}(Q_T)$, $i = 1, \dots, N$, such that for all $u \in W_{p(\cdot)}(Q_T)$

$$\langle \Phi, u \rangle = \int_{Q_T} \left(u \Phi_0 + \sum_{i=1}^N u_{x_i} \Phi_i \right) dx dt.$$

1.2.1 Problem 1: Strong solution of evolution equations with $p(x, t)$ -Laplacian

First, we study the Dirichlet problem for a class of parabolic equations with variable nonlinearity

$$\begin{cases} \partial_t u - \operatorname{div} \left(|\nabla u|^{p(x, t)-2} \nabla u \right) = f(x, t) & \text{in } Q_T \stackrel{\text{def}}{=} \Omega \times (0, T), \\ u = 0 \text{ on } \Gamma \stackrel{\text{def}}{=} \partial\Omega \times (0, T), \quad u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (1.2.1)$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with the boundary $\partial\Omega \in C^2$. The exponent p is a given function whose properties will be described in main results.

Concerning the problem (1.2.1), we are interested in the existence of **strong solutions** u and its global higher regularity and integrability properties. By the strong solution we mean a solution whose time derivative is not a distribution but an element of a Lebesgue space, and the flux has better integrability properties than the properties prompted by the energy equality (the rigorous formulation is given in Definition 1.3.1).

The local higher regularity and integrability properties for the problem (1.2.1) are intrinsic since their validity does not depend upon the problem data and geometry of the domain. So, a natural question arises “Does the regularity or integrability of a weak or strong solution improve if the initial data $(u(\cdot, 0), p(\cdot, \cdot), f(\cdot, \cdot), \partial\Omega)$ are more regular?”

To answer the above question, we study the global higher integrability and second-order regularity properties of the strong solution when the initial data posses better regularity properties.

1.2.2 **Problem 2:** Double phase parabolic problem with variable growth

Secondly, we study the following double phase parabolic problem with the homogeneous Dirichlet boundary conditions:

$$\begin{cases} u_t - \operatorname{div} \left(|\nabla u|^{p(z)-2} \nabla u + a(z) |\nabla u|^{q(z)-2} \nabla u \right) = F(z, u) & \text{in } Q_T, \\ u = 0 \text{ on } \Gamma_T, \quad u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (1.2.2)$$

where $z = (x, t)$ denotes the point in the cylinder $Q_T = \Omega \times (0, T]$ and $\Gamma_T = \partial\Omega \times (0, T)$ is the lateral boundary of the cylinder and $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, $N \geq 2$ and $0 < T < \infty$. The nonlinear source has the form $F(z, v) = f_0(z) + b(z)|v|^{\sigma(z)-2}v$. Equations of the type (1.2.2) are often termed “the double phase equations” because the flux function $(|\nabla u|^{p(z)-2} + a(z)|\nabla u|^{q(z)-2})\nabla u$ includes two terms with different properties.

Concerning the problem (1.2.2), we are interested in the existence of **strong solution** u with “better integrability properties”. By the better integrability properties here we mean the gradient of strong solution u has Lebesgue integrability with bigger exponent $q(\cdot)$ even if the modulating coefficient $a(\cdot)$ vanishes on a set of nonzero measure.

In this regard, we find conditions on the functions f_0, a, b, σ and u_0 sufficient for the existence of a unique strong solution by studying global regularity and integrability properties of the regularized flux function.

1.2.3 **Problem 3:** Doubly nonlinear equation for $p(x)$ -homogeneous operators

Thirdly, we investigate the following doubly nonlinear equation driven by a general class of Leray-Lions type operators

$$\begin{cases} \partial_t(\beta(u)) - \operatorname{div} a(x, \nabla u) = \mathcal{F}(x, t, u), & u > 0 & \text{in } Q_T; \\ u = 0 & & \text{on } \Gamma; \\ u(\cdot, 0) = u_0 & & \text{in } \Omega, \end{cases} \quad (\text{DNE})$$

with the following nonlinear time derivative, and sub-homogeneous and non-monotone forcing terms

$$\beta(u) = \frac{q}{2q-1} u^{2q-1} \quad \text{and} \quad \mathcal{F}(x, t, u) = f(x, u) + h(x, t)u^{q-1}$$

where $q > 1$, $T > 0$, with $\Omega \subset \mathbb{R}^N$, $N \geq 1$ a smooth bounded domain, and h belongs to $L^\infty(Q_T)$.

Problem (DNE) involves a class of variational operators $a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ defined as, for any $(x, \xi) \in \Omega \times \mathbb{R}^N$:

$$a(x, \xi) = (a_j(x, \xi))_j \stackrel{\text{def}}{=} \left(\frac{1}{p(x)} \partial_{\xi_j} A(x, \xi) \right)_j = \frac{1}{p(x)} \nabla_{\xi} A(x, \xi)$$

where $p \in C^1(\overline{\Omega})$ and $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^+$ is continuous, differentiable with respect to ξ and satisfies:

(A₁) $\xi \rightarrow A(\cdot, \xi)$ is $p(x)$ -homogeneous *i.e.* $A(x, t\xi) = t^{p(x)}A(x, \xi)$, for any $t \in \mathbb{R}^+$, $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

(A₂) For $j \in \llbracket 1, N \rrbracket$, $a_j(x, 0) = 0$, $a_j \in C^1(\Omega \times \mathbb{R}^N \setminus \{0\}) \cap C(\Omega \times \mathbb{R}^N)$ and there exist two constants $\gamma, \Gamma > 0$ such that for all $x \in \Omega$, $\xi \in \mathbb{R}^N \setminus \{0\}$ and $\eta \in \mathbb{R}^N$:

$$\begin{aligned} \sum_{i,j=1}^N \frac{\partial a_j}{\partial \xi_i}(x, \xi) \eta_i \eta_j &\geq \gamma |\xi|^{p(x)-2} |\eta|^2; \\ \sum_{i,j=1}^N \left| \frac{\partial a_j}{\partial \xi_i}(x, \xi) \right| &\leq \Gamma |\xi|^{p(x)-2}. \end{aligned}$$

The assumption (A₂) gives the convexity of $\xi \mapsto A(x, \xi)$ and growth estimates, for any $(x, \xi) \in \Omega \times \mathbb{R}^N$:

$$\frac{\gamma}{p(x)-1} |\xi|^{p(x)} \leq A(x, \xi) \leq \frac{\Gamma}{p(x)-1} |\xi|^{p(x)}; \quad |a(x, \xi)| \leq C |\xi|^{p(x)-1};$$

and the homogeneity assumption (A₁) implies that $A(x, \xi) = a(x, \xi) \cdot \xi$ for any $(x, \xi) \in \Omega \times \mathbb{R}^N$.

Next, we impose the condition below to insure qualitative properties as regularity and the validity of Hopf Lemma.

(A₃) There exists $C > 0$ such that for any $(x, \xi) \in \Omega \times \mathbb{R}^N \setminus \{0\}$:

$$\sum_{i,j=1}^N \left| \frac{\partial a_i}{\partial x_j}(x, \xi) \right| \leq C |\xi|^{p(x)-1} (1 + |\ln(|\xi|)|).$$

More precisely, from the condition (A₃) we derive the Strong Maximum Principle (see [262]) and the $C^{1,\alpha}$ -regularity of weak solutions (see Remark 5.3 in [118] and Remark 3.1 in [146]).

Example: Prototype examples of operators a satisfying (A₁)-(A₃) are given below: for any $(x, \xi) \in \Omega \times \mathbb{R}^N$ and $p \in C^{1,\beta}(\overline{\Omega})$ by:

$$A(x, \xi) = \sum_{j=1}^J \left(g_j(x) \left(\sum_{i \in P_j} \xi_i^2 \right)^{\frac{p(x)}{2}} \right)$$

where $(P_j)_{j \in J}$ is a partition of $\llbracket 1, N \rrbracket$, $g_j \in C^1(\overline{\Omega}) \cap C^{0,\beta}(\overline{\Omega})$ and $g_j(x) \geq c > 0$ for any $j \in J$. In particular for $A(x, \xi) = |\xi|^{p(x)}$, (DNE) can be classified as S.D.E. if $2q < p^-$ and F.D.E. if $2q > p^+$.

(A₄) $A(x, \frac{\xi+\eta}{2}) \leq \zeta(x)(A(x, \xi) + A(x, \eta))^{1-s(x)} \left(A(x, \xi) + A(x, \eta) - 2A(x, \frac{\xi+\eta}{2}) \right)^{s(x)}$
 where for any $x \in \Omega$, $s(x) = \min\{1, p(x)/2\}$ and $\zeta(x) = \left(1 - 2^{1-p(x)}\right)^{-s(x)}$ if $p(x) < 2$
 or $\zeta(x) = \frac{1}{2}$ if $p(x) \geq 2$.

The condition (A₄) reformulates the local form of Morawetz-type inequality producing convergence properties.

Concerning the conditions on the functions f and h in forcing term, we assume:

(f_0) $f : \bar{\Omega} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function such that $f(x, 0) \equiv 0$ and f is positive on $\Omega \times \mathbb{R}^+ \setminus \{0\}$.

(f_1) For any $x \in \Omega$, $s \mapsto \frac{f(x, s)}{s^{q-1}}$ is non-increasing in $\mathbb{R}^+ \setminus \{0\}$.

(f_2) The mapping $x \mapsto \delta^{1-q}(x)f(x, \delta(x))$ belongs to $L^2(\Omega_\varepsilon)$ for some $\varepsilon > 0$ where $\Omega_\varepsilon \stackrel{\text{def}}{=} \{x \in \Omega \mid \delta(x) < \varepsilon\}$.

and

(\mathbf{H}_h) there exists $\underline{h} \in L^\infty(\Omega) \setminus \{0\}$, $\underline{h} \geq 0$ such that $h(t, x) \geq \underline{h}(x)$ for a.e in Q_T .

Example: Function f satisfying (f_0)-(f_2) is given by for any $(x, s) \in \Omega \times \mathbb{R}^+$, $f(x, s) = g(x)\delta^\gamma(x)s^\beta$ where $g \in L^\infty(\Omega)$ is a non-negative function, $\beta \in [0, q-1)$ and $\beta + \gamma > q - \frac{3}{2}$.

Concerning the problem (DNE), we are interested in the existence of a unique weak solution and its stabilization properties. The notion of the weak solution u for the problem (DNE) is understood in the following sense:

$$\partial_t(\beta(u)) \in L^2(Q_T), \quad u \in L^\infty(0, T; W_0^{1,p(\cdot)}(\Omega))$$

(a rigorous formulation is given by Definition 1.4.4).

To prove the existence of a weak solution, we alter our viewpoint towards our main problem (DNE). Precisely, we formulate an equivalent problem by replacing $\partial_t(\beta(u))$ to $\partial_t(u^q)u^{q-1}$ in our main problem (DNE) (see below (E)) such that

$$\partial_t(u^q) \in L^2(Q_T) \text{ and } u \in L^\infty(Q_T) \Rightarrow \partial_t(\beta(u)) \in L^2(Q_T) \quad (\text{weakly in } L^2(Q_T)).$$

To study the new equivalent problem, we develop a new version of Picone identity for $p(x)$ -homogeneous operators. Using this and semi-discretization in time method, we settled the question of existence of unique weak solution of the equivalent problem (E).

To answer the second question of stabilization of weak solution, we seek help from semigroup theory by shifting the nonlinearity in the time derivative term to diffusion term (see (1.4.32)).

1.3 State of the art

Problem 1: : Strong solution of evolution equations with $p(x, t)$ -Laplacian

Equation (1.2.1) falls into the class of equations with variable nonlinearity or non-standard growth, which have been intensively studied in the last decades. If the variable exponent $p \not\equiv 2$, equation (1.2.1) becomes degenerate or singular at the points where $|\nabla u| = 0$, which prevents one from expecting the existence of classical solutions. The solution of problem (1.2.1) is understood in the weak sense. Before starting the state of the art for the problem (1.2.1), we distinguish the notion of weak and strong solutions as follows.

Definition 1.3.1. *A function u is called **weak solution** of problem (1.2.1), if*

$$(i) \quad u \in W_{p(\cdot)}(Q_T), \quad u_t \in W'(Q_T),$$

(ii) for every $\psi \in W_{p(\cdot)}(Q_T)$ with $\psi_t \in W'(Q_T)$

$$\int_{Q_T} u_t \psi \, dz + \int_{Q_T} |\nabla u|^{p(x,t)-2} \nabla u \cdot \nabla \psi \, dz = \int_{Q_T} f \psi \, dz,$$

(iii) for every $\phi \in C_0^1(\Omega)$

$$\int_{\Omega} (u(x,t) - u_0(x)) \phi \, dx \rightarrow 0 \quad \text{as } t \rightarrow 0,$$

(iv) the weak solution u is called **strong solution** of problem (1.2.1) if

$$u_t \in L^2(Q_T), \quad |\nabla u| \in L^\infty(0, T; L^{p(\cdot)}(\Omega)).$$

The existence of a unique weak solution to problem (1.2.1) can be proven under the minimal requirements on the regularity of the data. We refer to [26, 34] for the results on existence and uniqueness of weak solutions for a single equation of the type (1.2.1), to [111] for systems of equations with the homogeneous Dirichlet boundary conditions, and to [116] for the case of the non-homogeneous boundary conditions. Precisely, we have the following result:

Proposition 1.3.1 ([26, 34, 111]). *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with the Lipschitz-continuous boundary. Assume that $p : Q_T \rightarrow \mathbb{R}$ satisfies the conditions*

$$\frac{2N}{N+2} < p^- \leq p(x,t) \leq p^+, \quad p \in C_{\log}(\overline{Q_T}).$$

Then for every $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$ problem (1.2.1) has a unique weak solution $u \in C^0([0, T]; L^2(\Omega)) \cap W_{p(\cdot)}(Q_T)$ with $u_t \in W'(Q_T)$. The solution satisfies the estimate

$$\operatorname{ess\,sup}_{t \in (0, T)} \|u\|_{2, \Omega} + \int_{Q_T} |\nabla u|^{p(x,t)} \, dx \, dt \leq C \quad (1.3.1)$$

with a constant C depending only on N , p^\pm , $\|f\|_{2, Q_T}$ and $\|u_0\|_{2, \Omega}$.

Concerning the study of regularity of weak solution a lot of attention has been paid by researchers. Let $\Omega' \Subset \Omega$, $\epsilon \in (0, T)$, $Q' = \Omega' \times (\epsilon, T)$, and let u be a weak solution of equation (1.2.1). It is known that u possesses the property of higher integrability of the gradient: for every Ω' and $\epsilon > 0$ there exists a constant $\delta > 0$ such that $|\nabla u|^{p(\cdot)+\delta} \in L^1(Q')$ and $\| |\nabla u|^{p+\delta} \|_{1, Q'} \leq C$ with a constant C depending on ϵ and the distance between $\partial\Omega$ and $\partial\Omega'$ - see [35, 56, 267] and [265] for global estimates in Reifenberg domains. The weak solutions are locally Hölder-continuous, provided that the variable exponent p is log-continuous [13, 55, 259]. Moreover, if the variable exponent p is Hölder-continuous, then ∇u is locally Hölder-continuous and $u \in C_{x,t}^{1,1/2}(Q')$ see [55, 260]. These local regularity properties are intrinsic for every weak solution of equation (1.2.1) and are completely defined by the nonlinear structure of the equation.

The issues of local higher regularity of solutions of systems of parabolic equations with non-standard growth have been addressed for the first time in paper [2]. Among other results, it was proven that the solutions of a system of equations with $p(x, t)$ -growth conditions with the exponent p Hölder-continuous in t and Lipschitz-continuous in x possess the property of local higher integrability and Hölder-continuity of the spatial gradient, as well as the property of local higher differentiability of the solutions.

The existence of strong solutions of problem (1.2.1) and their global regularity properties have already been addressed in a number of works but all known results refer to the singular equation (1.2.1) with $\frac{2N}{N+2} \leq p(x, t) \leq 2$, or to the equations with the exponent p non-increasing in t . It is known [27, 28, 233] that the weak solution becomes a strong solution with $u_t \in L^2(Q_T)$ and $|\nabla u|^{p(\cdot, \cdot)} \in L^\infty(0, T; L^1(\Omega))$, provided that $|\nabla u_0|^{p(\cdot, 0)} \in L^1(\Omega)$, $f \in L^2(Q_T)$, $p_t \in L^\infty(Q_T)$ and either $p_t \leq 0$ a.e. in Q_T , or $|p_t| \leq C$ a.e. in Q_T and $p \leq 2$. Further, if $|\nabla p| + |p_t| \leq C$ a.e. in Q_T , $u_0 \in W_0^{1,2}(\Omega)$ and $p \leq 2$ in Q_T , then $|D_{x_i x_j}^2 u|^{p(x,t)} \in L^1(Q_T)$ [27, 28], or $D_{x_i x_j}^2 u \in L^2(\Omega \times (\epsilon, T))$ for every $\epsilon \in (0, T)$ [27, 29]. The strong solution may be Hölder or even Lipschitz continuous in t in the cylinders $\Omega \times (\epsilon, T)$ with $\epsilon > 0$, [233, 247]. It is proven in [32] that if the initial function possesses a second-order regularity with respect to x and satisfies certain compatibility conditions, $|f|^{p(\cdot, \cdot)} \in L^1(Q_T)$ and $f_t \in L^2(Q_T)$, then the singular equation with the Lipschitz-continuous exponent $p \leq 2$ in a convex C^2 domain has a unique strong solution such that

$$|u_t|^{p(\cdot, \cdot)} \in L^1(Q_T), \quad |\nabla u_t|^{p(\cdot, \cdot)} \in L^1(Q_T), \quad |D_{x_i x_j}^2 u|^{p(\cdot, \cdot)} \in L^1(Q_T), \quad p' = \frac{p}{p-1}.$$

Stronger global regularity properties are known in the case of constant $p > 1$. It is shown in [95] that if $f \in L^2(Q_T)$, $u_0 \in W_0^{1,p}(\Omega)$, and $\partial\Omega$ is subject to minimal regularity assumptions, then

$$u_t \in L^2(Q_T), \quad |\nabla u|^{p-2} \nabla u \in (L^2(0, T; W^{1,2}(\Omega)))^N, \quad u \in L^\infty(0, T; W_0^{1,p}(\Omega))$$

and the corresponding norms are bounded through the norms of the data. The authors of [95] show that problem (1.2.1) with $p = \text{const}$ admits an approximable solution, i.e. a solution obtained as the limit of the sequence of smooth solutions of the same problem with smooth right-hand sides and initial data. The approximable solution inherits the regularity properties of the smooth approximations. We refer to [95] for a review of the previous results on the global regularity in the case of constant p .

Problem 2: Double phase parabolic problem with variable growth

Equations (1.2.2) with $p \neq q$ are also referred to as the equations with the (p, q) -growth because of the gap between the coercivity and growth conditions: if $p \leq q$ and $0 \leq a(x) \leq L$, then for every $\xi \in \mathbb{R}^N$

$$|\xi|^p \leq (|\xi|^{p-2} + a(x)|\xi|^{q-2})|\xi|^2 \leq C(1 + |\xi|^q), \quad C = \text{const} > 0.$$

These equations fall into the class of equations with nonstandard growth conditions which have been actively studied during the last decades in the cases of constant or variable exponents p and q . We refer to the recent works [20, 82, 97, 98, 115, 133, 166, 202, 213, 222, 263] and references therein for a review of results on the solvability of stationary problems and the regularity properties of solutions.

Results on the existence of solutions to the evolution double phase equations can be found in papers [57, 236, 237]. These works deal with the Dirichlet problem for systems of parabolic equations of the form

$$u_t - \operatorname{div} a(x, t, \nabla u) = 0, \quad (1.3.2)$$

where the flux $a(x, t, \nabla u)$ is assumed to satisfy the (p, q) -growth conditions and certain regularity assumptions. As a partial case, the class of equations (1.3.2) includes equation (1.2.2) with constant exponents $p \leq q$ and a nonnegative bounded coefficient $a(x, t)$. It is shown in [57, Th.1.6] that if

$$2 \leq p \leq q < p + \frac{4}{N+2},$$

then problem (1.2.2) with $F \equiv 0$ has a **very weak solution**

$$u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L_{loc}^q(0, T; W_{loc}^{1,q}(\Omega)) \quad \text{with} \quad u_t \in L^{\frac{p}{q-1}}(0, T; W^{-1, \frac{p}{q-1}}(\Omega)),$$

provided that $u_0 \in W_0^{1,r}(\Omega)$, $r = \frac{p(q-1)}{p-1}$. Moreover, $|\nabla u|$ is bounded on every strictly interior cylinder $Q'_T \Subset Q_T$ separated away from the parabolic boundary of Q_T . In [236] these results were extended to the case

$$\frac{2N}{N+2} < p < 2, \quad p \leq q < p + \frac{4}{N+2}.$$

Paper [237] deals with **weak solutions** of systems of equations of the type (1.3.2) with (p, q) growth conditions. When applied to problem (1.2.2) with constant $p, q, b \equiv 0$ and $a(\cdot, t) \in C^\alpha(\Omega)$ with some $\alpha \in (0, 1)$ for a.e. $t \in (0, T)$, the result of [237] guarantees the existence of a weak solution

$$u \in L^p(0, T; W_0^{1,p}(\Omega)) \cap L_{loc}^q(0, T; W_{loc}^{1,q}(\Omega)) \cap L^\infty(0, T; L^2(\Omega)),$$

provided that the exponents p and q obey the inequalities

$$\frac{2N}{N+2} < p < q < p + \frac{\alpha \min\{2, p\}}{N+2}.$$

The proofs of the existence theorems in [57, 236, 237] rely on the property of local higher integrability of the gradient, $|\nabla u|^{p+\delta} \in L^1(Q'_T)$ for every sub-cylinder $Q'_T \Subset Q_T$. The maximal possible value of $\delta > 0$ indicates the admissible gap between the exponents p and q and vary in dependence on the type of the solution.

Equation (1.2.2) with constant exponents p and q furnishes a prototype of the equations recently studied in papers [58, 125, 153, 201] in the context of weak or variational solutions.

The proofs of existence also use the local higher integrability of the gradient, but for the existence of variational solutions a weaker assumption on the gap $q - p$ is required.

Problem 3: Doubly nonlinear equation for $p(x)$ -homogeneous operators

Equation (DNE) belongs to the class of Doubly nonlinear equations (D.N.E.) with variable nonlinearity of type (1.1.2). Depending upon the exponents in both nonlinear diffusion and nonlinear time derivative term, (DNE) is classified into several categories. For $p = 2$, $q = 1$ and $\mathcal{F} = 0$ equation (DNE) is known as standard heat equation. For $q = 1$, $p \in (1, \infty)$ and $\mathcal{F} = 0$ equation (DNE) reduces to p -Laplace equation, while for $p = 2$, $m := 2q - 1 \in (0, \infty)$ and $\mathcal{F} = 0$, (DNE) is called Porous media equation. A vast amount of results is available in the literature concerning the above types of equations, so to inhibit the vastness of section we limit ourselves to the class of equations where double nonlinearity is involved.

In literature, various types of tools and techniques are present to deal with the D.N.E. of type (1.1.2). Concerning the existence of solution to D.N.E. of type (1.1.2), we refer to the work [8, 12, 59, 60, 250]. In [12], Akagi and Stefanelli studied the following D.N.E.

$$\partial_t b(u) - \nabla \cdot (a(\nabla u)) \ni f$$

where $b \subset \mathbb{R} \times \mathbb{R}$ and $a \subset \mathbb{R}^N \times \mathbb{R}^N$ are maximal monotone graphs satisfying the polynomial growth conditions for instance $b(u) = u^m$ and $a(\xi) = |\xi|^{p-2}\xi$. First, by using a nonlinear transformation $v = b(u)$, they transformed the original equation into an equivalent dual problem which reads as $-\nabla \cdot (a(\nabla b^{-1}(v))) \ni f - \partial_t v$ and then by using the method of elliptic regularization (Weighted Energy Dissipation approach), they constructed the sequence of minimizers of suitable energy functionals (for more details see equation (1.2) in [12]) whose limits converge to solution of the equivalent dual problem. For D.N.E. involving p -Laplacian operator, one only changes the viewpoint in the sense that the nonlinearity is shifted from

$$\partial_t u^m - \operatorname{div}(|\nabla u|^{p-2} \nabla u) = f \quad \text{to} \quad \partial_t v - \operatorname{div}(|\nabla v^{\frac{1}{m}}|^{p-2} \nabla v^{\frac{1}{m}}) = f. \quad (1.3.3)$$

In [59, 60], Bögelin et al. studied a more general D.N.E. of type (1.1.2)

$$\partial_t b(u) - \operatorname{div} f_\xi(x, u, \nabla u) = -f_u(x, u, \nabla u)$$

where f satisfies suitable convexity and coercivity conditions. By introducing a new notion of solution called variational solutions and nonlinear version of minimizing movement method (finite time discretization), they proved the existence of a variational solution u . Precisely, a function u is called a variational solution of (1.1.2), if the following inequality holds:

$$\frac{1}{p} \int_{Q_T} (|\nabla u|^p - |\nabla v|^p) \, dz + B(u(T), v(T)) \leq \int_{Q_T} \partial_t v (v^m - u^m) \, dz + B(u(0), v(0))$$

where

$$B(u, v) = \int_{\Omega} \left(\frac{1}{m+1} (v^{m+1} - u^{m+1}) - u^m (u - v) \right) \, dx$$

and v is a suitable positive test function. They also proved the existence of distributional and weak solution when $f(x, u, \xi)$ grow naturally as a polynomial of order p as $\xi \rightarrow \infty$. In [250], Tsutsumi has studied the D.N.E. of type (1.1.2) in the presence of an absorption term and using approximation method he proved the existence of a mild/weak/strong solution depending upon the regularity of the initial data and nonlinearity exponents m and p .

For the study related to the D.N.E. of type (1.1.1), we refer to the work [9–11, 174, 240, 241] for existence results, [128, 252] for Harnack type inequalities and [176, 219, 238, 253] for local and global behavior of solutions. The non-homogeneous variant of the model (1.1.1) together with multi valued source/sink terms can also be interpreted as the limiting case (when $m \rightarrow 1$) of the climate Energy Balance Models (see [50, 52, 107]).

Recently, the study of D.N.E. involving variable exponent growth are getting into substantial attention. To explore the questions of existence (local or global), regularity or behaviour of solutions for D.N.E. with variable exponent we refer to [8, 9, 23, 30, 31, 33, 34]. The authors in [34] have studied the following class of D.N.E. involving the $p(x, t)$ -laplacian and lower order terms

$$\partial_t u = \sum_{i=1}^N D_i \left(a_i |D_i(|u|^{m(x)-1} u)|^{p_i(x,t)-2} D_i(|u|^{m(x)-1} u) \right) + b|u|^{\sigma(x,t)-2} u$$

with given exponents m, p_i and σ . Using a nonlinear transformation $v = |u|^{m(x)-1} u$, they transformed the original equation into the D.N.E.

$$\partial_t (|v|^{\frac{1}{m(x)}} \text{sign}(v)) = \sum_{i=1}^N D_i \left(a_i |D_i v|^{p_i(x,t)-2} D_i v \right) + b|v|^{\frac{\sigma(x,t)-1}{m(x)}-1} v \quad (1.3.4)$$

similar to (1.1.2) and by using the Galerkin method, they proved the existence of a weak solution. The authors in [8, 9] have also studied the D.N.E. involving $p(x)$ -Laplacian operator and proved the existence of weak solution using Legendre-Fenchel transforms of convex functionals and an energy method. For uniqueness and comparison theorem for the solutions of D.N.E. with non-standard growth conditions we refer to [23, 34]. For localization, blow up and extinction in finite time for the solutions of D.N.E. of type (1.3.4), we refer to [30, 31].

We also recall the state of the art for Picone identity which is one of the main tool in studying Problem 3. Picone identity plays an important role for proving several qualitative properties of differential operators. In [215], M. Picone consider the following homogeneous second order linear differential system

$$\begin{cases} (a_1(x)u')' + a_2(x)u = 0 \\ (b_1(x)v')' + b_2(x)v = 0 \end{cases}$$

and proved for differentiable functions $u, v \neq 0$ the pointwise relation:

$$\left(\frac{u}{v} (a_1 u' v - b_1 u v') \right)' = (b_2 - a_2) u^2 + (a_1 - b_1) u'^2 + b_1 \left(u' - \frac{v' u}{v} \right)^2 \quad (1.3.5)$$

and in [216], extended (1.3.5) to the Laplace operator, *i.e.* for differentiable functions $u \geq 0, v > 0$ one has ¹

$$|\nabla u|^2 \geq \left\langle \nabla \left(\frac{u^2}{v} \right), \nabla v \right\rangle. \quad (1.3.6)$$

Later in [14], Allegretto and Huang extended (1.3.6) to the p -Laplacian operator with $1 < p < \infty$. Precisely, for differentiable functions $v > 0$ and $u \geq 0$ we have

$$|\nabla u|^p - |\nabla v|^{p-2} \nabla v \cdot \nabla \left(\frac{u^p}{v^{p-1}} \right) \geq 0. \quad (1.3.7)$$

As an immediate consequence, they obtained a wide array of applications including the simplicity of the eigenvalues, Sturmian comparison principles, oscillation theorems, Hardy, Barta's inequalities and some profound results for p -Laplacian equations and systems. This work motivated a lot of generalization of the Picone's identity and in this regard, various attempts have been made to generalize Picone identity for different types of differential operators see [61, 100] and the reference therein. In [109, 110], Díaz and Saá proved the following inequality

$$\int_{\Omega} \left(\frac{(-\Delta)_p w_1^{1/p}}{w_1^{(p-1)/p}} - \frac{(-\Delta)_p w_2^{1/p}}{w_2^{(p-1)/p}} \right) (w_1 - w_2) \geq 0$$

for $w_i \in L^\infty(\Omega), w_i \geq 0$ a.e. on $\Omega, w_i^{1/p} \in W^{1,p}(\Omega), (-\Delta)_p w_i^{1/p} \in L^\infty(\Omega)$ for $i = 1, 2, w_1 = w_2$ over $\partial\Omega$ and $w_i/w_j \in L^\infty(\Omega)$ for $i \neq j, i, j = 1, 2$. This inequality turns out to be equivalent with the convexity of a p -power type energy functional, as suggested in [71] for $p = 2$, and generalized in [127] to any constant $p \in (1, \infty)$. In applications to quasilinear elliptic operators (with p constant, $1 < p < \infty$), this equivalence played a decisive role in the works [155] and [245]. In [81], Chaib proved the above inequality in \mathbb{R}^N , and pointed out the connection between the Díaz-Saá inequality and the generalized Picone inequality (1.3.7). Later on, in [64], Brasco and Franzina extended the above Díaz-Saá inequality in p - q form. Precisely, for every pair u, v of positive differentiable functions the following holds:

$$\frac{1}{p} \left\langle \nabla A(\nabla u), \nabla \left(\frac{v^q}{u^{q-1}} \right) \right\rangle \leq A(\nabla v)^{\frac{q}{p}} A(\nabla u)^{\frac{p-q}{p}}$$

where $q \in (1, p], A : \mathbb{R}^N \rightarrow [0, \infty)$ is a C^1 positively p -homogeneous convex function.

1.4 New contributions

In this section, we provide the details of new tools and techniques which are developed to tackle the **Problems 1, 2 and 3** mentioned in Section 1.2 and main results with a glimpse of the proof.

¹Equation (1.3.6) is known as "identity", even if it is inequality, because of the two terms can be written as

$$\left| \nabla u - \frac{u}{v} \nabla v \right|^2 = |\nabla u|^2 + \frac{u^2}{v^2} |\nabla v|^2 - 2 \frac{u}{v} \nabla u \cdot \nabla v = |\nabla u|^2 - \nabla \left(\frac{u^2}{v} \right) \cdot \nabla v$$

which is indeed positive.

1.4.1 Problem 1: Strong solution of evolution equations with $p(x, t)$ -Laplacian

We study the following Dirichlet problem for a class of parabolic equations with variable nonlinearity

$$\begin{cases} \partial_t u - \operatorname{div} \left(|\nabla u|^{p(x,t)-2} \nabla u \right) = f(x, t) & \text{in } Q_T \stackrel{\text{def}}{=} \Omega \times (0, T), \\ u = 0 \text{ on } \Gamma \stackrel{\text{def}}{=} \partial\Omega \times (0, T), \quad u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (1.4.1)$$

1.4.1.1 Main tools

First, we derive a special interpolation inequality that yields the global integrability of $|\nabla u|^{p(x,t)+\delta}$ with some $\delta > 0$ independent of u , and provides an estimate on the term with the logarithmic growth. The interpolation inequality is also used in the proof of $W^{1,2}(Q_T)$ -regularity of the flux in the degenerate problem (1.4.1) and the counterpart problems with regularized fluxes. Define

$$\gamma_\epsilon(z, \mathbf{s}) = \beta_\epsilon^{\frac{p(z)-2}{2}}(\mathbf{s}) \equiv (\epsilon^2 + |\mathbf{s}|^2)^{\frac{p(z)-2}{2}}, \quad \epsilon > 0, \quad \mathbf{s} \in \mathbb{R}^N.$$

Theorem 1.4.1. *Let $\partial\Omega \in C^1$, $u \in C^1([0, T]; C^2(\bar{\Omega}))$ and $u = 0$ on $\partial\Omega \times [0, T]$. Assume that $p(\cdot) : Q_T \mapsto [p^-, p^+]$ such that $p \in C^0(\bar{Q}_T)$ with the modulus of continuity ω ,*

$$\begin{aligned} \frac{2N}{N+2} < p^-, \quad \operatorname{ess\,sup}_{Q_T} |\nabla p| = L, \\ \int_{Q_T} \gamma_\epsilon(z, \nabla u) |u_{xx}|^2 dz < \infty, \quad \sup_{(0,T)} \|u(t)\|_{2,\Omega}^2 = M_0, \quad \int_{Q_T} |\nabla u|^{p(z)} dz = M_1. \end{aligned}$$

Then for every

$$\frac{2}{N+2} = r_* < r < r^* = \frac{4p^-}{p^-(N+2) + 2N}$$

and every $\delta \in (0, 1)$ the function u satisfies the inequality

$$\int_{Q_T} \beta_\epsilon^{\frac{p(z)+r-2}{2}} (\nabla u) |\nabla u|^2 dz \leq \delta \int_{Q_T} \gamma_\epsilon(z, \nabla u) |u_{xx}|^2 dz + C \left(1 + \int_{Q_T} |\nabla u|^{p(z)} dz \right). \quad (1.4.2)$$

with an independent of u constant C .

The proof of above lemma is slightly technical. We will only highlight here the crucial point of the proof. Using uniform continuity of the exponent p in Q_T , it is enough to prove the estimate (1.4.2) in Ω and for a fixed $t \in (0, T)$ i.e. for every $\delta \in (0, 1)$

$$\int_{\Omega} \beta_\epsilon^{\frac{p(x)+r-2}{2}} (\nabla u) |\nabla u|^2 dx \leq \delta \int_{\Omega} \gamma_\epsilon(x, \nabla u) |u_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u|^{p(x)} dx \right)$$

with an independent of u constant C . A straightforward computation via Green's formula and a set of logarithmic inequalities (for more details see (4.2.18), Page 119, Chapter 4) leads

us to the following estimate

$$\begin{aligned} \int_{\Omega} \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) |\nabla u|^2 dx &\leq C_0 + \delta \int_{\Omega} \gamma_{\epsilon}(x, \nabla u) |u_{xx}|^2 dx \\ &+ C_{\delta} \int_{\Omega} u^2 \beta_{\epsilon}^{\frac{p(x)+2r-2}{2}} (\nabla u) dx + C_1 \int_{\Omega} |u| \beta_{\epsilon}^{\frac{p(x)+r-1+\nu}{2}} (\nabla u) dx \end{aligned} \quad (1.4.3)$$

for some $\nu \in (0, 1)$ and it remains to estimate last terms of (1.4.3). For this, the continuity of p allows us to choose a special finite cover $\{\Omega_i\}_i$ so small such that for every $i = 1, 2, \dots, K$

$$p_i^+ - p_i^- + r \left(1 + \frac{2N}{p^-(N+2)} \right) < \frac{4}{N+2}.$$

for any r between lower and upper exponent *i.e.* $r \in (r_*, r^*)$. Then, finally by using interpolation inequalities of Gagliardo-Nirenberg-Sobolev type with a suitable set of exponents, we get our final result. For a detailed proof, we refer to Theorem 4.2.3, Page 118, Chapter 4.

Secondly, we derive the trace-interpolation inequality used to estimate the traces of $|\nabla u|^{p(z)}$ on the lateral boundary of the cylinder Q_T . These estimates turns out to be useful in the study of the non-convex domains also.

Theorem 1.4.2. *Let us assume that p and u satisfy the conditions of Theorem 1.4.1. Then for every $\lambda \in (0, 1)$*

$$\begin{aligned} \int_{\partial\Omega \times (0, T)} (\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} |\nabla u|^2 dS dt &\leq \lambda \int_{Q_T} (\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} |u_{xx}|^2 dz \\ &+ C \left(1 + \int_{Q_T} |\nabla u|^{p(z)} dz \right) \end{aligned}$$

with an independent of u constant C .

The proof of the above lemma follows by Lemma 1.5.1.9 in [162], Green formula and logarithmic inequalities depending upon the size of the gradient.

1.4.1.2 Main results

The main result of **Problem 1** are given in the following theorems.

Theorem 1.4.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with the boundary $\partial\Omega \in C^2$. Assume that the exponent p satisfies*

$$\begin{aligned} \frac{2N}{N+2} < p^- \leq p(x, t) \leq p^+, \quad p \in C_{\log}(\overline{Q_T}). \\ \text{ess sup}_{Q_T} |\nabla p| \leq C_* < \infty, \quad \text{ess sup}_{Q_T} |p_t| \leq C^* \end{aligned}$$

with nonnegative finite constants C_* , C^* . Let

$$f \in L^2(0, T; W_0^{1,2}(\Omega)), \quad u_0 \in L^2(\Omega) \cap W_0^{1,q_0(\cdot)}(\Omega) \text{ with } q_0(x) = \max\{2, p(x, 0)\}.$$

Then the weak solution u of problem (1.4.1) is a strong solution. The function u satisfies estimate (1.3.1) and

$$\|u_t\|_{2,Q_T}^2 + \operatorname{ess\,sup}_{(0,T)} \int_{\Omega} |\nabla u|^{q(x,t)} dx \leq C \quad (1.4.4)$$

with the exponent $q(x,t) = \max\{2, p(x,t)\}$ and a constant C depends upon $N, \partial\Omega, T, p^{\pm}, C_*, C^*, \|u_0\|, \|f\|$.

Theorem 1.4.4. *Under the conditions of Theorem 1.4.3,*

(i) *The strong solution u possesses the property of higher integrability of the gradient:*

$$\int_{Q_T} |\nabla u|^{p(x,t)+\delta} dxdt \leq C_{\delta} \quad \text{for every } 0 < \delta < \frac{4p^-}{p^-(N+2)+2N} \quad (1.4.5)$$

with a finite constant C_{δ} depending on δ and the same quantities as the constant C in (1.4.4).

(ii) *Moreover,*

$$\begin{aligned} D_{x_i x_j}^2 u &\in L_{loc}^{p(\cdot)}(Q_T \cap \{(x,t) : p(x,t) < 2\}), \quad \text{if } N \geq 2, \\ D_{x_i} \left(|\nabla u|^{\frac{p(x,t)-2}{2}} D_{x_j} u \right) &\in L^2(Q_T) \quad \text{if } N \geq 3, \text{ or } N = 2 \text{ and } p^- > \frac{6}{5}, \end{aligned}$$

$i, j = 1, 2, \dots, N$, and the corresponding norms are bounded by constants depending only on the data.

Remark 1.4.1. *Due to the fact that estimate (1.4.5) is global in time and space, it is new even in the case of constant p . We refer to [114] for a detailed insight into this issue, in particular, to [114, Lemma 5.4].*

The same existence and regularity results are obtained for the solution of problem (1.4.1) with the regularized flux function $(\epsilon^2 + |\nabla u|^2)^{\frac{p(x,t)-2}{2}} \nabla u$, $\epsilon > 0$.

Let us give an outline of the proof of Theorem 1.4.3. The solution of problem (1.4.1) is constructed as the limit of the sequence of solutions of the following family of regularized non-degenerate parabolic problems

$$\begin{cases} \partial_t u - \operatorname{div}((\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} \nabla u) = f(z) & \text{in } Q_T, \\ u = 0 & \text{on } \Gamma_T = \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega. \end{cases} \quad (1.4.6)$$

for a given a parameter $\epsilon > 0$ and new regularized flux given by $(\epsilon^2 + |\nabla u|^2)^{\frac{p-2}{2}} \nabla u$, $\epsilon \in (0, 1)$.

For every fixed ϵ , a solution of problem (1.4.6) is constructed as the limit of the sequence of

finite-dimensional Galerkin's approximations $\{u_\epsilon^{(m)}\}$. The functions $u_\epsilon^{(m)}(x, t)$ are sought in the form

$$u_\epsilon^{(m)}(x, t) = \sum_{j=1}^m u_j^{(m)}(t) \phi_j(x),$$

where $\phi_j \in W_0^{1,2}(\Omega)$ and $\lambda_j > 0$ are the eigenfunctions and the corresponding eigenvalues of the problem

$$(\nabla \phi_j, \nabla \psi)_{2,\Omega} = \lambda(\phi_j, \psi)_{2,\Omega} \quad \forall \psi \in W_0^{1,2}(\Omega).$$

The coefficients $u_j^{(m)}(t)$ are defined as the solutions of the Cauchy problem for the system of m ordinary differential equations

$$\begin{aligned} (u_j^{(m)})'(t) &= - \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} \cdot \nabla \phi_j \, dx + \int_{\Omega} f \phi_j \, dx, \\ u_j^{(m)}(0) &= (u_0, \phi_j)_{2,\Omega}, \quad j = 1, 2, \dots, m, \end{aligned} \quad (1.4.7)$$

where the functions

$$u_0^{(m)} = \sum_{j=1}^m (u_0, \phi_j)_{2,\Omega} \phi_j \in \text{span}\{\phi_1, \phi_2, \dots, \phi_m\},$$

are chosen so that $u_0^{(m)} \rightarrow u_0$ in $W_0^{1,q(x,0)}(\Omega)$, $q(x, 0) = \max\{2, p(x, 0)\}$.

Using Carathéodory Theorem, we prove the existence of a solution $(u_1^{(m)}, u_2^{(m)}, \dots, u_m^{(m)})$ of the system of ODE (1.4.7) on an interval $(0, T_m)$ and for every finite m system (1.4.7) and this solution can be continued on the arbitrary interval $(0, T)$ because of the uniform estimate in ϵ and m proved in coming results.

To pass limits in the sequence of finite dimensional approximations $u_\epsilon^{(m)}$, we derive uniform a priori estimates in ϵ and m simultaneously. This is where the difference between the cases of constant and variable exponent p becomes obvious: in the latter case the estimates involve the expression $|\nabla p|(\epsilon^2 + |\nabla u|^2)^{\frac{p}{2}} |\ln(\epsilon^2 + |\nabla u|^2)|$, not included into the basic energy estimate (1.3.1). The integration by parts formula (see (4.1.7), Chapter 4) and the choice of eigen functions ϕ_j reveal the following a priori estimates for $u_\epsilon^{(m)}$:

$$\sup_{(0,T)} \|u_\epsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2 \, dz \leq e^T (\|f\|_{2,Q_T}^2 + \|u_0\|_{2,\Omega}^2) := L_0.$$

$$\int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p(z)} \, dz \leq \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} \, dz \leq L_1$$

where constants L_0, L_1 independent of ϵ and m , and $\epsilon \in (0, 1)$. For a detailed proof, we refer to Lemma 4.2.1, Page 115, Chapter 4.

Let us denote \mathbf{n} by the exterior normal vector to $\partial\Omega$. Repetitive usage of Green formula via the elemental properties of the eigen functions ϕ_j under suitable conditions on initial

data, boundary $\partial\Omega$ and variable exponent implies the following inequality relation: For a.e. $t \in (0, T)$ and any $\delta > 0$

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + (\min\{p^-, 2\} - 1 - \delta) \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2 dx \\
 & \leq C_0 \int_{\Omega} |\nabla u_\epsilon^{(m)}|^2 (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \ln^2(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2) dx \\
 & \quad - \int_{\partial\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \left(\Delta u_\epsilon^{(m)} (\nabla u_\epsilon^{(m)} \cdot \mathbf{n}) - \nabla u_\epsilon^{(m)} \cdot \nabla (\nabla u_\epsilon^{(m)} \cdot \mathbf{n}) \right) dx \\
 & \quad + C_1 \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + C_2 \|f(t)\|_{W_0^{1,2}(\Omega)}^2
 \end{aligned} \tag{1.4.8}$$

with constants C_i , $i = 0, 1, 2$, depending on the data and δ , but independent of m and ϵ . For a detailed explanation, we refer to Lemma 4.2.2, Page 116, Chapter 4.

The study of higher regularity of solutions usually involves “differentiation” of the equation. In the case of non-constant p this leads to appearance of the term $|\nabla u|^{p(x,t)} \ln |\nabla u|$ (as in (1.4.8)), which can not be controlled through the usual energy estimates for the weak solution of equation (1.2.1) unless $p(x, t) \leq 2$. The main issue is to get rid of the restriction $p(x, t) \leq 2$ in the proof of existence of strong solutions and in the study of their higher regularity.

1.4.1.3 A priori estimates and existence of strong solution

To control the R.H.S. of (1.4.8), we use the interpolation inequalities proved in Section 1.4.1.1 which further entails the global higher integrability of the gradients of the finite-dimensional approximations: instead of the natural order of integrability $p(z)$ prompted by the equation, the gradients are integrable in Q_T with the power $p(z) + \delta$ (estimate (1.4.5)). By combining the interpolation inequality for global higher integrability of the gradients (Theorem 1.4.1) and trace interpolation inequality (Theorem 1.4.2) we obtain a complete derivation of the following uniform a priori estimates of the type (1.4.4) for Galerkin’s approximations:

$$\begin{aligned}
 \sup_{(0,T)} \|\nabla u_\epsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2 dz \\
 \leq C e^{C'T} \left(1 + \|\nabla u_0\|_{2,\Omega}^2 + \|f\|_{L^2(0,T;W_0^{1,2}(\Omega))}^2 \right),
 \end{aligned}$$

$$\int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p(z)+r} dz \leq C'' \quad \text{for any } 0 < r < \frac{4p^-}{p^-(N+2) + 2N}$$

and

$$\|(u_\epsilon^{(m)})_t\|_{2,Q_T}^2 + \sup_{(0,T)} \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} dx \leq C' \left(1 + \int_{\Omega} |\nabla u_0|^{p(x,0)} dx \right) + \|f\|_{2,Q_T}^2$$

with constants C , C' , C'' independent of m and ϵ . For a complete derivation, we refer to Lemma 4.2.3, Page 118, Chapter 4.

Using the previous uniform estimates, the weak convergence of the sequence $\{\nabla u_\epsilon^{(m)}\}$, and

the monotonicity of the flux $\gamma_\epsilon((x, t), \mathbf{s})\mathbf{s} = (\epsilon^2 + |\mathbf{s}|^2)^{\frac{p(x,t)-2}{2}}\mathbf{s}$ in (1.4.6) implies the existence of unique weak solution of the regularized problem (1.4.6) with $\partial_t u_\epsilon \in L^2(Q_T)$ and the global higher regularity properties. However, these uniform estimates do not ensure that $\gamma_\epsilon((x, t), \nabla u_\epsilon^{(m)})\nabla u_\epsilon^{(m)} \rightarrow \gamma_\epsilon((x, t), \nabla u_\epsilon)\nabla u_\epsilon$ a.e. in Q_T , even in the case of constant p .

For this, we prove a.e. convergence of the sequence of $\nabla u_\epsilon^{(m)}$ to ∇u_ϵ , which yields a.e. convergence of fluxes. The proof relies on the convexity of the function $\gamma_\epsilon((x, t), \mathbf{s})|\mathbf{s}|^2$ with respect to \mathbf{s} , the weak convergence of the sequence $\nabla u_\epsilon^{(m)}$ to ∇u_ϵ , and the convergence of the integrals of $\gamma_\epsilon((x, t), \nabla u_\epsilon^{(m)})|\nabla u_\epsilon^{(m)}|^2$ to the integral of $\gamma_\epsilon((x, t), \nabla u_\epsilon)|\nabla u_\epsilon|^2$. The pointwise convergence of fluxes of Galerkin's approximations and the uniform a priori estimates allow one to show that the limit of the sequence of regularized fluxes $(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(x,t)-2}{4}}\nabla u_\epsilon^{(m)}$ belongs to $(L^2(0, T; W^{1,2}(\Omega)))^N$. The difference between the cases $N \geq 3$ and $N = 2$ is explained by the convexity properties of the function $\gamma_\epsilon((x, t), \mathbf{s})|\mathbf{s}|^2$ with $\epsilon > 0$. It is strictly convex with respect to \mathbf{s} if $p > \frac{6}{5}$, which is true for $N \geq 3$ because $p^- > \frac{2N}{N+2}$, but in the case $N = 2$ leads to the additional restriction. Precisely, we prove

Theorem 1.4.5. *Let the conditions of Theorem 1.4.3 be fulfilled.*

(i) *If $N \geq 3$ or $N = 2$ and $p^- > \frac{6}{5}$, then $\nabla u_\epsilon^{(m)} \rightarrow \nabla u_\epsilon$ a.e. in Q_T .*

(ii) *Under the conditions of item (i) $\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon)D_i u_\epsilon \in L^2(0, T; W^{1,2}(\Omega))$, $i = 1, 2, \dots, N$, and*

$$\|\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon)D_i u_\epsilon\|_{L^2(0, T; W^{1,2}(\Omega))} \leq M, \quad i = 1, 2, \dots, N,$$

with an independent of ϵ constant M .

(iii) *If $N \geq 2$ and $p^- > \frac{2N}{N+2}$, then $D_{ij}^2 u_\epsilon \in L_{loc}^{p(\cdot)}(Q_T \cap \{z : p(z) < 2\})$, $i, j = 1, 2, \dots, N$, and*

$$\sum_{i,j=1}^N \|D_{ij}^2 u_\epsilon\|_{p(\cdot), Q_T \cap \{z : p(z) < 2\}} \leq M'$$

with an independent of ϵ constant M' .

The proof of stronger convergence properties of the sequence $\nabla u_\epsilon^{(m)}$ stems from the Theorem 2.1 and Corollary 2.1, [224] on the convergence of sequences of functionals. The proof of the main result (Theorem 1.4.3) is based on the same ideas as the proofs in the case of the regularized problems (1.4.6). The difference in the arguments is due to the necessity of passing to the limit with respect to ϵ , which changes the nonlinear structure of the equation. The second order regularity and global higher integrability in Theorem 1.4.4 are the byproduct of previous uniform estimates with respect to m and ϵ , and convexity and almost everywhere convergence of regularized flux.

1.4.2 Problem 2: Doubly phase parabolic problem with variable growth

In this subsection, we study the following double phase parabolic problem with the homogeneous Dirichlet boundary conditions:

$$\begin{cases} u_t - \operatorname{div} \left(|\nabla u|^{p(z)-2} \nabla u + a(z) |\nabla u|^{q(z)-2} \nabla u \right) = F(z, u) & \text{in } Q_T, \\ u = 0 \text{ on } \Gamma_T, \quad u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (1.4.9)$$

where the nonlinear source has the form

$$F(z, v) = f_0(z) + b(z) |v|^{\sigma(z)-2} v. \quad (1.4.10)$$

Here $a \geq 0$, b , p , q , σ and f_0 are given functions of the variables $z \in Q_T$.

1.4.2.1 Main tools

In this part, we present estimates on the gradient trace on $\partial\Omega$ for the functions from variable Sobolev spaces. This property turns out to be the key element in the proof of the existence theorems for problem (1.4.9) and the regularized problem (1.4.19).

Until the end of this subsection, the notation $p(\cdot)$, $q(\cdot)$, $a(\cdot)$ is used for functions not related to the exponents and coefficient in (1.4.9) and (1.4.19). Let us accept the notation

$$\begin{aligned} \beta_\epsilon(\mathbf{s}) &= \epsilon^2 + |\mathbf{s}|^2, \\ \varphi_\epsilon(z, \mathbf{s}) &= (\epsilon^2 + |\mathbf{s}|^2)^{\frac{p(z)-2}{2}} + a(z) (\epsilon^2 + |\mathbf{s}|^2)^{\frac{q(z)-2}{2}}, \quad \mathbf{s} \in \mathbb{R}^N, \quad z \in Q_T, \quad \epsilon \in (0, 1). \end{aligned} \quad (1.4.11)$$

With certain abuse of notation, we will denote by $\varphi_\epsilon(x, \mathbf{s})$ the same function but with the exponents p , q and the coefficient a depending on the variable $x \in \Omega$.

Lemma 1.4.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded domain with the boundary $\partial\Omega \in C^2$, and $a \in W^{1,\infty}(\Omega)$ be a given nonnegative function. Assume that $v \in W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and denote*

$$K = \int_{\partial\Omega} a(x) (\epsilon^2 + |\nabla v|^2)^{\frac{p(x)-2}{2}} (\Delta v (\nabla v \cdot \mathbf{n}) - \nabla(\nabla v \cdot \mathbf{n}) \cdot \nabla v) \, dS,$$

where \mathbf{n} stands for the exterior normal to $\partial\Omega$. There exists a constant $L = L(\partial\Omega)$ such that

$$K \leq L \int_{\partial\Omega} a(x) (\epsilon^2 + |\nabla v|^2)^{\frac{p(x)-2}{2}} |\nabla v|^2 \, dS.$$

Proof of Lemma 1.4.1 follows from the well-known assertions, see, e.g., [180, Ch.1, Sec.1.5] for the case $a \equiv 1$, $N \geq 2$, or [28, Lemma A.1] for the case of an arbitrary dimension. For more details see Lemma 4.3.2, Chapter 4.

Theorem 1.4.6. *Let $\partial\Omega \in C^2$, $u \in C^2(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$. Assume that $p(\cdot)$ satisfies the conditions of Lemma 1.4.1, $a(\cdot)$ be a non-negative function on $\overline{\Omega}$ with $a, q \in W^{1,\infty}(\Omega)$ and*

$$q : \Omega \mapsto [q^-, q^+] \subset \left(\frac{2N}{N+2}, \infty \right), \quad \|\nabla q\|_{\infty, \Omega} \leq L < \infty, \quad \|\nabla a\|_{\infty, \Omega} \leq L_0 < \infty.$$

If for a.e. $x \in \Omega$

$$q(x) < p(x) + r \text{ with } \frac{2}{N+2} < r < \frac{4p^-}{p^-(N+2) + 2N},$$

then for every $\lambda \in (0, 1)$

$$\int_{\partial\Omega} \varphi_\epsilon(x, \nabla u) |\nabla u|^2 dS \leq \lambda \int_{\Omega} \varphi_\epsilon(x, \nabla u) |u_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u|^{p(x)} dx \right)$$

with a constant C depending on λ and the constants p^\pm , N , L , L_0 , but independent of u .

The proof of the this result follows from [162, Lemma 1.5.1.9], Cauchy inequality and Green Formula. For a detailed explanation see Theorem 1.4.6, Chapter 4.

Now, we construct a sequence of finite-dimensional approximations for the initial function u_0 in the same basis $\{\phi_j\}$ as in the Galerkin's approximations for the solution of the regularized problem (1.4.19). In the nondegenerate case, $q(x, 0) \leq 2$ in Ω , this sequence is obtained in a standard way, while in the case $\sup_{\Omega} q(x, 0) > 2$ the choice of the sequence becomes an independent problem. We construct it as a sequence of finite-dimensional approximations of the solution of the degenerate double phase elliptic equation (see (4.3.21)) with variable exponents $r(x) = \max\{2, p(x, 0)\}$ and $s(x) = \max\{2, q(x, 0)\}$, and the right-hand side depending on u_0 . This problem is solved with the method of Galerkin in the framework of Musielak-Orlicz spaces. Let $\sup q(x, 0) > 2$. We approximate the initial function u_0 by the sequence of finite-dimensional approximations for the solution of the elliptic problem

$$\beta(x, u)u - \operatorname{div}(\alpha(x, \nabla u)\nabla u) = f - \operatorname{div} \Phi \quad \text{in } \Omega, \quad u = 0 \text{ on } \partial\Omega \quad (1.4.12)$$

where

$$\begin{aligned} \beta(x, u)u &= |u|^{r(x)-2}u + a_0(x)|u|^{s(x)-2}u, \\ \alpha(x, \nabla u)\nabla u &= |\nabla u|^{r(x)-2}\nabla u + a_0(x)|\nabla u|^{s(x)-2}\nabla u, \quad a_0(x) = a(x, 0), \\ r(x) &= \max\{2, p(x, 0)\} \geq 2, \quad s(x) = \max\{2, q(x, 0)\}, \end{aligned}$$

and

$$f = \beta(x, u_0)u_0, \quad \Phi = \alpha(x, \nabla u_0)\nabla u_0. \quad (1.4.13)$$

For a detailed study of above elliptic problem, we refer to Section 4.3.4, Chapter 4.

1.4.2.2 Main results

Let $p, q : Q_T \mapsto \mathbb{R}$ be measurable functions satisfying the conditions

$$\begin{aligned} \frac{2N}{N+2} < p_- \leq p(z) \leq p_+ \text{ in } \overline{Q}_T, \\ \frac{2N}{N+2} < q_- \leq q(z) \leq q_+ \text{ in } \overline{Q}_T, \quad p^\pm, q^\pm = \text{const.} \end{aligned} \quad (1.4.14)$$

Moreover, let us assume that $p, q \in W^{1,\infty}(Q_T)$ as functions of variables $z = (x, t)$: there exist positive constants C^*, C^{**}, C_*, C_{**} such that

$$\begin{aligned} \text{ess sup}_{Q_T} |\nabla p| \leq C_* < \infty, \quad \text{ess sup}_{Q_T} |p_t| \leq C^*, \\ \text{ess sup}_{Q_T} |\nabla q| \leq C_{**} < \infty, \quad \text{ess sup}_{Q_T} |q_t| \leq C^{**}. \end{aligned} \quad (1.4.15)$$

The modulating coefficient $a(\cdot)$ is assumed to satisfy the following conditions:

$$a(z) \geq 0 \text{ in } \overline{Q}_T, \quad a \in C([0, T]; W^{1,\infty}(\Omega)), \quad \text{ess sup}_{Q_T} |a_t| \leq C_a, \quad C_a = \text{const.} \quad (1.4.16)$$

We do not impose any condition on the null set of the function a in \overline{Q}_T and do not distinguish between the cases of degenerate and singular equations. It is possible that $p(z) < 2$ and $q(z) > 2$ at the same point $z \in Q_T$.

Definition 1.4.1. A function $u : Q_T \mapsto \mathbb{R}$ is called **strong solution** of problem (1.4.9) if

- (i) $u \in \mathcal{W}_{q(\cdot)}(Q_T)$, $u_t \in L^2(Q_T)$, $|\nabla u| \in L^\infty(0, T; L^{s(\cdot)}(\Omega))$ with $s(z) = \max\{2, p(z)\}$,
- (ii) for every $\psi \in \mathcal{W}_{q(\cdot)}(Q_T)$ with $\psi_t \in L^2(Q_T)$

$$\int_{Q_T} u_t \psi \, dz + \int_{Q_T} (|\nabla u|^{p(z)-2} + a(z)|\nabla u|^{q(z)-2}) \nabla u \cdot \nabla \psi \, dz = \int_{Q_T} F(z, u) \psi \, dz,$$

- (iii) for every $\phi \in C_0^1(\Omega)$

$$\int_{\Omega} (u(x, t) - u_0(x)) \phi \, dx \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

The main results are given in the following theorems.

Theorem 1.4.7. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with the boundary $\partial\Omega \in C^2$. Assume that $p(\cdot), q(\cdot)$ satisfy conditions (1.4.14), (1.4.15), and there exists a constant

$$r \in (0, r^*), \quad r^* = \frac{4p^-}{p^-(N+2) + 2N},$$

such that

$$p(z) \leq q(z) \leq p(z) + \frac{r}{2} \text{ in } \overline{Q}_T.$$

If $a(\cdot)$ satisfies conditions (1.4.16) and $b \equiv 0$, then for every $f_0 \in L^2(0, T; W_0^{1,2}(\Omega))$ and $u_0 \in W_0^{1,2}(\Omega)$ with

$$\int_{\Omega} \left(|\nabla u_0|^2 + |\nabla u_0|^{p(x,0)} + a(x,0) |\nabla u_0|^{q(x,0)} \right) dx = K < \infty \quad (1.4.17)$$

problem (1.4.9) has a unique strong solution u . This solution satisfies the estimate

$$\|u_t\|_{2,Q_T}^2 + \operatorname{ess\,sup}_{(0,T)} \int_{\Omega} \left(|\nabla u|^{s(z)} + a(z) |\nabla u|^{q(z)} \right) dx + \int_{Q_T} |\nabla u|^{p(z)+r} dz \leq C \quad (1.4.18)$$

with the exponent $s(z) = \max\{2, p(z)\}$ and a constant C which depends on $N, \partial\Omega, T, p^{\pm}, q^{\pm}, r$, the constants in conditions (1.4.15), (1.4.16), $\|f_0\|_{L^2(0,T;W_0^{1,2}(\Omega))}$ and K .

Theorem 1.4.8. *Let in the conditions of Theorem 1.4.7, $b \neq 0$.*

(i) *Assume that b, σ are measurable bounded functions defined on Q_T ,*

$$\begin{aligned} \|\nabla b\|_{\infty, Q_T} < \infty, \quad \|\nabla \sigma\|_{\infty, Q_T} < \infty, \\ 2 \leq \sigma^- \leq \sigma^+ < 1 + \frac{p^-}{2}, \quad \sigma^- = \operatorname{ess\,inf}_{Q_T} \sigma(z), \quad \sigma^+ = \operatorname{ess\,sup}_{Q_T} \sigma(z). \end{aligned}$$

Then for every $f_0 \in L^2(0, T; W_0^{1,2}(\Omega))$ and $u_0 \in W_0^{1,2}(\Omega)$ satisfying condition (1.4.17) problem (1.4.9) has at least one strong solution u . The solution u satisfies estimate (1.4.18) with the constant depending on the same quantities as in the case $b \equiv 0$ and on $\|\nabla b\|_{\infty, Q_T}, \|\nabla \sigma\|_{\infty, Q_T}, \sigma^{\pm}, \operatorname{ess\,sup}_{Q_T} |b|$.

(ii) *The strong solution is unique if $p(\cdot), q(\cdot)$ satisfy the conditions of Theorem 1.4.7 and either $\sigma \equiv 2$, or $b(z) \leq 0$ in Q_T .*

A solution of problem (1.4.9) is obtained as the limit of the family of solutions of the nondegenerate problems with the regularized fluxes

$$\left((\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} + a(z) (\epsilon^2 + |\nabla u|^2)^{\frac{q(z)-2}{2}} \right) \nabla u, \quad \epsilon > 0.$$

Given $\epsilon > 0$, let us consider the following family of regularized double phase parabolic equations:

$$\begin{cases} \partial_t u - \operatorname{div}(\varphi_{\epsilon}(z, \nabla u) \nabla u) = F(z, u) & \text{in } Q_T, \\ u = 0 & \text{on } \Gamma_T, \\ u(0, \cdot) = u_0 & \text{in } \Omega, \quad \epsilon \in (0, 1), \end{cases} \quad (1.4.19)$$

where $F(z, u)$ is defined in (1.4.10) and $\varphi_{\epsilon}(z, \nabla u) \nabla u$ is the regularized flux function defined in (1.4.11).

Let $\epsilon > 0$ be a fixed parameter. The sequence $\{u_\epsilon^{(m)}\}$ of finite-dimensional Galerkin's approximations for the solutions of the regularized problem (1.4.19) is sought in the form

$$u_\epsilon^{(m)}(x, t) = \sum_{j=1}^m u_j^{(m)}(t) \phi_j(x)$$

where $\phi_j \in W_0^{1,2}(\Omega)$ and $\lambda_j > 0$ are the eigenfunctions and the corresponding eigenvalues of the problem

$$(\nabla \phi_j, \nabla \psi)_{2,\Omega} = \lambda_j (\phi_j, \psi)_{2,\Omega} \quad \forall \psi \in W_0^{1,2}(\Omega). \quad (1.4.20)$$

The systems $\{\phi_j\}$ and $\{\lambda_j^{-\frac{1}{2}} \phi_j\}$ are the orthogonal bases of $L^2(\Omega)$ and $W_0^{1,2}(\Omega)$. The coefficients $u_j^{(m)}(t)$ are characterized as the solutions of the Cauchy problem for the system of m ordinary differential equations

$$\begin{cases} (u_j^{(m)})'(t) = - \int_{\Omega} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) \nabla u_\epsilon^{(m)} \cdot \nabla \phi_j \, dx + \int_{\Omega} F(z, u_\epsilon^{(m)}) \phi_j \, dx, \\ u_j^{(m)}(0) = (u_0^{(m)}, \phi_j)_{2,\Omega}, \quad j = 1, 2, \dots, m, \end{cases} \quad (1.4.21)$$

where φ_ϵ is defined in (1.4.11) and the functions $u_0^{(m)}$ are chosen in such a way that

$$\begin{aligned} u_0^{(m)} &= \sum_{j=1}^m (u_0, \phi_j)_{2,\Omega} \phi_j \in \text{span}\{\phi_1, \phi_2, \dots, \phi_m\}, \\ u_0^{(m)} &\rightharpoonup u_0 \quad \begin{array}{l} \text{in } W_0^{1,2}(\Omega) \quad \text{if } \max_{\overline{\Omega}} q(x, 0) \leq 2, \\ \text{in } W_0^{1,r(\cdot)}(\Omega) \quad \text{if } \max_{\overline{\Omega}} q(x, 0) > 2, \text{ where } r(x) = \max\{2, q(x, 0)\}. \end{array} \end{aligned} \quad (1.4.22)$$

By the Carathéodory existence theorem, for every finite m system (1.4.21) has a solution $(u_1^{(m)}, u_2^{(m)}, \dots, u_m^{(m)})$ in the extended sense on an interval $(0, T_m)$, the functions $u_i^{(m)}(t)$ are absolutely continuous and differentiable a.e. in $(0, T_m)$. The a priori estimates (1.4.23), (1.4.25), show that for every m the function $u_\epsilon^{(m)}(x, T_m)$ belongs to $\text{span}\{\phi_1, \dots, \phi_m\}$ and satisfies the estimate

$$\begin{aligned} \|\nabla u_\epsilon^{(m)}(\cdot, T_m)\|_{2,\Omega}^2 + \int_{\Omega} \left(|\nabla u_\epsilon^{(m)}(x, T_m)|^{p(x, T_m)} + a(x, T_m) |\nabla u_\epsilon^{(m)}(x, T_m)|^{q(x, T_m)} \right) dx \\ \leq C + \|f_0\|_{2, Q_T}^2 + \|\nabla u_0^{(m)}\|_{2,\Omega}^2 + \int_{\Omega} \left(|\nabla u_0^{(m)}|^{p(x, 0)} + a(x, 0) |\nabla u_0^{(m)}|^{q(x, 0)} \right) dx \end{aligned}$$

with a constant C independent of m and ϵ . Since $a(\cdot, 0)$ is uniformly bounded in Ω , the sequence $\{u_0^{(m)}\}$ according to (1.4.22) and $|\nabla u_0|$ satisfies inequality (1.4.17), this estimate allows one to continue each of $u_\epsilon^{(m)}$ to the maximal existence interval $(0, T)$. In the case $\sup q(x, 0) \leq 2$, the embedding $W_0^{1,2}(\Omega) \subset W_0^{1,q(\cdot,0)}(\Omega)$ allows us to take $u_0^{(m)} = \sum_{i=1}^m u_i^{(m)}(0) \phi_i$ and in case of $\sup q(x, 0) > 2$ we approximate the initial function u_0 by the sequence of finite-dimensional approximations for the solution of the elliptic problem (1.4.12).

Now, we derive a priori estimates on the approximate solutions and their derivatives. For the convenience of presentation, we separate the cases when $b \equiv 0$ and the source function is

independent of the solution, and $b \neq 0$. Since no restriction on the sign of b is imposed, in the latter case derivation of the a priori estimates requires additional restrictions on the range of the exponent p .

For a.e. $t \in (0, T)$ $u_\epsilon^{(m)}$ satisfies the estimates:

$$\sup_{t \in (0, T)} \|u_\epsilon^{(m)}(\cdot, t)\|_{2, \Omega}^2 + \int_{Q_T} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 dz \leq C_0 e^T (\|f_0\|_{2, Q_T}^2 + \|u_0\|_{2, \Omega}^2),$$

The above estimates is obtained by multiplying j^{th} equation of (1.4.21) by $u_j^{(m)}(t)$ and then by summing up the results for $j = 1, 2, \dots, m$ via Cauchy and Grönwall inequality. For a detailed proof see Lemma 4.3.4, Page 149, Chapter 4.

The next a priori estimates involve higher-order derivatives of the approximate solutions. This is where we make use of the interpolation inequalities to obtain the global higher integrability of the gradient which, in turn, yields uniform boundedness of the $L^{q(\cdot)}(Q_T)$ -norms of the gradients of the approximate solutions.

$$\begin{aligned} \sup_{(0, T)} \|\nabla u_\epsilon^{(m)}(\cdot, t)\|_{2, \Omega}^2 + \int_{Q_T} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |(u_\epsilon^{(m)})_{xx}|^2 dz \\ \leq C e^{C'T} \left(1 + \|\nabla u_0\|_{2, \Omega}^2 + \|f_0\|_{L^2(0, T; W_0^{1, 2}(\Omega))}^2 \right) \end{aligned} \quad (1.4.23)$$

and

$$\int_{Q_T} |\nabla u_\epsilon^{(m)}|^{q(z)} dz + \int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p(z)+r} dz \leq C'' \quad \text{for any } 0 < r < \frac{4p^-}{p^-(N+2) + 2N}. \quad (1.4.24)$$

The estimates (1.4.23) and (1.4.24) are obtained by multiply each of equations in (1.4.21) by $\lambda_j u_j^{(m)}$ and sum up the results for $j = 1, 2, \dots, m$ using Green formula, Interpolation inequalities (Theorem 1.4.1 and Theorem 1.4.6). For a detailed explanation see Lemma 4.3.5 and Lemma 4.3.6, Page 150, Chapter 4.

Finally, by multiplying (1.4.21) with $(u_j^{(m)})_t$ and summing over $j = 1, 2, \dots, m$ using Cauchy inequality we obtain

$$\begin{aligned} \|(u_\epsilon^{(m)})_t\|_{2, Q_T}^2 + \sup_{(0, T)} \int_{\Omega} \left((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} + a(z) (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)}{2}} \right) dx \\ \leq C_{00} \left(1 + \int_{\Omega} (|\nabla u_0|^{p(x, 0)} + a(x, 0) |\nabla u_0|^{q(x, 0)}) dx \right) + \|f_0\|_{2, Q_T}^2 \end{aligned} \quad (1.4.25)$$

with an independent of m and ϵ constant C_{00} , C_0 , C , C' , C'' . For a detailed proof see Lemma 4.3.7, Page 156, Chapter 4. The similar kind of a priori estimates are derived in case when the equation contains the nonlinear source *i.e.* $b \neq 0$. The difference in the arguments consists in the necessity to estimate the integrals of the terms $b|u_\epsilon^{(m)}|^{\sigma(z)}$, $b|u_\epsilon^{(m)}|^{\sigma(z)-2} u_\epsilon^{(m)} \Delta u_\epsilon^{(m)}$, $b|u_\epsilon^{(m)}|^{\sigma(z)-2} u_\epsilon^{(m)} u_{\epsilon t}^{(m)}$. For more details see Lemma 4.3.8, Lemma 4.3.9 and Lemma 4.3.10, Chapter 4.

These uniform in m and ϵ estimates enable one to extract a subsequence $u_\epsilon^{(m)}$ (for which we keep the same name), and functions u_ϵ , η_ϵ , χ_ϵ such that

$$\begin{aligned} u_\epsilon^{(m)} &\rightharpoonup u_\epsilon \quad \star\text{-weakly in } L^\infty(0, T; L^2(\Omega)), \quad (u_\epsilon^{(m)})_t \rightharpoonup (u_\epsilon)_t \text{ in } L^2(Q_T), \\ \nabla u_\epsilon^{(m)} &\rightharpoonup \nabla u_\epsilon \text{ in } (L^{p(\cdot)}(Q_T))^N, \quad \nabla u_\epsilon^{(m)} \rightharpoonup \nabla u_\epsilon \text{ in } (L^{q(\cdot)}(Q_T))^N, \\ (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} &\rightharpoonup \eta_\epsilon \text{ in } (L^{q'(\cdot)}(Q_T))^N, \\ (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)-2}{2}} \nabla u_\epsilon^{(m)} &\rightharpoonup \chi_\epsilon \text{ in } (L^{q'(\cdot)}(Q_T))^N. \end{aligned}$$

In the third line we make use of the uniform estimate

$$\int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)(p(z)-1)}{2(q(z)-1)}} dz \leq C \left(1 + \int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p(z)+r} dz \right) \leq C.$$

Now, by using the fact that if $U_m \rightharpoonup U$ in $L^{q(\cdot)}(Q_T)$, then for every $V \in L^{q(\cdot)}(Q_T)$ we have

$$a(z)V \in L^{q(\cdot)}(Q_T) \quad \text{and} \quad \int_{Q_T} aU_m V dz \rightarrow \int_{Q_T} aUV dz$$

and the same arguments in Theorem 1.4.3, we show first that $u_\epsilon^{(m)}$ converges to a strong solution u_ϵ of the regularized problem (1.4.19). The proof relies on the compactness and monotonicity of the fluxes. Existence of a solution to problem (1.4.9) is established in a similar way. We show that the solutions of the regularized problem (1.4.19) converge (up to a subsequence) to a solution of the problem (1.4.9). For more details, we refer to the proof in Theorem 4.3.5 and Theorem 4.3.6, Chapter 4.

Remark 1.4.2. *Under the assumption of the Theorem 1.4.7 or Theorem 1.4.8 and, in addition $f_0 \in L^1(0, T; L^\infty(\Omega))$ and $u_0 \in L^\infty(\Omega)$, the strong solution of the problem (1.4.9) is bounded and satisfies the estimate*

$$\|u(\cdot, t)\|_{\infty, \Omega} \leq e^{C_1 t} \|u_0\|_{\infty, \Omega} + e^{C_1 t} \int_0^t e^{-C_1 \tau} \|f_0(\cdot, \tau)\|_{\infty, \Omega} d\tau$$

where $C_1 = 0$ if $b(z) \leq 0$ in Q_T , or $C_1 = \|b\|_{\infty, Q_T}$ if $\sigma \equiv 2$ (see [34, Ch.4, Sec.4.3, Th.4.3]).

1.4.3 Problem 3: Doubly nonlinear equation for $p(x)$ -homogeneous operators

The aim of this part is to study the following Doubly nonlinear parabolic problem mentioned in **Problem 3**

$$\begin{cases} \partial_t(\beta(u)) - \operatorname{div} a(x, \nabla u) = \mathcal{F}(x, t, u), & u > 0 & \text{in } Q_T; \\ u = 0 & & \text{on } \Gamma; \\ u(\cdot, 0) = u_0 & & \text{in } \Omega, \end{cases} \quad (\text{DNE})$$

with the following nonlinear time derivative, and sub-homogeneous and non-monotone forcing terms

$$\beta(u) = \frac{q}{2q-1} u^{2q-1} \quad \text{and} \quad \mathcal{F}(x, t, u) = f(x, u) + h(x, t)u^{q-1}.$$

Denote weighted spaces with the notation $\delta(x) \stackrel{\text{def}}{=} \text{dist}(x, \partial\Omega)$:

$$L_\delta^\infty(\Omega) \stackrel{\text{def}}{=} \{w : \Omega \rightarrow \mathbb{R} \mid \text{measurable, } \frac{w}{\delta(\cdot)} \in L^\infty(\Omega)\}$$

endowed with the norm $\|w\|_\delta = \sup_\Omega \left| \frac{w(x)}{\delta(x)} \right|$ and for $r > 0$:

$$\mathcal{M}_\delta^r(\Omega) \stackrel{\text{def}}{=} \{w : \Omega \rightarrow \mathbb{R}^+ \mid \text{measurable, } \exists c > 0, \frac{1}{c} \leq \frac{w^r}{\delta(x)} \leq c\}.$$

1.4.3.1 Main tools

First, we derive a new version of the Picone identity involving quasilinear elliptic operators with variable exponent with a glimpse of the proof. Precisely, we consider a continuous operator $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ such that $(x, \xi) \rightarrow A(x, \xi)$ is differentiable with respect to variable ξ , and satisfies (A_1) and a weaker condition (than (A_2))

(A_0) $\xi \rightarrow A(x, \xi)$ is strictly convex for any $x \in \Omega$.

Theorem 1.4.9 (Picone identity). *Let $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a continuous and differentiable function satisfying (A_0) and (A_1) . Let $v_0, v \in L^\infty(\Omega)$ belonging to*

$$\dot{V}_+^r \stackrel{\text{def}}{=} \{v : \Omega \rightarrow (0, +\infty) \mid v^{1/r} \in W_0^{1,p(x)}(\Omega)\}$$

for some $r \geq 1$. Then

$$\frac{1}{p(x)} \left\langle \partial_\xi A(x, \nabla v_0^{1/r}), \nabla \left(\frac{v}{v_0^{(r-1)/r}} \right) \right\rangle \leq A^{\frac{r}{p(x)}}(x, \nabla v^{1/r}) A^{\frac{(p(x)-r)}{p(x)}}(x, \nabla v_0^{1/r}) \quad (1.4.26)$$

where $\langle \cdot, \cdot \rangle$ is the inner scalar product and the above inequality is strict if $r > 1$ or $\frac{v}{v_0} \not\equiv \text{Const} > 0$.

To prove the above identity, first we transform the variable homogeneity to constant homogeneity by defining a new class of operators $N_r : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^+$ as

$$N_r(x, \xi) := A^{\frac{r}{p(x)}}(x, \xi) \text{ for any } r \geq 1$$

and the notion of strict ray-convexity:

Definition 1.4.2. *Let X be a real vector space. Let \dot{V} be a non empty cone in X . A function $J : \dot{V} \rightarrow \mathbb{R}$ is ray-strictly convex if for all $v_1, v_2 \in \dot{V}$ and for all $\theta \in (0, 1)$*

$$J((1 - \theta)v_1 + \theta v_2) \leq (1 - \theta)J(v_1) + \theta J(v_2)$$

where the inequality is always strict unless $v_1 = Cv_2$ for some $C > 0$.

By using the convexity and $p(x)$ -homogeneity of the operator A , we prove: for any $x \in \Omega$ the map $\xi \rightarrow N_r(x, \xi) \stackrel{\text{def}}{=} A(x, \xi)^{r/p(x)}$ is positively r -homogeneous and ray-strictly convex. Moreover for $r > 1$, $\xi \rightarrow N_r(\cdot, \cdot)$ is strictly convex (for a detailed proof see Proposition 4.4.1, Page 171, Chapter 4). Now, by exploiting the convexity of the function $N_r(x, \xi)$ with a multiple change of variables we get our Picone identity. As a first application of Picone identity, we extend the famous Díaz-Saá inequality to the class of variable exponent operators. This inequality is strongly linked to the strict convexity of some associated homogeneous energy type functional (see Proposition 4.4.2, Chapter 4 and [151]).

Theorem 1.4.10 (Díaz-Saá inequality). *Let $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a continuous and differentiable function satisfying (A_0) and (A_1) and define $a(x, \xi) = (a_i(x, \xi))_i \stackrel{\text{def}}{=} \left(\frac{1}{p(x)} \partial_{\xi_i} A(x, \xi) \right)_i$. Assume in addition that there exists $\Lambda > 0$ such that*

$$a \in C^1(\Omega \times (\mathbb{R}^N \setminus \{0\}))^N \quad \text{and} \quad \sum_{i,j=1}^N \left| \frac{\partial a_i(x, \xi)}{\partial \xi_j} \right| \leq \Lambda |\xi|^{p(x)-2}$$

for all $(x, \xi) \in \Omega \times \mathbb{R}^N \setminus \{0\}$. Then, we have in the sense of distributions, for any $r \in [1, p^-]$

$$\int_{\Omega} \left(- \frac{\operatorname{div}(a(x, \nabla w_1))}{w_1^{r-1}} + \frac{\operatorname{div}(a(x, \nabla w_2))}{w_2^{r-1}} \right) (w_1^r - w_2^r) dx \geq 0 \quad (1.4.27)$$

for any $w_1, w_2 \in W_0^{1,p(x)}(\Omega)$, positive in Ω such that $\frac{w_1}{w_2}, \frac{w_2}{w_1} \in L^\infty(\Omega)$. Moreover, if the equality occurs in (1.4.27), then w_1/w_2 is constant in Ω . If $p(x) \not\equiv r$ in Ω then even $w_1 = w_2$ holds in Ω .

The proof of the above inequality follows from the Young's inequality and the relation $A(x, \xi) = a(x, \xi) \cdot \xi$ in the Picone identity (1.4.26).

For more application of Picone identity in the study of various anisotropic quasilinear elliptic problems, we refer to Sections 4.4.4 and 4.4.6, Chapter 4.

We study the existence and regularity results for the elliptic problem associated to (DNE). Precisely, we study the following problem

$$\begin{cases} v^{2q-1} - \lambda \nabla \cdot a(x, \nabla v) = h_0(x) v^{q-1} + \lambda f(x, v), & v > 0 \quad \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.4.28)$$

The notion of weak solution of (1.4.28) is defined as follows:

Definition 1.4.3. *A weak solution of (1.4.28) is any nonnegative and nontrivial function $v \in W_0^{1,p(x)}(\Omega) \cap L^{2q}(\Omega)$ such that for any $\phi \in W_0^{1,p(x)}(\Omega) \cap L^{2q}(\Omega)$*

$$\int_{\Omega} v^{2q-1} \phi dx + \lambda \int_{\Omega} a(x, \nabla v) \cdot \nabla \phi dx = \int_{\Omega} h_0 v^{q-1} \phi dx + \lambda \int_{\Omega} f(x, v) \phi dx.$$

The next theorem gives the existence and the uniqueness of the weak solution of (1.4.28).

Theorem 1.4.11. *Assume that A satisfies (A_1) - (A_3) and f satisfies (f_0) and (f_1) . Then, for any $q \in (1, p^-)$, $\lambda > 0$ and $h_0 \in L^\infty(\Omega) \setminus \{0\}$, $h_0 \geq 0$, there exists a weak solution $v \in C^1(\overline{\Omega}) \cap \mathcal{M}_\delta^1(\Omega)$ to (1.4.28).*

Moreover, let v_1, v_2 be two weak solutions to (1.4.28) with $h_1, h_2 \in L^\infty(\Omega) \setminus \{0\}$, $h_1, h_2 \geq 0$ respectively, we have with the notation $t^+ \stackrel{\text{def}}{=} \max\{0, t\}$:

$$\|(v_1^q - v_2^q)^+\|_{L^2} \leq \|(h_1 - h_2)^+\|_{L^2}. \quad (1.4.29)$$

To prove the existence of a weak solution in above result, we investigate the problem (1.4.28) via variational methods and prove the existence of global minimizer of the energy functional $\mathcal{J} : W_0^{1,p(x)}(\Omega) \cap L^{2q}(\Omega) \rightarrow \mathbb{R}$ defined by:

$$\mathcal{J}(v) = \frac{1}{2q} \int_{\Omega} v^{2q} dx + \lambda \int_{\Omega} \frac{A(x, \nabla v)}{p(x)} dx - \frac{1}{q} \int_{\Omega} h_0 (v^+)^q dx - \lambda \int_{\Omega} F(x, v) dx$$

where $F(x, z)$ is the primitive of $f(x, z)$ w.r.t variable z . By constructing a function $w \in W_0^{1,p(x)}(\Omega) \cap L^{2q}(\Omega)$ satisfying $\mathcal{J}(w) < 0$ together with the non-negativity of the potential h_0 implies the non-negativity and non-triviality of the global minimizer. Concerning the $C^{1,\alpha}$ regularity and $\mathcal{M}_\delta^1(\Omega)$ boundary behavior of the weak solution we seek assistance of preliminary Hölder regularity results (see Proposition 4.5.2, Chapter 4 and Theorem 1.2 in [118]), Strong maximum Principle and Hopf lemma (see Lemma 4.5.2. Chapter 4). Furthermore, the Picone identity (Theorem 1.4.9) with the following choice of test functions

$$\phi = \left(v_1 - \frac{v_2^q}{v_1^{q-1}} \right)^+ \quad \text{and} \quad \Psi = \left(v_2 - \frac{v_1^q}{v_2^{q-1}} \right)^-$$

reveals the contraction properties and uniqueness of weak solution. The choice of test functions while applying Picone identity plays a significant role in the computations and their inclusion in the energy space $W_0^{1,p(x)}(\Omega) \cap L^{2q}(\Omega)$ is justified by the boundary behavior of weak solution v_1, v_2 . The contraction property (1.4.29) illustrate the continuous and monotone dependency of the weak solution of elliptic problem (1.4.28) with respect to the potentials (or coefficients).

Now by exploiting the regularity or boundary behavior of the weak solution of (1.4.28), we study the following perturbed problem induced by the operator \mathcal{T}_q which is associated to the parabolic equation (1.4.32):

$$\begin{cases} u + \lambda \mathcal{T}_q u = h_0, & u > 0 & \text{in } \Omega; \\ u = 0 & & \text{on } \partial\Omega. \end{cases} \quad (1.4.30)$$

We prove existence of weak solution, uniqueness and accretivity results for (1.4.30) (see Corollary 4.5.1, Page 212, Chapter 4). We observe that if u_0 is the weak solution of (E), then $v_0 = u_0^q$ is the weak solution of (1.4.30). In addition to this, by using approximating method, we also extend the existence and regularity results for elliptic problems (1.4.28) and (1.4.30) for a larger class of potentials $h_0 \in L^2(\Omega)$. For more details, we refer to Theorems 4.5.5 and Corollary 4.5.2, Section 4.5.2.2, Chapter 4.

1.4.3.2 Main results

The study of (DNE) is naturally concerned with the investigation of the following associated parabolic problem:

$$\begin{cases} v^{q-1} \partial_t(v^q) - \nabla \cdot a(x, \nabla v) = \mathcal{F}(x, t, v), & v > 0 & \text{in } Q_T; \\ v = 0 & & \text{on } \Gamma; \\ v(0, \cdot) = v_0 & & \text{in } \Omega, \end{cases} \quad (\text{E})$$

with $\mathcal{F}(x, t, u) = f(x, u) + h(x, t)u^{q-1}$ and then we further prove that a weak solution of associated parabolic problem (E) is also a weak solution of the main problem (DNE).

Using an identical approach based on nonlinear accretive operators theory as in [39, 146, 147], we introduce $\mathcal{T}_q := \mathcal{D}(\mathcal{T}_q) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ be the operator with the parameter q defined by

$$\mathcal{T}_q u = -u^{(1-q)/q} \left(\nabla \cdot a(x, \nabla(u^{1/q})) + f(x, u^{1/q}) \right)$$

and the associated domain

$$\mathcal{D}(\mathcal{T}_q) = \{w : \Omega \rightarrow \mathbb{R}^+ \mid \text{measurable, } w^{1/q} \in W_0^{1,p(x)}(\Omega) \cap L^{2q}(\Omega), \mathcal{T}_q w \in L^2(\Omega)\}.$$

For understanding the difficulties in solving the associated parabolic problem (E), we first study the problem (E) for $p(x)$ -Laplacian operator and then later generalize it to a class of $p(x)$ -homogeneous operator. Here we directly focused on generalized operator as mentioned in the problem (E) and for the study related to $p(x)$ -Laplacian operator, we refer to Section 4.4.5, Chapter 4.

Before stating the main result for the problem (E), we introduce the notion of weak solution as follows:

Definition 1.4.4. *Let $T > 0$, a weak solution to (E) is any positive function $v \in L^\infty(0, T; W_0^{1,p(x)}(\Omega)) \cap L^\infty(Q_T)$ such that $\partial_t(v^q) \in L^2(Q_T)$ satisfying for any $\phi \in L^2(Q_T) \cap L^1(0, T; W_0^{1,p(x)}(\Omega))$ and for any $t \in (0, T]$*

$$\int_0^t \int_\Omega \partial_t(v^q) v^{q-1} \phi \, dx ds + \int_0^t \int_\Omega a(x, \nabla v) \cdot \nabla \phi \, dx ds = \int_0^t \int_\Omega \mathcal{F}(x, s, v) \phi \, dx ds$$

and $v(0, \cdot) = v_0$ a.e. in Ω .

We prove the following result for (E):

Theorem 1.4.12. *Let $T > 0$ and $q \in (1, p^-)$. Assume A satisfies (A_1) - (A_3) , f satisfies (f_0) - (f_2) and Then, for any $h \in L^\infty(Q_T)$ satisfying (\mathbf{H}_h) and for any initial data $v_0 \in \mathcal{M}_\delta^1(\Omega) \cap W_0^{1,p(x)}(\Omega)$, there exists a solution in sense of Definition 1.4.4. More precisely, we have:*

- (i) there exists $c > 0$ such that for any $t \in [0, T]$, $\frac{1}{c}\delta(x) \leq u(t, x) \leq c\delta(x)$ a.e. in Ω ;
- (ii) Assume in addition A satisfies (A_4) for any $x \in \Omega$ and for any $\xi, \eta \in \mathbb{R}^N$. Then, $v \in C([0, T]; \mathcal{W})$.

The monotone and continuous dependence of weak solution with respect to the initial data and potential (or coefficients) in forcing terms is obtained by the following theorem relaxing the assumptions on v_0 and h . More precisely, we show:

Theorem 1.4.13. *Let v, w be two solutions of (E) in sense of Definition 1.4.4 with respect to the initial data $v_0, w_0 \in L^{2q}(\Omega)$, $v_0, w_0 \geq 0$ and $h, \tilde{h} \in L^2(Q_T)$. Then, for any $t \in [0, T]$:*

$$\|v^q(t) - w^q(t)\|_{L^2(\Omega)} \leq \|v_0^q - w_0^q\|_{L^2(\Omega)} + \int_0^t \|h(s) - \tilde{h}(s)\|_{L^2(\Omega)} ds. \quad (1.4.31)$$

In Theorem 1.4.12, the uniqueness of weak solution is the consequence of Theorem 1.4.13. In sense of Definition 1.4.4, a solution of (E) belongs to $L^\infty(Q_T)$, hence $\frac{q}{2q-1}\partial_t(v^{2q-1}) = v^{q-1}\partial_t(v^q) \in L^2(Q_T)$ holds in weak sense and we deduce the existence of a solution of (DNE).

Another important result of this part is to study the convergence of the weak solution to a steady state. For this, we shift the nonlinearity in the time derivative term to the diffusion term (as in (1.3.3)) in the associated parabolic problem and we prove the following result:

Theorem 1.4.14. *Under the assumptions of Theorem 1.4.12, for any u_0 such that $u_0^{1/q} \in \mathcal{M}_\delta^1(\Omega) \cap W_0^{1,p(x)}(\Omega)$, there exists a unique weak solution $u \in L^\infty(Q_T)$ of*

$$\begin{cases} \partial_t u + \mathcal{T}_q u = h, & u > 0 & \text{in } Q_T; \\ u = 0 & & \text{on } \Gamma; \\ u(0, \cdot) = u_0 & & \text{in } \Omega, \end{cases} \quad (1.4.32)$$

in the sense that:

- (i) $u^{1/q}$ belongs to $L^\infty(0, T; \mathcal{W})$, $\partial_t u \in L^2(Q_T)$;
- (ii) there exists $c > 0$ such that for any $t \in [0, T]$, $\frac{1}{c}\delta^q(x) \leq u(t, x) \leq c\delta^q(x)$ a.e. in Ω ;
- (iii) u satisfies, for any $t \in [0, T]$:

$$\begin{aligned} \int_0^t \int_\Omega \partial_t u \psi \, dx ds + \int_0^t \int_\Omega a(x, \nabla u^{1/q}) \cdot \nabla (u^{\frac{1-q}{q}} \psi) \, dx ds \\ = \int_0^t \int_\Omega f(x, u^{1/q}) u^{\frac{1-q}{q}} \psi \, dx ds + \int_0^t \int_\Omega h(s, x) \psi \, dx ds, \end{aligned} \quad (1.4.33)$$

for any ψ such that

$$|\psi|^{1/q} \in L^\infty(0, T; L_\delta^\infty(\Omega)) \text{ and } \frac{|\nabla \psi|}{\delta^{q-1}(\cdot)} \in L^1(0, T; L^{p(x)}(\Omega)).$$

Moreover, u belongs to $C([0, T]; L^r(\Omega))$ for any $r \in [1, +\infty)$.

Now, we start giving the main ingredients of the proof of the results stated above. To prove the main result Theorem 1.4.12 (and simultaneously Theorem 1.4.14), we use the technique of semi-discretization in time. We stress here that the general form of operators requires to sharply exploit the Picone's identity and demands new compelling estimates to prove the qualitative properties of the weak solution. In this regard, the integrability of the quotient in the choices of test functions forces conditions on the regularity or boundary behaviour of weak solution.

To apply the time-discretization method in our main result Theorem 1.4.12, we define the approximation of the potential h as: let $n^* \in \mathbb{N}$ and set $\Delta_t = T/n^*$. For $n \in \{1, 2, \dots, n^*\} := \llbracket 1, n^* \rrbracket$ we define $t_n = n\Delta_t$ and for $t \in [t_{n-1}, t_n)$ and $x \in \Omega$

$$h_{\Delta_t}(t, x) = h^n(x) \stackrel{\text{def}}{=} \frac{1}{\Delta_t} \int_{t_{n-1}}^{t_n} h(s, x) ds \quad (1.4.34)$$

such that $h_{\Delta_t} \rightarrow h$ in $L^2(Q_T)$. Now, by using Theorem 1.4.11, we define a sequence $\{v_n\}$ such that $v_n \in C^{1,\alpha}(\bar{\Omega}) \cap \mathcal{M}_\delta^1(\Omega)$ is the weak solution of the following implicit Euler scheme via a approximation of h defined in (1.4.34):

$$\left\{ \begin{array}{ll} \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right) v_n^{q-1} - \nabla \cdot a(x, \nabla v_n) = h^n v_n^{q-1} + f(x, v_n) & \text{in } \Omega; \\ v_n > 0 & \text{in } \Omega; \\ v_n = 0 & \text{on } \partial\Omega, \end{array} \right.$$

and two sequences of approximate functions in t :

$$v_{\Delta_t}(x, t) = v_n(x) \quad \text{and} \quad \tilde{v}_{\Delta_t}(x, t) = \frac{t - t_{n-1}}{\Delta_t} (v_n^q(x) - v_{n-1}^q(x)) + v_{n-1}^q(x)$$

which satisfy

$$v_{\Delta_t}^{q-1} \partial_t \tilde{v}_{\Delta_t} - \nabla \cdot a(x, \nabla v_{\Delta_t}) = f(x, v_{\Delta_t}) + h^n v_{\Delta_t}^{q-1}. \quad (1.4.35)$$

To prove the boundary behavior of the parabolic problem (E), first we show there exists $c > 0$ such that

$$\frac{1}{c} \delta(x) \leq v_{\Delta_t}(x, t), \quad \tilde{v}_{\Delta_t}^{1/q}(x, t) \leq c \delta(x) \quad \text{for all } (x, t) \in \Omega \times [0, T]. \quad (1.4.36)$$

In this regard, we construct a subsolution \underline{w} and supersolution \bar{w} in $C^1(\bar{\Omega}) \cap \mathcal{M}_\delta^1(\Omega)$ of suitable quasilinear elliptic equations (for more details see Step 2, Page 216, Chapter 4) such that $v_n \in [\underline{w}, \bar{w}]$ for every $n \in \llbracket 0, n^* \rrbracket$. The existence results (Theorems 1.4.11, 4.4.12 and 4.4.13) in the light of Picone identity facilitate the construction of subsolution \underline{w} and supersolution \bar{w} with $C^{1,\alpha}(\bar{\Omega}) \cap \mathcal{M}_\delta^1(\Omega)$ regularity. By choosing a suitable set of test functions in the Picone identity, and using interpolations inequalities and Arzela-Ascoli Theorem, we

show the following uniform estimates for v_{Δ_t} and \tilde{v}_{Δ_t}

$$\begin{cases} \partial_t \tilde{v}_{\Delta_t} \text{ is bounded in } L^2(Q_T) \text{ uniformly in } \Delta_t, \\ (v_{\Delta_t}), (\tilde{v}_{\Delta_t}^{1/q}) \text{ is bounded in } L^\infty(0, T; W_0^{1,p(x)}(\Omega)) \text{ uniformly in } \Delta_t, \\ \tilde{v}_{\Delta_t} \rightarrow v^q \text{ in } C([0, T]; L^r(\Omega)) \text{ and } v_{\Delta_t} \rightarrow v \text{ in } L^\infty(0, T; L^r(\Omega)), \\ v_{\Delta_t}, \tilde{v}_{\Delta_t}^{1/q} \xrightarrow{*} v \text{ in } L^\infty(0, T; W_0^{1,p(x)}(\Omega)) \text{ as } \Delta_t \rightarrow 0 \text{ and } \partial_t \tilde{v}_{\Delta_t} \rightarrow \partial_t(v^q) \text{ in } L^2(Q_T). \end{cases} \quad (1.4.37)$$

Finally by using (1.4.36), classical compactness argument with [Step 4, Proof of Theorem 1.1, [146]], convergence properties for v_{Δ_t} and \tilde{v}_{Δ_t} in (1.4.37) and ellipticity and growth conditions (A_1) - (A_3) , we pass to the limits $\Delta_t \rightarrow 0$ in (1.4.35) to get the existence of weak solution in the sense of Definition 1.4.4.

Theorem 1.4.13, is proved by taking

$$\phi = \frac{(v + \epsilon)^q - (w + \epsilon)^q}{(v + \epsilon)^{q-1}} \quad \text{and} \quad \Psi = \frac{(w + \epsilon)^q - (v + \epsilon)^q}{(w + \epsilon)^{q-1}}$$

as a test functions in the Definition (1.4.4) and passing limits $\epsilon \rightarrow 0$ using Lebesgue dominated convergence Theorem and regularity of weak solution.

Remark 1.4.3. *We observe that if v is the weak solution of (E) then $w = v^q$ is the weak solution of (1.4.32) in the sense of Definition 1.4.33 (see proof of Theorem 4.5.7 for a detailed explanation).*

Based on the accretive property of \mathcal{T}_q in $L^2(\Omega)$ (see Theorem 4.5.5 and Corollary 4.5.2, Page 213, Chapter 4) and additional regularity on initial data, we obtain the following stabilization result for the weak solutions to (E):

Theorem 1.4.15. *Under the assumptions of Theorem 1.4.12, let v be the weak solution of (E) with the initial data $v_0 \in \mathcal{M}_\delta^1(\Omega) \cap W_0^{1,p(x)}(\Omega)$. Assume that $h \in L^\infty([0, +\infty) \times \Omega)$ satisfying (\mathbf{H}_h) on $[0, +\infty) \times \Omega$ and there exists $h_\infty \in L^\infty(\Omega)$ such that*

$$t^{1+\eta} \|h(t, \cdot) - h_\infty\|_{L^2} = O(1) \quad \text{at infinity for some } \eta > 0. \quad (1.4.38)$$

Then, for any $r \in [1, \infty)$

$$\|v^q(t, \cdot) - v_{stat}^q\|_{L^r} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

where v_{stat} is the unique solution of associated stationary problem with the potential $h_\infty \in L^\infty(\Omega)$.

Now we study the convergence of weak solution of the D.N.E. to a steady state. To this goal, our approach is to use the semigroup theory. Due to the general class of operators, additional technical computations are needed and performed with the help of the above Picone's identity. With both autonomous and non-autonomous terms and the large class of considered operators,

(DNE) covers a large spectrum of physical situations.

We start by proving the existence of unique weak solution $v \in C^1(\bar{\Omega}) \cap \mathcal{M}_\delta^1(\Omega)$ of the following stationary problems associated to the (E) and (1.4.32) via minimization method

$$\begin{cases} -\nabla \cdot a(x, \nabla v) = b(x)v^{q-1} + f(x, v) & \text{in } \Omega; \\ v \geq 0 & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (S)$$

In the same way, we obtain the existence of a unique solution u in $\dot{V}_+^q \cap \mathcal{M}_\delta^{1/q}(\Omega)$ of the following problem

$$\begin{cases} \mathcal{T}_q u = b & \text{in } \Omega; \\ u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (1.4.39)$$

To prove the stabilization property, we divide the proof into two cases when the potential h is a function of x only and when the potential h is a function of both x and t .

For the Case 1, we introduce the family $\{S(t); t \geq 0\}$ defined on $\mathcal{M}_\delta^{1/q}(\Omega) \cap \dot{V}_+^q$ such that $w(t) = S(t)w_0$ where w is the solution obtained by Theorem 1.4.14 for $h = h_\infty$. Uniqueness and regularity results of the weak solution w implies that $\{S(t); t \geq 0\}$ a semi group on $\mathcal{M}_\delta^{1/q}(\Omega) \cap \dot{V}_+^q$. In the light of Remark 1.4.3, we notice that $v = (S(t)w_0)^{1/q}$ is the solution of (E) in the sense of Definition 1.4.4 with $h = h_\infty$ and the initial data $w_0^{1/q}$.

Let $T > 0$ and v be the solution of (E) obtained by Theorem 1.4.12 with $h \equiv h_\infty$ and the initial data v_0 , then $u(t) = v(t)^q = S(t)u_0$ with $u_0 = v_0^q$ is a weak solution of (1.4.32). Then we construct a subsolution \underline{w} and a supersolution \bar{w} of the stationary problem (S) with h_∞ such that $\underline{w} \leq v_0 \leq \bar{w}$. Define $\underline{u}(t) = S(t)\underline{w}^q$ and $\bar{u}(t) = S(t)\bar{w}^q$ the solutions to (1.4.32). Subsolution \underline{u} and supersolution \bar{u} are obtained by the iterative scheme defined in the proof of existence of weak solution (see (4.5.23)) with initial data $v_0 = \underline{w}$ and $v_0 = \bar{w}$ respectively. Hence, by using the monotonicity of the map $t \rightarrow \underline{u}(t)$, and $t \rightarrow \bar{u}(t)$, continuity property of semigroup in $L^2(\Omega)$ and (1.4.31) insures for any $t \geq 0$,

$$\underline{w}^q \leq \underline{u}(t) \leq u(t) \leq \bar{u}(t) \leq \bar{w}^q \text{ a. e. in } \Omega. \quad (1.4.40)$$

$$\underline{u}_\infty = \lim_{s \rightarrow \infty} S(t+s)(\underline{w}^q) = S(t)(\lim_{s \rightarrow \infty} S(s)(\underline{w}^q)) = S(t)\underline{u}_\infty$$

where $\underline{u}_\infty = \lim_{t \rightarrow \infty} \underline{u}(t)$ and $\bar{u}_\infty = \lim_{t \rightarrow \infty} \bar{u}(t)$. Analogously we have $\bar{u}_\infty = S(t)\bar{u}_\infty$. We deduce \underline{u}_∞ and \bar{u}_∞ are solutions of (1.4.39) with $b = h_\infty$ and by uniqueness, we have $u_{\text{stat}} \stackrel{\text{def}}{=} \underline{u}_\infty = \bar{u}_\infty$ where u_{stat} is the stationary solution of perturbed parabolic problem (1.4.39). Therefore from (1.4.40) and interpolation inequality $\|\cdot\|_r \leq \|\cdot\|_\infty^\theta \|\cdot\|_2^{1-\theta}$, we conclude

$$\|u(t) - u_{\text{stat}}\|_{L^r} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

for any $r \geq 1$.

For the case 2: From (1.4.38), for any ε and for some $\eta' \in (0, \eta)$, there exists $t_0 > 0$ large enough such that for any $t \geq t_0$:

$$t^{1+\eta'} \|h(t, \cdot) - h_\infty\|_{L^2} \leq \varepsilon.$$

Let $T > 0$ and v be the solution of (E) obtained by Theorem 1.4.12 with potential h and the initial data $v_0 = u_0^{1/q}$ and we set $u = v^q$. Since $v \in \mathcal{M}_\delta^1(\Omega)$, we can define $\tilde{u}(t) = S(t+t_0)u_0 = S(t)u(t_0)$. Then, by (1.4.31) and uniqueness, we have for any $t > 0$:

$$\|u(t+t_0, \cdot) - \tilde{u}(t, \cdot)\|_{L^2} \leq \int_0^t \|h(s+t_0, \cdot) - h_\infty\|_{L^2} ds \leq \frac{\varepsilon}{t_0^{\eta'}} \leq \varepsilon.$$

By Case 1, we have $\tilde{u}(t) \rightarrow u_{\text{stat}}$ in $L^2(\Omega)$ as $t \rightarrow \infty$. Therefore, we obtain

$$\|u(t) - u_{\text{stat}}\|_{L^2} \rightarrow 0 \text{ as } t \rightarrow \infty$$

and by using interpolation inequality we conclude the proof of Theorem 1.4.15.

Quasilinear Elliptic problem involving Kirchhoff and Choquard non-linearity

In the last few years, nonlinear and nonlocal partial differential equations (PDEs) have attracted a lot of mathematicians due to their appearance in variety of real world phenomenon. In particular, the study of nonlocal elliptic and parabolic PDEs play a vital role in the modelling of various natural processes. Out of many interesting research questions, the fundamental questions are the well posedness of the model, existence and multiplicity (versus uniqueness) of solutions, and in this direction, a considerable amount of results has been obtained in both nonlinear and nonlocal setting.

Recently, a lot of attention has been paid to the study of PDEs involving nonlinear operator like p -Laplacian and its higher order elliptic variants, in the presence of nonlocal terms (like Kirchhoff type non-linearity, Choquard type non-linearity), which give rise to the nonlocal effect in the equation. The importance of studying these type of PDEs provoked from various physical models such as Kirchhoff's model of studying transverse oscillation of the stretched string [177,178], Pekar's model for the quantum theory of the polaron at rest [214], Choquard's model of an electron trapped in its own hole [192], plasma theory of electromagnetic waves [51], Bose-Einstein condensation [103] and many more.

The main theme of this part of the thesis, is to study the existence and multiplicity results for the quasilinear elliptic problems involving the nonlocal Kirchhoff term and exponential non-linearity of Choquard type in the limiting case of Sobolev embedding (*i.e.* $p = n$). We primarily focused on the n -Laplacian and Polyharmonic operators with subcritical and critical exponential non-linearity, that arise out of several Orlicz type embeddings proved by Adams, Trudinger and Moser. For the existence of solution for Kirchhoff problem with Choquard non-linearity, we seek help of variational method in the light of Adams-Moser-Trudinger inequali-

ties, Hardy-Littlewood-Sobolev inequalities, compactness via higher integrability lemma and Mountain pass lemma. To answer the question of multiplicity, we study the convex-concave problem (involving an extra sublinear sign changing term) with nonlocal Kirchhoff term and exponential non-linearity of Choquard type, by Nehari manifold technique. Finally, to study the system of Kirchhoff equations with exponential nonlinearity of Choquard type, we prove a new singular and non-singular version of Adams, Moser and Trudinger inequalities and via variational method we prove the existence of a weak solution.

This chapter includes the results of the following research articles:

- (i) R. Arora, J. Giacomoni, T. Mukherjee and K. Sreenadh, *n-Kirchhoff-Choquard equations with exponential nonlinearity*, *Nonlinear Analysis*, **186** (2019), 113-144.
- (ii) R. Arora, J. Giacomoni, T. Mukherjee, K. Sreenadh, *Polyharmonic Kirchhoff type Choquard equations involving exponential nonlinearity with singular weights*, *Nonlinear Analysis*, **196** (2020), 111779, 24 pp.
- (iii) R. Arora, J. Giacomoni, T. Mukherjee K. Sreenadh, *Adams-Moser-Trudinger inequality in cartesian product of Sobolev space and its applications*, *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, **114** (2020), no. 3, Paper No. 111, 26 pp.

Turning to the layout of this chapter: In section 2.1, we introduce a short description of nonlinear operator and related functions spaces. In Section 2.2, we discuss the source of interest and motivations to study the nonlocal problems and the state of the art. In this regard, we start by stating several inequalities of Adams, Moser, and Trudinger and then present a state of the art for Kirchhoff type problems, Choquard non-linearity and Nehari manifold method. In Section 2.3, we present our main problems. In section 2.4, we study Kirchhoff equations and systems with exponential non-linearity of Choquard type and singular weights. We state our main results and present the main ingredients of their proofs whose expository part of proofs are given in Chapter 5.

2.1 Nonlinear operators and Function spaces

For $m \in \mathbb{N}$, $1 \leq p < \infty$ and $u \in C^m$, the vectorial polyharmonic operator Δ_p^m is defined by induction as

$$\Delta_p^m u = \begin{cases} \nabla \cdot \{ \Delta^{j-1} (|\nabla \Delta^{j-1} u|^{p-2} \nabla \Delta^{j-1} u) \} & \text{if } m = 2j - 1, \\ \Delta^j (|\Delta^j u|^{p-2} \Delta^j u) & \text{if } m = 2j. \end{cases}$$

The symbol $\nabla^m u$ denotes the m^{th} -order gradient of u and is defined as

$$\nabla^m u = \begin{cases} \nabla \Delta^{(m-1)/2} u & \text{if } m \text{ is odd,} \\ \Delta^{m/2} u & \text{if } m \text{ is even} \end{cases}$$

where Δ and ∇ denotes the usual Laplacian and gradient operator respectively, and $\nabla^m u \cdot \nabla^m v$ denotes the product of two vectors when m is odd and the product of two scalars when m is

even.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. The Sobolev space $W^{m,p}(\Omega)$ defined as

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : |\nabla^\alpha u|^p \in L^1(\Omega) \forall |\alpha| \leq m\}$$

endowed with the norm

$$\|u\|_{W^{m,p}(\Omega)} = \left(\sum_{|\alpha| \leq m} \int_{\Omega} |\nabla^\alpha u|^p dx \right)^{\frac{1}{p}}$$

is a Banach space. For $1 \leq p < \infty$, we define $W_0^{m,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in $W^{m,p}(\Omega)$. From Poincaré inequality, we can also define an equivalent norm on $W_0^{m,p}(\Omega)$ as

$$\|u\|_{W_0^{m,p}(\Omega)} = \left(\int_{\Omega} |\nabla^m u|^p dx \right)^{\frac{1}{p}}.$$

In the special case $p = 2$, $W_0^{m,2}(\Omega)$ (or $H_0^m(\Omega)$) becomes a Hilbert space with the inner product

$$\langle u, v \rangle = \int_{\Omega} \nabla^m u \cdot \nabla^m v dx.$$

2.2 Motivation and state of the art

2.2.1 Adams, Moser and Trudinger inequalities

The classical Sobolev space embedding says that

$$W_0^{m,p}(\Omega) \hookrightarrow L^{p^*}(\Omega) \text{ if } n > mp \text{ where } p^* = \frac{np}{n - mp}$$

or equivalently

$$\sup_{\|u\|_{W_0^{m,p}(\Omega)} \leq 1} \int_{\Omega} |u|^r < \infty \text{ for all } 1 \leq r \leq p^*$$

and in the limiting case $mp = n$, $W_0^{m, \frac{n}{m}}(\Omega) \hookrightarrow L^r(\Omega)$ for all $1 \leq r < \infty$ but not embedded in $L^\infty(\Omega)$. The maximal exponent p^* is called as Sobolev critical exponent. Hence, a natural question in connection with Orlicz space embeddings is to find a function $\phi : \mathbb{R} \rightarrow \mathbb{R}^+$ with maximal growth such that

$$\sup_{\|u\|_{W_0^{m, \frac{n}{m}}(\Omega)} \leq 1} \int_{\Omega} \phi(u) dx < \infty.$$

In this connection, in 1960's, Pohozaev [218] and in 1967's, Trudinger [249] independently answered the question for $m = 1$ and $p = n$, using the maximal growth function $\phi(t) = \exp(|t|^{\frac{n}{n-1}}) - 1$.

Later on, in 1971, Moser [207] and in 1984, Cherrier [92, 93] improved the result by proving the inequality in $W_0^{1,n}(\Omega)$ and $W^{1,n}(\Omega)$ respectively with sharp exponents in the maximal growth function and proved the following result:

Theorem 2.2.1. For $n \geq 2$, $u \in W_0^{1,n}(\Omega)$

$$\sup_{\|u\|_{W_0^{1,n}(\Omega)} \leq 1} \int_{\Omega} \exp(\alpha|u|^{\frac{n}{n-1}}) dx < \infty \quad \text{if and only if } \alpha \leq \alpha_n$$

and for $u \in W^{1,n}(\Omega)$

$$\sup_{\|u\|_{W^{1,n}(\Omega)} \leq 1} \int_{\Omega} \exp(\tilde{\alpha}|u|^{\frac{n}{n-1}}) dx < \infty \quad \text{if and only if } \tilde{\alpha} \leq \frac{\alpha_n}{2^{\frac{1}{n-1}}}$$

here $\alpha_n = n\omega_{n-1}^{\frac{1}{n-1}}$ ω_{n-1} = surface area of n -dimensional sphere \mathbb{S}^{n-1} .

From Theorem 2.2.1, it is easy to see that the embedding $W_0^{1,n}(\Omega) \ni u \mapsto \exp(|u|^\beta) \in L^1(\Omega)$ is compact for all $\beta \in [1, \frac{n}{n-1})$ and is continuous for $\beta = \frac{n}{n-1}$. Consequently the map $T : W_0^{1,n}(\Omega) \rightarrow L^q(\Omega)$, for $q \in [1, \infty)$, defined by $T(u) := \exp(|u|^{\frac{n}{n-1}})$, is continuous with respect to the norm topology.

In a further extend, Adams [3] generalized the Moser's inequality to higher order Sobolev spaces by proving the following inequality which is known as Adams-Moser-Trudinger inequality:

Theorem 2.2.2. Let Ω be a bounded domain in \mathbb{R}^n and $n, m \in \mathbb{N}$ satisfying $m < n$. Then for all $0 \leq \zeta \leq \zeta_{n,m}$ and $u \in W_0^{m, \frac{n}{m}}(\Omega)$ we have

$$\sup_{\|\nabla^m u\|_{L^{\frac{n}{m}}(\Omega)} \leq 1} \int_{\Omega} \exp(\zeta|u|^{\frac{n}{n-m}}) dx < \infty,$$

where $\zeta_{n,m}$ is sharp and given by

$$\zeta_{n,m} = \begin{cases} \frac{n}{\omega_{n-1}} \left(\frac{\pi^{n/2} 2^m \Gamma(\frac{m+1}{2})}{\Gamma(\frac{n-m+1}{2})} \right)^{\frac{n}{n-m}} & \text{when } m \text{ is odd,} \\ \frac{n}{\omega_{n-1}} \left(\frac{\pi^{n/2} 2^m \Gamma(\frac{m}{2})}{\Gamma(\frac{n-m}{2})} \right)^{\frac{n}{n-m}} & \text{when } m \text{ is even.} \end{cases}$$

Using the interpolation of Hardy inequality and Moser-Trudinger inequality, Adimurthi-Sandeep [7] established the singular Moser-Trudinger inequality for functions in $W_0^{1,n}(\Omega)$. This was consequently extended by Lam-Lu [183] for functions in $W_0^{m, \frac{n}{m}}(\Omega)$ while proving the following singular Adams-Moser-Trudinger inequality:

Theorem 2.2.3. Let $0 \leq \alpha < n$, Ω be a bounded domain in \mathbb{R}^n and $n, m \in \mathbb{N}$ satisfying $m < n$. Then for all $0 \leq \kappa \leq \kappa_{\alpha,n,m} = (1 - \frac{\alpha}{n}) \zeta_{n,m}$ we have

$$\sup_{u \in W_0^{m, \frac{n}{m}}(\Omega), \|\nabla^m u\|_{L^{\frac{n}{m}}(\Omega)} \leq 1} \int_{\Omega} \frac{\exp(\kappa|u|^{\frac{n}{n-m}})}{|x|^\alpha} dx < \infty.$$

If $\kappa > \kappa_{\alpha,n,m}$ then the above supremum is infinite (i.e. $\kappa_{\alpha,n,m}$ is sharp).

We recall two results from [196] which are known as Lions's higher integrability Lemma.

Lemma 2.2.4. *Let $\{v_k\}$ be a sequence in $W_0^{1,n}(\Omega)$ such that $\|v_k\| = 1$ converging weakly to a non zero $v \in W_0^{1,n}(\Omega)$. Then for every $p \leq (1 - \|v\|^n)^{-\frac{1}{n-1}}$,*

$$\sup_k \int_{\Omega} \exp\left(p\alpha_n |v_k|^{\frac{n}{n-1}}\right) < +\infty.$$

Lemma 2.2.5. *Let $\{v_k\}$ be a sequence in $W_0^{m,2}(\Omega)$ such that $\|v_k\| = 1$ converging weakly to a non zero $v \in W_0^{m,2}(\Omega)$. Then for every $p < (1 - \|v\|^2)^{-1}$,*

$$\sup_k \int_{\Omega} \exp\left(p\zeta_{m,2m} |v_k|^2\right) < +\infty.$$

In recent years, numerous generalizations, extensions and applications of the Moser-Trudinger and Adams-Trudinger-Moser inequalities have been widely explored. A huge amount of literature is available which are devoted to study these kinds of inequalities. We refer readers to [3, 7, 183, 207] for such topics and the survey article [184] including the references within. In the field of geometric analysis curvature for instance prescribed mean curvature problem, Yamabe's problem and partial differential equations where the nonlinear term behaves like $\exp\left(|t|^{\frac{n}{n-m}}\right)$ as $t \rightarrow \infty$, these inequalities play a vital role to carry out the analysis.

2.2.2 Kirchhoff problems

The starting point of studying Kirchhoff problems goes back to 1883, when Kirchhoff established a model governed by the equation

$$(I) : \rho u_{tt} - M\left(\int_0^L |u_x|^2 dx\right) u_{xx} = 0$$

for all $x \in (0, L)$ and $t \geq 0$ and $M(s) = \frac{P_0}{h} + \frac{E}{2L}s$ with the following interpretation of the constants: $u(x, t)$ is the lateral displacement at the coordinate x and time t , L is the length of the string, h is the area of the cross section, E is the Young's modulus of the material, ρ is the mass density and P_0 is the initial axial tension. The model (I) depicts that the transverse oscillations of stretched string with nonlocal flexural rigidity depend continuously on the Sobolev deflection norm of u via M . This model is an extension of classical D'Alembert wave equation, by considering the effects of change in the length of the string during the vibration. Further details and the physical phenomena described by the Kirchhoff's classical theory can be found in [177], [178].

The degenerate Kirchhoff problems *i.e.* $M(0) = 0$ are also very interesting and challenging from a mathematical point of view. The degeneracy in the model (I) implies that the base tension of the string is zero and M measures the change of the tension on the string caused by the change by its length during the vibration. The presence of the nonlinear coefficient

M is crucial and must be considered when the changes in the tension during the motion cannot be neglected. The early classical studies dedicated to Kirchhoff equations were given by Bernstein [53] and Pohozaev [217]. However, model (I) received much attention only after the paper by Lions [195], where an abstract framework to the problem was proposed.

After the appearance of (I), several physicists considered such equations for their research on nonlinear vibrations from both theoretical and experimental points of view. More general versions of these problems are also termed as the Kirchhoff equations and have been extensively studied by many researchers till date. Such equations also appear in biological systems where the function u describes a phenomenon which depends on the average of itself (such as a population density). Consider the following problem

$$(K) : \quad -M\left(\int_{\Omega} |\nabla^m u|^p dx\right) \Delta_p^m u = \lambda f(x, u) + a(x)|u|^{p^*-2}u \text{ in } \Omega \quad u = 0 \text{ on } \partial\Omega$$

where the function f has a suitable growth. Due to the presence of the nonlocal term M , the equation (K) is no longer a pointwise identity which makes the study of such problems more tricky. For $p = 2, m = 1, \lambda = 1$ and $a(x) = 0$, Alves et. al in [18] considered the nonlocal Kirchhoff Laplacian problem (K) with f satisfying sub-critical growth condition at ∞ , and using the truncation arguments and variational method showed the existence of a weak solution. In [99], Corrêa and Figueiredo studied the existence of positive solutions for Kirchhoff equations involving p-laplacian operator with critical or super critical Sobolev type non-linearity and $a(x) = 0$. In [17], Alves et. al considered the above nonlocal problem with $a(x) \not\equiv 0$ and using the Mountain-pass Lemma and the compactness analysis of local Palais-Smale sequences, showed the existence of solutions for large λ . Later on, in [124] for Laplacian operator and in [159] for n -Laplacian operator authors have studied the Kirchhoff problem with critical exponential growth non-linearity. Problems involving polyharmonic operators and polynomial type critical growth non-linearities have been broadly studied by many researchers till now, see [134, 135, 157]. We cite [158, 181, 182] and references therein for existence results on polyharmonic equations with exponential type non-linearity. We cite [4, 16, 18, 90, 122, 123, 186, 204, 220, 254] as references where the Kirchhoff equations for different kind of operators and non-linearities have been treated with no attempt to provide the complete list.

2.2.3 Exponential non-linearity of Choquard type

Let us consider the problem

$$(C) : \quad -\Delta u + V(x)u = (|x|^{-\mu} * F(x, u))f(x, u) \text{ in } \mathbb{R}^n$$

where $\mu \in (0, n)$, F is the primitive of f with respect to the second variable and V, f are continuous functions satisfying certain assumptions. The starting point of studying such problems was the work by S. Pekar (see [214]) in 1954 where he used such equations to

describe the quantum theory of a polaron at rest. Later, P. Choquard (see [192]) in 1976 used it to model an electron trapped in its own hole. The problem (C) also appears when we look for standing waves of the nonlinear and nonlocal Schrödinger equation, which is known to influence the propagation of electromagnetic waves in plasma [51]. Moreover, such problems play a key role in the Bose-Einstein condensation ([103]). To deal with the Choquard non-linearity term, the following Hardy-Littlewood-Sobolev inequality [193] and doubly weighted Hardy-Littlewood-Sobolev [239] inequality play a vital role:

Proposition 2.2.6. (*Hardy-Littlewood-Sobolev inequality*) *Let $t, r > 1$ and $0 < \mu < n$ with $1/t + \mu/n + 1/r = 2$, $f \in L^t(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$. There exists a sharp constant $C(t, n, \mu, r)$, independent of f, h such that*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)h(y)}{|x-y|^\mu} dx dy \leq C(t, n, \mu, r) \|f\|_{L^t(\mathbb{R}^n)} \|h\|_{L^r(\mathbb{R}^n)}. \quad (2.2.1)$$

If $t = r = \frac{2n}{2n-\mu}$ then

$$C(t, n, \mu, r) = C(n, \mu) = \pi^{\frac{n}{2}} \frac{\Gamma(\frac{n}{2} - \frac{\mu}{2})}{\Gamma(n - \frac{\mu}{2})} \left\{ \frac{\Gamma(\frac{n}{2})}{\Gamma(n)} \right\}^{-1 + \frac{\mu}{n}}.$$

In this case there is an equality in (2.2.1) if and only if $f \equiv (\text{constant})h$ and

$$h(x) = A(\gamma^2 + |x - a|^2)^{\frac{-(2n-\mu)}{2}}$$

for some $A \in \mathbb{C}$, $0 \neq \gamma \in \mathbb{R}$ and $a \in \mathbb{R}^n$.

Proposition 2.2.7. (*Doubly weighted Hardy-Littlewood-Sobolev inequality*) *Let $t, r > 1$ and $0 < \mu < n$ with $\alpha + \beta \geq 0$, $\frac{1}{t} + \frac{\mu + \alpha + \beta}{n} + \frac{1}{r} = 2$, $\alpha < \frac{n}{t'}$, $\beta < \frac{n}{r'}$, $f \in L^t(\mathbb{R}^n)$ and $h \in L^r(\mathbb{R}^n)$, where t' and r' denotes the Hölder conjugate of t and r respectively. Then there exists a constant $C(\alpha, \beta, t, n, \mu, r) > 0$ which is independent of f, h such that*

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{f(x)h(y)}{|x-y|^\mu |y|^\alpha |x|^\beta} dx dy \leq C(\alpha, \beta, t, n, \mu, r) \|f\|_{L^t(\mathbb{R}^n)} \|h\|_{L^r(\mathbb{R}^n)}.$$

For recent results involving different kinds of operators and growth conditions, we refer the readers to survey paper on Choquard equations by Moroz and Schaftingen [206], and Tuhina and Sreenadh [208] which cover as extensively as possible the existing literature on this topic.

In the light of Hardy-Littlewood-Sobolev inequalities, Lü [197] studied the following Choquard equation involving Kirchhoff operator

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + (1 + \mu g(x))u = (|x|^{-\alpha} * |u|^p) |u|^{p-2}u \text{ in } \mathbb{R}^3$$

where $a > 0$, $b \geq 0$ are constants, $\alpha \in (0, 3)$, $p \in (2, 6 - \alpha)$, $\mu > 0$ is a parameter and g is a nonnegative continuous potential satisfying some conditions. By using the Nehari manifold

method and the concentration compactness principle [196], he established the existence of ground state solutions when μ is large enough and studied the concentration behavior of these solutions as $\mu \rightarrow +\infty$. Recently, Li et al. [188] studied the existence and the concentration of sign-changing solutions to a class of Kirchhoff-type systems with Choquard-type nonlinearity in \mathbb{R}^3 using minimization argument on the sign-changing Nehari manifold. Pucci et al. [220] also studied the existence of nonnegative solutions of a Schrödinger-Choquard-Kirchhoff type fractional p -equation via variational methods.

An important question that arises now is the case of critical dimension n . But there is not much literature concerning problem (C) when $n = 2$ except the articles by Alves et al. [15] where authors have studied a singularly perturbed nonlocal Schrödinger equation using variational methods in \mathbb{R}^2 .

2.2.4 Nehari Manifold method

Let X be a Banach space and $\mathcal{I} : X \rightarrow \mathbb{R}$ be a C^1 functional. Let u be a non-trivial critical point of the energy function \mathcal{I} *i.e.* $\mathcal{I}'(u) = 0$, then u is necessarily contained in the set

$$\mathcal{N} := \{u \in X \setminus \{0\} : \langle \mathcal{I}'(u), u \rangle = 0\}$$

where \mathcal{I}' is the Fréchet derivative of energy functional \mathcal{I} . The set \mathcal{N} is the natural constraint set for the problem to find non-trivial critical point of \mathcal{I} . The set \mathcal{N} is known as Nehari manifold named after the work of Z. Nehari. We refer to [212] and [243] for a more detailed study.

In [113], authors have studied the associated fiber maps $\Phi_u : \mathbb{R}^+ \rightarrow \mathbb{R}$ defined as $\Phi_u(t) = \mathcal{I}(tu)$ in order to study the geometry of Nehari manifold. We observed that $u \in \mathcal{N}$ if and only if $\Phi'_u(1) = 0$. More generally, $tu \in \mathcal{N}$ if and only if $\Phi'_u(t) = 0$ which means that the elements in \mathcal{N} corresponds to critical point of the fiber maps. Thus, it is natural to split the Nehari manifold \mathcal{N} into three disjoint sets corresponding to local maxima, local minima and saddle points of Φ_u and defined as

$$\mathcal{N}^\pm := \{u \in \mathcal{N} : \Phi''_u(1) \leq 0\} \text{ and } \mathcal{N}^0 := \{u \in \mathcal{N} : \Phi''_u(1) = 0\}.$$

The main idea is to minimize the associated energy functional \mathcal{I} on the Nehari manifold and show that the minimizers are actually the critical points of the energy function \mathcal{I} *i.e.* the Lagrange multipliers is zero. In the last few decades, several authors such as in [19, 22, 72, 73, 113, 164, 255–257] used the Nehari manifold and associated fiber maps approach to study the multiplicity results for semilinear problems involving polynomial type nonlinearity and sign changing weight functions. In [255], Wu studied the multiplicity of weak solution semilinear elliptic equations

$$(K) \quad -M \left(\int_{\Omega} |\nabla u|^p dx \right) \Delta_p u = g(x)|u|^{p-1}u + \lambda f(x)|u|^{q-1}u \quad \text{in } \Omega, \quad q < 1, \quad p < 2^*$$

with sign changing functions f, g and using the method of Nehari manifold proved the existence of at least two weak solution for $M \equiv 1 \equiv g$ and $p = 2$, and in [72] for $p = 2$ and $M \equiv 1$. In [85], authors have studied the Kirchhoff problem (K) for $p = 2$ involving sign changing non-linearities and proved the multiplicity of weak solution via Nehari manifold method and fibering map analysis.

In [113], authors studied the combined effects of convex-concave non-linearities in quasilinear elliptic problems involving p -Laplacian and established the multiplicity of weak solution for Dirichlet boundary conditions and in [19, 164] for Dirichlet-Neumann or Neumann boundary conditions. Problem involving n -Laplace operator with exponential type nonlinearity has been addressed in [159–161]. For the systems of equations involving exponential nonlinearity, we refer to [200] for Laplacian and [149] for fractional laplacian operator.

With extensive research interest for and abundant physical applications of mathematical equations involving nonlocal Kirchhoff term and Choquard non-linearity, it is then natural to investigate the Kirchhoff equation involving a Choquard non-linearity from both mathematical and physical points of view. Motivating from above works, we investigate a class of Kirchhoff equations and systems in the limiting case of Sobolev embedding involving exponential nonlinearity of Choquard type and sign changing nonlinearity for different kind of operators that has not investigated in former contributions.

2.3 Presentation of problems

In this section, we present our main problems of Kirchhoff equations and systems involving exponential nonlinearity of Choquard type, singular weights and sign changing nonlinearity.

2.3.1 Problem 1: Kirchhoff Choquard equation with exponential non-linearity

(a): First, we study the following n -Kirchhoff equation with exponential non-linearity of Choquard type

$$(KC) \begin{cases} -M\left(\int_{\Omega} |\nabla u|^n dx\right) \Delta_n u = \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy\right) f(x, u), & u > 0 & \text{in } \Omega, \\ u = 0 & & \text{on } \partial\Omega, \end{cases}$$

where $\mu \in (0, n)$, Ω is a smooth bounded domain in \mathbb{R}^n , $n \geq 2$, the function F denotes the primitive of f with respect to the second variable (vanishing at 0) and M denotes the Kirchhoff term.

(b): Secondly, we study the higher order elliptic variant of **Problem 1 (a)**. Precisely, we study the following Polyharmonic Kirchhoff type Choquard equation with exponential non-linearity and singular weights:

$$(PKC) \begin{cases} -M \left(\int_{\Omega} |\nabla^m u|^2 dx \right) \Delta^m u = \left(\int_{\Omega} \frac{F(y, u)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u)}{|x|^\alpha} dx, & \text{in } \Omega, \\ u = \nabla u = \dots = \nabla^{m-1} u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $m \in \mathbb{N}$, $n = 2m$, $\mu \in (0, n)$, $0 < \alpha < \min\{\frac{n}{2}, n - \mu\}$, Ω is a bounded domain in \mathbb{R}^n with smooth boundary and the function F denotes the primitive of f with respect to the second variable and M denotes the Kirchhoff term.

Concerning both problems (KC) and (PKC) , we are interested in the existence of a weak solution (in a sense of Definition 2.4.2 and 2.4.14) in the light of Adams, Moser and Trudinger inequalities, and Hardy-Littlewood-Sobolev inequalities and variational techniques.

2.3.2 Problem 2: Kirchhoff Choquard equation with convex-concave type nonlinearity

(a): First, we study the following n -Kirchhoff-Choquard equation with a convex-concave type non-linearity:

$$(\mathcal{P}_{\lambda, M}) \begin{cases} -M(\|u\|^n) \Delta_n u = (|x|^{-\mu} * F(u))f(u) + \lambda h(x)|u|^{q-1}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega \end{cases}$$

where $\mu \in (0, n)$, Ω is a smooth bounded domain in \mathbb{R}^n , $f(s) = s|s|^p \exp(|s|^\beta)$, $0 < q < n - 1 < 2n - 1 < p + 1$, $\beta \in (1, \frac{n}{n-1}]$ and $F(t) = \int_0^t f(s) ds$, $M(t) = at + b$ where $a, b > 0$ and $h \in L^r(\Omega)$, with $r = \frac{p+2}{p-q+1}$, satisfying $h^+ \not\equiv 0$.

(b) Secondly, we study the higher order elliptic variant of **Problem 2 (a)**. Precisely, we investigate the existence of weak solutions of a Kirchhoff type Choquard equation for higher order elliptic operators with convex-concave sign changing non-linearity:

$$(\mathcal{P}_{\lambda, M}) \begin{cases} -M \left(\int_{\Omega} |\nabla^m u|^2 dx \right) \Delta^m u = \lambda h(x)|u|^{q-1}u + \left(\int_{\Omega} \frac{F(u)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f(u)}{|x|^\alpha} & \text{in } \Omega, \\ u = \nabla u = \dots = \nabla^{m-1} u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in \mathbb{R}^n , $n = 2m$, $m \in \mathbb{N}$, $f(s) = s|s|^p \exp(|s|^\gamma)$, $0 < q < 1$, $2 < p$, $\gamma \in (1, 2)$ and $F(t) = \int_0^t f(s) ds$. In this case, we assume $M(t) = at + b$ where $a, b > 0$ and $h \in L^r(\Omega)$ where $r = \frac{p+2}{q+1}$ is such that $h^+ \not\equiv 0$.

Due to the combination of sub homogeneous and super homogeneous term, and corresponding geometry of the energy functionals, we expect the existence of multiple solutions. With respect to the parameter λ , we are concerned to establish

- (i) multiplicity of solutions for subcritical case $\beta \in (0, \frac{n}{n-1})$ and existence of a solution for critical case $\beta = \frac{n}{n-1}$ in the problem $(\mathcal{P}_{\lambda, M})$.
- (ii) existence of multiple solution for the subcritical case $\gamma \in (1, 2)$ in the problem $(\mathcal{P}_{\lambda, M})$.

2.3.3 **Problem 3:** Kirchhoff systems involving exponential non-linearity of Choquard type

In the continuation of previous problems, we study the following doubly nonlocal system of n -Kirchhoff Choquard equations with exponential non-linearity

$$(KCS) \begin{cases} -m(\|(u, v)\|^n) \Delta_n u = \left(\int_{\Omega} \frac{F(y, u, v)}{|x-y|^\mu} dy \right) f_1(x, u, v), & u > 0 & \text{in } \Omega, \\ -m(\|(u, v)\|^n) \Delta_n v = \left(\int_{\Omega} \frac{F(y, u, v)}{|x-y|^\mu} dy \right) f_2(x, u, v), & v > 0 & \text{in } \Omega, \\ u, v = 0 & & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in \mathbb{R}^n , $0 < \mu < n$, function $F : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuously differentiable with respect to second and third variable and of the form $F(x, t, s) = h(x, t, s) \exp(|t|^{\frac{n}{n-1}} + |s|^{\frac{n}{n-1}})$ such that

$$f_1(x, t, s) := \frac{\partial F}{\partial t}(x, t, s), \quad f_2(x, t, s) := \frac{\partial F}{\partial s}(x, t, s)$$

and m denotes the Kirchhoff term.

Concerning the above system of Kirchhoff equations, we are interested in the Moser-Trudinger type inequalities for systems and existence of a non-trivial weak solution under the minimal assumptions on Kirchhoff and Choquard term, and cover both degenerate and non-degenerate cases for the Kirchhoff term.

2.4 New contributions and extensions

In this section, we state the main existence and multiplicity results for the problems presented in Section 2.3 and highlight the main difficulties and ideas for the proof. First we study the existence and multiplicity results for Kirchhoff-Choquard problem involving n -Laplacian operator and by analyzing the crucial points, we extend our study to Kirchhoff-Choquard problem involving higher order elliptic operators. To study the Kirchhoff-Choquard problems, we investigate the variational framework and seek help of the following mountain pass lemma (see [21] or Theorem 2, [203]).

Theorem 2.4.1. *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$. Suppose there exists a neighbourhood U of 0 in E and a positive constant α which satisfy the following conditions:*

- (i) $I(0) = 0$,
- (ii) $I(u) \geq \alpha$ on the boundary of U ,
- (iii) There exists an $e \notin U$ such that $I(e) < \alpha$.

Then for the constant

$$c = \inf_{\gamma \in \Lambda} \max_{u \in \gamma} I(u) \geq \alpha$$

there exists a sequence u_n in E such that

$$I(u_n) \rightarrow c, \quad I'(u_n) \rightarrow 0$$

where $\Lambda = \{g \in C([0, 1], E) : g(0) = 0, g(1) = e\}$.

2.4.1 n-Kirchhoff Choquard equation with exponential non-linearity

In this subsection, we study the existence results for **Problem 1(a)** and denote

$$\|u\| := \left(\int_{\Omega} |\nabla u|^n dx \right)^{1/n}.$$

We start by stating the assumptions on the Kirchhoff and Choquard term present in the **Problem 1(a)**. The function $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function satisfying the following conditions:

(m1) There exists $m_0 > 0$ such that $M(t) \geq m_0$ for all $t \geq 0$ and $\mathcal{M}(t) = \int_0^t M(s) ds$ satisfies

$$\mathcal{M}(t + s) \geq \mathcal{M}(t) + \mathcal{M}(s), \text{ for all } t, s \geq 0.$$

(m2) There exist constants $b_1, b_2 > 0$ and $\hat{t} > 0$ such that for some $r \in \mathbb{R}$

$$M(t) \leq b_1 + b_2 t^r, \text{ for all } t \geq \hat{t}.$$

(m3) The function $\frac{M(t)}{t}$ is non-increasing for $t > 0$.

The assumption (m1) implies the Kirchhoff term M is non-degenerate and its primitive satisfies the super additivity property.

Example 1: An example of a function M satisfying (m1) – (m3) is $M(t) = d_0 + d_1 t^\beta$ for $\beta < 1$ and $d_0, d_1 \geq 0$.

The function $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x, t) = h(x, t) \exp(|t|^{\frac{n}{n-1}})$. In the frame of problem (KC), $h \in C(\bar{\Omega} \times \mathbb{R})$ satisfies the following conditions:

(h1) $h(x, t) = 0$ for $t \leq 0$ and $h(x, t) > 0$ for $t > 0$.

(h2) For any $\epsilon > 0$, $\lim_{t \rightarrow \infty} \sup_{x \in \bar{\Omega}} h(x, t) \exp(-\epsilon |t|^{\frac{n}{n-1}}) = 0$ and $\lim_{t \rightarrow \infty} \inf_{x \in \bar{\Omega}} h(x, t) \exp(\epsilon |t|^{\frac{n}{n-1}}) = \infty$.

(h3) There exists $\ell > \max\{n - 1, \frac{n(r+1)}{2}\}$ such that $t \rightarrow \frac{f(x, t)}{t^\ell}$ is increasing on $\mathbb{R}^+ \setminus \{0\}$, uniformly in $x \in \Omega$ where r is specified in (m2).

(h4) There exist $T, T_0 > 0$ and $\gamma_0 > 0$ such that $0 < t^{\gamma_0} F(x, t) \leq T_0 f(x, t)$ for all $|t| \geq T$ and uniformly in $x \in \Omega$.

The condition (h2) implies that the function f has critical growth of exponential type in the sense of Theorem 2.2.1.

Example 2: An example of a function f satisfying (h1)–(h4) is $f(x, t) = t^{\beta_1} \exp(t^p) \exp(t^{\frac{n}{n-1}})$ for $t \geq 0$ and $f(x, t) = 0$ for $t < 0$ where $0 \leq p < \frac{n}{n-1}$ and $\beta_1 > l - 1$.

Definition 2.4.2. We call a function $u \in W_0^{1,n}(\Omega)$ to be a solution of (KC) if

$$M(\|u\|^n) \int_{\Omega} |\nabla u|^{n-2} \nabla u \cdot \nabla \varphi \, dx = \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \right) f(x, u) \varphi \, dx, \text{ for all } \varphi \in W_0^{1,n}(\Omega). \quad (2.4.1)$$

Concerning the problem (KC), we prove the following main result:

Theorem 2.4.3. Assume (m1)–(m3) and (h1)–(h4) holds. Assume in addition

$$(\text{Compactness condition}) \quad \lim_{s \rightarrow +\infty} \frac{s f(x, s) F(x, s)}{\exp\left(2|s|^{\frac{n}{n-1}}\right)} = \infty, \text{ uniformly in } x \in \bar{\Omega}. \quad (2.4.2)$$

Then the problem (KC) admits a positive weak solution.

The condition (2.4.2) is required to prove the existence of a strongly convergent subsequence of a Palais-Smale sequence if Palais-Smale sequence lies below a critical level. The assumption (2.4.2) depicts that “the perturbation term $h(x, t)$ ” should not be too small. For a detailed analysis on the perturbation term $h(x, t)$, we refer to the seminal work of Adimurthi [4].

We define the energy functional $E : W_0^{1,n}(\Omega) \rightarrow \mathbb{R}$ associated to the problem (KC) as

$$E(u) = \frac{1}{n} \mathcal{M}(\|u\|^n) - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \right) F(x, u) \, dx.$$

Under the assumptions on f , Moser-Trudinger inequality (Theorem 2.2.1) and Hardy- Littlewood -Sobolev inequality (Proposition 2.2.6) imply that E is well defined in $W_0^{1,n}(\Omega)$. For a detailed explanation, we refer to Page 231, Chapter 5. Also $E \in C^1(W_0^{1,n}(\Omega), \mathbb{R})$. Naturally, the critical points of E corresponds to weak solutions of (KC) and for any $u \in W_0^{1,n}(\Omega)$ we have

$$\langle E'(u), \varphi \rangle = M(\|u\|^n) \int_{\Omega} |\nabla u|^{n-2} \nabla u \cdot \nabla \varphi \, dx - \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \right) f(x, u) \varphi \, dx$$

for all $\varphi \in W_0^{1,n}(\Omega)$.

To show, the energy functional E satisfies the conditions of mountain pass theorem, we first study the mountain pass geometry of the energy functional E :

Lemma 2.4.4. Assume (m1), (m2) and (h1)–(h4) holds. Then, E has the mountain pass geometry around 0 i.e.

- (i) there exists $R_0 > 0, \eta > 0$ such that $E(u) \geq \eta$ for all $u \in W_0^{1,n}(\Omega)$ such that $\|u\| = R_0$.
 (ii) there exists a $v \in W_0^{1,n}(\Omega)$ with $\|v\| > R_0$ such that $E(v) < 0$.

The proof of the mountain pass geometry of E requires precise estimates of Kirchhoff and Choquard term with exponential non-linearities. The Choquard term is handled by choosing $\|u\| = R_0$ small enough with R_0 depending upon the sharp exponent α_n in Moser-Trudinger and μ in Hardy-Littlewood-Sobolev inequalities, and Kirchhoff term by the non-degeneracy assumption (m1) and the growth condition (m2) (with a detailed proof we refer to Lemma 5.1.9, Page 233, Chapter 5).

Let $\Gamma = \{\gamma \in C([0, 1], W_0^{1,n}(\Omega)) : \gamma(0) = 0, E(\gamma(1)) < 0\}$ and define the Mountain Pass critical level as

$$l^* = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} E(\gamma(t)).$$

Then by using Ekeland principle and deformation lemma (Theorem 2.4.1), we have the existence of minimizing Palais-Smale sequence $u_n \in W_0^{1,n}(\Omega)$ such that

$$E(u_n) \rightarrow l^*, \quad E'(u_n) \rightarrow 0.$$

Moreover, the non-degeneracy of the Kirchhoff term and suitable lower growth rate of the function f in the Choquard term (precisely (h3)) implies that every Palais-Smale sequence is bounded in $W_0^{1,n}(\Omega)$ (for a detailed proof see Lemma 5.1.10, Page 234, Chapter 5)

To prove the existence of non-trivial weak solution, we need to identify the mountain pass critical level below which the Palais-Smale condition is satisfied. In this regard, we prove the following lemma identifying the first critical level:

Lemma 2.4.5. *If (2.4.2) holds, then*

$$0 < l^* < \frac{1}{n} \mathcal{M} \left(\left(\frac{2n - \mu}{2n} \alpha_n \right)^{n-1} \right).$$

To prove the above result, implicitly, we consider the sequence of Moser functions $\{w_k\}$ defined as dilations and truncations of the fundamental solution:

$$w_k(x) = \frac{1}{\omega_{n-1}^{\frac{1}{n}}} \begin{cases} (\log k)^{\frac{n-1}{n}}, & 0 \leq |x| \leq \frac{\rho}{k}, \\ \frac{\log\left(\frac{\rho}{|x|}\right)}{(\log k)^{\frac{1}{n}}}, & \frac{\rho}{k} \leq |x| \leq \rho, \\ 0, & |x| \geq \rho \end{cases}$$

such that $\text{supp}(w_k) \subset B_\rho(0)$ and by using the geometric properties of the energy functional E and (2.4.2), we show that there exists a $k \in \mathbb{N}$ large enough such that

$$\max_{t \in [0, \infty)} E(tw_k) < \frac{1}{n} \mathcal{M} \left(\left(\frac{2n - \mu}{2n} \alpha_n \right)^{n-1} \right)$$

and which is sufficient to prove our claim (for a detailed explanation, we refer to Page 236, Chapter 5).

Now, to study the compactness of Palais Smale sequences for the energy functional E , we prove a set of convergence lemmas which prevent concentration phenomenon of the Palais-Smale sequence to occur and imply the weak convergence of Choquard term involving exponential non-linearities. The following lemma is very crucial for the convergence of Palais-Smale sequences to a weak solution and appeals new estimates for the convergence of the Choquard term.

Lemma 2.4.6. *If $\{u_k\}$ denotes a Palais Smale sequence then up to a subsequence, there exists $u \in W_0^{1,n}(\Omega)$ such that*

$$u_k \rightharpoonup u \text{ weakly in } W_0^{1,n}(\Omega), \quad |\nabla u_k|^{n-2} \nabla u_k \rightharpoonup |\nabla u|^{n-2} \nabla u \text{ weakly in } (L^{\frac{n}{n-1}}(\Omega))^n. \quad (2.4.3)$$

Moreover, for all $\phi \in C_c^\infty(\Omega)$

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(x, u_k) \phi \, dx \rightarrow \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \right) f(x, u) \phi \, dx,$$

and

$$\left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) F(x, u_k) \rightarrow \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \right) F(x, u) \text{ in } L^1(\Omega).$$

Due to the nonlinear nature of the operator $(-\Delta)_n$ the proof of (2.4.3) involves a delicate analysis of the Palais-Smale sequence $\{u_k\}$ for the energy functional E over $W_0^{1,n}(\Omega)$. Here, we sketch the main ingredient of the proof. Showing the boundedness property of the Palais-Smale sequence, there exists $u \in W_0^{1,n}(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $W_0^{1,n}(\Omega)$ and a non-negative radon measure ν and $v \in (L^{\frac{n}{n-1}}(\Omega))^n$ such that up to a subsequence

$$|u_k|^n + |\nabla u_k|^n \xrightarrow{*} \nu \text{ in } C(\bar{\Omega})^* \text{ and } |\nabla u_k|^{n-2} \nabla u_k \rightarrow v \text{ weakly in } (L^{\frac{n}{n-1}}(\Omega))^n \text{ as } k \rightarrow \infty.$$

Now to prove our claim, it is enough to show that $\nabla u_k \rightarrow \nabla u$ a.e. in Ω and $v = |\nabla u|^{n-2} \nabla u$. For this, we set $\sigma > 0$ and $X_\sigma = \{x \in \bar{\Omega} : \nu(B_r(x) \cap \bar{\Omega}) \geq \sigma, \text{ for all } r > 0\}$ and in order to prevent energy concentration phenomenon we show the following:

- (i) X_σ must be a finite set i.e $X_\sigma = \{x_1, x_2, \dots, x_m\}$
- (ii) For any $\sigma > 0$ below the ‘‘first concentration energy level’’ i.e. $\sigma < \frac{1}{2} \left(\frac{(2n-\mu)\alpha_n}{2n} \right)^{n-1}$ and for any K compact subset of $\bar{\Omega} \setminus X_\sigma$ we have

$$\lim_{k \rightarrow \infty} \int_K \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(x, u_k) u_k \, dx = \int_K \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \right) f(x, u) u \, dx.$$

- (iii) For $\delta > 0$ fixed small enough such that $B(x_i, \delta) \cap B(x_j, \delta) = \emptyset$ if $i \neq j$ and $\Omega_\delta = \{x \in \bar{\Omega} : |x - x_j| \geq \delta, j = 1, 2, \dots, m\}$

$$\int_{\Omega_\delta} (|\nabla u_k|^{n-2} \nabla u_k - |\nabla u|^{n-2} \nabla u) (\nabla u_k - \nabla u) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

To prove (i), we proceed by contradiction. Consider a sequence of distinct points $\{x_k\}$ in X_σ such that for all $r > 0$, $\nu(B_r(x_k) \cap \bar{\Omega}) \geq \sigma$ for all k . This implies that $\nu(\{x_k\}) \geq \sigma$ for all k , hence $\nu(X_\sigma) = +\infty$. But this is a contradiction to

$$\nu(X_\sigma) = \lim_{k \rightarrow \infty} \int_{X_\sigma} |u_k|^n + |\nabla u_k|^n dx \leq C.$$

To show (ii), we choose a finite covering $\{B_{r_i}(x_i)\}_{i \in I}$ of K such that

$$\int_{B_{\frac{r_i}{2}}(x_i) \cap \bar{\Omega}} |u_k|^n + |\nabla u_k|^n \leq \sigma(1 - \epsilon). \quad (2.4.4)$$

for large enough $k \in \mathbb{N}$, $\epsilon > 0$ small enough and $I = \{0, 1, \dots, l\}$ is the index set. Using (2.4.4), Vitali convergence Theorem and asymptotic growth of f , we get

$$\|f(\cdot, u_k)\|_{L^q(B_{r_0}(x_0) \cap \bar{\Omega})} \leq C_0, \quad \int_{\Omega} \frac{F(y, u)}{|x - y|^\mu} dy \in L^\infty(\Omega) \quad (2.4.5)$$

for some appropriate $q > 1$ and

$$\int_{B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}} \left| \left(\int_{\Omega} \frac{F(y, u)}{|x - y|^\mu} dy \right) (f(x, u_k)u_k - f(x, u)u) \right| dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

To complete the proof of (ii), we vigorously exploit the semigroup property of the Riesz potential in the light of Moser-Trudinger inequality in $W_0^{1,n}(\Omega)$ and $W^{1,n}(\Omega)$, and prove that

$$\begin{aligned} & \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k) - F(y, u)}{|x - y|^\mu} dy \right) \chi_{B_{\frac{r_0}{2}} \cap \bar{\Omega}}(x) f(x, u_k)u_k dx \\ & \leq \left(\int_{\Omega} \left(\int_{\Omega} \frac{|F(y, u_k) - F(y, u)| dy}{|x - y|^\mu} \right) |F(x, u_k) - F(x, u)| dx \right)^{\frac{1}{2}} \\ & \quad \times \left(\int_{\Omega} \left(\int_{\Omega} \chi_{B_{\frac{r_0}{2}} \cap \bar{\Omega}}(y) \frac{f(y, u_k)u_k}{|x - y|^\mu} dy \right) \chi_{B_{\frac{r_0}{2}} \cap \bar{\Omega}}(x) f(x, u_k)u_k dx \right)^{\frac{1}{2}} \rightarrow 0. \end{aligned} \quad (2.4.6)$$

Precisely, boundedness property of Palais-Smale sequence u_k and Lebesgue dominated convergence theorem implies the first integral in the R.H.S. of (2.4.6) tends to 0 and the second integral is uniformly bounded because σ lies below the first concentration energy level and (2.4.5).

To prove (iii) and weak convergence in Choquard terms, we choose special type of test functions $\phi_1 = \psi_\epsilon u_k$ and $\phi_2 = \psi_\epsilon u$ in the following inequality

$$\left| M(\|u_k\|^n) \int_{\Omega} |\nabla u_k|^{n-2} \nabla u_k \nabla \phi - \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x - y|^\mu} dy \right) f(x, u_k) \phi dx \right| \leq \epsilon_k \|\phi\|$$

where $\psi \in C_c^\infty(\mathbb{R}^n)$, $\psi = 0$ in $\Omega \setminus \Omega_{\frac{\delta}{2}}$, $\psi = 1$ in Ω_δ and with the help of strict convexity of the function $g(t) = |t|^n$ and ϵ -Young inequality (the classical arguments of Lemma 4 in [203]) we get

$$\lim_{k \rightarrow \infty} \sup M(\|u_k\|^n) \int_{\Omega} |\nabla u_k|^{n-2} \nabla u_k \cdot \nabla \psi_\epsilon(u_k - u) \leq 0. \quad (2.4.7)$$

Now, by using the the fact that $M(\|u_k\|^n)$ is bounded and $u_k \rightharpoonup u$ in $W_0^{1,n}(\Omega)$, we get

$$\lim_{k \rightarrow \infty} M(\|u_k\|^n) \int_{\Omega} |\nabla u|^{n-2} \nabla u \cdot (\nabla u - \nabla u_k) \psi_{\epsilon} \rightarrow 0. \quad (2.4.8)$$

Finally, by combining the estimate (2.4.7), (2.4.8) and by taking $K = \Omega_{\frac{\delta}{2}}$ in (ii), we get $\nabla u_k \rightarrow \nabla u$ a.e. in Ω and hence the our claim (2.4.3) (for a detailed insight see Lemma 5.1.12, Lemma 5.1.13 and Lemma 5.1.14, Chapter 5).

Now, we sketch the main ingredient of the proof of our main existence result Theorem 2.4.3: For a Palais Smale sequence $\{u_k\}$ at the level l^* and using Lemma 2.4.6, there exists a $u_0 \in W_0^{1,n}(\Omega)$ such that up to a subsequence $u_k \rightharpoonup u_0$ weakly in $W_0^{1,n}(\Omega)$ as $k \rightarrow \infty$. Now, we divide the proof into two main steps:

Step 1: u_0 is the weak solution.

The proof of this step is rather technical. First, we define the associated Nehari manifold as

$$\mathcal{N} = \{u \in W_0^{1,n}(\Omega) \setminus \{0\} : \langle E'(u), u \rangle = 0\} \text{ and } l^{**} = \inf_{u \in \mathcal{N}} E(u).$$

Using the fact that $l^* \leq l^{**}$ (see Lemma 5.1.15, Chapter 5), weak lower semi-continuity of the norm, by the monotonicity and continuity of Kirchhoff term and from convergence Lemma 2.4.6, we prove the following inequalities:

$$\langle E'(u_0), u_0 \rangle \geq 0, \quad E(u_0) \leq l^*. \quad (2.4.9)$$

Now to discard the case $E(u_0) < l^*$ we discover a contradiction. Precisely, if $E(u_0) < l^*$ holds then the monotonicity of Kirchhoff term gives

$$\lim_{k \rightarrow \infty} \mathcal{M}(\|u_k\|^n) > \mathcal{M}(\|u_0\|^n) \Rightarrow \tau^n := \lim_{k \rightarrow \infty} \|u_k\|^n > \|u_0\|^n.$$

Now we define the sequence $v_k = \frac{u_k}{\|u_k\|}$ and $v_0 = \frac{u_0}{\tau}$ then $v_k \rightharpoonup v_0$ weakly in $W_0^{1,n}(\Omega)$ and $\|v_0\| < 1$. From Lions higher integrability lemma (Lemma 2.2.4) and (2.4.9) we obtain

$$\int_{\Omega} \exp\left(\frac{2n}{2n-\mu} q |u_k|^{\frac{n}{n-1}}\right) \leq C, \quad \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^{\mu}} dy\right) f(x, u_k)(u_k - u_0) dx \rightarrow 0.$$

for some constant $C > 0$. Using $\langle E'(u_k), u_k - u_0 \rangle \rightarrow 0$, $M(t) \geq m_0$ and monotonicity of the operator, we get $u_k \rightarrow u$ in $W_0^{1,n}(\Omega)$ which further conclude that $\|u_k\| \rightarrow \|u_0\|$ and a contradiction to the fact $\|v_0\| < 1$. Therefore $E(u_0) = l^*$ and $\mathcal{M}(\tau^n) = \mathcal{M}(\|u_0\|^n)$ which shows that $\tau^n = \|u_0\|^n$ and hence claim.

Step 2: Positivity of u_0

The upper bound of the mountain pass critical level (Lemma 2.4.5) and Choquard convergence estimates (Lemma 2.4.6) induces the uniform bound of $\|u_k\|$ and L^q norm of $f(x, u_k)$ for some $q \geq \frac{2n}{2n-\mu}$, which further in the light of positivity of mountain pass critical level l^* implies $u_0 \not\equiv 0$ and by testing equation (2.4.1) with $\phi = u_0^-$, we get $u_0 \geq 0$ in Ω . Finally, elliptic regularity results and strong maximum principle implies $u_0 > 0$. A detailed proof is given in Theorem 5.1.5, Chapter 5.

2.4.2 Multiplicity of solutions via Nehari Manifold method

In this subsection, we state results and their main ideas of the proof for the problem $(\mathcal{P}_{\lambda,M})$ mentioned as **Problem 2(a)** in Section 2.3. and denote

$$\|u\| := \left(\int_{\Omega} |\nabla u|^n dx \right)^{1/n}.$$

We start by defining the notion of solution for $(\mathcal{P}_{\lambda,M})$ as:

Definition 2.4.7. *A function $u \in W_0^{1,n}(\Omega)$ is said to be weak solution of $(\mathcal{P}_{\lambda,M})$ if $\forall \phi \in W_0^{1,n}(\Omega)$ we have*

$$M(\|u\|^n) \int_{\Omega} |\nabla u|^{n-2} \nabla u \cdot \nabla \phi dx = \lambda \int_{\Omega} h(x) u^{q-1} u \phi dx + \int_{\Omega} (|x|^{-\mu} * F(u)) f(u) \phi dx.$$

The energy functional $\mathcal{J}_{\lambda,M} : W_0^{1,n}(\Omega) \rightarrow \mathbb{R}$ associated to the problem $(\mathcal{P}_{\lambda,M})$ is defined as

$$\mathcal{J}_{\lambda,M}(u) = \frac{1}{n} \mathcal{M}(\|u\|^n) - \frac{\lambda}{q+1} \int_{\Omega} h(x) |u|^{q+1} dx - \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(u)) F(u) dx$$

where $|x|^{-\mu} * F(u)$ denotes $\int_{\Omega} \frac{F(u(y))}{|x-y|^{\mu}} dy$, and F, \mathcal{M} are anti-derivatives of f, M (vanishing at 0) respectively. Using Hardy-Littlewood-Sobolev inequality (Proposition 2.2.6) and Moser-Trudinger inequality (Theorem 2.2.1), we can see that the energy functional $\mathcal{J}_{\lambda,M}$ is well defined.

Using the Nehari manifold technique, we show existence and multiplicity of solutions of the problem $(\mathcal{P}_{\lambda,M})$ with respect to the parameter λ . Precisely, we show the following main results in the subcritical case (local multiplicity):

Theorem 2.4.8. *Let $\beta \in \left(1, \frac{n}{n-1}\right)$. Then there exists λ_0 such that $(\mathcal{P}_{\lambda,M})$ admits at least two non-trivial solutions for $\lambda \in (0, \lambda_0)$.*

In the critical case, we show the following existence result:

Theorem 2.4.9. *Let $\beta = \frac{n}{n-1}$, then there exists $\lambda_1 > 0$ such that for $\lambda \in (0, \lambda_1)$, $(\mathcal{P}_{\lambda,M})$ admits a non-trivial solution.*

In order to prove the existence of weak solutions to $(\mathcal{P}_{\lambda,M})$, we establish the existence of minimizers of $\mathcal{J}_{\lambda,M}$ under the natural constraint of the Nehari Manifold:

$$N_{\lambda,M} := \{u \in W_0^{1,n}(\Omega) \mid \langle \mathcal{J}'_{\lambda,M}(u), u \rangle = 0\}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $W_0^{1,n}(\Omega)$ and $W^{-1,n}(\Omega)$. Therefore, $u \in N_{\lambda,M}$ if and only if

$$\|u\|^n M(\|u\|^n) - \lambda \int_{\Omega} h(x) u^{q+1} dx - \int_{\Omega} (|x|^{-\mu} * F(u)) f(u) u dx = 0.$$

For $u \in W_0^{1,n}(\Omega)$, we define the fiber map $\Phi_{u,M} : \mathbb{R}^+ \rightarrow \mathbb{R}$ as

$$\Phi_{u,M}(t) = \mathcal{J}_{\lambda,M}(tu).$$

By observing the fact that, the Nehari Manifold is closely related to the the maps $\Phi_{u,M}$ by the relation $tu \in N_{\lambda,M}$ iff $\Phi'_{u,M}(t) = 0$, we study the geometry of the energy functional on the following components of the Nehari Manifold:

$$N_{\lambda,M}^{\pm} := \{u \in N_{\lambda,M} : \Phi''_{u,M}(1) \leq 0\} = \{tu \in W_0^{1,n}(\Omega) : \Phi'_{u,M}(t) = 0, \Phi''_{u,M}(t) \leq 0\},$$

$$N_{\lambda,M}^0 := \{u \in N_{\lambda,M} : \Phi''_{u,M}(1) = 0\} = \{tu \in W_0^{1,n}(\Omega) : \Phi'_{u,M}(t) = 0, \Phi''_{u,M}(t) = 0\}.$$

Due to the presence of the sign changing non-linearity, we define $H(u) = \int_{\Omega} h|u|^{q+1} dx$ and study the behaviour of fibering maps $\Phi_{u,M}$ according to the sign of $H(u)$. We split the study into two different cases $u \in H_0^-$ and $u \in H^+$ where

$$H^+ := \{u \in W_0^{1,n}(\Omega) : H(u) > 0\}, \quad H_0^- := \{u \in W_0^{1,n}(\Omega) : H(u) \leq 0\}.$$

We define the map $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$\psi_u(t) = t^{n-1-q} M(\|tu\|^n) \|u\|^n - t^{-q} \int_{\Omega} (|x|^{-\mu} * F(tu)) f(tu) u dx$$

and observing the fact that $tu \in N_{\lambda,M}$ if and only if $t > 0$ is a solution of $\psi_u(t) = \lambda \int_{\Omega} h(x)|u|^{q+1} dx$. In this regard, we prove the following result which totally describes the geometry of the fiber map:

- Lemma 2.4.10.** (i) For any $u \in H_0^- \setminus \{0\}$ and $\lambda > 0$, there exists a unique $t^* > 0$ such that $t^*u \in N_{\lambda,M}^-$. Moreover, $\Phi_{u,M}$ is increasing on $(0, t^*)$ and decreasing on (t^*, ∞) .
 (ii) For any $\lambda \in (0, \lambda_0)$ and $u \in H^+$, there exist $t_*, t_1, t_2 > 0$ such that $t_1u \in N_{\lambda,M}^+$ and $t_2u \in N_{\lambda,M}^-$ for any and $t_1 < t_* < t_2$.

As an application of Lemma 2.4.10, we also prove the non-existence of non-trivial solutions in the degenerate component $N_{\lambda,M}^0$ i.e. $N_{\lambda,M}^0 = \{0\}$. (For a detailed proof we refer to Lemma 5.1.20, Page 251, Chapter 5)

Now to prove the existence of first solution, we extract a nearly minimizing sequence $\{u_k\}$ in $N_{\lambda,M} \setminus \{0\}$ in the following sense:

$$\begin{cases} \mathcal{J}_{\lambda,M}(u_k) \leq \theta + \frac{1}{k}, & \theta = \inf_{u \in N_{\lambda,M}} \mathcal{J}_{\lambda,M}(u); \\ \mathcal{J}_{\lambda,M}(v) \geq \mathcal{J}_{\lambda,M}(u_k) - \frac{1}{k} \|u_k - v\| & \forall v \in N_{\lambda,M}. \end{cases}$$

For extraction of the sequence, we study the geometric structure of the energy functional $\mathcal{J}_{\lambda,M}$. Precisely, we prove $\mathcal{J}_{\lambda,M}$ is bounded below with precise bounds of θ and then Ekeland variational principle implies the required sequence. Now to show that sequence $\{u_k\}$ obtained

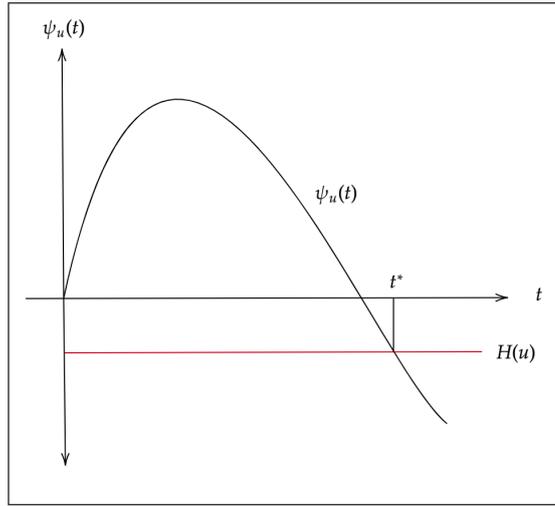


Figure 2.1: Geometry of $\psi_u(t)$ for $u \in H_0^-$

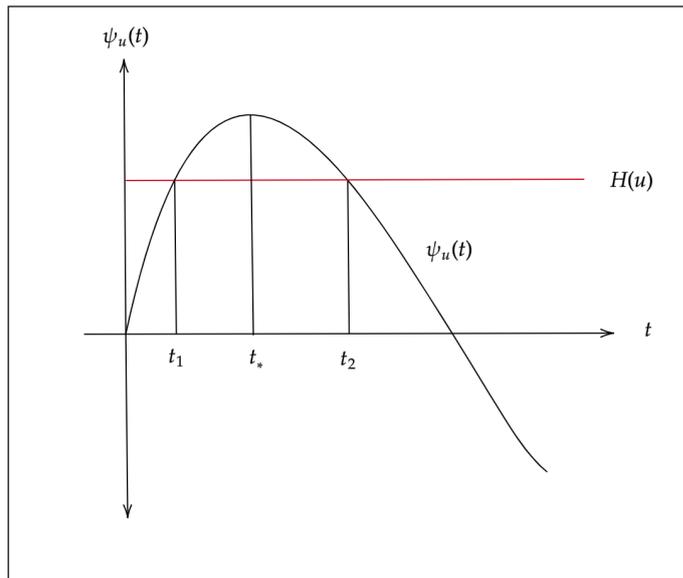


Figure 2.2: Geometry of $\psi_u(t)$ for $u \in H^+$

above satisfies $\|\mathcal{J}'_{\lambda,M}(u_k)\|_* \rightarrow 0$ as $k \rightarrow \infty$ in both subcritical and critical case, we prove the following preliminary lemma that gives a local parameterization of $N_{\lambda,M}$ via the Implicit Function theorem:

Lemma 2.4.11. *Let $\lambda \in (0, \lambda_0)$. Then given any $u \in N_{\lambda,M} \setminus \{0\}$, there exists $\epsilon > 0$ and a differentiable function $\xi : B(0, \epsilon) \subset W_0^{1,n}(\Omega) \rightarrow \mathbb{R}$ such that $\xi(0) = 1$, and $\xi(w)(u-w) \in N_{\lambda,M}$*

and for all $w \in W_0^{1,n}(\Omega)$

$$\langle \xi'(0), w \rangle = \frac{n(2a\|u\|^n + b) \int_{\Omega} |\nabla(u)|^{n-2} \nabla u \cdot \nabla w \, dx - \lambda(q+1) \int_{\Omega} h(x)|u|^{q-1}uw \, dx - \langle S(u), w \rangle}{a(2n-1-q)\|u\|^{2n} + b(n-1-q)\|u\|^n + R(u)}$$

Moreover, for $u \in N_{\lambda,M}^- \setminus \{0\}$, there exists $\epsilon > 0$ and a differentiable function $\xi^- : B(0, \epsilon) \subset W_0^{1,n}(\Omega) \rightarrow \mathbb{R}$ such that $\xi^-(0) = 1$, and $\xi^-(w)(u-w) \in N_{\lambda,M}^-$ and for all $w \in W_0^{1,n}(\Omega)$

$$\begin{aligned} & \langle (\xi^-)'(0), w \rangle \\ &= \frac{n(2a\|u\|^n + b) \int_{\Omega} |\nabla(u)|^{n-2} \nabla u \cdot \nabla w \, dx - \lambda(q+1) \int_{\Omega} h(x)|u|^{q-1}uw \, dx - \langle S(u), w \rangle}{a(2n-1-q)\|u\|^{2n} + b(n-1-q)\|u\|^n + R(u)} \end{aligned}$$

where

$$R(u) = \int_{\Omega} (|x|^{-\mu} * F(u))(qf(u) - f'(u) \cdot u) \cdot u \, dx - \int_{\Omega} (|x|^{-\mu} * f(u) \cdot u)f(u)u \, dx$$

and

$$\langle S(u), w \rangle = \int_{\Omega} (|x|^{-\mu} * F(u))(f'(u)u + f(u))w \, dx + \int_{\Omega} (|x|^{-\mu} * f(u)u)f(u)w \, dx.$$

Using the assumption (m1) of the Kirchhoff term and the property $\|\mathcal{J}'_{\lambda,M}(u_k)\|_* \rightarrow 0$ as $k \rightarrow \infty$, we prove the existence of a positive solution in subcritical case in $N_{\lambda,M}^+ \cap H^+$. For the second solution, we re-investigate the geometry of energy functional $\mathcal{J}_{\lambda,M}$ over $N_{\lambda,M}^-$ in the light of Lemmas 2.4.10-2.4.11 and extract a nearly minimizing sequence $\{v_k\}$ in $N_{\lambda,M}^-$ with $\theta^- = \min_{v \in N_{\lambda,M}^-} \mathcal{J}_{\lambda,M}(v)$ and $\|\mathcal{J}'_{\lambda,M}(v_k)\|_* \rightarrow 0$ as $k \rightarrow \infty$ using the fact that $N_{\lambda,M}^-$ is closed. For a detailed explanation of the proof, we refer to Lemma 5.1.28, Page 259, Chapter 5.

