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Zeina Ghazo Hanna. Combinatorial and geometric cycles. Combinatorics [math.CO]. Université de Bretagne occidentale - Brest; Université Libanaise, 2020. English. NNT : 2020BRES0006 . tel-03162197

HAL Id: tel-03162197

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- EIFFEL EXCELLENCE -
PROGRAMME

THESE DE DOCTORAT EN CO-TUTELLE

DE L'UNIVERSITE DE BRETAGNE OCCIDENTALE

ECOLE DOCTORALE N° 601

*Mathématiques et Sciences et Technologies
de l'Information et de la Communication*

Spécialité : *Mathématiques et leurs interactions*

ET L'UNIVERSITE LIBANAISE

ECOLE DOCTORALE

de Sciences et de Technologie

Par

Zeina GHAZO HANNA

Cycles combinatoires et géométriques.

Thèse présentée et soutenue à Brest, le 31 Janvier 2020

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Acknowledgments

The completion of my PhD thesis couldn't have been possible without the participation and assistance of many people whose names may not be all stated. Their contributions are sincerely appreciated and gratefully acknowledged.

However, I would particularly like to express my deep appreciation to my thesis directors Prof. Amine El Sahili, and Prof. Paul Baird, and co-director Dr. Ali Fardoun, for the continuous support during my PhD study and research work, for their patience, motivation, enthusiasm, and immense knowledge.

I also give my best regards to Prof. Maurice Pouzet and Prof. Laurent Najman for reviewing my thesis in details despite of their busy schedule, and to the examiners Prof. Gilles Burel, Prof. Hassane Abbas and Dr. Maydoun Mortada.

A special thank you also goes to Dr. Mohammad Wehbe with whom I enjoyed a research side-work.

I am also grateful to my parents and my sister, for their caring, encouragements, and deep understanding, handling my mood swings caused by the pressure of the work.

I would also like to thank my friends and colleagues in Lebanon and Brest, who in one way or another shared their support either morally or materially; A special endearment goes to Maria Abi Aad for the moments we shared, and her wise life advices, to Ali Janbain who offered his help writing the algorithm of Chapter 5, to Nasab Yassine who was like my un-biological sister during my stay in Brest, and to Ziad El Khoury Hanna who helped me writing the algorithms in chapters 2 and 4, and had always been there when I needed a hand.

I would also express my deep gratitude toward the Lebanese National High Conservatory of Music, to whom I am affiliated as a piano teacher, allowing me to travel abroad several months during these three years of PhD.

Finally, I want to thank the Lebanese University for the grant it offered, Campus France for the EIFFEL excellence scholarship, KALMA and LMBA laboratories where I spent most of my time doing research, and the two doctoral schools of the Lebanese University and the University of Brest (UBO).

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Introduction

This thesis has two principal components, both concerned with the study of paths and cycles in different contexts. The first part focuses on Hamiltonian paths and cycles of a given type in a tournament. Here, type refers to the orientation of successive edges. Our main objective will be to compute the number of paths and cycles of a given type in a tournament and more generally to compute the number of copies of a given digraph in a tournament. To achieve this end, we will exploit combinatorial techniques, involving decomposition of paths and cycles into blocks. The second part of this work derives from a quadratic difference equation on a graph with a parameter which one may interpret as curvature. On a cyclic graph (cycle), this yields numerical sequences with rich algebraic interconnections. Complex sequences may be interpreted as walks in the plane which, in particular cases, exploit the sides of a regular n -sided polygon. We will see how unexpected phenomena appears when $n \geq 12$.

A path $P = v_1v_2\dots v_s$ is a graph where $V(P) = \{v_1, v_2, \dots, v_s\}$ and $E(P) = \{v_iv_{i+1}, 1 \leq i \leq s-1\}$. Similarly, a cycle $C = v_1v_2\dots v_s$ is a graph where $V(C) = \{v_1, v_2, \dots, v_s\}$ and $E(C) = \{v_iv_{i+1}, 1 \leq i \leq s-1\} \cup \{v_1v_s\}$. An oriented path P (resp. cycle C) is a digraph whose underlying graph is a path (resp. cycle). A directed path P (resp. cycle C) is an oriented path (resp. cycle) whose arcs all have the same direction. An antidirected path P (resp. cycle C) is an oriented path (resp. cycle) whose arcs have successively opposite directions. A tournament is a complete oriented graph. Given a tournament T , an oriented Hamiltonian path P (resp. cycle C) in T is an oriented path (resp. cycle) in T passing through all the vertices of T .

The existence and the enumeration of Hamiltonian paths and cycles in tournaments has for many decades intrigued combinatorists.

The first and oldest result in this area concerns the existence of directed paths. Proved by Redei [12] in 1934, it affirms that every tournament contains a directed Hamiltonian path. Later on, the question concerning the existence of any oriented Hamiltonian path in tournaments arose: First of all, Grunbaüm [7] proved in 1969 that every tournament

(with the exception of 3 cases, which are the directed 3-cycle, the regular tournament on 5 vertices, and the Paley tournament on 7 vertices) contains an antidirected Hamiltonian path. Then after several attempts and partial results, the problem was completely settled by Havet and Thomassé [8] in 1993, who proved that every tournament contains all types of oriented Hamiltonian paths, with the exception of the same 3 cases discovered by Grunbaüm. Regarding oriented cycles, we first have Camion’s famous result about the existence of directed cycles: A tournament T is strong if and only if it contains a directed Hamiltonian cycle, where a strong tournament is one where we can find a directed path between any two of its vertices. Then, Havet [9] proved in 1999 that every tournament contains all types of oriented Hamiltonian cycles, except possibly the Hamiltonian circuit (directed cycle) that can’t be found in a non strong tournament.

By considering a certain type of oriented Hamiltonian path (resp. cycle), that is, by fixing a certain orientation of their arcs, one wants to know how many copies of such paths (resp. cycles) can be found in a tournament. Up until the present day, no exact values of these numbers have been found. However, it is possible to find bounds for directed Hamiltonian paths (resp. cycles). The oldest result in this direction was given more than seventy years ago by Szele [14], who gave lower and upper bounds for the maximum number $P(n)$ of directed Hamiltonian paths in a tournament on n vertices:

$$\frac{n!}{2^{n-1}} \leq P(n) \leq c_1 \frac{n!}{2^{\frac{3}{4}n}},$$

where c_1 is a positive constant independent of n . It is easy to verify that the minimum number of directed Hamiltonian paths in a tournament is 1, and this value corresponds to the transitive tournament, denoted by TTn , which is a tournament of order n for which we can label the vertices in such a way that all its arcs become forward. Using Camion’s result, we notice that a transitive tournament is not strong, since it doesn’t contain a directed Hamiltonian cycle. However, in the case of strong tournaments, the minimum number of directed Hamiltonian paths increases exponentially with n . For instance, consider the nearly-transitive tournament, which is obtained from a transitive tournament TTn by reversing the orientation of the arc joining the extremities of the unique directed Hamiltonian path of TTn . This tournament is strong and the number of its directed Hamiltonian paths is $2^{n-2} + 1$. In 1972, Moon [10] gave upper and lower bounds for the minimum number $p(n)$ of directed Hamiltonian paths in a strong tournament of order n :

$$\alpha^{n-1} \leq p(n) \leq \begin{cases} 3 \cdot \beta^{n-3} & \text{for } n \equiv 0 \pmod{3} \\ \beta^{n-1} & \text{for } n \equiv 1 \pmod{3} \\ 9 \cdot \beta^{n-5} & \text{for } n \equiv 2 \pmod{3} \end{cases}$$

where $\alpha = 6^{\frac{1}{4}} \approx 1.565$ and $\beta = 5^{\frac{1}{3}} \approx 1.710$. In 2006, after finding an interesting characterization of strong tournaments, Busch [3] improved this result by proving that the value of the minimum number $p(n)$ for strong tournaments on n vertices is exactly the upper bound given by Moon.

For the minimum number of directed Hamiltonian cycles in tournaments, it's obviously zero, since any tournament that is not strong does not contain a directed Hamiltonian cycle (by Camion's theorem). Considering strong tournaments, Thomassen [15] gave in 1980 an extension of the previously mentioned result of Moon, and using this extension, he was able to prove that every 2-connected tournament of order n contains at least $\alpha^{\frac{n}{32}-1}$ distinct directed Hamiltonian cycles. In a proof analogous to the one given in the above mentioned result of Szele, one can prove that the maximum number $C(n)$ of directed Hamiltonian cycles in a tournament of order n is

$$C(n) \geq \frac{(n-1)!}{2^n}.$$

However, in 1968, in his classic book on tournaments [11], Moon discussed the question of exhibiting tournaments having a big number of Hamiltonian cycles, and he observed that it seems difficult to give explicit tournaments with such a large number of directed Hamiltonian cycles.

In respect of oriented paths that are not directed paths, Rosenfeld [13] proved in 1974 that in any tournament, the number of antidirected Hamiltonian paths starting with a forward arc is equal to the number of antidirected Hamiltonian paths starting with a backward arc, which can be stated as: the number of antidirected Hamiltonian paths in any tournament T is equal to the number of antidirected Hamiltonian paths in the complement of T , denoted by \bar{T} , where the complement of T is the tournament obtained from T by reversing the orientation of all its arcs.

In Chapter 2, we generalize Rosenfeld's result to any type of oriented Hamiltonian path and cycle. That is, we prove that a tournament T and its complement \bar{T} contain the same number of oriented Hamiltonian paths (resp. cycles) of any given type.

To be more specific, let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{Z}^s$; $\alpha_i \cdot \alpha_{i+1} < 0$, $\forall i = 1, \dots, s-1$. An oriented path P is of type $P(\alpha_1, \alpha_2, \dots, \alpha_s)$ if P is formed by s consecutive blocks (maximal directed subpaths) I_1, I_2, \dots, I_s such that $length(I_i) = |I_i| = |\alpha_i|$, with $\alpha_i > 0$ if the arcs of the block of length $|\alpha_i|$ are forward, and $\alpha_i < 0$ if the arcs of the block are backward. This is similar for cycles: let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s$; $\alpha_i \cdot \alpha_{i+1} < 0$, $\forall i = 1, \dots, s-1$, and $\alpha_s \cdot \alpha_1 < 0$. An oriented cycle C is of type $C(\alpha_1, \dots, \alpha_s)$ if C is formed by s consecutive blocks I_1, I_2, \dots, I_s , with $|I_i| = |\alpha_i|$, and $\alpha_i > 0$ if the arcs of the block are forward, and $\alpha_i < 0$ otherwise.

In a tournament T , $\mathcal{P}_T(\alpha_1, \dots, \alpha_s)$ (resp. $\mathcal{C}_T(\alpha_1, \dots, \alpha_s)$) is defined to be the set of ori-

ented paths (resp. cycles) in T of type $P(\alpha_1, \dots, \alpha_s)$ (resp. $C(\alpha_1, \dots, \alpha_s)$) and $f_T(\alpha_1, \dots, \alpha_s)$ (resp. $g_T(\alpha_1, \dots, \alpha_s)$) is the cardinal of this set, that is, the number of oriented paths (resp. cycles) of type $P(\alpha_1, \dots, \alpha_s)$ (resp. $C(\alpha_1, \dots, \alpha_s)$) in T .

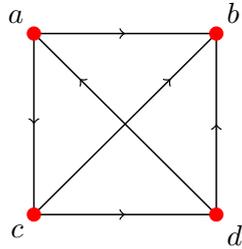
For $\alpha = (\alpha_1, \dots, \alpha_s)$ in \mathbb{Z}^s , we denote by $-\alpha$ the tuple $(-\alpha_1, \dots, -\alpha_s)$ and by $\bar{\alpha}$ the tuple $(\alpha_s, \alpha_{s-1}, \dots, \alpha_1)$. A tuple α is symmetric if $\alpha = -\bar{\alpha}$. An oriented path P (resp. cycle C) is symmetric if there exists an α -symmetric tuple, such that P (resp. C) is of type $P(\alpha)$ (resp. $C(\alpha)$). Otherwise, the path P (resp. cycle C) is non-symmetric.

We first prove the following:

Theorem 2.1.2. *Let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s$; $\alpha_i \cdot \alpha_{i+1} < 0$, $\alpha_1 \geq 0$, and let T be a tournament of order n , where $n = \sum_{i=1}^s |\alpha_i| + 1$. Then:*

$$f_T(\alpha) = f_T(-\alpha).$$

The above theorem means that a tournament and its complement contains the same number of oriented Hamiltonian paths of any given type, generalizing Rosenfeld's aforementioned result. For example, consider the following tournament on 4 vertices:



There are 3 paths of type $P(2, -1)$ which are $acbd$, $cdba$, and $dabc$ and also 3 paths of type $P(-2, 1)$ which are $adcb$, $cadb$, and $dcab$.

We then establish the same result for cycles: First, let T be a tournament on n vertices. An oriented Hamiltonian cycle C in T is said to be generated by an oriented path $P = x_1x_2 \dots x_n$ if C is the union of P and the arc joining x_1 and x_n in T . We write $C = C_P$. Now remark that given an oriented Hamiltonian path $P = x_1x_2 \dots x_n$ of a certain type $P(\alpha)$ in a tournament T , then the type of the cycle C_P in T can take 2 different values, $C(\beta)$ or $C(\beta')$, whether (x_1, x_n) or (x_n, x_1) is an arc in T . On the other hand, given an oriented cycle of some type $C(\beta)$, it can be generated by many paths of the same type $P(\alpha)$, which we can enumerate using the notion of "period". Distinguishing between the different cases of symmetric and non-symmetric paths and cycles, this enables the determination of a relation between $f_T(\alpha)$, $g_T(\beta)$ and $g_T(\beta')$, then, using Theorem 2.1.2, we obtain the following:

Theorem 2.3.1. *Let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s$; $\alpha_i \cdot \alpha_{i+1} < 0$, $\alpha_1 \cdot \alpha_s < 0$, $\alpha_1 \geq 0$, and let T be a tournament of order n , where $n = \sum_{i=1}^s |\alpha_i|$. Then:*

$$g_T(\alpha) = g_T(-\alpha).$$

By noticing that any digraph of maximal degree $\Delta \leq 2$ is a union of oriented cycles and paths, we are able to further generalize Rosenfeld's result, using Theorems 2.1.2 and 2.3.1:

Theorem 2.4.4. *Let T be a tournament and let H be a digraph with $\Delta(G(H)) \leq 2$. Then the number of copies of H in T and its complement \bar{T} is the same.*

In [13], Rosenfeld proved that for any tournament T of order n , where $T \neq TTn$, the number of its antidirected Hamiltonian paths is less than the number of antidirected Hamiltonian paths in TTn .

In Chapter 3, we consider the enumeration of oriented Hamiltonian paths in transitive tournaments. First, we construct a combinatorial function giving us the exact number of oriented Hamiltonian paths of any given type in a transitive tournament:

Let $\mathcal{K} = \{(\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{Z}^s, \alpha_i \cdot \alpha_{i+1} < 0, \forall 1 \leq i \leq s-1, s \in \mathbb{N}^*\}$.

Let \mathcal{F} be the following mapping:

$$\begin{aligned} \mathcal{F} \quad \mathcal{K} &\longrightarrow \mathbb{N} \\ (\alpha_1, \alpha_2, \dots, \alpha_s) &\longrightarrow \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_s), \end{aligned}$$

defined by the recurrence relation:

$$\begin{aligned} \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_s) = & \mathcal{F}(\alpha_1 * 1, \alpha_2, \dots, \alpha_s) + \mathcal{F}(\alpha_1, \alpha_2 * 1, \dots, \alpha_s) \\ & + \dots + \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_s * 1) \end{aligned}$$

where $\alpha_i * 1 = \alpha_i - 1$ if $\alpha_i > 0$ and $\alpha_i * 1 = \alpha_i + 1$ if $\alpha_i < 0$, and satisfying:

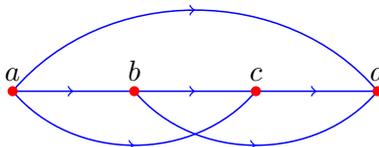
1. $\forall t \in \mathbb{N}^*, \mathcal{F}(0, \alpha_2, \dots, \alpha_t) = \mathcal{F}(\alpha_2, \dots, \alpha_t)$,
2. $\forall t' \in \mathbb{N}^*, \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_{t'}, 0) = \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_{t'})$,
3. $\forall r \in \mathbb{N}^*, \mathcal{F}(\alpha_1, \dots, \alpha_r, 0, \alpha_{r+2}, \dots, \alpha_s) = \mathcal{F}(\alpha_1, \dots, \alpha_r + \alpha_{r+2}, \dots, \alpha_s)$,
4. $\forall \alpha \in \mathbb{Z}^*, \mathcal{F}(\alpha) = 1$.

We call \mathcal{F} the path-function. We prove the following:

Theorem 3.1.1. *Let TTn be a transitive tournament of order n , and $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathcal{K}$, such that $\sum_{i=1}^s |\alpha_i| = n - 1$. Then:*

$$f_{TTn}(\alpha) = \begin{cases} \mathcal{F}(\alpha) & \text{if } \alpha \text{ is non symmetric,} \\ \frac{\mathcal{F}(\alpha)}{2} & \text{if } \alpha \text{ is symmetric.} \end{cases}$$

For example, consider the transitive tournament $TT4$:



In this tournament, there are 3 paths of type $P(2, -1)$ which are $abdc$, $acdb$ and $bcda$, and if we compute $\mathcal{F}(2, -1)$ we also get 3. (Note that the tuple $(2, -1)$ is non symmetric).

Subsequently, we study some properties of \mathcal{F} . Using "Python", we build a program to compute all possible values of $\mathcal{F}(a_1, \dots, a_s)$, $1 \leq s \leq p$, for a given $p = \sum_{i=1}^s a_i$. Analysing the results for $3 \leq p \leq 18$, we are led to state a conjecture about paths in transitive tournaments, and to discuss an interesting property about antidirected Hamiltonian paths in TTn :

Conjecture 3.3.1. *Let TTn be a transitive tournament on n vertices.*

Then $\forall \alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathcal{K}$, and $\sum_{i=1}^s \alpha_i = n - 1$, if n is even we have:

$$f_{TTn}(1, -1, 1, \dots, -1, 1) \geq f_{TTn}(\alpha_1, \alpha_2, \dots, \alpha_s),$$

where $(1, -1, 1, \dots, -1, 1)$ has $n - 1$ components, while if n is odd, we have:

$$f_{TTn}(1, -2, 1, -1, 1, \dots, -1, 1) \geq f_{TTn}(\alpha_1, \alpha_2, \dots, \alpha_s),$$

where the number of components of $(1, -2, 1, \dots, 1)$ is $n - 2$.

If the conjecture is true, it means that in transitive tournaments of even order, the number of antidirected Hamiltonian paths starting with a forward arc is the maximum of the numbers of oriented Hamiltonian paths starting with a forward block, of any type.

The second part of the thesis has a distinctly geometric flavour. In contrast to the first part, we will see that *Eulerian* digraphs provide a model for some of our constructions. Furthermore, algebraic properties of polynomials play an essential role in our proofs.

Consider a finite graph $G = (V, E)$, where V is the set of its vertices, and E the set of its edges, and a real valued function $\gamma : V \rightarrow \mathbb{R}$, and consider the following quadratic equation

$$\forall x \in V, \quad \gamma(\Delta\varphi(x))^2 = (\nabla\varphi)^2(x), \quad (1)$$

where $\varphi : V \rightarrow \mathbb{C}$ is a complex valued function.

The term $\Delta\varphi(x)$ is the *Laplacian* of φ at vertex x :

$$\Delta\varphi(x) = \frac{1}{d(x)} \sum_{y \sim x} (\varphi(y) - \varphi(x)),$$

where $d(x)$ is the degree of x , and

$$(\nabla\varphi)^2(x) = \frac{1}{d(x)} \sum_{y \sim x} (\varphi(y) - \varphi(x))^2$$

is the symmetric square of the derivative.

The *geometric spectrum* of G is defined to be the set of all functions $\gamma : V \rightarrow [-\infty, 1]$, $\gamma(x) < 1$ if $d(x) \geq 3$, such that there exists a non constant function $\varphi : V \rightarrow \mathbb{C}$, satisfying equation (1). Consider the Euclidean space \mathbb{R}^N , ($N \in \mathbb{N}$, $N \geq 2$). A *framework* in \mathbb{R}^N is a graph $G = (V, E)$ realized as a subset of this space, where edges $vw \in E$ are straight lines segments joining the vertices $v, w \in V$. The framework $G = (V, E)$ in \mathbb{R}^N is *invariant* if there exists a function γ lying in its geometric spectrum, such that this framework satisfies equation (1) with φ being the restriction to its vertices of an orthogonal projection in \mathbb{C} , independently of any similarity transformation in \mathbb{R}^n of this framework.

The quadratic equation (1) has been first introduced in [1]. For frameworks arising from the 1-skeleton of a regular simplex in the Euclidean 3-space, Eastwood and Penrose [5] show that equation (1) is satisfied. This property is generalized to other (invariant) frameworks by Baird [1]: by methods of linear algebra, he addresses the problem of when a given graph can be realized as an invariant framework in an Euclidean space of dimension greater than or equal to three; some of these frameworks are called the "configured stars", where a star consists of an internal vertex v_0 adjacent to n external vertices v_1, v_2, \dots, v_n , with no other connections. Moreover, equation (1) arises from its smooth counterpart for a hypersurface in Euclidean space [2]:

Any regular polytope in Euclidean space provides an example of an invariant framework, with γ varying from polytope to polytope. For example, for the cube $\gamma = 0$ and for the tetrahedron $\gamma = 3/4$. When one considers an orthogonal projection of a smooth hypersurface in the Euclidean space, remarkably we find the same phenomena, namely that a smooth

version of the equation is satisfied independently of the projection, where $\gamma = -1/H^2$ with H being the mean curvature.

So-called *body-bar frameworks* have been studied in respect of their rigidity properties. For such frameworks, the bars (edges) can rotate freely at each vertex and one is interested to know if they can be deformed without changing the lengths of the bars, see for example [4]. Frameworks in the context of equation (1), in particular satisfying the invariance property, have only been considered recently, see [1], however, the examples of the regular polytopes can be deduced from [5], and the case of the cube goes back to the *Theorem of Axonometry of Gauss* [6].

In Chapter 4 of this thesis, we are interested in studying solutions to equation (1) on cyclic graphs (cycles), with γ constant.

So let $C = (V, E)$ be a cyclic graph on N vertices, and set $V = \{0, 1, 2, \dots, N-1\}$, then $\forall j \in \mathbb{Z}/N\mathbb{Z}$, the vertex j has $j-1$ and $j+1$ as neighbors. For γ a real constant, equation (1) takes the form:

$$\frac{\gamma}{2}(2\varphi(j) - \varphi(j-1) - \varphi(j+1))^2 = (\varphi(j) - \varphi(j-1))^2 + (\varphi(j) - \varphi(j+1))^2 \quad \forall j \in \mathbb{Z}/N\mathbb{Z} \quad (2)$$

where $j \pm 1$ are calculated modulo N .

A *quadratic cyclic sequence (QCS) of order N* is a function $\varphi : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ solution to equation (2).

If we define the *increment* $u_j = \varphi(j+1) - \varphi(j)$, then the above equation reads

$$\frac{\gamma}{2}(u_j - u_{j-1})^2 - u_j^2 - u_{j-1}^2 = 0. \quad (3)$$

which affirms the vanishing of a linear combination of the elementary symmetric quadratic polynomials $u_j^2 + u_{j-1}^2$ and $u_j u_{j-1}$ in the two variables u_j and u_{j-1} .

Using the Cauchy-Schwartz inequality, it follows from equation (3) that for a given $j \in \{0, 1, \dots, N-1\}$, if u_j, u_{j-1} are both real, we have

$$u_j^2 + u_{j-1}^2 = \frac{\gamma}{2}(u_j - u_{j-1})^2 \leq \gamma(u_j^2 + u_{j-1}^2).$$

In particular, for there to exist a non-constant *real* solution to (2), then necessarily $\gamma \geq 1$. Equally, if $\gamma < 1$, then for any three successive terms of the QCS, at least one must be complex with non zero imaginary part. We will also see that for the real case, γ can be at most equal 2, and that this value corresponds to the case when u_j or u_{j-1} equal zero; otherwise, $\gamma < 2$.

INTRODUCTION

First, we examine real QCS, showing how they arise from polynomials with positive integer coefficients. An example of an integer QCS of order 10 is given by

$$(0, 9, 3, 12, 6, 10, 4, 8, 2, 6)$$

It satisfies (2) with $\gamma = 26/25$. Moreover, it may not be unique, for example the following is another sequence of order 10 with $\gamma = 26/25$:

$$(0, 9, 3, 7, 1, 10, 4, 8, 2, 6) .$$

We will see how a given polynomial can give rise to different sequences coming from *legitimate* orderings (given in the following theorem) of a corresponding set of increments:

Theorem 4.1.3. (Construction of real quadratic cyclic sequences): *Let $q(x) = a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \dots + a_1x + a_0$ ($n \geq 2$) be any polynomial with integer coefficients all strictly positive. Multiply by $x + 1$ to obtain the new polynomial*

$$\begin{aligned} p(x) &:= b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_1x + b_0 \\ &= a_{n-2}x^{n-1} + (a_{n-2} + a_{n-3})x^{n-2} + \dots + (a_1 + a_0)x + a_0 . \end{aligned}$$

Let y be any real root of $p(x)$ (necessarily negative). Then a quadratic cyclic sequence

$$(x_0, x_1, x_2, \dots, x_{N-1})$$

of order $N = 2 \sum_k a_k$ is constructed by arbitrarily prescribing x_0 and then requiring increments $u_j = x_{j+1} - x_j$ of successive terms to be taken from the set $\{1, y, y^2, \dots, y^{n-1}\}$ in such a way that each increment y^k occurs precisely b_k times and any two adjacent increments have powers that differ by ± 1 , including the powers corresponding to the elements u_0 and u_{N-1} . This is always possible and up to these constraints, the ordering is arbitrary. The constant γ in (2) is given by $\gamma = 2(1 + y^2)/(1 - y)^2 < 2$.

Conversely, up to addition of a constant, cyclic permutations and order reversal, a multiple of any real cyclic sequence with $\gamma \neq 2$ arises in this way from such a polynomial $p(x)$, well-defined up to replacement of $p(x)$ by $\tilde{p}(x) := x^{\deg p} p(1/x)$.

The cyclic sequences with $\gamma = 2$ are characterized as those made up of connected segments of order ≥ 2 on which the sequence is constant. The cyclic sequences with $\gamma = 1$ are, up to normalization, equivalent to $(0, 1, 0, 1, \dots, 0, 1)$; they arise by taking the root $y = -1$ of $p(x)$.

The two real sequences given in the examples above arise from different legitimate orderings. Furthermore, we will see how one can actually capture the legitimate orderings by *Eulerian walks* in a corresponding digraph, (see Chapter 4).

It is possible to combine real QCSs with common γ to obtain new ones, and we show how such concatenation of sequences is reflected in the defining polynomials:

Proposition 4.3.4. *Let $p_1(x)$ and $p_2(x)$ be polynomials of degree $m - 1$ and $n - 1$ (resp.) having the form given in Theorem 4.1.3 which define real QCS with $\gamma \neq 2$, of orders M and N (resp.) deriving from a common fundamental increment y . Then there exists a real QCS of order $M + N$ with defining polynomial $p(x) = p_1(x) + x^k p_2(x)$, where $0 \leq k \leq m$, obtained by concatenation of two real QCS with defining polynomials $p_1(x)$ and $p_2(x)$ resp..*

Suppose that the function γ is not necessarily constant. Consider a cyclic framework in the Euclidean space \mathbb{R}^2 . We will show that it is invariant in \mathbb{R}^2 if and only if it is realized in the plane as a polygon with sides of equal length. With the requirement of γ to be constant, the absolute angle $\alpha \in [0, \pi]$ between each two consecutive edges is constant, and we have

$$\gamma = \frac{2 \cos \alpha}{1 + \cos \alpha}.$$

So now, one can picture a complex QCS (which is a solution $\varphi : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ to equation (2) with $\gamma \in [-\infty, 1]$) as a closed polygon in the plane with edges the straight line segments $[\varphi(j), \varphi(j + 1)]$, each having the same length, and with constant absolute angle between the edges. We will be interested in such polygons and how they will be defined by polynomial equations.

Complex QCS which arise from polynomials will be called *algebraic*. In this case, a *legitimate* polynomial $p(x)$ (one defining a complex QCS) determines a closed polygonal walk in the plane with exterior angle either $+\theta$ or $-\theta$ for some fixed angle θ (the turning angle), where $\theta = \pi - \alpha$, α being the constant absolute angle between every two consecutive edges of the polygon.

As we will see, complex algebraic QCS exist with turning angle not a rational multiple of 2π , (Chapter 4, Fig.4.2). In the case when $\theta = 2\pi m/n$, $m < n$, m and n relatively prime, necessarily the n 'th cyclotomic polynomial $\Phi_n(x)$ must divide $p(x)$:

Theorem 4.2.1. *Let φ be a complex QCS with increment $y = e^{2m\pi i/n}$ (m, n relatively prime, $m < n$).*

(i) *When $n = 2k$ ($k \geq 2$) is even, φ is determined by a polynomial of the form*

$$p(x) = (x + 1)\Phi_n(x)q(x)$$

where $q(x)$ is a polynomial of degree $\leq n - 2 - \deg \Phi_n(x)$. When $n = 2$, $p(x) = a(x + 1)$ for some positive integer a .

INTRODUCTION

(ii) When n is odd, φ is determined by a polynomial of the form

$$p(x) = \Phi_n(x)q(x)$$

where $q(x)$ is a polynomial of degree $\leq n - 1 - \deg \Phi_n(x)$. In the case when n is prime, then

$$p(x) = a(x^{n-1} + x^{n-2} + \cdots + x + 1)$$

for some positive integer a .

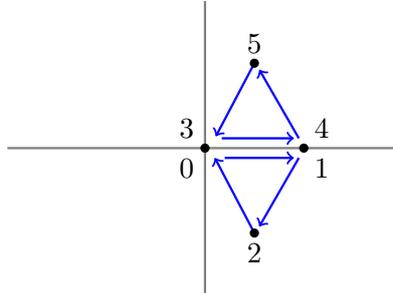
Conversely, polynomials of the above type (with $q(x)$ to be made precise on a case by case basis) yield a corresponding QCS.

Then, we discuss the conditions on the coefficients of $q(x)$ of the above theorem.

An example of a complex QCS with $\gamma = 2/3$ and $n = 6$ is given by

$$\left(0, 1, \frac{1}{2} - \frac{\sqrt{3}}{2}i, 0, 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$$

with corresponding walk illustrated below, where we label the vertices in sequential order. In

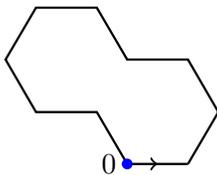


this example, we see that the complex QCS has indeed a corresponding polygonal walk, with 6 edges of equal length 1, and whose constant absolute angle is $\alpha = \frac{\pi}{3}$. The polygonal walk is defined by $(1, y^2, y, 1, y, y^2)$ with $y = e^{\frac{2i\pi}{3}}$, and the turning angle θ is $\theta = \pi - \alpha = \frac{2\pi}{3}$. The corresponding legitimate polynomial is $p(x) = 2(1 + x + x^2)$; y is its root.

The problem of which polynomials give rise to complex QCS turns out to be challenging when n becomes large and bears on the following geometric question:

One wishes to construct a polygonal path starting at the origin, with directed edges taken from the edges of a closed regular polygon with exterior angle $2\pi/n$. One may use edges as often as one likes, but at each step, the turning angle must be either $+2\pi/n$ or $-2\pi/n$ (one turns left or right through an angle $2\pi/n$). In the case when the polygonal path is closed and n is even, must each edge occur with its parallel counterpart oriented in the opposite direction?

For example, the following illustration is of a polygonal path with directed edges taken from the edges of a closed regular polygon with exterior angle $2\pi/6$:



In this path, the edge 1 is used twice, so is its parallel counterpart oriented in the opposite direction, -1 . Also, the edge $e^{i\pi/3}$ is used once, so is the edge $e^{i4\pi/3}$, and the edge $e^{i2\pi/3}$ twice, just like $e^{i5\pi/3}$.

Another issue is whether, to complete a circuit, all edges of the polygon are required (now for n even or odd). We prove the following:

Proposition 4.2.10. *Any complex algebraic QCS with turning angle $2\pi m/n$ (m, n relatively prime with $m < n$) either with $n \leq 11$, or $n = 2^r$ ($r > 0$), or $n = 2p$ (p an odd prime), must use all increments $\{y^0, y^1, \dots, y^{n-1}\}$ ($y = e^{2\pi mi/n}$). Furthermore if n is even with the same hypotheses, then for each occurrence of the increment y^k , there is also an occurrence of the increment $y^{\frac{n}{2}+k} = -y^k$. In particular, the corresponding polygonal walk in the plane contains each edge with its oppositely orientated counterpart.*

We will see that the above properties fail to hold when $n \geq 12$, for $n \neq 2^r$ ($r > 0$) and $n \neq 2p$ (p an odd prime), (see Chapter 4, Fig.4.1): We build a computer program using "Rust" to find paths of smallest lengths which violate these properties.

Finally, in the last section of Chapter 4, we study the unicity of the edges used to construct a polygonal walk that is not necessarily closed, with turning angle $2\pi/n$, of given length and end point:

We study what we refer to as 2-step walks in the plane with constant turning angle; We consider walks as discussed above with constant turning angle θ , and look at the corresponding walk that occurs by taking two steps at a time. In the case when $\theta = \pi/2$, this yields the standard planar walk, whereby one moves on the integer lattice in the plane and at each step one chooses one of the four neighbouring points. Moreover, unicity of the edges used to construct the polygonal walk occurs (Theorem 4.5.4).

In the case when $\theta = \pi/3$, then the corresponding 2-step walk takes place on the triangular lattice, but is no longer a Markovian process, that is a subsequent steps depends on the previous step and we find restrictions on the points that can be attained. Furthermore, unicity of the edges used to attain a certain point for a given path length, no longer holds (Example 4.5.8).

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Chapter 1

Basic definitions, historical overview and preliminary results

This thesis has two principal components, both concerned with the study of paths and cycles in different contexts. The first part focuses on Hamiltonian paths and cycles of a given type in a tournament. Here, type refers to the orientation of successive edges. Our main objective will be to compute their number in a tournament, for any given type. The second part of this work derives from a geometric equation on a graph. On a cyclic graph (cycle), solutions to this equation are numerical sequences, we call quadratic cyclic sequences, with rich algebraic interconnections that we will study.

In this introductory chapter, we recall in §1.1 elementary definitions related to graphs and digraphs. In §1.2, we give a historical overview about Hamiltonian paths and cycles in tournaments, giving a background to the work that will be done in Chapters 2 and 3, and introduce in §1.3 basic definitions concerning the types of oriented paths and cycles, needed for our work. In §1.4, we talk about the quadratic difference equation defined on a graph, frameworks in \mathbb{R}^N , and the geometric spectrum of a graph, with some related results, and introduce in §1.5 the materials needed to study the quadratic cyclic sequences in Chapter 4.

1.1 Generalities in Graph Theory

1.1.1 Graphs

Definition 1.1.1. (Graphs)

A graph G is a pair $(V(G), E(G))$ where $V(G)$ is a finite, non empty set of elements, called vertices of G , and $E(G)$ is a set of pairs of vertices, called edges of G , i.e. $E(G) \subseteq \{\{x, y\}, x, y \in V(G)\}$.

1.1. GENERALITIES IN GRAPH THEORY

Dots and lines are used to represent vertices and edges respectively, and instead of writing a set $\{x, y\}$ for an edge, we can simply write xy .

If $xy \in E(G)$ then x and y are said to be adjacent, and we write $y \sim x$. The edge xy is said to be incident to x (and to y).

The order of G is $v(G) = |V(G)|$, and $e(G) = |E(G)|$ is called the size of G .

Definition 1.1.2. (Complete / empty graphs)

A graph G is said to be complete if $xy \in E(G), \forall x, y \in V(G)$.

A complete graph of order n is denoted by K_n .

A graph G is said to be empty if $xy \notin E(G), \forall x, y \in V(G)$.

Definition 1.1.3. (Subgraphs)

Let G be a graph.

A graph H is said to be a subgraph of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

A graph H is said to be an induced subgraph of G if $V(H) \subseteq V(G)$ and $\forall x, y \in V(H), xy \in E(H) \Leftrightarrow xy \in E(G)$.

A graph H is said to be a spanning subgraph of G if $V(H) = V(G)$ and $E(H) \subseteq E(G)$.

If $S \subseteq V(G)$, then the subgraph of G induced by S is denoted by $\langle S \rangle$.

If $S \subseteq V(G)$, then $G - S$ is the subgraph of G obtained from G by removing S and all the edges incident to the vertices in S .

Definition 1.1.4. (Paths and cycles)

A path $P = v_1v_2\dots v_s$ of a graph G is a subgraph of G such that $V(P) = \{v_1, v_2, \dots, v_s\} \subseteq V(G)$ and $E(P) = \{v_iv_{i+1}, 1 \leq i \leq s-1\}$.

A cycle $C = v_1v_2\dots v_s$ of a graph G is a subgraph of G such that $V(C) = \{v_1, v_2, \dots, v_s\} \subseteq V(G)$ and $E(C) = \{v_iv_{i+1}, 1 \leq i \leq s-1\} \cup \{v_1v_s\}$.

A Hamiltonian path P of a graph G is a spanning path of G , and a Hamiltonian cycle C of a graph G is a spanning cycle of G .

The length of a path P , denoted by $l(P)$, is equal to $l(P) = e(P) = v(P) - 1$, and the length of a cycle C , denoted by $l(C)$, is equal to $l(C) = e(C) = v(C)$.

If $P = v_1v_2\dots v_s$ is a path, then v_1 and v_s are called the extremities of P and P is called a v_1v_s -path. For $u, v \in V(P)$, $P_{[u,v]}$ denotes the subpath of P with extremities u and v .

Definition 1.1.5. (Connected graphs)

A graph G is said to be connected if $\forall x, y \in V(G)$, there exists an xy -path. Otherwise G is said to be disconnected, and a connected component of G is a maximal connected subgraph of G .

Definition 1.1.6. (Degrees)

Given a graph G and $v \in V(G)$, the neighborhood of v in G , denoted by $N(v)$ is the set of all $x \in V(G)$ such that $xv \in E(G)$.

The degree of v in G denoted by $d(v)$ is equal to $|N(v)|$, and $\max_{v \in V(G)} d(v)$ and $\min_{v \in V(G)} d(v)$ are denoted by $\Delta(G)$ and $\delta(G)$ respectively.

We recall Euler's famous theorem:

Theorem 1.1.7. (Euler) *Let G be a graph, then*

$$\sum_{v \in V(G)} d(v) = 2e(G).$$

Definition 1.1.8. (Regular graphs)

A graph G is said to be k -regular if $d(v) = k, \forall v \in V(G)$.

A complete graph G of order n is $(n - 1)$ -regular.

Definition 1.1.9. (Isomorphic graphs)

Two graphs G and G' are said to be isomorphic if there exists a bijection $f : V(G) \rightarrow V(G')$ such that $xy \in E(G) \Leftrightarrow f(x)f(y) \in E(G')$.

Such a bijection is called an isomorphism, and we write $G \sim G'$.

1.1.2 Digraphs and tournaments

Definition 1.1.10. (Digraphs)

A digraph D is a couple $(V(D), E(D))$ where $V(D)$ is a non empty set of elements, called vertices of D , and $E(D)$ is a set of couples of vertices in D , i.e. $E(D) \subseteq V \times V$.

The elements of $E(D)$ are called arcs. Let $e = (a, b) \in E(D)$ then e is an oriented arc from a to b where a is called the tail of e and b is the head of e and we say that a dominates b or b is dominated by a .

Definition 1.1.11. (Underlying graphs)

Let D be a digraph. The underlying graph of a digraph, denoted by $G(D)$ is the graph obtained from D by removing the orientations of its arcs.

Remarks:

1. All the definitions given about graphs in the previous section are similarly defined for digraphs through their underlying graph.
2. Unless stated, all the digraphs we will work with are simple digraphs, that is no multiple arcs (two or more arcs with same head and tail), no directed cycles of order 2 (see 1.1.20), and no loops (arc whose head and tail are the same) are allowed.

Definition 1.1.12. (Tournaments)

A digraph D is said to be a tournament if and only if $G(D)$ is complete. Clearly, an induced subdigraph of a tournament is a tournament.

Definition 1.1.13. (Complement of a tournament)

The complement of a tournament T is a tournament obtained from T by reversing all the orientations of its arcs. It is denoted by \bar{T} .

Definition 1.1.14. (In-degrees and out-degrees)

Let D be a digraph and let $v \in V(D)$. We define the following:

The set $N^+(v) = \{u \in V(D) / (v, u) \in E(D)\}$ is the out-neighborhood of v in D .

The set $N^-(v) = \{u \in V(D) / (u, v) \in E(D)\}$ is the in-neighborhood of v in D .

The out-degree of v in D is $d^+(v)$ where $d^+(v) = |N^+(v)|$.

The in-degree of v in D is $d^-(v)$ where $d^-(v) = |N^-(v)|$.

$$\Delta^+(D) = \max_{v \in V(D)} d^+(v),$$

$$\delta^+(D) = \min_{v \in V(D)} d^+(v),$$

$$\Delta^-(D) = \max_{v \in V(D)} d^-(v),$$

$$\delta^-(D) = \min_{v \in V(D)} d^-(v).$$

Note that $\forall v \in D, d_D^+(v) + d_D^-(v) = d_{G(D)}(v)$, where $G(D)$ is the underlying graph of D . We have the following property:

Proposition 1.1.15. *Let D be a digraph, then*

$$\sum_{v \in V(D)} d^+(v) = e(D) = \sum_{v \in V(D)} d^-(v).$$

Definition 1.1.16. (Balanced digraph)

A digraph D is said to be a balanced digraph if $\forall x \in V(D)$, $d^+(x) = d^-(x)$.

Definition 1.1.17. (Sources and sinks)

Let D be a digraph. A vertex $x \in V(D)$ is said to be a source if $d^-(x) = 0$, and x is said to be a sink if $d^+(x) = 0$.

Definition 1.1.18. (Regular tournaments)

A tournament T of odd order n is said to be regular if $d^+(v) = (n - 1)/2$, $\forall v \in T$. This obviously implies that $d^-(v) = (n - 1)/2$, $\forall v \in T$, since $d^+(v) + d^-(v) = d(v) = n - 1$, as T is a tournament.

Definition 1.1.19. (Oriented paths)

Let D be a digraph.

An oriented path $P = v_1v_2\dots v_s$ of D is a subdigraph of D in which $V(P) = \{v_1, v_2, \dots, v_s\} \subseteq V(D)$ and $G(P)$ is a path in $G(D)$.

A directed path $P = v_1v_2\dots v_s$ of D is an oriented path of D in which $V(P) = \{v_1, v_2, \dots, v_s\} \subseteq V(D)$ and $E(P) = \{(v_i, v_{i+1}), 1 \leq i \leq s - 1\}$; that is, all the arcs of P have the same direction.

If $P = v_1v_2\dots v_s$ is a directed path of D , it is called a v_1v_s -directed path, v_1 and v_s are called the origin and destination of P respectively.

An antidirected path $P = v_1v_2\dots v_s$ of D is an oriented path of D in which $V(P) = \{v_1, v_2, \dots, v_s\} \subseteq V(D)$ and $\forall 1 \leq i \leq s - 2$, the directions of the arcs $\langle \{v_i, v_{i+1}\} \rangle$ and $\langle \{v_{i+1}, v_{i+2}\} \rangle$ are opposite.

An oriented Hamiltonian path P of a digraph D is an oriented path of D such that $V(P) = V(D)$.

Definition 1.1.20. (Oriented cycles)

Let D be a digraph.

An oriented cycle $C = v_1v_2\dots v_s$ of D is a subdigraph of D in which $V(C) = \{v_1, v_2, \dots, v_s\} \subseteq V(D)$ and $G(C)$ is a cycle in $G(D)$.

A directed cycle $C = v_1v_2\dots v_s$ of D is an oriented cycle of D in which $V(C) = \{v_1, v_2, \dots, v_s\} \subseteq V(D)$ and $E(C) = \{(v_i, v_{i+1}), 1 \leq i \leq s - 1\} \cup \{(v_s, v_1)\}$; that is, all the arcs of C have the same direction.