In the critical case, we study the following compactness result for a Palais-Smale below a prescribed critical level:

Lemma 2.4.12. *There exists $C = C(p, q, n) > 0$ such that for any $\{u_k\} \subset W_0^{1,n}(\Omega)$ satisfying*

$$\mathcal{J}'_{\lambda,M}(u_k) \rightarrow 0 \quad \text{and} \quad \mathcal{J}_{\lambda,M}(u_k) \rightarrow c \leq \frac{m_0}{2n} \left(\frac{\alpha_n}{2^{\frac{1}{n-1}}} \left(\frac{2n-\mu}{2n} \right) \right)^{n-1} - C\lambda^{\frac{2(p+2)}{2p+3-q}} \quad \text{as } k \rightarrow \infty$$

is relatively compact in $W_0^{1,n}(\Omega)$.

Here, we shortly sketch the main details of the proof: Using the boundedness of the Palais-Smale sequence $\{u_k\}$, there exist two radon measures ν_1, ν_2 and $u \in W_0^{1,n}(\Omega)$ such that

$$u_k \rightharpoonup u \text{ in } W_0^{1,n}(\Omega), |\nabla u_k|^n \rightarrow \nu_1 \text{ and } (|x|^{-\mu} * F(u_k))f(u_k)u_k \rightarrow \nu_2.$$

Again, we prove that the concentration phenomenon cannot occur at level c: For this first we derive following relations between two measures and convergence in Choquard term

$$\nu_2(A) \geq m_0\nu_1(A) \geq m_0 \left(\frac{\alpha_n}{2^{\frac{1}{n-1}}} \left(\frac{2n-\mu}{2n} \right) \right)^{n-1},$$

$$\frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(u_k)) f(u_k) u_k \, dx \rightarrow \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(u)) f(u) u \, dx + \frac{\nu_2(A)}{2}$$

where $A = \overline{\Omega} \setminus B$ is a finite set with

$$B = \left\{ x \in \overline{\Omega} : \exists r = r_x > 0, \nu_1(B_r \cap \Omega) < \left(\frac{\alpha_n}{2^{\frac{1}{n-1}}} \left(\frac{2n - \mu}{2n} \right) \right)^{n-1} \right\}.$$

Then by using similar arguments as in Lemma 2.4.6, we get a necessary condition for the upper bound of critical level to get Palais Smale condition to hold. A detailed proof is presented in Lemma 5.1.29, Page 260, Chapter 5.

2.4.3 Extensions: Polyharmonic Kirchhoff Choquard equations with singular weights

In this subsection, we study the **Problem 1(b)** and **Problem 2(b)**. We denote

$$\|u\| = \left(\int_{\Omega} |\nabla^m u|^2 \, dx \right)^{\frac{1}{2}}.$$

Problem 1(b)

We assume the following conditions on M and f for the **Problem 1(b)**. The function $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function satisfying the conditions (m1) – (m3) and the function $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ which governs the Choquard term is given by $f(x, t) = h(x, t) \exp(t^2)$, where $h \in C(\Omega \times \mathbb{R})$ satisfies (h1), (h4) and the following growth conditions:

(h5) For any $\epsilon > 0$, $\limsup_{t \rightarrow \infty} \sup_{x \in \overline{\Omega}} h(x, t) \exp(-\epsilon t^2) = 0$ and $\liminf_{t \rightarrow \infty} \inf_{x \in \overline{\Omega}} h(x, t) \exp(\epsilon t^2) = \infty$.

(h6) There exists $\ell > \max\{1, r + 1\}$ such that $\frac{f(x, t)}{t^\ell}$ is increasing for each $t > 0$ uniformly in $x \in \Omega$, where r is specified in (m2).

Example 2.4.13. A typical example of f satisfying (h1), (h4)–(h6) is $f(x, t) = t^{\beta+1} \exp(t^p) \exp(t^2)$ for $t \geq 0$ and $f(x, t) = 0$ for $t < 0$ where $0 \leq p < 2$ and $\beta > l - 1$.

The notion of weak solution for (PKC) is given as follows.

Definition 2.4.14. A weak solution of (PKC) is a function $u \in W_0^{m,2}(\Omega)$ such that for all $\varphi \in W_0^{m,2}(\Omega)$, it satisfies

$$M(\|u\|^2) \int_{\Omega} \nabla^m u \cdot \nabla^m \varphi \, dx = \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|y|^\alpha |x - y|^\mu} \, dy \right) \frac{f(x, u)}{|x|^\alpha} \varphi \, dx.$$

The problem (PKC) has a variational structure and the energy functional $\mathcal{J} : W_0^{m,2}(\Omega) \rightarrow \mathbb{R}$ associated to (PKC) is given by

$$\mathcal{J}(u) = \frac{1}{2} \mathcal{M}(\|u\|^2) - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|y|^\alpha |x - y|^\mu} \, dy \right) \frac{F(x, u)}{|x|^\alpha} \, dx.$$

The energy functional \mathcal{J} is well defined due to Doubly weighted Hardy-Littlewood-Sobolev inequality (Proposition 2.2.7) and Adams-Moser-Trudinger inequality (Theorem 2.2.2). We prove the following main result concerning the problem (PKC).

Theorem 2.4.15. *Let M satisfies (m1)-(m3) and f satisfies (h1), (h4)-(h6). Assume in addition*

$$\text{(Compactness condition)} \quad \lim_{s \rightarrow +\infty} \frac{sf(x, s)F(x, s)}{\exp(2s^2)} = \infty, \text{ uniformly in } x \in \bar{\Omega}. \quad (2.4.10)$$

Then the problem (PKC) admits a non-trivial weak solution.

We establish the existence of a nontrivial weak solution for the problem (PKC) using the same techniques introduced in section above. The presence of higher order derivatives and singular weights in Choquard term require a precise investigation of the mountain pass geometry of the energy functional \mathcal{J} and mountain pass critical level in the light of Adams functions, Adams-Moser-Trudinger (Theorem 2.2.2) and doubly weighted Hardy-Littlewood-Sobolev (Proposition 2.2.7) inequalities. First, we study the mountain pass geometry of the energy functional \mathcal{J} around 0 and using the properties of the nonlocal term M and the exponential growth of f , we prove that every Palais Smale sequence is bounded. For more details, we refer to Lemma 5.2.7, Lemma 5.2.8, Section 5.2.2.1, Chapter 5.

To study the compactness of Palais Smale sequences for \mathcal{J} , identify the mountain pass first critical level with the help of Adams functions which play an equivalent role of Moser functions and establish the convergence of weighted Choquard term for Palais-Smale sequences whose energy level is strictly below the first critical level.

We define the mountain pass critical level as

$$l^* = \inf_{\vartheta \in \Gamma} \max_{t \in [0,1]} \mathcal{J}(\vartheta(t)),$$

where $\Gamma = \{\vartheta \in C([0,1], W_0^{m,2}(\Omega)) : \vartheta(0) = 0, \mathcal{J}(\vartheta(1)) < 0\}$. Using Adams functions and [182, Lemma 5, p. 895], we construct a sequence of test functions to analyze the first critical level. Let \mathcal{B} denotes the unit ball and \mathcal{B}_l is the ball with center 0 and radius l in \mathbb{R}^n . Without loss of generality, we can assume that $\mathcal{B}_l \subset \Omega$, then we have the following result: For $l \in (0, 1)$, there exists

$$U_l \in \{u \in W_0^{m,2}(\Omega) : u|_{\mathcal{B}_l} = 1\}$$

such that

$$\|U_l\|^2 = C_{m,2}(\mathcal{B}_l; \mathcal{B}) \leq \frac{\zeta_{m,2m}}{n \log\left(\frac{1}{l}\right)}$$

where $C_{m,2}(K, E)$ is the conductor capacity of K in E whenever E is an open set and K is relatively compact subset of E and $C_{m,2}(K; E) \stackrel{\text{def}}{=} \inf\{\|u\|^2 : u \in C_0^\infty(E), u|_K = 1\}$.

Let $\sigma > 0$, $k \in \mathbb{N}$, then we consider the Adams functions

$$A_k(x) = \begin{cases} \left(\frac{n \log(k)}{\zeta_{m,2m}} \right)^{\frac{1}{2}} U_{\frac{1}{k}} \left(\frac{x}{\sigma} \right) & \text{if } |x| < \sigma, \\ 0 & \text{if } |x| \geq \sigma, \end{cases}$$

with $A_k(0) = \left(\frac{n \log(k)}{\zeta_{m,2m}} \right)^{\frac{1}{2}}$ and $\|A_k\| \leq 1$.

By using the geometric characteristics of the energy functional \mathcal{J} , we identify the mountain pass critical level strictly below which Palais-Smale condition holds. In this regard, we prove the following result:

Theorem 2.4.16. *Under the assumption (2.4.10), (m1) – (m3) and (h1), (h4) – (h6),*

$$0 < l^* < \frac{1}{2} \mathcal{M} \left(\frac{2n - (2\alpha + \mu)}{2n} \zeta_{m,2m} \right).$$

We also prove two similar convergence lemma (as in the previous subsection) which are essential, while passing to the weak limits in the Choquard term. The presence of singular weights with exponential non-linearity make the proof a bit trickier and involving:

Lemma 2.4.17. *Let $\{u_k\} \subset W_0^{m,2}(\Omega)$ be a Palais Smale sequence for \mathcal{J} at $c \in \mathbb{R}$ then there exists a $u \in W_0^{m,2}(\Omega)$ such that as $k \rightarrow \infty$ (up to a subsequence)*

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{f(x, u_k)}{|x|^{\alpha}} \phi dx \rightarrow \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{f(x, u)}{|x|^{\alpha}} \phi dx$$

for all $\phi \in C_c^{\infty}(\Omega)$ and

$$\left(\int_{\Omega} \frac{F(y, u_k)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{F(x, u_k)}{|x|^{\alpha}} \rightarrow \left(\int_{\Omega} \frac{F(y, u)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{F(x, u)}{|x|^{\alpha}} \text{ in } L^1(\Omega).$$

Finally, the proof of the main result Theorem 2.4.15 follows from the Higher integrability lemma (Lemma 2.2.5), Lemma 2.4.17 and precise estimates of the mountain pass critical level (Theorem 2.4.16). For a detailed proof we refer to Section 5.2.2, Chapter 5.

Problem 2(b):

For the problem $(\mathcal{P}_{\lambda, \mathcal{M}})$, the energy functional $\mathcal{J}_{\lambda, \mathcal{M}} : W_0^{m,2}(\Omega) \rightarrow \mathbb{R}$ associated to the problem $(\mathcal{P}_{\lambda, \mathcal{M}})$ is defined as

$$\mathcal{J}_{\lambda, \mathcal{M}}(u) = \frac{1}{2} \mathcal{M}(\|u\|^2) - \frac{\lambda}{q+1} \int_{\Omega} h(x)|u|^{q+1} dx - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{F(u)}{|x|^{\alpha}} dx$$

where F and \mathcal{M} are primitive of f and M respectively vanishing at 0. Similarly as definition 2.4.14

Definition 2.4.18. A function $u \in W_0^{m,2}(\Omega)$ is said to be a weak solution of $(\mathcal{P}_{\lambda,\mathcal{M}})$ if for all $\phi \in W_0^{m,2}(\Omega)$, it satisfies

$$M(\|u\|^2) \int_{\Omega} \nabla^m u \cdot \nabla^m \phi \, dx = \lambda \int_{\Omega} h(x) |u|^{q-1} u \phi \, dx + \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^\mu |y|^\alpha} \, dy \right) \frac{f(u)}{|x|^\alpha} \phi \, dx.$$

For the problem $(\mathcal{P}_{\lambda,\mathcal{M}})$, we have the following result concerning the subcritical case:

Theorem 2.4.19. *There exists a $\lambda_0 > 0$ such that for $\gamma \in (1, 2)$ and $\lambda \in (0, \lambda_0)$, $(\mathcal{P}_{\lambda,\mathcal{M}})$ admits atleast two solutions.*

To prove the existence and multiplicity result of the convex-concave problem, we follow the same approach of Nehari manifold method as we done above for n -Laplace operator. The singular weights in the Choquard term is handled by doubly weighted Hardy-Littlewood-Sobolev inequality (Proposition 2.2.7) and exponential non-linearity by Adams-Moser-Trudinger inequality (Lemma 2.2.4). The presence of singular weights with Choquard type exponential non-linearity demands careful analysis of the geometry of the energy functional and fiber maps. To avoid the recurrence of similar ideas and computations, we refer the precise results and their complete proofs in Section 5.2.3, Chapter 5.

Remark 2.4.1. *We conjecture that for critical case, the multiplicity results holds by additional estimates on the level of minimizing sequence.*

2.4.4 Adams-Moser-Trudinger inequalities for Cartesian product of Sobolev space

To study the Kirchhoff system with exponential non-linearity of Choquard type in **Problem 3**, Section 2.3, first we need to prove the non-singular version of Moser-Trudinger and Adams-Moser-Trudinger inequalities in higher dimensional product spaces. Let

$$\mathcal{Y} := W_0^{m, \frac{n}{m}}(\Omega) \times W_0^{m, \frac{n}{m}}(\Omega)$$

be the Banach space endowed with the norm

$$\|(u, v)\|_{\mathcal{Y}} := \left(\|u\|_{W_0^{m, \frac{n}{m}}(\Omega)}^{\frac{n}{m}} + \|v\|_{W_0^{m, \frac{n}{m}}(\Omega)}^{\frac{n}{m}} \right)^{\frac{m}{n}}$$

where $\|u\|_{W_0^{m, \frac{n}{m}}(\Omega)}^{\frac{n}{m}} := \int_{\Omega} |\nabla^m u|^{\frac{n}{m}} \, dx$. We prove the following result:

Theorem 2.4.20. *For $(u, v) \in \mathcal{Y}$, $n, m \in \mathbb{N}$ such that $n \geq 2m$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain, we have*

$$\int_{\Omega} \exp \left(\Theta \left(|u|^{\frac{n}{n-m}} + |v|^{\frac{n}{n-m}} \right) \right) \, dx < \infty$$

for any $\Theta > 0$. Moreover,

$$\sup_{\|(u,v)\|_{\mathcal{Y}}=1} \int_{\Omega} \exp \left(\Theta \left(|u|^{\frac{n}{n-m}} + |v|^{\frac{n}{n-m}} \right) \right) \, dx < \infty \text{ if and only if } \Theta \leq \frac{\zeta_{n,m}}{2_{n,m}}$$

where $2_{n,m} = 2^{\frac{n-2m}{n-m}}$ and $\xi_{n,m}$ are defined in Theorem 2.2.2.

As an consequence the sharp Theorem 2.4.20, we prove the following version of Lions' Lemma [196] in the product space \mathcal{Y} .

Theorem 2.4.21. *Let $(u_k, v_k) \in \mathcal{Y}$ such that $\|(u_k, v_k)\|_{\mathcal{Y}} = 1$ for all k and $(u_k, v_k) \rightharpoonup (u, v) \neq (0, 0)$ weakly in \mathcal{Y} . Then for all $p < \frac{\zeta_{n,m}}{2_{n,m}(1 - \|(u, v)\|_{\mathcal{Y}}^{\frac{m}{n-m}})}$,*

$$\sup_{k \in \mathbb{N}} \int_{\Omega} \exp\left(p\left(|u_k|^{\frac{n}{n-m}} + |v_k|^{\frac{n}{n-m}}\right)\right) dx < \infty.$$

Next, we prove the singular version of Moser-Trudinger inequality in the Cartesian product of Sobolov spaces when $m = 1$.

Theorem 2.4.22. *For $(u, v) \in \mathcal{Y} = W_0^{1,n}(\Omega) \times W_0^{1,n}(\Omega)$, $n \geq 2$, $\lambda \in [0, n)$ and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, we have*

$$\int_{\Omega} \frac{\exp(\beta(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}}))}{|x|^{\lambda}} dx < \infty$$

for any $\beta > 0$. Moreover,

$$\sup_{\|(u,v)\|_{\mathcal{Y}}=1} \int_{\Omega} \frac{\exp(\beta(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}}))}{|x|^{\lambda}} dx < \infty \text{ if and only if } \frac{2_n \beta}{\alpha_n} + \frac{\lambda}{n} \leq 1$$

where $2_n := 2_{n,1} = 2^{\frac{n-2}{n-1}}$ and α_n is defined in Theorem 2.2.1.

Similarly we can prove singular and non-singular Moser-Trudinger inequalities in the product space $\mathcal{Z} := W^{1,n}(\Omega) \times W^{1,n}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain endowed with the norm

$$\|(u, v)\|_{\mathcal{Z}} := \left(\|u\|_{W^{1,n}(\Omega)}^n + \|v\|_{W^{1,n}(\Omega)}^n \right)^{\frac{1}{n}}$$

where $\|u\|_{W^{1,n}(\Omega)}^n := \int_{\Omega} (|u|^n + |\nabla u|^n) dx$. Precisely we establish the following result.

Theorem 2.4.23. *For $(u, v) \in \mathcal{Z}$, $n \geq 2$, $\lambda \in [0, n)$ and $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, we have*

$$\int_{\Omega} \frac{\exp(\tilde{\beta}(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}}))}{|x|^{\lambda}} dx < \infty$$

for any $\tilde{\beta} > 0$. Moreover,

$$\sup_{\|(u,v)\|_{\mathcal{Z}}=1} \int_{\Omega} \frac{\exp(\tilde{\beta}(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}}))}{|x|^{\lambda}} dx < \infty \text{ if and only if } \frac{2\tilde{\beta}}{\alpha_n} + \frac{\lambda}{n} \leq 1$$

where α_n is defined in Theorem 2.2.1.

The main ingredients in the proof of above results are Lemma 5.3.1, Page 287, Chapter 5, generalized Hölder inequality and existing Adams, Moser and Trudinger inequalities mentioned in Section 2.2.1. Moreover, to prove the sharpness of the exponent in various Theorem, we construct the sequence of functions using Adams and Moser functions such that supremum of the required integrals is not finite. For detailed proof the above inequalities, we refer to Section 5.3, Page 286, Chapter 5.

2.4.5 Kirchhoff systems with Choquard non-linearity

In this part, we study the system of Kirchhoff equations mentioned in **Problem 3**, Section 2.3. We start by stating the assumptions required for the existence of a solution.

Let $m : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying (m1), (m3) and

(m4) There exist constants $c_0, c_1, c_2 > 0$ and $\tilde{t} > 0$ such that for some $r, z \in \mathbb{R}^+$

$$m(t) \geq c_0 \quad \text{or} \quad m(t) \geq t^z, \quad \text{for all } t \geq 0$$

and

$$m(t) \leq c_1 + c_2 t^r, \quad \text{for all } t \geq \tilde{t}.$$

The assumption (m4) covers both degenerate and non-degenerate cases for the Kirchhoff term.

Let the function $F : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuously differentiable with respect to second and third variable and of the form $F(x, t, s) = h(x, t, s) \exp(|t|^{\frac{n}{n-1}} + |s|^{\frac{n}{n-1}})$ such that

$$f_1(x, t, s) := \frac{\partial F}{\partial t}(x, t, s) = h_1(x, t, s) \exp(|t|^{\frac{n}{n-1}} + |s|^{\frac{n}{n-1}}),$$

$$f_2(x, t, s) := \frac{\partial F}{\partial s}(x, t, s) = h_2(x, t, s) \exp(|t|^{\frac{n}{n-1}} + |s|^{\frac{n}{n-1}}).$$

We assume h_i 's for $i = 1, 2$ are continuous functions satisfying the following conditions-

(f1) $h_i(x, t, s) = 0$ when either $t \leq 0$ or $s \leq 0$ and $h_i(x, t, s) > 0$ when $t, s > 0$, for all $x \in \Omega$ and $i = 1, 2$.

(f2) For any $\epsilon > 0$ and $i = 1, 2$

$$\lim_{t, s \rightarrow \infty} \sup_{x \in \Omega} h_i(x, t, s) \exp(-\epsilon(|t|^{\frac{n}{n-1}} + |s|^{\frac{n}{n-1}})) = 0,$$

$$\lim_{t, s \rightarrow \infty} \inf_{x \in \Omega} h_i(x, t, s) \exp(\epsilon(|t|^{\frac{n}{n-1}} + |s|^{\frac{n}{n-1}})) = \infty.$$

(f3) There exists

$$l > \begin{cases} \max \left\{ n-1, \frac{n(r+1)}{2} \right\} & \text{when } m \text{ is non-degenerate,} \\ \max \left\{ n-1, \frac{n(z+1)}{2}, \frac{n(r+1)}{2} \right\} & \text{when } m \text{ is degenerate.} \end{cases}$$

such that the maps $t \mapsto \frac{f_1(x,t,s)}{|t|^l}$, $s \mapsto \frac{f_2(x,t,s)}{|s|^l}$ are increasing functions of t (uniformly in s and x) and s (uniformly in t and x) respectively.

(f4) There exist $q, s_0, t_0, M_0 > 0$ such that $s^q F(x, t, s) \leq M_0 f_2(x, t, s)$ for all $s \geq s_0$ and $t^q F(x, t, s) \leq M_0 f_1(x, t, s)$ for all $t \geq t_0$ uniformly in $x \in \Omega$.

(f5) There exists a γ satisfying $\frac{n-2}{2} < \gamma$ such that $\lim_{(t,s) \rightarrow (0,0)} \frac{f_i(x,t,s)}{s^{\gamma+t^\gamma}} = 0$ holds for $i = 1, 2$.

The assumptions (f1) – (f5) for the problem (KCS) are the system analogue of the assumptions (h1) – (h4) for the problem (KC). The assumption (f2) implies that functions f_1 and f_2 have critical growth.

Let $\mathcal{P} := W_0^{1,n}(\Omega) \times W_0^{1,n}(\Omega)$ endowed with the graph norm

$$\|(u, v)\| := \left(\|u\|_{W_0^{1,n}(\Omega)}^n + \|v\|_{W_0^{1,n}(\Omega)}^n \right)^{\frac{1}{n}}$$

where $\|u\|_{W_0^{1,n}(\Omega)}^n := \int_{\Omega} |\nabla u|^n dx$. Following is the notion of weak solution for (KCS).

Definition 2.4.1. A function $(u, v) \in \mathcal{P}$ is said to be weak solution of (KCS) if for all $(\phi, \psi) \in \mathcal{P}$, it satisfies

$$\begin{aligned} m(\|u, v\|^n) & \left(\int_{\Omega} |\nabla u|^{n-2} \nabla u \nabla \phi dx + \int_{\Omega} |\nabla v|^{n-2} \nabla v \nabla \psi dx \right) \\ & = \int_{\Omega} \left(\int_{\Omega} \frac{F(x, u, v)}{|x-y|^\mu} dy \right) (f_1(x, u, v)\phi + f_2(x, u, v)\psi) dx. \end{aligned}$$

We define the energy functional J on \mathcal{P} as

$$J(u, v) = \frac{1}{n} M(\|u, v\|^n) - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u, v)}{|x-y|^\mu} dy \right) F(x, u, v) dx.$$

The energy functional J is well defined because of Hardy-Littlewood-Sobolev inequality (see Proposition 2.2.6) and new version of Adams, Moser and Trudinger inequalities (see Theorem 2.4.20).

Concerning the problem (KCS), we prove the following existence result:

Theorem 2.4.24. Let m satisfies (m1), (m3), (m4) and f satisfies (f1) – (f5) and the **Compactness condition**

$$\lim_{t,s \rightarrow \infty} \frac{(f_1(x, t, s)t + f_2(x, t, s)s)F(x, t, s)}{\exp(q(|t|^{\frac{n}{n-1}} + |s|^{\frac{n}{n-1}}))} = \infty \text{ uniformly in } x \in \bar{\Omega}. \quad (2.4.11)$$

for some $q \geq 2$. Then there exists a positive weak solution of the problem (KCS).

We generalize the approach of variational method used above for Kirchhoff-Choquard equation to system of Kirchhoff-Choquard equations. To prove the existence of solution of the problem (KCS), first we study the mountain pass geometry of the energy functional J around

$(0, 0)$ in which Choquard term is handled by new Adams-Moser-Trudinger inequality in the Cartesian product of Sobolev space (Theorem 2.4.20) and Hardy-Littlewood-Sobolev inequality (Proposition 2.2.6), and Kirchhoff term by non-degenerate or degenerate assumption and growth condition (m_4) (see Lemma 5.4.1, Lemma 5.4.2, Chapter 5). To analyze accurately the compactness of Palais Smale sequences for J , we show a series of Lemmas consisting of weak convergence of Palais-Smale sequence in appropriate energy space, convergence in Choquard term involving exponential non-linearity and to identify the first critical level under the **Compactness condition** (2.4.11) (see Theorem 5.4.3, Theorem 5.4.4 and Theorem 5.4.5 in Chapter 5). For a detailed insight, we refer to the results in Section 5.4.2, Chapter 5.

The results of this chapter can be extended in various directions. We have mentioned a few of them in Section 5.5, Chapter 5.

Non-local singular problems

Non-local elliptic equations involving general integral differential operators as the fractional Laplacian have been studied for many years by an important number of researchers and a vast amount of work is present in the literature dealing with existence and regularity results. This kind of problems appears in several physical models like combustion, crystals, dislocations in mechanical systems and many other problems where anomalous diffusion or/and interaction with long range come into picture. The study of fractional and singular problems have been investigated more recently and have an intrinsic mathematical interest since in the local setting, it appears in several physical models like non Newtonian flows in porous media, heterogeneous catalysts.

The main course of this chapter is to study singular problems involving nonlocal operators. The investigation of singular problems are divided into two parts depending upon the nature of the operator. The main crux of this chapter is twofold.

Firstly, we investigate the $\frac{1}{2}$ -Laplacian singular problem (P_λ) (see below) involving critical exponential non-linearity in one dimension. The study of the problem (P_λ) is motivated from the Moser-Trudinger inequality and specific representation of Green Formula in case of Half Laplacian operator. Using bifurcation theory in the framework of weighted spaces, sub-supersolution method and barrier arguments, we show the existence of a connected unbounded branch of classical solution (see Definition 3.4.2) that admits an asymptotic bifurcation point (*i.e.* bifurcation from infinity). For that we need to establish Hölder regularity and asymptotic behavior of the solution. In order to characterize the blow up behavior of weak solution at an asymptotic bifurcation point, we study the isolated singularities for the singular semi linear elliptic equation, and symmetry and monotonicity of classical solution via moving plane method and the narrow maximum principle in the context of nonlocal operators. In a different extent, global multiplicity of weak solution is proved via Variational method. Furthermore,

symmetry and monotonicity results are extended to a general class of nonlocal operators that is of independent interest.

Secondly, we study the nonlinear fractional Laplacian elliptic problem (see (P) below) involving purely singular nonlinearity and weights in a smooth bounded domain $\Omega \subset \mathbb{R}^N$. Using approximation method and Hardy inequality, we prove the existence of minimal weak solution (see Definition 3.4.1). In order to investigate the asymptotic behavior of minimal weak solution, we exploit the $C^{1,1}$ regularity of the boundary via barrier arguments. In addition, we prove a new weak comparison principle adopted in the setting of very weak solution and as a consequence, we prove the uniqueness and non-existence results depending upon the singular exponent δ and γ .

This chapter includes the results of following research articles:

- (i) R. Arora, J. Giacomoni, D. Goel and K. Sreenadh, *Positive solutions of 1-D Half-Laplacian equation with singular exponential nonlinearity*, *Asymptot. Anal.*, **1** (2019) 1-34.
- (ii) R. Arora, J. Giacomoni, D. Goel and K. Sreenadh, *Symmetry of solutions to singular fractional elliptic equations and applications*, *Comptes Rendus Mathématique*, Volume 358, issue 2 (2020), p. 237-243.
- (iii) R. Arora, J. Giacomoni and G. Warnault, *Nonlinear fractional Laplacian problem with singular nonlinearity and singular weights*, (submitted).

Turning to the layout of this chapter: In section 3.1, we introduce non-local operators and a framework of functions spaces. In section 3.2, we present our main problems of this Chapter. In section 3.3, we present a state of art on singular problems. In section 3.4, we develop new tools to tackle the problem mentioned in Section 3.2 and state the main results with a short glimpse of a proof.

3.1 Function spaces

In this section, we introduce the non-local operator $(-\Delta)_p^s$ known as p -fractional operator and a brief description of the function spaces. For $u \in \mathcal{S}$ the class of rapidly decaying C^∞ functions in \mathbb{R}^N , the p -fractional operator acting at u is defined as

$$(-\Delta)_p^s u = 2 \lim_{\epsilon \rightarrow 0} \int_{B_\epsilon^c(x)} \frac{[u(x) - u(y)]^{p-1}}{|x - y|^{N+sp}} dy$$

for $s \in (0, 1)$ and $p \in [1, \infty)$ with the notation $[a - b]^{p-1} = |a - b|^{p-2}(a - b)$.

For $p = 2$, it reduces to the well known linear fractional Laplacian operator $(-\Delta)^s$ defined as

$$(-\Delta)^s u(x) = \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

up to a normalising constant where P.V. denotes the Cauchy principal value. The fractional Laplacian operator has a long history in mathematics. In particular, it is known as the

infinitesimal generator of Lévy stable diffusion process and has been a classic topic in Fourier analysis, and nonlinear partial differential equations due to its appearance in real life models in phase transitions, anomalous diffusion, material science, finance, optimization, etc (see [74, 171] and their references within).

Let Ω be bounded domain and for a measurable function $u : \mathbb{R}^N \rightarrow \mathbb{R}$, denote

$$[u]_{s,p} := \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}}.$$

Define

$$W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\}$$

endowed with the norm

$$\|u\|_{s,p,\mathbb{R}^N} = \|u\|_{L^p(\mathbb{R}^N)} + [u]_{s,p}.$$

We also define

$$W_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

endowed with the norm

$$\|u\|_{s,p} = [u]_{s,p}.$$

We can equivalently define $W_0^{s,p}(\Omega)$ as the closure of $C_c^\infty(\Omega)$ in the norm $[\cdot]_{s,p}$, with continuous boundary of the domain of Ω (see Theorem 6, [126] and [162]) where

$$C_c^\infty(\Omega) := \{f : \mathbb{R}^N \rightarrow \mathbb{R} : f \in C^\infty(\mathbb{R}^N) \text{ and } \text{supp}(f) \subset \omega \Subset \Omega\}.$$

We also define

$$W_{loc}^{s,p}(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \mid u \in L^p(\omega), [u]_{s,p,\omega} < \infty, \text{ for all } \omega \Subset \Omega\}$$

where the localized Gagliardo seminorm is defined as

$$[u]_{s,p,\omega} := \left(\iint_{\omega^2} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{1/p}.$$

The nonlinear operator $(-\Delta)_p^s$ is the nonlocal analogue of p -Laplacian operator in the (weak) sense that $(1-s)(-\Delta)_p^s \rightarrow (-\Delta)_p$ as $s \rightarrow 1^-$ i.e. for any $u \in W_0^{1,p}(\Omega)$

$$\lim_{s \rightarrow 1^-} (1-s)[u]_{s,p}^p = K(p, N) \|\nabla u\|_{L^p(\Omega)}^p.$$

where $K(p, N)$ is defined in Proposition 2.2, [67]. For $p = 2$, it is worth mentioning that $W^{s,p}(\Omega)$ and $W_0^{s,p}(\Omega)$ turns out to be Hilbert spaces. For more details and motivations we refer to [77, 211].

Definition 3.1.1. For $\phi \in C_0(\overline{\Omega})$ with $\phi > 0$ in Ω , the set $C_\phi(\Omega)$ is defined as

$$C_\phi(\Omega) = \{u \in C_0(\overline{\Omega}) : \text{there exists } c \geq 0 \text{ such that } |u(x)| \leq c\phi(x), \text{ for all } x \in \Omega\},$$

endowed with the natural norm $\left\| \frac{u}{\phi} \right\|_{L^\infty(\Omega)}$.

Definition 3.1.2. *The positive cone of $C_\phi(\Omega)$ is the open convex subset of $C_\phi(\Omega)$ defined as*

$$C_\phi^+(\Omega) = \left\{ u \in C_\phi(\Omega) : \inf_{x \in \Omega} \frac{u(x)}{\phi(x)} > 0 \right\}.$$

We also define

$$X := \left\{ u : \mathbb{R}^N \rightarrow \mathbb{R} \mid \text{measurable, } u|_\Omega \in L^2(\Omega) \text{ and } \frac{(u(x) - u(y))}{|x - y|^{N+2s}} \in L^2(Q) \right\}$$

where $Q = \mathbb{R}^{2N} \setminus \Omega^c \times \Omega^c$ and $\Omega^c = \mathbb{R}^N \setminus \Omega$ endowed with the norm

$$\|u\|_X = \|u\|_{L^2(\Omega)} + \mathcal{C}_s \left(\int_Q \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} dx dy \right)^{\frac{1}{2}}.$$

Define the Hilbert space X_0 as

$$X_0 := \{u \in X : u = 0 \text{ in } \mathbb{R}^N \setminus \Omega\}$$

equipped with the inner product

$$\langle u, v \rangle = \mathcal{C}_s \int_Q \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+2s}} dx dy.$$

We also recall the Moser-Trudinger inequality in case of half-Laplacian operator which has been recently proved by Martinazzi [199] in Bessel potential spaces and by Takahashi [244] in Sobolev spaces using Green functions for fractional Laplacian operators .

Theorem 3.1.3. *Let Ω be an open bounded interval in \mathbb{R} . Then it holds*

$$\pi = \max \left\{ c : \sup_{\|u\|_{X_0} \leq 1} \int_\Omega e^{cu^2} dx < \infty \right\}.$$

3.2 Presentation of problems

In this section we present the main problems to be dealt in this chapter.

3.2.1 Problem 1: Half Laplacian singular problem

First, we study the following nonlocal singular problem in critical dimension $N = 1$:

$$(P_\lambda) \quad \begin{cases} (-\Delta)^{\frac{1}{2}} u = \lambda \left(\frac{1}{u^\delta} + f(u) \right), & u > 0 \quad \text{in } (-1, 1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1, 1) \end{cases}$$

where $f(t) = h(t)e^{t^\alpha}$, $1 \leq \alpha \leq 2$, $\delta > 0$, $\lambda \geq 0$ and $h(t)$ is the smooth perturbation of e^{t^α} of lower growth order. The study of above problem is motivated from the Moser inequality (see Theorem 3.1.3) and extended version of Green formula (see [84]) in case of Half-Laplacian operator. In the continuation of the work in [6] (where $N > 2s$ and $s \in (0, 1)$ is considered), we are interested in the detailed study of the nonlocal singular problems in 1-dimension involving exponential nonlinearity and with the following questions:

- (i) Does the solution of the problem (P_λ) exists, if yes, what is the notion of the solution? Is it symmetric or monotonic w.r.t $|x|$?
- (ii) If the solution exists, it is unique or there exist more than one solution?
- (iii) What is the boundary behavior and Hölder regularity of the solution?

To answer the above questions, we study the asymptotic behavior of the purely singular problems $((3.4.2), (P_\delta))$ in terms of first normalized eigenfunction of $(-\Delta)^{\frac{1}{2}}$ and seek help from bifurcation theory and sub-supersolutions method. Concerning the symmetry and monotonicity of solution we use moving plane method and narrow maximum principles. Multiplicity of solution is also discussed by variational methods.

3.2.2 **Problem 2:** Nonlinear fractional singular problem with singular weights

Secondly, we study the following nonlinear fractional elliptic and singular problem

$$(P) \begin{cases} (-\Delta)_p^s u = \frac{K_\delta(x)}{u^\gamma}, u > 0 & \text{in } \Omega; \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $C^{1,1}$ boundary, $s \in (0, 1)$, $p \in (1, +\infty)$, $\gamma > 0$ and K_δ satisfies $K_\delta \sim d^{-\delta}$ for some $\delta \in [0, sp)$, $d(x) = \text{dist}(x, \partial\Omega)$ for any $x \in \Omega$.

Concerning the problem (P) , we are interested in the existence of a weak solution (we adopt a weaker notion of solution with respect to other contributions due to the nonlinearity of the operator and absence of integration by parts formula). By the weak solution here we mean a solution u such that

$$u^\kappa \in W_0^{s,p}(\Omega) \text{ for some } \kappa \geq 1 \text{ and } \inf_K u > 0 \text{ for all } K \Subset \Omega \quad (3.2.1)$$

and for $\phi \in \bigcup_{\tilde{\Omega} \Subset \Omega} W_0^{s,p}(\tilde{\Omega})$, u satisfies

$$\iint_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)]^{p-1} (\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy = \int_\Omega \frac{K_\delta(x)}{u^\gamma} \phi dx.$$

Having in mind the condition (3.2.1), a function $u = 0$ in $\mathbb{R}^N \setminus \Omega$ satisfies $u \leq 0$ on $\partial\Omega$ in sense that for $\epsilon > 0$, $(u - \epsilon)^+ \in W_0^{s,p}(\Omega)$. This weak form of trace of solution emerged from the lack of regularity of solutions when γ is large (a rigorous formulation is given in Definition 3.4.1).

Subsequently, a next question arises is to find the optimal range of constant $\theta = \theta(\delta, \gamma) \geq 1$ such that $u^\theta \in W_0^{s,p}(\Omega)$. We designate this problem as to show the optimal Sobolev regularity of the weak solution.

To deal with above problems, we investigate the boundary behavior of the weak solution in terms of distance function. The Hölder regularity of the weak solution and the non-existence results are the byproduct of this investigation and are of independent interest.

3.3 A brief literature on singular problems

The study and understanding of existence, multiplicity, and regularity of weak and classical solutions to elliptic singular equations have been a matter of intensive research. The pioneering work of Crandall et al. [101] in local case ($s = 1, p = 2$) is the starting point of the study of singular problems. Later on, the perturbed equation of type

$$-\Delta u = \lambda a(x)u^{-\gamma} + Mu^{r-1}, \quad u > 0, \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \quad M \in \{0, 1\} \quad (3.3.1)$$

received much attention and studied by a large number of researchers in this field. The authors in [101] studied the purely singular problem ($a = \lambda = 1$ and $M = 0$) and proved the existence, uniqueness in $C^2(\Omega) \cap C(\bar{\Omega})$ for $\gamma > 0$ and boundary behavior of classical solution in terms of the first normalized and positive eigenfunction of $-\Delta$. In [185], authors proved the existence of unique solution u in $C^{2+\alpha}(\Omega) \cap C(\bar{\Omega})$ for purely singular problem with $a \in C^\alpha(\bar{\Omega})$, $M = 0$ and $\partial\Omega \in C^{2,\alpha}$. Moreover, they have also proved the $u \notin C^1(\bar{\Omega})$ if $\gamma > 1$ and $u \in W_0^{1,2}(\Omega)$ if and only if $\gamma < 3$.

In [96], authors showed the existence and non-existence of classical solutions of (3.3.1) depending upon the parameter λ with Ω satisfying $\partial\Omega \in C^3$ and $a = M = 1$. Using Nehari manifold method, Yijing et al. [261] proved the existence of two solutions in the subcritical range *i.e.* $1 < r < 2^*$ for $a \in L^2(\Omega)$ and $\lambda = M = 1$. Thereafter, problem (3.3.1) with critical exponent $r = 2^*$ and $a = 1, M = 1$ is studied by Haitao [165] and Hirano et al [170] when $0 < \gamma < 1$. In [165], the author proved the global multiplicity of weak solutions by combining sub-supersolution and variational methods (Perron's method), whereas in [170], authors used variational methods to prove the existence of two solutions. While in [167], authors studied the problem for all $\gamma > 0$, $a = 1 = M$, and established a global multiplicity result with a deep use of non-smooth analysis. In case of critical exponential growth for $N = 2$, authors in [5] and [102] proved the multiplicity results of the following singular equation

$$-\Delta u = \lambda \left(a(x)u^{-\gamma} + b(u)e^{u^\alpha} \right), \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

via shooting method combining with ODE analysis and global bifurcation theory due to P. H. Rabinowitz for $1 \leq \alpha \leq 2$, $0 < \gamma < 3$, $a \in L^\infty(\Omega)$ and $b(t)$ is a smooth perturbation of e^{t^α} as $t \rightarrow \infty$.

Pertaining to the case, when a has a singularity, Gomes [156] studied the purely singular problem and proved the existence and uniqueness of $C^1(\bar{\Omega})$ classical solution via Schauder fixed point Theorem and the required compactness is obtained by suitable estimates on the integral representation involving the Green function and in [106], Díaz et al. considered the case where a behaves as some negative power of the distance function. Here, regularity of gradient of u in Lorentz spaces is established.

For the quasilinear case ($p \neq 2$), authors in [142] studied

$$(-\Delta)_p u = \lambda u^{-\gamma} + u^{r-1}, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega \quad (3.3.2)$$

for $0 < \gamma < 1$ and $1 < r \leq p^*$. By employing variational methods and depending upon the range of parameter λ , they proved existence of multiple solution in $W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ for all $p > 1$ in subcritical range and $p \in (\frac{2N}{N+2}, 2] \cup (\frac{3N}{N+3}, 3)$ in critical case range (where the behavior of Talenti functions is proved). Here the authors have also proved $C^{1,\beta}(\overline{\Omega})$ regularity of weak solution for the problem (3.3.2) for any $p > 1$. In [143], authors proved the existence of a weak solution in $W^{1,p}(\Omega) \cap C(\overline{\Omega})$ if and only if $0 < \gamma < 2 + \frac{1}{p-1}$ using ODE techniques and shooting method for $r > p$. Thereafter, the authors in [63] and [79], studied the purely singular problem (by replacing λ to a function f in (3.3.2)) for p -Laplacian operator and proved existence, uniqueness and boundary behavior of weak solution (with different summability conditions on f). In [231], authors studied the quasilinear singular version in the presence of exponential non-linearity (limiting case of Sobolev embedding)

$$(-\Delta)_N u = \lambda \left(g(u) + b(u)e^{u^{\frac{N}{N-1}}} \right), \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega$$

where $g(u) \sim u^{-\gamma}$ for $0 < \gamma < 1$ and $b(t)$ is a smooth perturbation of $e^{\frac{N}{N-1}}$ of lower growth order. Using variational methods, they proved the existence of multiple solutions for $\lambda \in (0, \Lambda)$ and one solution for $\lambda = \Lambda$ (what we call global multiplicity result). For a more detailed analysis of semilinear and quasilinear elliptic equations with singular non-linearities involving different kind of non-linearities, we refer to [36, 41, 101, 105, 106, 130, 132, 136, 141, 160, 169, 198, 204] and surveys [137] and [168] with no intent to furnish an exhaustive list.

Recently, equations involving nonlocal operators attracted a large number of researchers, especially in the study of fractional powers of $(-\Delta)$ and equivalent $(-\Delta)_p$. This interest has been provoked and sustained by the applications of such results in mathematical physics and geometry (see [54, 211]). The study of singular problems involving the fractional Laplacian operator started in [46]. Precisely, the authors studied the following singular problem

$$(-\Delta)^s u = \lambda \left(\frac{K(x)}{u^\gamma} + Mf(x, u) \right), \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \quad (3.3.3)$$

and proved the existence of a weak solution for $f(x, u) = u^p$ and $p > 1$ via approximation method for $M = 0$ and Sattinger method for $M = 1$, and Sobolev regularity of the weak solution for the function $K \in L^q(\Omega)$ where q depends upon the singular exponent $\gamma > 0$. In the recent times, Adimurthi et al. [6] studied the problem (3.3.3) with $M = 0$, $N > 2s$ and $K(x) \sim d^{-\beta}(x)$ and complement the results of [46] by exploiting the integral formula with Green function. In particular, they obtained the boundary behaviour and deduce optimal Hölder regularity of the classical solution (in the sense of Definition 3.4.2). Using the asymptotic behavior near the boundary, authors obtained multiplicity of classical solutions for (3.3.3) with f satisfying subcritical growth conditions via global bifurcation method in weighted spaces.

Concerning the critical growth, authors in [139], studied the problem (3.3.3) for $K(x) = 1$, $\gamma > 0$, $N > 2s$, and $f(x, u) = u^{2^*_s-1}$ and proved the existence and multiplicity of weak

solutions in $C_{loc}^\alpha(\Omega) \cap L^\infty(\Omega)$ using non-smooth analysis and linking theorems. In [150], authors proved the multiplicity of energy solutions for the non-local singular problem (3.3.3) with Sobolev critical nonlinearity $f(x, u) = u^{2^*_s-1}$, $N > 2s$ and γ satisfies $\gamma(2s - 1) < 2s + 1$. For more results on nonlocal problems with singular non-linearity, interested readers can refer to [140, 144, 145, 208, 209]. For results regarding the symmetry and monotonicity of weak solution we refer to the latest work [87, 175] where the moving plane method and maximum principle for narrow domains (in the spirit of Alexandrov-Bakelman-Pucci type estimates) are proved for nonlocal operators.

The study of regularity results in case of non-local operators started a long back and is now rather well understood for $p = 2$. Proving sufficiently good regularity estimates up to the boundary has useful applications in obtaining multiplicity results, apart from being relevant itself. Consider the following non-local problem

$$(-\Delta)_p^s u = f \text{ in } \Omega \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega.$$

For $p = 2$, the interior regularity of the solutions is primarily resolved by Caffarelli et al. [75, 76] and boundary regularity is settled by Ros-Oton et al. in [228] for $f \in L^\infty(\Omega)$. For the general case $p \neq 2$, the situation is more delicate due to the nonlinear nature of the operator: the representation of Green formula and explicit tools to compute the $(-\Delta)_p^s$ are not available. The local Hölder regularity for viscosity solution is proved in [80, 189] without mentioning the dependence and optimality of Hölder exponent. In the degenerate case *i.e.* $p \geq 2$, Brasco et al. in [65] proved the existence of optimal Hölder exponent *i.e.* $u \in C_{loc}^{\frac{sp}{p-1}}$ when $f \in L^\infty(\Omega)$ and $\frac{sp}{p-1} < 1$. The proof of boundary regularity is more involved. The first work regarding the nonlinear case is Iannizzotto et al. in [172]. They proved the global Hölder regularity result, with an unspecified Hölder exponent via barrier argument and by combining it with the optimal interior regularity of [65] we have $u \in C^s(\mathbb{R}^N)$ when $p \geq 2$. The same is conjectured to hold in the case $p \in (1, 2)$, but the optimal (at least C^s) interior regularity is still an open problem.

More recently, by extending the work of [46], Canino et al. in [78] studied the following purely singular problem

$$(-\Delta)_p^s u = \frac{K(x)}{u^\gamma}, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^N \setminus \Omega \tag{3.3.4}$$

and by approximation method proved the existence and Sobolev regularity of the very weak solution depending upon the range of singular exponent γ . In [210], authors have studied the problem (3.3.4) perturbed with critical growth non-linearities $f(x, u) = u^\alpha$, $\alpha \leq p_s^* - 1$ for $0 < \gamma \leq 1$, $K(x) = 1$, $N > sp$ and proved the existence and multiplicity of very weak solution via the minimization method under the natural Nehari manifold constraint.

3.4 New contribution: Tools and main results

3.4.1 Problem 1: Half-Laplacian singular problem

In this part, first we introduce different notion of solutions (weak and classical) concerning the main problem (P_λ) and main tools required to study the qualitative properties of the weak or classical solutions and then we state the main results with a glimpse of a proof. We start by stating the assumption on the function h for $1 \leq \alpha \leq 2$:

- (H1) $h : [0, \infty) \rightarrow \mathbb{R}$ is a positive function of class C^2 in $(0, \infty)$ with $h(0) = 0$ and such that the map $t \rightarrow t^{-\delta} + h(t)e^{t^\alpha}$ is convex for all $t > 0$.
- (H2) For any $\epsilon > 0$, $\lim_{t \rightarrow \infty} h(t)e^{-\epsilon t^\alpha} = 0$ and $\lim_{t \rightarrow \infty} h(t)e^{\epsilon t^\alpha} = \infty$.

Motivating from the Definition 2.2 in [46], we define the following notion of (very) weak solution for the problem (P_λ) :

Definition 3.4.1. A function $u \in L^1(\mathbb{R})$ with $u \equiv 0$ on $\mathbb{R} \setminus (-1, 1)$ is said to be a weak solution of (P_λ) if $\inf_K u > 0$ for any compact set $K \subset (-1, 1)$ and for any $\phi \in \sigma$,

$$\int_{-1}^1 u (-\Delta)^{\frac{1}{2}} \phi = C_s \int_Q \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^2} dx dy = \lambda \int_{-1}^1 \left(\frac{1}{u^\delta} + h(u)e^{u^\alpha} \right) \phi dx \quad (3.4.1)$$

where

$$\sigma = \{ \psi : \mathbb{R} \rightarrow \mathbb{R} : \text{measurable}, (-\Delta)^{\frac{1}{2}} \psi \in L^\infty((-1, 1)) \text{ and } \phi \text{ has compact support in } (-1, 1) \}.$$

As in [6], we define the notion of classical solutions of (P_λ) (adding continuity property):

Definition 3.4.2. The set of classical solutions to (P_λ) is defined as

$$\mathcal{S} = \left\{ (\lambda, u) \in \mathbb{R}^+ \times C_0([-1, 1]) : u \text{ is a weak solution to } (P_\lambda) \text{ in } X_0 \right\}.$$

Remark 3.4.3. Regularity of a classical solution u (proved later in Theorem 3.4.9) for the problem (P_λ) implies $u \in C_{\phi_\delta}^+((-1, 1))$ (defined below). Indeed with the continuity property, we can use some comparison principle for nonlocal operator. Then by using Hardy's inequality (see [162, Corollary 1.4.4.10, p.23]) in (3.4.1) together with the fact that $C_c^\infty((-1, 1))$ is dense in X_0 , we obtain that $\frac{1}{u^\delta}$ belongs to dual space of X_0 for all $\delta > 0$ and hence in case of classical solution (3.4.1) holds for all $\phi \in X_0$.

Now, we recall the definition of an asymptotic bifurcation point and then state the result regarding existence of a global branch of classical solutions to (P_λ) .

Definition 3.4.4. A point $\Lambda_a \in [0, \infty)$ is said to be an asymptotic bifurcation point, if there exists a sequence $(\lambda_n, u_n) \in \mathcal{S}$ such that $\lambda_n \rightarrow \Lambda_a$ and $\|u_n\|_{L^\infty((-1, 1))} \rightarrow \infty$ as $n \rightarrow \infty$.

3.4.1.1 Preliminary results

This section is devoted to the development of preliminary results of independent interest for the study of the original problem (P_λ) . The first result is the study of existence and regularity (Theorem 3.4.6) of solutions to the purely singular problem (P_δ) (see below) which is used to construct the sub and supersolutions of the main problem (P_λ) . The second result is the study of behavior of isolated singularities in Brezis-Lions type problem for fractional Laplacian operator (see below (P_s)) which is used to identify the blow up behavior of weak solution of original problem (P_λ) at an asymptotic bifurcation point.

To analyze the asymptotic behavior of solution for the problem (P_λ) near boundary, the first key ingredient is to construct the barrier function as a solution of the following problem:

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u = \frac{1}{d(x)^\alpha \log^\beta\left(\frac{A}{d(x)}\right)} & \text{in } (-1, 1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1, 1). \end{cases} \quad (3.4.2)$$

For the operator $(-\Delta)^s$ with $N > 2s$, Abatangelo [1] studied the boundary behavior of the corresponding problem like (3.4.2) with $\beta = 0$ and $0 < \alpha < 1 + s$. We provide the extension in case $N = 1$ and $s = \frac{1}{2}$ in the following theorem:

Theorem 3.4.5. *Let A be a positive constant such that $A \geq 2$. Then the weak solution of (3.4.2) satisfies*

$$\begin{aligned} c_1 d(x)^{\frac{1}{2}} \leq u(x) \leq c_2 d(x)^{\frac{1}{2}} & \quad \text{for } 0 < \alpha < \frac{1}{2} \text{ and } \beta = 0, \\ c_3 d(x)^{1-\alpha} \leq u(x) \leq c_4 d(x)^{1-\alpha} & \quad \text{for } \frac{1}{2} < \alpha < \frac{3}{2} \text{ and } \beta = 0, \\ c_5 d(x)^{\frac{1}{2}} \log^{1-\beta}\left(\frac{A}{d(x)}\right) \leq u(x) \leq c_6 d(x)^{\frac{1}{2}} \log^{1-\beta}\left(\frac{A}{d(x)}\right) & \quad \text{for } \alpha = \frac{1}{2} \text{ and } 0 \leq \beta < 1. \end{aligned} \quad (3.4.3)$$

where $c_i, i = 1, 2, \dots, 6$ are constants.

The main essence in the proof of (3.4.3) is the use of following integral representation of weak solution u via Green function (see [84]) given by

$$u(x) = \int_{-1}^1 \frac{G(x, y)}{d(y)^\alpha \log^\beta(A/d(y))} dy \text{ where } G(x, y) \asymp \log\left(1 + \frac{d(x)^{\frac{1}{2}} d(y)^{\frac{1}{2}}}{|x - y|}\right) \quad \forall (x, y) \in (-1, 1)^2.$$

For a detailed proof we refer to Page 316, proof of Theorem 6.1.6, Chapter 6.

As in [6], we define the function ϕ_δ as

$$\phi_\delta = \begin{cases} \phi_1 & \text{if } 0 < \delta < 1, \\ \phi_1 \left(\log\left(\frac{2}{\phi_1}\right)\right)^{\frac{1}{2}} & \text{if } \delta = 1, \\ \phi_1^{\frac{2}{\delta+1}} & \text{if } \delta > 1 \end{cases} \quad (3.4.4)$$

where ϕ_1 is the normalized ($\|\phi_1\|_{L^\infty(\Omega)} = 1$) positive eigenfunction corresponding to the first eigenvalue of $(-\Delta)^{\frac{1}{2}}$ on X_0 . We recall that $\phi_1 \in C^{\frac{1}{2}}(\mathbb{R})$ and $\phi_1 \in C^+_{\frac{1}{2}}((-1, 1))$ (See

Proposition 1.1 and Theorem 1.2 of [226]).

Now, to construct appropriate sub and super solutions of the problem (P_λ) , we start by studying the following singular problem (P_δ) with the help of barrier function ϕ_δ

$$(P_\delta) \begin{cases} (-\Delta)^{\frac{1}{2}} u = \frac{1}{u^\delta}, & u > 0, & \text{in } (-1, 1), \\ u = 0 & & \text{in } \mathbb{R} \setminus (-1, 1). \end{cases}$$

In this regard we prove the following result:

Theorem 3.4.6. (i) For all $\delta > 0$, there exists a unique $u \in C_0([-1, 1])$ classical solution of (P_δ) . Moreover, $u \in C_{\phi_\delta}^+((-1, 1))$ where ϕ_δ is defined in (3.4.4).
(ii) The classical solution u to (P_δ) belongs to $C^\gamma(\mathbb{R})$ where γ is defined in Theorem 3.4.9.

To prove the above result, first we investigate the existence of the solution u_ϵ of approximated problem with the help of integral representation formula via Green function and secondly together with Theorem 3.4.5 and uniform estimates for the approximated solution u_ϵ , we pass to the limit $\epsilon \rightarrow 0$. For a detailed proof we refer to Page 319, proof of Theorem 6.1.7, Chapter 6.

To characterize the blow up behavior of weak solutions at an asymptotic bifurcation point Λ_a we study the behavior of solutions near isolated singularities as in Brezis-Lions problem (see [70]) for the fractional Laplacian operator. Precisely, we consider the following problem:

$$(P_s) \begin{cases} (-\Delta)^s u = g(u), & u \geq 0 & \text{in } \Omega', \\ u = 0 & & \text{in } \mathbb{R}^N \setminus \Omega, \\ u \in L^1(\Omega), & g(u) \in L_{loc}^t(\Omega'), \end{cases}$$

where $0 < s < 1$, $t > \frac{N}{2s} \geq 1$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with $0 \in \Omega$ and $\Omega' = \Omega \setminus \{0\}$. The notion of distributional solution for (P_s) is defined as follows:

Definition 3.4.7. A function u is said to be a distributional solution of (P_s) if $u \in L^1(\Omega)$ such that $g(u) \in L_{loc}^1(\Omega')$ and

$$\int_{\Omega} u(x)(-\Delta)^s \phi \, dx = \int_{\Omega} g(u)\phi \, dx$$

for all $\phi \in C_c^\infty(\Omega)$ with $\text{supp}(\phi) \subset \Omega'$.

In [89], authors have studied the problem (P_s) for a power type nonlinearity function $g(u) = u^p$ by showing the L^p integrability of the solution in Ω and scaling typical test functions. In the next theorem, we extend the result of Chen and Quaas ([89]) for the problem (P_s) for $N \geq 2s$ by using the same approach with precise estimates but considering a weaker notion of solution and for a large class of nonlinearities (in particular exponential growth nonlinearities).

Theorem 3.4.8. For $0 < s < 1$, let u be non-negative distributional solution of (P_s) such that $u \in L^1(\Omega)$, $g(u) \in L^t_{loc}(\Omega')$ for $t > \frac{N}{2s} \geq 1$. Then there exists $k \geq 0$ such that u is distributional solution of

$$(\mathcal{D}_s) \begin{cases} (-\Delta)^s u = g(u) + k\delta_0, & u \geq 0, & \text{in } \Omega, \\ u = 0 & & \text{in } \mathbb{R}^N \setminus \Omega, \\ g(u) \in L^1(\Omega), & & \end{cases}$$

i.e.

$$\int_{\Omega} u(-\Delta)^s \phi - g(u)\phi \, dx = k\phi(0) \text{ for all } \phi \in C_c^\infty(\Omega).$$

This result asserts that the due to the effect of fundamental solution problem (P_s) extend to the distributional equation (\mathcal{D}_s) containing dirac mass at the isolated singularity. To prove the above result, we follow the proof in [89] and define the distribution $P : C_c^\infty(\Omega) \rightarrow \mathbb{R}$ such that

$$P(\phi) = \int_{\Omega} u(-\Delta)^s \phi - g(u)\phi \, dx \text{ for all } \phi \in C_c^\infty(\Omega)$$

where $u \in L^1(\Omega)$ is a non-negative distributional solution of (P_s) and $g(u) \in L^1(\Omega)$ (see Theorem 6.1.19, Chapter 6) and then by using integration by parts formula and Theorem XXXV in [232], we infer that

- (i) $P(\phi) = 0$ for any $\phi \in C_c^\infty(\Omega)$ with $\text{supp}(\phi) \subset \Omega \setminus \{0\}$.
- (ii) There exists constants c_a such that

$$P(\phi) = \sum_{|a| < 1} c_a D^a \phi(0)$$

and $c_a = 0$ if $|a| \geq 1$

where $a = (a_1, a_2, \dots, a_N)$ with $a_i \in \mathbb{N}$, $|a| = \sum_{i=1}^N a_i$, $D^a = (\partial^{a_1} \phi, \partial^{a_2} \phi, \dots, \partial^{a_N} \phi)$.

3.4.1.2 Main results with a glimpse of the proof

To study the existence, multiplicity of solutions to (P_λ) , we seek assistance of global bifurcation theory due to P. H. Rabinowitz [221], Theorem 1.6 of [6] and Theorem 3.4.6. We establish the existence of an unbounded connected branch of solution to (P_λ) emanating from the trivial solution at $\lambda = 0$. Precisely, we prove the following result.

Theorem 3.4.9. Let h satisfy the hypothesis (H1) and (H2) and $\delta > 0$. Then the following holds:

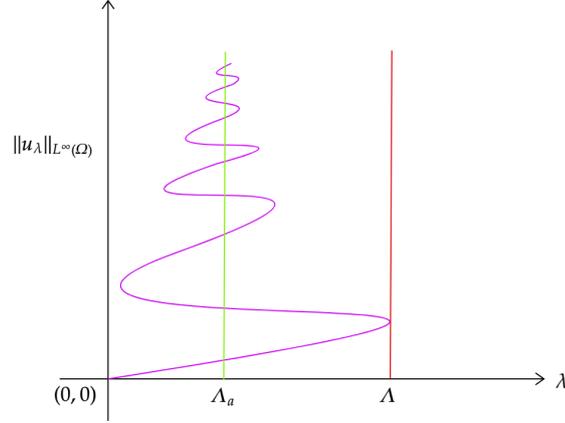


Figure 3.1: Bifurcations Diagram I

- (i) There exists $\Lambda \in [0, +\infty)$ and $\gamma > 0$ such that $\mathcal{S} \subset [0, \Lambda] \times (X_0 \cap C_{\phi_\delta}^+((-1, 1)) \cap C^\gamma(\mathbb{R}))$, where γ is defined as

$$\gamma = \begin{cases} \frac{1}{2} & \text{if } \delta < 1, \\ \frac{1}{2} - \epsilon & \text{if } \delta = 1, \text{ for all } \epsilon > 0 \text{ small enough,} \\ \frac{1}{\delta+1} & \text{if } \delta > 1. \end{cases}$$

and ϕ_δ is defined in (3.4.4).

- (ii) There exists a connected unbounded branch \mathcal{C} of solutions to (P_λ) in $\mathbb{R}^+ \times C_0([-1, 1])$, emanating from $(0, 0)$ such that for any $\lambda \in (0, \Lambda)$, there exists $(\lambda, u_\lambda) \in \mathcal{C}$ with u_λ being minimal solution to (P_λ) . Furthermore, as $\lambda \rightarrow \Lambda^-$, $u_\lambda \rightarrow u_\Lambda$ in X_0 , where u_Λ is a classical solution to (P_Λ) .
- (iii) The curve $(0, \Lambda) \ni \lambda \rightarrow u_\lambda \in C_0([-1, 1])$ is of class C^2 .
- (iv) (Bending and local multiplicity near Λ) $\lambda = \Lambda$ is a bifurcation point, that is, there exists a unique C^2 -curve $(\lambda(s), u(s)) \in \mathcal{C}$, where the parameter s varies in an open interval about the origin in \mathbb{R} , such that

$$\lambda(0) = \Lambda, u(0) = u_\Lambda, \lambda'(0) = 0, \lambda''(0) < 0.$$

- (v) (Asymptotic bifurcation point) \mathcal{C} admits an asymptotic bifurcation point Λ_a satisfying $0 \leq \Lambda_a \leq \Lambda$.

The above theorem assert the existence of a branch of solutions \mathcal{C} containing the minimal solution branch and bending back at classical solution (Λ, u_Λ) . According to assertion (v),

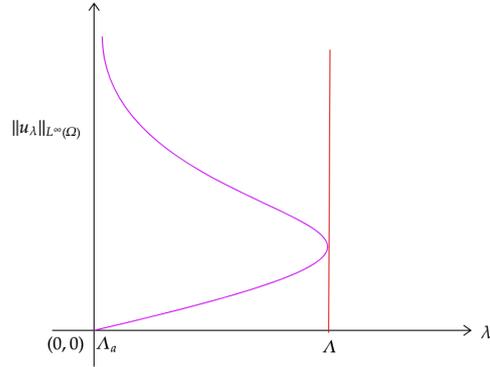


Figure 3.2: Bifurcations Diagram II

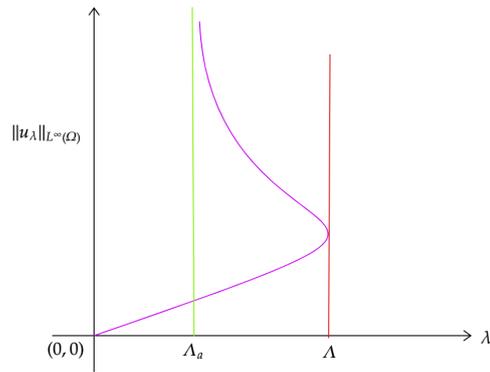


Figure 3.3: Bifurcations Diagram III

we have the possibility of different Bifurcation diagrams (see Figure 3.1, 3.2, 3.3). To prove the main existence result Theorem 3.4.9, we show

$$\Lambda \in (0, \infty) \text{ where } \Lambda := \sup\{\lambda > 0 : (P_\lambda) \text{ has a weak solution}\}.$$

First, we define $\underline{u}_\lambda := \lambda^{\frac{1}{\delta+1}}u$ and $\bar{u}_\lambda := \underline{u}_\lambda + MU$ where $U > 0$ is a weak solution of the problem $(-\Delta)^{\frac{1}{2}}U = 1$ in $(-1, 1)$, $U = 0$ in $\mathbb{R} \setminus (-1, 1)$ and u is the weak solution of the problem (P_δ) . Then there exists a λ_0 such that for all $\lambda \in (0, \lambda_0)$ (P_λ) admits a solution u_λ with $\underline{u}_\lambda \leq u \leq \bar{u}_\lambda$ where \underline{u}_λ and \bar{u}_λ act as a strict subsolution and supersolution respectively for (P_λ) . Indeed, we define the following iterative scheme, starting with $u_0 = \underline{u}_\lambda$ and $n \geq 1$

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u_n + \lambda C u_n - \frac{\lambda}{u_n^\delta} = \lambda C u_{n-1} + \lambda f(u_{n-1}), & u > 0 \text{ in } (-1, 1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1, 1). \end{cases}$$

Now, by taking into account the monotonicity of the operator $(-\Delta)^{\frac{1}{2}}u - \lambda u^{-\delta}$, Comparison Principle ([150, Lemma 2.2]) and Theorem 3.4.6, we prove the following uniform estimates

$$u_n \text{ is increasing, } \{u_n\} \subset X_0 \cap C_{\phi_\delta}^+((-1, 1)), \quad \underline{u}_\lambda \leq u_n \leq \bar{u}_\lambda, \quad \sup_{n \in \mathbb{N}} \|u_n\|_{C^\gamma(\mathbb{R})} \leq C_0$$

for some $C_0 = C_0(\delta, \lambda_0)$ large enough and γ is defined in Theorem 3.4.9. Finally, by using asymptotic behavior of $f(t)$ and passing as $n \rightarrow \infty$, we get the desired result. For a detailed explanation see Theorem 6.1.9, Chapter 6.

In order to get the asymptotic behavior of the branch of solution, we study the qualitative properties of solutions for the problem (P_λ) . In the light of narrow maximum principle (see Proposition 3.6, [175]) and the moving plane method, we derive the radial symmetry and monotonicity properties of the weak solutions with respect to $|x|$. More precisely, we prove the following result:

Theorem 3.4.10. *For $1 \leq \alpha \leq 2$, $\delta > 0$, let h satisfies (H1) – (H2), f is locally Lipschitz function. Then every positive solution $(\lambda, u) \in S$ of (P_λ) is symmetric and strictly decreasing in $|x|$ i.e. $u(x) > u(y)$ for all $|x| < |y|$ and $x, y \in (-1, 1)$.*

To prove Theorem 3.4.10, we follow the approach of moving plane method in [175]. We start by defining the antisymmetric function

$$v_h(x) := u_h(x) - u(x) \quad \text{where } u_h(x) = u(R_h(x)).$$

in the sense that $v_h(R_h(x)) = -v_h(x)$ where $R_h(x) := (2h - x)$ is the reflection of the point x about h . The proof of the above theorem emerge from the following observations:

(i) To claim positivity of the antisymmetric function v_h in $(-1, h)$ if $h < 0$ and in $(h, 1)$ if $h > 0$ is equivalent to show that the positive solution (λ, u) of the problem (P_λ) is strictly decreasing in the neighbourhood of $-1, 1$ with respect to $|x|$ i.e. $u(x) > u(y)$ for all $|h| < |x| < |y|$ and $x, y \in (-1, 1)$.

(ii) If $\inf\{|h| : v_h \geq 0 \text{ in } (-1, h) \text{ if } h < 0 \text{ and in } (h, 1) \text{ if } h > 0\} = 0$ then u is symmetric and strictly decreasing in $|x|$ i.e. $u(x) > u(y)$ for all $|x| < |y|$ and $x, y \in (-1, 1)$.

In this regard we divide the proof into two steps. In the first step, we prove the positivity of v_h in the neighbourhood of $-1, 1$ i.e.

$$v_h(x) \geq 0 \text{ in } (-1, h) \cap H_h^- \text{ if } h \leq 0 \text{ and in } (h, 1) \cap H_h^+ \text{ if } h > 0$$

where $H_h^\pm = \{x \in \mathbb{R} : x \gtrless h\}$ and h lies in the neighborhood of $x_0 \in \{-1, 1\}$. Clearly for $|h|$ sufficiently large, $v_h(x) \geq 0$. Suppose that $v_h < 0$ in $K \subset (-1, h) \cap H_h^-$ for some $h \leq 0$. Using

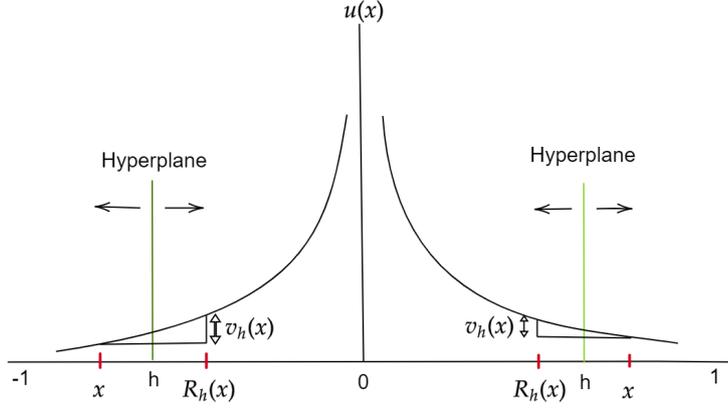


Figure 3.4: Moving plane in the neighbourhood of -1 and 1.

f is Lipschitz, Poincaré inequality and noting that $\text{supp}((-v_h)^+) \subset (-1, 2h + 1)$, we have

$$\begin{aligned} \int_{\mathbb{R}} \left((-\Delta)^{\frac{1}{4}}(u - u_h)^+ \right)^2 &\leq \langle (-\Delta)^{1/2}(-v_h), (-v_h)^+ \rangle \\ &= \lambda \int_{-1}^{2h+1} \left(\frac{1}{u^\delta} - \frac{1}{(u_h)^\delta} + f(u) - f(u_h) \right) (-v_h)^+ dx \\ &\leq C \int_K ((u - u_h)^+)^2 dx. \leq C(\text{diam}(K)) \int_{\mathbb{R}} \left((-\Delta)^{\frac{1}{4}}(u - u_h)^+ \right)^2. \end{aligned}$$

Then by choosing h close enough to -1 we get, $C(\text{diam}(K)) < 1$ and then $(-v_h)^+ = (u - u_h)^+ = 0$. Similarly in the case of $(h, 1) \cap H_h^+$ for $h > 0$. Now by moving the point in the neighborhood of -1 and 1 we obtain there exists $T > 0$ independent of u such that

$$\begin{cases} u(x-t) \text{ is non-increasing} & \forall (t, x) \in [0, T] \times (-1, h) \text{ if } h \leq 0, \\ u(x-t) \text{ is non-decreasing} & \forall (t, x) \in [0, T] \times (h, 1) \text{ if } h \geq 0. \end{cases} \quad (3.4.5)$$

Without loss of generality we can assume that $h \geq 0$ be the smallest value such that $v_h \geq 0$ in $(h, 1)$ and the mean value theorem implies v_h satisfies the following for some $\theta \in (0, 1)$

$$(-\Delta)^{\frac{1}{2}} v_h = c(x)v_h \text{ in } (h, 1). \quad (3.4.6)$$

where

$$c(x) = \begin{cases} \frac{f(u_h) - f(u)}{v_h} - \frac{\delta}{(\theta u + (1-\theta)u_h)^{\delta+1}} & \text{if } v_h \neq 0, \\ 0 & \text{if } v_h = 0. \end{cases}$$

Now, in the second step, we prove the strict positivity of the function v_h in the interior of $(-1, 1)$. For this, we start by proving $\text{ess inf}_{U_0} v_h > 0$ where $U_0 := B_r(x^*) \Subset (h, 1)$ via constructing a strictly positive subsolution \tilde{v}_h of the problem (3.4.6) in U_0 (a detailed construction is given in the proof of Theorem 6.1.10 by adapting the tools of maximum principle

for antisymmetric solution Proposition 3.5 in [175] and narrow maximum principle Proposition 3.6 in [175]). To prove $h = 0$ we argue by contradiction and suppose $h > 0$. Since h is the smallest value such that $v_h \geq 0$ in $(h, 1)$, so we claim that for a small $\epsilon > 0$ we have $v_{h-\epsilon} \geq 0$ in $(h-\epsilon, 1)$ and thus get a contradiction that h is the smallest value. Fix γ (to be determined later) and let $K \Subset (h, 1)$ such that $|(h, 1) \setminus K| \leq \frac{\gamma}{2}$. Then from above estimates, $v_h \geq r > 0$ in K and then by continuity $v_{h-\epsilon} > 0$ in K for ϵ small enough. Let λ_1 be the first eigenvalue of $(-\Delta)^s$ in $(h-\epsilon, 1) \setminus K$. Since $v_{h-\epsilon}$ satisfies (3.4.6) in $(h-\epsilon, 1) \setminus K$ and by taking $w := \mathbb{1}_{H_{h-\epsilon}^+} v_{h-\epsilon}^-$ such that $\text{supp}(w) \subset (h-\epsilon, 1) \setminus K$ as a test function, we obtain

$$\begin{aligned} \lambda_1((h-\epsilon, 1) \setminus K) \int_{(h-\epsilon, 1) \setminus K} |v_{h-\epsilon}^-|^2 dx &\leq - \langle (-\Delta)^{\frac{1}{2}} v_{h-\epsilon}, w \rangle \\ &= \int_{(h-\epsilon, 1) \setminus K} \frac{\delta v_{h-\epsilon} \mathbb{1}_{(h-\epsilon, 1) \setminus K} v_{h-\epsilon}^-}{(\theta u + (1-\theta)u_h)^{\delta+1}} dx + \int_{(h-\epsilon, 1) \setminus K} (-f(u_{h-\epsilon}) + f(u)) \mathbb{1}_{(h-\epsilon, 1) \setminus K} v_{h-\epsilon}^- dx \\ &\leq C_L \int_{(h-\epsilon, 1) \setminus K} |v_{h-\epsilon}^-|^2 dx. \end{aligned}$$

Since $\lambda_1(\Omega) \rightarrow \infty$ when $|\Omega| \rightarrow 0$ (see Lemma 2.1 in [175]) then by choosing γ small enough we get $v_{h-\epsilon} \geq 0$ in $(h-\epsilon, 1)$, which is a contradiction. Therefore $h = 0$ *i.e.* $u(-x) \geq u(x)$ and then by repeating the same proof for largest value of h over $(-1, h)$ we obtain $u(x) = u(-x)$ for all $x \in (-1, 1)$. Since $h = 0$, therefore (3.4.5) and u is strictly decreasing in $|x|$. For a detailed proof see Theorem 6.1.10, Chapter 6.

Assertion (v) in Theorem 3.4.9 asserts that the connected branch admits at least one asymptotic bifurcation point. Then by using Theorem 3.4.8, we show that the sequence of large solution converge to a singular solution if $\Lambda_a \neq 0$.

Theorem 3.4.11. *For $1 < \alpha \leq 2$, $\delta > 0$, assume $\Lambda_a > 0$ be an asymptotic bifurcation point as in the Definition 3.4.4. Then, for any sequence $(\lambda_k, u_k) \in \mathcal{S} \cap ((0, \Lambda) \times C_0([-1, 1]))$ such that $\lambda_k \rightarrow \Lambda_a$ and $\|u_k\|_{L^\infty((-1, 1))} \rightarrow \infty$, the following assertions holds:*

- (i) $0 \in \Omega$ is the only blow up point for a sequence u_k .
- (ii) $u_k \rightarrow u$ in $C_{loc}^s((-1, 1) \setminus \{0\})$ where u is a weak (singular) solution to (P_λ) . Moreover, $u(0) = \infty$, $u \in L^p((-1, 1))$ for any $1 \leq p < \infty$, $u \notin X_0$ and $\frac{1}{v^\delta} + f(u) \in L^1((-1, 1))$.

To prove the above result, we exploit the growth of the function f and monotonicity of the solution w.r.t. $|x|$ (Theorem 3.4.10) to prove

$$\sup_k \|u_k\|_{L^\infty((-1, 1) \setminus [\epsilon, \epsilon])} \leq c_\epsilon < \infty.$$

where c_ϵ is independent of k and which further implies "0" is the blow up point for the sequence u_k . By regularity of u_k , compact embeddings, Fatou's lemma and Vitali's convergence

theorem we obtain u satisfies (in the sense of Definition 6.1.11):

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u = \Lambda_a g(u) & \text{in } (-1, 1) \setminus \{0\}, \\ u \geq 0 & \text{in } (-1, 1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1, 1), \end{cases}$$

with $g(u) := u^{-\delta} + f(u) \in L^1(\Omega)$. Then by using Theorem 3.4.8 there exists $\mu \geq 0$ such that u satisfies (in the sense of Definition 6.1.1)

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u = \Lambda_a g(u) + \mu\delta_0 & \text{in } (-1, 1), \\ u \geq 0 & \text{in } (-1, 1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1, 1). \end{cases} \quad (3.4.7)$$

Suppose $\mu \neq 0$ in (3.4.7). Then by using integral representation of solution u , we get $u(x) \geq \log(|x|^{-\mu/\pi}) - C$ and since $\alpha > 1$,

$$f(u) \geq h\left(\log(|x|^{-\mu/\pi}) - C\right) \exp^{(\log(|x|^{-\mu/\pi}) - C)^\alpha} \geq h\left(\log(|x|^{-\mu/\pi}) - C\right) |x|^{-\mu p/\pi}$$

for all $p > 1$, $0 < |x| \leq |x_\rho|$ and $|x_\rho|$ small. Then by integrating $f(u)$ over a small ball B around 0, we obtain $\int_B f(u) = \infty$ which contradicts $f(u) \in L^1((-1, 1))$. Therefore $\mu = 0$ and (ii) is proved.

Previous results do not imply the multiplicity of solutions for all $\lambda \in (0, \Lambda)$. Via variational techniques, we next prove the global multiplicity result to the problem (P_λ) for all $\delta > 0$, under the following assumptions on growth of the function f .

- (K1) $h \in C^1(\mathbb{R}^+)$, $h(0) = 0$, $h(t) > 0$ for $t > 0$ and $f(t) = h(t)e^{t^2}$ is nondecreasing in t .
- (K2) For any $\epsilon > 0$, $\lim_{t \rightarrow \infty} (h(t) + h'(t))e^{-\epsilon t^2} = 0$ and $\lim_{t \rightarrow \infty} h(t)te^{\epsilon t^2} = \infty$ for some $0 \leq \epsilon < 1$.
- (K3) There exists $M_1, M_2, K > 0$ such that $F(t) = \int_0^t h(s)e^{s^2} ds < M_1(f(t) + 1)$ and $f'(t) \geq Kf(t) - M_2$ for all $t > 0$.

Example 1: An example of the function h satisfying the above conditions is $h(x) = x^k e^{x^\gamma}$, $k > 0$, $0 \leq \gamma < 2$.

We prove the following global multiplicity theorem.

Theorem 3.4.12. (a) *If f satisfies the assumption (K1)-(K3). There exists a $\Lambda > 0$ such that*

- (i) *For every $\lambda \in (0, \Lambda)$ the problem (P_λ) admits two solutions in $X_0 \cap C_{\phi_\delta}^+((-1, 1))$.*
- (ii) *For $\lambda = \Lambda$ there exists a solution in $X_0 \cap C_{\phi_\delta}^+((-1, 1))$.*
- (iii) *For $\lambda > \Lambda$, there exists no solution.*

(b) *Let $u \in X_0$ be any positive solution to (P_λ) where $\lambda \in (0, \Lambda]$, $\delta > 0$. Then $u \in C^\gamma(\mathbb{R})$ where γ is defined (6.1.5).*

To prove the Theorem 3.4.12, we followed the approach of [150]: To obtain the first solution, we use the standard Perron's method (which is a variational version of sub and supersolution method) on the functional J_λ (See (6.1.30)). To get a second solution, we use the assumption (K2) to guarantee that the energy level of the Palais Smale sequence is strictly below the first critical level. For that we seek help of Moser functions (See [244]) and then by using mountain-pass Lemma we prove the existence of a second solution. Notice that the Theorem 6.1.15 shows the existence of solutions in the energy space X_0 . We remark that the Hölder regularity proved in Theorem 3.4.12 is the optimal due to the behavior of the solution near the points -1 and 1 . We also remark that the Theorem 3.4.12 holds for the subcritical nonlinearities like $f(t) = h(t)e^{t^\alpha}$ with $1 \leq \alpha < 2$ as well. In this case, the proof is similar and Palais-Smale condition is satisfied.

3.4.1.3 Generalization of symmetry results and its application

In this part, we are interested to generalize Theorem 3.4.10. Precisely, we investigate the symmetry of positive solutions to a class of singular semilinear elliptic problem whose prototype is

$$(P) \quad \begin{cases} (-\Delta)^s u = \frac{1}{u^\delta} + f(u), & u > 0 & \text{in } \Omega, \\ u = 0 & & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $0 < s < 1$, $N \geq 2s$, $\Omega = B_r(0) \subset \mathbb{R}^N$, $\delta > 0$, $f(u)$ is a locally Lipschitz function. Precisely, we prove the following result:

Theorem 3.4.13. *Let $\delta > 0$ and f be a locally Lipschitz function. Then a classical solution u to (P) is radially symmetric and strictly decreasing in $|x|$.*

The proof of Theorem 3.4.13 involves the moving plane method adapted in the non local setting. In this regard, as in the local case, we need a maximum principle in narrow domains and a strong maximum principle to hold for equations of the type (P). The extension of these key tools is not straightforward due to the non local nature of $(-\Delta)^s$ and the presence of a singular nonlinearity in the right hand side. Besides this, we will take advantage of monotonicity properties of the nonlinear operator $(-\Delta)^s u - \frac{1}{u^\delta}$ and borrow some "local" maximum principle shown in [175].

In this regard, we introduce the following definitions:

Let $A_\lambda := \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_1 = \lambda\}$ and

$$\Sigma_\lambda := \begin{cases} \{x \in \mathbb{R}^N : x_1 < \lambda\} & \text{if } \lambda \leq 0, \\ \{x \in \mathbb{R}^N : x_1 > \lambda\} & \text{if } \lambda > 0, \end{cases}$$

for some $\lambda \in \mathbb{R}$ and $\mathcal{D}_\lambda(x) := (2\lambda - x_1, x_2, \dots, x_N)$ be the reflection of the point x about A_λ and $v_\lambda(x) := u_\lambda(x) - u(x)$ where $u_\lambda(x) = u(\mathcal{D}_\lambda(x))$ and let u be a classical solution of (P).

To prove radial symmetry and strict monotonicity of the solution u , it is enough to prove $v_\lambda(x) \geq 0$ for all $x \in B_r(0) \cap \Sigma_\lambda$ and $\lambda \in (-r, r)$, by moving hyperplane A_λ in a fixed direction. Since, if $v_\lambda(x) \geq 0$ for all $\lambda \in (-r, r)$ and $x \in B_r(0)$ holds then we can rotate and move the hyperplane A_λ in the direction close to fixed direction to get the desired result. Since λ is independent from the direction of movement of hyperplane A_λ , so we fix $\nu(x_0) = (1, 0, \dots, 0)$ (without loss of generality) as the direction of movement of hyperplane A_λ where ν denotes the unit outward normal vector at $x_0 = (r, 0, \dots, 0) \in \partial B_r(0)$. We divide the proof of above assertion into the following steps:

Step 1: $v_\lambda(x) \geq 0$ for all $x \in B_r(0) \cap \Sigma_\lambda$ and $|\lambda| \in [r_1, r)$ for some $r_1 > 0$:

By using the Poincaré inequality and Lipschitz property of the function f , we get $v_\lambda > 0$ in a region $\Sigma_\lambda \cap B_r$ for some $r - \epsilon_1 < |\lambda| < r$ and $\epsilon_1 > 0$. Now by rotating and moving the hyperplane A_λ in a direction close to the outward normal ν in any neighborhood of $x_0 \in \partial\Omega$ and repeating the above steps by taking into account that $x_0 \in \partial B_r(0)$, $\nu(x_0)$ is arbitrary and by using continuity of solution u we obtain, $v_\lambda(x) \geq 0$ for all $x \in B_r(0) \setminus B_{r_1}(0)$ and $|\lambda| \in [r_1, r)$ for some $r_1 > 0$.

From **Step 1**, we can assume that $\lambda = r_1$ be the smallest value such that $0 \leq r_1 < r$, $v_{r_1} \geq 0$ in $B_r \setminus B_{r_1}$ and satisfies

$$(-\Delta)^s v_{r_1}(x) - \frac{1}{u_{r_1}^\delta(x)} + \frac{1}{u^\delta(x)} = f(u_{r_1}) - f(u) \quad \text{in } B_r \setminus B_{r_1}. \quad (3.4.8)$$

Step 2: $r_1 = 0$.

To prove this, we adapt in our situation the maximum principles in nonlocal setting *i.e.* Proposition 3.5 (maximum principle in narrow domains) and Proposition 3.6 (strong maximum principle) in [175]. The main role of the above tools is to construct a strictly positive subsolution of the problem (3.4.8) in every compact subset of $B_r \setminus B_{r_1}$ so that $\text{ess inf}_R v_{r_1} > 0$ for every compact subset $R \subset B_r \setminus B_{r_1}$. For more details, we refer to the proof of Theorem 6.2.1, Page 340, Chapter 6. Now, by repeating the proof by moving hyperplane A_λ as in **Step 1** we obtain u is radially symmetric and the strict monotonicity property.

Next, we apply this main result in a different situation: Consider the problem

$$(Q) \quad \begin{cases} (-\Delta)^s u = \mu \left(\frac{1}{u^\delta} + f(u) \right), & u > 0 & \text{in } \Omega, \\ u = 0 & & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

where Ω is a bounded domain with C^2 boundary regularity. This concerns the existence of uniform a priori bound for classical solutions to (Q) when f has a subcritical growth. Similar type of result is also discussed in [6]. In the spirit of the work [121], we combine the monotonicity property of solutions near the boundary of Ω and a blow up technique with the help of a Liouville theorem. Precisely we prove:

Theorem 3.4.14. *Let $N > 2s$ and $\mu_0 > 0$. Let u be the classical solution of (Q) with*

$f(u) = u^p$ for $1 < p < \frac{N+2s}{N-2s}$ and $\mu \geq \mu_0$. Then $\|u\|_\infty \leq C_1$ with C_1 depending only on δ, p, Ω, μ_0 .

The second application concerns the asymptotic behaviour of large solutions with respect to the parameter μ . Let $s = \frac{1}{2}$, $n = 1$, $\Omega = B_r(0)$ and $f(u) = h(u) \exp(u^\alpha)$ for some $1 < \alpha \leq 2$ where h satisfies $\lim_{t \rightarrow \infty} h(t)e^{-\epsilon t^\alpha} = 0$ and $\lim_{t \rightarrow \infty} h(t)e^{\epsilon t^\alpha} = \infty$ for any $\epsilon > 0$. Then we have the following result that complements Theorem 3.4.11:

Theorem 3.4.15. *Let $\mu_0 > 0$ and u be the classical solution of (Q) for some $\mu \geq \mu_0$. Then for any $\epsilon > 0$, the following holds*

$$\|u\|_{L^\infty(B_r \setminus B_\epsilon)} \leq C_2(\delta, n, \epsilon, \mu_0).$$

In addition, we have the following blow up profile: Let $\{u_k\}$ be a sequence of solutions for the problem (Q) such that $\|u_k\|_{L^\infty(B_r)} \rightarrow \infty$, $\mu_k \rightarrow \tilde{\mu}$ with $\tilde{\mu} > 0$,

- (i) *There exists a singular solution \tilde{u} in $C_{loc}^s(B_r \setminus \{0\})$ such that $u_k - \tilde{u} \rightarrow 0$ in $L_{loc}^\infty(B_r \setminus \{0\})$.*
- (ii) *If $(u_k)_{k \in \mathbb{N}}$ has uniform bounded energy and $F(t) = O(f(t))$ as $t \rightarrow \infty$ where $F(t)$ is the antiderivative of f , then $\tilde{\mu} = 0$.*

For more details, we refer to Theorem 6.2.2 and Theorem 6.2.3, Page 342, Chapter 6.

3.4.2 **Problem 2:** Non-linear fractional singular problem with singular weights

In this part, we study the following nonlinear fractional elliptic and singular problem

$$(P) \begin{cases} (-\Delta)_p^s u = \frac{K_\delta(x)}{u^\gamma}, u > 0 & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $C^{1,1}$ boundary, $s \in (0, 1)$, $p \in (1, +\infty)$, $\gamma > 0$ and K_δ satisfies the growth condition: for any $x \in \Omega$

$$\frac{\mathcal{C}_1}{d^\delta(x)} \leq K_\delta(x) \leq \frac{\mathcal{C}_2}{d^\delta(x)} \quad (3.4.9)$$

for some $\delta \in [0, sp)$, where, for any $x \in \Omega$, $d(x) = \text{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|$.

Due to the nonlinearity of the operator and absence of integration by parts formula, we define the following notion of weak solution:

Definition 3.4.1. *A function $u \in W_{loc}^{s,p}(\Omega)$ is said to be a weak subsolution (resp. supersolution) of (P), if*

$$u^\kappa \in W_0^{s,p}(\Omega), \quad \inf_K u > 0 \text{ for all } K \Subset \Omega \text{ and for some } \kappa \geq 1$$

and

$$\iint_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)]^{p-1} (\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy \leq (\text{resp. } \geq) \int_{\Omega} \frac{K_{\delta}(x)}{u^{\gamma}} \phi dx$$

for all $\phi \in \mathbb{T} = \bigcup_{\tilde{\Omega} \in \Omega} W_0^{s,p}(\tilde{\Omega})$.

A function which is both sub and supersolution of (P) is called a weak solution.

Having in mind Lemma 3.5, [78], Lemma 3.3, [66] and the condition $u^{\kappa} \in W_0^{s,p}(\Omega)$, $\kappa \geq 1$ in definition 3.4.1, u satisfies the following definition of the boundary datum:

Definition 3.4.2. We say that a function $u = 0$ in $\mathbb{R}^N \setminus \Omega$ satisfies $u \leq 0$ on $\partial\Omega$ in sense that for $\epsilon > 0$, $(u - \epsilon)^+ \in W_0^{s,p}(\Omega)$.

3.4.2.1 New Tools

This part is devoted to the development of new tools to deal with the singular problem (P). In this regard, as a first preliminary tool we define the approximated problem (P_{ϵ}^{γ}) (see below) and study the existence of an increasing sequence of weak solutions of approximated problem (P_{ϵ}^{γ}) (see below Proposition 3.4.1).

To deal with the boundary behavior of the weak solution of the original problem (P), we also study the purely singular weight problem (S_0^{δ}) (see below). We define a new notion of weak energy solution and by using barrier arguments and exploiting the $C^{1,1}$ regularity of the boundary, we construct lower and super solutions of the purely singular weight problem near the boundary.

Tool 1:

For a fixed parameter $\epsilon > 0$, we define a sequence of function $K_{\epsilon,\delta} : \mathbb{R}^N \rightarrow \mathbb{R}_+$ as

$$K_{\epsilon,\delta}(x) = \begin{cases} (K_{\delta}^{-\frac{1}{\delta}}(x) + \epsilon^{\frac{\gamma+p-1}{sp-\delta}})^{-\delta} & \text{if } x \in \Omega, \\ 0 & \text{else,} \end{cases}$$

and $K_{\epsilon,\delta}$ is an increasing function as $\epsilon \downarrow 0$, $K_{\epsilon,\delta} \rightarrow K_{\delta}$ a.e. in Ω and there exist two positive constants $\mathcal{C}_3, \mathcal{C}_4$ such that, for any $x \in \Omega$,

$$\frac{\mathcal{C}_3}{\left(d(x) + \epsilon^{\frac{\gamma+p-1}{sp-\delta}}\right)^{\delta}} \leq K_{\epsilon,\delta}(x) \leq \frac{\mathcal{C}_4}{\left(d(x) + \epsilon^{\frac{\gamma+p-1}{sp-\delta}}\right)^{\delta}}.$$

Define the approximated problem as

$$(P_{\epsilon}^{\gamma}) \begin{cases} (-\Delta)_p^s u = \frac{K_{\epsilon,\delta}(x)}{(u + \epsilon)^{\gamma}} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Proposition 3.4.1. *For any $\epsilon > 0$ and $\gamma \geq 0$, there exists a unique weak solution $u_\epsilon \in W_0^{s,p}(\Omega) \cap C^{0,\ell}(\bar{\Omega})$ of the problem (P_ϵ^γ) i.e.*

$$\iint_{\mathbb{R}^{2N}} \frac{[u_\epsilon(x) - u_\epsilon(y)]^{p-1}(\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy = \int_{\Omega} \frac{K_{\epsilon,\delta}(x)}{(u + \epsilon)^\gamma} \phi dx \quad (3.4.10)$$

for all $\phi \in W_0^{s,p}(\Omega)$ and for some $\ell \in (0, 1)$. Moreover, the sequence $\{u_\epsilon\}_{\epsilon>0}$ satisfies $u_\epsilon > 0$ in Ω ,

$$u_{\epsilon_1}(x) < u_{\epsilon_2}(x) \text{ in } \Omega \text{ and } \epsilon_2 < \epsilon_1$$

and for any $\Omega' \Subset \Omega$, there exists $\sigma = \sigma(\Omega') > 0$ such that for any $\epsilon \in (0, 1)$:

$$\sigma \leq u_1(x) < u_\epsilon(x) \text{ in } \Omega'.$$

The proof follows from the maximum principle having in mind that $(-\Delta)_p^s \cdot + \frac{1}{(\cdot + \epsilon)^\delta}$ is a monotone operator in $W_0^{s,p}(\Omega)$ (see also Proposition 2.3, Lemma 2.4 in [78] and Theorem 1.1 in [172]).

Tool 2:

As a second tool, we study the following problem:

$$(S_0^\delta) \begin{cases} (-\Delta)_p^s u(x) = K_\delta(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

where $d^\delta(x)K_\delta(x) = O(1)$ and introduce the new notion of weak energy solution and corresponding vector space. Let $\Omega \subset \mathbb{R}^N$ be bounded. We define

$$\bar{W}^{s,p}(\Omega) := \left\{ u \in L_{loc}^p(\mathbb{R}^N) : \exists K \text{ s.t. } \Omega \Subset K, \|u\|_{W^{s,p}(K)} + \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1 + |x|)^{N+sp}} dx < \infty \right\}$$

where $\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + [u]_{s,p,\Omega}$. If Ω is unbounded, we define

$$\bar{W}_{loc}^{s,p}(\Omega) := \{u \in L_{loc}^p(\mathbb{R}^N) : u \in \bar{W}^{s,p}(\tilde{\Omega}), \text{ for any bounded } \tilde{\Omega} \subset \Omega\}.$$

Definition 3.4.3. (Weak energy Solution) *Let $f \in W^{-s,p'}(\Omega)$ and Ω be a bounded domain. We say that $u \in \bar{W}^{s,p}(\Omega)$ is a weak energy solution of $(-\Delta)_p^s(u) = f$ in Ω , if*

$$\iint_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)]^{p-1}(\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy = \int_{\Omega} f(x)\phi(x) dx$$

for all $\phi \in W_0^{s,p}(\Omega)$ and a function u is weak energy subsolution (resp. weak energy supersolution) of $(-\Delta)_p^s(u) = f$ in Ω , if

$$(-\Delta)_p^s(u) \leq (\text{resp. } \geq) f \quad E\text{-weakly in } \Omega$$

i.e.

$$\iint_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)]^{p-1}(\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy \leq (\text{resp. } \geq) \int_{\Omega} f(x)\phi(x) dx$$

for all $\phi \in W_0^{s,p}(\Omega)$, $\phi \geq 0$.

If Ω is unbounded we say that $u \in \overline{W}_{loc}^{s,p}(\Omega)$ is a energy solution or energy subsolution (or energy supersolution) of $(-\Delta)_p^s(u) = f$ in Ω , if it does so in any open bounded set $\Omega' \subset \Omega$.

In order to study the boundary behavior of minimal weak solution for purely singular weight problem (S_0^δ) , we construct explicitly lower and supersolutions of the weak solution to the approximated problem. For this, first we define the prototype of the barrier function in \mathbb{R}_+ (and \mathbb{R}_+^N) as a power type function and compute upper and lower estimates of $(-\Delta)_p^s$ acting on this function. Precisely, we have:

For any $\alpha \in (0, s)$, we begin by computing the upper and lower estimates in the half line $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$ of $(-\Delta)_p^s$ of the function $U_\lambda(x) := \left((x + \lambda^{\frac{1}{\alpha}})^+\right)^\alpha$, $\lambda \geq 0$ defined in \mathbb{R} . We recall the notation, for any $t \in \mathbb{R}$, $[t]^{p-1} = |t|^{p-2}t$.

Theorem 3.4.1. *Let $\lambda \geq 0$, $\alpha \in (0, s)$ and $p > 1$. Then, there exist two positive constants $C_1, C_2 > 0$ depending upon α, p and s such that*

$$C_1(x + \lambda^{\frac{1}{\alpha}})^{-\beta} \leq (-\Delta)_p^s U_\lambda(x) \leq C_2(x + \lambda^{\frac{1}{\alpha}})^{-\beta} \text{ pointwisely in } \mathbb{R}_+. \quad (3.4.11)$$

Moreover, for $\lambda > 0$, $U_\lambda \in \overline{W}_{loc}^{s,p}(\mathbb{R}_+)$ and for $\lambda = 0$, $U_\lambda \in \overline{W}_{loc}^{s,p}(\mathbb{R}_+)$ if $s - \frac{1}{p} < \alpha < s$.

To prove the estimate in (3.4.11), we explicitly estimate the $(-\Delta)_p^s$ of barrier functions. The crucial point in the proof of estimates is the positivity of constant C_1 , which plays a decisive role in further computations. These types of estimates are motivated from the fact that $(-\Delta)_p^s(x)_+^s = 0$. The proof of above result is rather technical and tricky (for a detailed proof see Theorem 6.3.5, Chapter 6).

We then consider the case of flat boundary of \mathbb{R}^N . Precisely, by extending the functions U_λ to V_λ defined on \mathbb{R}^N , we study the behavior of $(-\Delta)_p^s V_\lambda(x)$ on $\mathbb{R}_+^N := \{x \in \mathbb{R}^N : x_N > 0\}$ where $V_\lambda(x) := U_\lambda(x \cdot e_N) = U_\lambda(x_N)$. Let GL_N be the set of $N \times N$ invertible matrices, we prove

Corollary 3.4.1. *Let $\lambda \geq 0$, $\alpha \in (0, s)$, $A \in GL_N$ and $p > 1$. Let $\mathcal{J}_{\epsilon,A}$ be the function defined on \mathbb{R}_+^N by*

$$\mathcal{J}_{\epsilon,A}(x) = \int_{B_\epsilon(0)^c} \frac{[V_\lambda(x) - V_\lambda(x+z)]^{p-1}}{|Az|^{N+sp}} dz$$

for some $\epsilon > 0$.

Then, there exist two positive constants C_3 and C_4 depending on $\alpha, s, p, N, \|A\|_2, \|A^{-1}\|_2$ such that

$$C_3(x_N + \lambda^{1/\alpha})^{-\beta} \leq \lim_{\epsilon \rightarrow 0} \mathcal{J}_{\epsilon,A}(x) \leq C_4(x_N + \lambda^{1/\alpha})^{-\beta}$$

pointwisely in $\mathbb{R}_+^N \times GL_N$. In particular, for $A = I$, there exist two positive constants \tilde{C}_3 and \tilde{C}_4 independent of λ such that:

$$\tilde{C}_3(x_N + \lambda^{1/\alpha})^{-\beta} \leq (-\Delta)_p^s V_\lambda(x) \leq \tilde{C}_4(x_N + \lambda^{1/\alpha})^{-\beta} \text{ pointwisely in } \mathbb{R}_+^N.$$

Moreover, for $\lambda > 0$, $V_\lambda \in \overline{W}_{loc}^{s,p}(\mathbb{R}_+^N)$ and for $\lambda = 0$, $V_\lambda \in \overline{W}_{loc}^{s,p}(\mathbb{R}_+^N)$ if $s - \frac{1}{p} < \alpha < s$.

Next, in order to handle the case of smooth boundary portion we prove the above upper and lower estimates are preserved under a smooth change of variables via a $C^{1,1}$ diffeomorphism (close to identity) in the isomorphic image of a set close to the boundary of \mathbb{R}_+^N .

Theorem 3.4.2. *Let $\alpha \in (0, s)$ and $p > 1$. Let $\psi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a $C^{1,1}$ -diffeomorphism in \mathbb{R}^N such that $\psi = Id$ in $B_R(0)^c$, for some $R > 0$.*

Then, considering $W_\lambda(x) = U_\lambda(\psi^{-1}(x) \cdot e_N)$, there exist $\rho^ = \rho^*(\psi) > 0$ and $\lambda^* = \lambda^*(\psi) > 0$ such that for any $\rho \in (0, \rho^*)$, there exists a constant $\tilde{C} > 0$ independent of λ such that, for any $\lambda \in [0, \lambda^*]$,*

$$\frac{1}{\tilde{C}} W_\lambda(x)^{-\frac{\beta}{\alpha}} \leq (-\Delta)_p^s W_\lambda(x) \leq \tilde{C} W_\lambda(x)^{-\frac{\beta}{\alpha}} \quad E\text{-weakly in } \psi(\{X : 0 < X_N < \rho\}). \quad (3.4.12)$$

The crucial step for proving estimate (3.4.12) is to split the integral of nonlocal terms in different sub regions in the light of Lemma 2.5 and Lemma 3.4 in [172] and observing that proving (3.4.12) is equivalent to show that there exists a constant \tilde{C} independent of λ such that

$$\frac{1}{\tilde{C}} (X_N + \lambda^{1/\alpha})^{-\beta} \leq \lim_{\epsilon \rightarrow 0} H_\epsilon(x) \leq \tilde{C} (X_N + \lambda^{1/\alpha})^{-\beta}$$

for all $x \in \psi(\{X : 0 < X_N < \rho\})$ where

$$H_\epsilon(x) = \int_{(D_\epsilon(x))^c} \frac{[W_\lambda(x) - W_\lambda(y)]^{p-1}}{|x - y|^{N+sp}} dy \quad (3.4.13)$$

and $D_\epsilon(x) = \{y \in \mathbb{R}^N : |\psi^{-1}(x) - \psi^{-1}(y)| \leq \epsilon\}$ and $\epsilon > 0$. To estimate (3.4.13), we exploit the $C^{1,1}$ regularity of the diffeomorphism, Corollary 3.4.1 and the fact that $\psi = I$ outside $B_R(0)$ for some $R > 0$. (For precise details, we refer to Theorem 6.3.6, Page 355, Chapter 6).

Due to nonlocal feature of $(-\Delta)_p^s$, we extend the definition of the distance function d in Ω^c as follows

$$d_e(x) = \begin{cases} \text{dist}(x, \partial\Omega) & \text{if } x \in \Omega, \\ -\text{dist}(x, \partial\Omega) & \text{if } x \in (\Omega^c)_{\lambda^{\frac{1}{\alpha}}}, \\ -\lambda^{1/\alpha} & \text{otherwise,} \end{cases}$$

where $(\Omega^c)_\eta = \{x \in \Omega^c : \text{dist}(x, \partial\Omega) < \eta\}$. Hence we define, for some $\rho > 0$ and $\lambda > 0$:

$$\underline{w}_\rho(x) = \begin{cases} (d_e(x) + \lambda^{1/\alpha})_+^\alpha - \lambda & \text{if } x \in \Omega \cup (\Omega^c)_\rho, \\ -\lambda & \text{otherwise,} \end{cases} \quad (3.4.14)$$

$$\overline{w}_\rho(x) = \begin{cases} (d_e(x) + \lambda^{1/\alpha})_+^\alpha & \text{if } x \in \Omega \cup (\Omega^c)_\rho, \\ 0 & \text{otherwise.} \end{cases} \quad (3.4.15)$$

Now, we state and prove our main result for establishing the boundary behavior:

Theorem 3.4.3. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain with a $C^{1,1}$ boundary and $\alpha \in (0, s)$. Then, for some $\rho > 0$, there exist $(\lambda_*, \eta_*) \in \mathbb{R}_*^+ \times \mathbb{R}_*^+$ such that for any $\eta < \eta_*$, there exist positive constants C_5, C_6 such that for any $\lambda \in [0, \lambda_*]$:*

$$(-\Delta)_p^s \bar{w}_\rho \geq C_5(d(x) + \lambda^{1/\alpha})^{-\beta} \quad \text{and} \quad (-\Delta)_p^s \underline{w}_\rho \leq C_6(d(x) + \lambda^{1/\alpha})^{-\beta} \quad \text{E-weakly in } \Omega_\eta \quad (3.4.16)$$

where $\Omega_\eta = \{x \in \Omega : d(x) < \eta\}$. Moreover, for $\lambda > 0$, $\underline{w}_\rho, \bar{w}_\rho$ belong to $\overline{W}^{s,p}(\Omega_\eta)$.

We divide the proof into three main steps:

Step 1: Covering of Ω_η

In this step, we choose a special covering $\{B_{R_i}(x_i)\}_i$ of Ω_η and diffeomorphisms Φ_i with some local inclusion properties. Precisely, by using the geometry of $\partial\Omega$ and arguing as in Lemma 3.5 and Theorem 3.6 in [172], there exist a finite covering $\{B_{R_i}(x_i)\}_{i \in I}$ of $\partial\Omega$, $\eta^* = \eta^*(R_i)$, $i \in I$ and diffeomorphisms $\Phi_i \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$ such that for any $\eta \in (0, \eta^*)$, $i \in I$

$$\Omega_\eta \cap B_{R_i}(x_i) \Subset \Phi_i(B_{\tilde{\rho}} \cap \mathbb{R}_+^N), \quad d_e(\Phi_i(X)) = (X_N + \lambda^{1/\alpha})_+ - \lambda^{1/\alpha} \quad \forall X \in B_\rho$$

$$\Phi_i(X) = X \quad \text{for } X \in (B_{4\rho}(0))^c$$

with $0 < \tilde{\rho} < \rho < \rho^*$ where ρ^* is defined in Theorem 3.4.2 and for λ small enough $\lambda^{1/\alpha} < \rho$,

$$\Phi_i(B_\rho(0) \cap \{X_N \geq -\lambda^{1/\alpha}\}) \subset \Omega \cup (\Omega^c)_\rho.$$

Using the finite covering, it is sufficient to prove the (3.4.16) in any of set $\Omega_\eta \cap B_{R_i}(x_i)$ with $x_i \in \partial\Omega$.

Step 2: For $x \in \Omega_\eta \cap B_{R_i}(x_i)$ with $x_i \in \partial\Omega$ the following holds:

$$\lim_{\epsilon \rightarrow 0} g_{\epsilon,1}(x) \leq c_3(d(x) + \lambda^{1/\alpha})^{-\beta} \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} g_{\epsilon,2}(x) \geq c_4(d(x) + \lambda^{1/\alpha})^{-\beta} \quad \text{E-weakly in } \Omega_\eta \cap B_{R_i}(x_i)$$

where $g_{\epsilon,1}$ and $g_{\epsilon,2}$ be two functions defined by

$$g_{\epsilon,1}(x) = \int_{D_\epsilon(x)} \frac{[\underline{w}_\rho(x) - \underline{w}_\rho(y)]^{p-1}}{|x-y|^{N+sp}} dy$$

and

$$g_{\epsilon,2}(x) = \int_{D_\epsilon(x)} \frac{[\bar{w}_\rho(x) - \bar{w}_\rho(y)]^{p-1}}{|x-y|^{N+sp}} dy$$

and $D_\epsilon(x) = \{y : |\Phi^{-1}(x) - \Phi^{-1}(y)| > \epsilon\}$.

As above, it suffices to obtain suitable uniform bounds on compact sets of $g_{\epsilon,1}$ and $g_{\epsilon,2}$. These uniform bounds are attained by the regularity and inclusion properties mentioned in Step 1, and using Theorem 3.4.2.

Step 3: $\underline{w}_\rho, \bar{w}_\rho \in \overline{W}^{s,p}(\Omega_\eta)$

It is sufficient to claim

$$\underline{w}_\rho, \bar{w}_\rho \in W^{s,p}(K) \text{ for } K = \Omega_{\eta_1} \cup (\Omega^c)_{\eta_2}$$

where $0 < \eta < \eta_1$ and $\eta_2 > 0$. For $x_i \in \partial\Omega$, $\eta_0 \in (\eta, \eta^*)$, let $\{B_{R_i}(x_i)\}_{i \in I}$ be the finite covering of Ω_{η_0} and $\Xi_i \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$ such that

$$B_{R_i}(x_i) \Subset \Xi_i(B_{\xi_0}), \quad d_e(\Xi_i(X)) = (X_N + \lambda^{1/\alpha})_+ - \lambda^{1/\alpha}, \quad \forall X \in B_{\xi_0} \quad (3.4.17)$$

for some $\xi_0 > 0$. The existence of finite covering $\{B_{R_i}(x_i)\}_{i \in I}$ and diffeomorphism Ξ_i can be proved by using Step 1. For any $i \in I$, there exists a subset J^i of I such that $i \notin J^i$ and

$$B_{R_i}(x_i) \cap B_{R_j}(x_j) \neq \emptyset \quad \forall j \in J^i. \quad (3.4.18)$$

Now for any $i \in I$ and $j \in J^i$, define $K_i := B_{\tau_i}(x_i) \subset B_{R_i}(x_i)$ for $\tau_i < R_i$ such that

$$K_i \cap K_j \neq \emptyset \quad \text{and} \quad \min_{i \in I} \min_{j \in J^i} \text{dist}(K_j \setminus B_{R_i}(x_i), K_i) > 0. \quad (3.4.19)$$

By using (3.4.18) and (3.4.19), we choose η_1 and η_2 small enough such that

$$K = \Omega_{\eta_1} \cup \Omega_{\eta_2}^c \subset \bigcup_{i \in I} K_i$$

and using (3.4.17), we obtain

$$\begin{aligned} \Omega_{\eta_1} \cap K_i &\subset \Omega_{\eta_1} \cap B_{R_i}(x_i) \Subset \Xi_i(B_{\xi_0} \cap \mathbb{R}_+^N), \\ \Omega_{\eta_2}^c \cap K_i &\subset \Omega_{\eta_2}^c \cap B_{R_i}(x_i) \Subset \Xi_i(B_{\xi_1} \cap \mathbb{R}_+^N), \\ d_e(\Xi_i(X)) &= (X_N + \lambda^{1/\alpha})_+ - \lambda^{1/\alpha}, \quad \forall X \in \Xi_i^{-1}(K_i) \subset B_{\xi_0} \end{aligned} \quad (3.4.20)$$

for some $\eta_1 < \eta^*$ and $\eta_2 > 0$ such that $0 < \xi_1 < \frac{\lambda^{1/\alpha}}{2}$. Set $\widehat{K}_i = K_i \cap K$. Then

$$\begin{aligned} \int_{K \times K} &= \sum_{i \in I} \sum_{i_1 \in I} \int_{\widehat{K}_i} \int_{\widehat{K}_{i_1}} = \sum_{i \in I} \left(\sum_{i_1 \in J^i} \int_{\widehat{K}_i} \int_{\widehat{K}_{i_1} \cap B_{R_i}^c(x_i)} + \sum_{i_1 \in I \setminus J^i} \int_{\widehat{K}_i} \int_{\widehat{K}_{i_1}} \right. \\ &\quad \left. + \sum_{i_1 \in J^i} \int_{\widehat{K}_i} \int_{\widehat{K}_{i_1} \cap B_{R_i}(x_i)} + \int_{\widehat{K}_i} \int_{\widehat{K}_i} \right) := \int_{\mathcal{Q}_1} + \int_{\mathcal{Q}_2} + \int_{\mathcal{Q}_3} + \int_{\mathcal{Q}_4}. \end{aligned}$$

Now by estimating integrand over $\mathcal{Q}_1, \mathcal{Q}_2$ we use (3.4.18)-(3.4.19) and to estimate over $\mathcal{Q}_3, \mathcal{Q}_4$, we perform change of variables using the diffeomorphisms Ξ_i , and (3.4.20) and by observing the fact that $X_N, Y_N > -\min\{\xi_0, \xi_1\} > -\frac{\lambda^{1/\alpha}}{2}$ for all $X, Y \in \Xi_i^{-1}(\widehat{K}_i)$, we obtain $\underline{w}_\rho \in \overline{W}^{s,p}(\Omega_\eta)$. Similarly, we can prove $\bar{w}_\rho \in \overline{W}^{s,p}(\Omega_\eta)$.

3.4.2.2 Main results and glimpse of proof

Here we describe our results with main ingredients of the proof. First, we establish the following very weak comparison principle:

Theorem 3.4.4. *For $0 \leq \delta < 1 + s - \frac{1}{p}$, $\gamma \geq 0$, let u be a subsolution of (P) and \tilde{v} be a supersolution of (P) in the sense of Definition 3.4.1. Then $u \leq \tilde{v}$ a.e. in Ω .*

We start by defining the energy functional J_ϵ on $W_0^{s,p}(\Omega)$ as, for $\epsilon > 0$

$$J_\epsilon(w) := \frac{1}{p} \iint_{\mathbb{R}^{2N}} \frac{|w(x) - w(y)|^p}{|x - y|^{N+sp}} dx dy - \int_{\Omega} K_\delta(x) G_\epsilon(w) dx$$

where G_ϵ is the primitive such that $G_\epsilon(1) = 0$ of the function g_ϵ defined by

$$g_\epsilon(t) = \begin{cases} \min \left\{ \frac{1}{t^\gamma}, \frac{1}{\epsilon} \right\} & \text{if } t > 0, \\ \frac{1}{\epsilon} & \text{if } t \leq 0. \end{cases}$$

Claim: There exists a minimizer of the energy functional J_ϵ in

$$\mathcal{L} := \{ \phi \in W_0^{s,p}(\Omega) : 0 \leq \phi \leq \tilde{v} \text{ a.e. in } \Omega \}.$$

The restriction on δ ensure the weakly lower semicontinuity of the energy functional J_ϵ in $W_0^{s,p}(\Omega)$ and $\frac{1}{d^\delta(x)} \in W^{-s,p'}(\Omega)$ via Hardy inequality. Precisely, let $\{w_n\} \subset W_0^{s,p}(\Omega)$ be such that $w_n \rightharpoonup w$ in $W_0^{s,p}(\Omega)$. Let $\nu \in (0, 1)$ small enough such that $\frac{1-\nu}{p} + \frac{\nu}{q} + \frac{1}{r} = 1$ where $q < p_s^* := \frac{Np}{N-sp}$ if $N > sp$ and $(s(1-\nu) - \delta)r > -1$ (since $\delta < 1 + s - \frac{1}{p}$). Hence $x \mapsto d^{s(1-\nu)-\delta}(x) \in L^r(\Omega)$ and by using Hölder and Hardy inequalities (see Theorem 1.4.4.4 and Corollary 1.4.4.10 in [162]), we obtain

$$\begin{aligned} \int_{\Omega} \frac{|w_n - w|}{d^\delta(x)} dx &= \int_{\Omega} \left(\frac{|w_n - w|}{d^s(x)} \right)^{1-\nu} |w_n - w|^\nu d^{s(1-\nu)-\delta}(x) dx \\ &\leq C \|w_n - w\|_{s,p}^{1-\nu} \|w_n - w\|_{L^q(\Omega)}^\nu \end{aligned}$$

for some constant $C > 0$ independent of w_n and w . Since $W_0^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for $q < p_s^*$, $\|w_n - w\|_{s,p}$ is uniformly bounded in n and $\|w_n - w\|_{L^q(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$. Finally, gathering the weakly lower semicontinuity of $[\cdot]_{s,p}$ and G_ϵ globally Lipschitz, we deduce that J_ϵ is weakly lower semicontinuous in $W_0^{s,p}(\Omega)$ and admits a minimizer w_0 on \mathcal{L} . By density results and Fatou's lemma, we have for any $\phi \in W_0^{s,p}(\Omega)$ with $\phi \geq 0$,

$$\int_{\mathbb{R}^{2N}} \frac{[w_0(x) - w_0(y)]^{p-1} (\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy \geq \int_{\Omega} K_\delta(x) g_\epsilon(w_0) \phi dx. \quad (3.4.21)$$

Now, by using $w_0 \in W_0^{s,p}(\Omega)$ and $w_0 \geq 0$, we get for any $\epsilon_1 > 0$

$$\text{supp}((u - w_0 - \epsilon_1)^+) \subset \text{supp}((u - \epsilon_1)^+) \text{ and } (u - w_0 - \epsilon_1)^+ \in W_0^{s,p}(\Omega).$$

Since u is a subsolution of (P) then for any $\phi \in \mathbb{T}$, we get

$$\int_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)]^{p-1}(\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy \leq \int_{\Omega} K_{\delta}(x) \frac{\phi(x)}{u^{\gamma}} dx. \quad (3.4.22)$$

Then, by subtracting (3.4.21) and (3.4.22), using the following inequality, for $a, b \in \mathbb{R}$ with $|a| + |b| > 0$,

$$(|a|^{p-2}a - |b|^{p-2}b)(a - b) \geq C(|a| + |b|)^{p-2}|a - b|^2$$

and by choosing ϵ_1 such that $\epsilon_1^{\gamma} > \epsilon$ and $\phi = (u - w_0 - \epsilon_1)$ as a test function, we obtain $u \leq w_0 + \epsilon_1 \leq \tilde{v} + \epsilon_1$ in Ω and letting $\epsilon_1 \rightarrow 0$, we get our proof.

Next, we prove the existence result:

Theorem 3.4.5. *Let Ω be a bounded domain with Lipschitz boundary $\partial\Omega$ and $\delta \in (0, sp)$. Then,*

- (i) *for $\delta - s(1 - \gamma) \leq 0$, then there exists a minimal weak solution u to (P) in $W_0^{s,p}(\Omega)$.*
- (ii) *for $\delta - s(1 - \gamma) > 0$, there exist a minimal weak solution u to (P) and a constant θ_0 such that*

$$u^{\theta} \in W_0^{s,p}(\Omega) \text{ if } \theta \geq \theta_0 \text{ and } \theta_0 > \max \left\{ 1, \frac{p + \gamma - 1}{p}, \Lambda \right\}$$

$$\text{where } \Lambda := \frac{(sp-1)(p-1+\gamma)}{p(sp-\delta)}.$$

We describe the main ingredients of the existence results and Sobolev regularity depending upon s and singular exponents δ, γ .

Let $u_{\epsilon} \in W_0^{s,p}(\Omega)$ be the weak solution of (P_{ϵ}^{γ}) satisfying (3.4.10) for all $\phi \in W_0^{s,p}(\Omega)$. It suffices to verify the sequences $\{u_{\epsilon}\}_{\epsilon}$ in the case $\delta - s(1 - \gamma) \leq 0$ and $\{u_{\epsilon}^{\theta}\}$ for a suitable parameter $\theta > 1$ in the case $\delta - s(1 - \gamma) > 0$ are bounded in $W_0^{s,p}(\Omega)$ and the convergence of the right-hand side in (3.4.10). Precisely,

The condition implies $\gamma < 1$ hence taking $\phi = u_{\epsilon}$ in (3.4.10) and applying Hölder and Hardy inequalities (see Theorem 1.4.4.4 and Corollary 1.4.4.10 in [162]), we obtain

$$[u_{\epsilon}]_{s,p}^p \leq C_2 \int_{\Omega} d^{s(1-\gamma)-\delta}(x) \left(\frac{u_{\epsilon}}{d^s(x)} \right)^{1-\gamma} dx \leq C \| \frac{u_{\epsilon}}{d^s} \|_{L^p(\Omega)}^{1-\gamma} \leq C [u_{\epsilon}]_{s,p}^{1-\gamma} \quad (3.4.23)$$

which implies $\|u_{\epsilon}\|_{s,p} \leq C < \infty$.

Case 2: $\delta - s(1 - \gamma) > 0$

Let $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the function defined as $\Phi(t) = t^{\theta}$ for some $\theta > \max \left\{ 1, \frac{p+\gamma-1}{p}, \Lambda \right\}$. Using the convexity of the function Φ and Lemma 3.3 in [66], we obtain: for any $\epsilon > 0$

$$\iint_{\mathbb{R}^{2N}} \frac{[\Phi(u_{\epsilon})(x) - \Phi(u_{\epsilon})(y)]^{p-1}(\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy \leq \int_{\Omega} \frac{K_{\epsilon,\delta}(x)}{(u_{\epsilon} + \epsilon)^{\gamma}} |\Phi'(u_{\epsilon})|^{p-2} \Phi'(u_{\epsilon}) \phi dx \quad (3.4.24)$$

for all nonnegative functions $\phi \in W_0^{s,p}(\Omega)$. Since $u_\epsilon \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ and Φ is locally Lipschitz, therefore $\Phi(u_\epsilon) \in W_0^{s,p}(\Omega)$. Then by choosing $\phi = \Phi(u_\epsilon)$ as a test function in (3.4.24), we get

$$\iint_{\mathbb{R}^{2N}} \frac{|\Phi(u_\epsilon)(x) - \Phi(u_\epsilon)(y)|^p}{|x - y|^{N+sp}} dx dy \leq C_2 \int_{\Omega} \frac{1}{d^\delta(x)} \frac{|\Phi'(u_\epsilon)|^{p-2} \Phi'(u_\epsilon) \Phi(u_\epsilon)}{u_\epsilon^\gamma} dx. \quad (3.4.25)$$

Now, for any $\epsilon > 0$, there exists a constant C independent of ϵ such that

$$\frac{|\Phi'(u_\epsilon)|^{p-2} \Phi'(u_\epsilon) \Phi(u_\epsilon)}{u_\epsilon^\gamma} \leq C(\Phi(u_\epsilon))^{\frac{\theta p - (p+\gamma-1)}{\theta}} \quad (3.4.26)$$

where $\frac{\theta p - (p+\gamma-1)}{\theta} > 0$ since $\theta > \frac{p+\gamma-1}{p}$. By combining (3.4.25)-(3.4.26) and applying Hölder and Hardy inequalities, we conclude that $\{\Phi(u_\epsilon)\}_{\epsilon>0}$ is bounded in $W_0^{s,p}(\Omega)$. Based on these uniform estimates, we pass to the limits $\epsilon \rightarrow 0$ to complete the rest of the proof. Finally, for any $\epsilon > 0$, $u_\epsilon \leq v$ a.e. in Ω where v is another weak solution of (P) . Indeed, v is a weak supersolution in sense of Definition 3.4.1 of the problem (P_ϵ^γ) hence weak comparison principle in Theorem 3.4.4 and passing limits $\epsilon \rightarrow 0$, we obtain that u is a minimal solution.

Remark 3.4.1. *The proof of Case 1 holds assuming $\Lambda \leq 1$ and $\gamma < 1$. Indeed, $d^{s(1-\gamma)-\delta} \in L^{\frac{p}{p-1+\gamma}}(\Omega)$ and we obtain (3.4.23).*

Now to study the boundary behavior of the main problem (P) with respect to distance function, we state and prove existence and boundary behavior of weak minimal solution of purely singular weight problem. For that, we consider the sequence of function $\{\tilde{K}_{\lambda,\delta}\}_{\lambda \geq 0}$ where $\delta \in (0, sp)$, $\tilde{K}_{\lambda,\delta} : \mathbb{R}^N \rightarrow \mathbb{R}_+$ such that

$$\tilde{K}_{\lambda,\delta}(x) = \begin{cases} (K_\delta^{-\frac{1}{\delta}}(x) + \lambda^{\frac{p-1}{sp-\delta}})^{-\delta} & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega, \end{cases}$$

satisfying $\tilde{K}_{\lambda,\delta} \nearrow K_\delta$ a.e. in Ω as $\lambda \rightarrow 0^+$, and there exist two positive constants $\mathcal{D}_3, \mathcal{D}_4$ such that

$$\frac{\mathcal{D}_3}{\left(d(x) + \lambda^{\frac{p-1}{sp-\delta}}\right)^\delta} \leq \tilde{K}_{\lambda,\delta}(x) \leq \frac{\mathcal{D}_4}{\left(d(x) + \lambda^{\frac{p-1}{sp-\delta}}\right)^\delta}. \quad (3.4.27)$$

Gathering Proposition 3.4.1, Theorem 3.4.5 and Remark 3.4.1, we have the following result for the following approximated problem (noting $\gamma = 0$ in Proposition 3.4.1):

$$(S_\lambda^\delta) \begin{cases} (-\Delta)_p^s u = \tilde{K}_{\lambda,\delta} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Theorem 3.4.6. *Let Ω be a bounded domain with Lipschitz boundary. Then there exists a increasing sequence of weak solution $\{u_\lambda\}_{\lambda>0} \subset W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ of (S_λ^δ) such that*

$$\iint_{\mathbb{R}^{2N}} \frac{[u_\lambda(x) - u_\lambda(y)]^{p-1} (\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy = \int_{\Omega} \tilde{K}_{\lambda,\delta}(x) \phi dx.$$

for all $\phi \in W_0^{s,p}(\Omega)$ and a minimal weak solution u of (S_0^δ) such that $u_\lambda^{\theta_1} \rightarrow u^{\theta_1}$ in $W_0^{s,p}(\Omega)$ and

$$\iint_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)]^{p-1} (\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx dy = \int_{\Omega} K_\delta(x) \varphi dx$$

for all $\varphi \in \mathbb{T}$ where $\theta_1 = \begin{cases} 1 & \text{if } 0 < \delta < 1 + s - \frac{1}{p}, \\ \theta_2 & \text{otherwise,} \end{cases}$ and $\theta_2 > \max\{\frac{(p-1)(sp-1)}{p(sp-\delta)}, 1\}$.

Let $\lambda_{s,p}$ be the first eigenvalue and $\varphi_{s,p}$ be a positive eigenfunction for the operator $(-\Delta)_p^s$. Then, by using Proposition 2.10 in [172], we get, for any $\delta \in (0, sp)$, there exists a constant κ_1 such that for any $\lambda \geq 0$

$$\kappa_1 d^s(x) \leq u_\lambda(x) \text{ for any } x \in \Omega \quad (3.4.28)$$

and from Theorem 3.2 and Remark 3.3 in [65], we deduce that for any $\eta > 0$, there exists $\kappa_\eta > 0$ independent of λ such that

$$\|u_\lambda\|_{L^\infty(\Omega \setminus \Omega_\eta)} \leq \kappa_\eta. \quad (3.4.29)$$

Now, we prove the sharp estimates for both upper and lower boundary behavior of the minimal weak solution for problem (S_0^δ) for different range of δ . In this regard, we prove the following result:

Theorem 3.4.7. *Let Ω be a bounded domain with $C^{1,1}$ boundary and u be minimal weak solution of the problem (S_0^δ) . Then, we have*

(i) For $\delta \in (s, sp)$, there exists a positive constant Υ_1 such that for any $x \in \Omega$,

$$\frac{1}{\Upsilon_1} d^{\frac{sp-\delta}{p-1}}(x) \leq u(x) \leq \Upsilon_1 d^{\frac{sp-\delta}{p-1}}(x).$$

(ii) For $\delta \in (0, s]$, for any $\epsilon > 0$, there exist positive constants Υ_2 and $\Upsilon_3 = \Upsilon_3(\epsilon)$ such that for any $x \in \Omega$:

$$\Upsilon_2 d^s(x) \leq u(x) \leq \Upsilon_3 d^{s-\epsilon}(x).$$

Here, we describe the main ingredients of the proof. Let u_λ be the solution of (S_λ^δ) for $\lambda < \lambda^*$, $\eta < \eta^*$ and $\rho > 0$ given by Theorem 3.4.3.

To prove (i): We define, for some $\eta > 0$,

$$\underline{u}^{(\lambda)} = \min\left\{\kappa_2 \left(\frac{\eta}{2}\right)^{s-\alpha}, \left(\frac{\mathcal{D}_3}{C_6}\right)^{\frac{1}{p-1}}\right\} \underline{w}_\rho = \underline{c}_\eta \underline{w}_\rho$$

and

$$\bar{u}^{(\lambda)} = \max\left\{\left(\frac{2}{\eta}\right)^\alpha \kappa_{\frac{\eta}{2}}, \left(\frac{\mathcal{D}_4}{C_5}\right)^{\frac{1}{(p-1)}}\right\} \bar{w}_\rho = \bar{c}_\eta \bar{w}_\rho$$

where \bar{w}_ρ and \underline{w}_ρ satisfies (3.4.16), $0 < \kappa_2 < \kappa_1$, C_5, C_6 are defined in (3.4.16), κ_1 and $\kappa_{\frac{\eta}{2}}$ are defined in (3.4.28) and (3.4.29) respectively and $\mathcal{D}_3, \mathcal{D}_4$ are defined in (3.4.27). Note \underline{c}_η and

\bar{c}_η are independent of λ .

Hence for any $\lambda > 0$, u_λ satisfies

$$\underline{u}^{(\lambda)}(x) \leq u_\lambda(x) \leq \bar{u}^{(\lambda)}(x) \text{ for } x \in \Omega \setminus \Omega_{\frac{\eta}{2}}, \text{ and } \underline{u}^{(\lambda)}(x) \leq 0 = u_\lambda(x) = \bar{u}^{(\lambda)}(x) \text{ for } x \in \Omega^c. \quad (3.4.30)$$

Then by applying weak comparison principle (Theorem 3.4.4) in Ω_η implies $\underline{u}^{(\lambda)} \leq u_\lambda \leq \bar{u}^{(\lambda)}$. Hence, from (3.4.30) and passing $\lambda \rightarrow 0$, we deduce (i).

Now we prove (ii) i.e. the case $\delta \leq s$. Since (6.3.44) holds, it remains to obtain the upper bound estimate.

Let $\tilde{u}_\lambda \in W_0^{s,p}(\Omega)$ be the weak solution of $(S_\lambda^{\tilde{\delta}})$ with $\tilde{\delta} = s + \epsilon(p-1) > s$ and for $\epsilon > 0$. Then, choosing a suitable constant $c_\epsilon > 0$ independent of λ , $\tilde{u}^{(\lambda)} = c_\epsilon \tilde{u}_\lambda$ is a weak supersolution of (S_λ^δ) . Hence by Theorem 3.4.4, we have $u_\lambda \leq \tilde{u}^{(\lambda)}$ in Ω . We pass to the limit as $\lambda \rightarrow 0$ and using (i) with $\tilde{u}(x) = \lim_{\lambda \rightarrow 0} \tilde{u}^{(\lambda)}(x)$, we get, for $\epsilon > 0$, $u(x) \leq \bar{c}_{\eta,\epsilon} d^{s-\epsilon}(x)$ for $x \in \Omega$.

Remark 3.4.2. *The boundary behavior in Theorem 3.4.7 (ii) is not optimal. We conjecture that $u \sim d^s$.*

Concerning the Hölder regularity of the weak solution of the problem (P), we prove the following result:

Theorem 3.4.8. *Let Ω be a bounded domain with $C^{1,1}$ boundary and u be minimal weak solution of (P). Then there exist constant $C_1, C_2 > 0$ and $0 < \omega_1 < s$, $0 < \omega_2 \leq \frac{sp-\delta}{\gamma+p-1}$ such that*

(i) $0 < \frac{\delta}{s} + \gamma \leq 1$, then $C_1 d^s(x) \leq u \leq C_2 d^{s-\epsilon}(x)$ in Ω and for every $\epsilon > 0$

$$u \in \begin{cases} C^{s-\epsilon}(\mathbb{R}^N) & \text{if } 2 \leq p < \infty, \\ C^{\omega_1}(\mathbb{R}^N) & \text{if } 1 < p < 2. \end{cases}$$

(ii) $\frac{\delta}{s} + \gamma > 1$ then $C_1 d^{\frac{(sp-\delta)}{\gamma+p-1}}(x) \leq u \leq C_2 d^{\frac{(sp-\delta)}{\gamma+p-1}}(x)$ in Ω and

$$u \in \begin{cases} C^{\frac{(sp-\delta)}{\gamma+p-1}}(\mathbb{R}^N) & \text{if } 2 \leq p < \infty, \\ C^{\omega_2}(\mathbb{R}^N) & \text{if } 1 < p < 2. \end{cases}$$

Remark 3.4.3. *The Hölder regularity of the minimal weak solution in the case of $p \geq 2$ and $s \in (0, 1)$ is optimal.*

Glimpse of the proof:

Let u be the minimal solution of the problem (P) and $\{u_\lambda\}_\lambda$ be its approximated sequence of solution for the approximated problem (P_λ^γ) . First, we prove the boundary behavior of the minimal weak solution by dividing the proof into two cases:

Case 1: $\frac{\delta}{s} + \gamma \leq 1$

Let \tilde{u} and \dot{u} are weak solution of the problem (S_0^ζ) for $\zeta = \delta + \gamma s \leq s$ and $\zeta = \delta + \gamma(s - \epsilon)$ respectively for $\epsilon \in (0, s)$ and then from Theorem 3.4.7 (ii) there exist constants $c_i > 0$ such that

$$c_1 d^s(x) \leq \tilde{u}(x) \leq c_2 d^{s-\epsilon}(x), \quad c_3 d^s(x) \leq \dot{u}(x) \leq c_4 d^{s-\epsilon}(x) \text{ in } \Omega$$

and \dot{u}, \tilde{u} satisfies

$$(-\Delta)_p^s (C_* \dot{u}) \leq \frac{K_\delta(x)}{\bar{u}^\gamma} \text{ and } \frac{K_\delta(x)}{\tilde{u}^\gamma} \leq (-\Delta)_p^s (C^* \tilde{u})$$

where $C_* = \left(\frac{c_1}{c_2 c_4^\gamma}\right)^{\frac{1}{p-1}}$ and $C^* = \left(\frac{c_2}{c_1 c_3^\gamma}\right)^{\frac{1}{p-1}}$ and C_1, C_2 are defined in (3.4.9). Then by applying Theorem 3.4.4, we get

$$C_1 d^s(x) \leq u(x) \leq C_2 d^{s-\epsilon}(x) \text{ in } \Omega$$

for every $\epsilon > 0$, $C_1 = c_1 C_*$ and $C_2 = c_2 C^*$.

Case 2: $\frac{\delta}{s} + \gamma > 1$

Let $\lambda > 0$ and $u_\lambda \in W_0^{s,p}(\Omega)$ be the solution of the problem (P_λ^γ) for $\lambda < \lambda^*$ given in Theorem 3.4.3. Now by applying applying Theorem 3.2 and Remark 3.3 in [65] and by repeating the same arguments as in (3.4.28), we obtain

$$\varkappa d^s(x) \leq u_\lambda(x) \text{ in } \Omega \quad (3.4.31)$$

for some \varkappa independent of λ and for any $\eta > 0$, there exists $\varkappa_\eta > 0$ independent of λ such that

$$\|u_\lambda\|_{L^\infty(\Omega \setminus \Omega_\eta)} \leq \varkappa_\eta. \quad (3.4.32)$$

For $\alpha = \frac{sp-\delta}{p+\gamma-1}$ and $0 < \eta < \eta^*$, define

$$\underline{u}^{[\lambda]} = c_\eta \underline{w}_\rho \text{ and } \bar{u}^{[\lambda]} = \dot{c}_\eta \bar{w}_\rho \text{ such that } c_\eta \leq \left(\frac{\eta}{2}\right)^{s-\alpha} \varkappa \text{ and } \dot{c}_\eta \geq \left(\frac{4}{\eta}\right)^\alpha \varkappa_{\frac{\eta}{2}}$$

where $\underline{w}_\rho, \bar{w}_\rho, \varkappa, \varkappa_{\frac{\eta}{2}}$ and η^* are defined in (3.4.14), (3.4.15), (3.4.31), (3.4.32) and Theorem 3.4.3 respectively. We note that c_η, \dot{c}_η are independent of λ and for any $\lambda > 0$, $\underline{u}^{[\lambda]}$ and $\bar{u}^{[\lambda]}$ satisfy

$$\underline{u}^{[\lambda]}(x) \leq u_\lambda(x) \leq \bar{u}^{[\lambda]}(x) \text{ for } x \in \Omega \setminus \Omega_{\frac{\eta}{2}} \text{ and } \underline{u}^{[\lambda]}(x) \leq u_\lambda(x) \leq \bar{u}^{[\lambda]}(x) \text{ for } x \in \Omega^c. \quad (3.4.33)$$

Using the definition of \underline{w}_ρ and \bar{w}_ρ in (3.4.14) and (3.4.15) respectively and estimates in (3.4.16), we can choose η small enough (independent of λ) such that

$$(-\Delta)_p^s \bar{u}^{[\lambda]} \geq \frac{K_{\lambda,\delta}(x)}{(\bar{u}^{[\lambda]} + \lambda)^\gamma} \text{ and } (-\Delta)_p^s \underline{u}^{[\lambda]} \leq \frac{K_{\lambda,\delta}(x)}{(\underline{u}^{[\lambda]} + \lambda)^\gamma} \text{ weakly in } \Omega_\eta$$

Since $\underline{u}^{[\lambda]}, \bar{u}^{[\lambda]} \in \overline{W}^{s,p}(\Omega_\eta)$ and $u_\lambda \in L^\infty(\Omega) \cap W_0^{s,p}(\Omega) \subset \overline{W}^{s,p}(\Omega_\eta)$, Proposition 2.10 in [172] in Ω_η implies $\underline{u}^{[\lambda]} \leq u_\lambda \leq \bar{u}^{[\lambda]}$ in Ω_η . Hence, from (3.4.33) and passing $\lambda \rightarrow 0$,

$$C_1 d^{\frac{sp-\delta}{p+\gamma-1}} \leq u \leq C_2 d^{\frac{sp-\delta}{\gamma+p-1}} \text{ in } \Omega.$$

where $C_1 = c_\eta$ and $C_2 = \dot{c}_\eta$.

Interior and Boundary regularity: To prove this, first we claim the following:

Claim: For all $x_0 \in \Omega$ and $R_0 = \frac{d(x_0)}{2}$ there exists universally $C_\Omega > 0$, $0 < \omega_3 < s$ and $0 < \omega_4 \leq \frac{sp-\delta}{p+\gamma-1}$ such that

$$\text{For } 1 < p < 2: \quad \|u\|_{C^{\omega_3}(B_{R_0}(x_0))} \leq C_\Omega \text{ for } \frac{\delta}{s} + \gamma \leq 1, \quad \|u\|_{C^{\omega_4}(B_{R_0}(x_0))} \leq C_\Omega \text{ for } \frac{\delta}{s} + \gamma \geq 1$$

and

$$\text{for } 2 \leq p < \infty: \quad \|u\|_{C^{s-\epsilon}(B_{R_0}(x_0))} \leq C_\Omega \text{ for } \frac{\delta}{s} + \gamma \leq 1, \quad \|u\|_{C^{\frac{sp-\delta}{p+\gamma-1}}(B_{R_0}(x_0))} \leq C_\Omega \text{ for } \frac{\delta}{s} + \gamma \geq 1.$$

Let $x_0 \in \Omega$, $R_0 = \frac{d(x_0)}{2}$ such that $B_{R_0}(x_0) \subset B_{2R_0}(x_0) \subset \Omega$ and $u \in W^{s,p}(B_{2R_0}(x_0)) \cap L^\infty(B_{2R_0}(x_0))$ be the weak solution of (P), then it satisfies

$$(-\Delta)_p^s u \leq \frac{C_2}{C_1^\gamma} \frac{1}{R_0^{\gamma s + \delta}} \text{ for } 0 < \frac{\delta}{s} + \gamma \leq 1, \quad (-\Delta)_p^s u \leq \frac{C_2}{C_1^\gamma} \frac{1}{R_0^{\gamma(\frac{sp-\delta}{\gamma+p-1}) + \delta}} \text{ for } \frac{\delta}{s} + \gamma > 1 \text{ in } B_{R_0}(x_0)$$

where C_2 is defined in (3.4.9). Then, by using Corollary 5.5, [172] for $1 < p < 2$ we obtain: there exist $0 < \omega_3 < s$ and $0 < \omega_4 \leq \frac{sp-\delta}{p+\gamma-1}$ such that

$$\|u\|_{C^{\omega_3}(\overline{B_{R_0}(x_0)})} \leq \mathbf{C}_1 \text{ if } 0 < \frac{\delta}{s} + \gamma \leq 1 \text{ and } \|u\|_{C^{\omega_4}(\overline{B_{R_0}(x_0)})} \leq \mathbf{C}_2 \text{ if } \frac{\delta}{s} + \gamma > 1 \quad (3.4.34)$$

and by using Theorem 1.4, [65] for $2 \leq p < \infty$ we get

$$\|u\|_{C^{s-\epsilon}(\overline{B_{R_0}(x_0)})} \leq \mathbf{C}_3 \text{ if } 0 < \frac{\delta}{s} + \gamma \leq 1 \text{ and } \|u\|_{C^{\frac{sp-\delta}{p+\gamma-1}}(\overline{B_{R_0}(x_0)})} \leq \mathbf{C}_4 \text{ if } \frac{\delta}{s} + \gamma > 1. \quad (3.4.35)$$

where \mathbf{C}_i 's are independent of the choice of point x_0 (and R_0).

Now, to prove the regularity estimate in Ω (and then the whole \mathbb{R}^N) since $u = 0$ in $\mathbb{R}^N \setminus \Omega$, it is enough to extend (3.4.34) and (3.4.35) on $\cup_{x_0 \in \Omega_\eta} \overline{B_{2R}(x_0)} \setminus B_{R_0}(x_0)$ where $\eta > 0$ small enough and $\Omega_\eta = \{x : d(x) < \eta\}$. In this regard, let $x, y \in \Omega_\eta$ with $|x - y| \geq \max\left\{\frac{d(x)}{2}, \frac{d(y)}{2}\right\}$. Then for a constant $C_1 > 0$ large enough, we get

$$\frac{|u(x) - u(y)|}{|x - y|^{s-\epsilon}} \leq \frac{|u(x)|}{|x - y|^{s-\epsilon}} + \frac{|u(y)|}{|x - y|^{s-\epsilon}} \leq 2^s \left(\frac{u(x)}{d^{s-\epsilon}(x)} + \frac{u(y)}{d^{s-\epsilon}(y)} \right) \leq C_1, \quad (3.4.36)$$

$$\begin{aligned} \frac{|u(x) - u(y)|}{|x - y|^{\frac{(sp-\delta)}{(\gamma+p-1)}}} &\leq \frac{|u(x)|}{|x - y|^{\frac{(sp-\delta)}{(\gamma+p-1)}}} + \frac{|u(y)|}{|x - y|^{\frac{(sp-\delta)}{(\gamma+p-1)}}} \\ &\leq 2^{\frac{(sp-\delta)}{(\gamma+p-1)}} \left(\frac{u(x)}{d^{\frac{(sp-\delta)}{(\gamma+p-1)}}(x)} + \frac{u(y)}{d^{\frac{(sp-\delta)}{(\gamma+p-1)}}(y)} \right) \leq C_1. \end{aligned} \quad (3.4.37)$$

Then, finally by combining (3.4.34)-(3.4.37), we get our claim and which completes the proof.

Remark 3.4.4. In case of local operator, i.e. p -Laplacian operator, the optimal condition of Sobolev regularity in Theorem 1.4, [138] coincide with the our condition for $s = 1$.

Corollary 3.4.2. For $\frac{\delta}{s} + \gamma > 1$ and Ω be a bounded domain with $C^{1,1}$ boundary. Then the minimal weak solution u of the problem (P) has the optimal Sobolev regularity:

$$u \in W_0^{s,p}(\Omega) \quad \text{if and only if} \quad \Lambda < 1$$

and

$$u^\rho \in W_0^{s,p}(\Omega) \quad \text{if and only if} \quad 1 \leq \Lambda < \rho.$$

The proof of above result follows from testing the equation (3.4.10) with the approximated solution of the problem (P_ϵ^γ). Precisely, by taking $\phi = u_\epsilon$ in (3.4.10), we obtain

$$\|u_\epsilon\|_{s,p} = \int_{\Omega} K_{\epsilon,\delta}(x) u_\epsilon^{1-\gamma} dx \leq \int_{\Omega} d^{(1-\gamma)\frac{(sp-\delta)}{p+\gamma-1}-\delta}(x) dx \leq C$$

if $(1-\gamma)(sp-\delta) > (\delta-1)(p+\gamma-1) \Leftrightarrow sp(\gamma-1) + \delta p < (p+\gamma-1) \Leftrightarrow \Lambda < 1$.

Similarly, by taking $\phi = u_\epsilon^\theta$ in (3.4.10) and using Proposition 3.4.1, we obtain for $\theta > \Lambda > 1$

$$\|u_\epsilon^\theta\|_{s,p} \leq \int_{\Omega} K_{\epsilon,\delta}(x) u_\epsilon^{(\theta-1)(p-1)+\theta-\gamma} dx \leq \int_{\Omega} d^{(\theta p - (p-1+\gamma))\frac{(sp-\delta)}{p+\gamma-1}-\delta}(x) dx \leq C.$$

Now, by passing limits $\epsilon \rightarrow 0$ in (3.4.10), we get the minimal solution $u \in W_0^{s,p}(\Omega)$ if $\Lambda < 1$ and $u^\theta \in W_0^{s,p}(\Omega)$ if $\theta > \Lambda > 1$. The only if statement follows from the Hardy inequality and the boundary behavior of the weak solution. Precisely, if $\Lambda \geq 1$, then $u \notin W_0^{s,p}(\Omega)$. Indeed, we have

$$\|u\|_{s,p} \geq C \int_{\Omega} \left| \frac{u(x)}{d^s(x)} \right|^p dx \geq C \int_{\Omega} d^{\frac{p(sp-\delta)}{p+\gamma-1}-sp}(x) dx = +\infty.$$

In the same way, if $\theta \in [1, \Lambda]$, then

$$\|u^\theta\|_{s,p} \geq C \int_{\Omega} \left| \frac{u^\theta(x)}{d^s(x)} \right|^p dx \geq C \int_{\Omega} d^{\frac{\theta p(sp-\delta)}{p+\gamma-1}-sp}(x) dx = \infty$$

and we deduce $u^\theta \notin W_0^{s,p}(\Omega)$.

As a consequence of comparison principle, we have the following uniqueness and nonexistence result:

Corollary 3.4.3. For $0 < \delta < 1 + s - \frac{1}{p}$, the minimal weak solution u is a unique weak solution of the problem (P).

Theorem 3.4.9. Let $\delta \geq sp$. Then there doesn't exist any weak solution of the problem (P) in the sense of definition 3.4.1.

From above, the non-existence result is optimal and corresponds to the limitation of the use of Hardy inequality. The proof of the above results follows from the weak comparison principle, Hardy inequality and boundary behavior of approximated weak solution u_ϵ of the approximated problem. For more details, we refer to proof of Theorem 6.3.3, Page 370, Chapter 6.

Volume II

Main results with detailed proof

Parabolic problems with nonstandard growth

This work is done jointly with Jacques Giacomoni, Guillaume Warnault and Sergey Shmarev.

Abstract : In this chapter, we study the qualitative properties of the parabolic problems with non-standard growth conditions. The main purpose of this chapter is three fold. Firstly, we derive conditions on the initial data for the existence of strong solution of evolution equations with $p(x, t)$ -Laplacian and prove the global higher integrability, higher differentiability and second order regularity of the strong solution. Secondly, we study the double phase parabolic equation with variable growth and nonlinear source term. We prove the existence of strong solution with global higher integrability and regularity properties. Thirdly, we derive the Picone identity for the $p(x)$ -homogeneous operators and as applications of this identity, we extend Díaz-Saá inequality for non-standard growth operators and study some boundary value problems comprising of variable exponent operators and non-standard growth conditions. Using this, we study the Doubly non-linear parabolic equations involving $p(x)$ -Laplacian operator and prove the existence, uniqueness, regularity and contraction properties of the weak solution. By generalizing the above results to $p(x)$ -homogeneous operator of Leray-Lions type, we study the stabilization property of the weak solution.

4.1 Functions spaces

Prior to formulating the results, we introduce the variable Lebesgue and Sobolev space. We limit ourselves to collecting the most basic facts of the theory and refer to [112] for a detailed insight, see also [34, Ch.1] and [111]. Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ is a bounded domain with Lipschitz-continuous boundary $\partial\Omega$. Let $\mathcal{P}(\Omega)$ be the set of all measurable function $p : \Omega \rightarrow [1, \infty[$ in N -dimensional Lebesgue measure. Let us define the functional

$$A_{p(\cdot)}(f) = \int_{\Omega} |f(x)|^{p(x)} dx. \quad (4.1.1)$$

The set

$$L^{p(\cdot)}(\Omega) = \{f : \Omega \rightarrow \mathbb{R} : f \text{ is measurable on } \Omega, A_{p(\cdot)}(f) < \infty\}$$

equipped with the Luxemburg norm

$$\|f\|_{L^{p(x)}} = \inf \left\{ \lambda > 0 : A_{p(\cdot)} \left(\frac{f}{\lambda} \right) \leq 1 \right\}$$

is a reflexive and separable Banach space and $C_0^\infty(\Omega)$ is dense in $L^{p(x)}(\Omega)$. The modular $A_{p(\cdot)}(f)$ is lower semicontinuous. We also recall some well-known properties on $L^{p(x)}$ spaces (see [223]). Throughout the chapter, we assume that

$$1 < p^- \stackrel{\text{def}}{=} \min_{\Omega} p(x) \leq p(x) \leq p^+ \stackrel{\text{def}}{=} \max_{\Omega} p(x) < \infty.$$

Proposition 4.1.1. *Let $p \in L^\infty(\Omega)$. Then for any $u \in L^{p(x)}(\Omega)$ we have:*

- (i) $A_{p(\cdot)}(u/\|u\|_{L^{p(x)}}) = 1$.
- (ii) $\|u\|_{L^{p(x)}} \rightarrow 0$ if and only if $A_{p(\cdot)}(u) \rightarrow 0$.
- (iii) $L^{p'(x)}(\Omega)$ is the dual space of $L^{p(x)}(\Omega)$ where we denote by p' the conjugate exponent of p defined as

$$p'(x) = \frac{p(x)}{p(x) - 1}.$$

Proposition 4.1.1 (i) implies that: if $\|u\|_{L^{p(x)}} \geq 1$,

$$\|u\|_{L^{p(x)}}^{p^-} \leq A_{p(\cdot)}(u) \leq \|u\|_{L^{p(x)}}^{p^+} \quad (4.1.2)$$

and if $\|u\|_{L^{p(x)}} \leq 1$

$$\|u\|_{L^{p(x)}}^{p^+} \leq A_{p(\cdot)}(u) \leq \|u\|_{L^{p(x)}}^{p^-}. \quad (4.1.3)$$

Moreover, we have also the generalized Hölder inequality: for p measurable function in Ω , there exists a constant $C = C(p^+, p^-) \geq 1$ such that for any $f \in L^{p(x)}(\Omega)$ and $g \in L^{p_c(x)}(\Omega)$

$$\int_{\Omega} |fg| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) \|f\|_{p(\cdot), \Omega} \|g\|_{p'(\cdot), \Omega} \leq 2 \|f\|_{p(\cdot), \Omega} \|g\|_{p'(\cdot), \Omega}, \quad (4.1.4)$$

Let p_1, p_2 are two bounded measurable functions in Ω such that $1 < p_1(x) \leq p_2(x)$ a.e. in Ω , then $L^{p_1(\cdot)}(\Omega)$ is continuously embedded in $L^{p_2(\cdot)}(\Omega)$ and

$$\forall u \in L^{p_2(\cdot)}(\Omega) \quad \|u\|_{L^{p_1(x)}} \leq C(|\Omega|, p_1^\pm, p_2^\pm) \|u\|_{L^{p_2(x)}}.$$

The variable exponent Sobolev space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the set of functions

$$W_0^{1,p(\cdot)}(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \mid u \in L^{p(\cdot)}(\Omega) \cap W_0^{1,1}(\Omega), |\nabla u| \in L^{p(\cdot)}(\Omega)\}$$

equipped with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(x)}} + \|\nabla u\|_{L^{p(x)}}.$$

It is known that $C_c^\infty(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ and the Poincaré inequality holds

$$\|u\|_{p(\cdot),\Omega} \leq C \|\nabla u\|_{p(\cdot),\Omega}. \quad (4.1.5)$$

if $p \in C_{\log}(\overline{\Omega})$, *i.e.*, the exponent p is continuous in $\overline{\Omega}$ with the logarithmic modulus of continuity:

$$|p(x_1) - p(x_2)| \leq \omega(|x_1 - x_2|), \quad (4.1.6)$$

where $\omega(\tau)$ is a nonnegative function satisfying the condition

$$\limsup_{\tau \rightarrow 0^+} \omega(\tau) \ln \left(\frac{1}{\tau} \right) = C < \infty.$$

By $W'(\Omega)$ we denote the dual of $W_0^{1,p(\cdot)}(\Omega)$, which is the set of bounded linear functionals over $W_0^{1,p(\cdot)}(\Omega)$: $\Phi \in W'(\Omega)$ iff there exist $\Phi_0 \in L^{p'(\cdot)}(\Omega)$, $\Phi_i \in L^{p'(\cdot)}(\Omega)$, $i = 1, \dots, N$, such that for all $u \in W_0^{1,p(\cdot)}(\Omega)$

$$\langle \Phi, u \rangle = \int_{\Omega} \left(u \Phi_0 + \sum_{i=1}^N u_{x_i} \cdot \Phi_i \right) dx.$$

For the study of parabolic problem with spaces of functions depending on $(x, t) \in Q_T$, we define the following spaces:

$$\begin{aligned} V_{p(\cdot,t)}(\Omega) &= \{u : \Omega \rightarrow \mathbb{R} \mid u \in L^2(\Omega) \cap W_0^{1,1}(\Omega), |\nabla u|^{p(x,t)} \in L^1(\Omega)\}, \quad t \in (0, T), \\ W_{p(\cdot)}(Q_T) &= \{u : (0, T) \rightarrow V_{p(\cdot,t)}(\Omega) \mid u \in L^2(Q_T), |\nabla u|^{p(x,t)} \in L^1(Q_T)\}. \end{aligned}$$

The dual $W'(Q_T)$ of the space $W_{p(\cdot)}(Q_T)$ is defined as follows: $\Phi \in W'(Q_T)$ iff there exists $\Phi_0 \in L^2(Q_T)$, $\Phi_i \in L^{p'(x,t)}(\Omega)$, $i = 1, \dots, N$, such that for all $u \in W_{p(\cdot)}(Q_T)$

$$\langle \Phi, u \rangle = \int_{Q_T} \left(u \Phi_0 + \sum_{i=1}^N u_{x_i} \Phi_i \right) dx dt.$$

Let $C_{\log}(\overline{Q_T})$ be the set of functions satisfying condition (4.1.6) in the closure of the cylinder Q_T . If $u \in W_{p(\cdot)}(Q_T)$, $u_t \in W'(Q_T)$ and $p(x, t) \in C_{\log}(\overline{Q_T})$, then

$$\int_{Q_T} uu_t dz = \frac{1}{2} \int_{\Omega} u^2(x, t) dx \Big|_{t=0}^{t=T}. \quad (4.1.7)$$

4.2 Strong solution: Existence, global higher integrability and differentiability, second order regularity

In this part, we study the sufficient condition on f, u_0 for the existence of strong solution of the following Dirichlet problem for the class of parabolic equations with variable non-linearity

$$\begin{cases} \partial_t u - \operatorname{div} (|\nabla u|^{p(x,t)-2} \nabla u) = f(x, t) & \text{in } Q_T = \Omega \times (0, T), \\ u = 0 \text{ on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) \text{ in } \Omega, \end{cases} \quad (4.2.1)$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with the boundary $\partial\Omega \in C^2$.

4.2.1 Statement of main results

We will distinguish between the weak and strong solutions of problem (4.2.1) defined as follows.

Definition 4.2.1. *A function u is called **weak solution** of problem (4.2.1), if*

- (i) $u \in W_{p(\cdot)}(Q_T)$, $u_t \in W'(Q_T)$,
- (ii) for every $\psi \in W_{p(\cdot)}(Q_T)$ with $\psi_t \in W'(Q_T)$

$$\int_{Q_T} u_t \psi \, dx \, dt + \int_{Q_T} |\nabla u|^{p(x,t)-2} \nabla u \cdot \nabla \psi \, dx \, dt = \int_{Q_T} f \psi \, dx \, dt, \quad (4.2.2)$$

- (iii) for every $\phi \in C_0^1(\Omega)$

$$\int_{\Omega} (u(x, t) - u_0(x)) \phi \, dx \rightarrow 0 \quad \text{as } t \rightarrow 0,$$

- (iv) the weak solution u is called **strong solution** of problem (4.2.1) if

$$u_t \in L^2(Q_T), \quad |\nabla u| \in L^\infty(0, T; L^{p(\cdot)}(\Omega)).$$

The existence of a unique weak solution to problem (4.2.1) can be proven under the minimal requirements on the regularity of the data.

Proposition 4.2.1 ([26, 34, 111]). *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with the Lipschitz-continuous boundary. Assume that $p : Q_T \rightarrow \mathbb{R}$ satisfies the conditions*

$$\frac{2N}{N+2} < p^- \leq p(x, t) \leq p^+, \quad p \in C_{\log}(\bar{Q}_T), \quad (4.2.3)$$

where $p^- := \min_{Q_T} p(x, t)$ and $p^+ := \max_{Q_T} p(x, t)$. Then for every $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$ problem (4.2.1) has a unique weak solution $u \in C^0([0, T]; L^2(\Omega)) \cap W_{p(\cdot)}(Q_T)$ with $u_t \in W'(Q_T)$. The solution satisfies the estimate

$$\operatorname{ess\,sup}_{t \in (0, T)} \|u\|_{2, \Omega} + \int_{Q_T} |\nabla u|^{p(x, t)} dx dt \leq C \quad (4.2.4)$$

with a constant C depending only on N , p^\pm , $\|f\|_{2, Q_T}$ and $\|u_0\|_{2, \Omega}$.

We are interested in the global regularity of weak solutions in the case when the problem data, f , u_0 , p , Ω , possess better regularity properties. The main result of this section is given in the following theorem.

Theorem 4.2.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with the boundary $\partial\Omega \in C^2$. Assume that $p(x, t)$ satisfies conditions (4.2.3) and*

$$\operatorname{ess\,sup}_{Q_T} |\nabla p| \leq C_* < \infty, \quad \operatorname{ess\,sup}_{Q_T} |p_t| \leq C^*$$

with nonnegative finite constants C_* , C^* . Let

$$f \in L^2(0, T; W_0^{1, 2}(\Omega)), \quad u_0 \in L^2(\Omega) \cap W_0^{1, q_0(\cdot)}(\Omega) \text{ with } q_0(x) = \max\{2, p(x, 0)\}.$$

- (i) *The weak solution $u(x, t)$ of problem (4.2.1) is a strong solution. The function $u(x, t)$ satisfies estimate (4.2.4) and*

$$\|u_t\|_{2, Q_T}^2 + \operatorname{ess\,sup}_{(0, T)} \int_{\Omega} |\nabla u|^{q(x, t)} dx \leq C \quad (4.2.5)$$

with the exponent $q(x, t) = \max\{2, p(x, t)\}$ and a constant C depends upon N , $\partial\Omega$, T , p^\pm , C_* , C^* , $\|u_0\|$, $\|f\|$.

- (ii) *The solution $u(x, t)$ possesses the property of higher integrability of the gradient:*

$$\int_{Q_T} |\nabla u|^{p(x, t) + \delta} dx dt \leq C_\delta \quad \text{for every } 0 < \delta < \frac{4p^-}{p^-(N+2) + 2N}$$

with a finite constant C_δ depending on δ and the same quantities as the constant C in (4.2.5).

- (iii) *Moreover,*

$$D_{x_i x_j}^2 u \in L_{loc}^{p(\cdot)}(Q_T \cap \{(x, t) : p(x, t) < 2\}), \quad \text{if } N \geq 2,$$

$$D_{x_i} \left(|\nabla u|^{\frac{p(x, t) - 2}{2}} D_{x_j} u \right) \in L^2(Q_T) \quad \text{if } N \geq 3, \text{ or } N = 2 \text{ and } p^- > \frac{6}{5},$$

$i, j = 1, 2, \dots, N$, and the corresponding norms are bounded by constants depending only on the data.

Notation. Throughout the section, the symbol C represents the constants which can be calculated or estimates using the known quantities, but whose exact value is not crucial for the argument and may change from line to line even inside the same formula. We use the notation z for the points of the cylinder Q_T : $z = (x, t) \in \Omega \times (0, T) = Q_T$. The notation D_i is used for the spatial derivative with respect to x_i . We also use the shorthand notation

$$|u_{xx}|^2 \equiv \sum_{i,j=1}^N |D_{ij}^2 u|^2$$

and omit the arguments of the variable exponent p wherever it does not cause confusion.

4.2.2 Regularized problem

Given a parameter $\epsilon > 0$, let us consider the family of regularized nondegenerate parabolic problems

$$\begin{cases} \partial_t u - \operatorname{div}((\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} \nabla u) = f(z) & \text{in } Q_T, \\ u = 0 & \text{on } \Gamma_T = \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & \text{in } \Omega. \end{cases} \quad (4.2.6)$$

4.2.2.1 Galerkin's approximations

For every fixed ϵ , a solution of problem (4.2.6) can be constructed as the limit of the sequence of finite-dimensional Galerkin's approximations $\{u_\epsilon^{(m)}\}$. The functions $u_\epsilon^{(m)}(x, t)$ are sought in the form

$$u_\epsilon^{(m)}(x, t) = \sum_{j=1}^m u_j^{(m)}(t) \phi_j(x), \quad (4.2.7)$$

where $\phi_j \in W_0^{1,2}(\Omega)$ and $\lambda_j > 0$ are the eigenfunctions and the corresponding eigenvalues of the problem

$$(\nabla \phi_j, \nabla \psi)_{2,(\Omega)} = \lambda(\phi_j, \psi)_{2,\Omega} \quad \forall \psi \in W_0^{1,2}(\Omega).$$

The systems $\{\phi_j\}$ and $\{\lambda_j^{-\frac{1}{2}} \phi_j\}$ form the orthogonal bases in $L^2(\Omega)$ and $W_0^{1,2}(\Omega)$. The coefficients $u_j^{(m)}(t)$ are defined as the solutions of the Cauchy problem for the system of m ordinary differential equations

$$\begin{aligned} (u_j^{(m)})'(t) &= - \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} \cdot \nabla \phi_j \, dx + \int_{\Omega} f \phi_j \, dx, \\ u_j^{(m)}(0) &= (u_0, \phi_j)_{2,\Omega}, \quad j = 1, 2, \dots, m, \end{aligned} \quad (4.2.8)$$

where the functions

$$u_0^{(m)} = \sum_{j=1}^m (u_0, \phi_j)_{2,\Omega} \phi_j \in \text{span}\{\phi_1, \phi_2, \dots, \phi_m\},$$

are chosen so that

$$u_0^{(m)} \rightarrow u_0 \text{ in } W_0^{1,q(x,0)}(\Omega), \quad q(x,0) = \max\{2, p(x,0)\}.$$

By the Carathéodory Theorem, for every finite m system (4.2.8) has a solution $(u_1^{(m)}, u_2^{(m)}, \dots, u_m^{(m)})$ on an interval $(0, T_m)$. This solution can be continued on the arbitrary interval $(0, T)$ because of the uniform estimate $\sup_{(0, T_m)} \|\nabla u_\epsilon^{(m)}(\cdot, t)\|_{q(\cdot), \Omega} \leq M$ with $q(x, t) = \max\{2, p(x, t)\}$, which follows from (4.2.29) and (4.2.33).

4.2.2.2 Basic a priori estimates

Lemma 4.2.1. *Let Ω and p satisfy the conditions of Theorem 4.2.1. If $f \in L^2(Q_T)$ and $u_0 \in L^2(\Omega)$, then $u_\epsilon^{(m)}$ satisfy the estimates*

$$\sup_{(0, T)} \|u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2 dz \leq e^T (\|f\|_{2,Q_T}^2 + \|u_0\|_{2,\Omega}^2) := L_0. \quad (4.2.9)$$

Proof. Multiplying j th equation of (4.2.8) by $u_j^{(m)}(t)$ and summing up the results for $j = 1, 2, \dots, m$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_\epsilon^{(m)}\|_{2,\Omega}^2 &= \sum_{j=1}^m u_j^{(m)}(t) (u_j^{(m)})'(t) \\ &= - \sum_{j=1}^m \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} \cdot \nabla \phi_j u_j^{(m)}(t) dx + \sum_{j=1}^m \int_{\Omega} f \phi_j u_j^{(m)}(t) dx \\ &= - \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2 dx + \int_{\Omega} f u_\epsilon^{(m)} dx. \end{aligned}$$

Applying the Cauchy inequality to the last term of the right-hand side we transform this inequality into the form

$$\frac{1}{2} \frac{d}{dt} \|u_\epsilon^{(m)}\|_{2,\Omega}^2 + \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2 dx \leq \frac{1}{2} \|f_0\|_{2,\Omega}^2 + \frac{1}{2} \|u_\epsilon^{(m)}\|_{2,\Omega}^2.$$

The last inequality can be written as

$$\frac{1}{2} \frac{d}{dt} \left(e^{-t} \|u_\epsilon^{(m)}\|_{L^2(\Omega)}^2 \right) + e^{-t} \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2 dx \leq \frac{e^{-t}}{2} \|f_0\|_{2,\Omega}^2.$$

Integration of the last inequality in t gives

$$\sup_{(0, T)} \|u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2 dx dt \leq C e^T \left(\|f_0\|_{2,Q_T}^2 + \|u_0\|_{2,\Omega}^2 \right)$$

with a constant C which does not depend on $u_\epsilon^{(m)}$. \square

Corollary 4.2.1. *Let $\epsilon \in (0, 1)$. Under the conditions of Lemma 4.2.1*

$$\int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p(z)} dz \leq \int_{Q_T} \left(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2 \right)^{\frac{p(z)}{2}} dz \leq L_1 \quad (4.2.10)$$

with a constant L_1 independent of ϵ and m .

Proof. The assertion immediately follows from (4.2.9) and the inequalities

$$|\nabla u_\epsilon^{(m)}|^{p(z)} \leq \left(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2 \right)^{\frac{p(z)}{2}} \leq \begin{cases} 2 \left(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2 \right)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2 & \text{if } |\nabla u_\epsilon^{(m)}| \geq \epsilon, \\ (2\epsilon^2)^{\frac{p(z)}{2}} \leq 2^{\frac{p^+}{2}} & \text{otherwise.} \end{cases} \quad (4.2.11)$$

\square

Let us denote \mathbf{n} by the exterior normal vector to $\partial\Omega$.

Lemma 4.2.2. *Let $\partial\Omega \in C^2$, $p(z)$ satisfies (4.2.3) and*

$$\operatorname{ess\,sup}_{Q_T} |\nabla p| \leq C_* < \infty, \quad u_0 \in W_0^{1,2}(\Omega), \quad f \in L^2(0, T; W_0^{1,2}(\Omega)).$$

Then the following inequality holds: for a.e. $t \in (0, T)$ and any $\delta > 0$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + (\min\{p^-, 2\} - 1 - \delta) \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2 dx \\ & \leq C_0 \int_{\Omega} |\nabla u_\epsilon^{(m)}|^2 (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \ln^2(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2) dx \\ & \quad - \int_{\partial\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \left(\Delta u_\epsilon^{(m)} (\nabla u_\epsilon^{(m)} \cdot \mathbf{n}) - \nabla u_\epsilon^{(m)} \cdot \nabla (\nabla u_\epsilon^{(m)} \cdot \mathbf{n}) \right) dx \\ & \quad + C_1 \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + C_2 \|f(t)\|_{W_0^{1,2}(\Omega)}^2 \end{aligned} \quad (4.2.12)$$

with constants C_i , $i = 0, 1, 2$, depending on the data and δ , but independent of m and ϵ .

Proof. Multiplying each of equations in (4.2.8) by $\lambda_j u_j^{(m)}$, $j = 1, 2, \dots, m$, and summing up the results we obtain the equality

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla u_\epsilon^{(m)}\|_{2,\Omega}^2 = \sum_{j=1}^m \lambda_j (u_j^{(m)})'(t) u_j^{(m)}(t) \\ & = \sum_{j=1}^m \lambda_j u_j^{(m)} \int_{\Omega} \operatorname{div} \left((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} \right) \phi_j dx + \sum_{j=1}^m \lambda_j u_j^{(m)} \int_{\Omega} f(x, t) \phi_j dx \\ & = - \int_{\Omega} \operatorname{div} \left((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} \right) \Delta u_\epsilon^{(m)} dx + \int_{\Omega} f \Delta u_\epsilon^{(m)} dx. \end{aligned} \quad (4.2.13)$$

Since $u_\epsilon^{(m)} \in C^3(\Omega)$ and $\partial\Omega \in C^2$, the first term on the right-hand can be transformed by means of the Green formula:

$$\begin{aligned}
 & - \int_{\Omega} \operatorname{div} \left((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} \right) \Delta u_\epsilon^{(m)} dx \\
 &= - \int_{\Omega} \left(\sum_{k=1}^N (u_\epsilon^{(m)})_{x_k x_k} \right) \left(\sum_{i=1}^N \left((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} (u_\epsilon^{(m)})_{x_i} \right)_{x_i} \right) dx \\
 &= - \int_{\partial\Omega} \Delta u_\epsilon^{(m)} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} (\nabla u_\epsilon^{(m)} \cdot \mathbf{n}) dS \\
 &\quad + \int_{\Omega} \sum_{k,i=1}^N (u_\epsilon^{(m)})_{x_k x_k x_i} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} (u_\epsilon^{(m)})_{x_i} dx \\
 &= - \int_{\partial\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \sum_{k,i=1}^N \left((u_\epsilon^{(m)})_{x_k x_k} (u_\epsilon^{(m)})_{x_i} n_i - (u_\epsilon^{(m)})_{x_k x_i} (u_\epsilon^{(m)})_{x_i} n_k \right) dS \\
 &\quad - \int_{\Omega} \sum_{k,i=1}^N (u_\epsilon^{(m)})_{x_k x_i} \left((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} (u_\epsilon^{(m)})_{x_i} \right)_{x_k} dx \\
 &= - \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2 dx + J_1 + J_2 + J_{\partial\Omega}
 \end{aligned}$$

where

$$\begin{aligned}
 J_1 &:= \int_{\Omega} (2 - p(z)) (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2} - 1} \left(\sum_{k=1}^N \left(\nabla u_\epsilon^{(m)} \cdot \nabla (u_\epsilon^{(m)})_{x_k} \right)^2 \right) dx, \\
 J_2 &= - \int_{\Omega} \sum_{k,i=1}^N (u_\epsilon^{(m)})_{x_k x_i} (u_\epsilon^{(m)})_{x_i} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \frac{p x_k}{2} \ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2) dx, \\
 J_{\partial\Omega} &= - \int_{\partial\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \left(\Delta u_\epsilon^{(m)} (\nabla u_\epsilon^{(m)} \cdot \mathbf{n}) - \nabla u_\epsilon^{(m)} \cdot \nabla (\nabla u_\epsilon^{(m)} \cdot \mathbf{n}) \right) dS.
 \end{aligned}$$

Substitution into (4.2.13) leads to the inequality

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \|\nabla u_\epsilon^{(m)}\|_{2,\Omega}^2 + \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2 dx \\
 &= J_1 + J_2 + J_{\partial\Omega} - \int_{\Omega} \nabla f \cdot \nabla u_\epsilon^{(m)} dx \\
 &\leq J_1 + J_2 + J_{\partial\Omega} + \frac{1}{2} \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + \frac{1}{2} \|f(t)\|_{W_0^{1,2}(\Omega)}^2.
 \end{aligned}$$

The term J_1 is absorbed in the left-hand side because

$$\begin{aligned}
 J_1 &= \int_{\{x \in \Omega: p(z) \geq 2\}} (2 - p(z)) \dots + \int_{\{x \in \Omega: p(z) < 2\}} (2 - p(z)) \dots \\
 &\leq \int_{\{x \in \Omega: p(z) < 2\}} (2 - p(z)) (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2} - 1} \left(\sum_{k=1}^N \left(\nabla u_\epsilon^{(m)} \cdot \nabla (u_\epsilon^{(m)})_{x_k} \right)^2 \right) dx,
 \end{aligned}$$

and

$$|J_1| \leq \max\{0, 2 - p^-\} \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2 dx.$$

The term J_2 is estimated in the following way: by the Cauchy inequality, for every $\delta > 0$

$$\begin{aligned} |J_2| &\leq \|\nabla p\|_{\infty, Q_T} \int_{\Omega} \left(\sum_{i,k=1}^N |(u_{\epsilon}^{(m)})_{x_i x_k}| (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{4}} \right) \\ &\quad \times \left(|\nabla u_{\epsilon}^{(m)}| (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{4}} |\ln(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)| \right) dx \\ &\leq \delta \int_{\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_{\epsilon}^{(m)})_{xx}|^2 dx \\ &\quad + C \int_{\Omega} |\nabla u_{\epsilon}^{(m)}|^2 (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{2}} \ln^2(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2) dx \end{aligned}$$

with a constant $C = C(C^*, N, \delta)$. Choosing $\delta \in (0, 1)$ so small that $\min\{2, p^-\} > 1 + \delta$ and collecting in the right-hand side all terms which contain $(u^{(m)})_{xx}$ we obtain (4.2.12) because

$$1 - \delta - \max\{0, 2 - p^-\} = \begin{cases} 1 - \delta & \text{if } p^- \geq 2, \\ p^- - 1 - \delta & \text{if } p^- < 2 \end{cases} = \min\{p^-, 2\} - 1 - \delta.$$

□

4.2.3 Interpolation inequalities

In this section, we derive first the interpolation inequality which yields the property of higher integrability of the gradient of the finite-dimensional approximations $u_{\epsilon}^{(m)}$ of the solutions of problems (4.2.6). We prove next an estimate on the trace of $\nabla u_{\epsilon}^{(m)}$ on $\partial\Omega$, which turns out to be useful in the study of the nonconvex domains. Both estimates will be applied to obtain upper bounds for the terms on the right-hand side of (4.2.12).

With certain abuse of notation, throughout the section we denote by $p(x)$ or $p(x, t)$ given exponents defined on Ω or Q_T and not related to the exponent in equation (4.2.1). Let us accept the notation

$$\begin{aligned} \beta_{\epsilon}(\mathbf{s}) &= \epsilon^2 + |\mathbf{s}|^2, \quad \epsilon > 0, \quad \mathbf{s} \in \mathbb{R}^N, \quad x \in \Omega, \\ \gamma_{\epsilon}(x, \mathbf{s}) &= \beta_{\epsilon}^{\frac{p(x)-2}{2}}(\mathbf{s}) \equiv (\epsilon^2 + |\mathbf{s}|^2)^{\frac{p(x)-2}{2}}. \end{aligned}$$

Lemma 4.2.3. *Let $\partial\Omega \in C^1$, $u \in C^2(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$. Assume that*

$$\begin{aligned} p : \Omega &\mapsto [p^-, p^+], \quad p^{\pm} = \text{const}, \\ \frac{2N}{N+2} &< p^-, \quad p(\cdot) \in C^0(\overline{\Omega}), \quad \text{ess sup}_{\Omega} |\nabla p| = L, \\ \int_{\Omega} \gamma_{\epsilon}(x, \nabla u) |u_{xx}|^2 dx &< \infty, \quad \int_{\Omega} u^2 dx = M_0, \quad \int_{\Omega} |\nabla u|^{p(x)} dx = M_1. \end{aligned} \tag{4.2.14}$$

Then for every

$$\frac{2}{N+2} =: r_* < r < r^* := \frac{4p^-}{p^-(N+2) + 2N} \tag{4.2.15}$$

and every $\delta \in (0, 1)$

$$\int_{\Omega} \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) |\nabla u|^2 dx \leq \delta \int_{\Omega} \gamma_{\epsilon}(x, \nabla u) |u_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u|^{p(x)} dx \right) \quad (4.2.16)$$

with an independent of u constant $C = C(\partial\Omega, \delta, p^{\pm}, N, r, M_0, M_1)$.

Proof. Let us fix some $r \in (r_*, r^*)$. By the Green formula

$$\begin{aligned} \int_{\Omega} \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) |\nabla u|^2 dx &= \int_{\Omega} \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) \nabla u \cdot \nabla u dx \\ &= \int_{\partial\Omega} u \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) \nabla u \cdot \mathbf{n} dS - \int_{\Omega} u \operatorname{div} \left(\beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) \nabla u \right) dx \\ &= - \int_{\Omega} u \operatorname{div} \left(\beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) \nabla u \right) dx =: -J, \end{aligned}$$

where \mathbf{n} stands for the outer normal to $\partial\Omega$. A straightforward computation leads to the representation

$$\begin{aligned} J &= \int_{\Omega} u \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) \Delta u dx \\ &\quad + \int_{\Omega} (p(x) + r - 2) u \beta_{\epsilon}^{\frac{p(x)+r-2}{2} - 1} (\nabla u) \sum_{i=1}^n \left(u_{x_i} \sum_{j=1}^n u_{x_j} u_{x_i x_j} \right) dx \\ &\quad + \int_{\Omega} u \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) \ln(\beta_{\epsilon}(\nabla u)) \nabla u \cdot \nabla p dx, \end{aligned}$$

whence

$$|J| \leq C \int_{\Omega} |u| \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) |u_{xx}| dx + L \int_{\Omega} |u| \beta_{\epsilon}^{\frac{p(x)+r-1}{2}} (\nabla u) |\ln \beta_{\epsilon}(\nabla u)| dx \quad (4.2.17)$$

with $C = C(n, p^{\pm}, r)$ and $|u_{xx}|^2 = \sum_{i,j=1}^n |D_{ij}^2 u|^2$. For every constant $0 < \rho < \min\{1, p^- + r - 1\}$ and $0 < \nu < 1$, the integrand of the last term in (4.2.17) admits the estimate

$$\begin{aligned} \beta_{\epsilon}^{\frac{p(x)+r-1}{2}} (\nabla u) |\ln \beta_{\epsilon}(\nabla u)| &\leq \begin{cases} \beta_{\epsilon}^{\frac{p(x)+r-1-\rho}{2}} (\nabla u) \left(\beta_{\epsilon}^{\frac{\rho}{2}} (\nabla u) |\ln \beta_{\epsilon}(\nabla u)| \right) & \text{if } \beta_{\epsilon}(\nabla u) \leq 1, \\ \beta_{\epsilon}^{\frac{p(x)+r-1+\nu}{2}} (\nabla u) \left(\beta_{\epsilon}^{-\frac{\nu}{2}} (\nabla u) |\ln \beta_{\epsilon}(\nabla u)| \right) & \text{if } \beta_{\epsilon}(\nabla u) > 1, \end{cases} \\ &\leq C(\rho) + C(\nu) \beta_{\epsilon}^{\frac{p(x)+r-1+\nu}{2}} (\nabla u), \end{aligned} \quad (4.2.18)$$

which allows one to continue (4.2.17) as follows:

$$\begin{aligned} |J| &\leq C \int_{\Omega} |u| \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) |u_{xx}| dx + C' \left(\int_{\Omega} |u| dx + \int_{\Omega} |u| \beta_{\epsilon}^{\frac{p(x)+r-1+\nu}{2}} (\nabla u) dx \right) \\ &\leq C \int_{\Omega} |u| \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) |u_{xx}| dx + C' \int_{\Omega} |u| \beta_{\epsilon}^{\frac{p(x)+r-1+\nu}{2}} (\nabla u) dx + M_0^{1/2} + C'' =: I. \end{aligned}$$

Using Young's inequality we finally estimate: for every $\delta \in (0, 1)$

$$\begin{aligned} |I| &= C \int_{\Omega} \left(|u| \beta_{\epsilon}^{\frac{p(x)+r-1}{2} - \frac{p(x)}{4}} (\nabla u) \right) \left(\beta_{\epsilon}^{\frac{p(x)-2}{4}} (\nabla u) |u_{xx}| \right) dx + C' \int_{\Omega} |u| \beta_{\epsilon}^{\frac{p(x)-1+r+\nu}{2}} (\nabla u) dx + \widehat{C} \\ &\leq \delta \int_{\Omega} \gamma_{\epsilon}(x, \nabla u) |u_{xx}|^2 dx + C_{\delta} \int_{\Omega} u^2 \beta_{\epsilon}^{\frac{p(x)+2r-2}{2}} (\nabla u) dx + C' \int_{\Omega} |u| \beta_{\epsilon}^{\frac{p(x)+r-1+\nu}{2}} (\nabla u) dx + \widehat{C} \\ &\equiv \delta I_0 + C_{\delta} I_1 + C' I_2 + \widehat{C}. \end{aligned}$$

Let $\{\Omega_i\}_{i=1}^K$ be a finite cover of Ω such that

$$\Omega_i \subset \Omega, \quad \partial\Omega_i \in C^2, \quad p_i^+ = \max_{\Omega_i} p(x), \quad p_i^- = \min_{\Omega_i} p(x).$$

For any $r_* < r < r^*$, the continuity of $p(x)$ allows us to choose Ω_i so small that for every $i = 1, 2, \dots, K$

$$p_i^+ - p_i^- + r \left(1 + \frac{2N}{p^-(N+2)} \right) < \frac{4}{N+2}. \quad (4.2.19)$$

To estimate the terms I_1 and I_2 we represent them in the form

$$I_j = \sum_{i=1}^K I_j^{(i)}, \quad I_1^{(i)} = \int_{\Omega_i} u^2 \beta_{\epsilon}^{\frac{p(x)+2(r-1)}{2}} (\nabla u) dx, \quad I_2^{(i)} = \int_{\Omega_i} |u| \beta_{\epsilon}^{\frac{p(x)+r-1+\nu}{2}} (\nabla u) dx.$$

Recall that $\nu \in (0, 1)$. By the Young inequality, for any $\lambda > 0$

$$\begin{aligned} I_2^{(i)} &\leq \lambda \int_{\Omega_i} \beta_{\epsilon}^{\frac{p(x)+r}{2}} (\nabla u) dx + C_{\lambda} \int_{\Omega_i} |u|^{\frac{p(x)+r}{1-\nu}} dx \\ &\leq \lambda \int_{\Omega_i} \beta_{\epsilon}^{\frac{p(x)+r}{2}} (\nabla u) dx + C_{\lambda} \left(1 + \int_{\Omega_i} |u|^{\frac{p_i^++r}{1-\nu}} dx \right) \\ &= \lambda \left(\int_{\Omega_i \cap \{|\nabla u| > 1\}} \beta_{\epsilon}^{\frac{p(x)+r}{2}} (\nabla u) dx + \int_{\Omega_i \cap \{|\nabla u| \leq 1\}} \dots \right) + C_{\lambda} \left(1 + \int_{\Omega_i} |u|^{\frac{p_i^++r}{1-\nu}} dx \right). \end{aligned}$$

For $\epsilon \in (0, 1)$

$$\beta_{\epsilon}^{\frac{p(x)+r}{2}} (\nabla u) = \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) (\epsilon^2 + |\nabla u|^2) \leq \begin{cases} 2\beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) |\nabla u|^2 & \text{if } |\nabla u| > 1, \\ (1 + \epsilon^2)^{\frac{p_i^++r}{2}} & \text{otherwise,} \end{cases}$$

which entails the estimate

$$I_2^{(i)} \leq 2\lambda \int_{\Omega} \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) |\nabla u|^2 dx + C_{\lambda} \int_{\Omega_i} |u|^{\frac{p_i^++r}{1-\nu}} dx + C.$$

The second integral on the right-hand side is estimated by the Gagliardo-Nirenberg inequality:

$$\|u\|_{\sigma, \Omega_i}^{\sigma} \leq C_1 \|\nabla u\|_{p_i^-, \Omega_i}^{\sigma\theta} \|u\|_{2, \Omega_i}^{\sigma(1-\theta)} + C_2 \|u\|_{2, \Omega_i}^{\sigma} \leq C_1' \|\nabla u\|_{p_i^-, \Omega_i}^{\sigma\theta} + C_2 M_0^{\frac{\sigma}{2}}, \quad C_1' = C_1 M_0^{\frac{\sigma}{2}(1-\theta)},$$

with

$$\sigma = \frac{p_i^+ + r}{1 - \nu} > p_i^+ + r > p_i^-, \quad \theta = \frac{p_i^-}{\sigma} \in (0, 1), \quad \frac{1}{\sigma} = \left(\frac{1}{p_i^-} - \frac{1}{N} \right) \theta + \frac{1 - \theta}{2}.$$

Such a choice of the parameters σ, θ is possible if

$$\nu = 1 - \frac{p_i^+ + r}{p_i^-} \frac{N}{N+2} \quad \text{with } r_* < r < r^*.$$

Gathering the estimates on $I_2^{(i)}$ and using the Young inequality we finally obtain: for every $\lambda \in (0, 1)$

$$\begin{aligned} I_2 &\leq 2\lambda K \int_{\Omega} \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) |\nabla u|^2 dx + C_{\lambda} \sum_{i=1}^K \int_{\Omega_i} |\nabla u|^{p_i^-} dx + C \\ &\leq 2\lambda K \int_{\Omega} \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) |\nabla u|^2 dx + C'_{\lambda} \int_{\Omega} |\nabla u|^{p(x)} dx + C' \\ &= 2\lambda K \int_{\Omega} \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) |\nabla u|^2 dx + C'', \quad C'' = C''(N, \lambda, p_i^{\pm}, r, |\Omega|, M_0, M_1). \end{aligned}$$

To estimate $I_1^{(i)}$ we first use the Young inequality: since $\frac{2N}{N+2} < p_i^-$ by assumption, then for every $\tilde{\lambda} \in (0, 1)$

$$I_1^{(i)} \leq C_{\tilde{\lambda}} \int_{\Omega_i} |u|^{p_i^- \frac{N+2}{N}} dx + \tilde{\lambda} \int_{\Omega_i} \beta_{\epsilon}^{\frac{\kappa}{2}} (\nabla u) dx, \quad \kappa = (p(x) + 2(r-1)) \frac{p_i^- \frac{N+2}{2N}}{p_i^- \frac{N+2}{2N} - 1}. \quad (4.2.20)$$

To estimate the second integral, let us claim that $0 < \kappa < p(x) + r$ on Ω_i , i.e.,

$$0 < p(x) + 2(r-1) < \frac{p(x) + r}{p_i^-} \left(p_i^- - \frac{2N}{N+2} \right).$$

In this double inequality the first one is fulfilled by the choice of r :

$$0 = \frac{2N}{N+2} + 2(r_* - 1) < p^- + 2(r-1) \leq p(x) + 2(r-1).$$

The second inequality is fulfilled if

$$p_i^+ + 2(r-1) < \frac{p_i^- + r}{p_i^-} \left(p_i^- - \frac{2N}{N+2} \right) \quad \Leftrightarrow \quad (p_i^+ - p_i^-) + r < 2 - \frac{p_i^- + r}{p_i^-} \frac{2N}{N+2},$$

which is true because of (4.2.19) and the condition $r < r^*$. By the Young inequality

$$\begin{aligned} \beta_{\epsilon}^{\frac{\kappa}{2}} (\nabla u) &\leq 1 + \beta_{\epsilon}^{\frac{p(x)+r}{2}} (\nabla u) \leq 1 + \begin{cases} (2\epsilon^2)^{\frac{p(x)+r}{2}} & \text{if } |\nabla u| \leq \epsilon, \\ 2\beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) |\nabla u|^2 & \text{if } |\nabla u| > \epsilon \end{cases} \\ &\leq C + 2\beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) |\nabla u|^2. \end{aligned}$$

It remains to estimate the first integral in (4.2.20). By the Gagliardo-Nirenberg inequality

$$\int_{\Omega_i} |u|^{p_i^- \frac{N+2}{N}} dx \leq C_1(M_0) \|\nabla u\|_{p_i^-, \Omega_i}^{\theta p_i^- \frac{N+2}{N}} + C_2(M_0)$$

with

$$\theta = \frac{\frac{1}{2} - \frac{N}{p_i^-(N+2)}}{\frac{N+2}{2N} - \frac{1}{p_i^-}} = \frac{N}{N+2} \in (0, 1),$$

whence

$$I_1 \leq 2\tilde{\lambda} \int_{\Omega} \beta_{\epsilon}^{\frac{p(x)+r-2}{2}} (\nabla u) |\nabla u|^2 dx + C' \int_{\Omega} |\nabla u|^{p(x)} dx + C''.$$

Gathering the estimates of I_i for $|I|$ and choosing $\lambda, \tilde{\lambda}$ so small that $2\lambda K + 2\tilde{\lambda} < 1$, we arrive at the desired estimate (4.2.16). \square

The assertion of Lemma 4.2.3 easily extends to functions defined on the cylinder Q_T . Let us recall the notation $z = (x, t) \in Q_T = \Omega \times (0, T)$ and re-define

$$\gamma_{\epsilon}(z, \mathbf{s}) = \beta_{\epsilon}^{\frac{p(z)-2}{2}}(\mathbf{s}) \equiv (\epsilon^2 + |\mathbf{s}|^2)^{\frac{p(z)-2}{2}}, \quad \epsilon > 0, \quad \mathbf{s} \in \mathbb{R}^N.$$

Theorem 4.2.2. *Let $\partial\Omega \in C^1$, $u \in C^1([0, T]; C^2(\bar{\Omega}))$ and $u = 0$ on $\partial\Omega \times [0, T]$. Assume that*

$$\begin{aligned} p(z) : Q_T &\mapsto [p^-, p^+], \quad p^{\pm} = \text{const}, \\ p(\cdot) &\in C^0(\bar{Q}_T) \text{ with the modulus of continuity } \omega, \\ \frac{2N}{N+2} &< p^-, \quad \text{ess sup}_{Q_T} |\nabla p| = L, \end{aligned} \quad (4.2.21)$$

$$\int_{Q_T} \gamma_{\epsilon}(z, \nabla u) |u_{xx}|^2 dz < \infty, \quad \sup_{(0, T)} \|u(t)\|_{2, \Omega}^2 = M_0, \quad \int_{Q_T} |\nabla u|^{p(z)} dz = M_1.$$

Then for every

$$\frac{2}{N+2} = r_* < r < r^* = \frac{4p^-}{p^-(N+2) + 2N}$$

and every $\delta \in (0, 1)$ the function u satisfies the inequality

$$\int_{Q_T} \beta_{\epsilon}^{\frac{p(z)+r-2}{2}} (\nabla u) |\nabla u|^2 dz \leq \delta \int_{Q_T} \gamma_{\epsilon}(z, \nabla u) |u_{xx}|^2 dz + C \left(1 + \int_{Q_T} |\nabla u|^{p(z)} dz \right). \quad (4.2.22)$$

with an independent of u constant $C = C(N, \partial\Omega, T, \delta, p^{\pm}, \omega, r, M_0, M_1)$.

Proof. Since the exponent $p(z)$ is uniformly continuous in \bar{Q}_T , then for any $r_* < r < r^*$ there exists a finite cover of Q_T composed of the cylinders $Q^{(i)} = \Omega_i \times (t_{i-1}, t_i)$, $i = 1, 2, \dots, K$, such that

$$t_0 = 0, \quad t_K = T, \quad t_i - t_{i-1} = \rho, \quad Q_T \subset \bigcup_{i=1}^K Q^{(i)}, \quad \partial\Omega_i \in C^2,$$

$$p_i^+ = \max_{Q^{(i)}} p(z), \quad p_i^- = \min_{Q^{(i)}} p(z),$$

$$p_i^+ - p_i^- + r \left(1 + \frac{2N}{p^-(N+2)} \right) < \frac{4}{N+2}, \quad i = 1, 2, \dots, K.$$

For a.e. $t \in (0, T)$ the function $u(x, t)$ satisfies inequality (4.2.16). Integrating this inequality over the interval (t_{i-1}, t_i) and summing the results gives (4.2.22). \square

Remark 4.2.1. If $p = \text{const} > \frac{2N}{N+2}$ and $u(z)$ satisfies conditions (4.2.21), then inequalities (4.2.19) and (4.2.22) hold for every $r_* < r < r^*$.

Lemma 4.2.4. Let $\partial\Omega$ be a Lipschitz-continuous surface and $\|\nabla p\|_{\infty, \Omega} = L$. There exists a constant $\delta = \delta(\partial\Omega)$ such that for every $u \in W^{1,p(\cdot)}(\Omega)$

$$\delta \int_{\partial\Omega} |u|^{p(x)} dS \leq C \int_{\Omega} \left(|u|^{p(x)-1} |\nabla u| + |u|^{p(x)} |\ln |u|| + |u|^{p(x)} \right) dx \quad (4.2.23)$$

with a constant $C = C(p^+, L, N, \partial\Omega)$.

Proof. By [162, Lemma 1.5.1.9] there exists $\delta > 0$ and $\mu \in (C^\infty(\bar{\Omega}))^N$ such that $\mu \cdot \mathbf{n} \geq \delta$ a.e. on $\partial\Omega$. By the Green formula

$$\begin{aligned} \delta \int_{\partial\Omega} |u|^{p(x)} dS &\leq \int_{\partial\Omega} |u|^{p(x)} (\mu \cdot \mathbf{n}) dS = \int_{\Omega} \operatorname{div}(|u|^{p(x)} \mu) dx \\ &= \int_{\Omega} \left(p(x) |u|^{p(x)-2} u (\nabla u \cdot \mu) + |u|^{p(x)} \ln |u| (\nabla p \cdot \mu) + |u|^{p(x)} \operatorname{div} \mu \right) dx \\ &\leq p^+ \max_{\Omega} |\mu| \int_{\Omega} |u|^{p(x)-1} |\nabla u| dx + \|\nabla p\|_{\infty, \Omega} \max_{\Omega} |\mu| \int_{\Omega} |u|^{p(x)} |\ln |u|| dx \\ &\quad + \max_{\Omega} |\operatorname{div} \mu| \int_{\Omega} |u|^{p(x)} dx \\ &\leq C(p^+, L, N, \Omega) \int_{\Omega} \left(|u|^{p(x)-1} |\nabla u| + |u|^{p(x)} |\ln |u|| + |u|^{p(x)} \right) dx. \end{aligned}$$

□

Lemma 4.2.5. Under the conditions of Lemma 4.2.4, for every $\lambda \in (0, 1)$ and $\epsilon \in (0, 1)$

$$\int_{\partial\Omega} |u|^{p(x)} dS \leq \lambda \int_{\Omega} (\epsilon^2 + |u|^2)^{\frac{p(x)-2}{2}} |\nabla u|^2 + C_1 \int_{\Omega} |u|^{p(x)} dx + C_2 \int_{\Omega} |u|^{p(x)} |\ln |u|| dx + C_3 \quad (4.2.24)$$

with constants C_i , $i = 1, 2, 3$, depending on $N, p^\pm, L, \partial\Omega, \lambda$, but independent of u .

Proof. By the Cauchy inequality, for every $\lambda \in (0, 1)$

$$\begin{aligned} |u|^{p-1} |\nabla u| &= \left(\lambda (\epsilon^2 + |u|^2)^{\frac{p-2}{2}} |\nabla u|^2 \right)^{\frac{1}{2}} \left(\lambda^{-1} (\epsilon^2 + |u|^2)^{\frac{2-p}{2}} |u|^{2(p-1)} \right)^{\frac{1}{2}} \\ &\leq \lambda (\epsilon^2 + |u|^2)^{\frac{p-2}{2}} |\nabla u|^2 + \frac{1}{\lambda} (\epsilon^2 + |u|^2)^{1-\frac{p}{2}} |u|^{2(p-1)} \\ &\leq \lambda (\epsilon^2 + |u|^2)^{\frac{p-2}{2}} |\nabla u|^2 + \frac{1}{\lambda} (\epsilon^2 + |u|^2)^{1-\frac{p}{2}+(p-1)} \\ &= \lambda (\epsilon^2 + |u|^2)^{\frac{p-2}{2}} |\nabla u|^2 + \frac{1}{\lambda} (\epsilon^2 + |u|^2)^{\frac{p}{2}} \\ &\leq \lambda (\epsilon^2 + |u|^2)^{\frac{p-2}{2}} |\nabla u|^2 + C(1 + |u|^p). \end{aligned}$$

Inequality (4.2.24) follows now from (4.2.23). □

Lemma 4.2.6. *Let $\partial\Omega \in C^2$. Assume that the functions $p(x)$ and $u(x)$ satisfy the conditions of Lemma 4.2.3. Then for every $\lambda \in (0, 1)$*

$$\int_{\partial\Omega} |\nabla u|^{p(x)} dS \leq \lambda \int_{\Omega} (\epsilon^2 + |\nabla u|^2)^{\frac{p(x)-2}{2}} |u_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u|^{p(x)} dx \right) \quad (4.2.25)$$

with a constant C depending on λ and the constants p^\pm , L , M_0 , M_1 in (4.2.14) and $\partial\Omega$, but independent of u .

Proof. Applying (4.2.24) to u_{x_i} we obtain

$$\begin{aligned} \int_{\partial\Omega} |\nabla u|^{p(x)} dS &\leq \lambda \int_{\Omega} (\epsilon^2 + |\nabla u|^2)^{\frac{p(x)-2}{2}} |u_{xx}|^2 dx \\ &+ C_1 \int_{\Omega} |\nabla u|^{p(x)} dx + C_2 \int_{\Omega} |\nabla u|^{p(x)} \ln |\nabla u| dx + C_3 \end{aligned} \quad (4.2.26)$$

with independent of u constants M, L, K . For every $0 < \theta < p^-$ and r from inequality (4.2.16)

$$|\nabla u|^{p(x)} |\ln |\nabla u|| \leq \begin{cases} |\nabla u|^{p^- - \theta} (|\nabla u|^\theta |\ln |\nabla u||) \leq C'(p^-, \theta) & \text{if } |\nabla u| \leq 1, \\ |\nabla u|^{p(x)+r} (|\nabla u|^{-r} |\ln |\nabla u||) \leq C''(p^-, r) |\nabla u|^{p(x)+r} & \text{if } |\nabla u| \geq 1. \end{cases}$$

Thus, there exists a constant C such that

$$|\nabla u|^{p(x)} |\ln |\nabla u|| \leq C(1 + |\nabla u|^{p(x)+r}) \text{ in } \Omega$$

and (4.2.25) follows from (4.2.26), (4.2.11) and (4.2.16). \square

Theorem 4.2.3. *Let us assume that $p(z)$ and $u(z)$ satisfy the conditions of Theorem 4.2.2. Then for every $\lambda \in (0, 1)$*

$$\int_{\partial\Omega \times (0, T)} |\nabla u|^{p(z)} dS dt \leq \lambda \int_{Q_T} (\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} |u_{xx}|^2 dz + C \left(1 + \int_{Q_T} |\nabla u|^{p(z)} dz \right)$$

with an independent of u constant $C = C(\lambda, N, p^\pm, \partial\Omega, T, L, M_0, M_1)$.

Corollary 4.2.1. *Under the conditions of Theorem 4.2.3*

$$\int_{\partial\Omega \times (0, T)} (\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} |\nabla u|^2 dS dt \leq \lambda \int_{Q_T} (\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} |u_{xx}|^2 dz + C \left(1 + \int_{Q_T} |\nabla u|^{p(z)} dz \right)$$

with an independent of u constant C .

Proof. The inequality is an immediate byproduct of Theorem 4.2.3 and the inequality

$$(\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} |\nabla u|^2 \leq (\epsilon^2 + |\nabla u|^2)^{\frac{p(z)}{2}} \leq C(1 + |\nabla u|^{p(z)}).$$

\square

4.2.4 A priori estimates

We are in position to estimate every term on the right-hand side of (4.2.12).

(a) By (4.2.18) and Lemma 4.2.3

$$\begin{aligned} \int_{\Omega} |\nabla u_{\epsilon}^{(m)}|^2 \gamma_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) \ln^2(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2) dx &\leq C \left(1 + \int_{\Omega} \beta_{\epsilon}^{\frac{p(z)+r-2}{2}} (\nabla u_{\epsilon}^{(m)}) |\nabla u_{\epsilon}^{(m)}|^2 dx \right) \\ &\leq \delta_1 \int_{\Omega} \gamma_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |(u_{\epsilon}^{(m)})_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u_{\epsilon}^{(m)}|^{p(z)} dx \right) \end{aligned} \quad (4.2.27)$$

with an arbitrary $\delta_1 > 0$.

(b) The term

$$I_{\partial\Omega} = - \int_{\partial\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{2}} \left(\Delta u_{\epsilon}^{(m)} (\nabla u_{\epsilon}^{(m)} \cdot \mathbf{n}) - \nabla u_{\epsilon}^{(m)} \cdot \nabla (\nabla u_{\epsilon}^{(m)} \cdot \mathbf{n}) \right)$$

is estimated with the use of Lemma 4.2.5 and the following known assertion.

Lemma 4.2.7 (Lemma A.1, [28]). *If $\partial\Omega \in C^2$ and $u \in W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$, then*

$$|I_{\partial\Omega}| \leq L \int_{\partial\Omega} (\epsilon^2 + |\nabla u|^2)^{\frac{p(z)-2}{2}} |\nabla u|^2 dS$$

with a constant $L = L(\partial\Omega)$. Moreover, $I_{\partial\Omega} \geq 0$ if $\partial\Omega$ is convex.

Gathering Lemmas 4.2.6 and 4.2.7 we arrive at the following estimate: for a.e. $t \in (0, T)$

$$\int_{\partial\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{2}} |\nabla u_{\epsilon}^{(m)}|^2 dS \leq \delta_2 \int_{\Omega} \gamma_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |(u_{\epsilon}^{(m)})_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u_{\epsilon}^{(m)}|^{p(z)} dx \right) \quad (4.2.28)$$

with an arbitrary $\delta_2 > 0$ and a constant C independent of ϵ and m .

Lemma 4.2.8. *Under the conditions of Lemma 4.2.2*

$$\begin{aligned} \sup_{(0,T)} \|\nabla u_{\epsilon}^{(m)}(t)\|_{2,\Omega}^2 + \int_{Q_T} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_{\epsilon}^{(m)})_{xx}|^2 dz \\ \leq C e^{C'T} \left(1 + \|\nabla u_0\|_{2,\Omega}^2 + \|f\|_{L^2(0,T;W_0^{1,2}(\Omega))}^2 \right) \end{aligned} \quad (4.2.29)$$

and

$$\int_{Q_T} |\nabla u_{\epsilon}^{(m)}|^{p(z)+r} dz \leq C'' \quad \text{for any } 0 < r < \frac{4p^-}{p^-(N+2) + 2N} \quad (4.2.30)$$

with constants C, C', C'' independent of m and ϵ .

Proof. Substitution of estimates (4.2.27), (4.2.28) into (4.2.12) leads to the differential inequality

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + (\min\{2, p^-\} - \delta - \delta_1 - \delta_2 - 1) \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2 \\ \leq C_0 + C_1 \int_{\Omega} |\nabla u_\epsilon^{(m)}|^{p(z)} dx + C_2 \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 + C_3 \|f(t)\|_{W_0^{1,2}(\Omega)}^2 \end{aligned} \quad (4.2.31)$$

with constants C_i , $i = 0, 1, 2, 3$, depending on the data but independent of m and ϵ , and arbitrary positive δ_1, δ_2 . Choosing δ_i so small that $\min\{2, p^-\} - (1 + \delta + \delta_1 + \delta_2) = \mu > 0$, multiplying by $e^{-2C_2 t}$ and dropping the second term on the left-hand side, we transform (4.2.31) into the differential inequality for $\|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2$:

$$\frac{d}{dt} \left(e^{-2C_2 t} \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 \right) \leq C e^{-2C_2 t} \left(1 + \int_{\Omega} |\nabla u_\epsilon^{(m)}|^{p(z)} dx + \|f(t)\|_{W_0^{1,2}(\Omega)}^2 \right).$$

Integrating in t and using (4.2.9) and (4.2.10), we finally obtain: for every $t \in (0, T)$

$$\begin{aligned} \|\nabla u_\epsilon^{(m)}(t)\|_{2,\Omega}^2 &\leq C e^{2C_2 T} \left(\|\nabla u_0\|_{2,\Omega}^2 + e^T \left(1 + \|u_0\|_{2,\Omega}^2 + \|f\|_{2,Q_T}^2 \right) + \|\nabla f\|_{2,Q_T}^2 \right) \\ &\leq C e^{C'T} \left(1 + \|u_0\|_{W_0^{1,2}(\Omega)}^2 + \|f\|_{L^2(0,T;W_0^{1,2}(\Omega))}^2 \right). \end{aligned}$$

Now we substitute this estimate into (4.2.31) and integrate the result in t . Plugging (4.2.10), we arrive at the inequality

$$\int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2 dz \leq C e^{C'T} \left(1 + \|\nabla u_0\|_{2,\Omega}^2 + \|f\|_{L^2(0,T;W_0^{1,2}(\Omega))}^2 \right).$$

Estimate (4.2.30) follows then from Theorem 4.2.2. It is sufficient to prove (4.2.30) for $r \in (r_*, r^*)$ with r_*, r^* defined in (4.2.15). Fix some $r \in (r_*, r^*)$, define $Q_T^+ = Q_T \cap \{p(z) + r \geq 2\}$, $Q_T^- = Q_T \cap \{p(z) + r < 2\}$ and represent

$$\int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p+r} dz = \int_{Q_T^+} |\nabla u_\epsilon^{(m)}|^{p+r} dz + \int_{Q_T^-} \dots \equiv I_+ + I_-.$$

Then

$$I_+ \leq \int_{Q_T^+} \beta_\epsilon^{\frac{p+r-2}{2}} (\nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 dz \leq \int_{Q_T} \beta_\epsilon^{\frac{p+r-2}{2}} (\nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 dz$$

and estimate on I_+ follows. To estimate I_- , set $B_+ = Q_T^- \cap \{z : |\nabla u_\epsilon^{(m)}| \geq \epsilon\}$, $B_- = Q_T^- \cap \{z : |\nabla u_\epsilon^{(m)}| < \epsilon\}$. Then

$$\begin{aligned} I_- &= \int_{B_+ \cup B_-} |\nabla u_\epsilon^{(m)}|^{p+r} dz = \int_{B_+} (|\nabla u_\epsilon^{(m)}|^2)^{\frac{p+r-2}{2}} |\nabla u_\epsilon^{(m)}|^2 dz + \int_{B_-} \epsilon^{p+r} dz \\ &\leq 2^{\frac{2-r-p^-}{2}} \int_{B_+} \beta_\epsilon^{\frac{p+r-2}{2}} (\nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 dz + \epsilon^{p^-+r} T |\Omega| \\ &\leq C \left(1 + \int_{Q_T} \beta_\epsilon^{\frac{p+r-2}{2}} (\nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 dz \right). \end{aligned}$$

Gathering these estimates and applying Theorem 4.2.2 we obtain (4.2.30) with $r \in (r_*, r^*)$. The case $r \in (0, r_*]$ follows then by the Young inequality. \square

Remark 4.2.2. *Inequality (4.2.30) entails the inequality*

$$\int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)+r}{2}} dz \leq C, \quad \epsilon \in (0, 1), \quad t \in (0, r^*), \quad (4.2.32)$$

with an independent of ϵ constant C .

Lemma 4.2.9. *Let the conditions of Lemma 4.2.2 be fulfilled and*

$$\operatorname{ess\,sup}_{Q_T} |p_t| \leq C^* < \infty.$$

Then the following estimate holds:

$$\|(u_\epsilon^{(m)})_t\|_{2, Q_T}^2 + \sup_{(0, T)} \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} dx \leq C \left(1 + \int_{\Omega} |\nabla u_0|^{p(x, 0)} dx \right) + \|f\|_{2, Q_T}^2 \quad (4.2.33)$$

with an independent of m and ϵ constant C .

Proof. Multiplying (4.2.8) with $(u_j^{(m)})_t$ and summing over $j = 1, 2, \dots, m$ we obtain the equality

$$\int_{\Omega} (u_\epsilon^{(m)})_t^2 dx + \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(x, t)-2}{2}} \nabla u_\epsilon^{(m)} \cdot \nabla (u_\epsilon^{(m)})_t dx = \int_{\Omega} f (u_\epsilon^{(m)})_t dx. \quad (4.2.34)$$

It is straightforward to check that

$$\begin{aligned} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} \cdot \nabla (u_\epsilon^{(m)})_t &= \frac{d}{dt} \left(\frac{(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}}}{p(z)} \right) \\ &\quad + \frac{p_t(z) (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}}}{p^2(z)} \left(1 - \frac{p(z)}{2} \ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2) \right). \end{aligned}$$

With the use of this identity we rewrite (4.2.34) in the form

$$\begin{aligned} \int_{\Omega} (u_\epsilon^{(m)})_t^2 dx + \frac{d}{dt} \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} dx \\ = - \int_{\Omega} \frac{p_t (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}}}{p^2(z)} \left(1 - \frac{p(z)}{2} \ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2) \right) + \int_{\Omega} f (u_\epsilon^{(m)})_t dx. \end{aligned} \quad (4.2.35)$$

The terms on the right-hand side of (4.2.35) are estimated separately. For the first term, we use (4.2.10) and (4.2.27):

$$\begin{aligned} \left| \int_{\Omega} \frac{p_t (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}}}{p^2} \left(1 - \frac{p}{2} \ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2) \right) \right| &\leq C_1 \left(1 + \int_{\Omega} |\nabla u_\epsilon^{(m)}|^{p(z)} dx \right) \\ &\quad + C_2 \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} \ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2) dx. \end{aligned} \quad (4.2.36)$$

The second term is estimated by the Cauchy inequality:

$$\left| \int_{\Omega} f(u_{\epsilon}^{(m)})_t dx \right| \leq \frac{1}{2} \|(u_{\epsilon}^{(m)})_t\|_{2,\Omega}^2 + \frac{1}{2} \|f\|_{2,\Omega}^2. \quad (4.2.37)$$

Estimate (4.2.33) follows after substitution of (4.2.36), (4.2.37) into (4.2.35) and integration of the resulting inequality in t : for every $t \in (0, T)$

$$\|(u_{\epsilon}^{(m)})_t\|_{2,Q_t}^2 + 2 \int_{\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)}{2}} dx \leq C \left(1 + \int_{\Omega} (\epsilon^2 + |\nabla u_0^{(m)}(x)|^2)^{\frac{p(x,0)}{2}} dx \right) + \|f\|_{2,Q_t}^2.$$

□

4.2.5 Strong solution of the regularized problem

In this section, we prove that the regularized problem (4.2.6) has a unique strong solution. We show first the existence of a weak solution with $u_{\epsilon t} \in L^2(Q_T)$ and then prove that this solution possesses extra regularity properties and, thus, is the strong solution.

4.2.5.1 Existence and uniqueness of weak solution

Theorem 4.2.4. *Let u_0 , f , p and $\partial\Omega$ satisfy the conditions of Theorem 4.2.1. Then for every $\epsilon \in (0, 1)$ problem (4.2.6) has a unique solution u_{ϵ} which satisfies the estimates*

$$\begin{aligned} \|u_{\epsilon}\|_{W_{p(\cdot)}(Q_T)} \leq C_0, \quad \operatorname{ess\,sup}_{(0,T)} \|u_{\epsilon}(t)\|_{2,\Omega}^2 + \|u_{\epsilon t}\|_{2,Q_T}^2 \leq C_0 \\ \operatorname{ess\,sup}_{(0,T)} \int_{\Omega} |\nabla u_{\epsilon}|^{q(z)} dx \leq C_0, \quad q(z) = \max\{2, p(z)\}, \end{aligned} \quad (4.2.38)$$

with a constant C_0 depending on the data but not on ϵ . Moreover, u_{ϵ} possesses the property of global higher integrability of the gradient: for every

$$\delta \in (0, r^*), \quad r^* = \frac{4p^-}{p^-(N+2) + 2N},$$

there exists a constant $C = C\left(\partial\Omega, N, p^{\pm}, \delta, \|u_0\|_{W_0^{1,q(\cdot,0)}(\Omega)}, \|f\|_{L^2(0,T;W_0^{1,2}(\Omega))}\right)$ such that

$$\int_{Q_T} |\nabla u_{\epsilon}|^{p(z)+\delta} dz \leq C. \quad (4.2.39)$$

Remark 4.2.3. *Due to the fact that estimate (4.2.39) is global in time and space, it is new even in the case of constant p . We refer to [114] for a detailed insight into this issue, in particular, to [114, Lemma 5.4].*

Let $\epsilon > 0$ be a fixed parameter, Ω be a bounded domain with the boundary $\partial\Omega \in C^2$ boundary, and let $u_{\epsilon}^{(m)}$ be the sequence of Galerkin approximations defined in (4.2.7). Under the assumptions

$$u_0 \in W_0^{1,2}(\Omega), \quad f \in L^2((0, T); W_0^{1,2}(\Omega)), \quad \|\nabla p\|_{\infty, Q_T} \leq C_*, \quad \|p_t\|_{\infty, Q_T} \leq C^*$$

the functions $u_\epsilon^{(m)}$ exist and satisfy estimates (4.2.9), (4.2.10), (4.2.29), (4.2.30) and (4.2.33). These uniform in m and ϵ estimates allow one to choose a subsequence $u_\epsilon^{(m)}$ (for which we keep the same notation), and functions $u_\epsilon, \eta_\epsilon$ such that

$$\begin{aligned} u_\epsilon^{(m)} &\rightharpoonup u_\epsilon \quad \star\text{-weakly in } L^\infty(0, T; L^2(\Omega)), \\ u_{\epsilon t}^{(m)} &\rightharpoonup u_{\epsilon t} \text{ in } L^2(Q_T), \\ \nabla u_\epsilon^{(m)} &\rightharpoonup \nabla u_\epsilon \text{ in } (L^{p(\cdot)}(Q_T))^N, \\ (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} &\rightharpoonup \eta_\epsilon \text{ in } (L^{p'(\cdot)}(Q_T))^N \end{aligned} \tag{4.2.40}$$

The assumption $p^- > \frac{2N}{N+2}$ yields the inclusions

$$W_0^{1,p(\cdot,t)}(\Omega) \subset W_0^{1,p^-}(\Omega) \hookrightarrow L^2(\Omega).$$

Since $u_\epsilon^{(m)}$ and $(u_\epsilon^{(m)})_t$ are uniformly bounded in $L^\infty(0, T; W_0^{1,p^-}(\Omega))$ and $L^\infty(0, T; L^2(\Omega))$, it follows from the compactness lemma [235, Sec.8, Corollary 4] that the sequence $\{u_\epsilon^{(m)}\}$ is relatively compact in $C([0, T]; L^2(\Omega))$, *i.e.*, there exists a subsequence $\{u_\epsilon^{(m_k)}\}$, which we assume coinciding with $\{u_\epsilon^{(m)}\}$, such that $u_\epsilon^{(m)} \rightarrow u_\epsilon$ in $C([0, T]; L^2(\Omega))$ and a.e. in Q_T .

Let us define

$$\mathcal{P}_m = \left\{ \psi : \psi = \sum_{i=1}^m \psi_i(t) \phi_i(x), \psi_i \in C^1[0, T] \right\}.$$

Fix some $m \in \mathbb{N}$. By the method of construction $u_\epsilon^{(m)} \in \mathcal{P}_m$. Since $\mathcal{P}_k \subset \mathcal{P}_m$ for $k < m$, then for every $\xi_k \in \mathcal{P}_k$ with $k \leq m$

$$\int_{Q_T} u_{\epsilon t}^{(m)} \xi_k dz + \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} \cdot \nabla \xi_k dz = \int_{Q_T} f \xi_k dz. \tag{4.2.41}$$

Let $\xi \in W_{p(\cdot)}(Q_T)$. Take a sequence $\{\xi_k\}$ such that $\xi_k \in \mathcal{P}_k$ and $\xi_k \rightarrow \xi \in W_{p(\cdot)}(Q_T)$. Passing to the limit as $m \rightarrow \infty$ with a fixed k , and then letting $k \rightarrow \infty$, from the above equality we infer that

$$\int_{Q_T} u_{\epsilon t} \xi dz + \int_{Q_T} \eta_\epsilon \cdot \nabla \xi dz = \int_{Q_T} f \xi dz \tag{4.2.42}$$

for all $\xi \in W_{p(\cdot)}(Q_T)$. To identify the limit vector η_ϵ we use the classical argument based on monotonicity of the function $\gamma_\epsilon(z, \mathbf{s}) \mathbf{s} \equiv (\epsilon^2 + |\mathbf{s}|^2)^{\frac{p(z)-2}{2}} \mathbf{s} : \mathbb{R}^N \mapsto \mathbb{R}^N$.

Lemma 4.2.10. *For all $z \in Q_T$, $\xi, \zeta \in \mathbb{R}^N$, ($\zeta \neq \xi$) and $\epsilon > 0$*

$$(\gamma_\epsilon(z, \zeta) \zeta - \gamma_\epsilon(z, \xi) \xi) \cdot (\zeta - \xi) \geq 0. \tag{4.2.43}$$

Proof. Let $\zeta \neq \xi$. The straightforward computation shows that

$$\begin{aligned}
 & (\gamma_\epsilon(z, \xi)\xi - \gamma_\epsilon(z, \zeta)\zeta) \cdot (\xi - \zeta) \\
 &= \int_0^1 \frac{d}{d\theta} (\epsilon^2 + |\theta\xi + (1-\theta)\zeta|^2)^{\frac{p(z)-2}{2}} (\theta\xi + (1-\theta)\zeta) d\theta \cdot (\xi - \zeta) \\
 &= \int_0^1 (\epsilon^2 + |\theta\xi + (1-\theta)\zeta|^2)^{\frac{p(z)-2}{2}} \left[(p(z) - 2) \cos^2(\widehat{\mu, \nu}) + 1 \right] d\theta |\xi - \zeta|^2 \\
 &\geq |\xi - \zeta|^2 \begin{cases} \epsilon^{p(z)-2} & \text{if } p(z) \geq 2, \\ (p(z) - 1) \int_0^1 (\epsilon^2 + |\theta\xi + (1-\theta)\zeta|^2)^{\frac{p(z)-2}{2}} d\theta & \text{if } p(z) \in (1, 2), \end{cases}
 \end{aligned}$$

where μ, ν are the unit vectors $\mu = \frac{\xi - \zeta}{|\xi - \zeta|}$, $\nu = \frac{\zeta + \theta(\xi - \zeta)}{|\zeta + \theta(\xi - \zeta)|}$. \square

Equality (4.2.41) is true for $\xi_k = u_\epsilon^{(m)}$. By virtue of (4.2.43), for every $\psi \in \mathcal{P}_k$ with $k \leq m$

$$\begin{aligned}
 0 &= \int_{Q_T} (u_\epsilon^{(m)})_t u_\epsilon^{(m)} dz + \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2 dz - \int_{Q_T} f u_\epsilon^{(m)} dz \\
 &\geq \int_{Q_T} (u_\epsilon^{(m)})_t u_\epsilon^{(m)} dz + \int_{Q_T} (\epsilon^2 + |\nabla \psi|^2)^{\frac{p(z)-2}{2}} \nabla \psi \cdot \nabla (u_\epsilon^{(m)} - \psi) dz \\
 &\quad + \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} \cdot \nabla \psi dz - \int_{Q_T} f u_\epsilon^{(m)} dz.
 \end{aligned}$$

Let us pass to the limit as $m \rightarrow \infty$. Using the limit relations (4.2.40), the fact that $u_\epsilon^{(m)} (u_\epsilon^{(m)})_t \rightharpoonup u_\epsilon u_{\epsilon t}$ as the product of weakly and strongly convergent sequences, and substituting (4.2.42) into the resulting inequality, we find that for every $\psi \in \mathcal{P}_k$

$$\begin{aligned}
 0 &\geq \int_{Q_T} u_\epsilon u_{\epsilon t} dz + \int_{Q_T} (\epsilon^2 + |\nabla \psi|^2)^{\frac{p(z)-2}{2}} \nabla \psi \cdot \nabla (u_\epsilon - \psi) dz + \int_{Q_T} \eta_\epsilon \cdot \nabla \psi dz - \int_{Q_T} f u_\epsilon dz \\
 &= \int_{Q_T} \left((\epsilon^2 + |\nabla \psi|^2)^{\frac{p(z)-2}{2}} \nabla \psi - \eta_\epsilon \right) \cdot \nabla (u_\epsilon - \psi) dz.
 \end{aligned}$$

Since $\bigcup_{k=1}^\infty \mathcal{P}_k$ is dense in $W_{p(\cdot)}(Q_T)$, the last inequality also holds for every $\psi \in W_{p(\cdot)}(Q_T)$.

Let us take $\psi = u_\epsilon + \lambda \xi$ with $\lambda > 0$ and an arbitrary $\xi \in W_{p(\cdot)}(Q_T)$. Then

$$\lambda \int_{Q_T} \int_{Q_T} \left((\epsilon^2 + |\nabla (u_\epsilon + \lambda \xi)|^2)^{\frac{p(z)-2}{2}} \nabla (u_\epsilon + \lambda \xi) - \eta_\epsilon \right) \cdot \nabla \xi dz \leq 0.$$

Simplifying and letting $\lambda \rightarrow 0$ we find that

$$\int_{Q_T} \left((\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon - \eta_\epsilon \right) \cdot \nabla \xi dz \leq 0 \quad \forall \xi \in W_{p(\cdot)}(Q_T),$$

which is possible only if

$$\int_{Q_T} \left((\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon - \eta_\epsilon \right) \cdot \nabla \xi dz = 0 \quad \forall \xi \in W_{p(\cdot)}(Q_T).$$

Thus, the limit function u_ϵ satisfies identity (4.2.2) with the regularized flux $(\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon$. The initial condition for u_ϵ is fulfilled by continuity because $u_\epsilon \in C^0([0, T]; L^2(\Omega))$.

Uniqueness of the weak solution is an immediate byproduct of monotonicity of the function $\gamma_\epsilon(z, \mathbf{s})$. Let u_1, u_2 be two different strong solutions of problem (4.2.6). Combining equalities (4.2.2) for u_i with the test-function $u_1 - u_2$, using (4.2.43) and the formula of integration by parts (4.1.7) we find that

$$\|u_1 - u_2\|_{2, \Omega}^2(t) \leq 0 \quad \text{for a.e. } t \in (0, T),$$

whence $u_1 = u_2$ a.e. in Q_T .

Let us prove estimates (4.2.38), (4.2.39). The uniform with respect to ϵ estimates (4.2.29) and (4.2.33) allow us to choose a subsequence of $\{u_\epsilon^{(m)}\}$ which satisfies (4.2.40) and also $|\nabla u_\epsilon^{(m)}|^{q(x,t)} \rightarrow |\nabla u_\epsilon|^{q(x,t)}$ \star -weakly in $L^\infty(0, T; L^1(\Omega))$, $q(x, t) = \max\{p(x, t), 2\}$. Estimate (4.2.38) follows now from the lower semicontinuity of the norm and the modular $\rho_{r(\cdot)}(s) = \int_\Omega |s|^{r(x)} dx$ with $r(x) \in C^0(\bar{\Omega})$, $r(x) \in [1, r^+]$, $r^+ < \infty$ (see [112, Th. 3.2.9]). For every $\delta \in (0, r^*)$, inequality (4.2.39) follows in the same way from the uniform estimate (4.2.30).

4.2.5.2 Second-order regularity

Lemma 4.2.11. *If $p^- \geq \max\left\{\frac{2N}{N+2}, \frac{6}{5}\right\}$, the function $h(\mathbf{s}) = \gamma_\epsilon(z, \mathbf{s})|\mathbf{s}|^2$ is strictly convex with respect to \mathbf{s} .*

Proof. Fix two points $\xi, \zeta \in \mathbb{R}^N$, $\xi \neq \zeta$, and consider the function

$$F(\tau) = \gamma_\epsilon(z, \tau\xi + (1-\tau)\zeta) |\tau\xi + (1-\tau)\zeta|^2, \quad \tau \in [0, 1].$$

Let us accept the notation $\sigma = |\tau\xi + (1-\tau)\zeta|^2$ and $\eta = \frac{\xi - \zeta}{|\xi - \zeta|}$. The straightforward computation gives

$$F''(\tau) = |\xi - \zeta|^2 (\epsilon^2 + \sigma)^{\frac{p-2}{2}-2} \left[(p\sigma + 2\epsilon^2)(\sigma + \epsilon^2) + (p-2)(p\sigma + 4\epsilon^2)(\tau\xi + (1-\tau)\zeta, \eta)^2 \right].$$

Obviously, $F''(\tau) > 0$ if $p(z) \geq 2$. Let $1 < p(z) < 2$. Since $(\tau\xi + (1-\tau)\zeta, \eta)^2 \leq \sigma$, we obtain:

$$\begin{aligned} F''(\tau) &\geq |\xi - \zeta|^2 (\epsilon^2 + \sigma)^{\frac{p-2}{2}-2} \left[(p\sigma + 2\epsilon^2)(\sigma + \epsilon^2) + (p-2)(p\sigma + 4\epsilon^2)\sigma \right] \\ &= |\xi - \zeta|^2 (\epsilon^2 + \sigma)^{\frac{p-2}{2}-2} \left[p(p-1)\sigma^2 + (5p-6)\sigma\epsilon^2 + 2\epsilon^4 \right], \end{aligned}$$

whence $F'' > 0$ for all $\xi \neq \zeta$ and $\epsilon \geq 0$, provided that $p^- \geq \frac{6}{5}$.

□

The proof of stronger convergence properties of the sequence $\nabla u_\epsilon^{(m)}$ stems from the following general result on the convergence of sequences of functionals. For convenience, we formulate it in the form already adapted to our problem.

Proposition 4.2.1 (Theorem 2.1, Corollary 2.1, [224]). *Let $\mathcal{F}_m(z, \mathbf{s}) : Q_T \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a sequence of nonnegative functions, convex with respect to \mathbf{s} for every $z \in Q_T$ and locally uniformly convergent to a function $\mathcal{F}_0(z, \mathbf{s})$ as $m \rightarrow \infty$, which is essentially convex with respect to \mathbf{s} for every $z \in Q_T$. Assume that $\mathcal{F}_m(z, \mathbf{s}) \geq a(|\mathbf{s}|^\alpha + 1)$ with some constants $a > 0$, $\alpha > 1$. If $v_m \in (L^s(Q_T))^N$, $v_m \rightharpoonup v_0$ in $(L^s(Q_T))^N$, $s > 1$, and*

$$\int_{Q_T} \mathcal{F}_m(z, v_m) dz \rightarrow \int_{Q_T} \mathcal{F}_0(z, v_0) dz < \infty,$$

then

$$\int_{Q_T} |v_m - v_0|^\alpha dz \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Theorem 4.2.5. *Let the conditions of Theorem 4.2.4 be fulfilled.*

(i) *If $N \geq 3$ or $N = 2$ and $p^- > \frac{6}{5}$, then*

$$\nabla u_\epsilon^{(m)} \rightarrow \nabla u_\epsilon \text{ a.e. in } Q_T.$$

(ii) *Under the conditions of item (i) $\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon) D_i u_\epsilon \in L^2(0, T; W^{1,2}(\Omega))$, $i = 1, 2, \dots, N$, and*

$$\|\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon) D_i u_\epsilon\|_{L^2(0, T; W^{1,2}(\Omega))} \leq M, \quad i = 1, 2, \dots, N,$$

with an independent of ϵ constant M .

(iii) *If $N \geq 2$ and $p^- > \frac{2N}{N+2}$, then $D_{ij}^2 u_\epsilon \in L_{loc}^{p(\cdot)}(Q_T \cap \{z : p(z) < 2\})$, $i, j = 1, 2, \dots, N$, and*

$$\sum_{i,j=1}^N \|D_{ij}^2 u_\epsilon\|_{p(\cdot), Q_T \cap \{z : p(z) < 2\}} \leq M'$$

with an independent of ϵ constant M' .

Proof. (i) It is already shown that $\nabla u_\epsilon^{(m)} \rightharpoonup \nabla u_\epsilon$ in $L^{p(\cdot)}(Q_T)$ as $m \rightarrow \infty$. By Lemma 4.2.11 the function $\gamma_\epsilon(z, \mathbf{s})|\mathbf{s}|^2$ is strictly convex with respect to \mathbf{s} . According to (4.2.11)

$$\left(1 + 2^{\frac{p^+}{2}}\right) + \gamma_\epsilon(z, \mathbf{s})|\mathbf{s}|^2 \geq 1 + |\mathbf{s}|^{p(z)} \geq |\mathbf{s}|^{p^-}. \quad (4.2.44)$$

By virtue of the energy equalities (4.2.41), (4.2.42) and the limit relations (4.2.40)

$$\begin{aligned} \int_{Q_T} \gamma_\epsilon(z, \nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 dz &= - \int_{Q_T} u_{\epsilon t}^{(m)} u_\epsilon^{(m)} dz + \int_{Q_T} f u_\epsilon^{(m)} dz \\ &\rightarrow - \int_{Q_T} u_{\epsilon t} u_\epsilon dz + \int_{Q_T} f u_\epsilon dz = \int_{Q_T} \gamma_\epsilon(z, \nabla u_\epsilon) |\nabla u_\epsilon|^2 dz \quad \text{as } m \rightarrow \infty. \end{aligned}$$

Now we apply Proposition 4.2.1 with $\mathcal{F}_m(z, \mathbf{s}) = \gamma_\epsilon(z, \mathbf{s})|\mathbf{s}|^2 + M$ and a sufficiently large positive constant M . It follows that $\nabla u_\epsilon^{(m)} \rightarrow \nabla u_\epsilon$ a.e. in Q_T , whence

$$\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon^{(m)})\nabla u_\epsilon^{(m)} \rightarrow \gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon)\nabla u_\epsilon \quad \text{a.e. in } Q_T. \quad (4.2.45)$$

(ii) According to (4.2.29) and (4.2.32), for every $i, j = 1, 2, \dots, N$

$$\begin{aligned} & \left\| D_i \left(\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon^{(m)}) D_j u_\epsilon^{(m)} \right) \right\|_{2, Q_T}^2 \\ & \leq C \left(\int_{Q_T} \gamma_\epsilon(z, \nabla u_\epsilon^{(m)}) |u_{\epsilon xx}^{(m)}|^2 dz + \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} |\ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)| dz \right) \\ & \leq C' \left(1 + \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} |\ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)| dz \right) \\ & \leq C \int_{Q_T} \gamma_\epsilon(z, \nabla u_\epsilon^{(m)}) |u_{\epsilon xx}^{(m)}|^2 dz + C'' \left(1 + \int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p(z)+\mu} dz \right) \\ & \leq M, \quad M = M(\|u_0\|_{W_0^{1,2}(\Omega)}, \|f\|_{L^2(0,T;W_0^{1,2}(\Omega))}, N, p^\pm, \omega, \partial\Omega), \end{aligned}$$

whence the existence of a subsequence $\{u_\epsilon^{(m_k)}\}$ (we may assume that it coincides with the whole sequence) such that

$$D_i \left(\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon^{(m)}) D_j u_\epsilon^{(m)} \right) \rightharpoonup \eta_{ij} \in L^2(Q_T) \quad \text{as } m \rightarrow \infty.$$

By (4.2.30) there exists $\delta > 0$ such that

$$\|\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon^{(m)}) D_j u_\epsilon^{(m)}\|_{2+\delta, Q_T} \leq \|(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{4}}\|_{2+\delta, Q_T} \leq C$$

with a constant C independent of m and ϵ . Since $\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon^{(m)}) D_j u_\epsilon^{(m)}$ are uniformly bounded in $L^{2+\delta}(Q_T)$ and converge pointwise due to (4.2.45), it follows from the Vitali convergence theorem that

$$\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon^{(m)}) D_j u_\epsilon^{(m)} \rightarrow \gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon) D_j u_\epsilon \quad \text{in } L^2(Q_T).$$

For every $\phi \in C^\infty(Q_T)$ with $\text{supp } \phi \Subset Q_T$ and $i, j = 1, \dots, N$

$$\begin{aligned} \left(D_i \left(\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon^{(m)}) D_j u_\epsilon^{(m)} \right), \phi \right)_{2, Q_T} &= - \left(\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon^{(m)}) D_j u_\epsilon^{(m)}, D_i \phi \right)_{2, Q_T} \\ &\rightarrow - \left(\gamma_\epsilon^{\frac{1}{2}}(z, \nabla u_\epsilon) D_j u_\epsilon, D_i \phi \right)_{2, Q_T} \quad \text{as } m \rightarrow \infty. \end{aligned}$$

Thus, it is necessary that

$$\eta_{ij} = D_i \left((\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{p(z)-2}{4}} D_j u_\epsilon \right) \in L^2(Q_T) \quad \text{and} \quad \|\eta_{ij}\|_{2, Q_T}^2 \leq M.$$

(iii) Let us denote $Q_T^- = Q_T \cap \{z : p(z) < 2\}$. By Young's inequality, (4.2.10) and (4.2.29), for every $D \Subset Q_T^-$

$$\begin{aligned} \int_D |D_{ij}^2 u_\epsilon^{(m)}|^{p(z)} dz &= \int_D \left(\gamma_\epsilon(z, \nabla u_\epsilon^{(m)}) |D_{ij}^2 u_\epsilon^{(m)}|^2 \right)^{\frac{p(z)}{2}} \gamma_\epsilon^{-\frac{p(z)}{2}}(z, \nabla u_\epsilon^{(m)}) dz \\ &\leq \int_D \gamma_\epsilon(z, \nabla u_\epsilon^{(m)}) |D_{ij}^2 u_\epsilon^{(m)}|^2 dz + \int_D (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} dz \leq C \end{aligned}$$

with a constant C independent of ϵ , m and D . It follows that there exists $\chi \in L^{p(\cdot)}(Q_T^-)$ such that $D_{ij}^2 u_\epsilon^{(m)} \rightharpoonup \chi$ in $L^{p(\cdot)}(D)$ (up to a subsequence). Since $\nabla u_\epsilon^{(m)} \rightharpoonup \nabla u_\epsilon$ in $L^{p(\cdot)}(Q_T)$, for every $\phi \in C^\infty(Q_T^-)$ with $\text{supp } \phi \Subset Q_T^-$

$$(\chi, \phi)_{2, Q_T} = \lim_{m \rightarrow \infty} (D_{ij}^2 u_\epsilon^{(m)}, \phi)_{2, Q_T} = - \lim_{m \rightarrow \infty} (D_i u_\epsilon^{(m)}, D_j \phi)_{2, Q_T} = (D_i u_\epsilon, D_j \phi)_{2, Q_T}.$$

It is necessary that $\chi = D_{ij}^2 u_\epsilon$, and $\|D_{ij}^2 u_\epsilon\|_{p(\cdot), D} \leq C$ by the lower semicontinuity of the modular. \square

Remark 4.2.4. Let u_ϵ be a solution of problem (4.2.6). The regularity of the regularized flux

$$D_{x_j} \left((\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{p(z)-2}{4}} D_{x_i} u_\epsilon \right) \in L^2(Q_T)$$

leads to the local fractional differentiability of ∇u , see [114, Ch.6] for the case of constant p .

4.2.6 Strong solution of the degenerate problem

4.2.6.1 Existence and uniqueness of strong solutions

Let $\{u_\epsilon\}$ be the family of strong solutions of the regularized problems (4.2.6). The uniform in ϵ estimates (4.2.38) allow us to choose a sequence $\{u_{\epsilon_k}\}$ and functions $u \in W_{p(\cdot)}(Q_T)$, $u_t \in L^2(Q_T)$, $\eta \in (L^{p'(\cdot)}(Q_T))^N$ with the following properties:

$$\begin{aligned} u_{\epsilon_k} &\rightarrow u \quad \star\text{-weakly in } L^\infty(0, T; L^2(\Omega)), \\ u_{\epsilon_k t} &\rightharpoonup u_t \text{ in } L^2(Q_T), \\ \nabla u_{\epsilon_k} &\rightharpoonup \nabla u \text{ in } (L^{p(\cdot)}(Q_T))^N, \\ \gamma_{\epsilon_k}(z, \nabla u_{\epsilon_k}) \nabla u_{\epsilon_k} &\rightharpoonup \eta \text{ in } (L^{p'(\cdot)}(Q_T))^N. \end{aligned}$$

Moreover, $u \in C([0, T]; L^2(\Omega))$. Each of u_{ϵ_k} satisfies the identity

$$\int_{Q_T} u_{\epsilon_k t} \xi dz + \int_{Q_T} \gamma_{\epsilon_k}(z, \nabla u_{\epsilon_k}) \nabla u_{\epsilon_k} \cdot \nabla \xi dz = \int_{Q_T} f \xi dz \quad \forall \xi \in W_{p(\cdot)}(Q_T), \quad (4.2.46)$$

which yields

$$\int_{Q_T} u_t \xi dz + \int_{Q_T} \eta \cdot \nabla \xi dz = \int_{Q_T} f \xi dz \quad \forall \xi \in W_{p(\cdot)}(Q_T). \quad (4.2.47)$$

To identify η , we use the monotonicity argument. Take $\xi = u_{\epsilon_k}$ in (4.2.46):

$$\int_{Q_T} u_{\epsilon_k t} u_{\epsilon_k} dz + \int_{Q_T} \gamma_{\epsilon_k}(z, \nabla u_{\epsilon_k}) \nabla u_{\epsilon_k} \cdot \nabla u_{\epsilon_k} dz = \int_{Q_T} f u_{\epsilon_k} dz. \quad (4.2.48)$$

By virtue of monotonicity, for every $\phi \in W_{p(\cdot)}(Q_T)$

$$\begin{aligned} \int_{Q_T} \gamma_{\epsilon_k}(z, \nabla u_{\epsilon_k}) \nabla u_{\epsilon_k} \cdot \nabla u_{\epsilon_k} dz &\geq \int_{Q_T} \gamma_{\epsilon_k}(z, \nabla \phi) \nabla \phi \cdot \nabla (u_{\epsilon_k} - \phi) dz + \int_{Q_T} \gamma_{\epsilon_k}(z, \nabla u_{\epsilon_k}) \nabla u_{\epsilon_k} \cdot \nabla \phi dz \\ &= \int_{Q_T} (\gamma_{\epsilon_k}(z, \nabla \phi) - |\nabla \phi|^{p-2}) \nabla \phi \cdot \nabla (u_{\epsilon_k} - \phi) dz + \int_{Q_T} \gamma_{\epsilon_k}(z, \nabla u_{\epsilon_k}) \nabla u_{\epsilon_k} \cdot \nabla \phi dz \\ &\quad + \int_{Q_T} |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla (u_{\epsilon_k} - \phi) dz \equiv J_1 + J_2 + J_3, \end{aligned}$$

where

$$J_2 \rightarrow \int_{Q_T} \eta \cdot \nabla \phi dz, \quad J_3 \rightarrow \int_{Q_T} |\nabla \phi|^{p-2} \nabla \phi \cdot \nabla (u - \phi) dz \quad \text{as } k \rightarrow \infty.$$

Since $|(\gamma_{\epsilon_k}(z, \nabla \phi) - |\nabla \phi|^{p-2}) \nabla \phi| \rightarrow 0$ a.e. in Q_T as $k \rightarrow \infty$, and the integrand of J_1 has the majorant

$$\left| (\gamma_{\epsilon_k}(z, \nabla \phi) - |\nabla \phi|^{p-2}) \nabla \phi \right| \leq 2(1 + |\nabla \phi|^2)^{\frac{p(z)}{2}} \leq C \left(1 + |\nabla \phi|^{p(z)}\right),$$

$J_1 \rightarrow 0$ by the dominated convergence theorem. Combining (4.2.47) with (4.2.48) and letting $k \rightarrow \infty$ we find that for every $\phi \in W_{p(\cdot)}(Q_T)$

$$\int_{Q_T} \left(|\nabla \phi|^{p(z)-2} \nabla \phi - \eta \right) \cdot \nabla (u - \phi) dz \geq 0.$$

Choosing $\phi = u + \lambda \zeta$ with $\lambda > 0$ and $\zeta \in W_{p(\cdot)}(Q_T)$, simplifying and letting $\lambda \rightarrow 0^+$, we obtain the inequality

$$\int_{Q_T} \left(|\nabla u|^{p(z)-2} \nabla u - \eta \right) \cdot \nabla \zeta dz \geq 0 \quad \forall \zeta \in W_{p(\cdot)}(Q_T),$$

which means that in (4.2.47) η coincides with $|\nabla u|^{p(z)-2} \nabla u$. Since $u \in C([0, T]; L^2(\Omega))$, the initial condition is fulfilled by continuity.

By virtue of (4.2.38), (4.2.39), the subsequence convergent to the solution may be chosen so that $|\nabla u_{\epsilon_k}|^{q(x,t)} \rightarrow |\nabla u|^{q(x,t)}$ \star -weakly in $L^\infty(0, T; L^1(\Omega))$. Estimate (4.2.5) follows then from the lower semicontinuity of the modular exactly as in the proof of (4.2.38).

Uniqueness of the constructed strong solution of problem (4.2.1) stems from the monotonicity of the mapping $\gamma_0(z, \mathbf{s})\mathbf{s}$ and the formula of integration by parts.

4.2.6.2 Higher integrability of the gradient

Let us fix $\delta \in (0, r^*)$. According to (4.2.39) $\|\nabla u_{\epsilon_k}\|_{p(\cdot)+\delta, Q_T} \leq C_\delta$ with an independent of ϵ_k constant C , which allows one to choose a subsequence (for which we use the same notation), such that

$$\nabla u_{\epsilon_k} \rightharpoonup \nabla u \text{ in } L^{p(\cdot)+\delta}(Q_T).$$

By the property of lower semicontinuity of the modular

$$\int_{Q_T} |\nabla u|^{p(z)+\delta} dz \leq \liminf_{k \rightarrow \infty} \int_{Q_T} |\nabla u_{\epsilon_k}|^{p(z)+\delta} dz \leq C_\delta$$

with the constant C_δ from (4.2.39).

4.2.6.3 Second-order regularity of strong solutions

Let us assume that $p^- > \max\left\{\frac{2N}{N+2}, \frac{6}{5}\right\}$ and show that $\nabla u_\epsilon \rightarrow \nabla u$ a.e. in Q_T . Consider the sequence of nonnegative functions

$$F_{\epsilon_k}(z, \mathbf{s}) = \gamma_{\epsilon_k}(z, \mathbf{s})|\mathbf{s}|^2.$$

$F_\epsilon(x, \mathbf{s})$ are strictly convex with respect to \mathbf{s} (by Lemma 4.2.11) and satisfy inequality (4.2.44). It is already shown that $\nabla u_{\epsilon_k} \rightarrow \nabla u$ in $L^{p(\cdot)}(Q_T)$. According to (4.2.46), (4.2.47)

$$\int_{Q_T} F_{\epsilon_k}(z, \nabla u_{\epsilon_k}) dz \rightarrow \int_{Q_T} F_0(z, \nabla u) dz \text{ as } k \rightarrow \infty$$

and $F_\epsilon(z, \mathbf{s}) \rightarrow F_0(z, \mathbf{s}) = |\mathbf{s}|^p$ as $\epsilon \rightarrow 0$ locally uniformly with respect to $(z, \mathbf{s}) \in Q_T \times \mathbb{R}^N$. Indeed:

$$\begin{aligned} |(\epsilon^2 + |\mathbf{s}|^2)^{\frac{p-2}{2}} |\mathbf{s}|^2 - |\mathbf{s}|^p| &= |\mathbf{s}|^2 \left| \int_0^1 \frac{d}{d\theta} (\theta \epsilon^2 + |\mathbf{s}|^2)^{\frac{p-2}{2}} d\theta \right| = |\mathbf{s}|^2 \epsilon^2 \frac{|p-2|}{2} \int_0^1 (\theta \epsilon^2 + |\mathbf{s}|^2)^{\frac{p-4}{2}} d\theta \\ &\leq \frac{|p-2|}{2} \begin{cases} \epsilon^2 (1 + |\mathbf{s}|^2)^{\frac{p-2}{2}} & \text{if } p \geq 2, \\ \frac{2\epsilon^{p^-}}{p^-} & \text{if } 1 < p < 2. \end{cases} \end{aligned}$$

By Proposition 4.2.1, $\nabla u_{\epsilon_k} \rightarrow \nabla u$ a.e. in Q_T .

Let us fix $i, j \in \{1, 2, \dots, N\}$. By Theorem 4.2.5

$$\|D_j \left(\gamma_{\epsilon_k}^{\frac{1}{2}}(z, \nabla u_{\epsilon_k}) D_i u_{\epsilon_k} \right)\|_{2, Q_T} \leq C$$

uniformly in ϵ_k , therefore there exists $\eta_{ij} \in L^2(Q_T)$ such that $D_j \left(\gamma_{\epsilon_k}^{\frac{1}{2}}(z, \nabla u_{\epsilon_k}) D_i u_{\epsilon_k} \right) \rightharpoonup \eta_{ij}$ in $L^2(Q_T)$. The pointwise convergence $\nabla u_{\epsilon_k} \rightarrow \nabla u$ yields the pointwise convergence $\gamma_{\epsilon_k}^{\frac{1}{2}}(z, \nabla u_{\epsilon_k}) \nabla u_{\epsilon_k} \rightarrow |\nabla u|^{\frac{p(z)-2}{2}} \nabla u$, by virtue of (4.2.39) $\|\gamma_{\epsilon_k}^{\frac{1}{2}}(z, \nabla u_{\epsilon_k}) \nabla u_{\epsilon_k}\|_{2+\delta, Q_T}$ are uniformly bounded for some $\delta > 0$. It follows from the Vitali convergence theorem that $\gamma_{\epsilon_k}^{\frac{1}{2}}(z, \nabla u_{\epsilon_k}) \nabla u_{\epsilon_k} \rightarrow |\nabla u|^{\frac{p(z)-2}{2}} \nabla u$ in $L^2(Q_T)$. It follows that $\eta_{ij} = D_j \left(|\nabla u|^{\frac{p(z)-2}{2}} D_i u \right)$: for every $\phi \in C^\infty(\overline{Q_T})$, $\text{supp } \phi \Subset Q_T$,

$$\begin{aligned} -(\eta_{ij}, \phi)_{2, Q_T} &= -\lim_{k \rightarrow \infty} \left(D_j \left(\gamma_{\epsilon_k}^{\frac{1}{2}}(z, \nabla u_{\epsilon_k}) D_i u_{\epsilon_k} \right), \phi \right)_{2, Q_T} \\ &= \lim_{k \rightarrow \infty} \left(\gamma_{\epsilon_k}^{\frac{1}{2}}(z, \nabla u_{\epsilon_k}) D_i u_{\epsilon_k}, D_j \phi \right)_{2, Q_T} = \left(|\nabla u|^{\frac{p(z)-2}{2}} D_i u, D_j \phi \right)_{2, Q_T}. \end{aligned}$$

Let $N \geq 2$ and $p^- > \frac{2N}{N+2}$. Assume that $p^- < 2$ and, thus, $Q_T^- = Q_T \cap \{z : p(z) < 2\} \neq \emptyset$. Arguing as in the proof of Theorem 4.2.5 we find that for every $D \Subset Q_T^-$

$$\begin{aligned} \int_D |D_{ij}^2 u_\epsilon|^{p(z)} dz &= \int_D \left(\gamma_\epsilon(z, \nabla u_\epsilon) |D_{ij}^2 u_\epsilon|^2 \right)^{\frac{p(z)}{2}} \gamma_\epsilon^{-\frac{p(z)}{2}}(z, \nabla u_\epsilon^{(m)}) dz \\ &\leq \int_D \gamma_\epsilon(z, \nabla u_\epsilon) |D_{ij}^2 u_\epsilon|^2 dz + \int_D (\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{p(z)}{2}} dz \leq C \end{aligned}$$

with a constant C independent of ϵ and D . It follows that $D_{ij}^2 u_{\epsilon_k} \rightharpoonup \zeta \in L^{p(\cdot)}(D)$ (up to a subsequence). Because of the weak convergence $\nabla u_{\epsilon_k} \rightharpoonup \nabla u$ in $L^{p(\cdot)}(Q_T)$, it is necessary that $\zeta = D_{ij}^2 u$. The estimate $\|D_{ij}^2 u\|_{p(\cdot), D} \leq C$ follows from the uniform estimate on $D_{ij}^2 u_\epsilon$.

4.3 Double phase parabolic problem with variable growth

In this section, we study the following parabolic problem with the homogeneous Dirichlet boundary conditions:

$$\begin{cases} u_t - \operatorname{div} \left(|\nabla u|^{p(z)-2} \nabla u + a(z) |\nabla u|^{q(z)-2} \nabla u \right) = F(z, u) & \text{in } Q_T, \\ u = 0 & \text{on } \Gamma_T, \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases} \quad (4.3.1)$$

where $z = (x, t)$ denotes the point in the cylinder $Q_T = \Omega \times (0, T]$ and $\Gamma_T = \partial\Omega \times (0, T)$ is the lateral boundary of the cylinder, $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, $N \geq 2$ and $0 < T < \infty$. The nonlinear source has the form

$$F(z, v) = f_0(z) + b(z) |v|^{\sigma(z)-2} v. \quad (4.3.2)$$

Here $a \geq 0$, b , p , q , σ and f_0 are given functions of the variables $z \in Q_T$.

4.3.1 Assumptions and main results

Let $p, q : Q_T \mapsto \mathbb{R}$ be measurable functions satisfying the conditions

$$\begin{aligned} \frac{2N}{N+2} < p_- \leq p(z) \leq p_+ & \text{ in } \overline{Q_T}, \\ \frac{2N}{N+2} < q_- \leq q(z) \leq q_+ & \text{ in } \overline{Q_T}, \quad p^\pm, q^\pm = \text{const.} \end{aligned} \quad (4.3.3)$$

Moreover, let us assume that $p, q \in W^{1,\infty}(Q_T)$ as functions of variables $z = (x, t)$: there exist positive constants C^* , C^{**} , C_* , C_{**} such that

$$\begin{aligned} \operatorname{ess\,sup}_{Q_T} |\nabla p| \leq C_* < \infty, \quad \operatorname{ess\,sup}_{Q_T} |p_t| \leq C^*, \\ \operatorname{ess\,sup}_{Q_T} |\nabla q| \leq C_{**} < \infty, \quad \operatorname{ess\,sup}_{Q_T} |q_t| \leq C^{**}. \end{aligned} \quad (4.3.4)$$

The modulating coefficient $a(\cdot)$ is assumed to satisfy the following conditions:

$$a(z) \geq 0 \text{ in } \overline{Q_T}, \quad a \in C([0, T]; W^{1, \infty}(\Omega)), \quad \operatorname{ess\,sup}_{Q_T} |a_t| \leq C_a, \quad C_a = \text{const.} \quad (4.3.5)$$

We do not impose any condition on the null set of the function a in $\overline{Q_T}$ and do not distinguish between the cases of degenerate and singular equations. It is possible that $p(z) < 2$ and $q(z) > 2$ at the same point $z \in Q_T$.

Definition 4.3.1. A function $u : Q_T \mapsto \mathbb{R}$ is called **strong solution** of problem (4.3.1) if

- (i) $u \in \mathcal{W}_{q(\cdot)}(Q_T)$, $u_t \in L^2(Q_T)$, $|\nabla u| \in L^\infty(0, T; L^{s(\cdot)}(\Omega))$ with $s(z) = \max\{2, p(z)\}$,
- (ii) for every $\psi \in \mathcal{W}_{q(\cdot)}(Q_T)$ with $\psi_t \in L^2(Q_T)$

$$\int_{Q_T} u_t \psi \, dz + \int_{Q_T} (|\nabla u|^{p(z)-2} + a(z)|\nabla u|^{q(z)-2}) \nabla u \cdot \nabla \psi \, dz = \int_{Q_T} F(z, u) \psi \, dz, \quad (4.3.6)$$

- (iii) for every $\phi \in C_0^1(\Omega)$

$$\int_{\Omega} (u(x, t) - u_0(x)) \phi \, dx \rightarrow 0 \quad \text{as } t \rightarrow 0.$$

The main results are given in the following theorems.

Theorem 4.3.1. Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with the boundary $\partial\Omega \in C^2$. Assume that $p(\cdot)$, $q(\cdot)$ satisfy conditions (4.3.3), (4.3.4), and there exists a constant

$$r \in (0, r^*), \quad r^* = \frac{4p^-}{p^-(N+2) + 2N},$$

such that

$$p(z) \leq q(z) \leq p(z) + \frac{r}{2} \text{ in } \overline{Q_T}. \quad (4.3.7)$$

If $a(\cdot)$ satisfies conditions (4.3.5) and $b \equiv 0$, then for every $f_0 \in L^2(0, T; W_0^{1,2}(\Omega))$ and $u_0 \in W_0^{1,2}(\Omega)$ with

$$\int_{\Omega} (|\nabla u_0|^2 + |\nabla u_0|^{p(x,0)} + a(x,0)|\nabla u_0|^{q(x,0)}) \, dx = K < \infty \quad (4.3.8)$$

problem (4.3.1) has a unique strong solution u . This solution satisfies the estimate

$$\|u_t\|_{2, Q_T}^2 + \operatorname{ess\,sup}_{(0, T)} \int_{\Omega} (|\nabla u|^{s(z)} + a(z)|\nabla u|^{q(z)}) \, dx + \int_{Q_T} |\nabla u|^{p(z)+r} \, dz \leq C \quad (4.3.9)$$

with the exponent $s(z) = \max\{2, p(z)\}$ and a constant C which depends on $N, \partial\Omega, T, p^\pm, q^\pm, r$, the constants in conditions (4.3.4), (4.3.5), $\|f_0\|_{L^2(0, T; W_0^{1,2}(\Omega))}$ and K .

Theorem 4.3.2. *Let in the conditions of Theorem 4.3.1, $b \neq 0$.*

(i) *Assume that b, σ are measurable bounded functions defined on Q_T ,*

$$\begin{aligned} \|\nabla b\|_{\infty, Q_T} < \infty, \quad \|\nabla \sigma\|_{\infty, Q_T} < \infty, \\ 2 \leq \sigma^- \leq \sigma^+ < 1 + \frac{p^-}{2}, \quad \sigma^- = \operatorname{ess\,inf}_{Q_T} \sigma(z), \quad \sigma^+ = \operatorname{ess\,sup}_{Q_T} \sigma(z). \end{aligned}$$

Then for every $f_0 \in L^2(0, T; W_0^{1,2}(\Omega))$ and $u_0 \in W_0^{1,2}(\Omega)$ satisfying condition (4.3.8) problem (4.3.1) has at least one strong solution u . The solution u satisfies estimate (4.3.9) with the constant depending on the same quantities as in the case $b \equiv 0$ and on $\|\nabla b\|_{\infty, Q_T}, \|\nabla \sigma\|_{\infty, Q_T}, \sigma^\pm, \operatorname{ess\,sup}_{Q_T} |b|$.

(ii) *The strong solution is unique if $p(\cdot), q(\cdot)$ satisfy the conditions of Theorem 4.3.1 and either $\sigma \equiv 2$, or $b(z) \leq 0$ in Q_T .*

4.3.2 Auxiliary propositions

Until the end of this section, the notation $p(\cdot), q(\cdot), a(\cdot)$ is used for functions not related to the exponents and coefficient in (4.3.1) and (4.3.16).

Lemma 4.3.1 (Lemma 1.32, [34]). *Let $\partial\Omega \in \operatorname{Lip}$ and $p \in C^0(\overline{Q_T})$. Assume that $u \in L^\infty(0, T; L^2(\Omega)) \cap W_0^{1,p(\cdot)}(Q_T)$ and*

$$\operatorname{ess\,sup}_{(0,T)} \|u(\cdot, t)\|_{2,\Omega}^2 + \int_{Q_T} |\nabla u|^{p(z)} dz = M < \infty.$$

Then

$$\|u\|_{p(\cdot), Q_T} \leq C, \quad C = C(M, p^\pm, N, \omega),$$

where ω is the modulus of continuity of the exponent $p(\cdot)$.

The proof in [34] is given for the case $\Omega = B_R(x_0)$. To adapt it to the general case, it is sufficient to consider the zero continuation of u to a circular cylinder containing Q_T .

Let us accept the notation

$$\begin{aligned} \beta_\epsilon(\mathbf{s}) &= \epsilon^2 + |\mathbf{s}|^2, \\ \varphi_\epsilon(z, \mathbf{s}) &= (\epsilon^2 + |\mathbf{s}|^2)^{\frac{p(z)-2}{2}} + a(z)(\epsilon^2 + |\mathbf{s}|^2)^{\frac{q(z)-2}{2}}, \quad \mathbf{s} \in \mathbb{R}^N, \quad z \in Q_T, \quad \epsilon \in (0, 1). \end{aligned} \tag{4.3.10}$$

With certain abuse of notation, we will denote by $\varphi_\epsilon(x, \mathbf{s})$ the same function but with the exponents p, q and the coefficient a depending on the variable $x \in \Omega$.

Lemma 4.3.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$ be a bounded domain with the boundary $\partial\Omega \in C^2$, and $a \in W^{1,\infty}(\Omega)$ be a given nonnegative function. Assume that $v \in W^{3,2}(\Omega) \cap W_0^{1,2}(\Omega)$ and denote*

$$K = \int_{\partial\Omega} a(x)(\epsilon^2 + |\nabla v|^2)^{\frac{p(x)-2}{2}} (\Delta v (\nabla v \cdot \mathbf{n}) - \nabla(\nabla v \cdot \mathbf{n}) \cdot \nabla v) dS, \quad (4.3.11)$$

where \mathbf{n} stands for the exterior normal to $\partial\Omega$. There exists a constant $L = L(\partial\Omega)$ such that

$$K \leq L \int_{\partial\Omega} a(x)(\epsilon^2 + |\nabla v|^2)^{\frac{p(x)-2}{2}} |\nabla v|^2 dS.$$

Lemma 4.3.2 follows from the well-known assertions, see, e.g., [180, Ch.1, Sec.1.5] for the case $a \equiv 1$, $N \geq 2$, or [28, Lemma A.1] for the case of an arbitrary dimension. Fix an arbitrary point $\xi \in \partial\Omega$ and introduce the local coordinate system $\{y\}$ with the origin ξ . The system is chosen so that y_N coincides with the direction \mathbf{n} . There is a neighborhood of ξ where $\partial\Omega$ is represented in the form $y_N = \omega(y_1, \dots, y_{N-1})$ with a twice differentiable function ω . In the local coordinates

$$I_{\partial\Omega} \equiv \Delta v (\nabla v \cdot \mathbf{n}) - \nabla(\nabla v \cdot \mathbf{n}) \cdot \nabla v = \sum_{i=1}^{N-1} \left(D_{y_i y_i}^2 w D_{y_N} w - D_{y_i y_N}^2 w D_{y_i} w \right),$$

where $w(y) = v(x)$, and

$$I_{\partial\Omega}(\xi) = - (D_{y_N} w(0))^2 \sum_{i=1}^{N-1} D_{y_i y_i}^2 \omega(0) = - (\nabla v(\xi) \cdot \mathbf{n})^2 \sum_{i=1}^{N-1} D_{y_i y_i}^2 \omega(0).$$

Since ω is two times differentiable, then $|I_{\partial\Omega}(\xi)| \leq C |\nabla v(\xi)|^2$ with a constant C depending only on N and $\sup |D_{y_i y_j}^2 \omega(y)|$. Estimate (4.3.11) follows because $\xi \in \partial\Omega$ is arbitrary.

Lemma 4.3.3. *Let $\partial\Omega$ be a Lipschitz-continuous surface and $a(\cdot)$ be a nonnegative function on $\bar{\Omega}$. Assume that $a, q \in W^{1,\infty}(\Omega)$, with*

$$\|\nabla q\|_{\infty, \Omega} \leq L < \infty, \quad \|\nabla a\|_{\infty, \Omega} \leq L_0 < \infty.$$

There exists a constant $\delta = \delta(\partial\Omega)$ such that for every $u \in W^{1,q(\cdot)}(\Omega)$

$$\begin{aligned} & \delta \int_{\partial\Omega} a(x)(\epsilon^2 + |u|^2)^{\frac{q(x)-2}{2}} |u|^2 dS \\ & \leq C \int_{\Omega} \left(a(x)|u|^{q(x)-1} |\nabla u| + a(z)|u|^{q(x)} |\ln |u|| + |u|^{q(x)} + 1 \right) dx \end{aligned} \quad (4.3.12)$$

with a constant $C = C(q^+, L, L_0, N, \Omega)$.

Proof. By [162, Lemma 1.5.1.9] there exists $\delta > 0$ and $\mu \in (C^\infty(\bar{\Omega}))^N$ such that $\mu \cdot \mathbf{n} \geq \delta$ a.e. on $\partial\Omega$. By the Green formula

$$\begin{aligned} \delta \int_{\partial\Omega} a(x)|u|^{q(x)} dS &\leq \int_{\partial\Omega} a(x)|u|^{q(x)}(\mu \cdot \mathbf{n}) dS = \int_{\Omega} \operatorname{div}(a(x)|u|^{q(x)}\mu) dx \\ &= \int_{\Omega} \left[a(x) \left(q(x)|u|^{q(x)-2}u(\nabla u \cdot \mu) + |u|^{q(x)} \ln |u|(\nabla q \cdot \mu) + |u|^{q(x)} \operatorname{div} \mu \right) + |u|^{q(x)}(\nabla a \cdot \mu) \right] dx \\ &\leq q^+ \max_{\Omega} |\mu| \int_{\Omega} a(x)|u|^{q(x)-1} |\nabla u| dx + \|\nabla q\|_{\infty, \Omega} \max_{\Omega} |\mu| \int_{\Omega} a(x)|u|^{q(x)} |\ln |u|| dx \\ &\quad + \max_{\Omega} |\operatorname{div} \mu| \int_{\Omega} a(x)|u|^{q(x)} dx + \max_{\Omega} |\mu| \|\nabla a\|_{L^\infty(\Omega)} \int_{\Omega} |u|^{q(x)} dx \\ &\leq C \int_{\Omega} \left(a(x)|u|^{q(x)-1} |\nabla u| + a(x)|u|^{q(x)} |\ln |u|| + |u|^{q(x)} \right) dx \end{aligned}$$

with $C = C(N, q^+, L, L_0, \Omega)$. This inequality implies (4.3.12) because

$$a(x)(\epsilon^2 + |u|^2)^{\frac{q(x)-2}{2}} |u|^2 \leq a(x)(\epsilon^2 + |u|^2)^{\frac{q(x)}{2}} \leq C + a(x)|u|^{q(x)}$$

with an independent of u constant C . \square

Corollary 4.3.1. *Under the conditions of Lemma 4.3.3, for every $\lambda \in (0, 1)$ and $\epsilon \in (0, 1)$*

$$\begin{aligned} \int_{\partial\Omega} a(x)(\epsilon^2 + |u|^2)^{\frac{q(x)-2}{2}} |u|^2 dS &\leq \lambda \int_{\Omega} a(x)(\epsilon^2 + |u|^2)^{\frac{q(x)-2}{2}} |\nabla u|^2 dx \\ &\quad + L_0 \int_{\Omega} |u|^{q(x)} dx + L \int_{\Omega} a(z)|u|^{q(x)} |\ln |u|| dx + K \end{aligned} \quad (4.3.13)$$

with independent of u constants K, L, L_0 .

Proof. We transform the first term on the right-hand side of (4.3.12) using the Cauchy inequality:

$$\begin{aligned} a|u|^{q-1} |\nabla u| &\leq (a(\epsilon^2 + |u|^2)^{\frac{q-2}{2}} |\nabla u|^2)^{\frac{1}{2}} (a(\epsilon^2 + |u|^2)^{\frac{q}{2}})^{\frac{1}{2}} \\ &\leq \lambda a(\epsilon^2 + |u|^2)^{\frac{q-2}{2}} |\nabla u|^2 + Ca(\epsilon^2 + |u|^2)^{\frac{q}{2}}. \end{aligned}$$

\square

Theorem 4.3.3. *Let $\partial\Omega \in C^2$, $u \in C^2(\bar{\Omega})$ and $u = 0$ on $\partial\Omega$. Assume that $a(\cdot)$ satisfies the conditions of Lemma 4.3.3, $p(\cdot)$ satisfies the conditions of Lemma 4.2.3, and*

$$q : \Omega \mapsto [q^-, q^+] \subset \left(\frac{2N}{N+2}, \infty \right), \quad q \in W^{1, \infty}(\Omega), \quad \operatorname{ess\,sup}_{\Omega} |\nabla q| = L.$$

If for a.e. $x \in \Omega$

$$q(x) < p(x) + r \quad \text{with} \quad \frac{2}{N+2} < r < \frac{4p^-}{p^-(N+2) + 2N},$$

then for every $\lambda \in (0, 1)$

$$\int_{\partial\Omega} \varphi_\epsilon(x, \nabla u) |\nabla u|^2 dS \leq \lambda \int_{\Omega} \varphi_\epsilon(x, \nabla u) |u_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u|^{p(x)} dx \right) \quad (4.3.14)$$

with a constant C depending on λ and the constants p^\pm, N, L, L_0 , but independent of u .

Proof. Applying (4.3.13) to $|\nabla u|$ we obtain

$$\begin{aligned} \int_{\partial\Omega} a(x)(\epsilon^2 + |\nabla u|^2)^{\frac{q(x)-2}{2}} |\nabla u|^2 dS &\leq \lambda \int_{\Omega} a(x)(\epsilon^2 + |\nabla u|^2)^{\frac{q(x)-2}{2}} |u_{xx}|^2 dx \\ &\quad + L_0 \int_{\Omega} |\nabla u|^{q(x)} dx + L \int_{\Omega} |\nabla u|^{q(x)} \ln |\nabla u| dx + K \end{aligned} \quad (4.3.15)$$

with independent of u constants L, K, L_0 . Choose $0 < r_1 < r_2 < r^*$ so small that $q(x) + r_1 < p(x) + r_2$ and

$$\begin{aligned} |\nabla u|^{q(x)} |\ln |\nabla u|| &\leq \begin{cases} |\nabla u|^{q(x)+r_1} (|\nabla u|^{-r_1} |\ln |\nabla u||) \leq C(r_1, q^+) |\nabla u|^{q(x)+r_1} & \text{if } |\nabla u| \geq 1, \\ |\nabla u|^{q^-} |\ln |\nabla u|| \leq C(q^-) & \text{if } |\nabla u| \in (0, 1). \end{cases} \\ &\leq C \left(1 + |\nabla u|^{q(x)+r_1} \right) \end{aligned}$$

with a constant C independent of u . Thus, there exists a constant C such that

$$|\nabla u|^{q(x)} |\ln |\nabla u|| \leq C(1 + |\nabla u|^{q(x)+r_1}) \leq C(1 + |\nabla u|^{p(x)+r_2}) \text{ in } \Omega.$$

Using this inequality and then applying Lemma 4.2.3 we continue (4.3.15) as follows:

$$\begin{aligned} &\int_{\partial\Omega} a(x)(\epsilon^2 + |\nabla u|^2)^{\frac{q(x)-2}{2}} |\nabla u|^2 dS \\ &\leq \lambda \int_{\Omega} a(x)(\epsilon^2 + |\nabla u|^2)^{\frac{q(x)-2}{2}} |u_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u|^{p(x)+r_2} dx \right) \\ &\leq \lambda \int_{\Omega} a(x)(\epsilon^2 + |\nabla u|^2)^{\frac{q(x)-2}{2}} |u_{xx}|^2 dx + \lambda \int_{\Omega} (\epsilon^2 + |\nabla u|^2)^{\frac{p(x)-2}{2}} |u_{xx}|^2 dx \\ &\quad + C \left(1 + \int_{\Omega} |\nabla u|^{p(x)} dx \right) \\ &= \lambda \int_{\Omega} \varphi_\epsilon(x, \nabla u) |u_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u|^{p(x)} dx \right). \end{aligned}$$

Adding to this inequality the inequality corresponding to $q = p$ and $a \equiv 1$, we arrive at (4.3.14). \square

4.3.3 Regularized problem

Given $\epsilon > 0$, let us consider the following family of regularized double phase parabolic equations:

$$\begin{cases} \partial_t u - \operatorname{div}(\varphi_\epsilon(z, \nabla u) \nabla u) = F(z, u) & \text{in } Q_T, \\ u = 0 & \text{on } \Gamma_T, \\ u(0, \cdot) = u_0 & \text{in } \Omega, \epsilon \in (0, 1), \end{cases} \quad (4.3.16)$$

where $F(z, u)$ is defined in (4.3.2) and $\varphi_\epsilon(z, \nabla u) \nabla u$ is the regularized flux function.

Let $\epsilon > 0$ be a fixed parameter. The sequence $\{u_\epsilon^{(m)}\}$ of finite-dimensional Galerkin's approximations for the solutions of the regularized problem (4.3.16) is sought in the form

$$u_\epsilon^{(m)}(x, t) = \sum_{j=1}^m u_j^{(m)}(t) \phi_j(x) \quad (4.3.17)$$

where $\phi_j \in W_0^{1,2}(\Omega)$ and $\lambda_j > 0$ are the eigenfunctions and the corresponding eigenvalues of the problem

$$(\nabla \phi_j, \nabla \psi)_{2,\Omega} = \lambda_j (\phi_j, \psi)_{2,\Omega} \quad \forall \psi \in W_0^{1,2}(\Omega). \quad (4.3.18)$$

The systems $\{\phi_j\}$ and $\{\lambda_j^{-\frac{1}{2}} \phi_j\}$ are the orthogonal bases of $L^2(\Omega)$ and $W_0^{1,2}(\Omega)$. The coefficients $u_j^{(m)}(t)$ are characterized as the solutions of the Cauchy problem for the system of m ordinary differential equations

$$\begin{cases} (u_j^{(m)})'(t) = - \int_{\Omega} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) \nabla u_\epsilon^{(m)} \cdot \nabla \phi_j \, dx + \int_{\Omega} F(z, u_\epsilon^{(m)}) \phi_j \, dx, \\ u_j^{(m)}(0) = (u_0^{(m)}, \phi_j)_{2,\Omega}, \quad j = 1, 2, \dots, m, \end{cases} \quad (4.3.19)$$

where φ_ϵ is defined in (4.3.10) and the functions $u_0^{(m)}$ are chosen in such a way that

$$\begin{aligned} u_0^{(m)} &= \sum_{j=1}^m (u_0, \phi_j)_{2,\Omega} \phi_j \in \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_m\}, \\ u_0^{(m)} &\rightharpoonup u_0 \quad \begin{cases} \text{in } W_0^{1,2}(\Omega) & \text{if } \max_{\overline{\Omega}} q(x, 0) \leq 2, \\ \text{in } W_0^{1,r(\cdot)}(\Omega) & \text{if } \max_{\overline{\Omega}} q(x, 0) > 2, \text{ where } r(x) = \max\{2, q(x, 0)\}. \end{cases} \end{aligned} \quad (4.3.20)$$

By the Carathéodory existence theorem, for every finite m system (4.3.19) has a solution $(u_1^{(m)}, u_2^{(m)}, \dots, u_m^{(m)})$ in the extended sense on an interval $(0, T_m)$, the functions $u_i^{(m)}(t)$ are absolutely continuous and differentiable a.e. in $(0, T_m)$. The a priori estimates (4.3.47), (4.2.33) in the case $b \equiv 0$, and (4.3.56), (4.3.57) in the case $b \not\equiv 0$, show that for every m the function $u_\epsilon^{(m)}(x, T_m)$ belongs to $\operatorname{span}\{\phi_1, \dots, \phi_m\}$ and satisfies the estimate

$$\begin{aligned} \|\nabla u_\epsilon^{(m)}(\cdot, T_m)\|_{2,\Omega}^2 &+ \int_{\Omega} \left(|\nabla u_\epsilon^{(m)}(x, T_m)|^{p(x, T_m)} + a(x, T_m) |\nabla u_\epsilon^{(m)}(x, T_m)|^{q(x, T_m)} \right) dx \\ &\leq C + \|f_0\|_{2, Q_T}^2 + \|\nabla u_0^{(m)}\|_{2,\Omega}^2 + \int_{\Omega} \left(|\nabla u_0^{(m)}|^{p(x, 0)} + a(x, 0) |\nabla u_0^{(m)}|^{q(x, 0)} \right) dx \end{aligned}$$

with a constant C independent of m and ϵ . Since $a(\cdot, 0)$ is uniformly bounded in Ω , the sequence $\{u_0^{(m)}\}$ according to (4.3.20) and $|\nabla u_0|$ satisfies inequality (4.3.8), this estimate allows one to continue each of $u_\epsilon^{(m)}$ to the maximal existence interval $(0, T)$.

4.3.4 The choice of the sequence $\{u_0^{(m)}\}$

In the case $\sup q(x, 0) \leq 2$ the embedding $W_0^{1,2}(\Omega) \subset W_0^{1,q(\cdot,0)}(\Omega)$ allows us to take $u_0^{(m)} = \sum_{i=1}^m u_i^{(m)}(0)\phi_i$. Let $\sup q(x, 0) > 2$. We approximate the initial function u_0 by the sequence of finite-dimensional approximations for the solution of the elliptic problem

$$\beta(x, u)u - \operatorname{div}(\alpha(x, \nabla u)\nabla u) = f - \operatorname{div} \Phi \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega \quad (4.3.21)$$

where

$$\begin{aligned} \beta(x, u)u &= |u|^{r(x)-2}u + a_0(x)|u|^{s(x)-2}u, \\ \alpha(x, \nabla u)\nabla u &= |\nabla u|^{r(x)-2}\nabla u + a_0(x)|\nabla u|^{s(x)-2}\nabla u, \quad a_0(x) = a(x, 0), \\ r(x) &= \max\{2, p(x, 0)\} \geq 2, \quad s(x) = \max\{2, q(x, 0)\}, \end{aligned}$$

and

$$f = \beta(x, u_0)u_0, \quad \Phi = \alpha(x, \nabla u_0)\nabla u_0. \quad (4.3.22)$$

It is assumed that the exponents $p(x, 0)$, $q(x, 0)$ and the coefficient $a_0(x)$ satisfy conditions (4.3.3), (4.3.4), (4.3.5), (4.2.19). Since $u_0 \in W_0^{1,2}(\Omega)$ and satisfies condition (4.3.8), then $u_0 \in L^{s(\cdot)}(\Omega)$ by virtue of Sobolev type embedding and condition (4.2.19) on the gap between $p(x, 0)$ and $q(x, 0)$.

A natural analytic framework for the study of problem (4.3.21) is provided by the Musielak-Orlicz spaces. We introduce these spaces following [119, Sec.1], see also [82, 83]. Let us define the function $\mathcal{H} : \Omega \times [0, \infty) \mapsto [0, \infty)$ as

$$\mathcal{H}(x, t) = t^{r(x)} + a_0(x)t^{s(x)}. \quad (4.3.23)$$

The function \mathcal{H} is a generalized N -function: for every $t \geq 0$ $\mathcal{H}(\cdot, t)$ is measurable in Ω , for a.e. $x \in \Omega$ the function $\mathcal{H}(x, \cdot)$ is even and convex, $\mathcal{H}(x, 0) = 0$, $\mathcal{H}(x, t) > 0$ for $t \neq 0$ and

$$\lim_{t \rightarrow 0} \frac{\mathcal{H}(x, t)}{t} = 0, \quad \lim_{t \rightarrow \infty} \frac{\mathcal{H}(x, t)}{t} = \infty.$$

The function \mathcal{H} satisfies condition (Δ_2) : there is a positive constant K such that

$$\mathcal{H}(x, 2t) \leq K\mathcal{H}(x, t) \quad \text{for } x \in \Omega, t \geq 0.$$

The set

$$L^{\mathcal{H}}(\Omega) = \left\{ u : \Omega \mapsto \mathbb{R} \mid u \text{ is measurable, } \rho_{\mathcal{H}}(u) = \int_{\Omega} \mathcal{H}(x, |u|) dx < \infty \right\}$$

equipped with the Luxemburg norm

$$\|u\|_{\mathcal{H}} = \inf \left\{ \lambda > 0 : \rho_{\mathcal{H}} \left(\frac{u}{\lambda} \right) \leq 1 \right\}$$

becomes a Banach space. The function

$$\mathcal{H}^*(x, s) = \sup_{t \geq 0} (st - \mathcal{H}(x, t)), \quad x \in \Omega, s \geq 0,$$

is called the complementary to \mathcal{H} function in the sense of Young. For \mathcal{H} defined by (4.3.23)

$$\mathcal{H}^*(x, t) = (r(x) - 1)t^{r(x)} + (s(x) - 1)a_0(x)t^{s(x)}, \quad x \in \Omega, t \geq 0.$$

The function \mathcal{H}^* also satisfies condition (Δ_2) , and \mathcal{H} is the complementary function to \mathcal{H}^* . The following properties hold:

Proposition 4.3.1 (Propositions 1.1, 1.3, [119]). *For every $u \in L^{\mathcal{H}}(\Omega)$, $v \in L^{\mathcal{H}^*}(\Omega)$*

- (i) $\int_{\Omega} \mathcal{H}(x, |u|) dx < 1 \Leftrightarrow \|u\|_{\mathcal{H}} < 1$,
- (ii) $u_n \rightarrow u$ in $L^{\mathcal{H}}(\Omega) \Leftrightarrow \int_{\Omega} \mathcal{H}(x, |u_n|) dx \rightarrow \int_{\Omega} \mathcal{H}(x, |u|) dx$ as $n \rightarrow \infty$,
- (iii) $|\langle v, u \rangle_{\mathcal{H}^*, \mathcal{H}}| = \left| \int_{\Omega} uv dx \right| \leq 2\|u\|_{\mathcal{H}}\|v\|_{\mathcal{H}^*}$,
- (iv) $\mathcal{H}_t(x, |t|)|_{t=u} := r(x)|u|^{r(x)-2}u + a_0(x)s(x)|u|^{s(x)-2}u \in L^{\mathcal{H}^*}(\Omega)$,
- (v) $st \leq \epsilon \mathcal{H}(x, t) + C(\epsilon) \mathcal{H}^*(x, s)$ for all $\epsilon > 0$, $x \in \Omega$, $s, t \geq 0$, and equality holds if $s = \mathcal{H}_t(x, t)$ and $\epsilon = 1$,
- (vi) $\mathcal{H}(x, t) \leq \mathcal{H}_t(x, t) \leq \mathcal{H}(x, 2t)$ for $x \in \Omega$ and $t \geq 0$.

By $\mathcal{V}(\Omega)$ we denote the Musielak-Sobolev space

$$\mathcal{V}(\Omega) = \left\{ u \in L^{\mathcal{H}}(\Omega) : |\nabla u| \in L^{\mathcal{H}}(\Omega) \right\}$$

with the norm

$$\|u\|_{\mathcal{V}} = \|u\|_{\mathcal{H}} + \|\nabla u\|_{\mathcal{H}}.$$

Let us define the space $\mathcal{V}_0(\Omega)$ as the closure of $C_0^{\infty}(\Omega)$ with respect to the norm $\|u\|_{\mathcal{V}}$. By [119, Propositions 1.7, 1.8] the space $\mathcal{V}_0(\Omega)$ is a separable and reflexive Banach space.

Definition 4.3.2. *A function $u \in \mathcal{V}_0(\Omega)$ is called weak solution of problem (4.3.21) if for every $\phi \in \mathcal{V}_0(\Omega)$*

$$\int_{\Omega} (\beta(x, u)u\phi + \alpha(x, \nabla u)\nabla u \cdot \nabla \phi) dx = \int_{\Omega} (f\phi + \Phi \cdot \nabla \phi) dx.$$

We want to construct a solution of problem (4.3.21) as the limit of the sequence of finite-dimensional approximations in the same basis we use to approximate the solution of the evolution problems. Let us define the operator

$$\langle \mathcal{A}(u), v \rangle = \int_{\Omega} \beta(x, u)uv dx + \int_{\Omega} \alpha(x, \nabla u)\nabla u \cdot \nabla v dx \quad \forall u, v \in \mathcal{V}(\Omega).$$

Solvability of problem (4.3.21) will follow from the following properties of the operator \mathcal{A} .

(a) $\mathcal{A} : \mathcal{V}_0 \mapsto \mathcal{V}^* \equiv \left\{ v \in L^{\mathcal{H}^*}(\Omega) \mid |\nabla v| \in L^{\mathcal{H}^*}(\Omega) \right\}$. By the properties of $\rho_{\mathcal{H}}(\cdot)$, for every $u, v \in \mathcal{V}_0(\Omega)$ we have $\beta(x, u)u \in L^{\mathcal{H}^*}(\Omega)$, $\alpha(x, \nabla u)\nabla u \in (L^{\mathcal{H}^*}(\Omega))^N$ and

$$\begin{aligned} |\langle \mathcal{A}(u), v \rangle_{\mathcal{V}^*, \mathcal{V}}| &= \left| \int_{\Omega} \beta(x, u)uv \, dx + \int_{\Omega} \alpha(x, \nabla u)\nabla u \cdot \nabla v \, dx \right| \\ &\leq 2 (\|\beta(x, u)u\|_{\mathcal{H}^*} \|v\|_{\mathcal{H}} + \|\alpha(x, \nabla u)\nabla u\|_{\mathcal{H}^*} \|\nabla v\|_{\mathcal{H}}) \\ &\leq C (\|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}} + \|\nabla u\|_{\mathcal{H}} \|\nabla v\|_{\mathcal{H}}) \leq C \|u\|_{\mathcal{V}(\Omega)} \|v\|_{\mathcal{V}(\Omega)}. \end{aligned}$$

(b) \mathcal{A} is strictly monotone: for every $\xi, \zeta \in \mathbb{R}^N$, $y, z \in \mathbb{R}$, and $x \in \Omega$

$$\begin{aligned} (\alpha(x, \xi)\xi - \alpha(x, \zeta)\zeta, \xi - \zeta) &\geq 2^{-r(x)} |\xi - \zeta|^{r(x)} + a(x)2^{-s(x)} |\xi - \zeta|^{s(x)}, \\ (\beta(x, y)y - \beta(x, z)z, y - z) &\geq 2^{-r(x)} |y - z|^{r(x)} + a(x)2^{-s(x)} |y - z|^{s(x)}. \end{aligned} \quad (4.3.24)$$

(c) \mathcal{A} is hemicontinuous: for all $u, v, w \in \mathcal{V}_0(\Omega)$ the function $\lambda \mapsto \langle \mathcal{A}(u + \lambda v), w \rangle_{\mathcal{V}^*, \mathcal{V}}$ is continuous.

We look for a solution of problem (4.3.21) as the limit of the sequence $u_m = \sum_{i=1}^m c_i \phi_i \in \mathcal{P}_m$, where ϕ_i are the eigenfunction for the $(-\Delta)$ operator normalized by the condition $\|\phi_i\|_{2, \Omega} = 1$, $\mathcal{P}_m = \text{span}\{\phi_1, \dots, \phi_m\}$. The set $\mathcal{P}_m \subset \mathcal{V}_0(\Omega)$ is isomorphic to the space \mathbb{R}^m equipped with the usual scalar product $(x, y) = \sum_{i=1}^m x_i y_i$ and the norm $|x|_m^2 = \sum_{i=1}^m x_i^2$. The constant vector $c = (c_1, \dots, c_m) \in \mathbb{R}^m$ is the solution of the system of nonlinear algebraic equations $\mathcal{G}_i(c) = 0$, where

$$\mathcal{G}_i(c) \equiv \langle \mathcal{A}(u_m), \phi_i \rangle_{\mathcal{V}^*, \mathcal{V}} - \langle f, \phi_i \rangle_{\mathcal{H}^*, \mathcal{H}} - \langle \Phi, \nabla \phi_i \rangle_{\mathcal{H}^*, \mathcal{H}}, \quad i = 1, 2, \dots, m. \quad (4.3.25)$$

Solvability of the system $\mathcal{G}_i(c) = 0$, $i = 1, 2, \dots, m$, follows from the Brouwer fixed point principle in the form [196, Ch.1, Lemma 4.3]. Relations (4.3.25) define the mapping $c \mapsto \mathcal{G}(c)$ from \mathbb{R}^m into itself. The mapping \mathcal{G} is continuous and $(\mathcal{G}(c), c) \geq 0$, provided $|c|_m = \rho$ with a sufficiently large $\rho > 0$. Multiplying each of equations (4.3.25) by c_i , summing up and using Young's inequality we obtain:

$$\begin{aligned} \langle \mathcal{A}(u_m), u_m \rangle_{\mathcal{V}^*, \mathcal{V}} &= \rho_{\mathcal{H}}(u_m) + \rho_{\mathcal{H}}(|\nabla u_m|), \\ \langle \Phi, \nabla u_m \rangle_{\mathcal{H}^*, \mathcal{H}} &= \int_{\Omega} \left(|\nabla u_0|^{r(x)-2} + a_0 |\nabla u_0|^{s(x)-2} \right) \nabla u_0 \cdot \nabla u_m \, dx \\ &\leq \int_{\Omega} \left(|\nabla u_0|^{r(x)-1} |\nabla u_m| + \left(a_0^{\frac{1}{s(x)}} |\nabla u_0| \right)^{s(x)-1} \left(a_0^{\frac{1}{s(x)}} |\nabla u_m| \right) \right) dx \quad (4.3.26) \\ &\leq \delta \rho_{\mathcal{H}}(|\nabla u_m|) + C_{\delta} \rho_{\mathcal{H}}(|\nabla u_0|), \\ \langle f, u_m \rangle_{\mathcal{H}^*, \mathcal{H}} &\leq \delta \rho_{\mathcal{H}}(u_m) + C_{\delta} \rho_{\mathcal{H}}(u_0) \end{aligned}$$

Gathering these relations with $\delta = \frac{1}{2}$ and using Young's inequality once again (recall that $r(x) \geq 2$) we have: for the sufficiently large $|c|_m = \rho \equiv \rho(f, \Phi)$

$$\begin{aligned}
 (\mathcal{G}(c), c) &\geq \frac{1}{2} (\rho_{\mathcal{H}}(u_m) + \rho_{\mathcal{H}}(|\nabla u_m|)) - C' (\rho_{\mathcal{H}}(|\nabla u_0|) + \rho_{\mathcal{H}}(|\nabla u_0|)) \\
 &\geq -C'' + \frac{1}{2} \|\nabla u_m\|_{2,\Omega}^2 - C' (\rho_{\mathcal{H}}(u_0) + \rho_{\mathcal{H}}(|\nabla u_0|)) \\
 &\geq -C'' + \frac{1}{2\widehat{C}} \|u_m\|_{2,\Omega}^2 - C' (\rho_{\mathcal{H}}(u_0) + \rho_{\mathcal{H}}(|\nabla u_0|)) \\
 &= \frac{|c|_m^2}{2\widehat{C}} - C'' - C' (\rho_{\mathcal{H}}(u_0) + \rho_{\mathcal{H}}(|\nabla u_0|)) \geq 0,
 \end{aligned} \tag{4.3.27}$$

where \widehat{C} is the constant from the Poincaré inequality (4.1.5) with $r = 2$. Thus, if f, Φ are defined by (4.3.22) and $f, |\Phi| \in L^{\mathcal{H}^*}(\Omega)$, then problem (4.3.25) has a unique solution $u_m \in \mathcal{P}_m$ in a ball $\|u_m\|_{2,\Omega}^2 = |c|_m^2 \leq \rho^2$.

Theorem 4.3.4. *Assume that the exponents $p(\cdot, 0)$, $q(\cdot, 0)$ and the coefficient $a_0(\cdot)$ satisfy conditions (4.3.3), (4.3.4), (4.3.5), (4.2.19). If $u_0 \in \mathcal{V}_0(\Omega)$, then problem (4.3.21) has a unique weak solution $u \in \mathcal{V}_0(\Omega)$.*

Proof. Let $\{u_m\} \subset \mathcal{V}_0(\Omega)$ be the sequence of the approximate solutions. By (4.3.25), (4.3.26)

$$\rho_{\mathcal{H}}(u_m) + \rho_{\mathcal{H}}(|\nabla u_m|) \leq C (\rho_{\mathcal{H}}(u_0) + \rho_{\mathcal{H}}(|\nabla u_0|)) \tag{4.3.28}$$

with an independent of m constant C . Since $\mathcal{V}_0(\Omega)$ is separable, $\mathcal{V}_0(\Omega) \subset W_0^{1,r(\cdot)}(\Omega)$, the embedding $W_0^{1,r(\cdot)}(\Omega) \subset L^{s(\cdot)}(\Omega)$ is compact and $L^{s(\cdot)}(\Omega) \subset L^{\mathcal{H}}(\Omega)$, there is a subsequence (we assume that it coincides with the whole sequence) and functions $\eta \in (L^{\mathcal{H}^*}(\Omega))^N$, $u \in L^{\mathcal{H}}(\Omega)$ such that

$$\begin{aligned}
 \text{(i)} \quad &u_m \rightarrow u \text{ in } L^{\mathcal{H}}(\Omega), \quad \nabla u_m \rightharpoonup \nabla u \text{ in } L^{\mathcal{H}}(\Omega), \\
 \text{(ii)} \quad &\beta(\cdot, u_m)u_m \rightarrow \beta(\cdot, u)u \text{ in } L^{\mathcal{H}^*}(\Omega), \\
 \text{(iii)} \quad &\alpha(\cdot, \nabla u_m)\nabla u_m \rightharpoonup \eta \text{ in } (L^{\mathcal{H}^*}(\Omega))^N.
 \end{aligned} \tag{4.3.29}$$

The claim (4.3.29) (ii) follows from the generalized Lebesgue dominated convergence theorem. Since $u_m \rightarrow u$ in $L^{\mathcal{H}}(\Omega)$ and $\beta(\cdot, \cdot)$ is a Carathéodory function, then

$$\mathcal{H}^*(x, |\beta(x, u_m)u_m - \beta(x, u)u|) \rightarrow 0 \text{ for a.e. } x \in \Omega \tag{4.3.30}$$

(up to a subsequence). Using the convexity, (Δ_2) property of the generalized N -functions \mathcal{H} and \mathcal{H}^* , and Proposition 4.3.1, we obtain

$$\begin{aligned}
 0 &\leq \mathcal{H}^*(x, |\beta(x, u_m)u_m - \beta(x, u)u|) \leq C (\mathcal{H}^*(x, |\beta(x, u_m)u_m|) + \mathcal{H}^*(x, |\beta(x, u)u|)) \\
 &\leq C \left(\mathcal{H}^*(x, |\mathcal{H}_t(x, |t|)|) \Big|_{t=u_m} + \mathcal{H}^*(x, |\mathcal{H}_t(x, |t|)|) \Big|_{t=u} \right) \\
 &= C (\mathcal{H}_t(x, |t|) \Big|_{t=u_m} u_m - \mathcal{H}(x, |u_m|) + \mathcal{H}_t(x, |t|) \Big|_{t=u} u - \mathcal{H}(x, |u|)) \\
 &\leq C(K - 1) (\mathcal{H}(x, |u_m|) + \mathcal{H}(x, |u|)).
 \end{aligned} \tag{4.3.31}$$

According to Proposition 4.3.1 (ii) and (4.3.29) (i) we have $\rho_{\mathcal{H}}(u_m) \rightarrow \rho_{\mathcal{H}}(u)$ and $\mathcal{H}(x, |u|) \in L^1(\Omega)$. The claim (4.3.29) follows now from (4.3.30) and (4.3.31).

For every $m \in \mathbb{N}$ and $\phi_k \in \mathcal{P}_k$ with $k \leq m$

$$\langle \mathcal{A}(u_m), \phi_k \rangle_{\mathcal{V}^*, \mathcal{V}} = \langle f, \phi_k \rangle_{\mathcal{H}^*, \mathcal{H}} + \langle \Phi, \nabla \phi_k \rangle_{\mathcal{H}^*, \mathcal{H}}.$$

Letting $m \rightarrow \infty$ we obtain

$$\langle \beta(x, u)u, \phi_k \rangle_{\mathcal{H}^*, \mathcal{H}} + \langle \eta, \nabla \phi_k \rangle_{\mathcal{H}^*, \mathcal{H}} = \langle f, \phi_k \rangle_{\mathcal{H}^*, \mathcal{H}} + \langle \Phi, \nabla \phi_k \rangle_{\mathcal{H}^*, \mathcal{H}}.$$

By the monotonicity of $\alpha(\cdot, \nabla u_m) \nabla u_m$, for every $\psi \in \mathcal{P}_m$

$$\langle \alpha(x, \nabla u_m) \nabla u_m, \nabla u_m \rangle_{\mathcal{H}^*, \mathcal{H}} \geq \langle \alpha(x, \nabla u_m) \nabla u_m, \nabla \psi \rangle_{\mathcal{H}^*, \mathcal{H}} + \langle \alpha(x, \nabla \psi) \nabla \psi, \nabla u_m - \nabla \psi \rangle_{\mathcal{H}^*, \mathcal{H}},$$

thence

$$\begin{aligned} \langle \alpha(x, \nabla u_m) \nabla u_m, \nabla \psi \rangle_{\mathcal{H}^*, \mathcal{H}} + \langle \alpha(x, \nabla \psi) \nabla \psi, \nabla u_m - \nabla \psi \rangle_{\mathcal{H}^*, \mathcal{H}} &\leq \langle \alpha(x, \nabla u_m) \nabla u_m, \nabla u_m \rangle_{\mathcal{H}^*, \mathcal{H}} \\ &= \langle f, u_m \rangle_{\mathcal{H}^*, \mathcal{H}} + \langle \Phi, \nabla u_m \rangle_{\mathcal{H}^*, \mathcal{H}} - \langle \beta(x, u_m) u_m, u_m \rangle_{\mathcal{H}^*, \mathcal{H}}. \end{aligned}$$

The right-hand side and both terms on the left-hand side of this inequality have limits as $m \rightarrow \infty$, whence

$$\langle \alpha(x, \nabla \psi) \nabla \psi - \eta, \nabla u - \nabla \psi \rangle_{\mathcal{H}^*, \mathcal{H}} \leq 0$$

for every $\psi \in \mathcal{P}_l$ with any finite l . It follows that the same is true for every $\psi \in \mathcal{V}_0(\Omega)$. To identify η we take $\psi = u + \lambda \zeta$ with $\lambda > 0$ and $\zeta \in \mathcal{V}_0(\Omega)$. Simplifying the resulting inequality, sending $\lambda \rightarrow 0^+$ and using hemicontinuity of $\alpha(x, \xi) \xi$ we find that

$$\langle \alpha(x, \nabla u) \nabla u - \eta, \nabla \zeta \rangle_{\mathcal{H}^*, \mathcal{H}} \leq 0,$$

which is impossible unless this relation is the equality. Uniqueness of the weak solution follows from (4.3.24). \square

The constructed solution u of problem (4.3.21) is unique and $\mathcal{P}_m \ni u_m \rightarrow u$ in $\mathcal{V}_0(\Omega)$. On the other hand, u_0 is another solution of the same problem, therefore $u = u_0$ a.e. in Ω . By (4.3.28) and due to the choice of the exponents r, s , for every $\epsilon \in (0, 1)$

$$\begin{aligned} \|\nabla u_m\|_{2, \Omega}^2 + \int_{\Omega} \left[(\epsilon^2 + |\nabla u_m|^2)^{\frac{p(x,0)-2}{2}} |\nabla u_m|^2 + a(x,0) (\epsilon^2 + |\nabla u_m|^2)^{\frac{q(x,0)-2}{2}} |\nabla u_m|^2 \right] dx \\ \leq C \left(1 + \|\nabla u_0\|_{2, \Omega}^2 + \mathcal{F}(u_0, 0) \right) \leq C' \end{aligned} \tag{4.3.32}$$

with an independent of m and ϵ constants C, C' .

4.3.5 A priori estimates

 (i) A priori estimates I: the case $b \equiv 0$

Lemma 4.3.4. *Let Ω be a bounded domain with the boundary $\partial\Omega \in \text{Lip}$, $p(\cdot), q(\cdot)$ satisfy (4.3.3), $a(\cdot)$ satisfies (4.3.5), $u_0 \in L^2(\Omega)$ and $f_0 \in L^2(Q_T)$. If $b \equiv 0$, then $u_\epsilon^{(m)}$ satisfies the estimates*

$$\sup_{t \in (0, T)} \|u_\epsilon^{(m)}(\cdot, t)\|_{2, \Omega}^2 + \int_{Q_T} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 dz \leq C_1 e^T (\|f_0\|_{2, Q_T}^2 + \|u_0\|_{2, \Omega}^2) \quad (4.3.33)$$

and

$$\int_{Q_T} \left(|\nabla u_\epsilon^{(m)}|^{p(z)} + a(z) |\nabla u_\epsilon^{(m)}|^{q(z)} \right) dz \leq C_2 \int_{Q_T} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 dz + C_3 \quad (4.3.34)$$

where the constants C_i are independent of ϵ and m .

Proof. By multiplying j^{th} equation of (4.3.19) by $u_j^{(m)}(t)$ and then by summing up the results for $j = 1, 2, \dots, m$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u_\epsilon^{(m)}(\cdot, t)\|_{2, \Omega}^2 &= \sum_{j=1}^m u_j^{(m)}(t) (u_j^{(m)})'(t) = - \sum_{j=1}^m u_j^{(m)}(t) \int_{\Omega} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) \nabla u_\epsilon^{(m)} \cdot \nabla \phi_j dx \\ &\quad + \sum_{j=1}^m \int_{\Omega} f_0(x, t) \phi_j(x) u_j^{(m)}(t) dx \\ &= - \int_{\Omega} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 dx + \int_{\Omega} f_0(x, t) u_\epsilon^{(m)} dx. \end{aligned} \quad (4.3.35)$$

Using the Cauchy inequality, we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_\epsilon^{(m)}(\cdot, t)\|_{2, \Omega}^2 + \int_{\Omega} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 dx \leq \frac{1}{2} \|f_0(\cdot, t)\|_{2, \Omega}^2 + \frac{1}{2} \|u_\epsilon^{(m)}(\cdot, t)\|_{2, \Omega}^2. \quad (4.3.36)$$

Now, rewriting the last inequality in the equivalent form

$$\frac{1}{2} \frac{d}{dt} \left(e^{-t} \|u_\epsilon^{(m)}(\cdot, t)\|_{L^2(\Omega)}^2 \right) + e^{-t} \int_{\Omega} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 dx \leq \frac{e^{-t}}{2} \|f_0(\cdot, t)\|_{2, \Omega}^2$$

and integrating with respect to t , we arrive at the inequality

$$\sup_{t \in (0, T)} \|u_\epsilon^{(m)}(\cdot, t)\|_{L^2(\Omega)}^2 + \int_{Q_T} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 dx dt \leq C e^T \left(\|f_0\|_{2, Q_T}^2 + \|u_0\|_{2, \Omega}^2 \right) \quad (4.3.37)$$

where the constant C is independent of ϵ and m . Since $a(\cdot)$ is a nonnegative bounded function, the second assertion follows from (4.3.37) and the inequality

$$\begin{aligned} a(z)|\nabla u_\epsilon^{(m)}|^{q(z)} &\leq a(z) \left(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2 \right)^{\frac{q(z)}{2}} \\ &\leq \begin{cases} 2a(z) \left(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2 \right)^{\frac{q(z)-2}{2}} |\nabla u_\epsilon^{(m)}|^2 & \text{if } |\nabla u_\epsilon^{(m)}| \geq \epsilon, \\ (2\epsilon^2)^{\frac{q(z)}{2}} a(z) \leq 2^{\frac{q^+}{2}} a(z) & \text{otherwise.} \end{cases} \end{aligned} \quad (4.3.38)$$

□

Lemma 4.3.5. *Let Ω be a bounded domain with C^2 boundary. Assume that $p(\cdot), q(\cdot)$ satisfies (4.3.3), (4.3.4), (4.3.7) and $a(\cdot)$ satisfy (4.3.5). If $u_0 \in W_0^{1,2}(\Omega)$, $f_0 \in L^2((0, T); W_0^{1,2}(\Omega))$ and $b \equiv 0$, then for a.e. $t \in (0, T)$ the following inequality holds:*

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u_\epsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + C_0 \int_{\Omega} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |(u_\epsilon^{(m)})_{xx}|^2 dx \\ \leq C_1 \left(1 + \int_{\Omega} |\nabla u_\epsilon^{(m)}|^{p(z)} dx + \|\nabla u_\epsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \|f_0(\cdot, t)\|_{W_0^{1,2}(\Omega)}^2 \right) \end{aligned} \quad (4.3.39)$$

with independent of m and ϵ constants $0 < C_0 < \min\{p^- - 1, 1\}$ and $C_1 > 0$.

Proof. Let us multiply each of equations in (4.3.19) by $\lambda_j u_j^{(m)}$ and sum up the results for $j = 1, 2, \dots, m$:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u_\epsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 &= \sum_{j=1}^m \lambda_j (u_j^{(m)})'(t) u_j^{(m)}(t) \\ &= \sum_{j=1}^m \lambda_j u_j^{(m)} \int_{\Omega} \operatorname{div}(\varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) \nabla u_\epsilon^{(m)}) \phi_j dx + \sum_{j=1}^m \lambda_j u_j^{(m)} \int_{\Omega} f_0(x, t) \phi_j dx \\ &= - \int_{\Omega} \operatorname{div}(\varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) \nabla u_\epsilon^{(m)}) \Delta u_\epsilon^{(m)} dx + \int_{\Omega} f_0(x, t) \Delta u_\epsilon^{(m)} dx. \end{aligned} \quad (4.3.40)$$

Since $\partial\Omega \in C^2$, then $u_\epsilon^{(m)}(\cdot, t) \in C^3(\Omega) \cap C^1(\bar{\Omega})$ as a combination of solutions of problem (4.3.18). Therefore the first term on the right-hand of (4.3.40) can be transformed by means

of the Green formula:

$$\begin{aligned}
 & - \int_{\Omega} \operatorname{div} \left(\varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) \nabla u_{\epsilon}^{(m)} \right) \Delta u_{\epsilon}^{(m)} dx \\
 &= - \int_{\Omega} \left(\sum_{k=1}^N (u_{\epsilon}^{(m)})_{x_k x_k} \right) \left(\sum_{i=1}^N \left(\varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) (u_{\epsilon}^{(m)})_{x_i} \right)_{x_i} \right) dx \\
 &= - \int_{\partial\Omega} \Delta u_{\epsilon}^{(m)} \varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) (\nabla u_{\epsilon}^{(m)} \cdot \mathbf{n}) dS + \int_{\Omega} \sum_{k,i=1}^N (u_{\epsilon}^{(m)})_{x_k x_k x_i} \varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) (u_{\epsilon}^{(m)})_{x_i} dx \\
 &= - \int_{\partial\Omega} \varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) \sum_{k,i=1}^N \left((u_{\epsilon}^{(m)})_{x_k x_k} (u_{\epsilon}^{(m)})_{x_i} n_i - (u_{\epsilon}^{(m)})_{x_k x_i} (u_{\epsilon}^{(m)})_{x_i} n_k \right) dS \\
 &\quad - \int_{\Omega} \sum_{k,i=1}^N (u_{\epsilon}^{(m)})_{x_k x_i} \left(\varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) (u_{\epsilon}^{(m)})_{x_i} \right)_{x_k} dx \\
 &= - \int_{\Omega} \varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |(u_{\epsilon}^{(m)})_{xx}|^2 dx + J_1 + J_2 + J_{\partial\Omega} + J_a,
 \end{aligned}$$

where $\mathbf{n} = (n_1, \dots, n_N)$ is the outer normal vector to $\partial\Omega$,

$$\begin{aligned}
 J_1 &:= \int_{\Omega} (2 - p(z)) (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{2}-1} \left(\sum_{k=1}^N \left(\nabla u_{\epsilon}^{(m)} \cdot \nabla (u_{\epsilon}^{(m)})_{x_k} \right)^2 \right) dx \\
 &\quad + \int_{\Omega} (2 - q(z)) a(z) (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{q(z)-2}{2}-1} \left(\sum_{k=1}^N \left(\nabla u_{\epsilon}^{(m)} \cdot \nabla (u_{\epsilon}^{(m)})_{x_k} \right)^2 \right) dx, \\
 J_2 &= - \int_{\Omega} \sum_{k,i=1}^N (u_{\epsilon}^{(m)})_{x_k x_i} (u_{\epsilon}^{(m)})_{x_i} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{2}} \frac{p_{x_k}}{2} \ln(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2) dx \\
 &\quad - \int_{\Omega} \sum_{k,i=1}^N (u_{\epsilon}^{(m)})_{x_k x_i} (u_{\epsilon}^{(m)})_{x_i} a(z) (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{q(z)-2}{2}} \frac{q_{x_k}}{2} \ln(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2) dx, \\
 J_{\partial\Omega} &= - \int_{\partial\Omega} \varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) \left(\Delta u_{\epsilon}^{(m)} (\nabla u_{\epsilon}^{(m)} \cdot \mathbf{n}) - \nabla u_{\epsilon}^{(m)} \cdot \nabla (\nabla u_{\epsilon}^{(m)} \cdot \mathbf{n}) \right) dS, \\
 J_a &= - \int_{\Omega} \sum_{i,k=1}^N a_{x_k} (u_{\epsilon}^{(m)})_{x_i} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{q(z)-2}{2}} (u_{\epsilon}^{(m)})_{x_k x_i}.
 \end{aligned}$$

Substitution into (4.3.40) leads to the inequality

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \|\nabla u_{\epsilon}^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{\Omega} \varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |(u_{\epsilon}^{(m)})_{xx}|^2 dx \\
 &= J_1 + J_2 + J_{\partial\Omega} + J_a - \int_{\Omega} \nabla f_0 \cdot \nabla u_{\epsilon}^{(m)} dx \\
 &\leq J_1 + J_2 + J_{\partial\Omega} + J_a + \frac{1}{2} \|\nabla u_{\epsilon}^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \frac{1}{2} \|f_0(\cdot, t)\|_{W_0^{1,2}(\Omega)}^2.
 \end{aligned} \tag{4.3.41}$$

The terms on the right-hand side of (4.3.41) are estimated in three steps.

Step 1: estimate on J_1 . Since $a(z) \geq 0$ and $p(z) < q(z)$ in Q_T , the term J_1 is merged in the left-hand side. Indeed:

$$\begin{aligned} J_1 &= \int_{\{x \in \Omega: p(z) \geq 2\}} (2 - p(z)) \dots + \int_{\{x \in \Omega: p(z) < 2\}} (2 - p(z)) \dots \\ &\quad + \int_{\{x \in \Omega: q(z) \geq 2\}} (2 - q(z)) \dots + \int_{\{x \in \Omega: q(z) < 2\}} (2 - q(z)) \dots \\ &\leq \int_{\{x \in \Omega: p(z) < 2\}} (2 - p(z)) (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}-1} \left(\sum_{k=1}^N (\nabla u_\epsilon^{(m)} \cdot \nabla (u_\epsilon^{(m)})_{x_k})^2 \right) dx \\ &\quad + \int_{\{x \in \Omega: q(z) < 2\}} (2 - q(z)) a(z) (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)-2}{2}-1} \left(\sum_{k=1}^N (\nabla u_\epsilon^{(m)} \cdot \nabla (u_\epsilon^{(m)})_{x_k})^2 \right) dx, \end{aligned}$$

whence

$$\begin{aligned} |J_1| &\leq \max\{0, 2 - p^-\} \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2 dx \\ &\quad + \max\{0, 2 - q^-\} \int_{\Omega} a(z) (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)-2}{2}} |(u_\epsilon^{(m)})_{xx}|^2 dx. \end{aligned}$$

Step 2: estimate on J_2 . By the Cauchy inequality, for every $\delta_0 > 0$

$$\begin{aligned} |J_2| &\leq \frac{1}{2} \|\nabla p\|_{\infty, \Omega} \int_{\Omega} \left((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{4}} \sum_{k,i=1}^N |(u_\epsilon^{(m)})_{x_k x_i}| \right) \\ &\quad \times \left(|(u_\epsilon^{(m)})_{x_i}| \ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2) (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{4}} \right) dx \\ &\quad + \frac{1}{2} \|\nabla q\|_{\infty, \Omega} \int_{\Omega} \left((a(z))^{\frac{1}{2}} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)-2}{4}} \sum_{k,i=1}^N |(u_\epsilon^{(m)})_{x_k x_i}| \right) \\ &\quad \times \left((a(z))^{\frac{1}{2}} |(u_\epsilon^{(m)})_{x_i}| \ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2) (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)-2}{4}} \right) dx \\ &\leq \delta_0 \int_{\Omega} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) \sum_{k,i=1}^N |(u_\epsilon^{(m)})_{x_k x_i}|^2 dx \\ &\quad + C_1 \int_{\Omega} \ln^2(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2) \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 dx \end{aligned} \tag{4.3.42}$$

with a constant $C_1 = C_1(C^*, C^{**}, N, \delta_0)$. Let us denote

$$\mathcal{M} = C_1 \int_{\Omega} \ln^2(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2) \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 dx.$$

For $\mu_1 \in (0, 1)$ and $y > 0$ the following inequality holds:

$$y^{\frac{p}{2}} \ln^2 y \leq \begin{cases} y^{\frac{p+\mu_1}{2}} (y^{-\frac{\mu_1}{2}} \ln^2(y)) \leq C(\mu_1, p^+) (y^{\frac{p+\mu_1}{2}}) & \text{if } y \geq 1, \\ y^{\frac{p^-}{2}} \ln^2(y) \leq C(p^-) & \text{if } y \in (0, 1). \end{cases} \tag{4.3.43}$$

Let

$$r_* = \frac{2}{N+2} \text{ and } r^* = \frac{4p^-}{p^-(N+2) + 2N}. \quad (4.3.44)$$

Take the numbers r_1, r_2 such that

$$r_1 \in (r_*, r^*), \quad r_2 \in (0, 1), \quad q(z) + r_2 \leq p(z) + r_1 < p(z) + r^*$$

and estimate \mathcal{M} applying (4.3.43):

$$\mathcal{M} \leq C \left(1 + \int_{\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)+r_1-2}{2}} |\nabla u_{\epsilon}^{(m)}|^2 dx + \int_{\Omega} a(z) (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{q(z)+r_2-2}{2}} |\nabla u_{\epsilon}^{(m)}|^2 dx \right)$$

with a constant $C = C(C_1, r_1, r_2)$. Let us transform the integrand of the second integral using the following inequality:

$$\begin{aligned} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{q(z)+r_2-2}{2}} |\nabla u_{\epsilon}^{(m)}|^2 &\leq (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{q(z)+r_2}{2}} \leq 1 + (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)+r_1}{2}} \\ &\leq 1 + \begin{cases} (2\epsilon^2)^{\frac{p(z)+r_1}{2}} & \text{if } |\nabla u_{\epsilon}^{(m)}| < \epsilon, \\ 2(\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)+r_1-2}{2}} |\nabla u_{\epsilon}^{(m)}|^2 & \text{if } |\nabla u_{\epsilon}^{(m)}| \geq \epsilon. \end{cases} \end{aligned} \quad (4.3.45)$$

Using (4.3.45) and the interpolation inequality of Lemma 4.2.3 we finally obtain

$$\begin{aligned} \mathcal{M} &\leq C \left(1 + \int_{\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)+r_1-2}{2}} |\nabla u_{\epsilon}^{(m)}|^2 dx \right) \\ &\leq \delta_1 \int_{\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_{\epsilon}^{(m)})_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u_{\epsilon}^{(m)}|^{p(z)} dx \right) \end{aligned} \quad (4.3.46)$$

with any $\delta_1 \in (0, 1)$ and $C = C(\delta_1)$. Gathering (4.3.42) and (4.3.46), we finally obtain:

$$|J_2| \leq (\delta_0 + \delta_1) \int_{\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_{\epsilon}^{(m)})_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u_{\epsilon}^{(m)}|^{p(z)} dx \right)$$

with a constant C depending on δ_i and $\|a(\cdot, t)\|_{\infty, \Omega}$, but independent of ϵ and m .

Step 3: estimates on J_a and $J_{\partial\Omega}$. Let $\rho \in (r_*, r^*)$ be such that $2q(z) - p(z) < p(z) + \rho < p(z) + r^*$. Applying Young's inequality and (4.3.45) we obtain the estimate

$$\begin{aligned} |J_a| &\leq \int_{\Omega} \sum_{i,k=1}^N |a_{x_k}| |(u_{\epsilon}^{(m)})_{x_i}| (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{q(z)-2}{2}} |(u_{\epsilon}^{(m)})_{x_k x_i}| dx \\ &\leq \|\nabla a\|_{\infty, \Omega} \int_{\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{2q(z)-p(z)}{4}} \left((\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{4}} |(u_{\epsilon}^{(m)})_{xx}| \right) dx \\ &\leq \tilde{\delta} \int_{\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_{\epsilon}^{(m)})_{xx}|^2 dx + C(\tilde{\delta}) \int_{\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{2q(z)-p(z)}{2}} dx \\ &\leq \tilde{\delta} \int_{\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_{\epsilon}^{(m)})_{xx}|^2 dx + C' \left(1 + \int_{\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)+\rho}{2}} dx \right) \\ &\leq \tilde{\delta} \int_{\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_{\epsilon}^{(m)})_{xx}|^2 dx + C'' \left(1 + \int_{\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)+\rho-2}{2}} |\nabla u_{\epsilon}^{(m)}|^2 dx \right) \end{aligned}$$

where $C'' = C''(\|\nabla a\|_{\infty, \Omega}, N, q)$ is independent of ϵ and m . By Lemma 4.2.3 we obtain

$$|J_a| \leq \delta_2 \int_{\Omega} (\epsilon^2 + |\nabla u_{\epsilon}^{(m)}|^2)^{\frac{p(z)-2}{2}} |(u_{\epsilon}^{(m)})_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u_{\epsilon}^{(m)}|^{p(z)} dx \right)$$

for any $\delta_2 \in (0, 1)$ and a constant C independent of ϵ and m .

To estimate $J_{\partial\Omega}$ we use Lemma 4.3.2 and Theorem 4.3.3:

$$\begin{aligned} |J_{\partial\Omega}| &\leq \left| \int_{\partial\Omega} \varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) \left(\Delta u_{\epsilon}^{(m)} (\nabla u_{\epsilon}^{(m)} \cdot \mathbf{n}) - \nabla u_{\epsilon}^{(m)} \cdot \nabla (\nabla u_{\epsilon}^{(m)} \cdot \mathbf{n}) \right) dS \right| \\ &\leq C \int_{\partial\Omega} \varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |\nabla u_{\epsilon}^{(m)}|^2 dS \\ &\leq \delta_3 \int_{\Omega} \varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |(u_{\epsilon}^{(m)})_{xx}|^2 dx + C \left(1 + \int_{\Omega} |\nabla u_{\epsilon}^{(m)}|^{p(z)} dx \right) \end{aligned}$$

with an arbitrary $\delta_3 \in (0, 1)$ and C depending upon $\delta_3, p, q, a, \partial\Omega$ and their differential properties, but not on ϵ and m . To complete the proof and obtain (4.3.39), we gather the estimates of $J_1, J_2, J_a, J_{\partial\Omega}$ and choose δ_i so small that

$$\min\{1, p^- - 1\} - \sum_{i=0}^3 \delta_i = \eta > 0.$$

□

Lemma 4.3.6. *Under the conditions of Lemma 4.3.5*

$$\begin{aligned} \sup_{(0,T)} \|\nabla u_{\epsilon}^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{Q_T} \varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |(u_{\epsilon}^{(m)})_{xx}|^2 dz \\ \leq C e^{C'T} \left(1 + \|\nabla u_0\|_{2,\Omega}^2 + \|f_0\|_{L^2(0,T;W_0^{1,2}(\Omega))}^2 \right) \end{aligned} \quad (4.3.47)$$

and

$$\int_{Q_T} |\nabla u_{\epsilon}^{(m)}|^{q(z)} dz + \int_{Q_T} |\nabla u_{\epsilon}^{(m)}|^{p(z)+r} dz \leq C'' \quad \text{for any } 0 < r < \frac{4p^-}{p^-(N+2) + 2N} \quad (4.3.48)$$

with constants C, C', C'' independent of m and ϵ .

Proof. Multiplying (4.3.39) by $e^{-2C_1 t}$ and simplifying, we obtain the following differential inequality:

$$\frac{d}{dt} \left(e^{-2C_1 t} \|\nabla u_{\epsilon}^{(m)}(\cdot, t)\|_{2,\Omega}^2 \right) \leq C e^{-2C_1 t} \left(1 + \int_{\Omega} |\nabla u_{\epsilon}^{(m)}|^{p(z)} dx + \|f_0(\cdot, t)\|_{W_0^{1,2}(\Omega)}^2 \right).$$

Integrating it with respect to t and taking into account (4.3.33), (4.3.34) we arrive at the following estimate: for every $t \in [0, T]$

$$\begin{aligned} \|\nabla u_{\epsilon}^{(m)}(\cdot, t)\|_{2,\Omega}^2 &\leq C e^{2C_1 T} \left(\|\nabla u_0\|_{2,\Omega}^2 + e^T \left(1 + \|u_0\|_{2,\Omega}^2 + \|f_0\|_{2,Q_T}^2 \right) + \|\nabla f\|_{2,Q_T}^2 \right) \\ &\leq C e^{C'T} \left(1 + \|u_0\|_{W_0^{1,2}(\Omega)}^2 + \|f_0\|_{L^2(0,T;W_0^{1,2}(\Omega))}^2 \right). \end{aligned}$$

Substitution of the above estimate into (4.3.39) gives

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla u_\epsilon^{(m)}(\cdot, t)\|_{2, \Omega}^2 + C_0 \int_{\Omega} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |(u_\epsilon^{(m)})_{xx}|^2 dx \\ \leq C_1 \left(1 + \int_{\Omega} |\nabla u_\epsilon^{(m)}|^{p(z)} dx + \|\nabla u_0\|_{2, \Omega}^2 + \|f_0(\cdot, t)\|_{W_0^{1,2}(\Omega)}^2 \right). \end{aligned}$$

Integrating it with respect to t and using (4.3.34) to estimate the integral of $|\nabla u_\epsilon^{(m)}|^{p(z)}$ on the right-hand side, we obtain

$$\int_{Q_T} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |(u_\epsilon^{(m)})_{xx}|^2 dz \leq C e^{C'T} \left(1 + \|\nabla u_0\|_{2, \Omega}^2 + \|f_0\|_{L^2(0, T; W_0^{1,2}(\Omega))}^2 \right).$$

To prove estimate (4.3.48), we make use of Theorem 4.2.2. Let us fix a number $r \in (r_*, r^*)$ with r_*, r^* defined in (4.3.44). Split the cylinder Q_T into the two parts $Q_T^+ = Q_T \cap \{p(z) + r \geq 2\}$, $Q_T^- = Q_T \cap \{p(z) + r < 2\}$ and represent

$$\int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p(z)+r} dz = \int_{Q_T^+} |\nabla u_\epsilon^{(m)}|^{p(z)+r} dz + \int_{Q_T^-} |\nabla u_\epsilon^{(m)}|^{p(z)+r} dz \equiv I_+ + I_-.$$

Since

$$I_+ \leq \int_{Q_T^+} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)+r-2}{2}} |\nabla u_\epsilon^{(m)}|^2 dz \leq \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)+r-2}{2}} |\nabla u_\epsilon^{(m)}|^2 dz,$$

the estimate on I_+ follows immediately from Theorem 4.2.2 and (4.3.47). To estimate I_- , we set $B_+ = Q_T^- \cap \{z : |\nabla u_\epsilon^{(m)}| \geq \epsilon\}$, $B_- = Q_T^- \cap \{z : |\nabla u_\epsilon^{(m)}| < \epsilon\}$. The estimate on I_- follows from Theorem 4.2.2 and (4.3.47) because

$$\begin{aligned} I_- &= \int_{B_+ \cup B_-} |\nabla u_\epsilon^{(m)}|^{p(z)+r} dz = \int_{B_+} (|\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)+r-2}{2}} |\nabla u_\epsilon^{(m)}|^2 dz + \int_{B_-} \epsilon^{p(z)+r} dz \\ &\leq 2^{\frac{2-r-p^-}{2}} \int_{B_+} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)+r-2}{2}} |\nabla u_\epsilon^{(m)}|^2 dz + \epsilon^{p^-+r} T |\Omega| \\ &\leq C \left(1 + \int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)+r-2}{2}} |\nabla u_\epsilon^{(m)}|^2 dz \right). \end{aligned}$$

By combining the above estimates, using the Young inequality, and applying (4.3.47), (4.3.34) and Theorem 4.2.2 we obtain (4.3.48) with $r \in (r_*, r^*)$:

$$\begin{aligned} \int_{Q_T} |\nabla u_\epsilon^{(m)}|^{q(z)} dz + \int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p(z)+r} dz \leq 1 + \int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p(z)+r} dz \\ \leq C \left(1 + \int_{Q_T} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |(u_\epsilon^{(m)})_{xx}|^2 dz + \int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p(z)} dz \right) \leq C. \end{aligned}$$

If $r \in (0, r_*]$, the required inequality follows from Young's inequality. \square

Remark 4.3.1. Under the conditions of Lemma 4.3.6

$$\int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)+r}{2}} dz \leq C, \quad \epsilon \in (0, 1), \quad (4.3.49)$$

with an independent of ϵ and m constant C .

Corollary 4.3.2. Let condition (4.3.7) be fulfilled. Under the conditions of Lemma 4.3.6

$$\|(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)}\|_{q'(\cdot), Q_T} \leq C \quad (4.3.50)$$

with a constant C independent of m and ϵ .

Proof. Condition (4.3.7) entails the inequality

$$\frac{q(z)(p(z)-1)}{q(z)-1} \leq q(z) \leq p(z) + r.$$

By Young's inequality, the assertion follows then from (4.3.49):

$$\int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)(p(z)-1)}{2(q(z)-1)}} dz \leq C \left(1 + \int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p(z)+r} dz \right) \leq C.$$

□

Lemma 4.3.7. Assume that in the conditions of Lemma 4.3.5 $u_0 \in W_0^{1,2}(\Omega) \cap W_0^{1,q(\cdot,0)}(\Omega)$.

Then

$$\begin{aligned} \|(u_\epsilon^{(m)})_t\|_{2,Q_T}^2 + \sup_{(0,T)} \int_{\Omega} \left((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} + a(z)(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)}{2}} \right) dx \\ \leq C \left(1 + \int_{\Omega} (|\nabla u_0|^{p(x,0)} + a(x,0)|\nabla u_0|^{q(x,0)}) dx \right) + \|f_0\|_{2,Q_T}^2 \end{aligned} \quad (4.3.51)$$

with an independent of m and ϵ constant C , which depends on the constants in conditions (4.3.4).

Proof. By multiplying (4.3.19) with $(u_j^{(m)})_t$ and summing over $j = 1, 2, \dots, m$ we obtain the equality

$$\int_{\Omega} (u_\epsilon^{(m)})_t^2 dx + \int_{\Omega} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) \nabla u_\epsilon^{(m)} \cdot \nabla (u_\epsilon^{(m)})_t dx = \int_{\Omega} f_0 (u_\epsilon^{(m)})_t dx. \quad (4.3.52)$$

Using the identity

$$\begin{aligned} a(z)(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)-2}{2}} \nabla u_\epsilon^{(m)} \cdot \nabla (u_\epsilon^{(m)})_t &= \frac{d}{dt} \left(\frac{a(z)(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)}{2}}}{q(z)} \right) \\ &+ \frac{a(z)q_t(z)(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)}{2}}}{q^2(z)} \left(1 - \frac{q(z)}{2} \ln((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)) \right) - \frac{a_t(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)}{2}}}{q(z)} \end{aligned}$$

we rewrite (4.3.52) as

$$\begin{aligned}
 & \| (u_\epsilon^{(m)})_t(\cdot, t) \|_{2, \Omega}^2 + \frac{d}{dt} \int_{\Omega} \left((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} + a(z) (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)}{2}} \right) dx \\
 &= \int_{\Omega} f_0(u_\epsilon^{(m)})_t dx - \int_{\Omega} \frac{p_t (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}}}{p^2(z)} \left(1 - \frac{p(z)}{2} \ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2) \right) dx \\
 &\quad - \int_{\Omega} \frac{a(z) q_t(z) (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)}{2}}}{p^2(z)} \left(1 - \frac{q(z)}{2} \ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2) \right) dx \\
 &\quad + \int_{\Omega} \frac{a_t (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)}{2}}}{q(z)} dx \\
 &\equiv \int_{\Omega} f_0(u_\epsilon^{(m)})_t dx + \mathcal{J}_1 + \mathcal{J}_2 + \mathcal{J}_3.
 \end{aligned} \tag{4.3.53}$$

The first term on the right-hand side of (4.3.53) is estimated by the Cauchy inequality:

$$\left| \int_{\Omega} f_0(u_\epsilon^{(m)})_t dx \right| \leq \frac{1}{2} \| (u_\epsilon^{(m)})_t(\cdot, t) \|_{2, \Omega}^2 + \frac{1}{2} \| f_0(\cdot, t) \|_{2, \Omega}^2.$$

To estimate \mathcal{J}_i we use (4.3.33), (4.3.34), (4.3.43), (4.3.46) and (4.3.48). Fix two numbers $r_1 \in (r_*, r^*)$, $r_2 \in (0, 1)$ such that

$$q(z) + r_2 < p(z) + r_1 < p(z) + r^*.$$

Then

$$\begin{aligned}
 \sum_{i=1}^3 |\mathcal{J}_i| &\leq C_1 \left(1 + \int_{\Omega} |\nabla u_\epsilon^{(m)}|^{p(z)} dx + \int_{\Omega} |\nabla u_\epsilon^{(m)}|^{q(z)} dx \right) \\
 &\quad + C_2 \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} \ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2) dx \\
 &\quad + C_3 \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)}{2}} \ln(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2) dx \\
 &\leq C_4 \left(1 + \int_{\Omega} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)+r_1}{2}} dx \right).
 \end{aligned}$$

The required inequality (4.3.51) follows after gathering the above estimates, integrating the result in t and applying (4.3.32). \square

(ii) A priori estimates II: the case $b \neq 0$

We proceed to derive a priori estimates in the case when the equation contains the nonlinear source. The difference in the arguments consists in the necessity to estimate the integrals of the terms $b|u_\epsilon^{(m)}|^{\sigma(z)}$, $b|u_\epsilon^{(m)}|^{\sigma(z)-2}u_\epsilon^{(m)}\Delta u_\epsilon^{(m)}$, $b|u_\epsilon^{(m)}|^{\sigma(z)-2}u_\epsilon^{(m)}u_{\epsilon t}^{(m)}$.

1) Let us multiply j^{th} equation of (4.3.19) by $u_j^{(m)}$ and sum up. In the result we arrive at equality (4.3.35) with the right-hand side containing the additional term

$$\mathcal{I}_0 \equiv \int_{\Omega} b(z)|u_\epsilon^{(m)}|^{\sigma(z)} dx.$$

Let $2(\sigma^+ - 1) < p^-$. Using the inequalities of Young and Poincaré we find that for every $t \in (0, T)$

$$\begin{aligned} |\mathcal{I}_0| &\leq B \left(1 + \int_{\Omega} |u_{\epsilon}^{(m)}|^{2(\sigma^+-1)} dx + \int_{\Omega} |u_{\epsilon}^{(m)}|^2 dx \right) \\ &\leq C_{\delta} + \delta \int_{\Omega} |u_{\epsilon}^{(m)}|^{p^-} dx + \int_{\Omega} |u_{\epsilon}^{(m)}|^2 dx \\ &\leq C'_{\delta} + \widehat{C} \delta \int_{\Omega} |\nabla u_{\epsilon}^{(m)}|^{p^-} dx + C \int_{\Omega} |u_{\epsilon}^{(m)}|^2 dx \\ &\leq C''_{\delta} + \widehat{C} \delta \int_{\Omega} |\nabla u_{\epsilon}^{(m)}|^{p(z)} dx + C \int_{\Omega} |u_{\epsilon}^{(m)}|^2 dx \end{aligned}$$

where $\delta \in (0, 1)$ is an arbitrary constant and \widehat{C} is the constant from inequality (4.1.5) with $r = p^-$. We plug this estimate into (4.3.36) and use (4.3.38) with $a \equiv 1$ and q substituted by p . Choosing δ sufficiently small, we transform (4.3.36) to the form

$$\frac{1}{2} \frac{d}{dt} \|u_{\epsilon}^{(m)}(\cdot, t)\|_{2, \Omega}^2 + (1 - C\delta) \int_{\Omega} \varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |\nabla u_{\epsilon}^{(m)}|^2 dx \leq C' \left(1 + \|f_0(\cdot, t)\|_{2, \Omega}^2 + \|u_{\epsilon}^{(m)}(\cdot, t)\|_{2, \Omega}^2 \right).$$

Integrating this inequality in t we obtain the following counterpart of Lemma 4.3.4.

Lemma 4.3.8. *Assume that $a(\cdot)$, $p(\cdot)$, $q(\cdot)$, u_0 , f_0 satisfy the conditions of Lemma 4.3.4. If σ, b are measurable and bounded functions in Q_T and $1 < \sigma^- \leq \sigma^+ < 1 + \frac{p^-}{2}$, then*

$$\sup_{t \in (0, T)} \|u_{\epsilon}^{(m)}(\cdot, t)\|_{2, \Omega}^2 + \int_{Q_T} \varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |\nabla u_{\epsilon}^{(m)}|^2 dz \leq C_1 e^T (\|f_0\|_{2, Q_T}^2 + \|u_0\|_{2, \Omega}^2) + C_0 \quad (4.3.54)$$

and

$$\int_{Q_T} \left(|\nabla u_{\epsilon}^{(m)}|^{p(z)} + a(z) |\nabla u_{\epsilon}^{(m)}|^{q(z)} \right) dx dt \leq C_2 \int_{Q_T} \varphi_{\epsilon}(z, \nabla u_{\epsilon}^{(m)}) |\nabla u_{\epsilon}^{(m)}|^2 dz + C_3 \quad (4.3.55)$$

with independent of ϵ and m constants C_i .

2) Estimate on $\|\nabla u_{\epsilon}^{(m)}(t)\|_{2, \Omega}$. We follow the proof of Lemma 4.3.5: multiplying each of equations in (4.3.19) by $\lambda_j u_j^{(m)}$ and summing the results we arrive at equality (4.3.40) with the additional term in the right-hand side. The new term can be transformed by means of integration by parts in Ω :

$$\begin{aligned} \mathcal{I}_1 &= \int_{\Omega} b(z) |u_{\epsilon}^{(m)}|^{\sigma(z)-2} u_{\epsilon}^{(m)} \Delta u_{\epsilon}^{(m)} dx \\ &\leq \int_{\Omega} (\sigma(z) - 1) |b(z)| |u_{\epsilon}^{(m)}|^{\sigma(z)-2} |\nabla u_{\epsilon}^{(m)}|^2 dx \\ &\quad + \int_{\Omega} |u_{\epsilon}^{(m)}|^{\sigma(z)-1} |\nabla b| |\nabla u_{\epsilon}^{(m)}| dx + \int_{\Omega} |b(z)| |u_{\epsilon}^{(m)}|^{\sigma(z)-1} |\ln ||u_{\epsilon}^{(m)}|| | |\nabla u_{\epsilon}^{(m)}| |\nabla \sigma| dx \\ &\equiv \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3. \end{aligned}$$

To estimate \mathcal{K}_3 we assume that the functions $|b|$ and $|\nabla\sigma|$ are bounded a.e. in Q_T and then apply the Cauchy inequality, (4.3.43), and the Poincaré inequality: if $2(\sigma^+ - 1) < p^-$, there exists a constant $\mu > 0$ such that $2(\sigma^+ - 1) + \mu \leq p^-$

$$\begin{aligned} \mathcal{K}_3 &\leq C \left(1 + \|\nabla u_\epsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{\Omega} |u_\epsilon^{(m)}|^{2(\sigma(z)-1+\mu)} dx \right) \\ &\leq C \left(1 + \|\nabla u_\epsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{\Omega} |u_\epsilon^{(m)}|^{2(\sigma^+-1)+\mu} dx \right) \\ &\leq C' \left(1 + \|\nabla u_\epsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{\Omega} |\nabla u_\epsilon^{(m)}|^{2(\sigma^+-1)+\mu} dx \right) \\ &\leq C'' \left(1 + \|\nabla u_\epsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{\Omega} |\nabla u_\epsilon^{(m)}|^{p(z)} dx \right). \end{aligned}$$

\mathcal{K}_2 is estimated likewise: if $|\nabla b|$ is bounded a.e. in Q_T and $2(\sigma^+ - 1) < p^-$, then

$$\mathcal{K}_2 \leq C \left(1 + \|\nabla u_\epsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{\Omega} |u_\epsilon^{(m)}|^{2(\sigma(z)-1)} dx \right) \leq C' \left(1 + \|\nabla u_\epsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{\Omega} |\nabla u_\epsilon^{(m)}|^{p(z)} dx \right).$$

To estimate \mathcal{K}_1 we assume that $\sigma^- \geq 2$ and notice that the restriction on p^- and σ^+ imposed to estimate \mathcal{K}_2 and \mathcal{K}_3 yields

$$4 \leq 2\sigma^- \leq 2\sigma^+ < 2 + p^- \quad \Rightarrow \quad p^- > 2 \quad \Rightarrow \quad \sigma^+ < 1 + \frac{p^-}{2} < p^-.$$

Using this observation and the Young inequality we estimate \mathcal{K}_1 as follows:

$$\begin{aligned} \mathcal{K}_1 &\leq C \left(\int_{\Omega} |\nabla u_\epsilon^{(m)}|^{p(z)} dx + \int_{\Omega} |u_\epsilon^{(m)}|^{p(z)\frac{\sigma(z)-2}{p(z)-2}} dx \right) \\ &\leq C \left(1 + \int_{\Omega} |\nabla u_\epsilon^{(m)}|^{p(z)} dx + \int_{\Omega} |u_\epsilon^{(m)}|^{p(z)\frac{\sigma^+-2}{p^--2}} dx \right) \\ &\leq C' \left(1 + \int_{\Omega} |\nabla u_\epsilon^{(m)}|^{p(z)} dx + \int_{\Omega} |u_\epsilon^{(m)}|^{p(z)} dx \right). \end{aligned}$$

Following the proof of Lemma 4.3.6 and taking into account the estimates on \mathcal{K}_i we arrive at the inequality

$$\begin{aligned} \sup_{(0,T)} \|\nabla u_\epsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{Q_T} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |(u_\epsilon^{(m)})_{xx}|^2 dz \\ \leq C e^{C'T} \left(1 + \|\nabla u_0\|_{2,\Omega}^2 + \|f_0\|_{L^2(0,T;W_0^{1,2}(\Omega))}^2 \right) \\ + C'' e^{C'T} \left(\int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p(z)} dz + \int_{Q_T} |u_\epsilon^{(m)}|^{p(z)} dz \right) \end{aligned}$$

with new constants C, C', C'' which do not depend on ϵ and m . The last term on the right-hand side of this inequality is estimated by virtue of Lemma 4.3.1 and estimates (4.3.54), (4.3.55).