An antidirected cycle $C = v_1v_2\dots v_s$ of D is an oriented cycle of D in which $V(C) = \{v_1, v_2, \dots, v_s\} \subseteq V(D)$ and $\forall 1 \leq i \leq s - 2$, the directions of the arcs $\langle \{v_i, v_{i+1}\} \rangle$ and $\langle \{v_{i+1}, v_{i+2}\} \rangle$ are opposite, and so are the directions of $\langle \{v_{s-1}, v_s\} \rangle$ and $\langle \{v_s, v_1\} \rangle$.

An oriented Hamiltonian cycle C of a digraph D is an oriented cycle of D such that $V(C) = V(D)$.

Definition 1.1.21. (Strongly connected digraphs)

Let D be a digraph and let $x, y \in V(D)$.

The vertex y is reachable from x in D if there exists an xy -directed path in D .

A digraph D is called strongly connected (or simply strong) if $\forall x, y \in V(D)$ with $x \neq y$, y is reachable from x .

Let D be a strong digraph and let U be a set of vertices of D . If $D - U$ is not strong, or is trivial, then U is called a separator of D .

For $k \geq 1$, D is k -strongly connected, if it has order at least $k + 1$ and no set of $k - 1$ vertices is a separator of D .

The strong connectivity of D , denoted by $K(D)$ is the maximal integer k such that D is k -strongly connected.

Definition 1.1.22. (Isomorphic digraphs)

Two digraphs D and D' are said to be isomorphic if there exists a bijection $f : V(D) \rightarrow V(D')$ such that $(x, y) \in E(D) \Leftrightarrow (f(x), f(y)) \in E(D')$.

We write $D \sim D'$.

Definition 1.1.23. (Transitive and nearly-transitive tournaments)

A tournament T of order $n \geq 3$ is said to be transitive, denoted by TTn , if its vertices can be enumerated in such a way that $V(T) = \{v_1, v_2, \dots, v_n\}$ and $E(T) = \{(v_i, v_j); i < j\}$. The vertex v_1 is then the unique source, and v_n is the unique sink.

A tournament T of order $n \geq 3$ is said to be nearly-transitive, if it's obtained from a transitive tournament TTn by reversing the orientation of the arc (v_1, v_n) , where v_1 and v_n are the source and sink of TTn respectively.

Remark: After giving an enumeration for the vertices of a tournament T of order n , say $V(T) = \{v_1, v_2, \dots, v_n\}$, an arc $e = (v_i, v_j)$ will be referred to as a forward arc if $i < j$, and as a backward arc otherwise.

1.2 Overview: Hamiltonian paths and cycles in tournaments

Since the middle of the last century, Hamiltonian paths and cycles have intrigued combinatorists, including the existence, the parity and the enumeration of such paths and cycles in tournaments.

The first result in this area concerns the existence of directed Hamiltonian paths. It was given by Redei in 1934:

Theorem 1.2.1. ([27]) *Every tournament contains a directed Hamiltonian path.*

Proof. Let T be a tournament of order n and let $P = v_1v_2\dots v_s$ be a longest directed path in T , we need to prove that $s = n$. In fact, if $s < n$ then $\exists v \in V(T)$ such that $v \notin V(P)$. Since T is a tournament, then either $(v_1, v) \in E(T)$ or $(v, v_1) \in E(T)$. If $(v, v_1) \in E(T)$ then $vv_1v_2\dots v_s$ is a path longer than P , contradiction. So $(v_1, v) \in E(T)$. Again, if $(v, v_2) \in E(T)$ then $v_1vv_2\dots v_s$ is a path longer than P , contradiction. So $(v_2, v) \in E(T)$. By simple induction, we may show that $(v_s, v) \in E(T)$ but in this case $v_1v_2\dots v_s v$ is a path longer than P , contradiction. So $s = n$. \square

Later on, the question concerning the existence of any oriented Hamiltonian path in tournaments arose. This problem was partially treated by many researchers, and the first result was given by Grünbaum [17] in 1969, concerning the existence of antidirected Hamiltonian paths in tournaments, which are oriented paths whose arcs have successively opposite directions: he proved that with the exception of 3 cases, which are the directed 3-cycle, the regular tournament on 5 vertices, and the Paley tournament on 7 vertices, every tournament contains an antidirected Hamiltonian path. Few years later, after giving an easier proof of Grünbaum's result, Rosenfeld [28] conjectured that there exists some integer N , $N > 7$, such that every tournament on n vertices, $n > N$, contains all types of oriented Hamiltonian paths. In 1986, Thomason [32] proved this fact for some N large enough, and for a tournament of order $(n + 1)$. Many graph theorists kept trying to prove Rosenfeld's conjecture, even under some conditions, until the problem was completely settled by Havet and Thomassé in 1993:

Theorem 1.2.2. ([18]) *Every tournament contains all types of oriented Hamiltonian paths, with the exception of the directed 3-cycle, the regular tournament on 5 vertices, and the Paley tournament on 7 vertices.*

However, the proof is complicated and relies on computer analysis to enumerate the enormous number of different cases that arise. No combinatorial proof has yet been found. In 2004, El Sahili [11] proved Rosenfeld's conjecture for any integer n , and for a tournament of order $2n - 2$, using the notion of maximal forests in a tournament.

Concerning the existence of directed Hamiltonian cycles, we have Camion's famous theorem for strong tournaments [7]:

Theorem 1.2.3. *A tournament T is strong if and only if it contains a directed Hamiltonian cycle.*

Proof. Let T be a strong tournament of order n . Since T is strongly connected then it contains a directed cycle. In fact let $e = (x, y) \in E(T)$ then \exists a yx -directed path P as T is strong, so $P \cup (x, y)$ is a directed cycle in T . Let $C = v_1v_2\dots v_s$ be a longest directed cycle in T , we need

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to show that $s = n$. Suppose $s < n$ then $\exists v \in V(T)$ such that $v \notin V(C)$. If $\exists i, j$ such that (v_i, v) and $(v, v_j) \in E(T)$, then $\exists 1 \leq k \leq s$ such that (v_k, v) and $(v, v_{k+1}) \in E(T)$ if $k \neq s$, or (v_s, v) and $(v, v_1) \in E(T)$ if $k = s$, but then $v_1 \dots v_k v v_{k+1} \dots v_n$ or $v_2 \dots v_s v v_1$ is a directed cycle longer than C , contradiction. So $(v, v_i) \in E(T) \forall 1 \leq i \leq s$ or $(v_i, v) \in E(T) \forall 1 \leq i \leq s$. Suppose $(v, v_i) \in E(T) \forall 1 \leq i \leq s$, but T is strongly connected, then \exists a $v_i v$ -directed path $\forall 1 \leq i \leq s$, so let P be a directed $v_i v$ -path, $i \neq s$, and let $v_k \in P \cap C$ such that $l(P_{[v_k, v]})$ is minimum. Now set $C' = v_1 \dots v_k P_{[v_k, v]} v_{k+1} \dots v_n$. Since $P_{[v_k, v]} \cap C = \{v_k\}$ then C' is a directed cycle, with $l(C') > l(C)$, contradiction. Suppose $(v_i, v) \in E(T) \forall 1 \leq i \leq s$, we reach a contradiction too using the same arguments as previously done. So $s = n$.

On the other hand, it's easy to remark that every tournament T that contains a directed Hamiltonian cycle is strong, since this cycle induces an xy -directed path, $\forall x, y \in V(T)$. \square

Then, as for paths, the existence of any oriented cycle in tournaments was treated. In 1971, Grünbaum [17] conjectured that every tournament T of even order $n = 2k$, $n \geq 10$, contains an antidirected Hamiltonian cycle, and in 1974, Rosenfeld [29] proved this conjecture for every order $2k \geq 28$. Finally, In 1999, Havet proved the following:

Theorem 1.2.4. ([19]) *Every tournament contains all types of oriented Hamiltonian cycles, except possibly the Hamiltonian circuit (directed cycle) in a reducible tournament, which is a tournament that is not strong.*

Other than the existence of oriented Hamiltonian paths and cycles, researchers were interested in guessing their parity in any tournament. The parity of paths was first studied by Redei:

Theorem 1.2.5. ([27]) *The number of directed Hamiltonian paths in any tournament is always odd.*

However the proof of this simple statement was surprisingly hard and original.

Later on, on 1973, Forcade [15] proved that except for the symmetric paths (defined later in this chapter), the parity of the number of oriented Hamiltonian paths of a given type in a tournament depends only on the orientation of the arcs of these paths, and on the number of the vertices of the tournament.

As a result, to study the parity of some types of non-symmetric Hamiltonian paths in a tournament of order n , one may study this parity in the transitive tournament TT_n , which is a tournament of order n whose all arcs are forward with respect to some enumeration of its vertices. This property of transitive tournaments makes counting oriented Hamiltonian paths easier, and due to Forcade's theorem, many graph theorists investigated in the parity and the enumeration of some types of non-symmetric oriented Hamiltonian paths, in transitive tournaments, and the last result was given by El Sahili and A. Aad [13] who proved that in any tournament, the number of antisymmetric Hamiltonian paths is always odd.

Concerning the parity of oriented cycles, Forcade [15] similarly proved that the parity of the number of oriented Hamiltonian cycles of a given type in a tournament depends on the orientations of the arcs of these cycles and on the number of Hamiltonian circuits in the tournament. Although some researchers gave an algorithm for computing the parity of some kinds of oriented Hamiltonian cycles in any digraph, as did Bjorklund et. al. [5], no explicit formula about this parity has been established yet. However, a partial result about this parity was given recently by El Sahili and A. Aad [12], who proved that the number of Hamiltonian circuits and the number of antidirected Hamiltonian paths starting with a forward arc, in a tournament of odd order n , $n \geq 5$, have the same parity.

Now we consider enumerating Hamiltonian paths and cycles in tournaments: Consider a certain type of oriented paths (resp. cycles) on n vertices, that is, by fixing a certain orientation of their arcs. How many copies of such paths (resp. cycles) can be found in a tournament of order n ? Till now, no exact value of these numbers was given.

However, one can bound the number of the *directed* Hamiltonian paths and cycles in tournaments, and work on characterizing the tournaments having the minimum or the maximum number of such paths and cycles.

The first result through this investigation was given by Szele [30], who gave lower and upper bounds for the maximum number $P(n)$ of directed Hamiltonian paths in a tournament on n vertices, and which was considered to be an introduction to the probabilistic methods in graph theory:

$$\frac{n!}{2^{n-1}} \leq P(n) \leq c_1 \frac{n!}{2^{\frac{3}{4}n}},$$

where c_1 is a positive constant independent of n . Then, the upper bound of $P(n)$ was improved by Alon [1]:

$$P(n) \leq c_2 \cdot n^{\frac{3}{2}} \frac{n!}{2^{n-1}}.$$

Concerning tournaments having the maximum number of directed Hamiltonian paths, an interesting characterisation was given recently by El Sahili and Maria A.A.:

Theorem 1.2.6. ([14]) *The tournaments having the maximum number of directed Hamiltonian paths must be strong.*

For the minimum number of directed Hamiltonian paths in a tournament, we can easily prove the following:

Theorem 1.2.7. *Let T be a tournament of order n . The number of Hamiltonian paths in T is equal to 1 if and only if T is transitive.*

Proof. For the sufficient condition, suppose that T is transitive. Let $P = v_1v_2\dots v_n$ be a directed Hamiltonian path of T . Suppose that T contains another directed Hamiltonian path

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$P' = v_{i_1}v_{i_2}\dots v_{i_n}$. As T is transitive, $(v_{i_{k-1}}, v_{i_k}) \in E(T) \Rightarrow i_{k-1} < i_k$, thus $1 \leq i_1 < i_2 < \dots < i_{n-1} < i_n \leq n$ and therefore $i_j = j \forall 1 \leq j \leq n$, so $P' = v_1v_2\dots v_n$ and we conclude that the number of directed Hamiltonian paths in T is equal to 1.

For the necessary condition, suppose that the number of directed Hamiltonian paths in T is equal to 1 and let $P = v_1v_2\dots v_n$ be this Hamiltonian path. We need to prove that $E(T) = \{(v_i, v_j), i < j\}$. Suppose to the contrary that T contains a backward arc. If $(v_n, v_1) \in E(T)$ then $v_2v_3\dots v_nv_1$ is another directed Hamiltonian path, contradiction. So $(v_1, v_n) \in E(T)$. Let v_j be the tail of a backward arc with maximal index. If $(v_j, v_1) \in E(T)$ then $j < n$ and $(v_{j-1}, v_{j+1}) \in E(T)$ due to the maximality of j , so $v_jv_1\dots v_{j-1}v_{j+1}\dots v_n$ is another directed Hamiltonian path, contradiction. Thus $(v_1, v_j) \in E(T)$. Let $(v_j, v_i) \in E(T)$ be a backward arc such that i is minimal. We have $i > 1$ because $(v_1, v_j) \in E(T)$, then $(v_{i-1}, v_j) \in E(T)$ due to the minimality of i . But in this case, if $j < n$ then $v_1\dots v_{i-1}v_jv_i\dots v_{j-1}v_{j+1}\dots v_n$ is another directed Hamiltonian path, and if $j = n$ then $v_1\dots v_{i-1}v_nv_i\dots v_{n-1}$ is another directed Hamiltonian path, which leads to a contradiction in both cases. Thus $E(T) = \{(v_i, v_j), i < j\}$ and T is transitive. \square

So, a transitive tournament contains the minimum number of directed Hamiltonian paths.

Using Theorem 1.2.3, we can easily notice that a transitive tournament is not strong, since it doesn't contain a directed Hamiltonian cycle. However, we can observe that in the case of strong tournaments, the minimum number of directed Hamiltonian paths increases exponentially with the order of the tournament. For example, consider the nearly-transitive tournament, which is obtained from a transitive tournament TTn by reversing the orientation of the arc joining the extremities of the unique directed Hamiltonian path of TTn . This tournament is strong (by Theorem 1.2.3) for it contains a directed Hamiltonian cycle formed by the directed Hamiltonian path of TTn union the new arc. We have the following property:

Theorem 1.2.8. *Let T be a nearly-transitive tournament of order n . Then the number h_n of directed Hamiltonian paths in T is*

$$h_n = 2^{n-2} + 1.$$

Proof. Let's order the vertices of T , $V(T) = \{v_1, v_2, \dots, v_n\}$, such that v_n dominates v_1 , but all other arcs forward with respect to this enumeration. First consider the directed Hamiltonian paths of T that don't use the arc (v_n, v_1) . There is only one path that satisfies this condition, that is $P = v_1v_2\dots v_n$ as if we skip some of these vertices while constructing such a path, we can't go back and reach them again since we can't use the backward arc (v_n, v_1) . Now let's consider the directed Hamiltonian paths of T that use the arc (v_n, v_1) . We can choose any set of vertices (that could be empty) from the set $\{v_2, v_3, \dots, v_{n-1}\}$, whose cardinal is equal to $n-2$, say $v_{i_1}, v_{i_2}, \dots, v_{i_s}$, $i_1 \leq i_2 \leq \dots \leq i_s$ and start the path with a path $P' = v_{i_1}v_{i_2}\dots v_{i_s}$ then

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use the backward arc (v_n, v_1) to reach all the remaining vertices by respecting an increasing order of the indexes. The number of such Hamiltonian paths is equal to $\sum_{k=0}^{n-2} \binom{n-2}{k} = 2^{n-2}$. So the total number of directed Hamiltonian paths in T is equal to $2^{n-2} + 1$. \square

In 1972, Moon [21] gave upper and lower bounds for the minimum number $p(n)$ of directed Hamiltonian paths in a strong tournament of order n :

$$\alpha^{n-1} \leq p(n) \leq \begin{cases} 3 \cdot \beta^{n-3} & \text{for } n \equiv 0 \pmod{3} \\ \beta^{n-1} & \text{for } n \equiv 1 \pmod{3} \\ 9 \cdot \beta^{n-5} & \text{for } n \equiv 2 \pmod{3} \end{cases}$$

where $\alpha = 6^{\frac{1}{4}} \approx 1.565$ and $\beta = 5^{\frac{1}{3}} \approx 1.710$. In 2006, after finding an interesting characterization of strong tournaments, Busch [6] improved this result by giving the exact value of the minimum number $p(n)$ for strong tournaments on n vertices:

$$p(n) = \begin{cases} 3 \cdot \beta^{n-3} & \text{for } n \equiv 0 \pmod{3} \\ \beta^{n-1} & \text{for } n \equiv 1 \pmod{3} \\ 9 \cdot \beta^{n-5} & \text{for } n \equiv 2 \pmod{3} \end{cases}$$

Later on, researchers tried to characterize the strong tournaments having the minimum number of directed Hamiltonian paths. An explicit construction of such tournaments was given by Moon and Yang [23]. They constructed tournaments that they called chains, which are obtained from a succession of nearly transitive tournaments, identifying the bottom node of each one of them with the top node of the following one, and adding arcs directed in one way, following the succession of the nearly-transitive tournaments, and they proved that some kinds of these chains, which they called special chains, contain the minimum number $p(n)$ of directed Hamiltonian paths, and that they are the only tournaments verifying this minimum.

Concerning the minimum number of directed Hamiltonian cycles in tournaments, we first remark that it's equal zero, since any tournament that is not strong does not contain a directed Hamiltonian cycle (by Theorem 1.2.3). Considering strong tournaments, Thomassen [31] gave in 1980 an extension of the previously mentioned result of Moon, proving that if T is a strong tournament of order n , and A is a subset of the vertices of T , then T has at least α^{n-1} directed paths such that each one of them contains all the vertices of A , starts and terminates in A , and such that for any two of these paths, the vertices of A occur in different order. Using this extension, he was able to prove that every 2-connected tournament of order n contains at least $\alpha^{\frac{n}{32}-1}$ distinct directed Hamiltonian cycles. Thomassen also gave a lower bound for the minimum number of directed Hamiltonian cycles in a tournament of connectivity 1, with given minimal in-degree. In fact, he proved first that a strong tournament with minimal

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in-degree greater than k has at least 2^{k-1} directed Hamiltonian paths starting at any vertex, and as a result, he proved that a tournament of connectivity 1, and of in-degree greater than k , $k \geq 2$, has at least 2^{k-1} directed Hamiltonian cycles.

On the other hand, many graph theorists tried to construct tournaments having a large number of directed Hamiltonian cycles. With the same reasoning adopted to prove the above mentioned result of Szele, one can prove that the maximum number $C(n)$ of directed Hamiltonian cycles in a tournament of order n is

$$C(n) \geq \frac{(n-1)!}{2^n}.$$

However Moon observed that it seems difficult to give explicit tournaments with such a large number of directed Hamiltonian cycles. In 1968, in his classic book on tournaments [22], Moon discusses the question of exhibiting tournaments having a large number of Hamiltonian cycles. He poses the question (attributed to Moser) of constructing a tournament on n vertices having at least $(\frac{n}{3e})^n$ directed Hamiltonian cycles, and in 2015, Calkin, Novik and Ushijima-Mwesigwa [26] showed that Moser could have even asked for more. They constructed what we call a triangular tournament, which is a tournament obtained from three tournaments T_1 , T_2 , and T_3 , by orienting all edges from T_1 to T_2 , T_2 to T_3 , T_3 to T_1 , and it is denoted by $C_3(T_1, T_2, T_3)$. By Theorem 1.2.3, we can clearly see that such tournament is strong. They proved that the number of directed Hamiltonian cycles in a triangular tournament of order n can be exactly computed using the Stirling numbers of the second kind and the fact that every directed Hamiltonian cycle C in the triangular tournament $C_3(T_1, T_2, T_3)$ corresponds to a k -path cover of this tournament, where k is the number of times that C visits each subtournament T_i . They showed that in the case where the order of each subtournament T_i is equal to $\frac{n}{3}$, the number of directed Hamiltonian cycles happens to be greater than $(\frac{n}{3e})^n$.

Concerning enumerating oriented Hamiltonian paths and cycles that are not directed, Rosenfeld [29] proved in 1974 that the number of antidirected Hamiltonian paths starting with a forward arc is equal to the number of antidirected Hamiltonian paths starting with a backward arc, in any tournament, which can be stated as: the number of antidirected Hamiltonian paths in any tournament T is equal to the number of antidirected Hamiltonian paths in the complement of T , denoted by \bar{T} .

On the other hand, for some positive integer n , Désiré André [9] studied in 1881 the notion of alternating permutations of n distinct labelled letters. A permutation of n letters labelled $\{a_1, a_2, \dots, a_n\}$ is a given order of these letters, and for a fixed permutation of the a'_i 's, one defines the sequence of relative integers s_1, s_2, \dots, s_{n-1} such that $\forall 1 \leq i \leq n-1$, $s_i = a_i - a_{i+1}$. If every two consecutive elements s'_i 's of this sequence have opposite signs, then the corresponding permutation of the a'_i 's is said to be alternating, and, after proving that the number of alternating permutations of n distinct labelled letters is always even, Désiré André

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denoted this number by $2An$. Moreover, the number An is given by [25]:

$$\sum \frac{An}{n!} x^n = \sec a + \tan x.$$

Interestingly, for the particular case of a transitive tournament TTn , after labelling its vertices by v_1, v_2, \dots, v_n , such that all its arcs are forward with respect to this enumeration, it is easy to see that each antidirected Hamiltonian path in this tournament corresponds to exactly two alternating permutations of the n indexes, since each path has two enumerations of its vertices. Based on this, in his paper, Rosenfeld [29] proved that in a transitive tournament TTn , the number of distinct antidirected Hamiltonian paths is equal to An if n is even, and $2An$ if n is odd. Furthermore, he proved that for any tournament T of order n , where $T \neq TTn$, the number of its antidirected Hamiltonian paths is less than the number of antidirected Hamiltonian paths in TTn .

Moreover, in the same paper [29], Rosenfeld proved that the number of antidirected Hamiltonian cycles in any tournament T of even order $2k$ is not greater than $A(2k - 1)$ and that if $T \neq TT(2k)$, then the number of antidirected Hamiltonian cycles in T is less than the number of those in the transitive tournament $TT(2k)$.

1.3 Types of oriented paths and cycles

Since the orientations of the arcs of an oriented path (resp. cycle) is arbitrary, we may assign to each path (resp. cycle) a type. In this section, we will give the explicit definitions concerning the notion of the types of paths and cycles, give some related elementary results and introduce basic notations. Most of the contents in this section were given in [12]. They will be widely used throughout the work that will be done in Chapter 2 and Chapter 3.

Let $\mathbb{K}_s = \{(\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{Z}^s, s \geq 1, \alpha_i \alpha_{i+1} < 0, \forall 1 \leq i \leq s - 1\}$.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{K}_s$.

Definition 1.3.1. (Type of a path.)

An oriented path P is said to be of type $P(\alpha_1, \alpha_2, \dots, \alpha_s)$ if P is formed by s blocks (i.e. maximal directed subpaths) I_1, I_2, \dots, I_s such that $l(I_i) = |I_i| = |\alpha_i|$ and with x_i, y_i being the ends of the block I_i , $I_i \cap I_{i+1} = \{y_i\} = \{x_{i+1}\}$, the following condition is verified: $\forall i = 1, \dots, s, \alpha_i > 0 \iff I_i$ is directed from x_i to y_i .

Example 1.3.2. The following digraph is an oriented path $P = v_1 v_2 \dots v_9$ on 9 vertices, of type $P(2, -3, 1, -2)$, formed by 4 blocks:



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We note $\text{end}(I_i) = \{x_i, y_i\}$, and we write $P = I_1 I_2 \dots I_s$.

For $u, v \in I_i$, $I_i[u, v]$ denotes the subpath of I_i of ends u and v .

If the path is Hamiltonian in a tournament of order n , then we have

$$\sum_{i=1}^s |\alpha_i| = n - 1.$$

This notation can be extended by allowing α_i to be 0, by considering:

- $P(\alpha_1, \dots, \alpha_i, 0, \alpha_{i+2}, \dots, \alpha_s) = P(\alpha_1, \dots, \alpha_i + \alpha_{i+2}, \dots, \alpha_s)$ (remark that in this case, α_i and α_{i+2} have the same sign),
- $P(0, \alpha_2, \dots, \alpha_s) = P(\alpha_2, \dots, \alpha_s)$,
- $P(\alpha_1, \dots, \alpha_{s-1}, 0) = P(\alpha_1, \dots, \alpha_{s-1})$.

Remark 1.3.3. Note that a path $P = v_1 v_2 \dots v_r$ that is of type $P(\alpha_1, \dots, \alpha_s)$ with respect to this enumeration is also of type $P(-\alpha_s, \dots, -\alpha_1)$ with respect to the other enumeration $v_r v_{r-1} \dots v_1$ denoting it, so we remark that any path has at most two types.

Two paths P and P' in a tournament T are equal if they have the same set of arcs, i.e.

$$E(P) = E(P').$$

Definition 1.3.4. Let T be a tournament. The set $\mathcal{P}_T(\alpha_1, \dots, \alpha_s)$ is defined to be the set of oriented paths in T of type $P(\alpha_1, \dots, \alpha_s)$ and $f_T(\alpha_1, \alpha_2, \dots, \alpha_s)$ denotes the cardinal of this set. That is, $f_T(\alpha_1, \alpha_2, \dots, \alpha_s)$ is the number of paths in T of type $P(\alpha_1, \dots, \alpha_s)$.

Particularly, $f_T(n - 1)$ is the number of directed Hamiltonian paths in a tournament T of order n .

For $\alpha = (\alpha_1, \dots, \alpha_s)$ in \mathbb{Z}^s , we denote by $-\alpha$ the s -tuple $(-\alpha_1, \dots, -\alpha_s)$ and by $\bar{\alpha}$ the s -tuple $(\alpha_s, \alpha_{s-1}, \dots, \alpha_1)$.

We first give two properties about the sets of oriented paths in a tournament T .

Proposition 1.3.5. *Let α and $\beta \in \mathbb{K}^s$. We have:*

$$\mathcal{P}_T(\alpha) = \mathcal{P}_T(\beta) \iff \alpha = \beta \text{ or } \alpha = -\bar{\beta}.$$

Proof. Let $\alpha = (\alpha_1, \dots, \alpha_s)$ and $\beta = (\beta_1, \dots, \beta_t)$. For the necessary condition, let $P \in \mathcal{P}(\alpha) = \mathcal{P}(\beta)$; P is simultaneously of type $P(\alpha)$ and $P(\beta)$. Since s and t both refer to the

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number of blocks in P , then $s = t$.

P is of type $P(\alpha) \Rightarrow P = I_1 I_2 \dots I_s$, where $|I_i| = |\alpha_i|$, $\text{end}(I_i) = \{x_i, y_i\}$ and $\text{end}(P) = \{x_1, y_s\}$. Similarly, P is of type $P(\beta) \Rightarrow P = I'_1 I'_2 \dots I'_s$, where $|I'_i| = |\beta_i|$, $\text{end}(I'_i) = \{x'_i, y'_i\}$ and $\text{end}(P) = \{x'_1, y'_s\}$.

Observe that whenever $P = J_1 J_2 \dots J_k$, then $\forall 2 \leq i < t \leq k$, $J_i \cap J_t \neq \emptyset \iff t = i + 1$. Moreover, $J_i \cap J_t = \{y_l\}$; $i < t \Rightarrow i = l$ and $t = i + 1$.

The ends of P are the vertices whose degree is one. Since P has exactly two such vertices, then $\{x_1, y_s\} = \{x'_1, y'_s\}$. We have two cases:

- $x_1 = x'_1 \Rightarrow y_s = y'_s$. Since I_1 and I'_1 are the only blocks that contain x_1 and x'_1 respectively, then $I_1 = I'_1$. Similarly, $I_s = I'_s$. Moreover, $\{y_1\} = I_1 \cap I_2 = I'_1 \cap I_2$. By the previous observation, since I'_2 is the only block that intersects with I'_1 , then $I_2 = I'_2$ and $\{y_1\} = \{y'_1\}$. We proceed similarly to prove that $I_i = I'_i \forall i = 3, \dots, s - 1$. Hence, $\alpha_i = \beta_i \forall i = 1, \dots, s$. Thus $\alpha = \beta$.
- $x_1 = y'_s \Rightarrow x'_1 = y_s$. This implies that $I_1 = I'_s$ and $I_s = I'_1$. Again, using the above observation, we obtain that $I_2 = I'_{s-1}, \dots, I_{s-1} = I'_2$. Hence, $(\alpha_1, \dots, \alpha_s) = (-\beta_s, \dots, -\beta_1) \Rightarrow \alpha = -\bar{\beta}$.

For the sufficient condition, if $\alpha = \beta$, then the result is trivial. Suppose that $\alpha = -\bar{\beta}$ and let $P \in \mathcal{P}(\alpha)$; $P = x_1 x_2 \dots x_n$. The path $P' = x_n x_{n-1} \dots x_1$ is of type $P(-\alpha_s, \dots, -\alpha_1) = P(\beta)$, and $P = P'$ since $E(P) = E(P')$. Hence, P is also of type $P(\beta)$, which proves that $\mathcal{P}(\alpha) \subseteq \mathcal{P}(\beta)$. We may similarly prove that $\mathcal{P}(\beta) \subseteq \mathcal{P}(\alpha)$, which concludes our proof. \square

Proposition 1.3.6. *Let T be a tournament of order n .*

The sets $\mathcal{P}_T(\alpha)$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{K}_s$, $\sum_{i=1}^s |\alpha_i| = n - 1$, form a partition of the set \mathcal{P}_T of all the oriented Hamiltonian paths in T .

Proof. Let R_P be the binary relation defined on the set \mathcal{P}_T by:

$P_1 R_P P_2 \Leftrightarrow P_1$ and P_2 belong to the same set $\mathcal{P}_T(\alpha)$, for some $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{Z}^s$, $\alpha_i \cdot \alpha_{i+1} < 0$, $\forall 1 \leq i \leq s - 1$ and $\sum_{i=1}^s |\alpha_i| = n - 1$.

Clearly, R_P is an equivalence relation, thus the equivalence classes of \mathcal{P}_T with respect to R_P , which are the sets $\mathcal{P}_T(\alpha)$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{K}_s$, $\sum_{i=1}^s |\alpha_i| = n - 1$, form a partition of the set \mathcal{P}_T . \square

Let $\mathbb{K}'_s = \{(\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{Z}^s, s \geq 1, \alpha_i \cdot \alpha_{i+1} < 0, \forall 1 \leq i \leq s - 1, \alpha_s \cdot \alpha_1 < 0\}$.

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{K}'_s$.

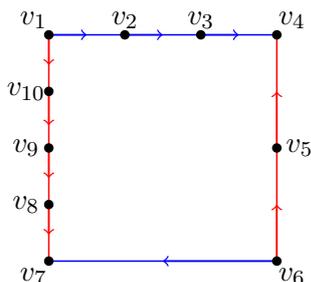
Definition 1.3.7. (Type of a cycle.)

An oriented cycle C is said to be of type $C(\alpha_1, \dots, \alpha_s)$ if C is formed by s blocks I_1, I_2, \dots, I_s ,

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with $\text{end}(I_i) = \{x_i, y_i\}$, $|I_i| = |\alpha_i|$ and $I_i \cap I_{i+1} = \{y_i\} = \{x_{i+1}\}$, $1 \leq i \leq s-1$ and $I_s \cap I_1 = \{y_s\} = \{x_1\}$, such that $\forall i = 1, \dots, s$, $\alpha_i > 0 \iff I_i$ is directed from x_i to y_i .

Example 1.3.8. The following digraph is a cycle $C = v_1v_2 \dots v_{10}$ on 10 vertices, of type $C(3, -2, 1, -4)$, formed by 4 blocks:



We write $C = I_1I_2 \dots I_s$.

If the cycle is Hamiltonian in a tournament of order n , then we have

$$\sum_{i=1}^s |\alpha_i| = n.$$

Note that for cycles, if $s \neq 1$, (i.e. the cycle is not a directed one), then s must be even, since the blocks I_1 and I_s should have opposite orientations.

As we did for paths, we may also allow α_i to be 0 for cycles, by considering:

- $C(\alpha_1, \alpha_2, \dots, \alpha_{i-1}, 0, \alpha_{i+1}, \dots, \alpha_s) = C(\alpha_1, \alpha_2, \dots, \alpha_{i-1} + \alpha_{i+1}, \dots, \alpha_s)$
(α_{i-1} and α_{i+1} have same sign),
- $C(0, \alpha_2, \dots, \alpha_s) = C(\alpha_2 + \alpha_s, \alpha_3, \dots, \alpha_{s-1})$
(α_2 and α_s have same sign),
- $C(\alpha_1, \alpha_2, \dots, \alpha_{s-1}, 0) = C(\alpha_1 + \alpha_{s-1}, \alpha_2, \dots, \alpha_{s-2})$
(α_1 and α_{s-1} have same sign).

Remark 1.3.9. Note that a cycle $C = v_1v_2 \dots v_r = I_1I_2 \dots I_s$ that is of type $C(\alpha_1, \dots, \alpha_s)$ with respect to the enumeration $v_1v_2 \dots v_r$ is also of type $P(\alpha_i, \alpha_{i+1}, \dots, \alpha_s, \alpha_1, \dots, \alpha_{i-1} \forall 2 \leq i \leq s)$, (since C can be read as $C = I_iI_{i+1} \dots I_sI_1 \dots I_{i-1}$), and also of type $P(-\alpha_s, \dots, -\alpha_1)$ with respect to the other enumeration $v_1v_rv_{r-1} \dots v_2$ denoting it, and thus it is also of type $C(-\alpha_i, -\alpha_{i-1}, \dots, -\alpha_1, -\alpha_s, \dots, -\alpha_{i+1}) \forall 1 \leq i \leq s-1$. So we remark that any cycle formed by s blocks has at most $2 \cdot s$ types.

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Moreover, as for paths, two cycles C and C' in a tournament T are equal if they have the same set of arcs, that is,

$$E(C) = E(C').$$

Definition 1.3.10. Let T be a tournament. The set $\mathcal{C}_T(\alpha_1, \dots, \alpha_s)$ is defined to be the set of oriented cycles of T of type $C(\alpha_1, \dots, \alpha_s)$ and $g_T(\alpha_1, \dots, \alpha_s)$ denotes the cardinal of this set. That is, $g_T(\alpha_1, \dots, \alpha_s)$ is the number of cycles of type $C(\alpha_1, \dots, \alpha_s)$ in T .

Particularly, $g_T(n)$ is the number of directed Hamiltonian cycles in a tournament T of order n .

Now, we also give two properties about the sets of oriented cycles in a tournament T .

Proposition 1.3.11. *We have:*

$$\begin{aligned} \mathcal{C}_T(\alpha) = \mathcal{C}_T(\beta) \iff & \beta = (\alpha_i, \alpha_{i+1}, \dots, \alpha_s, \alpha_1, \dots, \alpha_{i-1}) \\ & \text{or } \beta = (-\alpha_i, -\alpha_{i-1}, \dots, -\alpha_1, -\alpha_s, \dots, -\alpha_{i+1}), \end{aligned}$$

for some $1 \leq i \leq s$.

Proof. The proof is similar to the proof of Proposition 1.3.5. □

Proposition 1.3.12. *Let T be a tournament of order n .*

The sets $\mathcal{C}_T(\alpha)$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{K}'_s$, $\sum_{i=1}^s |\alpha_i| = n$, form a partition of the set \mathcal{C}_T of all the oriented Hamiltonian cycles in T .

Proof. Let R_C be the binary relation defined on the set \mathcal{C}_T by:

$C_1 R_C C_2 \iff C_1$ and C_2 belong to the same set $\mathcal{C}_T(\alpha)$, for some $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{Z}^s$, $\alpha_i \cdot \alpha_{i+1} < 0$, $\forall 1 \leq i \leq s-1$, $\alpha_1 \cdot \alpha_s < 0$, and $\sum_{i=1}^s |\alpha_i| = n$.

Clearly, R_C is an equivalence relation, thus the equivalence classes of \mathcal{C}_T with respect to R_C , which are the sets $\mathcal{C}_T(\alpha)$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{K}'_s$, and $\sum_{i=1}^s |\alpha_i| = n$, form a partition of the set \mathcal{C}_T . □

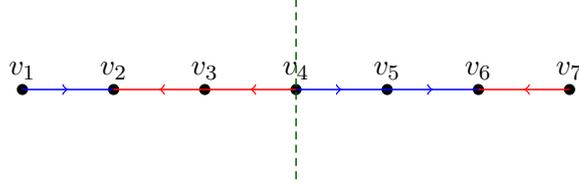
An s -tuple $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{Z}^s$ is said to be symmetric if $\alpha = -\bar{\alpha}$. That is, if $(\alpha_1, \alpha_2, \dots, \alpha_s) = (-\alpha_s, -\alpha_{s-1}, \dots, -\alpha_1)$.

Definition 1.3.13. (Symmetric paths and cycles.)

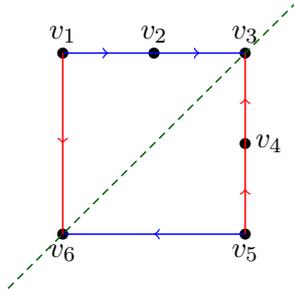
An oriented path P (resp. cycle C) is said to be symmetric if there exists a tuple α that is symmetric, such that P (resp. C) is of type $P(\alpha)$ (resp. $C(\alpha)$). Otherwise, the path P (resp. cycle C) is non-symmetric.

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Example 1.3.14. The following path $P = v_1v_2 \dots v_7$ is of type $P(1, -2, 2, -1)$ and the 4-tuple $(1, -2, 2, -1)$ is symmetric, so P is symmetric.



The following cycle C is of type $C(2, -2, 1, -1)$ with respect to the enumeration $v_1v_2 \dots v_6$ of its vertices, but $(2, -2, 1, -1)$ is not symmetric. However, C is also of type $C(-2, 1, -1, 2)$ with respect to the enumeration $v_3v_4 \dots v_7v_1v_2$, and $(-2, 1, -1, 2)$ is symmetric, so C is symmetric.



Definition 1.3.15. (Generated cycles.)

Let T be a tournament on n vertices $\{x_1, x_2, \dots, x_n\}$. An oriented cycle C in T is said to be generated by an oriented path $P = x_1x_2 \dots x_n$ if $C = P \cup \langle \{x_1, x_n\} \rangle$. That is, if C is the cycle obtained from P by adding the arc in T between x_1 and x_n . We write $C = C_P$.

From now on, we will write uv instead of $\langle \{u, v\} \rangle$.

Definition 1.3.16. We define the relation \mathcal{R} on the set of oriented paths in T by:

$$PRP' \iff C_P = C_{P'}.$$

The relation \mathcal{R} is obviously an equivalence relation, and so is \mathcal{R}_α , the restriction of \mathcal{R} on the set $\mathcal{P}_T(\alpha)$. We can easily prove the following:

Proposition 1.3.17. *Let $P = v_1v_2 \dots v_n$ and P' be two oriented paths in a tournament T of order n . Then PRP' if and only if $P = P'$ or $P' = v_iv_{i+1} \dots v_nv_1v_2 \dots v_{i-1}$ for some $2 \leq i \leq n$.*

Proof. First, suppose that PRP' , then $C_P = C_{P'}$. Let $P = v_1v_2 \dots v_n$. $P' = C_{P'} - uv = C_P - uv$, where u and v are the ends of P' . This implies that $uv = v_{i-1}v_i$ for some $2 \leq i \leq n$, or $uv = v_nv_1$. Thus, $P' = P$ or $P' = v_iv_{i+1} \dots v_nv_1 \dots v_{i-2}v_{i-1}$, for some $2 \leq i \leq n$.

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Conversely, suppose that $P = v_1v_2 \dots v_n$ and $P' = v_iv_{i+1} \dots v_nv_1 \dots v_{i-1}$, then $C_P = P \cup v_nv_1$ and $C_{P'} = P' \cup v_{i-1}v_i$. It follows that $C_P = C_{P'}$ and PRP' . \square

Remark 1.3.18. Let $P = v_1v_2 \dots v_n$ be an oriented path in a tournament T , of some type $P(\alpha) = P(\alpha_1, \dots, \alpha_s)$, and let $C = C_P$ be the cycle generated by P in T which is of some type $C(\beta)$. We will see what are the different values β could take:

- **Case 1:** s is even. Then if $\alpha_1 > 0$ (which means $\alpha_s < 0$), we have $\beta = (\alpha_1 + 1, \alpha_2, \dots, \alpha_s)$ or $\beta = (\alpha_1, \dots, \alpha_{s-1}, \alpha_s - 1)$ whether (v_n, v_1) or $(v_1, v_n) \in E(T)$ respectively, while if $\alpha_1 < 0$ (i.e. $\alpha_s > 0$), then $\beta = (\alpha_1 - 1, \alpha_2, \dots, \alpha_s)$ or $\beta = (\alpha_1, \dots, \alpha_{s-1}, \alpha_s + 1)$ whether (v_1, v_n) or $(v_n, v_1) \in E(T)$ respectively.
- **Case 2:** s is odd. Then if $\alpha_1 > 0$ (which means $\alpha_s > 0$ also), we have $\beta = (-1, \alpha_1, \dots, \alpha_s)$ or $\beta = (\alpha_s + 1 + \alpha_1, \alpha_2, \dots, \alpha_{s-1})$ whether (v_1, v_n) or $(v_n, v_1) \in E(T)$ respectively, while if $\alpha_1 < 0$ (and so is α_s), then $\beta = (1, \alpha_1, \dots, \alpha_s)$ or $\beta = (\alpha_s - 1 + \alpha_1, \alpha_2, \dots, \alpha_{s-1})$ whether (v_n, v_1) or $(v_1, v_n) \in E(T)$ respectively.

So we remark that every oriented path P in a tournament T may generate 2 types of cycles, that we will denote by $C(\beta)$ and $C(\beta')$ later in Chapter 2.

Remark 1.3.19. If a path P has the type $P(\alpha) = P(\alpha_1, \dots, \alpha_s)$ where α is symmetric, then the cycle C_P generated by P cannot be symmetric.

In fact, if P has the type $P(\alpha) = P(\alpha_1, \dots, \alpha_s)$ and α is symmetric, thus $\alpha_1 = -\alpha_s$, so α_1 and α_s have opposite signs, which means that s should be even. Thus by the previous remark, C_P has one of these types: $C(\alpha_1 + 1, \dots, \alpha_s)$ or $C(\alpha_1 - 1, \dots, \alpha_s)$ or $C(\alpha_1, \dots, \alpha_s - 1)$ or $C(\alpha_1, \dots, \alpha_s + 1)$. But in all these cases, and due to the fact that α is symmetric, C_P cannot be written as a succession of blocks having the type $C(\beta)$ where β is symmetric, thus the cycle C_P cannot be symmetric.

Let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s$.

Definition 1.3.20. (Period.)

An integer $1 \leq r \leq s$ is said to be a period of α if $[i \equiv j \pmod{r} \Rightarrow \alpha_{i_s} = \alpha_{j_s}]$ where i_s is the unique integer in $\{1, 2, \dots, s\}$ such that $i \equiv i_s \pmod{s}$.

Let $r(\alpha) = \min\{r; r \text{ is a period of } \alpha\}$.

Proposition 1.3.21. r is a period of $\alpha \iff r(\alpha)$ divides r .

Proof. If $r(\alpha)$ divides r , let $i, j \in \mathbb{N}$ such that $i \equiv j \pmod{r}$, then $i \equiv j \pmod{r(\alpha)}$, and since $r(\alpha)$ is a period of α , $\alpha_{i_s} = \alpha_{j_s}$. This implies that r is a period of α .

Conversely, suppose that r is a period of α . We may write $r = q \cdot r(\alpha) + b$; $0 \leq b < r(\alpha)$.

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If $b \neq 0$, let $i \leq j$ be two integers such that $i \equiv j \pmod{b}$. $j = i + b \cdot k$ for some $k \Rightarrow j = i + (r - q \cdot r(\alpha)) \cdot k = i + k \cdot r - q \cdot k \cdot r(\alpha)$. Set $i' = i + k \cdot r$, $i' \equiv i \pmod{r} \Rightarrow \alpha_{i'_s} = \alpha_{i_s}$ and $i' \equiv j \pmod{r(\alpha)} \Rightarrow \alpha_{i'_s} = \alpha_{j_s}$. It follows that $\alpha_{i_s} = \alpha_{j_s}$, then b is a period of α , which is a contradiction. \square

As a consequence, $r(\alpha)$ divides s , since s is a trivial period of α .

$$\text{Let } t(\alpha) = \frac{s}{r(\alpha)}.$$

Proposition 1.3.22. *Let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s$ and $\alpha' = \bar{\alpha} = (\alpha_s, \dots, \alpha_1)$, then $r(\alpha) = r(\alpha')$ and $t(\alpha) = t(\alpha')$.*

Proof. Let $\alpha' = (\alpha'_1, \alpha'_2, \dots, \alpha'_s) = (\alpha_s, \dots, \alpha_1)$, and $r(\alpha') = r'$, $r(\alpha) = r$.