Lemma 4.3.9. *Let in the conditions of Lemma 4.3.8, $2 \leq \sigma^- \leq \sigma^+ < 1 + \frac{p^-}{2}$ holds. If $\|\nabla b\|_{\infty, Q_T} < \infty$ and $\|\nabla \sigma\|_{\infty, Q_T} < \infty$, then*

$$\begin{aligned} \sup_{(0,T)} \|\nabla u_\epsilon^{(m)}(\cdot, t)\|_{2,\Omega}^2 + \int_{Q_T} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |(u_\epsilon^{(m)})_{xx}|^2 dz \\ \leq C e^{C'T} \left(\widehat{C} + \|u_0\|_{W_0^{1,2}(\Omega)}^2 + \|f_0\|_{L^2(0,T;W_0^{1,2}(\Omega))}^2 \right) \end{aligned} \quad (4.3.56)$$

with an independent of ϵ and m constants C , C' , and a constant \widehat{C} depending only on T , and the quantities on the right-hand sides of (4.3.33), (4.3.34).

3) Estimate on $\|u_{\epsilon t}^{(m)}\|_{2,Q_T}$. We follow the proof of Lemma 4.3.7. Multiplying (4.3.19) by $(u_\epsilon^{(m)})_t$ and summing the results we obtain equality (4.3.53) with the additional term on the right-hand side:

$$\mathcal{M}_0 \equiv \int_{\Omega} b(z) |u_\epsilon^{(m)}|^{\sigma(z)-2} u_\epsilon^{(m)} (u_\epsilon^{(m)})_t dx.$$

By Young's inequality

$$\mathcal{M}_0 \leq C \int_{\Omega} |u_\epsilon^{(m)}|^{2(\sigma(z)-1)} dx + \frac{1}{2} \int_{\Omega} (u_\epsilon^{(m)})_t^2 dx.$$

Combining this inequality with (4.3.51) and taking into account the inequality $2(\sigma(z) - 1) < p(z)$ following from the inequality $2(\sigma^+ - 1) < p^-$, we obtain

$$\begin{aligned} \frac{1}{2} \|(u_\epsilon^{(m)})_t\|_{2,Q_T}^2 + \sup_{(0,T)} \int_{\Omega} \left((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} + a(z) (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)}{2}} \right) dx \\ \leq C \left(1 + \int_{\Omega} (|\nabla u_0|^{p(x,0)} + a(x,0) |\nabla u_0|^{q(x,0)}) dx \right) + \|f_0\|_{2,Q_T}^2 \\ + C' \left(1 + \int_{Q_T} |u_\epsilon|^{p(z)} dz \right). \end{aligned}$$

The last integral on the right-hand side is estimated by virtue of Lemma 4.3.1 and the estimates of Lemma 4.3.8.

Lemma 4.3.10. *Let the conditions of Lemma 4.3.9 be fulfilled. Then*

$$\begin{aligned} \frac{1}{2} \|(u_\epsilon^{(m)})_t\|_{2,Q_T}^2 + \sup_{(0,T)} \int_{\Omega} \left((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} + a(z) (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)}{2}} \right) dx \\ \leq C \left(1 + \int_{\Omega} (|\nabla u_0|^{p(x,0)} + a(x,0) |\nabla u_0|^{q(x,0)}) dx \right) + \|f_0\|_{2,Q_T}^2 + C' \end{aligned} \quad (4.3.57)$$

with constants C , C' independent of ϵ and m .

4.3.6 Existence and uniqueness of strong solution

In this section, we prove that the regularized problem (4.3.16) and the degenerate problem (4.3.1) have strong solutions and derive conditions of uniqueness of these solutions.

4.3.6.1 Regularized problem

Theorem 4.3.5. *Let u_0, f, p, q, a and $\partial\Omega$ satisfy the conditions of Theorem 4.3.1. Then for every $\epsilon \in (0, 1)$ problem (4.3.16) has a unique solution u_ϵ which satisfies the estimates*

$$\begin{aligned} \|u_\epsilon\|_{\mathcal{W}_{q(\cdot)}(Q_T)} &\leq C_0, \\ \operatorname{ess\,sup}_{(0,T)} \|u_\epsilon(\cdot, t)\|_{2,\Omega}^2 + \|u_{\epsilon t}\|_{2,Q_T}^2 + \operatorname{ess\,sup}_{(0,T)} \|\nabla u_\epsilon(\cdot, t)\|_{2,\Omega}^2 \\ &+ \operatorname{ess\,sup}_{(0,T)} \int_{\Omega} \left((\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)}{2}} + a(z)(\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)}{2}} \right) dx \leq C_0 \end{aligned} \quad (4.3.58)$$

with a constant C_0 depending on the data but not on ϵ . Moreover, u_ϵ possesses the property of global higher integrability of the gradient: for every

$$\delta \in (0, r^*), \quad r^* = \frac{4p^-}{p^-(N+2) + 2N},$$

there exists a constant $C = C(\partial\Omega, N, p^\pm, \delta, \|u_0\|_{W_0^{1,2}(\Omega)}, \|f\|_{L^2(0,T;W_0^{1,2}(\Omega))})$ such that

$$\int_{Q_T} |\nabla u_\epsilon|^{p(z)+\delta} dz \leq C. \quad (4.3.59)$$

Proof. Let $\epsilon \in (0, 1)$ be a fixed parameter. Under the assumptions of Theorem 4.3.1, there exists a sequence of Galerkin approximations $u_\epsilon^{(m)}$ defined by formulas (4.3.17) which satisfies estimates (4.3.33), (4.3.34), (4.3.47), (4.3.48), (4.3.50) and (4.3.51). These uniform in m and ϵ estimates enable one to extract a subsequence $u_\epsilon^{(m)}$ (for which we keep the same name), and functions $u_\epsilon, \eta_\epsilon, \chi_\epsilon$ such that

$$\begin{aligned} u_\epsilon^{(m)} &\rightharpoonup u_\epsilon \quad \star\text{-weakly in } L^\infty(0, T; L^2(\Omega)), \quad (u_\epsilon^{(m)})_t \rightharpoonup (u_\epsilon)_t \text{ in } L^2(Q_T), \\ \nabla u_\epsilon^{(m)} &\rightharpoonup \nabla u_\epsilon \text{ in } (L^{p(\cdot)}(Q_T))^N, \quad \nabla u_\epsilon^{(m)} \rightharpoonup \nabla u_\epsilon \text{ in } (L^{q(\cdot)}(Q_T))^N, \\ (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{p(z)-2}{2}} \nabla u_\epsilon^{(m)} &\rightharpoonup \eta_\epsilon \text{ in } (L^{q'(\cdot)}(Q_T))^N, \\ (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)-2}{2}} \nabla u_\epsilon^{(m)} &\rightharpoonup \chi_\epsilon \text{ in } (L^{q'(\cdot)}(Q_T))^N. \end{aligned} \quad (4.3.60)$$

In the third line we make use of the uniform estimate

$$\int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon^{(m)}|^2)^{\frac{q(z)(p(z)-1)}{2(q(z)-1)}} dz \leq C \left(1 + \int_{Q_T} |\nabla u_\epsilon^{(m)}|^{p(z)+r} dz \right) \leq C,$$

which follows from (4.3.7) and (4.3.48). The functions $u_\epsilon^{(m)}$ and $(u_\epsilon^{(m)})_t$ are uniformly bounded in $L^\infty(0, T; W_0^{1,p^-}(\Omega))$ and $L^2(0, T; L^2(\Omega))$ respectively, and $W_0^{1,q(\cdot,t)}(\Omega) \subseteq W_0^{1,q^-}(\Omega) \hookrightarrow L^2(\Omega)$. By [235, Sec.8, Corollary 4] the sequence $\{u_\epsilon^{(m)}\}$ is relatively compact in $C([0, T]; L^2(\Omega))$, i.e., there exists a subsequence $\{u_\epsilon^{(m_k)}\}$, which we assume coinciding with $\{u_\epsilon^{(m)}\}$, such that $u_\epsilon^{(m)} \rightarrow u_\epsilon$ in $C([0, T]; L^2(\Omega))$ and a.e. in Q_T . Let us define

$$\mathcal{P}_m = \left\{ \phi : \phi = \sum_{i=1}^m \psi_i(t) \phi_i(x), \psi_i \text{ are absolutely continuous in } [0, T] \right\}.$$

Fix some $m \in \mathbb{N}$. By the method of construction $u_\epsilon^{(m)} \in \mathcal{P}_m$. Since $\mathcal{P}_k \subset \mathcal{P}_m$ for $k < m$, then for every $\xi_k \in \mathcal{P}_k$ with $k \leq m$

$$\int_{Q_T} u_{\epsilon t}^{(m)} \xi_k dz + \int_{Q_T} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) \nabla u_\epsilon^{(m)} \cdot \nabla \xi_k dz = \int_{Q_T} f_0 \xi_k dz. \quad (4.3.61)$$

Let $\xi \in \mathcal{W}_{q(\cdot)}(Q_T)$. The space $C^\infty([0, T]; C_0^\infty(\Omega))$ is dense in $\mathcal{W}_{q(\cdot)}(Q_T)$, therefore there exists a sequence $\{\xi_k\}$ such that $\xi_k \in \mathcal{P}_k$ and $\xi_k \rightarrow \xi \in \mathcal{W}_{q(\cdot)}(Q_T)$. If $U_m \rightharpoonup U$ in $L^{q'(\cdot)}(Q_T)$, then for every $V \in L^{q(\cdot)}(Q_T)$ we have

$$a(z)V \in L^{q(\cdot)}(Q_T) \quad \text{and} \quad \int_{Q_T} aU_m V dz \rightarrow \int_{Q_T} aUV dz.$$

Using this fact we pass to the limit as $m \rightarrow \infty$ in (4.3.61) with a fixed k , and then letting $k \rightarrow \infty$, we conclude that

$$\int_{Q_T} u_{\epsilon t} \xi dz + \int_{Q_T} \eta_\epsilon \cdot \nabla \xi dz + \int_{Q_T} a(z) \chi_\epsilon \cdot \nabla \xi dz = \int_{Q_T} f_0 \xi dz \quad (4.3.62)$$

for all $\xi \in \mathcal{W}_{q(\cdot)}(Q_T)$. To identify the limit vectors η_ϵ and χ_ϵ we use the classical argument based on monotonicity. The flux function $\varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) \nabla u_\epsilon^{(m)}$ is monotone:

$$(\varphi_\epsilon(z, \xi)\xi - \varphi_\epsilon(z, \zeta)\zeta, \xi - \zeta) \geq 0 \quad \text{for all } \xi, \zeta \in \mathbb{R}^N, z \in Q_T, \epsilon > 0, \quad (4.3.63)$$

see, e.g., Lemma 4.2.10 for the proof. By virtue of (4.3.63), for every $\psi \in \mathcal{P}_m$

$$\begin{aligned} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 &= \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) \nabla u_\epsilon^{(m)} \cdot (\nabla u_\epsilon^{(m)} - \nabla \psi) + \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) \nabla u_\epsilon^{(m)} \cdot \nabla \psi \\ &= (\varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) \nabla u_\epsilon^{(m)} - \varphi_\epsilon(z, \nabla \psi) \nabla \psi) \cdot (\nabla u_\epsilon^{(m)} - \nabla \psi) \\ &\quad + \varphi_\epsilon(z, \nabla \psi) \nabla \psi \cdot (\nabla u_\epsilon^{(m)} - \nabla \psi) + \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) \nabla u_\epsilon^{(m)} \cdot \nabla \psi \\ &\geq \varphi_\epsilon(z, \nabla \psi) \nabla \psi \cdot (\nabla u_\epsilon^{(m)} - \nabla \psi) + \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) \nabla u_\epsilon^{(m)} \cdot \nabla \psi. \end{aligned} \quad (4.3.64)$$

By taking $\xi_k = u_\epsilon^{(m)}$ in (4.3.61) we obtain: for every $\psi \in \mathcal{P}_k$ with $k \leq m$

$$\begin{aligned} 0 &= \int_{Q_T} (u_\epsilon^{(m)})_t u_\epsilon^{(m)} dz + \int_{Q_T} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) |\nabla u_\epsilon^{(m)}|^2 dz - \int_{Q_T} f_0 u_\epsilon^{(m)} dz \\ &\geq \int_{Q_T} (u_\epsilon^{(m)})_t u_\epsilon^{(m)} dz + \int_{Q_T} \varphi_\epsilon(z, \nabla \psi) \nabla \psi \cdot \nabla (u_\epsilon^{(m)} - \psi) dz \\ &\quad + \int_{Q_T} \varphi_\epsilon(z, \nabla u_\epsilon^{(m)}) \nabla u_\epsilon^{(m)} \cdot \nabla \psi dz - \int_{Q_T} f_0 u_\epsilon^{(m)} dz. \end{aligned}$$

Notice that $(u_\epsilon^{(m)}, (u_\epsilon^{(m)})_t)_{2, Q_T} \rightarrow (u_{\epsilon t}, u_\epsilon)_{2, Q_T}$ as $m \rightarrow \infty$ as the product of weakly and strongly convergent sequences. This fact together with (4.3.60) means that each term of the

last inequality has a limit as $m \rightarrow \infty$. Letting $m \rightarrow \infty$ and using (4.3.62), we find that for every $\psi \in \mathcal{P}_k$

$$\begin{aligned} 0 &\geq \int_{Q_T} u_\epsilon u_{\epsilon t} dz + \int_{Q_T} \varphi_\epsilon(z, \nabla \psi) \nabla \psi \cdot \nabla (u_\epsilon - \psi) dz + \int_{Q_T} (\eta_\epsilon + a(z)\chi_\epsilon) \cdot \nabla \psi dz - \int_{Q_T} f_0 u_\epsilon dz \\ &= \int_{Q_T} \left((\epsilon^2 + |\nabla \psi|^2)^{\frac{p(z)-2}{2}} \nabla \psi - \eta_\epsilon \right) \cdot \nabla (u_\epsilon - \psi) dz \\ &\quad + \int_{Q_T} a(z) \left((\epsilon^2 + |\nabla \psi|^2)^{\frac{q(z)-2}{2}} \nabla \psi - \chi_\epsilon \right) \cdot \nabla (u_\epsilon - \psi) dz. \end{aligned}$$

By the density of $\bigcup_{k=1}^{\infty} \mathcal{P}_k$ in $\mathcal{W}_{q(\cdot)}(Q_T)$, the last inequality also holds for every $\psi \in \mathcal{W}_{q(\cdot)}(Q_T)$. Take $\psi = u_\epsilon + \lambda \xi$ with a constant $\lambda > 0$ and an arbitrary $\xi \in \mathcal{W}_{q(\cdot)}(Q_T)$. Then

$$\begin{aligned} \lambda \left[\int_{Q_T} \left((\epsilon^2 + |\nabla (u_\epsilon + \lambda \xi)|^2)^{\frac{p(z)-2}{2}} \nabla (u_\epsilon + \lambda \xi) - \eta_\epsilon \right) \cdot \nabla \xi dz \right. \\ \left. + \int_{Q_T} a(z) \left((\epsilon^2 + |\nabla (u_\epsilon + \lambda \xi)|^2)^{\frac{q(z)-2}{2}} \nabla (u_\epsilon + \lambda \xi) - \chi_\epsilon \right) \cdot \nabla \xi dz \right] \leq 0. \end{aligned}$$

Simplifying and letting $\lambda \rightarrow 0$ we find that

$$\int_{Q_T} (\varphi_\epsilon(z, \nabla u_\epsilon) \nabla u_\epsilon - (\eta_\epsilon + a(z)\chi_\epsilon)) \cdot \nabla \xi dz \leq 0 \quad \forall \xi \in \mathcal{W}_{q(\cdot)}(Q_T),$$

which is possible only if

$$\int_{Q_T} (\varphi_\epsilon(z, \nabla u_\epsilon) \nabla u_\epsilon - (\eta_\epsilon + a(z)\chi_\epsilon)) \cdot \nabla \xi dz = 0 \quad \forall \xi \in \mathcal{W}_{q(\cdot)}(Q_T),$$

The initial condition for u_ϵ is fulfilled by continuity because $u_\epsilon \in C([0, T]; L^2(\Omega))$.

Uniqueness of the weak solution is an immediate byproduct of monotonicity. Let u, v are two solutions of problem (4.3.16). Take an arbitrary $\tau \in (0, T]$. Choosing $u - v$ for the test function in equalities (4.3.6) for u and v in the cylinder $Q_\tau = \Omega \times (0, \tau)$, subtracting the results and applying (4.3.63) we arrive at the inequality

$$\frac{1}{2} \|u - v\|_{2, \Omega}^2(\tau) = \int_{Q_\tau} (u - v)(u - v)_t dz \leq 0.$$

It follows that $u(x, \tau) = v(x, \tau)$ a.e. in Ω for every $\tau \in [0, T]$.

Estimates (4.3.58) follow from the uniform in m estimates on the functions $u_\epsilon^{(m)}$ and their derivatives, the properties of weak convergence (4.3.60) and lower semicontinuity of the modular. Inequality (4.3.49) yields that for every $\delta \in (0, r^*)$ the sequence $\{\nabla u_\epsilon^{(m)}\}$ contains a subsequence which converges to ∇u_ϵ weakly in $(L^{p(\cdot)+\delta}(Q_T))^N$, whence (4.3.59). \square

Theorem 4.3.6. *Let in the conditions of Theorem 4.3.5, $b \neq 0$.*

(i) Assume that b, σ are measurable and bounded functions in Q_T

$$\|\nabla b\|_{\infty, Q_T} < \infty, \quad \|\nabla \sigma\|_{\infty, Q_T} < \infty, \quad 2 \leq \sigma^- \leq \sigma^+ < 1 + \frac{p^-}{2}.$$

Then for every $\epsilon \in (0, 1)$ problem (4.3.16) has at least one strong solution u , which satisfies estimates (4.3.58), (4.3.59).

(ii) The solution is unique if either $\sigma \equiv 2$, or $b(z) \leq 0$ in Q_T and $\sigma^- \geq 1$.

Proof. The proof is an imitation of the proof of Theorem 4.3.5. The estimates of Lemmas 4.3.8, 4.3.9, 4.3.10 allows one to extract a subsequence $\{u_\epsilon^{(m_k)}\}$ with the convergence properties (4.3.60). Let u_ϵ be the pointwise limit of the sequence $\{u_\epsilon^{(m_k)}\}$. We have to show that for every $\phi \in L^2(Q_T)$

$$\int_{Q_T} |u_\epsilon^{(m_k)}|^{\sigma(z)-2} u_\epsilon^{(m_k)} \phi \, dz \rightarrow \int_{Q_T} |u_\epsilon|^{\sigma(z)-2} u_\epsilon \phi \, dz.$$

The sequence $v_{m_k} = |u_\epsilon^{(m_k)}|^{\sigma(z)-2} u_\epsilon^{(m_k)}$ converges a.e. in Q_T to $|u_\epsilon|^{\sigma(z)-2} u_\epsilon$ and is uniformly bounded in $L^2(Q_T)$ because

$$\begin{aligned} \int_{Q_T} v_{m_k}^2 \, dz &= \int_{Q_T} |u_\epsilon^{(m_k)}|^{2(\sigma(z)-1)} \, dz \leq C \left(1 + \int_{Q_T} |u_\epsilon^{(m_k)}|^{p^-} \, dz \right) \\ &\leq C \left(1 + \int_{Q_T} |\nabla u_\epsilon^{(m_k)}|^{p^-} \, dz \right) \leq C \left(1 + \int_{Q_T} |u_\epsilon^{(m_k)}|^{p(z)} \, dz \right) \leq C'. \end{aligned}$$

It follows that there is $v \in L^2(Q_T)$ such that $v_{m_k} \rightharpoonup v$ in $L^2(Q_T)$ and by virtue of pointwise convergence it is necessary that $v = |u_\epsilon|^{\sigma(z)-2} u_\epsilon$ a.e. in Q_T .

Assume that $u_1, u_2 \in \mathcal{W}_{q(\cdot)}(Q_T)$ are two strong solutions of problem (4.3.16). The function $u_1 - u_2$ is an admissible test-function in the integral identities (4.3.6) for u_i . Combining these identities and using (4.3.63) we arrive at the inequality

$$\begin{aligned} \frac{1}{2} \|u_1 - u_2\|_{2, \Omega}^2(t) &\leq \frac{1}{2} \|u_1 - u_2\|_{2, \Omega}^2(t) + \int_0^t \int_{\Omega} (\varphi_\epsilon(z, \nabla u_1) \nabla u_1 - \varphi_\epsilon(z, \nabla u_2) \nabla u_2) \cdot \nabla(u_1 - u_2) \, dz \\ &= \int_0^t \int_{\Omega} b(z) \left(|u_1|^{\sigma(z)-2} u_1 - |u_2|^{\sigma(z)-2} u_2 \right) (u_1 - u_2) \, dz. \end{aligned}$$

If $\sigma \equiv 2$, this inequality takes the form

$$\frac{1}{2} \|u_1 - u_2\|_{2, \Omega}^2(t) \leq B \int_0^t \|u_1 - u_2\|_{2, \Omega}^2(\tau) \, d\tau, \quad t \in (0, T), \quad B = \operatorname{ess\,sup}_{Q_T} b(z),$$

whence $\|u_1 - u_2\|_{2, \Omega}(t) = 0$ in $(0, T)$ by Grönwall's inequality. Let $b(z) \leq 0$ in Q_T . For $\sigma(z) \geq 1$ the function $|s|^{\sigma(z)-2} s$ is monotone increasing as a function of s , therefore

$$\left(|u_1|^{\sigma(z)-2} u_1 - |u_2|^{\sigma(z)-2} u_2 \right) (u_1 - u_2) \geq 0 \text{ a.e. in } Q_T$$

and

$$\frac{1}{2}\|u_1 - u_2\|_{2,\Omega}^2(t) \leq 0 \quad \text{in } (0, T).$$

□

4.3.6.2 Degenerate problem. Proof of Theorems 4.3.1, 4.3.2

Let $\{u_\epsilon\}$ be the family of strong solutions of the regularized problems (4.3.16) satisfying estimates (4.3.58). These uniform in ϵ estimates enable one to extract a sequence $\{u_{\epsilon_k}\}$ and find functions $u \in \mathcal{W}_{q(\cdot)}(Q_T)$, $\eta, \chi \in (L^{q'(\cdot)}(Q_T))^N$ with the following properties:

$$\begin{aligned} u_{\epsilon_k} &\rightharpoonup u \quad \star\text{-weakly in } L^\infty(0, T; L^2(\Omega)), & u_{\epsilon_k t} &\rightharpoonup u_t \text{ in } L^2(Q_T), \\ \nabla u_{\epsilon_k} &\rightharpoonup \nabla u \text{ in } (L^{q(\cdot)}(Q_T))^N, \\ (\epsilon_k^2 + |\nabla u_{\epsilon_k}|^2)^{\frac{p(z)-2}{2}} \nabla u_{\epsilon_k} &\rightharpoonup \eta \text{ in } (L^{q'(\cdot)}(Q_T))^N, \\ (\epsilon_k^2 + |\nabla u_{\epsilon_k}|^2)^{\frac{q(z)-2}{2}} \nabla u_{\epsilon_k} &\rightharpoonup \chi \text{ in } (L^{q'(\cdot)}(Q_T))^N. \end{aligned}$$

In the third line we make use of the uniform estimate

$$\int_{Q_T} (\epsilon^2 + |\nabla u_\epsilon|^2)^{\frac{q(z)(p(z)-1)}{2(q(z)-1)}} dz \leq C \left(1 + \int_{Q_T} |\nabla u_\epsilon|^{p(z)+r} dz \right) \leq C,$$

which follows from (4.3.7) and (4.3.59). Moreover, $u \in C([0, T]; L^2(\Omega))$. Each of u_{ϵ_k} satisfies the identity

$$\int_{Q_T} u_{\epsilon_k t} \xi dz + \int_{Q_T} \varphi_{\epsilon_k}(z, \nabla u_{\epsilon_k}) \nabla u_{\epsilon_k} \cdot \nabla \xi dz = \int_{Q_T} f_0 \xi dz \quad \forall \xi \in \mathcal{W}_{q(\cdot)}(Q_T), \quad (4.3.65)$$

which yields

$$\int_{Q_T} u_t \xi dz + \int_{Q_T} (\eta + a(z)\chi) \cdot \nabla \xi dz = \int_{Q_T} f_0 \xi dz \quad \forall \xi \in \mathcal{W}_{q(\cdot)}(Q_T). \quad (4.3.66)$$

To identify η and χ we use the monotonicity argument. Take $\xi = u_{\epsilon_k}$ in (4.3.65):

$$\int_{Q_T} u_{\epsilon_k t} u_{\epsilon_k} dz + \int_{Q_T} \varphi_{\epsilon_k}(z, \nabla u_{\epsilon_k}) \nabla u_{\epsilon_k} \cdot \nabla u_{\epsilon_k} dz = \int_{Q_T} f_0 u_{\epsilon_k} dz. \quad (4.3.67)$$

According to (4.3.64), for every $\phi \in \mathcal{W}_{q(\cdot)}(Q_T)$

$$\begin{aligned} \int_{Q_T} \varphi_{\epsilon_k}(z, \nabla u_{\epsilon_k}) \nabla u_{\epsilon_k} \cdot \nabla u_{\epsilon_k} dz &\geq \int_{Q_T} (\varphi_{\epsilon_k}(z, \nabla \phi) - (|\nabla \phi|^{p-2} + a(z)|\nabla \phi|^{q-2}) \nabla \phi \cdot \nabla (u_{\epsilon_k} - \phi)) dz \\ &+ \int_{Q_T} \varphi_{\epsilon_k}(z, \nabla u_{\epsilon_k}) \nabla u_{\epsilon_k} \cdot \nabla \phi dz + \int_{Q_T} (|\nabla \phi|^{p-2} + a(z)|\nabla \phi|^{q-2}) \nabla \phi \cdot \nabla (u_{\epsilon_k} - \phi) dz \\ &\equiv J_{1,k} + J_{2,k} + J_{3,k}, \end{aligned}$$

where

$$\begin{aligned} J_{2,k} &\rightarrow \int_{Q_T} (\eta + a(z)\chi) \cdot \nabla \phi dz, \\ J_{3,k} &\rightarrow \int_{Q_T} (|\nabla \phi|^{p-2} + a(z)|\nabla \phi|^{q-2}) \nabla \phi \cdot \nabla (u - \phi) dz \quad \text{as } k \rightarrow \infty. \end{aligned}$$

Since $\left|(\varphi_{\epsilon_k}(z, \nabla\phi)\nabla\phi - (|\nabla\phi|^{p-2} + a^{\frac{q-1}{q}}(z)|\nabla\phi|^{q-2})\nabla\phi\right| \rightarrow 0$ a.e. in Q_T as $k \rightarrow \infty$, and because the integrand of $J_{1,k}$ has the majorant

$$\begin{aligned} & \left|((\epsilon_k^2 + |\nabla\phi|^2)^{\frac{p-2}{2}} - |\nabla\phi|^{p-2})\nabla\phi\right|^{p'} + \left|a^{\frac{q-1}{q}}(z)((\epsilon_k^2 + |\nabla\phi|^2)^{\frac{q-2}{2}} - |\nabla\phi|^{q-2})\nabla\phi\right|^{q'} \\ & \leq C \left((1 + |\nabla\phi|^2)^{\frac{p(z)}{2}} + a(z)(1 + |\nabla\phi|^2)^{\frac{q(z)}{2}} \right) \\ & \leq C \left(1 + |\nabla\phi|^{p(z)} + a(z)|\nabla\phi|^{q(z)} \right), \end{aligned}$$

then $J_{1,k} \rightarrow 0$ by the dominated convergence theorem. Combining (4.3.66) with (4.3.67) and letting $k \rightarrow \infty$ we find that for every $\phi \in \mathcal{W}_{q(\cdot)}(Q_T)$

$$\int_{Q_T} \left((|\nabla\phi|^{p(z)-2} + a(z)|\nabla\phi|^{q(z)-2})\nabla\phi - (\eta + a(z)\chi) \right) \cdot \nabla(u - \phi) dz \geq 0.$$

Choosing $\phi = u + \lambda\zeta$ with $\lambda > 0$ and $\zeta \in \mathcal{W}_{q(\cdot)}(Q_T)$, simplifying, and then letting $\lambda \rightarrow 0^+$, we obtain the inequality

$$\int_{Q_T} \left((|\nabla u|^{p(z)-2}\nabla u + a(z)|\nabla u|^{q(z)-2}\nabla u) - (\eta + a(z)\chi) \right) \cdot \nabla\zeta dz \geq 0 \quad \forall \zeta \in \mathcal{W}_{q(\cdot)}(Q_T).$$

Since the sign of ζ is arbitrary, the previous relation is the equality. It follows that in (4.3.66) $\eta + a(z)\chi$ can be substituted by $|\nabla u|^{p(z)-2}\nabla u + a(z)|\nabla u|^{q(z)-2}\nabla u$. Since $u \in C([0, T]; L^2(\Omega))$, the initial condition is fulfilled by continuity. Estimates (4.3.9) follow from the uniform in ϵ estimates of Theorem 4.3.5 and the lower semicontinuity of the modular exactly as in the proof of Theorem 4.3.5. Uniqueness of a strong solution is an immediate consequence of the monotonicity. Theorem 4.3.1 is proven.

To prove Theorem 4.3.2 we only have to check that $|u_{\epsilon_k}|^{\sigma(z)-2}u_{\epsilon_k} \rightharpoonup |u|^{\sigma(z)-2}u$ in $L^2(Q_T)$ (up to a subsequence). This is done as in the case of the regularized problem.

Remark 4.3.2. *Under the assumption of the Theorem 4.3.1 or Theorem 4.3.2 and, in addition $f_0 \in L^1(0, T; L^\infty(\Omega))$ and $u_0 \in L^\infty(\Omega)$, the strong solution of the problem (4.3.1) is bounded and satisfies the estimate*

$$\|u(\cdot, t)\|_{\infty, \Omega} \leq e^{C_1 t} \|u_0\|_{\infty, \Omega} + e^{C_1 t} \int_0^t e^{-C_1 \tau} \|f_0(\cdot, \tau)\|_{\infty, \Omega} d\tau$$

where $C_1 = 0$ if $b(z) \leq 0$ in Q_T , or $C_1 = \|b\|_{\infty, Q_T}$ if $\sigma \equiv 2$ (see [34, Ch.4, Sec.4.3, Th.4.3]).

4.4 A Picone identity for variable exponent operators and its applications

In this section, we prove the Picone identity for a general class of nonlinear operator and derive some of its applications by studying the qualitative properties of elliptic and parabolic equations. Precisely, we consider a continuous operator $A: \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ such that $(x, \xi) \rightarrow A(x, \xi)$ is differentiable with respect to variable ξ and satisfies:

(A0) $\xi \rightarrow A(x, \xi)$ is strictly convex for any $x \in \Omega$.

(A1) $\xi \rightarrow A(x, \xi)$ is positively $p(x)$ -homogeneous i.e. $A(x, t\xi) = t^{p(x)}A(x, \xi)$, $\forall t \in \mathbb{R}^+$, $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$.

Remark 4.4.1. From the assumptions of A , we deduce $A(x, \xi) > 0$ for $\xi \neq 0$ and for any $x \in \Omega$.

4.4.1 Main results

By using the convexity and the $p(x)$ -homogeneity of the operator A , we prove the following extension of the Picone identity:

Theorem 4.4.1 (Picone identity). *Let $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a continuous and differentiable function satisfying (A0) and (A1). Let $v_0, v \in L^\infty(\Omega)$ belonging to $\dot{V}_+^r \stackrel{\text{def}}{=} \{v : \Omega \rightarrow (0, +\infty) \mid v^{\frac{1}{r}} \in W_0^{1,p(x)}(\Omega)\}$ for some $r \geq 1$. Then*

$$\frac{1}{p(x)} \left\langle \partial_\xi A(x, \nabla v_0^{1/r}), \nabla \left(\frac{v}{v_0^{(r-1)/r}} \right) \right\rangle \leq A^{\frac{r}{p(x)}}(x, \nabla v^{1/r}) A^{\frac{(p(x)-r)}{p(x)}}(x, \nabla v_0^{1/r})$$

where $\langle \cdot, \cdot \rangle$ is the inner scalar product and the above inequality is strict if $r > 1$ or $\frac{v}{v_0} \neq \text{Const} > 0$.

From the above Picone identity, we can show an extension of the famous Diaz-Saa inequality to the class of variable exponent operators as a first application. This inequality is strongly linked to the strict convexity of some associated homogeneous energy type functional.

Theorem 4.4.2 (Diaz-Saa inequality). *Let $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ is a continuous and differentiable function satisfying (A0) and (A1) and define $a(x, \xi) = (a_i(x, \xi))_i \stackrel{\text{def}}{=} \left(\frac{1}{p(x)} \partial_{\xi_i} A(x, \xi) \right)_i$. Assume in addition that there exists $\Lambda > 0$ such that*

$$a \in C^1(\Omega \times (\mathbb{R}^N \setminus \{0\}))^N \quad \text{and} \quad \sum_{i,j=1}^N \left| \frac{\partial a_i(x, \xi)}{\partial \xi_j} \right| \leq \Lambda |\xi|^{p(x)-2}$$

for all $(x, \xi) \in \Omega \times \mathbb{R}^N \setminus \{0\}$. Then, we have in the sense of distributions, for any $r \in [1, p^-]$

$$\int_{\Omega} \left(- \frac{\text{div}(a(x, \nabla w_1))}{w_1^{r-1}} + \frac{\text{div}(a(x, \nabla w_2))}{w_2^{r-1}} \right) (w_1^r - w_2^r) dx \geq 0 \quad (4.4.1)$$

for any $w_1, w_2 \in W_0^{1,p(x)}(\Omega)$, positive in Ω such that $\frac{w_1}{w_2}, \frac{w_2}{w_1} \in L^\infty(\Omega)$. Moreover, if the equality occurs in (4.4.1), then w_1/w_2 is constant in Ω . If $p(x) \neq r$ in Ω then even $w_1 = w_2$ holds in Ω .

As a second application, we investigate the solvability of the following boundary problems involving quasilinear elliptic operators with variable exponent:

$$\begin{cases} -\Delta_{p(x)}u + g(x, u) = f(x, u) & \text{in } \Omega; \\ u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.4.2)$$

The extended Picone identity can be reformulated as in Lemma 4.4.1 below. Together with the strong maximum principle and elliptic regularity, this identity can be used to prove the uniqueness of weak solutions to elliptic equations as (4.4.2). In particular, we establish the following result:

Theorem 4.4.3. *Let $f, g : \Omega \times [0, \infty) \rightarrow \mathbb{R}^+$ be defined as $f(x, t) = h(x)t^{q(x)-1}$ and $g(x, t) = l(x)t^{s(x)-1}$ with $1 \leq q, s \in C(\overline{\Omega})$ such that*

- $q_+ < p^- < s_-$ and $q_- \geq 1$;
- $h, l \in L^\infty(\Omega)$, positive functions such that $x \rightarrow \frac{h(x)}{l(x)} \in L^\infty(\Omega)$.

Then, there exists a weak solution u to (4.4.2), i.e. u belongs to $W_0^{1,p(x)}(\Omega) \cap L^{s(x)}(\Omega)$ and satisfies for any $\phi \in W_0^{1,p(x)}(\Omega) \cap L^{s(x)}(\Omega)$:

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \phi \, dx = \int_{\Omega} (f(x, u) - g(x, u)) \phi \, dx.$$

Furthermore $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and $0 \leq u^{s_- - q_+} \leq \max\{\|\frac{h}{l}\|_{L^\infty}, 1\}$ a.e. in Ω .

Assume in addition that $x \rightarrow \frac{l(x)}{h(x)}$ belongs to $L^\infty(\Omega)$, then $u \in C_d^0(\overline{\Omega})^+ \stackrel{\text{def}}{=} \{v \in C_0(\overline{\Omega}) \mid \exists c_1, c_2 \in \mathbb{R}_^+ : c_1 \leq \frac{v}{\text{dist}(x, \partial\Omega)} \leq c_2\}$ and is the unique weak solution to (4.4.2).*

We remark that Theorem 4.4.3 does not require any subcritical growth condition for g to establish existence and uniqueness of the weak solution to (4.4.2). As a third application of Picone identity, we study the following Doubly nonlinear equation (D.N.E. for short) driven by $p(x)$ -Laplacian:

$$\begin{cases} \frac{q}{2q-1} \partial_t(u^{2q-1}) - \Delta_{p(x)}u = f(x, u) + h(t, x)u^{q-1} & \text{in } Q_T; \\ u > 0 & \text{in } Q_T; \\ u = 0 & \text{on } \Gamma; \\ u(0, \cdot) = u_0 & \text{in } \Omega \end{cases} \quad (4.4.3)$$

where $q \in (1, p^-)$, $Q_T = (0, T) \times \Omega$ and $\Gamma = (0, T) \times \partial\Omega$ for some $T > 0$. We suppose that $h \in L^\infty(Q_T)$ and nonnegative. The assumptions on f are given by

(f0) $f : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a function such that $f(x, 0) = 0$ for all $x \in \Omega$ and f is positive on $\Omega \times \mathbb{R}^+ \setminus \{0\}$.

(f1) for any $x \in \Omega, s \rightarrow \frac{f(x, s)}{s^{q-1}}$ is nonincreasing in $\mathbb{R}^+ \setminus \{0\}$.

Remark 4.4.2. Conditions (f0) and (f1) imply there exist positive constant C_1, C_2 such that for any $(x, s) \in \Omega \times \mathbb{R}^+$:

$$0 \leq f(x, s) \leq C_1 + C_2 s^{q-1},$$

i.e. f has a strict subhomogeneous growth.

We set \mathcal{R} the operator defined by $\mathcal{R}v = \frac{-\Delta_{p(x)}(v^{1/q})}{v^{(q-1)/q}} - \frac{f(x, v^{1/q})}{v^{(q-1)/q}}$ and the associated domain

$$\mathcal{D}(\mathcal{R}) = \{v : \Omega \rightarrow (0, \infty) : v^{1/q} \in W_0^{1,p(x)}(\Omega), v \in L^2(\Omega), \mathcal{R}v \in L^2(\Omega)\}.$$

Note that $\mathcal{D}(\mathcal{R})$ contains for instance solutions to (4.4.28). One can also easily check that solutions to (4.4.29) belong to $\overline{\mathcal{D}(\mathcal{R})}^{L^2(\Omega)}$. In the sequel, we denote $X^+ \stackrel{\text{def}}{=} \{x \in X \mid x \geq 0\}$ the associated positive cone of a given real vector space X .

In order to establish existence and properties of weak solutions to (4.4.3), we investigate the following related parabolic problem:

$$\begin{cases} v^{q-1} \partial_t(v^q) - \Delta_{p(x)} v = h(t, x) v^{q-1} + f(x, v) & \text{in } Q_T; \\ v > 0 & \text{in } Q_T; \\ v = 0 & \text{on } \Gamma; \\ v(0, \cdot) = v_0(x) > 0 & \text{in } \Omega. \end{cases} \quad (4.4.4)$$

The notion of weak solution for (4.4.4) is given as follows:

Definition 4.4.1. A weak solution to (4.4.4) is any positive function $v \in L^\infty(0, T; W_0^{1,p(x)}(\Omega)) \cap L^\infty(Q_T) \cap C(0, T; L^r(\Omega))$ for any $r \geq 1$ such that $\partial_t(v^q) \in L^2(Q_T)$ and for any $\phi \in C_0^\infty(Q_T)$ satisfies

$$\begin{aligned} \int_0^T \int_\Omega \partial_t(v^q) v^{q-1} \phi \, dx dt + \int_0^T \int_\Omega |\nabla v|^{p(x)-2} \nabla v \cdot \nabla \phi \, dx dt \\ = \int_0^T \int_\Omega h(t, x) v^{q-1} \phi \, dx dt + \int_0^T \int_\Omega f(x, v) \phi \, dx dt. \end{aligned} \quad (4.4.5)$$

Concerning (4.4.4), we prove the following results:

Theorem 4.4.4. Let $T > 0, v_0 \in C_d^0(\overline{\Omega})^+ \cap W_0^{1,p(x)}(\Omega)$. In addition, there exists $h_0 \in L^\infty(\Omega)$, $h_0 \not\equiv 0$ and $h(t, x) \geq h_0(x) \geq 0$ for a.e. $x \in \Omega$, for a.e. $t \geq 0$. Assume in addition $q \in (1, p^-)$ and f satisfies (f0)-(f1) and

(f2) The mapping $x \mapsto \delta^{1-q}(x) f(x, \delta(x))$ belongs to $L^2(\Omega_\varepsilon)$ for some $\varepsilon > 0$ where $\Omega_\varepsilon \stackrel{\text{def}}{=} \{x \in \Omega \mid \delta(x) < \varepsilon\}$.

Then there exists a weak solution to (4.4.4).

Based on the accretivity of \mathcal{R} with domain $\mathcal{D}(\mathcal{R})$, we show the following result providing a contraction property for weak solutions to (4.4.4) under suitable conditions on initial data:

Theorem 4.4.5. *Let v_1 and v_2 are weak solutions of (4.4.4) with initial data $u_0, v_0 \in C_d^0(\bar{\Omega})^+ \cap W_0^{1,p(x)}(\Omega)$ and such that $u_0^q, v_0^q \in \overline{\mathcal{D}(\mathcal{R})}^{L^2(\Omega)}$ and $h, g \in L^\infty(Q_T)$, such that $h \geq h_0$, $g \geq g_0$ with h_0, g_0 as in Theorem 4.4.4. Then, for any $0 \leq t \leq T$,*

$$\|(v_1^q(t) - v_2^q(t))^+\|_{L^2} \leq \|(u_0^q - v_0^q)^+\|_{L^2} + \int_0^t \|(h(s) - g(s))^+\|_{L^2} ds. \quad (4.4.6)$$

Furthermore, using a similar approach as in [62], we consider for $\epsilon > 0$ the perturbed operator $\mathcal{R}_\epsilon v = \frac{-\Delta_{p(x)}(v^{1/q})}{(v+\epsilon)^{(q-1)/q}} - \frac{f(x, v^{1/q})}{(v+\epsilon)^{(q-1)/q}}$. If $p^- \geq 2$, we can prove (as in Proposition 2.6 in [62]) that

$$\overline{\mathcal{D}(\mathcal{R}_\epsilon)}^{L^2(\Omega)} \supset \dot{V}_+^q \cap C_{dq}^0(\bar{\Omega})^+.$$

Arguing as in Theorem 4.4.5 with the operator \mathcal{R}_ϵ instead of \mathcal{R} and passing to the limit as $\epsilon \rightarrow 0^+$, we get:

Corollary 4.4.1. *Assume $p^- \geq 2$. Let v_1 and v_2 are weak solutions of (4.4.4) with initial data $u_0, v_0 \in C_d^0(\bar{\Omega})^+ \cap W_0^{1,p(x)}(\Omega)$. Then Theorem 4.4.5 holds.*

From Theorem 4.4.5, we derive the following comparison principle from which uniqueness of the weak solution to problem (4.4.4) follows:

Corollary 4.4.2. *Let u and v are the weak solutions of (4.4.4) with initial data u_0, v_0 satisfying conditions in Theorem 4.4.5 or Corollary 4.4.1. Assume $u_0 \leq v_0$ and $h, g \in L^\infty(Q_T)$, $h_0 \in L^\infty(\Omega)$ such that and $0 < h_0 \leq h \leq g$. Then $u \leq v$.*

Remark 4.4.3. *If $v \in L^\infty(Q_T)^+$ then from Proposition 9.5 in [69] we obtain $\frac{q}{2q-1} \partial_t(v^{2q-1}) = v^{q-1} \partial_t(v^q) = qv^{2q-2} \partial_t v$ in weak sense.*

From the above remark, under assumptions given in Theorem 4.4.4, we obtain the existence of weak solutions to (4.4.3) satisfying the monotonicity properties in Theorem 4.4.5 and Corollaries 4.4.1, 4.4.2. In the previous applications, the condition (A1) plays a crucial role to get suitable convexity property of energy functionals. We also study a quasilinear elliptic problem where this condition is not satisfied. Precisely, given $\epsilon > 0$, we study the following nonhomogeneous quasilinear elliptic problem:

$$\begin{cases} -\operatorname{div}(|\nabla u|^2 + \epsilon u^2)^{\frac{p(x)-2}{2}} \nabla u - (|\nabla u|^2 + \epsilon u^2)^{\frac{p(x)-2}{2}} \epsilon u = g(x, u) & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega; \\ u > 0 & \text{in } \Omega \end{cases} \quad (4.4.7)$$

where g satisfies (f_0) and (\tilde{g}) for some $m \in [1, p^-]$:

(\tilde{g}) For any $x \in \Omega$, $s \rightarrow \frac{g(x, s)}{s^{m-1}}$ is decreasing in $\mathbb{R}^+ \setminus \{0\}$ and a.e. in Ω .

Then we prove the following result:

Theorem 4.4.6. *Assume that g satisfies (f0) and (\tilde{g}). Then for any ϵ , (4.4.7) admits one and only one positive weak solution. Furthermore, $u \in C^1(\overline{\Omega})$, $u > 0$ in Ω and $\frac{\partial u}{\partial \vec{n}} < 0$ on $\partial\Omega$.*

To get the uniqueness result contained in Theorem 4.4.6, we exploit the hidden convexity property of the associated energy functional in the interior of positive cone of $C^1(\overline{\Omega})$.

4.4.2 Picone identity

First we recall the notion of strict ray-convexity.

Definition 4.4.2. *Let X be a real vector space. Let \dot{V} be a non empty cone in X . A function $J : \dot{V} \rightarrow \mathbb{R}$ is ray-strictly convex if for all $v_1, v_2 \in \dot{V}$ and for all $\theta \in (0, 1)$*

$$J((1 - \theta)v_1 + \theta v_2) \leq (1 - \theta)J(v_1) + \theta J(v_2)$$

where the inequality is always strict unless $v_1 = Cv_2$ for some $C > 0$.

Then we have the following result:

Proposition 4.4.1. *Let A satisfying (A0) and (A1) and let $r \geq 1$. Then, for any $x \in \Omega$ the map $\xi \rightarrow N_r(x, \xi) \stackrel{\text{def}}{=} A(x, \xi)^{r/p(x)}$ is positively r -homogeneous and ray-strictly convex. For $r > 1$, $\xi \rightarrow N_r(x, \xi)$ is even strictly convex.*

Proof. We begin by the case $r = 1$. For any $t \in \mathbb{R}^+$, we have $N_1(x, t\xi) = tN_1(x, \xi)$. Furthermore,

$$A(x, (1 - t)\xi_1 + t\xi_2) \leq (1 - t)A(x, \xi_1) + tA(x, \xi_2) \leq \max\{A(x, \xi_1), A(x, \xi_2)\}$$

for any $x \in \Omega$, $\xi_1, \xi_2 \in \mathbb{R}^N$ and $t \in [0, 1]$. Therefore

$$N_1(x, (1 - t)\xi_1 + t\xi_2) \leq \max\{N_1(x, \xi_1), N_1(x, \xi_2)\} \tag{4.4.8}$$

and this inequality is always strict unless $\xi_1 = \lambda\xi_2$, for some $\lambda > 0$.

Now we prove that N_1 is subadditive.

Without loss of generality, we can assume that $\xi_1 \neq 0$ and $\xi_2 \neq 0$. Then we have $N_1(x, \xi_1) > 0$ and $N_1(x, \xi_2) > 0$. Therefore, from (4.4.8) and 1-homogeneity of $N_1(x, \xi)$ we obtain for any $t \in (0, 1)$:

$$N_1\left(x, (1 - t)\frac{\xi_1}{N_1(x, \xi_1)} + t\frac{\xi_2}{N_1(x, \xi_2)}\right) \leq 1.$$

We now fix t such that

$$\frac{1-t}{N_1(x, \xi_1)} = \frac{t}{N_1(x, \xi_2)} \quad \text{i.e.} \quad t = \frac{N_1(x, \xi_2)}{N_1(x, \xi_1) + N_1(x, \xi_2)} \leq 1.$$

Then we get

$$N_1\left(x, \frac{\xi_1 + \xi_2}{N_1(x, \xi_1) + N_1(x, \xi_2)}\right) \leq 1$$

and by 1-homogeneity of N_1 , we obtain

$$N_1(x, \xi_1 + \xi_2) \leq N_1(x, \xi_1) + N_1(x, \xi_2), \quad \text{i.e.} \quad N_1 \text{ is subadditive.}$$

Finally for $t \in (0, 1)$, $\xi_1 \neq \lambda \xi_2$, $\forall \lambda > 0$

$$N_1(x, (1-t)\xi_1 + t\xi_2) < N_1(x, (1-t)\xi_1) + N_1(x, t\xi_2) = (1-t)N_1(x, \xi_1) + tN_1(x, \xi_2).$$

This proves that $\xi \rightarrow N_1(x, \xi)$ is ray-strictly convex. Now consider the case $r > 1$. Since for any $x \in \Omega$, $\xi \rightarrow N_r^{1/r}(x, \xi) = N_1(x, \xi)$ is ray-strictly convex and thanks to the strict convexity of $t \rightarrow t^r$ on \mathbb{R}^+ , we deduce that $\xi \rightarrow N_r(x, \xi) = N_1^r(x, \xi)$ is strictly convex when $r > 1$. \square

From Proposition 4.4.1 and from the r -homogeneity of N_r , we easily deduce the following convexity property of the energy functional:

Proposition 4.4.2. *Under hypothesis of Proposition 4.4.1 and assume in addition A is continuous on $\bar{\Omega} \times \mathbb{R}^N$. Then, for $1 \leq r < p^-$:*

$$\dot{V}_+^r \cap L^\infty(\Omega) \ni v \rightarrow \int_{\Omega} A(x, \nabla(v^{1/r})) dx$$

is ray-strictly convex (if $r > 1$, it is even strictly convex).

Proof. We know that $\xi \rightarrow N_r(x, \xi) = A^{r/p(x)}(x, \xi)$ is r -positively homogeneous and strictly convex if $r > 1$ and for $r = 1$ this function is ray-strictly convex. For $v_1, v_2 \in \dot{V}_+^r$ and $\theta \in (0, 1)$ define $v = (1-\theta)v_1 + \theta v_2$ and we get

$$N_r\left(x, \frac{\nabla v}{v}\right) \leq (1-\theta) \frac{v_1}{v} N_r\left(x, \frac{\nabla v_1}{v_1}\right) + \theta \frac{v_2}{v} N_r\left(x, \frac{\nabla v_2}{v_2}\right).$$

By homogeneity,

$$N_r(x, \nabla(v^{1/r})) \leq (1-\theta)N_r(x, \nabla(v_1^{1/r})) + \theta N_r(x, \nabla(v_2^{1/r}))$$

and equality holds if and only if $v_1 = \lambda v_2$ for some $\lambda > 0$. Using the convexity of $t \rightarrow t^{p(x)/r}$ for $1 \leq r < p^-$ we obtain

$$\int_{\Omega} A(x, \nabla v^{1/r}) dx \leq (1-\theta) \int_{\Omega} A(x, \nabla v_1^{1/r}) dx + \theta \int_{\Omega} A(x, \nabla v_2^{1/r}) dx.$$

Moreover, if $p(x) \neq r$ equality holds if and only if $v_1 = v_2$. \square

From Proposition 4.4.1, we deduce the proof of Picone identity.

Proof of Theorem 4.4.1: Firstly, we deal with the case $r > 1$. Then from Proposition 4.4.1, for any $x \in \Omega$ the function $\xi \rightarrow N_r(x, \xi) = A(x, \xi)^{r/p(x)}$ is strictly convex. Let $\xi, \xi_0 \in \mathbb{R}^N \setminus \{0\}$ such that $\xi \neq \xi_0$ then

$$N_r(x, \xi) - N_r(x, \xi_0) > \langle \partial_\xi N_r(x, \xi_0), \xi - \xi_0 \rangle.$$

Setting $\tilde{a}(x, \xi) = \frac{1}{r} \partial_\xi N_r(x, \xi)$, we obtain:

$$N_r(x, \xi) - \langle \tilde{a}(x, \xi_0), \xi_0 \rangle > r \langle \tilde{a}(x, \xi_0), \xi - \xi_0 \rangle.$$

Let $v, v_0 > 0$ and replacing ξ, ξ_0 by ξ/v and ξ_0/v_0 respectively in the above expression, we get

$$N_r\left(x, \frac{\xi}{v}\right) > r \left\langle \tilde{a}\left(x, \frac{\xi_0}{v_0}\right), \frac{\xi}{v} - \frac{r-1}{r} \frac{\xi_0}{v_0} \right\rangle.$$

Taking $\xi = \nabla v$ and $\xi_0 = \nabla v_0$ and using $(r-1)$ -homogeneity of $\tilde{a}(x, \cdot)$,

$$N\left(x, \frac{\nabla v}{rv^{(r-1)/r}}\right) > \frac{1}{v_0^{(r-1)/r}} \left\langle \tilde{a}\left(x, \frac{\nabla v_0}{rv_0^{(r-1)/r}}\right), \nabla v - \frac{r-1}{r} \frac{\nabla v_0}{v_0} v \right\rangle$$

where the inequality is strict unless $\frac{\nabla v}{v} = \frac{\nabla v_0}{v_0}$.

Since $v^{1/r}, v_0^{1/r} \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$, we can write

$$\nabla(v^{1/r}) = \frac{\nabla v}{rv^{(r-1)/r}} \quad \text{and} \quad \nabla\left(\frac{v}{v_0^{(r-1)/r}}\right) = \frac{1}{v_0^{(r-1)/r}} \left(\nabla v - \frac{r-1}{r} \frac{\nabla v_0}{v_0} v \right)$$

and we obtain

$$N(x, \nabla v^{1/r}) > \left\langle \tilde{a}(x, \nabla v_0^{1/r}), \nabla\left(\frac{v}{v_0^{(r-1)/r}}\right) \right\rangle. \quad (4.4.9)$$

We have

$$\begin{aligned} \tilde{a}(x, \nabla v_0^{1/r}) &= \frac{1}{r} \partial_\xi N(x, \nabla v_0^{1/r}) = \frac{1}{r} \partial_\xi A^{r/p(x)}(x, \nabla v_0^{1/r}) \\ &= \frac{1}{p(x)} \partial_\xi A(x, \nabla v_0^{1/r}) A^{\frac{r-p(x)}{p(x)}}(x, \nabla v_0^{1/r}) \end{aligned}$$

and by replacing in (4.4.9) we obtain

$$A^{\frac{r}{p(x)}}(x, \nabla v^{1/r}) A^{\frac{p(x)-r}{p(x)}}(x, \nabla v_0^{1/r}) > \frac{1}{p(x)} \left\langle \partial_\xi A(x, \nabla v_0^{1/r}), \nabla\left(\frac{v}{v_0^{(r-1)/r}}\right) \right\rangle.$$

Now we deal with the case $r = 1$. Let $\xi, \xi_0 \in \mathbb{R}^N \setminus \{0\}$ such that for any $\lambda > 0$, $\xi \neq \lambda \xi_0$. Then, from Proposition 4.4.1, we have that

$$N(x, \xi) - N(x, \xi_0) \geq \langle \partial_\xi N(x, \xi_0), \xi - \xi_0 \rangle.$$

Taking $\xi = \nabla v$ and $\xi_0 = \nabla v_0$, we deduce

$$N(x, \nabla v) - N(x, \nabla v_0) \geq \langle \partial_\xi N(x, \nabla v_0), \nabla(v - v_0) \rangle$$

and

$$A^{\frac{1}{p(x)}}(x, \nabla v) A^{\frac{p(x)-1}{p(x)}}(x, \nabla v_0) \geq \frac{1}{p(x)} \langle \partial_\xi A(x, \nabla v_0), \nabla v \rangle$$

for any $x \in \Omega$ and the inequality is strict unless $v = \lambda v_0$ for some $\lambda > 0$. \square

The Picone identity also holds for anisotropic operators of the following type:

$$\sum_{i=1}^N \nabla_i (b_i(x, \nabla_i u)) = \sum_{i=1}^N \frac{\partial}{\partial x_i} \left(b_i \left(x, \frac{\partial u}{\partial x_i} \right) \right).$$

Precisely we have:

Corollary 4.4.3. *Let $B : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^N$ is a continuous and differentiable function such that $B(x, s) = (B_i(x, s))_{i=1,2,\dots,N}$ satisfying for any i , for any $x \in \Omega$, the map $s \rightarrow B_i(x, s)$ is $p_i(x)$ -homogeneous and strictly convex with $1 < p_i^- \leq p_i(\cdot) \leq p_i^+ < \infty$. For any i , we define $b_i(x, s) = \frac{1}{p_i(x)} \partial_s B_i(x, s)$. Then, for $v, v_0 \in \dot{V}_+^r \cap L^\infty(\Omega)$, we have*

$$\sum_{i=1}^N b_i(x, \partial_{x_i}(v_0^{1/r})) \partial_{x_i} \left(\frac{v}{v_0^{\frac{r-1}{r}}} \right) \leq \sum_{i=1}^N B_i^{\frac{r}{p_i(x)}}(x, \partial_{x_i}(v^{1/r})) B_i^{\frac{p_i(x)-r}{p_i(x)}}(x, \partial_{x_i}(v_0^{1/r})).$$

Proof. By taking $A(x, s) = B_i(x, s)$ in Theorem 4.4.1, we obtain $\forall i \in \{1, 2, \dots, N\}$

$$\frac{1}{p_i(x)} \partial_s B_i(x, \partial_{x_i}(v_0^{1/r})) \cdot \partial_{x_i} \left(\frac{v}{v_0^{\frac{r-1}{r}}} \right) \leq B_i^{\frac{r}{p_i(x)}}(x, \partial_{x_i}(v^{1/r})) \cdot B_i^{\frac{p_i(x)-r}{p_i(x)}}(x, \partial_{x_i}(v_0^{1/r}))$$

for all $v, v_0 \in \dot{V}_+^r \cap L^\infty(\Omega)$ and $i = 1, 2, \dots, N$.

Then by summing the expression over $i = 1, 2, \dots, N$, we obtain

$$\sum_{i=1}^N b_i(x, \partial_{x_i}(v_0^{1/r})) \cdot \partial_{x_i} \left(\frac{v}{v_0^{\frac{r-1}{r}}} \right) \leq \sum_{i=1}^N B_i^{\frac{r}{p_i(x)}}(x, \partial_{x_i}(v^{1/r})) \cdot B_i^{\frac{p_i(x)-r}{p_i(x)}}(x, \partial_{x_i}(v_0^{1/r})).$$

\square

4.4.3 An extension of the Diaz-Saa inequality

We prove the first application of Picone identity.

Proof of Theorem 4.4.2: The Picone identity implies

$$A^{r/p(x)}(x, \nabla w_1) A^{(p(x)-r)/p(x)}(x, \nabla w_2) \geq a(x, \nabla w_2) \cdot \nabla \left(\frac{w_1^r}{w_2^{r-1}} \right).$$

Using the Young inequality for $r \in [1, p^-]$, we get

$$\frac{r}{p(x)}(A(x, \nabla w_1) - A(x, \nabla w_2)) + A(x, \nabla w_2) \geq a(x, \nabla w_2) \cdot \nabla \left(\frac{w_1^r}{w_2^{r-1}} \right).$$

Noting that for any $\xi \in \mathbb{R}^N$, $A(x, \xi) = a(x, \xi) \cdot \xi$, we deduce

$$a(x, \nabla w_2) \cdot \nabla \left(w_2 - \frac{w_1^r}{w_2^{r-1}} \right) dx \geq \frac{r}{p(x)}(A(x, \nabla w_2) - A(x, \nabla w_1)). \quad (4.4.10)$$

Commuting w_1 and w_2 , we have

$$a(x, \nabla w_1) \cdot \nabla \left(w_1 - \frac{w_2^r}{w_1^{r-1}} \right) \geq \frac{r}{p(x)}(A(x, \nabla w_1) - A(x, \nabla w_2)). \quad (4.4.11)$$

Summing (4.4.10) and (4.4.11) and integrating over Ω yield

$$\int_{\Omega} a(x, \nabla w_1) \cdot \nabla \left(\frac{w_1^r - w_2^r}{w_1^{r-1}} \right) dx + \int_{\Omega} a(x, \nabla w_2) \cdot \nabla \left(\frac{w_2^r - w_1^r}{w_2^{r-1}} \right) \geq 0.$$

The rest of the proof is the consequence of Proposition 4.4.2. \square

Diaz-Saa inequality also holds for anisotropic operators. Here we require that $\xi \rightarrow B_i(x, \xi)$ is $p_i(x)$ -homogeneous and strictly convex and $b_i(x, \xi) = \frac{1}{p_i(x)} \partial_i B_i(x, \xi)$ where $r \in \mathbb{R}$, $1 \leq r \leq \min_{i=1,2,\dots,N} \{(p_i)_-\}$.

Corollary 4.4.4. *Under the assumptions of Corollary 4.4.3 and in addition that there exist $\Lambda > 0$ such that for each i , $\left| \frac{\partial b_i}{\partial s}(x, s) \right| \leq \Lambda |s|^{p(x)-2}$. Then we have in the sense of distributions, for $r \in [1, \min_i \{(p_i)_-\}]$ and $v, v_0 \in \dot{V}_+^r \cap L^\infty(\Omega)$:*

$$\sum_{i=1}^N \int_{\Omega} \left(-\frac{\partial_{x_i}(b_i(x, \partial_{x_i} v))}{v^{r-1}(x)} + \frac{\partial_{x_i}(b_i(x, \partial_{x_i} v_0))}{v_0^{r-1}(x)} \right) (v^r - v_0^r) dx \geq 0$$

Proof. We apply Theorem 4.4.2. For $A = B_i : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ and by replacing ∇ by ∂_{x_i} . \square

4.4.4 Application of Picone identity to quasilinear elliptic equations

The aim of this section is to establish Theorem 4.4.3.

4.4.4.1 Preliminary results

The first lemma is the Picone identity in the context of the $p(x)$ -Laplacian operator.

Lemma 4.4.1. *Let $r \in [1, p^-]$ and $u, v \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ two positive functions. Then for any $x \in \Omega$*

$$|\nabla u|^{p(x)} + |\nabla v|^{p(x)} \geq |\nabla v|^{p(x)-2} \nabla v \cdot \nabla \left(\frac{u^r}{v^{r-1}} \right) + |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \left(\frac{v^r}{u^{r-1}} \right).$$

Following the proof of Theorem 1.1 in [262], we first prove the following comparison principle:

Lemma 4.4.2. *Let $\lambda \geq 0$ and $u, v \in W_0^{1,p(x)}(\Omega) \cap L^{\alpha(x)}(\Omega)$ two nonnegative functions for some function $\alpha \in \mathcal{P}(\Omega)$ satisfying $1 < \alpha_- \leq \alpha_+ < \infty$. Assume for any $\phi \in W_0^{1,p(x)}(\Omega)$, $\phi \geq 0$:*

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \phi + u^{\alpha(x)-1} \chi_{u \geq \lambda} \phi \, dx \geq \int_{\Omega} |\nabla v|^{p(x)-2} \nabla v \cdot \nabla \phi + v^{\alpha(x)-1} \chi_{v \geq \lambda} \phi \, dx$$

where

$$\chi_{v \geq \lambda}(x) = \begin{cases} 1 & \text{if } \lambda \leq v < \infty; \\ 0 & \text{if } 0 \leq v < \lambda, \end{cases}$$

and $u \geq v$ a.e. in $\partial\Omega$. Then $u \geq v$ a.e. in Ω .

Proof. Let $\phi = (v - u)^+ \in W_0^{1,p(x)}(\Omega)$ and $\Omega_1 = \{x \in \Omega : u(x) < v(x)\}$. Then

$$\begin{aligned} 0 \leq & - \int_{\Omega_1} (|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v) \cdot \nabla (u - v) \, dx \\ & - \int_{\Omega_1} (u^{\alpha(x)-1} \chi_{u \geq \lambda} - v^{\alpha(x)-1} \chi_{v \geq \lambda})(u - v) \, dx \leq 0 \end{aligned}$$

from which we obtain $u \geq v$ a.e. in Ω . \square

Using lemma 4.4.2, we show the following strong maximum principle:

Lemma 4.4.3. *Let $h, l \in L^\infty(\Omega)$ be nonnegative functions, $h > 0$ and $k : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$. Let $\alpha, \beta \in \mathcal{P}(\Omega)$ be two functions such that $1 < \beta_- \leq \beta_+ < \alpha_- \leq \alpha_+ < \infty$. Let $u \in C^1(\bar{\Omega})$ be nonnegative and a nontrivial solution to*

$$\begin{cases} -\Delta_{p(x)} u + l(x) u^{\alpha(x)-1} = h(x) u^{\beta(x)-1} + k(x, u) & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.4.12)$$

Assume in addition either

(c1) $\frac{l}{h} \in L^\infty(\Omega)$

or

(c2) $k : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying $\liminf_{t \rightarrow 0^+} \frac{k(x, t)}{t^{\alpha(x)-1}} > \|l\|_{L^\infty}$ uniformly in x .

Then u is positive in Ω .

Proof. We follow the idea of the proof of Theorem 1.1 in [262]. For the reader's convenience we have included the detailed proof. We rewrite our equation (4.4.12) under condition (c1) as follows:

$$-\Delta_{p(x)} u + l(x) u^{\alpha(x)-1} \chi_{u \geq \lambda} \geq h(x) u^{\beta(x)-1} (1 - \chi_{u \geq \lambda}) \left(1 - \frac{l(x)}{h(x)} u^{\alpha(x)-\beta(x)} \right),$$

since $\frac{l}{h} \in L^\infty(\Omega)$, we choose $\lambda \in (0, 1)$ small enough such that for any $u(x) \leq \lambda$, we have $1 - \frac{l(x)}{h(x)}u^{\alpha(x)-\beta(x)} \geq 1 - \|\frac{l}{h}\|_{L^\infty(\Omega)}\lambda^{\alpha-\beta_+} \geq 0$.

Assuming condition **(c2)**, we have

$$-\Delta_{p(x)}u + l(x)u^{\alpha(x)-1}\chi_{u \geq \lambda} \geq k(x, u) - (1 - \chi_{u \geq \lambda})l(x)u^{\alpha(x)-1}.$$

We choose λ small enough such that for any $u(x) \leq \lambda$, we have $k(x, u) - l(x)u^{\alpha(x)-1} \geq 0$. Hence under both conditions, we get for any $x \in \Omega$,

$$-\Delta_{p(x)}u + l(x)u^{\alpha(x)-1}\chi_{u \geq \lambda} \geq 0.$$

Suppose that there exists x_1 such that $u(x_1) = 0$ then using the fact that u is nontrivial, we can find $x_2 \in \Omega$ and a ball $B(x_2, 2C)$ in Ω such that $x_1 \in \partial B(x_2, 2C)$ and $u > 0$ in $B(x_2, 2C)$. Let $a = \inf\{u(x) : |x - x_2| = C\}$ then $a > 0$ and choosing x_2 close enough to x_1 such that $0 < a < \lambda$ and $\nabla u(x_1) = 0$ since $u(x_1) = 0$.

Denote the annulus $P = \{x \in \Omega : C < |x - x_2| < 2C\}$. We define $p_1 = p(x_1)$, $M = \sup\{|\nabla p(x)| : x \in P\}$, $b = 8M + 2$, $l_1 = -b \ln(\frac{a}{C}) + \frac{2(N-1)}{C}$ and

$$j(t) = \frac{a}{e^{\frac{l_1 t}{p_1-1}} - 1} \left(e^{\frac{l_1 t}{p_1-1}} - 1 \right) \quad \forall t \in [0, C].$$

We have

$$\frac{a}{C}e^{\frac{-l_1 C}{p_1-1}} < j'(0) \leq j'(t) \leq j'(C) < \frac{a}{C}e^{\frac{l_1 C}{p_1-1}}$$

and then

$$\left(\frac{a}{C}\right)^3 \leq j'(t) \leq 1 \quad \forall t \in [0, C]. \quad (4.4.13)$$

We choose $C < 1$ and using $\nabla u(x_1) = 0$, $\frac{a}{C} < 1$ small enough such that for any $x \in P$

$$\frac{p(x) - 1}{p_1 - 1} \geq \frac{1}{2}. \quad (4.4.14)$$

Without loss of generality we can take $x_2 = 0$ and we set $r = |x - x_2| = |x|$, $t = 2C - r$. For $t \in [0, C]$ and $r \in [C, 2C]$, denote $w(r) = j(2C - r) = j(t)$, then

$$w'(r) = -j'(t), \quad w''(t) = j''(t).$$

From (4.4.13) and (4.4.14), we obtain

$$\begin{aligned} \operatorname{div}(|\nabla w|^{p(x)-2}\nabla w) &= (p(x) - 1)(j'(t))^{p(x)-2}j''(t) - \frac{N-1}{r}(j'(t))^{p(x)-1} \\ &\quad - (j'(t))^{p(x)-1} \ln(j'(t)) \sum_{i=1}^n \frac{\partial p}{\partial x_i} \cdot \frac{x_i}{r} \\ &\geq (j'(t))^{p(x)-1} \left(\frac{1}{2}l_1 + M \ln(j'(t)) - \frac{N-1}{r} \right) \end{aligned}$$

$$\geq -\ln\left(\frac{a}{C}\right)(j'(t))^{p(x)-1} \geq 0.$$

Since $j(t) < a < \lambda$, we deduce

$$-\operatorname{div}(|\nabla w|^{p(x)-2}\nabla w) + w^{\alpha(x)-1}\chi_{w \geq \lambda} \leq 0.$$

On ∂P , $w(C) = j(C) = a \leq u(x)$ and $w(2C) = j(0) = 0 \leq u(x)$. Then by Lemma 4.4.2, we obtain $w \leq u$ on P . Finally,

$$\begin{aligned} \lim_{s \rightarrow 0^+} \frac{u(x_1 + s(x_2 - x_1)) - u(x_1)}{s} &\geq \lim_{s \rightarrow 0^+} \frac{w(x_1 + s(x_2 - x_1)) - w(x_1)}{s} \\ &= j'(0) > 0 \end{aligned}$$

which contradicts $\nabla u(x_1) = 0$. Therefore, $u > 0$ in Ω . \square

Remark 4.4.4. Conditions (c1) and (c2) can be replaced by the condition that there exists t_0 such that $h(x)t^{\beta(x)-1} + k(x,t) - l(x)t^{\alpha(x)-1} \geq 0$ for all $0 < t < t_0$ and $x \in \Omega$.

Lemma 4.4.4. Under the same conditions of h, l, k as in Lemma 4.4.3, let $u \in C^1(\overline{\Omega})$ be the nonnegative and nontrivial solution of (4.4.12), $x_1 \in \partial\Omega$, $u(x_1) = 0$ and Ω satisfies the interior ball condition at x_1 , then $\frac{\partial u}{\partial \vec{n}}(x_1) < 0$ where \vec{n} is the outward unit normal vector at x_1 .

Proof. Choose $C > 0$ small enough such that $B(x_2, 2C) \subset \Omega$, $x_1 \in \partial B(x_2, 2C)$. Then $x_2 = x_1 + 2C\vec{n}$, where \vec{n} is the outward normal at x_1 . Denote $P = \{x \in \Omega : C < |x - x_2| < 2C\}$ and by choosing a such that $0 < a < \lambda$, then by Lemma 4.4.3, there exist a subsolution $w \in C^1(\overline{P}) \cap C^2(P)$ of (4.4.12) in P and w satisfies $w \leq u$ in P with $w(x_1) = 0$, $\frac{\partial w}{\partial \vec{n}}(x_1) < 0$. Hence, we get $\frac{\partial u}{\partial \vec{n}}(x_1) \leq \frac{\partial w}{\partial \vec{n}}(x_1) < 0$. \square

4.4.4.2 Proof of Theorem 4.4.3

Proof of Theorem 4.4.3: We perform the proof along five steps. First we introduce notations. Define $F, G : \Omega \times \mathbb{R} \rightarrow \mathbb{R}^+$ as follows:

$$F(x, t) = \begin{cases} \frac{h(x)}{q(x)} t^{q(x)} & \text{if } 0 \leq t < \infty; \\ 0 & \text{if } -\infty < t < 0, \end{cases}$$

and

$$G(x, t) = \begin{cases} \frac{l(x)}{s(x)} t^{s(x)} & \text{if } 0 \leq t < \infty; \\ 0 & \text{if } -\infty < t < 0. \end{cases}$$

We also extend the domain of f and g to all $\Omega \times \mathbb{R}$ by setting

$$f(x, t) = \frac{\partial F}{\partial t}(x, t) = 0 \text{ and } g(x, t) = \frac{\partial G}{\partial t}(x, t) = 0 \text{ for } (x, t) \in \Omega \times (-\infty, 0).$$

Define the energy functional $\mathcal{E} : W_0^{1,p(x)}(\Omega) \cap L^{s(x)}(\Omega) \rightarrow \mathbb{R}$ by

$$\mathcal{E}(u) = \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} dx + \int_{\Omega} G(x, u(x)) dx - \int_{\Omega} F(x, u(x)) dx. \quad (4.4.15)$$

Step 1 : Existence of a global minimizer

Since $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ (see Theorem 3.3.1 and Theorem 8.2.4 in [112]), the functional \mathcal{E} is well-defined for every function $u \in W_0^{1,p(x)}(\Omega) \cap L^{s(x)}(\Omega)$.

For $\|u\|_{W_0^{1,p(x)}}$ large enough: by (4.1.2) or (4.1.3)

$$\begin{aligned} \mathcal{E}(u) &\geq \int_{\Omega} \frac{|\nabla u|^{p(x)}}{p(x)} - \int_{\Omega} \frac{h(x)}{q(x)} |u|^{q(x)} \geq \frac{1}{p^-} \|\nabla u\|_{L^{p(x)}}^{p^-} - C\rho_q(u) \\ &\geq \frac{1}{p^-} \|u\|_{W_0^{1,p(x)}}^{p^-} - C\|u\|_{W_0^{1,p(x)}}^{\tilde{q}} \end{aligned}$$

where $\tilde{q} = \begin{cases} q_- & \text{if } \|u\|_{L^{p(x)}} \leq 1 \\ q_+ & \text{if } \|u\|_{L^{p(x)}} > 1 \end{cases}$. Since $p^- > q_+$, this implies

$$\mathcal{E}(u) \rightarrow \infty \text{ as } \|u\|_{W_0^{1,p(x)}} \rightarrow +\infty.$$

We argue similarly when $\|u\|_{L^{s(x)}} \rightarrow \infty$ and we deduce \mathcal{E} is coercive. The continuity of \mathcal{E} on $W_0^{1,p(x)}(\Omega) \cap L^{s(x)}(\Omega)$ is given by Theorem 3.2.8 and 3.2.9 of [112]. Hence we get the existence of at least one global minimizer, say u_0 , to (4.4.15).

Step 2: Claim: $u_0 \geq 0$ and $u_0 \not\equiv 0$

Since u_0 is a global minimizer of \mathcal{E} then $\mathcal{E}(u_0^+) \geq \mathcal{E}(u_0)$ where $u_0^+ = \max\{u_0, 0\} \in W_0^{1,p(x)}(\Omega)$. Set $\Omega^- = \{x \in \Omega : u_0(x) < 0\}$. We have

$$\begin{aligned} \mathcal{E}(u_0) &= \int_{\Omega} \frac{|\nabla u_0|^{p(x)}}{p(x)} dx + \int_{\Omega} G(x, u_0(x)) dx - \int_{\Omega} F(x, u_0(x)) dx \\ &= \mathcal{E}(u_0^+) + \int_{\Omega^-} \frac{|\nabla u_0|^{p(x)}}{p(x)} dx \end{aligned}$$

which implies $\int_{\Omega^-} \frac{|\nabla u_0|^{p(x)}}{p(x)} = 0$ i.e. $\nabla u_0(x) = 0$ a.e. in Ω^- then by (4.1.2) and (4.1.3) we have $u_0 = 0$ a.e in Ω^- . This implies that $u_0 \geq 0$.

In order to show that $u_0 \not\equiv 0$ in Ω , we construct a function v in $W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ such that $\mathcal{E}(v) < 0 = \mathcal{E}(0)$. Precisely, consider $v = t\phi$ where $\phi \in C_c^1(\Omega)$, $\phi \geq 0$, $\phi \not\equiv 0$ in Ω and for $0 < t \leq 1$ small enough, we have

$$\mathcal{E}(v) \leq t^{q_+} (c_1 t^{p^- - q_+} + c_2 t^{s^- - q_+} - c_3)$$

where for any $i \in \{1, 2, 3\}$, c_i are suitable constants independent of t . Hence, choosing t small enough the right-hand side is negative and we conclude that $\mathcal{E}(t\phi) < 0 = \mathcal{E}(0)$ which implies

$u_0 \neq 0$.

Step 3: u_0 satisfies the equation in (4.4.2)

Since u_0 is a global minimizer and \mathcal{E} is C^1 on $W_0^{1,p(x)}(\Omega) \cap L^{s(x)}(\Omega)$, then for any $\phi \in W_0^{1,p(x)}(\Omega) \cap L^{s(x)}(\Omega)$, we have

$$\langle \mathcal{E}'(u_0), \phi \rangle = \int_{\Omega} |\nabla u_0|^{p(x)-2} \nabla u_0 \cdot \nabla \phi \, dx - \int_{\Omega} f(x, u_0) \phi \, dx + \int_{\Omega} g(x, u_0) \phi \, dx = 0.$$

Step 4: Regularity and positivity of weak solutions

First we prove that all nonnegative weak solutions of (4.4.2) belongs to $L^\infty(\Omega)$ which yields $C^{1,\alpha}(\overline{\Omega})$ regularity.

Let $K(x, t) = h(x)t^{q(x)-1} - l(x)t^{s(x)-1}$ and $\Lambda \stackrel{\text{def}}{=} \max \left\{ \left\| \frac{h}{l} \right\|_{L^\infty}, 1 \right\}^{1/(s_- - q_+)}$

Then it is not difficult to show that for any $t \geq \Lambda$, $K(x, t) \leq 0$. Let u be a nonnegative function satisfying weakly the equation in (4.4.2). Then for any $\phi \in W_0^{1,p(x)}(\Omega) \cap L^{s(x)}(\Omega)$,

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \phi \, dx = \int_{\Omega} (h(x)u^{q(x)-1} - l(x)u^{s(x)-1})\phi(x) \, dx.$$

Taking the testing function $\phi(x) = (u - \Lambda)^+$, we get

$$\int_{\Omega} |\nabla (u - \Lambda)^+|^{p(x)} \leq 0.$$

By using (4.1.3), we deduce $\|(u - \Lambda)^+\|_{W_0^{1,p(x)}} = 0$ which implies $u(x) \leq \Lambda$. From Theorem 1.2 in [118], we get $u \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. Furthermore assuming $x \rightarrow \frac{l(x)}{h(x)}$ belongs to $L^\infty(\Omega)$, Lemma 4.4.3 yields $u > 0$ in Ω .

Step 5: Uniqueness of the positive solution of (4.4.2)

Let u, v be two positive solutions of (4.4.2). Thus for any $\phi, \tilde{\phi} \in W_0^{1,p(x)}(\Omega) \cap L^{s(x)}(\Omega)$,

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \phi \, dx = \int_{\Omega} (h(x)u^{q(x)-1} - l(x)u^{s(x)-1})\phi(x) \, dx$$

and

$$\int_{\Omega} |\nabla v|^{p(x)-2} \nabla v \cdot \nabla \tilde{\phi} \, dx = \int_{\Omega} (h(x)v^{q(x)-1} - l(x)v^{s(x)-1})\tilde{\phi}(x) \, dx.$$

By the previous steps, u and v belong to $C^1(\overline{\Omega})$ and Lemma 4.4.4 implies $u, v \in C_d^0(\overline{\Omega})^+$.

Hence taking the testing functions as $\phi = \frac{(u^{p^-} - v^{p^-})^+}{u^{p^- - 1}}$ and $\tilde{\phi} = \frac{(v^{p^-} - u^{p^-})^-}{v^{p^- - 1}} \in W_0^{1,p(x)}(\Omega)$

(with the following notation $t^- \stackrel{\text{def}}{=} \max\{0, -t\}$) and from Lemma 4.4.1 we obtain

$$\begin{aligned} 0 &\leq \int_{\{u>v\}} (|\nabla u|^{p(x)-2} \nabla u - |\nabla v|^{p(x)-2} \nabla v) \cdot \nabla (u - v) \, dx \\ &= \int_{\{u>v\}} h(x)(u^{q(x)-p^-} - v^{q(x)-p^-})(u^{p^-} - v^{p^-}) \, dx \end{aligned}$$

$$+ \int_{\{u>v\}} l(x)(v^{s(x)-p^-} - u^{s(x)-p^-})(u^{p^-} - v^{p^-}) dx.$$

Since $q_+ \leq p^- \leq s_-$, the both terms in right-hand side are nonpositive. This implies $v(x) \geq u(x)$ a.e in Ω .

Finally reversing the role of u and v , we get $u = v$. \square

Remark 4.4.5. *Theorem 4.4.3 still holds when the condition $\frac{l}{h} \in L^\infty(\Omega)$ is replaced by $p^+ < s_-$ and using strong maximum principle in [262].*

4.4.5 Application to Doubly nonlinear equation

In this section, we establish Theorems 4.4.4 and 4.4.5. To this aim, we use a time semi-discretization method associated to (4.4.4). With the help of accurate energy estimates about the related quasilinear elliptic equation and passing to the limit as the discretization parameter goes to 0, we prove the existence and the properties of weak solutions to (4.4.3). In the subsection below, we study the associated elliptic problem.

4.4.5.1 Study of the quasilinear elliptic problem associated to D.N.E.

Consider the following problem

$$\begin{cases} v^{2q-1} - \lambda \Delta_{p(x)} v = h_0(x)v^{q-1} + \lambda f(x, v) & \text{in } \Omega; \\ v > 0 & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.4.16)$$

Assume $h_0 \in L^\infty(\Omega)^+$ and f satisfies (f0)-(f1). Then from (f1), we have

$$(f3) \lim_{s \rightarrow +\infty} \frac{f(x, s)}{s^{p^- - 1}} = 0 \text{ uniformly in } x \in \Omega.$$

Therefore, for any $\epsilon > 0$, there exists a positive constant C_ϵ such that for any $(x, s) \in \Omega \times \mathbb{R}^+$:

$$0 \leq f(x, s) \leq C_\epsilon + \epsilon s^{p^- - 1}. \quad (4.4.17)$$

We have the following preliminary result about (4.4.16):

Theorem 4.4.7. *Let $\lambda > 0$, $q \in (1, p^-]$, $f : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying (f0) and (f3) and $h_0 \in L^\infty(\Omega)^+$. Then there exists a weak solution $v \in C^1(\overline{\Omega})$ to (4.4.16), i.e. for any $\phi \in \mathbf{W} \stackrel{\text{def}}{=} W_0^{1,p(x)}(\Omega) \cap L^{2q}(\Omega)$*

$$\int_{\Omega} v^{2q-1} \phi dx + \lambda \int_{\Omega} |\nabla v|^{p(x)-2} \nabla v \cdot \nabla \phi dx = \int_{\Omega} h_0 v^{q-1} \phi dx + \lambda \int_{\Omega} f(x, v) \phi dx. \quad (4.4.18)$$

In addition, if (f1) holds then $v \in C_d^0(\overline{\Omega})^+$. Moreover if $v_1, v_2 \in C_d^0(\overline{\Omega})^+$ are two weak solutions to (4.4.16) corresponding to $h_0 = h_1, h_2 \in L^\infty(\Omega)^+$ respectively, then we have

$$\|(v_1^q - v_2^q)^+\|_{L^2} \leq \|(h_1 - h_2)^+\|_{L^2}. \quad (4.4.19)$$

Remark 4.4.6. (4.4.19) implies the uniqueness of the weak solution to (4.4.16) in $C_d^0(\bar{\Omega})^+$.

Proof. We perform the proof into several steps.

Step 1: Existence of a weak solution

Consider the energy functional \mathcal{J} defined on \mathbf{W} equipped with $\|\cdot\|_{\mathbf{W}} = \|\cdot\|_{W_0^{1,p(x)}} + \|\cdot\|_{L^{2q}}$

$$\mathcal{J}(v) = \frac{1}{2q} \int_{\Omega} v^{2q} dx + \lambda \int_{\Omega} \frac{|\nabla v|^{p(x)}}{p(x)} dx - \frac{1}{q} \int_{\Omega} h_0 D(v) dx - \lambda \int_{\Omega} F(x, v) dx \quad (4.4.20)$$

where

$$D(t) = \begin{cases} t^q & \text{if } 0 \leq t < \infty; \\ 0 & \text{if } -\infty < t < 0, \end{cases} \quad \text{and } F(x, t) = \begin{cases} \int_0^t f(x, s) ds & \text{if } 0 \leq t < \infty; \\ 0 & \text{if } -\infty < t < 0. \end{cases}$$

We also extend the domain of f to all of $\Omega \times \mathbb{R}$ by setting $f(x, t) = \frac{\partial F}{\partial t}(x, t) = 0$ for $(x, t) \in \Omega \times (-\infty, 0)$. From (4.4.17), Hölder inequality (4.1.4) and since $W_0^{1,p(x)} \hookrightarrow L^{p^-}(\Omega)$, we obtain

$$\begin{aligned} \mathcal{J}(v) &\geq \frac{1}{2q} \|v\|_{L^{2q}}^{2q} + \lambda \|v\|_{W_0^{1,p(x)}}^{p^-} - \frac{1}{q} \|h_0\|_{L^2} \|v\|_{L^{2q}}^q - \lambda C_{\epsilon} \int_{\Omega} |v| dx - \lambda \frac{\epsilon}{p^-} \int_{\Omega} |v|^{p^-} dx \\ &\geq \frac{1}{q} \|v\|_{L^{2q}}^q \left(\frac{1}{2} \|v\|_{L^{2q}}^q - \|h_0\|_{L^2} \right) + \lambda \|v\|_{W_0^{1,p(x)}}^{p^-} ((1 - \epsilon) \|v\|_{W_0^{1,p(x)}}^{p^- - 1} - \tilde{C}). \end{aligned}$$

Then by choosing ϵ small enough we conclude the coercivity of \mathcal{J} on \mathbf{W} and \mathcal{J} is also continuous on \mathbf{W} therefore we deduce the existence of a global minimizer v_0 to \mathcal{J} .

Furthermore we note

$$\mathcal{J}(v_0) \geq \mathcal{J}(v_0^+) + \frac{1}{2q} \int_{\Omega} (v_0^-)^{2q} dx + \lambda \int_{\Omega} \frac{|\nabla v_0^-|^{p(x)}}{p(x)} dx$$

which implies $v_0 \geq 0$.

Now we claim that $v_0 \not\equiv 0$ in Ω . Since $\mathcal{J}(0) = 0$, it is sufficient to prove the existence of $\tilde{v} \in \mathbf{W}$ such that $\mathcal{J}(\tilde{v}) < 0$. For that take $\tilde{v} = t\phi$ where $\phi \in C_c^1(\Omega)$ is nonnegative function such that $\phi \not\equiv 0$ and $t > 0$ small enough.

Since v_0 is a global minimizer for the differentiable functional \mathcal{J} , we have that v_0 satisfies (4.4.18) *i.e.* v_0 is a weak solution to (4.4.16). From Corollary 4.4.7 we infer that $v_0 \in L^{\infty}(\Omega)$. Then by using Theorem 4.4.12, we obtain, $v_0 \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$.

From (f1) and Lemma 4.4.3 (with condition (c2)), we obtain $v_0 > 0$ and by Lemma 4.4.4 we get $\frac{\partial v_0}{\partial \bar{n}} < 0$ on $\partial\Omega$. Therefore, v_0 belongs to $C_d^0(\bar{\Omega})^+$.

Step 2: Contraction property (4.4.19)

Let v_1 and v_2 two positive weak solutions of (4.4.16) such that $v_1, v_2 \in C_d^0(\bar{\Omega})^+$. For any $\phi, \Psi \in \mathbf{W}$:

$$\int_{\Omega} v_1^{2q-1} \phi dx + \lambda \int_{\Omega} |\nabla v_1|^{p(x)-2} \nabla v_1 \cdot \nabla \phi dx = \int_{\Omega} h_1 v_1^{q-1} \phi dx + \lambda \int_{\Omega} f(x, v_1) \phi dx$$

and

$$\int_{\Omega} v_2^{2q-1} \Psi \, dx + \lambda \int_{\Omega} |\nabla v_2|^{p(x)-2} \nabla v_2 \cdot \nabla \Psi \, dx = \int_{\Omega} h_2 v_2^{q-1} \Psi \, dx + \lambda \int_{\Omega} f(x, v_2) \Psi \, dx.$$

Since $v_1, v_2 \in C_d^0(\bar{\Omega})^+$, $\phi = \left(v_1 - \frac{v_2^q}{v_1^{q-1}}\right)^+$ and $\Psi = \left(v_2 - \frac{v_1^q}{v_2^{q-1}}\right)^-$ are well-defined and belong to \mathbf{W} . Subtracting the two above expressions and using (f1) together with Lemma 4.4.1 we obtain

$$\int_{\Omega} ((v_1^q - v_2^q)^+)^2 \, dx \leq \int_{\Omega} (h_1 - h_2)(v_1^q - v_2^q)^+ \, dx.$$

Finally, applying the Hölder inequality we get (4.4.19). \square

From Theorem 4.4.7, we deduce the accretivity of \mathcal{R} :

Corollary 4.4.5. *Let $\lambda > 0$, $q \in (1, p^-]$, $f : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying (f0)-(f1) and $h_0 \in L^\infty(\Omega)^+$. Consider the following problem*

$$\begin{cases} u + \lambda \mathcal{R}u = h_0(x) & \text{in } \Omega; \\ u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.4.21)$$

Then there exists a unique distributional solution $u \in \mathcal{D}(\mathcal{R}) \cap C^1(\bar{\Omega})$ of (4.4.21) i.e. $\forall \phi \in C_c^1(\Omega)$

$$\begin{aligned} \int_{\Omega} u_0 \phi \, dx + \lambda \int_{\Omega} |\nabla u_0^{1/q}|^{p(x)-2} \nabla u_0^{1/q} \cdot \nabla \left(\frac{\phi}{u_0^{(q-1)/q}} \right) \, dx \\ = \int_{\Omega} h_0 \phi \, dx + \lambda \int_{\Omega} \frac{f(x, u_0^{1/q})}{u_0^{(q-1)/q}} \phi \, dx. \end{aligned}$$

Moreover, if u_1 and u_2 are two distributional solutions of (4.4.21) in $\mathcal{D}(\mathcal{R}) \cap C^1(\bar{\Omega})$ associated to h_1 and h_2 respectively, then the operator \mathcal{R} satisfies

$$\|(u_1 - u_2)^+\|_{L^2} \leq \|(u_1 - u_2 + \lambda(\mathcal{R}u_1 - \mathcal{R}u_2))^+\|_{L^2}. \quad (4.4.22)$$

Proof. Define the energy functional \mathcal{E} on $\dot{V}_+^q \cap L^2(\Omega)$ as $\mathcal{E}(u) = \mathcal{J}(u^{1/q})$ where \mathcal{J} is defined in (4.4.20).

Let $\phi \in C_c^1(\Omega)$ and v_0 is the global minimizer of (4.4.20) which is also the weak solution of (4.4.16) and $u_0 = v_0^q$ then there exists $t_0 = t_0(\phi) > 0$ such that for $t \in (-t_0, t_0)$, $u_0 + t\phi > 0$. Hence we have

$$\begin{aligned} 0 \leq \mathcal{E}(u_0 + t\phi) - \mathcal{E}(u_0) &= \frac{1}{2q} \left(\int_{\Omega} (t\phi)^2 \, dx + \int_{\Omega} 2tu_0\phi \, dx \right) - \frac{1}{q} \int_{\Omega} ht\phi \, dx \\ &+ \lambda \left(\int_{\Omega} \frac{|\nabla(u_0 + t\phi)^{1/q}|^{p(x)}}{p(x)} \, dx - \int_{\Omega} \frac{|\nabla u_0^{1/q}|^{p(x)}}{p(x)} \, dx \right) \end{aligned}$$

$$- \lambda \left(\int_{\Omega} F(x, (u_0 + t\phi)^{1/q}) dx - \int_{\Omega} F(x, u_0^{1/q}) dx \right).$$

Then divide by t and passing to the limits $t \rightarrow 0$ we obtain $u_0 = v_0^q$ is the distributional solution of (4.4.21). Finally (4.4.22) and uniqueness follow from (4.4.19). \square

We now generalize some above results for a larger class of potentials h_0 :

4.4.5.2 Further results for (4.4.16) and uniqueness

Theorem 4.4.8. *Let $\lambda > 0$, $f : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying (f0)-(f1) and $h_0 \in L^2(\Omega)^+$ and $q \in (1, p^-]$. Then there exists a positive weak solution $v \in \mathbf{W}$ of (4.4.16) in the sense of (4.4.18). Moreover assuming that h_0 belongs to $L^\nu(\Omega)$ for some $\nu > \max\{1, \frac{N}{p^-}\}$, $v \in L^\infty(\Omega)$.*

Proof. Let $h_n \in C_c^1(\Omega)$ such that $h_n \geq 0$ and $h_n \rightarrow h$ in $L^2(\Omega)$. Define $(v_n) \subset C^{1,\alpha}(\bar{\Omega}) \cap C_d^0(\bar{\Omega})^+$ as for a fixed n , v_n is the unique positive weak solution of (4.4.16) with $h_0 = h_n$ i.e. v_n satisfies: for $\phi \in \mathbf{W}$

$$\int_{\Omega} v_n^{2q-1} \phi dx + \lambda \int_{\Omega} |\nabla v_n|^{p(x)-2} \nabla v_n \cdot \nabla \phi dx = \int_{\Omega} h_n v_n^{q-1} \phi dx + \lambda \int_{\Omega} f(x, v_n) \phi dx. \quad (4.4.23)$$

Since $(a-b)^{2q} \leq (a^q - b^q)^2$ for any $q \geq 1$, (4.4.19) implies for any $n, p \in \mathbb{N}^*$

$$\|(v_n - v_p)^+\|_{L^{2q}} \leq \|(h_n - h_p)^+\|_{L^2}^q$$

thus we deduce that (v_n) converges to $v \in L^{2q}(\Omega)$.

We infer that the limit v does not depend on the choice of the sequence (h_n) . Indeed, consider $\tilde{h}_n \neq h_n$ such that $\tilde{h}_n \rightarrow h_0$ in $L^2(\Omega)$ and \tilde{v}_n the positive solution of (4.4.16) corresponding to \tilde{h}_n which converges to \tilde{v} .

Then, for any $n \in \mathbb{N}$, (4.4.19) implies

$$\|(v_n^q - \tilde{v}_n^q)^+\|_{L^2} \leq \|(h_n - \tilde{h}_n)^+\|_{L^2}$$

and passing to the limit we get $\tilde{v} \geq v$ and then by reversing the role of v and \tilde{v} we obtain $v = \tilde{v}$.

So define, for any $n \in \mathbb{N}^*$, $h_n = \min\{h, n\}$. Thus (v_n) is nondecreasing and for any $n \in \mathbb{N}^*$, $v_n \leq v$ a.e. in Ω which implies $v \geq v_1 > 0$ in Ω .

We choose $\phi = v_n$ in (4.4.23). Applying the Hölder inequality and (4.4.17), we obtain

$$\begin{aligned} \lambda \int_{\Omega} |\nabla v_n|^{p(x)} dx &\leq \|h_n\|_{L^2} \|v_n\|_{L^{2q}}^q + \lambda C_\epsilon \|v_n\|_{L^1} + \lambda \epsilon \|v_n\|_{L^{p^-}}^{p^-} \\ &\leq C + \lambda \epsilon \|v_n\|_{L^{p^-}}^{p^-}. \end{aligned}$$

Assume $\|\nabla v_n\|_{L^{p(x)}} \geq 1$. Since $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^-}(\Omega)$ and by (4.1.2) we deduce for some positive constant $C > 0$:

$$\lambda \int_{\Omega} |\nabla v_n|^{p(x)} dx \leq C + \lambda \epsilon C \int_{\Omega} |\nabla v_n|^{p(x)} dx.$$

Choosing ϵ small enough and gathering with the case $\|\nabla v_n\|_{L^{p(x)}} \leq 1$, we conclude (v_n) is uniformly bounded in $W_0^{1,p(x)}(\Omega)$ and $L^{p^-}(\Omega)$. Hence v_n converges weakly to v in $W_0^{1,p(x)}(\Omega)$ and by monotonicity of (v_n) strongly in $L^{p^-}(\Omega)$ and in $L^{2q}(\Omega)$. Taking now $\phi = v_n - v$ in (4.4.23), from (4.4.17) with $\epsilon = 1$ and by Hölder inequality

$$\left| \int_{\Omega} f(x, v_n)(v_n - v) dx \right| \leq C \|v_n - v\|_{L^{2q}} + \|v_n\|_{L^{p^-}}^{p^- - 1} \|v_n - v\|_{L^{p^-}} \rightarrow 0$$

and

$$\int_{\Omega} h_n v_n^{q-1}(v_n - v) dx \rightarrow 0 \quad \text{and} \quad \int_{\Omega} v_n^{2q-1}(v_n - v) dx \rightarrow 0.$$

Finally (4.4.23) becomes

$$\int_{\Omega} |\nabla v_n|^{p(x)-2} \nabla v_n \cdot \nabla(v_n - v) dx \rightarrow 0.$$

Then, since $v_n \rightharpoonup v$ in $W_0^{1,p(x)}(\Omega)$

$$\int_{\Omega} (|\nabla v_n|^{p(x)-2} \nabla v_n - |\nabla v|^{p(x)-2} \nabla v) \cdot \nabla(v_n - v) dx \rightarrow 0.$$

Lemma A.2 and Remark A.3 of [147] give the strong convergence of v_n to v in $W_0^{1,p(x)}(\Omega)$. Since (v_n^{2q-1}) and $(h_n v_n^{q-1})$ are uniformly bounded in $L^{2q/(2q-1)}(\Omega)$ and by (4.4.17), $f(x, v_n)$ is uniformly bounded in $L^{2q/q-1}(\Omega)$ and $f(x, v_n) \rightarrow f(x, v)$ a.e. in Ω . Then by Lebesgue dominated convergence theorem we have (up to a subsequence), for $\phi \in \mathbf{W}$

$$\int_{\Omega} v_n^{2q-1} \phi dx \rightarrow \int_{\Omega} v^{2q-1} \phi dx, \quad \int_{\Omega} h_n v_n^{q-1} \phi dx \rightarrow \int_{\Omega} h v^{q-1} \phi dx$$

and

$$\int_{\Omega} f(x, v_n) \phi dx \rightarrow \int_{\Omega} f(x, v) \phi dx.$$

Finally we pass to the limit in (4.4.23) and we obtain v is a weak solution of (4.4.16). To conclude corollary 4.4.7 implies $v \in L^\infty(\Omega)$. \square

Remark 4.4.7. Let v_1, v_2 are the weak solutions of (4.4.16) obtained by Theorem 4.4.8 corresponding to $h_1, h_2 \in L^2(\Omega)^+, h_1 \not\equiv h_2$ respectively. Then

$$\|(v_1^q - v_2^q)^+\|_{L^2} \leq \|(h_1 - h_2)^+\|_{L^2}.$$

Remark 4.4.8. As in **Step 1** of the proof of Theorem 4.4.7, we can alternatively prove the existence of a weak solution by global minimization method.

Under the hypothesis of Theorem 4.4.8 and with the help of Picone identity, the following theorem gives the uniqueness of the solution to (4.4.16).

Theorem 4.4.9. Let v, \tilde{v} be respectively a subsolution and supersolution to (4.4.16) for $h \in L^{p_0}(\Omega)$, $p_0 \geq 2$, $h \geq 0$ and f satisfies (f0) and (f1). Then $v \leq \tilde{v}$.

Proof. We have for any nonnegative $\phi, \Psi \in \mathbf{W}$

$$\int_{\Omega} v^{2q-1} \phi \, dx + \lambda \int_{\Omega} |\nabla v|^{p(x)-2} \nabla v \cdot \nabla \phi \, dx \leq \int_{\Omega} h v^{q-1} \phi \, dx + \lambda \int_{\Omega} f(x, v) \phi \, dx$$

and

$$\int_{\Omega} \tilde{v}^{2q-1} \Psi \, dx + \lambda \int_{\Omega} |\nabla \tilde{v}|^{p(x)-2} \nabla \tilde{v} \cdot \nabla \Psi \, dx \geq \int_{\Omega} h \tilde{v}^{q-1} \Psi \, dx + \lambda \int_{\Omega} f(x, \tilde{v}) \Psi \, dx.$$

Subtracting the above inequalities with test functions $\phi = \left(\frac{(v+\epsilon)^q - (\tilde{v}+\epsilon)^q}{(v+\epsilon)^{q-1}} \right)^+$ and

$\Psi = \left(\frac{(\tilde{v}+\epsilon)^q - (v+\epsilon)^q}{(\tilde{v}+\epsilon)^{q-1}} \right)^- \in \mathbf{W}$ for $\epsilon \in (0, 1)$, we obtain

$$\begin{aligned} & \int_{\{v > \tilde{v}\}} \left(\frac{v^{2q-1}}{(v+\epsilon)^{q-1}} - \frac{\tilde{v}^{2q-1}}{(\tilde{v}+\epsilon)^{q-1}} \right) ((v+\epsilon)^q - (\tilde{v}+\epsilon)^q) \, dx \\ & + \lambda \int_{\{v > \tilde{v}\}} |\nabla(v+\epsilon)|^{p(x)-2} \nabla(v+\epsilon) \cdot \nabla \left(\frac{(v+\epsilon)^q - (\tilde{v}+\epsilon)^q}{(v+\epsilon)^{q-1}} \right) \, dx \\ & + \lambda \int_{\{v > \tilde{v}\}} |\nabla(\tilde{v}+\epsilon)|^{p(x)-2} \nabla(\tilde{v}+\epsilon) \cdot \nabla \left(\frac{(\tilde{v}+\epsilon)^q - (v+\epsilon)^q}{(\tilde{v}+\epsilon)^{q-1}} \right) \, dx \quad (4.4.24) \\ & \leq \int_{\{v > \tilde{v}\}} h \left(\frac{v^{q-1}}{(v+\epsilon)^{q-1}} - \frac{\tilde{v}^{q-1}}{(\tilde{v}+\epsilon)^{q-1}} \right) ((v+\epsilon)^q - (\tilde{v}+\epsilon)^q) \, dx \\ & + \lambda \int_{\{v > \tilde{v}\}} \left(\frac{f(x, v)}{(v+\epsilon)^{q-1}} - \frac{f(x, \tilde{v})}{(\tilde{v}+\epsilon)^{q-1}} \right) ((v+\epsilon)^q - (\tilde{v}+\epsilon)^q) \, dx. \end{aligned}$$

Since $\frac{\tilde{v}}{\tilde{v}+\epsilon} \leq \frac{v}{v+\epsilon} < 1$ in $\{v > \tilde{v}\}$, then we obtain

$$\begin{aligned} & \left(\frac{v^{2q-1}}{(v+\epsilon)^{q-1}} - \frac{\tilde{v}^{2q-1}}{(\tilde{v}+\epsilon)^{q-1}} \right) ((v+\epsilon)^q - (\tilde{v}+\epsilon)^q) \\ & = \left(v^q \left(\frac{v}{v+\epsilon} \right)^{q-1} - \tilde{v}^q \left(\frac{\tilde{v}}{\tilde{v}+\epsilon} \right)^{q-1} \right) ((v+\epsilon)^q - (\tilde{v}+\epsilon)^q) \\ & \leq v^q ((v+\epsilon)^q - (\tilde{v}+\epsilon)^q) \leq v^q (v+\epsilon)^q \leq v^q (v+1)^q. \end{aligned}$$

In the same fashion, we have

$$0 \leq h \left(\frac{v^{q-1}}{(v+\epsilon)^{q-1}} - \frac{\tilde{v}^{q-1}}{(\tilde{v}+\epsilon)^{q-1}} \right) ((v+\epsilon)^q - (\tilde{v}+\epsilon)^q) \leq h (v+\epsilon)^q \leq h (v+1)^q.$$

Moreover, as $\epsilon \rightarrow 0$

$$\left(\frac{v^{2q-1}}{(v+\epsilon)^{q-1}} - \frac{\tilde{v}^{2q-1}}{(\tilde{v}+\epsilon)^{q-1}} \right) ((v+\epsilon)^q - (\tilde{v}+\epsilon)^q) \rightarrow (v^q - \tilde{v}^q)^2$$

and

$$h \left(\frac{v^{q-1}}{(v+\epsilon)^{q-1}} - \frac{\tilde{v}^{q-1}}{(\tilde{v}+\epsilon)^{q-1}} \right) ((v+\epsilon)^q - (\tilde{v}+\epsilon)^q) \rightarrow 0$$

a.e. in Ω . Then by Lebesgue dominated convergence theorem we have

$$\begin{aligned} \int_{\{v>\tilde{v}\}} \left(\frac{v^{2q-1}}{(v+\epsilon)^{q-1}} - \frac{\tilde{v}^{2q-1}}{(\tilde{v}+\epsilon)^{q-1}} \right) ((v+\epsilon)^q - (\tilde{v}+\epsilon)^q) dx \\ \rightarrow \int_{\{v>\tilde{v}\}} (v^q - \tilde{v}^q)^2 dx \end{aligned}$$

and

$$\int_{\{v>\tilde{v}\}} h \left(\frac{v^{q-1}}{(v+\epsilon)^{q-1}} - \frac{\tilde{v}^{q-1}}{(\tilde{v}+\epsilon)^{q-1}} \right) ((v+\epsilon)^q - (\tilde{v}+\epsilon)^q) dx \rightarrow 0.$$

Then by using Fatou's Lemma and (f0), we have

$$\begin{aligned} - \liminf_{\epsilon \rightarrow 0} \int_{\{v>\tilde{v}\}} \frac{f(x, v)}{(v+\epsilon)^{q-1}} (\tilde{v}+\epsilon)^q dx &\leq - \int_{\{v>\tilde{v}\}} \frac{f(x, v)}{v^{q-1}} \tilde{v}^q dx, \\ - \liminf_{\epsilon \rightarrow 0} \int_{\{v>\tilde{v}\}} \frac{f(x, \tilde{v})}{(\tilde{v}+\epsilon)^{q-1}} (v+\epsilon)^q dx &\leq - \int_{\{v>\tilde{v}\}} \frac{f(x, \tilde{v})}{\tilde{v}^{q-1}} v^q dx \end{aligned}$$

and

$$\begin{aligned} \int_{\{v>\tilde{v}\}} f(x, v)(v+\epsilon) dx &\rightarrow \int_{\{v>\tilde{v}\}} f(x, v)v dx, \\ \int_{\{v>\tilde{v}\}} f(x, \tilde{v})(\tilde{v}+\epsilon) dx &\rightarrow \int_{\{v>\tilde{v}\}} f(x, \tilde{v})\tilde{v} dx. \end{aligned}$$

By Lemma 4.4.1 we have,

$$\begin{aligned} \int_{\{v>\tilde{v}\}} |\nabla(v+\epsilon)|^{p(x)-2} \nabla(v+\epsilon) \cdot \nabla \left(\frac{(v+\epsilon)^q - (\tilde{v}+\epsilon)^q}{(v+\epsilon)^{q-1}} \right) dx \\ + \int_{\{v>\tilde{v}\}} |\nabla(\tilde{v}+\epsilon)|^{p(x)-2} \nabla(\tilde{v}+\epsilon) \cdot \nabla \left(\frac{(\tilde{v}+\epsilon)^q - (v+\epsilon)^q}{(\tilde{v}+\epsilon)^{q-1}} \right) dx \\ \geq 0. \end{aligned}$$

Then by combining above estimates and taking $\limsup_{\epsilon \rightarrow 0}$ in (4.4.24), we get by (f1)

$$0 \leq \int_{\{v>\tilde{v}\}} (v^q - \tilde{v}^q)^2 dx \leq \lambda \int_{\{v>\tilde{v}\}} \left(\frac{f(x, v)}{v^{q-1}} - \frac{f(x, \tilde{v})}{\tilde{v}^{q-1}} \right) (v^q - \tilde{v}^q) dx \leq 0.$$

It implies $\tilde{v} \geq v$. □

Corollary 4.4.6. *Let $\lambda > 0$, $f : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ satisfying (f0)-(f1) and $h_0 \in L^2(\Omega)^+ \cap L^\gamma(\Omega)$ where $\gamma > \max\{1, \frac{N}{p}\}$. Then there exists a unique positive distributional solution $u \in \mathcal{D}(\mathcal{R}) \cap L^\infty(\Omega)$ of (4.4.21) in the same sense as in Corollary 4.4.5.*

Moreover if u_1 and u_2 are two positive distributional solutions of (4.4.21) for $h_1, h_2 \in L^2(\Omega)^+$ then \mathcal{R} satisfies

$$\|(u_1 - u_2)^+\|_{L^2} \leq \|(u_1 - u_2 + \lambda(\mathcal{R}u_1 - \mathcal{R}u_2))^+\|_{L^2}. \quad (4.4.25)$$

Proof. Define the functional energy \mathcal{E} on $\dot{V}_+^q \cap L^2(\Omega)$ as $\mathcal{E}(u) = \mathcal{J}(u^{1/q})$ where \mathcal{J} is given by (4.4.20).

By Theorem 4.4.8, Remark 4.4.8 and Theorem 4.4.9, v_0 is the unique positive solution of (4.4.16) and then unique global minimizer of \mathcal{J} . We take $u_0 = v_0^q$ and proceed as the proof of Corollary 4.4.5 and we obtain $u_0 = v_0^q$ is a distributional solution of (4.4.21). Finally Remark 4.4.7 gives (4.4.25). \square

4.4.5.3 Existence of a weak solution to (4.4.3)

In light of Remark 4.4.3, we consider the problem (4.4.4) and establish the existence of weak solution when $v_0 \in C_d^0(\bar{\Omega})^+ \cap W_0^{1,p(x)}(\Omega)$.

Proof of Theorem 4.4.4: Let $n^* \in \mathbb{N}^*$ and set $\Delta_t = T/n^*$. For $0 \leq n \leq n^*$, we define $t_n = n\Delta_t$.

Step 1 : Approximation of h

For $n \in \{1, 2, \dots, n^*\}$, we define for $t \in [t_{n-1}, t_n)$ and $x \in \Omega$

$$h_{\Delta_t}(t, x) = h^n(x) \stackrel{\text{def}}{=} \frac{1}{\Delta_t} \int_{t_{n-1}}^{t_n} h(s, x) ds.$$

Then by Jensen inequality,

$$\begin{aligned} \|h_{\Delta_t}\|_{L^2(Q_T)}^2 &= \Delta_t \sum_{n=1}^N \|h^n\|_{L^2(\Omega)}^2 = \Delta_t \sum_{n=1}^N \left\| \frac{1}{\Delta_t} \int_{t_{n-1}}^{t_n} h(s, x) ds \right\|_{L^2(\Omega)}^2 \\ &\leq \sum_{n=1}^N \int_{t_{n-1}}^{t_n} \|h(s, \cdot)\|_{L^2(\Omega)}^2 ds \leq \|h\|_{L^2(Q_T)}^2. \end{aligned}$$

Hence $h_{\Delta_t} \in L^2(Q_T)$ and $h^n \in L^2(\Omega)$ and let $\epsilon > 0$, then there exists a function $h_\epsilon \in C_0^1(Q_T)$ such that $\|h - h_\epsilon\|_{L^2(Q_T)} < \frac{\epsilon}{3}$.

Hence,

$$\|(h_\epsilon)_{\Delta_t} - h_{\Delta_t}\|_{L^2(Q_T)} \rightarrow 0.$$

Since $\|h_\epsilon - (h_\epsilon)_{\Delta_t}\|_{L^2(Q_T)} \rightarrow 0$ as $\Delta_t \rightarrow 0$ then for small enough Δ_t we have

$$\|h_{\Delta_t} - h\|_{L^2(Q_T)} \leq \|(h_\epsilon)_{\Delta_t} - h_{\Delta_t}\|_{L^2(Q_T)} + \|h_\epsilon - (h_\epsilon)_{\Delta_t}\|_{L^2(Q_T)} + \|h - h_\epsilon\|_{L^2(Q_T)} < \epsilon.$$

Hence $h_{\Delta_t} \rightarrow h$ in $L^2(Q_T)$.

Step 2: Time discretization of (4.4.4)

Define the following implicit Euler scheme and for $n \geq 1$, v_n is the weak solution of

$$\left\{ \begin{array}{ll} \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right) v_n^{q-1} - \Delta_{p(x)} v_n = h^n v_n^{q-1} + f(x, v_n) & \text{in } \Omega; \\ v_n > 0 & \text{in } \Omega; \\ v_n = 0 & \text{on } \partial\Omega. \end{array} \right. \quad (4.4.26)$$

Note that the sequence $(v_n)_{n=1,2,\dots,n^*}$ is well-defined. Indeed for $n = 1$ the existence and the uniqueness of $v_1 \in C^{1,\alpha}(\bar{\Omega}) \cap C_d^0(\bar{\Omega})^+$ follows from Theorems 4.4.7 and 4.4.9 with $h = \Delta_t h^1 + v_0^q \in L^\infty(\Omega)^+$. Hence by induction we obtain in the same way the existence and the uniqueness of the solution v_n for any $n = 2, 3, \dots, n^*$ where $v_n \in C^{1,\alpha}(\bar{\Omega}) \cap C_d^0(\bar{\Omega})^+$.

Step 3: Existence of a subsolution and supersolution

Now we construct a subsolution and a supersolution \underline{w} and \bar{w} of (4.4.26) such that for each $n \in \{0, 1, 2, \dots, n^*\}$, v_n satisfies $0 < \underline{w} \leq v_n \leq \bar{w}$.

Rewrite (4.4.26) as

$$v_n^{2q-1} - \Delta_t \Delta_{p(x)} v_n = (\Delta_t h^n + v_{n-1}^q) v_n^{q-1} + \Delta_t f(x, v_n). \quad (4.4.27)$$

Then following arguments in the proof of Theorems 4.4.7 and 4.4.9, from Theorem 4.4.12 and from Lemma 4.4.4, for any $\mu > 0$ there exists a unique weak solution, $w_\mu \in C^{1,\alpha}(\bar{\Omega}) \cap C_d^0(\bar{\Omega})^+$, to

$$\begin{cases} -\Delta_{p(x)} w = \mu(h_0 w^{q-1} + f(x, w)) & \text{in } \Omega; \\ w > 0 & \text{in } \Omega; \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.4.28)$$

Let $\mu_1 < \mu_2$ and w_{μ_1}, w_{μ_2} be weak solutions of (4.4.28). Then,

$$\begin{aligned} \int_{\Omega} |\nabla w_{\mu_1}|^{p(x)-2} \nabla w_{\mu_1} \cdot \nabla \phi \, dx &= \mu_1 \int_{\Omega} (h_0 w_{\mu_1}^{q-1} + f(x, w_{\mu_1})) \phi \, dx, \\ \int_{\Omega} |\nabla w_{\mu_2}|^{p(x)-2} \nabla w_{\mu_2} \cdot \nabla \psi \, dx &= \mu_2 \int_{\Omega} (h_0 w_{\mu_2}^{q-1} + f(x, w_{\mu_2})) \psi \, dx. \end{aligned}$$

Subtracting the last two equations with $\phi = \frac{(w_{\mu_1}^q - w_{\mu_2}^q)^+}{w_{\mu_1}^{q-1}}$ and $\psi = \frac{(w_{\mu_2}^q - w_{\mu_1}^q)^-}{w_{\mu_2}^{q-1}} \in W_0^{1,p(x)}(\Omega)$ we obtain, by Lemma 4.4.1 and (f1), $w_{\mu_1} \leq w_{\mu_2}$.

Then by using Theorems 4.4.12 and 4.4.13, we can choose μ small enough such that $\|w_\mu\|_{C^{1,\alpha}(\bar{\Omega})} \leq C_{\mu_0}$ for all $\mu \leq \mu_0$ and $\|w_\mu\|_{L^\infty(\Omega)} \rightarrow 0$ as $\mu \rightarrow 0$. Therefore $\{w_\mu : \mu \leq \mu_0\}$ is uniformly bounded and equicontinuous in $C^1(\bar{\Omega})$ and by Arzela Ascoli theorem $\|w_\mu\|_{C^1(\bar{\Omega})} \rightarrow 0$ as $\mu \rightarrow 0$ up to a subsequence. Then by mean value theorem we can choose μ small enough such that there exists $\underline{w} \in C^{1,\alpha}(\bar{\Omega}) \cap C_d^0(\bar{\Omega})^+$ such that $0 < \underline{w} \stackrel{\text{def}}{=} w_\mu \leq v_0$. Also \underline{w} is the subsolution of (4.4.27) for $n = 1$ *i.e.*

$$\begin{aligned} \int_{\Omega} \underline{w}^{2q-1} \phi \, dx + \Delta_t \int_{\Omega} |\nabla \underline{w}|^{p(x)-2} \nabla \underline{w} \cdot \nabla \phi \, dx &\leq \Delta_t \int_{\Omega} (h^1 \underline{w}^{q-1} + f(x, \underline{w})) \phi \, dx \\ &\quad + \int_{\Omega} v_0^q \underline{w}^{q-1} \phi \, dx \end{aligned}$$

for all $\phi \in W_0^{1,p(x)}(\Omega)$ and $\phi \geq 0$. We also recall v_1 satisfies

$$\begin{aligned} \int_{\Omega} v_1^{2q-1} \psi \, dx + \Delta_t \int_{\Omega} |\nabla v_1|^{p(x)-2} \nabla v_1 \cdot \nabla \psi \, dx &= \Delta_t \int_{\Omega} (h^1 v_1^{q-1} + f(x, v_1)) \psi \, dx \\ &\quad + \int_{\Omega} v_0^q v_1^{q-1} \psi \, dx \end{aligned}$$

for all $\psi \in W_0^{1,p(x)}(\Omega)$.

By Theorem 4.4.9, we obtain, $\underline{w} \leq v_1$ and then by induction a subsolution \underline{w} such that $0 < \underline{w} \leq v_n$ for all $n = 0, 1, 2, \dots, n^*$.

Now we construct a supersolution. For that, we consider the following problem:

$$\begin{cases} -\Delta_{p(x)} w = \|h\|_{L^\infty} w^{q-1} + f(x, w) + K & \text{in } \Omega; \\ w > 0 & \text{in } \Omega; \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.4.29)$$

As above, there exists a unique weak solution to (4.4.29), $\bar{w}_K \in C^1(\bar{\Omega}) \cap C_d^0(\bar{\Omega})^+$. Let w_K be the unique weak solution of

$$\begin{cases} -\Delta_{p(x)} w_K = K & \text{in } \Omega; \\ w_K = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.4.30)$$

From Theorem 4.4.13, $w_K \geq CK^{1/(p^+-1+\nu)} \text{dist}(x, \partial\Omega)$ where $\nu \in (0, 1)$ and $\|w_K\|_{L^\infty(\Omega)} \rightarrow \infty$ as $K \rightarrow \infty$. Then by weak comparison principle we can choose K large enough such that there exists such that $v_0 \leq w_K < \bar{w} \stackrel{\text{def}}{=} \bar{w}_K$. We easily check that \bar{w} is the supersolution of (4.4.27) for $n = 1$ i.e.

$$\begin{aligned} \int_{\Omega} \bar{w}^{2q-1} \phi \, dx + \Delta_t \int_{\Omega} |\nabla \bar{w}|^{p(x)-2} \nabla \bar{w} \cdot \nabla \phi \, dx &\geq \Delta_t \int_{\Omega} (h^1 \bar{w}^{q-1} + K + f(x, \bar{w})) \phi \, dx \\ &+ \int_{\Omega} v_0^q \bar{w}^{q-1} \phi \, dx \end{aligned}$$

for all $\phi \in W_0^{1,p(x)}(\Omega)$ and $\phi \geq 0$. From Theorem 4.4.9, we get $\bar{w} \geq v_1$ and then by induction we have $\bar{w} \geq v_n$ for all $n \in \{1, 2, \dots, n^*\}$.

Step 4: Energy estimates

Define the function for $n = 1, \dots, n^*$ and $t \in [t_{n-1}, t_n)$

$$v_{\Delta_t}(t) = v_n \quad \text{and} \quad \tilde{v}_{\Delta_t}(t) = \frac{t - t_{n-1}}{\Delta_t} (v_n^q - v_{n-1}^q) + v_{n-1}^q$$

which satisfies

$$v_{\Delta_t}^{q-1} \frac{\partial \tilde{v}_{\Delta_t}}{\partial t} - \Delta_{p(x)} v_{\Delta_t} = f(x, v_{\Delta_t}) + h^n v_{\Delta_t}^{q-1}. \quad (4.4.31)$$

Multiplying the equation (4.4.26) by $\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}}$ and summing from $n = 1$ to $n' \leq n^*$, we get

$$\begin{aligned} \sum_{n=1}^{n'} \int_{\Omega} \Delta_t \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right)^2 \, dx + \sum_{n=1}^{n'} \int_{\Omega} |\nabla v_n|^{p(x)-2} \nabla v_n \cdot \nabla \left(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}} \right) \, dx \\ = \sum_{n=1}^{n'} \int_{\Omega} h^n (v_n^q - v_{n-1}^q) \, dx + \sum_{n=1}^{n'} \int_{\Omega} \frac{f(x, v_n)}{v_n^{q-1}} (v_n^q - v_{n-1}^q) \, dx. \end{aligned}$$

Then from Young inequality we have,

$$\begin{aligned} & \sum_{n=1}^{n'} \int_{\Omega} \Delta_t \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right)^2 dx + \sum_{n=1}^{n'} \int_{\Omega} |\nabla v_n|^{p(x)-2} \nabla v_n \cdot \nabla \left(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}} \right) dx \\ & \leq \sum_{n=1}^{n'} \Delta_t \|h^n\|_{L^2}^2 + \frac{1}{4} \sum_{n=1}^{n'} \int_{\Omega} \Delta_t \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right)^2 dx \\ & \quad + \sum_{n=1}^{n'} \Delta_t \left\| \frac{f(x, v_n)}{v_n^{q-1}} \right\|_{L^2}^2 + \frac{1}{4} \sum_{n=1}^{n'} \int_{\Omega} \Delta_t \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right)^2 dx \end{aligned}$$

i.e.

$$\begin{aligned} & \frac{1}{2} \sum_{n=1}^{n'} \int_{\Omega} \Delta_t \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right)^2 dx + \sum_{n=1}^{n'} \int_{\Omega} |\nabla v_n|^{p(x)-2} \nabla v_n \cdot \nabla \left(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}} \right) dx \\ & \leq \sum_{n=1}^{n'} \Delta_t \|h^n\|_{L^2}^2 + \sum_{n=1}^{n'} \Delta_t \left\| \frac{f(x, v_n)}{v_n^{q-1}} \right\|_{L^2}^2. \end{aligned}$$

Using $\underline{w} \leq v_n \leq \bar{w}$ and (f1)-(f2), we obtain $\frac{f(x, v_n)}{v_n^{q-1}}$ in uniformly bounded in $L^2(\Omega)$. Then by **Step 1**, we obtain

$$\left(\frac{\partial \tilde{v}_{\Delta_t}}{\partial t} \right) \text{ is bounded in } L^2(Q_T) \text{ uniformly in } \Delta_t. \quad (4.4.32)$$

Now from Lemma 4.4.1, we have

$$\begin{aligned} |\nabla v_n|^{p(x)-2} \nabla v_n \cdot \nabla \left(\frac{v_{n-1}^q}{v_n^{q-1}} \right) & \leq |\nabla v_{n-1}|^q |\nabla v_n|^{p(x)-q} \\ & \leq \frac{q}{p(x)} |\nabla v_{n-1}|^{p(x)} + \frac{(p(x) - q)}{p(x)} |\nabla v_n|^{p(x)}. \end{aligned}$$

Then we obtain for any $n' \geq 1$

$$\begin{aligned} & \sum_{n=1}^{n'} \Delta_t \|h^n\|_{L^2}^2 + \sum_{n=1}^{n'} \Delta_t \left\| \frac{f(x, v_n)}{v_n^{q-1}} \right\|_{L^2}^2 \\ & \geq \sum_{n=1}^{n'} \int_{\Omega} |\nabla v_n|^{p(x)-2} \nabla v_n \cdot \nabla \left(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}} \right) dx \\ & \geq \sum_{n=1}^{n'} \left[\int_{\Omega} |\nabla v_n|^{p(x)} dx - \int_{\Omega} \frac{q}{p(x)} |\nabla v_{n-1}|^{p(x)} dx \right. \\ & \quad \left. - \int_{\Omega} \frac{(p(x) - q)}{p(x)} |\nabla v_n|^{p(x)} dx \right] \\ & \geq q \int_{\Omega} \frac{|\nabla v_{n'}|^{p(x)}}{p(x)} dx - q \int_{\Omega} \frac{|\nabla v_0|^{p(x)}}{p(x)} dx \end{aligned}$$

which implies that

$$(v_{\Delta_t}) \text{ is bounded in } L^\infty(0, T; W_0^{1, p(x)}(\Omega)) \text{ uniformly in } \Delta_t. \quad (4.4.33)$$

Since

$$\begin{aligned} \nabla(\tilde{v}_{\Delta_t}^{1/q}) &= \frac{1}{q} \zeta \nabla v_n \left(\zeta + (1 - \zeta) \left(\frac{v_{n-1}}{v_n} \right)^q \right)^{(1-q)/q} \\ &\quad + (1 - \zeta) \nabla v_{n-1} \left((1 - \zeta) + \zeta \left(\frac{v_n}{v_{n-1}} \right)^q \right)^{(1-q)/q} \end{aligned}$$

where $\zeta = \frac{t - t_{n-1}}{\Delta_t}$, then we conclude that

$$(\tilde{v}_{\Delta_t}^{1/q}) \text{ is bounded in } L^\infty(0, T; W_0^{1,p(x)}(\Omega)) \text{ uniformly in } \Delta_t. \quad (4.4.34)$$

Since $\left(\frac{v_n}{v_{n-1}} \right)$ is uniformly bounded in $L^\infty(\Omega)$, $v_{\Delta_t} \xrightarrow{*} v$ and $\tilde{v}_{\Delta_t}^{1/q} \xrightarrow{*} \tilde{v}$ in $L^\infty(0, T; W_0^{1,p(x)}(\Omega))$. Furthermore using (4.4.32), we have

$$\sup_{t \in (0, T)} \|\tilde{v}_{\Delta_t}^{1/q} - v_{\Delta_t}\|_{L^{2q}(\Omega)}^{2q} \leq \sup_{t \in (0, T)} \|\tilde{v}_{\Delta_t} - v_{\Delta_t}^q\|_{L^2(\Omega)}^2 \leq \Delta_t \rightarrow 0 \text{ as } \Delta_t \rightarrow 0. \quad (4.4.35)$$

It follows from (4.4.35) that $v = \tilde{v}$. By mean value theorem and (4.4.32), we get that $(\tilde{v}_{\Delta_t})_{\Delta_t}$ is equicontinuous in $C(0, T; L^r(\Omega))$ for $1 < r \leq 2$. Thus using $\underline{w}^q \leq \tilde{v}_{\Delta_t} \leq \bar{w}^q$ together with the interpolation inequality $\|\cdot\|_r \leq \|\cdot\|_\infty^\alpha \|\cdot\|_2^{1-\alpha}$, with $\frac{1}{r} = \frac{\alpha}{\infty} + \frac{1-\alpha}{2}$, we obtain that $(\tilde{v}_{\Delta_t})_{\Delta_t}$ and $(\tilde{v}_{\Delta_t}^{1/q})_{\Delta_t}$ is equicontinuous in $C(0, T; L^r(\Omega))$ for any $1 < r < +\infty$. Again using interpolation inequality and Sobolev embedding, we get as $\Delta_t \rightarrow 0^+$ and up to a subsequence that for all $r > 1$

$$\tilde{v}_{\Delta_t} \rightarrow v^q \text{ in } C(0, T; L^r(\Omega)), \quad (4.4.36)$$

and

$$v_{\Delta_t} \rightarrow v \text{ in } L^\infty(0, T; L^r(\Omega)). \quad (4.4.37)$$

From (4.4.32) and (4.4.36), we obtain

$$\frac{\partial \tilde{v}_{\Delta_t}}{\partial t} \rightarrow \frac{\partial v^q}{\partial t} \text{ in } L^2(Q_T). \quad (4.4.38)$$

Step 5 : v satisfies (4.4.5)

Multiplying (4.4.31) by $(v_{\Delta_t} - v)$ and integrating by parts, we get

$$\begin{aligned} &\int_0^T \int_\Omega v_{\Delta_t}^{q-1} \frac{\partial \tilde{v}_{\Delta_t}}{\partial t} (v_{\Delta_t} - v) \, dxdt + \int_0^T \int_\Omega |\nabla v_{\Delta_t}|^{p(x)-2} \nabla v_{\Delta_t} \cdot \nabla (v_{\Delta_t} - v) \, dxdt \\ &= \int_0^T \int_\Omega f(x, v_{\Delta_t}) (v_{\Delta_t} - v) \, dxdt + \int_0^T \int_\Omega h^n v_{\Delta_t}^{q-1} (v_{\Delta_t} - v) \, dxdt. \end{aligned}$$

From (4.4.37) and (4.4.38), we have

$$\left| \int_0^T \int_\Omega v_{\Delta_t}^{q-1} \frac{\partial \tilde{v}_{\Delta_t}}{\partial t} (v_{\Delta_t} - v) \, dxdt \right| + \left| \int_0^T \int_\Omega h^n v_{\Delta_t}^{q-1} (v_{\Delta_t} - v) \, dxdt \right| = o_{\Delta_t}(1)$$

and from (4.4.33), (4.4.34), (4.4.37) and Lebesgue Dominated convergence theorem,

$$\int_0^T \int_{\Omega} f(x, v_{\Delta_t})(v_{\Delta_t} - v) dx = o_{\Delta_t}(1).$$

Then we obtain

$$\int_0^T \int_{\Omega} |\nabla v_{\Delta_t}|^{p(x)-2} \nabla v_{\Delta_t} \cdot \nabla (v_{\Delta_t} - v) dx \rightarrow 0 \text{ as } \Delta_t \rightarrow 0^+.$$

Then from [Step 4, Proof of Theorem 1.1, [146]] and from classical compactness argument we get

$$|\nabla v_{\Delta_t}|^{p(x)-2} \nabla v_{\Delta_t} \rightarrow |\nabla v|^{p(x)-2} \nabla v \text{ in } (L^{p(x)/(p(x)-1)}(Q_T))^N. \quad (4.4.39)$$

From (4.4.35) and (4.4.36) we have,

$$\begin{aligned} \|v_{\Delta_t}^{q-1} - v^{q-1}\|_{L^2(Q_T)} &\leq \|v_{\Delta_t}^{q-1} - v^{q-1}\|_{L^\infty(0,T;L^2)} \\ &\leq \|v_{\Delta_t}^{q-1} - v^{q-1}\|_{L^\infty(0,T;L^{\frac{2q}{q-1}})} \\ &\leq \|v_{\Delta_t}^q - v^q\|_{L^\infty(0,T;L^2)} \\ &\leq \|v_{\Delta_t}^q - \tilde{v}_{\Delta_t}\|_{L^\infty(0,T;L^2)} + \|\tilde{v}_{\Delta_t} - v^q\|_{L^\infty(0,T;L^2)} \rightarrow 0 \end{aligned} \quad (4.4.40)$$

as $\Delta_t \rightarrow 0$. By Hölder inequality we have for $\phi \in C_c^\infty(Q_T)$

$$\begin{aligned} &\int_0^T \int_{\Omega} \left(v_{\Delta_t}^{q-1} \frac{\partial \tilde{v}_{\Delta_t}}{\partial t} - \frac{\partial v^q}{\partial t} v^{q-1} \right) \phi dx \\ &= \int_0^T \int_{\Omega} v_{\Delta_t}^{q-1} \left(\frac{\partial \tilde{v}_{\Delta_t}}{\partial t} - \frac{\partial v^q}{\partial t} \right) \phi dx + \int_0^T \int_{\Omega} \frac{\partial v^q}{\partial t} (v_{\Delta_t}^{q-1} - v^{q-1}) \phi dx \\ &\leq \|v_{\Delta_t}^{q-1} \phi\|_{L^2(Q_T)} \left\| \left(\frac{\partial \tilde{v}_{\Delta_t}}{\partial t} - \frac{\partial v^q}{\partial t} \right) \right\|_{L^2(Q_T)} + \|v_{\Delta_t}^{q-1} - v^{q-1}\|_{L^2(Q_T)} \left\| \phi \frac{\partial \tilde{v}_{\Delta_t}}{\partial t} \right\|_{L^2(Q_T)} \end{aligned}$$

and

$$\begin{aligned} &\int_0^T \int_{\Omega} (h^n v_{\Delta_t}^{q-1} - h v^{q-1}) \phi dx \\ &= \int_0^T \int_{\Omega} h^n (v_{\Delta_t}^{q-1} - v^{q-1}) \phi dx + \int_0^T \int_{\Omega} (h^n - h) v^{q-1} \phi dx \\ &\leq \|h^n \phi\|_{L^2(Q_T)} \|v_{\Delta_t}^{q-1} - v^{q-1}\|_{L^2(Q_T)} + \|v^{q-1} \phi\|_{L^2(Q_T)} \|h^n - h\|_{L^2(Q_T)}. \end{aligned}$$

Then from (4.4.32), (4.4.37), (4.4.38), (4.4.40) and **Step 1** we obtain

$$\begin{aligned} &\int_0^T \int_{\Omega} \left(v_{\Delta_t}^{q-1} \frac{\partial \tilde{v}_{\Delta_t}}{\partial t} - \frac{\partial v^q}{\partial t} v^{q-1} \right) \phi dx \rightarrow 0, \\ &\int_0^T \int_{\Omega} (h^n v_{\Delta_t}^{q-1} - h v^{q-1}) \phi dx \rightarrow 0 \text{ as } \Delta_t \rightarrow 0. \end{aligned} \quad (4.4.41)$$

From (4.4.37) we have $f(x, v_{\Delta_t}) \rightarrow f(x, v)$ pointwise and from (4.4.33) together with (4.4.34) we have $\int_{\Omega} f(x, v_{\Delta_t}) \phi \, dx$ is bounded uniformly in Δ_t . Then by Lebesgue dominated convergence theorem we have

$$\int_0^T \int_{\Omega} (f(x, v_{\Delta_t}) - f(x, v)) \phi \, dx \rightarrow 0 \quad \text{as } \Delta_t \rightarrow 0. \quad (4.4.42)$$

Then finally gathering (4.4.39), (4.4.41) and (4.4.42), we conclude by passing to the limits in equation (4.4.31) that v is weak solution of (4.4.4). \square

Remark 4.4.9. *All the results in Section 4.4.5.1, Section 4.4.2 and Theorem 4.4.4 hold if we replace the assumption (f1) by $h \geq c > 0$.*

Proof of Theorem 4.4.5: For a given function g , let $\|g\|_{2+} \stackrel{\text{def}}{=} \|[g]^+\|_{L^2(\Omega)}$. For $z \in \mathcal{D}(\mathcal{R})$ and $r, k \in L^\infty(Q_T)^+$ satisfying assumptions in Theorem 4.4.5, set

$$\phi(t, s) = \|r(t) - k(s)\|_{2+} \quad \forall (t, s) \in [0, T] \times [0, T],$$

for $t \in [-T, T]$

$$b(t, r, k) = \|u_0^q - z\|_{2+} + \|v_0^q - z\|_{2+} + |t| \|\mathcal{R}z\|_{2+} + \int_0^{t^+} \|r(\tau)\|_{2+} d\tau + \int_0^{t^-} \|k(\tau)\|_{2+} d\tau$$

and

$$\psi(t, s) = b(t - s, r, k) + \begin{cases} \int_0^s \phi(t - s + \tau, \tau) d\tau & \text{if } 0 \leq s \leq t \leq T \\ \int_0^t \phi(\tau, s - t + \tau) d\tau & \text{if } 0 \leq t \leq s \leq T \end{cases}$$

is a solution of

$$\begin{cases} \frac{\partial \psi}{\partial t}(t, s) + \frac{\partial \psi}{\partial s}(t, s) = \phi(t, s) & \text{on } (t, s) \in [0, T] \times [0, T]; \\ \psi(t, 0) = b(t, r, k) & \text{on } t \in [0, T]; \\ \psi(0, s) = b(-s, r, k) & \text{on } s \in [0, T]. \end{cases} \quad (4.4.43)$$

Define the following iterative scheme, $u^0 = u_0^q$ and for $n \geq 1$, u^n is the solution of

$$\begin{cases} \frac{u^n - u^{n-1}}{\Delta_t} + \mathcal{R}u^n = h^n & \text{in } \Omega; \\ u^n = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.4.44)$$

Note that the sequence $\{u^n\}_{n=1,2,\dots,N}$ is well defined. Indeed for $n = 1$ the existence and the uniqueness of $u^1 \in \mathcal{D}(\mathcal{R})$ follows from Corollary 4.4.5 with $h = \Delta_t h^1 + u^0 \in L^\infty(\Omega)^+$ and $\lambda = \Delta_t$. Hence by induction we obtain in the same way the existence of the solution u^n for any $n = 2, 3, \dots, N$ where $u^n \in \mathcal{D}(\mathcal{R})$.

Moreover let denote by (u_ϵ^n) the solution of (4.4.44) with $\Delta_t = \epsilon$, $h = r$, $r^n = \frac{1}{\epsilon} \int_{(n-1)\epsilon}^{n\epsilon} r(\tau, \cdot) d\tau$

and (u_η^m) the solution of (4.4.44) with $\Delta_t = \eta, h = k, k^m = \frac{1}{\eta} \int_{(m-1)\eta}^{m\eta} k(\tau, \cdot) d\tau$ respectively i.e we have

$$\begin{cases} \frac{u_\epsilon^n - u_\epsilon^{n-1}}{\epsilon} + \mathcal{R}u_\epsilon^n = r^n; \\ \frac{u_\eta^m - u_\eta^{m-1}}{\eta} + \mathcal{R}u_\eta^m = k^m. \end{cases} \quad (4.4.45)$$

For $(n, m) \in \mathbb{N}^*$, multiplying the equation in (4.4.45) by $\frac{\epsilon\eta}{\epsilon + \eta}$ and then subtracting the two expressions we get,

$$\frac{\eta}{\eta + \epsilon}(u_\epsilon^n - u_\epsilon^{n-1}) + \frac{\eta\epsilon}{\eta + \epsilon}(\mathcal{R}u_\epsilon^n - \mathcal{R}u_\eta^m) - \frac{\epsilon}{\eta + \epsilon}(u_\eta^m - u_\eta^{m-1}) = \frac{\eta\epsilon}{\eta + \epsilon}(r^n - k^m).$$

Then we infer that

$$\begin{aligned} u_\epsilon^n - u_\eta^m + \frac{\epsilon\eta}{\epsilon + \eta}(\mathcal{R}u_\epsilon^n - \mathcal{R}u_\eta^m) &= \frac{\epsilon\eta}{\epsilon + \eta}(r^n - k^m) + \frac{\eta}{\epsilon + \eta}(u_\epsilon^{n-1} - u_\eta^m) \\ &\quad + \frac{\epsilon}{\epsilon + \eta}(u_\epsilon^n - u_\eta^{m-1}). \end{aligned}$$

Let $\Phi_{n,m}^{\epsilon,\eta} = \|u_\epsilon^n - u_\eta^m\|_{2+}$ and since \mathcal{R} satisfies (4.4.25) and setting $\lambda = \frac{\epsilon\eta}{\epsilon + \eta}$, we get

$$\begin{aligned} \Phi_{n,m}^{\epsilon,\eta} &= \|u_\epsilon^n - u_\eta^m\|_{2+} \leq \|u_\epsilon^n - u_\eta^m + \frac{\epsilon\eta}{\epsilon + \eta}(\mathcal{R}u_\epsilon^n - \mathcal{R}u_\eta^m)\|_{2+} \\ &\leq \frac{\epsilon\eta}{\epsilon + \eta} \|r^n - k^m\|_{2+} + \frac{\eta}{\epsilon + \eta} \|u_\epsilon^{n-1} - u_\eta^m\|_{2+} + \frac{\epsilon}{\epsilon + \eta} \|u_\epsilon^n - u_\eta^{m-1}\|_{2+}. \end{aligned}$$

Then by elementary calculations, we get

$$\Phi_{n,0}^{\epsilon,\eta} = \|u_\epsilon^n - u_\eta\|_{2+} \leq b(t_n, r_\epsilon, k_\eta)$$

and

$$\Phi_{0,m}^{\epsilon,\eta} \leq b(-s_m, r_\epsilon, k_\eta).$$

Then by using above computations we get , $\Phi_{n,m}^{\epsilon,\eta} \leq \psi_{n,m}^{\epsilon,\eta}$ where $\psi_{n,m}^{\epsilon,\eta}$ satisfies

$$\psi_{n,m}^{\epsilon,\eta} = \frac{\epsilon\eta}{\epsilon + \eta} \|(r^n - k^m)\|_{2+} + \frac{\eta}{\epsilon + \eta} \|\psi_{n-1,m}^{\epsilon,\eta}\|_{2+} + \frac{\epsilon}{\epsilon + \eta} \|\psi_{n,m-1}^{\epsilon,\eta}\|_{2+}$$

and $\psi_{n,0}^{\epsilon,\eta} = b(t_n, r_\epsilon, k_\eta)$ and $\psi_{0,m}^{\epsilon,\eta} = b(-s_m, r_\epsilon, k_\eta)$.

For $(t, s) \in (t_{n-1}, t_n) \times (s_{m-1}, s_m)$, set $\phi^{\epsilon,\eta}(t, s) = \|r_\epsilon(t) - k_\eta(s)\|_{2+}$,

$$\psi^{\epsilon,\eta} = \psi_{n,m}^{\epsilon,\eta}, \quad b_{\epsilon,\eta}(t, r, k) = b(t_n, r_\epsilon, k_\eta), \quad b_{\epsilon,\eta}(-s, r, k) = b(-s_m, r_\epsilon, k_\eta).$$

Then by elementary calculations $\psi^{\epsilon,\eta}$ satisfies the following discrete version of (4.4.43),

$$\begin{cases} \frac{\psi^{\epsilon,\eta}(t, s) - \psi^{\epsilon,\eta}(t - \epsilon, s)}{\epsilon} + \frac{\psi^{\epsilon,\eta}(t, s) - \psi^{\epsilon,\eta}(t, s - \eta)}{\eta} = \phi^{\epsilon,\eta}(t, s); \\ \psi^{\epsilon,\eta}(t, 0) = b_{\epsilon,\eta}(t, r, k); \\ \psi^{\epsilon,\eta}(0, s) = b_{\epsilon,\eta}(s, r, k). \end{cases}$$

Since $r_\epsilon \rightarrow r$ in $L^2(Q_T)$ then $b_{\epsilon,\eta}(\cdot, r, k) \rightarrow b(\cdot, r, k)$ in $L^\infty([0, T])$ and $\phi^{\epsilon,\eta} \rightarrow \phi$ in $L^\infty([0, T] \times [0, T])$ and we deduce that $\rho_{\epsilon,\eta} = \|\psi^{\epsilon,\eta} - \psi\|_{L^\infty([0, T] \times [0, T])} \rightarrow 0$ (for more details see for instance [[43], Chapter 4, Lemma 4.3, page 136] and [[43], Chapter 4, Proof of Theorem 4.1, page 138]). Therefore,

$$\|u_\epsilon(t) - u_\eta(s)\|_{2^+} = \Phi^{\epsilon,\eta}(t, s) \leq \psi^{\epsilon,\eta}(t, s) \leq \psi(t, s) + \rho_{\epsilon,\eta}.$$

Since $u_\epsilon(t) = v_\epsilon^q(t)$ and $u_\eta(t) = v_\eta^q(t)$, we obtain

$$\|v_\epsilon^q(t) - v_\eta^q(s)\|_{2^+} = \Phi^{\epsilon,\eta}(t, s) \leq \psi^{\epsilon,\eta}(t, s) \leq \psi(t, s) + \rho_{\epsilon,\eta}. \quad (4.4.46)$$

From Theorem 4.4.4, v_ϵ^q and v_η^q satisfies $0 < \underline{w} < v_\epsilon, v_\eta < \bar{w}$ where \underline{w}, \bar{w} are subsolution and supersolution defined in (4.4.28) and (4.4.29) and $v_\epsilon^q \rightarrow v_1^q$ and $v_\eta^q \rightarrow v_2^q$ *a.e.* in Ω where v_1 and v_2 are weak solutions of (4.4.4) with initial data u_0, v_0 respectively. Since $v_\epsilon^q \rightarrow v_1^q$ and $v_\eta^q \rightarrow v_2^q$ in $L^\infty(0, T; L^2(\Omega))$ and passing to the limit in (4.4.46) as $\epsilon, \eta \rightarrow 0$ with $t = s$ we get

$$\begin{aligned} \|v_1^q(t) - v_2^q(t)\|_{2^+} &\leq \|v_1^q(t) - v_\epsilon^q(t)\|_{2^+} + \|v_\eta^q(t) - v_2^q(t)\|_{2^+} + \|v_\epsilon^q(t) - v_\eta^q(t)\|_{2^+} \\ &\leq \|u_0^q - z\|_{2^+} + \|v_0^q - z\|_{2^+} + \int_0^t \|r(\gamma) - k(\gamma)\|_{2^+} d\gamma. \end{aligned}$$

Then (4.4.6) follows since we can choose z arbitrary close to v_0^q and with $r = h, k = g$. \square

4.4.6 An application to nonhomogeneous operators

In this section, we prove Theorem 4.4.6. To this aim, we first study the properties of a related energy functional. Let $m \geq 1$ and $K : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^+$ be a continuous differentiable function which satisfies the following conditions:

(k1) $K \in C^1(\Omega \times \mathbb{R}^N) \cap C^2(\Omega \times \mathbb{R}^N \setminus \{0\})$.

(k2) Ellipticity condition: $\exists k_1 \geq 0$ and $\gamma \in (0, \infty)$ such that

$$\sum_{i,j=1}^N \frac{\partial^2 K}{\partial \xi_i \partial \xi_j}(x, \xi) \eta_i \eta_j \geq \gamma(k_1 + |\xi|)^{m-2} |\eta|^2.$$

(k3) Growth condition: $\exists k_2 \geq 0$ and $\Gamma \in (0, \infty)$ such that

$$\sum_{i,j=1}^N \left| \frac{\partial^2 K}{\partial \xi_i \partial \xi_j}(x, \xi) \right| \leq \Gamma(k_2 + |\xi|)^{m-2}$$

for all $\xi \in \mathbb{R}^N \setminus \{0\}$ and $\eta \in \mathbb{R}^N$.

Remark 4.4.10. From the assumption **(k2)**, it follows that K is strictly convex and from **(k1)**-**(k3)** there exists some positive constant γ_1 and γ_2 with $0 < \gamma_1 \leq \gamma_2 < +\infty$ and some nonnegative constants Γ_1 and Γ_2 such that

$$\gamma_1|\xi|^m - \Gamma_1 \leq K(x, \xi) \leq \gamma_2|\xi|^m + \Gamma_2$$

for $x \in \Omega$ and $\xi \in \mathbb{R}^N \setminus \{0\}$.

Consider the associated functional \mathcal{J}_m defined by

$$\mathcal{J}_m(u) \stackrel{\text{def}}{=} \int_{\Omega} \frac{|u|^{p(x)}}{p(x)} K\left(x, \frac{\nabla u}{u}\right)^{\frac{p(x)}{m}} dx.$$

for any positive function $u \in W_0^{1,p(x)}(\Omega)$. Now we extend Lemma 2.4 in [246] as follows:

Theorem 4.4.10. Let $K : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^+$ satisfying **(k1)**-**(k3)** for some $m \in [1, p^-]$. Then, the function $\mathcal{E} : \dot{V}_+^m \cap L^\infty(\Omega) \rightarrow \mathbb{R}^+$, defined by $\mathcal{E}(u) \stackrel{\text{def}}{=} \mathcal{J}_m(u^{1/m})$, is ray-strictly convex (even strictly convex if $p(\cdot) \not\equiv m$).

Proof. We observe that for $u \in \dot{V}_+^m \cap L^\infty(\Omega)$

$$\mathcal{E}(u) = \int_{\Omega} \frac{1}{p(x)} \left(u K\left(x, \frac{\nabla u}{mu}\right) \right)^{\frac{p(x)}{m}} dx.$$

Therefore, since for $1 \leq m \leq p^-$, $t \rightarrow t^{p(x)/m}$ is convex in \mathbb{R}^+ (even strictly convex if $p(x) > m$) it is enough to prove that

$$\dot{V}_+^m \ni u \rightarrow u K\left(x, \frac{\nabla u}{mu}\right)$$

is ray-strictly convex. To achieve this goal, let $\theta \in (0, 1)$ and $u_1, u_2 \in \dot{V}_+^m$ then by using the strict convexity of K we obtain, for $x \in \Omega$

$$\begin{aligned} & ((1-\theta)u_1 + \theta u_2) K\left(x, \frac{(1-\theta)\nabla u_1 + \theta\nabla u_2}{m((1-\theta)u_1 + \theta u_2)}\right) \\ &= ((1-\theta)u_1 + \theta u_2) K\left(x, \frac{(1-\theta)u_1}{((1-\theta)u_1 + \theta u_2)} \frac{\nabla u_1}{mu_1} + \frac{\theta u_2}{((1-\theta)u_1 + \theta u_2)} \frac{\nabla u_2}{mu_2}\right) \\ &\leq ((1-\theta)u_1 + \theta u_2) \left(\frac{(1-\theta)u_1}{((1-\theta)u_1 + \theta u_2)} K\left(x, \frac{\nabla u_1}{mu_1}\right) \right. \\ &\quad \left. + \frac{\theta u_2}{((1-\theta)u_1 + \theta u_2)} K\left(x, \frac{\nabla u_2}{mu_2}\right) \right) \\ &= (1-\theta)u_1 K\left(x, \frac{\nabla u_1}{mu_1}\right) + \theta u_2 K\left(x, \frac{\nabla u_2}{mu_2}\right). \end{aligned}$$

The above inequality is always strict unless $\frac{\nabla u_1}{u_1} = \frac{\nabla u_2}{u_2}$, i.e. $u_1/u_2 \equiv \text{Const}$. \square

Proof of Theorem 4.4.6: Consider the functional $\mathcal{J}_\epsilon : W_0^{1,p(x)}(\Omega) \rightarrow \mathbb{R}$, defined by

$$\mathcal{J}_\epsilon(u) = \int_{\Omega} \frac{(|\nabla u|^2 + \epsilon u^2)^{p(x)/2}}{p(x)} dx - \int_{\Omega} G(x, u) dx$$

where the potential $G(x, t)$ defined as

$$G(x, t) = \begin{cases} \int_0^t g(x, s) ds & \text{if } 0 \leq t < \infty; \\ 0 & \text{if } -\infty < t < 0. \end{cases}$$

Assumptions (f_0) , (\tilde{g}) and Remark 4.4.10 ensure that \mathcal{J}_ϵ is well defined, coercive and continuous. Then there exists at least one global minimizer of \mathcal{J}_ϵ on $W_0^{1,p(x)}(\Omega)$, say u_0 . We can easily prove that u_0 is nonnegative and nontrivial.

Since \mathcal{J}_ϵ is differentiable, we deduce that u_0 is a weak solution of (4.4.7). Now from Theorems 4.4.11 and 4.4.12 in Appendix A, we obtain that any weak solution u to (4.4.7) belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$ and $u > 0$ in Ω and $\frac{\partial u}{\partial \vec{n}} < 0$ on $\partial\Omega$. Therefore any weak solution belongs to $C_d^0(\overline{\Omega})^+$.

Now we prove that u_0 is the unique weak solution to (4.4.7). Let $W : \dot{V}_+^m \rightarrow \mathbb{R}$ defined by

$$W(u) = \mathcal{J}_\epsilon(u^{1/m}) = \int_{\Omega} \frac{(|\nabla(u^{1/m})|^2 + \epsilon(u^{1/m})^2)^{p(x)/2}}{p(x)} dx - \int_{\Omega} G(x, u^{1/m}) dx.$$

The assumption (\tilde{g}) together with Theorem 4.4.10 with $K(x, \xi) = (\epsilon + |\xi|^2)^{m/2}$ imply that W is strictly convex.

Let u_1 a weak solution to (4.4.7). Then setting $v_0 \stackrel{\text{def}}{=} u_0^m$, $v_1 \stackrel{\text{def}}{=} u_1^m \in \dot{V}_+^m$ and $t \in [0, 1]$, we define $\xi(t) \stackrel{\text{def}}{=} J_\epsilon(((1-t)v_0 + tv_1)^{1/m})$. Since u_0 and u_1 belong to $C_d^0(\overline{\Omega})^+$, ξ is differentiable in $[0, 1]$. From the convexity of \mathcal{E} , we have for any $t \in [0, 1]$

$$\xi'(0) \leq \xi'(t) \leq \xi'(1). \quad (4.4.47)$$

Since u_0 and u_1 are weak solutions to (4.4.7), $\xi'(0) = \xi'(1) = 0$ and from (4.4.47) we get that ξ is constant which contradicts the strict convexity of \mathcal{E} unless $u_0 \equiv u_1$. \square

4.4.7 Additional results

In this section, we recall the following regularity of weak solutions of quasilinear elliptic differential equation

$$\begin{cases} \operatorname{div} A(x, u, Du) + B(x, u, Du) = 0 & \text{on } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.4.48)$$

Now we recall the boundedness and $C^{0,\alpha}(\overline{\Omega})$ regularity results of weak solutions of (4.4.48) satisfying the following growth conditions:

$$\begin{aligned} A(x, u, z)z &\geq a_0|z|^{p(x)} - b|u|^{r(x)} - c; \\ |A(x, u, z)| &\leq a_1|z|^{p(x)-1} + b|u|^{\sigma(x)} + c; \\ |B(x, u, z)| &\leq a_2|z|^{\alpha(x)} + b|u|^{r(x)-1} + c \end{aligned} \quad (4.4.49)$$

where a_0, a_1, a_2, b, c are positive constants and p^* is the Sobolev embedding exponent of p and

$$p(x) \leq r(x) < p^*(x), \quad \sigma(x) = \frac{p(x) - 1}{p(x)} r(x) \quad \text{and} \quad \alpha(x) = \frac{r(x) - 1}{r(x)} p(x). \quad (4.4.50)$$

Theorem 4.4.11. ([120], Theorem 4.1 and Theorem 4.4) *Let (4.4.49)-(4.4.50) hold and $p \in \mathcal{P}^{log}(\Omega)$. If $u \in W^{1,p(x)}(\Omega)$ is a weak solution of (4.4.48), then $u \in C^{0,\alpha}(\overline{\Omega})$.*

Theorem 4.4.12 below ensures $C^{1,\alpha}(\overline{\Omega})$ regularity to weak solutions of (4.4.48) under the additional assumptions on p , A and B :

Assumptions (\mathbf{A}_k): $A = (A_1, A_2, \dots, A_n) \in C(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N, \mathbb{R}^N)$. For every $(x, u) \in \overline{\Omega} \times \mathbb{R}$, $A(x, u, \cdot) \in C^1(\mathbb{R}^N \setminus \{0\}, \mathbb{R}^N)$, there exist a nonnegative constants $k_1, k_2, k_3 \geq 0$, a nonincreasing continuous function $\lambda : [0, \infty) \rightarrow (0, \infty)$ and a nondecreasing continuous function $\Lambda : [0, \infty) \rightarrow (0, \infty)$ such that for all $x, x_1, x_2 \in \overline{\Omega}$, $u, u_1, u_2 \in \mathbb{R}, \eta \in \mathbb{R}^N \setminus \{0\}$ and $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^N$, the following conditions are satisfied

$$\begin{aligned} A(x, u, 0) &= 0, \\ \sum_{i,j} \frac{\partial A_j(x, u, \eta)}{\partial \eta_i} (x, u, \eta) \xi_i \xi_j &\geq \lambda(|u|) (k_1 + |\eta|^2)^{(p(x)-2)/2} |\xi|^2, \\ \sum_{i,j} \left| \frac{\partial A_j(x, u, \eta)}{\partial \eta_i} (x, u, \eta) \right| &\leq \Lambda(|u|) (k_2 + |\eta|^2)^{(p(x)-2)/2} \quad \text{and} \\ |A(x_1, u_1, \eta) - A(x_2, u_2, \eta)| &\leq \Lambda(\max\{|u_1|, |u_2|\}) (|x_1 - x_2|^{\beta_1} + |u_1 - u_2|^{\beta_2}) \\ &\quad \times [(k + |\eta|^2)^{(p(x_1)-2)/2} + (k + |\eta|^2)^{(p(x_2)-2)/2}] |\eta| (1 + |\log(k_3 + |\eta|^2)|). \end{aligned}$$

Assumption (\mathbf{B}): $B : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$, the function $B(x, u, \eta)$ is measurable in x and is continuous in (u, η) , and

$$|B(x, u, \eta)| \leq \Lambda(|u|) (1 + |\eta|^{p(x)}), \quad \forall (x, u, \eta) \in \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N.$$

Theorem 4.4.12. ([118], Theorem 1.2) *Let assumptions (\mathbf{A}_k), (\mathbf{B}) hold. Assume p belongs to $C^{0,\beta}(\overline{\Omega})$, for some $\beta \in (0, 1)$. Suppose that Ω satisfy ($\mathbf{\Omega}$). If $u \in W_0^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ is a weak solution of (4.4.48), then $u \in C^{1,\alpha}(\overline{\Omega})$ where $\alpha \in (0, 1)$ and $\|u\|_{C^{1,\alpha}(\overline{\Omega})}$ depends upon $p^-, p^+, \lambda(M), \Lambda(M), \beta_1, \beta_2, M, \Omega$ where $M \stackrel{\text{def}}{=} \|u\|_{L^\infty(\Omega)}$.*

In the next theorem, we recall some results contained in Lemma 2.1 of [117] and Lemma 3.2 of [146]. Set $\varrho = \frac{p^-}{2|\Omega|^{1/N} C_0}$ where C_0 is the best embedding constant of $W_0^{1,1}(\Omega) \subset L^{\frac{N}{N-1}}(\Omega)$.

Theorem 4.4.13. *Let $K > 0$ and $w_K \in W_0^{1,p(x)}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ be the weak solution of (4.4.30).*

Then for $K \geq \varrho$, $\|w_K\|_{L^\infty(\Omega)} \leq C_1 K^{1/(p^- - 1)}$, $w_K(x) \geq C_2 K^{1/(p^+ - 1 + \varsigma)} \text{dist}(x, \partial\Omega)$ where $\varsigma \in (0, 1)$ and for $K < \varrho$, $\|w_K\|_{L^\infty(\Omega)} \leq C_3 K^{1/(p^+ - 1)}$ where C_1, C_2 and C_3 depends upon p^+, p^-, N, Ω . Moreover if $K_1 < K_2$ then $w_{K_1} \leq w_{K_2}$.

Next we prove a slight extension of Proposition A.2 in [146].

Proposition 4.4.3. *Let $p \in C(\bar{\Omega})$ and $q \in (1, p^-]$. Assume $u \in \mathbf{W}$ satisfying for any $\Psi \in \mathbf{W}$:*

$$\int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \Psi \, dx = \int_{\Omega} h u^{q-1} \Psi \, dx \quad (4.4.51)$$

where $h \in L^2(\Omega) \cap L^r(\Omega)$ with $r > \max\{1, \frac{N}{p^-}\}$. Then $u \in L^\infty(\Omega)$.

First we prove a regularity lemma.

Lemma 4.4.5. *Let $u \in W_0^{1,p}(\Omega)$ satisfying for any B_R , $R < R_0$, and for all $\sigma \in (0, 1)$, and any $k \geq k_0 > 0$*

$$\int_{A_{k,\sigma R}} |\nabla u|^p \, dx \leq C \left[\int_{A_{k,R}} \left| \frac{u-k}{R(1-\sigma)} \right|^{p^*} dx + k^\alpha |A_{k,R}| + |A_{k,R}|^{\frac{p}{p^*}+\varepsilon} + k^\beta |A_{k,R}|^{\frac{p}{p^*}+\varepsilon} + \left(\int_{A_{k,R}} \left| \frac{u-k}{R(1-\sigma)} \right|^{p^*} dx \right)^{\frac{p}{p^*}} |A_{k,R}|^\delta \right]$$

where $A_{k,R} = \{x \in B_R \cap \Omega \mid u(x) > k\}$, $0 < \alpha < p^* = \frac{Np}{N-p}$, $\beta \in (1, p]$ and $\varepsilon, \delta > 0$. Then $u \in L^\infty(\Omega)$.

Proof. A similar result exists in [131] or in [147] without the term $k^\beta |A_{k,R}|^{\frac{p}{p^*}+\varepsilon}$. For the reader's convenience, we include the complete proof.

Let $x_0 \in \bar{\Omega}$, B_R the ball centred in x_0 . We define $K_R \stackrel{\text{def}}{=} B_R \cap \Omega$ and we set

$$r_j = \frac{R}{2} + \frac{R}{2^{j+1}}, \quad \tilde{r}_j = \frac{r_j + r_{j+1}}{2} \quad \text{and} \quad k_j = k \left(1 - \frac{1}{2^{j+1}} \right) \quad \text{for any } j \in \mathbb{N}.$$

Define also

$$I_j = \int_{A_{k_j, r_j}} |u(x) - k_j|^{p^*} \, dx \quad \text{and} \quad \varphi(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 0 & \text{if } t \geq \frac{3}{4} \end{cases}$$

satisfying $\varphi \in C^1([0, +\infty))$ and $0 \leq \varphi \leq 1$. We set $\varphi_j(x) = \varphi\left(\frac{2^{j+1}}{R}(|x| - \frac{R}{2})\right)$. Hence $\varphi_j = 1$ on $B_{r_{j+1}}$ and $\varphi_j = 0$ on $\mathbb{R}^N \setminus B_{\tilde{r}_{j+1}}$.

We have

$$\begin{aligned} I_{j+1} &= \int_{A_{k_{j+1}, r_{j+1}}} |u(x) - k_{j+1}|^{p^*} \, dx = \int_{A_{k_{j+1}, r_{j+1}}} |u(x) - k_{j+1}|^{p^*} \varphi_j(x)^{p^*} \, dx \\ &\leq \int_{K_R} (u(x) - k_{j+1})^+ \varphi_j(x)^{p^*} \, dx. \end{aligned}$$

Since $u \in W_0^{1,p}(\Omega)$, $(u - k_{j+1})^+ \varphi_j \in W_0^{1,p}(K_R)$,

$$\begin{aligned} I_{j+1} &\lesssim \left(\int_{K_R} |\nabla((u - k_{j+1})^+ \varphi_j)|^p \, dx \right)^{\frac{p^*}{p}} \\ &\lesssim \left(\int_{A_{k_{j+1}, \tilde{r}_j}} |\nabla u|^p \, dx + \int_{A_{k_{j+1}, \tilde{r}_j}} (u - k_{j+1})^p \, dx \right)^{\frac{p^*}{p}} \end{aligned}$$

where we use the notation $f \lesssim g$ in the sense there exists a constant $c > 0$ such that $f \leq cg$. Since $\tilde{r}_j < r_j$, we have

$$\begin{aligned} I_{j+1} &\lesssim \left(2^{jp^*} \int_{A_{k_{j+1}, r_j}} |u - k_{j+1}|^{p^*} dx + k_{j+1}^\alpha |A_{k_{j+1}, r_j}| + |A_{k_{j+1}, r_j}|^{\frac{p}{p^*} + \varepsilon} \right. \\ &\quad \left. + k_{j+1}^\beta |A_{k_{j+1}, r_j}|^{\frac{p}{p^*} + \varepsilon} + 2^{jp} \left(\int_{A_{k_{j+1}, r_j}} |u - k_{j+1}|^{p^*} dx \right)^{\frac{p}{p^*}} |A_{k_{j+1}, r_j}|^\delta \right. \\ &\quad \left. + \int_{A_{k_{j+1}, r_j}} |u - k_{j+1}|^{p^*} dx \right)^{\frac{p^*}{p}}. \end{aligned} \quad (4.4.52)$$

Moreover, for any j , $k_j \leq k_{j+1}$, this implies

$$I_j \geq \int_{A_{k_{j+1}, r_j}} |u - k_j|^{p^*} dx \geq \int_{A_{k_{j+1}, r_j}} |k_j - k_{j+1}|^{p^*} dx = |A_{k_{j+1}, r_j}| |k_{j+1} - k_j|^{p^*}.$$

Then, for any $k > k_0$ and $j \in \mathbb{N}$

$$|A_{k_{j+1}, r_j}| + k_{j+1}^{p^*} |A_{k_{j+1}, r_j}| \lesssim 2^{jp^*} I_j$$

where the constant in the notation depends only on k_0 , p and α . From the previous inequality, we deduce

$$k_{j+1}^\beta |A_{k_{j+1}, r_j}|^{\frac{p}{p^*} + \varepsilon} \leq k_{j+1}^{p + \varepsilon p^*} |A_{k_{j+1}, r_j}|^{\frac{p}{p^*} + \varepsilon} \lesssim 2^{j(p + \varepsilon p^*)} I_j^{\frac{p}{p^*} + \varepsilon}.$$

Replacing in (4.4.52), we obtain

$$I_{j+1} \lesssim \left(2^{jp^*} I_j + 2^{j(p + \varepsilon p^*)} I_j^{\frac{p}{p^*} + \varepsilon} + 2^{j(p + \delta p^*)} I_j^{\frac{p}{p^*} + \delta} \right)^{\frac{p^*}{p}}. \quad (4.4.53)$$

Setting $M = \frac{p}{p^*} \max\{p^*, p + \varepsilon p^*, p + \delta p^*\}$ and $\theta = \min\{1 - \frac{p}{p^*}, \varepsilon, \delta\}$ and noting

$$I_j \leq \int_{K_R} (|u - k_j|^+)^{p^*} dx \leq \int_{K_R} |u|^{p^*} \leq \|u\|_{W_0^{1,p}}^{p^*},$$

(4.4.53) becomes

$$I_{j+1} \lesssim 2^{jM} I_j^{1 + \frac{\theta p^*}{p}}$$

where the constant depends on $\|u\|_{W_0^{1,p}}$, k_0 , α and p . We conclude with Lemma 4.7 in Chapter 2 of [191].