Let l, p be two integers such that $p \equiv l \pmod{r}$. We would like to prove that $\alpha'_{p_s} = \alpha'_{l_s}$. We have $\alpha'_{p_s} = \alpha_k$; $k = s - p_s + 1$, i.e. $k \equiv -p + 1 \pmod{s}$ thus $k \equiv -p + 1 \pmod{r}$. Also, $\alpha'_{l_s} = \alpha_j$; $j = s - l_s + 1$, i.e. $j \equiv -l + 1 \pmod{s}$ thus $j \equiv -l + 1 \pmod{r}$. Now $p \equiv l \pmod{r} \Rightarrow -p + 1 \equiv -l + 1 \pmod{r} \Rightarrow k \equiv j \pmod{r} \Rightarrow \alpha_k = \alpha_j \Rightarrow \alpha'_{p_s} = \alpha'_{l_s}$.

Thus, r is a period of α' and $r' \leq r$. Similarly, we prove that $r \leq r'$. Hence, $r = r'$. Consequently, $t(\alpha) = t(\alpha')$. \square

We may also easily remark that $t(\alpha) = t(-\alpha)$, and prove that

$$t(\alpha) = t(\alpha_i, \alpha_{i+1}, \dots, \alpha_s, \alpha_1, \dots, \alpha_{i-1}),$$

and as a result we have $\forall 1 \leq i \leq s$:

$$\begin{aligned} t(\alpha) &= t(-\alpha_i, -\alpha_{i+1}, \dots, -\alpha_s, -\alpha_1, \dots, -\alpha_{i-1}) \\ &= t(\alpha_i, \alpha_{i-1}, \dots, \alpha_1, \alpha_s, \dots, \alpha_{i+1}) \\ &= t(-\alpha_i, -\alpha_{i-1}, \dots, -\alpha_1, -\alpha_s, \dots, -\alpha_{i+1}). \end{aligned}$$

Let $C = I_1 I_2 \dots I_s$ be an oriented cycle of type $C(\beta) = C(\beta_1, \dots, \beta_s)$ and let $r = r(\beta)$.

Definition 1.3.23. (Similar blocks.)

For $1 \leq i < j \leq s$, I_i and I_j are said to be similar if $j \equiv i \pmod{r}$. This is equivalent to say that $j - i$ is a period of β , by Proposition 1.3.21.

For every $1 \leq i \leq s$, there are $t(\beta) - 1$ blocks similar to I_i . It follows that if I_i and I_j are similar, then $\beta_i = \beta_j$, and for any nonnegative integer k , $I_{[i+k]_s}$ and $I_{[j+k]_s}$ are similar.

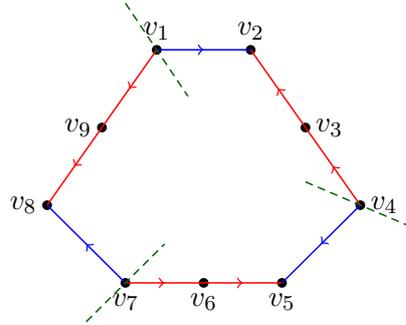
Definition 1.3.24. (Clone vertices.)

If $C = I_1 I_2 \dots I_s$ is an oriented cycle of type $C(\beta) = C(\beta_1, \dots, \beta_s)$, then two vertices $u, v \in C$ are said to be clones (with respect to C) if:

- u and v belong to similar blocks, say I_i and I_j .
- $l(I_i[x_i, u]) = l(I_j[x_j, v])$.
It obviously follows that $l(I_i[u, y_i]) = l(I_j[v, y_j])$.

Remark. If $C \in \mathcal{C}(\beta)$, then each vertex of C has $t(\beta) - 1$ clones.

Example 1.3.25. Consider the cycle $C = v_1 v_2 \dots v_9$ of type $C(\beta) = C(1, -2, 1, -2, 1, -2)$:



- C is generated by 9 paths, each starting from one of its the vertices. For example, the path $P = v_2 v_3 \dots v_9 v_1$ generates C .
- C has 6 blocks. C can be divided into three similar parts, each one formed by two blocks. So the period $r(\beta) = 2$ and $t(\beta) = 3$.
- The blocks $I_2 = v_2 v_3 v_4$, $I_4 = v_4 v_5 v_6 v_7$ and $I_6 = v_6 v_7 v_8 v_9 v_1$ are similar blocks. The vertices v_3, v_6 and v_9 are clone vertices.

In order to simplify the work that will be done in the first section of Chapter 2, we may extend the above definitions and notations given for paths and cycles in tournaments, to enumerations of the vertices of a tournament, so we introduce the following:

Let $\alpha = (\alpha_1, \dots, \alpha_s)$; $\alpha_i \in \mathbb{Z}$, $\alpha_i \cdot \alpha_{i+1} < 0 \forall i = 1, \dots, s - 1$, and let T be a tournament on

$$n \geq \sum_{i=1}^s |\alpha_i| + 1 \text{ vertices.}$$

Definition 1.3.26. (Type of an enumeration.)

An enumeration $E = v_1 v_2 \dots v_r$ of some vertices of T is said to be of type $E(\alpha_1, \dots, \alpha_s)$ if the path $P = v_1 v_2 \dots v_r$ is of type $P(\alpha_1, \dots, \alpha_s)$ with respect to this enumeration.

Definition 1.3.27. Consider a tournament T of order n , then $\mathcal{E}_T(\alpha_1, \dots, \alpha_s)$ is defined to be the set of enumerations of any $m = \sum_{i=1}^s |\alpha_i| + 1$ vertices of T , $m \leq n$, of type $E(\alpha_1, \dots, \alpha_s)$. We denote by $e_T(\alpha_1, \alpha_2, \dots, \alpha_s)$ the cardinal of this set.

Remark that, unlike the case of paths, where every path has two types, if two enumerations E and E' have different types, then $E \neq E'$.

In fact, we have the following property:

Proposition 1.3.28. Let T be a tournament of order n , and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s)$, $\beta = (\beta_1, \beta_2, \dots, \beta_{s'})$, $\sum_{i=1}^s |\alpha_i| \leq n$, and $\sum_{i=1}^{s'} |\beta_i| \leq n$, we have:

$$\mathcal{E}_T(\alpha) = \mathcal{E}_T(\beta) \iff \alpha = \beta.$$

Proof. The sufficient condition is trivial.

For the necessary condition, let $E \in \mathcal{E}_T(\alpha) = \mathcal{E}_T(\beta)$, then $s = s'$, and set $E = v_1 v_2 \dots v_r$. Since $E \in \mathcal{E}_T(\alpha)$ then the path $P = v_1 v_2 \dots v_r$ is of type $P(\alpha)$ with respect to this enumeration. Similarly, since $E \in \mathcal{E}_T(\beta)$ then $P = v_1 v_2 \dots v_r$ is of type $P(\beta)$ with respect to this enumeration. Thus $\alpha = \beta$. \square

Proposition 1.3.29. Let T be a tournament of order n .

The sets $\mathcal{E}_T(\alpha)$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{Z}^s$, $\alpha_i \cdot \alpha_{i+1} < 0$, $s \geq 1$ that verify $\sum_{i=1}^s |\alpha_i| = n - 1$, form a partition of the set \mathcal{E}_T of all the enumerations on n vertices of T .

Proof. Let R_E be the binary relation defined on the set \mathcal{E}_T by:

$E_1 R_E E_2 \iff E_1$ and E_2 belong to the same set $\mathcal{E}_T(\alpha)$, for some $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{Z}^s$, $\alpha_i \cdot \alpha_{i+1} < 0$, $s \geq 1$, and $\sum_{i=1}^s |\alpha_i| = n - 1$.

Clearly, R_E is an equivalence relation, thus the equivalence classes of \mathcal{E}_T with respect to R_E , which are the sets $\mathcal{E}_T(\alpha)$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{Z}^s$, $\alpha_i \cdot \alpha_{i+1} < 0$, $s \geq 1$, and $\sum_{i=1}^s |\alpha_i| = n - 1$, form a partition of the set \mathcal{E}_T . \square

1.4 Quadratic difference equation and invariant frameworks

1.4.1 The quadratic difference equation associated to a graph

Consider a finite connected graph $G = (V(G), E(G))$, where $V(G)$ is the set of its vertices, and $E(G)$ the set of its edges, and a real valued function $\gamma : V(G) \rightarrow \mathbb{R}$. For more simplicity, we will denote $V(G)$ by V and $E(G)$ by E . Consider the following quadratic equation

$$\forall x \in V, \quad \gamma(\Delta\varphi(x))^2 = (\nabla\varphi)^2(x), \tag{1.1}$$

1.4. QUADRATIC DIFFERENCE EQUATION AND INVARIANT FRAMEWORKS

where $\varphi : V \rightarrow \mathbb{C}$ is a complex valued function. The term $\Delta\varphi(x)$ is defined to be the *Laplacian* of φ at vertex x ,

$$\Delta\varphi(x) = \frac{1}{d(x)} \sum_{y \sim x} (\varphi(y) - \varphi(x)),$$

where $d(x)$ is the degree of x , and

$$(\nabla\varphi)^2(x) = \frac{1}{d(x)} \sum_{y \sim x} (\varphi(y) - \varphi(x))^2$$

is the symmetric square of the derivative. Equation (1.1) is then equivalent to

$$\forall x \in V, \quad \frac{\gamma(x)}{d(x)} \left(\sum_{y \sim x} (\varphi(y) - \varphi(x)) \right)^2 = \sum_{y \sim x} (\varphi(y) - \varphi(x))^2. \quad (1.2)$$

Remark 1.4.1. Equation (1.2) is invariant (for fixed γ) under affine linear transformations and conjugation in the complex plane:

$$\varphi \mapsto a\varphi + b \quad \text{and} \quad \varphi \mapsto \bar{\varphi}, \quad \forall a, b \in \mathbb{C} \text{ with } a \neq 0. \quad (1.3)$$

and we refer to these freedoms as a *normalization*.

Recall the Cauchy-Schwarz inequality, for any set $\{a_1, \dots, a_n\}$ of non null complex numbers:

$$\left| \sum_{j=1}^n a_j \right|^2 \leq n \sum_{j=1}^n |a_j|^2$$

with equality if and only if $a_1 = a_2 = \dots = a_n$. When the n numbers a_i , $1 \leq i \leq n$, are real, the Cauchy-Schwartz inequality is written as

$$\left(\sum_{i=1}^n a_i \right)^2 \leq n \cdot \sum_{i=1}^n a_i^2.$$

It follows from (1.2) that for a given $x \in V$, if $\varphi(y) - \varphi(x)$ is real $\forall y \sim x$, we have

$$\sum_{y \sim x} (\varphi(y) - \varphi(x))^2 = \frac{\gamma(x)}{d(x)} \left(\sum_{y \sim x} (\varphi(y) - \varphi(x)) \right)^2 \leq \gamma(x) \sum_{y \sim x} (\varphi(y) - \varphi(x))^2.$$

In particular, for there to exist a non-constant *real* solution φ to (1.2), necessarily $\gamma(x) \geq 1$ $\forall x \in V$. If $\gamma(x) < 1$, for some $x \in V$, then there exists a vertex $y \sim x$ such that $\varphi(y) - \varphi(x)$

is a complex number, with $\text{Im}(\varphi(y) - \varphi(x)) \neq 0$. We allow $\gamma(x)$ to take the value $-\infty$ when $\sum_{y \sim x} (\varphi(y) - \varphi(x)) = 0$.

1.4.2 Invariant frameworks in \mathbb{R}^n and geometric spectrum of a graph

We consider the Euclidean space \mathbb{R}^N , ($N \in \mathbb{N}$, $N \geq 2$).

Definition 1.4.2. A *framework* in the Euclidean space \mathbb{R}^N , is a graph $G = (V, E)$ realized as a subset of this space, where edges $vw \in E$ are straight lines segments joining the vertices $v, w \in V$.

Let $G = (V, E)$ be a framework in \mathbb{R}^N .

Definition 1.4.3. The *geometric spectrum* of G is the set of all functions $\gamma : V \rightarrow [-\infty, 1]$, $\gamma(x) < 1$ if $d(x) \geq 3$, such that there exists a non constant function $\varphi : V \rightarrow \mathbb{C}$, satisfying equation (1.2).

Definition 1.4.4. A framework $G = (V, E)$ in \mathbb{R}^N is said to be *invariant* if there exists a function γ lying in its geometric spectrum, such that this framework satisfies the quadratic equation (1.2) with φ being the restriction to its vertices of an orthogonal projection in \mathbb{C} , independently of any similarity transformation in \mathbb{R}^n of this framework.

The quadratic equation (1.2) has been introduced in [3], where invariant frameworks were actually studied. All we had at first was the *Theorem of Axonometry of Gauss*:

Theorem 1.4.5. ([16]) *Suppose that a cube in \mathbb{R}^3 is orthogonally projected into \mathbb{R}^2 and normalised so that a particular vertex v_0 is mapped to the origin. If α , β , and γ are the images of the three neighbouring vertices of v_0 on the complex plane \mathbb{C} , then*

$$\alpha^2 + \beta^2 + \gamma^2 = 0. \tag{1.4}$$

Conversely, if equation (1.4) is satisfied, then one can find a cube whose orthogonal projection is given this way.

Proof. First, let's consider a cube in \mathbb{R}^3 of sides' length ℓ , and a vertex v_0 of the cube with neighboring vertices v_1 , v_2 and v_3 , normalized such that v_0 is mapped to the origin. The three vectors $V_1 = (x_1, y_1, z_1)$, $V_2 = (x_2, y_2, z_2)$ and $V_3 = (x_3, y_3, z_3)$ in \mathbb{R}^3 corresponding to the coordinates of the three vertices v_1 , v_2 and v_3 , are the columns of a 3×3 matrix A .

Since V_1 , V_2 and V_3 are pairwise orthonormal, then we can verify that $A^T.A = \ell.I_3$, where

I_3 is the 3×3 identity matrix, that is

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} = \begin{pmatrix} \ell & 0 & 0 \\ 0 & \ell & 0 \\ 0 & 0 & \ell \end{pmatrix}. \quad (1.5)$$

Equation (1.5) then implies that $A \cdot A^T = \ell \cdot I_3$, so that

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} \cdot \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix} = \begin{pmatrix} \ell & 0 & 0 \\ 0 & \ell & 0 \\ 0 & 0 & \ell \end{pmatrix}. \quad (1.6)$$

Henceforth, we have

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2 = \ell \quad \text{and} \quad x_1 y_1 + x_2 y_2 + x_3 y_3 = 0, \quad (1.7)$$

(i.e. the vectors (x_1, x_2, x_3) and (y_1, y_2, y_3) are orthonormal.)

Writing $\alpha = x_1 + iy_1$, $\beta = x_2 + iy_2$ and $\gamma = x_3 + iy_3$, equation (1.4) is then satisfied.

Conversely, given three complex numbers $\alpha = x_1 + iy_1$, $\beta = x_2 + iy_2$ and $\gamma = x_3 + iy_3$ for which equation (1.4) is satisfied, then equation (1.7) is satisfied, where $\ell = \|X\| = \|Y\|$, $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3)$.

The vectors X and Y are then orthonormal in \mathbb{R}^2 , so if we consider the vector $Z = (z_1, z_2, z_3)$ in \mathbb{R}^3 , where

$$Z = \frac{X \wedge Y}{\ell},$$

the vectors X, Y and Z form an orthonormal basis in \mathbb{R}^3 , so equation (1.6) is satisfied.

Hence, by multiplying (1.6) by A^T on the left side, A on the right side, equation (1.5) is also satisfied, and the vectors $V_1 = (x_1, y_1, z_1)$, $V_2 = (x_2, y_2, z_2)$ and $V_3 = (x_3, y_3, z_3)$ in \mathbb{R}^3 are pairwise orthonormal, hence α , β and γ are the orthogonal projection in \mathbb{R}^2 of three vertices v_1 , v_2 and v_3 of a cube, neighboring some vertex v_0 of the cube. \square

It can be noticed that any regular polytope in Euclidean space \mathbb{R}^3 can provide an example of an invariant framework, with γ varying from polytope to polytope. For example, for the cube, $\gamma = 0$, and it is a result of Theorem 1.4.5:

Theorem 1.4.6. *Let $G = (V, E)$ be a framework in \mathbb{R}^3 . If G is a cube then it is an invariant framework, satisfying the quadratic equation (1.2) for $\gamma = 0$.*

Proof. Consider a vertex v_0 of the cube with neighboring vertices v_1 , v_2 and v_3 , normalized such that v_0 is mapped to the origin. Let the three vectors $V_1 = (x_1, y_1, z_1)$, $V_2 = (x_2, y_2, z_2)$

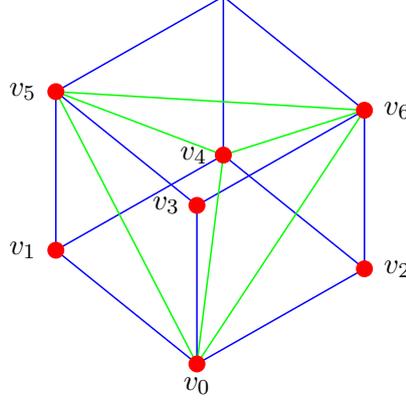


Figure 1.1: A cube and a tetrahedron in \mathbb{R}^3

and $V_3 = (x_3, y_3, z_3)$ in \mathbb{R}^3 be the coordinates of the three vertices v_1 , v_2 and v_3 . Let φ be the orthogonal projection from \mathbb{R}^3 in \mathbb{C} , restricted to the vertices of the cube. Vertex v_0 being at the origin, we have $\varphi(v_0) = 0$. By Theorem 1.4.5, for $\alpha = \varphi(v_1) = x_1 + iy_1$, $\beta = \varphi(v_2) = x_2 + iy_2$, and $\gamma = \varphi(v_3) = x_3 + iy_3$ we have

$$\alpha^2 + \beta^2 + \gamma^2 = 0.$$

Hence equation (1.2) on vertex v_0 , which has the form:

$$\begin{aligned} & \frac{\gamma(v_0)}{3} (\varphi(v_1) - \varphi(v_0) + \varphi(v_2) - \varphi(v_0) + \varphi(v_3) - \varphi(v_0))^2 \\ &= (\varphi(v_1) - \varphi(v_0))^2 + (\varphi(v_2) - \varphi(v_0))^2 + (\varphi(v_3) - \varphi(v_0))^2 \end{aligned}$$

is satisfied for $\gamma = 0 = \text{constant}$. Thus, the cube is an invariant framework in \mathbb{R}^3 . \square

A second example is the tetrahedron, for which we have $\gamma = 3/4$:

Theorem 1.4.7. *Let $G = (V, E)$ be a framework in \mathbb{R}^3 . If G is a tetrahedron then it is an invariant framework, satisfying the quadratic equation (1.2) for $\gamma = 3/4$.*

Proof. We will picture the tetrahedron as being inside a cube and use Theorem 1.4.5.

Consider a vertex v_0 of the cube with neighboring vertices v_1 , v_2 and v_3 , normalized such that v_0 is mapped to the origin. Let the three vectors $V_1 = (x_1, y_1, z_1)$, $V_2 = (x_2, y_2, z_2)$ and $V_3 = (x_3, y_3, z_3)$ in \mathbb{R}^3 be the coordinates of the three vertices v_1 , v_2 and v_3 .

Let v_4 , v_5 and v_6 be the vertices of the cube of coordinates $V_4 = (x_4, y_4, z_4)$, $V_5 = (x_5, y_5, z_5)$ and $V_6 = (x_6, y_6, z_6)$ respectively, such that $V_4 = V_1 + V_2$, $V_5 = V_1 + V_3$ and $V_6 = V_2 + V_3$, and construct the tetrahedron on the vertices v_0, v_4, v_5 and v_6 , as illustrated in

Figure 1.1. We have:

$$V_4 = (x_1+x_2, y_1+y_2, z_1+z_2), V_5 = (x_1+x_3, y_1+y_3, z_1+z_3), \text{ and } V_6 = (x_2+x_3, y_2+y_3, z_2+z_3).$$

Let φ be the orthogonal projection from \mathbb{R}^3 in \mathbb{C} , restricted to the vertices of the cube. Vertex v_0 being at the origin, we have $\varphi(v_0) = 0$. Then for $\alpha = \varphi(v_1) = x_1 + iy_1$, $\beta = \varphi(v_2) = x_2 + iy_2$, and $\gamma = \varphi(v_3) = x_3 + iy_3$, we have

$$\varphi(v_4) = \varphi(v_1) + \varphi(v_2) = \alpha + \beta,$$

$$\varphi(v_5) = \varphi(v_1) + \varphi(v_3) = \alpha + \gamma,$$

$$\varphi(v_6) = \varphi(v_2) + \varphi(v_3) = \beta + \gamma.$$

Moreover, by Theorem 1.4.5 we have

$$\alpha^2 + \beta^2 + \gamma^2 = 0.$$

Now, equation (1.2) on vertex v_0 of the tetrahedron, has the form:

$$\begin{aligned} & \frac{\gamma(v_0)}{3} (\varphi(v_3) - \varphi(v_0) + \varphi(v_4) - \varphi(v_0) + \varphi(v_5) - \varphi(v_0))^2 \\ &= (\varphi(v_3) - \varphi(v_0))^2 + (\varphi(v_5) - \varphi(v_0))^2 + (\varphi(v_5) - \varphi(v_0))^2, \end{aligned}$$

so we have:

$$\frac{\gamma(v_0)}{3} (2\alpha + 2\beta + 2\gamma)^2 = (\alpha + \beta)^2 + (\alpha + \gamma)^2 + (\beta + \gamma)^2.$$

Since $\alpha^2 + \beta^2 + \gamma^2 = 0$, we can verify that equation (1.2) is satisfied for $\gamma = 3/4 = \text{constant}$, hence the tetrahedron is an invariant framework in \mathbb{R}^3 . \square

The above examples of the regular polytopes can be deduced from [10], where Eastwood and Penrose studied frameworks arising from the 1-skeleton of a regular simplex in the Euclidean space \mathbb{R}^3 , showing that equation (1.2) is satisfied.

Then, this property is generalized to other invariant frameworks by Baird [3]: by methods of linear algebra, he addresses the problem of when a given graph can be realized as an invariant framework in an Euclidean space of dimension greater than or equal to 3. For example, some of these frameworks are called the *configured stars*, which are some special cases of the stars, where a star consists of an internal vertex v_0 adjacent to n external vertices v_1, v_2, \dots, v_n , with no other connections. Every star is invariant at each of its external vertices of degree 1, with γ taking on the value 1. However, it is not always the case at the internal vertex v_0 . But each configured star is invariant:

Theorem 1.4.8. ([3]) Let $W = (\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n)$ define a configured star in \mathbb{R}^N and let $\varphi : \mathbb{R}^N \rightarrow \mathbb{C}$ be the orthogonal projection $\varphi(y_1, y_2, \dots, y_N) = y_1 + iy_2$. Then if $z_\ell = \varphi(\vec{x}_\ell) = x_{\ell 1} + ix_{\ell 2}$, we have:

$$\frac{\sigma}{n(\sigma + \rho)} \left(\sum_{\ell=1}^n z_\ell \right)^2 = \sum_{\ell=1}^n z_\ell^2,$$

where σ and ρ are real constants, called the star invariants. In particular, $\gamma = \sigma/(\sigma + \rho)$ is real and depends only on the star invariants.

On the other hand, equation (1.2) arises from its smooth counterpart for a hypersurface in Euclidean space [4]: When one considers an orthogonal projection of a smooth hypersurface in the Euclidean space, remarkably we find the same phenomena, namely that a smooth version of the equation is satisfied independently of the projection, with $\gamma = -1/H^2$ where H is the mean curvature:

Theorem 1.4.9. [4] Let M^n be a smooth hypersurface in \mathbb{R}^{n+1} ($n \geq 1$) and let g denote the metric in M^n induced from the standard metric on \mathbb{R}^n . Let $\varphi : (M^n, g) \rightarrow \mathbb{C}$ be any orthogonal projection, then:

$$(\Delta\varphi)^2 = -H^2(\nabla\varphi)^2,$$

where H is the mean curvature of M^n .

1.5 Quadratic cyclic sequences and cyclic graphs

In this section, we will give some definitions and properties that we will use in Chapter 4 in order to study solutions to equation (1.2) associated to cyclic graphs.

Definition 1.5.1. A cyclic graph $C = (V, E)$ is a finite connected graph with $d(x) = 2$, $\forall x \in V$.

That is, as defined in Chapter 1, it's a graph where we can label the vertices such that $V = \{v_1, v_2, \dots, v_N\}$ and $E = \{v_i v_{i+1}, 1 \leq i \leq N-1\} \cup \{v_1 v_N\}$.

Let us rewrite equation (1.2) for such a graph. Let $C = (V, E)$ be a cyclic graph on N vertices, and set $V = \{0, 1, 2, \dots, N-1\}$. Then, $\forall j \in \mathbb{Z}/N\mathbb{Z}$, the vertex j has $j-1$ and $j+1$ as neighbors, and equation (1.2) has the form

$$\frac{\gamma(j)}{2} (\varphi(j) - \varphi(j-1) + \varphi(j) - \varphi(j+1))^2 = (\varphi(j) - \varphi(j-1))^2 + (\varphi(j) - \varphi(j+1))^2 \quad \forall j \in \mathbb{Z}/N\mathbb{Z}. \quad (1.8)$$

In this thesis, we are interested in the case when $\gamma : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{R}$ is constant.

Definition 1.5.2. For a given integer $N \geq 2$, a *quadratic cyclic sequence (QCS) of order N* is a function $\varphi : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ satisfying the quadratic difference relation

$$\frac{\gamma}{2}(2\varphi(j) - \varphi(j-1) - \varphi(j+1))^2 = (\varphi(j) - \varphi(j-1))^2 + (\varphi(j) - \varphi(j+1))^2 \quad \forall j \in \mathbb{Z}/N\mathbb{Z} \quad (1.9)$$

for some real number γ where $j \pm 1$ are calculated modulo N .

If we define the *increment* $u_j = \varphi(j+1) - \varphi(j)$, then the above equation reads

$$\frac{\gamma}{2}(u_j - u_{j-1})^2 - u_j^2 - u_{j-1}^2 = 0. \quad (1.10)$$

We recall that equation (1.8) is invariant (for fixed γ) under affine linear transformations and conjugation in the complex plane, which we refer to as *normalization*:

$$\varphi \mapsto a\varphi + b \quad \text{and} \quad \varphi \mapsto \bar{\varphi}, \quad \forall a, b \in \mathbb{C} \text{ with } a \neq 0. \quad (1.11)$$

Since there exists such a transformation mapping any pair of distinct points to any other pair of distinct points, we can normalize a QCS so that two distinct terms take on two distinct specified values.

Equation (1.8) is also invariant under *cyclic permutations* and *order reversal* of the sequence $S = (\varphi(0), \varphi(1), \dots, \varphi(N-1))$, where a cyclic permutation of S is one of the form $(\varphi(i), \varphi(i+1), \dots, \varphi(N-1), \varphi(0), \dots, \varphi(i-1))$ for some $0 \leq i \leq N-1$, and the order reversal of S has the form $(\varphi(N-1), \varphi(N-2), \dots, \varphi(1), \varphi(0))$.

As we mentioned previously, using the Cauchy-Schwartz inequality, it follows from (1.10) that for a given $j \in \{0, 1, \dots, N-1\}$, if u_j, u_{j-1} are both real, we have

$$u_j^2 + u_{j-1}^2 = \frac{\gamma}{2}(u_j - u_{j-1})^2 \leq \gamma(u_j^2 + u_{j-1}^2).$$

In particular, for there to exist a non-constant *real* solution to (1.9), we necessarily have $\gamma \geq 1$. Equally, if $\gamma < 1$, then for any three successive terms of the QCS, at least one must be complex with non zero imaginary part.

An example of an integer QCS of order 10 is given by

$$(0, 9, 3, 12, 6, 10, 4, 8, 2, 6)$$

This satisfies (1.9) with $\gamma = 26/25$. On applying the normalization (1.11), a QCS is defined up to addition and multiplication by a constant. Given a rational sequence, we may therefore multiply through by the smallest common multiple of the denominators, subtract the value

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of the first term and finally divide by any common factor in the subsequent numerators, to obtain an integer sequence $\varphi : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{N}$ with $\varphi(0) = 0$ and with no common factor. Even then it may not be unique, for example the following is another sequence of order 10 with $\gamma = 26/25$:

$$(0, 9, 3, 7, 1, 10, 4, 8, 2, 6) .$$

Let's reconsider the case when the function γ is not necessarily constant, and consider a cyclic framework in \mathbb{R}^2 . We want to find the conditions on this framework in order for it to be invariant.

Since we are in the plane \mathbb{R}^2 , any orthogonal projection φ of this framework in \mathbb{C} is the identity, and all its similarity transformations are given by the normalization (1.11), so the function γ in equation (1.8) is invariant under any similarity transformation. So to characterize an invariant cyclic framework in \mathbb{R}^2 , we only have to find the conditions on that framework so that γ belongs to its geometric spectrum, that is, γ takes values in $[-\infty, 1]$.

We have the following characterization: a cyclic framework in \mathbb{R}^2 is invariant if and only if it is embedded in the plane with sides of equal length. In fact, we have the following proposition:

Proposition 1.5.3. *The edges $[\varphi(j), \varphi(j+1)]$, $j \in \mathbb{Z}/N\mathbb{Z}$, of a polygon corresponding to a solution $\varphi : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ of (1.8), with $\gamma : \mathbb{Z}/N\mathbb{Z} \rightarrow [-\infty, 1]$, all have the same length.*

Proof. Consider three successive non-identical terms $(\varphi(j-1), \varphi(j), \varphi(j+1))$ of a polygon corresponding to a solution $\varphi : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ that satisfies equation (1.8) for $\gamma : \mathbb{Z}/N\mathbb{Z} \rightarrow [-\infty, 1]$. By normalization we can suppose that $\varphi(j) = 0$ and $\varphi(j-1) = 1$. Suppose that $\varphi(j+1) = z = x + iy$, x and y real numbers. At the term j , equation (1.8) takes the form:

$$\frac{\gamma(j)}{2}(1+z)^2 = 1+z^2 .$$

Suppose that $z \neq -1$. Then the requirement that $\gamma(j)$ be real is equivalent to

$$\text{either } \operatorname{Im}(z) = 0 \quad \text{or} \quad |z| = 1 .$$

In fact, since $z \neq -1$, then:

$$\frac{\gamma(j)}{2}(1+z)^2 = 1+z^2 \Leftrightarrow \gamma(j) = 2 - \frac{4z}{1+z^2+2z} .$$

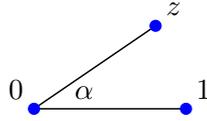
By replacing z with $x + iy$, a short calculation shows that:

$$\gamma(j) \text{ is real} \Leftrightarrow y = 0 \text{ or } 1 - x^2 - y^2 = 0,$$

1.5. QUADRATIC CYCLIC SEQUENCES AND CYCLIC GRAPHS

hence $\text{Im}(z) = 0$ or $|z| = 1$. If z is real and $z \neq \pm 1$, then $\gamma(j) > 1$, which is not allowed. Otherwise, we always have $|z| = 1$, and the result follows.

Conversely, consider a function $\varphi : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$, and a closed polygon in \mathbb{C} whose edges are $[\varphi(j), \varphi(j+1)]$, $j \in \mathbb{Z}/N\mathbb{Z}$, such that they all have the same length ℓ . Normalize such that $\ell = 1$. Also suppose by normalization that $\varphi(j-1) = 0$, $\varphi(j) = 1$ and $\varphi(j+1) = z$.



If $z \neq -1$, then $\varphi(j-1)$, $\varphi(j)$ and $\varphi(j+1)$ satisfy equation (1.8) for

$$\gamma(j) = \frac{2(1+z^2)}{(1+z)^2},$$

where $\gamma(j) \in]-\infty, 1]$. In fact, since $|z| = 1$, replacing z by $e^{i\alpha}$, $\alpha \in [0, \pi]$ being the absolute angle between the two edges $[\varphi(j-1), \varphi(j)]$ and $[\varphi(j), \varphi(j+1)]$, we get

$$\gamma(j) = \frac{2 \cos \alpha}{1 + \cos \alpha} = \frac{2 \cos \theta}{\cos \theta - 1} \quad (1.12)$$

where $\theta = \pi - \alpha$ is the exterior angle, so γ is real and clearly belongs to $]-\infty, 1]$. The two limiting cases $\alpha = 0$ and $\alpha = \pi$ correspond to $\gamma(j) = 1$ and $\gamma(j) = -\infty$, respectively.

As a consequence, a solution $\varphi : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ of (1.8) with $\gamma : \mathbb{Z}/N\mathbb{Z} \rightarrow]-\infty, 1]$ corresponds to a polygon in the plane with sides of equal length. \square

In our work, we are interested in such polygons corresponding to a complex QCS, that is, we suppose henceforth that γ is constant.

So, we can picture a complex QCS with $\gamma \in]-\infty, 1]$ as a closed polygon in the plane with edges the straight line segments $[\varphi(j), \varphi(j+1)]$, each having the same length and where the absolute angle α between each two consecutive edges is constant, and we have

$$\gamma = \frac{2 \cos \alpha}{1 + \cos \alpha}.$$

An example of a complex QCS with $\gamma = 2/3$ of order 6 is given by

$$\left(0, 1, \frac{1}{2} - \frac{\sqrt{3}}{2}i, 0, 1, \frac{1}{2} + \frac{\sqrt{3}}{2}i\right).$$

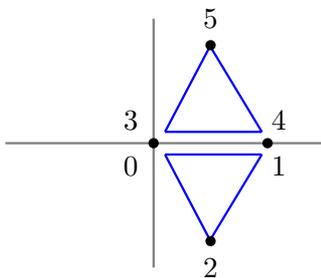


Figure 1.2: Complex cyclic sequence

It can be represented by a polygon illustrated in Fig.1.2, where we label the vertices in sequential order. We can see that all edges have same length with constant absolute angle $\alpha = \frac{\pi}{3}$ between them.

Chapter 2

On the number of oriented Hamiltonian paths and cycles in a tournament and its complement

Rosenfeld [29] proved in 1974 that in a tournament T , the number of antidirected Hamiltonian paths starting with a forward arc is equal to the number of antirected Hamiltonian paths starting with a backward arc.

In §2.1 of this chapter, we generalize Rosenfeld's result, proving that in a tournament T , the number of oriented Hamiltonian paths of a given type $P(\alpha)$ is equal to the number of oriented Hamiltonian paths of type $P(-\alpha)$. That way, we would have shown that every tournament T and its complement \bar{T} contain the same number of oriented Hamiltonian paths of any given type.

In §2.2, and given any tournament T , we find a relation between the number of oriented Hamiltonian paths of some type $P(\alpha)$ in T and the number of oriented Hamiltonian cycles of types $C(\beta)$ and $C(\beta')$ in T , where $C(\beta)$ and $C(\beta')$ are the two types of cycles that can be generated by a path of type $P(\alpha)$ in T .

In §2.3, and using both results of §2.1 and §2.2, we are able to establish the main result of §2.1 for oriented cycles: We prove that a tournament and its complement contain the same number of oriented Hamiltonian cycles of any given type.

In §2.4 of this chapter, since a digraph of maximal degree $\Delta \leq 2$ is a union of oriented paths and cycles, we are able to generalize the two main results of §2.1 and §2.3 to any digraph of maximal degree 2: Let T be a tournament and H a digraph with maximal degree $\Delta(G(H)) \leq 2$. The number of copies of H in T and in \bar{T} is the same.

2.1 OHP in T and \overline{T} : a generalization of Rosenfeld's result

Rosenfeld's result [29] given in 1974 can be stated as follows:

Theorem 2.1.1. *Let T be a tournament of order n . We have:*

$$f_T(\underbrace{1, -1, 1, \dots, \pm 1}_{n-1 \text{ components}}) = f_T(\underbrace{-1, 1, \dots, \mp 1}_{n-1 \text{ components}}).$$

In this section, we generalize Rosenfeld's result, by proving the following:

Theorem 2.1.2. *Let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{K}_s$, $\alpha_1 \geq 0$, and let T be a tournament of order n ; $n = \sum_{i=1}^s |\alpha_i| + 1$. We have:*

$$f_T(\alpha) = f_T(-\alpha).$$

Before going through the proof of our statement, we will give some related results and propositions:

Let T be a tournament of order n and let \mathcal{P}_T and \mathcal{E}_T be the sets of all oriented Hamiltonian paths and of all enumerations on n vertices of T respectively.

Proposition 2.1.3. *We have:*

$$|\mathcal{E}_T| = 2 \cdot |\mathcal{P}_T|.$$

Proof. Each enumeration $v_1 v_2 \dots v_n$ of the n vertices of T corresponds to an oriented Hamiltonian path $P = v_1 v_2 \dots v_n$ in T .

Let Φ be the correspondence:

$$\Phi : \mathcal{E}_T \longrightarrow \mathcal{P}_T,$$

such that $\forall E = v_1 v_2 \dots v_n \in \mathcal{E}_T$, $\Phi(E)$ is the path $P = v_1 v_2 \dots v_n$.

Obviously, Φ is a surjective mapping. Let $E = v_1 v_2 \dots v_n \in \mathcal{E}_T$ and let's find all the other enumerations $E' = v'_1 v'_2 \dots v'_n$ such that $\Phi(E') = \Phi(E)$.

Let $P = \Phi(E)$ and $P' = \Phi(E')$. Since $\Phi(E') = \Phi(E)$ thus $P = v_1 v_2 \dots v_n = v'_1 v'_2 \dots v'_n = P'$, which implies that $P = P'$ with $(v'_1, v'_2, \dots, v'_n) = (v_1, v_2, \dots, v_n)$ or $(v'_1, v'_2, \dots, v'_n) = (v_n, v_{n-1}, \dots, v_1)$. Thus $E' = v_1 v_2 \dots v_n = E$ and $E' = v_n v_{n-1} \dots v_1$ verify $\Phi(E') = \Phi(E)$, and they are unique.

Thus we deduce that for every oriented Hamiltonian path in \mathcal{P}_T corresponds two enumerations E and E' in \mathcal{E}_T such that $\Phi(E) = \Phi(E') = P$, and the result follows. \square

Let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s$; $\alpha_i \cdot \alpha_{i+1} < 0$, $\alpha_1 \geq 0$, such that $\sum_{i=1}^s |\alpha_i| = n - 1$.

Proposition 2.1.4. *If α is symmetric, then we have $|\mathcal{E}_T(\alpha)| = 2 \cdot |\mathcal{P}_T(\alpha)|$, while $|\mathcal{E}_T(\alpha)| = |\mathcal{P}_T(\alpha)|$ otherwise.*

Proof. Consider the correspondence:

$$\Phi_\alpha : \mathcal{E}_T(\alpha) \longrightarrow \mathcal{P}_T(\alpha),$$

such that $\Phi_\alpha(E) = \Phi(E)$, $\forall E \in \mathcal{E}_T(\alpha)$.

The mapping Φ_α is clearly surjective. In fact let $E, E' \in \mathcal{E}_T(\alpha)$, $E = v_1 v_2 \dots v_n$ and $E' = v'_1 v'_2 \dots v'_n$, with $\Phi(E) = P = v_1 v_2 \dots v_n$ and $\Phi(E') = P' = v'_1 v'_2 \dots v'_n$. If $E = E'$ then $(v_1, v_2, \dots, v_n) = (v'_1, v'_2, \dots, v'_n)$, thus $P = P'$ so Φ' is an application. Let $P = v_1 v_2 \dots v_n \in \mathcal{P}_T(\alpha)$ then $P \in \mathcal{P}_T(\alpha)$ with respect to the enumeration $E = v_1 v_2 \dots v_n$ or $P \in \mathcal{P}_T(\alpha)$ with respect to the enumeration $E' = v_n v_{n-1} \dots v_1$, so in both cases we can find an enumeration in $\mathcal{E}_T(\alpha)$ such that $\Phi'(E) = P$, thus Φ' is surjective.

Let $E = v_1 v_2 \dots v_n \in \mathcal{E}_T(\alpha)$ and let's find all the other enumerations $E' = v'_1 v'_2 \dots v'_n \in \mathcal{E}_T(\alpha)$ such that $\Phi_\alpha(E') = \Phi_\alpha(E)$.

Let $P = \Phi_\alpha(E)$ and $P' = \Phi_\alpha(E')$. Since $\Phi_\alpha(E') = \Phi_\alpha(E)$ thus $P = v_1 v_2 \dots v_n = v'_1 v'_2 \dots v'_n = P'$, so $P = P'$ with $(v'_1, v'_2, \dots, v'_n) = (v_1, v_2, \dots, v_n)$ or $(v'_1, v'_2, \dots, v'_n) = (v_n, v_{n-1}, \dots, v_1)$. The first case implies that $E' = v_1 v_2 \dots v_n = E$, while for the second case, $E' = v_n v_{n-1} \dots v_1$, but we need to verify that $E' = v_n v_{n-1} \dots v_1 \in \mathcal{E}_T(\alpha)$. Since $E = v_1 v_2 \dots v_n \in \mathcal{E}_T(\alpha)$ then $E' = v_n v_{n-1} \dots v_1 \in \mathcal{E}_T(-\bar{\alpha})$, so if α is symmetric, then by Proposition 1.3.28, $\mathcal{E}_T(-\bar{\alpha}) = \mathcal{E}_T(\alpha)$ thus $E' = v_n v_{n-1} \dots v_1 \in \mathcal{E}_T(\alpha)$, while if α is not symmetric, then by Proposition 1.3.28, $\mathcal{E}_T(-\bar{\alpha}) \neq \mathcal{E}_T(\alpha)$ which implies that $\mathcal{E}_T(-\bar{\alpha}) \cap \mathcal{E}_T(\alpha) = \emptyset$ by Proposition 1.3.29, thus $E' = v_n v_{n-1} \dots v_1 \notin \mathcal{E}_T(\alpha)$, so the second case is impossible for α not symmetric.

As a result, $\forall P \in \mathcal{P}_T(\alpha)$, $|\Phi_\alpha^{-1}(P)| = 1$ if α is not symmetric, and $|\Phi_\alpha^{-1}(P)| = 2$ otherwise. Now, if $P \neq P'$, then $\Phi_\alpha^{-1}(P) \cap \Phi_\alpha^{-1}(P') = \emptyset$, hence:

$$|\mathcal{E}_T(\alpha)| = \sum_{P \in \mathcal{P}_T(\alpha)} |\Phi_\alpha^{-1}(P)| \text{ which is equal to } \sum_{P \in \mathcal{P}_T(\alpha)} 1 = |\mathcal{P}_T(\alpha)| \text{ if } \alpha \text{ not symmetric, and is equal to } \sum_{P \in \mathcal{P}_T(\alpha)} 2 = 2 \cdot |\mathcal{P}_T(\alpha)| \text{ otherwise, and the result follows. } \quad \square$$

We may now give the **proof of Theorem 2.1.2**:

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Proof. First of all remark that if α is symmetric, so is $-\alpha$ and vice versa. Thus to prove that $f_T(\alpha) = f_T(-\alpha)$, and using Proposition 2.1.4, it is enough to prove that $e_T(\alpha) = e_T(-\alpha)$.

The proof will be done by induction on s .

If $s = 1$, $\alpha = (\alpha_1) = (n - 1)$ and $-\alpha = (-\alpha_1) = (1 - n)$. Since every directed Hamiltonian path $P = v_1 v_2 \dots v_n$ in T corresponds to two enumerations $E = v_1 v_2 \dots v_n$ and $E' = v_n v_{n-1} \dots v_1$ of types $E(\alpha) = E(n - 1)$ and $E(-\alpha) = E(1 - n)$ respectively, and vice versa, thus $|\mathcal{E}_T(\alpha)| = |\mathcal{E}_T(-\alpha)| = |\mathcal{P}_T(\alpha)|$ and we have $e_T(\alpha) = e_T(-\alpha)$.

Suppose that the result is true when α has s components, i.e. if $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s$; $\alpha_i \cdot \alpha_{i+1} < 0$, $\alpha_1 \geq 0$, and T is a tournament of order $n = \sum_{i=1}^s |\alpha_i| + 1$, we have: $e_T(\alpha_1, \dots, \alpha_s) = e_T(-\alpha_1, \dots, -\alpha_s)$, and let's prove the result for $s + 1$ components.