For this it suffices to prove that I_0 is small enough. Indeed $u \in L^{p^*}(\Omega)$ implies

$$I_0 = \int_{A_{\frac{k}{2}, R}} |u - \frac{k}{2}|^{p^*} dx \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

Hence for k large enough, $I_0 \leq C^{-\frac{1}{\eta}} (2^M)^{-\frac{1}{\eta^2}}$ with $\eta = \frac{\theta p^*}{p}$. Thus I_j converges to 0 as $j \rightarrow +\infty$ and

$$\int_{A_{k, \frac{R}{2}}} |u - k|^{p^*} dx = 0.$$

We deduce that $u \leq k$ on $K_{\frac{R}{2}}$. In the same way, we prove that $-u \leq k$ on $K_{\frac{R}{2}}$. Since $\bar{\Omega}$ is compact, we conclude that $u \in L^\infty(\Omega)$. \square

Proof of Proposition 4.4.3: We follow the idea of the proof of Theorem 4.1 in [120]. Let $x_0 \in \bar{\Omega}$, B_R the ball of radius R centered in x_0 and $K_R \stackrel{\text{def}}{=} \Omega \cap B_R$. We define

$$p^+ \stackrel{\text{def}}{=} \max_{K_R} p(x) \quad \text{and} \quad p^- \stackrel{\text{def}}{=} \min_{K_R} p(x)$$

and we choose R small enough such that $p^+ < (p^-)^*$ where

$$(p^-)^* \stackrel{\text{def}}{=} \begin{cases} \frac{Np^-}{N-p^-} & \text{if } p^- < N, \\ p^+ + 1 & \text{if } p^- \geq N. \end{cases}$$

Fix $(s, t) \in (\mathbb{R}_+^*)^2$, $t < s < R$ then $K_t \subset K_s \subset K_R$. Define $\varphi \in C^\infty(\Omega)$, $0 \leq \varphi \leq 1$ such that

$$\varphi = \begin{cases} 1 & \text{in } B_t, \\ 0 & \text{in } \mathbb{R}^N \setminus B_s \end{cases}$$

satisfying $|\nabla\varphi| \lesssim 1/(s-t)$. Let $k \geq 1$, using the same notations as previously $A_{k,\lambda} = \{y \in K_\lambda \mid u(y) > k\}$ and taking $\Psi = \varphi^{p^+}(u-k)^+ \in W_0^{1,p(x)}(\Omega)$ in (4.4.51), we obtain

$$\begin{aligned} \int_{A_{k,s}} |\nabla u|^{p(x)} \varphi^{p^+} dx + p^+ \int_{A_{k,s}} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \varphi^{p^+-1} (u-k)^+ dx \\ = \int_{A_{k,s}} hu^{q-1} \varphi^{p^+} (u-k) dx. \end{aligned} \quad (4.4.54)$$

Hence by Young inequality, for $\epsilon > 0$, we have

$$\begin{aligned} p^+ \int_{A_{k,s}} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \varphi^{p^+-1} (u-k) dx \leq \epsilon \int_{A_{k,s}} |\nabla u|^{p(x)} \varphi^{(p^+-1)\frac{p(x)}{p(x)-1}} dx \\ + c\epsilon^{-1} \int_{A_{k,s}} (u-k)^{p(x)} |\nabla \varphi|^{p(x)} dx. \end{aligned}$$

Since $|\nabla\varphi| \leq c/(s-t)$ and for any $x \in K_R$, $p^+ \leq (p^+-1)\frac{p(x)}{p(x)-1}$, we have $\varphi^{(p^+-1)\frac{p(x)}{p(x)-1}} \leq \varphi^{p^+}$. This implies

$$\begin{aligned} p^+ \int_{A_{k,s}} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \varphi \varphi^{p^+-1} (u-k) dx \leq \epsilon \int_{A_{k,s}} |\nabla u|^{p(x)} \varphi^{p^+} dx \\ + c\epsilon^{-1} \int_{A_{k,s}} \left(\frac{u-k}{s-t} \right)^{p(x)} dx. \end{aligned} \quad (4.4.55)$$

Using Hölder inequality, we estimate the right-hand side of (4.4.54) as follows:

$$\int_{A_{k,s}} hu^{q-1} \varphi^{p^+} (u-k) dx \leq \|h\|_{L^r} \left(\int_{A_{k,s}} u^{\frac{r(q-1)}{r-1}} (u-k)^{\frac{r}{r-1}} dx \right)^{\frac{r-1}{r}}.$$

Since $r > \frac{N}{p^-}$, we have $\frac{(p^-)^* r-1}{p^-} > 1$, applying once again the Hölder inequality and the Young inequality, we obtain

$$\int_{A_{k,s}} hu^{q-1} \varphi^{p^+} (u-k) dx \lesssim \left(\int_{A_{k,s}} u^{\frac{q(p^-)^*}{p^-}} dx + \int_{A_{k,s}} (u-k)^{\frac{q(p^-)^*}{p^-}} dx \right)^{\frac{p^-}{(p^-)^*}} |A_{k,s}|^\delta$$

where $\delta = \frac{r-1}{r} - \frac{p^-}{(p^-)^*} > 0$.

Set $A_{k,s,t} = \{x \in A_{k,s} \mid u(x) - k > s - t\}$ and its complement as $A_{k,s,t}^c$. Now we split the integrals in the right-hand side of (4.4.56) as follows:

$$\begin{aligned} \int_{A_{k,s,t}} \left(\frac{u-k}{s-t} \right)^{\frac{q(p^-)^*}{p^-}} (s-t)^{\frac{q(p^-)^*}{p^-}} dx + \int_{A_{k,s,t}^c} \left(\frac{u-k}{s-t} \right)^{\frac{q(p^-)^*}{p^-}} (s-t)^{\frac{q(p^-)^*}{p^-}} dx \\ \lesssim \int_{A_{k,s}} \left(\frac{u-k}{s-t} \right)^{(p^-)^*} dx + |A_{k,s}| \stackrel{\text{def}}{=} \mathcal{I} \end{aligned} \quad (4.4.56)$$

since $q < p^-$ and we also have

$$\int_{A_{k,s}} u^{\frac{q(p^-)^*}{p^-}} dx \lesssim \int_{A_{k,s}} (u-k)^{\frac{q(p^-)^*}{p^-}} + k^{\frac{q(p^-)^*}{p^-}} dx \lesssim \mathcal{I} + k^{\frac{q(p^-)^*}{p^-}} |A_{k,s}|.$$

In the same way, the second term in the right-hand side of (4.4.55) can be estimated as follows:

$$\int_{A_{k,s} \cap A_{k,s,t}} \left(\frac{u-k}{s-t} \right)^{p(x)} dx + \int_{A_{k,s} \cap A_{k,s,t}^c} \left(\frac{u-k}{s-t} \right)^{p(x)} dx \lesssim \mathcal{I}. \quad (4.4.57)$$

Finally plugging (4.4.55)-(4.4.57), we obtain for ε small enough

$$\int_{A_{k,s}} |\nabla u|^{p(x)} \varphi^{p^+} dx \lesssim \mathcal{I} + |A_{k,s}|^\delta (\mathcal{I} + k^{\frac{q(p^-)^*}{p^-}} |A_{k,s}|)^{\frac{p^-}{(p^-)^*}}$$

where the constant depends on p , R and ε . Moreover we have

$$\begin{aligned} (\mathcal{I} + k^{\frac{q(p^-)^*}{p^-}} |A_{k,s}|)^{\frac{p^-}{(p^-)^*}} &\lesssim \left(\int_{A_{k,s}} \left(\frac{u-k}{s-t} \right)^{(p^-)^*} dx \right)^{\frac{p^-}{(p^-)^*}} \\ &+ |A_{k,s}|^{\frac{p^-}{(p^-)^*}} + k^q |A_{k,s}|^{\frac{p^-}{(p^-)^*}}. \end{aligned}$$

To conclude, using the Young inequality, we obtain the following estimate:

$$\begin{aligned} \int_{A_{k,t}} |\nabla u|^{p^-} dx &\leq \int_{A_{k,s}} |\nabla u|^{p(x)} \varphi^{p^+} dx \lesssim \int_{A_{k,s}} \left(\frac{u-k}{s-t} \right)^{(p^-)^*} dx + 2|A_{k,s}| \\ &+ (1+k^q) |A_{k,s}|^{\frac{p^-}{(p^-)^*} + \delta} + |A_{k,s}|^\delta \left(\int_{A_{k,s}} \left(\frac{u-k}{s-t} \right)^{(p^-)^*} dx \right)^{\frac{p^-}{(p^-)^*}}. \end{aligned}$$

By Lemma 4.4.5, we deduce that u bounded in Ω . \square

Combining Theorem 4.1 of [120] and Proposition 4.4.3, we have the following corollary:

Corollary 4.4.7. *Let $p \in C(\bar{\Omega})$ and $q \in (1, p^-]$. Assume $u \in \mathbf{W}$ and nonnegative satisfying for any $\Psi \in \mathbf{W}$, $\Psi \geq 0$,*

$$\int_{\Omega} u^{2q-1} \Psi \, dx + \int_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla \Psi \, dx \leq \int_{\Omega} (f(x, u) + hu^{q-1}) \Psi \, dx$$

where f verifies for any $(x, t) \in \Omega \times \mathbb{R}^+$, $|f(x, t)| \leq c_1 + c_2 |t|^{s(x)-1}$ with $s \in C(\bar{\Omega})$ such that for any $x \in \bar{\Omega}$, $1 < s(x) < p^*(x)$ and $h \in L^2(\Omega) \cap L^r(\Omega)$ with $r > \max\{1, \frac{N}{p^-}\}$. Then $u \in L^\infty(\Omega)$.

4.5 Generalized doubly nonlinear equation: Local existence, uniqueness, regularity and stabilization

In the section, we study the existence, uniqueness and qualitative properties of the weak solutions of the following D.N.E. driven by a general quasilinear operator of Leray-Lions type:

$$\begin{cases} \frac{q}{2q-1} \partial_t (u^{2q-1}) - \nabla \cdot a(x, \nabla u) = f(x, u) + h(t, x)u^{q-1}, & u > 0 & \text{in } Q_T; \\ u = 0 & & \text{on } \Gamma; \\ u(0, \cdot) = u_0 & & \text{in } \Omega, \end{cases} \quad (\text{DNE})$$

where $T > 0$, $q > 1$, $Q_T \stackrel{\text{def}}{=} (0, T) \times \Omega$ with $\Omega \subset \mathbb{R}^N$, $N \geq 1$ a smooth bounded domain, $\Gamma \stackrel{\text{def}}{=} (0, T) \times \partial\Omega$ and h belongs to $L^\infty(Q_T)$. The main difference of this work with the previous sections is the doubly nonlinear feature together combined to the broad class of considered Leray-Lions type operators a . More precisely, problem (DNE) involves a class of variational operators $a : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ defined as, for any $(x, \xi) \in \Omega \times \mathbb{R}^N$:

$$a(x, \xi) = (a_j(x, \xi))_j \stackrel{\text{def}}{=} \left(\frac{1}{p(x)} \partial_{\xi_j} A(x, \xi) \right)_j = \frac{1}{p(x)} \nabla_{\xi} A(x, \xi)$$

where $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^+$ is continuous, differentiable with respect to ξ and satisfies:

(A₁) $\xi \rightarrow A(\cdot, \xi)$ is $p(x)$ -homogeneous i.e. $A(x, t\xi) = t^{p(x)} A(x, \xi)$, for any $t \in \mathbb{R}^+$, $\xi \in \mathbb{R}^N$ and a.e. $x \in \Omega$

with $p \in C^1(\bar{\Omega})$ satisfying

$$1 < p^- \stackrel{\text{def}}{=} \min_{x \in \bar{\Omega}} p(x) \leq p(x) \leq p^+ \stackrel{\text{def}}{=} \max_{x \in \bar{\Omega}} p(x) < \infty.$$

This class of operators a also satisfies ellipticity and growth conditions:

(A₂) For $j \in \llbracket 1, N \rrbracket$, $a_j(x, 0) = 0$, $a_j \in C^1(\Omega \times \mathbb{R}^N \setminus \{0\}) \cap C(\Omega \times \mathbb{R}^N)$ and there exist two constants $\gamma, \Gamma > 0$ such that for all $x \in \Omega$, $\xi \in \mathbb{R}^N \setminus \{0\}$ and $\eta \in \mathbb{R}^N$:

$$\begin{aligned} \sum_{i,j=1}^N \frac{\partial a_j}{\partial \xi_i}(x, \xi) \eta_i \eta_j &\geq \gamma |\xi|^{p(x)-2} |\eta|^2; \\ \sum_{i,j=1}^N \left| \frac{\partial a_j}{\partial \xi_i}(x, \xi) \right| &\leq \Gamma |\xi|^{p(x)-2}. \end{aligned}$$

Remark 4.5.1. *The assumption (A₂) gives the convexity of $\xi \mapsto A(x, \xi)$ and growth estimates, for any $(x, \xi) \in \Omega \times \mathbb{R}^N$:*

$$\frac{\gamma}{p(x)-1} |\xi|^{p(x)} \leq A(x, \xi) \leq \frac{\Gamma}{p(x)-1} |\xi|^{p(x)}; \quad |a(x, \xi)| \leq C |\xi|^{p(x)-1}; \quad (4.5.1)$$

and, see [248], for any $\xi, \eta \in \mathbb{R}^N$ and $x \in \Omega$, there exists a constant $\gamma_0 > 0$ depending on γ and p such that

$$\langle a(x, \xi) - a(x, \eta), \xi - \eta \rangle \geq \gamma_0 \begin{cases} |\xi - \eta|^{p(x)} & \text{if } p(x) > 2; \\ \frac{|\xi - \eta|^2}{(1 + |\xi| + |\eta|)^{2-p(x)}} & \text{if } p(x) \leq 2. \end{cases} \quad (4.5.2)$$

Moreover, the homogeneity assumption implies that $A(x, \xi) = a(x, \xi) \cdot \xi$ for any $(x, \xi) \in \Omega \times \mathbb{R}^N$.

Next, we impose the condition below to insure qualitative properties as regularity and the validity of Hopf Lemma.

(A₃) There exists $C > 0$ such that for any $(x, \xi) \in \Omega \times \mathbb{R}^N \setminus \{0\}$:

$$\sum_{i,j=1}^N \left| \frac{\partial a_i}{\partial x_j}(x, \xi) \right| \leq C |\xi|^{p(x)-1} (1 + |\ln(|\xi|)|).$$

Remark 4.5.2. *More precisely, from the condition (A₃) we derive the Strong Maximum Principle (see [262]) and the $C^{1,\alpha}$ -regularity of weak solutions (see Remark 5.3 in [118] and Remark 3.1 in [146]).*

Concerning the conditions on the functions f and h , we assume:

(f₀) $f : \bar{\Omega} \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function such that $f(x, 0) \equiv 0$ and f is positive on $\Omega \times \mathbb{R}^+ \setminus \{0\}$.

(f₁) For any $x \in \Omega$, $s \mapsto \frac{f(x,s)}{s^{q-1}}$ is nonincreasing in $\mathbb{R}^+ \setminus \{0\}$.

and

(H_h) there exists $\underline{h} \in L^\infty(\Omega) \setminus \{0\}$, $\underline{h} \geq 0$ such that $h(t, x) \geq \underline{h}(x)$ for a.e in Q_T .

The study of (DNE) is naturally concerned with the investigation of the following associated parabolic problem:

$$\begin{cases} v^{q-1} \partial_t(v^q) - \nabla \cdot a(x, \nabla v) = h(t, x)v^{q-1} + f(x, v) & \text{in } Q_T; \\ v \geq 0 & \text{in } Q_T; \\ v = 0 & \text{on } \Gamma; \\ v(0, \cdot) = v_0 & \text{in } \Omega. \end{cases} \quad (\text{E})$$

We further prove that a weak solution of (E) is also a weak solution of (DNE).

By denoting $\mathcal{W} \stackrel{\text{def}}{=} W_0^{1,p(x)}(\Omega)$ (we refer to [112, 223] for the definitions and properties of variables exponent Lebesgue and Sobolev spaces) and introducing weighted spaces with the notation $\delta(x) \stackrel{\text{def}}{=} \text{dist}(x, \partial\Omega)$:

$$L_\delta^\infty(\Omega) \stackrel{\text{def}}{=} \{w : \Omega \rightarrow \mathbb{R} \mid \text{measurable, } \frac{w}{\delta(\cdot)} \in L^\infty(\Omega)\}$$

endowed with the norm $\|w\|_\delta = \sup_\Omega \left| \frac{w(x)}{\delta(x)} \right|$ and for $r > 0$:

$$\mathcal{M}_\delta^r(\Omega) \stackrel{\text{def}}{=} \{w : \Omega \rightarrow \mathbb{R}^+ \mid \text{measurable, } \exists c > 0, \frac{1}{c} \leq \frac{w^r}{\delta(x)} \leq c\},$$

we introduce the notion of weak solution of (E) as follows:

Definition 4.5.1. *Let $T > 0$, a weak solution to (E) is any positive function $v \in L^\infty(0, T; \mathcal{W}) \cap L^\infty(Q_T)$ such that $\partial_t(v^q) \in L^2(Q_T)$ satisfying for any $\phi \in L^2(Q_T) \cap L^1(0, T; \mathcal{W})$ and for any $t \in (0, T]$*

$$\begin{aligned} \int_0^t \int_\Omega \partial_t(v^q)v^{q-1}\phi \, dx ds + \int_0^t \int_\Omega a(x, \nabla v) \cdot \nabla \phi \, dx ds \\ = \int_0^t \int_\Omega (h(s, x)v^{q-1} + f(x, v))\phi \, dx ds \end{aligned} \quad (4.5.3)$$

and $v(0, \cdot) = v_0$ a.e. in Ω .

Remark 4.5.3. *In sense of Definition 4.5.1, a solution of (E) belongs to $L^\infty(Q_T)$, hence $\frac{q}{2q-1} \partial_t(v^{2q-1}) = v^{q-1} \partial_t(v^q) \in L^2(Q_T)$ holds in weak sense and we deduce the existence of a solution of (DNE).*

Remark 4.5.4. *Prototype examples of operators a satisfying (A_1) - (A_3) are given below: for any $(x, \xi) \in \Omega \times \mathbb{R}^N$ and $p \in C^{1,\beta}(\bar{\Omega})$ by:*

$$A(x, \xi) = \sum_{j=1}^J \left(g_j(x) \left(\sum_{i \in P_j} \xi_i^2 \right)^{\frac{p(x)}{2}} \right)$$

where $(P_j)_{j \in J}$ is a partition of $\llbracket 1, N \rrbracket$, $g_j \in C^1(\bar{\Omega}) \cap C^{0,\beta}(\bar{\Omega})$ and $g_j(x) \geq c > 0$ for any $j \in J$. In particular for $A(x, \xi) = |\xi|^{p(x)}$, (DNE) can be classified as S.D.E. if $2q < p^-$ and F.D.E. if $2q > p^+$.

4.5.1 Main results

About the existence and properties of solutions of (E), we prove the following result:

Theorem 4.5.1. *Let $T > 0$ and $q \in (1, p^-)$. Assume A satisfies (A_1) - (A_3) , f satisfies (f_0) , (f_1) and*

(f_2) The mapping $x \mapsto \delta^{1-q}(x)f(x, \delta(x))$ belongs to $L^2(\Omega_\varepsilon)$ for some $\varepsilon > 0$ where $\Omega_\varepsilon \stackrel{\text{def}}{=} \{x \in \Omega \mid \delta(x) < \varepsilon\}$.

Then, for any $h \in L^\infty(Q_T)$ satisfying (\mathbf{H}_h) and for any initial data $v_0 \in \mathcal{M}_\delta^1(\Omega) \cap \mathcal{W}$, there exists a unique solution in sense of Definition 4.5.1.

More precisely, we have:

(i) Let v, w be two weak solutions of (E) with respect to the initial data $v_0, w_0 \in \mathcal{M}_\delta^1(\Omega) \cap \mathcal{W}$ and $h, g \in L^\infty(Q_T)$ satisfying (\mathbf{H}_h) . Then, for any $t \in [0, T]$:

$$\|(v^q(t) - w^q(t))^+\|_{L^2} \leq \|(v_0^q - w_0^q)^+\|_{L^2} + \int_0^t \|(h(s) - g(s))^+\|_{L^2} ds. \quad (4.5.4)$$

(ii) Assume in addition A satisfies, for any $x \in \Omega$ and for any $\xi, \eta \in \mathbb{R}^N$:

*(A_4) $A(x, \frac{\xi-\eta}{2}) \leq \zeta(x)(A(x, \xi) + A(x, \eta))^{1-s(x)} \left(A(x, \xi) + A(x, \eta) - 2A(x, \frac{\xi+\eta}{2}) \right)^{s(x)}$
where for any $x \in \Omega$, $s(x) = \min\{1, p(x)/2\}$ and $\zeta(x) = \left(1 - 2^{1-p(x)}\right)^{-s(x)}$ if $p(x) < 2$
or $\zeta(x) = \frac{1}{2}$ if $p(x) \geq 2$.*

Then, $v \in C([0, T]; \mathcal{W})$.

Remark 4.5.5. *The above result can be generalized in case $f(x, s) \stackrel{\text{def}}{=} \tilde{f}(x, s) + \hat{f}(x, s)$ where \tilde{f} satisfies (f_1) and $s \rightarrow \frac{\hat{f}(x, s)}{s^{q-1}}$ is Lipschitz with respect to the second variable, uniformly in $x \in \Omega$ with constant $\omega > 0$. Then if f satisfies additionally (f_0) , (f_2) and under same conditions for A and q , Theorem 4.5.1 still holds, (4.5.4) being replaced by*

$$\|(v^q(t) - w^q(t))^+\|_{L^2} \leq e^{\omega t} \|(v_0^q - w_0^q)^+\|_{L^2} + \int_0^t e^{\omega(t-s)} \|(h(s) - g(s))^+\|_{L^2} ds.$$

Similar results have been obtained in [108] in the case of the p -laplacian operator.

Remark 4.5.6. *Prototype example of functions f satisfying (f_0) - (f_2) is given by for any $(x, s) \in \Omega \times \mathbb{R}^+$, $f(x, s) = g(x)\delta^\gamma(x)s^\beta$ where $g \in L^\infty(\Omega)$ is a nonnegative function, $\beta \in [0, q-1)$ and $\beta + \gamma > q - \frac{3}{2}$.*

Remark 4.5.7. *The condition (A_4) reformulates the local form of Morawetz-type inequality producing convergence properties.*

In Theorem 4.5.1, the uniqueness of the solution in sense of Definition 4.5.1 is obtained by the following theorem relaxing the assumptions on v_0 and h . More precisely, we show:

Theorem 4.5.2. *Let v, w be two solutions of (E) in sense of Definition 4.5.1 with respect to the initial data $v_0, w_0 \in L^{2q}(\Omega)$, $v_0, w_0 \geq 0$ and $h, \tilde{h} \in L^2(Q_T)$. Then, for any $t \in [0, T]$:*

$$\|v^q(t) - w^q(t)\|_{L^2(\Omega)} \leq \|v_0^q - w_0^q\|_{L^2(\Omega)} + \int_0^t \|h(s) - \tilde{h}(s)\|_{L^2(\Omega)} ds. \quad (4.5.5)$$

Using a similar approach based on nonlinear accretive operators theory as in [39, 146, 147], we introduce $\mathcal{T}_q : \mathcal{D}(\mathcal{T}_q) \subset L^2(\Omega) \rightarrow L^2(\Omega)$ be the operator with the parameter q defined by

$$\mathcal{T}_q u = -u^{(1-q)/q} \left(\nabla \cdot a(x, \nabla(u^{1/q})) + f(x, u^{1/q}) \right)$$

and the associated domain

$$\mathcal{D}(\mathcal{T}_q) = \{w : \Omega \rightarrow \mathbb{R}^+ \mid \text{measurable, } w^{1/q} \in \mathcal{W} \cap L^{2q}(\Omega), \mathcal{T}_q w \in L^2(\Omega)\}.$$

Based on the accretive property of \mathcal{T}_q in $L^2(\Omega)$ (see Theorem 4.5.5 and Corollary 4.5.2) and additional regularity on initial data, we obtain the following stabilization result for the weak solutions to (E):

Theorem 4.5.3. *Under the assumptions of Theorem 4.5.1, let v be the weak solution of (E) with the initial data $v_0 \in \mathcal{M}_\delta^1(\Omega) \cap \mathcal{W}$. Assume that $h \in L^\infty([0, +\infty) \times \Omega)$ satisfying (\mathbf{H}_h) on $[0, +\infty) \times \Omega$ and there exists $h_\infty \in L^\infty(\Omega)$ such that*

$$t^{1+\eta} \|h(t, \cdot) - h_\infty\|_{L^2} = O(1) \quad \text{at infinity for some } \eta > 0. \quad (4.5.6)$$

Then, for any $r \in [1, \infty)$

$$\|v^q(t, \cdot) - v_{stat}^q\|_{L^r} \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

where v_{stat} is the unique solution of associated stationary problem with the potential $h_\infty \in L^\infty(\Omega)$.

Remark 4.5.8. *The stabilization in L^∞ -norm appeals new estimates linked to the T -accretivity of the operator \mathcal{T}_q in L^∞ and in L^1 (see Remark 1.6 and Theorem 2.1 in [49] and Theorem 1.18 in [163]).*

Remark 4.5.9. *In Theorem 4.5.3, we noticed that $v_0 \in \mathcal{M}_\delta^1(\Omega) \cap \mathcal{W}$ implies $v_0^q \in \overline{\mathcal{D}(\mathcal{T}_q)}^{L^2}$ (see Proposition 2.11 in [68]).*

4.5.2 Elliptic problem related to D.N.E.

In this section, we study a class of elliptic problem related to D.N.E. in order to prove Theorem 4.5.1. First we start with a direct application of Theorem 4.4.1 which provides a comparison principle, uniform estimates and uniqueness.

Lemma 4.5.1. *Let $A : \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}$ be a continuous and differentiable function satisfying (A_1) with $a(x, \xi) = \frac{1}{p(x)} \nabla_\xi A(x, \xi)$ such that $\xi \rightarrow A(x, \xi)$ is strictly convex for any $x \in \Omega$. Then, for $r \in [1, p^-)$, for any $w_1, w_2 \in \mathcal{W} \cap L^\infty(\Omega)$ two positive functions and for any $x \in \Omega$*

$$a(x, \nabla w_1) \cdot \nabla \left(\frac{w_1^r - w_2^r}{w_1^{r-1}} \right) + a(x, \nabla w_2) \cdot \nabla \left(\frac{w_2^r - w_1^r}{w_2^{r-1}} \right) \geq 0. \quad (4.5.7)$$

If the equality occurs in (4.5.7), then $w_1 \equiv w_2$ in Ω .

Proof. Let $w_1, w_2 \in \mathcal{W} \cap L^\infty(\Omega)$ such that $w_1, w_2 > 0$ in Ω . Then Theorem 4.4.1 yields

$$A^{r/p(x)}(x, \nabla w_1) A^{(p(x)-r)/p(x)}(x, \nabla w_2) \geq a(x, \nabla w_2) \cdot \nabla \left(\frac{w_1^r}{w_2^{r-1}} \right).$$

Then, by using Young inequality and the equality $A(x, \xi) = a(x, \xi) \cdot \xi$, we obtain

$$a(x, \nabla w_2) \cdot \nabla \left(w_2 - \frac{w_1^r}{w_2^{r-1}} \right) \geq \frac{r}{p(x)} (A(x, \nabla w_2) - A(x, \nabla w_1)). \quad (4.5.8)$$

Reversing the role of w_1 and w_2 :

$$a(x, \nabla w_1) \cdot \nabla \left(w_1 - \frac{w_2^r}{w_1^{r-1}} \right) \geq \frac{r}{p(x)} (A(x, \nabla w_1) - A(x, \nabla w_2))$$

and adding the above inequalities we obtain (4.5.7) and the rest of the proof follows from Theorem 5.2 in [146]. \square

4.5.2.1 L^∞ -potential

In this subsection, we study the following associated elliptic problem:

$$\begin{cases} v^{2q-1} - \lambda \nabla \cdot a(x, \nabla v) = h_0 v^{q-1} + \lambda f(x, v) & \text{in } \Omega; \\ v \geq 0 & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.5.9)$$

where $h_0 \in L^\infty(\Omega)$ and λ is a positive parameter. The notion of weak solution of (4.5.9) is defined as follows:

Definition 4.5.2. *A weak solution of (4.5.9) is any nonnegative and nontrivial function $v \in \mathbf{X} \stackrel{\text{def}}{=} \mathcal{W} \cap L^{2q}(\Omega)$ such that for any $\phi \in \mathbf{X}$*

$$\int_{\Omega} v^{2q-1} \phi \, dx + \lambda \int_{\Omega} a(x, \nabla v) \cdot \nabla \phi \, dx = \int_{\Omega} h_0 v^{q-1} \phi \, dx + \lambda \int_{\Omega} f(x, v) \phi \, dx. \quad (4.5.10)$$

The first theorem gives the existence and the uniqueness of the weak solution of (4.5.9).

Theorem 4.5.4. *Assume that A satisfies (A_1) - (A_3) and f satisfies (f_0) and (f_1) . Then, for any $q \in (1, p^-)$, $\lambda > 0$ and $h_0 \in L^\infty(\Omega) \setminus \{0\}$, $h_0 \geq 0$, there exists a weak solution $v \in C^1(\overline{\Omega}) \cap \mathcal{M}_\delta^1(\Omega)$ to (4.5.9).*

Moreover, let v_1, v_2 be two weak solutions to (4.5.9) with $h_1, h_2 \in L^\infty(\Omega) \setminus \{0\}$, $h_1, h_2 \geq 0$ respectively, we have with the notation $t^+ \stackrel{\text{def}}{=} \max\{0, t\}$:

$$\|(v_1^q - v_2^q)^+\|_{L^2} \leq \|(h_1 - h_2)^+\|_{L^2}. \quad (4.5.11)$$

Proof. Define the energy functional \mathcal{J} on \mathbf{X} :

$$\begin{aligned} \mathcal{J}(v) = & \frac{1}{2q} \int_{\Omega} v^{2q} dx + \lambda \int_{\Omega} \frac{A(x, \nabla v)}{p(x)} dx - \frac{1}{q} \int_{\Omega} h_0 (v^+)^q dx \\ & - \lambda \int_{\Omega} F(x, v) dx \end{aligned} \quad (4.5.12)$$

where $F(x, t) = \int_0^{t^+} f(x, s) ds$.

Note from (f_0) - (f_1) that there exists $C > 0$ large enough such that for any $(x, s) \in \Omega \times \mathbb{R}^+$

$$0 \leq f(x, s) \leq C(1 + s^{q-1}). \quad (4.5.13)$$

By (4.5.1) and (4.5.13), \mathcal{J} is well defined, continuous on \mathbf{X} and we have

$$\mathcal{J}(v) \geq \|v\|_{L^{2q}}^q \left(c_1 \|v\|_{L^{2q}}^q - c_2 \right) + \|v\|_{\mathcal{W}} \left(c_3 \|v\|_{\mathcal{W}}^{p^- - 1} - c_4 \right)$$

where the constants do not depend on u . Thus we deduce that \mathcal{J} is coercive on \mathbf{X} . Therefore we affirm that there exists $v_0 \in \mathbf{X}$ a global minimizer of \mathcal{J} .

Noting that, with the notation $t^- = t^+ - t$,

$$\mathcal{J}(v_0) \geq \mathcal{J}(v_0^+) + \frac{1}{2q} \int_{\Omega} (v_0^-)^{2q} dx + \lambda \int_{\Omega} \frac{A(x, \nabla v_0^-)}{p(x)} dx \geq \mathcal{J}(v_0^+)$$

we deduce $v_0 \geq 0$. Let $\phi \in C_c^1(\Omega)$ be a nonnegative and nontrivial function, thus for any $t > 0$

$$\mathcal{J}(t\phi) \leq t^q (c_1 t^q + c_2 t^{p^- - q} - c_3)$$

where the constants are independent of t and $c_3 > 0$ since $h_0 \not\equiv 0$. Hence for t small enough, $\mathcal{J}(t\phi) < 0$ and since $\mathcal{J}(0) = 0$, we deduce $v_0 \not\equiv 0$. The Gâteaux differentiability of \mathcal{J} insures that v_0 satisfies (4.5.10).

From Proposition 4.5.2, we deduce $v_0 \in L^\infty(\Omega)$ and Theorem 1.2 in [118] provides the $C^{1,\alpha}(\overline{\Omega})$ -regularity of v_0 for some $\alpha \in (0, 1)$.

By (f_0) and (f_1) , f satisfies $\lim_{s \rightarrow 0^+} f(x, s) s^{1-2q} = \infty$ uniformly in $x \in \Omega$, hence Lemma 4.5.2 implies $v_0 \in \mathcal{M}_\delta^1(\Omega)$.

Finally, let $v_1, v_2 \in \mathcal{M}_\delta^1(\Omega)$ be two weak solutions of (4.5.9) with respect to h_1 and h_2 respectively. Namely, for any $\phi, \Psi \in \mathbf{X}$, we have

$$\int_{\Omega} v_1^{2q-1} \phi \, dx + \lambda \int_{\Omega} a(x, \nabla v_1) \cdot \nabla \phi \, dx = \int_{\Omega} h_1 v_1^{q-1} \phi \, dx + \lambda \int_{\Omega} f(x, v_1) \phi \, dx$$

and

$$\int_{\Omega} v_2^{2q-1} \Psi \, dx + \lambda \int_{\Omega} a(x, \nabla v_2) \cdot \nabla \Psi \, dx = \int_{\Omega} h_2 v_2^{q-1} \Psi \, dx + \lambda \int_{\Omega} f(x, v_2) \Psi \, dx.$$

Subtracting above expressions by taking $\phi = \left(v_1 - \frac{v_2^q}{v_1^{q-1}}\right)^+$ and $\Psi = \left(v_2 - \frac{v_1^q}{v_2^{q-1}}\right)^-$ then by (f_1) and Lemma 4.5.1, we obtain

$$\begin{aligned} \int_{\Omega} ((v_1^q - v_2^q)^+)^2 \, dx &\leq \int_{\Omega} (h_1 - h_2)(v_1^q - v_2^q)^+ \, dx \\ &\leq \|(h_1 - h_2)^+\|_{L^2(\Omega)} \|(v_1^q - v_2^q)^+\|_{L^2} \end{aligned}$$

from which (4.5.11) follows. \square

Remark 4.5.10. *In the proof of Theorem 4.5.4, condition (f_1) is not optimal to obtain the existence of a minimizer and to apply Lemma 4.5.2. Indeed define a more general condition on f*

$$(f'_1) \quad \limsup_{s \rightarrow +\infty} \frac{f(x, s)}{s^{p^- - 1}} < \gamma \Lambda p_{\pm} \quad \text{uniformly in } x \in \Omega$$

where $p_{\pm} := \frac{p^-}{p^+(p^+ - 1)}$ and $\Lambda^{-1} \stackrel{\text{def}}{=} (\sup_{\|u\|_{\mathcal{W}}=1} (\|u\|_{L^{p^-}(\Omega)})^{p^-})^{p^-}$, condition (f'_1) is a sufficient condition to obtain the existence of a weak solution of (4.5.9). Moreover, to apply Lemma 4.5.2 we assume in addition that f satisfies:

$$(f''_1) \quad \liminf_{s \rightarrow 0^+} \frac{f(x, s)}{s^{2q-1}} > 1 \quad \text{uniformly in } x \in \Omega.$$

Remark 4.5.11. *Inequality (4.5.11) implies the uniqueness of the solution in the sense of Definition 4.5.2. Moreover to obtain (4.5.11), we use more precisely ϕ, ψ belong to $L^\infty(\Omega) \cap \mathcal{W}$. The uniqueness can be also obtained by using Theorem 4.5.10.*

Remark 4.5.12. *For $q = 1$, (4.5.9) becomes*

$$\begin{cases} v + \lambda \mathcal{T}_1 = h_0 & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.5.14)$$

For any $h_0 \in L^\infty(\Omega)$ and for any $f \in L^\infty(\bar{\Omega} \times \mathbb{R})$ satisfying (f_1) with $q = 1$, following the proof of Theorem 4.5.4, we get the existence of a unique weak solution $v_0 \in \mathcal{W} \cap L^2(\Omega)$ (not necessary nonnegative) in sense of Definition 4.5.2 with $\phi \in \mathcal{W} \cap L^2(\Omega)$.

Moreover, choosing as test function $\phi = (v_0 \pm M)^+$ where $M = \|h_0\|_{L^\infty} + \|f\|_{L^\infty}$, we deduce $v_0 \in L^\infty(\Omega)$ and hence for any $\lambda > 0$, $R(I + \lambda \mathcal{T}_1) = L^\infty(\Omega)$.

Moreover, let v_1 and v_2 be two solutions to (4.5.14) with $h_1, h_2 \in L^\infty(\Omega)$ respectively, we get from (4.5.2) and (f_1) : for any $\ell : \mathbb{R} \mapsto \mathbb{R}$ Lipschitz and nondecreasing function such that $\ell(0) = 0$:

$$\int_{\Omega} (\mathcal{T}_1 v_1 - \mathcal{T}_1 v_2) \ell(v_1 - v_2) dx \geq 0.$$

Thus, by section I.4. in [163], \mathcal{T}_1 is T -accretive in $L^1(\Omega)$ namely for any $h_1, h_2 \in L^\infty(\Omega)$ and respectively v_1, v_2 the solutions to (4.5.14), we have

$$\|(v_1 - v_2)^+\|_{L^1} \leq \|(h_1 - h_2)^+\|_{L^1}.$$

Finally, using Remark 1.6 in [49], \mathcal{T}_1 is T -accretive in $L^m(\Omega)$, for any $m \in [1, \infty]$ i.e

$$\|(v_1 - v_2)^+\|_{L^m} \leq \|(h_1 - h_2)^+\|_{L^m}, \quad m \in [1, \infty].$$

We point out that T -accretivity of \mathcal{T}_q , for $q > 1$, in $L^2(\Omega)$ is equivalent to

$$\int_{\Omega} (\mathcal{T}_q v_1 - \mathcal{T}_q v_2) \ell(v_1 - v_2) dx \geq 0$$

with the fixed choice $\ell(t) = t^+$.

In the way of Remark 4.5.12, Theorem 4.5.4 implies existence, uniqueness and accretivity results for the perturbed problem induced by the operator \mathcal{T}_q :

Corollary 4.5.1. *Assume A satisfies (A_1) - (A_3) and f verifies (f_0) and (f_1) . Then, for any $q \in (1, p^-)$, $\lambda > 0$ and $h_0 \in L^\infty(\Omega) \setminus \{0\}$, $h_0 \geq 0$, there exists a unique solution $u \in C^1(\bar{\Omega})$ of*

$$\begin{cases} u + \lambda \mathcal{T}_q u = h_0 & \text{in } \Omega; \\ u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.5.15)$$

Namely, u belongs to $\dot{V}_+^q \cap \mathcal{M}_\delta^{1/q}(\Omega)$ and satisfies:

$$\int_{\Omega} u \psi dx + \lambda \int_{\Omega} a(x, \nabla(u^{\frac{1}{q}})) \cdot \nabla(u^{\frac{1-q}{q}} \psi) - f(x, u^{\frac{1}{q}}) u^{\frac{1-q}{q}} \psi dx = \int_{\Omega} h_0 \psi dx \quad (4.5.16)$$

for any ψ such that

$$|\psi|^{1/q} \in L^\infty(\Omega) \quad \text{and} \quad \frac{|\nabla \psi|}{\delta^{q-1}(\cdot)} \in L^{p(x)}(\Omega). \quad (4.5.17)$$

Moreover, if u_1 and u_2 be two solutions of (4.5.15) corresponding to h_1 and h_2 respectively, then

$$\|(u_1 - u_2)^+\|_{L^2} \leq \|(u_1 - u_2 + \lambda(\mathcal{T}_q u_1 - \mathcal{T}_q u_2))^+\|_{L^2}. \quad (4.5.18)$$

Proof. Define the energy functional \mathcal{E} on $\dot{V}_+^q \cap L^2(\Omega)$ as $\mathcal{E}(u) = \mathcal{J}(u^{1/q})$ where \mathcal{J} is defined in (4.5.12).

Let v_0 is the weak solution of (4.5.9) and the global minimizer of (4.5.12). We set $u_0 = v_0^q$. Then, u_0 belongs to $\dot{V}_+^q \cap \mathcal{M}_\delta^{1/q}(\Omega)$.

Let ψ satisfying (4.5.17). Then there exists $t_0 > 0$ such that for $t \in (-t_0, t_0)$, $u_0 + t\psi > 0$. Hence we have $\mathcal{E}(u_0 + t\psi) \geq \mathcal{E}(u_0)$ for any $t \in (-t_0, t_0)$. Using Taylor expansion, dividing by t and passing to the limit as $t \rightarrow 0$ we deduce that u_0 verifies (4.5.16).

Consider $\tilde{u} \in \dot{V}_+^q \cap \mathcal{M}_\delta^{1/q}(\Omega)$ another solution satisfying (4.5.16). Thus $\tilde{v} = \tilde{u}^{1/q}$ verifies (4.5.10) for $\phi \in L_\delta^\infty(\Omega) \cap \mathcal{W}$. By Remark 4.5.11, we deduce $\tilde{v} = v_0$ and the uniqueness of the solution of (4.5.15). Finally (4.5.18) follows from (4.5.11). \square

4.5.2.2 Extensions for L^2 -potential

We now generalize existence results of subsection 4.5.2.1 for $h_0 \in L^2(\Omega)$ by approximation method.

Theorem 4.5.5. *Assume A satisfies (A_1) - (A_3) and f verifies (f_0) and (f_1) . Then, for any $q \in (1, p^-)$, $\lambda > 0$ and $h_0 \in L^2(\Omega) \setminus \{0\}$, $h_0 \geq 0$, there exists a positive weak solution $v \in \mathbf{X}$ of (4.5.9) in the sense of Definition 4.5.2. Moreover, if $h_0 \in L^r(\Omega)$ for some $r > \max\left\{1, \frac{N}{p^-}\right\}$, $v \in L^\infty(\Omega)$ and v is unique.*

Proof. Consider $h_n \in C_c^1(\Omega)$, $h_n \geq 0$ which converges to h in $L^2(\Omega)$. By Theorem 4.5.4, for any $n \geq 1$, define $v_n \in C^{1,\alpha}(\overline{\Omega}) \cap \mathcal{M}_\delta^1(\Omega)$ as the unique positive weak solution of (4.5.9) with $h_0 = h_n$.

For any $s > 1$ and $a, b \geq 0$, observe that

$$|a - b|^{2s} \leq (a^s - b^s)^2. \quad (4.5.19)$$

Hence (4.5.11) implies, for any $n, p \in \mathbb{N}^*$:

$$\|(v_n - v_p)^+\|_{L^{2q}} \leq \|(v_n^q - v_p^q)^+\|_{L^2}^q \leq \|(h_n - h_p)^+\|_{L^2}^q.$$

Thus we deduce that (v_n) converges to v in $L^{2q}(\Omega)$ and (v_n^q) converges to v^q in $L^2(\Omega)$.

Note that the limit v does not depend to the choice of the sequence (h_n) by (4.5.11). So define in particular, for any $n \in \mathbb{N}^*$, $h_n = \min\{h, n\}$. By (4.5.11), we deduce that (v_n) is nondecreasing and for any $n \in \mathbb{N}^*$,

$$v(x) \geq v_n(x) \geq v_1(x) \geq c\delta(x) > 0 \quad a.e. \text{ in } \Omega, \quad (4.5.20)$$

for some c independent of n .

From (4.5.1), (4.5.13) and using Hölder inequality, equation (4.5.10) with $\phi = v_n$ becomes

$$\begin{aligned} \frac{\lambda\gamma}{p^+ - 1} \int_{\Omega} |\nabla v_n|^{p(x)} dx &\leq \int_{\Omega} a(x, \nabla v_n) \cdot \nabla v_n dx \\ &\leq c (\|v_n\|_{L^{2q}}^q (\|h_n\|_{L^2} + 1) + \|v_n\|_{L^{2q}}) \\ &\leq c (\|v\|_{L^{2q}}^q (\sup_{n \in \mathbb{N}} \|h_n\|_{L^2} + 1) + \|v\|_{L^{2q}}) \end{aligned}$$

for some c independent on n . Hence we deduce that (v_n) is uniformly bounded in \mathcal{W} and v_n converges weakly to v in \mathcal{W} (up to a subsequence).

Now taking $\phi = v_n - v$ in (4.5.10), we obtain as $n \rightarrow \infty$

$$\left| \int_{\Omega} f(x, v_n)(v_n - v) dx \right| + \left| \int_{\Omega} h_n v_n^{q-1}(v_n - v) dx \right| + \left| \int_{\Omega} v_n^{2q-1}(v_n - v) dx \right| \rightarrow 0$$

which infers $\int_{\Omega} a(x, \nabla v_n) \cdot \nabla(v_n - v) dx \rightarrow 0$.

Since $v_n \rightharpoonup v$ in \mathcal{W} , we deduce that:

$$\int_{\Omega} (a(x, \nabla v_n) - a(x, \nabla v)) \cdot \nabla(v_n - v) dx \rightarrow 0.$$

Thus we infer that

$$\int_{\Omega} |\nabla(v_n - v)|^{p(x)} dx \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (4.5.21)$$

Indeed we split Ω into two parts: $\Omega^l = \{x \in \Omega : p(x) \leq 2\}$ and $\Omega^u = \{x \in \Omega : p(x) > 2\}$.

Since $\gamma_0 > 0$, (4.5.2) implies (4.5.21) directly on Ω^u . On Ω^l , we get from the Hölder inequality and (v_n) bounded in \mathcal{W} :

$$\begin{aligned} &\int_{\Omega^l} |\nabla(v_n - v)|^{p(x)} dx \\ &\leq c \left\| \frac{|\nabla(v_n - v)|^{p(x)}}{(|\nabla v| + |\nabla v_n|)^{r(x)}} \right\|_{L^{\frac{2}{p(x)}}(\Omega^l)} \left\| (|\nabla v| + |\nabla v_n|)^{r(x)} \right\|_{L^{\frac{2}{2-p(x)}}(\Omega^l)} \\ &\leq c_1 \left\| \frac{|\nabla(v_n - v)|^{p(x)}}{(|\nabla v| + |\nabla v_n|)^{r(x)}} \right\|_{L^{\frac{2}{p(x)}}(\Omega^l)} \stackrel{\text{def}}{=} c_1 \mathcal{N} \\ &\leq c_1 \left(\int_{\Omega^l} \frac{|\nabla(v_n - v)|^2 dx}{(|\nabla v| + |\nabla v_n|)^{2-p(x)}} dx \right)^{\hat{p}} \end{aligned}$$

where $r(x) = \frac{p(x)(2-p(x))}{2}$, $\hat{p} = \min\{1, \frac{p^+}{2}\}$ if $\mathcal{N} \leq 1$ and $\hat{p} = \frac{p^-}{2}$ otherwise.

Hence from (4.5.2), we conclude (4.5.21) in Ω^l and the convergence of (v_n) to v in \mathcal{W} . Then by using dominated convergence Theorem and classical compactness arguments, we obtain

$$a(x, \nabla v_n) \rightarrow a(x, \nabla v) \quad \text{in } \left(L^{\frac{p(x)}{p(x)-1}}(\Omega) \right)^N.$$

Finally passing to the limit in (4.5.10) satisfied by v_n and applying the dominated convergence Theorem, we obtain v is a weak solution of (4.5.9). The regularity arises from Proposition 4.5.2. \square

Next result is the extension of Corollary 4.4.5 for L^2 -potential.

Corollary 4.5.2. *Assume A satisfies (A_1) - (A_3) and f verifies (f_0) and (f_1) . Then, for any $q \in (1, p^-)$, $\lambda > 0$ and $h_0 \in L^2(\Omega) \cap L^r(\Omega) \setminus \{0\}$ for some $r > \max\{1, \frac{N}{p}\}$, $h_0 \geq 0$, there exists a solution u of (4.5.15). Namely, u belongs to $\dot{V}_+^q \cap L^\infty(\Omega)$ and satisfies (4.5.16) for any ψ verifying (4.5.17) and there exists $c > 0$ such that $u(x) \geq c\delta^q(x)$ a.e. in Ω .*

Proof. Noting that the existence of a weak solution $v_0 \in L^\infty(\Omega)$ of (4.5.9) for $h \in L^2(\Omega)$, can be obtained by global minimization method as in Theorem 4.5.4, we deduce from Theorem 4.5.10 that the solution obtained by Theorem 4.5.5 is a global minimizer.

Then we follow the same scheme as the proof of Corollary 4.5.1. We consider the functional energy \mathcal{E} defined on $\dot{V}_+^q \cap L^2(\Omega)$. We set $u_0 = v_0^q$. Then, u_0 belongs to $\dot{V}_+^q \cap L^\infty$ and (4.5.20) implies $u_0(x) \geq c\delta^q(x)$ a.e. in Ω .

Take ψ satisfying (4.5.17), then for t small enough, $\mathcal{E}(u_0 + t\psi) \geq \mathcal{E}(u_0)$. From classical arguments, we deduce that u_0 verifies (4.5.16). \square

4.5.3 Parabolic problem related to D.N.E.

In this section, we prove Theorems 4.5.1 by dividing the proof into three main steps: existence, uniqueness and regularity of weak solution. The proof of Theorem 4.5.1 (i) follows from the proof of Theorem 4.4.5 and using Lemma 4.5.1, Theorem 4.5.5 and Corollary 4.5.2. Thus we omit the proof.

4.5.3.1 Existence of a weak solution

In light of Remark 4.5.3 and improving Theorem 4.4.4 to $p(x)$ -homogeneous operator, we consider the problem (E) with $v_0 \in \mathcal{M}_\delta^1(\Omega) \cap \mathcal{W}$.

Theorem 4.5.6. *Under the assumptions of Theorem 4.5.1, there exists a solution v to (E) in sense of Definition 4.5.1. Furthermore v belongs to $C([0, T]; L^r(\Omega))$ for any $r \geq 1$ and there exists $C > 0$ such that, for any $t \in [0, T]$:*

$$\frac{1}{C}\delta(x) \leq v(t, x) \leq C\delta(x) \quad \text{a.e. in } \Omega. \quad (4.5.22)$$

Proof. The sketch of the proof is classical and in particular we follow the proof of Theorem 4.4.4. However, for the convenience of the readers, we give the entire proof due to the general form setting of the operator a which requires technical computations. We proceed in several steps:

Step 1: Semi-discretization in time of (E)

Let $n^* \in \mathbb{N}^*$ and set $\Delta_t = T/n^*$. For $n \in \llbracket 0, n^* \rrbracket$, we define $t_n = n\Delta_t$ and for $(t, x) \in [t_{n-1}, t_n) \times \Omega$:

$$h_{\Delta_t}(t, x) = h^n(x) \stackrel{\text{def}}{=} \frac{1}{\Delta_t} \int_{t_{n-1}}^{t_n} h(s, x) ds.$$

Thus $\|h_{\Delta_t}\|_{L^\infty(Q_T)} \leq \|h\|_{L^\infty(Q_T)}$ and let $\epsilon > 0$, then there exists a function $h_\epsilon \in C_0^1(Q_T)$ such that $h_\epsilon \rightarrow h$ in $L^2(Q_T)$. Since h_ϵ is uniformly continuous then $(h_\epsilon)_{\Delta_t} \rightarrow h_\epsilon$ in $L^2(Q_T)$ and by observing that $\|(h_\epsilon)_{\Delta_t} - h_{\Delta_t}\|_{L^2(Q_T)} \leq \|h_\epsilon - h\|_{L^2(Q_T)}$, then as $\Delta_t \rightarrow 0$,

$$\|h_{\Delta_t} - h\|_{L^2(Q_T)} \leq \|(h_\epsilon)_{\Delta_t} - h_{\Delta_t}\|_{L^2(Q_T)} + \|(h_\epsilon)_{\Delta_t} - h_\epsilon\|_{L^2(Q_T)} + \|h_\epsilon - h\|_{L^2(Q_T)} \rightarrow 0.$$

Applying Theorem 4.5.4 with $\lambda = \Delta_t$, $h_0 = \Delta_t h^n + v_{n-1}^q$, we define the implicit Euler scheme,

$$\begin{cases} \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right) v_n^{q-1} - \nabla \cdot a(x, \nabla v_n) = h^n v_n^{q-1} + f(x, v_n) & \text{in } \Omega; \\ v_n \geq 0 & \text{in } \Omega; \\ v_n = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.5.23)$$

where, for all $n \in \llbracket 1, n^* \rrbracket$, $v_n \in C^1(\bar{\Omega}) \cap \mathcal{M}_\delta^1(\Omega)$ is the weak solution in sense of Definition 4.5.2.

Step 2: Sub- and supersolution

In this step, we establish the existence of a subsolution \underline{w} and a supersolution \bar{w} of a suitable equations such that $v_n \in [\underline{w}, \bar{w}]$ for all $n \in \llbracket 0, n^* \rrbracket$.

As in Theorem 4.5.4, we prove, for any $\mu > 0$, there exists a unique weak solution, $\underline{w}_\mu \in C^1(\bar{\Omega}) \cap \mathcal{M}_\delta^1(\Omega)$, to

$$\begin{cases} -\nabla \cdot a(x, \nabla w) = \mu(\underline{h} w^{q-1} + f(x, w)) & \text{in } \Omega; \\ w \geq 0 & \text{in } \Omega; \\ w = 0 & \text{on } \partial\Omega, \end{cases} \quad (4.5.24)$$

where \underline{h} is defined in (\mathbf{H}_h) .

Let $\mu_1 < \mu_2$ and $\underline{w}_{\mu_1}, \underline{w}_{\mu_2}$ be weak solutions of (4.5.24). Then,

$$\begin{aligned} \int_{\Omega} a(x, \nabla \underline{w}_{\mu_1}) \cdot \nabla \phi \, dx &= \mu_1 \int_{\Omega} (\underline{h} \underline{w}_{\mu_1}^{q-1} + f(x, \underline{w}_{\mu_1})) \phi \, dx \\ \int_{\Omega} a(x, \nabla \underline{w}_{\mu_2}) \cdot \nabla \psi \, dx &= \mu_2 \int_{\Omega} (\underline{h} \underline{w}_{\mu_2}^{q-1} + f(x, \underline{w}_{\mu_2})) \psi \, dx. \end{aligned}$$

Summing the above equations with $\phi = \frac{(w_{\mu_1}^q - w_{\mu_2}^q)^+}{w_{\mu_1}^{q-1}}$ and $\psi = \frac{(w_{\mu_2}^q - w_{\mu_1}^q)^-}{w_{\mu_2}^{q-1}}$, then from (4.5.1) and (f_1) , we deduce $(\underline{w}_\mu)_\mu$ is nondecreasing. From Theorem 1.2 of [118] and Theorem 4.5.11 we obtain,

$$\|\underline{w}_\mu\|_{C^1(\bar{\Omega})} \leq \|\underline{w}_\mu\|_{C^{1,\alpha}(\bar{\Omega})} \leq C_{\mu_0}, \quad \|\underline{w}_\mu\|_{L^\infty} \rightarrow 0 \quad \mu \rightarrow 0 \quad (4.5.25)$$

for $\mu_0 > 0$ small enough, $\alpha \in (0, 1)$ and C_{μ_0} is independent of w_μ and α . Therefore, (4.5.25) implies $\{\underline{w}_\mu : \mu \leq \mu_0\}$ is uniformly bounded and equicontinuous in $C^1(\bar{\Omega})$. Applying Arzela-Ascoli Theorem, we obtain, up to a subsequence, $\underline{w}_\mu \rightarrow 0$ in $C^1(\bar{\Omega})$ as $\mu \rightarrow 0$. Then by Mean Value Theorem, we choose μ small enough such that $\underline{w} \stackrel{\text{def}}{=} \underline{w}_\mu \in C^1(\bar{\Omega}) \cap \mathcal{M}_\delta^1(\Omega)$ satisfies $0 < \underline{w} \leq v_0$.

Similarly, there exists $\bar{w}_\kappa \in C^1(\bar{\Omega}) \cap \mathcal{M}_\delta^1(\Omega)$ the weak solution of the following problem:

$$\begin{cases} -\nabla \cdot a(x, \nabla w) = \|h\|_{L^\infty(Q_T)} w^{q-1} + f(x, w) + \kappa & \text{in } \Omega; \\ w \geq 0 & \text{in } \Omega; \\ w = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.5.26)$$

By Theorem 4.5.11 and by comparison principle, we have for κ large enough that $\bar{w} \stackrel{\text{def}}{=} \bar{w}_\kappa \geq w_\kappa \geq v_0$ where w_κ is the weak solution of (4.5.49).

Rewrite (4.5.23) as follows

$$v_n^{2q-1} - \Delta_t \nabla \cdot a(x, \nabla v_n) = \Delta_t \left(h^n v_n^{q-1} + f(x, v_n) \right) + v_{n-1}^q v_n^{q-1}.$$

Since $\underline{w} \leq v_0 \leq \bar{w}$ and \underline{w}, \bar{w} are respectively a sub- and supersolution of the above equation for $n = 1$, Theorem 4.5.10 yields v_1 belongs to $[\underline{w}, \bar{w}]$ and by induction $v_n \in [\underline{w}, \bar{w}]$ for any $n \in \llbracket 1, n^* \rrbracket$.

Step 3: A priori estimates

Define the functions for $n \in \llbracket 1, n^* \rrbracket$ and $t \in [t_{n-1}, t_n)$

$$v_{\Delta_t}(t) = v_n \quad \text{and} \quad \tilde{v}_{\Delta_t}(t) = \frac{t - t_{n-1}}{\Delta_t} (v_n^q - v_{n-1}^q) + v_{n-1}^q$$

which satisfy

$$v_{\Delta_t}^{q-1} \partial_t \tilde{v}_{\Delta_t} - \nabla \cdot a(x, \nabla v_{\Delta_t}) = f(x, v_{\Delta_t}) + h^n v_{\Delta_t}^{q-1} \quad (4.5.27)$$

and by **Step 2**, there exists $c > 0$ independent of Δ_t such that for any $(t, x) \in Q_T$

$$\frac{1}{c} \delta(x) \leq v_{\Delta_t}, \quad \tilde{v}_{\Delta_t}^{1/q} \leq c \delta(x). \quad (4.5.28)$$

In (4.5.23), summing from 1 to $n' \in \llbracket 1, n^* \rrbracket$ and multiplying $\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}} \in \mathbf{X}$, Young's inequality implies

$$\begin{aligned} & \frac{1}{2} \sum_{n=1}^{n'} \int_{\Omega} \Delta_t \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right)^2 dx + \sum_{n=1}^{n'} \int_{\Omega} a(x, \nabla v_n) \cdot \nabla \left(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}} \right) dx \\ & \leq 2 \sum_{n=1}^{n'} \Delta_t \|h^n\|_{L^2}^2 + 2 \sum_{n=1}^{n'} \Delta_t \left\| \frac{f(x, v_n)}{v_n^{q-1}} \right\|_{L^2}^2. \end{aligned} \quad (4.5.29)$$

Since $v_n \in [\underline{w}, \bar{w}] \subset \mathcal{M}_\delta^1(\Omega)$, (4.5.13) and (f_2) insure that $\frac{f(x, v_n)}{v_n^{q-1}}$ is uniformly bounded in $L^2(\Omega)$ in Δ_t . Hence, combining (4.5.1), (4.5.8) and (4.5.29), we deduce, for any $n' \geq 1$:

$$\begin{aligned} & \int_{\Omega} \frac{c_1 |\nabla v_{n'}|^{p(x)} - c_2 |\nabla v_0|^{p(x)}}{p(x)} dx \leq \int_{\Omega} \frac{q}{p(x)} (A(x, \nabla v_{n'}) - A(x, \nabla v_0)) dx \\ & \leq \sum_{n=1}^{n'} \int_{\Omega} \frac{q}{p(x)} (A(x, \nabla v_n) - A(x, \nabla v_{n-1})) dx \\ & \leq \sum_{n=1}^{n'} \int_{\Omega} a(x, \nabla v_n) \cdot \nabla \left(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}} \right) dx \leq c_3 \end{aligned}$$

where the constants $c_1 = \frac{q\gamma}{p^+-1}$ and $c_2 = \frac{q\Gamma}{p^- - 1}$. The above inequality implies that

$$(v_{\Delta_t}) \text{ is bounded in } L^\infty(0, T; \mathcal{W}) \text{ uniformly in } \Delta_t \quad (4.5.30)$$

and from (4.5.29), we deduce

$$(\partial_t \tilde{v}_{\Delta_t}) \text{ is bounded in } L^2(Q_T) \text{ uniformly in } \Delta_t. \quad (4.5.31)$$

Moreover, for $\tilde{t} = \frac{t - t_{n-1}}{\Delta_t}$, we have

$$\nabla(\tilde{v}_{\Delta_t}^{\frac{1}{q}}) = \left(\tilde{t} + (1 - \tilde{t}) \left(\frac{v_{n-1}}{v_n} \right)^q \right)^{\frac{1-q}{q}} \left(\tilde{t} \nabla v_n + (1 - \tilde{t}) \left(\frac{v_{n-1}}{v_n} \right)^{q-1} \nabla v_{n-1} \right).$$

Hence we deduce from (4.5.30) and **Step 2** that

$$(\tilde{v}_{\Delta_t}^{1/q}) \text{ is bounded in } L^\infty(0, T; \mathcal{W}) \text{ uniformly in } \Delta_t.$$

Furthermore using (4.5.19), (4.5.31) implies

$$\sup_{[0, T]} \|\tilde{v}_{\Delta_t}^{1/q} - v_{\Delta_t}\|_{L^{2q}(\Omega)} \leq \sup_{[0, T]} \|\tilde{v}_{\Delta_t} - v_{\Delta_t}^q\|_{L^2(\Omega)} \leq o_{\Delta_t}(1). \quad (4.5.32)$$

Gathering (4.5.30)-(4.5.32), up to a subsequence, $v_{\Delta_t}, \tilde{v}_{\Delta_t}^{1/q} \xrightarrow{*} v$ in $L^\infty(0, T; \mathcal{W})$ as $\Delta_t \rightarrow 0$. From (4.5.28) and (4.5.31) we deduce that $(\tilde{v}_{\Delta_t})_{\Delta_t}$ is equicontinuous in $C([0, T]; L^r(\Omega))$ for any $r \in [1, +\infty)$. Moreover, from (4.5.19), we also deduce that $(\tilde{v}_{\Delta_t}^{1/q})_{\Delta_t}$ is uniformly equicontinuous in $C([0, T]; L^r(\Omega))$ for any $r \in [1, +\infty)$. Thus, by Arzela Theorem, we get up to a subsequence that for any $r \in [1, +\infty)$

$$\tilde{v}_{\Delta_t} \rightarrow v^q \text{ in } C([0, T]; L^r(\Omega)) \text{ and } v_{\Delta_t} \rightarrow v \text{ in } L^\infty(0, T; L^r(\Omega)), \quad (4.5.33)$$

hence (4.5.28) implies (4.5.22). From (4.5.31) and (4.5.33), we obtain

$$\partial_t \tilde{v}_{\Delta_t} \rightarrow \partial_t(v^q) \text{ in } L^2(Q_T). \quad (4.5.34)$$

Step 4: v satisfies (4.5.3)

From (4.5.33) and (4.5.34), we have as $\Delta_t \rightarrow 0^+$

$$\left| \int_{Q_T} v_{\Delta_t}^{q-1} (v_{\Delta_t} - v) \partial_t \tilde{v}_{\Delta_t} dxdt \right| + \left| \int_{Q_T} h^n v_{\Delta_t}^{q-1} (v_{\Delta_t} - v) dxdt \right| \rightarrow 0$$

and from (f_0) , (4.5.28) and (4.5.33), we obtain

$$\int_{Q_T} f(x, v_{\Delta_t}) (v_{\Delta_t} - v) dxdt \rightarrow 0 \text{ as } \Delta_t \rightarrow 0^+.$$

Then, multiplying (4.5.27) to $(v_{\Delta_t} - v)$ and passing to the limit, we obtain

$$\int_{Q_T} a(x, \nabla v_{\Delta_t}) \cdot \nabla (v_{\Delta_t} - v) dxdt \rightarrow 0 \text{ as } \Delta_t \rightarrow 0^+.$$

Since $v_{\Delta_t} \xrightarrow{*} v$ in $L^\infty(0, T; \mathcal{W})$ and from the above limit, we conclude

$$\int_{Q_T} (a(x, \nabla v_{\Delta_t}) - a(x, \nabla v)) \cdot \nabla (v_{\Delta_t} - v) dxdt \rightarrow 0 \text{ as } \Delta_t \rightarrow 0^+.$$

By (4.5.2) and classical compactness arguments, we get

$$a(x, \nabla v_{\Delta_t}) \rightarrow a(x, \nabla v) \text{ in } (L^{p(x)/(p(x)-1)}(Q_T))^N. \quad (4.5.35)$$

Now, we pass to the limit in (4.5.27). First we remark that $(v_{\Delta_t}^{q-1})$ converges to v^{q-1} in $L^2(Q_T)$. Indeed (4.5.19) and (4.5.32)-(4.5.33) imply as $\Delta_t \rightarrow 0$:

$$\begin{aligned} \|v_{\Delta_t}^{q-1} - v^{q-1}\|_{L^2(Q_T)}^{\frac{2q}{q-1}} &\leq C \int_{Q_T} |v_{\Delta_t}^{q-1} - v^{q-1}|^{\frac{2q}{q-1}} dxdt \\ &\leq C \int_{Q_T} |v_{\Delta_t}^q - v^q|^2 dxdt \\ &\leq C \sup_{[0, T]} \left(\|v_{\Delta_t}^q - \tilde{v}_{\Delta_t}\|_{L^2}^2 + \|\tilde{v}_{\Delta_t} - v^q\|_{L^2}^2 \right) \rightarrow 0. \end{aligned}$$

Hence plugging (4.5.31) and **Step 1**, we have in $L^2(Q_T)$:

$$v_{\Delta_t}^{q-1} \partial_t \tilde{v}_{\Delta_t} \rightarrow v^{q-1} \partial_t (v^q) \quad \text{and} \quad h_{\Delta_t} v_{\Delta_t}^{q-1} \rightarrow h v^{q-1}.$$

Thus, we deduce, for any $\phi \in L^2(Q_T)$ as $\Delta_t \rightarrow 0^+$:

$$\left| \int_{Q_T} \left(v_{\Delta_t}^{q-1} \partial_t \tilde{v}_{\Delta_t} - v^{q-1} \partial_t (v^q) \right) \phi dxdt \right| + \left| \int_{Q_T} \left(h_{\Delta_t} v_{\Delta_t}^{q-1} - h v^{q-1} \right) \phi dxdt \right| \rightarrow 0.$$

Furthermore from (4.5.13) and (4.5.28), $(f(x, v_{\Delta_t})\phi)$ is uniformly bounded in $L^2(Q_T)$ in Δ_t and by (4.5.33) we have $f(x, v_{\Delta_t})\phi \rightarrow f(x, v)\phi$ a.e in Q_T (up to a subsequence). Then, by dominated convergence Theorem we obtain

$$\int_{Q_T} f(x, v_{\Delta_t})\phi dxdt \rightarrow \int_{Q_T} f(x, v)\phi dxdt \text{ as } \Delta_t \rightarrow 0. \quad (4.5.36)$$

Finally gathering (4.5.35)-(4.5.36), we conclude that v satisfies (4.5.3) by passing to the limit in (4.5.27) for any $\phi \in L^2(Q_T) \cap L^1(0, T; \mathcal{W})$. \square

4.5.3.2 Uniqueness

Proof of Theorem 4.5.2. Let $\epsilon \in (0, 1)$, we take

$$\phi = \frac{(v + \epsilon)^q - (w + \epsilon)^q}{(v + \epsilon)^{q-1}} \quad \text{and} \quad \Psi = \frac{(w + \epsilon)^q - (v + \epsilon)^q}{(w + \epsilon)^{q-1}} \quad (4.5.37)$$

both belonging to $L^2(Q_T) \cap L^1(0, T; \mathcal{W})$, in

$$\begin{aligned} & \int_0^t \int_{\Omega} \partial_t(v^q)v^{q-1}\phi \, dxds + \int_0^t \int_{\Omega} a(x, \nabla v) \cdot \nabla \phi \, dxds \\ & \quad = \int_0^t \int_{\Omega} h(s, x)v^{q-1}\phi \, dxds + \int_0^t \int_{\Omega} f(x, v)\phi \, dxds, \\ & \int_0^t \int_{\Omega} \partial_t(w^q)w^{q-1}\psi \, dxds + \int_0^t \int_{\Omega} a(x, \nabla w) \cdot \nabla \psi \, dxds \\ & \quad = \int_0^t \int_{\Omega} \tilde{h}(s, x)w^{q-1}\psi \, dxds + \int_0^t \int_{\Omega} f(x, w)\psi \, dxds \end{aligned}$$

and summing the above equalities, we obtain $\mathbf{I}_\epsilon = \mathbf{J}_\epsilon$ where

$$\begin{aligned} \mathbf{I}_\epsilon &= \int_0^t \int_{\Omega} \left(\frac{\partial_t(v^q)v^{q-1}}{(v + \epsilon)^{q-1}} - \frac{\partial_t(w^q)w^{q-1}}{(w + \epsilon)^{q-1}} \right) ((v + \epsilon)^q - (w + \epsilon)^q) \, dxds \\ & \quad + \int_0^t \int_{\Omega} a(x, \nabla(v + \epsilon)) \cdot \nabla \left(\frac{(v + \epsilon)^q - (w + \epsilon)^q}{(v + \epsilon)^{q-1}} \right) \, dxds \\ & \quad + \int_0^t \int_{\Omega} a(x, \nabla(w + \epsilon)) \cdot \nabla \left(\frac{(w + \epsilon)^q - (v + \epsilon)^q}{(w + \epsilon)^{q-1}} \right) \, dxds \end{aligned}$$

and

$$\begin{aligned} \mathbf{J}_\epsilon &= \int_0^t \int_{\Omega} \left(\frac{hv^{q-1}}{(v + \epsilon)^{q-1}} - \frac{\tilde{h}w^{q-1}}{(w + \epsilon)^{q-1}} \right) ((v + \epsilon)^q - (w + \epsilon)^q) \, dxds \\ & \quad + \int_0^t \int_{\Omega} \left(\frac{f(x, v)}{(v + \epsilon)^{q-1}} - \frac{f(x, w)}{(w + \epsilon)^{q-1}} \right) ((v + \epsilon)^q - (w + \epsilon)^q) \, dxds. \end{aligned}$$

First we consider \mathbf{I}_ϵ . Since $\frac{w}{w+\epsilon}, \frac{v}{v+\epsilon} \leq 1$ and $v, w \in L^\infty(Q_T)$, we have

$$\left| \frac{\partial_t(v^q)v^{q-1}}{(v + \epsilon)^{q-1}} - \frac{\partial_t(w^q)w^{q-1}}{(w + \epsilon)^{q-1}} \right| |(v + \epsilon)^q - (w + \epsilon)^q| \leq C(|\partial_t(v^q)| + |\partial_t(w^q)|)$$

where C depends on the L^∞ norm of v and w . Moreover, as $\epsilon \rightarrow 0$

$$\left(\frac{\partial_t(v^q)v^{q-1}}{(v + \epsilon)^{q-1}} - \frac{\partial_t(w^q)w^{q-1}}{(w + \epsilon)^{q-1}} \right) ((v + \epsilon)^q - (w + \epsilon)^q) \rightarrow \frac{1}{2} \partial_t(v^q - w^q)^2$$

a.e. in Q_T . Then dominated convergence Theorem and Lemma 4.5.1 give

$$\lim_{\epsilon \rightarrow 0} \mathbf{I}_\epsilon \geq \frac{1}{2} \int_0^t \int_{\Omega} \partial_t(v^q - w^q)^2 \, dxds.$$

In the same way for \mathbf{J}_ϵ , dominated convergence Theorem implies

$$\begin{aligned} \int_0^t \int_\Omega \left(\frac{hv^{q-1}}{(v+\epsilon)^{q-1}} - \frac{\tilde{h}w^{q-1}}{(w+\epsilon)^{q-1}} \right) ((v+\epsilon)^q - (w+\epsilon)^q) dx ds \\ \rightarrow \int_0^t \int_\Omega (h - \tilde{h})(v^q - w^q) dx ds. \end{aligned}$$

Moreover Fatou's Lemma gives

$$\begin{aligned} \liminf_{\epsilon \rightarrow 0} \int_0^t \int_\Omega \frac{f(x, v)}{(v+\epsilon)^{q-1}} (w+\epsilon)^q dx ds &\geq \int_0^t \int_\Omega \frac{f(x, v)}{v^{q-1}} w^q dx ds, \\ \liminf_{\epsilon \rightarrow 0} \int_0^t \int_\Omega \frac{f(x, w)}{(w+\epsilon)^{q-1}} (v+\epsilon)^q dx ds &\geq \int_0^t \int_\Omega \frac{f(x, w)}{w^{q-1}} v^q dx ds. \end{aligned}$$

Hence gathering the three last limits and from (f_1) , we obtain

$$\liminf_{\epsilon \rightarrow 0} \mathbf{J}_\epsilon \leq \int_0^t \int_\Omega (h - \tilde{h})(v^q - w^q) dx ds.$$

Since $\mathbf{I}_\epsilon = \mathbf{J}_\epsilon$, we conclude using Hölder inequality that for any $t \in [0, T]$

$$\frac{1}{2} \int_0^t \int_\Omega \partial_t (v^q - w^q)^2 dx ds \leq \int_0^t \|h - \tilde{h}\|_{L^2(\Omega)} \|v^q - w^q\|_{L^2(\Omega)} ds$$

and by Grönwall Lemma (Lemma A.4 in [68]) we deduce (4.5.5). \square

Hence we conclude the uniqueness of the solution in sense of Definition 4.5.1 in Theorem 4.5.1:

Corollary 4.5.3. *Let v be a solution of (E) in sense of Definition 4.5.1 with the initial data $v_0 \in L^{2q}(\Omega)$, $v_0 \geq 0$ and $h \in L^2(Q_T)$. Then, v is unique.*

From Theorem 4.5.6 and Corollary 4.5.3, we deduce the existence result for the parabolic problem involving the operator \mathcal{T}_q :

Theorem 4.5.7. *Under the assumptions of Theorem 4.5.1, for any u_0 such that $u_0^{1/q} \in \mathcal{M}_\delta^1(\Omega) \cap \mathcal{W}$, there exists a unique weak solution $u \in L^\infty(Q_T)$ of*

$$\left\{ \begin{array}{ll} \partial_t u + \mathcal{T}_q u = h & \text{in } Q_T; \\ u > 0 & \text{in } Q_T; \\ u = 0 & \text{on } \Gamma; \\ u(0, \cdot) = u_0 & \text{in } \Omega, \end{array} \right. \quad (4.5.38)$$

in the sense that:

- $u^{1/q}$ belongs to $L^\infty(0, T; \mathcal{W})$, $\partial_t u \in L^2(Q_T)$;

- there exists $c > 0$ such that for any $t \in [0, T]$, $\frac{1}{c}\delta^q(x) \leq u(t, x) \leq c\delta^q(x)$ a.e. in Ω ;
- u satisfies, for any $t \in [0, T]$:

$$\begin{aligned} \int_0^t \int_{\Omega} \partial_t u \psi \, dx ds + \int_0^t \int_{\Omega} a(x, \nabla u^{1/q}) \cdot \nabla (u^{\frac{1-q}{q}} \psi) \, dx ds \\ = \int_0^t \int_{\Omega} f(x, u^{1/q}) u^{\frac{1-q}{q}} \psi \, dx ds + \int_0^t \int_{\Omega} h(s, x) \psi \, dx ds, \end{aligned} \quad (4.5.39)$$

for any ψ such that

$$|\psi|^{1/q} \in L^\infty(0, T; L^\infty(\Omega)) \text{ and } \frac{|\nabla \psi|}{\delta^{q-1}(\cdot)} \in L^1(0, T; L^{p(x)}(\Omega)). \quad (4.5.40)$$

Moreover, u belongs to $C([0, T]; L^r(\Omega))$ for any $r \in [1, +\infty)$.

Proof. Let v be the weak solution of (E) in sense of Definition 4.5.1 obtained by Theorem 4.5.6. Then, setting in (4.5.3) $u = v^q$ and choosing $\phi = \frac{\psi}{v^{q-1}}$ with ψ satisfying (4.5.40), we get the existence of a solution of (4.5.38).

Let us consider the uniqueness issue: let \tilde{u} be another solution of (4.5.38). We set $\tilde{v} = \tilde{u}^{1/q}$ and taking $\psi = v^{q-1}\phi$ with $\phi \in L^\infty(0, T; L^\infty(\Omega)) \cap L^1(0, T; \mathcal{W})$ in (4.5.39), we obtain that \tilde{v} verifies (4.5.3) with the additional condition $\phi \in L^\infty(0, T; L^\infty(\Omega))$. Since v, \tilde{v} verify (4.5.22), the test functions defined in (4.5.37) with v and \tilde{v} belong to $L^\infty(0, T; L^\infty(\Omega))$. Hence (4.5.5) holds and we conclude the uniqueness. \square

4.5.3.3 Regularity of weak solution

Theorem 4.5.8. *Under the assumptions of Theorem 4.5.1, assume in addition A satisfies (A_4) . Then, v the weak solution of (E) obtained by Theorem 4.5.6 belongs to $C([0, T]; \mathcal{W})$.*

Proof. The proof is similar as the proof of Theorem 1.1, Step 4 in [146]. However, the nonlinear term in time implies a specific approach in the computations. Hence for the reader's convenience, we include the complete proof.

We have $v \in L^\infty(0, T; \mathcal{W}) \cap C([0, T]; L^{p^-}(\Omega))$ and $p \in C^1(\overline{\Omega})$, Theorem 8.4.2 in [112] yields $\mathcal{W} \subset L^{p^-}(\Omega)$ with compact embedding. So we deduce $t \mapsto v(t)$ is weakly continuous in \mathcal{W} . Moreover, we consider the mapping $\mathcal{K}(v) = \int_{\Omega} \frac{A(x, \nabla v)}{p(x)} dx$ defined in \mathcal{W} . The convexity of A implies that \mathcal{K} is weakly lower semicontinuous. Thus for any $t_0 \in [0, T]$, we have

$$\mathcal{K}(v(t_0)) \leq \liminf_{t \rightarrow t_0} \mathcal{K}(v(t)). \quad (4.5.41)$$

In (4.5.23), summing from n' to n'' and multiplying by $\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}} \in \mathbf{X}$, we obtain

$$\sum_{n=n'}^{n''} \int_{\Omega} \Delta_t \left(\frac{v_n^q - v_{n-1}^q}{\Delta_t} \right)^2 dx + \sum_{n=n'}^{n''} \int_{\Omega} a(x, \nabla v_n) \cdot \nabla \left(\frac{v_n^q - v_{n-1}^q}{v_n^{q-1}} \right) dx$$

$$= \sum_{n=n'}^{n''} \int_{\Omega} h^n (v_n^q - v_{n-1}^q) dx + \sum_{n=n'}^{n''} \int_{\Omega} \frac{f(x, v_n)}{v_n^{q-1}} (v_n^q - v_{n-1}^q) dx.$$

As in **Step 4** of the proof of Theorem 4.5.6, after using Lemma 4.5.1 we pass to the limit as $n \rightarrow \infty$ and we get: for $t \in [t_0, T]$

$$\begin{aligned} \int_{t_0}^t \int_{\Omega} \partial_t (v^q)^2 dx ds + q\mathcal{K}(v(t)) &\leq \int_{t_0}^t \int_{\Omega} h \partial_t (v^q) dx ds + q\mathcal{K}(v(t_0)) \\ &+ \int_{t_0}^t \int_{\Omega} \frac{f(x, v)}{v^{q-1}} \partial_t (v^q) dx ds. \end{aligned} \quad (4.5.42)$$

Taking lim sup in (4.5.42) as $t \rightarrow t_0^+$ and by (4.5.41) we deduce

$$\lim_{t \rightarrow t_0^+} \mathcal{K}(v(t)) = \mathcal{K}(v(t_0))$$

and hence we get the right-continuity of \mathcal{K} .

Now, for $t > t_0$, let $\eta \in (0, t - t_0)$. We multiply (E) by $\tau_{\eta} v = \frac{v^q(\cdot + \eta, \cdot) - v^q}{\eta v^{q-1}} \in L^2(Q_T) \cap L^1(0, T; \mathcal{W})$ and integrate over $(t_0, t) \times \Omega$ and hence by using Theorem 4.4.1 and Young inequality, we obtain:

$$\begin{aligned} \int_{t_0}^t \int_{\Omega} v^{q-1} \partial_t (v^q) \tau_{\eta} v dx ds + \frac{q}{\eta} \int_{t_0}^t \mathcal{K}(v(s + \eta)) - \mathcal{K}(v(s)) ds \\ \geq \int_{t_0}^t \int_{\Omega} h v^{q-1} \tau_{\eta} v dx ds + \int_{t_0}^t \int_{\Omega} f(x, v) \tau_{\eta} v dx ds. \end{aligned} \quad (4.5.43)$$

Since $v \in L^{\infty}(0, T; \mathcal{W})$ and \mathcal{K} is right-continuous in \mathcal{W} , by dominated convergence Theorem, we have as $\eta \rightarrow 0^+$

$$\frac{1}{\eta} \int_{t_0}^{t_0+\eta} \mathcal{K}(v(s)) ds \rightarrow \mathcal{K}(v(t_0)) \quad \text{and} \quad \frac{1}{\eta} \int_t^{t+\eta} \mathcal{K}(v(s)) ds \rightarrow \mathcal{K}(v(t)).$$

Then (4.5.43) yields,

$$\begin{aligned} \int_{t_0}^t \int_{\Omega} \partial_t (v^q)^2 dx ds + q\mathcal{K}(v(t)) &\geq \int_{t_0}^t \int_{\Omega} h \partial_t (v^q) dx ds + q\mathcal{K}(v(t_0)) \\ &+ \int_{t_0}^t \int_{\Omega} \frac{f(x, v)}{v^{q-1}} \partial_t (v^q) dx ds. \end{aligned}$$

From (4.5.42), we have the equality for any $t, t_0 \in [0, T]$ in the above inequality and we deduce the left-continuity of \mathcal{K} .

By (A₄), the proof of corollary A.3 in [152] holds by considering \mathcal{K} as the semimodular. Then, we deduce that $\nabla v(t)$ converges to $\nabla v(t_0)$ in $L^{p(x)}(\Omega)^N$ as $t \rightarrow t_0$ and hence $v \in C([0, T]; \mathcal{W})$. \square

4.5.4 Stabilization

4.5.4.1 Stationary problem related to (E)

In the aim of studying the behaviour of global solution of the problem (E) as $t \rightarrow \infty$, we consider the following problem

$$\begin{cases} -\nabla \cdot a(x, \nabla v) = b(x)v^{q-1} + f(x, v) & \text{in } \Omega; \\ v \geq 0 & \text{in } \Omega; \\ v = 0 & \text{on } \partial\Omega, \end{cases} \quad (S)$$

where $b \in L^\infty(\Omega)$. The notion of weak solution of (S) is defined as follows:

Definition 4.5.3. *A weak solution to (S) is any nonnegative function $v \in \mathcal{W} \cap L^\infty(\Omega)$, $v \not\equiv 0$ such that for any $\phi \in \mathcal{W}$, v satisfies*

$$\int_{\Omega} a(x, \nabla v) \cdot \nabla \phi \, dx = \int_{\Omega} b v^{q-1} \phi \, dx + \int_{\Omega} f(x, v) \phi \, dx. \quad (4.5.44)$$

Theorem 4.5.9. *Assume that A satisfies (A_1) - (A_3) and (f_0) and (f_1) hold. Then, for any $q \in (1, p^-)$, $b \in L^\infty(\Omega) \setminus \{0\}$, $b \geq 0$, there exists a unique weak solution $v \in C^1(\overline{\Omega}) \cap \mathcal{M}_\delta^1(\Omega)$ to (S).*

Proof. Consider the energy functional \mathcal{L} defined on \mathcal{W} such that

$$\tilde{\mathcal{L}}(v) = \int_{\Omega} \frac{A(x, \nabla v)}{p(x)} \, dx - \frac{1}{q} \int_{\Omega} b(v^+)^q \, dx - \int_{\Omega} F(x, v) \, dx$$

where F is defined as in (4.5.12). By following the same arguments as in Theorem 4.5.4, we deduce the existence of nonnegative global minimizer v_0 to \mathcal{L} and the Gâteaux differentiability of $\tilde{\mathcal{L}}$ implies v_0 satisfies (4.5.44).

Combining Proposition 4.5.1 and Theorem 4.1 in [120], we deduce $v_0 \in L^\infty(\Omega)$. Then by Theorem 1.2 of [118], we obtain, $v_0 \in C^{1,\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. From Lemma 4.5.2, we deduce $v_0 > 0$ and v_0 belongs to $\mathcal{M}_\delta^1(\Omega)$.

Let \tilde{v}_0 another solution of (S). As previously, we deduce that $\tilde{v}_0 \in C^{1,\alpha}(\overline{\Omega}) \cap \mathcal{M}_\delta^1(\Omega)$.

We choose $\frac{v_0^q - \tilde{v}_0^q}{v_0^{q-1}}$ and $\frac{\tilde{v}_0^q - v_0^q}{\tilde{v}_0^{q-1}}$ as test functions in (4.5.44) satisfied by v_0 respectively \tilde{v}_0 , then adding the both equations we deduce from Lemma 4.5.1 and (f_1) :

$$\int_{\Omega} a(x, \nabla v_0) \cdot \nabla \left(\frac{v_0^q - \tilde{v}_0^q}{v_0^{q-1}} \right) + a(x, \nabla \tilde{v}_0) \cdot \nabla \left(\frac{\tilde{v}_0^q - v_0^q}{\tilde{v}_0^{q-1}} \right) \, dx \leq 0.$$

Applying once again Lemma 4.5.1, we obtain $v_0 = \tilde{v}_0$. □

Hence we obtain using the same way of the proof of Corollary 4.5.1:

Corollary 4.5.4. *Under the conditions of Theorem 4.5.9, there exists a unique solution u of the following problem*

$$\begin{cases} \mathcal{T}_q u = b & \text{in } \Omega; \\ u > 0 & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.5.45)$$

Namely, u belongs to $\dot{V}_+^q \cap \mathcal{M}_\delta^{1/q}(\Omega)$ and satisfies, for any ψ such that (4.5.17):

$$\int_{\Omega} a(x, \nabla u^{1/q}) \cdot \nabla (u^{\frac{1-q}{q}} \psi) dx - \int_{\Omega} \frac{f(x, u^{1/q})}{u^{(q-1)/q}} \psi dx = \int_{\Omega} b \psi dx.$$

4.5.4.2 Proof of Theorem 4.5.3

Proof of Theorem 4.5.3. We consider two cases:

Case 1: $h \equiv h_\infty$.

We introduce the family $\{S(t); t \geq 0\}$ on $\mathcal{M}_\delta^{1/q}(\Omega) \cap \dot{V}_+^q$ defined as $w(t) = S(t)w_0$ where w is the solution obtained by Theorem 4.5.7 (and Theorem 4.5.6) of

$$\begin{cases} \partial_t w + \mathcal{T}_q w = h_\infty & \text{in } Q_T; \\ w > 0 & \text{in } Q_T; \\ w = 0 & \text{on } \Gamma; \\ w(0, \cdot) = w_0 & \text{in } \Omega. \end{cases} \quad (4.5.46)$$

Thus $\{S(t); t \geq 0\}$ defines a semigroup on $\mathcal{M}_\delta^{1/q}(\Omega) \cap \dot{V}_+^q$. Indeed the uniqueness and properties of solution of (4.5.38) imply for any w_0 ,

$$S(t+s)w_0 = S(t)S(s)w_0, \quad S(0)w_0 = w_0 \quad (4.5.47)$$

and from (4.5.33) the map $t \rightarrow S(t)w_0$ is continuous from $[0, \infty)$ to $L^2(\Omega)$.

Note that $v = (S(t)w_0)^{1/q}$ is the solution of (E) in the sense of Definition 4.5.1 with $h = h_\infty$ and the initial data $w_0^{1/q}$.

Let $T > 0$ and v be the solution of (E) obtained by Theorem 4.5.6 with $h \equiv h_\infty$ and the initial data v_0 , hence we get $u(t) = v(t)^q = S(t)u_0$ with $u_0 = v_0^q$.

Let $\underline{w} = w_\mu$ be the solution of (4.5.24) and $\bar{w} = \bar{w}_\kappa$ be the solution of (4.5.26). Then, $\underline{w}, \bar{w} \in \mathcal{M}_\delta^1(\Omega)$ and for μ small enough and κ large enough, \underline{w} is a subsolution and \bar{w} a supersolution of (S) with $b = h_\infty$ such that $\underline{w} \leq v_0 \leq \bar{w}$.

We define $\underline{u}(t) = S(t)\underline{w}^q$ and $\bar{u}(t) = S(t)\bar{w}^q$ the solutions to (4.5.46). So \underline{u} and \bar{u} are obtained by the iterative scheme (4.5.23) with $v_0 = \underline{w}$ and $v_0 = \bar{w}$. Hence, by construction the map $t \rightarrow \underline{u}(t)$ is nondecreasing, the map $t \rightarrow \bar{u}(t)$ is nonincreasing and (4.5.4) insures for any $t \geq 0$,

$$\underline{w}^q \leq \underline{u}(t) \leq u(t) \leq \bar{u}(t) \leq \bar{w}^q \text{ a. e. in } \Omega. \quad (4.5.48)$$

We set $\underline{u}_\infty = \lim_{t \rightarrow \infty} \underline{u}(t)$ and $\bar{u}_\infty = \lim_{t \rightarrow \infty} \bar{u}(t)$. Then from (4.5.47), the continuity in $L^2(\Omega)$ and monotone convergence theorem, we get in $L^2(\Omega)$:

$$\underline{u}_\infty = \lim_{s \rightarrow \infty} S(t+s)(\underline{w}^q) = S(t)(\lim_{s \rightarrow \infty} S(s)(\underline{w}^q)) = S(t)\underline{u}_\infty$$

and analogously we have $\bar{u}_\infty = S(t)\bar{u}_\infty$. We deduce \underline{u}_∞ and \bar{u}_∞ are solutions of (4.5.45) with $b = h_\infty$ and by uniqueness, we have $u_{\text{stat}} \stackrel{\text{def}}{=} \underline{u}_\infty = \bar{u}_\infty$ where u_{stat} is the stationary solution of perturbed parabolic problem (4.5.45). Therefore from (4.5.48) and dominated convergence Theorem, we obtain

$$\|u(t) - u_{\text{stat}}\|_{L^2} \rightarrow 0 \text{ as } t \rightarrow \infty.$$

Finally, using (4.5.48) and interpolation inequality $\|\cdot\|_r \leq \|\cdot\|_\infty^\theta \|\cdot\|_2^{1-\theta}$, we conclude the above convergence for any $r \geq 1$.

Case 2: $h \neq h_\infty$.

From (4.5.6), for any ε and for some $\eta' \in (0, \eta)$, there exists $t_0 > 0$ large enough such that for any $t \geq t_0$:

$$t^{1+\eta'} \|h(t, \cdot) - h_\infty\|_{L^2} \leq \varepsilon.$$

Let $T > 0$ and v be the solution of (E) obtained by Theorem 4.5.6 with h and the initial data $v_0 = u_0^{1/q}$ and we set $u = v^q$.

Since v satisfies (4.5.22), we can define $\tilde{u}(t) = S(t+t_0)u_0 = S(t)u(t_0)$. Then, by (4.5.4) and uniqueness, we have for any $t > 0$:

$$\|u(t+t_0, \cdot) - \tilde{u}(t, \cdot)\|_{L^2} \leq \int_0^t \|h(s+t_0, \cdot) - h_\infty\|_{L^2} ds \leq \frac{\varepsilon}{t_0^{\eta'}} \leq \varepsilon.$$

By **Case 1**, we have $\tilde{u}(t) \rightarrow u_{\text{stat}}$ in $L^2(\Omega)$ as $t \rightarrow \infty$. Therefore, we obtain

$$\|u(t) - u_{\text{stat}}\|_{L^2} \rightarrow 0 \text{ as } t \rightarrow \infty$$

and by using interpolation inequality we conclude the proof of Theorem 4.5.3. \square

4.5.5 Additional results

In this section, we give extensions of technical results for the class of operator A or for some boundary value problems.

We begin by extending Theorem 4.4.9 using Lemma 4.4.1. Then, we obtain the comparison principle:

Theorem 4.5.10. *Assume A satisfies (A_1) - (A_3) and f satisfies (f_0) and (f_2) . Let $\underline{v}, \bar{v} \in \mathbf{X} \cap L^\infty(\Omega)$ be nonnegative functions respectively subsolution and supersolution to (4.5.9) for some $h \in L^r(\Omega)$, $r \geq 2$, $h \geq 0$. Then $\underline{v} \leq \bar{v}$.*

The proof is similar as the proof of Theorem 4.5.2 where the sub- and supersolution do not need to belong to $\mathcal{M}_\delta^1(\Omega)$. The proof is very similar and we omit it. In the next theorem, we extend Lemma 2.1 of [117] and Lemma 3.2 of [146] for $p(x)$ -homogeneous operators.

Theorem 4.5.11. *Assume A satisfies (A_1) - (A_3) . Let $\lambda > 0$ and $w_\lambda \in \mathcal{W} \cap C^{1,\alpha}(\overline{\Omega})$ be the positive weak solution of*

$$\begin{cases} -\nabla \cdot a(x, w_\lambda) = \lambda & \text{in } \Omega; \\ w_\lambda = 0 & \text{on } \partial\Omega. \end{cases} \quad (4.5.49)$$

Then, there exists $\lambda^ > 0$ such that w_λ satisfies*

- *for any $\lambda \geq \lambda^*$, $\|w_\lambda\|_{L^\infty(\Omega)} \leq C_1\lambda^{1/(p^- - 1)}$ and $w_\lambda(x) \geq C_2\lambda^{\frac{1}{p^+ - 1 + \varepsilon}}\delta(x)$ for some $\varepsilon \in (0, 1)$;*
- *for $\lambda < \lambda^*$, $\|w_\lambda\|_{L^\infty(\Omega)} \leq C_3\lambda^{1/(p^+ - 1)}$*

where the constants depend upon p^+, p^-, N, Ω and α . Moreover if $\lambda_1 < \lambda_2$ then $w_{\lambda_1} \leq w_{\lambda_2}$.

Now we state a Strong and Hopf maximum principle for variable exponent $p(x)$ -homogeneous operators and their proof follows from Lemma 4.5.2 and Lemma 4.4.4.

Lemma 4.5.2. *Let α, β be two measurable functions such that $1 < \beta_- \leq \beta_+ < \alpha_- \leq \alpha_+ < \infty$. Let $h, l \in L^\infty(\Omega)$ be nonnegative functions, $h > 0$ and $k : \Omega \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and A satisfies (A_1) - (A_2) . Consider $u \in C^1(\overline{\Omega})$ a nonnegative and nontrivial solution to*

$$\begin{cases} -\nabla \cdot a(x, \nabla u) + l(x)u^{\alpha(x)-1} = h(x)u^{\beta(x)-1} + k(x, u) & \text{in } \Omega; \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

If $\liminf_{t \rightarrow 0^+} k(x, t)t^{1-\alpha(x)} > \|l\|_{L^\infty}$ uniformly in $x \in \Omega$, then u is positive in Ω .

Furthermore, if Ω satisfies the interior ball condition for any $x \in \partial\Omega$, then $\frac{\partial u}{\partial \vec{n}}(x) < 0$ where \vec{n} is the outward unit normal vector at x .

We state a slight extension of Proposition 4.4.11 and Proposition 4.4.12.

Proposition 4.5.1. *Let $q \in [1, p^-)$. Assume A satisfies (A_1) - (A_3) and $u \in \mathbf{X}$ satisfying for any $\Psi \in \mathbf{X}$:*

$$\int_{\Omega} a(x, \nabla u) \cdot \nabla \Psi \, dx = \int_{\Omega} hu^{q-1}\Psi \, dx$$

where $h \in L^2(\Omega) \cap L^r(\Omega)$ with $r > \max\{1, \frac{N}{p}\}$. Then $u \in L^\infty(\Omega)$.

Proposition 4.5.2. *Under the assumptions of Proposition 4.5.1, consider $u \in \mathbf{X}$ a nonnegative function satisfying, for any $\Psi \in \mathbf{X}$, $\Psi \geq 0$:*

$$\int_{\Omega} u^{2q-1}\Psi \, dx + \int_{\Omega} a(x, \nabla u) \cdot \nabla \Psi \, dx \leq \int_{\Omega} (f(x, u) + hu^{q-1})\Psi \, dx$$

where f verifies for any $(x, t) \in \Omega \times \mathbb{R}^+$, $|f(x, t)| \leq c_1 + c_2|t|^{s(x)-1}$ with $s \in C(\overline{\Omega})$ such that for any $x \in \overline{\Omega}$, $1 < s(x) < p^(x)$ and $h \in L^2(\Omega) \cap L^r(\Omega)$ with $r > \max\{1, \frac{N}{p}\}$. Then $u \in L^\infty(\Omega)$.*

The proofs of above results follow the proofs of Theorem 4.1 in [120] and Proposition 4.4.11.

Kirchhoff equations and systems involving exponential non-linearity of Choquard type and singular weights

This work is done jointly with Jacques Giacomoni, Tuhina Mukherjee and Konijeti Sreenadh.

Abstract : In this chapter, we focus on Kirchhoff type Choquard equations and systems involving exponential non-linearity and singular weights. The main feature of this chapter is three fold. Firstly, we prove the existence of solution using the variational method in light of Adams, Moser and Trudinger inequalities and the mountain pass Lemma.

Secondly, we study the existence and multiplicity for the problem with an extra sublinear sign changing term by using the Nehari manifold technique. By analyzing the Fiberings maps and extracting the Palais-Smale sequence in the natural decomposition of the Nehari manifold, we prove the multiplicity of the weak solutions with respect to an unknown parameter in the subcritical case. In the critical case (for the second order operator), we again use the concentration compactness together with the accurate analysis of the energy levels on the Nehari manifold to determine potential concentration phenomenon for associated Palais-Smale sequence. Based on this analysis we show the existence of a relatively compact Palais-Smale sequence which yields atleast one solution.

Thirdly, we prove new singular and non-singular version of Adams, Moser and Trudinger inequalities in the Cartesian product of Sobolev space. As an application of these inequalities, we further study the system of Kirchhoff equation with exponential non-linearity of Choquard type for both non-degenerate and degenerate case.

5.1 n-Kirchhoff Choquard equation with exponential non-linearity

In this section, we study the following Kirchhoff equation with exponential non-linearity of Choquard type

$$(KC) \begin{cases} -M\left(\int_{\Omega} |\nabla u|^n dx\right) \Delta_n u = \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy\right) f(x, u), & u > 0 \quad \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\mu \in (0, n)$, Ω is a smooth bounded domain in \mathbb{R}^n , $n \geq 2$, the function F denotes the primitive of f with respect to the second variable (vanishing at 0). The function $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function satisfying the following conditions:

(m1) There exists $m_0 > 0$ such that $m(t) \geq m_0$ for all $t \geq 0$ and $\mathcal{M}(t) = \int_0^t M(s) ds$ satisfies

$$\mathcal{M}(t+s) \geq \mathcal{M}(t) + \mathcal{M}(s), \text{ for all } t, s \geq 0.$$

(m2) There exist constants $b_1, b_2 > 0$ and $\hat{t} > 0$ such that for some $r \in \mathbb{R}$

$$M(t) \leq b_1 + b_2 t^r, \text{ for all } t \geq \hat{t}.$$

(m3) The function $\frac{M(t)}{t}$ is non-increasing for $t > 0$.

Example 5.1.1. An example of a function satisfying (m1), (m2) and (m3) is $M(t) = m_0 + bt^\beta$ where $m_0 > 0$, $\beta < 1$ and $b \geq 0$.

Using (m3), one can easily deduce that the function

$$(m3)' \quad \frac{1}{n} \mathcal{M}(t) - \frac{1}{\theta} M(t)t \text{ is non-negative and non-decreasing for } t \geq 0 \text{ and } \theta \geq 2n.$$

The function $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x, t) = h(x, t) \exp(|t|^{\frac{n}{n-1}})$. In the frame of problem (KC), $h \in C(\bar{\Omega} \times \mathbb{R})$ satisfies the following conditions:

(h1) $h(x, t) = 0$ for $t \leq 0$ and $h(x, t) > 0$ for $t > 0$.

(h2) For any $\epsilon > 0$, $\lim_{t \rightarrow \infty} \sup_{x \in \bar{\Omega}} h(x, t) \exp(-\epsilon |t|^{\frac{n}{n-1}}) = 0$ and $\lim_{t \rightarrow \infty} \inf_{x \in \bar{\Omega}} h(x, t) \exp(\epsilon |t|^{\frac{n}{n-1}}) = \infty$.

(h3) There exists $\ell > \max\{n-1, \frac{n(r+1)}{2}\}$ such that $t \rightarrow \frac{f(x, t)}{t^\ell}$ is increasing on $\mathbb{R}^+ \setminus \{0\}$, uniformly in $x \in \Omega$ where r is specified in (m2).

(h4) There exist $T, T_0 > 0$ and $\gamma_0 > 0$ such that $0 < t^{\gamma_0} F(x, t) \leq T_0 f(x, t)$ for all $|t| \geq T$ and uniformly in $x \in \Omega$.

The condition (h3) implies that $\frac{f(x, t)}{t^{n-1}}$ is increasing for each $t > 0$ and $\lim_{t \rightarrow 0^+} \frac{f(x, t)}{t^{n-1}} = 0$ uniformly in $x \in \Omega$.

Example 5.1.2. An example of functions satisfying (h1) – (h4) is

$$f(x, t) = t^{\beta_0+(n-1)} \exp(t^p) \exp(|t|^{\frac{n}{n-1}})$$

for $t \geq 0$ and $f(x, t) = 0$ for $t < 0$ where $0 \leq p < \frac{n}{n-1}$ and $\beta_0 > 0$.

We also study the existence and multiplicity of solutions to the following Kirchhoff equation with a convex-concave type non-linearity:

$$(\mathcal{P}_{\lambda, M}) \begin{cases} -M \left(\int_{\Omega} |\nabla u|^n dx \right) \Delta_n u = (|x|^{-\mu} * F(u)) f(u) + \lambda h(x) |u|^{q-1} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u > 0 & \text{in } \Omega \end{cases}$$

where $\mu \in (0, n)$, Ω is a smooth bounded domain in \mathbb{R}^n , $f(u) = u|u|^p \exp(|u|^\beta)$, $0 < q < n-1 < 2n-1 < p+1 = \beta_0 + (n-1)$, $\beta \in \left(1, \frac{n}{n-1}\right)$ and $F(t) = \int_0^t f(s) ds$. We assume $M(t) = at + b$ where $a, b > 0$ and $h \in L^r(\Omega)$, with $r = \frac{p+2}{p-q+1}$, satisfying $h^+ \not\equiv 0$. Throughout this section, we denote

$$\|u\| := \left(\int_{\Omega} |\nabla u|^n dx \right)^{1/n}.$$

Definition 5.1.3. We call a function $u \in W_0^{1,n}(\Omega)$ to be a solution of (KC) if

$$M(\|u\|^n) \int_{\Omega} |\nabla u|^{n-2} \nabla u \cdot \nabla \varphi dx = \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \right) f(x, u) \varphi dx, \text{ for all } \varphi \in W_0^{1,n}(\Omega).$$

The energy functional $E : W_0^{1,n}(\Omega) \rightarrow \mathbb{R}$ associated to (KC) is given by

$$E(u) = \frac{1}{n} \mathcal{M}(\|u\|^n) - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \right) F(x, u) dx.$$

Under the assumptions on f , we get that for any $\epsilon > 0$, $p \geq 1$ and $0 \leq \beta_0 < \ell$, there exists $C(\epsilon, n, \mu) > 0$ such that for each $x \in \Omega$

$$|F(x, t)| \leq \epsilon |t|^{\beta_0+1} + C(\epsilon, n, \mu) |t|^p \exp((1+\epsilon)|t|^{\frac{n}{n-1}}), \text{ for all } t \in \mathbb{R}. \quad (5.1.1)$$

For any $u \in W_0^{1,n}(\Omega)$, by virtue of Sobolev embedding we get that $u \in L^q(\Omega)$ for all $q \in [1, \infty)$. This also implies that

$$F(x, u) \in L^q(\Omega) \text{ for any } q \geq 1. \quad (5.1.2)$$

Taking $t = r = \frac{2n}{2n-\mu}$ in Proposition 2.2.6 and using (5.1.2), we get that E is well defined. Also $E \in C^1(W_0^{1,n}(\Omega), \mathbb{R})$. Naturally, the critical points of E corresponds to weak solutions of (KC) and for any $u \in W_0^{1,n}(\Omega)$ we have

$$\langle E'(u), \varphi \rangle = M(\|u\|^n) \int_{\Omega} |\nabla u|^{n-2} \nabla u \cdot \nabla \varphi dx - \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \right) f(x, u) \varphi dx$$

for all $\varphi \in W_0^{1,n}(\Omega)$.

The energy functional $\mathcal{J}_{\lambda,M} : W_0^{1,n}(\Omega) \longrightarrow \mathbb{R}$ associated to the problem $(\mathcal{P}_{\lambda,M})$ is defined as

$$\mathcal{J}_{\lambda,M}(u) = \frac{1}{n} \mathcal{M}(\|u\|^n) - \frac{\lambda}{q+1} \int_{\Omega} h(x)|u|^{q+1} dx - \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(u))F(u) dx$$

where $|x|^{-\mu} * F(u)$ denotes $\int_{\Omega} \frac{F(u(y))}{|x-y|^{\mu}} dy$, F , \mathcal{M} are anti-derivatives of f , M (vanishing at 0) respectively and $f(s) = s|s|^p \exp(|s|^{\beta})$.

Definition 5.1.4. A function $u \in W_0^{1,n}(\Omega)$ is said to be weak solution of $(\mathcal{P}_{\lambda,M})$ if $\forall \phi \in W_0^{1,n}(\Omega)$ we have

$$M(\|u\|^n) \int_{\Omega} |\nabla u|^{n-2} \nabla u \cdot \nabla \phi dx = \lambda \int_{\Omega} h(x)u^{q-1}u\phi dx + \int_{\Omega} (|x|^{-\mu} * F(u))f(u)\phi dx.$$

5.1.1 Main results

The following theorem is the main result concerning (KC) :

Theorem 5.1.5. Assume $(m1)$ - $(m3)$ and $(h1)$ - $(h4)$ holds. Assume in addition

$$\lim_{s \rightarrow +\infty} \frac{sf(x,s)F(x,s)}{\exp\left(2|s|^{\frac{n}{n-1}}\right)} = \infty, \text{ uniformly in } x \in \bar{\Omega}. \quad (5.1.3)$$

Then the problem (KC) admits a weak solution.

Example 5.1.6. f defined by $f(x,t) = g(x)t^p \exp(t^{\frac{n}{n-1}})$ for $t \geq 0$, $x \in \Omega$ with $0 \not\equiv g \in L^{\infty}(\Omega)$ non-negative and $p > n - 1$ satisfy $(h1)$ - $(h4)$ and (5.1.3).

Using the Nehari manifold technique, we show existence and multiplicity of solutions of the problem $(\mathcal{P}_{\lambda,M})$ with respect to the parameter λ . Precisely, we show the following main results in the subcritical and critical case:

Theorem 5.1.7. Let $\beta \in \left(1, \frac{n}{n-1}\right)$. Then there exists λ_0 such that $(\mathcal{P}_{\lambda,M})$ admits at least two solutions for $\lambda \in (0, \lambda_0)$.

In the critical case, we show the following existence result.

Theorem 5.1.8. Let $\beta = \frac{n}{n-1}$, then there exists $\lambda_1 > 0$ such that for $\lambda \in (0, \lambda_1)$, $(\mathcal{P}_{\lambda,M})$ admits a solution.

5.1.2 Existence of a positive weak solution

In this section, we study the problem (KC) and for that we use the mountain pass theorem and analyze accurately the compactness of Palais-Smale sequences for E . First we prove the energy functional E possesses the mountain pass geometry.

Lemma 5.1.9. *Assume the assumptions (m1), (m2) and (h1)-(h4). Then, E has the mountain pass geometry around 0 i.e.*

- (i) *there exists $R_0 > 0, \eta > 0$ such that $E(u) \geq \eta$ for all $u \in W_0^{1,n}(\Omega)$ such that $\|u\| = R_0$.*
- (ii) *there exists a $v \in W_0^{1,n}(\Omega)$ with $\|v\| > R_0$ such that $E(v) < 0$.*

Proof. Let $u \in W_0^{1,n}(\Omega)$ such that $\|u\|$ small enough. Let $0 < \beta_0 < \ell$. Then from Proposition 2.2.6, (h3) and (5.1.1), for any $\epsilon > 0$ and $p > 1$ we know that there exists a $C(\epsilon) > 0$ such that

$$\begin{aligned}
 & \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \right) F(x, u) dx \leq C(n, \mu) \|F(x, u)\|_{L^{\frac{2n}{2n-\mu}}(\Omega)}^2 \\
 & \leq C(n, \mu) 2^{\frac{2n}{2n-\mu}} \left(\epsilon \int_{\Omega} |u|^{\frac{2n(\beta_0+1)}{2n-\mu}} + C(\epsilon) \int_{\Omega} |u|^{\frac{2pn}{2n-\mu}} \exp\left(\frac{2n(1+\epsilon)}{2n-\mu} |u|^{\frac{n}{n-1}}\right) \right)^{\frac{2n-\mu}{n}} \\
 & \leq C_1 \left(\epsilon \int_{\Omega} |u|^{\frac{2n(\beta_0+1)}{2n-\mu}} + C_2(\epsilon) \|u\|^{\frac{2pn}{2n-\mu}} \left(\int_{\Omega} \exp\left(\frac{4n(1+\epsilon)\|u\|^{\frac{n}{n-1}}}{2n-\mu} \left(\frac{|u|}{\|u\|}\right)^{\frac{n}{n-1}}\right) \right)^{\frac{1}{2}} \right)^{\frac{2n-\mu}{n}}
 \end{aligned} \tag{5.1.4}$$

where we used Sobolev and Hölder inequality. So if we choose $\epsilon > 0$ small enough and u such that $\frac{4n(1+\epsilon)\|u\|^{\frac{n}{n-1}}}{2n-\mu} \leq \alpha_n$ then using Theorem 2.2.1, Chapter 1, in (5.1.4) we get

$$\begin{aligned}
 \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \right) F(x, u) dx & \leq C_3 \left(\epsilon \|u\|^{\frac{2n(\beta_0+1)}{2n-\mu}} + C(\epsilon) \|u\|^{\frac{2pn}{2n-\mu}} \right)^{\frac{2n-\mu}{n}} \\
 & \leq C_4 \left(\epsilon \|u\|^{2(\beta_0+1)} + C(\epsilon) \|u\|^{2p} \right).
 \end{aligned}$$

Hence from (m1) and above estimate, we deduce that for $\|u\| = \rho$ where $\rho < \left(\frac{\alpha_n(2n-\mu)}{4n(1+\epsilon)}\right)^{\frac{n-1}{n}}$

$$E(u) \geq m_0 \frac{\|u\|^n}{n} - C_4 \left(\epsilon \|u\|^{2(\beta_0+1)} + C(\epsilon) \|u\|^{2p} \right).$$

Taking $\beta_0 > 0$ such that $2(\beta_0 + 1) > n$ and $2p > n$, we can choose ρ small enough so that $E(u) \geq \sigma$ for some $\sigma > 0$ (depending on ρ) when $\|u\| = \rho$. Furthermore, under the assumption (m2), for some $a_1, a_2 > 0$ and $t_0 > 0$ we have $M(t) \leq a_1 + a_2 t^r$ and

$$\mathcal{M}(t) \leq \begin{cases} a_0 + a_1 t + \frac{a_2 t^{r+1}}{r+1}, & r \neq -1 \\ a_0 + a_1 t + a_2 \ln t, & r = -1 \end{cases}$$

when $t \geq \hat{t}$ and where

$$a_0 = \begin{cases} \mathcal{M}(t_0) - a_1 t_0 - a_2 \frac{t_0^{r+1}}{r+1}, & r \neq -1 \\ \mathcal{M}(t_0) - a_1 t_0 - a_2 \ln t_0, & r = -1. \end{cases}$$

Let $u_0 \in W_0^{1,n}(\Omega)$ such that $u_0 \geq 0$ and $\|u_0\| = 1$. Then (h3) implies that there exists $K_1 \geq \max\{\frac{n}{2}, \frac{n(r+1)}{2}\}$ such that $F(x, s) \geq C_1 s^{K_1} - C_2$ for all $(x, s) \in \Omega \times [0, \infty)$ and for some positive constants C_1 and C_2 . Using this, we obtain

$$\begin{aligned} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, tu_0)}{|x-y|^\mu} dy \right) F(x, tu_0) dx &\geq \int_{\Omega} \int_{\Omega} \frac{(C_1(tu_0)^{K_1}(y) - C_2)(C_1(tu_0)^{K_1}(x) - C_2)}{|x-y|^\mu} dx dy \\ &= C_1^2 t^{2K_1} \int_{\Omega} \int_{\Omega} \frac{u_0^{K_1}(y) u_0^{K_1}(x)}{|x-y|^\mu} dx dy \\ &\quad - 2C_1 C_2 t^{K_1} \int_{\Omega} \int_{\Omega} \frac{u_0^{K_1}(y)}{|x-y|^\mu} dx dy + C_2^2 \int_{\Omega} \int_{\Omega} |x-y|^{-\mu} dx dy. \end{aligned}$$

Therefore from above we obtain

$$\begin{aligned} E(tu_0) &\leq \frac{\mathcal{M}(\|tu_0\|^n)}{n} - \int_{\Omega} \left(\int_{\Omega} \frac{F(y, tu_0)}{|x-y|^\mu} dy \right) F(x, tu_0) dx \\ &\leq C_3 + C_4 t^n + C_5 t^{n(r+1)} - C_4 t^{2K_1} + C_6 t^{K_1} \end{aligned}$$

where C_i 's are positive constants for $i = 4, 5, 6$. This implies that $E(tu_0) \rightarrow -\infty$ as $t \rightarrow \infty$. Thus there exists a $v_0 \in W_0^{1,n}(\Omega)$ with $\|v_0\| > \sigma$ such that $E(v_0) < 0$. \square

Let $\Gamma = \{\gamma \in C([0, 1], W_0^{1,n}(\Omega)) : \gamma(0) = 0, E(\gamma(1)) < 0\}$ and define the Mountain Pass critical level as

$$l^* = \inf_{\gamma \in \Gamma} \max_{t \in [0, 1]} E(\gamma(t)). \quad (5.1.5)$$

Then by using Ekeland principle and deformation lemma (Theorem 2.4.1), we have the existence of minimizing Palais-Smale sequence $u_n \in W_0^{1,n}(\Omega)$ such that

$$E(u_n) \rightarrow l^*, \quad E'(u_n) \rightarrow 0.$$

Lemma 5.1.10. *Every Palais Smale sequence is bounded in $W_0^{1,n}(\Omega)$.*

Proof. Let $\{u_k\} \subset W_0^{1,n}(\Omega)$ denotes a $(PS)_c$ sequence of E that is

$$E(u_k) \rightarrow c \text{ and } E'(u_k) \rightarrow 0 \text{ as } k \rightarrow \infty$$

for some $c \in \mathbb{R}$. This implies

$$\begin{aligned} \frac{\mathcal{M}(\|u_k\|^n)}{n} - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) F(x, u_k) dx &\rightarrow c \text{ as } k \rightarrow \infty, \\ \left| M(\|u_k\|^n) \int_{\Omega} |\nabla u_k|^{n-2} \nabla u_k \nabla \phi - \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(x, u_k) \phi dx \right| &\leq \epsilon_k \|\phi\| \end{aligned} \quad (5.1.6)$$

where $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$. In particular, taking $\phi = u_k$ we get

$$\left| M(\|u_k\|^n) \int_{\Omega} |\nabla u_k|^n - \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(u_k) u_k dx \right| \leq \epsilon_k \|u_k\|. \quad (5.1.7)$$

From the assumption (h3), there exists $\alpha > n$ such that $\alpha F(x, t) \leq tf(x, t)$ for any $t > 0$ and $x \in \Omega$ which yields

$$\alpha \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) F(u_k) dx \leq \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(u_k) u_k dx. \quad (5.1.8)$$

Using (5.1.6), (5.1.7) along with above inequality and (m3)', we get

$$\begin{aligned} E(u_k) - \frac{1}{2\alpha} \langle E'(u_k), u_k \rangle &= \frac{\mathcal{M}(\|u_k\|^n)}{n} - \frac{M(\|u_k\|^n) \|u_k\|^n}{2\alpha} \\ &\quad - \frac{1}{2} \left(\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) F(x, u_k) dx - \frac{1}{\alpha} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(x, u_k) u_k dx \right) \\ &\geq \frac{\mathcal{M}(\|u_k\|^n)}{n} - \frac{M(\|u_k\|^n) \|u_k\|^n}{2\alpha} \geq \left(\frac{1}{2n} - \frac{1}{2\alpha} \right) M(\|u_k\|^n) \|u_k\|^n \geq \left(\frac{1}{2n} - \frac{1}{2\alpha} \right) m_0 \|u_k\|^n. \end{aligned} \quad (5.1.9)$$

Also from (5.1.6) and (5.1.7) it follows that

$$E(u_k) - \frac{1}{2\alpha} \langle E'(u_k), u_k \rangle \leq C \left(1 + \epsilon_k \frac{\|u_k\|}{2\alpha} \right) \quad (5.1.10)$$

for some constant $C > 0$. Therefore from (5.1.9) and (5.1.10) we get that

$$\left(\frac{1}{2n} - \frac{1}{2\alpha} \right) m_0 \|u_k\|^n \leq C \left(1 + \epsilon_k \frac{\|u_k\|}{2\alpha} \right).$$

This implies that $\{u_k\}$ must be bounded in $W_0^{1,n}(\Omega)$. \square

To prove the existence of non-trivial weak solution, we need an essential upper bound on the mountain pass critical level which is given by following lemma:

Lemma 5.1.11. *If (5.1.3) holds, then*

$$0 < l^* < \frac{1}{n} \mathcal{M} \left(\left(\frac{2n-\mu}{2n} \alpha_n \right)^{n-1} \right).$$

Proof. It is easy to verify that $\|w_k\| = 1$ for all k . So we claim that there exists a $k \in \mathbb{N}$ such that

$$\max_{t \in [0, \infty)} E(tw_k) < \frac{1}{n} \mathcal{M} \left(\left(\frac{2n-\mu}{2n} \alpha_n \right)^{n-1} \right).$$

Suppose this is not true then for all $k \in \mathbb{N}$ there exists a $t_k > 0$ such that

$$\begin{aligned} \max_{t \in [0, \infty)} E(tw_k) = E(t_k w_k) &\geq \frac{1}{n} \mathcal{M} \left(\left(\frac{2n-\mu}{2n} \alpha_n \right)^{n-1} \right) \\ \text{and } \frac{d}{dt} (E(tw_k))|_{t=t_k} &= 0. \end{aligned} \quad (5.1.11)$$

From the proof of Lemma 5.1.10, $E(tw_k) \rightarrow -\infty$ as $t \rightarrow \infty$ uniformly in k . Then we infer that $\{t_k\}$ must be a bounded sequence in \mathbb{R} . From (5.1.11) and definition of $E(t_k w_k)$ we obtain

$$\frac{1}{n} \mathcal{M} \left(\left(\frac{2n - \mu}{2n} \alpha_n \right)^{n-1} \right) < \frac{\mathcal{M}(t_k^n)}{n}. \quad (5.1.12)$$

Since \mathcal{M} is monotone increasing, from (5.1.12) we get that

$$t_k^n \geq \left(\frac{2n - \mu}{2n} \alpha_n \right)^{n-1}. \quad (5.1.13)$$

From (5.1.13), we get

$$\frac{t_k}{\omega_{n-1}^{\frac{1}{n}}} (\log k)^{\frac{n-1}{n}} \rightarrow \infty \text{ as } k \rightarrow \infty. \quad (5.1.14)$$

Furthermore from (5.1.11), we have

$$\begin{aligned} M(t_k^n) t_k^n &= \int_{\Omega} \left(\int_{\Omega} \frac{F(y, t_k w_k)}{|x-y|^\mu} dy \right) f(x, t_k w_k) t_k w_k dx \\ &\geq \int_{B_{\rho/k}} f(x, t_k w_k) t_k w_k \int_{B_{\rho/k}} \frac{F(y, t_k w_k)}{|x-y|^\mu} dy dx. \end{aligned} \quad (5.1.15)$$

In addition, as in equation (2.11) p. 1943 in [15], it is easy to get that

$$\int_{B_{\rho/k}} \int_{B_{\rho/k}} \frac{dx dy}{|x-y|^\mu} \geq C_{\mu,n} \left(\frac{\rho}{k} \right)^{2n-\mu}$$

where $C_{\mu,n}$ is a positive constant depending on μ and n . From (5.1.3), we know that for each $d > 0$ there exists a s_d such that

$$s f(x, s) F(x, s) \geq d \exp \left(2|s|^{\frac{n}{n-1}} \right), \text{ whenever } s \geq s_d.$$

Since (5.1.14) holds, we can choose a $r_d \in \mathbb{N}$ such that

$$\frac{t_k}{\omega_{n-1}^{\frac{1}{n}}} (\log k)^{\frac{n-1}{n}} \geq s_d, \text{ for all } k \geq r_d.$$

Using these estimates in (5.1.15) and from (5.1.13), for d large enough we get that

$$M(t_k^n) t_k^n \geq d \exp \left((\log k) \left(\frac{2t_k^{\frac{n}{n-1}}}{\omega_{n-1}^{\frac{1}{n-1}}} \right) \right) C_{\mu,n} \left(\frac{\rho}{k} \right)^{2n-\mu} \geq d C_{\mu,n} \rho^{2n-\mu}.$$

Taking d large enough and since t_k^n is bounded, we arrive at a contradiction. This establishes our claim and we conclude the proof of the result. \square

Now, to prove the weak limit of the Palais-Smale sequence is the solution our problem (KC) , we prove a set of convergence lemmas:

Lemma 5.1.12. *If $\{u_k\}$ denotes a Palais Smale sequence then up to a subsequence, there exists $u \in W_0^{1,n}(\Omega)$ such that*

$$|\nabla u_k|^{n-2} \nabla u_k \rightharpoonup |\nabla u|^{n-2} \nabla u \text{ weakly in } (L^{\frac{n}{n-1}}(\Omega))^n. \quad (5.1.16)$$

Proof. From Lemma 5.1.10, we know that the sequence must be bounded in $W_0^1(\Omega)$. Consequently, up to a subsequence there exists $u \in W_0^{1,n}(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $W_0^{1,n}(\Omega)$ and strongly in $L^q(\Omega)$ for any $q \in [1, \infty)$ as $k \rightarrow \infty$. Also still upto a subsequence we can assume $u_k(x) \rightarrow u(x)$ pointwise a.e. for $x \in \Omega$. Therefore the sequence $|\nabla u_k|^{n-2} \nabla u_k$ must be bounded in $(L^{\frac{n}{n-1}}(\Omega))^n$ where $|\nabla u_k|^n$ is bounded in $L^1(\Omega)$. So there exists a non-negative radon measure ν such that up to a subsequence

$$|u_k|^n + |\nabla u_k|^n \rightarrow \nu \text{ in } (C(\bar{\Omega}))^* \text{ as } k \rightarrow \infty.$$

Moreover there exists $v \in (L^{\frac{n}{n-1}}(\Omega))^n$ such that,

$$|\nabla u_k|^{n-2} \nabla u_k \rightarrow v \text{ weakly in } (L^{\frac{n}{n-1}}(\Omega))^n \text{ as } k \rightarrow \infty.$$

Claim : $v = |\nabla u|^{n-2} \nabla u$.

To prove this, we set $\sigma > 0$ and $X_\sigma = \{x \in \bar{\Omega} : \nu(B_r(x) \cap \bar{\Omega}) \geq \sigma, \text{ for all } r > 0\}$ and divide the proof in two steps:

Step 1: X_σ must be a finite set.

Because if not, then there exists a sequence of distinct points $\{x_k\}$ in X_σ such that for all $r > 0$, $\nu(B_r(x_k) \cap \bar{\Omega}) \geq \sigma$ for all k . This implies that $\nu(\{x_k\}) \geq \sigma$ for all k , hence $\nu(X_\sigma) = +\infty$. But this is a contradiction to

$$\nu(X_\sigma) = \lim_{k \rightarrow \infty} \int_{X_\sigma} |u_k|^n + |\nabla u_k|^n dx \leq C.$$

So let $X_\sigma = \{x_1, x_2, \dots, x_m\}$.

Step 2: For $\sigma > 0$ such that $\sigma^{\frac{1}{n-1}} < \frac{\alpha_n}{2^{\frac{1}{n-1}}} \left(\frac{2n-\mu}{2n} \right)$, the for any K compact subset of $\bar{\Omega} \setminus X_\sigma$ we have

$$\lim_{k \rightarrow \infty} \int_K \left(\int_\Omega \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(x, u_k) u_k dx = \int_K \left(\int_\Omega \frac{F(y, u)}{|x-y|^\mu} dy \right) f(x, u) u dx. \quad (5.1.17)$$

To show this, let $x_0 \in K$ and $r_0 > 0$ be such that $\nu(B_{r_0}(x_0) \cap \bar{\Omega}) < \sigma$ that is $x_0 \notin X_\sigma$. Also we consider a $\psi \in C^\infty(\Omega)$ satisfying $0 \leq \psi(x) \leq 1$ for $x \in \Omega$, $\psi \equiv 1$ in $B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}$ and $\psi \equiv 0$ in $\bar{\Omega} \setminus (B_{r_0}(x_0) \cap \bar{\Omega})$. Then

$$\lim_{k \rightarrow \infty} \int_{B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}} |u_k|^n + |\nabla u_k|^n \leq \lim_{k \rightarrow \infty} \int_{B_{r_0}(x_0) \cap \bar{\Omega}} |u_k|^n + |\nabla u_k|^n \psi \leq \nu(B_{r_0}(x_0) \cap \bar{\Omega}) < \sigma.$$

Therefore for large enough $k \in \mathbb{N}$ and $\epsilon > 0$ small enough, it must be

$$\int_{B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}} |u_k|^n + |\nabla u_k|^n \leq \sigma(1 - \epsilon). \quad (5.1.18)$$

Now we estimate the following using (5.1.18) and Theorem 2.2.1, Chapter 2,

$$\begin{aligned}
 \int_{B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}} |f(x, u_k)|^q dx &= \int_{B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}} |h(x, u_k)|^q \exp\left(q|u_k|^{\frac{n}{n-1}}\right) dx \\
 &\leq C_\delta \int_{B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}} \exp\left((1+\epsilon)q|u_k|^{\frac{n}{n-1}}\right) dx \\
 &\leq C_\delta \int_{B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}} \exp\left((1+\epsilon)q\sigma^{\frac{1}{n-1}}(1-\epsilon)^{\frac{1}{n-1}} \left(\frac{|u_k|^n}{\int_{B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}} |u_k|^n + |\nabla u_k|^n}\right)^{\frac{1}{n-1}}\right) dx \leq C_0
 \end{aligned} \tag{5.1.19}$$

for some constant $C_0 > 0$ while choosing $q > 1$ such that $(1+\epsilon)q\sigma^{\frac{1}{n-1}} \leq \frac{\alpha_1}{2^{n-1}}$. Consider

$$\begin{aligned}
 &\int_{B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}} \left| \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(x, u_k) u_k - \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \right) f(x, u) u \right| dx \\
 &\leq \int_{B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}} \left| \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \right) (f(x, u_k) u_k - f(x, u) u) \right| dx \\
 &\quad + \int_{B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}} \left| \left(\int_{\Omega} \frac{F(y, u_k) - F(y, u)}{|x-y|^\mu} dy \right) f(x, u_k) u_k \right| dx \\
 &:= I_1 + I_2 \text{ (say)}.
 \end{aligned}$$

From (5.1.2), we know that $F(u) \in L^r(\Omega)$ for any $r \in [1, \infty)$. Since $\mu \in (0, n)$, $y \rightarrow |x-y|^{-\mu} \in L^{r_0}(\Omega)$ for all $r_0 \in (1, \frac{n}{\mu})$ uniformly in $x \in \Omega$ (since Ω is bounded). So using Hölder's inequality we get that

$$\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \in L^\infty(\Omega). \tag{5.1.20}$$

From the asymptotic growth of $f(x, t)$, it is easy to get that

$$\lim_{t \rightarrow \infty} \frac{f(x, t)t}{(f(x, t))^r} = 0 \text{ uniformly in } x \in \Omega, \text{ for all } r > 1. \tag{5.1.21}$$

Using (5.1.20) we get

$$I_1 \leq C \int_{B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}} |f(x, u_k) u_k - f(x, u) u| dx$$

where $C > 0$ is a constant. Because of (5.1.21) and (5.1.19), the family $\{f(x, u_k) u_k\}$ is equi-integrable over $B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}$. Also continuity of $f(x, t)$ gives that $f(x, u_k) u_k \rightarrow f(x, u) u$ pointwise a.e. in Ω as $k \rightarrow \infty$ and thus using Vitali's convergence theorem, it follows that $I_1 \rightarrow 0$ as $k \rightarrow \infty$. Next we show $I_2 \rightarrow 0$ as $k \rightarrow \infty$.

First by using the semigroup property of the Riesz potential we get that for some constant

$C > 0$ independent of k

$$\begin{aligned} & \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k) - F(y, u)}{|x - y|^{\mu}} dy \right) \chi_{B_{r_0/2} \cap \bar{\Omega}}(x) f(x, u_k) u_k dx \\ & \leq \left(\int_{\Omega} \left(\int_{\Omega} \frac{|F(y, u_k) - F(y, u)|}{|x - y|^{\mu}} dy \right) |F(x, u_k) - F(x, u)| dx \right)^{\frac{1}{2}} \\ & \quad \times \left(\int_{\Omega} \left(\int_{\Omega} \chi_{B_{r_0/2} \cap \bar{\Omega}}(y) \frac{f(y, u_k) u_k}{|x - y|^{\mu}} dy \right) \chi_{B_{r_0/2} \cap \bar{\Omega}}(x) f(x, u_k) u_k dx \right)^{\frac{1}{2}}. \end{aligned}$$

From (5.1.19) and since $\sigma^{\frac{1}{n-1}} < \frac{\alpha_n}{2^{n-1}} \left(\frac{2n-\mu}{2n} \right)$ we obtain

$$\begin{aligned} & \left(\int_{\Omega} \left(\int_{\Omega} \chi_{B_{r_0/2} \cap \bar{\Omega}}(y) \frac{f(y, u_k) u_k}{|x - y|^{\mu}} dy \right) \chi_{B_{r_0/2} \cap \bar{\Omega}}(x) f(x, u_k) u_k dx \right)^{\frac{1}{2}} \leq \\ & \quad \|\chi_{B_{r_0/2} \cap \bar{\Omega}} f(x, u_k) u_k\|_{L^{\frac{2n}{2n-\mu}}(\Omega)} \leq C. \end{aligned}$$

Now we claim that

$$\lim_{k \rightarrow \infty} \int_{\Omega} \left(\int_{\Omega} \frac{|F(y, u_k) - F(y, u)|}{|x - y|^{\mu}} dy \right) |F(x, u_k) - F(x, u)| dx = 0. \quad (5.1.22)$$

From (5.1.6), (5.1.7) and (5.1.8) we get that there exists a constant $C > 0$ such that

$$\begin{aligned} & \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x - y|^{\mu}} dy \right) F(x, u_k) dx \leq C, \\ & \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x - y|^{\mu}} dy \right) f(x, u_k) u_k dx \leq C. \end{aligned} \quad (5.1.23)$$

We argue as along equation (2.20) in Lemma 2.4 in [15]. Now using (5.1.23), (h4) and the semigroup property of the Riesz Potential we obtain,

$$\int_{\Omega} \int_{|u| \geq T} \frac{F(y, u)}{|x - y|^{\mu}} F(x, u) dy dx = o(T), \quad \int_{\Omega} \int_{|u_k| \geq T} \frac{F(y, u_k)}{|x - y|^{\mu}} F(x, u_k) dy dx = o(T), \quad (5.1.24)$$

$$\int_{\Omega} \int_{|u| \geq T} \frac{F(y, u_k)}{|x - y|^{\mu}} F(x, u) dy dx = o(T), \quad (5.1.25)$$

and

$$\int_{\Omega} \int_{|u_k| \geq T} \frac{F(y, u_k)}{|x - y|^{\mu}} F(x, u) dy dx = o(T) \text{ as } T \rightarrow \infty. \quad (5.1.26)$$

So,

$$\begin{aligned} & \int_{\Omega} \left(\int_{\Omega} \frac{|F(y, u_k) - F(y, u)|}{|x - y|^{\mu}} dy \right) |F(x, u_k) - F(x, u)| dx \leq \\ & 2 \int_{\Omega} \left(\int_{\Omega} \frac{\chi_{u_k \geq T}(y) F(y, u_k)}{|x - y|^{\mu}} dy \right) F(x, u_k) dx \\ & + 4 \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k) \chi_{u \geq T}(x) F(x, u)}{|x - y|^{\mu}} dy \right) dx + 4 \int_{\Omega} \left(\int_{\Omega} \frac{\chi_{u_k \geq T}(y) F(y, u_k) F(x, u)}{|x - y|^{\mu}} dy \right) dx \\ & + 2 \int_{\Omega} \left(\int_{\Omega} \frac{\chi_{u \geq T}(y) F(y, u)}{|x - y|^{\mu}} dy \right) F(x, u) dx \\ & + \int_{\Omega} \left(\int_{\Omega} \frac{|F(y, u_k) \chi_{u_k \leq T} - F(y, u) \chi_{u \leq T}|}{|x - y|^{\mu}} dy \right) |F(x, u_k) \chi_{u_k \leq T} - F(x, u) \chi_{u \leq T}| dx. \end{aligned}$$

Then from Lebesgue dominated convergence theorem the above integrand tends to 0 as $k \rightarrow \infty$. Hence using (5.1.24), (5.1.25) and (5.1.26), it is easy to conclude (5.1.22) and $I_2 \rightarrow 0$ as $k \rightarrow \infty$. This implies that

$$\lim_{k \rightarrow \infty} \int_{B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}} \left| \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(x, u_k) u_k - \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \right) f(x, u) u \right| dx = 0.$$

To conclude (5.1.17), we repeat this procedure over a finite covering of balls using the fact that K is compact. Lastly, the proof of (5.1.16) can be achieved by classical arguments as in the proof of Lemma 4 in [203]. \square

Lemma 5.1.13. *Let $\{u_k\} \subset W_0^{1,n}(\Omega)$ be a Palais Smale sequence for E at level l^* then there exists a $u_0 \in W_0^{1,n}(\Omega)$ such that as $k \rightarrow \infty$ (up to a subsequence)*

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(x, u_k) \phi dx \rightarrow \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|x-y|^\mu} dy \right) f(x, u_0) \phi dx, \text{ for all } \phi \in C_c^\infty(\Omega).$$

Proof. If $\{u_k\}$ is a Palais Smale sequence at l^* for E then it must satisfy (5.1.6) and (5.1.7). We remark that $E(u^+) \leq E(u)$ for each $u \in W_0^{1,n}(\Omega)$, then we can assume $u_k \geq 0$ for each $k \in \mathbb{N}$. From Lemma 5.1.10 we know that $\{u_k\}$ must be bounded in $W_0^{1,n}(\Omega)$ so there exists a $C_0 > 0$ such that $\|u_k\| \leq C_0$. Also there exists a $u_0 \in W_0^{1,n}(\Omega)$ such that up to a subsequence $u_k \rightharpoonup u_0$ in $W_0^{1,n}(\Omega)$, strongly in $L^q(\Omega)$ for all $q \in [1, \infty)$ and pointwise a.e. in Ω as $k \rightarrow \infty$. Let $\Omega' \subset\subset \Omega$ and $\varphi \in C_c^\infty(\Omega)$ such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in Ω' . With easy computations, we get that

$$\begin{aligned} \left\| \frac{\varphi}{1+u_k} \right\|^n &= \int_{\Omega} \left| \frac{\nabla \varphi}{1+u_k} - \varphi \frac{\nabla u_k}{(1+u_k)^2} \right|^n dx \\ &\leq 2^{n-1} (\|\varphi\|^n + \|u_k\|^n). \end{aligned}$$

This implies that $\frac{\varphi}{1+u_k} \in W_0^{1,n}(\Omega)$. So using $\frac{\varphi}{1+u_k}$ as a test function (5.1.6), we get the following estimate

$$\begin{aligned} \int_{\Omega'} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) \frac{f(x, u_k)}{1+u_k} dx &\leq \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) \frac{f(x, u_k) \varphi}{1+u_k} dx \\ &\leq \epsilon_k \left\| \frac{\varphi}{1+u_k} \right\| + \int_{\Omega} M(\|u_k\|^n) |\nabla u_k|^{n-2} \nabla u_k \nabla \left(\frac{\varphi}{1+u_k} \right) dx \\ &\leq \epsilon_k 2^{\frac{n-1}{n}} (\|\varphi\| + \|u_k\|) + M(\|u_k\|^n) \int_{\Omega} |\nabla u_k|^{n-2} \nabla u_k \left(\frac{\nabla \varphi}{1+u_k} - \varphi \frac{\nabla u_k}{(1+u_k)^2} \right) dx \\ &\leq \epsilon_k 2^{\frac{n-1}{n}} (\|\varphi\| + \|u_k\|) + M(\|u_k\|^n) \int_{\Omega} |\nabla u_k|^{n-1} (|\nabla \varphi| + |\nabla u_k|) dx \\ &\leq \epsilon_k 2^{\frac{n-1}{n}} (\|\varphi\| + \|u_k\|) + M(\|u_k\|^n) [\|\varphi\| \|u_k\|^{n-1} + \|u_k\|^n]. \end{aligned}$$

But using $\|u_k\| \leq C_0$ for all k and (m2), we infer that there must exists a $C_1 > 0$ such that

$$\int_{\Omega'} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) \frac{f(x, u_k)}{1+u_k} dx \leq C_1. \quad (5.1.27)$$

Also for the same reason, (5.1.7) gives that

$$\int_{\Omega'} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(x, u_k) u_k dx \leq C_2 \quad (5.1.28)$$

for some $C_2 > 0$. Gathering (5.1.27) and (5.1.28) we obtain

$$\begin{aligned} & \int_{\Omega'} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(x, u_k) dx \\ & \leq 2 \int_{\Omega' \cap \{u_k < 1\}} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) \frac{f(x, u_k)}{1+u_k} dx + \int_{\Omega' \cap \{u_k \geq 1\}} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) u_k f(x, u_k) dx \\ & \leq 2 \int_{\Omega'} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) \frac{f(x, u_k)}{1+u_k} dx + \int_{\Omega'} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) u_k f(x, u_k) dx \\ & \leq 2C_1 + C_2 := C_3. \end{aligned}$$

Thus the sequence $\{w_k\} := \left\{ \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(x, u_k) \right\}$ is bounded in $L^1_{\text{loc}}(\Omega)$ which implies that up to a subsequence, $w_k \rightharpoonup w$ in the *weak**-topology as $k \rightarrow \infty$, where w denotes a Radon measure. So for any $\phi \in C_c^\infty(\Omega)$ we get

$$\lim_{k \rightarrow \infty} \int_{\Omega} \int_{\Omega} \left(\frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(x, u_k) \phi dx = \int_{\Omega} \phi dw, \quad \forall \phi \in C_c^\infty(\Omega).$$

Since u_k satisfies (5.1.6), we get that

$$\int_E \phi dw = \lim_{k \rightarrow \infty} M(\|u_k\|) \int_E |\nabla u_k|^{n-2} \nabla u_k \nabla \phi dx, \quad \forall E \subset \Omega.$$

Together with Lemma 5.1.12, this implies that w is absolutely continuous with respect to the Lebesgue measure. Thus, Radon-Nikodym theorem asserts that there exists a function $g \in L^1_{\text{loc}}(\Omega)$ such that for any $\phi \in C_c^\infty(\Omega)$, $\int_{\Omega} \phi dw = \int_{\Omega} \phi g dx$. Therefore for any $\phi \in C_c^\infty(\Omega)$ we get

$$\lim_{k \rightarrow \infty} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(x, u_k) \phi dx = \int_{\Omega} \phi g dx = \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|x-y|^\mu} dy \right) f(x, u_0) \phi dx$$

which completes the proof. \square

Lemma 5.1.14. *Let $\{u_k\} \subset W_0^{1,n}(\Omega)$ be a Palais Smale sequence of E . Then there exists a $u \in W_0^{1,n}(\Omega)$ such that, up to a subsequence, $u_k \rightharpoonup u$ weakly in $W_0^{1,n}(\Omega)$ and*

$$\left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) F(x, u_k) \rightarrow \left(\int_{\Omega} \frac{F(y, u)}{|x-y|^\mu} dy \right) F(x, u) \text{ in } L^1(\Omega)$$

as $k \rightarrow \infty$.

Proof. Let $\{u_k\} \subset W_0^{1,n}(\Omega)$ be a Palais Smale sequence of E at level c . From Lemma 5.1.10 we know that $\{u_k\}$ must be bounded in $W_0^{1,n}(\Omega)$. Thus there exists a $u \in W_0^{1,n}(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $W_0^{1,n}(\Omega)$, $u_k \rightarrow u$ pointwise a.e. in \mathbb{R}^n and $u_k \rightarrow u$ strongly in $L^q(\Omega)$, $q \in [1, \infty)$ as $k \rightarrow \infty$. Also from (5.1.6), (5.1.7) and (5.1.8) we get that there exists a constant $C > 0$ such that (5.1.23) holds. Now the proof of main claim follows similarly the proof of (5.1.22) (see also equation (2.20) of Lemma 2.4 in [15]). \square

Now we define the associated Nehari manifold as

$$\mathcal{N} = \{u \in W_0^{1,n}(\Omega) \setminus \{0\} : \langle E'(u), u \rangle = 0\} \text{ and } l^{**} = \inf_{u \in \mathcal{N}} E(u)$$

and we show the mountain pass critical level lies below every local minimum value of the energy functional at the point of local minimum.

Lemma 5.1.15. *If (m3) holds then $l^* \leq l^{**}$.*

Proof. Let $u \in \mathcal{N}$ and $h : (0, +\infty) \rightarrow \mathbb{R}$ be defined as $h(t) = E(tu)$. Then

$$h'(t) = M(\|tu\|^n) \|u\|^n t^{n-1} - \int_{\Omega} \left(\int_{\Omega} \frac{F(y, tu)}{|x-y|^\mu} dy \right) f(x, tu) u \, dx.$$

Since u satisfies $\langle E'(u), u \rangle = 0$, we get

$$\begin{aligned} h'(t) &= \|u\|^{2n} t^{2n-1} \left(\frac{M(\|tu\|^n)}{t^n \|u\|^n} - \frac{M(\|u\|^n)}{\|u\|^n} \right) \\ &\quad + t^{2n-1} \left[\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u) f(x, u)}{|x-y|^\mu} dy - \int_{\Omega} \frac{F(y, tu) f(x, tu)}{|x-y|^\mu} dy \right) u^n(x) \, dx \right]. \end{aligned}$$

Claim: For any $x \in \Omega$

$$t \rightarrow tf(x, t) - nF(x, t) \text{ is increasing on } \mathbb{R}^+. \quad (5.1.29)$$

Indeed, from (h3), for $0 < t_1 < t_2$, we have

$$t_1 f(x, t_1) - nF(x, t_1) \leq t_1 f(x, t_1) - nF(x, t_2) + \frac{f(x, t_2)}{t_2^{n-1}} (t_2^n - t_1^n) \leq t_2 f(x, t_2) - nF(x, t_2).$$

Using this we get that $tf(x, t) - nF(x, t) \geq 0$ for $t \geq 0$ which implies that $t \rightarrow \frac{F(x, tu)}{t^n}$ is non-decreasing for $t > 0$. Therefore for $0 < t < 1$ and $x \in \Omega$, we get $\frac{F(x, tu)}{t^n} \leq F(x, u)$ and this implies

$$\begin{aligned} h'(t) &\geq \|u\|^{2n} t^{2n-1} \left(\frac{M(\|tu\|^n)}{\|tu\|^n} - \frac{M(\|u\|^n)}{\|u\|^n} \right) \\ &\quad + t^{2n-1} \left[\int_{\Omega} \left(\int_{\Omega} \left(F(y, u) - \frac{F(y, tu)}{t^n} \right) \frac{dy}{|x-y|^\mu} \right) \frac{f(x, tu)}{(tu(x))^{n-1}} u^n(x) \, dx \right]. \end{aligned}$$

This gives that $h'(t) \geq 0$ for $0 < t \leq 1$ and $h'(t) < 0$ for $t > 1$. Hence $E(u) = \max_{t \geq 0} E(tu)$. Now we define $g : [0, 1] \rightarrow W_0^{1,n}(\Omega)$ as $g(t) = (t_0 u)t$ where $t_0 > 1$ is such that $E(t_0 u) < 0$. So $g \in \Gamma$, where Γ is as defined in the definition of l^* . Therefore we obtain

$$l^* \leq \max_{t \in [0, 1]} E(g(t)) \leq \max_{t \geq 0} E(tu) = E(u).$$

Since $u \in \mathcal{N}$ is arbitrary, we get $l^* \leq l^{**}$. □

Now, we give the proof of our main result:

Proof of Theorem 5.1.5: Let $\{u_k\}$ denotes a Palais Smale sequence at the level l^* . Then $(u_k)_{k \in \mathbb{N}}$ can be obtained as a minimizing sequence associated to the variational problem (5.1.5). Then by Lemma 5.1.14 we know that there exists a $u_0 \in W_0^{1,n}(\Omega)$ such that up to a subsequence $u_k \rightharpoonup u_0$ weakly in $W_0^{1,n}(\Omega)$ as $k \rightarrow \infty$.

Step 1: $u \not\equiv 0$ and $u > 0$.

Suppose $u_0 \equiv 0$ then using Lemma 5.1.14, we infer that

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) F(x, u_k) dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

This gives that $\lim_{k \rightarrow \infty} E(u_k) = \frac{1}{n} \lim_{k \rightarrow \infty} \mathcal{M}(\|u_k\|^n) = l^*$ which implies in the light of Lemma 5.1.11 that for large enough k

$$\mathcal{M}(\|u_k\|^n) < \mathcal{M} \left(\left(\frac{2n-\mu}{2n} \alpha_n \right)^{n-1} \right).$$

Therefore since \mathcal{M} is non decreasing, we get

$$\frac{2n}{2n-\mu} \|u_k\|^{\frac{n}{n-1}} < \alpha_n.$$

Now, this implies that $\sup_k \int_{\Omega} f(x, u_k)^q dx < +\infty$ for some $q > \frac{2n}{2n-\mu}$ and along with Proposition 2.2.6, Theorem 2.2.1 and the Vitali's convergence theorem, we get

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(x, u_k) u_k dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Hence $\lim_{k \rightarrow \infty} \langle E'(u_k), u_k \rangle = 0$ gives $\lim_{k \rightarrow \infty} M(\|u_k\|^n) \|u_k\|^n = 0$. From (m1) we then obtain $\lim_{k \rightarrow \infty} \|u_k\|^n = 0$. Thus using Lemma 5.1.14, it must be that $\lim_{k \rightarrow \infty} E(u_k) = 0 = l^*$ which contradicts $l^* > 0$. Thus $u_0 \not\equiv 0$. Now, we show that $u_0 > 0$ in Ω . From Lemma 5.1.10 we know that $\{u_k\}$ must be bounded. Therefore there exists a constant $\tau > 0$ such that up to a subsequence $\|u_k\| \rightarrow \tau$ as $k \rightarrow \infty$. Since $E'(u_k) \rightarrow 0$, again up to a subsequence $|\nabla u_k|^{n-2} \nabla u_k \rightharpoonup |\nabla u_0|^{n-2} \nabla u_0$ weakly in $(L^{\frac{n}{n-1}}(\Omega))^n$. Furthermore, by Lemma 5.1.12 and by Lemma 5.1.13, we get as $k \rightarrow \infty$,

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) f(x, u_k) \varphi dx \rightarrow \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|x-y|^\mu} dy \right) f(x, u_0) \varphi dx$$

and

$$M(\tau^n) \int_{\Omega} |\nabla u_0|^{n-2} \nabla u_0 \nabla \varphi dx = \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|x-y|^\mu} dy \right) f(x, u_0) \varphi dx, \text{ for all } \varphi \in W_0^{1,n}(\Omega).$$

In particular, taking $\varphi = u_0^-$ in the above equation we get $M(\tau^n) \|u_0^-\| = 0$ which implies together with assumption (m1) that $u_0^- = 0$ a.e. in Ω . Therefore $u_0 \geq 0$ a.e. in Ω .

From Theorem 2.2.1, Chapter 1, we have $f(\cdot, u_0) \in L^q(\Omega)$ for $1 \leq q < \infty$. Also as in (5.1.20),

we can similarly get that $\int_{\Omega} \frac{F(y, u_0)}{|x-y|^\mu} dy \in L^\infty(\Omega)$. Hence $\left(\int_{\Omega} \frac{F(y, u_0)}{|x-y|^\mu} dy\right) f(x, u_0) \in L^q(\Omega)$ for $1 \leq q < \infty$. By elliptic regularity results, we finally get that $u_0 \in L^\infty(\Omega)$ and $u_0 \in C^{1,\gamma}(\bar{\Omega})$ for some $\gamma \in (0, 1)$. Therefore, $u_0 > 0$ in Ω follows from the strong maximum principle and $u_0 \not\equiv 0$.

Step 2:

$$M(\|u_0\|^n)\|u_0\|^n \geq \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|x-y|^\mu} dy\right) f(x, u_0)u_0 dx. \quad (5.1.30)$$

Arguing by contradiction, suppose that

$$M(\|u_0\|^n)\|u_0\|^n < \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|x-y|^\mu} dy\right) f(x, u_0)u_0 dx$$

which implies that $\langle E'(u_0), u_0 \rangle < 0$. For $t > 0$, using (5.1.29) we have that

$$\begin{aligned} \langle E'(tu_0), u_0 \rangle &\geq M(t^n\|u_0\|^n)t^{n-1}\|u_0\|^n - \frac{1}{n} \int_{\Omega} \left(\int_{\Omega} \frac{f(y, tu_0)tu_0(y)}{|x-y|^\mu} dy\right) f(x, tu_0)u_0 dx \\ &\geq m_0 t^{n-1}\|u_0\|^n - \frac{1}{n} \int_{\Omega} \left(\int_{\Omega} \frac{f(y, tu_0)tu_0(y)}{|x-y|^\mu} dy\right) f(x, tu_0)u_0 dx. \end{aligned}$$

Since (h3) gives that

$$\lim_{t \rightarrow 0^+} \frac{f(x, t)}{t^\gamma} = 0 \text{ uniformly in } x \in \Omega, \text{ for all } \gamma \in [0, n-1],$$

we can choose $t > 0$ sufficiently small so that $\langle E'(tu_0), u_0 \rangle > 0$. Thus there exists a $t_* \in (0, 1)$ such that $\langle E'(t_*u_0), u_0 \rangle = 0$ that is $t_*u_0 \in \mathcal{N}$. So using Lemma 5.1.15, $(m3)'$ and (5.1.29) we get

$$\begin{aligned} l^* \leq l^{**} &\leq E(t_*u_0) = E(t_*u_0) - \frac{1}{2n} \langle E'(t_*u_0), u_0 \rangle \\ &= \frac{\mathcal{M}(\|t_*u_0\|^n)}{n} - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, t_*u_0)}{|x-y|^\mu} dy\right) F(x, t_*u_0) dx - \frac{1}{2n} M(\|t_*u_0\|^n)\|t_*u_0\|^n \\ &\quad + \frac{1}{2n} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, t_*u_0)}{|x-y|^\mu} dy\right) f(x, t_*u_0)t_*u_0 dx \\ &< \frac{\mathcal{M}(\|u_0\|^n)}{n} - \frac{1}{2n} M(\|u_0\|^n)\|u_0\|^n \\ &\quad + \frac{1}{2n} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, t_*u_0)}{|x-y|^\mu} dy\right) (f(x, t_*u_0)t_*u_0 - nF(x, t_*u_0)) dx \\ &\leq \frac{\mathcal{M}(\|u_0\|^n)}{n} - \frac{1}{2n} M(\|u_0\|^n)\|u_0\|^n + \frac{1}{2n} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|x-y|^\mu} dy\right) (f(x, u_0)u_0 - nF(x, u_0)) \\ &\leq \liminf_{k \rightarrow \infty} \frac{\mathcal{M}(\|u_k\|^n)}{n} - \frac{1}{2n} M(\|u_k\|^n)\|u_k\|^n \\ &\quad + \frac{1}{2n} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy\right) (f(x, u_k)u_k - nF(x, u_k)) dx \\ &= \liminf_{k \rightarrow \infty} \left(E(u_k) - \frac{1}{2n} \langle E'(u_k), u_k \rangle \right) = l^*. \end{aligned}$$

This gives a contradiction, that is (5.1.30) holds true.

Step 3: $E(u_0) = l^*$.

From Lemma 5.1.14 we know that

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^\mu} dy \right) F(x, u_k) dx \rightarrow \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|x-y|^\mu} dy \right) F(x, u_0) dx.$$

Using this and the weakly lower semicontinuity of norms in $\lim_{k \rightarrow \infty} E(u_k) = l^*$, we obtain $E(u_0) \leq l^*$. If $E(u_0) < l^*$ then it must be

$$\lim_{k \rightarrow \infty} \mathcal{M}(\|u_k\|^n) > \mathcal{M}(\|u_0\|^n)$$

which implies that $\lim_{k \rightarrow \infty} \|u_k\|^n > \|u_0\|^n$, since \mathcal{M} is continuous and increasing. From this we get

$$\tau^n > \|u_0\|^n.$$

Moreover we get

$$\mathcal{M}(\tau^n) = n \left(l^* + \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|x-y|^\mu} dy \right) F(x, u_0) dx \right). \quad (5.1.31)$$

Now we define the sequence $v_k = \frac{u_k}{\|u_k\|}$ and $v_0 = \frac{u_0}{\tau}$ then $v_k \rightharpoonup v_0$ weakly in $W_0^{1,n}(\Omega)$ and $\|v_0\| < 1$. From Lemma 2.2.4 we have that

$$\sup_{k \in \mathbb{N}} \int_{\Omega} \exp(p|v_k|^{\frac{n}{n-1}}) < +\infty, \text{ for } 1 < p < \frac{\alpha_n}{(1 - \|v_0\|^n)^{\frac{1}{n-1}}}. \quad (5.1.32)$$

Also from (m3)', **Step 1** and Lemma 5.1.15 we obtain

$$E(u_0) \geq \frac{\mathcal{M}(\|u_0\|^n)}{n} - \frac{M(\|u_0\|^n)\|u_0\|^n}{2n} + \frac{1}{2n} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|x-y|^\mu} dy \right) (f(x, u_0)u_0 - nF(x, u_0)) dx \geq 0.$$

Using this with (5.1.31) we get that

$$\mathcal{M}(\tau^n) = nl^* - nE(u_0) + \mathcal{M}(\|u_0\|^n) < \mathcal{M} \left(\left(\frac{2n - \mu}{2n} \alpha_n \right)^{n-1} \right) + \mathcal{M}(\|u_0\|^n)$$

which implies together with (m1) that

$$\tau^n < \frac{\alpha_n^{n-1} \left(\frac{2n - \mu}{2n} \right)^{n-1}}{1 - \|v_0\|^n}.$$

Thus it is possible to find a $\tau_* > 0$ such that for $k \in \mathbb{N}$ large enough

$$\|u_k\|^{\frac{n}{n-1}} < \tau_* < \frac{\alpha_n \left(\frac{2n - \mu}{2n} \right)}{(1 - \|v_0\|^n)^{\frac{1}{n-1}}}.$$

Then we choose a $q > 1$ but close to 1 such that

$$\frac{2n}{2n - \mu} q \|u_k\|^{\frac{n}{n-1}} \leq \frac{2n}{2n - \mu} \tau_* < \frac{\alpha_n}{(1 - \|v_0\|^n)^{\frac{1}{n-1}}}.$$

Therefore from (5.1.32) we conclude that

$$\int_{\Omega} \exp\left(\frac{2n}{2n-\mu} q |u_k|^{\frac{n}{n-1}}\right) \leq C \quad (5.1.33)$$

for some constant $C > 0$. Using (5.1.33)

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|x-y|^{\mu}} dy \right) f(x, u_k) u_k dx \rightarrow \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|x-y|^{\mu}} dy \right) f(x, u_0) u_0 dx.$$

We conclude that $\|u_k\| \rightarrow \|u_0\|$ and we get a contradiction and claim in **Step 3** is proved. Now, by combining claims of the proof of **Step 1, 2** and **3**, the proof of Theorem 5.1.5 follows. \square

5.1.3 The Nehari Manifold method for Kirchhoff-Choquard equations

We observe that $\mathcal{J}_{\lambda, M}$ is only bounded below on suitable subsets of $W_0^{1, n}(\Omega)$. In order to prove the existence of weak solutions to $(\mathcal{P}_{\lambda, M})$, we establish the existence of minimizers of $\mathcal{J}_{\lambda, M}$ under the natural constraint of the Nehari Manifold:

$$N_{\lambda, M} := \{u \in W_0^{1, n}(\Omega) \mid \langle \mathcal{J}'_{\lambda, M}(u), u \rangle = 0\}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $W_0^{1, n}(\Omega)$ and $W^{-1, n}(\Omega)$. Therefore, $u \in N_{\lambda, M}$ if and only if

$$\|u\|^n M(\|u\|^n) - \lambda \int_{\Omega} h(x) u^{q+1} dx - \int_{\Omega} (|x|^{-\mu} * F(u)) f(u) u dx = 0.$$

Remark 5.1.16. We notice that $N_{\lambda, M}$ contains every solution of $(\mathcal{P}_{\lambda, M})$.

For $u \in W_0^{1, n}(\Omega)$, we define the fiber map $\Phi_{u, M} : \mathbb{R}^+ \rightarrow \mathbb{R}$ as

$$\Phi_{u, M}(t) = \mathcal{J}_{\lambda, M}(tu) = \frac{\mathcal{M}(\|tu\|^n)}{n} - \frac{\lambda}{q+1} \int_{\Omega} h(x) |tu|^{q+1} dx - \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(tu)) F(tu) dx,$$

$$\Phi'_{u, M}(t) = t^{n-1} \|u\|^n M(\|tu\|^n) - \lambda t^q \int_{\Omega} h(x) |u|^{q+1} dx - \int_{\Omega} (|x|^{-\mu} * F(tu)) f(tu) u dx$$

and

$$\begin{aligned} \Phi''_{u, M}(t) &= nt^{2n-2} \|u\|^{2n} M'(\|tu\|^n) + (n-1)t^{n-2} \|u\|^n M(\|tu\|^n) - \lambda qt^{q-1} \int_{\Omega} h(x) |u|^{q+1} dx \\ &\quad - \int_{\Omega} (|x|^{-\mu} * f(tu) \cdot u) f(tu) u dx - \int_{\Omega} (|x|^{-\mu} * F(tu)) f'(tu) u^2 dx. \end{aligned}$$

The Nehari Manifold is closely related to the the maps $\Phi_{u, M}$ by the relation $tu \in N_{\lambda, M}$ iff $\Phi'_{u, M}(t) = 0$. In particular, $u \in N_{\lambda, M}$ iff $\Phi'_{u, M}(1) = 0$. So we study the geometry of the energy functional on the following components of the Nehari Manifold:

$$N_{\lambda, M}^{\pm} := \{u \in N_{\lambda, M} : \Phi''_{u, M}(1) \leq 0\} = \{tu \in W_0^{1, n}(\Omega) : \Phi'_{u, M}(t) = 0, \Phi''_{u, M}(t) \leq 0\},$$

$$N_{\lambda,M}^0 := \{u \in N_{\lambda,M} : \Phi''_{u,M}(1) = 0\} = \{tu \in W_0^{1,n}(\Omega) : \Phi'_{u,M}(t) = 0, \Phi''_{u,M}(t) = 0\}.$$

We also define $H(u) = \int_{\Omega} h|u|^{q+1} dx$ and study the behaviour of fibering maps $\Phi_{u,M}$ according to the sign of $H(u)$. Let

$$H^+ := \{u \in W_0^{1,n}(\Omega) : H(u) > 0\}, \quad H_0^- := \{u \in W_0^{1,n}(\Omega) : H(u) \leq 0\}.$$

5.1.4 Analysis of Fiber Maps

Here we analyze accurately the geometry of the energy functional on the Nehari manifold. We split the study into two different cases $u \in H_0^-$ and $u \in H^+$. We define the map $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$\psi_u(t) = t^{n-1-q} M(\|tu\|^n) \|u\|^n - t^{-q} \int_{\Omega} (|x|^{-\mu} * F(tu)) f(tu) u dx$$

and observing the fact that $tu \in N_{\lambda,M}$ if and only if $t > 0$ is a solution of $\psi_u(t) = \lambda \int_{\Omega} h(x)|u|^{q+1} dx$. For the first case, we have the following result:

Lemma 5.1.17. *For any $u \in H_0^- \setminus \{0\}$ and $\lambda > 0$, there exists a unique t^* such that $t^*u \in N_{\lambda,M}^-$. Moreover, $\Phi_{u,M}$ is increasing on $(0, t^*)$ and decreasing on (t^*, ∞) .*

Proof. Since

$$\begin{aligned} \Phi'_{u,M}(t) &= t^{n-1} \|u\|^n M(\|tu\|^n) - \lambda t^q \int_{\Omega} h(x)|u|^{q+1} dx - \int_{\Omega} (|x|^{-\mu} * F(tu)) f(tu) u dx \\ &= t^q (\psi_u(t) - \lambda \int_{\Omega} h(x)|u|^{q+1} dx), \end{aligned}$$

therefore $tu \in N_{\lambda,M}$ iff $t > 0$ is a solution of $\psi_u(t) = \lambda \int_{\Omega} h(x)|u|^{q+1} dx$.

$$\begin{aligned} \psi'_u(t) &= (n-1-q)t^{n-2-q} M(\|tu\|^n) \|u\|^n + nt^{2n-2-q} M'(\|tu\|^n) \|u\|^{2n} \\ &\quad + \frac{q}{t^{q+1}} \int_{\Omega} (|x|^{-\mu} * F(tu)) f(tu) \cdot u dx - t^{-q} \left[\int_{\Omega} (|x|^{-\mu} * f(tu) \cdot u) f(tu) \cdot u dx \right. \\ &\quad \left. + \int_{\Omega} (|x|^{-\mu} * F(tu)) f'(tu) \cdot u^2 dx \right]. \end{aligned} \quad (5.1.34)$$

Due to the exponential growth of f , for large t we have $\psi'_u(t) < 0$ and since $u \in H_0^-$, there exists $t^* > 0$ such that $\psi_u(t^*) = \lambda \int_{\Omega} h(x)|u|^{q+1} dx$, i.e. $t^*u \in N_{\lambda,M}$.

If there exists another point t_1 such that $t^* < t_1$ and $\psi_u(t_1) = \lambda \int_{\Omega} h(x)|u|^{q+1} dx \leq 0$, i.e.

$$t_1^{n-1-q} (at_1^n \|u\|^n + b) \|u\|^n \leq t_1^{-q} \int_{\Omega} (|x|^{-\mu} * F(t_1 u)) f(t_1 u) u dx \quad (5.1.35)$$

and $\psi'_u(t_1) \geq 0$. Then by using $f'(t_1 u) t_1 u > (p+1)f(t_1 u)$ and $p > 2n-2-q$ we obtain from (5.1.35),

$$\psi'_u(t_1) < (2n-1-q) \left[t_1^{n-2-q} (at_1^n \|u\|^n + b) \|u\|^n - t_1^{-q-1} \int_{\Omega} (|x|^{-\mu} * F(t_1 u)) f(t_1 u) u dx \right] \leq 0.$$

Therefore $\psi'_u(t_1) < 0$ which yields a contradiction. Therefore there exists a unique t^* such that $\psi_u(t^*) = \lambda \int_{\Omega} h(x)|u|^{q+1} dx$. Also for $0 < t < t^*$, $\Phi'_{u,M}(t) = t^q(\psi_u(t) - \lambda \int_{\Omega} h(x)|u|^{q+1} dx) > 0$. Consequently, $\Phi_{u,M}$ is increasing in $(0, t^*)$ and decreasing on (t^*, ∞) . Therefore there exists a unique critical point of $\Phi_{u,M}$ which is also a global maximum point. Furthermore, since $\psi'_u(t) = \frac{(t\Phi''_{u,M}(t) - q\Phi'_{u,M}(t))}{t^q}$, we get $t^*u \in N_{\lambda,M}^-$. \square

For the second case, first we need the following result which characterizes the local minimum value of the function ψ_u at the local minimum point t_* is strictly greater than $\lambda H(u)$.

Lemma 5.1.18. *Let*

$$\Gamma := \left\{ u \in W_0^{1,n}(\Omega) : \|u\|^{3n/2} \leq \frac{B(u)}{2\sqrt{(2n-1-q)(n-1-q)ab}} \right\}$$

where $B(u) = \int_{\Omega} (|x|^{-\mu} * F(u))f'(u)(u)^2 + \int_{\Omega} (|x|^{-\mu} * f(u)u)f(u)u dx$. Then there exists a $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$, $\Gamma_0 > 0$ holds where

$$\Gamma_0 := \inf_{u \in \Gamma \setminus \{0\} \cap H^+} \left[B(u) - (2n-1) \int_{\Omega} (|x|^{-\mu} * F(u))f(u)u dx + nb\|u\|^n - \lambda(2n-1-q)H(u) \right]. \quad (5.1.36)$$

Proof. Step 1: Claim: $\inf_{u \in \Gamma \setminus \{0\} \cap H^+} \|u\| > 0$.