Let $\alpha = (\alpha_1, \dots, \alpha_s, \alpha_{s+1}) \in \mathbb{Z}^{s+1}$; $\alpha_i \cdot \alpha_{i+1} < 0$, $\alpha_1 \geq 0$, and T be a tournament of order $n = \sum_{i=1}^{s+1} |\alpha_i| + 1$.

We argue by induction on α_1 . If $\alpha_1 = 0$, then by the previous induction, $e_T(0, \alpha_2, \dots, \alpha_{s+1}) = e_T(\alpha_2, \dots, \alpha_{s+1}) = e_T(-\alpha_2, \dots, -\alpha_{s+1}) = e_T(0, -\alpha_2, \dots, -\alpha_{s+1})$.

So suppose that $\alpha_1 > 0$, and that the result is true when the first component is equal to $\alpha_1 - 1$, and let's prove it when the first component is equal to α_1 .

Let $X \subseteq V(T)$ such that $|X| = \alpha_1$. Set $T' = T - X$, and define the following sets:

$$A_X = \mathcal{E}_{\langle X \rangle}(\alpha_1 - 1) \times \mathcal{E}_{T'}(\alpha_2, \dots, \alpha_{s+1}),$$

$$A'_X = \{(E, E') \in A_X; E = v_1 \dots v_{\alpha_1}, E' = v_{\alpha_1+1} \dots v_n \text{ and } (v_{\alpha_1}, v_{\alpha_1+1}) \in E(T)\},$$

$$A''_X = \{(E, E') \in A_X; E = v_1 \dots v_{\alpha_1}, E' = v_{\alpha_1+1} \dots v_n, \text{ and } (v_{\alpha_1+1}, v_{\alpha_1}) \in E(T)\}.$$

Obviously we have: $A'_X \cap A''_X = \emptyset$, and $A_X = A'_X \cup A''_X$, thus

$$|A_X| = |A'_X| + |A''_X|.$$

Define the following two sets:

$$\mathcal{E}_X(\alpha_1, \dots, \alpha_{s+1}) = \{E = v_1 \dots v_{\alpha_1} v_{\alpha_1+1} \dots v_n \in \mathcal{E}_T(\alpha_1, \dots, \alpha_{s+1}); \{v_1, \dots, v_{\alpha_1}\} = X\},$$

$$\mathcal{E}_X(\alpha_1 - 1, \alpha_2 - 1, \alpha_3, \dots, \alpha_{s+1}) = \{E = v_1 \dots v_{\alpha_1} v_{\alpha_1+1} \dots v_n \in \mathcal{E}_T(\alpha_1 - 1, \alpha_2 - 1, \alpha_3, \dots, \alpha_{s+1}); \{v_1, \dots, v_{\alpha_1}\} = X\}.$$

We have that

$$|A'_X| = |\mathcal{E}_X(\alpha_1, \dots, \alpha_{s+1})|,$$

and that

$$|A''_X| = |\mathcal{E}_X(\alpha_1 - 1, \alpha_2 - 1, \alpha_3, \dots, \alpha_{s+1})|.$$

In fact, we just have to consider the correspondence

$$f : \mathcal{E}_X(\alpha_1, \dots, \alpha_{s+1}) \longrightarrow A'_X$$

such that $\forall E = v_1 \dots v_{\alpha_1} v_{\alpha_1+1} \dots v_n \in \mathcal{E}_X(\alpha_1, \dots, \alpha_{s+1})$, $f(E) = (v_1 \dots v_{\alpha_1}, v_{\alpha_1+1} \dots v_n)$, and the correspondence

$$f' : \mathcal{E}_X(\alpha_1 - 1, \alpha_2 - 1, \alpha_3, \dots, \alpha_{s+1}) \longrightarrow A''_X$$

where $\forall E = v_1 \dots v_{\alpha_1} v_{\alpha_1+1} \dots v_n \in \mathcal{E}_X(\alpha_1 - 1, \alpha_2 - 1, \alpha_3, \dots, \alpha_{s+1})$, $f'(E) = (v_1 \dots v_{\alpha_1}, v_{\alpha_1+1} \dots v_n)$, and verify that they are bijective mappings.

As a result,

$$|A_X| = |\mathcal{E}_X(\alpha_1, \dots, \alpha_{s+1})| + |\mathcal{E}_X(\alpha_1 - 1, \alpha_2 - 1, \alpha_3, \dots, \alpha_{s+1})|.$$

Now let's consider $-\alpha = (-\alpha_1, \dots, -\alpha_s)$, and let $X \subseteq V(T)$ such that $|X| = \alpha_1$. Set $T' = T - X$, and let's also define the following sets:

$$B_X = \mathcal{E}_{\langle X \rangle}(-\alpha_1 + 1) \times \mathcal{E}_{T'}(-\alpha_2, \dots, -\alpha_{s+1}),$$

$$B'_X = \{(E, E') \in B_X; E = v_1 \dots v_{\alpha_1}, E' = v_{\alpha_1+1} \dots v_n \text{ and } (v_{\alpha_1+1}, v_{\alpha_1}) \in E(T)\},$$

$$B''_X = \{(E, E') \in B_X; E = v_1 \dots v_{\alpha_1}, E' = v_{\alpha_1+1} \dots v_n \text{ and } (v_{\alpha_1}, v_{\alpha_1+1}) \in E(T)\}.$$

We also have: $B'_X \cap B''_X = \emptyset$, and $B_X = B'_X \cup B''_X$, thus

$$|B_X| = |B'_X| + |B''_X|.$$

Define the two following sets:

$$\mathcal{E}_X(-\alpha_1, \dots, -\alpha_{s+1}) = \{E = v_1 \dots v_{\alpha_1} v_{\alpha_1+1} v_n \in \mathcal{E}_T(-\alpha_1, \dots, -\alpha_{s+1}); \{v_1, \dots, v_{\alpha_1}\} = X\},$$

$$\mathcal{E}_X(-\alpha_1 + 1, -\alpha_2 + 1, -\alpha_3, \dots, -\alpha_{s+1}) = \{E = v_1 \dots v_{\alpha_1} v_{\alpha_1+1} v_n \in \mathcal{E}_T(-\alpha_1 + 1, -\alpha_2 + 1, -\alpha_3, \dots, -\alpha_{s+1}); \{v_1, \dots, v_{\alpha_1}\} = X\}.$$

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Similarly as before, we can prove that

$$\begin{aligned} |B'_X| &= |\mathcal{E}_X(-\alpha_1, \dots, -\alpha_{s+1})|, \\ |B''_X| &= |\mathcal{E}_X(-\alpha_1 + 1, -\alpha_2 + 1, -\alpha_3, \dots, -\alpha_{s+1})|, \end{aligned}$$

Hence

$$|B_X| = |\mathcal{E}_X(-\alpha_1, \dots, -\alpha_{s+1})| + |\mathcal{E}_X(-\alpha_1 + 1, -\alpha_2 + 1, -\alpha_3, \dots, -\alpha_{s+1})|.$$

On the other hand, we have that $|A_X| = e_X(\alpha_1 - 1) \cdot e_{T'}(\alpha_2, \dots, \alpha_{s+1})$, and $|B_X| = e_X(-\alpha_1 + 1) \cdot e_{T'}(-\alpha_2, \dots, -\alpha_{s+1})$, but since we have here less than $s + 1$ blocks, thus by induction, $e_X(\alpha_1 - 1) = e_X(-\alpha_1 + 1)$ and $e_{T'}(\alpha_2, \dots, \alpha_{s+1}) = e_{T'}(-\alpha_2, \dots, -\alpha_{s+1})$, As a result we get:

$$|A_X| = |B_X|.$$

Moreover, $\forall \beta = (\alpha_1, \dots, \alpha_{s+1})$ or $(\alpha_1 - 1, \alpha_2 - 1, \alpha_3, \dots, \alpha_{s+1})$ or $(-\alpha_1, \dots, -\alpha_{s+1})$ or $(-\alpha_1 + 1, -\alpha_2 + 1, -\alpha_3, \dots, -\alpha_{s+1})$, we have:

$$\mathcal{E}_T(\beta) = \sqcup_{X \subseteq V(T); |X|=\alpha_1} \mathcal{E}_X(\beta).$$

(The union is disjoint since if $X \neq X'$, any two enumerations E and E' of the vertices of X and X' respectively, differ).

Since $|A_X| = |B_X|$, then

$$\begin{aligned} & |\mathcal{E}_X(\alpha_1, \dots, \alpha_{s+1})| + |\mathcal{E}_X(\alpha_1 - 1, \alpha_2 - 1, \alpha_3, \dots, \alpha_{s+1})| \\ &= |\mathcal{E}_X(-\alpha_1, \dots, -\alpha_{s+1})| + |\mathcal{E}_X(-\alpha_1 + 1, -\alpha_2 + 1, -\alpha_3, \dots, -\alpha_{s+1})|. \end{aligned}$$

Doing the summation over all the sets $X \subseteq V(T)$, $|X| = \alpha_1$, we have:

$$\begin{aligned} & \sum_{X \subseteq V(T); |X|=\alpha_1} |\mathcal{E}_X(\alpha_1, \dots, \alpha_{s+1})| + \sum_{X \subseteq V(T); |X|=\alpha_1} |\mathcal{E}_X(\alpha_1 - 1, \alpha_2 - 1, \alpha_3, \dots, \alpha_{s+1})| \\ &= \sum_{X \subseteq V(T); |X|=\alpha_1} |\mathcal{E}_X(-\alpha_1, \dots, -\alpha_{s+1})| + \sum_{X \subseteq V(T); |X|=\alpha_1} |\mathcal{E}_X(-\alpha_1 + 1, -\alpha_2 + 1, -\alpha_3, \dots, -\alpha_{s+1})|, \end{aligned}$$

thus,

$$\begin{aligned} & |\mathcal{E}_T(\alpha_1, \dots, \alpha_{s+1})| + |\mathcal{E}_T(\alpha_1 - 1, \alpha_2 - 1, \alpha_3, \dots, \alpha_{s+1})| \\ &= |\mathcal{E}_T(-\alpha_1, \dots, -\alpha_{s+1})| + |\mathcal{E}_T(-\alpha_1 + 1, -\alpha_2 + 1, -\alpha_3, \dots, -\alpha_{s+1})|, \end{aligned}$$

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And this implies that

$$\begin{aligned} & e_T(\alpha_1, \dots, \alpha_{s+1}) + e_T(\alpha_1 - 1, \alpha_2 - 1, \alpha_3, \dots, \alpha_{s+1}) \\ &= e_T(-\alpha_1, \dots, -\alpha_{s+1}) + e_T(-\alpha_1 + 1, -\alpha_2 + 1, -\alpha_3, \dots, -\alpha_{s+1}). \end{aligned}$$

But by induction, since $\alpha_1 - 1 < \alpha_1$, we have that

$$e_T(\alpha_1 - 1, \alpha_2 - 1, \alpha_3, \dots, \alpha_{s+1}) = e_T(-\alpha_1 + 1, -\alpha_2 + 1, -\alpha_3, \dots, -\alpha_{s+1}),$$

So we finally get

$$e_T(\alpha_1, \dots, \alpha_{s+1}) = e_T(-\alpha_1, \dots, -\alpha_{s+1}),$$

which concludes the proof. \square

That way, we showed that every tournament and its complement contain the same number of oriented Hamiltonian paths of any given type.

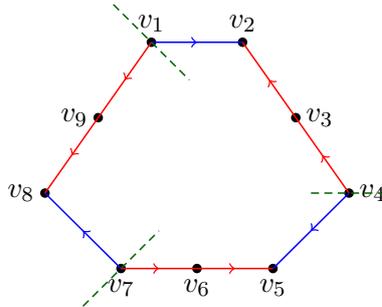
2.2 Oriented cycles and generating paths

Let T be a tournament. In this section we find a relation between $f_T(\alpha)$, $g_T(\beta)$ and $g_T(\beta')$, where $P(\alpha)$ is some type of oriented Hamiltonian paths in T , and $C(\beta)$ and $C(\beta')$ are the two types of cycles that can be generated by a path of type $P(\alpha)$ in the tournament T , (see Remark 1.3.18), and this result will be of great use in the next section.

We first start by the following theorem:

Theorem 2.2.1. *Let $P \in \mathcal{P}_T(\alpha)$ be an oriented Hamiltonian path in a tournament T and let C_P be the cycle generated by P in T , of type $C(\beta)$, such that C_P has at least 2 blocks (i.e. C_P is not a circuit). Then if C_P is non-symmetric, we have $|\bar{P}| = t(\beta)$, while if C_P is symmetric, then $|\bar{P}| = 2.t(\beta)$, where \bar{P} is the equivalence class of P with respect to \mathcal{R}_α .*

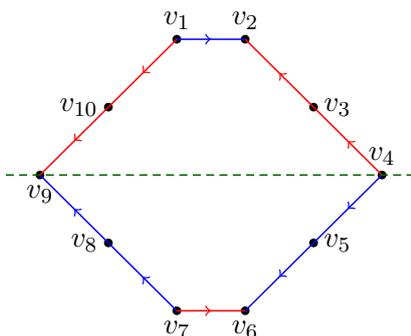
Example 2.2.2. The following cycle C is of type $C(\beta) = C(1, -2, 1, -2, 1, -2)$ with $t(\beta) = 3$:



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The path $v_3v_4 \dots v_9v_1v_2$ generates C and is of type $P(\alpha) = P(-1, 1, -2, 1, -2, 1)$. The paths $v_6 \dots v_9v_1 \dots v_5$ and $v_9v_1 \dots v_8$ are the only other paths generating C of type $P(\alpha)$. We have in total $t(\beta)$ paths of type $P(\alpha)$ generating C .

The following cycle C' is of type $C(\beta) = C(-2, 1, -2, 2, -1, 2)$ with $t(\beta) = 1$, and it is symmetric:



The path $v_9v_{10}v_1 \dots v_8$ generates C' and is of type $P(\alpha) = P(-2, 1, -2, 2, -1, 1)$. The path $v_9v_8 \dots v_1v_{10}$ is the only other path of type $P(\alpha)$ generating C' . We have in total $2.t(\beta)$ paths of type $P(\alpha)$ generating C' .

In [12], one actually proved that if C_P is non-symmetric, then $|\overline{P}| = t(\beta)$. In the following, we will present arguments useful for both the symmetric and the non-symmetric types.

Remark 2.2.3. If $P \in \mathcal{P}_T(\alpha)$ is an oriented Hamiltonian path in a tournament T of order n , and C_P the cycle generated by P in T , such that C_P is a Hamiltonian circuit, then $|\overline{P}| = n$, where \overline{P} is the equivalence class of P with respect to \mathcal{R}_α .

In fact, since C_P is a circuit, then P must be a directed path, also every Hamiltonian circuit is generated by exactly n directed Hamiltonian paths, starting each from a vertex of C_P . Note that if C_P is a circuit, say of type $C(\beta)$, (β in this case has 1 component), then $t(\beta) = 1$.

In order to prove Theorem 2.2.1, we first give the three following lemmas:

Lemma 2.2.4. *Let T be a tournament of order n , and let $C = v_1v_2 \dots v_n$ be a Hamiltonian cycle in T . Then C is symmetric if and only if $\forall 1 \leq i \leq n$, and for every Hamiltonian path $P = v_iv_{i+1} \dots v_nv_1 \dots v_{i-1}$, there exists $1 \leq i' \leq n$ such that $P = v_iv_{i+1} \dots v_nv_1 \dots v_{i-1}$ and $P' = v_iv_{i'-1} \dots v_1v_nv_{n-1} \dots v_{i'+1}$ have the same type with respect to these enumerations.*

Proof. For the necessary condition, since C is symmetric, we can suppose without loss of generality that $C = v_1v_2 \dots v_n$ is of type $C(\beta_1, \beta_2, \dots, \beta_s) = I_1I_2 \dots I_s$ with respect to this

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enumeration, where β is symmetric. We have $|I_j| = |\beta_j|$, and let $\text{end}(I_j) = \{x_j, y_j\}$, $\forall 1 \leq j \leq s$.

Suppose that $v_i \in I_j$, for some $1 \leq j \leq s$, and suppose without loss of generality that $\beta_j > 0$, (the case $\beta_j < 0$ is similar).

Let $i' = n - (i - 2)$, (assuming that if $i = 1$, $i' = n + 1$ simply denotes $i' = 1$), so $P' = v_{n-(i-2)}v_{n-(i-2)-1} \dots v_1v_n \dots v_{n-(i-2)+1}$. We will show that this value of i' satisfies the necessary condition.

In fact, let $x = l(I_{j[x_j, v_i]})$, then the path $P = v_iv_{i+1} \dots v_nv_1 \dots v_{i-1}$ is of type $P(\beta_j - x, \beta_{j+1}, \dots, \beta_s, \beta_1, \dots, \beta_{j-1}, x - 1)$ with respect to this enumeration.

Since β is symmetric, then $(\beta_1, \beta_2, \dots, \beta_j) = (-\beta_s, -\beta_{s-1}, \dots, -\beta_{s-(j-1)})$, so $|\beta_i| = |\beta_{s-i+1}| \forall 1 \leq i \leq j$, and since $l(C_{[v_1v_2 \dots v_i]}) = l(C_{[v_1v_nv_{n-1} \dots v_{n-(i-2)}]})$, we deduce that $v_{n-(i-2)} \in I_{s-(j-1)}$ and $l(I_{j[x_j, v_i]}) = l(I_{s-(j-1)[y_{s-(j-1)}, v_{n-(i-2)}]) = x$.

As a result, the path $P' = v_{n-(i-2)}v_{n-(i-2)-1} \dots v_1v_nv_{n-(i-2)+1}$ is of type $P(-\beta_{s-(j-1)} - x, -\beta_{s-(j-1)-1}, \dots, -\beta_1, -\beta_s, \dots, -\beta_{s-(j-1)+1}, x - 1)$ with respect to this enumeration.

But $(\beta_1, \beta_2, \dots, \beta_s) = (-\beta_s, -\beta_{s-1}, \dots, -\beta_1)$ (since β is symmetric), so we finally get $P' = v_{n-(i-2)}v_{n-(i-2)-1} \dots v_1v_nv_{n-(i-2)+1}$ is of type $P(\beta_j - x, \beta_{j+1}, \dots, \beta_s, \beta_1, \dots, \beta_{j-1}, x - 1)$ with respect to this enumeration.

For the sufficient condition, suppose to the contrary that C is non-symmetric but $\forall 1 \leq i \leq n$, and for every Hamiltonian path $P = v_iv_{i+1} \dots v_nv_1 \dots v_{i-1}$, there exist some i' , $1 \leq i' \leq n$, such that $P = v_iv_{i+1} \dots v_nv_1 \dots v_{i-1}$ and $P' = v_{i'}v_{i'-1} \dots v_1v_nv_{n-1} \dots v_{i'}$ have the same type with respect to these enumerations.

Suppose without loss of generality that $v_i \in I_1$, and that $\beta_1 > 0$. (The case $\beta_1 < 0$ is similarly treated). $P = v_iv_{i+1} \dots v_nv_1 \dots v_{i-1}$ is of type $P(\beta_1 - x, \beta_2, \dots, \beta_s, x - 1)$ with respect to this enumeration, for some $0 \leq x \leq \beta_1$. Thus $P' = v_{i'}v_{i'-1} \dots v_1v_nv_{n-1} \dots v_{i'+1}$ has the type $P(\beta_1 - x, \beta_2, \dots, \beta_s, x - 1)$ with respect to this enumeration. But the vertex $v_{i'}$ belongs to C , thus $v_{i'}$ belongs to a block I_j of C of length $|\beta_j|$, then $P' = v_{i'}v_{i'-1} \dots v_1v_nv_{n-1} \dots v_{i'+1}$ is of type $P(-\beta_j - y, -\beta_{j-1}, \dots, -\beta_1, -\beta_s, \dots, -\beta_{j+1}, y - 1)$ with respect to this enumeration, where $-\beta_j > 0$ in this case (since we should have $\beta_1 = -\beta_j - y$ and $\beta_1 > 0$), and $0 \leq y \leq -\beta_j$.

We get $(\beta_1 - x, \beta_2, \dots, \beta_s, x - 1) = (-\beta_j - y, -\beta_{j-1}, \dots, -\beta_1, -\beta_s, \dots, -\beta_{j+1}, y - 1)$, thus $x - 1 = y - 1$ so $x = y$. As a result, we have:

$$(\beta_1 - x, \beta_2, \dots, \beta_j, \beta_{j+1}, \dots, \beta_s, x - 1) = (-\beta_j - x, -\beta_{j-1}, \dots, -\beta_1, -\beta_s, \dots, -\beta_{j+1}, x - 1). \quad (2.1)$$

Equation (2.1) implies that $\beta_1 - x = -\beta_j - x$ and $\forall 2 \leq p \leq j$, $\beta_p = -\beta_{j-(p-1)}$, thus $\forall 1 \leq p \leq j$, $\beta_p = -\beta_{j-(p-1)}$, that is $\beta' = (\beta_1, \beta_2, \dots, \beta_j) = (-\beta_j, -\beta_{j-1}, \dots, -\beta_1)$, which means that β' is symmetric, and we can write β' as $(\beta_1, \beta_2, \dots, \beta_{\frac{j}{2}}, -\beta_{\frac{j}{2}}, \dots, -\beta_2, -\beta_1)$.

Also, equation (2.1) implies that $\forall 1 \leq p' \leq s - j$, $\beta_{j+p'} = -\beta_{s-(p'-1)}$, that is $\beta'' =$

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$(\beta_{j+1}, \dots, \beta_{s-1}, \beta_s) = (-\beta_s, -\beta_{s-1}, \dots, -\beta_{j+1})$, which means that β'' is symmetric, and we can write β'' as $(-\beta_s, -\beta_{s-1}, \dots, -\beta_{\frac{s-j}{2}}, \beta_{\frac{s-j}{2}}, \dots, \beta_{s-1}, \beta_s)$. So finally we have:

$$\begin{aligned} (\beta_1, \beta_2, \dots, \beta_j, \beta_{j+1}, \dots, \beta_s) = & (\beta_1, \beta_2, \dots, \beta_{\frac{j}{2}}, -\beta_{\frac{j}{2}}, \dots, -\beta_2, -\beta_1, -\beta_s, -\beta_{s-1}, \\ & \dots, -\beta_{\frac{s-j}{2}}, \beta_{\frac{s-j}{2}}, \dots, \beta_{s-1}, \beta_s), \end{aligned}$$

which is the type of the cycle C . If we consider

$$\beta^* = (-\beta_{\frac{j}{2}}, \dots, -\beta_2, -\beta_1, -\beta_s, -\beta_{s-1}, \dots, -\beta_{\frac{s-j}{2}}, \beta_{\frac{s-j}{2}}, \dots, \beta_{s-1}, \beta_s, \beta_1, \beta_2, \dots, \beta_{\frac{j}{2}}),$$

β^* is symmetric, and is also a type of the cycle C , thus C is symmetric, which leads to a contradiction since C is non-symmetric. \square

Let T be a tournament of order n , and let C be a Hamiltonian cycle in T , such that $C = v_1 v_2 \dots v_n$ is of type $C(\beta)$ with respect to this enumeration, where β is symmetric. We know by the proof of the necessary condition of Lemma 2.2.4 that $\forall 1 \leq i \leq n$, the Hamiltonian paths $P = v_i v_{i+1} \dots v_n v_1 \dots v_{i-1}$ and $P' = v_{n-(i-2)} v_{n-(i-2)-1} \dots v_1 v_n \dots v_{n-(i-2)+1}$ have the same type with respect to these enumerations.

Let \mathcal{A} be the set of all paths in T that generate the cycle C , have the form $v_i v_{i+1} \dots v_n v_1 \dots v_{i-1}$ and are of a certain type $P(\alpha)$ with respect to this enumeration, and let \mathcal{B} be the set of all paths that generate C , have the form $v_{n-(i-2)} v_{n-(i-2)-1} \dots v_1 v_n \dots v_{n-(i-2)+1}$ and also have the type $P(\alpha)$ with respect to this enumeration.

Lemma 2.2.5. *We have $\mathcal{A} \cap \mathcal{B} = \emptyset$, and $|\mathcal{A}| = |\mathcal{B}|$.*

Proof. Suppose to the contrary that $\exists P \in \mathcal{A} \cap \mathcal{B}$. Since $P \in \mathcal{A}$, then $\exists 1 \leq i \leq n$ such that $P = v_i v_{i+1} \dots v_n v_1 \dots v_{i-2} v_{i-1}$ and is of type $P(\alpha) = P(\alpha_1, \dots, \alpha_s)$ with respect to this enumeration. Since $P \in \mathcal{B}$ also, then $P = v_{i-1} v_{i-2} \dots v_1 v_n \dots v_{i+1} v_i$ is of type $P(\alpha) = P(\alpha_1, \dots, \alpha_s)$ with respect to this enumeration, which means that $P = v_i v_{i+1} \dots v_n v_1 \dots v_{i-2} v_{i-1}$ is of type $P(-\bar{\alpha}) = P(-\alpha_s, \dots, -\alpha_1)$. Thus, $(\alpha_1, \dots, \alpha_s) = (-\alpha_s, \dots, -\alpha_1)$ which means that α is symmetric. But, since $C = C_P$, and since P has the type $P(\alpha)$ where α is symmetric, then the cycle C cannot be symmetric by Remark 1.3.19, thus β cannot be symmetric, which leads to a contradiction since β is symmetric.

For the second part, consider the correspondence $f : \mathcal{A} \rightarrow \mathcal{B}$, such that for every $P = v_i v_{i+1} \dots v_n v_1 \dots v_{i-1}$ of type $P(\alpha)$ with respect to this enumeration in \mathcal{A} , corresponds the path $P' = v_{n-(i-2)} v_{n-(i-2)-1} \dots v_1 v_n \dots v_{n-(i-2)+1}$, which belongs to \mathcal{B} since it is of type $P(\alpha)$ with respect to this enumeration, by Lemma 2.2.4. The correspondence f is trivially a bijective mapping, so $|\mathcal{A}| = |\mathcal{B}|$ which concludes the proof. \square

The last lemma is a result that was proven implicitly in [12]:

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Lemma 2.2.6. *Let $P = v_1v_2 \dots v_n$ and $P' = v_iv_{i+1} \dots v_nv_1 \dots v_{i-1}$ be two distinct oriented Hamiltonian paths in a tournament T of order n , that generate a cycle $C = C_P = C_{P'}$ in T . Then P and P' have the same type with respect to these enumerations if and only if v_1 and v_i are clones.*

We may now give the **proof of Theorem 2.2.1**:

Proof. The set \overline{P} contains P as well as the paths P' in T that have the same type as the type of P and such that $C_{P'} = C_P$. Let $C_P = v_1v_2 \dots v_n$ such that C_P is of type $C(\beta)$, with respect to this enumeration and $P = v_iv_{i+1} \dots v_nv_1 \dots v_{i-1}$ for some $1 \leq i \leq n$. If we consider all the paths P' having the form $v_jv_{j+1} \dots v_nv_1 \dots v_{j-1}$, and that have the same type as P with respect to these enumerations, and such that $C_P = C_{P'}$, then by Lemma 2.2.6, the number of such paths is exactly the number of clones that an end of P could have, that is $t(\beta) - 1$.

Let \mathcal{A} be the set of paths that generate C_P and have the same type as the type of P , following the order $v_1v_2 \dots v_n$ of the vertices, and \mathcal{B} be the set of paths that generate C_P and have the same type as the type of P , following the order $v_1v_n \dots v_2$ of the vertices. We have $|\mathcal{A}| = t(\beta)$. Now we need to count the number of paths in \mathcal{B} .

If C_P is non-symmetric, then by Lemma 2.2.4, $\forall 1 \leq i \leq n$, the path $P' = v_iv_{i-1} \dots v_1v_n \dots v_{i+1}$ cannot have the same type of P with respect to this enumeration, thus the set \mathcal{B} is empty. As a result, $|\overline{P}| = |\mathcal{A}| = t(\beta)$.

If the cycle C_P is symmetric, (we may suppose w.l.o.g. that $C_P = v_1v_2 \dots v_n$ is of type $C(\beta)$ with respect to this enumeration, where β is symmetric) then by Lemma 2.2.4, the set \mathcal{B} is non empty, and by Lemma 2.2.5 we have that $|\mathcal{A}| = |\mathcal{B}|$ and that the sets \mathcal{A} and \mathcal{B} are disjoint, thus we deduce that $|\overline{P}| = |\mathcal{A}| + |\mathcal{B}| = t(\beta) + t(\beta) = 2t(\beta)$.

This concludes our proof. \square

Note that all of the above results of this section are true for any oriented paths and cycles that are not necessarily Hamiltonian, since any path or cycle defines a set of vertices, and hence a subtournament in which the path and the cycle are Hamiltonian.

The following lemma, proved in [12], is of practical use in the following theorem:

Lemma 2.2.7. [12] *Let $\alpha_1, \alpha_2, \dots, \alpha_s, \beta_1, \dots, \beta_s \in \mathbb{Z}$.*

If $(\alpha_1, \alpha_2, \dots, \alpha_s) = (\beta_i, \beta_{i+1}, \dots, \beta_s, \beta_1, \dots, \beta_{i-1})$, then for any non negative integer k , we have

$$\alpha_{k_s} = \beta_{[k+i-1]_s}.$$

Proof. $\alpha_{k_s} = \alpha_p$ for some $p \equiv k \pmod{s}$, $1 \leq p \leq s$. If $p \leq s - i + 1$, then $\alpha_p = \beta_{p+i-1}$. Thus, $\alpha_{k_s} = \alpha_p = \beta_{p+i-1} = \beta_{[k+i-1]_s}$. If $p \geq s - i + 2$, then $\alpha_p = \beta_{p-s+i-1}$. Hence $\alpha_{k_s} = \alpha_p = \beta_{p-s+i-1} = \beta_{[p+i-1]_s} = \beta_{[k+i-1]_s}$. \square

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Remark 2.2.8. We saw in the second case of Remark 1.3.18 that if $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{K}_s$, s is odd, T is a tournament of order $n = \sum_{i=1}^s |\alpha_i| + 1$, and P a Hamiltonian path of type $P(\alpha)$ in T , then the two types of cycles that can be generated by P have either $s - 1$ or $s + 1$ blocks. So if we call $C(\beta)$ and $C(\beta')$ these two types, then obviously, the sets $\mathcal{C}_T(\beta)$ and $\mathcal{C}_T(\beta')$ are different. Remark also that when s is odd, α is always not symmetric, because we can't have $\alpha_1 = -\alpha_s$ since α_1 and α_s have the same sign.

However, when s is even, it's a completely different story. We have the following result:

Theorem 2.2.9. *Let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{K}_s$, and T a tournament of order $n = \sum_{i=1}^s |\alpha_i| + 1$.*

We have:

$$\mathcal{C}_T(\beta) = \mathcal{C}_T(\beta') \iff \alpha \text{ is symmetric,}$$

where $C(\beta)$ and $C(\beta')$ are the two types of Hamiltonian cycles in T that can be generated by a Hamiltonian path of type $P(\alpha)$ in T .

Proof. The case where s is odd being completely settled by Remark 13, we may assume that s is even.

By the first case of Remark 1.3.18, if $\alpha_1 > 0$, then $\beta = (\beta_1, \beta_2, \dots, \beta_s) = (\alpha_1 + 1, \alpha_2, \dots, \alpha_s)$ and $\beta' = (\beta'_1, \beta'_2, \dots, \beta'_s) = (\alpha_1, \dots, \alpha_{s-1}, \alpha_s - 1)$, while if $\alpha_1 < 0$, then $\beta = (\beta_1, \beta_2, \dots, \beta_s) = (\alpha_1 - 1, \alpha_2, \dots, \alpha_s)$ and $\beta' = (\beta'_1, \beta'_2, \dots, \beta'_s) = (\alpha_1, \dots, \alpha_{s-1}, \alpha_s + 1)$.

We will treat the case where $\alpha_1 > 0$, and the other case is similar.

For the sufficient condition, suppose that α is symmetric, thus $\alpha = (\alpha_1, \dots, \alpha_s) = (-\alpha_s, \dots, -\alpha_1)$, which implies that $\mathcal{C}_T(\beta) = \mathcal{C}_T(\alpha_1 + 1, \dots, \alpha_s)$ is equal to $\mathcal{C}_T(-\alpha_s + 1, \dots, -\alpha_1)$. Moreover, this set is equal to the set $\mathcal{C}_T(\beta') = \mathcal{C}_T(\alpha_1, \dots, \alpha_s - 1)$, thus $\mathcal{C}_T(\beta) = \mathcal{C}_T(\beta')$.

For the necessary condition, suppose that $\mathcal{C}_T(\beta) = \mathcal{C}_T(\beta')$, i.e. $\mathcal{C}_T(\alpha_1 + 1, \alpha_2, \dots, \alpha_s) = \mathcal{C}_T(\alpha_1, \dots, \alpha_{s-1}, \alpha_s - 1)$. Thus, since α_1 is different from $\alpha_1 + 1$ and $-\alpha_1 - 1$, then $(\alpha_1, \dots, \alpha_s - 1)$ is equal to one of these tuples:

1. $(\alpha_i, \alpha_{i+1}, \dots, \alpha_s, \alpha_1 + 1, \alpha_2, \dots, \alpha_{i-1})$ for some $2 \leq i \leq s$
2. $(-\alpha_i, -\alpha_{i-1}, \dots, -\alpha_2, -\alpha_1 - 1, -\alpha_s, \dots, -\alpha_{i+1})$ for some $2 \leq i \leq s$

Suppose that the first case is true, i.e. $(\alpha_1, \dots, \alpha_s - 1) = (\alpha_i, \alpha_{i+1}, \dots, \alpha_s, \alpha_1 + 1, \alpha_2, \dots, \alpha_{i-1}) = (\beta_i, \beta_{i+1}, \dots, \beta_s, \beta_1, \beta_2, \dots, \beta_{i-1})$ for some $2 \leq i \leq s$.

First observe that

$$\beta_{[i]_s} = \begin{cases} \alpha_{[i]_s} & \text{if } 1 < i \leq s \\ \alpha_1 + 1 & \text{if } i = 1 \end{cases}.$$

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We have: $\alpha_1 = \alpha_{1_s} = \beta_{[1+i-1]_s}$ (by Lemma 2.2.7) = $\alpha_{[1+i-1]_s}$ (since otherwise we get $\alpha_1 = \alpha_1 + 1$ which is a contradiction) = $\beta_{[1+2(i-1)]_s}$ (also by Lemma 2.2.7) = $\alpha_{[1+2(i-1)]_s}$ (also so that we don't get $\alpha_1 = \alpha_1 + 1$, a contradiction). And so on, we may prove by induction that

$$\alpha_1 = \alpha_{[1+k(i-1)]_s}, \forall k \in \mathbb{N}^*, \forall i \geq 2. \quad (2.2)$$

Now, observe that $\alpha_1 = \alpha_i$, $\alpha_2 = \alpha_{i+1}$, \dots , $\alpha_{s-i+1} = \alpha_s$ and $\alpha_{s-i+2} = \alpha_1 + 1$.

Moreover, we can write $s - i + 2 = 1 + k'(i - 1) + \lambda.s = [1 + k'(i - 1)]_s$, with $k' = s - 1 \in \mathbb{N}^*$ and $\lambda = 2 - i \in \mathbb{Z}$.

It follows that $\alpha_{s-i+2} = \alpha_{[1+k'(i-1)]_s} = \alpha_1$ by (2.2). But $\alpha_{s-i+2} = \alpha_1 + 1$, thus we reach a contradiction. So the first case cannot occur.

Consider the second case. First suppose that $i \neq s$. We have $\alpha_1 = -\alpha_i$ for some $2 \leq i \leq s - 1$, $\alpha_2 = -\alpha_{i-1}$, $\alpha_3 = -\alpha_{i-2}$, \dots , $\alpha_{i-1} = \alpha_{i-((i-1)-1)} = -\alpha_2$ and $\alpha_i = -\alpha_1 - 1$. Thus $\alpha_1 = -\alpha_i = \alpha_1 + 1$ and we reach a contradiction. So the second case is impossible for $2 \leq i \leq s - 1$. If $i = s$, we have $\alpha_1 = -\alpha_s$, $\alpha_2 = -\alpha_{s-1}$, $\alpha_3 = -\alpha_{s-2}$ \dots , $\alpha_{s-1} = -\alpha_{s-((s-1)-1)} = -\alpha_2$ and $\alpha_s - 1 = -\alpha_1 - 1$ which also means that $\alpha_s = -\alpha_1$. Thus $(\alpha_1, \dots, \alpha_s) = (-\alpha_s, \dots, -\alpha_1)$ and as a result α is symmetric. \square

Let $\beta = (\beta_1, \beta_2, \dots, \beta_s) \in \mathbb{Z}^s$; s is even, and $\beta_i \beta_{i+1} < 0 \forall i = 1, \dots, s - 1$. Then $\forall 1 \leq i \leq s$, define $\beta_i * 1$ as:

$$\beta_i * 1 = \begin{cases} \beta_i - 1 & \text{if } \beta_i > 0 \\ \beta_i + 1 & \text{if } \beta_i < 0 \end{cases}$$

We are now ready to give the relation linking between the number of oriented Hamiltonian paths of some type $P(\alpha)$ in T and the number of oriented Hamiltonian cycles of types $C(\beta)$ and $C(\beta')$ in T :

Theorem 2.2.10. *Let T be a tournament of order n , and $(\beta_1, \dots, \beta_s) \in \mathbb{K}'_s$, s even, and $\sum_{i=1}^s |\beta_i| = n$. Then:*

*If $(\beta_1 * 1, \beta_2, \dots, \beta_s)$ is symmetric, we have:*

$$f_T(\beta_1 * 1, \beta_2, \dots, \beta_s) = g_T(\beta_1, \beta_2, \dots, \beta_s) \cdot t(\beta_1, \beta_2, \dots, \beta_s).$$

Otherwise, we have:

$$f_T(\beta_1 * 1, \beta_2, \dots, \beta_s) = \delta(\beta_1, \beta_2, \dots, \beta_s) \cdot g_T(\beta_1, \beta_2, \dots, \beta_s) \cdot t(\beta_1, \beta_2, \dots, \beta_s) \\ + \delta(\beta_1 * 1, \beta_2, \dots, \beta_s * 1) \cdot g_T(\beta_1 * 1, \beta_2, \dots, \beta_s * 1) \cdot t(\beta_1 * 1, \beta_2, \dots, \beta_s * 1)$$

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$$\text{where } \delta(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ notsymmetric and is not a circuit} \\ 2 & \text{if } \gamma \text{ is symmetric} \\ \frac{n}{t(\gamma)} & \text{if } \gamma \text{ is a circuit} \end{cases}$$

Proof. In order to prove this theorem, let us compute $f_T(\beta_1 * 1, \dots, \beta_s)$.

Consider the set $\mathcal{P}_T(\beta_1 * 1, \beta_2, \dots, \beta_s)$ and let $P = x_1 \dots x_n$ be an element of this set. C_P is either of type $C(\beta_1, \dots, \beta_s)$ or of type $C(\beta_1 * 1, \beta_2, \dots, \beta_s * 1)$ whether (x_n, x_1) or $(x_1, x_n) \in E(T)$.

Let $\mathcal{C}_T(\beta) = \{C_1, \dots, C_t\}$ be the set of cycles of type $C(\beta_1, \dots, \beta_s)$ in T , and let $\mathcal{C}_T(\beta') = \{C'_1, \dots, C'_r\}$ be the set of cycles of type $C(\beta_1 * 1, \beta_2, \dots, \beta_s * 1)$ in T .

We have two cases to consider:

1. $(\beta_1 * 1, \beta_2, \dots, \beta_s)$ is symmetric (which implies that s is even).

Then by Theorem 2.2.9, $\mathcal{C}_T(\beta) = \mathcal{C}_T(\beta')$. Hence we only have to consider one of them, say $\mathcal{C}_T(\beta)$, to avoid counting the same cycle twice in the following step.

Let $\mathcal{C}_T(\beta) = \{C_1, C_2, \dots, C_t\}$. We have that for all $C_i \in \mathcal{C}_T(\beta)$, there exists a subclass X_i of $\mathcal{P}_T(\beta_1 * 1, \beta_2, \dots, \beta_s)$ with respect to $\mathcal{R}_{(\beta_1 * 1, \beta_2, \dots, \beta_s)}$ such that every path in X_i generates C_i . Thus, by Theorem 2.2.1,

$$|X_i| = |\overline{P}| = t(\beta)$$

for some $P \in X_i$, since if $(\beta_1 * 1, \beta_2, \dots, \beta_s)$ is symmetric, none of β or β' can be symmetric, nor a circuit. Hence,

$$\begin{aligned} f_T(\beta_1 * 1, \beta_2, \dots, \beta_s) &= \sum_{i=1}^t |X_i| = \sum_{i=1}^t t(\beta) \\ &= t.t(\beta) = |\mathcal{C}_T(\beta)| . t(\beta) \\ &= g_T(\beta).t(\beta). \end{aligned}$$

2. $(\beta_1 * 1, \beta_2, \dots, \beta_s)$ is not symmetric.

Then by Theorem 2.2.9, $\mathcal{C}_T(\beta) \neq \mathcal{C}_T(\beta')$, thus $\mathcal{C}_T(\beta) \cap \mathcal{C}_T(\beta') = \emptyset$ (because the sets of every type of Hamiltonian cycles form a partition of the set of all oriented Hamiltonian cycles in T).

We have that for all $C_i \in \mathcal{C}_T(\beta)$, there exists a subclass X_i of $\mathcal{P}_T(\beta_1 * 1, \beta_2, \dots, \beta_s)$ with respect to $\mathcal{R}_{(\beta_1 * 1, \beta_2, \dots, \beta_s)}$ such that every path in X_i generates C_i . Thus by Theorem 2.2.1, and Remark 2.2.3, $|X_i| = |\overline{P}| = t(\beta)$ or n or $2.t(\beta)$ for some $P \in X_i$, whether β is not symmetric and is not a circuit, is a circuit, or is symmetric, so

$$|X_i| = |\overline{P}| = \delta(\beta).t(\beta).$$

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Similarly, $\forall C'_j \in \mathcal{C}_T(\beta')$, \exists a subclass X'_j of $\mathcal{P}_T(\beta_1 * 1, \beta_2, \dots, \beta_s)$ with respect to $\mathcal{R}_{(\beta_1 * 1, \beta_2, \dots, \beta_s)}$ such that every path in X'_j generates C'_j . Thus

$$|X'_j| = |\overline{P'}| = \delta(\beta').t(\beta'),$$

for some $P' \in X'_j$. Hence,

$$\begin{aligned} f_T(\beta_1 * 1, \beta_2, \dots, \beta_s) &= \sum_{i=1}^t |X_i| + \sum_{j=1}^r |X'_j| \\ &= \sum_{i=1}^t \delta(\beta).t(\beta) + \sum_{j=1}^r \delta(\beta').t(\beta') \\ &= t.\delta(\beta).t(\beta) + r.\delta(\beta').t(\beta') \\ &= |\mathcal{C}_T(\beta)|.\delta(\beta).t(\beta) + |\mathcal{C}_T(\beta')|.\delta(\beta').t(\beta') \\ &= g_T(\beta).\delta(\beta).t(\beta) + g_T(\beta').\delta(\beta').t(\beta'), \end{aligned}$$

and this concludes our proof. \square

Moreover, for the case when the type of oriented paths in a tournament T is symmetric, we have the following property:

Theorem 2.2.11. *Let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{K}_s$, α symmetric, and T a tournament of order $n = \sum_{i=1}^s |\alpha_i| + 1$. Let P be a Hamiltonian path in T of type $P(\alpha)$ and $C_P \in \mathcal{C}_T(\beta)$. Then we have:*

$$t(\beta) = 1.$$

Proof. Suppose that $\alpha_1 > 0$. Since α is symmetric, then s is even, and by Theorem 2.2.9, we can assume that $C_P \in \mathcal{C}_T(\alpha_1 + 1, \dots, \alpha_s) = \mathcal{C}_T(\beta)$. If $\alpha_1 < 0$, then also by Theorem 2.2.9, we can assume that $C_P \in \mathcal{C}_T(\alpha_1 - 1, \dots, \alpha_s)$, but we will treat the case $\alpha_1 > 0$, and the other case is similar.

Since α is symmetric then $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l, -\alpha_l, \dots, -\alpha_2, -\alpha_1)$ where $l = \frac{s}{2}$, and $\beta = (\alpha_1 + 1, \alpha_2, \dots, \alpha_l, -\alpha_l, \dots, -\alpha_2, -\alpha_1)$.

Set $r' = r(\beta)$, we have $t(\beta) = \frac{s}{r'}$. Suppose to the contrary that $t(\beta) > 1$.

We have 2 cases:

1. $t(\beta)$ is even. Set $t(\beta) = 2k$, thus β is divided into $2k$ tuples $(\beta_1, \dots, \beta_{r'})$.

Set a be the first component of the first tuple, we have $a = \alpha_1 + 1$. Set b be the last component of the last tuple ($2k^{\text{th}}$ tuple), we have $b = -\alpha_1$.

Since r' is a period, then the first component a' of the $(k+1)^{\text{th}}$ tuple is equal to a , and

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the last component b' of the k^{th} tuple is equal to b .

But since α is symmetric, $a' = -b'$ because $a' = \alpha_l$ and $b' = -\alpha_l$. Thus $a = -b$ which implies that $\alpha_1 + 1 = -(-\alpha_1) = \alpha_1$ and this leads to a contradiction. So $t(\beta)$ cannot be even.

2. $t(\beta)$ is odd. Set $t(\beta) = 2k + 1$, $k \geq 1$, thus β is divided into $2k + 1$ tuples $(\beta_1, \dots, \beta_{r'})$, by noting that the $(k + 1)^{\text{th}}$ tuple takes the form $(\beta_1, \dots, \beta_{\frac{r'}{2}}, -\beta_{\frac{r'}{2}}, \dots, -\beta_1)$ where $\beta_{\frac{r'}{2}} = \alpha_l$, since α is symmetric. Thus it is symmetric. (Obviously all the other $2k$ tuples have this form since they are all equal).

Set a be the first component of the first tuple, we have $a = \alpha_1 + 1$. Set b be the last component of the last tuple ($(2k + 1)^{\text{th}}$ tuple), we have $b = -\alpha_1$.

Since r' is a period, then the first component a' of the $(k + 1)^{\text{th}}$ tuple is equal to a , and the last component b' of the $(k + 1)^{\text{th}}$ tuple is equal to b .

But since the $(k + 1)^{\text{th}}$ tuple is symmetric, $a' = -b'$. Thus $a = -b$ which implies that $\alpha_1 + 1 = -(-\alpha_1) = \alpha_1$ and this leads to a contradiction. So $T(\beta)$ cannot be an odd integer strictly greater than 1.

Thus we conclude that $t(\beta) = 1$. □

And finally, with the same hypothesis of Theorem 2.2.10, we can deduce the following:

Corollary 2.2.12. *If $(\beta_1 * 1, \beta_2, \dots, \beta_s)$ is symmetric, Then:*

$$f_T(\beta_1 * 1, \beta_2, \dots, \beta_s) = g_T(\beta_1, \beta_2, \dots, \beta_s)$$

Proof. The result follows immediately from Theorem 2.2.10 and Theorem 2.2.11. □

2.3 OHC in T and \bar{T}

Based on Theorem 2.2.10, linking between the number of oriented Hamiltonian paths of some type, and the number of oriented Hamiltonian cycles that can be generated by these paths in a tournament, we are able to establish the main result of the first section, Theorem 2.1.2, for oriented cycles:

Theorem 2.3.1. *Let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{K}'_s$, $\alpha_1 \geq 0$, and let T be a tournament of order n ; $n = \sum_{i=1}^s |\alpha_i|$. We have:*

$$g_T(\alpha) = g_T(-\alpha).$$

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Proof. The proof will be done by induction on s .

If $s = 1$, $\alpha = (\alpha_1) = (n)$ and $-\alpha = (-\alpha_1) = (-n)$ and we have $g_T(n) = g_T(-n)$.

Suppose that the result is true for $s - 2$ blocks, $s > 2$, i.e. if $\alpha = (\alpha_1, \dots, \alpha_{s-2}) \in \mathbb{Z}^{s-2}$; $\alpha_i \cdot \alpha_{i+1} < 0$, $\alpha_1 \geq 0$, $s - 2$ is even, and T is a tournament of order n ; $n = \sum_{i=1}^{s-2} |\alpha_i|$, we have: $g_T(\alpha) = g_T(-\alpha)$, and let's prove the result for s blocks. Let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}^s$; $\alpha_i \cdot \alpha_{i+1} < 0$, $\alpha_1 \geq 0$, and let T be a tournament of order n ; $n = \sum_{i=1}^s |\alpha_i|$. We argue by induction on α_1 .

If $\alpha_1 = 0$, then by induction, $g_T(\alpha) = g_T(0, \alpha_2, \dots, \alpha_s) = g_T(\alpha_2 + \alpha_s, \alpha_3, \dots, \alpha_{s-1}) = g_T(-\alpha_2 - \alpha_s, -\alpha_3, \dots, -\alpha_{s-1}) = g_T(0, -\alpha_2, -\alpha_3, \dots, -\alpha_{s-1}, -\alpha_s) = g_T(-\alpha)$.

So suppose that the result is true when the first block is of length $\alpha_1 - 1$, and let's prove it when the first block is of length α_1 .

We will consider two cases:

1. The tuple $(\alpha_1 - 1, \alpha_2, \dots, \alpha_s)$ is not symmetric.

We have $\alpha_1 - 1 \geq 0$, thus by Theorem 2.2.10,

$$\begin{aligned} f_T(\alpha_1 - 1, \alpha_2, \dots, \alpha_s) &= \delta(\alpha_1, \dots, \alpha_s) \cdot t(\alpha_1, \dots, \alpha_s) \cdot g_T(\alpha_1, \dots, \alpha_s) \\ &\quad + \delta(\alpha_1 - 1, \alpha_2, \dots, \alpha_{s-1}, \alpha_s - 1) \cdot t(\alpha_1 - 1, \alpha_2, \dots, \alpha_{s-1}, \alpha_s - 1) \\ &\quad \cdot g_T(\alpha_1 - 1, \alpha_2, \dots, \alpha_{s-1}, \alpha_s - 1) \\ &= \delta(\beta_1) \cdot t(\beta_1) \cdot g_T(\beta_1) + \delta(\beta'_1) \cdot t(\beta'_1) \cdot g_T(\beta'_1) \end{aligned}$$

$$\text{where } \delta(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ notsymmetric and is not a circuit} \\ 2 & \text{if } \gamma \text{ is symmetric} \\ \frac{1}{t(\gamma)} & \text{if } \gamma \text{ is a circuit} \end{cases}$$

Now consider the tuple $(-\alpha_1 + 1, -\alpha_2, \dots, -\alpha_s)$ which is also not symmetric. We have $-\alpha_1 + 1 \leq 0$, thus by Theorem 2.2.10,

$$\begin{aligned} f_T(-\alpha_1 + 1, -\alpha_2, \dots, -\alpha_s) &= \delta(-\alpha_1, \dots, -\alpha_s) \cdot t(-\alpha_1, \dots, -\alpha_s) \cdot g_T(-\alpha_1, \dots, -\alpha_s) \\ &\quad + \delta(-\alpha_1 + 1, -\alpha_2, \dots, -\alpha_s + 1) \cdot t(-\alpha_1 + 1, -\alpha_2, \dots, -\alpha_s + 1) \\ &\quad \cdot g_T(-\alpha_1 + 1, -\alpha_2, \dots, -\alpha_s + 1) \\ &= \delta(\beta_2) \cdot t(\beta_2) \cdot g_T(\beta_2) + \delta(\beta'_2) \cdot t(\beta'_2) \cdot g_T(\beta'_2) \end{aligned}$$

$$\text{where } \delta(\gamma) = \begin{cases} 1 & \text{if } \gamma \text{ notsymmetric and is not a circuit} \\ 2 & \text{if } \gamma \text{ is symmetric} \\ \frac{1}{t(\gamma)} & \text{if } \gamma \text{ is a circuit} \end{cases}$$

Since $(-\alpha_1 + 1, -\alpha_2, \dots, -\alpha_s) = -(\alpha_1 - 1, \alpha_2, \dots, \alpha_s)$, then by Theorem 2.1.2 we have

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$f_T(\alpha_1 - 1, \alpha_2, \dots, \alpha_s) = f_T(-\alpha_1 + 1, -\alpha_2, \dots, -\alpha_s)$. As a result,

$$\delta(\beta_1) \cdot g_T(\beta_1) \cdot t(\beta_1) + \delta(\beta'_1) \cdot g_T(\beta'_1) \cdot t(\beta'_1) = \delta(\beta_2) \cdot g_T(\beta_2) \cdot t(\beta_2) + \delta(\beta'_2) \cdot g_T(\beta'_2) \cdot t(\beta'_2).$$

But, since $\beta_2 = -\beta_1$ and $\beta'_2 = -\beta'_1$ thus if β_1 is not symmetric and is not a circuit (resp. is a circuit, or is symmetric), so is β_2 , and similarly for β'_1 and β'_2 , so $\delta(\beta_1) = \delta(\beta_2)$, and $\delta(\beta'_1) = \delta(\beta'_2)$, and also by Proposition 1.3.22 we have $t(\beta_2) = t(\beta_1)$ and $t(\beta'_2) = t(\beta'_1)$. Moreover, since $\alpha_1 - 1 < \alpha_1$, then by induction $g_T(\beta'_1) = g_T(\beta'_2)$, hence we have

$$g_T(\beta_1) = g_T(\beta_2).$$

2. The tuple $(\alpha_1 - 1, \alpha_2, \dots, \alpha_s)$ is symmetric.

We have $\alpha_1 - 1 \geq 0$, thus by Corollary 2.2.12,

$$f_T(\alpha_1 - 1, \alpha_2, \dots, \alpha_s) = g_T(\alpha_1, \alpha_2, \dots, \alpha_s).$$

Now consider the tuple $(-\alpha_1 + 1, -\alpha_2, \dots, -\alpha_s)$ which is also symmetric. We have $-\alpha_1 + 1 \leq 0$, thus by Corollary 2.2.12,

$$f_T(-\alpha_1 + 1, -\alpha_2, \dots, -\alpha_s) = g_T(-\alpha_1, -\alpha_2, \dots, -\alpha_s).$$

Since by Theorem 2.1.2 we have

$$f_T(\alpha_1 - 1, \alpha_2, \dots, \alpha_s) = f_T(-\alpha_1 + 1, -\alpha_2, \dots, -\alpha_s),$$

we get

$$g_T(\alpha_1, \dots, \alpha_s) = g_T(-\alpha_1, \dots, -\alpha_s).$$

This concludes the proof. □

That way, we showed that a tournament T and its complement \bar{T} contain the same number of oriented Hamiltonian cycles of any given type.

2.4 Digraphs of maximal degree $\Delta \leq 2$

After establishing the two results in Theorem 2.1.2 and Theorem 2.3.1 about Hamiltonian paths and cycles in tournaments, one can go further, generalizing these facts to any digraph of maximal degree 2, that is: let H be a digraph with maximal degree $\Delta(G(H)) \leq 2$, then $f_T(H) = f_{\bar{T}}(H)$, where $f_T(H)$ is the number of copies of the digraph H in a tournament T .

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To this purpose, we first need to prove several lemmas:

Lemma 2.4.1. *Let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{K}'_s$, $\alpha_1 \geq 0$, and let T be a tournament of order n ; $n \geq \sum_{i=1}^s |\alpha_i| + 1$. We have:*

$$f_T(\alpha) = f_T(-\alpha).$$

Proof. Let $m = \sum_{i=1}^s |\alpha_i| + 1$. Every oriented path in T of type $P(\alpha)$ is a Hamiltonian path of type $P(\alpha)$ contained in a subtournament T' of T of order m . By Theorem 2.1.2, $f_{T'}(\alpha) = f_{T'}(-\alpha)$. Moreover, if we consider another subtournament T'' of T , of order m , $T'' \neq T'$, then $\mathcal{P}_{T'}(\alpha) \cap \mathcal{P}_{T''}(\alpha) = \emptyset$, because every Hamiltonian path in T' differs with a least one vertex from every Hamiltonian path in T'' .

So let $V(T) = \bigcup_{X \subseteq V(T), |X|=m} X$, we have:

$$f_T(\alpha) = \sum_{X \subseteq V(T), |X|=m} f_{\langle X \rangle}(\alpha) = \sum_{X \subseteq V(T), |X|=m} f_{\langle X \rangle}(-\alpha) = f_T(-\alpha),$$

and we get our result. □

Similarly, we may prove the same result for cycles in tournaments:

Lemma 2.4.2. *Let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{K}'_s$, $\alpha_1 \geq 0$, and let T be a tournament of order n ; $n \geq \sum_{i=1}^s |\alpha_i|$. We have:*

$$g_T(\alpha) = g_T(-\alpha).$$

Lemma 2.4.3. *Let T be a tournament, and let H be a digraph with $\Delta(G(H)) \leq 2$ and such that its connected components are mutually isomorphic. Then the number of copies of H in T and that in its complement \bar{T} are the same.*

Proof. Since H is a digraph with $\Delta(G(H)) \leq 2$ and such that its connected components are isomorphic, then $H = H_1 \cup H_2 \cup \dots \cup H_r$ where the digraphs H_i , $1 \leq i \leq r$, are its connected components, with $|H_i| = m \forall 1 \leq i \leq r$, and such that they are either all paths of the same type, say $P(\alpha)$, or all cycles of the same type, $C(\beta)$. If T contains a copy of H , then since the digraphs H_i , $1 \leq i \leq r$, are disjoint, the copy of every digraph H_i is a spanning subdigraph of a subtournament T_i of T , such that the subtournaments T_i , $1 \leq i \leq r$, are also disjoint, with $|V(T_i)| = m \forall 1 \leq i \leq r$.

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Let's consider r disjoint subtournaments of T , T_i , $1 \leq i \leq r$, all of order m , and suppose that T contains a copy of H such that $\forall 1 \leq i \leq r$, H_i has a copy in T_i . As $f_{T_i}(H_i)$ denotes the number of copies of H_i in the subtournament T_i , then the number of copies of H in T , such that the copy of H_i is a spanning subdigraph of T_i , is:

$$\prod_{i=1}^r f_{T_i}(H_i)$$

Now if we consider any permutation σ of the subtournaments T_i , and since all the digraphs H_i are isomorphic, then $\forall 1 \leq i \leq r$, if H_i has a copy in T_i , then H_i also has a copy in $T_{\sigma(i)}$. But, also since all the digraphs H_i are isomorphic, then the copies of H obtained in T such that the copy of each H_i is a spanning subdigraph of T_i are the same as the ones obtained in T such that the copy of each H_i is a spanning subdigraph of $T_{\sigma(i)}$.

Let's compute $f_T(H)$, the total number of copies of H in T .

Let $\mathcal{L} = \{(T_1, T_2, \dots, T_r); T_i \text{ subtournament of } T \forall 1 \leq i \leq r, T_i \cap T_j = \emptyset \forall 1 \leq i, j \leq r, |V(T_i)| = m\}$. We have:

$$f_T(H) = \sum_{(T_1, T_2, \dots, T_r) \in \mathcal{L}} \frac{\prod_{i=1}^r f_{T_i}(H_i)}{r!}$$

However, by Lemma 2.4.1 and Lemma 2.4.2, we have that $\forall 1 \leq i \leq r$,

$$f_{T_i}(H_i) = f_{\overline{T_i}}(H_i).$$

So let $\mathcal{L}' = \{(\overline{T_1}, \overline{T_2}, \dots, \overline{T_r}); T_i \text{ subtournament of } T \forall 1 \leq i \leq r, T_i \cap T_j = \emptyset \forall 1 \leq i, j \leq r, |V(T_i)| = m\}$, we get:

$$f_T(H) = \sum_{(T_1, T_2, \dots, T_r) \in \mathcal{L}} \frac{\prod_{i=1}^r f_{T_i}(H_i)}{r!} = \sum_{(\overline{T_1}, \overline{T_2}, \dots, \overline{T_r}) \in \mathcal{L}'} \frac{\prod_{i=1}^r f_{\overline{T_i}}(H_i)}{r!} = f_{\overline{T}}(H),$$

and the result follows. \square

We may now give our theorem:

Theorem 2.4.4. *Let T be a tournament and let H be a digraph with $\Delta(G(H)) \leq 2$. Then the number of copies of H in T and its complement \overline{T} is the same.*

Proof. Since $\Delta(G(H)) \leq 2$, then H is a disjoint union of directed paths and cycles. Write H as $H = \bigcup_{i=1}^t H^i$, where each H^i is a subdigraph of H whose all connected components are isomorphic, and which is maximal with this property. The connected components of each H^i

2.4. DIGRAPHS OF MAXIMAL DEGREE $\Delta \leq 2$

are either all paths of the same type or cycles of the same type. Note that the digraphs H^i , $1 \leq i \leq t$, are disjoint, and non-isomorphic.

If T contains a copy of H , then since the digraphs H^i , $1 \leq i \leq t$, are disjoint, the copy of every digraph H^i is a spanning subdigraph of a subtournament T^i of T , and such that the subtournaments T^i , $1 \leq i \leq t$, are also disjoint, with $|V(T^i)| = |V(H^i)| \forall 1 \leq i \leq t$.

As we did in the previous lemma, let's consider t disjoint subtournaments of T , T^i , $1 \leq i \leq t$, and such that $|V(T^i)| = |V(H^i)|$, and suppose that T contains a copy of H such that $\forall 1 \leq i \leq t$, H^i has a copy in T^i . The number of copies of H in T , such that the copy of H^i is a spanning subdigraph of T^i , is:

$$\prod_{i=1}^t f_{T^i}(H^i)$$

However, if we consider any permutation σ of the subtournaments T^i , and since all the digraphs H^i are non-isomorphic, then if T contains a copy of H such that $\forall 1 \leq i \leq t$, H^i has a copy in $T^{\sigma(i)}$, the copies of H obtained in T such that the copy of each H^i is a spanning subdigraph of T^i are all different from those obtained in T such that the copy of each H^i is a spanning subdigraph of $T^{\sigma(i)}$.

Let's compute the total number of copies of H in T , $f_T(H)$.

Let $\mathcal{L} = \{(T^1, T^2, \dots, T^t); T^i \text{ subtournament of } T \forall 1 \leq i \leq t, T^i \cap T^j = \emptyset \forall 1 \leq i, j \leq t, |V(T_i)| = |V(H^i)|\}$. We have:

$$f_T(H) = \sum_{(T^1, T^2, \dots, T^t) \in \mathcal{L}} \prod_{i=1}^t f_{T^i}(H^i)$$

However, by Lemma 2.4.3, since the connected components of each digraph H^i are isomorphic, we have that $\forall 1 \leq i \leq t$, $f_{T^i}(H^i) = f_{\overline{T^i}}(H^i)$.

So let $\mathcal{L}' = \{(\overline{T^1}, \overline{T^2}, \dots, \overline{T^t}); T^i \text{ subtournament of } T \forall 1 \leq i \leq t, T^i \cap T^j = \emptyset \forall 1 \leq i, j \leq t, |V(T^i)| = |V(H^i)|\}$, we get:

$$f_T(H) = \sum_{(T^1, T^2, \dots, T^t) \in \mathcal{L}} \prod_{i=1}^t f_{T^i}(H^i) = \sum_{(\overline{T^1}, \overline{T^2}, \dots, \overline{T^t}) \in \mathcal{L}'} \prod_{i=1}^t f_{\overline{T^i}}(H^i) = f_{\overline{T}}(H),$$

hence:

$$f_T(H) = f_{\overline{T}}(H),$$

and this concludes the proof. □

Remark 2.4.5. Let T be a tournament on $n + 1$ vertices, formed by a directed n -cycle $C = v_1v_2 \dots v_n$, with its internal edges, where these edges may have any orientations, and a vertex v of in-degree equal to zero (a source), adjacent to the n vertices of the cycle ($d_T^+(v) = n$). Then the complement \bar{T} of this tournament is formed by a directed n -cycle, $C' = v_1v_nv_{n-1} \dots v_2$, and its internal edges which have opposite orientations of those of $T\langle C \rangle$, and a vertex v of out-degree equal to zero (a sink) adjacent to all the vertices of C' . Also note that since C and C' are directed cycles, then $\forall x \in C, d_T^+(x) \leq n - 1$ and $\forall y \in C', d_{\bar{T}}^+(y) \leq n - 1$. Thus if we consider a digraph H on $n + 1$ vertices, formed by a vertex x and n out-neighbors of x , which is a digraph of maximal degree $\Delta(G(H)) = n$, the number of copies of H in T is equal to one, while there are no such copies in \bar{T} .

To conclude this chapter, and based on the remark above, we ask the following:

Let $f_T(H)$ denote the number of copies of a digraph H in a tournament T .

Problem 2.4.6. *Can we characterize the set \mathcal{H} of all digraphs H such that $f_T(H) = f_{\bar{T}}(H)$ for any tournament T ?*

Chapter 3

A computation of the number of oriented Hamiltonian paths in transitive tournaments

In this chapter, we are interested in counting oriented Hamiltonian paths in transitive tournaments.

Recall that a tournament T is transitive, denoted by TT_n , if there exists an enumeration of its vertices such that $V(T) = \{v_1, v_2, \dots, v_n\}$ and $E(T) = \{(v_i, v_j); i < j\}$. The vertices v_1 and v_n are the source and the sink respectively.

As we saw in Chapter 1, the number of directed Hamiltonian paths in TT_n is equal to 1, that is,

$$f_{TT_n}(n-1) = 1.$$

Also, since a transitive tournament TT_n and its complement are isomorphic, then $\forall \alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{K}_s$ such that $\sum_{i=1}^s |\alpha_i| = n-1$, we have:

$$f_{TT_n}(\alpha) = f_{TT_n}(-\alpha).$$

Moreover, we can easily compute the number of oriented Hamiltonian paths formed of two blocks in TT_n :

Proposition 3.0.1. *Let TT_n be the transitive tournament on n vertices, and $(\alpha_1, \alpha_2) \in \mathbb{K}_2$, $|\alpha_1| \neq |\alpha_2|$. Then we have:*

$$f_{TT_n}(\alpha_1, \alpha_2) = C_{n-1}^{|\alpha_1|}.$$

Proof. Let's enumerate the vertices of TTn such that $V(T) = \{v_1, v_2, \dots, v_n\}$ and $E(T) = \{(v_i, v_j); i < j\}$, and suppose without loss of generality that $\alpha_1 > 0$ since $f_{TTn}(\alpha) = f_{TTn}(-\alpha)$. Consider a Hamiltonian path P of TTn of type $P(\alpha_1, \alpha_2)$. Since $\alpha_1 > 0$ then the sink v_n should be at the end of the first block. Now to construct the first block of P , we have to choose any α_1 vertices from the remaining $n - 1$ vertices. Once the vertices are chosen, there is only one path P corresponding to this choice: in fact, since TTn is transitive, the vertices chosen to construct the first block must be aligned in an increasing order of indexes, and all remaining vertices of TTn should be aligned in a decreasing order of indexes to form the second block. Hence, the number of paths of type $P(\alpha_1, \alpha_2)$ in TTn is exactly the number of possible choices of α_1 vertices among $n - 1$, that is $C_{n-1}^{|\alpha_1|}$. \square

Remark 3.0.2. We have $f_{TTn}(\alpha_1, -\alpha_1) = \frac{C_{n-1}^{|\alpha_1|}}{2}$.

In fact, following the same construction given in the proof of Proposition 3.0.1, we may remark that if $|\alpha_1| = |\alpha_2|$, that is if (α_1, α_2) is symmetric, then each constructed path P of type $P(\alpha_1, \alpha_2)$ is in total counted twice, which clarify the necessity to divide the total number of paths of type $P(\alpha_1, \alpha_1)$ in TTn by 2.

As we have just noticed, a transitive tournament of order n has an interesting structure, allowing us to count easily the number of some types of oriented Hamiltonian paths. So we ask the following question:

Question 3.0.3. *Is there a way to compute the exact number of oriented Hamiltonian paths of any given type, in a transitive tournament?*

In this chapter, we will show that it is possible, by defining a combinatorial function that will allow us to calculate the number of any type of oriented Hamiltonian paths in TTn . We will also build a program to compute the values of this function, and will reach some interesting observations.

3.1 The path-function \mathcal{F} : computing the number of OHP in TTn

Let $\mathcal{K} = \{(\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{K}_s, s \in \mathbb{N}^*\}$.

Let \mathcal{F} be the following mapping:

$$\begin{aligned} \mathcal{F} \quad \mathcal{K} &\longrightarrow \mathbb{N} \\ (\alpha_1, \alpha_2, \dots, \alpha_s) &\longrightarrow \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_s), \end{aligned}$$

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defined by the recurrence relation:

$$\begin{aligned} \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_s) = & \mathcal{F}(\alpha_1 * 1, \alpha_2, \dots, \alpha_s) + \mathcal{F}(\alpha_1, \alpha_2 * 1, \dots, \alpha_s) \\ & + \dots + \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_s * 1) \end{aligned}$$

where $\alpha_i * 1 = \alpha_i - 1$ if $\alpha_i > 0$ and $\alpha_i * 1 = \alpha_i + 1$ otherwise, and satisfying:

1. $\forall t \in \mathbb{N}^*, \mathcal{F}(0, \alpha_2, \dots, \alpha_t) = \mathcal{F}(\alpha_2, \dots, \alpha_t),$
2. $\forall t' \in \mathbb{N}^*, \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_{t'}, 0) = \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_{t'}),$
3. $\forall r \in \mathbb{N}^*, \mathcal{F}(\alpha_1, \dots, \alpha_r, 0, \alpha_{r+2}, \dots, \alpha_s) = \mathcal{F}(\alpha_1, \dots, \alpha_r + \alpha_{r+2}, \dots, \alpha_s),$
4. $\forall \alpha \in \mathbb{Z}^*, \mathcal{F}(\alpha) = 1.$

We call \mathcal{F} the path-function.

Theorem 3.1.1. *Let TTn be a transitive tournament of order n , and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{K}_s$ such that $\sum_{i=1}^s |\alpha_i| = n - 1$. Then:*

$$f_{TTn}(\alpha) = \begin{cases} \mathcal{F}(\alpha) & \text{if } \alpha \text{ is not symmetric,} \\ \frac{\mathcal{F}(\alpha)}{2} & \text{if } \alpha \text{ is symmetric.} \end{cases}$$

In order to prove this theorem, we need the following:

Definition 3.1.2. Let $\alpha = (\alpha_1, \dots, \alpha_s)$ in \mathbb{Z}^s . For some $1 \leq i \leq s$, we define by α^i the tuple $(\alpha_1, \alpha_2, \dots, \alpha_i * 1, \alpha_{i+1}, \dots, \alpha_s)$, where $\alpha_i * 1 = \alpha_i - 1$ if $\alpha_i > 0$, and $\alpha_i * 1 = \alpha_i + 1$ otherwise.

Proposition 3.1.3. *We have the following property:*

$$\mathcal{P}_T(\alpha^i) = \mathcal{P}_T(\alpha^j) \iff i = j \text{ or } \alpha^j = -\overline{\alpha^i}.$$

The proof of this proposition is easy, as it is a direct consequence of Proposition 1.3.5.

Lemma 3.1.4. *Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{K}_s$, and i, j such that $1 \leq i < j \leq s$. Then:*

$$\alpha^i = -\overline{\alpha^j} \iff \alpha \text{ is symmetric and } i + j = s + 1.$$

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Proof. We have $\alpha^i = (\alpha_1, \dots, \alpha_i * 1, \dots, \alpha_j, \dots, \alpha_s)$ and $\alpha^j = (\alpha_1, \dots, \alpha_i, \dots, \alpha_j * 1, \dots, \alpha_s)$ thus $-\overline{\alpha^j} = (-\alpha_s, \dots, -(\alpha_j * 1), \dots, -\alpha_i, \dots, -\alpha_1)$.

For the sufficient condition, suppose that α is symmetric and $i + j = s + 1$ and let's prove that $\alpha^i = -\overline{\alpha^j}$. The i^{th} component of α^i is $\alpha_i * 1$, and since $i + j = s + 1$, then the i^{th} component of $-\overline{\alpha^j}$ is exactly $-(\alpha_j * 1)$, because the i^{th} component of $\overline{\alpha^j}$ is the $(s + 1 - i)^{\text{th}}$ component of α^j . Moreover, since α is symmetric, then $\alpha_i = -\alpha_{s+1-i} = -\alpha_j$ thus if we assume without loss of generality that $\alpha_i > 0$, we have $\alpha_i * 1 = \alpha_i + 1$, then $-(\alpha_j * 1) = -(\alpha_j - 1) = -\alpha_j + 1$. Now, again, since α is symmetric, then $(\alpha_1, \alpha_2, \dots, \alpha_s) = (-\alpha_s, -\alpha_{s-1}, \dots, -\alpha_1)$, and using what preceded, we get $(\alpha_1, \dots, \alpha_i + 1, \dots, \alpha_j, \dots, \alpha_s) = (-\alpha_s, \dots, -\alpha_j + 1, \dots, -\alpha_i, \dots, -\alpha_1)$ with $-\alpha_j + 1$ on the i^{th} position, thus $\alpha^i = -\overline{\alpha^j}$.

For the necessary condition, suppose that $\alpha^i = -\overline{\alpha^j}$. The i^{th} component of α^i is $\alpha_i * 1$. If $i + j \neq s + 1$, then the i^{th} component of $-\overline{\alpha^j}$ is equal to some $-\alpha_t$, $t \neq j$. On the other hand, the $(s + 1 - i)^{\text{th}}$ component of α^i is α_t , and the $(s + 1 - i)^{\text{th}}$ component of $-\overline{\alpha^j}$ is $-\alpha_i$ since α_i is not modified in α^j because $i \neq j$. As a result, since $\alpha^i = -\overline{\alpha^j}$, we get $\alpha_i * 1 = -\alpha_t = -(-\alpha_i) = \alpha_i$ which is a contradiction. So $i + j = s + 1$ which implies that $\alpha_j = -\alpha_i$ since the j^{th} component of α^i is α_j and the j^{th} component of $-\overline{\alpha^j}$ is $-\alpha_i$, and $\alpha^i = -\overline{\alpha^j}$. Thus if we suppose w.l.o.g. that $\alpha_i > 0$, then $\alpha_i * 1 = \alpha_i + 1$, then $-(\alpha_j * 1) = -(\alpha_j - 1) = -\alpha_j + 1$. Since $\alpha^i = -\overline{\alpha^j}$ then $(\alpha_1, \dots, \alpha_i + 1, \dots, \alpha_j, \dots, \alpha_s) = (-\alpha_s, \dots, -\alpha_j + 1, \dots, -\alpha_i, \dots, -\alpha_1)$ with $-\alpha_j + 1$ on the i^{th} position (since $i + j = s + 1$), and as a result $(\alpha_1, \dots, \alpha_s) = (-\alpha_s, \dots, -\alpha_1)$ so α is symmetric which concludes the proof. \square

Lemma 3.1.5. *Let $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{K}_s$, we have:*

$$\alpha^i \text{ is symmetric} \Rightarrow \alpha^j \text{ is not symmetric} \forall 1 \leq j \leq s, j \neq i.$$

Proof. Since α^i is symmetric, then $\alpha^i = -\overline{\alpha^i}$, hence we have $(\alpha_1, \dots, \alpha_i * 1, \dots, \alpha_s) = (-\alpha_s, \dots, -(\alpha_i * 1), \dots, -\alpha_1)$. Let $-\alpha_t$ be the i^{th} component of $-\overline{\alpha^i}$, so $\alpha_i * 1 = -\alpha_t$. Suppose that $\exists j \neq i$ such that α^j is symmetric, then $\alpha^j = -\overline{\alpha^j}$, that is, (if we suppose without loss of generality that $i < j$), we have that $(\alpha_1, \dots, \alpha_i, \dots, \alpha_j * 1, \dots, \alpha_s) = (-\alpha_s, \dots, -(\alpha_j * 1), \dots, -\alpha_i, \dots, -\alpha_1)$. We have two cases to consider: If α_j is not on the $(s + 1 - i)^{\text{th}}$ position of α , then the i^{th} component of $-\overline{\alpha^j}$ is $-\alpha_t$. Thus, $\alpha_i = -\alpha_t$, a contradiction. If α_j is on the $(s + 1 - i)^{\text{th}}$ position of α , (which means that α_i and α_j are of opposite signs because α has an even number of components since α^i is symmetric), then $-\alpha_j$ is on the i^{th} position of $-\overline{\alpha^j}$ (which implies that $\alpha_i * 1 = -\alpha_j$ that is, if we suppose w.l.o.g. $\alpha_i > 0$, $\alpha_i + 1 = -\alpha_j$ so $\alpha_i = -\alpha_j - 1$), and $-(\alpha_j * 1)$ is on the i^{th} position of $-\overline{\alpha^j}$ (which implies that $\alpha_i = -(\alpha_j * 1)$ i.e. $\alpha_i = -(\alpha_j - 1)$ so $\alpha_i = -\alpha_j + 1$), and we get a contradiction. \square

We may now give the **proof of Theorem 3.1.1**:

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Proof. We have:

$$\begin{aligned}\mathcal{F}(\alpha) &= \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_s) \\ &= \mathcal{F}(\alpha_1 * 1, \alpha_2, \dots, \alpha_s) + \mathcal{F}(\alpha_1, \alpha_2 * 1, \dots, \alpha_s) + \dots + \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_s * 1) \\ &= \mathcal{F}(\alpha^1) + \mathcal{F}(\alpha^2) + \dots + \mathcal{F}(\alpha^s).\end{aligned}$$

We consider 2 cases:

- The tuple α is symmetric (which implies that s is even).

Thus obviously, all α^i are not symmetric.

The proof will be done by induction on the order of the tournament TTn . The smallest case of symmetric α is $\alpha = (1, -1)$ which corresponds to a transitive tournament of order 3 (acyclic triangle). In this tournament, $f_{TT3}(1, -1) = 1$ while $\mathcal{F}(1, -1) = \mathcal{F}(0, -1) + \mathcal{F}(1, 0) = 1 + 1 = 2$. So $f_{TT3}(1, -1) = \frac{1}{2}\mathcal{F}(1, -1)$.

Suppose that the statement is true for transitive tournaments of order $l \leq n - 1$, and let's prove it for n .

By Lemma 3.1.4, we have that $\alpha^i = -\overline{\alpha^j} \iff i + j = s + 1$. As a result, by Proposition 3.1.3, if we consider the transitive tournament $\tilde{T} = TTn - \{v\}$ where v is the source of TTn , we have $\mathcal{P}_{\tilde{T}}(\alpha^1) = \mathcal{P}_{\tilde{T}}(\alpha^s)$, $\mathcal{P}_{\tilde{T}}(\alpha^2) = \mathcal{P}_{\tilde{T}}(\alpha^{s-1})$, \dots , $\mathcal{P}_{\tilde{T}}(\alpha^{\frac{s}{2}}) = \mathcal{P}_{\tilde{T}}(\alpha^{\frac{s}{2}+1})$, and the sets $\mathcal{P}_{\tilde{T}}(\alpha^1), \mathcal{P}_{\tilde{T}}(\alpha^2), \dots, \mathcal{P}_{\tilde{T}}(\alpha^{\frac{s}{2}})$ are pairwise different.

Moreover, each path of type $P(\alpha^i)$ for some $1 \leq i \leq s$ is a Hamiltonian path in \tilde{T} , and since all the sets of oriented Hamiltonian paths of a given type in a tournament T form a partition of the set of all oriented Hamiltonian paths in T , then if $\mathcal{P}_{\tilde{T}}(\alpha^i) \neq \mathcal{P}_{\tilde{T}}(\alpha^j)$, we have $\mathcal{P}_{\tilde{T}}(\alpha^i) \cap \mathcal{P}_{\tilde{T}}(\alpha^j) = \emptyset$, and as a result, $\mathcal{P}_{\tilde{T}}(\alpha^1), \mathcal{P}_{\tilde{T}}(\alpha^2), \dots, \mathcal{P}_{\tilde{T}}(\alpha^{\frac{s}{2}})$ are all pairwise disjoint, and similarly $\mathcal{P}_{\tilde{T}}(\alpha^{\frac{s}{2}+1}), \mathcal{P}_{\tilde{T}}(\alpha^{\frac{s}{2}+2}), \dots, \mathcal{P}_{\tilde{T}}(\alpha^s)$ are pairwise disjoint.

Consider the correspondence:

$$\begin{aligned}g : \quad \mathcal{P}_{TTn}(\alpha) &\longrightarrow \cup_{i=1}^{\frac{s}{2}} \mathcal{P}_{\tilde{T}}(\alpha^i) \\ P &\longrightarrow g(P) = P - \langle v \rangle \cup \langle \{x, y\} \rangle,\end{aligned}$$

where x is the predecessor of v on P , and y its successor on P if any.

Clearly, g is well defined. In fact, Let $P \in \mathcal{P}_{TTn}(\alpha_1, \alpha_2, \dots, \alpha_s)$, $P = I_1 I_2 \dots I_s$, and suppose that the source v is the origin of some block I_i of P of length α_i ($\alpha_i > 0$ since v is a source), and let $x \in I_{i-1}$ be the predecessor of v on P and $y \in I_i$ its successor. If $(y, x) \in E(TTn)$, then $g(P)$ is of type $P(\alpha_1, \dots, \alpha_{i-1}, \alpha_i - 1, \dots, \alpha_s)$, and if

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$(x, y) \in E(TTn)$, then $g(P)$ is of type $P(\alpha_1, \dots, \alpha_{i-1} + 1, \alpha_i, \dots, \alpha_s)$, and both of them belong to $\cup_{i=1}^{\frac{s}{2}} \mathcal{P}_{\tilde{T}}(\alpha^i)$, (if $g(P) \in \mathcal{P}_{\tilde{T}}(\alpha^i)$ for some $\frac{s}{2} + 1 \leq i \leq s$, then as previously mentioned, it belongs to a set $\mathcal{P}_{\tilde{T}}(\alpha^i)$ for some $1 \leq i \leq \frac{s}{2}$ and so it belongs to $\cup_{i=1}^{\frac{s}{2}} \mathcal{P}_{\tilde{T}}(\alpha^i)$). Moreover, it is obvious that g is a mapping.

The mapping g is a bijection:

It is surjective: Let $P' \in \cup_{i=1}^{\frac{s}{2}} \mathcal{P}_{\tilde{T}}(\alpha^i)$, then $\exists 1 \leq i \leq \frac{s}{2}$ such that $P' \in \mathcal{P}_{\tilde{T}}(\alpha^i)$. Suppose that $\alpha_i > 0$, then $P' \in \mathcal{P}_{\tilde{T}}(\alpha_1, \dots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \dots, \alpha_s)$ and let x be the origin of the block I_i of length $\alpha_i - 1$, and $y \in I_{i-1}$ its predecessor, and write $P' = P_1 \cup (x, y) \cup P_2$. Since v is a source, then (v, y) and $(v, x) \in E(TTn)$, thus $P = P_1 \cup (v, y) \cup (v, x) \cup P_2$ is of type $P(\alpha_1, \alpha_2, \dots, \alpha_s)$, and so it belongs to $\mathcal{P}_{TTn}(\alpha)$ with $g(P) = P'$. The case $\alpha_i < 0$ is similar.

Also, g is injective: Let P and P' be two paths in TTn of type $P(\alpha)$, such that $g(P) = g(P') \in \mathcal{P}_{\tilde{T}}(\alpha^i) = \mathcal{P}_{\tilde{T}}(\alpha_1, \dots, \alpha_{i-1}, \alpha_i * 1, \alpha_{i+1}, \dots, \alpha_s)$ for some $1 \leq i \leq \frac{s}{2}$. Suppose that $\alpha_i > 0$. So let $g(P) = I_1 I_2 \dots I_s = u_1 u_2 \dots u_r y x w_1 w_2 \dots w_t$ where x is the origin of the block I_i of length $\alpha_i - 1$, and $y \in I_{i-1}$ its predecessor, and the arc (x, y) replaced the arcs (v, x) and (v, y) in $P = u_1 u_2 \dots u_r y v x w_1 w_2 \dots w_t$. The path $g(P) = u_1 u_2 \dots u_r y x w_1 w_2 \dots w_t$ is of type $P(\alpha^i)$ with respect to this enumeration. Also, let $g(P') = I'_1 I'_2 \dots I'_s = u'_1 u'_2 \dots u'_r y' x' w'_1 w'_2 \dots w'_t$ where x' is the origin of the block I'_i of length $\alpha_i - 1$, and $y' \in I'_{i-1}$ its predecessor, and the arc (x', y') replaced the arcs (v, x') and (v, y') in $P' = u'_1 u'_2 \dots u'_r y' v x' w'_1 w'_2 \dots w'_t$. The path $g(P') = u'_1 u'_2 \dots u'_r y' x' w'_1 w'_2 \dots w'_t$ is of type $P(\alpha^i)$ with respect to this enumeration. Since $g(P) = g(P')$ then we either have $(u_1, u_2, \dots, u_r, y, x, w_1, w_2, \dots, w_t) = (u'_1, u'_2, \dots, u'_r, y', x', w'_1, w'_2, \dots, w'_t)$ or we have $(u_1, u_2, \dots, u_r, y, x, w_1, w_2, \dots, w_t) = (w'_t, w'_{t-1}, \dots, w'_1, x', y', u'_r, u'_{r-1}, \dots, u'_1)$. If the second case is true, then the path $g(P') = w'_t w'_{t-1} \dots w'_1 x' y' u'_r u'_{r-1} \dots u'_1$ is of type $P(\alpha^i)$ with respect to this enumeration, which is impossible since α^i is not symmetric. Thus, only the first case is true, and adding the arcs $(v, y) = (v, y')$ and $(v, x) = (v, x')$ we get $P = u_1 u_2 \dots u_r y v x w_1 w_2 \dots w_t = P' = u'_1 u'_2 \dots u'_r y' v x' w'_1 w'_2 \dots w'_t$. The case $\alpha_i < 0$ is similar.

Since g is a bijection, then $f_{TTn}(\alpha) = \sum_{i=1}^{\frac{s}{2}} f_{\tilde{T}}(\alpha^i)$.

Since by induction we have $f_{\tilde{T}}(\alpha^i) = \mathcal{F}(\alpha^i)$ because the order of \tilde{T} is $n - 1$ and all α^i are not symmetric, then

$$f_{TTn}(\alpha) = \sum_{i=1}^{\frac{s}{2}} f_{\tilde{T}}(\alpha^i) = \sum_{i=1}^{\frac{s}{2}} \mathcal{F}(\alpha^i) = \frac{1}{2} \sum_{i=1}^s \mathcal{F}(\alpha^i) = \frac{1}{2} \mathcal{F}(\alpha).$$

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- The tuple α is not symmetric.

The proof will also be done by induction on the order of the tournament TTn . The smallest case of not symmetric α is $\alpha = (2)$ (i.e. directed Hamiltonian paths) which corresponds also to the transitive tournament of order 3 (acyclic triangle). In this tournament, $f_{TT3}(2) = 1$ and $\mathcal{F}(2) = 1$ by the definition of the mapping \mathcal{F} . So $f_{TT3}(2) = \mathcal{F}(2)$. Suppose that the statement is true for transitive tournaments of order $l \leq n - 1$, and let's prove it for n .

Since α is not symmetric, we have by Lemma 3.1.4 that $\alpha^i \neq \overline{\alpha^j} \forall 1 \leq i, j \leq s$. As a result, if we consider the transitive tournament $\tilde{T} = TTn - \{v\}$ where v is the source of TTn , then also by Proposition 3.1.3, $\mathcal{P}_{\tilde{T}}(\alpha^i) \neq \mathcal{P}_{\tilde{T}}(\alpha^j) \forall 1 \leq i, j \leq s$, so we have $\mathcal{P}_{\tilde{T}}(\alpha^i) \cap \mathcal{P}_{\tilde{T}}(\alpha^j) = \emptyset \forall 1 \leq i, j \leq s$, hence $\mathcal{P}_{\tilde{T}}(\alpha^1), \mathcal{P}_{\tilde{T}}(\alpha^2), \dots, \mathcal{P}_{\tilde{T}}(\alpha^s)$ are all pairwise disjoint.

Consider the correspondence:

$$\begin{aligned} g' : \quad \mathcal{P}_{TTn}(\alpha) &\longrightarrow \cup_{i=1}^s \mathcal{P}_{\tilde{T}}(\alpha^i) \\ P &\longrightarrow g'(P) = P - \langle v \rangle \cup \langle \{x, y\} \rangle, \end{aligned}$$

where v is the source of TTn , x is the predecessor of v on P , and y its successor on P if any.