Let us suppose that it doesn't hold then there exists a sequence $\{u_k\} \subset \Gamma \setminus \{0\} \cap H^+$ such that $\|u_k\| \rightarrow 0$ and $\|u_k\|^{3n/2} \leq \frac{B(u_k)}{2\sqrt{(2n-1-q)(n-1-q)ab}}, \forall k$. Then by the Hardy-Littlewood-Sobolev inequality, we have

$$\begin{aligned} B(u_k) &= \int_{\Omega} (|x|^{-\mu} * F(u_k))f'(u_k)u_k^2 dx + \int_{\Omega} (|x|^{-\mu} * f(u_k)u_k)f(u_k)u_k dx \\ &\leq C(n, \mu) \left(\|f(u_k)u_k\|_{L^{2n/(2n-\mu)}(\Omega)}^2 + \|F(u_k)\|_{L^{2n/(2n-\mu)}(\Omega)} \|f'(u_k)(u_k)^2\|_{L^{2n/(2n-\mu)}(\Omega)} \right). \end{aligned}$$

Since $f(u) = u|u|^p \exp(|u|^\beta)$ and $f'(u) = ((p+1) + \beta|u|^\beta)|u|^p \exp(|u|^\beta)$, then we have

$$\begin{aligned} |B(u_k)| &\leq C(n, \mu) \left(\int_{\Omega} (|u_k|^{p+2} \exp(|u_k|^\beta))^{\frac{2n}{2n-\mu}} dx \right)^{\frac{2n-\mu}{n}} + C(n, \mu) \left(\int_{\Omega} (F(u_k))^{\frac{2n}{2n-\mu}} dx \right)^{\frac{2n-\mu}{2n}} \\ &\quad \times \left(\int_{\Omega} (((p+1) + \beta|u_k|^\beta)|u_k|^{p+2} \exp(|u_k|^\beta))^{\frac{2n}{2n-\mu}} dx \right)^{\frac{2n-\mu}{2n}}. \end{aligned}$$

Then using $F(t) \leq tf(t)$ and by the Hölder's inequality, we obtain

$$\begin{aligned} |B(u_k)| &\leq C_1 \left(\int_{\Omega} |u_k|^{\frac{2n\alpha'(p+2)}{2n-\mu}} dx \right)^{\frac{2n-\mu}{n\alpha'}} \cdot \left(\int_{\Omega} \exp\left(|u_k|^\beta \frac{2n\alpha}{2n-\mu}\right) dx \right)^{\frac{2n-\mu}{n\alpha}} \\ &\quad + C_2 \left(\int_{\Omega} |u_k|^{\frac{2n\alpha'(p+2)}{2n-\mu}} dx \right)^{\frac{2n-\mu}{2n\alpha'}} \cdot \left(\int_{\Omega} \exp\left(|u_k|^\beta \frac{2n\alpha}{2n-\mu}\right) dx \right)^{\frac{2n-\mu}{2n\alpha}} \times \end{aligned}$$

$$\begin{aligned} & \left[\left(\int_{\Omega} |u_k|^{\frac{2n\alpha'(p+2)}{2n-\mu}} dx \right)^{\frac{2n-\mu}{2n\alpha'}} \cdot \left(\int_{\Omega} \exp\left(|u_k|^{\beta} \frac{2n\alpha}{2n-\mu}\right) dx \right)^{\frac{2n-\mu}{2n\alpha}} \right. \\ & \left. + \left(\int_{\Omega} |u_k|^{\frac{2n\alpha'(p+\beta+2)}{2n-\mu}} dx \right)^{\frac{2n-\mu}{2n\alpha'}} \cdot \left(\int_{\Omega} \exp\left(|u_k|^{\beta} \frac{2n\alpha}{2n-\mu}\right) dx \right)^{\frac{2n-\mu}{2n\alpha}} \right]. \end{aligned}$$

Let α be such that $2n\alpha/(2n-\mu)\|u_k\|^{\beta} \leq \alpha_n$ and $v_k = \frac{u_k}{\|u_k\|}$, then by the Trudinger-Moser inequality we obtain

$$\begin{aligned} |B(u_k)| & \leq C_1 \left(\int_{\Omega} |u_k|^{\frac{2n\alpha'(p+2)}{2n-\mu}} dx \right)^{\frac{2n-\mu}{n\alpha'}} \cdot \left(\sup_{\|v_k\| \leq 1} \int_{\Omega} \exp(|v_k|^{\beta} \alpha_n) dx \right)^{\frac{2n-\mu}{n\alpha}} \\ & \quad + C_2 \left(\int_{\Omega} |u_k|^{\frac{2n\alpha'(p+2)}{2n-\mu}} dx \right)^{\frac{2n-\mu}{n\alpha'}} \cdot \left(\sup_{\|v_k\| \leq 1} \int_{\Omega} \exp(|v_k|^{\beta} \alpha_n) dx \right)^{\frac{2n-\mu}{n\alpha}} \times \\ & \quad \left[\left(\int_{\Omega} |u_k|^{\frac{2n\alpha'(p+2)}{2n-\mu}} dx \right)^{\frac{2n-\mu}{2n\alpha'}} + \left(\int_{\Omega} |u_k|^{\frac{2n\alpha'(p+\beta+2)}{2n-\mu}} dx \right)^{\frac{2n-\mu}{2n\alpha'}} \right]. \end{aligned}$$

Using the Sobolev embedding, it implies that

$$\begin{aligned} |B(u_k)| & \leq C_1(n, k, \beta, \mu) (\|u_k\|^{2(p+2)} + \|u_k\|^{(p+2)} (\|u_k\|^{(p+2)} + \|u_k\|^{(p+\beta+2)})) \\ & \leq C \|u_k\|^{(2p+4)} + \|u_k\|^{(2p+\beta+4)}. \end{aligned}$$

Hence using $u_k \in \Gamma \setminus \{0\}$ and by the Sobolev embedding theorem, we get $1 \leq C(\|u_k\|^{(2p+4-\frac{3n}{2})} + \|u_k\|^{(2p+\beta+4-\frac{3n}{2})})$ and $2p+4-\frac{3n}{2} > 0$ which is a contradiction as $\|u_k\| \rightarrow 0$ as $k \rightarrow \infty$. Therefore we have $\inf_{u \in \Gamma \setminus \{0\} \cap H^+} \|u\| > 0$.

Step 2: Claim: $0 < \inf_{u \in \Gamma \setminus \{0\} \cap H^+} \left\{ \int_{\Omega} (|x|^{-\mu} * f(u)u)(p+2-2n+\beta|u|^{\beta}) \exp(|u|^{\beta}) |u|^{p+2} dx \right\}$.

Since $F(s) \leq \frac{f(s)s}{p+2}$ then by the definition of Γ and from Step 1, we obtain $0 < \inf_{u \in \Gamma \setminus \{0\} \cap H^+} B(u)$ *i.e.*

$$\begin{aligned} 0 & < \inf_{u \in \Gamma \setminus \{0\} \cap H^+} \left\{ \int_{\Omega} (|x|^{-\mu} * F(u)) f'(u) u^2 + \int_{\Omega} (|x|^{-\mu} * f(u)u) f(u)u \right\} \\ & \leq \inf_{u \in \Gamma \setminus \{0\} \cap H^+} \left\{ \int_{\Omega} (|x|^{-\mu} * f(u)u) \left(f(u) \cdot u + f'(u) \frac{u^2}{p+2} \right) \right\} \\ & = \inf_{u \in \Gamma \setminus \{0\} \cap H^+} \left\{ \int_{\Omega} (|x|^{-\mu} * f(u)u) |u|^{p+2} \exp(|u|^{\beta}) \left(1 + \frac{(p+1) + \beta|u|^{\beta}}{p+2} \right) \right\}. \end{aligned}$$

Since $p+2-2n > 0$, we infer

$$0 < \inf_{u \in \Gamma \setminus \{0\} \cap H^+} \left\{ \int_{\Omega} (|x|^{-\mu} * f(u)u)(p+2-2n+\beta|u|^{\beta}) \exp(|u|^{\beta}) |u|^{p+2} dx \right\}.$$

Step 3: Claim: $\Gamma_0 > 0$. First,

$$\int_{\Omega} h(x) |u|^{q+1} \leq \left(\int_{\Omega} |h(x)|^{\gamma} \right)^{1/\gamma} \left(|u|^{(1+q)\gamma'} \right)^{1/\gamma'} \leq l \|u\|^{q+1}$$

where $l = \|h\|_{L^\gamma(\Omega)}$. Choosing

$$\lambda < \frac{bn}{(2n-1-q)l} M_0 := \lambda_0$$

where $M_0 = \inf_{u \in \Gamma \setminus \{0\} \cap H^+} \|u\|^{n-1-q} > 0$, we get that $\lambda l(2n-1-q)\|u\|^{1+q} < nb\|u\|^n$ for any $u \in \Gamma \setminus \{0\} \cap H^+$. Then for $u \in \Gamma \setminus \{0\} \cap H^+$ and $p+1 > 2n-1$,

$$\begin{aligned} B(u) + nb\|u\|^n - (2n-1) \int_{\Omega} (|x|^{-\mu} * F(u)) f(u) \cdot u - \lambda(2n-1-q)H(u) &\geq \\ \int_{\Omega} (|x|^{-\mu} * F(u)) (f'(u)u^2 - (2n-1)f(u) \cdot u) + \int_{\Omega} (|x|^{-\mu} * f(u) \cdot u) f(u) \cdot u \, dx & \\ + nb\|u\|^n - (2n-1-q)\lambda H(u) &> 0. \end{aligned}$$

Therefore $\Gamma_0 > 0$. □

Lemma 5.1.19. *For any $u \in H^+$, there exist $t_*, t_1, t_2 > 0$ and λ_0 such that $t_1 u \in N_{\lambda, M}^+$ and $t_2 u \in N_{\lambda, M}^-$ for any $\lambda \in (0, \lambda_0)$ and $t_1 < t_* < t_2$.*

Proof. For $0 \neq u \in H^+$, we have that $\psi_u(t) \rightarrow -\infty$ as $t \rightarrow \infty$ and for small $t > 0$, $\psi_u(t) > 0$. Then there exists at least a point of maximum of $\psi_u(t)$, say t_* , and $\psi'_u(t_*) = 0$, i.e.

$$\begin{aligned} (2n-1-q)t_*^{2n-2-q}a\|u\|^{2n} + (n-1-q)t_*^{n-2-q}b\|u\|^n + \frac{q}{t_*^{q+1}} \int_{\Omega} (|x|^{-\mu} * F(t_*u)) f(t_*u) u \, dx \\ = t_*^{-q} \left[\int_{\Omega} (|x|^{-\mu} * F(t_*u)) f'(t_*u) u^2 \, dx + \int_{\Omega} (|x|^{-\mu} * f(t_*u) u) f(t_*u) \cdot u \, dx \right]. \end{aligned}$$

This implies that

$$\begin{aligned} (2n-1-q)a\|t_*u\|^{2n} + (n-1-q)b\|t_*u\|^n + q \int_{\Omega} (|x|^{-\mu} * F(t_*u)) f(t_*u) t_*u \, dx \\ = \int_{\Omega} (|x|^{-\mu} * F(t_*u)) f'(t_*u) (t_*u)^2 \, dx + \int_{\Omega} (|x|^{-\mu} * f(t_*u) t_*u) f(t_*u) t_*u \, dx. \end{aligned}$$

Then we have

$$2\sqrt{(2n-1-q)a\|t_*u\|^{2n}b(n-1-q)\|t_*u\|^n} \leq B(t_*u)$$

from which it follows

$$\|t_*u\|^{3n/2} \leq \frac{B(t_*u)}{2\sqrt{(2n-1-q)(n-1-q)ab}}$$

where $B(u) = \int_{\Omega} (|x|^{-\mu} * F(u)) f'(u) u^2 + \int_{\Omega} (|x|^{-\mu} * f(u) u) f(u) u \, dx$. Using $\psi'_u(t_*) = 0$, we replace the value of $a\|t_*u\|^{2n}$ in the definition of $\psi_u(t)$ to obtain

$$\psi_u(t_*) = \frac{1}{(2n-1-q)t_*^{q+1}} \left[B(t_*u) - (2n-1) \int_{\Omega} (|x|^{-\mu} * F(t_*u)) f(t_*u) t_*u \, dx + nb\|t_*u\|^n \right]. \quad (5.1.37)$$

We notice from Lemma 5.1.18 and Equation (5.1.37) that for $u \in H^+ \setminus \{0\}$, there exists a $t_* > 0$, local maximum of ψ_u verifying $\psi_u(t_*) - \lambda H(u) > 0$ since $t_* u \in \Gamma \setminus \{0\} \cap H^+$. From $\psi_u(0) = 0$, $\psi_u(t_*) > \lambda H(u) > 0$ and $\lim_{t \rightarrow \infty} \psi_u(t) = -\infty$, there exists $t_1 = t_1(u) < t_* < t_2(u) = t_2$ such that $\psi_u(t_1) = \lambda \int_{\Omega} h(x)|u|^{q+1} dx = \psi_u(t_2)$ with $\psi'_u(t_1) > 0, \psi'_u(t_2) < 0$. Therefore, $t_1 u \in N_{\lambda, M}^+$ and $t_2 u \in N_{\lambda, M}^-$. We now prove that $t_1 u \in N_{\lambda, M}^+$ and $t_2 u \in N_{\lambda, M}^-$ are unique. If not then there exists $t_3 u \in N_{\lambda, M}^+$ and t_{**} such that $t_2 < t_{**} < t_3$ and $\psi'_u(t_{**}) = 0$ and $\psi_u(t_{**}) < \lambda H(u)$. But Lemma 5.1.18 induces that if $\psi'_u(t_{**}) = 0$ then $\psi_u(t_{**}) > \lambda H(u)$ which is a contradiction. \square

In the sequel, we will denote t_* the smallest critical point of ψ_u . As an application of Lemma 5.1.18 and using the geometry of energy functional $\mathcal{J}_{\lambda, M}$ on the Nehari manifold, we prove the non-existence of non-trivial solutions in $N_{\lambda, M}^0$.

Lemma 5.1.20. *If $\lambda \in (0, \lambda_0)$ then $N_{\lambda, M}^0 = \{0\}$.*

Proof. Suppose $u \neq 0$ and $u \in N_{\lambda, M}^0$. Then $\Phi'_{u, M}(1) = 0$ and $\Phi''_{u, M}(1) = 0$, i.e.

$$a\|u\|^{2n} + b\|u\|^n = \lambda H(u) + \int_{\Omega} (|x|^{-\mu} * F(u))f(u)u dx \text{ and} \quad (5.1.38)$$

$$(2n-1)a\|u\|^{2n} + (n-1)b\|u\|^n = \lambda q H(u) + B(u). \quad (5.1.39)$$

Let $u \in H^+ \cap N_{\lambda, M}^0$, then from (5.1.38) and (5.1.39) (by replacing the value $\lambda H(u)$), we obtain

$$2\sqrt{(2n-1-q)(n-1-q)ab}\|u\|^{3n} \leq B(u)$$

which implies $u \in \Gamma \setminus \{0\} \cap H^+$. Again from (5.1.38), (5.1.39) and substituting the value of $a\|u\|^{2n}$, we obtain

$$B(u) - (2n-1) \int_{\Omega} (|x|^{\mu} * F(u))f(u).u + nb\|u\|^n - \lambda(2n-1-q)H(u) = 0$$

which contradicts Lemma 5.1.18. If $u \in H_0^- \cap N_{\lambda, M}^0$ and from Lemma 5.1.17, "1" is the only critical point of $\Phi_{u, M}$ and $\Phi''_{u, M}(1) = 0$. But $u \in H_0^-$ implies that $\psi'_u(1) < 0$ and then $\phi''_{u, M}(1) < 0$ which is a contradiction and the lemma is proved. \square

5.1.5 Existence and multiplicity of weak solutions

We start this section, by studying the geometric structure of the energy functional $\mathcal{J}_{\lambda, M}$. Define

$$\theta = \inf_{u \in N_{\lambda, M}} \mathcal{J}_{\lambda, M}(u).$$

Theorem 5.1.21. $\mathcal{J}_{\lambda, M}(u)$ is bounded below and coercive on $N_{\lambda, M}$ such that $\theta \geq -C(q, n, b)\lambda^{\frac{n}{n-q-1}}$.

Proof. Let $u \in N_{\lambda, M}$. Then,

$$\begin{aligned} \mathcal{J}_{\lambda, M}(u) &= \frac{1}{n} \left[\frac{a}{2} \|u\|^{2n} + b \|u\|^n \right] - \frac{\lambda}{q+1} H(u) - \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(u)) F(u) \, dx \\ &= \frac{1}{n} \left[\frac{a}{2} \|u\|^{2n} + b \|u\|^n \right] - \frac{\lambda}{q+1} H(u) - \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(u)) F(u) \, dx \\ &\quad - \frac{1}{p+2} \left[a \|u\|^{2n} + b \|u\|^n - \lambda H(u) - \int_{\Omega} (|x|^{-\mu} * F(u)) f(u) u \, dx \right] \\ &= a \|u\|^{2n} \left(\frac{p+2-2n}{2n(p+2)} \right) + b \|u\|^n \left(\frac{p+2-n}{n(p+2)} \right) - \lambda \left(\frac{p+1-q}{(1+q)(p+2)} \right) H(u) \\ &\quad - \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(u)) \left(F(u) - \frac{2f(u)u}{p+2} \right) \, dx. \end{aligned}$$

Since $0 \leq F(u) \leq \frac{2}{p+2} f(u)u$ and $H(u) \leq l \|u\|^{q+1}$. Then by the Sobolev inequality we obtain

$$\mathcal{J}_{\lambda, M}(u) \geq a \|u\|^{2n} \left(\frac{p+2-2n}{2n(p+2)} \right) + b \|u\|^n \left(\frac{p+2-n}{n(p+2)} \right) - \lambda l \left(\frac{p+1-q}{(1+q)(p+2)} \right) \|u\|^{q+1}.$$

Therefore since $q < n-1$, $\mathcal{J}_{\lambda, M}$ is coercive on $N_{\lambda, M}$, *i.e.* $\mathcal{J}_{\lambda, M}(u) \rightarrow \infty$ as $\|u\| \rightarrow \infty$.

For $u \in N_{\lambda, M}$ we have also,

$$\begin{aligned} \mathcal{J}_{\lambda, M}(u) &= \frac{b}{n} \|u\|^n - \frac{\lambda}{q+1} H(u) - \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(u)) F(u) \, dx \\ &\quad + \frac{1}{2n} \left(\lambda H(u) + \int_{\Omega} (|x|^{-\mu} * F(u)) f(u) u \, dx - b \|u\|^n \right) \\ &= \frac{1}{2n} b \|u\|^n - \lambda \left(\frac{1}{q+1} - \frac{1}{2n} \right) H(u) + \frac{1}{2} \left(\int_{\Omega} (|x|^{-\mu} * F(u)) \left(\frac{f(u)u}{n} - F(u) \right) \, dx \right) \\ &\geq \frac{1}{2n} b \|u\|^n - \lambda \left(\frac{1}{q+1} - \frac{1}{2n} \right) H(u) \end{aligned}$$

since $\left(\frac{f(u)u}{n} - F(u) \right) \geq 0$. Then for $u \in H_0^-$, we get $\mathcal{J}_{\lambda, M}(u) \geq 0$.

Now for $u \in H^+$, setting $r = \frac{n}{1+q}$ and by the Sobolev embedding we obtain

$$\begin{aligned} \mathcal{J}_{\lambda, M}(u) &\geq \frac{b}{2n} \|u\|^n - \frac{\lambda(2n-1-q)}{2n(q+1)} H(u) \geq \frac{b}{2n} \|u\|^n - \frac{\lambda(2n-1-q)}{2n(q+1)} l \left(\int_{\Omega} |u|^n \, dx \right)^{1/r} \\ &= c_1 \|u\|^n - c_2 \|u\|^{q+1} \end{aligned}$$

where $c_1 = \frac{b}{2n}$ and $c_2 = c_2(\lambda)$.

We observe that the minimum of the function $g(x) = c_1 x^n - c_2 x^{q+1}$ is achieved at $x = \left(\frac{c_2(q+1)}{c_1 n} \right)^{\frac{1}{n-q-1}}$. Therefore,

$$\inf_{u \in N_{\lambda, M}} \mathcal{J}_{\lambda, M}(u) \geq g \left(\frac{c_2(q+1)}{c_1 n} \right)^{\frac{1}{n-q-1}} = \left(\frac{c_2^n}{c_1^{q+1}} \right)^{\frac{1}{n-1-q}} \left(\left(\frac{q+1}{n} \right)^{\frac{n}{n-1-q}} - \left(\frac{q+1}{n} \right)^{\frac{q+1}{n-1-q}} \right).$$

From this it follows that

$$\theta \geq -C(q, n, b) \lambda^{\frac{n}{n-q-1}}$$

where $C(q, n, b) > 0$. □

Now since $\mathcal{J}_{\lambda,M}$ is bounded below on $N_{\lambda,M}$, by the Ekeland variational principle we get a sequence $\{u_k\}_{k \in \mathbb{N}} \subset N_{\lambda,M} \setminus \{0\}$ such that

$$\begin{cases} \mathcal{J}_{\lambda,M}(u_k) \leq \theta + \frac{1}{k}; \\ \mathcal{J}_{\lambda,M}(v) \geq \mathcal{J}_{\lambda,M}(u_k) - \frac{1}{k}\|u_k - v\|, \quad \forall v \in N_{\lambda,M}. \end{cases} \quad (5.1.40)$$

The following result shows that minimizers for $\mathcal{J}_{\lambda,M}$ in any of the subsets of $N_{\lambda,M}$ are critical points for $\mathcal{J}_{\lambda,M}$.

Lemma 5.1.22. *Let u be a local minimizer for $\mathcal{J}_{\lambda,M}$ on any subsets of $N_{\lambda,M}$ such that $u \notin N_{\lambda,M}^0$. Then u is a critical point of $\mathcal{J}_{\lambda,M}$.*

Proof. Let u be a local minimizer for $\mathcal{J}_{\lambda,M}$. Then, in any case u is a minimizer for $\mathcal{J}_{\lambda,M}$ under the constraint $I_{\lambda,M}(u) := \langle \mathcal{J}'_{\lambda,M}(u), u \rangle = 0$. Hence, by the theory of Lagrange multipliers, there exists a $\mu \in \mathbb{R}$ such that $\mathcal{J}'_{\lambda,M} = \mu I'_{\lambda,M}(u)$. Thus $\langle \mathcal{J}'_{\lambda,M}(u), u \rangle = \mu \langle I'_{\lambda,M}(u), u \rangle = \mu \Phi''_{\lambda,M}(1) = 0$, but $u \notin N_{\lambda,M}^0$ and so $\Phi''_{\lambda,M}(1) \neq 0$. Hence $\mu = 0$. \square

Now, we prove a set of lemmas which are necessary to study the $(PS)_\theta$ condition and compactness of the minimizing sequence $\{u_k\}_{k \in \mathbb{N}}$, whose proof are totally based on the geometry of the energy functional $\mathcal{J}_{\lambda,M}$ on the Nehari manifold.

Lemma 5.1.23. *There exists a constant $C_0 > 0$ such that $\theta \leq -C_0$.*

Proof. Let $u \in H^+$, then $\exists t_1(u) > 0$ such that $t_1 u \in N_{\lambda,M}^+$ and $\psi_{u,M}(t_1) = \lambda H(u)$. In that case,

$$\begin{aligned} \mathcal{J}_{\lambda,M}(t_1 u) &= \frac{1}{n} \left(\frac{a}{2} \|t_1 u\|^{2n} + b \|t_1 u\|^n \right) - \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(t_1 u)) F(t_1 u) \, dx \\ &\quad - \frac{\lambda}{q+1} \int_{\Omega} h(x) |t_1 u|^{q+1} \, dx \\ &= \frac{1}{n} \left(\frac{a}{2} \|t_1 u\|^{2n} + b \|t_1 u\|^n \right) - \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(t_1 u)) F(t_1 u) \, dx \\ &\quad - \frac{1}{q+1} \left(a \|t_1 u\|^{2n} + b \|t_1 u\|^n - \int_{\Omega} (|x|^{-\mu} * F(t_1 u)) f(t_1 u) t_1 u \, dx \right). \end{aligned}$$

Since $\Phi'_{u,M}(t_1) = 0$, $\Phi''_{u,M}(t_1) > 0$ and from (5.1.34) we obtain

$$\begin{aligned} \mathcal{J}_{\lambda,M}(t_1 u) &= \frac{-(n-1-q)b \|t_1 u\|^n}{2n(q+1)} + \int_{\Omega} (|x|^{-\mu} * F(t_1 u)) \left(\frac{2n+q}{2n(q+1)} f(t_1 u) t_1 u \right. \\ &\quad \left. - \frac{1}{2} F(t_1 u) - \frac{f'(t_1 u)(t_1 u)^2}{2n(q+1)} \right) \, dx - \frac{1}{2n(q+1)} \int_{\Omega} (|x|^{-\mu} * f(t_1 u) t_1 u) f(t_1 u) t_1 u \, dx \\ &\leq \frac{-(n-1-q)b \|t_1 u\|^n}{2n(q+1)} + \int_{\Omega} (|x|^{-\mu} * F(t_1 u)) \left(\frac{2n+q}{2n(q+1)} - \frac{p+2}{2n(q+1)} \right) \end{aligned}$$

$$-\frac{p+1}{2n(q+1)})f(t_1u)t_1u \, dx - \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(t_1u))F(t_1u) \, dx.$$

Since $q < n - 1$ and $p + 1 > 2n - 1$ we set $2n + q - (2p + 3) \leq 3n - 1 - (4n - 1) < 0$ and then $\theta \leq \inf_{u \in N_{\lambda, M}^+ \cap H^+} \mathcal{J}_{\lambda, M}(u) \leq -C_0 < 0$. \square

Then by (5.1.40) and Lemma 5.1.23, we have for large k ,

$$\mathcal{J}_{\lambda, M}(u_k) \leq -\frac{C_0}{2}. \quad (5.1.41)$$

Also since $u_k \in N_{\lambda, M} \setminus \{0\}$ we have

$$\begin{aligned} \mathcal{J}_{\lambda, M}(u_k) &= a\|u_k\|^{2n} \left(\frac{p+2-2n}{2n(p+2)} \right) + b\|u_k\|^n \left(\frac{p+2-n}{n(p+2)} \right) - \lambda \left(\frac{p+1-q}{(1+q)(p+2)} \right) H(u_k) \\ &\quad - \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(u_k)) \left(F(u_k) - \frac{2f(u_k)u_k}{p+2} \right) \, dx. \end{aligned}$$

then together with (5.1.41), we have

$$-\lambda \left(\frac{p+1-q}{(1+q)(p+2)} \right) H(u_k) \leq -\frac{C_0}{2} \implies H(u_k) \geq \frac{C_0(p+2)(1+q)}{2\lambda(p+1-q)} C_0 > 0$$

i.e.

$$H(u_k) > C > 0 \text{ and } u_k \in N_{\lambda, M} \cap H^+ \text{ for } k \text{ large enough.} \quad (5.1.42)$$

Lemma 5.1.24. *Let $\lambda \in (0, \lambda_0)$ where $\lambda_0 = \frac{bn}{(2n-1-q)l} M_0$. Then given any $u \in N_{\lambda, M} \setminus \{0\}$, then there exists $\epsilon > 0$ and a differentiable function $\xi : B(0, \epsilon) \subset W_0^{1, n}(\Omega) \rightarrow \mathbb{R}$ such that $\xi(0) = 1$, and $\xi(w)(u - w) \in N_{\lambda, M}$ and for all $w \in W_0^{1, n}(\Omega)$*

$$\langle \xi'(0), w \rangle = \frac{n(2a\|u\|^n + b) \int_{\Omega} |\nabla(u)|^{n-2} \nabla u \cdot \nabla w \, dx - \lambda(q+1) \int_{\Omega} h(x)|u|^{q-1}uw \, dx - \langle S(u), w \rangle}{a(2n-1-q)\|u\|^{2n} + b(n-1-q)\|u\|^n + R(u)} \quad (5.1.43)$$

where

$$R(u) = \int_{\Omega} (|x|^{-\mu} * F(u))(qf(u) - f'(u) \cdot u) \cdot u \, dx - \int_{\Omega} (|x|^{-\mu} * f(u) \cdot u) f(u) u \, dx$$

and

$$\langle S(u), w \rangle = \int_{\Omega} (|x|^{-\mu} * F(u))(f'(u)u + f(u))w \, dx + \int_{\Omega} (|x|^{-\mu} * f(u)u) f(u)w \, dx.$$

Proof. Fix $u \in N_{\lambda, M} \setminus \{0\}$, define a function $G_u : \mathbb{R} \times W_0^{1, n}(\Omega) \rightarrow \mathbb{R}$ as

$$\begin{aligned} G_u(t, v) &= at^{2n-1-q}\|u - v\|^{2n} + bt^{n-1-q}\|u - v\|^n - \\ &\quad \frac{1}{t^q} \int_{\Omega} (|x|^{-\mu} * F(t(u - v)))f(t(u - v)) \cdot (u - v) \, dx - \lambda \int_{\Omega} h|u - v|^{q+1} \, dx. \end{aligned}$$

Then $G_u \in C^1(\mathbb{R} \times W_0^{1,n}(\Omega), \mathbb{R})$ and

$$G_u(1, 0) = a\|u\|^{2n} + b\|u\|^n - \int_{\Omega} (|x|^{-\mu} * F(u))f(u).u \, dx - \lambda \int_{\Omega} h|u|^{q+1} \, dx = \Phi'_{u,M}(1) = 0$$

and

$$\begin{aligned} \frac{\partial}{\partial t} G_u(1, 0) &= a(2n-1-q)\|u\|^{2n} + b(n-1-q)\|u\|^n + q \int_{\Omega} (|x|^{-\mu} * F(u))f(u).u - B(u) \\ &= \Phi''_{u,M}(1) \neq 0. \end{aligned}$$

Then by the implicit function theorem, there exists $\epsilon > 0$ and a differentiable function $\xi : B(0, \epsilon) \subset W_0^{1,n}(\Omega) \rightarrow \mathbb{R}$ such that $\xi(0) = 1$ and $G_u(\xi(w), w) = 0 \quad \forall w \in B(0, \epsilon)$ which is equivalent to $\langle \mathcal{J}'_{\lambda,M}(\xi(w)(u-w), \xi(w)(u-w)) \rangle = 0 \quad \forall w \in B(0, \epsilon)$. Thus, $\xi(w)(u-w) \in N_{\lambda,M}$ and differentiating

$$\begin{aligned} G_u(\xi(w), w) &= a(\xi(w))^{2n-1-q}\|u-w\|^{2n} + b(\xi(w))^{n-1-q}\|u-w\|^n \\ &\quad - \frac{1}{(\xi(w))^q} \int_{\Omega} (|x|^{-\mu} * F(\xi(w))(u-w))f(\xi(w)(u-w))(u-w) - \lambda \int_{\Omega} h(x)|u-w|^{q+1} = 0 \end{aligned}$$

with respect to w , we obtain (5.1.43). \square

Similarly we have:

Lemma 5.1.25. *Let $\lambda \in (0, \lambda_0)$ where $\lambda_0 = \frac{bn}{(2n-1-q)l}M_0$. Then there exists $u \in N_{\lambda,M}^- \setminus \{0\}$, then there exists $\epsilon > 0$ and a differentiable function $\xi^- : B(0, \epsilon) \subset W_0^{1,n}(\Omega) \rightarrow \mathbb{R}$ such that $\xi^-(0) = 1$, and $\xi^-(w)(u-w) \in N_{\lambda,M}^-$ and for all $w \in W_0^{1,n}(\Omega)$*

$$\begin{aligned} &\langle (\xi^-)'(0), w \rangle \\ &= \frac{n(2a\|u\|^n + b) \int_{\Omega} |\nabla(u)|^{n-2} \nabla u \cdot \nabla w \, dx - \lambda(q+1) \int_{\Omega} h(x)|u|^{q-1}uw \, dx - \langle S(u), w \rangle}{a(2n-1-q)\|u\|^{2n} + b(n-1-q)\|u\|^n + R(u)} \end{aligned}$$

where $R(u)$ and $S(u)$ are as in Lemma 5.1.24.

Proof. For any $u \in N_{\lambda,M}^-$, $\Phi'_{u,M}(1) = 0$ and $\Phi''_{u,M}(1) < 0$. This implies $u \in \Gamma \setminus \{0\}$. Then by Lemma 5.1.24 there exists $\epsilon > 0$ and a differentiable function $\xi^- : B(0, \epsilon) \subset W_0^{1,n}(\Omega) \rightarrow \mathbb{R}$ such that $\xi^-(0) = 1$, and $\xi^-(w)(u-w) \in N_{\lambda,M}^-$ for all $w \in B(0, \epsilon)$. Then by the continuity of $\mathcal{J}'_{\lambda,M}$ and ξ^- and by choosing ϵ small enough we have

$$\begin{aligned} \Phi''_{\xi^-(u)(u-w), M}(1) &= n\|\xi^-(u)(u-w)\|^{2n}M(\|\xi^-(u)(u-w)\|^n) + (n-1)\|\xi^-(u)(u-w)\|^nM(\|tu\|^n) \\ &\quad - \lambda q \int_{\Omega} h(x)|\xi^-(u)(u-w)|^{q+1} \, dx \\ &\quad - \int_{\Omega} (|x|^{-\mu} * f(\xi^-(u)(u-w)).\xi^-(u)(u-w))f(\xi^-(u)(u-w))\xi^-(u)(u-w) \, dx \\ &\quad - \int_{\Omega} (|x|^{-\mu} * F(\xi^-(u)(u-w)))f'(\xi^-(u)(u-w))(\xi^-(u)(u-w))^2 \, dx < 0 \end{aligned}$$

that implies $\xi^-(w)(u-w) \in N_{\lambda,M}^-$. \square

As an application of above lemmas, we prove that our minimizing sequence $\{u_k\}_{k \in \mathbb{N}}$ satisfy the $(PS)_\theta$ condition and using this, we prove the multiplicity result in the subcritical case.

Proposition 5.1.26. *Let $\lambda \in (0, \lambda_0)$ where $\lambda_0 = \frac{bn}{(2n-1-q)l}M_0$. Assume $u_k \in N_{\lambda, M}$ is satisfying (5.1.40). Then $\|\mathcal{J}'_{\lambda, M}(u_k)\|_* \rightarrow 0$ as $k \rightarrow \infty$.*

Proof. We divide the proof into three steps:

Step 1: $\liminf_{k \rightarrow \infty} \|u_k\| > 0$.

We know that from (5.1.42) that for large k , $H(u_k) \geq C > 0$, so by using Hölder inequality we obtain $C < H(u_k) \leq C_1 \|u_k\|^{q+1}$.

Step 2: We claim that

$$\liminf_{k \rightarrow \infty} (2n-1-q)a\|u_k\|^{2n} + b(n-1-q)\|u_k\|^n + q \int_{\Omega} (|x|^{-\mu} * F(u_k)) f(u_k) u_k \, dx - B(u_k) > 0.$$

Without loss of generality, we can assume that $u_k \in N_{\lambda, M}^+$ (if not replace u_k by $t_1(u_k)u_k$). Arguing by contradiction, suppose that there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that

$$0 \leq (2n-1-q)a\|u_k\|^{2n} + b(n-1-q)\|u_k\|^n + q \int_{\Omega} (|x|^{-\mu} * F(u_k)) f(u_k) u_k \, dx - B(u_k) = o_k(1).$$

From Step 1 and the above equation we obtain that $\liminf_{k \rightarrow \infty} B(u_k) > 0$ and $(2n-1-q)a\|u_k\|^{2n} + b(n-1-q)\|u_k\|^n \leq B(u_k)$ i.e. $u_k \in \Gamma \setminus \{0\}$ for all large k .

Since $u_k \in N_{\lambda, M}^+ \setminus \{0\}$

$$-nb\|u_k\|^n + \lambda(2n-1-q)H(u_k) + (2n-1) \int_{\Omega} (|x|^{-\mu} * F(u_k)) f(u_k) u_k \, dx - B(u_k) = o_k(1)$$

which is a contradiction since $\Gamma_0 > 0$.

Step 3: $\|\mathcal{J}'_{\lambda, M}(u_k)\|_* \rightarrow 0$ as $k \rightarrow \infty$.

By using Lemma 5.1.24, there exists a differentiable function $\xi_k : B(0, \epsilon_k) \rightarrow \mathbb{R}$ for some $\epsilon_k > 0$ such that $\xi_k(0) = 1$ and $\xi_k(w)(u_k - w) \in N_{\lambda, M} \quad \forall w \in B(0, \epsilon_k)$. Choose $0 < \rho < \epsilon_k$ and $f \in W_0^{1, n}(\Omega)$ such that $\|f\| = 1$. Let $w_\rho = \rho f$. Then $\|w\|_\rho = \rho < \epsilon_k$ and define $\eta_\rho = \xi_k(w_\rho)(u_k - w_\rho)$. Then from the Taylor expansion and (5.1.40), we obtain

$$\begin{aligned} \frac{1}{k} \|\eta_\rho - u_k\| &\geq \mathcal{J}_{\lambda, M}(u_k) - \mathcal{J}_{\lambda, M}(\eta_\rho) = \langle \mathcal{J}'_{\lambda, M}(\eta_\rho), u_k - \eta_\rho \rangle + o(\|u_k - \eta_\rho\|) \\ &= (1 - \xi_k(w_\rho)) \langle \mathcal{J}'_{\lambda, M}(\eta_\rho), u_k \rangle + \rho \xi_k(w_\rho) \langle \mathcal{J}'_{\lambda, M}(\eta_\rho), f \rangle + o(\|u_k - \eta_\rho\|). \end{aligned} \quad (5.1.44)$$

We also infer

$$\frac{1}{\rho} \|\eta_\rho - u_k\| = \left\| \frac{(\xi_k(w_\rho) - 1)}{\rho} u_k - \xi_k(w_\rho) f \right\| \rightarrow \|u_k \langle \xi'_k(0), f \rangle - f\| \quad \text{as } \rho \rightarrow 0.$$

Since $u_k \in N_{\lambda, M}$, we have also $\frac{1 - \xi_k(w_\rho)}{\rho} \langle \mathcal{J}'_{\lambda, M}(\eta_\rho), u_k \rangle \rightarrow 0$ as $\rho \rightarrow 0$.

Thus, dividing the expression in (5.1.44) by ρ and doing $\rho \rightarrow 0^+$, we get

$$\langle \mathcal{J}'_{\lambda, M}(u_k), f \rangle \leq \frac{\|f\|}{k} (\|u_k\| \|\xi'_k(0)\|_* + O(1))$$

which implies that

$$\|\mathcal{J}'_{\lambda,M}(u_k)\|_* \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

if $\|\xi'_k(0)\|_*$ is bounded uniformly in k . To prove that, using (5.1.36) and the boundedness of the sequence $\{u_k\}$ in $W_0^{1,n}(\Omega)$, we only need to show that for any $f \in W_0^{1,n}(\Omega)$, $\langle S(u_k), f \rangle$ is uniformly bounded in k . For the subcritical case, *i.e.* $\beta \in (0, \frac{n}{n-1})$, it holds since for any $\epsilon > 0$ and $q > 1$, there exists $C_{\epsilon,q,\beta} > 0$ such that

$$\exp(q|t|^\beta) \leq C_{\epsilon,q,\beta} \exp(\epsilon|t|^{\frac{n}{n-1}}), \quad \forall t \in \mathbb{R}.$$

Then by Theorem 2.2.1, Chapter 1, we obtain $\langle S(u_k), f \rangle \leq C\|f\|$ with $C > 0$ independent of k . Consider now the critical case, *i.e.* $\beta = \frac{n}{n-1}$. From the boundedness of $R(u_k)$ (see the statement of Lemma 5.1.24), it follows that

$$\begin{aligned} \sup_k \int_{\Omega} (|x|^{-\mu} * F(u_k)) f(u_k) u_k \, dx &< \infty, \\ \sup_k \int_{\Omega} (|x|^{-\mu} * F(u_k)) f'(u_k) u_k^2 \, dx &< \infty \end{aligned}$$

and

$$\sup_k \int_{\Omega} (|x|^{-\mu} * f(u_k) u_k) f(u_k) u_k \, dx < \infty.$$

Then for any $\phi \in C_c^\infty(\Omega)$, we have by Vitali's convergence theorem and up to a subsequence

$$\langle S(u_k), \phi \rangle \rightarrow \langle S(u_0), \phi \rangle \tag{5.1.45}$$

where u_0 is the weak limit of $(u_k)_{k \in \mathbb{N}}$ in $W_0^{1,n}(\Omega)$. From (5.1.45), we have that there exists $C > 0$ independent of k such that

$$|\langle S(u_k), \phi \rangle| \leq C\|\phi\|. \tag{5.1.46}$$

Using a density argument, we conclude that (5.1.46) holds for any $\phi \in W_0^{1,n}(\Omega)$. This completes the proof in the critical case. \square

The existence result for first positive solution in subcritical case in $N_{\lambda,M} \cap H^+$ is given by the following Theorem:

Theorem 5.1.27. *Let $\beta < \frac{n}{n-1}$ and let $\lambda \in (0, \lambda_0)$ where $\lambda_0 = \frac{bn}{(2n-1-q)l} M_0$. Then there exists a positive weak solution to $(\mathcal{P}_{\lambda,M})$ $u_\lambda \in N_{\lambda,M}^+ \cap H^+$ such that $\mathcal{J}_{\lambda,M}(u_\lambda) = \inf_{u \in N_{\lambda,M} \setminus \{0\}} \mathcal{J}_{\lambda,M}(u)$. Moreover $u_\lambda \in N_{\lambda,M}^+ \cap H^+$ is a non-negative local minimum for $\mathcal{J}_{\lambda,M}$ in $W_0^{1,n}(\Omega)$.*

Proof. Let u_k be a minimizing sequence satisfying $\mathcal{J}_{\lambda,M}(u_k) \rightarrow \theta$ as $k \rightarrow \infty$ and $\mathcal{J}_{\lambda,M}(v) \geq \mathcal{J}_{\lambda,M}(u_k) - \frac{1}{k}\|u_k - v\|$, $\forall v \in N_\lambda$. Using $\mathcal{J}_{\lambda,M}(|u|) \leq \mathcal{J}_{\lambda,M}(u)$ for any $u \in W_0^{1,n}(\Omega)$ and from the proof of the Ekeland principle (see [242, p. 51-53]), we can assume that u_k is non-negative. By using Proposition 5.1.26 we obtain $\{u_k\}$ is $(\text{PS})_\theta$ sequence. Then from Lemma 5.1.10 we

get $\{u_k\}$ is a bounded sequence in $W_0^{1,n}(\Omega)$. Also there exists a subsequence of $\{u_k\}$ (denoted by same sequence) and a non-negative u_λ such that $u_k \rightharpoonup u_\lambda$ weakly in $W_0^{1,n}(\Omega)$ and $u_k \rightarrow u_\lambda$ strongly in $L^r(\Omega)$ for $r \geq 1$ and $u_k \rightarrow u_\lambda$ a.e. in Ω . Then using $f(t) \leq C_{\epsilon,\beta} \exp(\epsilon t^{\frac{n}{n-1}})$ for $\epsilon > 0$ small enough and from Theorem 2.2.1, Chapter 1, we obtain that $f(u_k)$ and $(|x|^{-\mu} * F(u_k))$ are uniformly bounded in $L^q(\Omega)$ for all $q > 1$. Then again by Vitali's convergence theorem, we obtain

$$\left| \int_{\Omega} (|x|^{-\mu} * F(u_k)) f(u_k) (u_k - u_\lambda) dx \right| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

and by Proposition 5.1.26, we have $\langle \mathcal{J}'_{\lambda,M}(u_k), u_k - u_\lambda \rangle \rightarrow 0$. Then we conclude that

$$M(\|u_k\|^n) \int_{\Omega} |\nabla u_k|^{n-2} \nabla u_k \cdot \nabla (u_k - u_\lambda) dx \rightarrow 0. \quad (5.1.47)$$

On the other hand, using $u_k \rightarrow u_\lambda$ weakly and by boundedness of $M(\|u_k\|^n)$ we have

$$M(\|u_k\|^n) \int_{\Omega} |\nabla u_\lambda|^{n-2} \nabla u_\lambda \cdot \nabla (u_k - u_\lambda) dx \rightarrow 0. \quad (5.1.48)$$

Subtracting (5.1.48) from (5.1.47), we get,

$$M(\|u_k\|^n) \int_{\Omega} (|\nabla u_k|^{n-2} \nabla u_k - |\nabla u_\lambda|^{n-2} \nabla u_\lambda) \cdot \nabla (u_k - u_\lambda) dx \rightarrow 0.$$

Now by using this and following inequality,

$$|a_1 - a_2|^n \leq 2^{n-2} (|a_1|^{n-2} a_1 - |a_2|^{n-2} a_2) (a_1 - a_2) \text{ for all } a_1, a_2 \in \mathbb{R}^n$$

with $a_1 = \nabla u_k$ and $a_2 = \nabla u_\lambda$, we obtain

$$M(\|u_k\|^n) \int_{\Omega} |\nabla u_k - \nabla u_\lambda|^n dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Since $M(t) \geq b$, then we obtain $u_k \rightarrow u_\lambda$ strongly in $W_0^{1,n}(\Omega)$ and by Lemma 5.1.13

$$\int_{\Omega} (|x|^{-\mu} * F(u_k)) f(u_k) \phi dx \rightarrow \int_{\Omega} (|x|^{-\mu} * F(u_\lambda)) f(u_\lambda) \phi dx$$

and also

$$\int_{\Omega} h(x) u_k^{q-1} u_k \phi dx \rightarrow \int_{\Omega} h(x) u_\lambda^{q-1} u_\lambda \phi dx$$

for all $\phi \in W_0^{1,n}(\Omega)$. Therefore, u_λ satisfies $(\mathcal{P}_{\lambda,M})$ in weak sense and hence $u_\lambda \in N_{\lambda,M}$. Moreover, $\theta \leq \mathcal{J}_{\lambda,M}(u_\lambda) \leq \liminf_{k \rightarrow \infty} \mathcal{J}_{\lambda,M}(u_k) = \theta$. Hence u_λ is a minimizer for $\mathcal{J}_{\lambda,M}$ in $N_{\lambda,M}$.

Using (5.1.42), we have $\int_{\Omega} h(x) |u_\lambda|^{q+1} > 0$, then there exists $t_1(u_\lambda)$ such that $t_1(u_\lambda) u_\lambda \in N_{\lambda,M}^+$. We now claim that $t_1(u_\lambda) = 1$ i.e. $u_\lambda \in N_{\lambda,M}^+$. Suppose that $t_1(u_\lambda) < 1$ and then $t_2(u_\lambda) = 1$ and $u_\lambda \in N_{\lambda,M}^-$. Now $\mathcal{J}_{\lambda,M}(t_1(u_\lambda) u_\lambda) < \mathcal{J}_{\lambda,M}(u_\lambda) \leq \theta$ which yields a contradiction, since $t_1(u_\lambda) u_\lambda \in N_{\lambda,M}$. Thus, u_λ is non-negative and nontrivial. From the strong comparison

principle (see for instance [251]), we get $u_\lambda > 0$ in Ω . Since $u_\lambda \in N_{\lambda,M}^+ \cap H^+$ then we have a $t_*(u_\lambda)$ such that $1 = t_1(u_\lambda) < t_*(u_\lambda)$. Hence by the continuity of $u \rightarrow t_*(u)$, given $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that

$$(1 + \epsilon) < t_*(u_\lambda - w) \quad \text{for all } \|w\| < \delta_\epsilon$$

and from Lemma 5.1.24 we have, for $\delta > 0$ small enough, a continuously differentiable map $t : B(0, \delta) \rightarrow \mathbb{R}^+$ such that $t(w)(u_\lambda - w) \in N_{\lambda,M}$, $t(0) = 1$. Then we have

$$t_1(u_\lambda - w) = t(w) < 1 + \epsilon < t_*(u_\lambda - w)$$

for δ small enough. Since $t_*(u_\lambda - w) > 1$ for all $\|w\| < \delta$, we obtain

$$\mathcal{J}_{\lambda,M}(u_\lambda) \leq \mathcal{J}_{\lambda,M}(t_1(u_\lambda - w)(u_\lambda - w)) \leq \mathcal{J}_{\lambda,M}(u_\lambda - w), \quad \text{if } \|w\| < \delta$$

which implies that u_λ is a local minimizer for $\mathcal{J}_{\lambda,M}$. \square

Theorem 5.1.28. *Let $\beta < \frac{n}{n-1}$ and let $\lambda \in (0, \lambda_0)$ where $\lambda_0 = \frac{bn}{(2n-1-q)l}M_0$. Then $\mathcal{J}_{\lambda,M}$ achieve its minimizers on $N_{\lambda,M}^-$.*

Proof. Let $u \in N_{\lambda,M}^-$. Then

$$(2n-1)a\|u\|^{2n} + (n-1)b\|u\|^n - \lambda q H(u) - \int_{\Omega} (|x|^{-\mu} * f(u)u) f(u) \cdot u - \int_{\Omega} (|x|^{-\mu} * F(u)) f'(u) u^2 < 0.$$

Then (5.1.38) implies that

$$(2n-1-q)a\|u\|^{2n} + (n-1-q)b\|u\|^n + q \int_{\Omega} (|x|^{-\mu} * F(u)) f(u) \cdot u - \left(\int_{\Omega} (|x|^{-\mu} * f(u)u) f(u) \cdot u + \int_{\Omega} (|x|^{-\mu} * F(u)) f'(u) u^2 \right) < 0. \quad (5.1.49)$$

Using $p+1 > 2n$ it is easy to deduce from (5.1.49) that $\exists c > 0$, $\|u\| \geq c > 0$ for any $u \in N_{\lambda,M}^-$ from which it follows that $N_{\lambda,M}^-$ is a closed set. Also as in Lemma 5.1.18 we can prove that $N_{\lambda,M}^- \subset \Gamma$ and then $\inf_{u \in N_{\lambda,M}^-} B(u) \geq \tilde{c} > 0$. Therefore, for $\lambda < \lambda_0$ small enough,

$$\inf_{u \in N_{\lambda,M}^- \setminus \{0\}} B(u) + nb\|u\|^n - (2n-1-q)\lambda H(u) - (2n-1) \int_{\Omega} |x|^{-\mu} * F(u) f(u) u > 0. \quad (5.1.50)$$

Now taking $\theta^- = \min_{u \in N_{\lambda,M}^-} \mathcal{J}_{\lambda,M}(u) > -\infty$. From Ekeland variational principle, there exist $\{v_k\}_{k \in \mathbb{N}}$ a non-negative minimizing sequence such that

$$\mathcal{J}_{\lambda,M}(v_k) \leq \inf_{u \in N_{\lambda,M}^-} \mathcal{J}_{\lambda,M}(u) + \frac{1}{k} \quad \text{and} \quad \mathcal{J}_{\lambda,M}(u) \geq \mathcal{J}_{\lambda,M}(v_k) - \frac{1}{k} \|v_k - u\| \quad \forall u \in N_{\lambda,M}^-.$$

From $\mathcal{J}_{\lambda,M}(v_k) \rightarrow \theta^-$ as $k \rightarrow \infty$ and $v_k \in N_{\lambda,M}$, it is easy to prove that $\|v_k\| \leq C$ (as in Lemma 5.1.10). Indeed,

$$\left| a\|v_k\|^{2n} + b\|v_k\|^n - \lambda H(v_k) - \int_{\Omega} (|x|^{-\mu} * F(v_k))f(v_k)v_k \, dx \right| = o(\|v_k\|)$$

and

$$C + o(\|v_k\|) \geq \mathcal{J}_{\lambda,M}(v_k) - \frac{1}{2n} \langle \mathcal{J}'_{\lambda,M}(v_k), v_k \rangle \geq \frac{b}{2n} \|v_k\|^{2n} - C(\lambda) \|v_k\|^{q+1}$$

imply $\|v_k\| \leq C$. Thus we get $\|S(v_k)\|_* \leq C_1$ and from (5.1.50) we have $\|\xi_k^-(0)\|_* \leq C_2$. Now the rest of the proof can be done as in the proof of Theorem 5.1.27 with the help of Lemma 5.1.25. \square

Proof of Theorem 5.1.7 for $\beta \in \left(1, \frac{n}{n-1}\right)$: The proof follows from Theorem 5.1.27 and 5.1.28. \square

Now we establish the following compactness result in the critical case.

Lemma 5.1.29. *There exists $C = C(p, q, n) > 0$ such that for any $\{u_k\} \subset W_0^{1,n}(\Omega)$ satisfying*

$$\mathcal{J}'_{\lambda,M}(u_k) \rightarrow 0 \quad \text{and} \quad \mathcal{J}_{\lambda,M}(u_k) \rightarrow c \leq \frac{m_0}{2n} \left(\frac{\alpha_n}{2^{\frac{1}{n-1}}} \left(\frac{2n-\mu}{2n} \right) \right)^{n-1} - C\lambda^{\frac{2(p+2)}{2p+3-q}} \quad \text{as } k \rightarrow \infty$$

is relatively compact in $W_0^{1,n}(\Omega)$.

Proof. As in Lemma 5.1.10 we can prove that $\{u_k\}$ is bounded in $W_0^{1,n}(\Omega)$ and up to a subsequence $u_k \rightarrow u$ in $L^\alpha(\Omega)$ for all $\alpha \geq 1$, $u_k(x) \rightarrow u$ a.e in Ω and $\nabla u_k \rightarrow \nabla u$ a.e. in Ω and weakly in $L^n(\Omega)$. Also still up to a subsequence, there exist radon measures ν_1 and ν_2 such that $|\nabla u_k|^n \rightarrow \nu_1$ and $(|x|^{-\mu} * F(u_k))f(u_k)u_k \rightarrow \nu_2$ weakly as $k \rightarrow \infty$.

Let $B = \{x \in \bar{\Omega} : \exists r = r_x > 0, \nu_1(B_r \cap \Omega) < \left(\frac{\alpha_n}{2^{\frac{1}{n-1}}} \left(\frac{2n-\mu}{2n} \right) \right)^{n-1}\}$ and let $A = \bar{\Omega} \setminus B$. Then by Lemma 5.1.12 we can infer that A is a finite set, say $\{x_1, x_2, \dots, x_t\}$. Since $\mathcal{J}'_{\lambda,M}(u_k) \rightarrow 0$ and since $\{u_k\}$ is bounded in $W_0^{1,n}(\Omega)$, we have that $\forall \phi \in C_c^\infty(\Omega)$,

$$\begin{aligned} 0 = \lim_{k \rightarrow \infty} \langle \mathcal{J}'_{\lambda,M}(u_k), \phi \rangle &= \lim_{k \rightarrow \infty} \left[M(\|u_k\|^n) \int_{\Omega} |\nabla u_k|^{n-2} \nabla u_k \cdot \nabla \phi \, dx - \lambda \int_{\Omega} h(x) |u_k|^{q-1} u_k \phi \, dx \right. \\ &\quad \left. - \int_{\Omega} (|x|^{-\mu} * F(u_k))f(u_k)\phi \, dx \right], \end{aligned} \tag{5.1.51}$$

$$\begin{aligned} 0 = \lim_{k \rightarrow \infty} \langle \mathcal{J}'_{\lambda,M}(u_k), u_k \phi \rangle &= \lim_{k \rightarrow \infty} \left[M(\|u_k\|^n) \int_{\Omega} (|\nabla u_k|^{n-2} \nabla u_k \cdot \nabla \phi u_k \, dx + |\nabla u_k|^n \phi) \right. \\ &\quad \left. - \lambda \int_{\Omega} h(x) |u_k|^{q+1} \phi \, dx - \int_{\Omega} (|x|^{-\mu} * F(u_k))f(u_k)u_k \phi \, dx \right], \end{aligned} \tag{5.1.52}$$

$$\begin{aligned}
 0 &= \lim_{k \rightarrow \infty} \langle \mathcal{J}'_{\lambda, M}(u_k), u\phi \rangle = \lim_{k \rightarrow \infty} \left[M(\|u_k\|^n) \int_{\Omega} (|\nabla u_k|^{n-2} \nabla u_k \cdot \nabla \phi u + |\nabla u_k|^{n-2} \nabla u_k \cdot \nabla u \phi) dx \right. \\
 &\quad \left. - \int_{\Omega} (|x|^{-\mu} * F(u_k)) f(u_k) u \phi dx \right] - \lambda \int_{\Omega} h(x) |u|^q \phi dx.
 \end{aligned} \tag{5.1.53}$$

Substituting (5.1.53) in (5.1.52) and taking into account (5.1.51), we get $\forall \phi \in C_c^\infty(\Omega)$

$$\begin{aligned}
 \lim_{k \rightarrow \infty} \int_{\Omega} (|x|^{-\mu} * F(u_k)) f(u_k) u_k \phi &= \lim_{k \rightarrow \infty} M(\|u_k\|^n) \int_{\Omega} |\nabla u_k|^n \phi - |\nabla u_k|^{n-2} \nabla u_k \cdot \nabla u \phi dx \\
 &\quad + \int_{\Omega} (|x|^{-\mu} * F(u)) f(u) u \phi dx + o_k(1).
 \end{aligned} \tag{5.1.54}$$

Now we take the cut-off function $\psi_\delta \in C_c^\infty(\Omega)$ such that $\psi_\delta = 1$ in $B_\delta(x_j) \forall j = \{1, \dots, t\}$ and $\psi_\delta(x) = 0$ in $B_{2\delta}^c(x_j)$ with $|\psi_\delta| \leq 1$. Then by taking $\phi = \psi_\delta$ in (5.1.54) and since as $\delta \rightarrow 0$

$$\begin{aligned}
 0 &\leq \left| \int_{\Omega} (|\nabla u_k|^{n-2} \nabla u_k \cdot \nabla u) \psi_\delta dx \right| \leq \int_{\Omega} |\nabla u_k|^{n-1} |\nabla u| |\psi_\delta| dx \\
 &\leq \int_{\cup_j B_{2\delta}(x_j)} |\nabla u_k|^{n-1} |\nabla u| dx \leq \left(\int_{\Omega} |\nabla u_k|^n dx \right)^{n/(n-1)} \left(\int_{\cup_j B_{2\delta}(x_j)} |\nabla u|^n dx \right)^{1/n} \rightarrow 0,
 \end{aligned}$$

we deduce after letting $\delta \rightarrow 0$ that

$$\nu_2(A) \geq m_0 \nu_1(A) \geq m_0 \left(\frac{\alpha_n}{2^{\frac{1}{n-1}}} \left(\frac{2n - \mu}{2n} \right) \right)^{n-1}. \tag{5.1.55}$$

On the other hand, by using the same argument as in Lemma 5.1.12 (in particular see (5.1.16)) we can prove that for any compact set $K \subset \Omega_\delta = \bar{\Omega} \setminus \cup_{i=1}^t B_{2\delta}(x_i)$

$$\lim_{k \rightarrow \infty} \int_K (|x|^{-\mu} * F(u_k)) f(u_k) u_k dx = \int_K (|x|^{-\mu} * F(u)) f(u) u dx.$$

Thus, we obtain

$$\begin{aligned}
 nc &= \lim_{k \rightarrow \infty} n \mathcal{J}_{\lambda, M}(u_k) - \frac{1}{2} \langle \mathcal{J}'_{\lambda, M}(u_k), u_k \rangle = \lim_{k \rightarrow \infty} \left(\mathcal{M}(\|u_k\|^n) - \frac{1}{2} M(\|u_k\|^n) \|u_k\|^n \right) \\
 &\quad + \lim_{k \rightarrow \infty} \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(u_k)) (f(u_k) u_k - n F(u_k)) dx + \lambda \left(\frac{1}{2} - \frac{n}{q+1} \right) \int_{\Omega} h(x) |u_k|^{q+1} dx.
 \end{aligned}$$

Since

$$\begin{aligned}
 \int_{\Omega} (|x|^{-\mu} * F(u_k)) F(u_k) dx &\rightarrow \int_{\Omega} (|x|^{-\mu} * F(u)) F(u) dx, \\
 \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(u_k)) f(u_k) u_k dx &\rightarrow \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(u)) f(u) u dx + \frac{\nu_2(A)}{2},
 \end{aligned}$$

together with (5.1.55) it follows that

$$nc \geq \frac{m_0}{2} \left(\frac{\alpha_n}{2^{\frac{1}{n-1}}} \left(\frac{2n - \mu}{2n} \right) \right)^{n-1} + \lambda \left(\frac{1}{2} - \frac{n}{q+1} \right) \int_{\Omega} h(x) |u|^{q+1} dx - \frac{n}{2} \int_{\Omega} (|x|^{-\mu} * F(u)) F(u) dx$$

$$+ \frac{1}{2} \int_{\Omega} (|x|^{-\mu} * F(u)) f(u) u \, dx.$$

Consequently,

$$\begin{aligned} c &\geq \frac{m_0}{2n} \left(\frac{\alpha_n}{2^{\frac{1}{n-1}}} \left(\frac{2n-\mu}{2n} \right) \right)^{n-1} + \lambda \left(\frac{1}{2n} - \frac{1}{(q+1)} \right) \int_{\Omega} h u^{q+1} \, dx \\ &+ \left(\frac{1}{2n} - \frac{1}{2(p+1)} \right) \int_{\Omega} (|x|^{-\mu} * F(u)) f(u) u \, dx \\ &\geq \frac{m_0}{2n} \left(\frac{\alpha_n}{2^{\frac{1}{n-1}}} \left(\frac{2n-\mu}{2n} \right) \right)^{n-1} - \|h\|_{L^{r'}(\Omega)} \lambda \left(\frac{2n-1-q}{2n(q+1)} \right) \left(\int_{\Omega} u^{p+2} \, dx \right)^{\frac{q+1}{p+2}} \\ &+ c_1 \frac{2p+2-2n}{2n(2p+2)(p+2)} \left(\int_{\Omega} u^{p+2} \, dx \right)^2 \geq \frac{m_0}{2n} \left(\frac{\alpha_n}{2^{\frac{1}{n-1}}} \left(\frac{2n-\mu}{2n} \right) \right)^{n-1} - \inf_{t \in \mathbb{R}^+} \rho(t) \end{aligned}$$

with $r' = \left(1 - \frac{q+1}{p+2}\right)^{-1}$, $c_1 = c_1(\Omega) > 0$ and $\rho(t) = \|h\|_{L^{r'}(\Omega)} \lambda \left(\frac{2n-1-q}{2n(q+1)} \right) t^{\frac{q+1}{2(p+2)}} - \frac{(2p+2-2n)c_1}{2n(2p+2)(p+2)} t$.

Thus $c \geq \frac{m_0}{2n} \left(\frac{\alpha_n}{2^{\frac{1}{n-1}}} \left(\frac{2n-\mu}{2n} \right) \right)^{n-1} - \tilde{C} \lambda^{\frac{2(p+2)}{2p+3-q}}$ which completes the proof. \square

Now we prove Theorem 5.1.8 which concerns the critical case $\beta = \frac{n}{n-1}$.

Proof of Theorem 5.1.8: Let u_k be a nonnegative minimizing sequence for $\mathcal{J}_{\lambda, M}$ on $N_{\lambda, M} \setminus \{0\}$ satisfying (5.1.40) then u_k is bounded in $W_0^{1, n}(\Omega)$. Using Proposition 5.1.26 we get u_k is a Palais Smale sequence at level $\theta < \frac{m_0}{2n} \left(\frac{\alpha_n}{2^{\frac{1}{n-1}}} \left(\frac{2n-\mu}{2n} \right) \right)^{n-1} - \tilde{C} \lambda^{\frac{2(p+2)}{2p+3-q}}$. Taking λ small enough, using Lemma 5.1.23 and Lemma 5.1.29, $\{u_k\}$ admits a strongly convergent subsequence. Let $u \in W_0^{1, n}(\Omega)$ be the limit of this subsequence. Then arguing as in the proof of Theorems 5.1.27 and 5.1.28, we prove that u is a non-trivial weak solution of $(\mathcal{P}_{\lambda, M})$ and $\mathcal{J}_{\lambda, M}(u) = \theta$. By elliptic regularity and strong maximum principle, we infer that $u > 0$ in Ω . This completes the proof of Theorem 5.1.8. \square

5.2 Polyharmonic Kirchhoff problems involving exponential non-linearity of Choquard type with singular weights

In this section, we prove the existence of a non-trivial weak solution to the following Kirchhoff type Choquard equation with exponential non-linearity and singular weights:

$$(PKC) \begin{cases} -M \left(\int_{\Omega} |\nabla^m u|^2 \, dx \right) \Delta^m u = \left(\int_{\Omega} \frac{F(y, u)}{|y|^\alpha |x-y|^\mu} \, dy \right) \frac{f(x, u)}{|x|^\alpha} \, dx, & \text{in } \Omega, \\ u = \nabla u = \dots = \nabla^{m-1} u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $m \in \mathbb{N}$, $n = 2m$, $\mu \in (0, n)$, $0 < \alpha < \min\{\frac{n}{2}, n - \mu\}$, Ω is a bounded domain in \mathbb{R}^n with smooth boundary and the function F denotes the primitive of f with respect to the second variable.

Throughout this section, we assume the following conditions on M and f . The function $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a continuous function satisfying the following conditions:

5.2. Polyharmonic Kirchhoff problems involving exponential non-linearity of Choquard type with singular weights

- (m1) There exists $M_0 > 0$ such that $M(t) \geq M_0$ and $\mathcal{M}(t+s) \geq \mathcal{M}(t) + \mathcal{M}(s)$, for all $t, s \geq 0$ where $\mathcal{M}(t) = \int_0^t M(s) ds$ is the primitive of the function M vanishing at 0.
(m2) There exist constants $b_1, b_2 > 0$ and $\hat{t} > 0$ such that for some $k \in \mathbb{R}$

$$M(t) \leq b_1 + b_2 t^k, \text{ for all } t \geq \hat{t}.$$

- (m3) The function $\frac{M(t)}{t}$ is non-increasing for $t > 0$.

Using (m3), one can easily deduce that the function

$$(m3)' \quad \frac{1}{2} \mathcal{M}(t) - \frac{1}{\theta} M(t)t \text{ is non-negative and non-decreasing for } t \geq 0 \text{ and } \theta \geq 4.$$

Example 5.2.1. An example of a function satisfying (m1), (m2) and (m3) is $M(t) = M_0 + bt^\beta$ where $M_0, > 0, \beta < 1$ and $b \geq 0$. Also $M(t) = M_0 + \log(1+t)$ with $M_0 \geq 1$ verifies (m1)-(m3).

The function $f : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ which governs the Choquard term is given by $f(x, t) = h(x, t) \exp(t^2)$, where $h \in C(\Omega \times \mathbb{R})$ satisfies the following growth conditions:

- (h1) $h(x, t) = 0$ for all $t \leq 0$ and $h(x, t) > 0$ for $t > 0$.
(h2) For any $\epsilon > 0$, $\limsup_{t \rightarrow \infty} \sup_{x \in \bar{\Omega}} h(x, t) \exp(-\epsilon t^2) = 0$ and $\liminf_{t \rightarrow \infty} \inf_{x \in \bar{\Omega}} h(x, t) \exp(\epsilon t^2) = \infty$.
(h3) There exists $\ell > \max\{1, k + 1\}$ such that $\frac{f(x, t)}{t^\ell}$ is increasing for each $t > 0$ uniformly in $x \in \Omega$, where k is specified in (m2).
(h4) There exist $T, T_0 > 0$ and $\gamma_0 > 0$ such that $0 < t^{\gamma_0} F(x, t) \leq T_0 f(x, t)$ for all $|t| \geq T$ and uniformly in $x \in \Omega$.

The condition (h3) implies that $\frac{f(x, t)}{t}$ is increasing in $t > 0$ and $\lim_{t \rightarrow 0^+} \frac{f(x, t)}{t} = 0$ uniformly in $x \in \Omega$.

Example 5.2.2. A typical example of f satisfying (h1)–(h4) is $f(x, t) = t^{\beta+1} \exp(t^p) \exp(t^2)$ for $t \geq 0$ and $f(x, t) = 0$ for $t < 0$ where $0 \leq p < 2$ and $\beta > l - 1$.

Furthermore, using (h1) – (h3) we obtain that for any $\epsilon > 0, r > \beta_0 + 1$ where $0 \leq \beta_0 < \ell$, there exist constants $C_1, C_2 > 0$ (depending upon ϵ, n, m) such that for each $x \in \Omega$

$$0 \leq F(x, t) \leq C_1 |t|^{\beta_0+1} + C_2 |t|^r \exp((1 + \epsilon)t^2), \text{ for all } t \in \mathbb{R}. \quad (5.2.1)$$

We also study the existence of weak solutions of a Kirchhoff type Choquard equation with convex-concave sign changing non-linearity:

$$(\mathcal{P}_{\lambda, \mathcal{M}}) \begin{cases} -M \left(\int_{\Omega} |\nabla^m u|^2 dx \right) \Delta^m u = \lambda h(x) |u|^{q-1} u + \left(\int_{\Omega} \frac{F(u)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f(u)}{|x|^\alpha} & \text{in } \Omega, \\ u = \nabla u = \dots = \nabla^{m-1} u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $f(u) = u|u|^p \exp(|u|^\gamma)$, $0 < q < 1$, $2 < p$, $\gamma \in (1, 2)$ and $F(t) = \int_0^t f(s) ds$. In this case, we assume $M(t) = at + b$ where $a, b > 0$ and $h \in L^r(\Omega)$ where $r = \frac{p+2}{q+1}$ is such that $h^+ \not\equiv 0$. For any $u \in W_0^{m,2}(\Omega)$, by virtue of Sobolev embedding we get that $u \in L^q(\Omega)$ for all $q \in [1, \infty)$. This also implies that $F(x, u) \in L^q(\Omega)$ for any $q \geq 1$.

Throughout this section, we denote

$$\|u\| = \left(\int_{\Omega} |\nabla^m u|^2 dx \right)^{\frac{1}{2}}.$$

The problem (PKC) has a variational structure and the energy functional $\mathcal{J} : W_0^{m,2}(\Omega) \rightarrow \mathbb{R}$ associated to (PKC) is given by

$$\mathcal{J}(u) = \frac{1}{2} \mathcal{M}(\|u\|^2) - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{F(x, u)}{|x|^\alpha} dx. \quad (5.2.2)$$

The notion of weak solution for (PKC) is given as follows.

Definition 5.2.3. A weak solution of (PKC) is a function $u \in W_0^{m,2}(\Omega)$ such that for all $\varphi \in W_0^{m,2}(\Omega)$, it satisfies

$$M(\|u\|^2) \int_{\Omega} \nabla^m u \cdot \nabla^m \varphi dx = \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u)}{|x|^\alpha} \varphi dx. \quad (5.2.3)$$

For the problem $(\mathcal{P}_{\lambda, \mathcal{M}})$, the energy functional $\mathcal{J}_{\lambda, \mathcal{M}} : W_0^{m,2}(\Omega) \rightarrow \mathbb{R}$ associated to the problem $(\mathcal{P}_{\lambda, \mathcal{M}})$ is defined as

$$\mathcal{J}_{\lambda, \mathcal{M}}(u) = \frac{1}{2} \mathcal{M}(\|u\|^2) - \frac{\lambda}{q+1} \int_{\Omega} h(x) |u|^{q+1} dx - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{F(u)}{|x|^\alpha} dx$$

where F and \mathcal{M} are primitive of f and M respectively vanishing at 0 and $f(s) = s|s|^p \exp(|s|^\gamma)$.

Definition 5.2.4. A function $u \in W_0^{m,2}(\Omega)$ is said to be a weak solution of $(\mathcal{P}_{\lambda, \mathcal{M}})$ if for all $\phi \in W_0^{m,2}(\Omega)$, it satisfies

$$M(\|u\|^2) \int_{\Omega} \nabla^m u \cdot \nabla^m \phi dx = \lambda \int_{\Omega} h(x) |u|^{q-1} u \phi dx + \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f(u)}{|x|^\alpha} \phi dx.$$

5.2.1 Main results

We prove the following main result concerning the problem (PKC) .

Theorem 5.2.5. Let $(m1)$ - $(m3)$ and $(h1)$ - $(h4)$ holds. Assume in addition

$$\lim_{s \rightarrow +\infty} \frac{sf(x, s)F(x, s)}{\exp(2s^2)} = \infty, \text{ uniformly in } x \in \bar{\Omega}. \quad (5.2.4)$$

Then the problem (PKC) admits a non-trivial weak solution.

For the problem $(\mathcal{P}_{\lambda, \mathcal{M}})$, we have the following result:

Theorem 5.2.6. There exists a $\lambda_0 > 0$ such that for $\gamma \in (1, 2)$ and $\lambda \in (0, \lambda_0)$, $(\mathcal{P}_{\lambda, \mathcal{M}})$ admits atleast two solutions.

5.2.2 Existence result for weak solution

In this subsection, we establish the existence of a nontrivial weak solution for the problem (PKC). To prove this we study the mountain pass geometry of the energy functional \mathcal{J} and using the properties of the non-local term M and the exponential growth of f , we prove that every Palais Smale sequence is bounded. To study the compactness of Palais Smale sequences for \mathcal{J} , we obtain a bound for the mountain pass critical level with the help of Adams functions and establish the convergence of weighted Choquard term for Palais-Smale sequences.

5.2.2.1 Mountain pass geometry

In the following result, we show that the energy functional \mathcal{J} possesses the mountain pass geometry around 0 in the light of Adams-Moser and doubly weighted Hardy-Littlewood-Sobolev inequalities.

Lemma 5.2.7. *Under the assumptions (m1), (m2) and (h1)-(h3) the following assertions hold:*

- (i) *there exists $R_0 > 0, \eta > 0$ such that $\mathcal{J}(u) \geq \eta$ for all $u \in W_0^{m,2}(\Omega)$ such that $\|u\| = R_0$.*
- (ii) *there exists a $v \in W_0^{m,2}(\Omega)$ with $\|v\| > R_0$ such that $\mathcal{J}(v) < 0$.*

Proof. Using Proposition 2.2.7 with $t = r$ and $\beta = \alpha$ and (5.2.1), we obtain that for any $\epsilon > 0$ and $u \in W_0^{m,2}(\Omega)$, there exist constants $C_i > 0$ depending upon ϵ such that

$$\begin{aligned} & \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|y|^\alpha |x - y|^\mu} dy \right) \frac{F(x, u)}{|x|^\alpha} dx \leq C(m, \mu, \alpha) \|F(x, u)\|_{L^{\frac{2n}{2n-(2\alpha+\mu)}}}^2 \\ & \leq \left(C_1 \int_{\Omega} |u|^{\frac{2n(\beta_0+1)}{2n-(2\alpha+\mu)}} + C_2 \int_{\Omega} |u|^{\frac{2rn}{2n-(2\alpha+\mu)}} \exp\left(\frac{2n(1+\epsilon)}{2n-(2\alpha+\mu)} |u|^2\right) \right)^{\frac{2n-(2\alpha+\mu)}{n}} \\ & \leq \left(C_1 \int_{\Omega} |u|^{\frac{2n(\beta_0+1)}{2n-(2\alpha+\mu)}} + C_2 \|u\|^{\frac{2rn}{2n-(2\alpha+\mu)}} \left(\int_{\Omega} \exp\left(\frac{4n(1+\epsilon)\|u\|^2}{2n-(2\alpha+\mu)} \left(\frac{|u|}{\|u\|}\right)^2\right) \right)^{\frac{1}{2}} \right)^{\frac{2n-(2\alpha+\mu)}{n}} \end{aligned}$$

For small $\epsilon > 0$ and u such that $\frac{4n(1+\epsilon)\|u\|^2}{2n-(2\alpha+\mu)} \leq \zeta_{m,2m}$, using Theorem 2.2.2, Chapter 1, we obtain

$$\begin{aligned} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|y|^\alpha |x - y|^\mu} dy \right) \frac{F(x, u)}{|x|^\alpha} dx & \leq C_3 \left(\|u\|^{\frac{2n(\beta_0+1)}{2n-(2\alpha+\mu)}} + \|u\|^{\frac{2rn}{2n-(2\alpha+\mu)}} \right)^{\frac{2n-(2\alpha+\mu)}{n}} \\ & \leq C_4 (\|u\|^{2(\beta_0+1)} + \|u\|^{2r}). \end{aligned} \quad (5.2.5)$$

Then for $\|u\| < \rho = \left(\frac{\zeta_{m,2m}(2n-(2\alpha+\mu))}{4n(1+\epsilon)}\right)^{\frac{1}{2}}$, (m1) and (5.2.5) gives

$$\mathcal{J}(u) \geq M_0 \frac{\|u\|^2}{2} - C_4 \|u\|^{2(\beta_0+1)} - C_4 \|u\|^{2r}.$$

So we choose $\|u\| = R_0$ small enough so that $\mathcal{J}(u) \geq \eta$ for some $\eta > 0$ (depending on R_0) and hence (i) follows. Furthermore (m2) implies that

$$\mathcal{M}(t) \leq \begin{cases} b_0 + b_1 t + \frac{b_2 t^{k+1}}{k+1}, & k \neq -1 \\ b_0 + b_1 t + b_2 \ln t, & k = -1 \end{cases}$$

for $t \geq \hat{t}$ where

$$b_0 = \begin{cases} \mathcal{M}(\hat{t}) - b_1 \hat{t} - b_2 \frac{\hat{t}^{k+1}}{k+1}, & k \neq -1, \\ \mathcal{M}(\hat{t}) - b_1 \hat{t} - b_2 \ln \hat{t}, & k = -1. \end{cases}$$

Under the assumption (h3), there exists $K_1 \geq \max\{1, k+1\}$, $c_1, c_2 > 0$ such that $F(x, s) \geq c_1 s^{K_1} - c_2$ for all $(x, s) \in \Omega \times [0, \infty)$. Therefore for $v \in W_0^{m,2}(\Omega)$ such that $v \geq 0$ and $\|v\| = 1$ we get

$$\begin{aligned} & \int_{\Omega} \left(\int_{\Omega} \frac{F(y, tv)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{F(x, tv)}{|x|^\alpha} dx \geq \int_{\Omega} \int_{\Omega} \frac{(c_1(tv)^{K_1}(y) - c_2)(c_1(tv)^{K_1}(x) - c_2)}{|y|^\alpha |x|^\alpha |x-y|^\mu} dx dy \\ & = c_1^2 t^{2K_1} \int_{\Omega} \int_{\Omega} \frac{v^{K_1}(y)v^{K_1}(x)}{|y|^\alpha |x|^\alpha |x-y|^\mu} dx dy - 2c_1 c_2 t^{K_1} \int_{\Omega} \int_{\Omega} \frac{v^{K_1}(y)}{|y|^\alpha |x|^\alpha |x-y|^\mu} dx dy \\ & \quad + c_2^2 \int_{\Omega} \int_{\Omega} \frac{1}{|y|^\alpha |x|^\alpha |x-y|^\mu} dx dy. \end{aligned}$$

Then using above estimates in (5.2.2) for $k \neq -1$, we obtain

$$\mathcal{J}(tv) \leq c_3 + c_4 t^2 + c_5 t^{2(k+1)} - c_4 t^{2K_1} + c_6 t^{K_1}$$

and for $k = -1$

$$\mathcal{J}(tv) \leq c_3 + c_4 t^2 + c_5 \ln(t^2) - c_4 t^{2K_1} + c_6 t^{K_1}$$

where c_i 's are positive constants for $i = 3, \dots, 6$. Now by choosing t large enough, we obtain that there exists a $v \in W_0^{m,2}(\Omega)$ with $\|v\| > R_0$ such that $\mathcal{J}(v) < 0$. \square

Lemma 5.2.8. *Every Palais Smale sequence of \mathcal{J} is bounded in $W_0^{m,2}(\Omega)$.*

Proof. Let $\{u_k\} \subset W_0^{m,2}(\Omega)$ be a Palais Smale sequence for \mathcal{J} at level c (denoted by $(PS)_c$ for some $c \in \mathbb{R}$) i.e.

$$\mathcal{J}(u_k) \rightarrow c \text{ and } \mathcal{J}'(u_k) \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Then from (5.2.2) and (5.2.3), we obtain

$$\begin{aligned} & \frac{1}{2} \mathcal{M}(\|u_k\|^2) - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{F(x, u_k)}{|x|^\alpha} dx \rightarrow c \text{ as } k \rightarrow \infty, \\ & \left| M(\|u_k\|^2) \int_{\Omega} \nabla^m u_k \cdot \nabla^m \phi - \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u_k)}{|x|^\alpha} \phi dx \right| \leq \epsilon_k \|\phi\| \end{aligned} \tag{5.2.6}$$

for any $\phi \in W_0^{m,2}(\Omega)$, where $\epsilon_k \rightarrow 0$ as $k \rightarrow \infty$. By substituting $\phi = u_k$ we get

$$\left| M(\|u_k\|^2) \int_{\Omega} |\nabla^m u_k|^2 - \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u_k) u_k}{|x|^\alpha} dx \right| \leq \epsilon_k \|u_k\|. \quad (5.2.7)$$

Using assumption (h3), we get that there exists a $\theta > 2$ such that $\theta F(x, t) \leq t f(x, t)$ for any $t > 0$ and $x \in \Omega$ which implies

$$\theta \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{F(x, u_k)}{|x|^\alpha} dx \leq \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u_k) u_k}{|x|^\alpha} dx. \quad (5.2.8)$$

Now using (5.2.6), (5.2.7), (5.2.8) and $(m3)'$, we get

$$\begin{aligned} \mathcal{J}(u_k) - \frac{1}{2\theta} \langle \mathcal{J}'(u_k), u_k \rangle &= \frac{1}{2} \mathcal{M}(\|u_k\|^2) - \frac{1}{2\theta} M(\|u_k\|^2) \|u_k\|^2 \\ &\quad - \frac{1}{2} \left(\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{F(x, u_k)}{|x|^\alpha} dx + \frac{1}{2\theta} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u_k) u_k}{|x|^\alpha} dx \right) \\ &\geq \frac{1}{2} \mathcal{M}(\|u_k\|^2) - \frac{1}{2\theta} M(\|u_k\|^2) \|u_k\|^2 \geq \left(\frac{1}{2} - \frac{1}{2\theta} \right) M_0 \|u_k\|^2. \end{aligned} \quad (5.2.9)$$

Also (5.2.6) and (5.2.7) yields

$$\mathcal{J}(u_k) - \frac{1}{2\theta} \langle \mathcal{J}'(u_k), u_k \rangle \leq C \left(1 + \epsilon_k \frac{\|u_k\|}{2\theta} \right) \quad (5.2.10)$$

for some $C > 0$. Therefore (5.2.9) and (5.2.10) gives us the desired result. \square

5.2.2.2 Mountain pass critical level

To obtain bound for the mountain pass critical level in this subsection, we use Adams functions to construct a sequence of test functions. Let \mathcal{B} denotes the unit ball and \mathcal{B}_l is the ball with center 0 and radius l in \mathbb{R}^n . Without loss of generality, we can assume that $\mathcal{B}_l \subset \Omega$, then from [182, Lemma 5, p. 895], we have the following result- For $l \in (0, 1)$, there exists

$$U_l \in \{u \in W_0^{m,2}(\Omega) : u|_{\mathcal{B}_l} = 1\} \quad (5.2.11)$$

such that

$$\|U_l\|^2 = C_{m,2}(\mathcal{B}_l; \mathcal{B}) \leq \frac{\zeta_{m,2m}}{n \log\left(\frac{1}{l}\right)}$$

where $C_{m,2}(K, E)$ is the conductor capacity of K in E whenever E is an open set and K is relatively compact subset of E and $C_{m,2}(K; E) \stackrel{\text{def}}{=} \inf\{\|u\|^2 : u \in C_0^\infty(E), u|_K = 1\}$.

Let $\tilde{x} \in \Omega$ and $R \leq R_0 = \text{dist}(\tilde{x}, \partial\Omega)$. Then the Adams function \tilde{A}_r is defined as

$$\tilde{A}_r(x) = \begin{cases} \left(\frac{n \log\left(\frac{R}{r}\right)}{\zeta_{m,2m}} \right)^{\frac{1}{2}} U_{\frac{r}{R}}\left(\frac{x - \tilde{x}}{R}\right) & \text{if } |x - \tilde{x}| < R, \\ 0 & \text{if } |x - \tilde{x}| \geq R \end{cases}$$

where $0 < r < R$, $U_{l=\frac{r}{R}}$ is as in (5.2.11) and $\|\tilde{A}_r\| \leq 1$.

Let $\sigma > 0$ (to be chosen later), $\tilde{x} = 0$, $R = \sigma$ and $r = \frac{\sigma}{k}$ for $k \in \mathbb{N}$, then we define

$$A_k(x) \stackrel{\text{def}}{=} \tilde{A}_{\frac{\sigma}{k}}(x) = \begin{cases} \left(\frac{n \log(k)}{\zeta_{m,2m}} \right)^{\frac{1}{2}} U_{\frac{1}{k}} \left(\frac{x}{\sigma} \right) & \text{if } |x| < \sigma, \\ 0 & \text{if } |x| \geq \sigma. \end{cases}$$

Then $A_k(0) = \left(\frac{n \log(k)}{\zeta_{m,2m}} \right)^{\frac{1}{2}}$ and $\|A_k\| \leq 1$.

We define the mountain pass critical level as

$$l^* = \inf_{\vartheta \in \Gamma} \max_{t \in [0,1]} \mathcal{J}(\vartheta(t)). \quad (5.2.12)$$

where $\Gamma = \{\vartheta \in C([0,1], W_0^{m,2}(\Omega)) : \vartheta(0) = 0, \mathcal{J}(\vartheta(1)) < 0\}$. Now we analyze the first critical level and study the convergence of Palais-Smale sequence below this level.

Theorem 5.2.9. *Under the assumption (5.2.4),*

$$0 < l^* < \frac{1}{2} \mathcal{M} \left(\frac{2n - (2\alpha + \mu)}{2n} \zeta_{m,2m} \right).$$

Proof. We have observed in Lemma 5.2.7 for $u \in W_0^{m,2}(\Omega) \setminus \{0\}$, $\mathcal{J}(tu) \rightarrow -\infty$ as $t \rightarrow \infty$ and $l^* \leq \max_{t \in [0,1]} \mathcal{J}(tu)$ for $u \in W_0^{m,2}(\Omega) \setminus \{0\}$ satisfying $\mathcal{J}(u) < 0$. So it is enough to prove that there exists a $k \in \mathbb{N}$ such that

$$\max_{t \in [0, \infty)} \mathcal{J}(tA_k) < \frac{1}{2} \mathcal{M} \left(\frac{2n - (2\alpha + \mu)}{2n} \zeta_{m,2m} \right).$$

We establish the above claim by contradiction. Suppose this is not true, then for all $k \in \mathbb{N}$ there exists a $t_k > 0$ such that

$$\begin{aligned} \max_{t \in [0, \infty)} \mathcal{J}(tA_k) = \mathcal{J}(t_k A_k) &\geq \frac{1}{2} \mathcal{M} \left(\frac{2n - (2\alpha + \mu)}{2n} \zeta_{m,2m} \right) \\ \text{and } \frac{d}{dt}(\mathcal{J}(tA_k))|_{t=t_k} &= 0. \end{aligned} \quad (5.2.13)$$

From Lemma 5.2.7 and (5.2.13), we obtain $\{t_k\}$ must be a bounded sequence in \mathbb{R} and

$$\frac{1}{2} \mathcal{M} \left(\frac{2n - (2\alpha + \mu)}{2n} \zeta_{m,2m} \right) < \frac{1}{2} \mathcal{M}(t_k^2)$$

Then monotonicity of \mathcal{M} implies that

$$t_k^2 > \left(\frac{2n - (2\alpha + \mu)}{2n} \zeta_{m,2m} \right). \quad (5.2.14)$$

Consequently, by using (5.2.13) and choosing σ, k such that $B_{\sigma/k} \subset \Omega$, we obtain

$$\begin{aligned} M((\|t_k A_k\|)^2) \|t_k A_k\|^2 &= \int_{\Omega} \left(\int_{\Omega} \frac{F(y, t_k A_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, t_k A_k) t_k A_k}{|x|^\alpha} dx \\ &\geq \int_{B_{\frac{\sigma}{k}}} \left(\int_{B_{\frac{\sigma}{k}}} \frac{F(y, t_k A_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, t_k A_k) t_k A_k}{|x|^\alpha} dx. \end{aligned} \quad (5.2.15)$$

For a positive constant $C_{\mu,n}$ depending on μ and n , we obtain (see equation (2.11), page. 1943, [15])

$$\int_{B_{\frac{\sigma}{k}}} \int_{B_{\frac{\sigma}{k}}} \frac{dx dy}{|y|^\alpha |x|^\alpha |x-y|^\mu} \geq C_{\mu,n} \left(\frac{\sigma}{k}\right)^{2n-(2\alpha+\mu)}.$$

From (5.2.4), we know that for each $\rho > 0$ there exists a $s_\rho > 0$ such that

$$sf(x, s)F(x, s) \geq \rho \exp(2s^2), \text{ whenever } s \geq s_\rho.$$

Using this in (5.2.15), we obtain, for some $C > 0$

$$M(\|t_k A_k\|^2) t_k^2 \geq \rho \exp(2|t_k A_k(0)|^2) C_{\mu,n} \left(\frac{\sigma}{k}\right)^{2n-(2\alpha+\mu)} \geq C k^{\frac{2nt_k^2}{\xi_{m,2m}} - (2n-(2\alpha+\mu))}.$$

Now from (5.2.14), it follows that taking k large enough, we arrive at a contradiction. This completes the proof of the result. \square

Lemma 5.2.10. *Let $\{u_k\} \subset W_0^{m,2}(\Omega)$ be a Palais Smale sequence for \mathcal{J} at $c \in \mathbb{R}$ then there exists a $u_0 \in W_0^{m,2}(\Omega)$ such that as $k \rightarrow \infty$ (up to a subsequence)*

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u_k)}{|x|^\alpha} \phi dx \rightarrow \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u_0)}{|x|^\alpha} \phi dx$$

for all $\phi \in C_c^\infty(\Omega)$.

Proof. If $\{u_k\}$ is a Palais Smale sequence at l^* for \mathcal{J} satisfying (5.2.6) and (5.2.7). From Lemma 5.2.8, we obtain that $\{u_k\}$ is bounded in $W_0^{m,2}(\Omega)$ so there exists a $u_0 \in W_0^{m,2}(\Omega)$ such that up to a subsequence $u_k \rightharpoonup u_0$ weakly in $W_0^{m,2}(\Omega)$, strongly in $L^q(\Omega)$ for all $q \in [1, \infty)$ and pointwise a.e. in Ω as $k \rightarrow \infty$. Let $\Omega' \subset\subset \Omega$ and $\varphi \in C_c^\infty(\Omega)$ such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in Ω' then by taking φ as a test function in (5.2.6), we get the following estimate

$$\begin{aligned} \int_{\Omega'} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u_k)}{|x|^\alpha} dx &\leq \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u_k) \varphi}{|x|^\alpha} dx \\ &\leq \epsilon_k \|\varphi\| + M(\|u_k\|^2) \int_{\Omega} \nabla^m u_k \cdot \nabla^m \varphi dx \leq \epsilon_k \|\varphi\| + C \|u_k\| \|\varphi\|. \end{aligned}$$

By using $\|u_k\| \leq C_0$ for all k , we obtain the sequence $\{w_k\} := \left\{ \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u_k)}{|x|^\alpha} \right\}$ is bounded in $L^1_{\text{loc}}(\Omega)$ which implies that up to a subsequence, $w_k \rightarrow w$ in the *weak**-topology as $k \rightarrow \infty$, where w denotes a Radon measure. So for any $\phi \in C_c^\infty(\Omega)$ we get

$$\lim_{k \rightarrow \infty} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u_k)}{|x|^\alpha} \phi dx = \int_{\Omega} \phi dw, \forall \phi \in C_c^\infty(\Omega).$$

Since u_k satisfies (5.2.6), for any measurable set $E \subset \Omega$ and $\phi \in C_c^\infty(\Omega)$ such that $\text{supp } \phi \subset E$ we get that

$$w(E) = \int_E \phi dw = \lim_{k \rightarrow \infty} \int_E \int_{\Omega} \left(\frac{F(y, u_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u_k)}{|x|^\alpha} \phi dx$$

$$\begin{aligned}
 &= \lim_{k \rightarrow \infty} \int_{\Omega} \int_{\Omega} \left(\frac{F(y, u_k)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{f(x, u_k)}{|x|^{\alpha}} \phi dx = \lim_{k \rightarrow \infty} M(\|u_k\|^2) \int_{\Omega} \nabla^m u_k \cdot \nabla^m \phi dx \\
 &\leq C_1 \int_E \nabla^m u \cdot \nabla^m \phi dx
 \end{aligned}$$

where we used (m2) in the last inequality and weak convergence of u_k to u in $W_0^{m,2}(\Omega)$. This implies that w is absolutely continuous with respect to the Lebesgue measure. Thus, Radon-Nikodym theorem establishes that there exists a function $g \in L^1_{\text{loc}}(\Omega)$ such that for any $\phi \in C_c^\infty(\Omega)$, $\int_{\Omega} \phi dw = \int_{\Omega} \phi g dx$. Therefore for any $\phi \in C_c^\infty(\Omega)$ we get

$$\lim_{k \rightarrow \infty} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{f(x, u_k)}{|x|^{\alpha}} \phi dx = \int_{\Omega} \phi g dx = \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{f(x, u_0)}{|x|^{\alpha}} \phi dx$$

which completes the proof. \square

Lemma 5.2.11. *Let $\{u_k\} \subset W_0^{m,2}(\Omega)$ be a Palais Smale sequence of \mathcal{J} at $c \in \mathbb{R}$ and (h4) holds. Then there exists a $u \in W_0^{m,2}(\Omega)$ such that, up to a subsequence, $u_k \rightharpoonup u$ weakly in $W_0^{m,2}(\Omega)$ and*

$$\left(\int_{\Omega} \frac{F(y, u_k)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{F(x, u_k)}{|x|^{\alpha}} \rightarrow \left(\int_{\Omega} \frac{F(y, u)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{F(x, u)}{|x|^{\alpha}} \text{ in } L^1(\Omega) \quad (5.2.16)$$

as $k \rightarrow \infty$.

Proof. Let $\{u_k\} \subset W_0^{m,2}(\Omega)$ be a Palais Smale sequence of \mathcal{J} at level c then from Lemma 5.2.8 we know that $\{u_k\}$ must be bounded in $W_0^{m,2}(\Omega)$. Thus there exists a $u \in W_0^{m,2}(\Omega)$ such that $u_k \rightharpoonup u$ weakly in $W_0^{m,2}(\Omega)$, $u_k \rightarrow u$ pointwise a.e. in Ω and $u_k \rightarrow u$ strongly in $L^q(\Omega)$, for each $q \in [1, \infty)$ as $k \rightarrow \infty$. Also from (5.2.6), (5.2.7) and (5.2.8) we get that there exists a constant $C > 0$ such that

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{F(x, u_k)}{|x|^{\alpha}} dx \leq C \text{ and } \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{f(x, u_k)u_k}{|x|^{\alpha}} \leq C. \quad (5.2.17)$$

Consider

$$\begin{aligned}
 &\left| \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{F(x, u_k)}{|x|^{\alpha}} dx - \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{F(x, u)}{|x|^{\alpha}} dx \right| \\
 &\leq \left| \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k) - F(y, u)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{F(x, u_k)}{|x|^{\alpha}} dx \right| \\
 &\quad + \left| \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{F(x, u_k) - F(x, u)}{|x|^{\alpha}} dx \right| \stackrel{\text{def}}{=} I_1 + I_2.
 \end{aligned}$$

Using the semigroup property of the Riesz potential we can write

$$\begin{aligned}
 I_1 &\leq \left(\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k) - F(y, u)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{F(x, u_k) - F(x, u)}{|x|^{\alpha}} dx \right)^{\frac{1}{2}} \\
 &\quad \times \left(\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{F(x, u_k)}{|x|^{\alpha}} dx \right)^{\frac{1}{2}}. \quad (5.2.18)
 \end{aligned}$$

$$I_2 \leq \left(\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k) - F(y, u)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{F(x, u_k) - F(x, u)}{|x|^{\alpha}} dx \right)^{\frac{1}{2}} \times \left(\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{F(x, u)}{|x|^{\alpha}} dx \right)^{\frac{1}{2}} \quad (5.2.19)$$

Therefore, by using (5.2.18) and (5.2.19) we obtain,

$$I_1 + I_2 \leq 2C \left(\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k) - F(y, u)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{F(x, u_k) - F(x, u)}{|x|^{\alpha}} dx \right)^{\frac{1}{2}}$$

where we used (5.2.17) to get the last inequality. Now the proof of (5.2.16) follows similarly as the proof of (5.1.22) of Lemma 5.1.12. \square

Now we define the associated Nehari manifold as

$$\mathcal{N} = \{u \in W_0^{m,2}(\Omega) \setminus \{0\} : \langle \mathcal{J}'(u), u \rangle = 0\}, \quad l^{**} = \inf_{u \in \mathcal{N}} \mathcal{J}(u)$$

and we show the mountain pass critical level lies below every local minimum value of the energy functional at the point of local minimum.

Lemma 5.2.12. *If (m3) and (h3) holds then $l^* \leq l^{**}$.*

Proof. For $u \in \mathcal{N}$, we define a map $h : (0, +\infty) \rightarrow \mathbb{R}$ such that $h(t) = \mathcal{J}(tu)$. Then

$$h'(t) = M(\|tu\|^2)\|u\|^2t - \int_{\Omega} \left(\int_{\Omega} \frac{F(y, tu)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{f(x, tu)u}{|x|^{\alpha}} dx.$$

and since $u \in \mathcal{N}$, therefore

$$h'(t) = \|u\|^4 t^3 \left(\frac{M(\|tu\|^2)}{t^2 \|u\|^2} - \frac{M(\|u\|^2)}{\|u\|^2} \right) + t^3 \left[\int_{\Omega} \int_{\Omega} \left(\frac{F(y, u)f(x, u)}{|y|^{\alpha}|x-y|^{\mu} u(x)} dy - \int_{\Omega} \frac{F(y, tu)f(x, tu)}{|y|^{\alpha}|x-y|^{\mu} t^3 u(x)} dy \right) \frac{u^2(x)}{|x|^{\alpha}} dx \right].$$

From (h3), we get

$$t_1 f(x, t_1) - 2F(x, t_1) \leq t_1 f(x, t_1) - 2F(x, t_2) + 2 \frac{f(x, t_2)}{t_2} (t_2^2 - t_1^2) \leq t_2 f(x, t_2) - 2F(x, t_2).$$

for $0 < t_1 < t_2$. Using this we get that $tf(x, t) - 2F(x, t) \geq 0$ for $t \geq 0$ and for any $x \in \Omega$, $t \mapsto tf(x, t) - 2F(x, t)$ is increasing on \mathbb{R}^+ , which further implies that $t \mapsto \frac{F(x, tu)}{t^2}$ is non-decreasing for $t > 0$. Therefore for $0 < t < 1$ and $x \in \Omega$, we get $\frac{F(x, tu)}{t^2} \leq F(x, u)$ and (h3) gives that $\frac{f(x, u)}{u} \geq \frac{f(x, tu)}{tu}$ then

$$h'(t) \geq \|u\|^4 t^3 \left(\frac{M(\|tu\|^2)}{\|tu\|^2} - \frac{M(\|u\|^2)}{\|u\|^2} \right)$$

$$+ t^3 \left[\int_{\Omega} \left(\int_{\Omega} \left(F(y, u) - \frac{F(y, tu)}{t^2} \right) \frac{dy}{|y|^{\alpha}|x-y|^{\mu}} \right) \frac{f(x, tu)u^2(x)}{|x|^{\alpha}tu(x)} dx \right].$$

This gives that $h'(t) \geq 0$ for $0 < t \leq 1$ and similarly we can show that $h'(t) < 0$ for $t > 1$. Hence $\mathcal{J}(u) = \max_{t \geq 0} \mathcal{J}(tu)$. Now we define $g : [0, 1] \rightarrow W_0^{m,2}(\Omega)$ as $g(t) = (t_0 u)t$ where $t_0 > 1$ is such that $\mathcal{J}(t_0 u) < 0$. So $g \in \Gamma$, where Γ is as defined in the definition of l^* . Therefore,

$$l^* \leq \max_{t \in [0,1]} \mathcal{J}(g(t)) \leq \max_{t \geq 0} \mathcal{J}(tu) = \mathcal{J}(u).$$

and since $u \in \mathcal{N}$ is arbitrary, so we get $l^* \leq l^{**}$. \square

Now, we give the proof of our main result:

Proof of Theorem 5.2.5: Let $\{u_k\}$ be a $(PS)_{l^*}$ sequence at the critical level l^* and hence considered as a minimizing sequence associated to the variational problem (5.2.12). Then by Lemma 5.2.11, there exists a $u_0 \in W_0^{m,2}(\Omega)$ such that up to a subsequence $u_k \rightharpoonup u_0$ weakly in $W_0^{m,2}(\Omega)$ as $k \rightarrow \infty$.

Step 1: u_0 is non-trivial and $u_0 \geq 0$.

If $u_0 \equiv 0$ then using Lemma 5.2.11, we infer that

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{F(x, u_k)}{|x|^{\alpha}} dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Therefore $\lim_{k \rightarrow \infty} \mathcal{J}(u_k) = \frac{1}{2} \lim_{k \rightarrow \infty} \mathcal{M}(\|u_k\|^2) = l^*$ and then for large enough k Theorem 5.2.9 gives

$$\mathcal{M}(\|u_k\|^2) < \mathcal{M} \left(\frac{2n - (2\alpha + \mu)}{2n} \zeta_{m,2m} \right).$$

Then by monotonicity of \mathcal{M} , we obtain

$$\frac{2n}{2n - (2\alpha + \mu)} \|u_k\|^2 < \zeta_{m,2m}.$$

Now, this implies that we can choose a $q > \frac{2n}{2n - (2\alpha + \mu)}$ such that $\sup_k \int_{\Omega} |f(x, u_k)|^q dx < +\infty$. Using Proposition 2.2.7, Theorem 2.2.2, Chapter 1 and the Vitali's convergence theorem we conclude that

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{f(x, u_k)u_k}{|x|^{\alpha}} dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Hence $\lim_{k \rightarrow \infty} \langle \mathcal{J}'(u_k), u_k \rangle = 0$ which gives $\lim_{k \rightarrow \infty} M(\|u_k\|^2) \|u_k\|^2 = 0$. From (m1) we then obtain $\lim_{k \rightarrow \infty} \|u_k\|^2 = 0$. Thus using Lemma 5.2.11, it must be that $\lim_{k \rightarrow \infty} \mathcal{J}(u_k) = 0 = l^*$ which contradicts $l^* > 0$. Thus $u_0 \not\equiv 0$. Now we show that $u_0 \geq 0$ in Ω . From Lemma 5.2.8 we know that $\{u_k\}$ must be bounded. Therefore there exists a constant $\rho > 0$ such that up to a subsequence $\|u_k\| \rightarrow \rho$ as $k \rightarrow \infty$. Let $\varphi \in W_0^{m,2}(\Omega)$ then by Lemma 5.2.10 we have

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{f(x, u_k)}{|x|^{\alpha}} \varphi dx \rightarrow \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|y|^{\alpha}|x-y|^{\mu}} dy \right) \frac{f(x, u_0)}{|x|^{\alpha}} \varphi dx \text{ as } k \rightarrow \infty.$$

Since $\mathcal{J}'(u_k) \rightarrow 0$ and $u_k \rightharpoonup u_0$ weakly in $W_0^{m,2}(\Omega)$, we get

$$M(\rho^2) \int_{\Omega} \nabla^m u_0 \cdot \nabla^m \varphi \, dx = \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|y|^\alpha |x-y|^\mu} \, dy \right) \frac{f(x, u_0)}{|x|^\alpha} \varphi \, dx,$$

as $k \rightarrow \infty$. In particular, taking $\varphi = u_0^-$ in the above equation we get $M(\rho^2) \|u_0^-\|^2 = 0$ which implies together with assumption (m1) that $u_0^- = 0$ a.e. in Ω . Therefore $u_0 \geq 0$ a.e. in Ω .

Step 2: $\mathcal{J}_{\lambda, M} = l^*$.

To prove this, first we claim $M(\|u_0\|^2) \|u_0\|^2 \geq \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|y|^\alpha |x-y|^\mu} \, dy \right) \frac{f(x, u_0) u_0}{|x|^\alpha} \, dx$.

Arguing by contradiction, suppose that

$$M(\|u_0\|^2) \|u_0\|^2 < \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|y|^\alpha |x-y|^\mu} \, dy \right) \frac{f(x, u_0) u_0}{|x|^\alpha} \, dx$$

which implies that $\langle \mathcal{J}'(u_0), u_0 \rangle < 0$. For $t > 0$, using the map $t \mapsto tf(x, t) - 2F(x, t)$ is increasing on \mathbb{R}^+ , we have

$$\begin{aligned} \langle \mathcal{J}'(tu_0), u_0 \rangle &\geq M(\|tu_0\|^2) t \|u_0\|^2 - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{f(y, tu_0) tu_0(y)}{|y|^\alpha |x-y|^\mu} \, dy \right) \frac{f(x, tu_0) u_0}{|x|^\alpha} \, dx \\ &\geq M_0 t \|u_0\|^2 - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{f(y, tu_0) tu_0(y)}{|y|^\alpha |x-y|^\mu} \, dy \right) \frac{f(x, tu_0) u_0}{|x|^\alpha} \, dx. \end{aligned}$$

Since (h3) gives that

$$\lim_{t \rightarrow 0^+} \frac{f(x, t)}{t^\gamma} = 0 \text{ uniformly in } x \in \Omega, \text{ for all } \gamma \in [0, 1],$$

we can choose $t > 0$ sufficiently small so that $\langle \mathcal{J}'(tu_0), u_0 \rangle > 0$. Thus there exists a $t_* \in (0, 1)$ such that $\langle \mathcal{J}'(t_* u_0), u_0 \rangle = 0$ i.e. $t_* u_0 \in \mathcal{N}$. So using Lemma 5.2.12 and (m3)' we get

$$\begin{aligned} l^* &\leq l^{**} \leq \mathcal{J}(t_* u_0) = \mathcal{J}(t_* u_0) - \frac{1}{4} \langle \mathcal{J}'(t_* u_0), t_* u_0 \rangle \\ &= \frac{\mathcal{M}(\|t_* u_0\|^2)}{2} - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, t_* u_0)}{|y|^\alpha |x-y|^\mu} \, dy \right) \frac{F(x, t_* u_0)}{|x|^\alpha} \, dx \\ &\quad - \frac{1}{4} M(\|t_* u_0\|^2) \|t_* u_0\|^2 + \frac{1}{4} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, t_* u_0)}{|y|^\alpha |x-y|^\mu} \, dy \right) \frac{f(x, t_* u_0) t_* u_0}{|x|^\alpha} \, dx \\ &< \frac{\mathcal{M}(\|u_0\|^2)}{2} - \frac{1}{4} M(\|u_0\|^2) \|u_0\|^2 \\ &\quad + \frac{1}{4} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|y|^\alpha |x-y|^\mu} \, dy \right) \frac{f(x, u_0) u_0 - 2F(x, u_0)}{|x|^\alpha} \, dx \\ &= \liminf_{k \rightarrow \infty} \left(\mathcal{J}(u_k) - \frac{1}{4} \langle \mathcal{J}'(u_k), u_k \rangle \right) = l^*. \end{aligned}$$

This gives a contradiction and hence Claim holds. From Lemma 5.2.11 we know that

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^\alpha |x-y|^\mu} \, dy \right) \frac{F(x, u_k)}{|x|^\alpha} \, dx \rightarrow \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|y|^\alpha |x-y|^\mu} \, dy \right) \frac{F(x, u_0)}{|x|^\alpha} \, dx$$

and by using the weakly lower semicontinuity of norms in $\lim_{k \rightarrow \infty} \mathcal{J}(u_k) = l^*$, we obtain $\mathcal{J}(u_0) \leq l^*$. If $\mathcal{J}(u_0) < l^*$ then it must be $\lim_{k \rightarrow \infty} \mathcal{M}(\|u_k\|^2) > \mathcal{M}(\|u_0\|^2)$ which implies that

$\lim_{k \rightarrow \infty} \|u_k\|^2 > \|u_0\|^2$, since \mathcal{M} is continuous and increasing. From this we get $\rho^2 > \|u_0\|^2$. Moreover we have

$$\mathcal{M}(\rho^2) = \left(2l^* + \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{F(x, u_0)}{|x|^\alpha} dx \right). \quad (5.2.20)$$

Now we define the sequence $v_k = \frac{u_k}{\|u_k\|}$ and $v_0 = \frac{u_0}{\rho}$ such that $v_k \rightharpoonup v_0$ weakly in $W_0^{m,2}(\Omega)$ and $\|v_0\| < 1$. Then from Lemma 2.2.5 we obtain

$$\sup_{k \in \mathbb{N}} \int_{\Omega} \exp(p|v_k|^2) < +\infty, \text{ for } p < \frac{\zeta_{m,2m}}{(1 - \|v_0\|^2)}. \quad (5.2.21)$$

Also from $(m3)'$, Claim (1) and proof of Lemma 5.2.12 we obtain

$$\begin{aligned} \mathcal{J}(u_0) &= \frac{1}{2} \mathcal{M}(\|u_0\|^2) - \frac{1}{4} M(\|u_0\|^2) \|u_0\|^2 \\ &\quad + \frac{1}{4} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{(f(x, u_0)u_0 - 2F(x, u_0))}{|x|^\alpha} dx \geq 0. \end{aligned}$$

Using this with (5.2.20) and Theorem 5.2.9 we get that

$$\mathcal{M}(\rho^2) = 2l^* - 2\mathcal{J}(u_0) + \mathcal{M}(\|u_0\|^2) < \mathcal{M} \left(\frac{2n - (2\alpha + \mu)}{2n} \zeta_{m,2m} \right) + \mathcal{M}(\|u_0\|^2)$$

which implies together with (m1) that

$$\rho^2 < \frac{\zeta_{m,2m} \left(\frac{2n - (2\alpha + \mu)}{2n} \right)}{1 - \|v_0\|^2}.$$

Thus it is possible to find a $\rho_* > 0$ such that for $k \in \mathbb{N}$ large enough

$$\|u_k\|^2 < \rho_* < \frac{\zeta_{m,2m} (2n - (2\alpha + \mu))}{2n(1 - \|v_0\|^2)}.$$

Then we choose a $q > 1$ but close to 1 such that

$$\frac{2n}{2n - (2\alpha + \mu)} q \|u_k\|^2 \leq \frac{2n}{2n - (2\alpha + \mu)} \rho_* < \frac{\zeta_{m,2m}}{(1 - \|v_0\|^2)}.$$

Therefore from (5.2.21) we conclude that

$$\int_{\Omega} \exp \left(\frac{2n}{2n - (2\alpha + \mu)} q |u_k|^2 \right) \leq C \quad (5.2.22)$$

for some constant $C > 0$. Using (5.2.22) and ideas similar as in Lemma 5.2.11 we obtain

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u_k)u_k}{|x|^\alpha} dx \rightarrow \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u_0)u_0}{|x|^\alpha} dx.$$

We conclude that $\|u_k\| \rightarrow \|u_0\|$ and we get a contradiction to the fact that $\mathcal{J}(u_0) < l^*$. Hence $\mathcal{J}(u_0) = l^* = \lim_{k \rightarrow \infty} \mathcal{J}(u_k)$ and $\|u_k\| \rightarrow \rho$ implies $\rho = \|u_0\|$. Then finally we have,

$$M(\|u_0\|^2) \int_{\Omega} \nabla^m u_0 \cdot \nabla^m \varphi dx = \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f(x, u_0)}{|x|^\alpha} \varphi dx.$$

for all $\varphi \in W_0^{m,2}(\Omega)$ and which completes the proof of Theorem 5.2.5. \square

5.2.3 Nehari manifold method for Kirchhoff-Choquard equation with singular weights

In this subsection, we consider the problem $(\mathcal{P}_{\lambda,M})$ with Kirchhoff non-linearity of the form $M(t) = at + b$ where $a, b > 0$. We observe that $\mathcal{J}_{\lambda,M}$ is unbounded on $W_0^{m,2}(\Omega)$ but bounded below on suitable subsets of $W_0^{m,2}(\Omega)$. To show the existence of weak solutions to $(\mathcal{P}_{\lambda,M})$, we establish the existence of minimizers of $\mathcal{J}_{\lambda,M}$ under the natural constraint of the Nehari Manifold which contains every solution of $(\mathcal{P}_{\lambda,M})$. So we define the Nehari manifold as

$$N_{\lambda,M} := \{u \in W_0^{m,2}(\Omega) \setminus \{0\} \mid \langle \mathcal{J}'_{\lambda,M}(u), u \rangle = 0\}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $W_0^{m,2}(\Omega)$ and $W^{-m,2}(\Omega)$ i.e. $u \in N_{\lambda,M}$ if and only if

$$\|u\|^2 M(\|u\|^2) - \lambda \int_{\Omega} h(x)|u|^{q+1} dx - \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(u)u}{|x|^{\alpha}} dx = 0. \quad (5.2.23)$$

For $u \in W_0^{m,2}(\Omega)$, we define the fiber map $\Phi_{u,M}$ introduced by Drabek and Pohozaev in [113] as $\Phi_{u,M} : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that $\Phi_{u,M}(t) = \mathcal{J}_{\lambda,M}(tu)$. Thus we get

$$\Phi'_{u,M}(t) = t\|u\|^2 M(\|tu\|^2) - \lambda t^q \int_{\Omega} h(x)|u|^{q+1} dx - \int_{\Omega} \left(\int_{\Omega} \frac{F(tu)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(tu)u}{|x|^{\alpha}} dx$$

and

$$\begin{aligned} \Phi''_{u,M}(t) &= 2t^2\|u\|^4 M'(\|tu\|^2) + \|u\|^2 M(\|tu\|^2) - \lambda q t^{q-1} \int_{\Omega} h(x)|u|^{q+1} dx \\ &\quad - \int_{\Omega} \left(\int_{\Omega} \frac{f(tu)u}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(tu)u}{|x|^{\alpha}} dx - \int_{\Omega} \left(\int_{\Omega} \frac{F(tu)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f'(tu)u^2}{|x|^{\alpha}} dx. \end{aligned}$$

Since the fiber map introduced above are closely related to Nehari manifold by the relation $tu \in N_{\lambda,M}$ iff $\Phi'_{u,M}(t) = 0$, so we analyze the geometry of the energy functional on the following components of the Nehari Manifold:

$$N_{\lambda,M}^{\pm} := \{u \in N_{\lambda,M} : \Phi''_{u,M}(1) \leq 0\} = \{tu \in W_0^{m,2}(\Omega) \setminus \{0\} : \Phi'_{u,M}(t) = 0, \Phi''_{u,M}(t) \leq 0\},$$

$$N_{\lambda,M}^0 := \{u \in N_{\lambda,M} : \Phi''_{u,M}(1) = 0\} = \{tu \in W_0^{m,2}(\Omega) \setminus \{0\} : \Phi'_{u,M}(t) = 0, \Phi''_{u,M}(t) = 0\}.$$

Due to presence of sign changing non-linearity in $(\mathcal{P}_{\lambda,M})$, we also decompose $W_0^{m,2}(\Omega)$ into the following sets to study the behavior of fibering maps $\Phi_{u,M}$. We define $H(u) = \int_{\Omega} h(x)|u|^{q+1} dx$ and

$$H^+ := \{u \in W_0^{m,2}(\Omega) : H(u) > 0\}, \quad H^- := \{u \in W_0^{m,2}(\Omega) : H(u) \leq 0\}.$$

5.2.4 Fiber Map Analysis

In this section, we study the geometry of $\mathcal{J}_{\lambda,M}$ on the Nehari manifold. We split the study according to the decomposition of $N_{\lambda,M}$ and the sign of $H(u)$. Define $\psi : \mathbb{R}^+ \rightarrow \mathbb{R}$ such that

$$\psi_u(t) = t^{1-q} M(\|tu\|^2) \|u\|^2 - t^{-q} \int_{\Omega} \left(\int_{\Omega} \frac{F(tu)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(tu)u}{|x|^{\alpha}} dx.$$

and observing the fact that $tu \in N_{\lambda, M}$ if and only if $t > 0$ is a solution of $\psi_u(t) = \lambda \int_{\Omega} h(x)|u|^{q+1} dx$.

First, we need a priori estimates, which indicate the local minimum value of the function ψ_u at the local minimum point t_* is strictly greater than $\lambda H(u)$.

Lemma 5.2.13. *Let*

$$\Gamma := \left\{ u \in W_0^{m,2}(\Omega) : \|u\|^3 \leq \frac{B(u)}{2\sqrt{(3-q)ab(1-q)}} \right\}.$$

Then there exists a $\lambda_0 > 0$ such that for every $\lambda \in (0, \lambda_0)$, $\Gamma_0 > 0$ holds where

$$\Gamma_0 := \inf_{u \in \Gamma \setminus \{0\} \cap H^+} \left[B(u) - 3 \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f(u)u}{|x|^\alpha} dx + 2b\|u\|^2 - \lambda(3-q)H(u) \right].$$

Proof. We establish the proof through various steps.

Step 1: Claim: $\inf_{u \in \Gamma \setminus \{0\} \cap H^+} \|u\| > 0$.

We argue with contradiction, suppose there exists a sequence $\{u_k\} \subset \Gamma \setminus \{0\} \cap H^+$ such that $\|u_k\| \rightarrow 0$. Then using Proposition 2.2.7 and putting the value of $f(u) = u|u|^p \exp(|u|^\gamma)$ as well as $f'(u) = ((p+1) + \gamma|u|^\gamma)|u|^p \exp(|u|^\gamma)$ we obtain

$$\begin{aligned} |B(u_k)| &= \left| \int_{\Omega} \left(\int_{\Omega} \frac{F(u_k)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f'(u_k)u_k^2}{|x|^\alpha} dx + \int_{\Omega} \left(\int_{\Omega} \frac{f(u_k)u_k}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f(u_k)u_k}{|x|^\alpha} dx \right| \\ &\leq C_1 \left(\int_{\Omega} (|u_k|^{p+2} \exp(|u_k|^\gamma))^{\frac{2n}{2n-(2\alpha+\mu)}} dx \right)^{\frac{2n-(2\alpha+\mu)}{n}} + C_2 \left(\int_{\Omega} (F(u_k))^{\frac{2n}{2n-(2\alpha+\mu)}} dx \right)^{\frac{2n-(2\alpha+\mu)}{2n}} \\ &\quad \times \left(\int_{\Omega} (((p+1) + \gamma|u_k|^\gamma)|u_k|^{p+2} \exp(|u_k|^\gamma))^{\frac{2n}{2n-(2\alpha+\mu)}} dx \right)^{\frac{2n-(2\alpha+\mu)}{2n}}, \end{aligned}$$

where C_1, C_2 are positive constants independent of u_k . Now $(p+2)F(t) \leq tf(t)$ and Hölder's inequality implies that

$$\begin{aligned} |B(u_k)| &\leq C_1 \left(\int_{\Omega} |u_k|^{\frac{2n\delta'(p+2)}{2n-(2\alpha+\mu)}} dx \right)^{\frac{2n-(2\alpha+\mu)}{n\delta'}} \times \left(\int_{\Omega} \exp \left(|u_k|^\gamma \frac{2n\delta}{2n-(2\alpha+\mu)} \right) dx \right)^{\frac{2n-(2\alpha+\mu)}{n\delta}} \\ &\quad + C_2 \left(\int_{\Omega} |u_k|^{\frac{2n\delta'(p+2)}{2n-(2\alpha+\mu)}} dx \right)^{\frac{2n-(2\alpha+\mu)}{2n\delta'}} \times \left(\int_{\Omega} \exp \left(|u_k|^\gamma \frac{2n\delta}{2n-(2\alpha+\mu)} \right) dx \right)^{\frac{2n-(2\alpha+\mu)}{2n\delta}} \times \\ &\quad \left[\left(\int_{\Omega} |u_k|^{\frac{2n\delta'(p+2)}{2n-(2\alpha+\mu)}} dx \right)^{\frac{2n-(2\alpha+\mu)}{2n\delta'}} \times \left(\int_{\Omega} \exp \left(|u_k|^\gamma \frac{2n\delta}{2n-(2\alpha+\mu)} \right) dx \right)^{\frac{2n-(2\alpha+\mu)}{2n\delta}} \right. \\ &\quad \left. + \left(\int_{\Omega} |u_k|^{\frac{2n\delta'(p+\gamma+2)}{2n-(2\alpha+\mu)}} dx \right)^{\frac{2n-(2\alpha+\mu)}{2n\delta'}} \times \left(\int_{\Omega} \exp \left(|u_k|^\gamma \frac{2n\delta}{2n-(2\alpha+\mu)} \right) dx \right)^{\frac{2n-(2\alpha+\mu)}{2n\delta}} \right], \end{aligned}$$

where $\delta > 1$ (which depends on k) and δ' denotes its Hölder conjugate. Using Moser-Trudinger inequality for u_k with large enough k such that $\frac{2n\delta}{(2n-(2\alpha+\mu))} \|u_k\|^\gamma \leq \zeta_{m,2m}$ (such k can be

chosen because $\|u_k\| \rightarrow 0$ as $k \rightarrow \infty$) and $v_k = \frac{u_k}{\|u_k\|}$, we obtain

$$\begin{aligned} |B(u_k)| &\leq C_1 \left(\int_{\Omega} |u_k|^{\frac{2n\delta'(p+2)}{2n-(2\alpha+\mu)}} dx \right)^{\frac{2n-(2\alpha+\mu)}{n\delta'}} \times \left(\sup_{\|v_k\| \leq 1} \int_{\Omega} \exp(|v_k|^\gamma \zeta_{m,2m}) dx \right)^{\frac{2n-(2\alpha+\mu)}{n\delta}} \\ &\quad + C_2 \left(\int_{\Omega} |u_k|^{\frac{2n\delta'(p+2)}{2n-(2\alpha+\mu)}} dx \right)^{\frac{2n-(2\alpha+\mu)}{2n\delta'}} \times \left(\sup_{\|v_k\| \leq 1} \int_{\Omega} \exp(|v_k|^\gamma \zeta_{m,2m}) dx \right)^{\frac{2n-(2\alpha+\mu)}{n\delta}} \times \\ &\quad \left[\left(\int_{\Omega} |u_k|^{\frac{2n\delta'(p+2)}{2n-(2\alpha+\mu)}} dx \right)^{\frac{2n-(2\alpha+\mu)}{2n\delta'}} + \left(\int_{\Omega} |u_k|^{\frac{2n\delta'(p+\gamma+2)}{2n-(2\alpha+\mu)}} dx \right)^{\frac{2n-(2\alpha+\mu)}{2n\delta'}} \right]. \end{aligned}$$

Finally the Sobolev embedding gives the following upper bound.

$$|B(u_k)| \leq C_3 (\|u_k\|^{2(p+2)} + \|u_k\|^{(p+2)} (\|u_k\|^{(p+2)} + \|u_k\|^{(p+\gamma+2)})) \leq C \|u_k\|^{(2p+4)} + \|u_k\|^{(2p+\frac{\gamma}{2}+4)}.$$

Using $u_k \in \Gamma \setminus \{0\}$ we get $1 \leq C(\|u_k\|^{(2p+1)} + \|u_k\|^{(2p+\frac{\gamma}{2}+1)})$, which is a contradiction as $\|u_k\| \rightarrow 0$ as $k \rightarrow \infty$. Therefore we have $\inf_{u \in \Gamma \setminus \{0\} \cap H^+} \|u\| > 0$.

Step 2: Claim: $0 < \inf_{u \in \Gamma \setminus \{0\} \cap H^+} \left\{ \int_{\Omega} \int_{\Omega} \left(\frac{f(u)u}{|x-y|^\mu |y|^\alpha} dy \right) (p-2 + \gamma|u|^\gamma) \frac{\exp(|u|^\gamma) |u|^{p+2}}{|x|^\alpha} dx \right\}$.

Since $F(s) \leq \frac{f(s)s}{p+2}$, then by the definition of Γ and from Step 1, we obtain $0 < \inf_{u \in \Gamma \setminus \{0\} \cap H^+} B(u)$ *i.e.*

$$\begin{aligned} 0 &< \inf_{u \in \Gamma \setminus \{0\} \cap H^+} \left\{ \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f'(u)u^2}{|x|^\alpha} dx + \int_{\Omega} \left(\int_{\Omega} \frac{f(u)u}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f(u)u}{|x|^\alpha} dx \right\} \\ &\leq \inf_{u \in \Gamma \setminus \{0\} \cap H^+} \left\{ \int_{\Omega} \left(\int_{\Omega} \frac{f(u)u}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f(u)u + f'(u) \frac{u^2}{p+2}}{|x|^\alpha} dx \right\} \\ &= \inf_{u \in \Gamma \setminus \{0\} \cap H^+} \left\{ \int_{\Omega} \left(\int_{\Omega} \frac{f(u)u}{|x-y|^\mu |y|^\alpha} dy \right) \frac{|u|^{p+2} \exp(|u|^\gamma)}{|x|^\alpha} \left(1 + \frac{(p+1) + \gamma|u|^\gamma}{p+2} \right) \right\}. \end{aligned}$$

Since $p > 2$, we infer

$$0 < \inf_{u \in \Gamma \setminus \{0\} \cap H^+} \left\{ \int_{\Omega} \int_{\Omega} \left(\frac{f(u)u}{|x-y|^\mu |y|^\alpha} \right) (p-2 + \gamma|u|^\gamma) \frac{\exp(|u|^\gamma) |u|^{p+2}}{|x|^\alpha} dx \right\}.$$

Step 3: Claim: $\Gamma_0 > 0$. Firstly, we have

$$H(u) = \int_{\Omega} h(x) |u|^{q+1} \leq \left(\int_{\Omega} |h(x)|^\rho \right)^{1/\rho} \left(\int_{\Omega} |u|^{(1+q)\rho'} \right)^{1/\rho'} \leq l \|u\|^{q+1}, \quad (5.2.24)$$

where $l = \|h\|_{L^\rho(\Omega)}$ and $\rho > 1$ will be specified later. Choosing

$$\lambda < \frac{2b}{(3-q)l} M_0 := \lambda_0 \quad (5.2.25)$$

where $M_0 = \inf_{u \in \Gamma \setminus \{0\} \cap H^+} \|u\|^{1-q} > 0$, we get that $\lambda l (3-q) \|u\|^{1+q} < 2b \|u\|^2$ for any $u \in \Gamma \setminus \{0\} \cap H^+$. Then for $u \in \Gamma \setminus \{0\} \cap H^+$ and $p > 2$,

$$B(u) + 2b \|u\|^2 - 3 \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f(u)u}{|x|^\alpha} dx - \lambda (3-q) H(u)$$

$$\begin{aligned} &\geq \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f'(u)u^2 - 3f(u)u}{|x|^{\alpha}} dx + \int_{\Omega} \left(\int_{\Omega} \frac{f(u)u}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(u)u}{|x|^{\alpha}} dx \\ &\quad + 2b\|u\|^2 - (3-q)\lambda H(u) > 0. \end{aligned}$$

Therefore $\Gamma_0 > 0$. □

Lemma 5.2.14. *Let $\lambda > 0$. Then*

- (i) *For any $u \in H_0^- \setminus \{0\}$, there exists a unique t^* such that $t^*u \in N_{\lambda, M}^-$. Moreover, $\Phi_{u, M}$ is increasing on $(0, t^*)$ and decreasing on (t^*, ∞) .*
- (ii) *For any $u \in H^+$, there exists λ_0 and $t_*, t_1, t_2 > 0$ such that $t_1u \in N_{\lambda, M}^+$ and $t_2u \in N_{\lambda, M}^-$ for any $\lambda \in (0, \lambda_0)$ and $t_1 < t_* < t_2$.*

Proof. (i) For $u \in H_0^- \setminus \{0\}$

Since

$$\Phi'_{u, M}(t) = t^q(\psi_u(t) - \lambda \int_{\Omega} h(x)|u|^{q+1} dx),$$

so $tu \in N_{\lambda, M}$ iff $t > 0$ is a solution of $\psi_u(t) = \lambda \int_{\Omega} h(x)|u|^{q+1} dx$. We have

$$\begin{aligned} \psi'_u(t) &= (1-q)t^{-q}M(\|tu\|^2)\|u\|^2 + 2t^{2-q}M'(\|tu\|^2)\|u\|^4 \\ &\quad + \frac{q}{t^{q+1}} \int_{\Omega} \left(\int_{\Omega} \frac{F(tu)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(tu)u}{|x|^{\alpha}} dx - t^{-q} \left[\int_{\Omega} \left(\int_{\Omega} \frac{f(tu)u}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(tu)u}{|x|^{\alpha}} dx \right. \\ &\quad \left. + \int_{\Omega} \left(\int_{\Omega} \frac{F(tu)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f'(tu)u^2}{|x|^{\alpha}} dx \right]. \end{aligned} \tag{5.2.26}$$

Due to the presence of exponential non-linearity, for large t we have $\psi'_u(t) < 0$ and since $u \in H_0^-$, there exists a unique $t^* > 0$ such that $\psi_u(t^*) = \lambda \int_{\Omega} h(x)|u|^{q+1} dx$, i.e. $t^*u \in N_{\lambda, M}$. Suppose there exists an another point t_1 ($t^* < t_1$) such that $\psi_u(t_1) = \lambda \int_{\Omega} h(x)|u|^{q+1} \leq 0$, i.e.

$$t_1^{1-q}(at_1^2\|u\|^2 + b)\|u\|^2 \leq t_1^{-q} \int_{\Omega} \left(\int_{\Omega} \frac{F(t_1u)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(t_1u)u}{|x|^{\alpha}} dx \tag{5.2.27}$$

and $\psi'_u(t_1) \geq 0$. Then from (5.2.27) and by using $f'(t_1u)t_1u > (p+1)f(t_1u)$, $f(t)t \geq (p+2)F(t)$ we obtain,

$$\psi'_u(t_1) < (3-q) \left[t_1^{-q}(at_1^2\|u\|^2 + b)\|u\|^2 - t_1^{-q-1} \int_{\Omega} \left(\int_{\Omega} \frac{F(t_1u)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(t_1u)u}{|x|^{\alpha}} dx \right] \leq 0.$$

which is a contradiction. Also for $0 < t < t^*$, $\Phi'_{u, M}(t) = t^q(\psi_u(t) - \lambda \int_{\Omega} h(x)|u|^{q+1} dx) > 0$. Consequently, $\Phi_{u, M}$ is increasing in $(0, t^*)$ and also decreasing on (t^*, ∞) . Therefore t^* is unique critical point of $\Phi_{u, M}$ which is also a point of global maximum. Furthermore, since

$$\psi'_u(t) = \frac{\left(t\Phi''_{u, M}(t) - q\Phi'_{u, M}(t) \right)}{t^q}, \text{ therefore } t^*u \in N_{\lambda, M}^-.$$

(ii) For small $t > 0$, $\psi_u(t) > 0$ and $\psi_u(t) \rightarrow -\infty$ as $t \rightarrow \infty$ for $u \in H^+$. Then there exists at least one point t^* such that $\psi'_u(t^*) = 0$, *i.e.*

$$\begin{aligned} & (3-q)t_*^{2-q}a\|u\|^4 + (1-q)t_*^{-q}b\|u\|^2 + \frac{q}{t_*^{q+1}} \int_{\Omega} \left(\int_{\Omega} \frac{F(t_*u)}{|x-y|^\mu|y|^\alpha} dy \right) \frac{f(t_*u)u}{|x|^\alpha} dx \\ &= t_*^{-q} \left[\int_{\Omega} \left(\int_{\Omega} \frac{F(t_*u)}{|x-y|^\mu|y|^\alpha} dy \right) \frac{f'(t_*u)u^2}{|x|^\alpha} dx + \int_{\Omega} \left(\int_{\Omega} \frac{f(t_*u)u}{|x-y|^\mu|y|^\alpha} dy \right) \frac{f(t_*u)u}{|x|^\alpha} dx \right]. \end{aligned}$$

So by AM-GM inequality we obtain $2\sqrt{(3-q)ab(1-q)}\|t_*u\|^3 \leq B(t_*u)$ where

$$B(u) = \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^\mu|y|^\alpha} dy \right) \frac{f'(u)u^2}{|x|^\alpha} dx + \int_{\Omega} \left(\int_{\Omega} \frac{f(u)u}{|x-y|^\mu|y|^\alpha} dy \right) \frac{f(u)u}{|x|^\alpha} dx.$$

Using $\psi'_u(t_*) = 0$, we replace the value of $a\|t_*u\|^4$ in the definition of $\psi_u(t)$ to obtain

$$\psi_u(t_*) = \frac{1}{(3-q)t_*^{q+1}} \left[B(t_*u) - 3 \int_{\Omega} \left(\int_{\Omega} \frac{F(t_*u)}{|x-y|^\mu|y|^\alpha} dy \right) \frac{f(t_*u)t_*u}{|x|^\alpha} dx + 2b\|t_*u\|^2 \right]. \quad (5.2.28)$$

From Lemma 5.2.13 and (5.2.28), we notice that for $u \in H^+ \setminus \{0\}$, there exists a $t_* > 0$, local maximum of ψ_u verifying $\psi_u(t_*) - \lambda H(u) > 0$ since $t_*u \in \Gamma \setminus \{0\} \cap H^+$. From $\psi_u(0) = 0$, $\psi_u(t_*) > \lambda H(u) > 0$ and $\lim_{t \rightarrow \infty} \psi_u(t) = -\infty$, there exists $t_1 = t_1(u) < t_* < t_2(u) = t_2$ such that $\psi_u(t_1) = \lambda \int_{\Omega} h(x)|u|^{q+1} dx = \psi_u(t_2)$ with $\psi'_u(t_1) > 0$, $\psi'_u(t_2) < 0$. Therefore, $t_1u \in N_{\lambda,M}^+$ and $t_2u \in N_{\lambda,M}^-$. Now we show that $t_1u \in N_{\lambda,M}^+$ and $t_2u \in N_{\lambda,M}^-$ are unique. Suppose not, then there exists $t_3 > 0$ such that $t_3u \in N_{\lambda,M}^+$ and t_{**} such that $t_2 < t_{**} < t_3$, $\psi'_u(t_{**}) = 0$ and $\psi_u(t_{**}) < \lambda H(u)$. Our Lemma 5.2.13 then induces that if $\psi'_u(t_{**}) = 0$ then $\psi_u(t_{**}) > \lambda H(u)$ which is a contradiction. \square

We will denote t_* as the smallest critical point of ψ_u in the sequel. As a consequence of Lemma 5.2.13 and geometry of the energy functional $\mathcal{J}_{\lambda,M}$ on Nehari manifold, we also prove that 0 is the only inflection point of the map $\Phi_{u,M}$ *i.e.* $N_{\lambda,M}^0 = \{0\}$.

Lemma 5.2.15. *If $\lambda \in (0, \lambda_0)$ then $N_{\lambda,M}^0 = \emptyset$.*

Proof. Let $u \in N_{\lambda,M}^0$ then u satisfies

$$a\|u\|^4 + b\|u\|^2 = \lambda H(u) + \int_{\Omega} \int_{\Omega} \left(\frac{F(u)}{|x-y|^\mu|y|^\alpha} dy \right) \frac{f(u)u}{|x|^\alpha} dx \quad \text{and} \quad (5.2.29)$$

$$3a\|u\|^4 + b\|u\|^2 = \lambda q H(u) + B(u). \quad (5.2.30)$$

Let $u \in H^+ \cap N_{\lambda,M}^0$, then substituting the value $\lambda H(u)$ from (5.2.29) into (5.2.30), we obtain

$$2\sqrt{(3-q)(1-q)ab}\|u\|^3 \leq B(u)$$

which implies $u \in \Gamma \setminus \{0\} \cap H^+$. Again substituting the value of $a\|u\|^4$ from (5.2.29) into (5.2.30), we obtain

$$B(u) - 3 \int_{\Omega} \int_{\Omega} \left(\frac{F(u)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(u) \cdot u}{|x|^{\alpha}} dx + 2b\|u\|^2 - \lambda(3-q)H(u) = 0$$

which contradicts Lemma 5.2.13. If $u \in H_0^- \cap N_{\lambda,M}^0$ then Case 1 implies that "1" is the only critical point of $\Phi_{u,M}$ and $\Phi_{u,M}''(1) < 0$ which is a contradiction to the fact that $u \in N_{\lambda,M}^0$. \square

5.2.5 Existence and multiplicity of weak solution

In this section, we first study the geometric structure of the energy functional $\mathcal{J}_{\lambda,M}$ over $N_{\lambda,M}$ and achieves its minimum, with the help of lower and upper bound estimates on θ , where $\theta = \inf_{u \in N_{\lambda,M}} \mathcal{J}_{\lambda,M}(u)$.

Theorem 5.2.16. $\mathcal{J}_{\lambda,M}$ is bounded below and coercive on $N_{\lambda,M}$. Moreover $\theta \geq -C\lambda^{\frac{2}{1-q}}$ where C depends on q, b .

Proof. Let $u \in N_{\lambda,M}$ i.e. $\Phi_{u,M}'(1) = 0$. Then,

$$\begin{aligned} \mathcal{J}_{\lambda,M}(u) &= a\|u\|^4 \left(\frac{p-2}{4(p+2)} \right) + b\|u\|^2 \left(\frac{p}{2(p+2)} \right) - \lambda \left(\frac{p+1-q}{(1+q)(p+2)} \right) H(u) \\ &\quad - \frac{1}{2} \int_{\Omega} \int_{\Omega} \left(\frac{F(u)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{F(u) - \frac{2f(u)u}{p+2}}{|x|^{\alpha}} dx. \end{aligned}$$

Since $0 \leq F(u) \leq \frac{2}{p+2}f(u)u$ and $q < 1$, (5.2.24) and Sobolev embedding implies that $\mathcal{J}_{\lambda,M}$ is coercive on $N_{\lambda,M}$ that is as $\|u\| \rightarrow \infty$,

$$\mathcal{J}_{\lambda,M}(u) \geq a\|u\|^4 \left(\frac{p-2}{4(p+2)} \right) + b\|u\|^2 \left(\frac{p}{2(p+2)} \right) - \lambda \left(\frac{p+1-q}{(1+q)(p+2)} \right) \|u\|^{q+1} \rightarrow \infty.$$

Similarly, we have

$$\begin{aligned} \mathcal{J}_{\lambda,M}(u) &= \frac{b}{2}\|u\|^2 - \frac{\lambda}{q+1}H(u) - \frac{1}{2} \int_{\Omega} \int_{\Omega} \left(\frac{F(u)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{F(u)}{|x|^{\alpha}} dx \\ &\quad + \frac{1}{4} \left(\lambda H(u) + \int_{\Omega} \int_{\Omega} \left(\frac{F(u)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(u)u}{|x|^{\alpha}} dx - b\|u\|^2 \right) \\ &\geq \frac{1}{4}b\|u\|^2 - \lambda \left(\frac{1}{q+1} - \frac{1}{4} \right) H(u). \end{aligned}$$

Then for $u \in H_0^-$, we get $\mathcal{J}_{\lambda,M}(u) \geq 0$ and for $u \in H^+$, the Sobolev embedding implies

$$\begin{aligned} \mathcal{J}_{\lambda,M}(u) &\geq \frac{b}{4}\|u\|^2 - \frac{\lambda(3-q)}{4(q+1)}H(u) \geq \frac{b}{4}\|u\|^2 - \frac{\lambda(3-q)l}{4(q+1)} \left(\int_{\Omega} |u|^{(1+q)\rho'} dx \right)^{1/\rho'} \\ &= b_3\|u\|^2 - b_4\|u\|^{q+1} \end{aligned}$$

where $b_3 = \frac{b}{4}$ and $b_4 = \frac{\lambda(3-q)}{4(q+1)}$. So by finding the minimum of function $g(x) = b_3x^2 - b_4x^{q+1}$, we can conclude that $\mathcal{J}_{\lambda,M}$ is bounded below on $N_{\lambda,M}$. \square

Lemma 5.2.17. *There exists a constant $C_0 > 0$ such that $\theta \leq -C_0$.*

Proof. Let $u \in H^+$, then from the fibering map analysis we know that there exists a $t_1(u) > 0$ such that $t_1 u \in N_{\lambda, M}^+ \cap H^+$ and $\psi_{u, M}(t_1) = \lambda H(u)$. Since $\Phi_{u, M}''(t_1) > 0$, from (5.2.26) we obtain

$$\frac{q-3}{m} a \|t_1 u\|^4 < (1-q) b \|t_1 u\|^2 - B(t_1 u) + q \int_{\Omega} \left(\int_{\Omega} \frac{F(t_1 u)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f(t_1 u) t_1 u}{|x|^\alpha} dx. \quad (5.2.31)$$

Using $\Phi_{u, M}'(t_1) = 0$, we get that

$$\begin{aligned} \mathcal{J}_{\lambda, M}(t_1 u) &= \frac{1}{2} \left(\frac{a}{2} \|t_1 u\|^4 + b \|t_1 u\|^2 \right) - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(t_1 u)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{F(t_1 u)}{|x|^\alpha} dx \\ &\quad - \frac{1}{q+1} \left(a \|t_1 u\|^4 + b \|t_1 u\|^2 - \int_{\Omega} \left(\int_{\Omega} \frac{F(t_1 u)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f(t_1 u) t_1 u}{|x|^\alpha} dx \right). \end{aligned}$$

In that case, by (5.2.31) we obtain,

$$\begin{aligned} \mathcal{J}_{\lambda, M}(t_1 u) &= \frac{-(1-q)}{4(q+1)} b \|t_1 u\|^2 + \int_{\Omega} \left(\int_{\Omega} \frac{F(t_1 u)}{|x-y|^\mu |y|^\alpha} dy \right) \left(\frac{4+q}{4(q+1)} \frac{f(t_1 u) t_1 u}{|x|^\alpha} \right. \\ &\quad \left. - \frac{1}{2} \frac{F(t_1 u)}{|x|^\alpha} - \frac{f'(t_1 u) (t_1 u)^2}{4(q+1) |x|^\alpha} \right) dx - \frac{1}{4(q+1)} \int_{\Omega} \left(\int_{\Omega} \frac{f(t_1 u) t_1 u}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f(t_1 u) t_1 u}{|x|^\alpha} dx \\ &\leq \frac{-(1-q)}{4(q+1)} b \|t_1 u\|^2 + \int_{\Omega} \left(\int_{\Omega} \frac{F(t_1 u)}{|x-y|^\mu |y|^\alpha} dy \right) \left(\frac{4+q}{4(q+1)} - \frac{(p+2)}{4(q+1)} \right. \\ &\quad \left. - \frac{(p+1)}{4(q+1)} \right) \frac{f(t_1 u) t_1 u}{|x|^\alpha} dx - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(t_1 u)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{F(t_1 u)}{|x|^\alpha} dx. \end{aligned}$$

Since $1+q-2p < 0$ therefore $\theta \leq \inf_{u \in N_{\lambda, M}^+ \cap H^+} \mathcal{J}_{\lambda, M}(u) \leq -C_0 < 0$. \square

Using Theorem 5.2.16 and Ekeland variational principle, we know that there exists a sequence $\{u_k\}_{k \in \mathbb{N}} \subset N_{\lambda, M}$ such that

$$\begin{cases} \mathcal{J}_{\lambda, M}(u_k) \leq \theta + \frac{1}{k}; \\ \mathcal{J}_{\lambda, M}(v) \geq \mathcal{J}_{\lambda, M}(u_k) - \frac{1}{k} \|u_k - v\|, \quad \forall v \in N_{\lambda, M}. \end{cases} \quad (5.2.32)$$

Then by (5.2.32) and Lemma 5.2.17, we have for large k ,

$$\mathcal{J}_{\lambda, M}(u_k) \leq -\frac{C_0}{2}. \quad (5.2.33)$$

Also since $u_k \in N_{\lambda, M}$ we have

$$\begin{aligned} \mathcal{J}_{\lambda, M}(u_k) &= a \|u_k\|^4 \left(\frac{p-2}{4(p+2)} \right) + b \|u_k\|^2 \left(\frac{p}{2(p+2)} \right) - \lambda \left(\frac{p+1-q}{(1+q)(p+2)} \right) H(u_k) \\ &\quad - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(u_k)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{F(u_k) - \frac{2f(u_k)u_k}{p+2}}{|x|^\alpha} dx. \end{aligned}$$

This together with (5.2.33) gives

$$-\lambda \left(\frac{p+1-q}{(1+q)(p+2)} \right) H(u_k) \leq -\frac{C_0}{2} \implies H(u_k) \geq \frac{C_0(p+2)(1+q)}{2\lambda(p+1-q)} > 0$$

i.e.

$$H(u_k) > C > 0, \text{ for large } k \text{ and } u_k \in N_{\lambda, M} \cap H^+. \quad (5.2.34)$$

The following result shows that minimizers for $\mathcal{J}_{\lambda, M}$ in any subset of the decomposition of $N_{\lambda, M}$ are critical points of $\mathcal{J}_{\lambda, M}$ and the proof follows from the Lagrange multipliers rule (see Lemma 5.1.22).

Lemma 5.2.18. *Let u be a local minimizer for $\mathcal{J}_{\lambda, M}$ on any subsets of $N_{\lambda, M}$ such that $u \notin N_{\lambda, M}^0$. Then u is a critical point of $\mathcal{J}_{\lambda, M}$.*

Now, we prove a set of lemmas which are necessary to study the $(PS)_\theta$ condition and compactness of the minimizing sequence $\{u_k\}_{k \in \mathbb{N}}$ and whose proof are totally based on the geometry of the energy functional $\mathcal{J}_{\lambda, M}$ on the Nehari manifold.

Lemma 5.2.19. *Let $\lambda > 0$ such that $\lambda \in (0, \lambda_0)$. Then for any $u \in N_{\lambda, M} \setminus \{0\}$, there exists a $\epsilon > 0$ and a differentiable function $\xi : B(0, \epsilon) \subset W_0^{m, 2}(\Omega) \rightarrow \mathbb{R}$ such that*

$$\xi(0) = 1 \text{ and } \xi(w)(u - w) \in N_{\lambda, M}$$

for all $w \in W_0^{m, 2}(\Omega)$. Moreover

$$\langle \xi'(0), w \rangle = \frac{2(2a\|u\|^2 + b) \int_{\Omega} \nabla^m u \cdot \nabla^m w \, dx - \lambda(q+1) \int_{\Omega} h(x)|u|^{q-1}uw \, dx - \langle S(u), w \rangle}{a(3-q)\|u\|^4 + b(1-q)\|u\|^2 + R(u)}$$

where

$$R(u) = \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^\mu|y|^\alpha} \right) \frac{qf(u) - f'(u)u}{|x|^\alpha} \, dx - \int_{\Omega} \left(\int_{\Omega} \frac{f(u)u}{|x-y|^\mu|y|^\alpha} \, dy \right) \frac{f(u)u}{|x|^\alpha} \, dx$$

and

$$\langle S(u), w \rangle = \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^\mu|y|^\alpha} \, dy \right) \frac{f'(u)u + f(u)}{|x|^\alpha} w \, dx + \int_{\Omega} \left(\int_{\Omega} \frac{f(u)u}{|x-y|^\mu|y|^\alpha} \, dy \right) \frac{f(u)}{|x|^\alpha} w \, dx.$$

Proof. For $u \in N_{\lambda, M}$, we define a continuous differentiable function $G_u : \mathbb{R} \times W_0^{m, 2}(\Omega) \rightarrow \mathbb{R}$ such that

$$G_u(t, v) = at^{3-q}\|u-v\|^4 + bt^{1-q}\|u-v\|^2 - \frac{1}{t^q} \int_{\Omega} \left(\int_{\Omega} \frac{F(t(u-v))}{|x-y|^\mu|y|^\alpha} \, dy \right) \frac{f(t(u-v))(u-v)}{|x|^\alpha} \, dx - \lambda \int_{\Omega} h(x)|u-v|^{q+1} \, dx.$$

Then $G_u(1, 0) = \Phi'_u(1) = 0$ and $\frac{\partial}{\partial t} G_u(1, 0) = \phi''_u(1) \neq 0$. Hence by the implicit function theorem, there exists $\epsilon > 0$ and a differentiable function $\xi : B(0, \epsilon) \subset W_0^{m, 2}(\Omega) \rightarrow \mathbb{R}$ such

that $\xi(0) = 1$ and $G_u(\xi(w), w) = 0 \quad \forall w \in B(0, \epsilon)$ which is equivalent to $\langle \mathcal{J}'_{\lambda, M}(\xi(w))(u - w), \xi(w)(u - w) \rangle = 0 \quad \forall w \in B(0, \epsilon)$. Thus, $\xi(w)(u - w) \in N_{\lambda, M}$ and differentiating $G_u(\xi(w), w) = 0$ with respect to w , we obtain the required claim. \square

Similarly, by following the proof of Lemma 5.1.25 and using Lemma 5.2.19, we have the following result.

Lemma 5.2.20. *Let $\lambda > 0$ satisfies (5.2.25) then given any $u \in N_{\lambda, M}^- \setminus \{0\}$, then there exists $\epsilon > 0$ and a differentiable function $\xi^- : B(0, \epsilon) \subset W_0^{m, 2}(\Omega) \rightarrow \mathbb{R}$ such that*

$$\xi^-(0) = 1 \text{ and } \xi^-(w)(u - w) \in N_{\lambda, M}^-$$

and for all $w \in W_0^{m, 2}(\Omega)$

$$\langle (\xi^-)'(0), w \rangle = \frac{2(2a\|u\|^2 + b) \int_{\Omega} \nabla^m u \cdot \nabla^m w \, dx - \lambda(q+1) \int_{\Omega} h(x)|u|^{q-1}uw \, dx - \langle S(u), w \rangle}{a(3-q)\|u\|^4 + b(1-q)\|u\|^2 + R(u)}$$

where $R(u)$ and $S(u)$ are as in Lemma 5.2.19.

Concerning the $(PS)_{\theta}$ condition, we have the following result.

Proposition 5.2.21. *Let $\lambda > 0$ such that $\lambda \in (0, \lambda_0)$ and $u_k \in N_{\lambda, M}$ satisfies (5.2.32). Then $\|\mathcal{J}'_{\lambda, M}(u_k)\|_* \rightarrow 0$ as $k \rightarrow \infty$.*

Proof. Step 1: $\liminf_{k \rightarrow \infty} \|u_k\| > 0$.

We know that $\{u_k\}$ satisfies (5.2.34) for large k , thus $H(u_k) \geq C > 0$ for large k . So by using Hölder inequality we obtain $C < H(u_k) \leq C_1 \|u_k\|^{q+1}$.

Step 2: We claim that

$$\liminf_{k \rightarrow \infty} \left[(3-q)a\|u_k\|^4 + b(1-q)\|u_k\|^2 + q \int_{\Omega} \left(\int_{\Omega} \frac{F(u_k)}{|x-y|^{\mu}|y|^{\alpha}} \right) \frac{f(u_k)u_k}{|x|^{\alpha}} \, dx - B(u_k) \right] > 0.$$

Without loss of generality, we can assume that $u_k \in N_{\lambda, M}^+$ (if not replace u_k by $t_1(u_k)u_k$). Arguing by contradiction, suppose that there exists a subsequence of $\{u_k\}$, still denoted by $\{u_k\}$, such that

$$0 \leq (3-q)a\|u_k\|^4 + b(1-q)\|u_k\|^2 + q \int_{\Omega} \left(\int_{\Omega} \frac{F(u_k)}{|x-y|^{\mu}|y|^{\alpha}} \right) \frac{f(u_k)u_k}{|x|^{\alpha}} \, dx - B(u_k) = o_k(1).$$

From Step 1 and the above equation we obtain that $\liminf_{k \rightarrow \infty} B(u_k) > 0$ and

$$(3-q)a\|u_k\|^4 + b(1-q)\|u_k\|^2 \leq B(u_k)$$

i.e. $u_k \in \Gamma \setminus \{0\}$ for all large k .

Since $u_k \in N_{\lambda, M}^+$, we get

$$-2b\|u_k\|^2 + \lambda(3-q)H(u_k) + 3 \int_{\Omega} \left(\int_{\Omega} \frac{F(u_k)}{|x-y|^{\mu}|y|^{\alpha}} \right) \frac{f(u_k)u_k}{|x|^{\alpha}} \, dx - B(u_k) = o_k(1)$$

which is a contradiction since $\Gamma_0 > 0$. The remaining proof follows similarly as the proof of Proposition 5.1.26. \square

5.2.5.1 Existence of local minimum of $\mathcal{J}_{\lambda, \mathcal{M}}$ in $N_{\lambda, \mathcal{M}}$

Theorem 5.2.22. *Let $1 < \gamma < 2$ and $\lambda > 0$ satisfies (5.2.25). Then there exists a weak solution $u_\lambda \in N_{\lambda, \mathcal{M}}^+ \cap H^+$ to $(\mathcal{P}_{\lambda, \mathcal{M}})$ such that $\mathcal{J}_{\lambda, \mathcal{M}}(u_\lambda) = \inf_{u \in N_{\lambda, \mathcal{M}} \setminus \{0\}} \mathcal{J}_{\lambda, \mathcal{M}}(u)$ and $u_\lambda \in N_{\lambda, \mathcal{M}}^+ \cap H^+$ is a local minimum for $\mathcal{J}_{\lambda, \mathcal{M}}$ in $W_0^{m, 2}(\Omega)$.*

Proof. Let $\{u_k\} \subset N_{\lambda, \mathcal{M}}$ be a minimizing sequence satisfying $\mathcal{J}_{\lambda, \mathcal{M}}(u_k) \rightarrow \theta$ as $k \rightarrow \infty$ and $\mathcal{J}_{\lambda, \mathcal{M}}(v) \geq \mathcal{J}_{\lambda, \mathcal{M}}(u_k) - \frac{1}{k} \|u_k - v\|$, $\forall v \in N_\lambda$ (as in (5.2.32)). Then by Theorem 5.2.16 we obtain $\{u_k\}$ is a bounded sequence in $W_0^{m, 2}(\Omega)$. Also there exists a subsequence of $\{u_k\}$ (denoted by same sequence) and u_λ such that $u_k \rightharpoonup u_\lambda$ weakly in $W_0^{m, 2}(\Omega)$, $u_k \rightarrow u_\lambda$ strongly in $L^r(\Omega)$ for $r \geq 1$ and $u_k \rightarrow u_\lambda$ a.e. in Ω as $k \rightarrow \infty$. Then using $f(t) \leq C_{\epsilon, \gamma} \exp(\epsilon t^2)$ for $\epsilon > 0$ small enough and Theorem 2.2.2, Chapter 1 with $n = 2m$, we obtain that $f(u_k)$ and $F(u_k)$ are uniformly bounded in $L^q(\Omega)$ for all $q > 1$. Then by Proposition 2.2.7 and Vitali's convergence theorem, we obtain

$$\left| \int_{\Omega} \left(\int_{\Omega} \frac{F(u_k)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f(u_k)(u_k - u_\lambda)}{|x|^\alpha} dx \right| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Thus by Proposition 5.2.21, we have $\langle \mathcal{J}'_{\lambda, \mathcal{M}}(u_k), (u_k - u_\lambda) \rangle \rightarrow 0$. Then we conclude that

$$M(\|u_k\|^2) \int_{\Omega} \nabla^m u_k \cdot \nabla^m (u_k - u_\lambda) dx \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (5.2.35)$$

On the other hand, using $u_k \rightharpoonup u_\lambda$ weakly and by boundedness of $M(\|u_k\|^2)$ we have

$$M(\|u_k\|^2) \int_{\Omega} \nabla^m u_\lambda \cdot \nabla^m (u_k - u_\lambda) dx \rightarrow 0 \text{ as } k \rightarrow \infty. \quad (5.2.36)$$

Subtracting (5.2.36) from (5.2.35), we get,

$$M(\|u_k\|^2) \int_{\Omega} (\nabla^m u_k - \nabla^m u_\lambda) \cdot \nabla^m (u_k - u_\lambda) dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

which gives

$$M(\|u_k\|^2) \int_{\Omega} |\nabla^m u_k - \nabla^m u_\lambda|^2 dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Since $M(t) \geq M_0$, we obtain $u_k \rightarrow u_\lambda$ strongly in $W_0^{m, 2}(\Omega)$. By Lemma 5.2.10

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(u_k)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f(u_k)}{|x|^\alpha} \phi dx \rightarrow \int_{\Omega} \left(\int_{\Omega} \frac{F(u_\lambda)}{|x-y|^\mu |y|^\alpha} dy \right) \frac{f(u_\lambda)}{|x|^\alpha} \phi dx$$

and also

$$\int_{\Omega} h(x) |u_k|^{q-1} u_k \phi dx \rightarrow \int_{\Omega} h(x) |u_\lambda|^{q-1} u_\lambda \phi dx$$

for all $\phi \in W_0^{m, 2}(\Omega)$. Therefore, u_λ satisfies $(\mathcal{P}_{\lambda, \mathcal{M}})$ in weak sense and hence $u_\lambda \in N_{\lambda, \mathcal{M}}$. Moreover, $\theta \leq \mathcal{J}_{\lambda, \mathcal{M}}(u_\lambda) \leq \liminf_{k \rightarrow \infty} \mathcal{J}_{\lambda, \mathcal{M}}(u_k) = \theta$. Hence u_λ is a minimizer for $\mathcal{J}_{\lambda, \mathcal{M}}$ in $N_{\lambda, \mathcal{M}}$.

Using (5.2.34), we have $\int_{\Omega} h(x)|u_{\lambda}|^{q+1} > 0$. Then there exists a $t_1(u_{\lambda}) > 0$ such that $t_1(u_{\lambda})u_{\lambda} \in N_{\lambda, \mathcal{M}}^+$. We now claim that $t_1(u_{\lambda}) = 1$ i.e. $u_{\lambda} \in N_{\lambda, \mathcal{M}}^+$. Suppose not then $t_2(u_{\lambda}) = 1$ and $u_{\lambda} \in N_{\lambda, \mathcal{M}}^-$. Now $\mathcal{J}_{\lambda, \mathcal{M}}(t_1(u_{\lambda})u_{\lambda}) < \mathcal{J}_{\lambda, \mathcal{M}}(u_{\lambda}) \leq \theta$ which yields a contradiction, since $t_1(u_{\lambda})u_{\lambda} \in N_{\lambda, \mathcal{M}}$. The proof for u_{λ} being a local minimum for $\mathcal{J}_{\lambda, \mathcal{M}}$ in $W_0^{m,2}(\Omega)$ follows exactly as the proof of Theorem 5.1.27. \square

Theorem 5.2.23. *Let $1 < \gamma < 2$ and $\lambda > 0$ satisfies (5.2.25). Then $\mathcal{J}_{\lambda, \mathcal{M}}$ achieves its minimizer on $N_{\lambda, \mathcal{M}}^-$.*

Proof. Let $u \in N_{\lambda, \mathcal{M}}^-$. Then

$$\begin{aligned} 3a\|u\|^4 + b\|u\|^2 - \lambda q H(u) - \int_{\Omega} \left(\int_{\Omega} \frac{f(u)u}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(u)u}{|x|^{\alpha}} dx \\ - \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f'(u)u^2}{|x|^{\alpha}} dx < 0. \end{aligned}$$

This along with (5.2.23) gives us

$$\begin{aligned} (3-q)a\|u\|^4 + (1-q)b\|u\|^2 + q \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(u)u}{|x|^{\alpha}} dx \\ - \int_{\Omega} \left(\int_{\Omega} \frac{f(u)u}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(u)u}{|x|^{\alpha}} dx - \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f'(u)u^2}{|x|^{\alpha}} dx < 0. \end{aligned}$$

This implies that $N_{\lambda, \mathcal{M}}^- \subset \Gamma$ and then following step 1 of Lemma 5.2.13 we get that $\exists c > 0$, $\|u\| \geq c > 0$ for any $u \in N_{\lambda, \mathcal{M}}^-$ from which it follows that $N_{\lambda, \mathcal{M}}^-$ is a closed set. Also this gives $\inf_{u \in N_{\lambda, \mathcal{M}}^- \setminus \{0\}} B(u) \geq \tilde{c} > 0$. Therefore, for $\lambda < \lambda_0$ small enough,

$$\inf_{u \in N_{\lambda, \mathcal{M}}^- \setminus \{0\}} B(u) + 2b\|u\|^2 - (3-q)\lambda H(u) - 3 \int_{\Omega} \left(\int_{\Omega} \frac{F(u)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(u)u}{|x|^{\alpha}} dx > 0.$$

Now let $\theta^- = \min_{u \in N_{\lambda, \mathcal{M}}^- \setminus \{0\}} \mathcal{J}_{\lambda, \mathcal{M}}(u) > -\infty$ then from Ekeland variational principle, we know that there exist $\{v_k\}_{k \in \mathbb{N}}$ a minimizing sequence such that

$$\mathcal{J}_{\lambda, \mathcal{M}}(v_k) \leq \inf_{u \in N_{\lambda, \mathcal{M}}^-} \mathcal{J}_{\lambda, \mathcal{M}}(u) + \frac{1}{k} \text{ and } \mathcal{J}_{\lambda, \mathcal{M}}(u) \geq \mathcal{J}_{\lambda, \mathcal{M}}(v_k) - \frac{1}{k}\|v_k - u\| \quad \forall u \in N_{\lambda, \mathcal{M}}^-.$$

From $\mathcal{J}_{\lambda, \mathcal{M}}(v_k) \rightarrow \theta^-$ as $k \rightarrow \infty$ and $v_k \in N_{\lambda, \mathcal{M}}$, it is easy to prove that $\|v_k\| \leq C$ (as in Lemma 5.2.8). Indeed,

$$\left| a\|v_k\|^4 + b\|v_k\|^2 - \lambda H(v_k) - \int_{\Omega} \left(\int_{\Omega} \frac{F(v_k)}{|x-y|^{\mu}|y|^{\alpha}} dy \right) \frac{f(v_k)v_k}{|x|^{\alpha}} dx \right| = o(\|v_k\|)$$

and

$$C + o(\|v_k\|) \geq \mathcal{J}_{\lambda, \mathcal{M}}(v_k) - \frac{1}{4} \langle \mathcal{J}'_{\lambda, \mathcal{M}}(v_k), v_k \rangle \geq \frac{b}{4} \|v_k\|^{2n} - C(\lambda) \|v_k\|^{q+1}$$

implies that $\|v_k\| \leq C$. Thus we get $\|S(v_k)\|_* \leq C_1$ and from (5.2.5.1) we have $\|\xi_k^-(0)\|_* \leq C_2$. Now the rest of the proof follows as in the proof of Theorem 5.2.22 with the help of Lemma 5.2.20. \square

Proof of Theorem 5.2.6 : The proof follows from Theorem 5.2.22 and Theorem 5.2.23. \square

5.3 Adams-Moser-Trudinger inequalities for Cartesian product of Sobolev space

5.3.1 Main results

In this subsection, we first establish the non-singular version of Moser-Trudinger and Adams-Moser-Trudinger inequalities in higher dimensional product spaces. Let

$$\mathcal{Y} := W_0^{m, \frac{n}{m}}(\Omega) \times W_0^{m, \frac{n}{m}}(\Omega)$$

be the Banach space endowed with the norm

$$\|(u, v)\|_{\mathcal{Y}} := \left(\|u\|_{W_0^{m, \frac{n}{m}}(\Omega)}^{\frac{n}{m}} + \|v\|_{W_0^{m, \frac{n}{m}}(\Omega)}^{\frac{n}{m}} \right)^{\frac{m}{n}}$$

where $\|u\|_{W_0^{m, \frac{n}{m}}(\Omega)}^{\frac{n}{m}} := \int_{\Omega} |\nabla^m u|^{\frac{n}{m}} dx$.

We prove the following result:

Theorem 5.3.1. *For $(u, v) \in \mathcal{Y}$, $n, m \in \mathbb{N}$ such that $n \geq 2m$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain, we have*

$$\int_{\Omega} \exp\left(\Theta \left(|u|^{\frac{n}{n-m}} + |v|^{\frac{n}{n-m}}\right)\right) dx < \infty$$

for any $\Theta > 0$. Moreover,

$$\sup_{\|(u, v)\|_{\mathcal{Y}}=1} \int_{\Omega} \exp\left(\Theta \left(|u|^{\frac{n}{n-m}} + |v|^{\frac{n}{n-m}}\right)\right) dx < \infty, \text{ provided } \Theta \leq \frac{\zeta_{n,m}}{2_{n,m}} \quad (5.3.1)$$

where $2_{n,m} = 2^{\frac{n-2m}{n-m}}$. Furthermore if $\Theta > \frac{\zeta_{n,m}}{2_{n,m}}$, then there exists a pair $(u, v) \in \mathcal{Y}$ with $\|(u, v)\|_{\mathcal{Y}} = 1$ such that the supremum in (5.3.1) is infinite.

As an consequence of Theorem 5.3.1, we prove the following version of Lions' Lemma [196] in the product space \mathcal{Y} .

Theorem 5.3.2. *Let $(u_k, v_k) \in \mathcal{Y}$ such that $\|(u_k, v_k)\|_{\mathcal{Y}} = 1$ for all k and $(u_k, v_k) \rightharpoonup (u, v) \neq (0, 0)$ weakly in \mathcal{Y} . Then for all $p < \frac{\zeta_{n,m}}{2_{n,m}(1 - \|(u, v)\|_{\mathcal{Y}}^{\frac{n}{m}})^{\frac{m}{n-m}}}$,*

$$\sup_{k \in \mathbb{N}} \int_{\Omega} \exp\left(p \left(|u_k|^{\frac{n}{n-m}} + |v_k|^{\frac{n}{n-m}}\right)\right) dx < \infty.$$

Next, we prove the singular version of Moser-Trudinger inequality in the Cartesian product of Sobolev spaces when $m = 1$.

Theorem 5.3.3. For $(u, v) \in \mathcal{Y} = W_0^{1,n}(\Omega) \times W_0^{1,n}(\Omega)$, $n \geq 2$, $\lambda \in [0, n)$ and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, we have

$$\int_{\Omega} \frac{\exp(\beta(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}}))}{|x|^\lambda} dx < \infty$$

for any $\beta > 0$. Moreover,

$$\sup_{\|(u,v)\|_{\mathcal{Y}}=1} \int_{\Omega} \frac{\exp(\beta(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}}))}{|x|^\lambda} dx < \infty \text{ if and only if } \frac{2_n \beta}{\alpha_n} + \frac{\lambda}{n} \leq 1$$

where $2_n := 2_{n,1} = 2^{\frac{n-2}{n-1}}$.

Similarly we can prove singular and non-singular Moser-Trudinger inequalities in the product space $\mathcal{Z} := W^{1,n}(\Omega) \times W^{1,n}(\Omega)$ where $\Omega \subset \mathbb{R}^n$ is a bounded domain endowed with the norm

$$\|(u, v)\|_{\mathcal{Z}} := \left(\|u\|_{W^{1,n}(\Omega)}^n + \|v\|_{W^{1,n}(\Omega)}^n \right)^{\frac{1}{n}}$$

where $\|u\|_{W^{1,n}(\Omega)}^n := \int_{\Omega} (|u|^n + |\nabla u|^n) dx$. Precisely we establish the following result.

Theorem 5.3.4. For $(u, v) \in \mathcal{Z}$, $n \geq 2$, $\lambda \in [0, n)$ and $\Omega \subset \mathbb{R}^n$ be a smooth bounded domain, we have

$$\int_{\Omega} \frac{\exp(\tilde{\beta}(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}}))}{|x|^\lambda} dx < \infty$$

for any $\tilde{\beta} > 0$. Moreover,

$$\sup_{\|(u,v)\|_{\mathcal{Z}}=1} \int_{\Omega} \frac{\exp(\tilde{\beta}(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}}))}{|x|^\lambda} dx < \infty \text{ if and only if } \frac{2\tilde{\beta}}{\alpha_n} + \frac{\lambda}{n} \leq 1.$$

5.3.2 Proof of the main results

Lemma 5.3.1. If $a, b > 0$ such that $a + b = 1$ then $a^\alpha + b^\alpha \leq 2^{1-\alpha}$ for all $0 < \alpha < 1$.

Proof. Let $r : (0, 1] \times (0, 1] \rightarrow \mathbb{R}$ such that $r(a, b) = a^\alpha + b^\alpha$ and $a + b = 1$ then

$$\tilde{r}(a) := r(a, 1-a) = a^\alpha + (1-a)^\alpha$$

and

$$\frac{d}{da} \tilde{r}(a) = \alpha(a^{\alpha-1} - (1-a)^{\alpha-1}) = 0$$

gives $a = \frac{1}{2}$, which is the point of maximum (since $\frac{d}{da} \left(\frac{d}{da} \tilde{r} \right) (a) \Big|_{a=\frac{1}{2}} < 0$). Therefore the maximum value of \tilde{r} in $(0, 1]$ is $2^{1-\alpha}$. \square

Proof of Theorem 5.3.1:

We denote $\|\cdot\| := \|\cdot\|_{W_0^{m,\frac{n}{m}}(\Omega)}$. Without loss of generality, let $(u, v) \in \mathcal{Y} \setminus \{(0, 0)\}$ be such that $\|(u, v)\|_{\mathcal{Y}} = 1$. If either $u \equiv 0$ or $v \equiv 0$, the result follows from Theorem 2.2.2, Chapter 1.