As in the previous case, we can prove that g' is a surjective mapping. However, g' is not always injective. In fact, let $P' \in \cup_{i=1}^s \mathcal{P}_{\tilde{T}}(\alpha^i)$, and let's find how many $P \in \mathcal{P}_{TTn}(\alpha)$ there exist, such that $g'(P) = P'$. We consider two cases:

1. All α^i are not symmetric.

Then following the same arguments as given in the first case to prove that g' is injective, we may prove that P' has only one antecedent in $\mathcal{P}_{TTn}(\alpha)$ thus g' is injective. So g' is a bijection.

Since g' is a bijection, then $f_{TTn}(\alpha) = \sum_{i=1}^s f_{\tilde{T}}(\alpha^i)$.

Since by induction we have $f_{\tilde{T}}(\alpha^i) = \mathcal{F}(\alpha^i)$ (the order of \tilde{T} is $n - 1$ and all α^i are not symmetric), then

$$f_{TTn}(\alpha) = \sum_{i=1}^s f_{\tilde{T}}(\alpha^i) = \sum_{i=1}^s \mathcal{F}(\alpha^i) = \mathcal{F}(\alpha).$$

2. There exists $1 \leq i_0 \leq s$ such that α^{i_0} is symmetric.

Then by Lemma 3.1.5, all α^j , $1 \leq j \neq i_0 \leq s$, are not symmetric.

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Since $P' \in \cup_{i=1}^s \mathcal{P}_{\tilde{T}}(\alpha^i)$, then $\exists 1 \leq i \leq s$ such that $P' \in \mathcal{P}_{\tilde{T}}(\alpha^i)$. Suppose that $\alpha_i > 0$, then $P' \in \mathcal{P}_{\tilde{T}}(\alpha_1, \dots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \dots, \alpha_s)$. (The case $\alpha_i < 0$ is similar).

Write $P' = I_1 I_2 \dots I_s = u_1 u_2 \dots u_r y x w_1 w_2 \dots w_t$ where x is the origin of the block I_i of length $\alpha_i - 1$, and $y \in I_{i-1}$ its predecessor.

If $i = i_0$, then α^i is symmetric, then P' is also of type $P(\alpha^i)$ with respect to the other enumeration: $w_t w_{t-1} \dots w_1 x y u_r u_{r-1} \dots u_1$ and we rewrite P' with respect to this enumeration as $P' = I'_1 I'_2 \dots I'_s = z_1 z_2 \dots z_r y' x' v_1 v_2 \dots v_t$ where x' is the origin of the block I'_i of length $\alpha_i - 1$, and $y' \in I'_{i-1}$ its predecessor. So now, we consider the two paths $P_1 = u_1 u_2 \dots u_r y v x w_1 w_2 \dots w_t$ where the arc (x, y) is replaced by the arcs (v, x) and (v, y) , and $P_2 = z_1 z_2 \dots z_r y' v x' v_1 v_2 \dots v_t$ where the arc (x', y') is replaced by the arcs (v, x') and (v, y') . They are distinct and both of type $P(\alpha)$, and they are the only ones such that $g'(P_1) = g'(P_2) = P'$.

If $i \neq i_0$, then α^i is not symmetric, thus P' can't be of type $P(\alpha^i)$ with respect to the other enumeration: $w_t w_{t-1} \dots w_1 x y u_r u_{r-1} \dots u_1$. So there is only one path $P = u_1 u_2 \dots u_r y v x w_1 w_2 \dots w_t$ where the arc (x, y) in P' is replaced by the arcs (v, x) and (v, y) , that is of type $P(\alpha)$, and such that $g'(P) = P'$.

As a result, we have: $f_{TTn}(\alpha) = 2 \cdot f_{\tilde{T}}(\alpha^{i_0}) + \sum_{i \neq i_0, i=1}^s f_{\tilde{T}}(\alpha^i)$.

By induction, $f_{\tilde{T}}(\alpha^{i_0}) = \frac{1}{2} \mathcal{F}(\alpha^{i_0})$, because α^{i_0} is symmetric, and $\forall i \neq i_0$, $f_{\tilde{T}}(\alpha^i) = \mathcal{F}(\alpha^i)$, because α^i is not symmetric, thus

$$f_{TTn}(\alpha) = \sum_{i=1}^s \mathcal{F}(\alpha^i) = \mathcal{F}(\alpha).$$

□

3.2 Some properties of \mathcal{F}

In this section, we will study the path-function \mathcal{F} , and the properties that may arise.

As we previously mentioned, we have $f_{TTn}(\alpha) = f_{TTn}(-\alpha) \forall \alpha \in \mathbb{K}_s$. For that reason, we will redefine the mapping f by removing the signs of the components of the tuples.

Let $\mathcal{N} = \{(\alpha_1, \dots, \alpha_s) \in \mathbb{N}^{s*}, s \in \mathbb{N}^*\}$, \mathcal{F} is defined by:

$$\begin{aligned} \mathcal{F}: \quad \mathcal{N} &\longrightarrow \mathbb{N} \\ (\alpha_1, \alpha_2, \dots, \alpha_s) &\longrightarrow \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_s), \end{aligned}$$

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under the recurrence relation:

$$\begin{aligned} \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_s) = & \mathcal{F}(\alpha_1 - 1, \alpha_2, \dots, \alpha_s) + \mathcal{F}(\alpha_1, \alpha_2 - 1, \dots, \alpha_s) \\ & + \dots + \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_s - 1) \end{aligned}$$

verifying the properties:

1. $\forall t \in \mathbb{N}^*, \mathcal{F}(0, \alpha_2, \dots, \alpha_t) = \mathcal{F}(\alpha_2, \dots, \alpha_t),$
2. $\forall t' \in \mathbb{N}^*, \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_{t'}, 0) = \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_{t'}),$
3. $\forall r \in \mathbb{N}^*, \mathcal{F}(\alpha_1, \dots, \alpha_r, 0, \alpha_{r+2}, \dots, \alpha_s) = \mathcal{F}(\alpha_1, \dots, \alpha_r + \alpha_{r+2}, \dots, \alpha_s),$
4. $\forall \alpha \in \mathbb{N}^*, \mathcal{F}(\alpha) = 1.$

Remark 3.2.1. With this definition of \mathcal{F} , $\forall \alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{K}_s$, the number computed now by \mathcal{F} is $\mathcal{F}(|\alpha_1|, |\alpha_2|, \dots, |\alpha_s|)$, which will be either $f_{TTn}(\alpha)$ or $f_{TTn}(-\alpha)$ if α is non symmetric (resp. either $\frac{f_{TTn}(\alpha)}{2}$ or $\frac{f_{TTn}(-\alpha)}{2}$ if α is symmetric).

Note that this does not exclude the fact that the sets $\mathcal{P}_{TTn}(\alpha)$ and $\mathcal{P}_{TTn}(-\alpha)$ can be either disjoint or the same, since the sets $\mathcal{P}_T(\alpha)$, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{K}_s$, $\sum_{i=1}^s |\alpha_i| = n - 1$, form a partition of the set \mathcal{P}_T of all the oriented Hamiltonian paths in a tournament T of order n .

Obviously, $\mathcal{F}(a_1, a_2, \dots, a_s) = \mathcal{F}(a_s, a_{s-1}, \dots, a_1)$, $\forall \alpha = (\alpha_1, \dots, \alpha_s) \in \mathcal{N}$, since \mathcal{F} is a symmetric function. But, we also have the following properties:

Remark 3.2.2. For all $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathcal{N}$, $s \geq 2$, and for all $m \in \mathbb{N}^*$ we have:

$$\mathcal{F}(m) < \mathcal{F}(\alpha_1, \alpha_2, \dots, \alpha_s),$$

Since by the definition of \mathcal{F} we have $\mathcal{F}(m) = 1$ and $\mathcal{F}(\alpha_1, \dots, \alpha_s) > 1$ for $s > 1$.

Proposition 3.2.3. For all $m, n, m', n' \in \mathbb{N}^*$, $m + n = m' + n'$, we have:

- $\mathcal{F}(m, n) = C_{m+n}^m,$
- $\mathcal{F}(m, n) < \mathcal{F}(m', n') \iff mn < m'n'.$

Proof. The first statement is a direct result of Proposition 3.0.1, Remark 3.0.2 and Theorem 3.1.1. For the second statement, we know that for $m, n, m', n' \in \mathbb{N}^*$ such that $m + n = m' + n' = p$ and $mn < m'n'$ we have $C_p^m = C_p^n < C_p^{m'} = C_p^{n'}$. This is equivalent to $\mathcal{F}(m, n) < \mathcal{F}(m', n')$, since $\mathcal{F}(m, n) = C_{m+n}^m$. \square

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Proposition 3.2.4. *For all $m, n, t, a \in \mathbb{N}^*$, $a + m + n = a + t$, we have:*

$$\mathcal{F}(a, t) < \mathcal{F}(a, m, n).$$

Proof. The proof will be done by induction on $p = a + t = a + m + n$, that is by induction on the order $p + 1$ of a transitive tournament.

The smallest tournament in which we can compare $\mathcal{F}(a, t)$ and $\mathcal{F}(a, m, n)$ has 4 vertices, that is $p = 3$, with $a = 1$, $t = 2$, and $m = n = 1$.

We have $\mathcal{F}(1, 2) = \mathcal{F}(2) + \mathcal{F}(1, 1)$, while $\mathcal{F}(1, 1, 1) = \mathcal{F}(1, 1) + \mathcal{F}(2) + \mathcal{F}(1, 1)$ and the inequality follows.

Suppose that the inequality is true for a tournament of order less than or equal to p , and let T be a tournament of order $p + 1$, with $a + t = a + m + n = p$.

Note that we necessarily have m and $n < t$.

We have $\mathcal{F}(a, t) = \mathcal{F}(a - 1, t) + \mathcal{F}(a, t - 1)$, and $\mathcal{F}(a, m, n) = \mathcal{F}(a - 1, m, n) + \mathcal{F}(a, m - 1, n) + \mathcal{F}(a, m, n - 1)$.

First, we have $\mathcal{F}(a - 1, t) < \mathcal{F}(a - 1, m, n)$. In fact, if $a - 1 \neq 0$, it is true by induction, while if $a - 1 = 0$, it follows from Remark 3.2.2 that $\mathcal{F}(t) < \mathcal{F}(m, n)$.

Now, if $m > 1$ (resp. $n > 1$), we have by induction that $\mathcal{F}(a, t - 1) < \mathcal{F}(a, m - 1, n)$ (resp. $\mathcal{F}(a, t - 1) < \mathcal{F}(a, m, n - 1)$), and the inequality follows.

If $m = n = 1$ (which implies $t = 2$), then $\mathcal{F}(a, t - 1) = \mathcal{F}(a, 1) < \mathcal{F}(a, m - 1, n) + \mathcal{F}(a, m, n - 1) = \mathcal{F}(a + 1) + \mathcal{F}(a, 1)$, and we get the desired inequality. \square

Proposition 3.2.5. *For all $m, n, m', n', a \in \mathbb{N}^*$, $a + m + n = a + m' + n'$, we have:*

- $\mathcal{F}(a, m, n) < \mathcal{F}(a, m', n') \iff (mn < m'n') \text{ or } (mn = m'n', m < n)$.
- $\mathcal{F}(m, a, n) < \mathcal{F}(m', a, n') \iff (mn < m'n')$.

Proof. We begin by the first statement.

Suppose first that $mn = m'n'$ with $m < n$.

Since $m + n = m' + n'$ then $m = n'$ and $n = m'$. So we need to compare $\mathcal{F}(a, m, n)$ and $\mathcal{F}(a, n, m)$. The proof will be done by induction on $p = m + n$ (so by induction on the order $p + 1$ of the tournament).

If $p = 4$ (tournament on 5 vertices) then the initial step is to compare $\mathcal{F}(1, 1, 2)$ and $\mathcal{F}(1, 2, 1)$. $\mathcal{F}(1, 1, 2) = \mathcal{F}(1, 2) + \mathcal{F}(3) + \mathcal{F}(1, 1, 1)$ and $\mathcal{F}(1, 2, 1) = \mathcal{F}(2, 1) + \mathcal{F}(1, 1, 1) + \mathcal{F}(1, 2)$. Since $\mathcal{F}(3) < \mathcal{F}(2, 1)$ by Remark 3.2.2, the result follows.

Suppose it's true till $p - 1$, i.e. for a tournament of order less than or equal p . So let T be a tournament of order $p + 1$, $a + m + n = p$, $m < n$.

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We have $\mathcal{F}(a, m, n) = \mathcal{F}(a, m - 1, n) + \mathcal{F}(a, m, n - 1) + \mathcal{F}(a - 1, m, n)$ and $\mathcal{F}(a, n, m) = \mathcal{F}(a, n, m - 1) + \mathcal{F}(a, n - 1, m) + \mathcal{F}(a - 1, n, m)$.

By induction, $m - 1 < n$ so $\mathcal{F}(a, m - 1, n) < \mathcal{F}(a, n, m - 1)$ (note that if $m = 1$, then $\mathcal{F}(a, m - 1, n) = \mathcal{F}(a + n) = 1 < \mathcal{F}(a, n, m - 1) = \mathcal{F}(a, n)$ by Remark 3.2.2), and $m \leq n - 1$ so $\mathcal{F}(a, m, n - 1) \leq \mathcal{F}(a, n - 1, m)$, and if $a - 1 \neq 0$, $\mathcal{F}(a - 1, m, n) < \mathcal{F}(a - 1, n, m)$, while if $a - 1 = 0$ then $\mathcal{F}(a - 1, m, n) = \mathcal{F}(m, n) = \mathcal{F}(n, m) = \mathcal{F}(a - 1, n, m)$. Hence $\mathcal{F}(a, m, n) < \mathcal{F}(a, n, m)$.

Suppose now that $mn < m'n'$.

Since by the first case, $\mathcal{F}(a, n, m) < \mathcal{F}(a, m, n)$ if $n < m$ and $\mathcal{F}(a, m', n') < \mathcal{F}(a, n', m')$ if $m' < n'$, let us suppose that $n \leq m$ and $m' \leq n'$, and prove that $\mathcal{F}(a, m, n) < \mathcal{F}(a, m', n')$.

Remark that $mn < m'n'$ and $m + n = m' + n'$ both imply that $m - n \geq 2$, because we can't find m' and n' such that $m + n = m' + n'$ and $mn < m'n'$ when $m = n$ or $m = n + 1$ because $m.n$ is maximal. We also have $m', n' > n$ and $m', n' < m$.

Let $p = a + m + n = a + m' + n'$. We will also do the proof by induction on p .

The smallest tournament to satisfy $mn < m'n'$ has 6 vertices (i.e. $p = 5$). That is for $m = 3$, $n = 1$, $m' = 2$, $n' = 2$, $a = 1$.

We have $\mathcal{F}(1, 3, 1) = \mathcal{F}(3, 1) + \mathcal{F}(1, 2, 1) + \mathcal{F}(1, 3)$ and $\mathcal{F}(1, 2, 2) = \mathcal{F}(2, 2) + \mathcal{F}(1, 1, 2) + \mathcal{F}(1, 2, 1)$. By Proposition 3.2.3 we have $\mathcal{F}(3, 1) < \mathcal{F}(2, 2)$ and by a simple calculation we have $\mathcal{F}(1, 3) < \mathcal{F}(1, 1, 2)$, and then we get $\mathcal{F}(1, 3, 1) < \mathcal{F}(1, 2, 2)$.

Suppose that the statement is true till $p - 1$, that is for a tournament of order less than or equal p . Let T be a tournament of order $p + 1$, $a + m + n = a + m' + n' = p$, with $mn < m'n'$. We have:

$\mathcal{F}(a, m, n) = \mathcal{F}(a - 1, m, n) + \mathcal{F}(a, m - 1, n) + \mathcal{F}(a, m, n - 1)$ and $\mathcal{F}(a, m', n') = \mathcal{F}(a - 1, m', n') + \mathcal{F}(a, m' - 1, n') + \mathcal{F}(a, m', n' - 1)$.

We have $\mathcal{F}(a - 1, m, n) < \mathcal{F}(a - 1, m', n')$ (by induction if $a > 1$ and by Proposition 3.2.3 if $a = 1$). Now, $m > n$ implies that

$$(m - 1)n = \frac{(p - a)^2 - (m - n - 1)^2}{4} \quad \text{and} \quad m(n - 1) = \frac{(p - a)^2 - (m - n + 1)^2}{4}.$$

We will consider two cases:

- Case 1: $m' < n'$. Hence $(m' - 1)n' = \frac{(p - a)^2 - (n' - m' + 1)^2}{4}$ and $m'(n' - 1) = \frac{(p - a)^2 - (n' - m' - 1)^2}{4}$. As a result, $(m - 1)n < m'(n' - 1)$ and $m(n - 1) < (m' - 1)n'$, so by induction we have $\mathcal{F}(a, m - 1, n) < \mathcal{F}(a, m', n' - 1)$ (since $m - 1 \geq n$ and $m' \leq n' - 1$) and $\mathcal{F}(a, m, n - 1) < \mathcal{F}(a, m' - 1, n')$ (by induction if $n - 1 \neq 0$ and by Proposition 3.2.4 if $n - 1 = 0$). So we finally get $\mathcal{F}(a, m, n) < \mathcal{F}(a, m', n')$.

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- Case 2: $m' = n'$. Thus $(m' - 1)n' = m'(n' - 1) = \frac{(p - a)^2 - 1}{4}$.
 If $m - n > 2$ then $m - n - 1 > 1$, so $(m - 1)n < m'(n' - 1)$ thus $\mathcal{F}(a, m - 1, n) < \mathcal{F}(a, n' - 1, m')$ (by induction, since $m - 1 \geq n$ and $n' - 1 \leq m'$) $< \mathcal{F}(a, m', n' - 1)$ (by the case treated above). And since $m(n - 1) < (m' - 1)n'$ then $\mathcal{F}(a, m, n - 1) < \mathcal{F}(a, m' - 1, n')$ (by induction if $n - 1 \neq 0$ and by Proposition 3.2.4 if $n - 1 = 0$). We finally get $\mathcal{F}(a, m, n) < \mathcal{F}(a, m'n')$.
 If $m - n = 2$ then $m = n + 2$, and since $m > n' = m' > n$ (bc $m + n = n' + m'$ and $mn < m'n'$) then $n' = m' = n + 1$. We have: $\mathcal{F}(a, m, n) = \mathcal{F}(a, n + 2, n) = \mathcal{F}(a - 1, n + 2, n) + \mathcal{F}(a, n + 1, n) + \mathcal{F}(a, n + 2, n - 1)$ and $\mathcal{F}(a, m', n') = \mathcal{F}(a, n + 1, n + 1) = \mathcal{F}(a - 1, n + 1, n + 1) + \mathcal{F}(a, n, n + 1) + \mathcal{F}(a, n + 1, n)$. Now $(n + 2)(n - 1) = n^2 + n - 2$ and $n(n + 1) = n^2 + n$ so $(n + 2)(n - 1) < n(n + 1)$, so $\mathcal{F}(a, n + 2, n - 1) < \mathcal{F}(a, n, n + 1)$ (by induction if $n - 1 \neq 0$ and by Proposition 3.2.4 if $n - 1 = 0$). Finally we get $\mathcal{F}(a, m, n) < \mathcal{F}(a, m'n')$.

For the necessary condition, suppose that $\mathcal{F}(a, m, n) < \mathcal{F}(a, m', n')$ with $mn \geq m'n'$ and let's prove $mn = m'n'$ with $m < n$. If $mn > m'n'$ then $\mathcal{F}(a, m, n) > \mathcal{F}(a, m', n')$ by the sufficient condition, which is a contradiction. If $mn = m'n'$ and since $m + n = m' + n'$ we have many cases: If $m = m', n = n'$ then $\mathcal{F}(a, m, n) = \mathcal{F}(a, m', n')$, a contradiction. If $m = n', n = m'$ then if $m > n$, $\mathcal{F}(a, m, n) > \mathcal{F}(a, m', n')$ by the sufficient condition, a contradiction too. Thus $m < n$.

Now we prove the second statement.

For the sufficient condition, the proof is similar to the proof of the previous one for $mn < m'n'$. Only note that if $a = 1$, we will have in the induction that $\mathcal{F}(m, a - 1, n) = \mathcal{F}(m + n)$ and $\mathcal{F}(m', a - 1, n') = \mathcal{F}(m' + n')$, so $\mathcal{F}(m, a - 1, n) = \mathcal{F}(m', a - 1, n') = 1$. But this won't cause a problem in proving $\mathcal{F}(m, a, n) < \mathcal{F}(m', a, n')$ for $mn < m'n'$ because the other terms will lead strict inequalities. For the necessary condition, suppose that $\mathcal{F}(m, a, n) < \mathcal{F}(m', a, n')$ and let's prove that $mn < m'n'$. If $mn > m'n'$ then $\mathcal{F}(m, a, n) > \mathcal{F}(m', a, n')$ by the sufficient condition, a contradiction. If $mn = m'n'$ and since $m + n = m' + n'$, then either $m = m', n = n'$ or $m = n', n = m'$. But in both cases we get $\mathcal{F}(m, a, n) = \mathcal{F}(m', a, n')$, a contradiction. Thus $mn < m'n'$. \square

Proposition 3.2.6. *For all $m, n, a, b \in \mathbb{N}^*$ we have:*

$$m < n \text{ and } a < b \Rightarrow \mathcal{F}(m, a, b, n) > \mathcal{F}(m, b, a, n).$$

Proof. The proof is also done by induction on $p = m + a + b + n$. The initial step is to compare $\mathcal{F}(1, 2, 1, 2)$ and $\mathcal{F}(1, 1, 2, 2)$, i.e. for $p = 6$, $m = 1$, $n = 2$, $a = 1$, $b = 2$. We have

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$\mathcal{F}(1, 2, 1, 2) = \mathcal{F}(2, 1, 2) + \mathcal{F}(1, 1, 1, 2) + \mathcal{F}(1, 4) + \mathcal{F}(1, 2, 1, 1)$ and $\mathcal{F}(1, 1, 2, 2) = \mathcal{F}(1, 2, 2) + \mathcal{F}(3, 2) + \mathcal{F}(1, 1, 1, 2) + \mathcal{F}(1, 1, 2, 1)$. By Proposition 3.2.3 we have $\mathcal{F}(1, 4) < \mathcal{F}(3, 2)$ and by Proposition 3.2.5 we have $\mathcal{F}(2, 1, 2) < \mathcal{F}(1, 2, 2)$, moreover $\mathcal{F}(1, 1, 2, 1) = \mathcal{F}(1, 2, 1, 1)$, and we get $\mathcal{F}(1, 2, 1, 2) < \mathcal{F}(1, 1, 2, 2)$.

Suppose that the statement is true till $p - 1$ (tournament having at most p vertices). So let T be a tournament on $p + 1$ vertices, $m + n + a + b = p$, and suppose $m < n$, $a < b$.

We have $\mathcal{F}(m, a, b, n) = \mathcal{F}(m - 1, a, b, n) + \mathcal{F}(m, a - 1, b, n) + \mathcal{F}(m, a, b - 1, n) + \mathcal{F}(m, a, b, n - 1)$ and $\mathcal{F}(m, b, a, n) = \mathcal{F}(m - 1, b, a, n) + \mathcal{F}(m, b - 1, a, n) + \mathcal{F}(m, b, a - 1, n) + \mathcal{F}(m, b, a, n - 1)$.

We have the following inequalities:

- $\mathcal{F}(m, a - 1, b, n) > \mathcal{F}(m, b, a - 1, n)$. In fact, $a - 1 < b$, so: If $a - 1 \neq 0$, by induction we have $\mathcal{F}(m, a - 1, b, n) > \mathcal{F}(m, b, a - 1, n)$. If $a - 1 = 0$ then $\mathcal{F}(m, a - 1, b, n) = \mathcal{F}(m + b, n)$ and $\mathcal{F}(m, b, a - 1, n) = \mathcal{F}(m, b + n)$. However, $(m + b) + n = m + (b + n)$ and $(m + b)n = mn + bn > mn + bm = m(b + n)$, so by Proposition 3.2.3 we have $\mathcal{F}(m + b, n) > \mathcal{F}(m, b + n)$.
- $\mathcal{F}(m - 1, a, b, n) > \mathcal{F}(m - 1, b, a, n)$. In fact, if $m - 1 > 0$ then it's true by induction. If $m - 1 = 0$ then since $b > a$, we have $\mathcal{F}(a, b, n) > \mathcal{F}(b, a, n)$ by Proposition 3.2.5.
- $\mathcal{F}(m, a, b - 1, n) \geq \mathcal{F}(m, b - 1, a, n)$. In fact, if $a < b - 1$ then by induction it's true. If $a = b - 1$, then $\mathcal{F}(m, a, b - 1, n) = \mathcal{F}(m, a, a, n) = \mathcal{F}(m, b - 1, a, n)$.
- $\mathcal{F}(m, a, b, n - 1) \geq \mathcal{F}(m, b, a, n - 1)$. In fact, if $m < n - 1$ it's true by induction. If $m = n - 1$ then $\mathcal{F}(m, a, b, n - 1) = \mathcal{F}(m, a, b, m) = \mathcal{F}(m, b, a, m) = \mathcal{F}(m, b, a, n - 1)$.

Then using all these inequalities, we get the result. \square

On the other hand, using a program (which will be discussed in the next section) for computing the function \mathcal{F} , one may prove the following statements wrong, despite many of them might seem to be true:

- For $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{N}^{s*}$ and $\beta = (\beta_1, \dots, \beta_t) \in \mathbb{N}^{t*}$, $2 \leq s < t$, $\sum_{i=1}^s \alpha_i = \sum_{i=1}^t \beta_i$, then $\mathcal{F}(\alpha) < \mathcal{F}(\beta)$.
Counter-example: $\mathcal{F}(3, 3) = 20 = \mathcal{F}(1, 1, 4)$, $\mathcal{F}(3, 4) = 35 > \mathcal{F}(1, 1, 5) = 27$.
- For $a, m, n, m', n' \in \mathbb{N}^*$ with $a + m + n = a + m' + n'$ then $mn < m'n' \Rightarrow \mathcal{F}(a, m, n) < \mathcal{F}(a, m', n')$.
Counter-example: $\mathcal{F}(1, 2, 4) = 85 > \mathcal{F}(3, 1, 3) = 69$ while $2 \cdot 4 = 8 < 3 \cdot 3 = 9$.

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- For $a, m, n, m', n' \in \mathbb{N}^*$ with $a + m + n = a + m' + n'$ then $mn < m'n' \Rightarrow \mathcal{F}(m, a, n) < \mathcal{F}(a, m', n')$.

Counter-examples: $\mathcal{F}(2, 11, 5) = 637924 > \mathcal{F}(11, 3, 4) = 631787$, $\mathcal{F}(2, 12, 5) = 1015988 > \mathcal{F}(12, 3, 4) = 984503$, while $2.5 = 10 < 3, 4 = 12$). However, for the same m, n, m', n' but for $a < 11$, the statement is true, for example: $\mathcal{F}(2, 7, 5) = 65857 < \mathcal{F}(7, 3, 4) = 74502$, $\mathcal{F}(2, 8, 5) = 125411 < \mathcal{F}(8, 3, 4) = 135927$, $\mathcal{F}(2, 9, 5) = 225589 < \mathcal{F}(9, 3, 4) = 236027$, $\mathcal{F}(2, 10, 5) = 387023 < \mathcal{F}(10, 3, 4) = 393107$.

So we remark that the first term is smaller than the second term, but the difference between them decreases as a tends to 10, and then for $a = 11$, the second term becomes bigger, and the difference between the two terms will increase, as a gets bigger.

We may also notice that the number in the middle of the tuple plays a role: in the first tuple it is a , while in the second tuple it is m' , and at some point where a becomes big enough with respect to m' , the first term becomes bigger than the second one.

- For $a, a', m, n, m', n' \in \mathbb{N}^*$ with $a + m + n = a' + m' + n'$ then $amn < a'm'n' \Rightarrow \mathcal{F}(a, m, n) < \mathcal{F}(a', m', n')$.

Counter-example: $\mathcal{F}(6, 7, 3) = 835549 > \mathcal{F}(4, 4, 8) = 614823$ with $6.7.3 = 126 < 4.4.8 = 128$. However, $\mathcal{F}(6, 3, 7) = 529957 < \mathcal{F}(4, 8, 4) = 806651$.

So putting the big number in the middle of the first tuple and the small number in the middle of the second tuple is what helped to find the counter-example.

But the difference should be big enough. For example: $\mathcal{F}(5, 7, 2) = 65857 < \mathcal{F}(4, 4, 6) = 150723$ and $\mathcal{F}(5, 2, 7) = 33606 < \mathcal{F}(4, 6, 4) = 178751$ with $5.7.2 = 70 > 4.4.6 = 96$. Also, $\mathcal{F}(4, 6, 1) = 2190 < \mathcal{F}(3, 3, 5) = 6566$ and $\mathcal{F}(4, 1, 6) = 791 < \mathcal{F}(3, 5, 3) = 8051$ with $4.6.1 = 24 < 3.3.5 = 45$.

- Let $(\alpha_1, \dots, \alpha_s)$ and $(\beta_1, \dots, \beta_s) \in \mathbb{N}^s$, α_i and $\beta_i \neq 0$, $\forall 1 \leq i \leq s$, and $\sum_{i=1}^s \alpha_i = \sum_{i=1}^s \beta_i$. If $(\beta_1, \beta_2, \dots, \beta_s) \neq (\alpha_1, \alpha_2, \dots, \alpha_s)$ and $(\beta_1, \beta_2, \dots, \beta_s) \neq (\alpha_s, \alpha_{s-1}, \dots, \alpha_1)$, then $\mathcal{F}(\alpha_1, \dots, \alpha_s) \neq \mathcal{F}(\beta_1, \dots, \beta_s)$.

Counter-example: $\mathcal{F}(1, 3, 1) = \mathcal{F}(2, 1, 2) = 19$, $\mathcal{F}(1, 2, 3, 1) = \mathcal{F}(2, 1, 2, 2) = 315$, $\mathcal{F}(2, 4, 2) = \mathcal{F}(3, 2, 3) = 379$.

However, we know nothing yet about the validity of many statements, as for example:

- For $a_1, \dots, a_s, a'_1, \dots, a'_s, m, n, m', n' \in \mathbb{N}^*$ we have $\mathcal{F}(a_1, \dots, a_s, m, n) < \mathcal{F}(a_1, \dots, a_s, m', n') \iff (mn < m'n') \text{ or } (mn = m'n', m < n)$.
- For $a, b, m, n, a', b' \in \mathbb{N}^*$, we have $(m < n) \text{ and } (ab < a'b') \Rightarrow \mathcal{F}(m, a, b, n) < \mathcal{F}(m, a', b', n)$.

The function \mathcal{F} which seems to be rich in properties, allows us to construct a program computing the number of oriented paths in a transitive tournament. By comparing the different

cases, we are led to state a conjecture about paths in transitive tournaments, and discuss an interesting property about antirected Hamiltonian paths in TTn .

3.3 Algorithmic approach to \mathcal{F} and some observations

In this last section, we first introduce the program, built using Python, to compute the values of the path-function \mathcal{F} defined in the previous section.

Listing 3.1: Program that computes the values of the function \mathcal{F} .

```
1
2 """
3 This routine represents the mapping " $\mathcal{F}$ ".
4 It reduces a tuple of positive integers into a single
5 positive integer according to the recurrence relation
6 and the 4 properties satisfied by " $\mathcal{F}$ ".
7 """
8
9 def f(a):
10     l = len(a)
11
12     assert (l > 0), "f() is undefined"
13     assert (l > 1 or a[0] > 0), "f(0) is undefined"
14
15     if l == 1:
16         return 1
17     elif a[0] == 0:
18         return f(a[1:])
19     elif a[-1] == 0:
20         return f(a[:-1])
21     else:
22         try:
23             i = a.index(0)
24             return f(
25                 a[:i - 1] +
26                 [a[i - 1] + a[i + 1]] +
27                 a[i + 2:]
28             )
29         except ValueError:
30             # If 'a' doesn't contain any zeros
31             return sum(
32                 f(a[:i] + [a[i] - 1] + a[i + 1:])
33                 for i in range(1)
34             )
35
36 import time
37 import sys
```

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```
38
39 # Change the next line to your needs
40 a = [1, 2, 1, 1]
41
42 start_time = time.time()
43 ans = f(a)
44 end_time = time.time()
45
46 print('{} => {}'.format(a, ans))
47 print(
48     'Took: {:.3} seconds'.format(end_time - start_time),
49     file=sys.stderr
50 )
```

In order to investigate more about the properties of the path-function \mathcal{F} , we create a new program, based on the previous one, allowing us to compute all the possible values of $\mathcal{F}(a_1, \dots, a_s)$, $1 \leq s \leq p$, for a given $p = \sum_{i=1}^s a_i$.

Listing 3.2: Program that computes all values of $\mathcal{F}(a_1, \dots, a_s)$, $1 \leq s \leq p$, $p = \sum_{i=1}^s a_i$.

```
1
2 """
3 Define the memorization system
4 """
5
6 class memoize(dict):
7     def __init__(self, f):
8         self.f = f
9
10        # Will hold some statistics
11        self.total = 0
12        self.miss = 0
13
14        def __call__(self, a):
15            self.total += 1
16            return self[tuple(a)]
17
18        def __missing__(self, a):
19            self.miss += 1
20            result = self.f(list(a))
21            # Save the result for a
22            self[a] = result
23            # and for the reverse of a, since they are always equal
24            self[a[::-1]] = result
25            return result
26
27
```

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```
28
29 """
30 This routine represents the mapping " $\mathcal{F}$ ".
31 It reduces a tuple of positive integers into a single
32 positive integer according to the recurrence relation
33 and the 4 properties satisfied by " $\mathcal{F}$ ".
34 To significantly improve the performance (speed) of this
35 routine, a memorization system is used.
36 It allows a fast answer lookup for known tuples (already seen
37 before), by saving all the results in a lookup table with
38 their corresponding tuples and their reverse (since they get
39 reduced to the same result).
40 """
41
42 @memoize
43 def f(a):
44     l = len(a)
45
46     assert (l > 0), "f() is undefined"
47     assert (l > 1 or a[0] > 0), "f(0) is undefined"
48
49     if l == 1:
50         return 1
51     elif a[0] == 0:
52         return f(a[1:])
53     elif a[-1] == 0:
54         return f(a[:-1])
55     else:
56         try:
57             i = a.index(0)
58             return f(
59                 a[:i - 1] +
60                 [a[i - 1] + a[i + 1]] +
61                 a[i + 2:]
62             )
63         except ValueError:
64             # If 'a' doesn't contain any zeros
65             return sum(
66                 f(a[:i] + [a[i] - 1] + a[i + 1:])
67                 for i in range(1)
68             )
69
70
71 """
72 This routine generates a list of all possible tuples with
73 cardinality less than or equal to 'p', having the sum of
74 all its elements equal to 'p'.
75 """
76
```

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```
77
78 def all_permutations(p):
79     if p > 0:
80         yield [p]
81         for s in range(p - 1, 0, -1):
82             for a in all_permutations(p - s):
83                 yield [s] + a
84
85
86 """
87 This routine prints the results in a pretty format.
88 """
89
90 def print_results(p, results):
91     iw = len(str(len(results)))
92     aw = 3 * p
93     for (i, (a, ans)) in enumerate(results):
94         print('{:>{iw}}: {:<{aw}} => {:,}'.format(
95             i + 1, str(a), ans, iw=iw, aw=aw
96         ))
97
98
99 """
100 This is the main routine that combines all the previous ones.
101 It generates all the permutations and their corresponding
102 results from 'f', sorts them in ascending order
103 and then prints them along with some execution statistics.
104 """
105
106 def main(p):
107     import time
108     import sys
109
110     start_time = time.time()
111     results = [(a, f(a)) for a in all_permutations(p)]
112     f_time = time.time()
113
114     results.sort(key=lambda r: r[1])
115     print_results(p, results)
116     end_time = time.time()
117
118     print(
119         'Cache hit: {:.2%}'.format(1 - f.miss / f.total),
120         file=sys.stderr
121     )
122     print(
123         'Took to apply f: {:.3} seconds'.format(
124             f_time - start_time
125         ),
```

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```

126     file=sys.stderr
127 )
128 print(
129     'Took to sort and print: {:.3} seconds'.format(
130         end_time - f_time
131     ),
132     file=sys.stderr
133 )
134 print(
135     'Took in total: {:.3} seconds'.format(
136         end_time - start_time
137     ),
138     file=sys.stderr
139 )
140
141 # Change the next line to your needs
142 main(6)

```

Now we give the first lists of numbers computed using the above program, giving all the possible values of $\mathcal{F}(a_1, \dots, a_s)$, $1 \leq s \leq p$, for $p = \sum_{i=1}^s a_i$, and where $3 \leq p \leq 8$. (The answers are in ascending order).

For $p = 3$:

1: [3] => 1	3: [1, 2] => 3
2: [2, 1] => 3	4: [1, 1, 1] => 5

For $p = 4$:

1: [4] => 1	4: [2, 2] => 6	7: [1, 2, 1] => 11
2: [3, 1] => 4	5: [2, 1, 1] => 9	8: [1, 1, 1, 1] => 16
3: [1, 3] => 4	6: [1, 1, 2] => 9	

For $p = 5$:

1: [5] => 1	9: [1, 3, 1] => 19
2: [4, 1] => 5	10: [2, 2, 1] => 26
3: [1, 4] => 5	11: [1, 2, 2] => 26
4: [3, 2] => 10	12: [2, 1, 1, 1] => 35
5: [2, 3] => 10	13: [1, 1, 1, 2] => 35
6: [3, 1, 1] => 14	14: [1, 2, 1, 1] => 40
7: [1, 1, 3] => 14	15: [1, 1, 2, 1] => 40
8: [2, 1, 2] => 19	16: [1, 1, 1, 1, 1] => 61

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For $p = 6$:

1: [6]	=> 1	17: [1, 1, 1, 3]	=> 64
2: [5, 1]	=> 6	18: [2, 2, 2]	=> 71
3: [1, 5]	=> 6	19: [1, 3, 1, 1]	=> 78
4: [4, 2]	=> 15	20: [1, 1, 3, 1]	=> 78
5: [2, 4]	=> 15	21: [2, 1, 1, 2]	=> 90
6: [4, 1, 1]	=> 20	22: [2, 1, 2, 1]	=> 99
7: [3, 3]	=> 20	23: [1, 2, 1, 2]	=> 99
8: [1, 1, 4]	=> 20	24: [2, 2, 1, 1]	=> 111
9: [1, 4, 1]	=> 29	25: [1, 1, 2, 2]	=> 111
10: [3, 1, 2]	=> 34	26: [1, 2, 2, 1]	=> 132
11: [2, 1, 3]	=> 34	27: [2, 1, 1, 1, 1]	=> 155
12: [3, 2, 1]	=> 50	28: [1, 1, 1, 1, 2]	=> 155
13: [1, 2, 3]	=> 50	29: [1, 1, 2, 1, 1]	=> 169
14: [2, 3, 1]	=> 55	30: [1, 2, 1, 1, 1]	=> 181
15: [1, 3, 2]	=> 55	31: [1, 1, 1, 2, 1]	=> 181
16: [3, 1, 1, 1]	=> 64	32: [1, 1, 1, 1, 1, 1]	=> 272

For $p = 7$:

1: [7]	=> 1	18: [4, 1, 1, 1]	=> 105
2: [6, 1]	=> 7	19: [1, 1, 1, 4]	=> 105
3: [1, 6]	=> 7	20: [3, 3, 1]	=> 125
4: [5, 2]	=> 21	21: [1, 3, 3]	=> 125
5: [2, 5]	=> 21	22: [1, 4, 1, 1]	=> 133
6: [5, 1, 1]	=> 27	23: [1, 1, 4, 1]	=> 133
7: [1, 1, 5]	=> 27	24: [3, 2, 2]	=> 155
8: [4, 3]	=> 35	25: [2, 2, 3]	=> 155
9: [3, 4]	=> 35	26: [2, 3, 2]	=> 181
10: [1, 5, 1]	=> 41	27: [3, 1, 1, 2]	=> 189
11: [4, 1, 2]	=> 55	28: [2, 1, 1, 3]	=> 189
12: [2, 1, 4]	=> 55	29: [3, 1, 2, 1]	=> 203
13: [3, 1, 3]	=> 69	30: [1, 2, 1, 3]	=> 203
14: [4, 2, 1]	=> 85	31: [2, 1, 3, 1]	=> 217
15: [1, 2, 4]	=> 85	32: [1, 3, 1, 2]	=> 217
16: [2, 4, 1]	=> 99	33: [3, 2, 1, 1]	=> 245
17: [1, 4, 2]	=> 99	34: [1, 1, 2, 3]	=> 245

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35: [2, 3, 1, 1]	=> 259	50: [1, 1, 2, 1, 2]	=> 477
36: [1, 1, 3, 2]	=> 259	51: [2, 1, 1, 2, 1]	=> 531
37: [2, 2, 1, 2]	=> 315	52: [1, 2, 1, 1, 2]	=> 531
38: [2, 1, 2, 2]	=> 315	53: [2, 2, 1, 1, 1]	=> 573
39: [1, 3, 2, 1]	=> 315	54: [1, 1, 1, 2, 2]	=> 573
40: [1, 2, 3, 1]	=> 315	55: [1, 2, 1, 2, 1]	=> 589
41: [3, 1, 1, 1, 1]	=> 323	56: [1, 2, 2, 1, 1]	=> 643
42: [1, 1, 1, 1, 3]	=> 323	57: [1, 1, 2, 2, 1]	=> 643
43: [1, 1, 3, 1, 1]	=> 365	58: [2, 1, 1, 1, 1, 1]	=> 791
44: [1, 3, 1, 1, 1]	=> 407	59: [1, 1, 1, 1, 1, 2]	=> 791
45: [1, 1, 1, 3, 1]	=> 407	60: [1, 1, 2, 1, 1, 1]	=> 875
46: [2, 2, 2, 1]	=> 413	61: [1, 1, 1, 2, 1, 1]	=> 875
47: [1, 2, 2, 2]	=> 413	62: [1, 2, 1, 1, 1, 1]	=> 917
48: [2, 1, 1, 1, 2]	=> 449	63: [1, 1, 1, 1, 2, 1]	=> 917
49: [2, 1, 2, 1, 1]	=> 477	64: [1, 1, 1, 1, 1, 1, 1]	=> 1,385

For $p = 8$:

1: [8]	=> 1	21: [1, 5, 2]	=> 161
2: [7, 1]	=> 8	22: [1, 5, 1, 1]	=> 208
3: [1, 7]	=> 8	23: [1, 1, 5, 1]	=> 208
4: [6, 2]	=> 28	24: [4, 3, 1]	=> 245
5: [2, 6]	=> 28	25: [1, 3, 4]	=> 245
6: [6, 1, 1]	=> 35	26: [3, 4, 1]	=> 259
7: [1, 1, 6]	=> 35	27: [1, 4, 3]	=> 259
8: [1, 6, 1]	=> 55	28: [4, 2, 2]	=> 295
9: [5, 3]	=> 56	29: [2, 2, 4]	=> 295
10: [3, 5]	=> 56	30: [4, 1, 1, 2]	=> 350
11: [4, 4]	=> 70	31: [2, 1, 1, 4]	=> 350
12: [5, 1, 2]	=> 83	32: [4, 1, 2, 1]	=> 370
13: [2, 1, 5]	=> 83	33: [1, 2, 1, 4]	=> 370
14: [4, 1, 3]	=> 125	34: [3, 2, 3]	=> 379
15: [3, 1, 4]	=> 125	35: [2, 4, 2]	=> 379
16: [5, 2, 1]	=> 133	36: [2, 1, 4, 1]	=> 412
17: [1, 2, 5]	=> 133	37: [1, 4, 1, 2]	=> 412
18: [5, 1, 1, 1]	=> 160	38: [3, 1, 1, 3]	=> 448
19: [1, 1, 1, 5]	=> 160	39: [3, 3, 2]	=> 461
20: [2, 5, 1]	=> 161	40: [2, 3, 3]	=> 461

3.3. ALGORITHMIC APPROACH TO \mathcal{F} AND SOME OBSERVATIONS

41: [4, 2, 1, 1]	=> 470	78: [1, 3, 1, 1, 2]	=> 1,351
42: [1, 1, 2, 4]	=> 470	79: [3, 2, 1, 1, 1]	=> 1,421
43: [3, 1, 3, 1]	=> 496	80: [1, 1, 1, 2, 3]	=> 1,421
44: [1, 3, 1, 3]	=> 496	81: [2, 2, 2, 2]	=> 1,456
45: [2, 4, 1, 1]	=> 512	82: [1, 3, 1, 2, 1]	=> 1,457
46: [1, 1, 4, 2]	=> 512	83: [1, 2, 1, 3, 1]	=> 1,457
47: [4, 1, 1, 1, 1]	=> 595	84: [2, 1, 2, 1, 2]	=> 1,513
48: [1, 1, 1, 1, 4]	=> 595	85: [2, 3, 1, 1, 1]	=> 1,519
49: [1, 4, 2, 1]	=> 632	86: [1, 1, 1, 3, 2]	=> 1,519
50: [1, 2, 4, 1]	=> 632	87: [1, 2, 3, 1, 1]	=> 1,667
51: [3, 3, 1, 1]	=> 664	88: [1, 1, 3, 2, 1]	=> 1,667
52: [1, 1, 3, 3]	=> 664	89: [1, 3, 2, 1, 1]	=> 1,735
53: [1, 1, 4, 1, 1]	=> 685	90: [1, 1, 2, 3, 1]	=> 1,735
54: [3, 1, 2, 2]	=> 728	91: [3, 1, 1, 1, 1, 1]	=> 1,856
55: [2, 2, 1, 3]	=> 728	92: [1, 1, 1, 1, 1, 3]	=> 1,856
56: [3, 2, 1, 2]	=> 784	93: [2, 2, 1, 1, 2]	=> 1,889
57: [2, 1, 2, 3]	=> 784	94: [2, 1, 1, 2, 2]	=> 1,889
58: [1, 4, 1, 1, 1]	=> 785	95: [2, 1, 2, 2, 1]	=> 2,051
59: [1, 1, 1, 4, 1]	=> 785	96: [1, 2, 2, 1, 2]	=> 2,051
60: [2, 3, 1, 2]	=> 812	97: [2, 2, 1, 2, 1]	=> 2,107
61: [2, 1, 3, 2]	=> 812	98: [1, 2, 1, 2, 2]	=> 2,107
62: [1, 3, 3, 1]	=> 880	99: [1, 1, 3, 1, 1, 1]	=> 2,144
63: [3, 2, 2, 1]	=> 1,016	100: [1, 1, 1, 3, 1, 1]	=> 2,144
64: [1, 2, 2, 3]	=> 1,016	101: [2, 2, 2, 1, 1]	=> 2,261
65: [3, 1, 1, 1, 2]	=> 1,051	102: [1, 1, 2, 2, 2]	=> 2,261
66: [2, 1, 1, 1, 3]	=> 1,051	103: [1, 3, 1, 1, 1, 1]	=> 2,312
67: [3, 1, 2, 1, 1]	=> 1,099	104: [1, 1, 1, 1, 3, 1]	=> 2,312
68: [1, 1, 2, 1, 3]	=> 1,099	105: [2, 1, 1, 1, 1, 2]	=> 2,590
69: [2, 2, 3, 1]	=> 1,100	106: [1, 2, 2, 2, 1]	=> 2,701
70: [1, 3, 2, 2]	=> 1,100	107: [2, 1, 2, 1, 1, 1]	=> 2,780
71: [2, 1, 3, 1, 1]	=> 1,141	108: [1, 1, 1, 2, 1, 2]	=> 2,780
72: [1, 1, 3, 1, 2]	=> 1,141	109: [2, 1, 1, 2, 1, 1]	=> 2,890
73: [2, 3, 2, 1]	=> 1,168	110: [1, 1, 2, 1, 1, 2]	=> 2,890
74: [1, 2, 3, 2]	=> 1,168	111: [2, 1, 1, 1, 2, 1]	=> 2,990
75: [3, 1, 1, 2, 1]	=> 1,253	112: [1, 2, 1, 1, 1, 2]	=> 2,990
76: [1, 2, 1, 1, 3]	=> 1,253	113: [1, 2, 1, 2, 1, 1]	=> 3,194
77: [2, 1, 1, 3, 1]	=> 1,351	114: [1, 1, 2, 1, 2, 1]	=> 3,194

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115: [2, 2, 1, 1, 1, 1]	=> 3,268	122: [1, 1, 1, 1, 1, 1, 2]	=> 4,529
116: [1, 1, 1, 1, 2, 2]	=> 3,268	123: [1, 1, 2, 1, 1, 1, 1]	=> 4,985
117: [1, 2, 1, 1, 2, 1]	=> 3,526	124: [1, 1, 1, 1, 2, 1, 1]	=> 4,985
118: [1, 1, 2, 2, 1, 1]	=> 3,526	125: [1, 1, 1, 2, 1, 1, 1]	=> 5,095
119: [1, 2, 2, 1, 1, 1]	=> 3,736	126: [1, 2, 1, 1, 1, 1, 1]	=> 5,263
120: [1, 1, 1, 2, 2, 1]	=> 3,736	127: [1, 1, 1, 1, 1, 2, 1]	=> 5,263
121: [2, 1, 1, 1, 1, 1, 1]	=> 4,529	128: [1, 1, 1, 1, 1, 1, 1, 1]	=> 7,936

Referring to the above data, and going on till $p = 18$, we may observe that $\forall 3 \leq p \leq 18$, $\mathcal{F}(1, 1, \dots, 1)$ (where 1 is repeated p times) always corresponds to the maximal value between all $\mathcal{F}(a_1, \dots, a_s)$, where $1 \leq s \leq p$, and $p = \sum_{i=1}^s a_i$. Also remark that $\mathcal{F}(1, 2, 1, \dots, 1)$ is the next biggest value, and $\mathcal{F}(1, 2, 1, \dots, 1)$ is always bigger than $\frac{\mathcal{F}(1, 1, \dots, 1)}{2}$. One may wonder if these properties hold for any p .