We set $\alpha = \frac{m}{n-m}$, $a = \|u\|_{\frac{n}{m}}$ and $b = \|v\|_{\frac{n}{m}}$ then Lemma 5.3.1 gives us that

$$\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}} + \frac{\|v\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}} \leq 1$$

where $2_{n,m} = 2^{\frac{n-2m}{n-m}}$.

Case 1: Let $\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}} + \frac{\|v\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}} < 1$.

Then there exists $1 < c := c(u, v) < \infty$ such that

$$\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}} + \frac{\|v\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}} + \frac{1}{c} = 1.$$

Using the generalized Hölder's inequality and $\Theta \leq \frac{\zeta_{n,m}}{2_{n,m}}$ we obtain

$$\begin{aligned} & \int_{\Omega} \exp(\Theta(|u|^{\frac{n}{n-m}} + |v|^{\frac{n}{n-m}})) \\ & \leq |\Omega|^{\frac{1}{c}} \left(\int_{\Omega} \exp(\Theta 2_{n,m} \left(\frac{|u|}{\|u\|} \right)^{\frac{n}{n-m}}) \right)^{\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}}} \left(\int_{\Omega} \exp(\Theta 2_{n,m} \left(\frac{|v|}{\|v\|} \right)^{\frac{n}{n-m}}) \right)^{\frac{\|v\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}}} \\ & \leq C \left(\int_{\Omega} \exp(\zeta_{n,m} \left(\frac{|u|}{\|u\|} \right)^{\frac{n}{n-m}}) \right)^{\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}}} \left(\int_{\Omega} \exp(\zeta_{n,m} \left(\frac{|v|}{\|v\|} \right)^{\frac{n}{n-m}}) \right)^{\frac{\|v\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}}} \end{aligned} \quad (5.3.2)$$

where C is a positive constant depending on $|\Omega|$ but independent of u, v .

Case 2: $\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}} + \frac{\|v\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}} = 1$.

Applying the Hölder's inequality and $\Theta \leq \frac{\zeta_{n,m}}{2_{n,m}}$ we obtain

$$\begin{aligned} & \int_{\Omega} \exp(\Theta(|u|^{\frac{n}{n-m}} + |v|^{\frac{n}{n-m}})) \\ & \leq \left(\int_{\Omega} \exp(\Theta 2_{n,m} \left(\frac{|u|}{\|u\|} \right)^{\frac{n}{n-m}}) \right)^{\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}}} \left(\int_{\Omega} \exp(\Theta 2_{n,m} \left(\frac{|v|}{\|v\|} \right)^{\frac{n}{n-m}}) \right)^{\frac{\|v\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}}} \\ & \leq \left(\int_{\Omega} \exp(\zeta_{n,m} \left(\frac{|u|}{\|u\|} \right)^{\frac{n}{n-m}}) \right)^{\frac{\|u\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}}} \left(\int_{\Omega} \exp(\zeta_{n,m} \left(\frac{|v|}{\|v\|} \right)^{\frac{n}{n-m}}) \right)^{\frac{\|v\|_{\frac{n}{m}}^{\frac{n}{n-m}}}{2_{n,m}}}. \end{aligned} \quad (5.3.3)$$

Now by combining (5.3.2), (5.3.3) and taking supremum over $\|(u, v)\|_{\mathcal{Y}} = 1$, we obtain the desired inequality (5.3.1). For the remaining part of the proof, we assume that $0 \in \Omega$ and

seek use of the Adams function to construct a sequence of test functions. Let us denote $\mathcal{B}(0, l) \stackrel{\text{def}}{=} \mathcal{B}_l$ as a ball with center 0 and radius l in \mathbb{R}^n then without loss of generality, we can assume that $B(0, l) \subset \Omega$ for $l \in (0, 1)$. We recall the following result (see [182]): For $l \in (0, 1)$, there exists

$$U_l \in \{u \in W_0^{m, \frac{n}{m}}(\Omega) : u|_{\mathcal{B}_l} = 1\}$$

such that

$$\|U_l\|_{\frac{n}{m}} = C_{m, \frac{n}{m}}(\mathcal{B}_l; \mathcal{B}_1) \leq \left(\frac{\zeta_{n, m}}{n \log\left(\frac{1}{l}\right)} \right)^{\frac{n-m}{m}}$$

where $C_{m, \frac{n}{m}}(K, E)$ is the conductor capacity of K in E whenever E is an open set and K is relatively compact subset of E and $C_{m, \frac{n}{m}}(K; E) \stackrel{\text{def}}{=} \inf\{\|u\|_{\frac{n}{m}} : u \in C_0^\infty(E), u|_K = 1\}$. Let us set $\sigma > 0$ and $l = \frac{1}{k}$, for $k \in \mathbb{N}$. Also we define

$$A_k(x) = \begin{cases} \left(\frac{n \log(k)}{\zeta_{n, m}} \right)^{\frac{n-m}{n}} U_{\frac{1}{k}}\left(\frac{x}{\sigma}\right) & \text{if } |x| < \sigma; \\ 0 & \text{if } |x| \geq \sigma. \end{cases}$$

Then we have $A_k(x)|_{\mathbb{B}_{\frac{\sigma}{k}}} = \left(\frac{n \log(k)}{\zeta_{n, m}} \right)^{\frac{n-m}{n}}$ and $\|A_k\| \leq 1$, Now we consider

$$Z_k = c_1 w_k \quad \text{and} \quad V_k = c_2 w_k$$

where $w_k(x) = \frac{A_k}{\|A_k\|}$ and $c_1, c_2 \in \mathbb{R}^+$ verifies

$$c_1^{\frac{n}{m}} + c_2^{\frac{n}{m}} = 1 \quad \text{and} \quad c_1^{\frac{n}{n-m}} + c_2^{\frac{n}{n-m}} = 2_{n, m}$$

which implies that $\text{supp}(w_k) \subset B_\sigma(0)$ and $\|w_k\| = 1$ for all k . The existence of c_1, c_2 can be proved using Lemma 5.3.1. Thus we obtain

$$\begin{aligned} \|Z_k, V_k\|_{\mathcal{Y}} &= \left(\|Z_k\|_{\frac{n}{m}}^{\frac{n}{m}} + \|V_k\|_{\frac{n}{m}}^{\frac{n}{m}} \right)^{\frac{m}{n}} = \left(c_1^{\frac{n}{m}} \|w_k\|_{\frac{n}{m}}^{\frac{n}{m}} + c_2^{\frac{n}{m}} \|w_k\|_{\frac{n}{m}}^{\frac{n}{m}} \right)^{\frac{m}{n}} \\ &= \|w_k\| \left(c_1^{\frac{n}{m}} + c_2^{\frac{n}{m}} \right)^{\frac{m}{n}} = 1. \end{aligned}$$

So if $\Theta > \frac{\zeta_{n, m}}{2_{n, m}}$, then for some $\epsilon > 0$, $\Theta = (1 + \epsilon) \frac{\zeta_{n, m}}{2_{n, m}}$ which gives that

$$\begin{aligned} \int_{\Omega} \exp(\Theta(|U_k|^{\frac{n}{n-m}} + |V_k|^{\frac{n}{n-m}})) &\geq \int_{B_{\frac{\sigma}{k}}} \exp\left((1 + \epsilon) \frac{\zeta_{n, m}}{2_{n, m}} (|w_k|^{\frac{n}{n-m}} (c_1^{\frac{n}{n-m}} + c_2^{\frac{n}{n-m}})) \right) \\ &= \int_{B_{\frac{\sigma}{k}}} k^{n(1+\epsilon)} \geq C_3 k^\epsilon \rightarrow \infty \quad \text{as } k \rightarrow \infty. \end{aligned}$$

This completes the proof. □

Proof of Theorem 5.3.2 : Using Brezis-Lieb lemma, it is easy to see that

$$\lim_{k \rightarrow \infty} \|(u_k - u), (v_k - v)\|_{\mathcal{Y}}^{\frac{n}{m}} = 1 - \|(u, v)\|_{\mathcal{Y}}^{\frac{n}{m}}$$

and

$$|u_k|^{\frac{n}{n-m}} \leq \left(|u_k - u|^{\frac{n}{n-m}} + |u|^{\frac{n}{n-m}} \right) + C(|u_k - u|^{\frac{m}{n-m}} |u| + |u|^{\frac{m}{n-m}} |u_k - u|)$$

where $C \stackrel{\text{def}}{=} C(n, m) > 0$. Now for any $\epsilon > 0$, from Young's inequality we have that

$$ab \leq \frac{m}{n} (\epsilon a)^{\frac{n}{m}} + \frac{n-m}{n} \left(\frac{b}{\epsilon} \right)^{\frac{n}{n-m}}.$$

This gives us

$$\begin{aligned} |u_k|^{\frac{n}{n-m}} &\leq \left((1 + C_1 \epsilon^{\frac{n}{m}} + C_1 \epsilon^{\frac{-n}{n-m}}) |u_k - u|^{\frac{n}{n-m}} + (1 + C_1 \epsilon^{\frac{-n}{m}} + C_1 \epsilon^{\frac{-n}{n-m}}) |u|^{\frac{n}{n-m}} \right) \\ &:= C_{1,\epsilon} |u_k - u|^{\frac{n}{n-m}} + C'_{1,\epsilon} |u|^{\frac{n}{n-m}} \quad (\text{say}). \end{aligned}$$

Similarly we also have

$$|v_k|^{\frac{n}{n-m}} \leq C_{1,\epsilon} |v_k - v|^{\frac{n}{n-m}} + C'_{1,\epsilon} |v|^{\frac{n}{n-m}}.$$

Therefore by using Hölder inequality and above estimates we obtain,

$$\begin{aligned} \int_{\Omega} \exp \left(p(|u_k|^{\frac{n}{n-m}} + |v_k|^{\frac{n}{n-m}}) \right) dx &\leq \left(\int_{\Omega} \exp \left(p C_{1,\epsilon} r_1 \left(|u_k - u|^{\frac{n}{n-m}} + |v_k - v|^{\frac{n}{n-m}} \right) \right) dx \right)^{\frac{1}{r_1}} \\ &\quad \cdot \left(\int_{\Omega} \exp \left(p C'_{1,\epsilon} r_2 \left(|u|^{\frac{n}{n-m}} + |v|^{\frac{n}{n-m}} \right) \right) dx \right)^{\frac{1}{r_2}} \\ &\leq C(n, m, u, v, r_2) \left(\int_{\Omega} \exp \left(p C_{1,\epsilon} r_1 (\|(u_k - u), (v_k - v)\|_{\mathcal{Y}})^{\frac{n}{n-m}} \right. \right. \\ &\quad \left. \left. \left(\left(\frac{|u_k - u|}{\|(u_k - u), (v_k - v)\|_{\mathcal{Y}}} \right)^{\frac{n}{n-m}} + \left(\frac{|v_k - v|}{\|(u_k - u), (v_k - v)\|_{\mathcal{Y}}} \right)^{\frac{n}{n-m}} \right) \right) dx \right)^{\frac{1}{r_1}} \end{aligned}$$

where r_1 and r_2 are Hölder conjugate to each other and $C(n, m, u, v, r_2)$ is a positive constant independent of k . Now since $C_{1,\epsilon} \rightarrow 1$ as $\epsilon \rightarrow 0$, by choosing $\epsilon > 0$ small enough and $r_1 > 1$ very close to 1 such that

$$p r_1 C_{1,\epsilon} (1 - \|(u, v)\|_{\mathcal{Y}}^{\frac{n}{m}})^{\frac{m}{n-m}} < \frac{\zeta_{n,m}}{2_{n,m}}$$

we get the desired result, by using Theorem 5.3.1. \square

To prove the following Singular Moser-Trudinger inequality in cartesian product of Sobolev space taking $m = 1$ and using the idea of Theorem 2.1 in [7].

Proof of Theorem 5.3.3:

We denote $\|\cdot\| := \|\cdot\|_{W_0^{1,n}(\Omega)}$ in this proof. Let $(u, v) \in \mathcal{Y}$ be such that $\|(u, v)\|_{\mathcal{Y}} = 1$,

$\lambda \in (0, n)$ and $\beta > 0$. Then following two cases arise:

Case 1: Let $\frac{\beta 2_n}{\alpha_n} + \frac{\lambda}{n} < 1$ then we choose $t > 1$ such that

$$\frac{\beta 2_n}{\alpha_n} + \frac{\lambda t}{n} = 1.$$

Now by using Cauchy-Schwarz inequality and Theorem 5.3.1, we obtain

$$\int_{\Omega} \frac{\exp(\beta(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}}))}{|x|^{\lambda}} \leq \left(\int_{\Omega} \exp\left(\frac{\alpha_n}{2_n}(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}})\right) \right)^{\frac{\beta 2_n}{\alpha_n}} \cdot \left(\int_{\Omega} \frac{1}{|x|^{\frac{\lambda t}{t}}} \right)^{\frac{\lambda t}{n}} \leq C$$

where C is a constant independent of u, v .

Case 2: Let $\frac{\beta 2_n}{\alpha_n} + \frac{\lambda}{n} = 1$. Then from standard symmetrization and density arguments we can reduce to the case Ω being a ball $B(0, R)$ with centre origin and radius R and u, v being positive smooth and radial functions. Therefore

$$\int_{B(0,R)} (|\nabla u|^n + |\nabla v|^n) dx = \omega_{n-1} \int_0^R ((u'(r))^n + (v'(r))^n) r^{n-1} dr \quad (5.3.4)$$

and

$$\int_{B(0,R)} \frac{\exp\left(\frac{s\alpha_n}{2_n}(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}})\right)}{|x|^{(1-s)n}} dx = \int_0^R \exp\left(\frac{s\alpha_n}{2_n}(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}})\right) r^{sn-1} dr$$

where $s = \frac{\beta 2_n}{\alpha_n}$ so that $\lambda = (1-s)n$. Now we set

$$\tilde{u}(r) = s^{\frac{n-1}{n}} u(r^{\frac{1}{s}}) \text{ and } \tilde{v}(r) = s^{\frac{n-1}{n}} v(r^{\frac{1}{s}}) \text{ for } r \in [0, R^s].$$

Therefore

$$\begin{aligned} \int_0^R ((u'(r))^n + (v'(r))^n) r^{n-1} dr &= \int_0^{R^s} ((\tilde{u}'(r))^n + (\tilde{v}'(r))^n) r^{n-1} dr, \\ \int_0^R \exp\left(\frac{s\alpha_n}{2_n}(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}})\right) r^{sn-1} dr &= \frac{1}{s} \int_0^{R^s} \exp\left(\frac{\alpha_n}{2_n}(|\tilde{u}|^{\frac{n}{n-1}} + |\tilde{v}|^{\frac{n}{n-1}})\right) r^{n-1} dr. \end{aligned} \quad (5.3.5)$$

Now by combining (5.3.4)-(5.3.5) and taking supremum, we obtain

$$\begin{aligned} &\sup_{\|(u,v)\|_{\mathcal{Y}}=1} \int_{B(0,R)} \frac{\exp\left(\frac{s\alpha_n}{2_n}(|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}})\right)}{|x|^{(1-s)n}} dx \\ &\leq \sup_{\|(\tilde{u},\tilde{v})\|_{\mathcal{Y}}=1} \frac{R^{s(n-1)}}{s} \int_0^{R^s} \exp\left(\frac{\alpha_n}{2_n}(|\tilde{u}|^{\frac{n}{n-1}} + |\tilde{v}|^{\frac{n}{n-1}})\right) dr < \infty \end{aligned}$$

which is the desired inequality. For the remaining part of the proof, we assume $0 \in \Omega$ and define

$$w_k(x) = \frac{1}{\omega_{n-1}^{\frac{1}{n}}} \begin{cases} (\log k)^{\frac{n-1}{n}}, & 0 \leq |x| \leq \frac{\rho}{k}, \\ \frac{\log\left(\frac{\rho}{|x|}\right)}{(\log k)^{\frac{1}{n}}}, & \frac{\rho}{k} \leq |x| \leq \rho, \\ 0, & |x| \geq \rho \end{cases}$$

such that $\text{supp}(w_k) \subset B_\rho(0)$ and $\|w_k\| = 1$ for all k . Let $c_1, c_2 \in \mathbb{R}^+$ such that $c_1^n + c_2^n = 1$ and $c_1^{\frac{n}{n-1}} + c_2^{\frac{n}{n-1}} = 2^{\frac{n-2}{n-1}}$ (The existence of c_1, c_2 can be proved by taking the maximum of function mentioned in Lemma 5.3.1).

Also we define

$$U_k = c_1 w_k \quad \text{and} \quad V_k = c_2 w_k$$

such that

$$\|U_k, V_k\|_{\mathcal{Y}} = (\|U_k\|^n + \|V_k\|^n)^{\frac{1}{n}} = (c_1^n \|w_k\|^n + c_2^n \|w_k\|^n)^{\frac{1}{n}} = \|w_k\| (c_1^n + c_2^n)^{\frac{1}{n}} = 1.$$

Now let $\beta > \left(1 - \frac{\lambda}{n}\right) \frac{\alpha_n}{2n}$, then for some $\epsilon > 0$, $\beta = (1 + \epsilon) \left(1 - \frac{\lambda}{n}\right) \frac{\alpha_n}{2n}$ and

$$\begin{aligned} \int_{\Omega} \frac{\exp\left(\beta(|U_k|^{\frac{n}{n-1}} + |V_k|^{\frac{n}{n-1}})\right)}{|x|^\lambda} &\geq \int_{B_{\frac{\rho}{k}}} \frac{\exp\left((1 + \epsilon) \left(1 - \frac{\lambda}{n}\right) \frac{\alpha_n}{2n} |w_k|^{\frac{n}{n-1}} (c_1^{\frac{n}{n-1}} + c_2^{\frac{n}{n-1}})\right)}{|x|^\lambda} \\ &\geq \int_{B_{\frac{\rho}{k}}} k^{n(1+\epsilon)(1-\frac{\lambda}{n})+\lambda} \geq C_3 k^{\epsilon(n-\lambda)} \rightarrow \infty \quad \text{as } k \rightarrow \infty. \end{aligned}$$

□

Proof of Theorem 5.3.4: The proof can be done by following the same steps as in Theorems 5.3.1 and 5.3.3. □

5.4 Kirchhoff systems with Choquard non-linearity

In this section, we study the following system of n -Kirchhoff Choquard equations with exponential non-linearity

$$(KCS) \begin{cases} -m(\|(u, v)\|^n) \Delta_n u = \left(\int_{\Omega} \frac{F(y, u, v)}{|x-y|^\mu} dy \right) f_1(x, u, v), & u > 0 \quad \text{in } \Omega, \\ -m(\|(u, v)\|^n) \Delta_n v = \left(\int_{\Omega} \frac{F(y, u, v)}{|x-y|^\mu} dy \right) f_2(x, u, v), & v > 0 \quad \text{in } \Omega, \\ u, v = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in \mathbb{R}^n , $0 < \mu < n$. Let $m : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a continuous function satisfying the following conditions:

- (m1) $M(t+s) \geq M(t) + M(s)$ for all $t, s \geq 0$ where $M(t)$ is the primitive of the function m .
- (m2) There exist constants $c_0, c_1, c_2 > 0$ and $\tilde{t} > 0$ such that for some $r, z \in \mathbb{R}^+$

$$m(t) \geq c_0 \quad \text{or} \quad m(t) \geq t^z, \quad \text{for all } t \geq 0$$

and

$$m(t) \leq c_1 + c_2 t^r, \quad \text{for all } t \geq \tilde{t}.$$

- (m3) The map $t \mapsto \frac{m(t)}{t}$ is non-increasing for $t > 0$.

We remark that the assumption (m2) covers both degenerate as well as non-degenerate case of the Kirchhoff term.

Example 1: An example of a function m satisfying (m1) – (m3) is $m(t) = d_0 + d_1 t^\beta$ for $\beta < 1$ and $d_0, d_1 \geq 0$.

Let the function $F : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be continuously differentiable with respect to second and third variable and of the form $F(x, t, s) = h(x, t, s) \exp(|t|^{\frac{n}{n-1}} + |s|^{\frac{n}{n-1}})$ such that

$$f_1(x, t, s) := \frac{\partial F}{\partial t}(x, t, s) = h_1(x, t, s) \exp(|t|^{\frac{n}{n-1}} + |s|^{\frac{n}{n-1}}),$$

$$f_2(x, t, s) := \frac{\partial F}{\partial s}(x, t, s) = h_2(x, t, s) \exp(|t|^{\frac{n}{n-1}} + |s|^{\frac{n}{n-1}}).$$

We assume h_i 's for $i = 1, 2$ are continuous functions satisfying the following conditions-

- (f1) $h_i(x, t, s) = 0$ when either $t \leq 0$ or $s \leq 0$ and $h_i(x, t, s) > 0$ when $t, s > 0$, for all $x \in \Omega$ and $i = 1, 2$.
- (f2) For any $\epsilon > 0$ and $i = 1, 2$

$$\lim_{t, s \rightarrow \infty} \sup_{x \in \overline{\Omega}} h_i(x, t, s) \exp(-\epsilon(|t|^{\frac{n}{n-1}} + |s|^{\frac{n}{n-1}})) = 0,$$

$$\lim_{t, s \rightarrow \infty} \inf_{x \in \overline{\Omega}} h_i(x, t, s) \exp(\epsilon(|t|^{\frac{n}{n-1}} + |s|^{\frac{n}{n-1}})) = \infty.$$

- (f3) There exists

$$l > \begin{cases} \max \left\{ n-1, \frac{n(r+1)}{2} \right\} & \text{when } m \text{ is non-degenerate,} \\ \max \left\{ n-1, \frac{n(z+1)}{2}, \frac{n(r+1)}{2} \right\} & \text{when } m \text{ is degenerate.} \end{cases}$$

such that the maps $t \mapsto \frac{f_1(x, t, s)}{|t|^l}$, $s \mapsto \frac{f_2(x, t, s)}{|s|^l}$ are increasing functions of t (uniformly in s and x) and s (uniformly in t and x) respectively.

- (f4) There exist $q, s_0, t_0, M_0 > 0$ such that $s^q F(x, t, s) \leq M_0 f_2(x, t, s)$ for all $s \geq s_0$ and $t^q F(x, t, s) \leq M_0 f_1(x, t, s)$ for all $t \geq t_0$ uniformly in $x \in \Omega$.
- (f5) There exists a γ satisfying $\frac{n-2}{2} < \gamma$ such that $\lim_{(t, s) \rightarrow (0, 0)} \frac{f_i(x, t, s)}{s^\gamma + t^\gamma} = 0$ holds for $i = 1, 2$.

Let $\mathcal{P} := W_0^{1, n}(\Omega) \times W_0^{1, n}(\Omega)$ endowed with the graph norm

$$\|(u, v)\| := \left(\|u\|_{W_0^{1, n}(\Omega)}^n + \|v\|_{W_0^{1, n}(\Omega)}^n \right)^{\frac{1}{n}}$$

where $\|u\|_{W_0^{1, n}(\Omega)}^n := \int_{\Omega} |\nabla u|^n dx$. The study of the elliptic system (KCS) is motivated by Theorems 5.3.1 and 5.3.2. Following is the notion of weak solution for (KCS).

Definition 5.4.1. A function $(u, v) \in \mathcal{P}$ is said to be weak solution of (KCS) if for all $(\phi, \psi) \in \mathcal{P}$, it satisfies

$$\begin{aligned} m(\|u, v\|^n) & \left(\int_{\Omega} |\nabla u|^{n-2} \nabla u \nabla \phi dx + \int_{\Omega} |\nabla v|^{n-2} \nabla v \nabla \psi dx \right) \\ & = \int_{\Omega} \left(\int_{\Omega} \frac{F(x, u, v)}{|x-y|^\mu} dy \right) (f_1(x, u, v)\phi + f_2(x, u, v)\psi) dx. \end{aligned}$$

We define the energy functional J on \mathcal{P} as

$$J(u, v) = \frac{1}{n} M(\|u, v\|^n) - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u, v)}{|x-y|^\mu} dy \right) F(x, u, v) dx. \quad (5.4.1)$$

Using assumption (f1) – (f3), we get that for any $\epsilon > 0, p \geq 1$ and $1 \leq k < l + 1$ there exist constant C_1, C_2 such that for any $(x, t, s) \in \Omega \times \mathbb{R}^2$

$$|F(x, t, s)| \leq C_1(|s|^k + |t|^k) + C_2(|s|^p + |t|^p) \exp((1 + \epsilon)(|s|^{\frac{n}{n-1}} + |t|^{\frac{n}{n-1}})). \quad (5.4.2)$$

Then by using Sobolev embedding and Hardy-Littlewood Sobolev inequality, we obtain $F(u, v) \in L^q(\Omega \times \Omega)$ for any $q \geq 1$ and the energy functional J is well defined in \mathcal{P} .

5.4.1 Main result

Theorem 5.4.1. Let m satisfies (m1) – (m3) and f satisfies (f1) – (f5) and

$$\lim_{t, s \rightarrow \infty} \frac{(f_1(x, t, s)t + f_2(x, t, s)s)F(x, t, s)}{\exp(q(|t|^{\frac{n}{n-1}} + |s|^{\frac{n}{n-1}}))} = \infty \text{ uniformly in } x \in \bar{\Omega}.$$

for some $q > 2$. Then there exists a positive weak solution of the problem (KCS).

5.4.2 Mountain Pass geometry and Analysis of Palais-Smale sequence

In this section we show that the energy functional J satisfies the mountain pass geometry and derive the integral estimates of Choquard term by exploiting the weak convergence of Palais-Smale squence in appropriate spaces.

Lemma 5.4.1. Assume m and f satisfies (m2) and (f1) – (f3) respectively then

- (i) There exists $\rho > 0$ such that $J(u, v) \geq \sigma$ when $\|(u, v)\| = \rho$, for some $\sigma > 0$.
- (ii) There exists a $(\tilde{u}, \tilde{v}) \in \mathcal{P}$ such that $J(\tilde{u}, \tilde{v}) < 0$ and $\|(\tilde{u}, \tilde{v})\| > \rho$.

Proof. Let $(u, v) \in \mathcal{P}$ such that $\|(u, v)\| = \rho$ (to be determined later). Then from (5.4.2), Proposition 2.2.6, Sobolev embedding, Hölder inequality, we have for any $\epsilon > 0, p \geq 1$ and

$1 \leq k < l + 1$ we have

$$\begin{aligned}
 & \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u, v)}{|x-y|^{\mu}} dy \right) F(x, u, v) dx \leq C(n, \mu) \|F(x, u, v)\|_{L^{\frac{2n}{2n-\mu}}(\Omega)}^2 \\
 & \leq \left[C_1 \left(\int_{\Omega} |u|^k + |v|^k \right)^{\frac{2n}{2n-\mu}} \right. \\
 & \quad \left. + C_2 \left(\int_{\Omega} (|u|^p + |v|^p)^{\frac{2n}{2n-\mu}} \exp \left(\frac{(1+\epsilon)2n}{2n-\mu} (|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}}) \right) \right)^{\frac{2n-\mu}{n}} \right. \\
 & \leq \left[C_1 (\|(u, v)\|)^{\frac{2nk}{2n-\mu}} \right. \\
 & \quad \left. + C_2 (\|(u, v)\|)^{\frac{2np}{2n-\mu}} \left(\int_{\Omega} \exp \left(\frac{(1+\epsilon)4n\|(u, v)\|^{\frac{n}{n-1}}}{2n-\mu} \left(\frac{|u|^{\frac{n}{n-1}} + |v|^{\frac{n}{n-1}}}{\|(u, v)\|^{\frac{n}{n-1}}} \right) \right) \right)^{\frac{1}{2}} \right]^{\frac{2n-\mu}{n}}.
 \end{aligned}$$

If we choose $\epsilon > 0$ and ρ such that $\frac{4n(1+\epsilon)\rho^{\frac{n}{n-1}}}{2n-\mu} \leq \frac{\alpha_n}{2n}$, then by using Theorem 5.3.1 in above we obtain,

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u, v)}{|x-y|^{\mu}} dy \right) F(x, u, v) dx \leq C_3 \|(u, v)\|^{2k} + C_4 \|(u, v)\|^{2p}. \quad (5.4.3)$$

Niw by using (5.4.3) and (m2) (for non-degenerate Kirchhoff term), we get

$$J(u, v) \geq c_0 \frac{\|(u, v)\|^n}{n} - C_3 \|(u, v)\|^{2k} - C_4 \|(u, v)\|^{2p}.$$

So choosing $k > n/2$, $p > n/2$ and ρ small enough such that $J(u, v) \geq \sigma$ when $\|(u, v)\| = \rho$ for some $\sigma > 0$ depending on ρ . Similarly for degenerate Kirchhoff term we get,

$$J(u, v) \geq \frac{\|(u, v)\|^{n(z+1)}}{n} - C_3 \|(u, v)\|^{2k} - C_4 \|(u, v)\|^{2p}$$

and we can choose $2k > n(z+1)$, $2p > n(z+1)$ and $\tilde{\rho}$ small enough such that $\|(u, v)\| = \tilde{\rho}$ and $J(u, v) \geq \tilde{\sigma}$ for some $\tilde{\sigma}$ depending upon $\tilde{\rho}$.

Furthermore, again by using (m2), there exist constant c_i , $i = 1, 2, 3$ such that

$$M(\|(u, v)\|^n) \leq \begin{cases} \frac{c_1}{(r+1)} \|(u, v)\|^{n(r+1)} + c_2 \|(u, v)\|^n + c_3, & r \neq 1, \\ c_1 \ln(\|(u, v)\|^n) + c_2 \|(u, v)\|^n + c_3 & r = 1, \end{cases} \quad (5.4.4)$$

for $\|(u, v)\| \geq \tilde{t}$ where

$$c_3 = \begin{cases} M(\tilde{t}) - c_2 \tilde{t} - \frac{c_1}{(r+1)} \tilde{t}^{r+1}, & r \neq 1, \\ M(\tilde{t}) - c_2 \tilde{t} - c_1 \ln(\tilde{t}) & r = 1. \end{cases}$$

Let $(u_0, v_0) \in \mathcal{P}$ such that $u_0 \geq 0, v_0 \geq 0$ and $\|(u_0, v_0)\| = 1$. Then by using (f3), there exists $p_i \geq 0$, $i = 1, 2, 3$ and $K > \frac{n(r+1)}{2}$ such that $F(x, t, s) \geq p_1 |t|^K + p_2 |s|^K - p_3$ and

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, \xi u_0, \xi v_0)}{|x-y|^{\mu}} dy \right) F(x, \xi u_0, \xi v_0) dx \geq C_5 \xi^{2K} - C_6 \xi^K + C_7. \quad (5.4.5)$$

Finally by combining (5.4.4) and (5.4.5), we obtain $J(\xi u_0, \xi v_0) \rightarrow -\infty$ as $\xi \rightarrow \infty$. Hence there exists $(\tilde{u}, \tilde{v}) \in \mathcal{P}$ such that $J(\tilde{u}, \tilde{v}) < 0$ and $\|(\tilde{u}, \tilde{v})\| > \rho$. \square

Define the Mountain Pass critical level as

$$l^* = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)) \text{ where } \Gamma = \{\gamma \in C([0,1], \mathcal{P}) : \gamma(0) = 0, J(\gamma(1)) < 0\}$$

and then by using Ekeland principle and deformation lemma (Theorem 2.4.1), we have the existence of minimizing Palais-Smale sequence $(u_n, v_n) \in \mathcal{P}$ such that

$$J(u_n, v_n) \rightarrow l^*, J'(u_n, v_n) \rightarrow 0.$$

To analyze accurately the compactness of Palais Smale sequences for J , we show a series of Lemmas, starting with every Palais-Smale sequence is bounded in energy space.

Lemma 5.4.2. *Every Palais-Smale sequence is bounded in \mathcal{P} .*

Proof. Let (u_k, v_k) be a Palais-Smale sequence such that $J(u_k, v_k) \rightarrow c$ and $J'(u_k, v_k) \rightarrow 0$ as $k \rightarrow \infty$ for some $c \in \mathbb{R}$. Therefore we have:

$$\left| \frac{M(\|(u_k, v_k)\|^n)}{n} - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k, v_k)}{|x-y|^\mu} dy \right) F(x, u_k, v_k) dx \right| \rightarrow c \quad (5.4.6)$$

and

$$\left| m(\|(u_k, v_k)\|^n) \left(\int_{\Omega} |\nabla u_k|^{n-2} \nabla u_k \nabla \phi dx + \int_{\Omega} |\nabla v_k|^{n-2} \nabla v_k \nabla \psi dx \right) - \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k, v_k)}{|x-y|^\mu} dy \right) (f_1(x, u_k, v_k) \phi + f_2(x, u_k, v_k) \psi) dx \right| \leq \epsilon_k \|(\phi, \psi)\|. \quad (5.4.7)$$

Now by using (f3) and (m3), there exists $\eta > \frac{n}{2}, \theta \geq 2n$ such that

$$\eta F(x, t, s) \leq t f_1(x, t, s) \text{ and } \eta F(x, t, s) \leq s f_2(x, t, s) \text{ for all } (x, t, s) \in \Omega \times \mathbb{R}^2$$

and

$$\frac{1}{n} M(t) - \frac{1}{\theta} m(t) t \text{ in nonnegative and nondecreasing for } t \geq 0.$$

Then by taking $\phi = u_k$ and $\psi = v_k$ in (5.4.7) along with (m2) (for both degenerate and non-degenerate Kirchhoff terms) and above inequalities, we obtain

$$\begin{aligned}
 J(u_k, v_k) - \frac{\langle J'(u_k, v_k), (u_k, v_k) \rangle}{4\eta} &= \frac{M(\|(u_k, v_k)\|^n)}{n} - \frac{m(\|(u_k, v_k)\|^n)}{4\eta} \|(u_k, v_k)\|^n \\
 &+ \frac{1}{4\eta} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k, v_k)}{|x-y|^\mu} dy \right) (f_1(x, u_k, v_k)u_k + f_2(x, u_k, v_k)v_k - 2\eta F(x, u_k, v_k)) dx \\
 &\geq \frac{M(\|(u_k, v_k)\|^n)}{n} - \frac{m(\|(u_k, v_k)\|^n)}{4\eta} \|(u_k, v_k)\|^n \\
 &\geq \left(\frac{1}{2n} - \frac{1}{4\eta} \right) m(\|(u_k, v_k)\|^n) \|(u_k, v_k)\|^n \\
 &\geq \begin{cases} c_0 \left(\frac{1}{2n} - \frac{1}{4\eta} \right) \|(u_k, v_k)\|^n \\ \text{or} \\ \left(\frac{1}{2n} - \frac{1}{4\eta} \right) \|(u_k, v_k)\|^{n+z}. \end{cases}
 \end{aligned} \tag{5.4.8}$$

Also, from (5.4.6) and (5.4.7), we get for some constant $C > 0$

$$J(u_k, v_k) - \frac{\langle J'(u_k, v_k), (u_k, v_k) \rangle}{4\eta} \leq C \left(1 + \epsilon_k \frac{\|(u_k, v_k)\|}{4\eta} \right). \tag{5.4.9}$$

Therefore, by combining (5.4.8) and (5.4.9), we obtain $\{(u_k, v_k)\}$ is bounded in \mathcal{P} . \square

Lemma 5.4.3. *Let $\{(u_k, v_k)\}$ be a Palais-Smale sequence then up to a subsequence*

$$\left. \begin{aligned} |\nabla u_k|^{n-2} \nabla u_k &\rightharpoonup |\nabla u|^{n-2} \nabla u \\ |\nabla v_k|^{n-2} \nabla v_k &\rightharpoonup |\nabla v|^{n-2} \nabla v \end{aligned} \right\} \text{weakly in } W_0^{1,n}(\Omega).$$

Proof. From Lemma 5.4.2, we know that every Palais-Smale sequence is bounded in \mathcal{P} . So there exist $u, v \in W_0^{1,n}(\Omega)$ such that up to a subsequence

$$\begin{cases} u_k \rightharpoonup u, v_k \rightharpoonup v \text{ weakly in } W_0^{1,n}(\Omega). \\ u_k \rightarrow u, v_k \rightarrow v \text{ strongly in } L^q(\Omega) \forall q \geq 1 \text{ and a.e. in } \Omega. \end{cases}$$

Since $|u_k|^n + |\nabla u_k|^n$ and $|v_k|^n + |\nabla v_k|^n$ is bounded in $L^1(\Omega)$, so there exist two radon measures μ_1, μ_2 and two functions $u_1, v_1 \in (L^{\frac{n}{n-1}}(\Omega))^n$ such that upto a subsequence

$$|u_k|^n + |\nabla u_k|^n \rightarrow \mu_1 \text{ and } |v_k|^n + |\nabla v_k|^n \rightarrow \mu_2 \text{ in the sense of measure and}$$

$$|\nabla u_k|^{n-2} \nabla u_k \rightharpoonup u_1, |\nabla v_k|^{n-2} \nabla v_k \rightharpoonup v_1 \text{ weakly in } (L^{\frac{n}{n-1}}(\Omega))^n \text{ as } k \rightarrow \infty.$$

We set $\sigma_1, \sigma_2 > 0$ such that $\frac{2n}{2n-\mu}(\sigma_1 + \sigma_2)^{\frac{1}{n-1}} < \frac{\alpha_n}{2}$ and $X_{\sigma_i} = \{x \in \bar{\Omega} : \mu_i(B_r(x) \cap \bar{\Omega}) \geq \sigma_i\}$, for all $r > 0\}$ for $i = 1, 2$. Then X_{σ_i} must be finite sets. Now, by using the same arguments as in Lemma 5.1.12, it is sufficient to prove the following:

Claim: For any open and relatively compact subset K of $\bar{\Omega} \setminus (X_{\sigma_1} \cup X_{\sigma_2})$

$$\lim_{k \rightarrow \infty} \int_K \left(\int_{\Omega} \frac{F(y, u_k, v_k)}{|x-y|^\mu} dy \right) f_1(x, u_k, v_k) u_k \rightarrow \lim_{k \rightarrow \infty} \int_K \left(\int_{\Omega} \frac{F(y, u, v)}{|x-y|^\mu} dy \right) f_1(x, u, v) u \quad (5.4.10)$$

and

$$\lim_{k \rightarrow \infty} \int_K \left(\int_{\Omega} \frac{F(y, u_k, v_k)}{|x-y|^\mu} dy \right) f_2(x, u_k, v_k) v_k \rightarrow \lim_{k \rightarrow \infty} \int_K \left(\int_{\Omega} \frac{F(y, u, v)}{|x-y|^\mu} dy \right) f_2(x, u, v) v. \quad (5.4.11)$$

Let $x_0 \in K$ and $r_i > 0$ be such that $\mu_i(B_{r_i}(x_0) \cap \bar{\Omega}) < \sigma_i$ and consider $\psi_i \in C^\infty(\Omega)$ satisfying $0 \leq \psi_i \leq 1$ for $x \in \Omega$, $\psi_i = 1$ in $B_{\frac{r_i}{2}}(x_0) \cap \bar{\Omega}$ and $\psi_i = 0$ in $\bar{\Omega} \setminus B_{r_i}(x_0)$ for $i = 1, 2$. Then

$$\lim_{k \rightarrow \infty} \int_{B_{\frac{r_1}{2}}(x_0) \cap \Omega} |u_k|^n + |\nabla u_k|^n dx \leq \lim_{k \rightarrow \infty} \int_{B_{r_1}(x_0) \cap \Omega} (|u_k|^n + |\nabla u_k|^n) \psi_1 dx = \mu_1(B_{r_1}(x_0) \cap \bar{\Omega}) < \sigma_1$$

and

$$\lim_{k \rightarrow \infty} \int_{B_{\frac{r_2}{2}}(x_0) \cap \Omega} |v_k|^n + |\nabla v_k|^n dx \leq \lim_{k \rightarrow \infty} \int_{B_{r_2}(x_0) \cap \Omega} (|v_k|^n + |\nabla v_k|^n) \psi_2 dx = \mu_2(B_{r_2}(x_0) \cap \bar{\Omega}) < \sigma_2.$$

Then by choosing $k \in \mathbb{N}$ large enough and $r_0 := \min\{r_1, r_2\}$ we get

$$\|(u_k, v_k)\|_{\mathcal{Z}(B_{\frac{r_0}{2}}(x_0) \cap \Omega)}^n := \int_{B_{\frac{r_0}{2}}(x_0) \cap \Omega} (|u_k|^n + |\nabla u_k|^n + |v_k|^n + |\nabla v_k|^n) < (\sigma_1 + \sigma_2). \quad (5.4.12)$$

Now by using (5.4.12), Theorem 5.3.4 with $\lambda = 0$ and choosing $\epsilon > 0$ small enough and $q > 1$ such that $\frac{2nq}{2n-\mu}(1+\epsilon)(\sigma_1 + \sigma_2)^{\frac{1}{n-1}} \leq \frac{\alpha_n}{2}$ we get the following estimates for $i = 1, 2$

$$\begin{aligned} & \int_{B_{\frac{r_0}{2}}(x_0) \cap \Omega} |f_i(x, u_k, v_k)|^{\frac{2nq}{2n-\mu}} dx \\ &= \int_{B_{\frac{r_0}{2}}(x_0) \cap \Omega} |h_i(x, u_k, v_k)|^{\frac{2nq}{2n-\mu}} \exp\left(\frac{2nq}{2n-\mu} (|u_k|^{\frac{n}{n-1}} + |v_k|^{\frac{n}{n-1}})\right) dx \\ &\leq C_\epsilon \int_{B_{\frac{r_0}{2}}(x_0) \cap \Omega} \exp\left(\frac{2nq(1+\epsilon)}{2n-\mu} (|u_k|^{\frac{n}{n-1}} + |v_k|^{\frac{n}{n-1}})\right) dx \\ &\leq C_\epsilon \int_{B_{\frac{r_0}{2}}(x_0) \cap \Omega} \exp\left(\frac{2nq}{2n-\mu} (1+\epsilon)(\sigma_1 + \sigma_2)^{\frac{1}{n-1}} \left(\frac{|u_k|^{\frac{n}{n-1}} + |v_k|^{\frac{n}{n-1}}}{\|(u_k, v_k)\|_{\mathcal{Z}(B_{\frac{r_0}{2}}(x_0) \cap \Omega)}^{\frac{n-1}{n}}}\right)\right) dx \leq \tilde{C}_\epsilon \end{aligned} \quad (5.4.13)$$

for some constant $\tilde{C}_\epsilon > 0$. First we prove (5.4.10), a similar proof provides (5.4.11). Consider

$$\begin{aligned}
 & \int_{B_{\frac{r_0}{2}}(x_0) \cap \Omega} \left| \left(\int_{\Omega} \frac{F(y, u_k, v_k)}{|x-y|^\mu} dy \right) f_1(x, u_k, v_k) u_k - \left(\int_{\Omega} \frac{F(y, u, v)}{|x-y|^\mu} dy \right) f_1(x, u, v) u \right| dx \\
 & \leq \int_{B_{\frac{r_0}{2}}(x_0) \cap \Omega} \left| \left(\int_{\Omega} \frac{F(y, u, v)}{|x-y|^\mu} dy \right) (f_1(x, u_k, v_k) u_k - f_1(x, u, v) u) \right| dx \\
 & \quad + \int_{B_{\frac{r_0}{2}}(x_0) \cap \Omega} \left| \left(\int_{\Omega} \frac{F(y, u_k, v_k) - F(y, u, v)}{|x-y|^\mu} dy \right) f_1(x, u_k, v_k) u_k \right| dx \\
 & := I_1 + I_2 \text{ (say)}.
 \end{aligned}$$

From (5.4.2), (5.4.13), Hölder's inequality and asymptotic growth of f_i we obtain that families $\{f_1(x, u_k, v_k) u_k\}$ and $\{f_2(x, u_k, v_k) v_k\}$ are equi-integrable over $B_{\frac{r_0}{2}}(x_0) \cap \bar{\Omega}$ and $\mu \in (0, n)$ gives

$$\int_{\Omega} \frac{F(y, u, v)}{|x-y|^\mu} dy \in L^\infty(\Omega). \quad (5.4.14)$$

Then (5.4.14) and Vitali's convergence theorem combined with pointwise convergence of $f_1(x, u_k, v_k) u_k \rightarrow f_1(x, u, v) u$ implies $I_1 \rightarrow 0$. Now we show that $I_2 \rightarrow 0$ as $k \rightarrow \infty$. Then by using semigroup property of the Riesz potential (see [205]) and (5.4.13) we get that for some constant $C > 0$ independent of k

$$\begin{aligned}
 & \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k, v_k) - F(y, u, v)}{|x-y|^\mu} dy \right) \chi_{B_{\frac{r_0}{2}} \cap \bar{\Omega}}(x) f_1(x, u_k, v_k) u_k dx \\
 & \leq \left(\int_{\Omega} \left(\int_{\Omega} \frac{|F(y, u_k, v_k) - F(y, u, v)|}{|x-y|^\mu} dy \right) |F(x, u_k, v_k) - F(x, u, v)| dx \right)^{\frac{1}{2}} \\
 & \quad \times \left(\int_{\Omega} \left(\int_{\Omega} \chi_{B_{\frac{r_0}{2}} \cap \bar{\Omega}}(y) \frac{f_1(y, u_k, v_k) u_k}{|x-y|^\mu} dy \right) \chi_{B_{\frac{r_0}{2}} \cap \bar{\Omega}}(x) f_1(x, u_k, v_k) u_k dx \right)^{\frac{1}{2}} \\
 & \leq C \left(\int_{\Omega} \left(\int_{\Omega} \frac{|F(y, u_k, v_k) - F(y, u, v)|}{|x-y|^\mu} dy \right) |F(x, u_k, v_k) - F(x, u, v)| dx \right)^{\frac{1}{2}}.
 \end{aligned}$$

Now we claim that

$$\lim_{k \rightarrow \infty} \int_{\Omega} \left(\int_{\Omega} \frac{|F(y, u_k, v_k) - F(y, u, v)|}{|x-y|^\mu} dy \right) |F(x, u_k, v_k) - F(x, u, v)| dx = 0. \quad (5.4.15)$$

From (5.4.6) and (5.4.7) we get that there exists a constant $C_1, C_2 > 0$ (independent of k) such that

$$\begin{aligned}
 & \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k, v_k)}{|x-y|^\mu} dy \right) F(x, u_k, v_k) dx \leq C_1, \\
 & \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k, v_k)}{|x-y|^\mu} dy \right) (f_1(x, u_k, v_k) u_k + f_2(x, u_k, v_k) v_k) dx \leq C_2.
 \end{aligned} \quad (5.4.16)$$

We argue as along equation (3.19) in Lemma 5.1.12. Consider

$$\begin{aligned}
 & \int_{\Omega} \left(\int_{\Omega} \frac{|F(y, u_k, v_k) - F(y, u, v)|}{|x - y|^{\mu}} dy \right) |F(x, u_k, v_k) - F(x, u, v)| dx \leq \\
 & \int_{\Omega} \left(\int_{\Omega} \frac{|F(y, u_k, v_k)\chi_A(y) - F(y, u, v)\chi_B(y)|}{|x - y|^{\mu}} dy \right) |F(x, u_k, v_k)\chi_A(x) - F(x, u, v)\chi_B(x)| dx \\
 & + 2 \int_{\Omega} \left(\int_{\Omega} \frac{(F(y, u_k, v_k)\chi_A(y) + F(y, u, v)\chi_B(y) + F(y, u, v)\chi_D(y))}{|x - y|^{\mu}} dy \right) F(x, u_k, v_k)\chi_C(x) dx \\
 & + 2 \int_{\Omega} \left(\int_{\Omega} \frac{(F(y, u_k, v_k)\chi_A(y) + F(y, u, v)\chi_B(y))}{|x - y|^{\mu}} dy \right) F(x, u, v)\chi_D(x) dx \\
 & + \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k, v_k)\chi_C(y)}{|x - y|^{\mu}} dy \right) F(x, u_k, v_k)\chi_C(x) dx \\
 & + \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u, v)\chi_D(y)}{|x - y|^{\mu}} dy \right) F(x, u, v)\chi_D(x) dx := I_3 + I_4 + I_5 + I_6 + I_7.
 \end{aligned}$$

where for a fixed $M > 0$

$$A = \{x \in \Omega : |u_k| \leq M \text{ and } |v_k| \leq M\}, \quad B = \{x \in \Omega : |u| \leq M \text{ and } |v| \leq M\},$$

$$C = \{x \in \Omega : |u_k| \geq M \text{ or } |v_k| \geq M\} \quad \text{and} \quad D = \{x \in \Omega : |u| \geq M \text{ or } |v| \geq M\}.$$

Now using (5.4.16), (f4), semigroup property of the Riesz Potential we obtain $I_j = o(M)$ for $j = 4, \dots, 7$, when M is large enough and from Lebesgue dominated convergence theorem we obtain $I_3 \rightarrow 0$ as $k \rightarrow \infty$. Hence (5.4.15) holds and $I_2 \rightarrow 0$ as $k \rightarrow \infty$. Now to conclude (5.4.10) and (5.4.11), we repeat this procedure over a finite covering of balls using the fact that K is compact. Now the remaining proof can be done by using the same arguments as in Lemma 5.1.12. \square

Lemma 5.4.4. *Let $\{(u_k, v_k)\}$ be a Palais-Smale sequence for the energy functional J . Then there exists $(u, v) \in \mathcal{P}$ such that upto a subsequence*

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(x, u_k, v_k)}{|x - y|^{\mu}} dy \right) f_i(x, u_k, v_k) \phi dx \rightarrow \int_{\Omega} \left(\int_{\Omega} \frac{F(x, u, v)}{|x - y|^{\mu}} dy \right) f_i(x, u, v) \phi dx$$

for all $\phi \in C_c^{\infty}(\Omega)$ and $i = 1, 2$ and

$$\left(\int_{\Omega} \frac{F(x, u_k, v_k)}{|x - y|^{\mu}} dy \right) F(x, u_k, v_k) \rightarrow \left(\int_{\Omega} \frac{F(x, u, v)}{|x - y|^{\mu}} dy \right) F(x, u, v) dx \quad \text{in } L^1(\Omega).$$

The proof of the above Lemma follows from similar arguments as in Lemma 5.1.13 and Lemma 5.1.14.

Now we define the associated Nehari Manifold as

$$N = \{(u, v) \in (W_0^{1,n}(\Omega) \setminus \{0\})^2 : \langle J'(u, v), (u, v) \rangle = 0\}$$

and we show that the mountain pass critical lies below every local minimum value of the energy functional at the point of local minimum and to prove the existence of non-trivial solution, we prove the essential upper bound on the critical level depending upon α_n, n, μ .

Lemma 5.4.5. *Let $l^{**} = \inf_{u \in N} J(u)$. Assume (m3), (f3) and for some $q > 2$*

$$\lim_{t,s \rightarrow \infty} \frac{(f_1(x,t,s)t + f_2(x,t,s)s)F(x,t,s)}{\exp(q(|t|^{\frac{n}{n-1}} + |s|^{\frac{n}{n-1}}))} = \infty \text{ uniformly in } x \in \bar{\Omega} \quad (5.4.17)$$

holds then

$$l^* \leq l^{**} \text{ and } 0 < l^* < \frac{1}{n} M \left(\left(\left(\frac{2n-\mu}{2n} \right) \frac{\alpha_n}{2n} \right)^{n-1} \right).$$

Proof. Let $(u, v) \in N$ and $h : (0, \infty) \rightarrow \mathbb{R}$ such that $h(t) = J((tu, tv))$. Then

$$h'(t) = m(\|(tu, tv)\|^n) t^{n-1} \|(u, v)\|^n - \int_{\Omega} \left(\int_{\Omega} \frac{F(y, tu, tv)}{|x-y|^\mu} dy \right) (f_1(x, tu, tv)u + f_2(x, tu, tv)v) dx.$$

Since $(u, v) \in N$, we get

$$\begin{aligned} h'(t) &= h'(t) - t^{2n-1} \langle J'(u, v), (u, v) \rangle = t^{2n-1} \left(\frac{m(\|(tu, tv)\|^n)}{t^n \|(u, v)\|^n} - \frac{m(\|(u, v)\|^n)}{\|(u, v)\|^n} \right) \|(u, v)\|^{2n} \\ &\quad + t^{2n-1} \left[\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u, v)}{|x-y|^\mu} dy \right) (f_1(x, u, v)u + f_2(x, u, v)v) dx \right. \\ &\quad \left. - \int_{\Omega} \left(\int_{\Omega} \frac{F(y, tu, tv)}{t^{2n}|x-y|^\mu} dy \right) (f_1(x, tu, tv)tu + f_2(x, tu, tv)tv) dx \right]. \end{aligned}$$

Now (f3) implies, for any $(x, s) \in \Omega \times \mathbb{R}^+$, the map $r \mapsto r f_1(x, r, s) - nF(x, r, s)$ and for any $(x, r) \in \Omega \times \mathbb{R}^+$, the map $s \mapsto s f_2(x, r, s) - nF(x, r, s)$ is increasing on \mathbb{R}^+ . Using this we get $r f_1(x, r, s) - nF(x, r, s) \geq 0$ and $s f_2(x, r, s) - nF(x, r, s) \geq 0$ for all $(x, r, s) \in \Omega \times \mathbb{R}^2$ which implies

$$t \mapsto \frac{F(x, tu, tv)}{t^n} \text{ is non-decreasing for } t > 0.$$

Then for $0 < t \leq 1$, $x \in \Omega$ and by using (m3) and (f3), we obtain

$$\begin{aligned} h'(t) &\geq t^{2n-1} \left(\frac{m(\|(tu, tv)\|^n)}{t^n \|(u, v)\|^n} - \frac{m(\|(u, v)\|^n)}{\|(u, v)\|^n} \right) \|(u, v)\|^{2n} \\ &\quad + t^{2n-1} \left[\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u, v)}{|x-y|^\mu} dy \right) \left(\left(\frac{f_1(x, u, v)u}{u^n} - \frac{f_1(x, tu, tv)tu}{(tu)^n} \right) u^n(x) \right. \right. \\ &\quad \left. \left. + \left(\frac{f_2(x, u, v)v}{v^n} - \frac{f_2(x, tu, tv)tv}{(tv)^n} \right) v^n(x) \right) \right] \geq 0. \end{aligned}$$

This gives that $h'(t) \geq 0$ for $0 < t \leq 1$ and $h'(t) < 0$ for $t > 1$. Hence $J(u, v) = \max_{t \geq 0} J(tu, tv)$. Now we define $g : [0, 1] \rightarrow \mathcal{P}$ as $g(t) = (t_0 u, t_0 v)t$ where $t_0 > 1$ is such that $J((t_0 u, t_0 v)) < 0$. So, $g \in \Gamma$ which gives

$$l^* \leq \max_{t \in [0, 1]} J(g(t)) \leq \max_{t \geq 0} J(tu, tv) = J(u, v).$$

Since $u \in N$ is arbitrary, we get $l^* \leq l^{**}$. For $u, v \neq 0$, $J(tu, tv) \rightarrow -\infty$ as $t \rightarrow \infty$ (from Lemma 5.4.1) and by definition $l^* \leq \max_{t \in [0, 1]} J(tu, tv)$ for $(u, v) \in (W_0^{1,n}(\Omega) \setminus \{0\})^2$ satisfying

$J(u, v) < 0$. So, it is enough to prove that there exists a $(w_1, w_2) \in \mathcal{P}$ such that $\|(w_1, w_2)\| = 1$ and

$$\max_{t \in [0, \infty)} J(tw_1, tw_2) < \frac{1}{n} M \left(\left(\left(\frac{2n - \mu}{2n} \right) \frac{\alpha_n}{2n} \right)^{n-1} \right). \quad (5.4.18)$$

To prove this, we consider the sequence of functions $\{(U_k, V_k)\}$ as defined in the proof of Theorem 5.3.3 such that $\text{supp}(U_k), \text{supp}(V_k) \subset B_\rho(0)$ and $\|(U_k, V_k)\| = 1$ for all k . So we claim that there exists a $k \in \mathbb{N}$ such that (5.4.18) is satisfied for $w_1 = U_k$ and $w_2 = V_k$.

We proceed by contradiction, suppose this is not true then for all $k \in \mathbb{N}$ there exists a $t_k > 0$ such that (5.4.18) does not hold *i.e.*

$$\max_{t \in [0, \infty)} J(tU_k, tV_k) = J(t_k U_k, t_k V_k) \geq \frac{1}{n} M \left(\left(\left(\frac{2n - \mu}{2n} \right) \frac{\alpha_n}{2n} \right)^{n-1} \right).$$

Since $J((tU_k, tV_k)) \rightarrow -\infty$ as $t \rightarrow \infty$ uniformly in k therefore $\{t_k\}$ must be a bounded sequence in \mathbb{R} . Then from (5.4.1), $\|U_k, V_k\| = 1$ and monotonicity of M , we obtain

$$\left(\frac{2n - \mu}{2n} \right) \frac{\alpha_n}{2n} \leq t_k^{\frac{n}{n-1}}. \quad (5.4.19)$$

Since $\frac{d}{dt}(J((tU_k, tV_k)))|_{t=t_k} = 0$ and $\int_{B_{\rho/k}} \int_{B_{\rho/k}} \frac{dx dy}{|x-y|^\mu} \geq C_{\mu, n} \left(\frac{\rho}{k}\right)^{2n-\mu}$ then by using (5.4.17), for $k \in \mathbb{N}$ large enough we obtain

$$\begin{aligned} m(t_k^n) t_k^n &= \int_{\Omega} \left(\int_{\Omega} \frac{F(y, t_k U_k, t_k V_k)}{|x-y|^\mu} dy \right) (f_1(x, t_k U_k, t_k V_k) t_k U_k + f_2(x, t_k U_k, t_k V_k) t_k V_k) dx \\ &\geq \int_{B_{\rho/k}} \left(\int_{B_{\rho/k}} \frac{F(y, t_k U_k, t_k V_k)}{|x-y|^\mu} dy \right) (f_1(x, t_k U_k, t_k V_k) t_k U_k + f_2(x, t_k U_k, t_k V_k) t_k V_k) dx. \\ &\geq \exp \left(q \left(c_1^{\frac{n}{n-1}} + c_2^{\frac{n}{n-1}} \right) \left(\frac{t_k^{\frac{n}{n-1}} (\log k)}{\omega_{n-1}^{\frac{1}{n-1}}} \right) \right) \int_{B_{\rho/k}} \int_{B_{\rho/k}} \frac{dx dy}{|x-y|^\mu} \\ &\geq \tilde{C}_{\mu, n} k \left(\frac{q \left(c_1^{\frac{n}{n-1}} + c_2^{\frac{n}{n-1}} \right) t_k^{\frac{n}{n-1}}}{\omega_{n-1}^{\frac{1}{n-1}}} - (2n - \mu) \right). \end{aligned}$$

Hence by using the fact that $(c_1^{\frac{n}{n-1}} + c_2^{\frac{n}{n-1}}) = 2n$, t_k^n is bounded, $q > 2$ and (5.4.19), we arrive at a contradiction by taking k large enough. \square

Proof of Theorem 5.4.1: Let $\{(u_k, v_k)\}$ denotes a Palais Smale sequence at the mountain pass critical level l^* . Then by Lemma 5.4.2 there exists a $u_0, v_0 \in W_0^{1, n}(\Omega)$ such that up to a subsequence $u_k \rightharpoonup u_0, v_k \rightharpoonup v_0$ weakly in $W_0^{1, n}(\Omega)$ as $k \rightarrow \infty$. We prove our main result in several steps.

Step 1: Positivity of u_0, v_0 .

If $u_0 = v_0 \equiv 0$ (or either one of them) then using Lemma 5.4.4, we infer that

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k, v_k)}{|x-y|^\mu} dy \right) F(x, u_k, v_k) dx \rightarrow 0 \text{ as } k \rightarrow \infty$$

and which further gives that $\lim_{k \rightarrow \infty} J(u_k, v_k) = \frac{1}{n} \lim_{k \rightarrow \infty} M(\|(u_k, v_k)\|^n) = l^*$. Now in the light of Lemma 5.4.5 and monotonicity of M , we obtain

$$\frac{2n}{2n - \mu} \|(u_k, v_k)\|^{\frac{n}{n-1}} < \frac{\alpha_n}{2n}$$

for large enough k . Now, this implies that $\sup_k \int_{\Omega} f_i(x, u_k, v_k)^q dx < +\infty$ for some $q > \frac{2n}{2n-\mu}$, $i = 1, 2$. Along with (5.4.2), Theorem 5.3.1, the Hardy-Littlewood-Sobolev inequality and the Vitali's convergence theorem, we also obtain

$$\int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k, v_k)}{|x-y|^\mu} dy \right) (f_1(x, u_k, v_k)u_k + f_2(x, u_k, v_k)v_k) dx \rightarrow 0 \text{ as } k \rightarrow \infty.$$

Hence $\lim_{k \rightarrow \infty} \langle J'((u_k, v_k)), (u_k, v_k) \rangle = 0$ gives $\lim_{k \rightarrow \infty} m(\|(u_k, v_k)\|^n) \|(u_k, v_k)\|^n = 0$. Now from (m2), we obtain $\lim_{k \rightarrow \infty} \|(u_k, v_k)\|^n = 0$. Thus using Lemma 5.4.4, it must be that $\lim_{k \rightarrow \infty} J(u_k, v_k) = 0 = l^*$ which contradicts $l^* > 0$. Thus $u_0, v_0 \neq 0$ and there exists a constant $\Upsilon > 0$ such that up to a subsequence $\|u_k\|^n + \|v_k\|^n \rightarrow \Upsilon^n$ as $k \rightarrow \infty$. Then from Lemma 5.4.3 and Lemma 5.4.4, we get as $k \rightarrow \infty$,

$$\begin{aligned} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k, v_k)}{|x-y|^\mu} dy \right) (f_1(x, u_k, v_k)\varphi + f_2(x, u_k, v_k)\psi) dx \rightarrow \\ \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0, v_0)}{|x-y|^\mu} dy \right) (f_1(x, u_0, v_0)\varphi + f_2(x, u_0, v_0)\psi) dx \end{aligned}$$

and

$$\begin{aligned} m(\Upsilon^n) \int_{\Omega} (|\nabla u_0|^{n-2} \nabla u_0 \nabla \varphi + |\nabla v_0|^{n-2} \nabla v_0 \nabla \psi) dx \\ = \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0, v_0)}{|x-y|^\mu} dy \right) (f_1(x, u_0, v_0)\varphi + f_2(x, u_0, v_0)\psi) dx, \text{ for all } \varphi, \psi \in W_0^{1,n}(\Omega). \end{aligned} \tag{5.4.20}$$

In particular, taking $\varphi = u_0^-$ and $\psi = 0$ (similarly $\varphi = 0$ and $\psi = v_0^-$) in (5.4.20) we get $m(\Upsilon^n) \|u_0^-\| = 0$ (similarly $m(\Upsilon^n) \|v_0^-\| = 0$) and together with assumption (m2) implies $u_0^- = 0$ ($v_0^- = 0$) a.e. in Ω . Therefore $u_0, v_0 \geq 0$ a.e. in Ω .

From Theorem 5.3.1 and Hölder inequality we get,

$$\left(\int_{\Omega} \frac{F(y, u_0, v_0)}{|x-y|^\mu} dy \right) (f_1(x, u_0, v_0) + f_2(x, u_0, v_0)) dx \in L^q(\Omega)$$

for $1 \leq q < \infty$. By elliptic regularity results and strong maximum principle, we finally get that $u_0, v_0 > 0$ in Ω .

Step 2: $m(\|u_0, v_0\|^n) \|(u_0, v_0)\|^n \geq \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0, v_0)}{|x-y|^\mu} dy \right) (f_1(x, u_0, v_0)u_0 + f_2(x, u_0, v_0)v_0) dx$.
Suppose by contradiction

$$m(\|u_0, v_0\|^n) \|(u_0, v_0)\|^n < \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0, v_0)}{|x-y|^\mu} dy \right) (f_1(x, u_0, v_0)u_0 + f_2(x, u_0, v_0)v_0) dx$$

which implies that $\langle J'(u_0, v_0), (u_0, v_0) \rangle < 0$. For $t > 0$ small enough, using (f3) and (f5) we have that

$$\begin{aligned} \langle J'(tu_0, tv_0), (u_0, v_0) \rangle &\geq m_0 t^{n-1} \|u_0, v_0\|^n \\ &- \frac{1}{2n} \int_{\Omega} \left(\int_{\Omega} \frac{f_1(y, tu_0, tv_0)tu_0 + f_2(x, tu_0, tv_0)tv_0}{|x-y|^\mu} dy \right) (f_1(x, tu_0, tv_0)u_0 + f_2(x, u_0, v_0)v_0) dx \\ &\geq m_0 t^{n-1} \|u_0, v_0\|^n - \frac{t^{2\gamma+1}}{2n} \int_{\Omega} \left(\int_{\Omega} \frac{((u_0^\gamma + v_0^\gamma)u_0 + (u_0^\gamma + v_0^\gamma)v_0)}{|x-y|^\mu} dy \right) ((u_0^\gamma + v_0^\gamma)u_0 + (u_0^\gamma + v_0^\gamma)v_0) dx \\ &\geq 0. \end{aligned}$$

Thus there exists a $t_* \in (0, 1)$ such that $\langle J'(t_*u_0, t_*v_0), (u_0, v_0) \rangle = 0$ i.e. $(t_*u_0, t_*v_0) \in N$. So using Lemma 5.4.5, (m3) and (f3) we get

$$\begin{aligned} l^* \leq l^{**} &\leq J((t_*u_0, t_*v_0)) = J(t_*u_0, t_*v_0) - \frac{1}{2n} \langle J'(t_*u_0, t_*v_0), (u_0, v_0) \rangle \\ &= \frac{M(\|t_*u_0, t_*v_0\|^n)}{n} - \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, t_*u_0, t_*v_0)}{|x-y|^\mu} dy \right) F(x, t_*u_0, t_*v_0) dx \\ &\quad - \frac{1}{2n} m(\|t_*u_0, t_*v_0\|^n) \|t_*u_0, t_*v_0\|^n \\ &\quad + \frac{1}{2n} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, t_*u_0, t_*v_0)}{|x-y|^\mu} dy \right) (f_1(x, t_*u_0, t_*v_0)t_*u_0 + f_2(x, t_*u_0, t_*v_0)dx \\ &< \frac{M(\|u_0, v_0\|^n)}{n} - \frac{1}{2n} m(\|(u_0, v_0)\|^n) \|(u_0, v_0)\|^n \\ &\quad + \frac{1}{2n} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, t_*u_0, t_*v_0)}{|x-y|^\mu} dy \right) (f_1(x, t_*u_0, t_*v_0)t_*u_0 + f_2(x, t_*u_0, t_*v_0) - nF(x, t_*u_0, t_*v_0)) dx \\ &\leq \frac{M(\|u_0, v_0\|^n)}{n} - \frac{1}{2n} m(\|(u_0, v_0)\|^n) \|u_0, v_0\|^n \\ &\quad + \frac{1}{2n} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0, v_0)}{|x-y|^\mu} dy \right) (f_1(x, u_0, v_0)u_0 + f_2(x, u_0, v_0) - nF(x, u_0, v_0)) dx \\ &\leq \liminf_{k \rightarrow \infty} \left(\frac{M(\|u_k, v_k\|^n)}{n} - \frac{1}{2n} m(\|(u_k, v_k)\|^n) \|(u_k, v_k)\|^n \right. \\ &\quad \left. + \frac{1}{2n} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k, v_k)}{|x-y|^\mu} dy \right) (f_1(x, u_k, v_k)u_k + f_2(x, u_k, v_k) - nF(x, u_k, v_k)) dx \right) \\ &= \liminf_{k \rightarrow \infty} \left(J(u_k, v_k) - \frac{1}{2n} \langle J'(u_k, v_k), (u_k, v_k) \rangle \right) = l^*. \end{aligned}$$

This gives a contradiction and completes the proof of **Step 2**. Similar arguments follows for the degenerate case also using (m3).

Step 3: $J(u_0, v_0) = l^*$.

Using the weakly lower semicontinuity of norms in $\lim_{k \rightarrow \infty} J(u_k, v_k) = l^*$ and Lemma 5.4.4 we obtain $J(u_0, v_0) \leq l^*$. If $J(u_0, v_0) < l^*$ then it must be $\lim_{k \rightarrow \infty} M(\|u_k, v_k\|^n) > M(\|u_0, v_0\|^n)$. Then continuity and monotonicity of M implies $\Upsilon^n > \|u_0, v_0\|^n$ and

$$M(\Upsilon^n) = n \left(l^* + \frac{1}{2} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0, v_0)}{|x-y|^\mu} dy \right) F(x, u_0, v_0) dx \right). \quad (5.4.21)$$

Define the sequence of functions

$$(\tilde{u}_k, \tilde{v}_k) = \left(\frac{u_k}{\|u_k, v_k\|}, \frac{v_k}{\|u_k, v_k\|} \right)$$

such that $\|\tilde{u}_k, \tilde{v}_k\| = 1$ and $\tilde{u}_k, \tilde{v}_k \rightharpoonup (\tilde{u}_0, \tilde{v}_0) = \left(\frac{u_0}{\Upsilon}, \frac{v_0}{\Upsilon}\right)$ weakly in \mathcal{P} and $\|u_0, v_0\| < \Upsilon$. From Theorem 5.3.2, we have that

$$\sup_{k \in \mathbb{N}} \int_{\Omega} \exp \left(p(|\tilde{u}_k|^{\frac{n}{n-1}} + |\tilde{v}_k|^{\frac{n}{n-1}}) \right) dx < +\infty, \text{ for } 1 < p < \frac{\alpha_n}{2_n(1 - \|\tilde{u}_0, \tilde{v}_0\|^n)^{\frac{1}{n-1}}}. \quad (5.4.22)$$

Then from (m3), Claim (1) and Lemma 5.4.5 we obtain

$$\begin{aligned} J(u_0, v_0) &= \frac{M(\|u_0, v_0\|^n)}{n} - \frac{m(\|u_0, v_0\|^n)\|u_0, v_0\|^n}{2n} \\ &\quad + \frac{1}{2n} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0, v_0)}{|x-y|^\mu} dy \right) (f_1(x, u_0, v_0)u_0 + f_2(x, u_0, v_0)v_0 - nF(x, u_0, v_0)) dx \geq 0. \end{aligned}$$

and from (5.4.21) we get

$$M(\Upsilon^n) = nl^* - nJ(u_0, v_0) + M(\|u_0, v_0\|^n) < M \left(\left(\left(\frac{2n-\mu}{2n} \right) \frac{\alpha_n}{2_n} \right)^{n-1} \right) + M(\|u_0, v_0\|^n)$$

which further implies together with (m1) that

$$\Upsilon^n < \frac{1}{1 - \|\tilde{u}_0, \tilde{v}_0\|^n} \left(\left(\frac{2n-\mu}{2n} \right) \frac{\alpha_n}{2_n} \right)^{n-1}.$$

Thus for $k \in \mathbb{N}$ large enough it is possible $b > 1$ but close to 1 such that

$$\frac{2n}{2n-\mu} b \|u_k, v_k\|^{\frac{n}{n-1}} \leq \frac{\alpha_n}{2_n(1 - \|\tilde{u}_0, \tilde{v}_0\|^n)^{\frac{1}{n-1}}}.$$

Therefore from (5.4.22) we conclude that

$$\int_{\Omega} \exp \left(\frac{2n}{2n-\mu} b (|u_k|^{\frac{n}{n-1}} + |v_k|^{\frac{n}{n-1}}) \right) \leq C$$

and

$$\begin{aligned} \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_k, v_k)}{|x-y|^\mu} dy \right) (f_1(x, u_k, v_k)u_k + f_2(x, u_k, v_k)v_k) dx \rightarrow \\ \int_{\Omega} \left(\int_{\Omega} \frac{F(y, u_0, v_0)}{|x-y|^\mu} dy \right) (f_1(x, u_0, v_0)u_0 + f_2(x, u_0, v_0)v_0) dx. \end{aligned}$$

This implies $(u_k, v_k) \rightarrow (u_0, v_0)$ strongly in \mathcal{P} and hence $J(u_0, v_0) = l^*$ which is a contradiction. Hence, $J(u_0, v_0) = l^* = \lim_{k \rightarrow \infty} J(u_k, v_k)$ and $\|(u_k, v_k)\| \rightarrow \Upsilon$ implies $\Upsilon = \|(u_0, v_0)\|$. Then finally we have

$$\begin{aligned} m(\|u_0, v_0\|^n) \left(\int_{\Omega} |\nabla u_0|^{n-2} \nabla u_0 \nabla \phi dx + \int_{\Omega} |\nabla v_0|^{n-2} \nabla v_0 \nabla \psi dx \right) \\ = \int_{\Omega} \left(\int_{\Omega} \frac{F(x, u_0, v_0)}{|x-y|^\mu} dy \right) (f_1(x, u_0, v_0)\phi + f_2(x, u_0, v_0)\psi) dx \end{aligned}$$

for all $\phi, \psi \in W_0^{1,n}(\Omega)$. This completes the proof. \square

5.5 Extensions and related problems

The results of this chapter can be extended in various directions. Let us mention here some obvious generalizations:

1: The class of system (KCS) can be extended to the following fractional Kirchhoff-Choquard system involving singular weights:

$$(F) \begin{cases} -m \left(\|(u, v)\|_{\dot{\mathcal{L}}^{\frac{n}{s}}} \right) \Delta_{n/s}^s u = \left(\int_{\Omega} \frac{F(y, u, v)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f_1(x, u, v)}{|x|^\alpha} & \text{in } \Omega, \\ -m \left(\|(u, v)\|_{\dot{\mathcal{L}}^{\frac{n}{s}}} \right) \Delta_{n/s}^s v = \left(\int_{\Omega} \frac{F(y, u, v)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f_2(x, u, v)}{|x|^\alpha} & \text{in } \Omega, \\ u, v = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $(-\Delta)_{n/s}^s$ is the n/s fractional Laplace operator, $s \in (0, 1)$, $n \geq 1$, $\mu \in (0, n)$, $0 < \alpha < \min\{\frac{n}{2}, n - \mu\}$, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $m : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $F : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continuous functions where F behaves like $\exp(|u|^{\frac{n}{n-s}} + |v|^{\frac{n}{n-s}})$ as $|u|, |v| \rightarrow \infty$.

We conjecture that the following Moser-Trudinger inequality holds in case fractional Sobolev space (counterpart of Theorem 5.3.1): Define $\mathcal{L} := X_0 \times X_0$ endowed with the norm

$$\|(u, v)\|_{\mathcal{L}} := \left(\|u\|_{X_0}^{n/s} + \|v\|_{X_0}^{n/s} \right)^{\frac{s}{n}}$$

where

$$X_0 := \{u \in W^{s, n/s}(\mathbb{R}^n) : u = 0 \text{ in } \mathbb{R}^n \setminus \Omega\}$$

endowed with the norm

$$\|u\|_{X_0} = \left(\int_{\mathbb{R}^{2n} \setminus (\Omega^c \times \Omega^c)} \frac{|u(x) - u(y)|^{\frac{n}{s}}}{|x - y|^{2n}} dx dy \right)^{s/n}$$

Theorem 5.5.1. For $(u, v) \in \mathcal{L}$, $n/s > 2$ and $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, we have

$$\int_{\Omega} \exp \left(\Pi \left(|u|^{\frac{n}{n-s}} + |v|^{\frac{n}{n-s}} \right) \right) dx < \infty$$

for any $\Pi > 0$. Moreover,

$$\sup_{\|(u, v)\|_{\mathcal{L}}=1} \int_{\Omega} \exp \left(\Pi \left(|u|^{\frac{n}{n-s}} + |v|^{\frac{n}{n-s}} \right) \right) dx < \infty, \text{ provided } \Pi \leq \frac{\alpha_{n,s}}{2_{n,s}} \quad (5.5.1)$$

where $\alpha_{n,s} = \frac{n}{\omega_{n-1}} \left(\frac{\Gamma(\frac{n-s}{2})}{\Gamma(\frac{s}{2}) 2^s \pi^{n/2}} \right)^{\frac{-n}{n-s}}$, $2_{n,s} = 2^{\frac{n-2s}{n-s}}$. Furthermore if $\Pi > \frac{\alpha_{n,s}^*}{2_{n,s}}$, then there exists a pair $(u, v) \in \mathcal{L}$ with $\|(u, v)\|_{\mathcal{L}} = 1$ such that the supremum in (5.5.1) is infinite.

Using Theorem 5.5.1, doubly weighted Hardy-Littlewood-Sobolev inequality, we can prove the existence and multiplicity of solutions for the problem (F).

2: We infer that similiar methods can be used to the following Kirchhoff-Choquard system for the Polyharmonic operator:

$$(P) \left\{ \begin{array}{l} -M \left(\int_{\Omega} (|\nabla^m u|^2 + |\nabla^m v|^2) dx \right) \Delta^m u = \left(\int_{\Omega} \frac{F(y, u, v)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f_1(x, u, v)}{|x|^\alpha}, \quad u > 0 \text{ in } \Omega, \\ -M \left(\int_{\Omega} (|\nabla^m u|^2 + |\nabla^m v|^2) dx \right) \Delta^m v = \left(\int_{\Omega} \frac{F(y, u, v)}{|y|^\alpha |x-y|^\mu} dy \right) \frac{f_2(x, u, v)}{|x|^\alpha}, \quad v > 0 \text{ in } \Omega, \\ \quad \quad \quad u = \nabla u = \dots = \nabla^{m-1} u = 0 \quad \quad \quad \text{on } \partial\Omega, \\ \quad \quad \quad v = \nabla v = \dots = \nabla^{m-1} v = 0 \quad \quad \quad \text{on } \partial\Omega, \end{array} \right.$$

where $n = 2m$, $\mu \in (0, n)$, $0 < \alpha < \min\{\frac{n}{2}, n - \mu\}$, $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, $M : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $F : \Omega \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is a continous functions where F behaves like $\exp(|u|^{\frac{n}{n-m}} + |v|^{\frac{n}{n-m}})$ as $|u|, |v| \rightarrow \infty$. Using Theorems 5.3.1, 5.3.2 and extension of Theorems 5.3.3 and 5.3.4 (which is an open question), we can study the system of Kirchhoff-Choquard equation for the Polyharmonic operator.

3: Another important open question is the Adams-Moser-Trundinger inequalities in Cartesian product of Sobolev space with unbounded domain (or in \mathbb{R}^n).

Non-local singular problem

This is a joint work with Jacques Giacomoni, Divya Goel, Konijeti Sreenadh and Guillaume Warnault.

Abstract : In this chapter, we study the non-local singular problems in the presence of exponential non-linearities and singular weights. The investigation of singular problems are divided into two parts depending upon the nature of the operator.

In the first part, we study the singular problems involving fractional Laplacian operator, precisely $\frac{1}{2}$ -Laplacian operator and exponential non-linearity. We establish the existence, multiplicity, regularity and asymptotic behavior of the positive solution in one dimension. We prove two results regarding the existence and multiplicity of solutions to the problem (P_λ) (see below). In the first result, existence and multiplicity (local) have been proved for classical solutions via bifurcation theory while in the latter global multiplicity result has been proved for critical exponential non-linearity by variational methods. An independent question of symmetry and monotonicity properties of classical solution has been answered using moving plane method and narrow maximum principle for $\frac{1}{2}$ -Laplacian operator and then extend it in the more general framework of $(-\Delta)^s$ operator and for all $0 < s < 1$. To characterize the behavior of large solutions, we further study isolated singularities for the singular semi linear elliptic equation in $\Omega \subset \mathbb{R}^N$ involving exponential growth non-linearity.

In the second part, we investigate the existence, uniqueness, nonexistence, and optimal Sobolev and Hölder regularity of weak solution to the nonlinear fractional elliptic problem involving singular nonlinearity and singular weights in smooth bounded domain $\Omega \subset \mathbb{R}^N$ (see (P) below). We prove the existence of weak solution in $W_{loc}^{s,p}(\Omega)$ via approximation method and as an application of new comparison principle, we prove the uniqueness of weak solution for $0 \leq \delta < 1 + s - \frac{1}{p}$ and also nonexistence of weak solution for $\delta \geq sp$. Moreover, by virtue of barrier arguments we study the behavior of minimal weak solution in terms of $\text{dist}(x, \partial\Omega)$

function. Consequently, we prove Hölder regularity up to the boundary for minimal weak solution.

6.1 Non-local singular problems with exponential non-linearities

In this section, we answer the questions of existence, local multiplicity and regularity of classical solution to the following problem:

$$(P_\lambda) \quad \begin{cases} (-\Delta)^{\frac{1}{2}}u = \lambda \left(\frac{1}{u^\delta} + f(u) \right), u > 0 & \text{in } (-1, 1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1, 1) \end{cases}$$

where $f(u) = h(u)e^{u^\alpha}$, $1 \leq \alpha \leq 2$, $\delta > 0$, $\lambda \geq 0$ and $h(t)$ is assumed to be a smooth perturbation of e^{t^α} as $t \rightarrow \infty$. We remark that in contrast to higher dimensions, there is no restriction on δ is required in dimension one.

6.1.1 Function spaces and main results

The fractional Laplacian $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(x) = 2 \text{ P.V. } \mathcal{C}_s \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}} dy,$$

where P.V. denotes the Cauchy principal value, $s \in (0, 1)$, $N \geq 2s$ and $\mathcal{C}_s = \pi^{-\frac{N}{2}} 2^{2s-1} s^{\frac{\Gamma(\frac{N+2s}{2})}{\Gamma(1-s)}}$, Γ being the Gamma function. When $N = 1$, $s = \frac{1}{2}$, $\mathcal{C}_s = \frac{1}{2\pi}$.

Before stating the results and outline of the main proofs, let us recall some definitions of function spaces from the work of [54] and define the notion of (very) weak solutions. Define

$$X := \left\{ u : \mathbb{R} \rightarrow \mathbb{R} \mid \text{measurable, } u|_{(-1,1)} \in L^2((-1,1)) \text{ and } \frac{(u(x) - u(y))}{|x - y|} \in L^2(Q) \right\}$$

where $Q = \mathbb{R}^2 \setminus (-1, 1)^c \times (-1, 1)^c$ and $(-1, 1)^c = \mathbb{R} \setminus (-1, 1)$ endowed with the norm

$$\|u\|_X = \|u\|_{L^2((-1,1))} + \mathcal{C}_s \left(\int_Q \frac{|u(x) - u(y)|^2}{|x - y|^2} dx dy \right)^{\frac{1}{2}}.$$

Define the Hilbert space X_0 as

$$X_0 := \{u \in X : u = 0 \text{ in } \mathbb{R} \setminus (-1, 1)\}$$

equipped with the inner product

$$\langle u, v \rangle = \mathcal{C}_s \int_Q \frac{(u(x) - u(y))(v(x) - v(y))}{|x - y|^2} dx dy$$

As in [46] we have the following definition of weak solutions to problem (P_λ) .

Definition 6.1.1. A function $u \in L^1(\mathbb{R})$ with $u \equiv 0$ on $\mathbb{R} \setminus (-1, 1)$ is said to be a weak solution of (P_λ) if $\inf_K u > 0$ for any compact set $K \subset (-1, 1)$ and for any $\phi \in \sigma$,

$$\int_{-1}^1 u (-\Delta)^{\frac{1}{2}} \phi = C_s \int_Q \frac{(u(x) - u(y))(\phi(x) - \phi(y))}{|x - y|^2} dx dy = \lambda \int_{-1}^1 \left(\frac{1}{u^\delta} + h(u)e^{u^\alpha} \right) \phi dx. \tag{6.1.1}$$

where

$$\sigma = \{ \psi \mid \psi : \mathbb{R} \rightarrow \mathbb{R}, \text{ measurable, } (-\Delta)^{\frac{1}{2}} \psi \in L^\infty((-1, 1)) \text{ and } \phi \text{ has compact support in } (-1, 1) \}.$$

Using the regularity theory of fractional Laplacian we define the set of classical solutions of (P_λ) :

Definition 6.1.2. The set of classical solutions to (P_λ) is defined as

$$\mathcal{S} = \{ (\lambda, u) \in \mathbb{R}^+ \times C_0([-1, 1]) : u \text{ is a weak solution to } (P_\lambda) \text{ in } X_0 \}.$$

Remark 6.1.3. Regularity of a classical solution u (proved later in Theorem 6.1.9) for the problem (P_λ) implies $u \in C_{\phi_\delta}^+((-1, 1))$ (defined below). Then by using Hardy's inequality (see [162, Corollary 1.4.4.10, p.23]) in (6.1.1) together with the fact that $C_c^\infty((-1, 1))$ is dense in X_0 , we obtain that $\frac{1}{u^\delta}$ belongs to dual space of X_0 for all $\delta > 0$ and hence (6.1.1) holds for all $\phi \in X_0$.

Definition 6.1.4. For $\phi \in C_0([-1, 1])$ with $\phi > 0$ in $(-1, 1)$, the set $C_\phi((-1, 1))$ is defined as

$$C_\phi((-1, 1)) = \{ u \in C_0([-1, 1]) : \text{there exists } c \geq 0 \text{ such that } |u(x)| \leq c\phi(x), \text{ for all } x \in (-1, 1) \},$$

endowed with the natural norm $\left\| \frac{u}{\phi} \right\|_{L^\infty((-1, 1))}$.

Definition 6.1.5. The positive cone of $C_\phi((-1, 1))$ is the open convex subset of $C_\phi((-1, 1))$ defined as

$$C_\phi^+((-1, 1)) = \left\{ u \in C_\phi((-1, 1)) : \inf_{x \in (-1, 1)} \frac{u(x)}{\phi(x)} > 0 \right\}.$$

To analyze the existence and regularity of solutions of (P_λ) , the key ingredient is to study the boundary behavior of the weak solution of the following problem:

$$(P) \begin{cases} (-\Delta)^{\frac{1}{2}} u = \frac{1}{d(x)^\alpha \log^\beta \left(\frac{A}{d(x)} \right)} & \text{in } (-1, 1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1, 1). \end{cases} \tag{6.1.2}$$

For the operator $(-\Delta)^s$ with $N > 2s$, Abatangelo [1] studied the boundary behavior of the corresponding problem like (6.1.2) with $\beta = 0$ and $0 < \alpha < 1 + s$. The case $N = 1$ and $s = \frac{1}{2}$

has been left open as the Green function for the Half-Laplacian in one dimension is different (see [84]) from that of $(-\Delta)^s$ with $N > 2s$. In this regard, we explore this case and prove the following theorem.

Theorem 6.1.6. *Let A be a positive constant such that $A \geq 2$. Then the weak solution of (6.1.2) satisfies*

$$\begin{aligned} c_1 d(x)^{\frac{1}{2}} \leq u(x) \leq c_2 d(x)^{\frac{1}{2}} & \quad \text{for } 0 < \alpha < \frac{1}{2} \text{ and } \beta = 0, \\ c_3 d(x)^{1-\alpha} \leq u(x) \leq c_4 d(x)^{1-\alpha} & \quad \text{for } \frac{1}{2} < \alpha < \frac{3}{2} \text{ and } \beta = 0, \\ c_5 d(x)^{\frac{1}{2}} \log^{1-\beta} \left(\frac{A}{d(x)} \right) \leq u(x) \leq c_6 d(x)^{\frac{1}{2}} \log^{1-\beta} \left(\frac{A}{d(x)} \right) & \quad \text{for } \alpha = \frac{1}{2} \text{ and } 0 \leq \beta < 1. \end{aligned} \quad (6.1.3)$$

where $c_i, i = 1, 2, \dots, 5$ are constants.

To get appropriate sub and supersolutions of the problem (P_λ) , we now turn our attention to the following pure singular problem (P_δ) .

$$(P_\delta) \begin{cases} (-\Delta)^{\frac{1}{2}} u = \frac{1}{u^\delta}, & u > 0, & \text{in } (-1, 1), \\ u = 0 & & \text{in } \mathbb{R} \setminus (-1, 1). \end{cases}$$

The barrier function ϕ_δ is defined as follows:

$$\phi_\delta = \begin{cases} \phi_1 & \text{if } 0 < \delta < 1, \\ \phi_1 \left(\log \left(\frac{2}{\phi_1} \right) \right)^{\frac{1}{2}} & \text{if } \delta = 1, \\ \phi_1^{\frac{2}{\delta+1}} & \text{if } \delta > 1, \end{cases} \quad (6.1.4)$$

where ϕ_1 is the normalized ($\|\phi_1\|_{L^\infty(\Omega)} = 1$) eigen function corresponding to the smallest eigen value of $(-\Delta)^{\frac{1}{2}}$ on X_0 . We recall that $\phi_1 \in C^{\frac{1}{2}}(\mathbb{R})$ and $\phi_1 \in C^+_{\frac{1}{d^{\frac{1}{2}}}}((-1, 1))$ (See Proposition 1.1 and Theorem 1.2 of [226]). For the problem (P_δ) , we are concerned about the existence, the asymptotic behavior and the regularity of the solution. In this regard we have the following result:

Theorem 6.1.7. (i) *For all $\delta > 0$, there exists a unique $u \in C_0([-1, 1])$ classical solution of (P_δ) . Moreover, $u \in X_0 \cap C^+_{\phi_\delta}((-1, 1))$ where ϕ_δ is defined in (6.1.4).*

(ii) *The classical solution u to (P_δ) belongs to $C^\gamma(\mathbb{R})$ with*

$$\gamma = \begin{cases} \frac{1}{2} & \text{if } \delta < 1, \\ \frac{1}{2} - \epsilon & \text{if } \delta = 1, \text{ for all } \epsilon > 0 \text{ small enough,} \\ \frac{1}{\delta+1} & \text{if } \delta > 1. \end{cases} \quad (6.1.5)$$

Now we will state some assumptions on the function h :

(H1) $h : [0, \infty) \rightarrow \mathbb{R}$ is a positive function of class C^2 in $(0, \infty)$ with $h(0) = 0$ and such that the map $t \rightarrow t^{-\delta} + h(t)e^{t^\alpha}$ is convex for all $t > 0$.

(H2) For any $\epsilon > 0$, $\lim_{t \rightarrow \infty} h(t)e^{-\epsilon t^\alpha} = 0$ and $\lim_{t \rightarrow \infty} h(t)e^{\epsilon t^\alpha} = \infty$.

First, we recall the definition of an asymptotic bifurcation point and then state the result regarding existence of a global branch of classical solutions to (P_λ) for $1 < \alpha \leq 2$.

Definition 6.1.8. A point $\Lambda_a \in [0, \infty)$ is said to be an asymptotic bifurcation point, if there exists a sequence $(\lambda_n, u_n) \in S$ such that $\lambda_n \rightarrow \Lambda_a$ and $\|u_n\|_{L^\infty((-1,1))} \rightarrow \infty$ as $n \rightarrow \infty$.

To study the existence, multiplicity of solutions to (P_λ) , we seek assistance of global bifurcation theory due to P. H. Rabinowitz [221] and proved the following result.

Theorem 6.1.9. Let h satisfy the hypothesis (H1) and (H2) and $\delta > 0$. Then the following holds:

- (i) There exists $\Lambda \in (0, +\infty)$ and $\gamma > 0$ such that $\mathcal{S} \subset [0, \Lambda] \times \left(X_0 \cap C_{\phi_\delta}^+((-1, 1)) \cap C^\gamma(\mathbb{R}) \right)$, where γ is defined in Theorem 6.1.7 and ϕ_δ is defined in (6.1.4).
- (ii) There exists a connected unbounded branch \mathcal{C} of solutions to (P_λ) in $\mathbb{R}^+ \times C_0([-1, 1])$, emanating from $(0, 0)$ such that for any $\lambda \in (0, \Lambda)$, there exists $(\lambda, u_\lambda) \in \mathcal{C}$ with u_λ being minimal solution to (P_λ) . Furthermore, as $\lambda \rightarrow \Lambda^-$, $u_\lambda \rightarrow u_\Lambda$ in X_0 , where u_Λ is a classical solution to (P_Λ) .
- (iii) The curve $(0, \Lambda) \ni \lambda \rightarrow u_\lambda \in C_0([-1, 1])$ is of class C^2 .
- (iv) (Bending and local multiplicity near $\lambda = \Lambda$) $\lambda = \Lambda$ is a bifurcation point, that is, there exists a unique C^2 -curve $(\lambda(s), u(s)) \in \mathcal{C}$, where the parameter s varies in an open interval about the origin in \mathbb{R} , such that

$$\lambda(0) = \Lambda, u(0) = u_\Lambda, \lambda'(0) = 0, \lambda''(0) < 0.$$

- (v) (Asymptotic bifurcation point) \mathcal{C} admits an asymptotic bifurcation point Λ_a satisfying $0 \leq \Lambda_a \leq \Lambda$.

Now we study the qualitative properties of solutions for the problem (P_λ) . In light of the maximum principle (see [175]) and the moving plane method, we derive the radial symmetry and monotonicity properties of the weak solutions with respect to $|x|$. More precisely, we prove the following result:

Theorem 6.1.10. For $1 \leq \alpha \leq 2$, $\delta > 0$, let h satisfies (H1) – (H2), f is Lipschitz function. Then every positive solution $(\lambda, u) \in S$ of (P_λ) is symmetric and strictly decreasing in $|x|$ i.e. $u(x) > u(y)$ for all $|x| < |y|$ and $x, y \in (-1, 1)$.

To prove Theorem 6.1.10, we have used tools of maximum principle proved in [175]. In particular we have used Lemma 3.2, Proposition 3.5 and a combination of both gives rise to a small volume maximum principle Proposition 3.6 of [175].

Assertion (v) in Theorem 6.1.9 conclude that the connected branch admits at least one asymptotic bifurcation point. To characterize the blow up behavior at $\lambda = \Lambda_a$ we study the behavior of isolated singularities as in Brezis-Lions problem (see [70]) for the fractional Laplacian operator.

We consider the following problem:

$$(P_s) \begin{cases} (-\Delta)^s u = g(u), u \geq 0 & \text{in } \Omega', \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ u \in L^1(\Omega), g(u) \in L^t_{loc}(\Omega'), \end{cases}$$

where $0 < s < 1$, $t > \frac{N}{2s} \geq 1$, $\Omega \subset \mathbb{R}^N$ be a bounded domain with $0 \in \Omega$ and $\Omega' = \Omega \setminus \{0\}$. The notion of distributional solution for (P_s) is defined as follows:

Definition 6.1.11. *A function u is said to be a distributional solution of (P_s) if $u \in L^1(\Omega)$ such that $g(u) \in L^1_{loc}(\Omega')$ and*

$$\int_{\Omega} u(x)(-\Delta)^s \phi \, dx = \int_{\Omega} g(u)\phi \, dx$$

for all $\phi \in C_c^\infty(\Omega)$ with $\text{supp}(\phi) \subset \Omega'$.

In [89], authors have studied the problem (P_s) by assuming the existence of classical solution u of (P_s) with polynomial type nonlinearity. In the next theorem, we extend the result of Chen and Quaas ([89]) for the problem (P_s) satisfying weaker assumption of distributional solution and for a larger class of nonlinearities (in particular exponential growth nonlinearity).

Theorem 6.1.12. *For $0 < s < 1$, let u be non-negative distributional solution of (P_s) such that $u \in L^1(\Omega)$, $g(u) \in L^t_{loc}(\Omega')$ for $t > \frac{N}{2s} \geq 1$. Then there exists $k \geq 0$ such that u is distributional solution of*

$$\begin{cases} (-\Delta)^s u = g(u) + k\delta_0, u \geq 0, & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \\ g(u) \in L^1(\Omega), \end{cases}$$

i.e.

$$\int_{\Omega} u(-\Delta)^s \phi - g(u)\phi \, dx = k\phi(0) \text{ for all } \phi \in C_c^\infty(\Omega).$$

As an application of Theorem 6.1.12, we characterize the asymptotic behavior of large solutions and prove the following result:

Theorem 6.1.13. For $1 < \alpha \leq 2$, $\delta > 0$, assume $\Lambda_\alpha > 0$ be an asymptotic bifurcation point as in the Definition 6.1.8. Then, for any sequence $(\lambda_k, u_k) \in \mathcal{S} \cap ((0, \Lambda) \times C_0([-1, 1]))$ such that $\lambda_k \rightarrow \Lambda_\alpha$ and $\|u_k\|_{L^\infty((-1, 1))} \rightarrow \infty$, the following assertions holds:

- (i) $0 \in \Omega$ is the only blow up point for a sequence u_k .
- (ii) $u_k \rightarrow u$ in $C_{loc}^s((-1, 1) \setminus \{0\})$ where u is a weak (singular) solution to (P_λ) . Moreover, $u(0) = \infty$, $u \in L^p((-1, 1))$ for any $1 \leq p < \infty$, $u \notin X_0$ and $\frac{1}{u^\delta} + f(u) \in L^1((-1, 1))$.

We have the following remark about the above theorem.

Remark 6.1.14. (a) Under the conditions of Theorem 6.1.13 in assertion (i), we expect concentration phenomena to hold for large solutions as $\lambda \rightarrow 0$.

- (b) Let G be the primitive of g defined as $g(t) = \frac{1}{t^\delta} + f(t)$ and assume that $G(t) = O(g(t))$ as $t \rightarrow \infty$. If the sequence of large solutions, say u_k , have bounded energy i.e.

$$J(u_k) = \frac{1}{2} \|u_k\|_{X_0}^2 - \int_{-1}^1 G(u_k) dx < C$$

where C is independent of k , then assertion (ii) cannot hold.

Due to the absence of the knowledge of the positioning of the bifurcation point, we may not have multiplicity of solution near $\lambda = 0$. So to prove the global multiplicity results, we approach to variational methods. Precisely, we proved the global multiplicity result to the problem (P_λ) for all $\delta > 0$, under the following assumptions on the function f .

- (K1) $h \in C^1(\mathbb{R}^+)$, $h(0) = 0$, $h(t) > 0$ for $t > 0$ and $f(t) = h(t)e^{t^2}$ is nondecreasing in t .
- (K2) For any $\epsilon > 0$, $\lim_{t \rightarrow \infty} (h(t) + h'(t))e^{-\epsilon t^2} = 0$ and $\lim_{t \rightarrow \infty} h(t)te^{\epsilon t^q} = \infty$ for some $0 \leq q < 1$.
- (K3) There exists $M_1, M_2, K > 0$ such that $F(t) = \int_0^t h(s)e^{s^2} ds < M_1(f(t) + 1)$ and $f'(t) \geq Kf(t) - M_2$ for all $t > 0$.

Example 1: An example of the function h satisfying the above conditions is $h(x) = x^k e^{x^\gamma}$, $k > 0$, $0 \leq \gamma < 2$.

We prove the following multiplicity theorem.

Theorem 6.1.15. (a) If f satisfies the assumption (K1)-(K5). There exists a $\Lambda > 0$ such that

- (i) For every $\lambda \in (0, \Lambda)$ the problem (P_λ) admits two solutions in $X_0 \cap C_{\phi_\delta}^+((-1, 1))$.
- (ii) For $\lambda = \Lambda$ there exists a solution in $X_0 \cap C_{\phi_\delta}^+((-1, 1))$.
- (iii) For $\lambda > \Lambda$, there exists no solution.

(b) Let $u \in X_0$ be any positive solution to (P_λ) where $\lambda \in (0, \Lambda]$, $\delta > 0$. Then $u \in C^\gamma(\mathbb{R})$ where γ is defined (6.1.5).

To prove the Theorem 6.1.15, we followed the approach of [150]. To obtain the first solution, we use the standard Perron's method on the functional J_λ (See (6.1.30)). To get a

second solution, we use the assumption (K2) to guarantee that the energy level of the Palais Smale sequence is below the first critical level. For that we seek help of Moser functions (See [244]) and then by using mountain-pass Lemma we prove the existence of a second solution. Notice that the Theorem 6.1.15 shows the existence of solution in the energy space X_0 . We remark that the Hölder regularity proved in theorem 6.1.15 is the optimal due to the behavior of the solution near the points -1 and 1 .

6.1.2 Global bifurcation result

In this section we first study the boundary behavior of the weak solution of (6.1.2). We further studied the pure singular problem (P_δ) and prove the Theorem 6.1.7 which deals with the existence and regularity of solutions of (P_δ) . In a same flow, we establish a global branch of classical solutions to (P_λ) .

Proposition 6.1.16. [84] *The Green function $G(x, y)$ associated to $(-\Delta)^{\frac{1}{2}}$ is the following:*

$$G(x, y) \asymp \log \left(1 + \frac{d(x)^{\frac{1}{2}} d(y)^{\frac{1}{2}}}{|x - y|} \right) \text{ for all } (x, y) \in (-1, 1) \times (-1, 1).$$

Proof of Theorem 6.1.6: Using the fact that $\frac{1}{d(y)^\alpha \log^\beta(A/d(y))} \in L^1(dx, d^{\frac{1}{2}})$, we have the following integral representation formula for the solution u to (6.1.2)

$$u(x) = \int_{-1}^1 \frac{G_B(x, y)}{d(y)^\alpha \log^\beta(A/d(y))} dy.$$

Therefore, from Proposition 6.1.16, up to multiplicative constants,

$$\frac{u(x)}{d(x)^{\frac{1}{2}}} \leq \int_{-1}^1 \log \left(1 + \frac{\sqrt{d(x)d(y)}}{|x - y|} \right) \frac{dy}{d(x)^{\frac{1}{2}} d(y)^\alpha \log^\beta(A/d(y))}. \quad (6.1.6)$$

Without loss of generality, we can assume $x \in [0, 1]$. Set $\varepsilon = d(x)$, $r = d(y)$ and $x = (1 - \varepsilon)$. Observe that the integral in (6.1.6) is symmetric around 0. Thus it is enough to consider the case $y \in [0, 1]$, from above transformations, we have $y = 1 - r$. To prove (6.1.3) we divide the proof in several steps.

Step 1: When $\alpha < \frac{1}{2}$ and $\beta = 0$. We rewrite

$$\begin{aligned} \frac{u(x)}{d(x)^{\frac{1}{2}}} &\leq \int_0^1 \log \left(1 + \frac{\sqrt{\varepsilon r}}{|r - \varepsilon|} \right) \frac{dr}{\varepsilon^{\frac{1}{2}} r^\alpha} = \int_0^{\frac{1}{\varepsilon}} \frac{\varepsilon^{\frac{1}{2}-\alpha}}{t^\alpha} \log \left(1 + \frac{\sqrt{t}}{|t - 1|} \right) dt \\ &= \left(\int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^{\frac{3}{2}} + \int_{\frac{3}{2}}^{\frac{1}{\varepsilon}} \right) \frac{\varepsilon^{\frac{1}{2}-\alpha}}{t^\alpha} \log \left(1 + \frac{\sqrt{t}}{|t - 1|} \right) dt. \end{aligned} \quad (6.1.7)$$

For the first integral we have

$$\int_0^{\frac{1}{2}} \frac{\varepsilon^{\frac{1}{2}-\alpha}}{t^\alpha} \log \left(1 + \frac{\sqrt{t}}{|t - 1|} \right) dt \leq C \int_0^{\frac{1}{2}} \frac{1}{t^\alpha} \log \left(1 + \frac{\sqrt{t}}{|t - 1|} \right) dt \leq C,$$

for some positive constant c_2 . For the second integral we have

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\varepsilon^{\frac{1}{2}-\alpha}}{t^\alpha} \log \left(1 + \frac{\sqrt{t}}{|t-1|} \right) dt \leq C \int_{\frac{1}{2}}^{\frac{3}{2}} \log \left(1 + \frac{\sqrt{t}}{|t-1|} \right) dt < 2C.$$

For the third integral we have

$$\int_{\frac{3}{2}}^{\frac{1}{\varepsilon}} \frac{\varepsilon^{\frac{1}{2}-\alpha}}{t^\alpha} \log \left(1 + \frac{\sqrt{t}}{|t-1|} \right) dt \leq C \int_{\frac{3}{2}}^{\frac{1}{\varepsilon}} \frac{\varepsilon^{\frac{1}{2}-\alpha}}{t^{\alpha+\frac{1}{2}}} dt \leq C.$$

It implies that there exists a positive constant c_2 (large enough) such that $u(x) \leq c_2 d(x)^{\frac{1}{2}}$. This affirms an upper bound of the solutions. For lower bound of the solutions, notice that the integrals in (6.1.7) works as a lower bound of $\frac{u(x)}{d(x)^{\frac{1}{2}}}$ (up to constants). Now we divide the proof in two cases:

Case 1: If $\varepsilon \geq \frac{1}{3}$ then

$$\frac{u(x)}{d(x)^{\frac{1}{2}}} \geq \int_0^{\frac{1}{3}} \frac{\varepsilon^{\frac{1}{2}-\alpha}}{t^\alpha} \log \left(1 + \frac{\sqrt{t}}{|t-1|} \right) dt \geq \left(\frac{1}{3} \right)^{\frac{1}{2}-\alpha} C \int_0^{\frac{1}{3}} t^{\frac{1}{2}-\alpha} dt = \frac{C}{\frac{3}{2}-\alpha} \left(\frac{1}{3} \right)^{2-2\alpha} > \frac{2C}{27}.$$

Case 2: If $\varepsilon < \frac{1}{3}$ then

$$\frac{u(x)}{d(x)^{\frac{1}{2}}} \geq \int_{\frac{3}{2}}^{\frac{1}{\varepsilon}} \frac{\varepsilon^{\frac{1}{2}-\alpha}}{t^\alpha} \log \left(1 + \frac{\sqrt{t}}{|t-1|} \right) dt \geq C \int_{\frac{1}{2\varepsilon}}^{\frac{1}{\varepsilon}} \frac{\varepsilon^{\frac{1}{2}}}{\sqrt{t}} dt = 2(1 - 1/\sqrt{2}).$$

It implies that there exists a positive constants c_1 (small enough) such that $c_1 d(x)^{\frac{1}{2}} \leq u(x)$.

Step 2: When $\alpha > \frac{1}{2}$ and $\beta = 0$. We rewrite

$$\frac{u(x)}{d(x)^{1-\alpha}} \leq \int_0^{\frac{1}{\varepsilon}} \frac{1}{t^\alpha} \log \left(1 + \frac{\sqrt{t}}{|t-1|} \right) dt = \left(\int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^{\frac{3}{2}} + \int_{\frac{3}{2}}^{\frac{1}{\varepsilon}} \right) \frac{1}{t^\alpha} \log \left(1 + \frac{\sqrt{t}}{|t-1|} \right) dt. \quad (6.1.8)$$

For the first integral,

$$\int_0^{\frac{1}{2}} \frac{1}{t^\alpha} \log \left(1 + \frac{\sqrt{t}}{|t-1|} \right) dt \leq \int_0^{\frac{1}{2}} t^{\frac{1}{2}-\alpha} dt < C.$$

By using the same estimation as in step 1, $\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{t^\alpha} \log \left(1 + \frac{\sqrt{t}}{|t-1|} \right) dt \leq C$. For the third integral

$$\int_{\frac{3}{2}}^{\frac{1}{\varepsilon}} \frac{1}{t^\alpha} \log \left(1 + \frac{\sqrt{t}}{|t-1|} \right) dt \leq \int_{\frac{3}{2}}^{\infty} \frac{1}{t^\alpha} \log \left(1 + \frac{\sqrt{t}}{|t-1|} \right) dt \leq C \int_{\frac{3}{2}}^{\infty} \frac{1}{t^{\frac{1}{2}+\alpha}} dt < C.$$

Observe that from the estimation of first and third integral is valid only when $\frac{1}{2} < \alpha < \frac{3}{2}$. For the lower bound, notice that integrals in (6.1.8) serve as lower bound as well. Hence

$$\frac{u(x)}{d(x)^{1-\alpha}} \geq \int_0^{\frac{1}{2}} \frac{1}{t^\alpha} \log \left(1 + \frac{\sqrt{t}}{|t-1|} \right) dt \geq \int_0^{\frac{1}{2}} t^{\frac{1}{2}-\alpha} dt > C.$$

Thus we can choose appropriate positive constant c_3 and c_4 such that

$$c_3 d(x)^{1-\alpha} \leq u(x) \leq c_4 d(x)^{1-\alpha}.$$

Step 3: When $\alpha = \frac{1}{2}$ and $0 \leq \beta < 1$. Clearly we can take $A = 2$. Then

$$\frac{u(x)}{d(x)^{\frac{1}{2}} \log^{1-\beta} \left(\frac{2}{d(x)} \right)} \leq \left(\int_0^{\frac{1}{2}} + \int_{\frac{1}{2}}^{\frac{3}{2}} + \int_{\frac{3}{2}}^{\frac{1}{\varepsilon}} \right) \frac{\log \left(1 + \frac{\sqrt{t}}{|t-1|} \right)}{t^{\frac{1}{2}} \log^{1-\beta} \left(\frac{2}{\varepsilon} \right) \log^\beta \left(\frac{2}{\varepsilon t} \right)} dt.$$

The first integral

$$\begin{aligned} \int_0^{\frac{1}{2}} \frac{\log \left(1 + \frac{\sqrt{t}}{|t-1|} \right)}{t^{\frac{1}{2}} \log^{1-\beta} \left(\frac{2}{\varepsilon} \right) \log^\beta \left(\frac{2}{\varepsilon t} \right)} dt &\leq \frac{C}{\log 2} \int_0^{\frac{1}{2}} \frac{dt}{\left(\log \left(\frac{1}{\varepsilon} \right) + \log \left(\frac{2}{t} \right) \right)^\beta} \\ &\leq C \int_0^{\frac{1}{2}} \frac{dt}{\log \left(\frac{2}{t} \right)^\beta} \leq C \int_0^{\frac{1}{2}} \frac{dt}{\log 4^\beta} \leq \frac{C}{\log 4}. \end{aligned}$$

For the second integral

$$\int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\log \left(1 + \frac{\sqrt{t}}{|t-1|} \right)}{t^{\frac{1}{2}} \log^{1-\beta} \left(\frac{2}{\varepsilon} \right) \log^\beta \left(\frac{2}{\varepsilon t} \right)} dt \leq C \int_{\frac{1}{2}}^{\frac{3}{2}} \log \left(1 + \frac{\sqrt{t}}{|t-1|} \right) dt < 2C.$$

For the third integral,

$$\begin{aligned} \int_{\frac{3}{2}}^{\frac{1}{\varepsilon}} \frac{\log \left(1 + \frac{\sqrt{t}}{|t-1|} \right)}{t^{\frac{1}{2}} \log^{1-\beta} \left(\frac{2}{\varepsilon} \right) \log^\beta \left(\frac{2}{\varepsilon t} \right)} dt &\leq \frac{C}{\log^{1-\beta} \left(\frac{2}{\varepsilon} \right)} \int_{\frac{3}{2}}^{\frac{1}{\varepsilon}} \frac{dt}{t \left(-\log \left(\frac{\varepsilon t}{2} \right) \right)^\beta} \\ &\leq \frac{C}{\log^{1-\beta} \left(\frac{2}{\varepsilon} \right)} \left(\log \frac{4}{3\varepsilon} \right)^{1-\beta} < C. \end{aligned}$$

For the lower bound, we again divide it in two cases:

Case 1: If $\varepsilon > \frac{1}{3}$ then

$$\frac{u(x)}{d(x)^{\frac{1}{2}} \log^{1-\beta} \left(\frac{2}{d(x)} \right)} \geq \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{\log \left(1 + \frac{\sqrt{t}}{|t-1|} \right)}{t^{\frac{1}{2}} \log^{1-\beta} \left(\frac{2}{\varepsilon} \right) \log^\beta \left(\frac{2}{\varepsilon t} \right)} dt \geq C \int_{\frac{1}{2}}^{\frac{3}{2}} \log \left(1 + \frac{\sqrt{t}}{|t-1|} \right) dt > C.$$

Case 2: If $\varepsilon < \frac{1}{3}$ then

$$\frac{u(x)}{d(x)^{\frac{1}{2}} \log^{1-\beta} \left(\frac{2}{d(x)} \right)} \geq \int_{\frac{3}{2}}^{\frac{1}{\varepsilon}} \frac{\log \left(1 + \frac{\sqrt{t}}{|t-1|} \right)}{t^{\frac{1}{2}} \log^{1-\beta} \left(\frac{2}{\varepsilon} \right) \log^\beta \left(\frac{2}{\varepsilon t} \right)} dt \geq \frac{C}{\log^{1-\beta} \left(\frac{2}{\varepsilon} \right)} \int_{\log \frac{3\varepsilon}{4}}^{\log \frac{1}{2}} \frac{dz}{(-z)^\beta} > C > 0.$$

It implies there exists suitable positive constant c_5 and c_6 such that

$$c_5 d(x)^{\frac{1}{2}} \log^{1-\beta} \left(\frac{A}{d(x)} \right) \leq u(x) \leq c_6 d(x)^{\frac{1}{2}} \log^{1-\beta} \left(\frac{A}{d(x)} \right).$$

□

We now study the pure singular problem.

Proof of Theorem 6.1.7 (i) The proof goes along the lines of [6, Theorem 1.2] for $s = \frac{1}{2}$. For the sake of completeness, we will give a short brief of the proof. Let us first consider the case $\delta < 1$. We introduce the following approximated problem:

$$(P_\delta^\epsilon) \begin{cases} (-\Delta)^{\frac{1}{2}} u = \frac{1}{(u+\epsilon)^\delta}, u > 0, & \text{in } (-1, 1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1, 1). \end{cases}$$

Following the same arguments and assertions as in proof of [6, Theorem 1.2], there exists a unique weak solution $u_\epsilon \in X_0 \cap C^{\frac{1}{2}}(\mathbb{R})$ to (P_δ^ϵ) , u_ϵ is a monotone increasing sequence as $\epsilon \rightarrow 0^+$ and there exists a constant $c > 0$ such that $u_\epsilon \geq c\phi_1$. Moreover,

$$\sup_{\epsilon > 0} \int_{\mathbb{R}} ((-\Delta)^{1/4} u_\epsilon)^2 dx < \infty.$$

To get an upper bound on u_ϵ , we will use the integral representation and the the Green's function $G(x, y)$. Clearly,

$$u_\epsilon(x) = \int_{-1}^1 \frac{G(x, y)}{(u_\epsilon(y) + \epsilon)^\delta} dy.$$

Then for a suitable positive constant C independent of ϵ , we have

$$\frac{u_\epsilon(x)}{d(x)^{\frac{1}{2}}} \leq C \int_{-1}^1 \frac{\log\left(1 + \frac{d(x)^{\frac{1}{2}}d(y)^{\frac{1}{2}}}{|x-y|}\right)}{d(x)^{\frac{1}{2}}(u_\epsilon(y) + \epsilon)^\delta} dy \leq C \int_{-1}^1 \frac{\log\left(1 + \frac{d(x)^{\frac{1}{2}}d(y)^{\frac{1}{2}}}{|x-y|}\right)}{d(x)^{\frac{1}{2}}d(y)^{\frac{\delta}{2}}} dy.$$

Utilizing the fact that $\delta < 1$ and the proof of Theorem 6.1.6 Step 1, we obtain that

$$u_\epsilon(x)/d(x)^{\frac{1}{2}} < \infty, \text{ for all } x \in (-1, 1).$$

Thus, we infer that $u = \lim_{\epsilon \rightarrow 0^+} u_\epsilon \leq c\phi_1$ and u is the unique weak solution to (P_δ) . Also,

$$c\phi_1 \leq u \leq C\phi_1$$

for some suitable constants c, C . This completes the proof of the theorem in case of $\delta < 1$. In a similar manner, for the case $\delta \geq 1$, we will follow the proof of Theorem 1.2 of [6] coupled with Theorem 6.1.6. Precisely, we will get unique solution of (P_δ) such that

$$\begin{aligned} \frac{1}{C_1} \phi_1 \log^{\frac{1}{2}}\left(\frac{2}{\phi_1}\right) &\leq u \leq C_1 \phi_1 \log^{\frac{1}{2}}\left(\frac{2}{\phi_1}\right), \text{ if } \delta = 1, \\ \frac{1}{C_2} \phi_1^{\frac{2}{\delta+1}} &\leq u \leq C_2 \phi_1^{\frac{2}{\delta+1}}, \text{ if } \delta > 1, \end{aligned}$$

for some appropriate positive constant C_1 and C_2 . For the Part (ii), the proof follows from Theorem 1.4 of [6]. We remark that all classical solutions belong to space X_0 as well. □

Define $\Lambda := \sup\{\lambda > 0 : (P_\lambda) \text{ has a weak solution}\}$.

Lemma 6.1.17. *It holds $0 < \Lambda < \infty$.*

Proof. Let u be the solution of (P_δ) given by Theorem 6.1.7 then $\underline{u}_\lambda = \lambda^{\frac{1}{\delta+1}}u$ is a solution of

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u = \frac{\lambda}{u^\delta}, u > 0 & \text{in } (-1, 1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1, 1). \end{cases}$$

Moreover, \underline{u}_λ is a strict subsolution of (P_λ) . Also, let $\bar{u}_\lambda = \underline{u}_\lambda + MU$ for some $M > 1$, where U is a solution of

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u = 1, u > 0, & \text{in } (-1, 1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1, 1). \end{cases}$$

There exists $\lambda_0 > 0$ such that \bar{u}_λ is a supersolution of (P_λ) for all $\lambda \leq \lambda_0$. Now we define the following iterative scheme for all $\lambda \leq \lambda_0$, starting with $u_0 = \underline{u}_\lambda$ and $(n \geq 1)$

$$\begin{cases} (-\Delta)^{\frac{1}{2}}u_n + \lambda C u_n - \frac{\lambda}{u_n^\delta} = \lambda C u_{n-1} + \lambda f(u_{n-1}), u > 0 & \text{in } (-1, 1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1, 1), \end{cases}$$

where $C = C(\lambda_0) > 0$ is large enough such that $t \rightarrow Ct + f(t)$ is non decreasing in $(0, \|\bar{u}_{\lambda_0}\|_{L^\infty})$. Taking into account monotonicity of the operator $(-\Delta)^{\frac{1}{2}}u - \lambda u^{-\delta}$, using the Comparison Principle ([150, Lemma 2.2]) and the proof of Theorem 6.1.7, we have that $\{u_n\}$ is increasing and $\{u_n\} \subset C^{\frac{1}{2}}(\mathbb{R}) \cap C^+_{\phi_\delta}((-1, 1))$. Furthermore, for all $\lambda \leq \lambda_0$, $\underline{u}_\lambda \leq u_n \leq \bar{u}_\lambda$. Using Theorem 6.1.7, we have $\sup_{n \in \mathbb{N}} \|u_n\|_{C^\gamma(\mathbb{R})} \leq C_0$ for some $C_0 = C_0(\delta, \lambda_0)$ large enough and γ is defined in Theorem 6.1.7. Hence $u_n \rightarrow u$ in $C(\mathbb{R})$ and u satisfies

$$(-\Delta)^{\frac{1}{2}}u = \frac{\lambda}{u^\delta} + f(u)$$

in the sense of distributions. Hence from the above arguments we get $\Lambda > 0$. From the superlinear behavior of $f(t)$ at infinity, we obtain that $\Lambda < \infty$. \square

Proof of Theorem 6.1.9 The proof follows from Theorem 1.6 of [6] (See also [102]). \square

Remark 6.1.18. *Consider the problem*

$$(P_\lambda^K) \quad \begin{cases} (-\Delta)^{\frac{1}{2}}u = \lambda \left(\frac{K(x)}{u^\delta} + f(u) \right), u > 0 & \text{in } (-1, 1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1, 1) \end{cases}$$

where $K \in C^{\nu}_{loc}((-1, 1))$, $\nu \in (0, 1)$ such that $\inf_{x \in (-1, 1)} K(x) > 0$ and satisfies for some $0 \leq \beta < 1$ and $c_1, c_2 > 0$ such that $c_1 d(x)^{-\beta} \leq K(x) \leq c_2 d(x)^{-\beta}$, for all $x \in (-1, 1)$. By modifying our barrier function ϕ_δ (see (6.1.4)) with respect to the growth of $K(x)$, we can prove Theorem 6.1.7. Subsequently, we can also prove Theorem 6.1.9 for the problem (P_λ^K) (same as [6, Theorem 1.6]).

6.1.3 Study of isolated singularities and qualitative properties

In this section, we study the qualitative properties as symmetry, monotonicity of solutions to the problem (P_λ) and asymptotic behavior of the connected branch \mathcal{C} . In order to describe the asymptotic behavior of large solutions, we first study Brezis-Lions problem in the setting of fractional Laplacian operator. In the spirit of Brezis and Lions work (see [70]), we classify the singularities of non-negative distributional solutions of fractional semilinear elliptic equation (P_s) for $N \geq 2s$. We assume that $g : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is continuous function with $g(0) = 0$ and Ω be a smooth bounded domain in \mathbb{R}^n .

Theorem 6.1.19. *Let u be nonnegative distributional solution of (P_s) in the sense of Definition 6.1.11 and $g(u) \in L^t_{loc}(\Omega')$ for $t > \frac{N}{2s} \geq 1$. Then $g(u) \in L^1(\Omega)$.*

Proof. Suppose by contradiction that $g(u) \notin L^1(\Omega)$. Since $g(u) \in L^t_{loc}(\Omega')$ then for any small $r > 0$ there exists a sequence $\{R_m\}_{m \in \mathbb{N}} \in (0, r)$ such that $R_m \rightarrow 0$ and

$$\int_{B_r(0) \setminus B_{R_m}(0)} g(u) \, dx = m. \quad (6.1.9)$$

Consider the problem

$$\begin{cases} (-\Delta)^s u_m = \chi_{\Omega \setminus B_{R_m}(0)} g(u), \quad u \geq 0 & \text{in } \Omega, \\ u_m = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \quad (6.1.10)$$

Since $\chi_{\Omega \setminus B_{R_m}(0)} g(u) \in L^t(\Omega)$ for $t > \frac{N}{2s}$, then there exists a sequence of classical solutions $\{u_m\}$ solving (6.1.10) such that $u_m \in L^\infty(\Omega) \cap C^\beta(\bar{\Omega})$ for some $\beta \in (0, 1)$ (see Proposition 1.4 in [227]).

Let Φ be the fundamental solution of $(-\Delta)^s$. *i.e.* $(-\Delta)^s \Phi = \delta_0$ in $\mathcal{D}'(\mathbb{R}^N)$, where

$$\Phi(x) = \begin{cases} \frac{\Gamma\left(\frac{N}{2} - s\right)}{2^{2s} \pi^{N/2} \Gamma(s)} \frac{1}{|x|^{N-2s}} & \text{if } N \neq 2s \\ \frac{-1}{\pi} \log(|x|) & \text{if } N = 2s. \end{cases}$$

Since $u \geq 0$ and u_m is bounded in $L^\infty(\Omega)$ therefore $\lim_{x \rightarrow 0} (u + \Phi)(x) = +\infty$ and for each $m \in \mathbb{N}$, there exists $r_m > 0$ such that $u + \Phi \geq u_m$ in $B_{r_m}(0) \setminus \{0\}$. Then by the weak comparison principle we obtain $u + \Phi \geq u_m$ in $\mathbb{R}^N \setminus \{0\}$.

Since $\lim_{y \rightarrow x} G(x, y) = +\infty$, there exists $r_1 > 0$ such that $G(x, y) \geq 1$ in $x, y \in B_{r_1}(0)$. Now by using (6.1.9) we obtain that,

$$\begin{aligned} u_m(x) &= \int_{\Omega} G(x, y) \chi_{\Omega \setminus B_{R_m}(0)} g(u) \, dy = \int_{\Omega \setminus B_{R_m}(0)} G(x, y) g(u) \, dy \\ &\geq \int_{B_{r_1}(0) \setminus B_{R_m}(0)} g(u) = m \rightarrow \infty \end{aligned}$$

which implies $u + \Phi = +\infty$ in $K \Subset B_{r_1}(0) \setminus \{0\}$, which is not possible. Therefore $g(u) \in L^1(\Omega)$. \square

Let $\xi : \mathbb{R}^N \rightarrow [0, 1]$ be radially symmetric increasing function such that $\xi \in C^\infty(\mathbb{R}^N)$ and

$$\xi(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R}^N \setminus B_2(0), \\ 0 & \text{if } x \in B_1(0). \end{cases}$$

Define $u_\epsilon = u\xi_\epsilon$ where $\xi_\epsilon(x) = \xi(\frac{x}{\epsilon})$ for all $x \in \mathbb{R}^N$. Then for any $x \in \Omega \setminus \{0\}$,

$$\begin{aligned} (-\Delta)^s u_\epsilon(x) &= C_{N,s} \int_{\mathbb{R}^N} \frac{u_\epsilon(x) - u_\epsilon(y)}{|x-y|^{N+2s}} dy \\ &= C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{u(x)\xi_\epsilon(x) - u(y)\xi_\epsilon(y)}{|x-y|^{N+2s}} dy \\ &= \xi_\epsilon(x)(-\Delta)^s u(x) + u(x)(-\Delta)^s \xi_\epsilon(x) \\ &\quad - C_{N,s} P.V. \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\xi_\epsilon(x) - \xi_\epsilon(y))}{|x-y|^{N+2s}} dy. \end{aligned} \tag{6.1.11}$$

Now we prove the following result:

Theorem 6.1.20. *Let $P : C_c^\infty(\Omega) \rightarrow \mathbb{R}$ be the operator such that*

$$P(\phi) = \int_{\Omega} u(-\Delta)^s \phi - g(u)\phi \, dx \quad \text{for all } \phi \in C_c^\infty(\Omega)$$

where $u \in L^1(\Omega)$ is a non-negative distributional solution of (P_s) and $g(u) \in L^1(\Omega)$. Then

- (i) $P(\phi) = 0$ for any $\phi \in C_c^\infty(\Omega)$ with $\text{supp}(\phi) \subset \Omega \setminus \{0\}$.
- (ii) There exists constants c_a such that

$$P(\phi) = \sum_{|a|=0}^{\infty} c_a D^a \phi(0)$$

where $a = (a_1, a_2, \dots, a_N)$ with $a_i \in \mathbb{N}$, $|a| = \sum_{i=1}^N a_i$, $D^a = (\partial^{a_1} \phi, \partial^{a_2} \phi, \dots, \partial^{a_N} \phi)$.

Proof. Let us prove assertion 1 and consider $\epsilon > 0$ small enough then we have

$$\begin{aligned} \left| \int_{\Omega} (u(-\Delta)^s \phi - g(u)\phi) \, dx \right| &\leq \int_{\Omega} |(u_\epsilon(-\Delta)^s \phi - g(u)\phi)| \, dx \\ &\quad + \left| \int_{\Omega} u(1 - \xi_\epsilon(x))(-\Delta)^s \phi \, dx \right|. \end{aligned}$$

Since $\phi \in C_c^\infty(\Omega)$ with $\text{supp}(\phi) \subset \Omega \setminus \{0\}$, then there exists $r > 0$ such that $\phi = 0$ in $B_r(0)$. Then by using integration by parts formula with $(-\Delta)^s u_\epsilon \in L^\infty(\Omega)$ (Lemma 2.2 in [88]) and (6.1.11) we obtain,

$$\begin{aligned} \left| \int_{\Omega} (u(-\Delta)^s \phi - g(u)\phi) \, dx \right| &\leq \left| \int_{\Omega} \phi(-\Delta)^s u_\epsilon - g(u)\phi \, dx \right| + \left| \int_{B_{2\epsilon}(0)} u(-\Delta)^s \phi \, dx \right| \\ &\leq \left| \int_{B_{2\epsilon}(0)} u(-\Delta)^s \phi \, dx \right| + \left| \int_{\Omega \setminus B_r(0)} (\xi_\epsilon(x)(-\Delta)^s u(x) + u(x)(-\Delta)^s \xi_\epsilon(x) - g(u))\phi \, dx \right| \\ &\quad + C_{n,s} \left| \int_{\Omega \setminus B_r(0)} \phi(x) \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\xi_\epsilon(x) - \xi_\epsilon(y))}{|x-y|^{N+2s}} dy \, dx \right|. \end{aligned}$$

For $x \in \Omega \setminus B_r(0)$, $y \in B_\epsilon(0)$ and $\epsilon < \frac{r}{4}$, we have

$$|(-\Delta)^s \xi_\epsilon(x)| = C_{N,s} \int_{B_{2\epsilon}(0)} \frac{1 - \xi_\epsilon(y)}{|x - y|^{N+2s}} dy \leq C \frac{|B_{2\epsilon}(0)|}{(r - \epsilon)^{N+2s}}.$$

Therefore,

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega \setminus B_r(0)} u(x) |\phi| |(-\Delta)^s \xi_\epsilon(x)| = 0. \quad (6.1.12)$$

Also,

$$\begin{aligned} & \left| \int_{\Omega \setminus B_r(0)} \phi(x) \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\xi_\epsilon(x) - \xi_\epsilon(y))}{|x - y|^{N+2s}} dy dx \right| \\ & \leq \|\phi\|_{L^\infty(\Omega)} \left| \int_{\Omega \setminus B_r(0)} u(x) \int_{B_{2\epsilon}(0)} \frac{|1 - \xi_\epsilon(y)|}{|x - y|^{N+2s}} dy dx \right. \\ & \quad \left. - \int_{\Omega \setminus B_r(0)} \left(\int_{B_{2\epsilon}(0)} \frac{u(y)|1 - \xi_\epsilon(y)|}{|x - y|^{N+2s}} dy \right) dx \right| \\ & \leq \frac{\|\phi\|_{L^\infty(\Omega)}}{(r - \epsilon)^{N+2s}} \left(C\epsilon^N \int_{\Omega \setminus B_r(0)} u(x) dx + |\Omega \setminus B_r(0)| \int_{B_{2\epsilon}(0)} u(y) dy \right) \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \end{aligned} \quad (6.1.13)$$

Since $\xi_\epsilon(x) = 1$ in $\Omega \setminus B_r(0)$ and u is distributional solution of (P_s) we obtain,

$$\int_{\Omega \setminus B_r(0)} (\xi_\epsilon(x)(-\Delta)^s u(x) - g(u))\phi dx = 0. \quad (6.1.14)$$

Therefore, by combining (6.1.12), (6.1.13) and (6.1.14) we obtain $P(\phi) = 0$ for all $\phi \in C_c^\infty(\Omega)$ with $\text{supp}(\phi) \subset \Omega \setminus \{0\}$. Since $u \in L^1(\Omega)$ and $g(u) \in L^1(\Omega)$, then P is a bounded linear functional on $C_c^\infty(\Omega)$. Therefore by using Theorem XXXV in [232], we obtain

$$P = \sum_{|a| \leq m} c_a D^a \delta_0$$

where $c_a \in \mathbb{R}$ and δ_0 denotes the Dirac mass at origin. *i.e.* for all $\phi \in C_c^\infty(\Omega)$

$$P(\phi) = \sum_{|a| \leq m} c_a D^a \phi(0). \quad (6.1.15)$$

□

Theorem 6.1.21. *Let P be a bounded linear functional satisfying (6.1.15). Then*

$$c_a = 0 \text{ for any } |a| \geq 1.$$

Proof. Let $\eta \in C^\infty(\mathbb{R}^N)$ with $\text{supp}(\eta) \subset \overline{B_1(0)}$ and $|a| \geq 1$ such that $D^a \eta(0) = c_a$ for every $|a| \leq m$ (see [70]). Define $\eta_\epsilon(x) = \eta(\frac{x}{\epsilon})$ for $x \in \mathbb{R}^N$, then from (6.1.15) we obtain,

$$P(\eta_\epsilon) = \sum_{|a| \leq m} c_a D^a \eta_\epsilon(0) = C \frac{c_a^2}{\epsilon^{|a|}} = \int_{\Omega} (u(-\Delta)^s \eta_\epsilon - g(u)\eta_\epsilon) dx. \quad (6.1.16)$$

Let $r > 0$ and divide the integral in (6.1.16) into two parts:

$$\left| \int_{\Omega} u(x)(-\Delta)^s \eta_{\epsilon}(x) \right| \leq \int_{\Omega \setminus B_r(0)} u(x) |(-\Delta)^s \eta_{\epsilon}(x)| dx + \int_{B_r(0)} u(x) |(-\Delta)^s \eta_{\epsilon}(x)| dx.$$

For $x \in \Omega \setminus B_r(0)$ with ϵ small enough, we obtain

$$|(-\Delta)^s \eta_{\epsilon}(x)| = \left| P.V. \int_{B_{\epsilon}(0)} \frac{\eta(\frac{x}{\epsilon}) - \eta(\frac{y}{\epsilon})}{|x-y|^{N+2s}} dy \right| \leq \frac{2\|\eta\|_{L^{\infty}(B_1(0))}|B_{\epsilon}(0)|}{(r-\epsilon)^{N+2s}} \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (6.1.17)$$

and for $x \in B_r(0)$ we obtain,

$$\left| \int_{B_r(0)} u(x)(-\Delta)^s \eta\left(\frac{x}{\epsilon}\right) dx \right| \leq \|(-\Delta)^s \eta\|_{L^{\infty}(B_r(0))} \int_{B_r(0)} u(x) dx \rightarrow 0 \text{ as } r \rightarrow 0. \quad (6.1.18)$$

independently of ϵ . Therefore by combining (6.1.17) and (6.1.18) with $\epsilon < r$, we obtain

$$\left| \int_{\Omega} u(x)(-\Delta)^s \eta_{\epsilon}(x) dx \right| \leq o(1) \text{ as } \epsilon \rightarrow 0. \quad (6.1.19)$$

Also, the second integral in (6.1.16) satisfies

$$\int_{\Omega} g(u)\eta_{\epsilon} dx \leq \|\eta\|_{L^{\infty}(\Omega)} \int_{B_{\epsilon}(0)} g(u) dx \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \quad (6.1.20)$$

From (6.1.16), (6.1.19) and (6.1.20), $c_a^2 \leq C_1 \epsilon^{|a|} o(1)$ as $\epsilon \rightarrow 0$. Therefore we have $c_a = 0$, for all $|a| \geq 1$ since ϵ is arbitrary. \square

Proof of Theorem 6.1.12 : Follows from combining Theorem 6.1.20 and Theorem 6.1.21. \square

Now we prove Theorems 6.1.10 and 6.1.13 concerning the qualitative properties of classical solution and asymptotic behavior of large solution for half Laplacian operator and $n = 1$:

Proof of Theorem 6.1.10: With the assistance of maximum principle in narrow domains (see [175]) and moving plane method, we prove the monotonicity and radial symmetry of classical solutions in Ω . Without loss of generality, we assume $\Omega = (-1, 1)$ and (λ, u) be classical solution of (P_{λ}) for $\lambda \leq \lambda_0$ (obtained from Theorem 6.1.9).

Define $R_h(x) := (2h - x)$ be the reflection of the point x about h and

$$v_h(x) := u_h(x) - u(x) \quad \text{where } u_h(x) = u(R_h(x)).$$

Step 1: Positivity of v_h near -1 and 1 :

Clearly for $|h|$ sufficiently large, $v_h(x) \geq 0$. Now we prove that $v_h(x) \geq 0$ in $(-1, h) \cap H_h^-$ if $h \leq 0$ and in $(h, 1) \cap H_h^+$ if $h > 0$ where $H_h^{\pm} = \{x \in \mathbb{R} : x \gtrless h\}$ and h lies in the neighborhood

of $x_0 \in \partial\Omega = \{-1, 1\}$. Suppose that $v_h < 0$ in $K \subset (-1, h) \cap H_h^-$ for some $h \leq 0$. Since f is Lipschitz and noting that $\text{supp}((-v_h)^+) \subset (-1, 2h + 1)$, we have

$$\begin{aligned} \langle (-\Delta)^{1/2}(-v_h), (-v_h)^+ \rangle &= \lambda \int_{-1}^{2h+1} \left(\frac{1}{u^\delta} - \frac{1}{(u_h)^\delta} + f(u) - f(u_h) \right) (-v_h)^+ dx \\ &\leq C \int_K ((u - u_h)^+)^2 dx. \end{aligned}$$

Then by Poincaré inequality, we obtain

$$\int_{\mathbb{R}} \left((-\Delta)^{\frac{1}{4}}(u - u_h)^+ \right)^2 \leq C \int_{-1}^{2h+1} ((-v_h)^+)^2 dx \leq C(\text{diam}(K)) \int_{\mathbb{R}} \left((-\Delta)^{\frac{1}{4}}(u - u_h)^+ \right)^2.$$

Then by choosing h close enough to -1 we get, $C(\text{diam}(K)) < 1$ and then $(-v_h)^+ = (u - u_h)^+ = 0$. Similarly in the case of $(h, 1) \cap H_h^+$ for $h > 0$. Now by moving the point in the neighborhood of -1 and 1 we obtain there exists $T > 0$ independent of u such that

$$\begin{cases} u(x-t) \text{ is non-increasing} & \forall (t, x) \in [0, T] \times (-1, h) \text{ if } h \leq 0, \\ u(x-t) \text{ is non-decreasing} & \forall (t, x) \in [0, T] \times (h, 1) \text{ if } h \geq 0. \end{cases} \quad (6.1.21)$$

Step 2: Positivity of v_h in interior of $(-1, 1)$:

In **Step 1**, we have proved that $v_h \geq 0$ in the neighborhood of -1 and 1 . So, without loss of generality we can assume that $h \geq 0$ be the smallest value such that $v_h \geq 0$ in $(h, 1)$. Then the mean value Theorem implies v_h satisfies the following for some $\theta \in (0, 1)$

$$(-\Delta)^{\frac{1}{2}}v_h + \frac{\delta v_h}{(\theta u + (1-\theta)u_h)^{\delta+1}} = f(u_h) - f(u) \text{ in } (h, 1). \quad (6.1.22)$$

Claim 1: For every compact subset $K \subset (h, 1)$, $\text{ess inf}_K v_h > 0$.

To establish our claim, we follow the proof of Proposition 3.6 in [175]. Since $v_h \not\equiv 0$ in $(h, 1)$ then for $x^* \in (h, 1)$, it is enough to prove that $\text{ess inf}_{B_r(x^*)} v_h > 0$ for r sufficiently small. From **Step 1**, $v_h \geq 0$ and $v_h(x) = -v_h(R_h(x))$ in H_h^+ , there exists a bounded set $B \subset H_h^+$ with $x^* \notin \bar{B}$ and

$$\tilde{\mu} := \inf_B v_h > 0. \quad (6.1.23)$$

Using Lemma 2.1 in [175], we fix r such that $U = B_{2r}(x^*)$ and

$$0 < r < \frac{1}{4} \text{dist}(x^*, B \cup (\mathbb{R} \setminus H_h^+)) \text{ and } \lambda_1(U) \geq C_L(f) \quad (6.1.24)$$

where $C_L(f)$ is the Lipschitz constant of f and $\lambda_1(U)$ is the first eigenvalue of $(-\Delta)^s$ in U . Now, in order to apply Proposition 3.5 in [175], we construct a subsolution of $(-\Delta)^{\frac{1}{2}}\tilde{v} = c(x)\tilde{v}$ in U where

$$c(x) = \begin{cases} \frac{f(u_h) - f(u)}{v_h} - \frac{\delta}{(\theta u + (1-\theta)u_h)^{\delta+1}} & \text{if } v_h \neq 0, \\ 0 & \text{if } v_h = 0. \end{cases}$$

Define

$$k : \mathbb{R} \rightarrow \mathbb{R}, \quad k(x) = m(x) - m(R_h(x)) + a[\mathbb{1}_B(x) - \mathbb{1}_B(R_h(x))]$$

where a will be determined later and $m \in C_c^2(\mathbb{R})$ such that $0 \leq m \leq 1$ on \mathbb{R} and

$$m(x) = \begin{cases} 1 & \text{if } |x - x^*| \leq r, \\ 0 & \text{if } |x - x^*| \geq 2r, \end{cases}$$

and satisfies $k(R_h(x)) = -k(x)$ on H_h^+ , $k = 0$ in $H_h^+ \setminus (U \cup B)$ and $k = a$ on B . Then by Proposition 2.3 in [175] we obtain,

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(m(x) - m(y))(\phi(x) - \phi(y))}{|x - y|^2} dx dy \leq C \int_U \phi(x) dx$$

for $\phi \in \tau$, $\phi \geq 0$ and $C = C(m)$ independent of ϕ . Since $\phi = 0$ in $\mathbb{R} \setminus U$, $(U \cap B) \cup (U \cap R_h(B)) = \emptyset$ and

$$m(R_h(x))\phi(x) = \mathbb{1}_B(x)\phi(x) = \mathbb{1}_{R_h(B)}(x)\phi(x) = 0 \text{ in } \mathbb{R}.$$

Then we have

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(k(x) - k(y))(\phi(x) - \phi(y))}{|x - y|^2} dx dy \leq C_a \int_U \phi(x) dx$$

where

$$C_a := C + \sup_{x \in U} \int_{R_h(U)} \frac{1}{|x - y|^2} dy - a \inf_{x \in U} \int_B \left(\frac{1}{|x - y|^2} - \frac{1}{|x - R_h(y)|^2} \right) dy.$$

Since $|x - y| \leq |x - R_h(y)|$ for all $x, y \in H_h^+$, $\bar{U} \subset H_h^+$ and continuity of the function $x \mapsto \int_B \left(\frac{1}{|x - y|^2} - \frac{1}{|x - R_h(y)|^2} \right) dy$ implies

$$\inf_U \int_B \left(\frac{1}{|x - y|^2} - \frac{1}{|x - R_h(y)|^2} \right) dy > 0.$$

Now by taking a sufficiently large enough such that $C_a \leq -C_L(f)$ and using $v_h \geq 0$ in U , we obtain

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(k(x) - k(y))(\phi(x) - \phi(y))}{|x - y|^2} dx dy &\leq -C_L(f) \int_U \phi(x) dx \\ &\leq \int_U \lambda \left(\frac{f(u_h) - f(u)}{v_h} - \frac{\delta}{(\theta u + (1 - \theta)u_h)^{\delta+1}} \right) k(x)\phi(x) dx. \end{aligned}$$

Then by using (6.1.23), (6.1.24) and Proposition 3.5 in [175], we obtain $\tilde{v}_h(x) := v_h(x) - \frac{\tilde{h}}{a}k(x) \geq 0$ a.e. in U so that $v_h(x) \geq \frac{\tilde{h}}{a}k(x) = \frac{\tilde{h}}{a} > 0$ a.e. in $B_r(x^*)$ and which completes the

Claim 1.

Claim 2: $h = 0$.

We argue by contradiction and suppose $h > 0$. Since h is the smallest value such that $v_h \geq 0$ in $(h, 1)$, so we claim that for a small $\epsilon > 0$ we have $v_{h-\epsilon} \geq 0$ in $(h - \epsilon, 1)$ and thus get a

contradiction that h is the smallest value. For this claim we follow the proof of Proposition 3.5 in [175]. Fix γ (to be determined later) and let $K \Subset (h, 1)$ such that $|(h, 1) \setminus K| \leq \frac{\gamma}{2}$. Then by using **Claim 1**, $v_h \geq r > 0$ in K and then by continuity $v_{h-\epsilon} > 0$ in K for ϵ small enough. Since $v_{h-\epsilon}$ satisfies (6.1.22) in $(h-\epsilon, 1) \setminus K$ and by taking $w := \mathbb{1}_{H_{h-\epsilon}^+} v_{h-\epsilon}^-$ such that $\text{supp}(w) \subset (h-\epsilon, 1) \setminus K$ as a test function, we obtain

$$\langle (-\Delta)^{\frac{1}{2}} v_{h-\epsilon}, w \rangle = \int_{(h-\epsilon, 1) \setminus K} \left(\frac{-\delta v_{h-\epsilon}}{(\theta u + (1-\theta)u_{h-\epsilon})^{\delta+1}} + f(u_{h-\epsilon}) - f(u) \right) w \, dx. \quad (6.1.25)$$

We observe that

$$[w + v_{h-\epsilon}]w = [\mathbb{1}_{H_{h-\epsilon}^+} v_{h-\epsilon}^+ + \mathbb{1}_{\mathbb{R} \setminus H_{h-\epsilon}^+} v_{h-\epsilon}] \mathbb{1}_{H_{h-\epsilon}^+} v_{h-\epsilon}^- = 0 \quad \text{in } \mathbb{R}$$

and therefore

$$\begin{aligned} [w(x) - w(y)]^2 + [v_{h-\epsilon}(x) - v_{h-\epsilon}(y)][w(x) - w(y)] \\ = - (w(x)[w(x) + v_{h-\epsilon}(y)] + w(y)[w(x) + v_{h-\epsilon}(x)]). \end{aligned} \quad (6.1.26)$$

Now using $|x - y| \leq |x - R_{h-\epsilon}(y)|$ for all $x, y \in H_{h-\epsilon}^+$, $R_{h-\epsilon}(\mathbb{R} \setminus H_{h-\epsilon}^+) = H_{h-\epsilon}^+$ and from (6.1.26), we obtain

$$\begin{aligned} \langle (-\Delta)^{\frac{1}{2}} w, w \rangle + \langle (-\Delta)^{\frac{1}{2}} v_{h-\epsilon}, w \rangle &= -2 \int_{H_{h-\epsilon}^+} \int_{\mathbb{R}} \frac{w(x)[w(y) + v_{h-\epsilon}(y)]}{|x - y|^2} \, dy \, dx \\ &= -2 \int_{H_{h-\epsilon}^+} \int_{\mathbb{R}} \frac{w(x)[\mathbb{1}_{H_{h-\epsilon}^+} v_{h-\epsilon}^+(y) + \mathbb{1}_{\mathbb{R} \setminus H_{h-\epsilon}^+} v_{h-\epsilon}(y)]}{|x - y|^2} \, dy \, dx \\ &= -2 \int_{H_{h-\epsilon}^+} \int_{H_{h-\epsilon}^+} w(x) \left(\frac{v_{h-\epsilon}^+(y)}{|x - y|^2} - \frac{v_{h-\epsilon}(y)}{|x - R_{h-\epsilon}(y)|^2} \right) \, dy \, dx \leq 0. \end{aligned} \quad (6.1.27)$$

Let λ_1 be the first eigenvalue of $(-\Delta)^s$ in $(h-\epsilon, 1) \setminus K$ and then by combining (6.1.25) and (6.1.27) we obtain,

$$\begin{aligned} \lambda_1((h-\epsilon, 1) \setminus K) \int_{(h-\epsilon, 1) \setminus K} |v_{h-\epsilon}^-|^2 \, dx &\leq \langle (-\Delta)^{\frac{1}{2}} w, w \rangle \leq - \langle (-\Delta)^{\frac{1}{2}} v_{h-\epsilon}, w \rangle \\ &= \int_{(h-\epsilon, 1) \setminus K} \frac{\delta v_{h-\epsilon} \mathbb{1}_{(h-\epsilon, 1) \setminus K} v_{h-\epsilon}^-}{(\theta u + (1-\theta)u_h)^{\delta+1}} \, dx + \int_{(h-\epsilon, 1) \setminus K} (-f(u_{h-\epsilon}) + f(u)) \mathbb{1}_{(h-\epsilon, 1) \setminus K} v_{h-\epsilon}^- \, dx \\ &\leq C_L \int_{(h-\epsilon, 1) \setminus K} |v_{h-\epsilon}^-|^2 \, dx. \end{aligned}$$

Since $\lambda_1(\Omega) \rightarrow \infty$ when $|\Omega| \rightarrow 0$ (see Lemma 2.1 in [175]) then by choosing γ small enough we get $v_{h-\epsilon} \geq 0$ in $(h-\epsilon, 1)$, which is a contradiction. Therefore $h = 0$ *i.e.* $u(-x) \geq u(x)$ and then by repeating the same proof for largest value of h over $(-1, h)$ we obtain $u(x) = u(-x)$ for all $x \in (-1, 1)$. Since $h = 0$, therefore (6.1.21) and **Claim 1** imply u is strictly decreasing in $|x|$. \square

Now we prove result describing the asymptotic behavior of connected branch \mathcal{C} :

Proof of Theorem 6.1.13: Let $(\lambda, u) \in \mathcal{S} \cap ((0, \Lambda) \times C_0([-1, 1]))$ be the solution of the problem (P_λ) . Then from Theorem 6.1.10 we obtain, u is decreasing with respect to $|x|$ then for every $\epsilon > 0$ there exists $\beta_1 > 0$ such that for any $x \in (-1, -\epsilon) \cup (\epsilon, 1)$, we have a measurable set M_ϵ satisfying

- (i) $|M_\epsilon| \geq \beta_1$ and $M_\epsilon \subset (-1 + \epsilon, 1 - \epsilon)$.
- (ii) $u(y) \geq u(x)$, $\forall y \in M_\epsilon$.

Then by multiplying ϕ_1 to the equation satisfied by u , we obtain

$$\lambda_1 \int_{-1}^1 u \phi_1 = \lambda \left(\int_{-1}^1 \frac{\phi_1}{u^\delta} dx + \int_{-1}^1 f(u) \phi_1 dx \right)$$

and for any $m \geq \frac{2\lambda_1}{\Lambda_a}$, there exists a $C > 0$,

$$mt - C \leq \frac{1}{t^\delta} + f(t), \quad t \in \mathbb{R}^+.$$

Then by using $u(y) \geq u(x)$, $\forall y \in M_\epsilon$ we obtain for $C_1 > 0$ large enough,

$$\left(m - \frac{2\lambda_1}{\Lambda_a} \right) u(x) \int_{M_\epsilon} \phi_1 dx \leq \left(m - \frac{2\lambda_1}{\Lambda_a} \right) \int_{-1}^1 u \phi_1 dx \leq C_1.$$

Together with $|M_\epsilon| \geq \beta_1$ it implies that $u(x) \leq C_2$ for all $x \in (-1, -\epsilon) \cup (\epsilon, 1)$. So,

$$\sup_{u \in \mathcal{S} \cap (\{\lambda \geq \lambda_0\} \times C_0([-1, 1]))} \|u\|_{L^\infty((-1, 1) \setminus [\epsilon, \epsilon])} \leq c_\epsilon < \infty. \quad (6.1.28)$$

Suppose there exists a sequence (λ_k, u_k) of solutions in $\mathcal{S} \cap ((0, \Lambda) \times C_0([-1, 1]))$ such that $\lambda_k \rightarrow \Lambda_a$ and $\|u_k\|_{L^\infty} \rightarrow \infty$ as $k \rightarrow \infty$, then (6.1.28) implies "0" is the blow up point. Hence by regularity of u_k and compact embedding we obtain $u_k \rightarrow u$ uniformly on compact subsets of $(-1, 1) \setminus \{0\}$. Since u_k satisfies (6.1.1), then from the proof of Remark 6.1.14, we obtain $\|g(u_k)\|_{L^1} \leq C_2$, where C_2 is independent of k . Then Fatou's lemma and Vitali's convergence theorem give $u \in L^p((-1, 1))$ and $\|u_k - u\|_{L^p((-1, 1))} \rightarrow 0$ for any $1 \leq p < \infty$. Now, by passing to the limit as $k \rightarrow \infty$ we obtain u satisfies (in the sense of Definition 6.1.11):

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u = \Lambda_a g(u) & \text{in } (-1, 1) \setminus \{0\}, \\ u \geq 0 & \text{in } (-1, 1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1, 1), \end{cases}$$

with $g(u) \in L^1(\Omega)$. Then by Theorems 6.1.19 and 6.1.12 there exists $\mu \geq 0$ such that u satisfies (in the sense of Definition 6.1.1)

$$\begin{cases} (-\Delta)^{\frac{1}{2}} u = \Lambda_a g(u) + \mu \delta_0 & \text{in } (-1, 1), \\ u \geq 0 & \text{in } (-1, 1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1, 1). \end{cases} \quad (6.1.29)$$

Suppose $\mu \neq 0$. Hence we have $u(x) = \Lambda_a g(u) * \Phi(x) + \mu \Phi(x) + l(x)$ where l is a s -harmonic function in $(-1, 1)$ and $\Phi(x) = \frac{-1}{\pi} \log(|x|)$. Therefore $u(x) \geq \log(|x|^{-\mu/\pi}) - C$ and since $\alpha > 1$,

$$f(u) \geq h \left(\log(|x|^{-\mu/\pi}) - C \right) \exp^{(\log(|x|^{-\mu/\pi}) - C)^\alpha} \geq h \left(\log(|x|^{-\mu/\pi}) - C \right) |x|^{-\mu p/\pi}$$

for all $p > 1$, $0 < |x| \leq |x_\rho|$ and $|x_\rho|$ small. Then by integrating $f(u)$ over a small ball B around 0, we obtain $\int_B f(u) = \infty$ which contradicts $f(u) \in L^1((-1, 1))$. Therefore $\mu = 0$. This completes the proof of Theorem 6.1.13. \square

6.1.4 Global multiplicity result via variational method

In this section, we will show the existence and multiplicity of solutions of (P_λ) by using variational methods. The energy functional corresponds to problem (P_λ) is defined as

$$J_\lambda(u) = \frac{1}{2} \|u\|^2 - \lambda \int_{-1}^1 (G(u) + F(u)) \, dx, \quad (6.1.30)$$

where

$$G(u) = \begin{cases} 0 & \text{if } u \leq 0 \text{ and } \delta > 0, \\ \frac{u^{1-\delta}}{1-\delta} & \text{if } u > 0 \text{ and } \delta \neq 1, \\ \ln u & \text{if } u > 0 \text{ and } \delta = 1. \end{cases}$$

Using the above theorem one can see that the functional J_λ is well defined.

Lemma 6.1.22. *For each $\lambda \in (0, \Lambda]$, (P_λ) admits a weak solution provided (K1) and (K2) holds.*

Proof. We use the classical Perron's method to proof the existence of a solution. Let $\underline{u} = \underline{u}_\lambda$ where \underline{u}_λ is defined in Lemma 6.1.17. Then \underline{u} is a strict subsolution of (P_λ) . Let $\lambda' \in (0, \Lambda)$ then it is easy to see that $u_{\lambda'}$ is a solution of $(P_{\lambda'})$ and forms a supersolution of (P_λ) . Note that such a λ' exists because of definition of Λ . Let $\bar{u} = u_{\lambda'}$ and $M := \{u \in X_0 \mid \underline{u} \leq u \leq \bar{u}\}$. Then M is closed, convex and J_λ is coercive and weakly semi lower continuous on M . It implies that u_n is a sequence in M such that $J_\lambda(u_n) \rightarrow \inf_{u \in M} J_\lambda(u) > -\infty$ when $n \rightarrow \infty$ and $u_n \leq \bar{u}$. It implies $\{u_n\}$ is bounded in X_0 . Then there exists $u_\lambda \in M$ such that (up to a subsequence) $u_n \rightharpoonup u_\lambda$ weakly in X_0 .

Claim: u_λ is weak solution of (P_λ) .

For $\phi \in X_0$ and $\varepsilon > 0$ small enough, define $v_\varepsilon = u_\lambda + \varepsilon \phi - \phi^\varepsilon + \phi_\varepsilon \in M$, where

$$\phi^\varepsilon = (u_\lambda + \varepsilon \phi - \bar{u})^+ \text{ and } \phi_\varepsilon = (u_\lambda + \varepsilon \phi - \underline{u})^-$$

By construction $\phi^\varepsilon, \phi_\varepsilon \in X_0 \cap L^\infty((-1, 1))$ and $u_\lambda + t(v_\varepsilon - u_\lambda) \in M$ for each $t \in (0, 1)$, we have

$$0 \leq \lim_{t \rightarrow 0^+} \frac{J_\lambda(u_\lambda + t(v_\varepsilon - u_\lambda)) - J_\lambda(u_\lambda)}{t}$$

$$= \int_Q (v_\varepsilon - u_\lambda)(-\Delta)^{\frac{1}{2}} u_\lambda \, dx - \int_{-1}^1 u_\lambda^{-\delta} (v_\varepsilon - u_\lambda) \, dx - \int_{-1}^1 f(u_\lambda)(v_\varepsilon - u_\lambda) \, dx.$$

Now using the same arguments as in proof of [150, Proposition 3.2] coupled with Lemma 6.1.23 and for the case $\lambda = \Lambda$ using the same assertions as in [150, Theorem 3.4], we have desired result. \square

Lemma 6.1.23. *Let $\lambda \in (0, \Lambda)$ and u_λ denotes the weak solution of (P_λ) obtained in Lemma 6.1.22. Then u_λ is a local minimum of the functional J_λ .*

Proof. The proof follows by using the same arguments as in [150, Lemma 3.3](see [104]), one can proof that u_λ is local minimum of the functional J_λ in X_0 topology. \square

Lemma 6.1.24. *There exists a positive weak solution of (P_Λ) and any weak solution of (P_λ) for $\lambda \in (0, \Lambda]$, belongs to $L^\infty((-1, 1)) \cap C_{\phi_\delta}^+((-1, 1))$ where ϕ_δ is defined in (6.1.4).*

Proof. See [150, Theorem 3.4, Proposition 4.1]. \square

The concern of this section is to prove the existence of a second solution for (P_λ) . Let u_λ is the first weak solution of (P_λ) in X_0 topology obtained in 6.1.22. Now, consider the following problem, which is (P_λ) translated by u_λ :

$$(\tilde{P}_\lambda) \quad \begin{cases} (-\Delta)^{\frac{1}{2}} u = \lambda \left((u + u_\lambda)^{-\delta} - (u_\lambda)^{-\delta} + f(u + u_\lambda) - f(u_\lambda) \right), u > 0 & \text{in } (-1, 1), \\ u = 0 & \text{in } \mathbb{R} \setminus (-1, 1). \end{cases}$$

For $x \in (-1, 1)$, define

$$\begin{aligned} \tilde{g}(x, s) &= \left((s + u_\lambda(x))^{-\delta} - (u_\lambda(x))^{-\delta} \right) \chi_{\mathbb{R}^+}(s), \quad \tilde{G}(x, t) = \int_0^t \tilde{g}(x, s) \, ds, \\ \tilde{f}(x, s) &= (f(s + u_\lambda(x)) - f(u_\lambda(x))) \chi_{\mathbb{R}^+}(s), \quad \tilde{F}(x, t) = \int_0^t \tilde{f}(x, s) \, ds. \end{aligned}$$

Let $\tilde{J}_\lambda : X_0 \rightarrow \mathbb{R}$ be the energy functional associated with (\tilde{P}_λ) defined as

$$\tilde{J}_\lambda(u) = \frac{\|u\|^2}{2} - \lambda \int_{-1}^1 \tilde{G}(x, u(x)) \, dx - \lambda \int_{-1}^1 \tilde{F}(x, u(x)) \, dx.$$

Remark 6.1.25. (i) *By Theorem 3.1.3, it can be easily shown that the map $X_0 \ni u \rightarrow \frac{1}{2}\|u\|^2 - \lambda \int_{-1}^1 \tilde{F}(x, u(x)) \, dx \in \mathbb{R}$ is a C^1 map. The map $X_0 \ni u \rightarrow \lambda \int_{-1}^1 \tilde{G}(x, u(x)) \, dx \in \mathbb{R}$ is locally Lipschitz. Therefore, \tilde{J}_λ is a sum of a C^1 and a Lipschitz functional. Hence, the generalized derivative of \tilde{J}_λ exist for all $u \in X_0$ and given by*

$$\tilde{J}_\lambda^0(u, \phi) = \limsup_{h \rightarrow 0, t \rightarrow 0} \frac{\tilde{J}_\lambda(u + h + t\phi) - \tilde{J}_\lambda(u + h)}{t}, \quad \phi \in X_0.$$

We say u is a generalized critical point if $\tilde{J}_\lambda^0(u, \phi) \geq 0$ for all $\phi \in X_0$.

(ii) For any $u \in X_0$,

$$J_\lambda(u^+ + u_\lambda) = J_\lambda(u_\lambda) + \tilde{J}_\lambda(u) - \frac{\|u^-\|^2}{2} - 4 \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{u^+(x)u^-(y)}{|x-y|^2} dx dy.$$

Since u_λ is a local minimum of J_λ , it follows that 0 is a local minimum of \tilde{J}_λ in X_0 -topology.

(iii) One can easily prove that if $u \geq 0$ then

$$\tilde{J}_\lambda^0(u, \phi) \leq \langle u_\lambda + u, \phi \rangle - \lambda \int_{-1}^1 (u_\lambda + u)^{-\delta} \phi dx - \lambda \int_{-1}^1 f(u_\lambda + u) \phi dx.$$

Now we will use the machinery of mountain pass Lemma and Ekeland variational principle to prove the existence of second solution. We will show the existence of solution in the following cone:

$$T = \{u \in X_0 : u \geq 0 \text{ a.e in } (-1, 1)\}.$$

Since 0 is local minimum of \tilde{J}_λ in X_0 topology, there exists a $\rho_0 > 0$ such that $\tilde{J}_\lambda(0) \leq \tilde{J}_\lambda(u)$ provided $\|u\| < \rho_0$. We distinguish two cases:

(ZA) (Zero Altitude): $\inf\{\tilde{J}_\lambda(u) : u \in T, \|u\| = \rho\} = \tilde{J}_\lambda(0) = 0$ for all $\rho \in (0, \rho_0)$.

(MP) (Mountain Pass) : There exists $\rho_1 \in (0, \rho_0)$ such that $\inf\{\tilde{J}_\lambda(u) : u \in T, \|u\| = \rho_1\} > \tilde{J}_\lambda(0)$.

Lemma 6.1.26. *Let (ZA) holds for some $\lambda \in (0, \Lambda)$. Then there exists a non-trivial generalized critical point $v_\lambda \in T$ for \tilde{J}_λ .*

Proof. Fix $\rho \in (0, \rho_0)$. By using the definition of infimum of there exist $\{u_n\} \subseteq T$ with $\|u_n\| = \rho$ and $\tilde{J}_\lambda(u_n) \leq 1/n$. Let $0 < \sigma < \frac{1}{2} \min\{\rho_0 - \rho, \rho\}$ and define the set

$$A = \{u \in T : \rho - \sigma \leq \|u\| \leq \rho + \sigma\}$$

which is closed in X_0 and \tilde{J}_λ is continuous on A . Now using the Ekeland Variational principle, we obtain the existence of a sequence $\{v_n\} \in A$ such that

$$\begin{cases} \tilde{J}_\lambda(v_n) \leq \tilde{J}_\lambda(u_n) \leq \frac{1}{n}, & \|u_n - v_n\| \leq \frac{1}{n}, \\ \tilde{J}_\lambda(v_n) \leq \tilde{J}_\lambda(z) + \frac{1}{n}\|z - v_n\| & \text{for all } z \in A. \end{cases} \quad (6.1.31)$$

For a given $z \in T$, we can choose $\varepsilon > 0$ such that $v_n + \varepsilon(z - v_n) \in A$. From (6.1.31), we obtain that

$$\frac{\tilde{J}_\lambda(v_n + \varepsilon(z - v_n)) - \tilde{J}_\lambda(v_n)}{\varepsilon} \geq -\frac{1}{n}\|z - v_n\|.$$

Taking $\varepsilon \rightarrow 0^+$ we get

$$\tilde{J}_\lambda^0(v_n, z - v_n) \geq -\frac{1}{n}\|z - v_n\| \text{ for all } z \in A.$$

From the remark 6.1.25, we deduce that for any $z \in A$,

$$\langle u_\lambda + v_n, z - v_n \rangle - \lambda \int_{-1}^1 (u_\lambda + v_n)^{-\delta} (z - v_n) dx - \lambda \int_{-1}^1 f(u_\lambda + v_n)(z - v_n) dx \geq -\frac{1}{n} \|z - v_n\|. \quad (6.1.32)$$

Since v_n is a bounded sequence in X_0 therefore, there exists $v_\lambda \in X_0$ such that $v_n \rightharpoonup v_\lambda$ weakly in X_0 as well as almost everywhere in $(-1, 1)$. We claim that v_λ is a weak solution of (\tilde{P}_λ) . For any $\psi \in C_c^\infty((-1, 1))$, set

$$\psi_{n,\varepsilon} = (v_n + \varepsilon\psi)^- \text{ and } z = v_n + \varepsilon\psi + \psi_{n,\varepsilon} = (v_n + \varepsilon\psi)^+ \in T.$$

Hence as a result of (6.1.32) and the choice of z , we have

$$\begin{aligned} & \langle u_\lambda + v_n, z - v_n \rangle - \lambda \int_{-1}^1 (u_\lambda + v_n)^{-\delta} (\varepsilon\psi + \psi_{n,\varepsilon}) dx - \lambda \int_{-1}^1 f(u_\lambda + v_n)(\varepsilon\psi + \psi_{n,\varepsilon}) dx \\ & \geq -\frac{1}{n} \|\varepsilon\psi + \psi_{n,\varepsilon}\|. \end{aligned}$$

Observe that $\psi_{n,\varepsilon} \rightarrow \psi_\varepsilon = (v_\lambda + \varepsilon\psi)^-$ a.e in $(-1, 1)$, $|\psi_{n,\varepsilon}| \leq \varepsilon|\psi|$ in $(-1, 1)$ and by using dominated convergence theorem one can easily show that $\psi_{n,\varepsilon} \rightarrow \psi_\varepsilon$ in $L^m((-1, 1))$ for all $m > 1$. Moreover, $\psi_{n,\varepsilon} \rightharpoonup \psi_\varepsilon$ weakly in X_0 . Using the same arguments as in [150, Lemma 4.2], we have

$$\langle u_\lambda + v_n, \varepsilon\psi + \psi_{n,\varepsilon} \rangle \leq \langle u_\lambda + v_\lambda, \varepsilon\psi + \psi_\varepsilon \rangle + o_n(1). \quad (6.1.33)$$

By Hardy's Inequality (See [162, Corollary 1.4.4.10, p.23]) and dominated convergence theorem,

$$\int_{-1}^1 (u_\lambda + v_n)^{-\delta} (\varepsilon\psi + \psi_{n,\varepsilon}) dx \rightarrow \int_{-1}^1 (u_\lambda + v_\lambda)^{-\delta} (\varepsilon\psi + \psi_\varepsilon) dx. \quad (6.1.34)$$

Taking into account the hypothesis (K2), Theorem 3.1.3 and Vitali's convergence theorem, we get

$$\int_{-1}^1 f(u_\lambda + v_n)\psi dx \rightarrow \int_{-1}^1 f(u_\lambda + v_\lambda)\psi dx.$$

Using the mean value Theorem, definition of $\psi_{n,\varepsilon}$ and the fact that $f' > 0$, we deduce that

$$\begin{aligned} f(u_\lambda + v_n)\psi_{n,\varepsilon} & \leq (f(u_\lambda - \varepsilon\psi) + f'(\xi_n)(v_n + \varepsilon\psi))\psi_{n,\varepsilon} \\ f(u_\lambda - \varepsilon\psi)\psi_{n,\varepsilon} & \leq f(u_\lambda - \varepsilon\psi)\varepsilon|\psi| \in L^1((-1, 1)). \end{aligned}$$

This on using dominated convergence theorem gives

$$\int_{-1}^1 f(u_\lambda + v_n)(\varepsilon\psi + \psi_{n,\varepsilon}) dx \rightarrow \int_{-1}^1 f(u_\lambda + v_\lambda)(\varepsilon\psi + \psi_\varepsilon) dx. \quad (6.1.35)$$

Using (6.1.33), (6.1.34) and (6.1.35), we obtain that

$$\langle u_\lambda + v_\lambda, \varepsilon\psi + \psi_\varepsilon \rangle - \int_{-1}^1 (u_\lambda + v_\lambda)^{-\delta} (\varepsilon\psi + \psi_\varepsilon) dx - \int_{-1}^1 f(u_\lambda + v_\lambda) (\varepsilon\psi + \psi_\varepsilon) dx \geq 0.$$

Employing the fact that u_λ is a weak solution of (P_λ) , we get

$$\begin{aligned} & \langle u_\lambda + v_\lambda, \psi \rangle - \int_{-1}^1 (u_\lambda + v_\lambda)^{-\delta} \psi dx - \int_{-1}^1 f(u_\lambda + v_\lambda) \psi dx \\ & \geq -\frac{1}{\varepsilon} \langle v_\lambda, \psi_\varepsilon \rangle + \frac{1}{\varepsilon} \int_{-1}^1 ((u_\lambda + v_\lambda)^{-\delta} - (u_\lambda)^{-\delta}) \psi_\varepsilon dx + \frac{1}{\varepsilon} \int_{-1}^1 (f(u_\lambda + v_\lambda) - f(u_\lambda)) \psi_\varepsilon dx \\ & \geq -\frac{1}{\varepsilon} \langle v_\lambda, \psi_\varepsilon \rangle + \frac{1}{\varepsilon} \int_{\Omega_\varepsilon} ((u_\lambda + v_\lambda)^{-\delta} - (u_\lambda)^{-\delta}) \psi_\varepsilon dx \end{aligned} \tag{6.1.36}$$

where the last inequality follows using the fact that f is an increasing function and $v_\lambda \geq 0$ and $\text{supp}(\psi_\varepsilon) =: \Omega_\varepsilon \subset (-1, 1)$. Keeping in mind that $u_\lambda^{-\delta} \phi \in L^1((-1, 1))$,

$$\frac{1}{\varepsilon} \int_{\Omega_\varepsilon} ((u_\lambda + v_\lambda)^{-\delta} - (u_\lambda)^{-\delta}) \psi_\varepsilon dx \leq \frac{2}{\varepsilon} \int_{-1}^1 u_\lambda^{-\delta} \psi = o(1).$$

Furthermore, trivial calculations gives

$$\begin{aligned} -\frac{1}{\varepsilon} \langle v_\lambda, \psi_\varepsilon \rangle & \geq \int_{\Omega_\varepsilon} \int_{\Omega_\varepsilon} \frac{(v_\lambda(x) - v_\lambda(y))(\psi(x) - \psi(y))}{|x - y|^2} dx dy \\ & + 2 \int_{\Omega_\varepsilon} \int_{\Omega_\varepsilon} \frac{(\psi(x) - \psi(y))(v_\lambda + \psi)(y)}{|x - y|^2} dx dy + 2 \int_{\Omega_\varepsilon} \int_{\Omega_\varepsilon} \frac{v_\lambda(x)\psi(x)}{|x - y|^2} dx dy. \end{aligned}$$

Letting $\varepsilon \rightarrow 0$ in (6.1.36), we deduce that, for all $\psi \in C_c^\infty((-1, 1))$,

$$\langle u_\lambda + v_\lambda, \psi \rangle - \int_{-1}^1 (u_\lambda + v_\lambda)^{-\delta} \psi dx - \int_{-1}^1 f(u_\lambda + v_\lambda) \psi dx \geq 0.$$

It implies that v_λ is a generalized critical point. Now we will show that $v_\lambda \not\equiv 0$. Note that $\|v_n\| \geq \max\{2\rho - \rho_0, 0\} \geq 0$, so it enough to show that $v_n \rightarrow v_\lambda$ strongly in X_0 . Let $z = v_\lambda$ in (6.1.32),

$$\begin{aligned} \|v_\lambda - v_n\|^2 & \leq \langle u_\lambda + v_\lambda, v_\lambda - v_n \rangle + \frac{1}{n} \|v_\lambda - v_n\| - \lambda \int_{-1}^1 (u_\lambda + v_n)^{-\delta} (v_\lambda - v_n) dx \\ & \quad - \lambda \int_{-1}^1 f(u_\lambda + v_n) (v_\lambda - v_n) dx. \end{aligned}$$

Observe that $(u_\lambda + v_n)^{-\delta} (v_\lambda - v_n) \rightarrow 0$ as $n \rightarrow \infty$ a.e on $(-1, 1)$, $u_\lambda \sim \phi_\delta$ in the neighborhood of -1 and 1 . In consequence of Hardy's inequality and Hölder inequality, for any measurable set $E \subset (-1, 1)$ and $\delta > 1$, we have

$$\int_E (u_\lambda + v_n)^{-\delta} (v_\lambda - v_n) dx \leq \int_E u_\lambda^{-\delta} |v_\lambda - v_n| dx \leq C \int_E \phi_\delta^{\frac{-2\delta}{\delta+1}} |v_\lambda - v_n| dx$$

$$\leq C \int_E d^{\frac{1-\delta}{2(\delta+1)}} \frac{|v_\lambda - v_n|}{d^{\frac{1}{2}}} dx \leq C \|v_\lambda - v_n\|_{X_0} \|d^{\frac{1-\delta}{2(\delta+1)}}\|_{L^2(E)}.$$

Thus in a consequence of Vitali's convergence theorem $\int_{-1}^1 (u_\lambda + v_n)^{-\delta} (v_\lambda - v_n) dx \rightarrow 0$. Rewrite

$$f(u_\lambda + v_n)(v_\lambda - v_n) = f(u_\lambda + v_n)(u_\lambda + v_\lambda) - f(u_\lambda + v_n)(u_\lambda + v_n).$$

Using the same arguments used for (6.1.35), one can easily show that

$$\int_{-1}^1 f(u_\lambda + v_n)(u_\lambda + v_\lambda) dx \rightarrow \int_{-1}^1 f(u_\lambda + v_\lambda)(u_\lambda + v_\lambda) dx.$$

Let $z_n = u_\lambda + v_n$ and $z_\lambda = u_\lambda + v_\lambda$ then $f(z_n)z_n \rightarrow f(z_\lambda)z_\lambda$ a.e in $(-1, 1)$. Let k be any integer such that $k > \|u_\lambda\|_\infty$. Using (K2),

$$\int_{\{z_n \geq k\}} f(z_n)z_n dx \leq C \int_{\{z_n \geq k\}} e^{\frac{3z_n^2}{2}} z_n dx \leq C \int_{\{z_n \geq k\}} e^{2z_n^2} z_n dx \leq C e^{-k^2} \int_{-1}^1 e^{3z_n^2} dx.$$

By means of the Hölder inequality and the relation $z_n^2 \leq 2(u_\lambda^2 + (z_n - u_\lambda)^2)$, we deduce that

$$\int_{\{z_n \geq k\}} f(z_n)z_n dx \leq C e^{-k^2} \int_{-1}^1 e^{6u_\lambda^2} e^{6v_n^2} dx \leq C e^{-k^2} \|e^{6u_\lambda^2}\|_{L^p((-1,1))} \|e^{6v_n^2}\|_{L^{p'}((-1,1))}. \quad (6.1.37)$$

Now for ρ_0 small enough, we can choose $p' > 1$ such that $6p'\|v_n\| \leq 12p'\rho_0 < \pi$. With the help of Trudinger-Moser inequality and (6.1.37), we have $\int_{\{z_n \geq k\}} f(z_n)z_n dx \leq C e^{-k^2}$ where C is independent of n . Hence for k large enough,

$$\int_{-1}^1 f(z_n)z_n dx \leq \int_{\{z_n \leq k\}} f(z_n)z_n dx + C e^{-k^2}.$$

Letting $n \rightarrow \infty$ and $k \rightarrow \infty$, $\limsup_{n \rightarrow \infty} \int_{-1}^1 f(z_n)z_n dx \leq \int_{-1}^1 f(z_\lambda)z_\lambda dx$. Using the fact that $v_n \rightharpoonup v_\lambda$ weakly in X_0 , we get $\langle u_\lambda + v_\lambda, v_\lambda - v_n \rangle \rightarrow 0$ as $n \rightarrow \infty$. Therefore, from all the calculations, we obtain that $\|v_\lambda - v_n\| \rightarrow 0$ as $n \rightarrow \infty$. \square

Now we will prove the existence of second solution if (MP) holds. Before this we will prove some preliminary results. We recall the definition of Moser function ω_n for half-Laplacian, which is recently given by Takahashi [244].

$$\omega_n(x) = \frac{1}{\sqrt{\pi}} \begin{cases} (\log n)^{\frac{1}{2}} & \text{if } |x| \leq \frac{1}{n}, \\ -\log(n)^{-\frac{1}{2}} \log |x| & \text{if } \frac{1}{n} \leq |x| \leq 1, \\ 0 & \text{if } |x| \geq 1. \end{cases}$$

Fix $x_0 \in \mathbb{R}$ and $r > 0$ such that $\omega_n^r(x) = \omega_n(\frac{x-x_0}{r})$ has support in $(-1, 1)$. Note that $\|\omega_n^r\| = 1$.

Lemma 6.1.27. *Assume (K1)-(K3). Then the following holds.*

(i) $\tilde{J}_\lambda(t\omega_n^r) \rightarrow -\infty$ as $t \rightarrow \infty$.

(ii) For a suitable $x_0 \in (-1, 1)$ and $r > 0$ small enough, $\sup_{t>0} \tilde{J}_\lambda(t\omega_n^r) < \frac{\pi}{2}$.

Proof. (i) Using (K2), there exist positive constants C_1 and C_2 such that

$$\tilde{F}(x, t) \geq C_1 e^{\frac{1}{2}(u_\lambda + t)^2} - C_2 - f(u_\lambda)t \text{ for } t \geq 0.$$

Hence for some $C > 0$,

$$\tilde{J}_\lambda(t\omega_n^r) \leq \frac{t^2}{2} - C\lambda \int_{B_{\frac{r}{n}}(x_0)} e^{\frac{t^2}{2}|\omega_n^r|^2} dx = \frac{t^2}{2} - \frac{C\lambda r}{n} e^{\frac{t^2 \log n}{\pi}} \rightarrow -\infty \text{ as } t \rightarrow \infty.$$

(ii) On the contrary, suppose that there exists a subsequence of \mathbb{N} such that $\sup_{t>0} \tilde{J}_\lambda(t\omega_n^r) \geq \frac{\pi}{2}$. That is, $\frac{t_n^2}{2} - \lambda \int_{-1}^1 \tilde{G}(x, t_n\omega_n) dx - \lambda \int_{-1}^1 \tilde{F}(x, t_n\omega_n) dx \geq \frac{\pi}{2}$. Using (K1), we have

$$\frac{t_n^2}{2} - \lambda \int_{-1}^1 \tilde{G}(x, t_n\omega_n) dx \geq \frac{\pi}{2}.$$

Clearly, $\tilde{g}(x, s) \leq 0$ for $x \in (-1, 1)$ and $s > 0$. For and $x \in B_r(x_0)$, applying Taylor's expansion, $\tilde{g}(x, s) = \frac{\delta s}{u_\lambda^{\delta+1}} + o(s^2)$. It implies that

$$\int_{-1}^1 \tilde{G}(x, t_n\omega_n) dx = \int_{B_r(x_0)} \tilde{G}(x, t_n\omega_n) dx \leq \int_{B_r(x_0)} \int_0^{t_n\omega_n} Cs ds dx \leq Ct_n^2 O((\log n)^{-1}).$$

As a result, we get

$$t_n^2 \geq \pi - O((\log n)^{-1}). \quad (6.1.38)$$

Since $\frac{d}{dt} \tilde{J}(t\omega_n^r)|_{t=t_n} = 0$, we get $t_n^2 - \lambda \int_{-1}^1 \tilde{g}(x, t_n\omega_n^r) t_n \omega_n^r dx = \lambda \int_{-1}^1 \tilde{f}(x, t_n\omega_n^r) t_n \omega_n^r dx$. Again using the fact that $|\tilde{g}(x, s)| \leq Cs$, we have

$$\int_{-1}^1 \tilde{f}(x, t_n\omega_n^r) t_n \omega_n^r dx \leq t_n^2 [1 + O((\log n)^{-1})]. \quad (6.1.39)$$

By definition of \tilde{f} and the fact that $\|\omega_n^r\|_{L^1((-1,1))} = O((\log n)^{-\frac{1}{2}})$, we have

$$\int_{-1}^1 \tilde{f}(x, t_n\omega_n^r) t_n \omega_n^r dx = \int_{-1}^1 f(u_\lambda + t_n\omega_n^r) t_n \omega_n^r dx - t_n O((\log n)^{-\frac{1}{2}}).$$

Now we will estimate $\int_{-1}^1 f(u_\lambda + t_n\omega_n^r) t_n \omega_n^r dx$ from below. Let $\mu = \min_{B_r(x_0)} u_\lambda$. Taking into account (K1), (K2), definition of ω_n^r and (6.1.38), we have

$$\begin{aligned} \int_{-1}^1 f(u_\lambda + t_n\omega_n^r) t_n \omega_n^r dx &\geq \int_{B_r(x_0)} h(u_\lambda + t_n\omega_n^r) e^{(u_\lambda + t_n\omega_n^r)^2} t_n \omega_n^r dx \\ &\geq Ch \left(\mu + \frac{t_n}{\sqrt{\pi}} (\log n)^{\frac{1}{2}} \right) e^{(\mu + \frac{t_n}{\sqrt{\pi}} (\log n)^{\frac{1}{2}})^2} t_n (\log n)^{\frac{1}{2}} \int_{B_{\frac{r}{n}}(x_0)} dx \\ &\geq \frac{Cr}{n} e^{-\left(\mu + \frac{t_n}{\sqrt{\pi}} (\log n)^{\frac{1}{2}}\right)^q} e^{2(\mu + \frac{t_n}{\sqrt{\pi}} (\log n)^{\frac{1}{2}})} e^{t_n^2 \frac{\log n}{\pi}} t_n (\log n)^{\frac{1}{2}} \left(\mu + \frac{t_n}{\sqrt{\pi}} (\log n)^{\frac{1}{2}} \right)^{-1} \\ &\geq Ce^{2(\mu + \frac{t_n}{\sqrt{\pi}} (\log n)^{\frac{1}{2}}) - (\mu + \frac{t_n}{\sqrt{\pi}} (\log n)^{\frac{1}{2}})^q}. \end{aligned}$$

Thus we obtain

$$\int_{-1}^1 \tilde{f}(x, t_n \omega_n^r) t_n \omega_n^r dx = C e^{2(\mu + \frac{t_n}{\sqrt{\pi}}(\log n)^{\frac{1}{2}}) - (\mu + \frac{t_n}{\sqrt{\pi}}(\log n)^{\frac{1}{2}})^q} - t_n O((\log n)^{-\frac{1}{2}}). \quad (6.1.40)$$

Since $(\log n) \rightarrow \infty$ and t_n is bounded away from 0 as $n \rightarrow \infty$, we obtain a contradiction from (6.1.39) and (6.1.40). \square

Lemma 6.1.28. *Let $\{u_k : \|u_k\| = 1\}$ be a sequence of X_0 functions converging weakly to a nonzero function u . Then for all $p < (1 - \|u\|)^{-1}$,*

$$\sup_{k \in \mathbb{N}} \int_{-1}^1 e^{\pi p |u_k|^2} dx < \infty.$$

Proof. See [149, Lemma 4.4]. \square

Lemma 6.1.29. *Assume (K1)-(K3) and fix $\lambda \in (0, \Lambda)$. Let (MP) holds then there exists a nontrivial generalized critical point v_λ of \tilde{J}_λ .*

Proof. Define the complete metric space

$$Y = \{\eta \in (C[0, 1], T) : \eta(0) = 0, \|\eta(1)\| > \rho_1, \tilde{J}_\lambda(\eta(1)) < \tilde{J}_\lambda(0) = 0\},$$

with metric space defined as $d(\eta, \eta') = \max_{t \in [0, 1]} \{\|\eta(t) - \eta'(t)\|\}$ for all $\eta, \eta' \in Y$. Fix $x_0 \in (-1, 1)$ and $r > 0$ such that Lemma 6.1.27 (ii) holds. Now choose $t_0 > 1$ such that $\tilde{J}_\lambda(t_0 \omega_n^r) < 0$. Note that existence of t_0 holds by Lemma 6.1.27 (i). Let $\eta(t) = t t_0 \omega_n^r$, $t > 0$. Then $\eta \in Y$. Define the mountain-pass critical level

$$\gamma_0 = \inf_{\eta \in Y} \max_{t \in [0, 1]} \tilde{J}_\lambda(\eta(t)).$$

From Lemma 6.1.27, we have $0 < \gamma_0 < \frac{\pi}{2}$. Define $\Psi : X \rightarrow \mathbb{R}$ as $\Psi(\eta) = \max_{t \in [0, 1]} \tilde{J}_\lambda(\eta(t))$, $\eta \in Y$.

Applying the Ekeland's variational principle, we get a sequence $\{\eta_k\} \subset Y$ such that

$$\Psi(\eta_k) < \gamma_0 + \frac{1}{k} \text{ and } \Psi(\eta_k) < \Psi(\eta) + \frac{1}{k} \|\eta - \eta_k\|, \text{ for all } \eta \in Y.$$

Denote $Z_k = \{t \in (0, 1) : \tilde{J}_\lambda(\eta_k(t)) = \max_{s \in [0, 1]} \tilde{J}_\lambda(\eta_k(s))\}$. Now using the arguments and assertions as in [150, Lemma 4.4], there exist $t_k \in Z_k$ such that if $v_k = \eta_k(t_k)$, then

- (i) $\tilde{J}_\lambda^0\left(v_k; \frac{w - v_k}{\max\{1, \|w - v_k\|\}}\right) \geq -\frac{1}{k}$ for all $w \in T$,
- (ii) $\tilde{J}_\lambda(v_k) \rightarrow \gamma_0$ as $k \rightarrow \infty$.

Taking $w = u_\lambda + 2v_k$ in (i), we obtain,

$$-\frac{1}{k} \max\{1, \|u_\lambda + v_k\|\} \leq \|u_\lambda + v_k\|^2 - \lambda \int_{-1}^1 \left((u_\lambda + v_k)^{1-\delta} + f(u_\lambda + v_k)(u_\lambda + v_k) \right) dx. \quad (6.1.41)$$

Fix $\varepsilon > 0$, using (K4), then there exists $C_\varepsilon > 0$ such that

$$\begin{aligned}\tilde{F}(x, v_k) &\leq \int_0^{u_\lambda + v_k} f(s) ds - f(u_\lambda)v_k \leq M(f(u_\lambda + v_k) + 1) - f(u_\lambda)v_k \\ &\leq \varepsilon f(u_\lambda + v_k)(u_\lambda + v_k) + C_\varepsilon - f(u_\lambda)v_k.\end{aligned}$$

Now using the fact that $\tilde{G}(x, v_k) \leq 0$, we have

$$\frac{1}{2}\|v_k\|^2 - \lambda \varepsilon f(u_\lambda + v_k)(u_\lambda + v_k) - C_\varepsilon + f(u_\lambda)v_k \leq \gamma_0 + o_k(1). \quad (6.1.42)$$

From (6.1.41) and (6.1.42), we have

$$\left(\frac{1}{2} - \varepsilon\right)\|v_k\|^2 < C + \lambda \int_{-1}^1 u_\lambda^{-\delta} v_k dx + \|u_\lambda\|^2 + \frac{\varepsilon}{k} \max\{1, \|u + \lambda + v_k\|\}.$$

With the help of Hardy's inequality, we obtain that $\{v_k\}$ is a bounded sequence in T . Therefore, there exists a $v_\lambda \in T$ such that $v_k \rightharpoonup v_\lambda$ in X_0 . From (i), we have

$$\tilde{J}_\lambda^0(v_k; w - v_k) \geq -\frac{1}{k}(1 + \|w\|), \quad (6.1.43)$$

for all $w \in T$. Using the same assertions and arguments as in proof of Lemma 6.1.26, one can easily prove that v_λ is a generalized critical point of (\tilde{P}_λ) . From (6.1.41), $\limsup_{k \rightarrow \infty} \int_{-1}^1 f(u_\lambda + v_k)(u_\lambda + v_k) < \infty$. Hence by Vitali's convergence theorem, $\int_{-1}^1 f(u_\lambda + v_k) dx \rightarrow \int_{-1}^1 f(u_\lambda + v_\lambda) dx$. Now using (K4) and generalized dominated convergence theorem $\int_{-1}^1 \tilde{F}(x, v_k) dx \rightarrow \int_{-1}^1 \tilde{F}(x, v_\lambda) dx$. Using the fact that $u_\lambda \sim \phi_\delta$, Hardy's inequality and similar arguments used above we can easily prove that $\int_{-1}^1 \tilde{G}(x, v_k) dx \rightarrow \int_{-1}^1 \tilde{G}(x, v_\lambda) dx$. Since $v_k \rightharpoonup v_\lambda$ weakly in X_0 ,

$$\tilde{J}(v_\lambda) \leq \liminf_{k \rightarrow \infty} \tilde{J}_\lambda(v_k) = \gamma_0.$$

Since $\lim_{k \rightarrow \infty} \tilde{J}_\lambda(v_k) = \gamma_0$ and if $v_k \rightarrow v_\lambda$ strongly in X_0 then $0 < \gamma_0 = \tilde{J}_\lambda(v_\lambda)$ implies $v_\lambda \neq 0$. Therefore, to show $v_\lambda \neq 0$, it is enough to show that $v_k \rightarrow v_\lambda$ strongly in X_0 . Let if possible then $v_k \not\rightarrow v_\lambda$ in X_0 then $\tilde{J}_\lambda(v_\lambda) < \gamma_0$, we can assume $\tilde{J}_\lambda(v_\lambda) = 0$ otherwise $v_\lambda \neq 0$. From Remark 6.1.25, we have $J_\lambda(u_\lambda + v_\lambda) = J_\lambda(u_\lambda)$. We can choose $\varepsilon > 0$ small enough so that

$$(\gamma_0 + J_\lambda(u_\lambda + v_\lambda) - J_\lambda(u_\lambda)) = \gamma_0(1 + \varepsilon) < \frac{\pi}{2}. \quad (6.1.44)$$

Define $\Theta = \lambda \int_{-1}^1 F(u_\lambda + v_\lambda) + G(u_\lambda + v_\lambda) dx$. Using the Remark 6.1.25, we have $\tilde{J}_\lambda(v_k) = J_\lambda(u_\lambda + v_k) - J - \lambda(u_\lambda)$. Therefore, $2(\gamma_0 + \Theta + J_\lambda(u_\lambda)) = \lim_{k \rightarrow \infty} \|u_\lambda + v_k\|^2$. Since $\tilde{J}_\lambda(v_\lambda) < \gamma_0$ then $\|u_\lambda + v_\lambda\|^2 < \lim_{k \rightarrow \infty} \|u_\lambda + v_k\|^2$. It gives that

$$0 < \|u_\lambda + v_\lambda\|^2 < 2(\gamma_0 + \Theta + J_\lambda(u_\lambda)). \quad (6.1.45)$$

Taking into account (6.1.44), (6.1.45) and the fact that $J_\lambda(u_\lambda + v_\lambda) = \frac{1}{2}\|u_\lambda + v_\lambda\|^2 - \Theta$, we deduce that

$$(1 + \varepsilon)\|u_\lambda + v_k\|^2 < \frac{\pi(\gamma_0 + \Theta + J_\lambda(u_\lambda))}{\gamma_0 + \Theta + J_\lambda(u_\lambda) + \frac{1}{2}\|u_\lambda + v_\lambda\|^2} = \pi \left(1 - \frac{\|u_\lambda + v_\lambda\|^2}{2(\gamma_0 + \Theta + J_\lambda(u_\lambda))} \right)^{-1}.$$

Now taking into mind (6.1.45), we can choose $p > 1$ such that

$$\frac{(1 + \varepsilon)\|u_\lambda + v_k\|^2}{p} \leq p < \left(1 - \frac{\|u_\lambda + v_\lambda\|^2}{2(\gamma_0 + \Theta + J_\lambda(u_\lambda))} \right)^{-1}.$$

Therefore, from Lemma 6.1.28, $\limsup_{k \rightarrow \infty} \int_{-1}^1 e^{(1+\varepsilon)(u_\lambda+v_k)^2} dx \leq \limsup_{k \rightarrow \infty} \int_{-1}^1 e^{\frac{\pi p (u_\lambda+v_k)^2}{\|u_\lambda+v_k\|^2}} dx < \infty$.

We write

$$\int_{-1}^1 f(u_\lambda + v_k)v_k dx = \int_{-1}^1 f(u_\lambda + v_k)(u_\lambda + v_k) dx - \int_{-1}^1 f(u_\lambda + v_k)u_\lambda dx.$$

From (K2), given $\varepsilon_1 < \varepsilon$ and $N \in \mathbb{N}$, for some $C > 0$, we have

$$\begin{aligned} \int_{-1}^1 f(u_\lambda + v_k)(u_\lambda + v_k) dx &= \left(\int_{v_k \leq N} + \int_{v_k > N} \right) f(u_\lambda + v_k)(u_\lambda + v_k) dx \\ &\leq \int_{v_k \leq N} f(u_\lambda + v_k)(u_\lambda + v_k) dx + C \int_{v_k > N} e^{(1+\varepsilon_1)(u_\lambda+v_k)^2} dx \\ &\leq \int_{v_k \leq N} f(u_\lambda + v_k)(u_\lambda + v_k) dx + Ce^{(\varepsilon_1-\varepsilon)N}. \end{aligned}$$

Now letting $k \rightarrow \infty$ and then $N \rightarrow \infty$, we obtain,

$$\limsup_{k \rightarrow \infty} \int_{-1}^1 f(u_\lambda + v_k)(u_\lambda + v_k) dx \leq \int_{-1}^1 f(u_\lambda + v_\lambda)(u_\lambda + v_\lambda) dx.$$

Hence

$$\limsup_{k \rightarrow \infty} \int_{-1}^1 f(u_\lambda + v_k)(v_k - v_\lambda) dx \leq 0. \quad (6.1.46)$$

On the other hand, since we assume $v_k \not\rightarrow v_\lambda$ then by using Remark 6.1.25, (6.1.43) with $w = v_\lambda$ and the fact that $v_k \rightarrow v_\lambda$, we have

$$0 < \nu \leq \|v_\lambda - v_k\|^2 \leq o(1) - \lambda \int_{-1}^1 f(u_\lambda + v_k)(v_\lambda - v_k) dx. \quad (6.1.47)$$

From (6.1.46) and (6.1.47), we obtain contradiction. Therefore, $v_\lambda \neq 0$. \square

Proof of Theorem 6.1.15: The proof follows from 6.1.22, Lemma 6.1.24, Lemma 6.1.26 along with Lemma 6.1.29. The proof of Hölder regularity follows straightaway from Lemma 6.1.24 and [6, Theorem 1.2] with $\beta = 0$. \square

Proof of Remark 6.1.14: Suppose there exists a sequence (λ_k, u_k) of solutions in $\mathcal{S} \cap ((0, \Lambda) \times C_0([-1, 1]))$ such that $\lambda_k \rightarrow \Lambda_a$ and $\|u_k\|_{L^\infty((-1, 1))} \rightarrow \infty$ as $k \rightarrow \infty$. Now we claim that $\int_{-1}^1 g(u_k) \leq C$ where C is independent of k . Suppose by contradiction that C depends upon k such that $C(k) \rightarrow \infty$ as $k \rightarrow \infty$. Since $g(u_k) \in L_{loc}^\infty((-1, 1) \setminus \{0\})$ uniformly in k , then for each $k \in \mathbb{N}$ there exists a sequence $\{\epsilon_k\}$ such that $\epsilon_k \rightarrow 0$,

$$\int_{B_r(0) \setminus B_{\epsilon_k}(0)} g(u_k) \, dx = n \rightarrow \infty \quad \text{as } k \rightarrow \infty. \quad (6.1.48)$$

and a sequence of classical solutions $\{v_k\}$ such that $v_k \in L^\infty((-1, 1)) \cap C^s(\mathbb{R})$ (see Proposition 1.1 in [226]) satisfying

$$\begin{cases} (-\Delta)^{\frac{1}{2}} v_k = \chi_{(-1, 1) \setminus B_{\epsilon_k}(0)} g(u_k), \, v_k \geq 0 & \text{in } (-1, 1), \\ v_k = 0 & \text{in } \mathbb{R} \setminus (-1, 1). \end{cases}$$

Since $\lim_{x \rightarrow 0} (u_k + \Phi)(x) = +\infty$ then for each k there exists a sequence $\{r_k\}$ such that $r_k \rightarrow 0$ as $k \rightarrow \infty$ and $u_k + \Phi \geq v_k$ in $B_{r_k}(0) \setminus \{0\}$. Then by weak comparison principle we obtain $u_k + \Phi \geq v_k$ in $\mathbb{R}^n \setminus \{0\}$.

Since $\lim_{y \rightarrow x} G(x, y) = +\infty$ then there exists $r_0 > 0$ such that $G(x, y) \geq 1$ in $x, y \in B_{r_0}(0)$. Now by using (6.1.48) we obtain that,

$$\begin{aligned} v_k(x) &= \int_{-1}^1 G(x, y) \chi_{(-1, 1) \setminus B_{\epsilon_k}(0)} g(u) \, dy = \int_{(-1, 1) \setminus B_{\epsilon_k}(0)} G(x, y) g(u) \, dy \\ &\geq \int_{B_{r_0}(0) \setminus B_{\epsilon_k}(0)} g(u) = n \rightarrow \infty \end{aligned}$$

and then it implies $u_k + \Phi = +\infty$ in $B_{r_0}(0)$ which is not possible. Therefore $\int_{-1}^1 g(u_k) \leq C$ where C is independent of k . Then combining with the fact that u_k have bounded energy and $G(t) \leq g(t)$ for large t , we obtain $\|u_k\|_{X_0} \leq C_2$ where C_2 is independent of k . Then there exists \tilde{u}_1 such that $u_k \rightharpoonup \tilde{u}_1$ in X_0 and by compact embedding $X_0 \hookrightarrow L^q((-1, 1))$, we obtain $\tilde{u}_1 = u \in X_0$ where u is the singular solution of (6.1.29) with $\mu = 0$. Then by theorem 3.1.3 and Remark 1.5 in [227], we obtain $g(u) \in L^p((-1, 1))$ and $u \in L^\infty((-1, 1))$, which is absurd. This completes the proof of Remark 6.1.14. \square

6.2 Generalization of symmetry results to fractional Laplacian operator

In this section, we are interested in the study of symmetry of positive solutions to a class of singular semilinear elliptic problem whose prototype is

$$(P) \quad \begin{cases} (-\Delta)^s u = \frac{1}{u^\delta} + f(u), \, u > 0 & \text{in } \Omega; \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $0 < s < 1$, $N \geq 2s$, $\Omega = B_r(0) \subset \mathbb{R}^N$, $\delta > 0$, $f(u)$ is a locally Lipschitz function.

6.2.1 Main results

Theorem 6.2.1. *Let $\delta > 0$ and f be a locally Lipschitz function. Then a classical solution u to (P) is radially symmetric and strictly decreasing in $|x|$.*

The proof of Theorem 6.2.1 involves the moving plane method adapted in the non local setting. In this regard, as in the local case, we need a maximum principle in narrow domains and a strong maximum principle to hold for equations of the type (P). The extension of these key tools is not straightforward due to the non local nature of $(-\Delta)^s$ and the presence of a singular nonlinearity in the right hand side. Besides this, we will take advantage of monotonicity properties of the nonlinear operator $(-\Delta)^s u - \frac{1}{u^\delta}$ and borrow some "local" maximum principle shown in [175]. In this regard, we introduce the following definitions:

Let $A_\lambda := \{x = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N : x_1 = \lambda\}$ and

$$\Sigma_\lambda := \begin{cases} \{x \in \mathbb{R}^N : x_1 < \lambda\} & \text{if } \lambda \leq 0, \\ \{x \in \mathbb{R}^N : x_1 > \lambda\} & \text{if } \lambda > 0 \end{cases}$$

for some $\lambda \in \mathbb{R}$ and $\mathcal{D}_\lambda(x) := (2\lambda - x_1, x_2, \dots, x_N)$ be the reflection of the point x about A_λ and $v_\lambda(x) := u_\lambda(x) - u(x)$ where $u_\lambda(x) = u(\mathcal{D}_\lambda(x))$.

Proof. Let u be a classical solution of (P). To prove radial symmetry and strict monotonicity of the solution u , it is enough to prove $v_\lambda(x) \geq 0$ for all $x \in B_r(0) \cap \Sigma_\lambda$ and $\lambda \in (-r, r)$, by moving hyperplane A_λ in a fixed direction. Since, if $v_\lambda(x) \geq 0$ for all $\lambda \in (-r, r)$ and $x \in B_r(0)$ holds then we can rotate and move the hyperplane A_λ in the direction close to fixed direction to get the desired result. Since λ is independent from the direction of movement of hyperplane A_λ , so we fix $\nu(x_0) = (1, 0, \dots, 0)$ (without loss of generality) as the direction of movement of hyperplane A_λ where ν denotes the unit outward normal vector at $x_0 = (r, 0, \dots, 0) \in \partial B_r(0)$. We divide the proof of above assertion into the following claims:

Claim 1: $v_\lambda(x) \geq 0$ for all $x \in B_r(0) \cap \Sigma_\lambda$ and $|\lambda| \in [r_1, r)$ for some $r_1 > 0$:

Suppose that $v_\lambda < 0$ in a region $P \subset \Sigma_\lambda \cap B_r$ for some $r - \epsilon_1 < |\lambda| < r$ and $\epsilon_1 > 0$. Then by using Poincaré inequality and since f is a Lipschitz function with Lipschitz constant C_L in the neighborhood of x_0 , we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \left((-\Delta)^{\frac{s}{2}} (u - u_\lambda)^+ \right)^2 &\leq \langle (-\Delta)^s (-v_\lambda), (-v_\lambda)^+ \rangle = \int_{B_r} \left(\frac{1}{u^\delta} - \frac{1}{(u_\lambda)^\delta} + f(u) - f(u_\lambda) \right) (-v_\lambda)^+ dx \\ &< C_L \int_P ((u - u_\lambda)^+)^2 dx \leq C(\text{diam}(P)) \int_{\mathbb{R}^n} \left((-\Delta)^{\frac{s}{2}} (u - u_\lambda)^+ \right)^2 dx. \end{aligned}$$

Then by choosing $\epsilon_1 > 0$ small enough such that $C(\text{diam}(P)) \leq 1$, one has $(-v_\lambda)^+ = (u - u_\lambda)^+ = 0$. Now by rotating and moving the hyperplane A_λ in a direction close to the outward normal ν in any neighborhood of $x_0 \in \partial\Omega$ and repeating the above steps by taking

into account that $x_0 \in \partial B_r(0)$, $\nu(x_0)$ is arbitrary and by using continuity of solution u we obtain, $v_\lambda(x) \geq 0$ for all $x \in B_r(0) \setminus B_{r_1}(0)$ and $|\lambda| \in [r_1, r]$ for some $r_1 > 0$.

Claim 2: $v_\lambda \geq 0$ for all $x \in B_r(0) \cap \Sigma_\lambda$ and $|\lambda| \in [0, r_1)$:

From **Claim 1**, we can assume that $\lambda = r_1$ be the smallest value such that $0 \leq r_1 < r$, $v_{r_1} \geq 0$ in $B_r \setminus B_{r_1}$ and satisfies

$$(-\Delta)^s v_{r_1}(x) - \frac{1}{u_{r_1}^\delta(x)} + \frac{1}{u^\delta(x)} = f(u_{r_1}) - f(u) \quad \text{in } B_r \setminus B_{r_1}. \quad (6.2.1)$$

Step 1: $\text{ess inf}_R v_{r_1} > 0$ for every compact subset $R \subset B_r \setminus B_{r_1}$.

To prove this, we adapt in our situation the maximum principles in non-local setting *i.e.* Proposition 3.5 (maximum principle in narrow domains) and Proposition 3.6 (strong maximum principle) in [175]. Since v_{r_1} is non-trivial in $B_r \setminus B_{r_1}$, then it is enough to prove that $\text{ess inf}_{B_{r_0}(x^*)} v_{r_1} > 0$ for all $x^* \in B_r \setminus B_{r_1}$ and r_0 sufficiently small. From **Claim 1**, $v_{r_1} \geq 0$ and $v_{r_1}(x) = -v_{r_1}(\mathcal{D}_{r_1}(x))$ in Σ_{r_1} then there exists a bounded set $Q \subset \Sigma_{r_1}$ with $x^* \notin \bar{Q}$ and $\tilde{\mu} := \inf_Q v_{r_1} > 0$. In the spirit of Lemma 2.1 in [175], we fix r_0 such that $U = B_{2r_0}(x^*)$ and $0 < r_0 < \frac{1}{4} \text{dist}(x^*, Q \cup (\mathbb{R}^N \setminus \Sigma_{r_1}))$ and $\lambda_1(U) \geq C_L(f)$ where $C_L(f)$ is the Lipschitz constant of f and $\lambda_1(U)$ is the first eigenvalue of $(-\Delta)^s$ in U . Now, we construct a subsolution of $(-\Delta)^s \tilde{u} = c(x)\tilde{u}$ in U where

$$c(x) = \begin{cases} \frac{f(u_{r_1}) - f(u)}{v_{r_1}} - \frac{\delta}{(\theta u + (1-\theta)u_{r_1})^{\delta+1}} & \text{if } v_{r_1} \neq 0, \\ 0 & \text{if } v_{r_1} = 0 \end{cases}$$

for some $\theta \in (0, 1)$. Define $k : \mathbb{R}^N \rightarrow \mathbb{R}$, $k(x) = m(x) - m(\mathcal{D}_{r_1}(x)) + a[\mathbb{1}_Q(x) - \mathbb{1}_Q(\mathcal{D}_{r_1}(x))]$ with $m \in C_c^2(\mathbb{R}^N)$, $0 \leq m \leq 1$ on \mathbb{R}^n , $m(x) = 1$ in $B_{r_0}(x^*)$, $m(x) = 0$ in $\mathbb{R}^N \setminus B_{2r_0}(x^*)$ and satisfies $k(\mathcal{D}_{r_1}(x)) = -k(x)$ on Σ_{r_1} , $k = 0$ in $\Sigma_{r_1} \setminus (U \cup Q)$ and $k = a$ on Q where the choice of a will be fixed later. Then by Proposition 2.3 in [175] we obtain, $\langle (-\Delta)^s m, \psi \rangle \leq C_1 \|\psi\|_{L^1(U)}$ for $\psi \in \tau$, $\psi \geq 0$ and $C_1 = C_1(m)$ independent of ψ . Since $\psi = 0$ in $\mathbb{R}^N \setminus U$, $(U \cap Q) \cup (U \cap \mathcal{D}_{r_1}(Q)) = \emptyset$ and $m(\mathcal{D}_{r_1}(x))\psi(x) = \mathbb{1}_Q(x)\psi(x) = \mathbb{1}_{\mathcal{D}_{r_1}(U)}(x)\psi(x) = 0$ in \mathbb{R}^N . Then we have $\langle (-\Delta)^s k, \psi \rangle \leq C_a \|\psi\|_{L^1(U)}$ where

$$C_a := C + \sup_{x \in U} \int_{\mathcal{D}_{r_1}(U)} \frac{1}{|x-y|^2} dy - a \inf_{x \in U} \int_Q \left(\frac{1}{|x-y|^2} - \frac{1}{|x-\mathcal{D}_{r_1}(y)|^2} \right) dy.$$

Since $|x-y| \leq |x-\mathcal{D}_{r_1}(y)|$ for all $x, y \in \Sigma_{r_1}$, $\bar{U} \subset \Sigma_{r_1}$ and then continuity of the function $x \mapsto \int_Q \left(\frac{1}{|x-y|^2} - \frac{1}{|x-\mathcal{D}_{r_1}(y)|^2} \right) dy$ implies $C_a \leq -C_L(f)$, by taking a sufficiently large. Since $v_{r_1} \geq 0$ in U , we obtain k is the required subsolution in U . Then Proposition 3.5 in [175], implies $\tilde{v}_{r_1}(x) := v_{r_1}(x) - \frac{\tilde{\mu}}{a} k(x) \geq 0$ a.e. in U which further gives $v_{r_1}(x) \geq \frac{\tilde{\mu}}{a} k(x) = \frac{\tilde{\mu}}{a} > 0$ a.e. in $B_{r_0}(x^*)$ and completes the proof of **Step 1**.

Step 2: $r_1 = 0$.

To prove this, we proceed by contradiction by assuming $r_1 > 0$. Since r_1 is the smallest value such that $v_{r_1} \geq 0$ in Σ_{r_1} , so we will prove that for a small $\epsilon > 0$ we have $v_{r_1-\epsilon} \geq 0$ in

$\Sigma_{r_1-\epsilon}$. This will provide the required contradiction that r_1 is the smallest value. Fix γ (to be determined later) and let $S \Subset \Sigma_{r_1}$ such that $|\Sigma_{r_1} \setminus S| \leq \frac{\gamma}{2}$. Then by using **Claim 1** and continuity of solution we get $v_{r_1-\epsilon} > 0$ in S for ϵ small enough. Since $v_{r_1-\epsilon}$ satisfies (6.2.1) in $\Sigma_{r_1-\epsilon} \setminus S$ then by using $|x - y| \leq |x - \mathcal{D}_{r_1-\epsilon}(y)|$ for all $x, y \in \Sigma_{r_1-\epsilon}$, $\mathcal{D}_{r_1-\epsilon}(\mathbb{R} \setminus \Sigma_{r_1-\epsilon}) = \Sigma_{r_1-\epsilon}$ and taking $w := \mathbb{1}_{\Sigma_{r_1-\epsilon}} v_{r_1-\epsilon}^-$ such that $\text{supp}(w) \subset \Sigma_{r_1-\epsilon} \setminus S$ as a test function, then after some straightforward computations we obtain

$$\begin{aligned} \langle (-\Delta)^s w, w \rangle + \langle (-\Delta)^s v_{r_1-\epsilon}, w \rangle &= -2 \int_{\Sigma_{r_1-\epsilon}} \int_{\mathbb{R}^n} \frac{w(x)[w(y) + v_{r_1-\epsilon}(y)]}{|x - y|^{N+2s}} dy dx \\ &= -2 \int_{\Sigma_{r_1-\epsilon}} \int_{\Sigma_{r_1-\epsilon}} w(x) \left(\frac{v_{r_1-\epsilon}^+(y)}{|x - y|^{N+2s}} - \frac{v_{r_1-\epsilon}(y)}{|x - \mathcal{D}_{r_1-\epsilon}(y)|^{N+2s}} \right) dy dx \leq 0. \end{aligned} \quad (6.2.2)$$

Let $\lambda_{1,\epsilon}^{r_1}$ be the first eigenvalue of $(-\Delta)^s$ in $\Sigma_{r_1-\epsilon} \setminus S$ and by mean value theorem together with (6.2.2) we get, for some $\theta \in (0, 1)$

$$\begin{aligned} \lambda_{1,\epsilon}^{r_1}(\Sigma_{r_1-\epsilon} \setminus S) \int_{\Sigma_{r_1-\epsilon} \setminus S} |v_{r_1-\epsilon}^-|^2 dx &\leq \langle (-\Delta)^s w, w \rangle \leq - \langle (-\Delta)^s v_{r_1-\epsilon}, w \rangle \\ &= \int_{\Sigma_{r_1-\epsilon} \setminus S} \frac{\delta v_{r_1-\epsilon} \mathbb{1}_{\Sigma_{r_1-\epsilon} \setminus S} v_{r_1-\epsilon}^-}{(\theta u + (1 - \theta)u_{r_1-\epsilon})^{\delta+1}} dx + \int_{\Sigma_{r_1-\epsilon} \setminus S} (-f(u_{r_1-\epsilon}) + f(u)) \mathbb{1}_{\Sigma_{r_1-\epsilon} \setminus S} v_{r_1-\epsilon}^- dx \\ &\leq C_L \int_{\Sigma_{r_1-\epsilon} \setminus S} |v_{r_1-\epsilon}^-|^2 dx. \end{aligned}$$

Then by Lemma 2.1 in [175]) and choosing γ small enough, we obtain $v_{r_1-\epsilon} \geq 0$ in $\Sigma_{r_1-\epsilon}$. Then $r_1 = 0$ and repeating the proof by moving hyperplane A_λ as in **Claim 1** we obtain u is radially symmetric. Now **Claim 1** gives further the strict monotonicity property. The proof is now complete. \square

Next, we apply this main result in a different situation: Consider the problem

$$(Q) \quad \begin{cases} (-\Delta)^s u = \mu \left(\frac{1}{u^\delta} + f(u) \right), & u > 0 & \text{in } \Omega, \\ u = 0 & & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

where Ω is a bounded domain with C^2 boundary regularity. This concerns the existence of uniform a priori bound for classical solutions to (Q) when f has a subcritical growth. In the spirit of the work [121], we combine the monotonicity property of solutions near the boundary of Ω and a blow up technique with the help of a Liouville theorem. Precisely we prove:

Theorem 6.2.2. *Let $N > 2s$ and $\mu_0 > 0$. Let u be the classical solution of (Q) with $f(u) = u^p$ for $1 < p < \frac{N+2s}{N-2s}$ and $\mu \geq \mu_0$. Then $\|u\|_\infty \leq C_1$ with C_1 depending only on δ, p, Ω, μ_0 .*

Proof. First we suppose that Ω is strictly convex then **Claim 1** in Theorem 6.2.1 combined with moving plane method gives boundary estimates and when Ω is not strictly convex, we

perform Kelvin transform near any boundary point (see [87], [6]). While, for interior estimates, we proceed by blow-up analysis. Precisely, assume there exists a sequence of bounded solution $(u_k)_{k \in \mathbb{N}}$ and a sequence of points $(x_k)_{k \in \mathbb{N}}$ such that $M_k = \sup_{y \in \Omega} u_k(y) = u_k(x_k) \rightarrow \infty$ as $k \rightarrow \infty$. Let λ_k is the sequence of positive numbers (to be determined later) and $y = \frac{x-x_k}{\lambda_k} \in \Omega_k$. From boundary estimates, notice that $\text{dist}(x_k, \partial\Omega) \geq c > 0$ uniformly in k . Define the blow up function $v_k(y) = \lambda_k^{\frac{2s}{p-1}} u_k(x)$ where $\lambda_k^{\frac{2s}{p-1}} M_k = 1$. We noticed that $\lambda_k \rightarrow 0$ as $k \rightarrow \infty$ (since $M_k \rightarrow \infty$) and for large k , $v_k(y)$ is well defined in $B_{\frac{m}{\lambda_k}}(0)$ and $\sup_{y \in B_{\frac{m}{\lambda_k}}(0)} v_k(y) = v^k(0) = 1$ where $0 < 2m \leq \inf_k \text{dist}(x_k, \partial\Omega)$. Accordingly, v_k satisfies

$$(-\Delta)^s v_k = \mu_k \left(\frac{\lambda_k^{\frac{2s(p+\delta)}{p-1}}}{v_k^\delta} + v_k^p \text{ in } B_{\frac{m}{\lambda_k}} \right).$$

Now passing to the limits we obtain, $v_k \rightarrow v$ in $C_{loc}^s(\mathbb{R}^N)$ and satisfies $(-\Delta)^s v = v^p$ in \mathbb{R}^N , $v(0) = 1$ and by using Liouville Theorem (see [87, Theorem 4]), we get a contradiction. \square

The second application concerns the asymptotic behaviour of large solutions with respect to the parameter μ . Let $s = \frac{1}{2}$, $n = 1$, $\Omega = B_r(0)$ and $f(u) = h(u) \exp(u^\alpha)$ for some $1 < \alpha \leq 2$ where h satisfies $\lim_{t \rightarrow \infty} h(t)e^{-\epsilon t^\alpha} = 0$ and $\lim_{t \rightarrow \infty} h(t)e^{\epsilon t^\alpha} = \infty$ for any $\epsilon > 0$. Then we have the following result:

Theorem 6.2.3. *Let $\mu_0 > 0$ and u be the classical solution of (Q) for some $\mu \geq \mu_0$. Then for any $\epsilon > 0$, the following holds*

$$\|u\|_{L^\infty(B_r \setminus B_\epsilon)} \leq C_2(\delta, n, \epsilon, \mu_0).$$

In addition, we have the following blow up profile: Let $\{u_k\}$ be a sequence of solutions for the problem (Q) such that $\|u_k\|_{L^\infty(B_r)} \rightarrow \infty$, $\mu_k \rightarrow \tilde{\mu}$ with $\tilde{\mu} > 0$,

- (i) *There exists a singular solution \tilde{u} in $C_{loc}^s(B_r \setminus \{0\})$ such that $u_k - \tilde{u} \rightarrow 0$ in $L_{loc}^\infty(B_r \setminus \{0\})$.*
- (ii) *If $(u_k)_{k \in \mathbb{N}}$ has uniform bounded energy and $F(t) = O(f(t))$ as $t \rightarrow \infty$ where $F(t)$ is the antiderivative of f , then $\tilde{\mu} = 0$.*

From Theorem 6.2.1 we know that the solutions are radial and radially decreasing, from this we only need to study the behavior near an isolated singularity. For that we exploit the Theorem 6.1.12.

Proof. Using Theorem 6.2.1, we obtain every classical solution of u of (Q) is radially symmetric and decreasing with respect to $|x|$. Then for every $\epsilon > 0$ there exists $\alpha_1 > 0$ such that for any $x \in B_r \setminus B_\epsilon$, we have a measurable set Z_ϵ satisfying $|Z_\epsilon| \geq \alpha_1$, $Z_\epsilon \subset B_r \setminus B_\epsilon$ and $u(y) \geq u(x)$, $\forall y \in B_\epsilon$. Then by multiplying ψ_1 (eigenfunction with respect to first eigenvalue

μ_1 of $(-\Delta)^s$ in B_r) to the equation satisfied by u , we obtain

$$\mu \int_{B_r} \frac{\psi_1}{u^\delta} dx + \int_{B_r} \exp(u^\alpha) \psi_1 dx = \mu_1 \int_{B_r} u \psi_1$$

and for any $m \geq \frac{\mu_1}{\mu}$, there exists a $C > 0$, $mt - C \leq \frac{1}{t^\delta} + \exp(t^\alpha)$, $t \in \mathbb{R}^+$. Then by using $u(y) \geq u(x)$, $\forall y \in B_\epsilon$ and $|Z_\epsilon| \geq \alpha_1$ it implies that $u(x) \leq C_2$ for all $x \in B_r \setminus B_\epsilon$ where C_2 is independent of u . Now we prove the blow up profile. From Theorem 6.2.1 and above estimates, we know that $(u_k)_k$ blows up only at 0. We deduce by regularity theory (see [6]) that the sequence $(u_k)_k$ converge to a singular solution u uniformly in $B_r \setminus \{0\}$. From Theorem 6.1.12 and the asymptotic growth of f , we prove that u is a singular solution of (Q). Finally assume that $(u_k)_k$ has uniform bounded energy. Then we easily get that u belongs to the energy space and from Moser-Trudinger inequality (see [247] or [148, Lemma 2.1]) and Remark 1.5 in [227] we obtain u is bounded which provides a contradiction and completes the proof. \square

6.3 Non-local fractional Laplacian singular problem with singular weights

In this part, we study the following nonlinear fractional elliptic and singular problem

$$(P) \begin{cases} (-\Delta)_p^s u = \frac{K_\delta(x)}{u^\gamma}, u > 0 & \text{in } \Omega; \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain with $C^{1,1}$ boundary, $s \in (0, 1)$, $p \in (1, +\infty)$, $\gamma > 0$ and K_δ satisfies the growth condition: for any $x \in \Omega$

$$\frac{C_1}{d^\delta(x)} \leq K_\delta(x) \leq \frac{C_2}{d^\delta(x)} \tag{6.3.1}$$

for some $\delta \in [0, sp)$, where, for any $x \in \Omega$, $d(x) = \text{dist}(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|$. The operator $(-\Delta)_p^s$ is known as fractional p -Laplacian operator and defined as

$$(-\Delta)_p^s u = 2 \lim_{\epsilon \rightarrow 0} \int_{B_\epsilon^c(x)} \frac{[u(x) - u(y)]^{p-1}}{|x - y|^{N+sp}} dy$$

with the notation $[a - b]^{p-1} = |a - b|^{p-2}(a - b)$.

In the case $p \neq 2$, the problem (P) is a non-linear and non-local one. The operator $(-\Delta)_p^s$ is degenerate if $p > 2$ and singular if $p < 2$. The operator $(-\Delta)_p^s$ is the nonlocal analogue of p -Laplacian operator in the sense that $(1-s)(-\Delta)_p^s \rightarrow (-\Delta)_p$ as $s \rightarrow 1^-$ and for $p = 2$, it reduces to fractional Laplacian operator which has a long history in mathematics. In particular, it is known as an infinitesimal generator of Lévy stable diffusion process in probability and has several appearance in real life models in phase transitions, crystal dislocations, anomalous diffusion, material science, water ets, etc (see [74, 171] and their reference within).

6.3.1 Function spaces and main results

Let Ω be bounded domain and for a measurable function $u : \mathbb{R}^N \rightarrow \mathbb{R}$, denote

$$[u]_{s,p} := \left(\iint_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{\frac{1}{p}}.$$

Define

$$W^{s,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\}$$

endowed with the norm

$$\|u\|_{s,p,\mathbb{R}^N} = \|u\|_{L^p(\mathbb{R}^N)} + [u]_{s,p}$$

where $\|\cdot\|_p$ denote the L^p norm. We also define

$$W_0^{s,p}(\Omega) := \{u \in W^{s,p}(\mathbb{R}^N) : u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}$$

endowed with the norm

$$\|u\|_{s,p} = [u]_{s,p}.$$

We can equivalently define $W_0^{s,p}$ as the closure of $C_c^\infty(\Omega)$ in the norm $[\cdot]_{s,p}$, with continuous boundary of the domain of Ω (see Theorem 6, [126]) where

$$C_c^\infty(\Omega) := \{f : \mathbb{R}^N \rightarrow \mathbb{R} : f \in C^\infty(\mathbb{R}^N) \text{ and } \text{supp}(f) \subset \omega \Subset \Omega\}.$$

We also define

$$W_{loc}^{s,p}(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \mid u \in L^p(\omega)\}, [u]_{s,p,\omega} < \infty, \text{ for all } \omega \Subset \Omega\}$$

where the localized Gagliardo seminorm is defined as

$$[u]_{s,p,\omega} := \left(\iint_{\omega^2} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} dx dy \right)^{1/p}.$$

Definition 6.3.1. A function $u \in W_{loc}^{s,p}(\Omega)$ is said to be a weak subsolution (resp. supersolution) of (P), if

$$u^\kappa \in W_0^{s,p}(\Omega) \text{ for some } \kappa \geq 1 \text{ and } \inf_K u > 0 \text{ for all } K \Subset \Omega$$

and

$$\iint_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)]^{p-1} (\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy \leq (\text{resp. } \geq) \int_\Omega \frac{K_\delta(x)}{u^\gamma} \phi dx$$

for all $\phi \in \mathbb{T} = \bigcup_{\tilde{\Omega} \Subset \Omega} W_0^{s,p}(\tilde{\Omega})$.

A function which is both sub and supersolution of (P) is called a weak solution to (P).

By virtue of the nonlinearity of the operator and the absence of integration by parts formula, such a notion of solution is considered. Before, stating our main results, we state some preliminary results proved in [66, 78]:

Proposition 6.3.1. (Lemma 3.5, [78]) For $\epsilon > 0$ and $q > 1$. Set

$$S_\epsilon^x := \{(x, y) : x \geq \epsilon, y \geq 0\}, \quad S_\epsilon^y := \{(x, y) : x \geq 0, y \geq \epsilon\}.$$

Then

$$|x^q - y^q| \geq \epsilon^{q-1}|x - y| \text{ for all } (x, y) \in S_\epsilon^x \cup S_\epsilon^y.$$

Proposition 6.3.2. (Lemma 3.3, [66]) Let $g \in L^q(\Omega)$ with $q > \frac{N}{sp}$ and $u \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ satisfying

$$\iint_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)]^{p-1}(\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy = \int_{\Omega} g\phi dx$$

for all $\phi \in W_0^{s,p}(\Omega)$. Then, for every C^1 convex function $\Phi : \mathbb{R} \rightarrow \mathbb{R}$, the composition $w = \Phi \circ u$ satisfies

$$\iint_{\mathbb{R}^{2N}} \frac{[w(x) - w(y)]^{p-1}(\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy \leq \int_{\Omega} g|\Phi'(u)|^{p-2}\Phi'(u)\phi dx.$$

for all nonnegative functions $\phi \in W_0^{s,p}(\Omega)$.

Having in mind Proposition 6.3.1 and the condition $u^\kappa \in W_0^{s,p}(\Omega)$, $\kappa \geq 1$ in definition 6.3.1, u satisfies the following definition of the boundary datum (see Proposition 1.5 in [78]):

Definition 6.3.2. We say that a function $u = 0$ in $\mathbb{R}^N \setminus \Omega$ satisfies $u \leq 0$ on $\partial\Omega$ in sense that for $\epsilon > 0$, $(u - \epsilon)^+ \in W_0^{s,p}(\Omega)$.

For a fixed parameter $\epsilon > 0$, we define a sequence of function $K_{\epsilon,\delta} : \mathbb{R}^N \rightarrow \mathbb{R}_+$ as

$$K_{\epsilon,\delta}(x) = \begin{cases} (K_\delta^{-\frac{1}{\delta}}(x) + \epsilon^{\frac{\gamma+p-1}{sp-\delta}})^{-\delta} & \text{if } x \in \Omega, \\ 0 & \text{else,} \end{cases}$$

and $K_{\epsilon,\delta}$ is an increasing function as $\epsilon \downarrow 0$, $K_{\epsilon,\delta} \rightarrow K_\delta$ a.e. in Ω and there exist two positive constants $\mathcal{C}_3, \mathcal{C}_4$ such that, for any $x \in \Omega$,

$$\frac{\mathcal{C}_3}{\left(d(x) + \epsilon^{\frac{\gamma+p-1}{sp-\delta}}\right)^\delta} \leq K_{\epsilon,\delta}(x) \leq \frac{\mathcal{C}_4}{\left(d(x) + \epsilon^{\frac{\gamma+p-1}{sp-\delta}}\right)^\delta}. \quad (6.3.2)$$

Define the approximated problem as

$$(P_\epsilon^\gamma) \begin{cases} (-\Delta)_p^s u = \frac{K_{\epsilon,\delta}(x)}{(u + \epsilon)^\gamma} & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Proposition 6.3.3. *For any $\epsilon > 0$ and $\gamma \geq 0$, there exists a unique weak solution $u_\epsilon \in W_0^{s,p}(\Omega) \cap C^{0,\ell}(\bar{\Omega})$ of the problem (P_ϵ^γ) i.e.*

$$\iint_{\mathbb{R}^{2N}} \frac{[u_\epsilon(x) - u_\epsilon(y)]^{p-1}(\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy = \int_{\Omega} \frac{K_{\epsilon,\delta}(x)}{(u + \epsilon)^\gamma} \phi dx \quad (6.3.3)$$

for all $\phi \in W_0^{s,p}(\Omega)$ and for some $\gamma_1 \in (0, 1)$. Moreover, the sequence $\{u_\epsilon\}_{\epsilon>0}$ satisfies $u_\epsilon > 0$ in Ω ,

$$u_{\epsilon_1}(x) \leq u_{\epsilon_2}(x) \text{ a.e. } x \in \Omega \text{ and } \epsilon_2 < \epsilon_1$$

and for any $\Omega' \Subset \Omega$, there exists $\sigma = \sigma(\Omega') > 0$ such that for any $\epsilon \in (0, 1)$:

$$\sigma \leq u_1(x) \leq u_\epsilon(x) \text{ a.e. } x \in \Omega'.$$

Proof. The proof follows from Proposition 2.3, Lemma 2.4 in [78] and Theorem 1.1 in [172]. □

Now we describe our main results. To prove the uniqueness and nonexistence result, we establish the following comparison principle:

Theorem 6.3.1. *For $0 \leq \delta < 1 + s - \frac{1}{p}$, $\gamma \geq 0$, let u be a subsolution of (P) and \tilde{v} be a supersolution of (P) in the sense of definition 6.3.1. Then $u \leq \tilde{v}$ a.e. in Ω .*

Next, we state the existence result:

Theorem 6.3.2. *Let Ω be a bounded domain with Lipschitz boundary $\partial\Omega$ and $\delta \in (0, sp)$. Then,*

- (i) *for $\delta - s(1 - \gamma) \leq 0$, then there exists a minimal weak solution $u \in W_0^{s,p}(\Omega)$ of the problem (P) ;*
- (ii) *for $\delta - s(1 - \gamma) > 0$, there exist a minimal weak solution u and a constant θ_0 such that*

$$u^\theta \in W_0^{s,p}(\Omega) \text{ if } \theta \geq \theta_0 \text{ and } \theta_0 > \max \left\{ 1, \frac{p + \gamma - 1}{p}, \Lambda \right\}$$

$$\text{where } \Lambda := \frac{(sp-1)(p-1+\gamma)}{p(sp-\delta)}.$$

As a consequence of comparison principle, we have the following uniqueness and nonexistence result:

Corollary 6.3.1. *For $0 < \delta < 1 + s - \frac{1}{p}$, the minimal weak solution u is a unique weak solution of the problem (P) .*

Theorem 6.3.3. *Let $\delta \geq sp$. Then there doesn't exist any weak solution of the problem (P) in the sense of definition 6.3.1.*

Now, we state the Hölder and optimal Sobolev regularity results:

Theorem 6.3.4. *Let Ω be a bounded domain with $C^{1,1}$ boundary and u be the minimal weak solution of (P). Then there exist constant $C_1, C_2 > 0$ and $0 < \omega_1 < s$, $0 < \omega_2 \leq \frac{sp-\delta}{\gamma+p-1}$ such that*

(i) *if $0 < \frac{\delta}{s} + \gamma \leq 1$, then $C_1 d^s \leq u \leq C_2 d^{s-\epsilon}$ in Ω and for every $\epsilon > 0$*

$$u \in \begin{cases} C^{s-\epsilon}(\mathbb{R}^N) & \text{if } 2 \leq p < \infty, \\ C^{\omega_1}(\mathbb{R}^N) & \text{if } 1 < p < 2. \end{cases}$$

(ii) *if $\frac{\delta}{s} + \gamma > 1$ then $C_1 d^{\frac{(sp-\delta)}{\gamma+p-1}} \leq u \leq C_2 d^{\frac{(sp-\delta)}{\gamma+p-1}}$ in Ω and*

$$u \in \begin{cases} C^{\frac{(sp-\delta)}{\gamma+p-1}}(\mathbb{R}^N) & \text{if } 2 \leq p < \infty, \\ C^{\omega_2}(\mathbb{R}^N) & \text{if } 1 < p < 2. \end{cases}$$

Corollary 6.3.2. *For $\frac{\delta}{s} + \gamma > 1$ and Ω be a bounded domain with $C^{1,1}$ boundary. Then the minimal weak solution u of the problem (P) has the optimal Sobolev regularity:*

$$u \in W_0^{s,p}(\Omega) \quad \text{if and only if} \quad \Lambda < 1$$

and

$$u^\theta \in W_0^{s,p}(\Omega) \quad \text{if and only if} \quad \theta > \Lambda > 1.$$

Remark 6.3.1. *In case of $\delta = 0$ and $\gamma > 0$, we extend the Sobolev regularity of minimal weak solution as compared to the Sobolev regularity in Theorem 3.6 in [78]. Precisely, $u \in W_0^{s,p}(\Omega)$ when $\gamma \leq 1$ or $\gamma > 1$ and $\Lambda < 1$, and $u^\rho \in W_0^{s,p}(\Omega)$ for $\rho > \Lambda$ when $\gamma > 1$ and $\Lambda \geq 1$*

6.3.2 Comparison principle and existence result

In this section, we prove the weak comparison principle and existence result concerning the problem (P).

Proof of Theorem 6.3.1: The proof is almost identical as the proofs of Lemma 4.1 and Theorem 4.2 in [78]. For the reader's convenience, we precise some details to explain the restriction on δ . More precisely, we need a minimizer belonging to $\mathcal{L} := \{\phi \in W_0^{s,p}(\Omega) : 0 \leq \phi \leq \tilde{v} \text{ a.e. in } \Omega\}$ of the following energy functional defined on $W_0^{s,p}(\Omega)$ as, for $\epsilon > 0$

$$J_\epsilon(w) := \frac{1}{p} \iint_{\mathbb{R}^{2N}} \frac{|w(x) - w(y)|^p}{|x - y|^{N+sp}} dx dy - \int_{\Omega} K_\delta(x) G_\epsilon(w) dx$$

where G_ϵ is the primitive such that $G_\epsilon(1) = 0$ of the function g_ϵ defined by

$$g_\epsilon(t) = \begin{cases} \min \left\{ \frac{1}{t^\gamma}, \frac{1}{\epsilon} \right\} & \text{if } t > 0, \\ \frac{1}{\epsilon} & \text{if } t \leq 0. \end{cases}$$

Let $\{w_n\} \subset W_0^{s,p}(\Omega)$ be such that $w_n \rightharpoonup w$ in $W_0^{s,p}(\Omega)$. Let $\nu \in (0, 1)$ small enough such that $\frac{1-\nu}{p} + \frac{\nu}{q} + \frac{1}{r} = 1$ where $q < p_s^* := \frac{Np}{N-sp}$ if $N > sp$ and $(s(1-\nu) - \delta)r > -1$ (since $\delta < 1 + s - \frac{1}{p}$).

Hence $x \mapsto d^{s(1-\nu)-\delta}(x) \in L^r(\Omega)$ and by using Hölder and Hardy inequalities (see Theorem 1.4.4.4 and Corollary 1.4.4.10 in [162]), we obtain

$$\begin{aligned} \int_{\Omega} \frac{|w_n - w|}{d^{\delta}(x)} dx &= \int_{\Omega} \left(\frac{|w_n - w|}{d^s(x)} \right)^{1-\nu} |w_n - w|^{\nu} d^{s(1-\nu)-\delta}(x) dx \\ &\leq C \|w_n - w\|_{s,p}^{1-\nu} \|w_n - w\|_{L^q(\Omega)}^{\nu} \end{aligned}$$

for some constant $C > 0$ independent of w_n and w .

Since $W_0^{s,p}(\Omega)$ is compactly embedded in $L^q(\Omega)$ for $q < p_s^*$, $\|w_n - w\|_{s,p}$ is uniformly bounded in n and $\|w_n - w\|_{L^q(\Omega)} \rightarrow 0$ as $n \rightarrow \infty$.

Finally, gathering the lower semicontinuity of $[\cdot]_{s,p}$ and G_{ϵ} globally Lipschitz, we deduce that J_{ϵ} is weakly lower semicontinuous in $W_0^{s,p}(\Omega)$ and admits a minimizer w_0 on \mathcal{L} .

The rest of the proof follows exactly the proofs of Lemma 4.1 and Theorem 4.2 in [78] and we obtain

$$u \leq w_0 \leq \tilde{v} \text{ in } \Omega.$$

By following the same idea of proof, we can prove it for $\gamma = 0$. \square

Now we prove our existence and uniqueness result:

Proof of Theorem 6.3.2: Let $u_{\epsilon} \in W_0^{s,p}(\Omega)$ be the weak solution of (P_{ϵ}^{γ}) . Adapting the proofs of Theorem 3.2 and 3.6 in [78], it is sufficient to verify the sequences $\{u_{\epsilon}\}_{\epsilon}$ in the case $\delta - s(1-\gamma) \leq 0$ and $\{u_{\epsilon}^{\theta}\}$ for a suitable parameter $\theta > 1$ in the case $\delta - s(1-\gamma) > 0$ are bounded in $W_0^{s,p}(\Omega)$ and the convergence of the right-hand side in (6.3.3).

Case 1: $\delta - s(1-\gamma) \leq 0$.

The condition implies $\gamma < 1$ hence taking $\phi = u_{\epsilon}$ in (6.3.3) and applying Hölder and Hardy inequalities (see Theorem 1.4.4.4 and Corollary 1.4.4.10 in [162]), we obtain

$$[u_{\epsilon}]_{s,p}^p \leq C_2 \int_{\Omega} d^{s(1-\gamma)-\delta}(x) \left(\frac{u_{\epsilon}}{d^s(x)} \right)^{1-\gamma} dx \leq C \| \frac{u_{\epsilon}}{d^s} \|_{L^p(\Omega)}^{1-\gamma} \leq C [u_{\epsilon}]_{s,p}^{1-\gamma} \quad (6.3.4)$$

which implies $\|u_{\epsilon}\|_{s,p} \leq C < \infty$.

Case 2: $\delta - s(1-\gamma) > 0$

Let $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be the function defined as $\Phi(t) = t^{\theta}$ for some

$$\theta > \max \left\{ 1, \frac{p+\gamma-1}{p}, \Lambda \right\}$$

For any $\epsilon > 0$, choosing $g = \frac{K_{\epsilon,\delta}}{(u_{\epsilon} + \epsilon)^{\gamma}} \in L^{\infty}(\Omega)$ and $w = \Phi \circ u_{\epsilon}$ in Proposition 6.3.2, we obtain

$$\iint_{\mathbb{R}^{2N}} \frac{[\Phi(u_{\epsilon})(x) - \Phi(u_{\epsilon})(y)]^{p-1} (\phi(x) - \phi(y))}{|x-y|^{N+sp}} dx dy \leq \int_{\Omega} \frac{K_{\epsilon,\delta}(x)}{(u_{\epsilon} + \epsilon)^{\gamma}} |\Phi'(u_{\epsilon})|^{p-2} \Phi'(u_{\epsilon}) \phi dx \quad (6.3.5)$$

for all nonnegative functions $\phi \in W_0^{s,p}(\Omega)$. Since $u_\epsilon \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ and Φ is locally Lipschitz, therefore $\Phi(u_\epsilon) \in W_0^{s,p}(\Omega)$. Then by choosing $\phi = \Phi(u_\epsilon)$ as a test function in (6.3.5), we get

$$\begin{aligned} \iint_{\mathbb{R}^{2N}} \frac{|\Phi(u_\epsilon)(x) - \Phi(u_\epsilon)(y)|^p}{|x-y|^{N+sp}} dx dy &\leq \int_{\Omega} \frac{K_{\epsilon,\delta}(x)}{(u_\epsilon + \epsilon)^\gamma} |\Phi'(u_\epsilon)|^{p-2} \Phi'(u_\epsilon) \Phi(u_\epsilon) dx \\ &\leq C_2 \int_{\Omega} \frac{1}{d^\delta(x)} \frac{|\Phi'(u_\epsilon)|^{p-2} \Phi'(u_\epsilon) \Phi(u_\epsilon)}{u_\epsilon^\gamma} dx. \end{aligned} \quad (6.3.6)$$

Now, for any $\epsilon > 0$, there exists a constant C independent of ϵ such that

$$\frac{|\Phi'(u_\epsilon)|^{p-2} \Phi'(u_\epsilon) \Phi(u_\epsilon)}{u_\epsilon^\gamma} \leq C (\Phi(u_\epsilon))^{\frac{\theta p - (p+\gamma-1)}{\theta}}. \quad (6.3.7)$$

where $\frac{\theta p - (p+\gamma-1)}{\theta} > 0$ since $\theta > \frac{p+\gamma-1}{p}$. By combining (6.3.6)-(6.3.7), we obtain applying Hölder and Hardy inequalities:

$$\begin{aligned} [\Phi(u_\epsilon)]_{s,p}^p &\leq C \int_{\Omega} d^{\frac{s(\theta p - (p+\gamma-1))}{\theta} - \delta(x)} \left(\frac{\Phi(u_\epsilon)}{d^s}(x) \right)^{\frac{\theta p - (p+\gamma-1)}{\theta}} dx \\ &\leq C \left(\int_{\Omega} d^{\frac{sp(\theta-\Lambda)-\theta}{\Lambda}}(x) dx \right)^{\frac{p+\gamma-1}{\theta p}} \left(\int_{\Omega} \left(\frac{\Phi(u_\epsilon)}{d^s(x)} \right)^p dx \right)^{\frac{\theta p - (p+\gamma-1)}{\theta p}} \\ &\leq C [\Phi(u_\epsilon)]_{s,p}^{\frac{\theta p - (p+\gamma-1)}{\theta}} \end{aligned}$$

and we conclude $\{\Phi(u_\epsilon)\}_{\epsilon>0}$ is bounded in $W_0^{s,p}(\Omega)$.

Finally, let $\tilde{\Omega} \Subset \Omega$, and $\phi \in W_0^{s,p}(\tilde{\Omega})$. By Proposition 6.3.3, there exists a constant $\eta_{\tilde{\Omega}}$ such that for any $\epsilon > 0$,

$$u_\epsilon(x) \geq \eta_{\tilde{\Omega}}, \text{ for a.e. in } \tilde{\Omega}.$$

By the previous inequality, we have

$$\left| \frac{K_{\epsilon,\delta}(x)\phi}{(u_\epsilon + \epsilon)^\gamma} \right| \leq \eta_{\tilde{\Omega}}^\gamma M |\phi|$$

where $M = \frac{1}{\text{dist}^\delta(\tilde{\Omega}, \Omega)}$, hence we get by Dominated convergence theorem:

$$\int_{\Omega} \frac{K_{\epsilon,\delta}(x)}{(u_\epsilon + \epsilon)^\gamma} \phi dx \rightarrow \int_{\Omega} \frac{K_\delta(x)}{u^\gamma} \phi dx$$

where $u := \lim_{\epsilon \rightarrow 0} u_\epsilon$. The rest of the proof follows exactly the end of the proofs of Theorem 3.2 and 3.6 in [78].

Finally, for any $\epsilon > 0$, $u_\epsilon \leq v$ a.e. in Ω where v is another weak solution of (P) . Indeed, v is a weak supersolution in sense of Definition 6.3.1 of the problem (P_ϵ^γ) hence Theorem 4.2 in [78] implies the inequality. Passing to the limit $\epsilon \rightarrow 0$ give u is a minimal solution. \square

Remark 6.3.2. *The proof of Case 1 holds assuming $\Lambda \leq 1$ and $\gamma < 1$. Indeed, $d^{s(1-\gamma)-\delta} \in L^{\frac{p}{p-1+\gamma}}(\Omega)$ and we obtain (6.3.4).*

Remark 6.3.3. In case of $\delta = 0$, the Sobolev regularity of the minimal weak solution in Theorem 6.3.2 coincides with the Sobolev regularity in Theorem 3.2 for $\gamma \leq 1$ and Theorem 3.6 in [78] for $\gamma > 1$ by taking $\theta = \frac{p+\gamma-1}{p}$.

Proof of Corollary 6.3.1

Let u_1, u_2 are two solution of the problem (P). Then by considering u_1 and u_2 as a subsolution and supersolution respectively in Theorem 6.3.1, we get $u_1 \leq u_2$ in Ω for $0 < \delta < 1 + s - \frac{1}{p}$. Now, by reversing the role of u_1 and u_2 , we obtain $u_1 = u_2$. \square

6.3.3 Estimates of distance functions

In this section, we construct explicit sub and supersolutions for the following problem

$$(S_0^\delta) \begin{cases} (-\Delta)_p^s u(x) = K_\delta(x) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Before that, we introduce the new notion of weak solution and corresponding vector space: Let $\Omega \subset \mathbb{R}^N$ be bounded. We define

$$\overline{W}^{s,p}(\Omega) := \left\{ u \in L_{loc}^p(\mathbb{R}^N) : \exists K \text{ s.t. } \Omega \Subset K, \|u\|_{W^{s,p}(K)} + \int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+sp}} dx < \infty \right\}$$

where $\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + [u]_{s,p,\Omega}$. If Ω is unbounded, we define

$$\overline{W}_{loc}^{s,p}(\Omega) := \{u \in L_{loc}^p(\mathbb{R}^N) : u \in \overline{W}^{s,p}(\tilde{\Omega}), \text{ for any bounded } \tilde{\Omega} \subset \Omega\}.$$

Definition 6.3.3. (Weak energy Solution) Let $f \in L^{p'}(\Omega)$ where p' is the conjugate exponent of p and Ω be a bounded domain. We say that $u \in \overline{W}^{s,p}(\Omega)$ is a weak energy solution of $(-\Delta)_p^s u = f$ in Ω , if

$$\iint_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)]^{p-1} (\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy = \int_{\Omega} f(x) \phi(x) dx$$

for all $\phi \in W_0^{s,p}(\Omega)$ and a function u is a weak energy subsolution (resp. weak energy supersolution) of $(-\Delta)_p^s u = f$ in Ω , if

$$(-\Delta)_p^s u \leq (\text{resp. } \geq) f \quad E\text{-weakly in } \Omega$$

that is

$$\iint_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)]^{p-1} (\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy \leq (\text{resp. } \geq) \int_{\Omega} f(x) \phi(x) dx$$

for all $\phi \in W_0^{s,p}(\Omega), \phi \geq 0$.

If Ω is unbounded we say that $u \in \overline{W}_{loc}^{s,p}(\Omega)$ is a weak energy solution (weak energy subsolution/weak energy supersolution) of $(-\Delta)_p^s(u) = (\leq / \geq) f$ in Ω , if it does so in any open bounded set $\Omega' \subset \Omega$.

For any $\alpha \in (0, s)$, we define

$$\beta := sp - \alpha(p - 1).$$

We start by computing the upper and lower estimates in the half line $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$ of $(-\Delta)_p^s$ of the function $U_\lambda(x) := \left((x + \lambda^{\frac{1}{\alpha}})^+\right)^\alpha$, $\lambda \geq 0$ defined in \mathbb{R} .

We recall the notation, for any $t \in \mathbb{R}$, $[t]^{p-1} = |t|^{p-2}t$.

Theorem 6.3.5. *Let $\lambda \geq 0$, $\alpha \in (0, s)$ and $p > 1$. Then, there exist two positive constants $C_1, C_2 > 0$ depending upon α, p and s such that*

$$C_1(x + \lambda^{\frac{1}{\alpha}})^{-\beta} \leq (-\Delta)_p^s U_\lambda(x) \leq C_2(x + \lambda^{\frac{1}{\alpha}})^{-\beta} \text{ pointwisely in } \mathbb{R}_+. \quad (6.3.8)$$

Moreover, for $\lambda > 0$, $U_\lambda \in \overline{W}_{loc}^{s,p}(\mathbb{R}_+)$ and for $\lambda = 0$, $U_\lambda \in \overline{W}_{loc}^{s,p}(\mathbb{R}_+)$ if $s - \frac{1}{p} < \alpha < s$.

Proof. Let $x \in \mathbb{R}_+$ and let $\epsilon \in \mathbb{R}$ such that $|\epsilon| < x$. We have

$$\begin{aligned} \int_{\mathbb{R} \setminus (x-|\epsilon|, x+|\epsilon|)} \frac{[U_\lambda(x) - U_\lambda(z)]^{p-1}}{|x-z|^{1+sp}} dz &= \left(\int_{-\infty}^{-\lambda^{\frac{1}{\alpha}}} \cdots + \int_{-\lambda^{\frac{1}{\alpha}}}^{x-|\epsilon|} \cdots + \int_{x+|\epsilon|}^{\infty} \cdots \right) \\ &= (x + \lambda^{1/\alpha})^{-\beta} \mathcal{P}_\epsilon(x) \end{aligned}$$

where, by the change of variable $y = \frac{z + \lambda^{1/\alpha}}{x + \lambda^{1/\alpha}}$:

$$\begin{aligned} \mathcal{P}_\epsilon(x) &:= (x + \lambda^{1/\alpha})^{sp} \int_{-\infty}^{-\lambda^{1/\alpha}} \frac{1}{|x-z|^{1+sp}} dz + \int_0^{1 - \frac{|\epsilon|}{x + \lambda^{1/\alpha}}} \frac{[1 - y^\alpha]^{p-1}}{|1 - y|^{1+sp}} dy \\ &\quad + \int_{1 + \frac{|\epsilon|}{x + \lambda^{1/\alpha} - |\epsilon|}}^{\infty} \frac{[1 - y^\alpha]^{p-1}}{|1 - y|^{1+sp}} dy + \int_{1 + \frac{|\epsilon|}{x + \lambda^{1/\alpha}}}^{1 + \frac{|\epsilon|}{x + \lambda^{1/\alpha} - |\epsilon|}} \frac{[1 - y^\alpha]^{p-1}}{|1 - y|^{1+sp}} dy \\ &:= \mathcal{P}_1(x) + \mathcal{P}_2(x, \epsilon) + \mathcal{P}_3(x, \epsilon) + \mathcal{P}_4(x, \epsilon). \end{aligned}$$

To conclude (6.3.8), it suffices to obtain a uniform estimate of \mathcal{P}_ϵ in \mathbb{R}_+ . First we note

$$\mathcal{P}_1(x) = (x + \lambda^{1/\alpha})^{sp} \int_{-\infty}^{-\lambda^{1/\alpha}} \frac{1}{|x-z|^{1+sp}} dz = \frac{1}{sp}. \quad (6.3.9)$$

Moreover, the change of variable $y \rightarrow \frac{1}{y}$ in \mathcal{P}_3 yields:

$$\mathcal{P}_3(x, \epsilon) = - \int_0^{1 - \frac{|\epsilon|}{x + \lambda^{1/\alpha}}} \frac{(1 - y^\alpha)^{p-1} y^{\beta-1}}{|1 - y|^{1+sp}} dy.$$

Hence

$$\mathcal{P}_{2,3}(x, \epsilon) := \mathcal{P}_2(x, \epsilon) + \mathcal{P}_3(x, \epsilon) = \int_0^{1 - \frac{|\epsilon|}{x + \lambda^{1/\alpha}}} \frac{(1 - y^\alpha)^{p-1} (1 - y^{\beta-1})}{|1 - y|^{1+sp}} dy. \quad (6.3.10)$$

We consider two cases to estimate $\mathcal{P}_{2,3}$:

Case 1: $\beta < 1$.

First, note in this case, $\mathcal{P}_{2,3}(x, \epsilon) \leq 0$, it suffices to estimate $\mathcal{P}_{2,3}$ from below.

There exists $\tilde{s} \in (s, 1)$ such that $\beta > \tilde{s}$ hence for any $y \in (0, 1)$:

$$y^{\beta-1} - 1 \leq y^{\tilde{s}-1} - 1, \quad (1 - y^\alpha) \leq (1 - y^{\tilde{s}}) \quad \text{and} \quad \frac{1}{(1 - y)^{1+sp}} \leq \frac{1}{(1 - y)^{1+\tilde{s}p}}.$$

Then by using the above estimates in (6.3.10), we obtain,

$$\begin{aligned} \mathcal{P}_{2,3}(x, \epsilon) &\geq \int_0^{1 - \frac{|\epsilon|}{x + \lambda^{1/\alpha}}} \frac{(1 - y^{\tilde{s}})^{p-1} (1 - y^{\tilde{s}-1})}{(1 - y)^{1+\tilde{s}p}} dy \\ &= \left[\frac{1}{\tilde{s}p} \frac{(1 - y^{\tilde{s}})^p}{(1 - y)^{\tilde{s}p}} \right]_0^{1 - \frac{|\epsilon|}{x + \lambda^{1/\alpha}}} = \frac{1}{\tilde{s}p} \left(\left(\frac{(x + \lambda^{1/\alpha})^{\tilde{s}} - (x + \lambda^{1/\alpha} - |\epsilon|)^{\tilde{s}}}{|\epsilon|^{\tilde{s}}} \right)^p - 1 \right) \\ &\geq -\frac{1}{\tilde{s}p}. \end{aligned} \tag{6.3.11}$$

Case 2: $\beta \geq 1$

In the same way, we note that $\mathcal{P}_{2,3}(x, \epsilon) \geq 0$. Now, for the upper bound, using $1 - y^\kappa \leq \max\{1, \kappa\}(1 - y)$ for any $y \in (0, 1)$ and $\kappa > 0$ we get:

$$\mathcal{P}_{2,3}(x, \epsilon) \leq \max\{1, \beta - 1\} \int_0^{1 - \frac{|\epsilon|}{x + \lambda^{1/\alpha}}} (1 - y)^{p(1-s)-1} dy \leq \frac{\max\{1, \beta - 1\}}{p(1-s)}. \tag{6.3.12}$$

Finally we estimate the last term \mathcal{P}_4 :

$$\begin{aligned} |\mathcal{P}_4(x, \epsilon)| &\leq \int_{1 + \frac{|\epsilon|}{x + \lambda^{1/\alpha}}}^{1 + \frac{|\epsilon|}{x + \lambda^{1/\alpha} - |\epsilon|}} \frac{|y^\alpha - 1|^{p-1}}{|1 - y|^{1+sp}} dy \leq \int_{1 + \frac{|\epsilon|}{x + \lambda^{1/\alpha}}}^{1 + \frac{|\epsilon|}{x + \lambda^{1/\alpha} - |\epsilon|}} \frac{|y^s - 1|^{p-1}}{|y - 1|^{1+sp}} dy \\ &\leq \int_{1 + \frac{|\epsilon|}{x + \lambda^{1/\alpha}}}^{1 + \frac{|\epsilon|}{x + \lambda^{1/\alpha} - |\epsilon|}} \frac{|y - 1|^{s(p-1)}}{|y - 1|^{1+sp}} dy = \frac{1}{s} \frac{(x + \lambda^{1/\alpha})^s - (x + \lambda^{1/\alpha} - |\epsilon|)^s}{|\epsilon|^s} := \frac{\xi_\epsilon(x)}{s}. \end{aligned} \tag{6.3.13}$$

Noting $\xi_\epsilon(x) \rightarrow 0$ *a.e.* in $x \in \mathbb{R}_+$, we deduce, combining (6.3.9)-(6.3.13), that there exist two constants C_1 and C_2 independent of x such that, for any $x \in \mathbb{R}_+$:

$$C_1 \leq \lim_{\epsilon \rightarrow 0} \mathcal{P}_\epsilon(x) \leq C_2.$$

Hence we deduce (6.3.8). More precisely, the constant C_1 and C_2 are given by

$$C_1 = \begin{cases} \frac{1}{p} \left(\frac{\tilde{s}-s}{\tilde{s}s} \right) & \text{if } \beta < 1, \\ \frac{1}{sp} & \text{if } \beta \geq 1, \end{cases} \quad \text{and} \quad C_2 = \begin{cases} \frac{1}{sp} & \text{if } \beta < 1, \\ \frac{1}{sp} + \frac{\max\{1, \beta-1\}}{p(1-s)} & \text{if } \beta \geq 1. \end{cases}$$

Finally the assertion, $U_\lambda \in \overline{W}_{loc}^{s,p}(\mathbb{R}_+)$ follows by showing $U_\lambda \in \overline{W}^{s,p}(a,b)$ for all $-\lambda^{1/\alpha} < a < b < \infty$. Indeed, using the symmetry of the integrand and changes of variable, we obtain

$$\begin{aligned}
 \iint_{[a,b]^2} \frac{|U_\lambda(x) - U_\lambda(y)|^p}{|x-y|^{1+sp}} dx dy &= \int_{a+\lambda^{1/\alpha}}^{b+\lambda^{1/\alpha}} \int_{a+\lambda^{1/\alpha}}^{b+\lambda^{1/\alpha}} \frac{|x^\alpha - y^\alpha|^p}{|x-y|^{1+sp}} dx dy \\
 &= 2 \int_{a+\lambda^{1/\alpha}}^{b+\lambda^{1/\alpha}} \int_{a+\lambda^{1/\alpha}}^x \frac{|x^\alpha - y^\alpha|^p}{|x-y|^{1+sp}} dy dx \\
 &= 2 \int_{a+\lambda^{1/\alpha}}^{b+\lambda^{1/\alpha}} x^{\alpha p - sp} \int_{\frac{a+\lambda^{1/\alpha}}{x}}^1 \frac{(1-t)^\alpha{}^p}{(1-t)^{1+sp}} dt dx \\
 &< 2 \int_{a+\lambda^{1/\alpha}}^{b+\lambda^{1/\alpha}} x^{\alpha p - sp} \int_0^1 \frac{(1-t)^p}{(1-t)^{1+sp}} dt dx < \infty
 \end{aligned} \tag{6.3.14}$$

for any $\alpha \in (0, s)$ if $\lambda > 0$ and $\alpha \in (s - \frac{1}{p}, s)$ if $\lambda = 0$. \square

Next, we study the behavior of $(-\Delta)_p^s V_\lambda(x)$ on $\mathbb{R}_+^N := \{x \in \mathbb{R}^N : x_N > 0\}$ where $V_\lambda(x) := U_\lambda(x \cdot e_N) = U_\lambda(x_N)$.

Let GL_N be the set of $N \times N$ invertible matrices, we have

Corollary 6.3.3. *Let $\lambda \geq 0$, $\alpha \in (0, s)$, $A \in GL_N$ and $p > 1$. Let $\mathcal{J}_{\epsilon,A}$ be the function defined on \mathbb{R}_+^N by*

$$\mathcal{J}_{\epsilon,A}(x) = \int_{B_\epsilon(0)^c} \frac{[V_\lambda(x) - V_\lambda(x+z)]^{p-1}}{|Az|^{N+sp}} dz$$

for some $\epsilon > 0$.

Then, there exist two positive constants C_3 and C_4 depending on $\alpha, s, p, N, \|A\|_2, \|A^{-1}\|_2$ such that

$$C_3(x_N + \lambda^{1/\alpha})^{-\beta} \leq \lim_{\epsilon \rightarrow 0} \mathcal{J}_{\epsilon,A}(x) \leq C_4(x_N + \lambda^{1/\alpha})^{-\beta} \tag{6.3.15}$$

pointwisely in $\mathbb{R}_+^N \times GL_N$. In particular, for $A = I$, there exist two positive constants \tilde{C}_3 and \tilde{C}_4 independent of λ such that:

$$\tilde{C}_3(x_N + \lambda^{1/\alpha})^{-\beta} \leq (-\Delta)_p^s V_\lambda(x) \leq \tilde{C}_4(x_N + \lambda^{1/\alpha})^{-\beta} \quad \text{pointwisely in } \mathbb{R}_+^N.$$

Moreover, for $\lambda > 0$, $V_\lambda \in \overline{W}_{loc}^{s,p}(\mathbb{R}_+^N)$ and for $\lambda = 0$, $V_\lambda \in \overline{W}_{loc}^{s,p}(\mathbb{R}_+^N)$ if $s - \frac{1}{p} < \alpha < s$.

Proof. As in the proof of Lemma 3.2 in [172], we define the elliptic coordinates for any $y \in \mathbb{R}^N \setminus \{0\}$ as $y = \rho w$ where $\rho > 0$ and $w \in \mathcal{E} := AS^{N-1}$. Hence we have $dy = \rho^{N-1} d\rho dw$ where dw is the surface of \mathcal{E} . We also define $e_A = {}^t(A^{-1})e_N$ and $E_A = \{x \in \mathbb{R}^N : x \cdot e_A > 0\}$ then we have

$$e_A \cdot w = (A^{-1}w)_N, \quad \forall w \in \mathcal{E}.$$

Let $x \in \mathbb{R}_+^N$, by the change of variable $z = \rho A^{-1}w$:

$$\begin{aligned} \mathcal{J}_{\epsilon,A}(x) &= |\det A|^{-1} \int_{\mathcal{E}} \frac{1}{|w|^{N+sp}} \int_{\epsilon}^{\infty} \frac{[U_{\lambda}(x_N) - U_{\lambda}(x_N + \rho(e_A \cdot w))]^{p-1}}{|\rho|^{1+sp}} d\rho dw \\ &= |\det A|^{-1} \left(\int_{\mathcal{E} \cap E_A} \int_{\epsilon}^{\infty} + \int_{\mathcal{E} \cap (E_A)^c} \int_{\epsilon}^{\infty} \right). \end{aligned}$$

Replacing ρ and w by $-\rho$ and $-w$ in the second integral in the right-hand side and noting $-w \in \mathcal{E} \cap E_A$, we get

$$\mathcal{J}_{\epsilon,A}(x) = |\det A|^{-1} \int_{\mathcal{E} \cap E_A} \frac{1}{|w|^{N+sp}} \int_{(-\epsilon, \epsilon)^c} \frac{[U_{\lambda}(x_N) - U_{\lambda}(x_N + \rho(e_A \cdot w))]^{p-1}}{|\rho|^{1+sp}} d\rho dw.$$

Now, the new change of variable $t = x_N + \rho(e_A \cdot w)$ yields in $\mathcal{J}_{\epsilon,A}$:

$$\mathcal{J}_{\epsilon,A}(x) = (x_N + \lambda^{1/\alpha})^{-\beta} |\det A|^{-1} \int_{\mathcal{E} \cap E_A} \frac{|e_A \cdot w|^{sp}}{|w|^{N+sp}} \mathcal{P}_{(e_A \cdot w)\epsilon}(x_N) dw.$$

Noting that

$$|\det A|^{-1} \int_{\mathcal{E} \cap E_A} \frac{|e_A \cdot w|^{sp}}{|w|^{N+sp}} dw = \frac{1}{2} \int_{S^{N-1}} \frac{|e_N \cdot v|^{sp}}{|Av|^{N+sp}} dv < \infty \quad (6.3.16)$$

we obtain (6.3.15) passing to the limit $\epsilon \rightarrow 0$ and using Theorem 6.3.5.

Finally, the assertion $V_{\lambda} \in \overline{W}_{loc}^{s,p}(\mathbb{R}_+^N)$ follows showing $V_{\lambda} \in \overline{W}^{s,p}(K)$ for any bounded set $K \Subset \mathbb{R}_+^N$ and using the computations in (6.3.14). \square

The next result gives the corresponding estimates of $(-\Delta)_p^s(x_N + \lambda^{1/\alpha})_+^{\alpha}$ under the smooth change of coordinates.

Theorem 6.3.6. *Let $\alpha \in (0, s)$ and $p > 1$. Let $\psi : \mathbb{R}^N \rightarrow \mathbb{R}^N$ be a $C^{1,1}$ -diffeomorphism in \mathbb{R}^N such that $\psi = Id$ in $B_R(0)^c$, for some $R > 0$.*

Then, considering $W_{\lambda}(x) = U_{\lambda}(\psi^{-1}(x) \cdot e_N)$, there exist $\rho^ = \rho^*(\psi) > 0$ and $\lambda^* = \lambda^*(\psi) > 0$ such that for any $\rho \in (0, \rho^*)$, there exists a constant $\tilde{C} > 0$ independent of λ such that, for any $\lambda \in [0, \lambda^*]$,*

$$\frac{1}{\tilde{C}} W_{\lambda}(x)^{-\frac{\beta}{\alpha}} \leq (-\Delta)_p^s W_{\lambda}(x) \leq \tilde{C} W_{\lambda}(x)^{-\frac{\beta}{\alpha}} \quad E\text{-weakly in } \psi(\{X : 0 < X_N < \rho\}). \quad (6.3.17)$$

Proof. Define, for any $x \in \psi(\mathbb{R}_+^N)$, $H(x) = 2 \lim_{\epsilon \rightarrow 0} H_{\epsilon}(x)$ where for $\epsilon > 0$,

$$H_{\epsilon}(x) = \int_{(D_{\epsilon}(x))^c} \frac{[W_{\lambda}(x) - W_{\lambda}(y)]^{p-1}}{|x - y|^{N+sp}} dy \quad (6.3.18)$$

and $D_{\epsilon}(x) = \{y \in \mathbb{R}^N : |\psi^{-1}(x) - \psi^{-1}(y)| \leq \epsilon\}$.

By change of variable, with the notations $x = \psi(X)$ and $A_X = D\psi(X)$, we have:

$$\begin{aligned} H_{\epsilon}(x) &= |\det A_X| \mathcal{J}_{\epsilon, A_X}(X) + \int_{(B_{\epsilon}(X))^c} \frac{[U_{\lambda}(X_N) - U_{\lambda}(Y_N)]^{p-1}}{|A_X(X - Y)|^{N+ps}} h(X, Y) dY \\ &= H_{\epsilon,1}(X) + H_{\epsilon,2}(X) \end{aligned}$$

where, by Lemma 3.4 in [172], there exists a constant C_ψ such that

$$\begin{aligned} |h(X, Y)| &= \left| \frac{|A_X(X - Y)|^{N+ps}}{|\psi(X) - \psi(Y)|^{N+sp}} |\det A_Y| - |\det A_X| \right| \\ &\leq C_\psi \min\{|X - Y|, 1\}. \end{aligned}$$

In order to apply Lemma 2.5 in [172], first we prove uniform estimates of H_ϵ on compact set of $\psi(\mathbb{R}_+^N)$. Since ψ is a $C^{1,1}$ -diffeomorphism such that $\psi = Id$ in $B_R(0)^c$ for some $R > 0$ therefore the mappings $X \mapsto |\det D\psi(X)|$ and $X \mapsto \|D\psi(X)\|_\infty$ are bounded on \mathbb{R}^N . More precisely, there exists a constant $c_\psi > 0$ such that for any $X \in \mathbb{R}^N$

$$\frac{1}{c_\psi} \leq |\det D\psi(X)| \leq c_\psi \quad \text{and} \quad \frac{1}{c_\psi} \leq \|D\psi(X)\|_\infty \leq c_\psi. \quad (6.3.19)$$

Hence plugging (6.3.16) and (6.3.19), we obtain $H_{\epsilon,1}$ is bounded in \mathbb{R}^N . Now, we give an estimate of $H_{\epsilon,2}$ in $\{X \in \mathbb{R}_+^N : 0 < X_N < 1\}$:

$$\begin{aligned} |H_{\epsilon,2}(X)| &\leq C_\psi \left(\int_{B_1(X) \setminus B_\epsilon(X)} \frac{|U_\lambda(X_N) - U_\lambda(Y_N)|^{p-1} |X - Y|}{|A_X(X - Y)|^{N+sp}} dY \right. \\ &\quad \left. + \int_{(B_1(X))^c} \frac{|U_\lambda(X_N) - U_\lambda(Y_N)|^{p-1}}{|A_X(X - Y)|^{N+sp}} dY \right) \\ &= C_\psi \left(H_{\epsilon,2}^*(X) + H_{\epsilon,2}^\diamond(X) \right). \end{aligned} \quad (6.3.20)$$

First, by Hölder regularity of the mapping $x \mapsto x^\alpha$, we have for any $X \in \mathbb{R}_+^N$:

$$H_{\epsilon,2}^\diamond(X) \leq C_\psi \int_1^\infty \frac{1}{t^{1+\beta}} dt \leq C_\psi. \quad (6.3.21)$$

For the first term, using polar coordinates $Y = X + \sigma w$ for $w \in S^{N-1}$, $\sigma > 0$, $X \in \mathbb{R}_+^N$ and by choosing $\epsilon < X_N$, we obtain from (6.3.19)

$$\begin{aligned} H_{\epsilon,2}^*(X) &\leq c_\psi \int_{S^{N-1}} \frac{1}{|w|^{N+sp-1}} \int_\epsilon^1 \frac{|U_\lambda(X_N) - U_\lambda(X_N + \sigma w_N)|^{p-1}}{|\sigma|^{sp}} d\sigma dw \\ &= c_\psi \int_{S^{N-1} \cap \{w_N > 0\}} |w|^{-N} \int_{(-\epsilon w_N, \epsilon w_N)^c \cap (-w_N, w_N)} \frac{|U_\lambda(X_N) - U_\lambda(X_N + t)|^{p-1}}{|t|^{sp}} dt dw \\ &\leq c_\psi \int_{S^{N-1} \cap \{w_N > 0\}} \int_{(-\epsilon w_N, \epsilon w_N)^c \cap (-1, 1)} \frac{|U_\lambda(X_N) - U_\lambda(X_N + t)|^{p-1}}{|t|^{sp}} dt dw \\ &= c_\psi \int_{S^{N-1} \cap \{w_N > 0\}} \mathcal{H}_{\epsilon w_N}(X_N) dw \end{aligned} \quad (6.3.22)$$

where for any $r \in \mathbb{R}_+$ and for $\vartheta \in (0, \min\{1, r\})$

$$\mathcal{H}_\vartheta(r) = \int_{(-\vartheta, \vartheta)^c \cap (-1, 1)} \frac{|U_\lambda(r) - U_\lambda(r + t)|^{p-1}}{|t|^{sp}} dt.$$

As previously, to estimate \mathcal{H}_ϑ , we split the integral as follows

$$\begin{aligned}\mathcal{H}_\vartheta(r) &= \int_{r-1}^{r-\vartheta} \frac{|U_\lambda(r) - U_\lambda(t)|^{p-1}}{|r-t|^{sp}} dt + \int_{r+\vartheta}^{r+1} \frac{|U_\lambda(r) - U_\lambda(t)|^{p-1}}{|r-t|^{sp}} dt \\ &= \mathcal{H}_{\vartheta,1}(r) + \mathcal{H}_{\vartheta,2}(r).\end{aligned}\quad (6.3.23)$$

For $\mathcal{H}_{\vartheta,1}$, we consider two cases: for $r \leq 1 - \lambda^{1/\alpha}$, we have

$$\mathcal{H}_{\vartheta,1}(r) = \int_{r-1}^{-\lambda^{1/\alpha}} \frac{|U_\lambda(r)|^{p-1}}{|r-t|^{sp}} dt + \int_{-\lambda^{1/\alpha}}^{r-\vartheta} \frac{|U_\lambda(r) - U_\lambda(t)|^{p-1}}{|r-t|^{sp}} dt.$$

Hence the first term in the right-hand side is bounded by

$$\begin{cases} \frac{1}{sp-1}(r + \lambda^{1/\alpha})^{1-\beta} & \text{if } sp > 1, \\ C(\alpha, s, p) & \text{if } sp \leq 1. \end{cases}\quad (6.3.24)$$

Using a change of variable in the second term of the right-hand side and for any $t \in (0, 1)$, $1 - t^\alpha \leq 1 - t^s \leq (1 - t)^s$, we get

$$\begin{aligned}(r + \lambda^{1/\alpha})^{1-\beta} \int_0^{1 - \frac{\vartheta}{r + \lambda^{1/\alpha}}} \frac{(1 - t^\alpha)^{p-1}}{(1 - t)^{sp}} dt &\leq (r + \lambda^{1/\alpha})^{1-\beta} \int_0^{1 - \frac{\vartheta}{r + \lambda^{1/\alpha}}} (1 - t)^{-s} dt \\ &\leq \frac{1}{1-s} (r + \lambda^{1/\alpha})^{1-\beta}.\end{aligned}\quad (6.3.25)$$

For $r > 1 - \lambda^{1/\alpha}$, we have

$$\mathcal{H}_{\vartheta,1}(r) \leq \int_{-\lambda^{1/\alpha}}^{r-\vartheta} \frac{|U_\lambda(r) - U_\lambda(t)|^{p-1}}{|r-t|^{sp}} dt \leq \frac{1}{1-s} (r + \lambda^{1/\alpha})^{1-\beta}.$$

In the same way for $\mathcal{H}_{\vartheta,2}$, since for any $t \geq 1$, $t^\alpha - 1 \leq t^s - 1 \leq (t - 1)^s$, we get:

$$\begin{aligned}\mathcal{H}_{\vartheta,2}(r) &\leq (r + \lambda^{1/\alpha})^{1-\beta} \int_{1 + \frac{\vartheta}{r + \lambda^{1/\alpha}}}^{1 + \frac{1}{r + \lambda^{1/\alpha}}} \frac{(t^\alpha - 1)^{p-1}}{(t - 1)^{sp}} dt \\ &\leq (r + \lambda^{1/\alpha})^{1-\beta} \int_{1 + \frac{\vartheta}{r + \lambda^{1/\alpha}}}^{1 + \frac{1}{r + \lambda^{1/\alpha}}} (t - 1)^{-s} dt \leq \frac{1}{1-s} (r + \lambda^{1/\alpha})^{s-\beta}.\end{aligned}\quad (6.3.26)$$

Then, by collecting the estimates (6.3.24)-(6.3.26), we obtain for any $r > 0$ and $\vartheta \in (0, \min\{1, r\})$

$$\mathcal{H}_\vartheta(r) \leq M(r + \lambda^{1/\alpha})^{-\beta} ((r + \lambda^{1/\alpha})^s + (r + \lambda^{1/\alpha}) + (r + \lambda^{1/\alpha})^\beta) \quad (6.3.27)$$

where M is positive constant depending upon α , s and p .

From (6.3.27), we deduce that $H_{\epsilon,2}^*$ and thus $H_{\epsilon,2}$ are bounded on compact sets of \mathbb{R}_+^N . Hence, H_ϵ converges to $\frac{1}{2}H$ in $L_{loc}^1(\psi(\mathbb{R}_+^N))$ and we apply Lemma 2.5 of [172] which implies that W_λ satisfies $(-\Delta)_p^s W_\lambda = H$ E-weakly in $\psi(\mathbb{R}_+^N)$.

Since (6.3.27) is independent of ϑ , then gathering (6.3.21), (6.3.22), (6.3.27) in (6.3.20), there

exist λ^* and ρ^* small enough, for any $\lambda \leq \lambda^*$ and $\rho \leq \rho^*$, there exists a constant \tilde{C} independent of λ and ϵ such that for any $X \in \{X : 0 < X_N < \rho\}$:

$$|H_{\epsilon,2}(X)| \leq \tilde{C}(1 + (X_N + \lambda^{1/\alpha})^{s-\beta}) \leq \frac{C_3}{2c_\psi}(X_N + \lambda^{1/\alpha})^{-\beta} \quad (6.3.28)$$

where C_3 is defined in (6.3.15).

Finally, by combining (6.3.15), (6.3.19) and (6.3.28), there exists a constant \tilde{C} independent of λ such that

$$\frac{1}{\tilde{C}}(X_N + \lambda^{1/\alpha})^{-\beta} \leq \lim_{\epsilon \rightarrow 0} H_\epsilon(x) \leq \tilde{C}(X_N + \lambda^{1/\alpha})^{-\beta}, \quad \forall x \in \psi(\{X : 0 < X_N < \rho\}) \quad (6.3.29)$$

and we deduce (6.3.17). \square

We extend the definition of the function d in Ω^c as follows

$$d_e(x) = \begin{cases} \text{dist}(x, \partial\Omega) & \text{if } x \in \Omega; \\ -\text{dist}(x, \partial\Omega) & \text{if } x \in (\Omega^c)_{\lambda^{\frac{1}{\alpha}}}; \\ -\lambda^{1/\alpha} & \text{otherwise.} \end{cases}$$

where $(\Omega^c)_\eta = \{x \in \Omega^c : \text{dist}(x, \partial\Omega) < \eta\}$. Hence we define, for some $\rho > 0$ and $\lambda > 0$:

$$\underline{w}_\rho(x) = \begin{cases} (d_e(x) + \lambda^{1/\alpha})_+^\alpha - \lambda & \text{if } x \in \Omega \cup (\Omega^c)_\rho \\ -\lambda & \text{otherwise,} \end{cases} \quad (6.3.30)$$

$$\overline{w}_\rho(x) = \begin{cases} (d_e(x) + \lambda^{1/\alpha})_+^\alpha & \text{if } x \in \Omega \cup (\Omega^c)_\rho, \\ 0 & \text{otherwise.} \end{cases} \quad (6.3.31)$$

Theorem 6.3.7. *Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain with a $C^{1,1}$ boundary and $\alpha \in (0, s)$. Then, for some $\rho > 0$, there exist $(\lambda_*, \eta_*) \in \mathbb{R}_*^+ \times \mathbb{R}_*^+$ such that for any $\eta < \eta_*$, there exist positive constants C_5, C_6 such that for any $\lambda \in [0, \lambda_*]$:*

$$(-\Delta)_p^s \overline{w}_\rho \geq C_5(d(x) + \lambda^{1/\alpha})^{-\beta} \quad \text{and} \quad (-\Delta)_p^s \underline{w}_\rho \leq C_6(d(x) + \lambda^{1/\alpha})^{-\beta} \quad E\text{-weakly in } \Omega_\eta \quad (6.3.32)$$

where $\Omega_\eta = \{x \in \Omega : d(x) < \eta\}$. Moreover, for $\lambda > 0$, $\underline{w}_\rho, \overline{w}_\rho$ belong to $\overline{W}^{s,p}(\Omega_\eta)$.

Proof. Since $\partial\Omega \in C^{1,1}$, then for every $x \in \partial\Omega$, there exist a neighbourhood N_x of x and a bijective map $\Psi_x : Q \mapsto N_x$ such that

$$\Psi_x \in C^{1,1}(\overline{Q}), \quad \Psi_x^{-1} \in C^{1,1}(\overline{N_x}), \quad \Psi_x(Q_+) = N_x \cap \Omega \quad \text{and} \quad \Psi_x(Q_0) = N_x \cap \partial\Omega$$

where $Q := \{X = (X', X_N) : |X'| < 1, |X_N| < 1\}$, $Q_+ := Q \cap \mathbb{R}_+^N$, $Q_0 := Q \cap \{X_N = 0\}$.

For any $x \in \partial\Omega$, $0 < \tilde{\rho} < \rho < \rho^*$ where ρ^* is defined in Theorem 6.3.6 and using the fact that $\partial\Omega$ is compact, there exist a finite covering $\{B_{R_i}(x_i)\}_{i \in I}$ of $\partial\Omega$ and $\eta^* = \eta^*(R_i)$, $i \in I$ such that for any $\eta \in (0, \eta^*)$

$$\Omega_\eta \subset \bigcup_{i \in I} B_{R_i}(x_i) \quad \text{and} \quad \forall i \in I, \quad \Psi_{x_i}^{-1}(B_{R_i}(x_i)) \subset B_{\tilde{\rho}}(0) \subset B_\rho(0). \quad (6.3.33)$$

Now by using the geometry of $\partial\Omega$ and arguing as in Lemma 3.5 and Theorem 3.6 in [172], there exist diffeomorphisms $\Phi_i \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$ for any $i \in I$ satisfying $\Phi_i = \Psi_{x_i}$ in $B_\rho(0)$ and $\Phi_i = Id$ in $(B_{4\rho}(0))^c$,

$$\Omega_\eta \cap B_{R_i}(x_i) \Subset \Phi_i(B_{\bar{\rho}} \cap \mathbb{R}_+^N), \quad d_e(\Phi_i(X)) = (X_N + \lambda^{1/\alpha})_+ - \lambda^{1/\alpha}, \quad \forall X \in B_\rho$$

and for λ small enough $\lambda^{1/\alpha} < \rho$,

$$\Phi_i(B_\rho(0) \cap \{X_N \geq -\lambda^{1/\alpha}\}) \subset \Omega \cup (\Omega^c)_\rho.$$

Using the finite covering, it is sufficient to prove the statement in any of set $\Omega_\eta \cap B_{R_i}(x_i)$ with $x_i \in \partial\Omega$ and for the sake of simplicity we can suppose $x_i = 0$, $\Phi_i = \Phi$ and $\Phi(0) = 0$.

Let $g_{\epsilon,1}$ and $g_{\epsilon,2}$ be two functions defined by

$$g_{\epsilon,1}(x) = \int_{D_\epsilon(x)} \frac{[\underline{w}_\rho(x) - \underline{w}_\rho(y)]^{p-1}}{|x-y|^{N+sp}} dy$$

and

$$g_{\epsilon,2}(x) = \int_{D_\epsilon(x)} \frac{[\overline{w}_\rho(x) - \overline{w}_\rho(y)]^{p-1}}{|x-y|^{N+sp}} dy$$

where $D_\epsilon(x) = \{y : |\Phi^{-1}(x) - \Phi^{-1}(y)| > \epsilon\}$.

As in the proof of Theorem 6.3.6, it suffices to obtain suitable uniform bounds on compact sets of $g_{\epsilon,1}$ and $g_{\epsilon,2}$. Hence Lemma 2.5 in [172] gives estimates (6.3.32).

Let $x \in B_{R_i}(0) \cap \Omega_\eta$, there exists $X \in B_{\bar{\rho}}(0) \cap \mathbb{R}_+^N$ such that $\Phi(X) = x$ and hence by change of variables and arguing as in Theorem 3.6 in [172], we obtain

$$\begin{aligned} g_{\epsilon,1}(x) &= \int_{(B_\epsilon(X))^c} \frac{[\underline{w}_\rho(\Phi(X)) - \underline{w}_\rho(\Phi(Y))]^{p-1}}{|\Phi(X) - \Phi(Y)|^{N+sp}} |\det D\Phi(Y)| dY \\ &= \int_{B_\rho(0) \setminus B_\epsilon(X)} + \int_{(B_\rho(0))^c} \\ &= \int_{(B_\epsilon(X))^c} \frac{[(X_N + \lambda^{1/\alpha})_+^\alpha - (Y_N + \lambda^{1/\alpha})_+^\alpha]^{p-1}}{|\Phi(X) - \Phi(Y)|^{N+sp}} |\det D\Phi(Y)| dY \\ &\quad + \int_{(B_\rho(0))^c} \frac{[\underline{w}_\rho(\Phi(X)) - \underline{w}_\rho(\Phi(Y))]^{p-1} - [U_\lambda(X_N) - U_\lambda(Y_N)]^{p-1}}{|\Phi(X) - \Phi(Y)|^{N+sp}} |\det D\Phi(Y)| dY \\ &= M_\epsilon(X) + M_{\underline{w}_\rho}(X) \end{aligned}$$

and similarly,

$$\begin{aligned} g_{\epsilon,2}(x) &= \int_{(B_\epsilon(X))^c} \frac{[\overline{w}_\rho(\Phi(X)) - \overline{w}_\rho(\Phi(Y))]^{p-1}}{|\Phi(X) - \Phi(Y)|^{N+sp}} |\det D\Phi(Y)| dY \\ &= \int_{B_\epsilon(X)} \frac{[(X_N + \lambda^{1/\alpha})_+^\alpha - (Y_N + \lambda^{1/\alpha})_+^\alpha]^{p-1}}{|\Phi(X) - \Phi(Y)|^{N+sp}} |\det D\Phi(Y)| dY \\ &\quad + \int_{(B_\rho(0))^c} \frac{[\overline{w}_\rho(\Phi(X)) - \overline{w}_\rho(\Phi(Y))]^{p-1} - [U_\lambda(X_N) - U_\lambda(Y_N)]^{p-1}}{|\Phi(X) - \Phi(Y)|^{N+sp}} |\det D\Phi(Y)| dY \\ &= M_\epsilon(X) + M_{\overline{w}_\rho}(X). \end{aligned}$$

From the Lipschitz continuity of Φ^{-1} , the α -Hölder continuity of U_λ , \underline{w}_ρ and \bar{w}_ρ , we obtain by using (6.3.19) for $w = \underline{w}_\rho$ or $w = \bar{w}_\rho$:

$$|M_w(X)| \leq c_\Phi \int_{(B_\rho(0))^c} \frac{2}{|X - Y|^{N+\beta}} dY \leq C(\Phi, \rho, \tilde{\rho}) \int_{\mathbb{R}^N} \frac{1}{(1 + |Y|)^{N+\beta}} dY \leq \mathcal{C} \quad (6.3.34)$$

where \mathcal{C} is a constant independent of X , λ and ϵ .

Now we deal with M_ϵ performing change of variables. We note M_ϵ coincides with H_ϵ in (6.3.18). Hence, using the estimate in (6.3.29), we get

$$c_3(d(x) + \lambda^{1/\alpha})^{-\beta} \leq \lim_{\epsilon \rightarrow 0} M_\epsilon(\Phi^{-1}(x)) \leq c_4(d(x) + \lambda^{1/\alpha})^{-\beta} \quad \text{E-weakly in } \Omega_\eta \cap B_{R_i}(0). \quad (6.3.35)$$

where c_3 and c_4 are positive constant depending upon α , N , s , p and Φ . By combining (6.3.34) and (6.3.35) for any $i \in I$, we obtain for all $x \in \Omega_\eta$

$$(-\Delta)_p^s \underline{w}_\rho(x) \leq c_3(d(x) + \lambda^{1/\alpha})^{-\beta} + \mathcal{C} \quad \text{E-weakly in } \Omega_\eta$$

and

$$c_4(d(x) + \lambda^{1/\alpha})^{-\beta} - \mathcal{C} \leq (-\Delta)_p^s \bar{w}_\rho(x) \quad \text{E-weakly in } \Omega_\eta.$$

Finally, we deduce the estimates (6.3.32) taking η and λ small enough.

To prove $\underline{w}_\rho, \bar{w}_\rho \in \overline{W}^{s,p}(\Omega_\eta)$ for $\lambda > 0$, it is sufficient to claim

$$\underline{w}_\rho, \bar{w}_\rho \in W^{s,p}(K), \quad K := \Omega_{\eta_1} \cup (\Omega^c)_{\eta_2}$$

for some $0 < \eta < \eta_1$ and $\eta_2 > 0$.

For $x_i \in \partial\Omega$, for $\eta_0 \in (0, \eta^*)$, let $\{B_{R_i}(x_i)\}_{i \in I}$ be the finite covering of Ω_{η_0} and $\Xi_i \in C^{1,1}(\mathbb{R}^N, \mathbb{R}^N)$ such that

$$B_{R_i}(x_i) \Subset \Xi_i(B_{\xi_0}), \quad d_e(\Xi_i(X)) = (X_N + \lambda^{1/\alpha})_+ - \lambda^{1/\alpha}, \quad \forall X \in B_{\xi_0} \quad (6.3.36)$$

for some $\xi_0 \in (0, \frac{\lambda^{1/\alpha}}{2})$. The existence of finite covering $\{B_{R_i}(x_i)\}_{i \in I}$ and diffeomorphisms Ξ_i are obtained as above by using (6.3.33).

For any $i \in I$, there exists a subset J^i of I such that

$$B_{R_i}(x_i) \cap B_{R_j}(x_j) \neq \emptyset \quad \forall j \in J^i. \quad (6.3.37)$$

The collection of sets $\{B_{R_j}(x_j)\}_{j \in J^i}$ satisfying (6.3.37) are called adjacent sets to $B_{R_i}(x_i)$. Now for any $i \in I$ and $j \in J^i$, define for some $\tau_i < R_i$

$$K_i := B_{\tau_i}(x_i) \subset B_{R_i}(x_i) \quad (6.3.38)$$

such that

$$\text{for any } i \in I, \quad K_i \cap K_j \neq \emptyset \quad \forall j \in J_i \quad \text{and} \quad \min_{i \in I} \left(\min_{j \in J^i} \text{dist}(K_j \setminus B_{R_i}(x_i), K_i) \right) > 0. \quad (6.3.39)$$

By using (6.3.38) and (6.3.39), we choose η_1 and η_2 small enough such that

$$K = \Omega_{\eta_1} \cup (\Omega^c)_{\eta_2} \subset \bigcup_{i \in I} K_i.$$

Now by using (6.3.36), we obtain, for any $i \in I$

$$\begin{aligned} \Omega_{\eta_1} \cap K_i &\subset \Omega_{\eta_1} \cap B_{R_i}(x_i) \Subset \Xi_i(B_{\xi_0} \cap \mathbb{R}_+^N), \\ (\Omega^c)_{\eta_2} \cap K_i &\subset (\Omega^c)_{\eta_2} \cap B_{R_i}(x_i) \Subset \Xi_i(B_{\xi_1} \cap \mathbb{R}_-^N) \\ \text{and } d_e(\Xi_i(X)) &= (X_N + \lambda^{1/\alpha})_+ - \lambda^{1/\alpha}, \quad \forall X \in \Xi_i^{-1}(K_i) \subset B_{\xi_0} \end{aligned} \quad (6.3.40)$$

for some $\eta_1 < \eta^*$ and $\eta_2 > 0$ such that $0 < \xi_1 < \frac{\lambda^{1/\alpha}}{2}$. Set $\widehat{K}_i = K_i \cap K$. Then, splitting $K \times K = \mathcal{Q} \cap (K \times K \setminus \mathcal{Q})$ where

$$\mathcal{Q} = \bigcup_{i \in I} \left(\widehat{K}_i \times \bigcup_{j \notin J^i} \widehat{K}_j \right) \cup \bigcup_{i \in I} \left(\widehat{K}_i \times \bigcup_{j \in J^i} \widehat{K}_j \cap (B_{R_i}(x_i))^c \right),$$

we obtain from (6.3.37)- (6.3.39)

$$\int_{\mathcal{Q}} \frac{|\underline{w}_\rho(x) - \underline{w}_\rho(y)|^p}{|x - y|^{N+sp}} dx dy = \int_{\mathcal{Q}} \frac{|(d(x) + \lambda^{\frac{1}{\alpha}})^\alpha - (d(y) + \lambda^{\frac{1}{\alpha}})^\alpha|^p}{|x - y|^{N+sp}} dx dy \leq C_{\Omega, \eta} \quad (6.3.41)$$

and for the second part, we perform change of variables using (6.3.40) and diffeomorphisms Ξ_i

$$\begin{aligned} &\int_{K \times K \setminus \mathcal{Q}} \frac{|\underline{w}_\rho(x) - \underline{w}_\rho(y)|^p}{|x - y|^{N+sp}} dx dy \\ &= \int_{\Xi_i^{-1}(\widehat{K}_i) \times \Xi_i^{-1}(\widehat{K}_i)} \frac{|(d(\Phi(X)) + \lambda^{\frac{1}{\alpha}})^\alpha - (d(\Phi(Y)) + \lambda^{\frac{1}{\alpha}})^\alpha|^p}{|\Phi_i(X) - \Phi_i(Y)|^{1+sp}} J_{\Xi_i}(X) J_{\Xi_i}(Y) dX dY \\ &\quad + \sum_{i \in I} \sum_{j \in J^i} \int_{\Xi_i^{-1}(\widehat{K}_i)} \int_{\Xi_i^{-1}(\widehat{K}_j \cap B_{R_i}(x_i))} \frac{|(d(\Phi(X)) + \lambda^{\frac{1}{\alpha}})^\alpha - (d(\Phi(Y)) + \lambda^{\frac{1}{\alpha}})^\alpha|^p}{|\Phi_i(X) - \Phi_i(Y)|^{1+sp}} J_{\Xi_i}(X) J_{\Xi_i}(Y) dX dY \\ &\leq C_{\Phi_i} \left(\int_{\Xi_i^{-1}(\widehat{K}_i) \times \Xi_i^{-1}(\widehat{K}_i)} \frac{|(X_N + \lambda^{\frac{1}{\alpha}})_+^\alpha - (Y_N + \lambda^{\frac{1}{\alpha}})_+^\alpha|^p}{|X_N - Y_N|^{N+sp}} dX dY \right. \\ &\quad \left. + \sum_{i \in I} \sum_{j \in J^i} \int_{\Xi_i^{-1}(\widehat{K}_i)} \int_{\Xi_i^{-1}(\widehat{K}_j \cap B_{R_i}(x_i))} \frac{|(X_N + \lambda^{\frac{1}{\alpha}})_+^\alpha - (Y_N + \lambda^{\frac{1}{\alpha}})_+^\alpha|^p}{|X_N - Y_N|^{N+sp}} dX dY \right). \end{aligned} \quad (6.3.42)$$

Hence by observing that $X_N, Y_N > -\min\{\xi_0, \xi_1\} > -\frac{\lambda^{1/\alpha}}{2}$ for all $X, Y \in \Xi_i^{-1}(\widehat{K}_i)$ and by using the same argument as in Theorem 6.3.5 and by combining (6.3.41) and (6.3.42), we obtain $\underline{w}_\rho \in \overline{W}^{s,p}(\Omega_\eta)$. Similarly, we can prove $\overline{w}_\rho \in \overline{W}^{s,p}(\Omega_\eta)$. \square

6.3.4 Sobolev and Hölder regularity

We consider the sequence of function $\{\tilde{K}_{\lambda,\delta}\}_{\lambda \geq 0}$ where $\delta \in (0, sp)$, $\tilde{K}_{\lambda,\delta} : \mathbb{R}^N \rightarrow \mathbb{R}_+$ such that

$$\tilde{K}_{\lambda,\delta}(x) = \begin{cases} (K_\delta^{-\frac{1}{\delta}}(x) + \lambda^{\frac{p-1}{sp-\delta}})^{-\delta} & \text{if } x \in \Omega, \\ 0 & \text{if } x \notin \Omega, \end{cases}$$

satisfying $\tilde{K}_{\lambda,\delta} \nearrow K_\delta$ a.e. in Ω as $\lambda \rightarrow 0^+$, and there exist two positive constants $\mathcal{D}_3, \mathcal{D}_4$ such that

$$\frac{\mathcal{D}_3}{\left(d(x) + \lambda^{\frac{p-1}{sp-\delta}}\right)^\delta} \leq \tilde{K}_{\lambda,\delta}(x) \leq \frac{\mathcal{D}_4}{\left(d(x) + \lambda^{\frac{p-1}{sp-\delta}}\right)^\delta}. \quad (6.3.43)$$

Gathering Proposition 6.3.3, Theorem 6.3.2 and Remark 6.3.2, we have the following result for the following approximated problem (noting $\gamma = 0$ in Proposition 6.3.3):

$$(S_\lambda^\delta) \begin{cases} (-\Delta)_p^s u = \tilde{K}_{\lambda,\delta} & \text{in } \Omega; \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Theorem 6.3.8. *Let Ω be a bounded domain with Lipschitz boundary. Then there exists a increasing sequence of weak solution $\{u_\lambda\}_{\lambda > 0} \subset W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$ of (S_λ^δ) such that*

$$\iint_{\mathbb{R}^{2N}} \frac{[u_\lambda(x) - u_\lambda(y)]^{p-1} (\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy = \int_\Omega \tilde{K}_{\lambda,\delta}(x) \phi dx.$$

for all $\phi \in W_0^{s,p}(\Omega)$ and a minimal weak solution u of (S_0^δ) such that $u_\lambda^{\theta_1} \rightarrow u^{\theta_1}$ in $W_0^{s,p}(\Omega)$ and

$$\iint_{\mathbb{R}^{2N}} \frac{[u(x) - u(y)]^{p-1} (\varphi(x) - \varphi(y))}{|x - y|^{N+sp}} dx dy = \int_\Omega K_\delta(x) \varphi dx$$

for all $\varphi \in \mathbb{T}$ where $\theta_1 = \begin{cases} 1 & \text{if } 0 < \delta < 1 + s - \frac{1}{p}, \\ \theta_2 & \text{otherwise,} \end{cases}$ and $\theta_2 > \max\left\{\frac{(p-1)(sp-1)}{p(sp-\delta)}, 1\right\}$.

Let $\lambda_{s,p}$ be the first eigenvalue and $\varphi_{s,p}$ be a positive eigenfunction for the operator $(-\Delta)_p^s$. There exists a constant $c > 0$ such that $\frac{1}{c}d^s(x) \leq \varphi_{s,p}(x) \leq cd^s(x)$ for any $x \in \Omega$. Hence, from (6.3.43), for any $\delta > 0$, choosing a constant $a > 0$ small enough, the following inequality holds for any $x \in \Omega$ and $\lambda \geq 0$:

$$(-\Delta)_p^s(a\varphi_{s,p}) \leq \tilde{K}_{\lambda,\delta}(x) \leq (-\Delta)_p^s u_\lambda.$$

Then, by using Proposition 2.10 in [172], we get, for any $\delta \in (0, sp)$, there exists a constant κ_1 such that for any $\lambda \geq 0$

$$\kappa_1 d^s(x) \leq u_\lambda(x) \text{ for any } x \in \Omega. \quad (6.3.44)$$

Moreover, we have the upper bound of u_λ in $\Omega \setminus \Omega_\eta$. For $\eta > 0$ small enough, we consider $\{B_{\frac{\eta}{4}}(x_i)\}_{i \in \{1, 2, \dots, m\}}$ a finite covering of $\overline{\Omega} \setminus \overline{\Omega_\eta}$ such that

$$\overline{\Omega} \setminus \overline{\Omega_\eta} \subset \bigcup_{i=1}^m B_{\frac{\eta}{4}}(x_i) \subset \Omega \setminus \Omega_{\frac{\eta}{2}}. \quad (6.3.45)$$

Then, from Theorem 3.2 and Remark 3.3 in [65], we deduce for any $i \in \{1, 2, \dots, m\}$

$$\begin{aligned} \|u_\lambda\|_{L^\infty(B_{\frac{\eta}{4}}(x_i))} \leq & C \left[\left(\int_{B_{\frac{\eta}{2}}(x_i)} |u_\lambda(x)|^p dx \right)^{\frac{1}{p}} + \left(\eta^{sp} \int_{\mathbb{R}^N \setminus B_{\frac{\eta}{4}}(x_i)} \frac{|u_\lambda(x)|^{p-1}}{|x-x_i|^{N+sp}} dx \right)^{\frac{1}{p-1}} \right. \\ & \left. + \left(\eta^{sp} \|\tilde{K}_{\lambda, \delta}\|_{L^\infty(B_{\frac{\eta}{2}}(x_i))} \right)^{\frac{1}{p-1}} \right] \end{aligned} \quad (6.3.46)$$

where C depends upon N, p and s . From the proof of Theorem 6.3.2, $\{u_\lambda^{\theta_1}\}_\lambda$ is uniformly bounded in $W_0^{s,p}(\Omega)$ and Sobolev embedding implies

$$\left(\int_{B_{\frac{\eta}{2}}(x_i)} |u_\lambda|^p \right)^{\frac{1}{p}} \leq c(1 + \|u_\lambda^{\theta_1}\|_{L^p(\Omega)}) \leq c(1 + \|u_\lambda^{\theta_1}\|_{s,p}) \leq c. \quad (6.3.47)$$

In the same way, the second term of the right hand-side is controlled, up to a constant independent of λ , by

$$\left(\eta^{sp} \int_{\Omega \setminus B_{\frac{\eta}{4}}(x_i)} \frac{|u_\lambda(x)|^{p-1}}{\eta^{N+sp}} dx \right)^{\frac{1}{p-1}} \leq \eta^{-\frac{N}{p-1}} \|u_\lambda\|_{L^{p-1}(\Omega)} \leq c. \quad (6.3.48)$$

For the last term, for any $x \in \Omega \setminus \Omega_{\frac{\eta}{2}}$, we have

$$|\tilde{K}_{\lambda, \delta}(x)| \leq \frac{\mathcal{D}_4}{\left(d(x) + \lambda^{\frac{p-1}{sp-\delta}}\right)^\delta} \leq c\eta^{-\delta} \leq c.$$

Each constant in the previous estimates are independent of λ but depends on η . Finally, plugging the three previous estimates into (6.3.46) we deduce that for any $\eta > 0$, there exists $\kappa_\eta > 0$ independent of λ such that

$$\|u_\lambda\|_{L^\infty(\Omega \setminus \Omega_\eta)} \leq \kappa_\eta. \quad (6.3.49)$$

Now, we prove the sharp estimates for both upper and lower boundary behavior of the minimal weak solution for problem (S_0^δ) for different range of δ . In this regard, we prove the following results with the help of comparison principle:

Theorem 6.3.9. *Let Ω be a bounded domain with $C^{1,1}$ boundary and u be minimal weak solution of the problem (S_0^δ) . Then, we have*

(i) For $\delta \in (s, sp)$, there exists a positive constant Υ_1 such that for any $x \in \Omega$,

$$\frac{1}{\Upsilon_1} d^{\frac{sp-\delta}{p-1}}(x) \leq u(x) \leq \Upsilon_1 d^{\frac{sp-\delta}{p-1}}(x).$$

(ii) For $\delta \in (0, s)$, for any $\epsilon > 0$, there exist positive constants Υ_2 and $\Upsilon_3 = \Upsilon_3(\epsilon)$ such that for any $x \in \Omega$:

$$\Upsilon_2 d^s(x) \leq u(x) \leq \Upsilon_3 d^{s-\epsilon}(x).$$

Proof. Let u_λ be the solution of (S_λ^δ) for $\lambda < \lambda^*$, $\eta < \eta^*$ and $\rho > 0$ given by Theorem 6.3.7. We begin to prove (i). Take $\alpha = \frac{sp-\delta}{p-1} < s$ implying $sp - \alpha(p-1) = \delta$ and we define, for some $\eta > 0$,

$$\underline{u}^{(\lambda)} = \min\left\{\kappa_2 \left(\frac{\eta}{2}\right)^{s-\alpha}, \left(\frac{\mathcal{D}_3}{C_6}\right)^{\frac{1}{p-1}}\right\} \underline{w}_\rho = \underline{c}_\eta \underline{w}_\rho$$

and

$$\bar{u}^{(\lambda)} = \max\left\{\left(\frac{2}{\eta}\right)^\alpha \kappa_{\frac{\eta}{2}}, \left(\frac{\mathcal{D}_4}{C_5}\right)^{\frac{1}{p-1}}\right\} \bar{w}_\rho = \bar{c}_\eta \bar{w}_\rho$$

where \bar{w}_ρ and \underline{w}_ρ satisfies (6.3.32), $0 < \kappa_2 < \kappa_1$, C_5, C_6 are defined in (6.3.32), κ_1 and $\kappa_{\frac{\eta}{2}}$ are defined in (6.3.44) and (6.3.49) respectively and $\mathcal{D}_3, \mathcal{D}_4$ are defined in (6.3.43). Note \underline{c}_η and \bar{c}_η are independent of λ .

Hence for any $\lambda > 0$, u_λ satisfies

$$\underline{u}^{(\lambda)}(x) \leq u_\lambda(x) \leq \bar{u}^{(\lambda)}(x) \text{ for } x \in \Omega \setminus \Omega_{\frac{\eta}{2}}, \text{ and } \underline{u}^{(\lambda)}(x) \leq 0 = u_\lambda(x) = \bar{u}^{(\lambda)}(x) \text{ for } x \in \Omega^c. \quad (6.3.50)$$

Precisely, from (6.3.44), (6.3.49) and the definitions of $\underline{w}_\rho, \bar{w}_\rho$ given by (6.3.30) and (6.3.31), we get for $x \in \Omega \setminus \Omega_{\frac{\eta}{2}}$

$$\begin{aligned} \underline{u}^{(\lambda)} &= \underline{c}_\eta \underline{w}_\rho \leq \kappa_2 \left(\frac{\eta}{2}\right)^{s-\alpha} \underline{w}_\rho \leq \kappa_1 \left(\frac{\eta}{2}\right)^{s-\alpha} d^\alpha(x) \leq \kappa_1 d^s(x) \leq u_\lambda(x), \\ u_\lambda(x) &\leq \kappa_{\frac{\eta}{2}} \leq \kappa_{\frac{\eta}{2}} \left(\frac{2}{\eta}\right)^\alpha d^\alpha(x) \leq \bar{c}_\eta \bar{w}_\rho = \bar{u}^{(\lambda)}. \end{aligned}$$

Moreover, from (6.3.32) and (6.3.43) and the choice of constants, we get $(-\Delta)_p^s \underline{u}^{(\lambda)} \leq (-\Delta)_p^s u_\lambda \leq (-\Delta)_p^s \bar{u}^{(\lambda)}$ weakly in $\Omega_{\frac{\eta}{2}}$ i.e. for any $\phi \in W_0^{s,p}(\Omega_\eta)$, $\phi \geq 0$:

$$\begin{aligned} &\iint_{\mathbb{R}^{2N}} \frac{[\underline{u}^{(\lambda)}(x) - \underline{u}^{(\lambda)}(y)]^{p-1} (\phi(x) - \phi(y))}{|x-y|^{N+sp}} dx dy \\ &\leq \iint_{\mathbb{R}^{2N}} \frac{[u_\lambda(x) - u_\lambda(y)]^{p-1} (\phi(x) - \phi(y))}{|x-y|^{N+sp}} dx dy \\ &\leq \iint_{\mathbb{R}^{2N}} \frac{[\bar{u}^{(\lambda)}(x) - \bar{u}^{(\lambda)}(y)]^{p-1} (\phi(x) - \phi(y))}{|x-y|^{N+sp}} dx dy. \end{aligned}$$

Since $\underline{w}_\rho, \bar{w}_\rho \in \overline{W}^{s,p}(\Omega_\eta)$ and $u_\lambda \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega) \subset \overline{W}^{s,p}(\Omega_\eta)$, Proposition 2.10 in [172] in Ω_η implies $\underline{u}^{(\lambda)} \leq u_\lambda \leq \bar{u}^{(\lambda)}$ in Ω_η . Hence, from (6.3.50) and passing $\lambda \rightarrow 0$, we deduce (i).

Now we prove (ii) i.e. the case $\delta \leq s$. Since (6.3.44) holds, it remains to obtain the upper bound estimate.

Let $\tilde{u}_\lambda \in W_0^{s,p}(\Omega)$ be the weak solution of $(S_\lambda^{\tilde{\delta}})$ with $\tilde{\delta} = s + \epsilon(p-1) > s$ and for $\epsilon > 0$. Then, choosing a suitable constant $c_\epsilon > 0$ independent of λ , $\tilde{u}^{(\lambda)} = c_\epsilon \tilde{u}_\lambda$ is a weak supersolution of (S_λ^δ) . Hence by Proposition 2.10 in [172], we have $u_\lambda \leq \tilde{u}^{(\lambda)}$ in Ω . We pass to the limit as $\lambda \rightarrow 0$ and using (i) with $\tilde{u}(x) = \lim_{\lambda \rightarrow 0} \tilde{u}^{(\lambda)}(x)$, we get, for $\epsilon > 0$, $u(x) \leq \bar{c}_{\eta,\epsilon} d^{s-\epsilon}(x)$ for $x \in \Omega$. \square

Concerning the Hölder regularity of the weak solution of the problem (P), we prove Theorem 6.3.4:

Proof of Theorem 6.3.4 Let u be the minimal solution of the problem (P). First, we prove the boundary behavior of the minimal weak solution by dividing the proof into two cases:

Case 1: $\frac{\delta}{s} + \gamma \leq 1$

Let \tilde{u} and \dot{u} are weak solution of the problem (S_0^ζ) for $\zeta = \delta + \gamma s \leq s$ and $\zeta = \delta + \gamma(s-\epsilon) < s$ respectively for $\epsilon \in (0, s)$. Then, from Theorem 6.3.9 (ii) there exist constants $c_i > 0$ such that

$$c_1 d^s(x) \leq \tilde{u}(x) \leq c_2 d^{s-\epsilon}(x), \quad c_3 d^s(x) \leq \dot{u}(x) \leq c_4 d^{s-\epsilon}(x) \text{ in } \Omega$$

and \dot{u}, \tilde{u} satisfies

$$(-\Delta)_p^s(C_* \dot{u}) = \frac{\mathcal{C}_1}{\mathcal{C}_2 c_4^\gamma} K_{\delta+\gamma(s-\epsilon)}(x) \leq \frac{\mathcal{C}_1}{d^\delta(x)(c_2 d^{s-\epsilon}(x))^\gamma} \leq \frac{\mathcal{C}_1}{d^\delta(x) \dot{u}^\gamma} \leq \frac{K_\delta(x)}{\dot{u}^\gamma}$$

and

$$\frac{K_\delta(x)}{\tilde{u}^\gamma} \leq \frac{\mathcal{C}_2}{d^\delta(x) \tilde{u}^\gamma} \leq \frac{\mathcal{C}_2}{d^\delta(x)(c_1 d^s(x))^\gamma} \leq \frac{\mathcal{C}_2}{\mathcal{C}_1 c_1^\gamma} K_{\delta+\gamma s}(x) = (-\Delta)_p^s(C^* \tilde{u})$$

where $C_* = \left(\frac{\mathcal{C}_1}{\mathcal{C}_2 c_4^\gamma}\right)^{\frac{1}{p-1}}$ and $C^* = \left(\frac{\mathcal{C}_2}{\mathcal{C}_1 c_1^\gamma}\right)^{\frac{1}{p-1}}$ and $\mathcal{C}_1, \mathcal{C}_2$ are defined in (6.3.1). Then by applying Theorem 6.3.1, we get

$$C_1 d^s(x) \leq u(x) \leq C_2 d^{s-\epsilon}(x) \text{ in } \Omega \tag{6.3.51}$$

for every $\epsilon > 0$, $C_1 = c_1 C_*$ and $C_2 = c_4 C^*$.

Case 2: $\frac{\delta}{s} + \gamma > 1$

Let $\lambda > 0$ and $u_\lambda \in W_0^{s,p}(\Omega)$ be the solution of the problem (P_λ^γ) for $\lambda < \lambda^*$ given in Theorem 6.3.7.

By considering the same cover of $\overline{\Omega \setminus \Omega_\eta}$ as in (6.3.45) and applying Theorem 3.2 and Remark

3.3 in [65], we obtain,

$$\begin{aligned} \|u_\lambda\|_{L^\infty(B_{\frac{\eta}{4}}(x_i))} &\leq C \left[\left(\int_{B_{\frac{\eta}{2}}(x_i)} |u_\lambda(x)|^p dx \right)^{\frac{1}{p}} + \left(\eta^{sp} \int_{\mathbb{R}^N \setminus B_{\frac{\eta}{4}}(x_i)} \frac{|u_\lambda(x)|^{p-1}}{|x-x_i|^{N+sp}} dx \right)^{\frac{1}{p-1}} \right. \\ &\quad \left. + \left(\eta^{sp} \left\| \frac{K_{\lambda,\delta}}{(u_\lambda + \lambda)^\gamma} \right\|_{L^\infty(B_{\frac{\eta}{2}}(x_i))} \right)^{\frac{1}{p-1}} \right] \end{aligned} \quad (6.3.52)$$

for any $i \in \{1, 2, \dots, m\}$ where C depends upon N , p and s . By repeating the same arguments as in (6.3.44), (6.3.47) and (6.3.48) we get that the first two terms in the right hand-side of (6.3.52) are bounded with bounds independent of λ and

$$\varkappa d^s(x) \leq u_\lambda(x) \text{ in } \Omega \quad (6.3.53)$$

for some $\varkappa > 0$ independent of λ . Now, by using above inequality, we estimate the last term in the right hand-side of (6.3.52): for any $x \in \Omega \setminus \Omega_{\frac{\eta}{2}}$, we have

$$\left| \frac{K_{\lambda,\delta}(x)}{(u_\lambda + \lambda)^\gamma} \right| \leq \frac{\mathcal{D}_4}{\left(d(x) + \lambda^{\frac{p-1}{sp-\delta}} \right)^\delta (\varkappa d^s(x))^\gamma} \leq c \eta^{-(\delta+\gamma s)} \leq c.$$

Finally, we deduce that for any $\eta > 0$, there exists $\varkappa_\eta > 0$ independent of λ such that

$$\|u_\lambda\|_{L^\infty(\Omega \setminus \Omega_\eta)} \leq \varkappa_\eta. \quad (6.3.54)$$

For $\alpha = \frac{sp-\delta}{p+\gamma-1}$ and $0 < \eta < \eta^*$, define

$$\underline{u}^{[\lambda]} = c_\eta \underline{w}_\rho \text{ and } \bar{u}^{[\lambda]} = \dot{c}_\eta \bar{w}_\rho \text{ such that } 0 < c_\eta \leq \left(\frac{\eta}{2} \right)^{s-\alpha} \varkappa \text{ and } \dot{c}_\eta \geq \left(\frac{2}{\eta} \right)^\alpha \varkappa_{\frac{\eta}{2}}$$

where \underline{w}_ρ , \bar{w}_ρ , \varkappa , $\varkappa_{\frac{\eta}{2}}$ and η^* are defined in (6.3.30), (6.3.31), (6.3.53), (6.3.54) and Theorem 6.3.7 respectively. We note that c_η , \dot{c}_η are independent of λ and for any $\lambda > 0$, $\underline{u}^{[\lambda]}$ and $\bar{u}^{[\lambda]}$ satisfy

$$\underline{u}^{[\lambda]}(x) \leq u_\lambda(x) \leq \bar{u}^{[\lambda]}(x) \text{ for } x \in \Omega \setminus \Omega_{\frac{\eta}{2}} \text{ and } \underline{u}^{[\lambda]}(x) \leq u_\lambda(x) \leq \bar{u}^{[\lambda]}(x) \text{ for } x \in \Omega^c. \quad (6.3.55)$$

Using the definition of \underline{w}_ρ and \bar{w}_ρ in (6.3.30) and (6.3.31) respectively and estimates in (6.3.32), we obtain

$$(\underline{u}^{[\lambda]} + \lambda) = c_\eta (d(x) + \lambda^{1/\alpha})^\alpha + \lambda (1 - c_\eta) \text{ and } (\bar{u}^{[\lambda]} + \lambda) = \dot{c}_\eta (d(x) + \lambda^{1/\alpha})^\alpha + \lambda \text{ in } \Omega$$

and

$$(-\Delta)_p^s \underline{u}^{[\lambda]} \leq \frac{c_\eta^{p-1} C_6}{(d(x) + \lambda^{1/\alpha})^{\delta+\alpha\gamma}} \leq c_\eta^{p-1} \frac{C_6 K_{\lambda,\delta}(x)}{C_3 (d(x) + \lambda^{1/\alpha})^{\alpha\gamma}} \text{ weakly in } \Omega_\eta, \quad (6.3.56)$$

$$(-\Delta)_p^s \bar{u}^{[\lambda]} \geq \frac{C_5 \dot{c}_\eta^{p-1}}{(d(x) + \lambda^{\frac{1}{\alpha}})^{\delta + \alpha\gamma}} \geq \dot{c}_\eta^{p-1} \frac{C_5 K_{\lambda, \delta}(x)}{C_4 (d(x) + \lambda^{\frac{1}{\alpha}})^{\alpha\gamma}} \quad \text{weakly in } \Omega_\eta \quad (6.3.57)$$

where C_5, C_6 and $\mathcal{C}_3, \mathcal{C}_4$ are defined in (6.3.32) and (6.3.2) respectively. Since $\dot{c}_\eta \rightarrow \infty$ as $\eta \rightarrow 0$ and $(\bar{u}^{[\lambda]} + \lambda) \geq \dot{c}_\eta (d(x) + \lambda^{\frac{1}{\alpha}})^\alpha$, we can choose η small enough (independent of λ) such that $C_5 \dot{c}_\eta^{\gamma+p-1} \geq \mathcal{C}_4$ and (6.3.57) reduced to

$$(-\Delta)_p^s \bar{u}^{[\lambda]} \geq \frac{K_{\lambda, \delta}(x)}{(\bar{u}^{[\lambda]} + \lambda)^\gamma} \quad \text{weakly in } \Omega_\eta.$$

Now to prove similar estimate for $\underline{u}^{[\lambda]}$, we divide the proof into two cases; for $x \in \Omega_\eta$:

Case (i): $c_\eta (d(x) + \lambda^{1/\alpha})^\alpha \geq \lambda (1 - c_\eta)$

In this case, we have $(\underline{u}^{[\lambda]} + \lambda)^{-\gamma} \geq (2c_\eta)^{-\gamma} (d(x) + \lambda^{1/\alpha})^{-\alpha\gamma}$ and by choosing η small enough such that $2^\gamma \dot{c}_\eta^{\gamma+p-1} \leq \frac{\mathcal{C}_3}{C_6}$, (6.3.56) reduced to

$$(-\Delta)_p^s \underline{u}^{[\lambda]} \leq \frac{2^\gamma \dot{c}_\eta^{\gamma+p-1} C_6}{\mathcal{C}_3} \frac{K_{\lambda, \delta}(x)}{(\underline{u}^{[\lambda]} + \lambda)^\gamma} \leq \frac{K_{\lambda, \delta}(x)}{(\underline{u}^{[\lambda]} + \lambda)^\gamma}.$$

Case (ii): $c_\eta (d(x) + \lambda^{1/\alpha})^\alpha \leq \lambda (1 - c_\eta)$

In this case, we have $(\underline{u}^{[\lambda]} + \lambda)^{-\gamma} \geq (2\lambda)^{-\gamma} (1 - c_\eta)^{-\gamma}$ and by choosing η small enough such that $c_\eta \leq 1$ and $C_6 \dot{c}_\eta^{p-1} \leq \mathcal{C}_3 (2\lambda^*)^{-\gamma} (1 - c_\eta)^{-\gamma}$, (6.3.56) reduced to,

$$(-\Delta)_p^s \underline{u}^{[\lambda]} \leq \frac{\dot{c}_\eta^{p-1} C_6}{\mathcal{C}_3} \frac{K_{\lambda, \delta}(x)}{(\underline{u}^{[\lambda]} + \lambda)^\gamma} (2\lambda)^\gamma (1 - c_\eta)^\gamma \leq \frac{K_{\lambda, \delta}(x)}{(\underline{u}^{[\lambda]} + \lambda)^\gamma}.$$

Therefore, in each case, we can choose η small enough (independent of λ) such that

$$(-\Delta)_p^s \underline{u}^{[\lambda]} \leq \frac{K_{\lambda, \delta}(x)}{(\underline{u}^{[\lambda]} + \lambda)^\gamma} \quad \text{weakly in } \Omega_\eta.$$

Since $\underline{u}^{[\lambda]}, \bar{u}^{[\lambda]} \in \overline{W}^{s,p}(\Omega_\eta)$ and $u_\lambda \in L^\infty(\Omega) \cap W_0^{s,p}(\Omega) \subset \overline{W}^{s,p}(\Omega_\eta)$, Proposition 2.10 in [172] in Ω_η implies $\underline{u}^{[\lambda]} \leq u_\lambda \leq \bar{u}^{[\lambda]}$ in Ω_η . Hence, from (6.3.55) and passing $\lambda \rightarrow 0$,

$$C_1 d^{\frac{sp-\delta}{p+\gamma-1}} \leq u \leq C_2 d^{\frac{sp-\delta}{\gamma+p-1}} \quad \text{in } \Omega. \quad (6.3.58)$$

where $C_1 = c_\eta$ and $C_2 = \dot{c}_\eta$.

Interior and boundary regularity: First we claim the following:

Claim: For all $x_0 \in \Omega$ and $R_0 = \frac{d(x_0)}{2}$ there exists universally $C_\Omega > 0$, $0 < \omega_1 < s$ and $0 < \omega_2 \leq \frac{sp-\delta}{p+\gamma-1}$ such that

$$\text{if } 1 < p < 2: \quad \|u\|_{C^{\omega_1}(B_{R_0}(x_0))} \leq C_\Omega \text{ for } \frac{\delta}{s} + \gamma \leq 1, \quad \|u\|_{C^{\omega_2}(B_{R_0}(x_0))} \leq C_\Omega \text{ for } \frac{\delta}{s} + \gamma \geq 1 \quad (6.3.59)$$

and

$$\text{if } 2 \leq p < \infty: \quad \|u\|_{C^{s-\epsilon}(B_{R_0}(x_0))} \leq C_\Omega \text{ for } \frac{\delta}{s} + \gamma \leq 1, \quad \|u\|_{C^{\frac{sp-\delta}{p+\gamma-1}}(B_{R_0}(x_0))} \leq C_\Omega \text{ for } \frac{\delta}{s} + \gamma \geq 1. \quad (6.3.60)$$

Let $x_0 \in \Omega$, $R_0 = \frac{d(x_0)}{2}$ such that $B_{R_0}(x_0) \subset B_{2R_0}(x_0) \subset \Omega$ and $u \in W^{s,p}(B_{2R_0}(x_0)) \cap L^\infty(B_{2R_0}(x_0))$ be the minimal weak solution of (P), then it satisfies

$$(-\Delta)_p^s u = \frac{K_\delta(x)}{u^\gamma} \leq \frac{C_2}{C_1^\gamma} \frac{1}{d^{\gamma s + \delta}} \leq \frac{C_2}{C_1^\gamma} \frac{1}{R_0^{\gamma s + \delta}} \text{ in } B_{R_0}(x_0) \text{ for } 0 < \frac{\delta}{s} + \gamma \leq 1$$

and

$$(-\Delta)_p^s u = \frac{K_\delta(x)}{u^\gamma} \leq \frac{C_2}{C_1^\gamma} \frac{1}{d^{\gamma(\frac{sp-\delta}{\gamma+p-1})+\delta}} \leq \frac{C_2}{C_1^\gamma} \frac{1}{R_0^{\gamma(\frac{sp-\delta}{\gamma+p-1})+\delta}} \text{ in } B_{R_0}(x_0) \text{ for } \frac{\delta}{s} + \gamma > 1$$

where C_2 is defined in (6.3.1). Then, by using Corollary 6.3.4 for $p \in (1, 2)$, (6.3.51) and (6.3.58) we obtain: there exist $\omega_1 \in (0, s)$ and $\omega_2 \in (0, \frac{sp-\delta}{p+\gamma-1}]$ such that if $0 < \frac{\delta}{s} + \gamma \leq 1$:

$$\begin{aligned} [u]_{C^{\omega_1}(B_{R_0}(x_0))} &\leq C R_0^{-\omega_1} \left(R_0^{\frac{(sp-\delta-\gamma s)}{p-1}} + \|u\|_{L^\infty(B_{2R_0}(x_0))} \right. \\ &\quad \left. + \left((2R_0)^{sp} \int_{(B_{2R_0}(x_0))^c} \frac{|u(y)|^{p-1}}{|x_0-y|^{N+sp}} dy \right)^{\frac{1}{p-1}} \right) \\ &\leq \mathbf{C}_1 \end{aligned}$$

and if $\frac{\delta}{s} + \gamma > 1$:

$$\begin{aligned} [u]_{C^{\omega_2}(B_{R_0}(x_0))} &\leq C R_0^{-\omega_2} \left(R_0^{\frac{(sp-\delta)}{\gamma+p-1}} + \|u\|_{L^\infty(B_{2R_0}(x_0))} \right. \\ &\quad \left. + \left((2R_0)^{sp} \int_{(B_{2R_0}(x_0))^c} \frac{|u(y)|^{p-1}}{|x_0-y|^{N+sp}} dy \right)^{\frac{1}{p-1}} \right) \\ &\leq \mathbf{C}_2. \end{aligned}$$

Furthermore, using Proposition 6.3.4 for $p \in [2, +\infty)$, we get for any $\epsilon > 0$

$$[u]_{C^{s-\epsilon}(B_{R_0/32}(x_0))} \leq \mathbf{C}_3 \text{ if } 0 < \frac{\delta}{s} + \gamma \leq 1 \quad \text{and} \quad [u]_{C^{\frac{sp-\delta}{p+\gamma-1}}(B_{R_0/32}(x_0))} \leq \mathbf{C}_4 \text{ if } \frac{\delta}{s} + \gamma > 1.$$

The constants \mathbf{C}_i are independent of the choice of point x_0 (and R_0) and since $u \in L^\infty(\Omega)$ we deduce (6.3.59) and (6.3.60) and by a covering argument for any $\Omega' \Subset \Omega$, we conclude

$$\text{if } 1 < p < 2: \quad \|u\|_{C^{\omega_1}(\Omega')} \leq C_{\Omega'} \text{ for } \frac{\delta}{s} + \gamma \leq 1, \quad \|u\|_{C^{\omega_2}(\Omega')} \leq C_{\Omega'} \text{ for } \frac{\delta}{s} + \gamma \geq 1 \quad (6.3.61)$$

and

$$\text{if } 2 \leq p < \infty: \quad \|u\|_{C^{s-\epsilon}(\Omega')} \leq C_{\Omega'} \text{ for } \frac{\delta}{s} + \gamma \leq 1, \quad \|u\|_{C^{\frac{sp-\delta}{p+\gamma-1}}(\Omega')} \leq C_{\Omega'} \text{ for } \frac{\delta}{s} + \gamma \geq 1. \quad (6.3.62)$$

Now, to prove the regularity estimate in Ω (and then the whole \mathbb{R}^N) since $u = 0$ in $\mathbb{R}^N \setminus \Omega$, it is sufficient from interior regularity that follows from (6.3.61), (6.3.62), to prove (6.3.61)

and (6.3.62) on Ω_η where $\eta > 0$ small enough.

In this regard, let $x, y \in \Omega_\eta$ and suppose without loss of generality $d(x) \geq d(y)$. Now two cases occur:

(I) either $|x - y| \leq \frac{d(x)}{2}$, in which case set $2R_0 = d(x)$ and $y \in B_{R_0}(x)$. Hence we apply (6.3.59) or (6.3.60) in $B_{R_0}(x)$ and we obtain the regularity.

(II) or $|x - y| \geq \frac{d(x)}{2} \geq \frac{d(y)}{2}$ in which case (6.3.51) and (6.3.58) ensures for a constant $C > 0$ large enough, we get

$$\frac{|u(x) - u(y)|}{|x - y|^{s-\epsilon}} \leq \frac{|u(x)|}{|x - y|^{s-\epsilon}} + \frac{|u(y)|}{|x - y|^{s-\epsilon}} \leq 2^s \left(\frac{u(x)}{d^{s-\epsilon}(x)} + \frac{u(y)}{d^{s-\epsilon}(y)} \right) \leq C, \quad (6.3.63)$$

$$\begin{aligned} \frac{|u(x) - u(y)|}{|x - y|^{\frac{(sp-\delta)}{(\gamma+p-1)}}} &\leq \frac{|u(x)|}{|x - y|^{\frac{(sp-\delta)}{(\gamma+p-1)}}} + \frac{|u(y)|}{|x - y|^{\frac{(sp-\delta)}{(\gamma+p-1)}}} \\ &\leq 2^{\frac{(sp-\delta)}{(\gamma+p-1)}} \left(\frac{u(x)}{d^{\frac{(sp-\delta)}{(\gamma+p-1)}}(x)} + \frac{u(y)}{d^{\frac{(sp-\delta)}{(\gamma+p-1)}}(y)} \right) \leq C. \end{aligned} \quad (6.3.64)$$

Then, finally by combining (6.3.61)-(6.3.64), we get our claim and the proof is complete. \square

Proof of Corollary 6.3.2:

For $\delta > s(1 - \gamma)$, let u_ϵ be the weak solution of the problem (P_ϵ^γ) . Then, using the boundary behavior of the approximating sequence u_ϵ and taking $\phi = u_\epsilon$ in (6.3.3), we obtain

$$\|u_\epsilon\|_{s,p} = \int_{\Omega} K_{\epsilon,\delta}(x) u_\epsilon^{1-\gamma} dx \leq \int_{\Omega} d^{(1-\gamma)\frac{(sp-\delta)}{p+\gamma-1}-\delta}(x) dx \leq C$$

if $(1 - \gamma)(sp - \delta) > (\delta - 1)(p + \gamma - 1) \Leftrightarrow sp(\gamma - 1) + \delta p < (p + \gamma - 1) \Leftrightarrow \Lambda < 1$.

Similarly, by taking $\phi = u_\epsilon^\theta$ in (6.3.3) and using Proposition 6.3.2, we obtain for $\theta > \Lambda > 1$

$$\|u_\epsilon^\theta\|_{s,p} \leq \int_{\Omega} K_{\epsilon,\delta}(x) u_\epsilon^{(\theta-1)(p-1)+\theta-\gamma} dx \leq \int_{\Omega} d^{(\theta p - (p-1+\gamma))\frac{(sp-\delta)}{p+\gamma-1}-\delta}(x) dx \leq C$$

Now, by passing limits $\epsilon \rightarrow 0$ in (6.3.3), we get the minimal solution $u \in W_0^{s,p}(\Omega)$ if $\Lambda < 1$ and $u^\theta \in W_0^{s,p}(\Omega)$ if $\theta > \Lambda > 1$.

The only if statement follows from the Hardy inequality and the boundary behavior of the weak solution. Precisely, if $\Lambda \geq 1$, then $u \notin W_0^{s,p}(\Omega)$. Indeed, we have

$$\|u\|_{s,p} \geq C \int_{\Omega} \left| \frac{u(x)}{d^s(x)} \right|^p dx \geq C \int_{\Omega} d^{\frac{p(sp-\delta)}{p+\gamma-1}-sp}(x) dx = +\infty.$$

In the same way, if $\theta \in [1, \Lambda]$, then

$$\|u^\theta\|_{s,p} \geq C \int_{\Omega} \left| \frac{u^\theta(x)}{d^s(x)} \right|^p dx \geq C \int_{\Omega} d^{\frac{\theta p(sp-\delta)}{p+\gamma-1}-sp}(x) dx = \infty$$

and we deduce $u^\theta \notin W_0^{s,p}(\Omega)$. \square

Remark 6.3.4. In case of local operator, i.e. p -Laplacian operator, the optimal condition of Sobolev regularity in Theorem 1.4, [138] coincide with the our condition for $s = 1$.

6.3.5 Non-existence result

Proof of Theorem 6.3.3: Let $\delta \geq sp$. We proceed by contradiction assuming there exist a weak solution $u_0 \in W_{loc}^{s,p}(\Omega)$ of the problem (P) and $\kappa_0 \geq 1$ such that $u_0^{\kappa_0} \in W_0^{s,p}(\Omega)$.

We choose $\Gamma \in (0, 1)$ and $\delta_0 < sp$ such that $\Gamma K_{\delta_0}(x) \leq K_\delta(x)$ and the constant Γ is independent of δ_0 for $\delta_0 \geq \delta_0^*$ with $\delta_0^* > 0$.

For $\epsilon > 0$, let $u_\epsilon \in W_0^{s,p}(\Omega) \cap C^{0,\ell}(\bar{\Omega})$ be the unique weak solution of

$$\iint_{\mathbb{R}^{2N}} \frac{[u_\epsilon(x) - u_\epsilon(y)]^{p-1}(\phi(x) - \phi(y))}{|x - y|^{N+sp}} dx dy = \int_{\Omega} \frac{\Gamma K_{\epsilon,\delta_0}(x)}{(u_\epsilon + \epsilon)^\gamma} \phi dx \quad (6.3.65)$$

for any $\phi \in W_0^{s,p}(\Omega)$.

By the continuity of u_ϵ , for given $\theta > 0$, there exists a $\eta = \eta(\epsilon, \theta) > 0$ such that $u_\epsilon \leq \frac{\theta}{2}$ in Ω_η . Since $u_0 \geq 0$, then $w := u_\epsilon - u_0 - \theta \leq -\frac{\theta}{2} < 0$ in Ω_η and

$$\text{supp}(w^+) \subset \text{supp}((u_\epsilon - \theta)^+) \subset \Omega \setminus \Omega_\eta.$$

We have $w^+ \in W_0^{s,p}(\tilde{\Omega}) \subset W_0^{s,p}(\Omega)$ for some $\tilde{\Omega}$ such that $\Omega \setminus \Omega_\eta \subset \tilde{\Omega} \Subset \Omega$. Hence, choosing w^+ as a test function in (6.3.65), we get

$$\iint_{\mathbb{R}^{2N}} \frac{[u_\epsilon(x) - u_\epsilon(y)]^{p-1}(w^+(x) - w^+(y))}{|x - y|^{N+sp}} dx dy = \int_{\Omega} \frac{\Gamma K_{\epsilon,\delta_0}(x)}{(u_\epsilon + \epsilon)^\gamma} w^+ dx \leq \int_{\Omega} \frac{\Gamma K_{\epsilon,\delta_0}(x)}{u_\epsilon^\gamma} w^+ dx. \quad (6.3.66)$$

Moreover, u_0 is a weak solution of (P) and taking $w^+ \in W_0^{s,p}(\tilde{\Omega})$ as test function in Definition 6.3.1 with u_0 , we have

$$\iint_{\mathbb{R}^{2N}} \frac{[u_0(x) - u_0(y)]^{p-1}(w^+(x) - w^+(y))}{|x - y|^{N+sp}} dx dy = \int_{\Omega} \frac{K_\delta(x)}{u_0^\gamma} w^+ dx \geq \int_{\Omega} \frac{\Gamma K_{\epsilon,\delta_0}(x)}{u_0^\gamma} w^+ dx. \quad (6.3.67)$$

By subtracting (6.3.67) and (6.3.66), we get

$$\begin{aligned} & \iint_{\mathbb{R}^{2N}} \frac{([u_\epsilon(x) - u_\epsilon(y)]^{p-1} - [u_0(x) - u_0(y)]^{p-1})(w^+(x) - w^+(y))}{|x - y|^{N+sp}} dx dy \\ & \leq \int_{\Omega} \left(\frac{\Gamma K_{\epsilon,\delta_0}(x)}{u_\epsilon^\gamma} - \frac{\Gamma K_{\epsilon,\delta_0}(x)}{u_0^\gamma} \right) w^+ dx \leq 0. \end{aligned} \quad (6.3.68)$$

Applying the following identity

$$[b]^{p-1} - [a]^{p-1} = (p-1)(b-a) \int_0^1 |a + t(b-a)|^{p-2} dt$$

with $a = u_0(x) - u_0(y)$ and $b = u_\epsilon(x) - u_\epsilon(y)$, we get

$$[u_\epsilon(x) - u_\epsilon(y)]^{p-1} - [u_0(x) - u_0(y)]^{p-1} = (p-1)Q(x, y)(w(x) - w(y)) \quad (6.3.69)$$

where

$$Q(x, y) = \int_0^1 |u_0(x) - u_0(y) + t(w(x) - w(y))|^{p-2} dt \geq 0.$$

Now by multiplying (6.3.69) with $(w^+(x) - w^+(y))$, we obtain

$$\begin{aligned} & ([u_\epsilon(x) - u_\epsilon(y)]^{p-1} - [u_0(x) - u_0(y)]^{p-1})(w^+(x) - w^+(y)) \\ &= (p-1)Q(x, y)(w(x) - w(y))(w^+(x) - w^+(y)) \geq 0 \end{aligned}$$

since the mapping $x \rightarrow x^+$ is nondecreasing.

From (6.3.68), we get $w^+ = (u_\epsilon - u_0 - \theta)^+ = 0$ a.e. in Ω . Since θ is arbitrary, we deduce $u_\epsilon \leq u_0$ in Ω . Using the estimates in **Case 2** of the proof of Theorem 6.3.4, we have

$$\eta c_\eta(d(x) + \epsilon^{\frac{\gamma+p-1}{sp-\delta_0}} \frac{sp-\delta_0}{\gamma+p-1} - \epsilon) \leq u_\epsilon \leq u_0 \text{ in } \Omega.$$

Now, by using Hardy inequality and $u_0^{\kappa_0} \in W_0^{s,p}(\Omega)$, we obtain

$$(\eta c_\eta)^{\kappa_0 p} \int_\Omega \left| \frac{\left((d(x) + \epsilon^{\frac{\gamma+p-1}{sp-\delta_0}} \frac{sp-\delta_0}{\gamma+p-1} - \epsilon) \right)^{\kappa_0}}{d^s(x)} \right|^p dx \leq \int_\Omega \left| \frac{u_0^{\kappa_0}}{d^s(x)} \right|^p dx < \infty.$$

Now, by choosing δ_0 close enough to sp and by taking $\epsilon \rightarrow 0$, we obtain that the left hand side is not finite, which is a contradiction and hence claim.

6.3.6 Appendix: Hölder regularity

In this section, we recall the local regularity results for the p-fractional Laplacian operator. We set for $R > 0$ and $y \in \mathbb{R}^N$

$$Q(u; y, R) = \|u\|_{L^\infty(B_R(y))} + \left(R^{sp} \int_{(B_R(y))^c} \frac{|u(x)|^{p-1}}{|x-y|^{N+sp}} dx \right)^{\frac{1}{p-1}}.$$

Corollary 6.3.4. (Corollary 5.5, [172]) *If $u \in \overline{W}^{s,p}(B_{2R_0}(y)) \cap L^\infty(B_{2R_0}(y))$ satisfies $(-\Delta)_p^s u \leq K$ weakly in $B_{2R_0}(y)$ for some $R_0 > 0$, then there exists universal constants $\omega \in (0, 1)$ and $C > 0$ with the following property:*

$$[u]_{C^\omega(B_{R_0}(x_0))} := \sup_{x, y \in B_{R_0}(x_0)} \frac{|u(x) - u(y)|}{|x - y|^\omega} \leq C[(KR_0^{sp})^{\frac{1}{p-1}} + Q(u; x_0, 2R_0)]R_0^{-\omega}.$$

Proposition 6.3.4. (Theorem 1.4, [65]) *Let $p \in [2, \infty)$ and $u \in W_{loc}^{s,p}(\Omega) \cap L_{loc}^\infty(\Omega) \cap L^{p-1}(\mathbb{R}^N)$ be a local weak solution of $(-\Delta)_p^s u = f$ in Ω with $f \in L_{loc}^\infty(\Omega)$. Then $u \in C_{loc}^\omega(\Omega)$ for every $0 < \omega < \min\{\frac{sp}{p-1}, 1\}$. More precisely, for every $0 < \omega < \min\{\frac{sp}{p-1}, 1\}$ and every ball $B_{4R}(x_0) \Subset \Omega$, there exists a constant $C = C(N, s, p, \omega)$ such that*

$$[u]_{C^\omega(B_{\frac{R}{8}}(x_0))} \leq C[(\|f\|_{L^\infty(B_R(x_0))})R^{sp})^{\frac{1}{p-1}} + Q(u; x_0, R)]R^{-\omega}.$$

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