On the other hand, if p is odd, (that is, the corresponding transitive tournament $TT(p+1)$ is of even order), then the tuple $(1, 1, \dots, 1)$ is not symmetric. If p is even, (that is, $TT(p+1)$ is of odd order), then the tuple $(1, 1, \dots, 1)$ is symmetric, and as a consequence, $(1, 2, 1, \dots, 1)$ is not symmetric.

Hence, using Theorem 3.1.1, we set the following conjecture:

Conjecture 3.3.1. *Let TTn be a transitive tournament on n vertices.*

Then $\forall \alpha = (\alpha_1, \alpha_2, \dots, \alpha_s) \in \mathbb{K}_s$, $\sum_{i=1}^s \alpha_i = n - 1$, if n is even we have:

$$f_{TTn}(1, -1, 1, \dots, -1, 1) \geq f_{TTn}(\alpha_1, \alpha_2, \dots, \alpha_s),$$

where $(1, -1, 1, \dots, -1, 1)$ has $n - 1$ components, while if n is odd, we have:

$$f_{TTn}(1, -2, 1, -1, 1, \dots, -1, 1) \geq f_{TTn}(\alpha_1, \alpha_2, \dots, \alpha_s),$$

where the number of components of $(1, -2, 1, \dots, 1)$ is $n - 2$.

If the conjecture is true, we can deduce that in a transitive tournament of even order, the number of antidirected Hamiltonian paths starting with a forward arc is the maximum of the numbers of oriented Hamiltonian paths starting with a forward block, for all given types.

It seems that the path-function \mathcal{F} holds many interesting properties that could tell a lot about the number of oriented Hamiltonian paths in a transitive tournament. Concerning oriented cycles:

Problem 3.3.2. *Can we find a combinatorial function \mathcal{G} , that allows us to compute the exact number of oriented Hamiltonian cycles of any given type in a transitive tournament?*

3.3. ALGORITHMIC APPROACH TO \mathcal{F} AND SOME OBSERVATIONS

Chapter 4

Real and complex quadratic cyclic sequences, and cyclic graphs.

We recall that for a given integer $N \geq 2$, a *quadratic cyclic sequence (QCS) of order N* is a function $\varphi : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ satisfying the quadratic difference relation

$$\frac{\gamma}{2}(2\varphi(j) - \varphi(j-1) - \varphi(j+1))^2 = (\varphi(j) - \varphi(j-1))^2 + (\varphi(j) - \varphi(j+1))^2 \quad \forall j \in \mathbb{Z}/N\mathbb{Z} \quad (4.1)$$

for some real number γ where $j \pm 1$ are calculated modulo N .

If we define the *increment* $u_j = \varphi(j+1) - \varphi(j)$, then the above equation reads

$$\frac{\gamma}{2}(u_j - u_{j-1})^2 - u_j^2 - u_{j-1}^2 = 0. \quad (4.2)$$

Moreover, equation (4.1) is invariant (for fixed γ) under affine linear transformations and conjugation in the complex plane:

$$\varphi \mapsto a\varphi + b \quad \text{and} \quad \varphi \mapsto \bar{\varphi}, \quad \forall a, b \in \mathbb{C} \text{ with } a \neq 0. \quad (4.3)$$

Equation (4.1) is also invariant under *cyclic permutations* and *order reversal* of the sequence $S = (\varphi(0), \varphi(1), \dots, \varphi(N-1))$. We refer to these four freedoms as *normalization*.

We examine in §4.1 real QCS, showing how they arise from polynomials with positive integer coefficients. In particular, we will see how a given polynomial can give rise to different sequences coming from *legitimate* orderings of a corresponding set of increments. It turns out that we can capture the legitimate orderings by Eulerian walks in a corresponding digraph, a model we discuss later on in §4.4.

Then, in §4.2, we study complex QCS which arise from polynomials. They will be called

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algebraic. In this case, a *legitimate* polynomial $p(x)$ determines a closed polygonal walk in the plane with exterior angle either $+\theta$ or $-\theta$ for some fixed angle θ (the turning angle). We will see that complex algebraic QCS exist with turning angle not a rational multiple of 2π . In the case when $\theta = 2\pi/n$, necessarily the n 'th cyclotomic polynomial $\Phi_n(x)$ must divide $p(x)$. The problem of which polynomials can arise turns out to be challenging when n becomes large:

One wishes to construct a polygonal path starting at the origin, with directed edges taken from the edges of a closed regular polygon with exterior angle $2\pi/n$. One may use edges as often as one likes, but at each step, the turning angle must be either $+2\pi/n$ or $-2\pi/n$. In the case when the polygonal path is closed and n is even, must each edge occur with its parallel counterpart oriented in the opposite direction? To complete a circuit, are all edges of the polygon required? (now for n even or odd). We will discuss these problems, showing that for $n \geq 12$, this is not necessarily true.

In §4.3, we see how it is possible to combine quadratic cyclic sequences with common γ to obtain new sequences, and show how this process of concatenation is reflected in the defining polynomials, and we finally study in §4.5 the unicity of the edges used to construct a walk with turning angle $2\pi/n$ of given length and end point. For walks on square, triangular and hexagonal lattices, unicity occurs when $n = 4$, but not in general when $n = 6$.

4.1 Construction of real QCS

In this section, we examine real QCS, showing how they arise from polynomials with positive integer coefficients.

By a *real* QCS, we mean one in which every term is real under some normalization. As we saw in Chapter 1, for such sequences, we must have $\gamma \geq 1$. Below, we will see that we necessarily have $\gamma \leq 2$. First we deal with the case when $\gamma = 2$.

Proposition 4.1.1. *Let $(x_0, x_1, x_2, \dots, x_{N-1})$ be a non-constant QCS (not necessarily real) which solves (4.1) for some real γ . Then $\gamma = 2$ if and only if for every three consecutive elements x_{j-1}, x_j, x_{j+1} , $j \in \mathbb{Z}/N\mathbb{Z}$, we have $x_{j-1} = x_j$ or $x_j = x_{j+1}$.*

Proof. Consider a segment of three successive terms x_{j-1}, x_j, x_{j+1} . On applying (4.1) at vertex x_j , we obtain the equation:

$$\begin{aligned} (x_{j+1} + x_{j-1} - 2x_j)^2 &= (x_{j-1} - x_j)^2 + (x_{j+1} - x_j)^2 \\ \Leftrightarrow (x_j - x_{j-1})(x_j - x_{j+1}) &= 0, \end{aligned}$$

so that necessarily, x_j is equal to one of its neighbours. Conversely, if every coefficient x_j has at most one adjacent coefficient taking on a different value, then by evaluating equation (4.1)

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on vertex j , we have

$$\frac{\gamma}{2}(x_{j+1} + x_{j-1} - 2x_j)^2 = (x_{j-1} - x_j)^2 + (x_{j+1} - x_j)^2,$$

and since we either have $x_j - x_{j-1} = 0$ or $x_j - x_{j+1} = 0$, and $(x_0, x_1, \dots, x_{N-1})$ is non-constant, then we get $\gamma = 2$. \square

Clearly, for all orders ≥ 4 , non-constant cyclic sequences with $\gamma = 2$ exist. We now treat the case when $\gamma \neq 2$. First, as a result of Proposition 4.1.1, we have the following:

Proposition 4.1.2. *If $X = (x_0, x_1, \dots, x_{N-1})$ is a QCS satisfying equation (4.1) with $\gamma \neq 2$, then either X is constant or no 2 consecutive terms of X can be equal.*

We now give the construction of real QCS:

Theorem 4.1.3. *Let $q(x) = a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \dots + a_1x + a_0$ ($n \geq 2$) be any polynomial with integer coefficients all strictly positive. Multiply by $x + 1$ to obtain the new polynomial*

$$\begin{aligned} p(x) &:= b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_1x + b_0 \\ &= a_{n-2}x^{n-1} + (a_{n-2} + a_{n-3})x^{n-2} + \dots + (a_1 + a_0)x + a_0. \end{aligned} \quad (4.4)$$

Let y be any real root of $p(x)$ (necessarily negative). Then a quadratic cyclic sequence

$$(x_0, x_1, x_2, \dots, x_{N-1})$$

of order $N = 2 \sum_{k=0}^{n-2} a_k$ is constructed by arbitrarily prescribing x_0 and then requiring increments $u_j = x_{j+1} - x_j$ of successive terms to be taken from the set $\{1, y, y^2, \dots, y^{n-1}\}$ in such a way that each increment y^k occurs precisely b_k times and any two adjacent increments have powers that differ by ± 1 , including the powers corresponding to the elements u_0 and u_{N-1} . This is always possible and up to these constraints, the ordering is arbitrary. The constant γ in (4.1) is given by $\gamma = 2(1 + y^2)/(1 - y)^2 < 2$.

Conversely, up to addition of a constant, cyclic permutations and order reversal, a multiple of any non-constant real cyclic sequence with $\gamma \neq 2$ arises in this way from such a polynomial $p(x)$, well-defined up to replacement of $p(x)$ by $\tilde{p}(x) := x^{\deg p} p(1/x)$.

The cyclic sequences with $\gamma = 2$ are characterized as those made up of connected segments of order ≥ 2 on which the sequence is constant. The cyclic sequences with $\gamma = 1$ are, up to normalization, equivalent to $(0, 1, 0, 1, \dots, 0, 1)$; they arise by taking the root $y = -1$ of $p(x)$.

We refer to the increment y as a *fundamental increment* associated to the sequence and the polynomial $p(x)$ as a *defining polynomial* of the sequence. As we see below y is only defined up to replacement by $1/y$ and $p(x)$ up to replacement by $x^{\deg p} p(1/x)$. The ordering of increments specified by the statement of the theorem will be referred to as *legitimate*.

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Remark 4.1.4. Since any real root y must be strictly negative and adjacent powers differ by one, it follows that a real cyclic sequence with $\gamma \neq 2$ oscillates. The length N of the sequence is given by $\sum_k b_k = 2 \sum_k a_k$, so that a non-trivial real QCS with $\gamma \neq 2$ always has even order (also a consequence of oscillation).

Example 4.1.5. Take $q(x) = x + 2$. Multiplication by $x + 1$ gives the polynomial $p(x) = x^2 + 3x + 2$ with real root $y = -2$. Arrange the powers of this root with appropriate multiplicity to give the legitimate sequence of increments $(1, y, 1, y, y^2, y) = (1, -2, 1, -2, 4, -2)$. We construct a real QCS of order 6 by first setting $x_0 = 0$ and then proceeding so that $u_0 = x_1 - x_0 = 1, u_1 = x_2 - x_1 = -2, \dots, u_5 = x_0 - x_5 = -2$. We thereby obtain the QCS $(0, 1, -1, 0, -2, 2)$ of order 6. We can normalize the sequence in such a way that the minimum value is 0 (that is by adding -2 to all the elements), and then applying a cyclic permutation, we can manage that the value 0 be the first term and we get: $(0, 4, 2, 3, 1, 2)$.

Example 4.1.6. Irrational sequences arise from irrational roots. For example, the polynomial $x^2 + 4x + 1$ has root $x = -2 + \sqrt{3}$. On multiplying by $x + 1$ we obtain the polynomial $p(x) = x^3 + 5x^2 + 5x + 1$. A legitimate sequence of increments is given by $(1, y, y^2, y, y^2, y, y^2, y, y^2, y^3, y^2, y)$ with $y = -2 + \sqrt{3}$. On calculating, we can now construct a real QCS of order 12; explicitly, it is given by $(0, 1, -1 + \sqrt{3}, 6 - 3\sqrt{3}, 4 - 2\sqrt{3}, 11 - 6\sqrt{3}, 9 - 5\sqrt{3}, 16 - 9\sqrt{3}, -10 + 6\sqrt{3}, -3 + 2\sqrt{3}, -5 + 3\sqrt{3}, 2 - \sqrt{3})$. All terms of this QCS lie in the interval $[0, 1]$. The value of the constant γ in (4.1) is given by $\gamma = 4/3$.

In order to prove Theorem 4.1.3, we give the two following lemmas.

Lemma 4.1.7. Consider a polynomial $p(x) = b_{n+1}x^{n+1} + b_nx^n + \dots + b_1x + b_0$, having strictly positive integer coefficients. Then $p(x) = (x + 1)q(x)$, where $q(x)$ has strictly positive integer coefficients if and only if we can associate to $p(x)$ a sequence $(u_0, u_1, \dots, u_{N-1})$ of $N = \sum_{k=0}^{n+1} b_k$ elements, whose successive terms are taken from the set $\{1, x, x^2, \dots, x^{n+1}\}$ in such a way that each x^k occurs precisely b_k times and any two adjacent elements have powers that differ by ± 1 , including the powers corresponding to the elements u_0 and u_{N-1} .

Proof. Suppose first that $p(x) = (x + 1)q(x)$, where $q(x) = a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0$ has strictly positive coefficients. That is, the coefficients of $p(x)$ are given by:

$$\begin{aligned} p(x) &= b_{n+1}x^{n+1} + b_nx^n + \dots + b_1x + b_0 \\ &= (x + 1)(a_nx^n + a_{n-1}x^{n-1} + \dots + a_1x + a_0) \\ &= a_nx^{n+1} + (a_n + a_{n-1})x^n + \dots + (a_1 + a_0)x + a_0. \end{aligned}$$

The polynomial $p(x)$ can be written as:

$$p(x) = a_n(x^{n+1} + x^n) + a_{n-1}(x^n + x^{n-1}) + \dots + a_1(x^2 + x) + a_0(x + 1).$$

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If n is even,

$$\begin{aligned} p(x) &= a_0(1+x) + a_2(x^2+x^3) + a_4(x^4+x^5) + \cdots + a_n(x^n+x^{n+1}) \\ &\quad + a_{n-1}(x^n+x^{n-1}) + a_{n-3}(x^{n-2}+x^{n-3}) + \cdots + a_1(x^2+x) \\ &= \sum_{k=0, k \text{ even}}^n a_k(x^k+x^{k+1}) + \sum_{k=1, k \text{ odd}}^{n-1} a_k(x^{k+1}+x^k). \end{aligned}$$

If n is odd,

$$\begin{aligned} p(x) &= a_0(1+x) + a_2(x^2+x^3) + a_4(x^4+x^5) + \cdots + a_{n-1}(x^{n-1}+x^n) \\ &\quad + a_n(x^{n+1}+x^n) + a_{n-2}(x^{n-1}+x^{n-2}) + \cdots + a_1(x^2+x) \\ &= \sum_{k=0, k \text{ even}}^{n-1} a_k(x^k+x^{k+1}) + \sum_{k=1, k \text{ odd}}^n a_k(x^{k+1}+x^k). \end{aligned}$$

Then we construct the following sequences:

If n is even:

$$\begin{aligned} S = & \left(\underbrace{1, x, \dots, 1, x}_{1, x \text{ repeated } a_0 \text{ times}}, \underbrace{x^2, x^3, \dots, x^2, x^3}_{x^2, x^3 \text{ repeated } a_2 \text{ times}}, \dots, \underbrace{x^n, x^{n+1}, \dots, x^n, x^{n+1}}_{x^n, x^{n+1} \text{ repeated } a_n \text{ times}}, \right. \\ & \left. \underbrace{x^n, x^{n-1}, \dots, x^n, x^{n-1}}_{x^n, x^{n-1} \text{ repeated } a_{n-1} \text{ times}}, \dots, \underbrace{x^4, x^3, \dots, x^4, x^3}_{x^4, x^3 \text{ repeated } a_3 \text{ times}}, \underbrace{x^2, x, \dots, x^2, x}_{x^2, x \text{ repeated } a_1 \text{ times}} \right), \end{aligned}$$

If n is odd:

$$\begin{aligned} S = & \left(\underbrace{1, x, \dots, 1, x}_{1, x \text{ repeated } a_0 \text{ times}}, \underbrace{x^2, x^3, \dots, x^2, x^3}_{x^2, x^3 \text{ repeated } a_2 \text{ times}}, \dots, \underbrace{x^{n-1}, x^n, \dots, x^{n-1}, x^n}_{x^{n-1}, x^n \text{ repeated } a_{n-1} \text{ times}}, \right. \\ & \left. \underbrace{x^{n+1}, x^n, \dots, x^{n+1}, x^n}_{x^{n+1}, x^n \text{ repeated } a_n \text{ times}}, \dots, \underbrace{x^4, x^3, \dots, x^4, x^3}_{x^4, x^3 \text{ repeated } a_3 \text{ times}}, \underbrace{x^2, x, \dots, x^2, x}_{x^2, x \text{ repeated } a_1 \text{ times}} \right), \end{aligned}$$

and since $\forall 1 \leq i \leq n, a_i > 0$, the above sequences satisfy the desired conditions.

Conversely, let $X = (u_0, u_1, \dots, u_{N-1}) = (x^{s_0}, x^{s_1}, \dots, x^{s_{N-1}})$ be a sequence associated to the polynomial $p(x) = b_{n+1}x^{n+1} + b_nx^n + \cdots + b_1x + b_0$ satisfying the above conditions. That is, each term of X is taken from the set $\{1, x, x^2, \dots, x^{n+1}\}$ in such a way that each x^k occurs precisely b_k times in X , and $\forall j \in \mathbb{Z}/N\mathbb{Z}, s_j - s_{j-1} = \pm 1$.

We can be more explicit about the form of $p(x)$. Since $\forall j \in \mathbb{Z}/N\mathbb{Z}, s_j - s_{j-1} = \pm 1$, then the number of *even* powers of x is equal to the number of *odd* powers of x in X , so -1 is a root of $p(x)$ and $(x+1)$ is a factor of $p(x)$. Moreover, for $n \geq 2$, we claim that $p(x)$ has the

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following form:

$$\begin{aligned}
 p(x) &= b_{n+1}x^{n+1} + (b_{n+1} + 1)x^n + 2x^{n-1} + 2x^{n-2} + \dots \\
 &\quad \dots + 2x^2 + (b_0 + 1)x + b_0 + \sum_{s=2}^n \beta_s(x^s + x^{s-1}) \\
 &= (x + 1)(b_{n+1}x^n + x^{n-1} + x^{n-2} + \dots + x + b_0) + (x + 1) \sum_{s=2}^n \beta_s x^{s-1}
 \end{aligned}$$

where b_{n+1}, b_0 are strictly positive integers and β_s are integers that are ≥ 0 . To see this, for each of the b_{n+1} occurrences of the maximum power x^{n+1} , we must have at least one more occurrence of x^n . Similarly for each of the b_0 occurrences of the minimum power x^0 , we must have one more occurrence of x^1 . Moreover, necessarily each of the intermediate powers must occur at least twice. However, we may have further oscillations between powers of x^s and x^{s-1} for $s = 2, \dots, n$, which are given by the coefficients β_s .

In the case when $n = 1$, then we must have $p(x) = (x + 1)(a_1x + a_0)$ for positive integers a_1, a_0 , and for $n = 0$, $p(x) = a_0(x + 1)$ for a positive integer a_0 . The polynomial $p(x)$ then has -1 as a root, and the coefficients of $q(x)$, such that $p(x) = (x + 1)q(x)$ are all strictly positive integers. This concludes the proof. \square

The second lemma is about a recurrence relation that determines each term of a real QCS as a function of the three previous terms.

Lemma 4.1.8. *Let $(x_0, x_1, \dots, x_{N-1})$ be a non-constant real QCS with $\gamma \neq 2$, then the increments $u_j = x_{j+1} - x_j$ are non-zero, and satisfy $u_0 + u_1 + \dots + u_{N-1} = 0$ and the recurrence relation:*

$$u_j = \begin{cases} \text{either} & u_{j-1}^2/u_{j-2} \\ \text{or} & u_{j-2}. \end{cases} \quad (4.5)$$

for all $j \in \mathbb{Z}/N\mathbb{Z}$. Conversely, any real sequence $(u_0, u_1, \dots, u_{N-1})$ with non-zero terms, whose sum is zero, and satisfying the above recurrence relation, determines a real QCS $(x_0, x_1, \dots, x_{N-1})$, by arbitrary prescribing x_0 and setting $x_{j+1} = u_j + x_j$ for $j = 0, \dots, N-1$.

Proof. Let $(x_0, x_1, \dots, x_{N-1})$ be a non-constant QCS with $\gamma \neq 2$. Consider a particular segment of the sequence consisting of four consecutive terms: $(x_{j-2}, x_{j-1}, x_j, x_{j+1})$. On normalizing, we can suppose this segment equivalent to $(u, 0, v, w)$, for some real numbers u, v, w . Using Proposition 4.1.2, we have $u \neq 0$. Similarly $v \neq 0$ and more generally, two successive terms are distinct so that the increments are all non-zero.

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On evaluating equation (4.1) at term $j - 1$, we obtain:

$$\gamma = \frac{2(u^2 + v^2)}{(u + v)^2}. \quad (4.6)$$

Now evaluate (4.1) at term j :

$$\gamma(w - 2v)^2 = 2((w - v)^2 + v^2).$$

On eliminating γ , we obtain the quadratic equation in w :

$$uw^2 + (u - v)^2w - v(u - v)^2 = 0.$$

This gives two possible values of w , which are $v(u - v)/u$ and $v - u$. To recover the general case, we set $u = x_{j-2} - x_{j-1}$, $v = x_j - x_{j-1}$, $w = x_{j+1} - x_{j-1}$. This gives the two values:

$$x_{j+1} = \begin{cases} \frac{x_{j-1}(x_j - x_{j-1}) + x_j(x_{j-2} - x_{j-1})}{x_{j-2} - x_{j-1}} \\ x_j + x_{j-1} - x_{j-2} \end{cases}$$

On subtracting x_j from both sides, we obtain the recurrence relation (4.5) for the increments.

Conversely, given a sequence $(u_0, u_1, \dots, u_{N-1})$ with non-zero terms whose sum is zero satisfying the recurrence relation (4.5), then the sequence must alternate in sign, for otherwise, if we have two consecutive terms of the same sign, then using the recurrence relation, all subsequent terms would have the same sign, contradicting $u_0 + u_1 + \dots + u_{N-1} = 0$. It is then easily checked that (4.2) is satisfied for $\gamma = 2(u_j^2 + u_{j-1}^2)/(u_j - u_{j-1})^2$ independently of j . In fact, we need to prove that

$$\frac{u_j^2 + u_{j-1}^2}{(u_j - u_{j-1})^2} = \frac{u_{j-1}^2 + u_{j-2}^2}{(u_{j-1} - u_{j-2})^2}.$$

The alternance in sign means that the denominators are non-zero. If $u_j = u_{j-2}$, the equality is obvious. If $u_j = \frac{u_{j-1}^2}{u_{j-2}}$, then we get:

$$\frac{u_j^2 + u_{j-1}^2}{(u_j - u_{j-1})^2} = \frac{\frac{u_{j-1}^4}{u_{j-2}^2} + u_{j-1}^2}{\left(\frac{u_{j-1}^2}{u_{j-2}} - u_{j-1}\right)^2} = \frac{u_{j-1}^2 \cdot (u_{j-1}^2 + u_{j-2}^2)}{u_{j-1}^2 \cdot (u_{j-1} - u_{j-2})^2},$$

which gives the desired equality. A corresponding real QCS $(x_0, x_1, \dots, x_{N-1})$ is determined

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by arbitrarily choosing x_0 and then setting $x_{j+1} = u_j + x_j$ for $j = 0, \dots, N-1$. The condition $u_0 + u_1 + \dots + u_{N-1} = 0$ means that $x_N = x_0$ and the sequence is cyclic as required. \square

We may now give the proof of our main theorem:

Proof of Theorem 4.1.3: Consider a polynomial $q(x) = a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \dots + a_1x + a_0$ ($n \geq 2$) with all coefficients strictly positive, and let's prove how one can construct a corresponding QCS by proceeding as in the statement of the theorem. Multiplying $q(x)$ by $x + 1$, we obtain the new polynomial

$$\begin{aligned} p(x) &:= b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_1x + b_0 \\ &= a_{n-2}x^{n-1} + (a_{n-2} + a_{n-3})x^{n-2} + \dots + (a_1 + a_0)x + a_0. \end{aligned}$$

Take any real root y of $p(x)$ if any. The root y is necessarily negative since $p(x)$ has positive coefficients. Using Lemma 4.1.7, we can construct the sequence

$$(u_0, u_1, u_2, \dots, u_{N-1})$$

of order $N = \sum_k b_k = 2 \sum_k a_k$, with successive terms taken from the set $\{1, y, y^2, \dots, y^{n-1}\}$ in such a way that each increment y^k occurs precisely b_k times and any two adjacent increments have powers that differ by ± 1 , including the powers corresponding to the elements u_0 and u_{N-1} .

It is easy to see that every three consecutive elements u_j , u_{j+1} and u_{j+2} verify the recurrence relation (4.5) of Lemma 4.1.8. In fact, write $u_j = y^{k_j}$, $u_{j+1} = y^{k_{j+1}}$ and $u_{j+2} = y^{k_{j+2}}$, we have four cases:

$$\begin{aligned} k_{j+1} &= k_j + 1 & \text{and} & & k_{j+2} &= k_j + 2, \\ k_{j+1} &= k_j - 1 & \text{and} & & k_{j+2} &= k_j - 2, \\ k_{j+1} &= k_j + 1 & \text{and} & & k_{j+2} &= k_j, \\ k_{j+1} &= k_j - 1 & \text{and} & & k_{j+2} &= k_j \end{aligned}$$

and we can check that in each of the above cases, the recurrence relation is satisfied. Moreover, we have that

$$\sum_{j=0}^{N-1} u_j = 0,$$

since $\sum_j u_j = p(y) = 0$ as y is a root of $p(x)$. So, by Lemma 4.1.8, there exists a real QCS $X = (x_0, x_1, x_2, \dots, x_{N-1})$ of order $N = \sum_k b_k = 2 \sum_k a_k$, determined by the sequence of increments $(u_0, u_1, \dots, u_{N-1})$, constructed by arbitrarily prescribing x_0 and then requiring $x_{j+1} = u_j + x_j$ for $j = 0, \dots, N-1$.

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Now, X being a QCS, we have by equation (4.2) that

$$\gamma = \frac{2(u_j^2 + u_{j-1}^2)}{(u_j - u_{j-1})^2},$$

independently of j . If we write $u_j = y^{k_j}$ and $u_{j-1} = y^{k_{j-1}}$ then whether $k_{j-1} = k_j - 1$ or $k_{j-1} = k_j + 1$, we can prove that

$$\gamma = \frac{2 \cdot (1 + y^2)}{(1 - y)^2},$$

which is clearly < 2 since $y < 0$.

Now let $(x_0, x_1, \dots, x_{N-1})$ be a non-constant real QCS with $\gamma \neq 2$. We will show how it arises from a polynomial with positive integer coefficients, as described in the statement of the theorem.

Normalize the sequence $(x_k)_k$ such that $x_0 = 0$ and $x_1 = 1$, (that is done by dividing $(x_k)_k$ by $(x_1 - x_0)$, since $x_1 \neq x_0$ by Proposition 4.1.2), and let $u_0 = x_1 - x_0 = 1$, and $u_1 = x_2 - x_1 = y < 0$. Note that $y \neq 0$ because by Proposition 4.1.2, no 2 consecutive terms of $(x_k)_k$ are equal, and y is negative since the sequence of increments $u_k = x_{k+1} - x_k$ must alternate in sign. In fact, using Lemma 4.1.8, the sequence $(u_k)_k$ of possible elements verifies the relation

$$u_k = \begin{cases} \text{either} & u_{k-1}^2/u_{k-2} \\ \text{or} & u_{k-2} \end{cases},$$

for all $k \in \mathbb{Z}/N\mathbb{Z}$. So if two consecutive increments have the same sign, and since u_k has the same sign as u_{k-2} , then all subsequent increments would have the same sign, and the sequence $(x_k)_k$ will be monotone increasing or decreasing, which is impossible since X is non constant and the sequence must be cyclic (that is, we must have $x_N = x_{N-1} + u_{N-1} \equiv x_0$).

It can be easily proved by induction that the above recurrence relation implies that $\forall k \in \mathbb{Z}/N\mathbb{Z}$, we have

$$u_{k+1} = \begin{cases} \text{either} & y \cdot u_k \\ \text{or} & u_k/y \end{cases},$$

so the sequence $(u_k)_k$ of possible increments of the QCS is:

$$\left(1, y, \begin{cases} y^2 \\ 1 \end{cases}, \begin{cases} y^3 \\ y \\ \frac{1}{y} \end{cases}, \begin{cases} y^4 \\ y^2 \\ 1 \\ \frac{1}{y^2} \end{cases}, \dots, \begin{cases} y \\ \frac{1}{y} \end{cases} \right). \quad (4.7)$$

For the case when $y = -1$, the sequence (4.7) is $(1, -1, 1, -1, \dots, 1, -1)$, which corresponds

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to a QCS equivalent to $(0, 1, 0, 1, \dots, 0, 1)$ with $\gamma = 1$. Furthermore, any non-constant QCS with $\gamma = 1$ is equivalent to one of this form, since if we take a segment (x_{k-1}, x_k, x_{k+1}) , then

$$(x_k - x_{k-1} + x_k - x_{k+1})^2 = 2(x_k - x_{k-1})^2 + 2(x_k - x_{k+1})^2 \Leftrightarrow (x_{k+1} - x_{k-1})^2 = 0,$$

so that $x_{k+1} = x_{k-1}$. So any non-constant QCS satisfying equation (4.1) with $\gamma = 1$ has a corresponding defining polynomial $p(x) = c \cdot (x + 1)$ where c is the number of repetition of 1 (of -1 also) in the normalized sequence of increments. The polynomial $p(x)$ has the form of the statement of the theorem, and the sequence of increments arises by taking the root -1 of $p(x)$.

Note that the defining polynomial of a non-constant QCS with $\gamma = 1$ is not unique, unlike the case $\gamma \neq 1$, which we will now discuss. (See Remark (4.1.9)).

Suppose now that $y \neq -1$. In particular, since y is real and negative, we cannot have $y^r = 1$ for any power $r \neq 0$, and $y^r \neq y^{r'} \forall r \neq r'$.

The sequence (4.7) implies that all terms must have the form y^r for some integer r and that powers of successive terms differ by ± 1 , that is, an occurrence of y^r must be followed by either y^{r+1} or y^{r-1} . Write the sequence of increments as $(y^{r_0}, y^{r_1}, \dots, y^{r_{N-1}})$ and let $t = \min\{r_j : 0 \leq j \leq N-1\}$. Now multiply through by y^{-t} to obtain $1 = y^0$ in some position, with all other powers of y greater than or equal to zero:

$$(y^{s_0}, y^{s_1}, \dots, 1, \dots, y^{s_{N-1}}), \tag{4.8}$$

where each $s_j \geq 0$ and $s_j - s_{j-1} = \pm 1, \forall j \in \mathbb{Z}/N\mathbb{Z}$.

But since the sequence is cyclic, then $\sum_{j=0}^{N-1} u_j = \sum_{j=0}^{N-1} (x_{j+1} - x_j) = 0$, which implies that $\sum_{j=0}^{N-1} y^{s_j} = y^{-t} \sum_{j=0}^{N-1} y^{r_j} = 0$ so that y satisfies a polynomial equation of the form:

$$p(x) := b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_1x + b_0 = 0$$

where b_s is the number of occurrences of y^s in (4.8) and $n-1$ is the maximal power that occurs (the choice of $n-1$ is to accord with later conventions). We can be more explicit about the form of $p(x)$: By the proof of Lemma 4.1.7 applied on the sequence (4.8), it follows that $x+1$ is a factor of $p(x)$, and for $n \geq 4$, $p(x)$ has the form:

$$\begin{aligned} p(x) &= (x+1)(b_{n-1}x^{n-2} + x^{n-3} + x^{n-4} + \dots + x + b_0) + (x+1) \sum_{s=2}^{n-2} \beta_s x^{s-1} \\ &= b_{n-1}x^{n-1} + (b_{n-1} + 1)x^{n-2} + 2x^{n-3} + 2x^{n-4} + \dots \\ &\quad \dots + 2x^2 + (b_0 + 1)x + b_0 \sum_{s=2}^{n-2} \beta_s (x^s + x^{s-1}) \end{aligned}$$

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where b_{n-1}, b_0 are strictly positive integers and β_s are integers that are ≥ 0 . In the case when $n = 3$, then we must have $p(x) = (x + 1)(a_1x + a_0)$ for positive integers a_1, a_0 , and for $n = 2$, $p(x) = a_0(x + 1)$ for a positive integer a_0 , (but this case is impossible since $y \neq -1$). The polynomial $p(x)$ is then equal to $(x + 1) \cdot q(x)$ where $q(x)$ has strictly positive integer coefficients, so has the form of the statement of the theorem.

We need to show that the above procedure is well-defined, that is, if we perform the various operations on the QCS (addition of a constant, multiplication by a constant, cyclic permutation, order reversal), the polynomial that results is well-defined. In fact, $p(x)$ is only defined up to replacement by $\tilde{p}(x) := x^{\deg p} p(1/x)$.

First note that all of the above operations on a QCS leave γ invariant. The expression for γ is deduced from (4.2) and (4.7):

$$\gamma = 2 \frac{(1 + y^2)}{(1 - y)^2}, \quad (4.9)$$

(where we can clearly see that $\gamma < 2$ since $y < 0$).

Thus, $y < 0$ is determined to be a root of the quadratic equation

$$(1 - y)^2 \gamma = 2(1 + y^2) \quad (4.10)$$

and the only other root is $1/y$. In fact, γ is invariant under $y \mapsto y^{-1}$. Thus for given γ the only two possible fundamental increments are roots y and $1/y$ of (4.10).

Addition of a constant makes no difference to the sequence of increments and so leaves the above construction of $p(x)$ invariant. However, as we now show, up to a multiple, cyclic permutations and order reversal have the same effect on the sequence of increments (see Example 4.1.13 below) which may modify $p(x)$.

As before, begin with a real QCS which we normalize as above:

$$\begin{aligned} (x_0, x_1, \dots, x_{N-1}) &\rightarrow (0, x_1 - x_0, x_2 - x_0, \dots, x_{N-1} - x_0) \\ &\rightarrow \left(0, 1, \frac{x_2 - x_0}{x_1 - x_0}, \frac{x_3 - x_0}{x_1 - x_0}, \dots, \frac{x_{N-1} - x_0}{x_1 - x_0} \right), \end{aligned}$$

where, in the previous notation, $y = \frac{x_2 - x_0}{x_1 - x_0} - 1 = \frac{x_2 - x_1}{x_1 - x_0}$. Then the sequence of increments has the form

$$\underline{u} := (u_0, u_1, \dots, u_{N-1}) = \left(1, \frac{x_2 - x_1}{x_1 - x_0}, \frac{x_3 - x_2}{x_1 - x_0}, \dots, \frac{x_{N-1} - x_{N-2}}{x_1 - x_0}, \frac{x_0 - x_{N-1}}{x_1 - x_0} \right),$$

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where for the moment we don't normalize to make y^0 the smallest power of y . Suppose we make a cyclic permutation to obtain the real QCS $(x_t, x_{t+1}, \dots, x_{N-1}, x_0, x_1, \dots, x_{t-1})$. Then the same procedure yields the sequence of increments

$$\begin{aligned} \underline{v} &= \left(1, \frac{x_{t+2} - x_{t+1}}{x_{t+1} - x_t}, \frac{x_{t+3} - x_{t+2}}{x_{t+1} - x_t}, \dots, \frac{x_0 - x_{N-1}}{x_{t+1} - x_t}, \frac{x_1 - x_0}{x_{t+1} - x_t}, \dots, \frac{x_t - x_{t-1}}{x_{t+1} - x_t} \right) \\ &= \left(\frac{x_1 - x_0}{x_{t+1} - x_t} \right) \left(\frac{x_{t+1} - x_t}{x_1 - x_0}, \frac{x_{t+2} - x_{t+1}}{x_1 - x_0}, \dots, \frac{x_0 - x_{N-1}}{x_1 - x_0}, 1, \frac{x_2 - x_1}{x_1 - x_0}, \dots, \frac{x_t - x_{t-1}}{x_1 - x_0} \right) \end{aligned}$$

which is a multiple of a cyclic permutation of \underline{u} .

Similarly, order reversal gives the QCS $(x_{N-1}, x_{N-2}, \dots, x_1, x_0)$ which, following the same normalization procedure, yields the corresponding sequence of increments

$$\left(\frac{x_1 - x_0}{x_{N-1} - x_{N-2}} \right) \left(\frac{x_{N-1} - x_{N-2}}{x_1 - x_0}, \frac{x_{N-2} - x_{N-3}}{x_1 - x_0}, \dots, \frac{x_2 - x_1}{x_1 - x_0}, 1, \frac{x_0 - x_{N-1}}{x_1 - x_0} \right)$$

which is a multiple of the sequence \underline{u} with order reversed together with a cyclic permutation. In each case we obtain a multiple of \underline{u} together with a cyclic permutation and/or with order reversal.

Now proceed as before to construct the polynomial $p(x)$. Thus some multiple of \underline{u} yields a sequence of powers of y (respectively $1/y$), the smallest being y^0 (respectively $(1/y)^0$). Suppose as in (4.8),

$$c\underline{u} = (y^{s_0}, y^{s_1}, \dots, y^{s_{N-1}}),$$

for some $c \in \mathbb{R} \setminus \{0\}$ with $\min\{s_j : 0 \leq j \leq N-1\} = 0$. Let $s_\ell = \max\{s_j : 0 \leq j \leq N-1\}$ and instead multiply \underline{u} by cy^{-s_ℓ} :

$$cy^{-s_\ell}\underline{u} = (y^{s_0-s_\ell}, y^{s_1-s_\ell}, \dots, y^{s_{N-1}-s_\ell}) = (\tilde{y}^{s_\ell-s_0}, \tilde{y}^{s_\ell-s_1}, \dots, \tilde{y}^{s_\ell-s_{N-1}})$$

where $\tilde{y} = 1/y$. Note that the smallest power of \tilde{y} in the sequence is \tilde{y}^0 . Furthermore, $p(x) = x^{s_0} + x^{s_1} + \dots + x^{s_{N-1}}$, so that the polynomial satisfied by \tilde{y} is given by

$$\tilde{p}(x) = x^{s_\ell-s_0} + x^{s_\ell-s_1} + \dots + x^{s_\ell-s_{N-1}} = x^{s_\ell}p(1/x)$$

where $s_\ell = \deg p$. In particular these are the only two multiples of \underline{u} which can be written as powers of y or $1/y$ for which the lowest power is 0. As already established, cyclic permutations and/or order reversal of the QCS yield a multiple of a cyclic permutation and/or order reversal of the sequence of increments \underline{u} . However, the number or occurrences b_k of the power y^k remains invariant by these operations (up to replacement of y by $1/y$ and $p(x)$ by $x^{\deg p}p(1/x)$).

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In particular $p(x)$ is well-defined up to replacement by $\tilde{p}(x) = x^{\deg p}p(1/x)$. \square

Remark 4.1.9. Note that when given a non-constant real QCS of order N with $\gamma \neq 2$, and when the fundamental increment y is different from -1 , (which is the case $\gamma \neq 1$), the polynomial $p(x)$ which we constructed in the proof of Theorem 4.1.3 is unique, because when $y \neq -1$, we can't have $y^r = y^{r'}$ for any powers $r \neq r'$. However, when $\gamma = 1$, the fundamental increment is $y = -1$ and we have $y^r = y^{r'} = 1$ for all r, r' even, and $y^r = y^{r'} = -1$ for all r, r' odd, so there may be many polynomials having the form (4.4), corresponding to the same QCS, as for example $p(x) = c.(x + 1)$ where c is the number of repetition of 1 (of -1 also) in the sequence of increments. More precisely, any polynomial $p(x) = (x + 1)q(x) = a_px^p + a_{p-1}x^{p-1} + \dots + a_1x + a_0$ such that $\sum_{i=0}^p a_i = N$, $q(x)$ having strictly positive integer coefficients, will do the work.

Definition 4.1.10. We define \mathcal{R} to be a binary relation on the set of all real QCS of some length N such that for any two real QCS $\varphi_1 = (x_0, x_1, \dots, x_{N-1})$ and $\varphi_2 = (y_0, y_1, \dots, y_{N-1})$, we have $\varphi_1 \mathcal{R} \varphi_2$ if and only if one can be obtained from the other by the following operations: multiplication by a constant; addition of a constant; cyclic permutation; order reversal; legitimate reordering of the sequence of increments (constructed following Theorem 4.1.3).

Clearly, the relation \mathcal{R} defined above is an equivalence relation, and any two sequences φ_1 and φ_2 such that $\varphi_1 \mathcal{R} \varphi_2$ are said to be *equivalent*.

Definition 4.1.11. Two polynomials with strictly positive integer coefficients $p_1(x)$ and $p_2(x)$ of common degree $n - 1$ form a *pair* if $p_2(x) = x^{n-1}p_1(1/x)$,

For example $p_1(x) = (x + 1)(2x + 3)$ and $p_2(x) = (x + 1)(3x + 2)$ form a pair.

Recall that by Theorem 4.1.3, up to normalization, the sequence of increments has the form $(y^{s_0}, y^{s_1}, \dots, y^{s_{N-1}})$ where all powers are integers ≥ 0 , and successive increments have powers differing by ± 1 and that there may be several different orderings possible of the elements of the sequence of increments. In §4.4 we will explore how to characterize these different orderings in terms of Eulerian digraphs.

In order to construct examples of real QCS, we follow the proof of Theorem 4.1.3. Given the polynomial $p(x) = (x + 1)q(x)$ of degree n , one constructs the corresponding QCS by first constructing a legitimate sequence of increments $(y^{s_0}, y^{s_1}, \dots, y^{s_{N-1}})$ where $s_j \in \{0, 1, \dots, n + 1\}$ and $s_j = s_{j-1} \pm 1$ and where the power y^j occurs b_j times.

Example 4.1.12. The example of the Introduction, in §1.5, is obtained by taking the polynomial $3x + 2$ with root $y = -2/3$. Multiply this by $x + 1$ to obtain the polynomial

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$p(x) = 3x^2 + 5x + 2$. Now proceed as follows:

$$\begin{array}{ll}
 3x^2 + 5x + 2 & \text{(defining polynomial)} \\
 \downarrow & \\
 (1, y, 1, y, y^2, y, y^2, y, y^2, y) & \text{(sequence of increments)} \\
 \parallel & \\
 (1, -\frac{2}{3}, 1, -\frac{2}{3}, \frac{4}{9}, -\frac{2}{3}, \frac{4}{9}, -\frac{2}{3}, \frac{4}{9}, -\frac{2}{3}) & \text{(seq. of increments for given root)} \\
 \downarrow & \\
 (0, 1, \frac{1}{3}, \frac{4}{3}, \frac{2}{3}, \frac{10}{9}, \frac{4}{9}, \frac{8}{9}, \frac{2}{9}, \frac{2}{3}) & \text{(corresponding cyclic sequence)} \\
 \downarrow & \\
 (0, 9, 3, 12, 6, 10, 4, 8, 2, 6) & \text{(normalized cyclic sequence)}
 \end{array}$$

The second sequence is obtained by taking a different legitimate ordering of the powers of y in the above construction, namely: $(1, y, y^2, y, 1, y, y^2, y, y^2, y)$. The two sequences are equivalent.

As explained in the proof of Theorem 4.1.3, the defining polynomial corresponding to a given sequence is not invariant under cyclic permutations.

Example 4.1.13. Perform a cyclic permutation on the QCS of Example 4.1.12:

$$\begin{array}{ll}
 (6, 10, 4, 8, 2, 6, 0, 9, 3, 12) & \text{(sequence)} \\
 \downarrow & \\
 (4, -6, 4, -6, 4, -6, 9, -6, 9, -6) & \text{(increments)} \\
 \downarrow & \\
 (1, -\frac{3}{2}, 1, -\frac{3}{2}, 1, -\frac{3}{2}, \frac{9}{4}, -\frac{3}{2}, \frac{9}{4}, -\frac{3}{2}) & \text{(normalization)} \\
 \downarrow & \\
 (1, y', 1, y', 1, y', y'^2, y', y'^2, y') & \text{where } y' = 1/y.
 \end{array}$$

Now the defining polynomial is given by $(x+1)(2x+3) = x^2p(1/x)$ which is the pair of $p(x)$, where $p(x)$ is the polynomial of Example 4.1.12.

4.2 Complex algebraic QCS

We return to equation (4.1), but in the first instance with $\gamma : \mathbb{Z}/N\mathbb{Z} \rightarrow [-\infty, 1]$ (not necessarily constant):

$$\frac{\gamma(j)}{2} (2\varphi(j) - \varphi(j-1) - \varphi(j+1))^2 = (\varphi(j) - \varphi(j-1))^2 + (\varphi(j) - \varphi(j+1))^2 \quad \forall j \in \mathbb{Z}/N\mathbb{Z} \quad (4.11)$$

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So we can picture $\varphi : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ as a closed polygon in the plane with edges the straight line segments $[\varphi(j), \varphi(j+1)]$. The condition that $\gamma(j) \leq 1$ for each j is imposed as a consequence of the Cauchy-Schwarz inequality discussed in the Introduction. We allow the limiting value $\gamma = 1$.

As we saw in Chapter 1, Proposition 1.5.3, a solution $\varphi : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ of (4.11) with $\gamma : \mathbb{Z}/N\mathbb{Z} \rightarrow [-\infty, 1]$ defines a polygon in \mathbb{C} whose edges $[\varphi(j), \varphi(j+1)]$, $j \in \mathbb{Z}/N\mathbb{Z}$ all have the same length, and conversely any polygon in \mathbb{C} with sides of equal length corresponds to a solution $\varphi : \mathbb{Z}/N\mathbb{Z} \rightarrow \mathbb{C}$ of (4.11) with $\gamma : \mathbb{Z}/N\mathbb{Z} \rightarrow [-\infty, 1]$. Moreover, the expression of $\gamma(j)$ is given by:

$$\gamma(j) = \frac{2 \cos \alpha_j}{1 + \cos \alpha_j} = \frac{2 \cos \theta_j}{\cos \theta_j - 1}$$

where $\alpha_j \in [0, \pi]$ is the absolute angle between the two edge $[\varphi(j-1), \varphi(j)]$ and $[\varphi(j), \varphi(j+1)]$, and $\theta_j = \pi - \alpha_j$ is the exterior angle. The two limiting cases $\alpha_j = 0$ and $\alpha_j = \pi$ correspond to $\gamma(j) = 1$ and $\gamma(j) = -\infty$, respectively.

In our work, we are interested in such polygons corresponding to a complex QCS, so we suppose that γ is constant. Hence, a complex QCS with $\gamma \in [-\infty, 1]$ will be pictured as a closed polygon in the plane with edges the straight line segments $[\varphi(j), \varphi(j+1)]$, each having the same length and where the absolute angle α between each two consecutive edges is constant. The expression of γ is given by

$$\gamma = \frac{2 \cos \alpha}{1 + \cos \alpha}.$$

Moreover, the exterior angle θ is then uniquely defined up to sign. In the following, we will see that such polygons will be defined by polynomial equations.

Up to normalization, we can suppose the length of each edge is 1. The fundamental increment y then has the form $y = e^{i\theta}$ and a complex QCS has corresponding sequence of increments $(y^{s_0}, y^{s_1}, \dots, y^{s_{N-1}})$ where $s_{j+1} = s_j \pm 1$, $\forall j \in \mathbb{Z}/N\mathbb{Z}$. We can also suppose that $s_0 = 0$ up to normalization, and that $s_j \geq 0$ for all $j = 0, \dots, N-1$. Then, since the sequence is cyclic then y satisfies the polynomial equation

$$b_{n-1}y^{n-1} + b_{n-2}y^{n-2} + \dots + b_1y + b_0 = 0,$$

where b_k is the number of occurrences of y^k in the sequence of increments. Call such a QCS *algebraic with turning angle θ* .

Remark 4.2.1. In general, a sequence of complex numbers corresponding to a solution to equation (4.11) for $\gamma : \mathbb{Z}/N\mathbb{Z} \rightarrow [-\infty, 1]$ is *algebraic* if it is determined by a polynomial equation, that is the edges of the corresponding polygon are powers of an increment y .

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Obviously, a complex QCS is always algebraic, since γ is constant, so that the exterior angle θ is constant and up to normalization the QCS has as corresponding sequence of increments $(y^{s_0}, y^{s_1}, \dots, y^{s_{N-1}})$ where $s_{j+1} = s_j \pm 1, \forall j \in \mathbb{Z}/N\mathbb{Z}$, where $y = e^{i\theta}$.

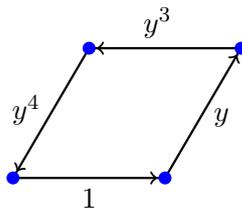
Example 4.2.2. An example of an algebraic sequence of complex numbers corresponding to a solution to equation (4.11) for $\gamma : \mathbb{Z}/N\mathbb{Z} \rightarrow [-\infty, 1]$, and which is not of constant turning angle, is the one defined by the polynomial

$$x^4 + x^3 + x + 1$$

with fundamental increment $y = e^{i\pi/3}$ (so $y^3 = -1$). We can take the sequence of increments

$$(1, y, y^3, y^4)$$

which determines a parallelogram with two different turning angles, and is a solution to equation (4.11) since edges have equal length 1, (by Proposition 1.5.3).



If we perturb the above parallelogram so the two parallel edges (different from 1 and -1) have angle which is an irrational multiple of 2π , the obtained sequence of complex numbers would still be a solution to equation (4.11), but is not algebraic.

A root of a polynomial with integer coefficients is called an *algebraic integer*. Algebraic integers exist of modulus 1 which are not roots of unity. We will return to complex algebraic QCS arising from such increments at the end of this section. However, in the first instance, we suppose that the fundamental increment y is a root of unity, that is $y^n = 1$, and that the sequence of increments $u_j = \varphi(j+1) - \varphi(j)$ is taken from a set $\{1, y, y^2, \dots, y^{n-1}\}$ where $y = e^{2m\pi i/n}$, where m and n are relatively prime with $m < n$. Note that since m and n are relatively prime, we can always suppose $m < n$. Also, since m and n are relatively prime, then n is the smallest positive integer for which $y^n = 1$. The condition the sequence be cyclic implies that y is a root of a polynomial of the form

$$p(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_1x + b_0,$$

where $b_k \geq 0$ represents the number of occurrences of y^k in the sequence of increments.

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Note that it may happen that some coefficients vanish. Our objective is to characterize the polynomials that determine a complex algebraic QCS with increment $y = e^{2m\pi i/n}$.

For $y = e^{2\pi i/n}$, the numbers $1, y, y^2, \dots, y^{n-1}$ are the n 'th roots of unity. A root of unity ν is *primitive* if $1, \nu, \nu^2, \dots, \nu^{n-1}$ are all distinct (the order of ν is n). Clearly, when m and n are relatively prime, then $y = e^{2m\pi i/n}$ is primitive since n is the smallest positive integer for which $y^n = 1$. The n 'th cyclotomic polynomial $\Phi_n(x)$ is the polynomial whose roots are the n 'th primitive roots of unity:

$$\Phi_n(x) := \prod_{\substack{1 \leq k \leq n \\ \gcd(n, k) = 1}} (x - e^{2\pi i k/n})$$

Then $\Phi_n(x)$ is irreducible over the integers and is the minimal polynomial over the integers of $y = e^{2m\pi i/n}$ (m, n relatively prime, $m < n$) [24].

Theorem 4.2.3. *Let φ be a complex QCS with increment $y = e^{2m\pi i/n}$ (m, n relatively prime, $m < n$).*

(i) *When $n = 2k$ ($k \geq 2$) is even, φ is determined by a polynomial of the form*

$$p(x) = (x + 1)\Phi_n(x)q(x)$$

where $q(x)$ is a polynomial of degree $\leq n - 2 - \deg \Phi_n(x)$ whose coefficients satisfy conditions discussed on a case by case basis below. When $n = 2$, $p(x) = a(x + 1)$ for some positive integer a .

(ii) *When n is odd, φ is determined by a polynomial of the form*

$$p(x) = \Phi_n(x)q(x)$$

where $q(x)$ is a polynomial of degree $\leq n - 1 - \deg \Phi_n(x)$ whose coefficients satisfy conditions discussed on a case by case basis below. In the case when n is prime, then

$$p(x) = a(x^{n-1} + x^{n-2} + \dots + x + 1)$$

for some positive integer a .

Conversely, polynomials of the above type (with $q(x)$ to be made precise) yield a corresponding QCS.

Proof. Let $p(x)$ be a defining polynomial of φ , so $p(y) = 0$. Consider a corresponding sequence of increments, then successive increments differ by a power of one, so each occurrence of y^s is followed by either y^{s-1} or y^{s+1} . Since y is a primitive root of unity, then all terms of the sequence of increments associated to $p(x)$ are taken from the set $\{1, y, \dots, y^{n-1}\}$, where

$1, y, \dots, y^{n-1}$ are all distinct. When $n = 2k$, if we set $y = -1$, the sequence of increments has the form $(1, -1, \dots, 1, -1)$ or $(-1, 1, \dots, -1, 1)$, hence $p(-1) = 0$, so $x + 1$ is a factor of $p(x)$, but this is not always the case when $n = 2k + 1$. In fact, the sequence of increments has any two successive terms y^{s_j} and $y^{s_{j+1}}$ taken from the set $\{1, y, \dots, y^{n-1}\}$ such that $s_{j+1} = s_j \pm 1 \pmod{n}$. In this sequence, we might have a transition from y^{n-1} to y^0 or not. In the latter case, $(x + 1)$ must be a factor of $p(x)$, whether n is odd or even, since powers of y must be alternatively even and odd. However, in the former case, $(x + 1)$ is a factor of $p(x)$ only when n is even, since the transition from y^{n-1} to $y^n = y^0$ is consistent with alternation from -1 to 1 when $y = -1$. On the other hand, for n being odd or even, y is a root of $p(x)$ so that the minimal polynomial over the integers of y must also divide $p(x)$. However, the minimal polynomial over the integers of $y = e^{2m\pi i/n}$, m and n relatively prime, $m < n$, is the n th cyclotomic polynomial $\Phi_n(x)$. In the case when n is prime, the n th cyclotomic polynomial is given by $x^{n-1} + x^{n-2} + \dots + x + 1$. \square

For background on cyclotomic polynomials see [24]. The first ones are given as follows

$$\begin{aligned}
 \Phi_1(x) &= x - 1 \\
 \Phi_2(x) &= x + 1 \\
 \Phi_3(x) &= x^2 + x + 1 \\
 \Phi_4(x) &= x^2 + 1 \\
 \Phi_5(x) &= x^4 + x^3 + x^2 + x + 1 \\
 \Phi_6(x) &= x^2 - x + 1 \\
 \Phi_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
 \Phi_8(x) &= x^4 + 1 \\
 \Phi_9(x) &= x^6 + x^3 + 1 \\
 \Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1 \\
 \Phi_{11}(x) &= x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
 \Phi_{12}(x) &= x^4 - x^2 + 1 \\
 \Phi_{13}(x) &= x^{12} + x^{11} + \dots + x^2 + x + 1 \\
 \Phi_{14}(x) &= x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 \\
 \Phi_{15}(x) &= x^8 - x^7 + x^5 - x^4 + x^3 - x + 1
 \end{aligned}$$

In general, if n is prime, then

$$\Phi_n(x) = x^{n-1} + x^{n-2} + \dots + x^2 + x + 1.$$

If $n = 2^r$ ($r > 0$) then

$$\Phi_{2^r}(x) = x^{2^{r-1}} + 1.$$

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If $n = 2p$ for p an odd prime, then

$$\Phi_{2p}(x) = 1 - x + x^2 - \dots + x^{p-1}.$$

In order to determine the coefficients of the polynomial $q(x)$ of Theorem 4.2.3 case by case, we need to find the conditions that the coefficients of the defining polynomial $p(x)$ should satisfy:

Call a sequence of powers $\underline{u} = (y^{s_0}, y^{s_1}, \dots, y^{s_{N-1}})$ *legitimate* if $s_{j+1} = s_j \pm 1 \pmod{n}$. Call a polynomial with non-negative integer coefficients *legitimate* if it determines a legitimate sequence of increments from one of its roots. First we establish an elementary recursive way to determine if a polynomial is legitimate. For ease of notation, write the sequence of increments as $y^{s_0}y^{s_1} \dots y^{s_{N-1}}$, i.e. omit brackets and commas.

Suppose that the sequence of increments $y^{s_0}y^{s_1} \dots y^{s_{N-1}}$ contains three successive terms $\dots y^k y^{k+1} y^k \dots$ or $\dots y^k y^{k-1} y^k, \dots$. We call such a subsequence an *elementary loop based at y^k* .

Lemma 4.2.4. *Let $p(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_1x + b_0$ be a legitimate polynomial with root y , and let $y^{s_0}y^{s_1} \dots y^{s_{N-1}}$ be a corresponding legitimate sequence of increments containing an elementary loop $\dots y^k y^{k+1} y^k \dots$ (respectively $\dots y^k y^{k-1} y^k \dots$) based at y^k . Then replacement of the loop by y^k yields a legitimate sequence of increments with corresponding (legitimate) polynomial*

$$p_1(x) = b_{n-1}x^{n-1} + \dots + b_{k+2}x^{k+2} + (b_{k+1} - 1)x^{k+1} + (b_k - 1)x^k + b_{k-1}x^{k-1} \dots + b_1x + b_0,$$

(respectively,

$$p_2(x) = b_{n-1}x^{n-1} + \dots + b_{k+1}x^{k+1} + (b_k - 1)x^k + (b_{k-1} - 1)x^{k-1} + b_{k-2}x^{k-2} \dots + b_1x + b_0.)$$

Furthermore, if $p(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_1x + b_0$ is a legitimate polynomial with $b_k \geq b_{k-1}, b_{k+1}$ with not all three coefficients equal, then necessarily, any corresponding legitimate sequence of increments contains a loop at y^k .

Proof. Suppose the sequence of increments contains a loop of the form $\dots y^k y^{k+1} y^k \dots$, so that the sequence contains a subsequence of the form

$$\dots \left. \begin{array}{l} y^{k+1} \\ y^{k-1} \end{array} \right\} y^k y^{k+1} y^k \left\{ \begin{array}{l} y^{k+1} \\ y^{k-1} \end{array} \right. \dots$$

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Then clearly replacement of this subsequence by the subsequence

$$\cdots \left. \begin{array}{c} y^{k+1} \\ y^{k-1} \end{array} \right\} y^k \left\{ \begin{array}{c} y^{k+1} \\ y^{k-1} \end{array} \right. \cdots$$

yields a legitimate sequence. Similarly for a loop of the form $\dots y^k y^{k-1} y^k \dots$

Suppose now that $b_k \geq b_{k-1}, b_{k+1}$ with not all three coefficients equal, and that there is no loop at y^k . Then each passage via y^k has one of the forms

$$\begin{aligned} & \dots y^{k-2} y^{k-1} y^k y^{k+1} y^{k+2} \dots \\ & \dots y^{k+2} y^{k+1} y^k y^{k-1} y^{k-2} \dots \\ & \dots y^{k+2} y^{k+1} y^k y^{k+1} y^{k+2} \dots \\ & \text{or } \dots y^{k-2} y^{k-1} y^k y^{k-1} y^{k-2} \dots \end{aligned}$$

which would imply in all cases that $b_{k-1}, b_{k+1} \geq b_k$, in contradiction to our hypothesis. \square

Call the process of removing a loop, that is replacing $\dots y^k y^{k+1} y^k \dots$ or $\dots y^k y^{k-1} y^k \dots$ by $\dots y^k \dots$, *reduction*.

Remark 4.2.5. In Lemma 4.2.4, the word *legitimate* is put between brackets because the root y of the original polynomial might no longer be a root of the reduced polynomial (the one obtained by reduction). So in particular, there is no guarantee that a reduced polynomial corresponds to a quadratic sequence which is *cyclic*, even if the initial polynomial does. But for ease of notation, we will keep calling a reduced polynomial *legitimate* when it admits a sequence of powers $(x^{s_0}, x^{s_1}, \dots, x^{s_{N-1}})$ where $s_{j+1} = s_j \pm 1 \pmod{n}$.

Remark 4.2.6. If, as in the above lemma, we have $b_k \geq b_{k-1}, b_{k+1}$ with not all three coefficients equal, without further information, it is not possible to know if a loop goes left or right, that is, takes the form $\dots y^k y^{k-1} y^k \dots$ or $\dots y^k y^{k+1} y^k \dots$. For example, if all loops at y^k have the form $\dots, y^k y^{k+1} y^k \dots$, then the following could arise with all inequalities possible:

$$\begin{aligned} & \dots y^{k-1} y^k y^{k+1} y^k y^{k-1} \dots & b_k = b_{k-1} > b_{k+1} \\ & \dots y^{k-1} y^k y^{k+1} y^k y^{k+1} y^k y^{k-1} \dots & b_k > b_{k-1} = b_{k+1} \\ & \dots y^{k-1} y^k y^{k+1} y^k y^{k+1} y^k y^{k+1} y^k y^{k-1} \dots & b_k > b_{k+1} > b_{k-1} \end{aligned}$$

Remark 4.2.7. Let $p(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_1x + b_0$ be a legitimate polynomial. If for some k , the condition " $b_k \geq b_{k-1}, b_{k+1}$ with not all three coefficients equal" is not satisfied,

it does not necessarily mean that there is no loop based at y^k in a legitimate sequence of increments. For example, for $y = e^{2i\pi/4}$, the sequence $1y^31y^3y^2y1yy^2y^3$ is legitimate having a loop based at y^1 of the form $y^1y^0y^1$ with corresponding legitimate polynomial $p(x) = 3x^3 + 2x^2 + 2x + 3$ where $b_0 = 3$, and $b_1 = b_2 = 2$.

Lemma 4.2.4 yields the following proposition:

Proposition 4.2.8. *Let $p(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_1x + b_0$ be a polynomial with non negative integer coefficients, admitting a root y . If there exists $0 \leq k \leq n - 1$ such that $b_k \geq b_{k-1}, b_{k+1}$, with not all three coefficients equal, then:*

$p(x)$ is legitimate

\Downarrow

$$p_1(x) = b_{n-1}x^{n-1} + \dots + b_{k+2}x^{k+2} + (b_{k+1} - 1)x^{k+1} + (b_k - 1)x^k + b_{k-1}x^{k-1} \dots + b_1x + b_0,$$

or

$$p_2(x) = b_{n-1}x^{n-1} + \dots + b_{k+1}x^{k+1} + (b_k - 1)x^k + (b_{k-1} - 1)x^{k-1} + b_{k-2}x^{k-2} \dots + b_1x + b_0$$

is legitimate.

Proof. If we suppose that $p(x)$ is legitimate, and since $b_k \geq b_{k-1}, b_{k+1}$ with not all three coefficients equal, then by Lemma 4.2.4, there exists a loop based at y^k in any corresponding sequence of increments, and any reduction at y^k yields a reduced legitimate polynomial having one of the two forms stated above. Conversely, suppose that $p_1(x) = b_{n-1}x^{n-1} + \dots + b_{k+2}x^{k+2} + (b_{k+1} - 1)x^{k+1} + (b_k - 1)x^k + b_{k-1}x^{k-1} \dots + b_1x + b_0$ is legitimate, then it admits a legitimate sequence of increments, that is a sequence $\underline{u} = (y^{s_0}, y^{s_1}, \dots, y^{s_{N-1}})$, where $s_{j+1} = s_j \pm 1 \pmod{n}$. If we replace some y^k in \underline{u} by $y^k y^{k+1} y^k$, we get a legitimate sequence of increments corresponding to the polynomial $p(x)$, so $p(x)$ is legitimate. The case is similar if $p_2(x)$ is legitimate, by replacing y^k in \underline{u} with $y^k y^{k-1} y^k$. \square

Even if $p(x)$ is legitimate, if we don't know whether a loop based at y^k goes left or right, we can't guess which of $p_1(x)$ or $p_2(x)$ is legitimate. However, using Proposition 4.2.8, in order to establish the legitimacy of a polynomial, it is enough to find a sequence of reductions which yields a legitimate polynomial, as we will see in the example below.

Example 4.2.9. The following sequence of reductions shows that the polynomial $p(x) = 2x^5 + 2x^4 + 3x^3 + 2x^2 + 2x + 3$ is legitimate. Indeed, consider the root $y = e^{\pi i/3}$ of $p(x)$.

Apply this sequence of reductions:

$$\begin{array}{c}
 2x^5 + 2x^4 + 3x^3 + 2x^2 + 2x + 3 \\
 \downarrow \\
 x^5 + 2x^4 + 2x^3 + x^2 + 2x + 2 \\
 \downarrow \\
 x^5 + x^4 + x^3 + x^2 + x + 1
 \end{array}$$

where at the first step, we have $b_3 \geq b_2, b_4$ (resp. $b_0 \geq b_1, b_5$) with not three coefficients equal, so we remove a loop based at y^3 and another based at y^0 . At the second step, we remove once again loops at y^3 and y^0 . The final polynomial $x^5 + x^4 + x^3 + x^2 + x + 1$ corresponds to the sequence $1yy^2y^3y^4y^5$ which is clearly legitimate. So by Proposition 4.2.8, the second polynomial $x^5 + 2x^4 + 2x^3 + x^2 + 2x + 2$ is legitimate, and once again by Proposition 4.2.8, we deduce that the original polynomial $p(x)$ is legitimate. More precisely, the third polynomial $x^5 + x^4 + x^3 + x^2 + x + 1$ came from reductions of the form $y^3y^4y^3$ and y^0yy^0 , so we insert these into the legitimate sequence $1yy^2y^3y^4y^5$ to obtain the legitimate sequence $1y1yy^2y^3y^4y^3y^4y^5$. Then the first reductions were of the form $y^3y^2y^3$ and $y^0y^5y^0$, so we now insert these into the sequence $1y1yy^2y^3y^4y^3y^4y^5$ to obtain the legitimate sequence $1y^51y1yy^2y^3y^2y^3y^4y^3y^4y^5$ which corresponds to the original polynomial. Hence the polynomial $p(x)$ admitting the root y defines a complex QCS.

Remark 4.2.10. There may be many reductions one can do to test the legitimacy of a polynomial, but even if the original polynomial is legitimate nothing guaranties that the reduced polynomial we get at the end is always legitimate. For example, in the above example, at the first step, one could have gone left at y^3 and left at y^0 , then at the second step again left at y^3 and left at y^0 to get the following:

$$\begin{array}{c}
 2x^5 + 2x^4 + 3x^3 + 2x^2 + 2x + 3 \\
 \downarrow \\
 2x^5 + x^4 + 2x^3 + 2x^2 + x + 2 \\
 \downarrow \\
 2x^5 + x^3 + 2x^2 + 1
 \end{array}$$

But this reduced polynomial is not legitimate, since it can't admit a legitimate sequence.

Now, to show that a polynomial is not legitimate, then using Proposition 4.2.8, one has to test all possible sequences of reductions where at each reduction we have two possibilities to consider (whether to go left or right). If they all yield to a reduced non legitimate polynomial, then the original polynomial is not legitimate (as in the following corollary).

Corollary 4.2.11. *Let $p(x) = b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_1x + b_0$ be a polynomial with $b_k \geq b_{k+1} + b_{k-1}$ for some $k \in \{0, \dots, n-1\}$ ($k \pm 1$ calculated modulo n). Then $p(x)$ is not legitimate.*

Proof. First of all, since $b_k \geq b_{k-1} + b_{k+1}$ so in particular, for any reduction at y^k we take, we will still have in every reduced polynomial $b_k \geq b_{k-1}, b_{k+1}$ with not all three coefficients equal. The polynomial $p(x)$ can be reduced recursively by removing loops at y^k , but whether we go left or right at each reduction, we will always reach at some point a reduced polynomial where the coefficient of x^{k+1} (respectively x^{k-1}) is zero, with the coefficient of x^k non-zero and greater than or equal to the coefficient of x^{k-1} (respectively x^{k+1}). Now, the last polynomial is clearly not legitimate. In fact, when the coefficient of x^{k+1} is zero, then in order to have a legitimate sequence, each y^k should be neighbored by y^{k-1} from left and right, that is the coefficient of x^{k-1} should be strictly greater than the coefficient of x^k , which is not the case. Similarly for when the coefficient of x^{k-1} is zero. So, if we suppose that $p(x)$ is legitimate, then for at least one of the sequences of reduction we took, the polynomial we get at each reduction is legitimate, so applying Proposition 4.2.8 recursively, the final polynomial should be legitimate which is not the case. Hence $p(x)$ is not legitimate. \square

Case by case analysis of the polynomials defining a complex algebraic QCS with fundamental increment $y = e^{2m\pi i/n}$ (m, n relatively prime, $m < n$). We apply Theorem 4.2.3.

$n = 3$: $p(x) = a(x^2 + x + 1)$, where a is a positive integer.

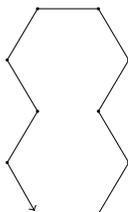
$n = 4$: $p(x) = a(x + 1)(x^2 + 1)$, where a is a positive integer.

$n = 5$: $p(x) = a(x^4 + x^3 + x^2 + x + 1)$, where a is a positive integer.

$n = 6$: $p(x) = (x + 1)(x^2 - x + 1)(ax^2 + bx + c) = (x^3 + 1)(ax^2 + bx + c) = ax^5 + bx^4 + cx^3 + ax^2 + bx + c$. First note that the coefficients a, b, c must be strictly positive, so as not to disconnect powers of y in the sequences of increments. What about other constraints? Clearly if $a = b = c > 0$, then we have a legitimate sequence of increments, namely the cyclic sequence $y^0y^1y^2y^3y^4y^5$ covered a times. Suppose then that the coefficients are not all equal, for example, suppose $a \geq b, c$ with one of the inequalities strict. By Corollary 4.2.11, we cannot have $a \geq b + c$. Suppose then that $a < b + c$. We claim this is legitimate. For example if $a \geq b, a > c$, then reduce the pair ab successively $a - c$ times until the sequence of coefficients $abcabc$ becomes $c(b - a + c)cc(b - a + c)c$. Now reduce the two pairs cc successively, until we have all coefficients equal to $b - a + c$ (> 0 by hypothesis). This is now legitimate, corresponding to the cyclic sequence $y^0y^1y^2y^3y^4y^5$ taken $b - a + c$ times. By symmetry we have the following characterization: *the polynomial $p(x)$ is legitimate if and only if all of a, b, c are positive integers and the maximum coefficient of $\{a, b, c\}$ is strictly less than the sum of the other two coefficients.*

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For example, take $a = b = 2$ and $c = 1$ to give $p(x) = 2x^5 + 2x^4 + x^3 + 2x^2 + 2x + 1$. One possible sequence of increments has the form $(1, y, y^2, y, y^2, y^3, y^4, y^5, y^4, y^5)$ with corresponding QCS:

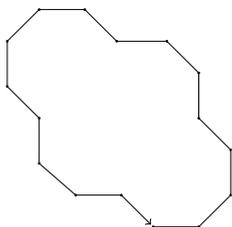


$n = 7$: $p(x) = a(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$, where a is a positive integer.

$n = 8$: $p(x) = (x+1)(x^4+1)(ax^2+bx+c) = ax^7 + (a+b)x^6 + (b+c)x^5 + cx^4 + ax^3 + (a+b)x^2 + (b+c)x + c$, where we clearly require $a, c, a+b, b+c > 0$. If $a, b, c > 0$, then the polynomial is legitimate. Specifically, we first reduce the sequence of coefficients $a(a+b)(b+c)ca(a+b)(b+c)c$ to $aaccaacc$ in the obvious way and then reduce to $aaaaaaaa$ if $a \leq c$ or to $cccccccc$ if $c \leq a$.

On the other hand, b may be zero or negative. If b is zero, then $p(x) = ax^7 + ax^6 + cx^5 + cx^4 + ax^3 + ax^2 + cx + c$ is clearly legitimate. If b is strictly negative and, say $a \geq c$, then this is again legitimate (similarly if $c \geq a$): Write $e = -b > 0$. Then the sequence of coefficients takes the form $a(a-e)(c-e)ca(a-e)(c-e)c$ (with $c-e > 0$). Now reduce e times at y^3 and y^7 to obtain the sequence of coefficients $(a-e)(a-e)(c-e)(c-e)(a-e)(a-e)(c-e)(c-e)$, which then reduces to the legitimate sequence $(c-e)(c-e)(c-e)(c-e)(c-e)(c-e)(c-e)(c-e)$.

It is worth illustrating the above construction with an example. The polynomial $p(x) = 3x^7 + 2x^6 + x^5 + 2x^4 + 3x^3 + 2x^2 + x + 2$ is legitimate ($a = 3, b = -1, c = 2$). To construct a corresponding sequence of increments, we work backwards from the cyclic sequence by the above procedure: $1yy^2y^3y^4y^5y^6y^7 \rightarrow 1yy^2y^3y^2y^3y^4y^5y^6y^7y^6y^7 \rightarrow 1yy^2y^3y^2y^3y^4y^3y^4y^5y^6y^7y^6y^71y^7$. The corresponding QCS is illustrated below.



To summarize: *the polynomial $p(x)$ is legitimate if the coefficients a, b, c are integers with $a, c, a+b, b+c > 0$.*

$n = 9$: $p(x) = (x^6 + x^3 + 1)(ax^2 + bx + c) = ax^8 + bx^7 + cx^6 + ax^5 + bx^4 + cx^3 + ax^2 + bx + c$. This is analogous to the case $n = 6$, with the same constraints on the coefficients a, b, c .

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$n = 10$: $p(x) = (x+1)(x^4 - x^3 + x^2 - x + 1)(a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0) = (x^5 + 1)(a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0) = a_4x^9 + a_3x^8 + a_2x^7 + a_1x^6 + a_0x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$. This is analogous to the case $n = 6$ but more complicated. Clearly a necessary condition is that a_1, a_2, a_3 and a_4 be strictly positive. Rather than give an exhaustive treatment of the different cases, it suffices to apply the recursive procedure given by Proposition 4.2.8.

$n = 11$: $p(x) = a(x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$ for a positive integer a .

$n = 12$: $p(x) = (x+1)(x^4 - x^2 + 1)(a_6x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0)$. This time $(x+1)\Phi_{12}(x)$ has a different form to the previous cases and we don't have an exhaustive description of the admissible coefficients. We discuss this case further below.

The above case by case analysis exhibits certain symmetry properties that we make precise in the following proposition. In Chapter 5, we take a different approach to these symmetry properties involving linear algebra.

Proposition 4.2.12. *Any complex algebraic QCS with turning angle $2\pi m/n$ (m, n relatively prime with $m < n$) either with $n \leq 11$, or $n = 2^r$ ($r > 0$), or $n = 2p$ (p an odd prime) must use all increments $\{y^0, y^1, \dots, y^{n-1}\}$ ($y = e^{2\pi mi/n}$). Furthermore if n is even with the same hypotheses, then for each occurrence of the increment y^k , there is also an occurrence of the increment $y^{\frac{n}{2}+k} = -y^k$. In particular, the corresponding polygonal walk in the plane contains each edge with its oppositely orientated counterpart.*

Proof. The proposition follows from the case by case analysis above. Specifically, for $n \leq 11$, the defining polynomial $p(x)$ has the form $p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1x + a_0$ with all $a_j > 0$ for $j = 0, \dots, n-1$, so that we must use all increments $\{y^0, y^1, \dots, y^{n-1}\}$. Furthermore, for $n = 2\ell$ even, up to $n = 10$, the defining polynomial $p(x)$ always has the form $p(x) = (x^\ell + 1)(a_{\ell-1}x^{\ell-1} + a_{\ell-2}x^{\ell-2} + \dots + a_1x + a_0)$, so that the coefficient of x^k is the same as the coefficient of $x^{\ell+k}$, for $k = 0, \dots, \ell-1$.

This continues to hold if $n = 2^r$ ($r > 0$) or $n = 2p$ (p an odd prime). For $n = 2^r$ the cyclotomic polynomial has the form $x^{2^{r-1}} + 1$. For $n = 2p$, p odd prime, the cyclotomic polynomial is given by $x^{p-1} - x^{p-2} + \dots - x + 1$, but $x+1$ must also be a factor of the defining polynomial since $2p$ is even, so in particular $(x+1)(x^{p-1} - x^{p-2} + \dots - x + 1) = x^p + 1$ is a factor. Hence, for $n = 2^r$ and $n = 2p$ (p odd prime), $(x^{\frac{n}{2}} + 1)$ is a factor of the defining polynomial, so for each occurrence of a y^k there is an occurrence of $y^{\frac{n}{2}+k}$. This obviously implies that if there is no occurrence of some increment y^k , then there is also no occurrence of the increment $y^{\frac{n}{2}+k}$, which disconnects the sequence of increments. Hence, we must use all increments $\{y^0, y^1, \dots, y^{n-1}\}$. \square

When $n = 12$, the properties of the above proposition no longer hold in general. That is, it can happen that we don't use all increments $\{y^0, y^1, \dots, y^{n-1}\}$ in a corresponding polygonal

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walk in the plane, or that in this walk, each edge does not necessarily have an oppositely orientated counterpart (the increments y^k and $y^{\frac{n}{2}+k}$ do not necessarily have the same number of occurrences).

Consider the cyclotomic polynomial $\Phi_{12}(x) = x^4 - x^2 + 1$. This has as a root the fundamental increment $y = e^{\pi i/6}$ corresponding to the exterior angle of a regular polygon of 12 sides. Clearly, a trivial legitimate polynomial is $p(x) = c(x^{11} + x^{10} + \dots + x + 1)$ where c is a strictly positive integer, for it admits the legitimate sequence of increments, namely the cyclic sequence $1yy^2 \dots y^{11}$ covered c times, and $p(x)$ is symmetric. Let $q(x) = 2x^6 + 2x^5 + 3x^4 + 3x^3 + 2x^2 + 2x + 1$. Then

$$q(x)\Phi_{12}(x) = 2x^{10} + 2x^9 + x^8 + x^7 + x^6 + x^5 + 2x^4 + x^3 + x^2 + 2x + 1,$$

has all coefficients strictly positive. However, it is not a legitimate polynomial. But we can now apply Lemma 4.1.7: since all coefficients of $q(x)\Phi_{12}(x)$ are strictly positive, multiplying by $x + 1$ we obtain the legitimate polynomial

$$p(x) = 2x^{11} + 4x^{10} + 3x^9 + 2x^8 + 2x^7 + 2x^6 + 3x^5 + 3x^4 + 2x^3 + 3x^2 + 3x + 1.$$

This defines a complex algebraic QCS with legitimate sequence of increments, say

$$y^0 y^1 y^2 y^3 y^4 y^5 y^6 y^7 y^8 y^9 y^{10} y^{11} y^{10} y^{11} y^{10} y^9 y^{10} y^9 y^8 y^7 y^6 y^5 y^4 y^5 y^4 y^3 y^2 y^1 y^2 y^1$$

for which symmetry no longer holds (left-hand path of Fig. 4.1).

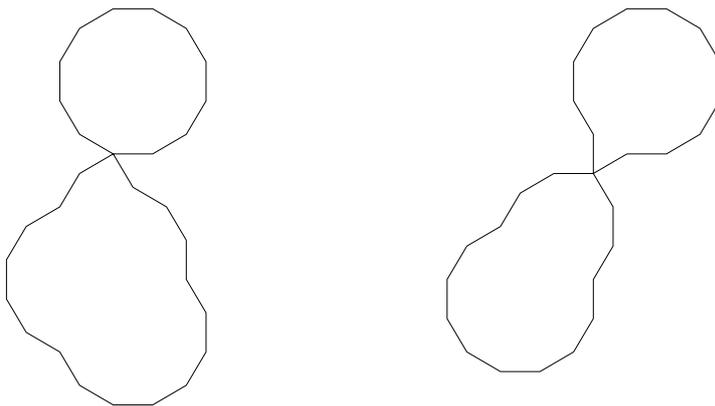


Figure 4.1: The two figures represent walks with turning angle $\pi/6$; the left is not symmetric; the right doesn't exploit all edges of a regular 12-gon.

Indeed, for $k = 5$ for example, the number of occurrences of y^5 (in the left-hand path) is 3 while the number of occurrences of $y^{\frac{n}{2}+k} = y^{11}$ is 2.

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It is also the case that for $n = 12$, all edges of the corresponding regular polygon are not required to complete a complex algebraic QCS with fundamental increment $y = e^{\pi i/6}$:

Let $q(x) = 2x^5 + 2x^4 + 3x^3 + 3x^2 + 2x + 2$. Then

$$\begin{aligned} q(x)\Phi_{12}(x) &= (2x^5 + 2x^4 + 3x^3 + 3x^2 + 2x + 2)(x^4 - x^2 + 1) \\ &= 2x^9 + 2x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + 2x + 2 \end{aligned}$$

is a polynomial of degree 9 with all coefficients strictly positive and with $y = e^{\pi i/6}$ as a root. Although it is not legitimate, multiplication by $x + 1$ once more leads to the legitimate polynomial

$$p(x) = 2x^{10} + 4x^9 + 3x^8 + 2x^7 + 2x^6 + 2x^5 + 2x^4 + 2x^3 + 3x^2 + 4x + 2,$$

from which we can construct a complex algebraic QCS which doesn't use the edge y^{11} (right-hand path of Fig. 4.1). Computer analysis (which we will discuss further below) shows that the right-hand path of Fig. 4.1 is the smallest length path for which this property holds, that is it has the minimal sum of the coefficients providing we don't use the edge y^{11} . Furthermore, we can show that there is no polynomial $q(x)$ of degree ≤ 4 for which $q(x)\Phi_{12}(x)$ has strictly positive coefficients, so that for $n = 12$, any corresponding QCS may omit at most one edge.

One can proceed in an ad hoc fashion for $n > 12$. One easily checks that the next case for which all edges are not required to complete a cycle is $n = 18$.

In fact, for $n = 13$ or $n = 17$, prime numbers, any legitimate polynomial has the form $p(x) = a(x^{n-1} + x^{n-2} + \dots + x + 1)$ where a is a strictly positive integer, so the two properties stated in Proposition 4.2.12 hold. For $n = 14 = 2 \cdot 7$ where 7 is prime, or $n = 16 = 2^4$, then by Proposition 4.2.12, the two properties also hold.

For $n = 15$, having as corresponding cyclotomic polynomial $\Phi_{15}(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$, we can show that there is no polynomial $q(x)$ of degree ≤ 4 such that $q(x)\Phi_{15}(x)$ has strictly positive coefficients, so that we can multiply by $(x + 1)$ to get a legitimate polynomial of degree < 14 , hence we must use all increments $\{y^{14}, y^{13}, \dots, y, 1\}$. However, note that it's the symmetry that don't always hold: if we consider the polynomial

$$q(x) = 2x^5 + 4x^4 + 5x^3 + 5x^2 + 4x + 2,$$

then multiplication by $\Phi_{15}(x)$ yields a polynomial having strictly positive integer coefficients, and once again multiplying by $(x + 1)$ we get a legitimate polynomial of degree 14 where there is no symmetry.

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For $n = 18$, the cyclotomic polynomial is $\Phi_{18}(x) = x^6 - x^3 + 1$. Multiplication by $2x^8 + 2x^7 + 2x^6 + 3x^5 + 3x^4 + 3x^3 + 2x^2 + 2x + 2$ yields the polynomial

$$2x^{14} + 2x^{13} + 2x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + 2x^2 + 2x + 2$$

with all coefficients positive. Multiplication by $x + 1$ yields the legitimate polynomial of degree 15 which determines a complex QCS with fundamental increment $y = e^{\pi i/9}$ of length 42 which doesn't use the edges y^{16} and y^{17} . Computer analysis shows that 2 is the most number of edges that can be omitted for $n = 18$ and furthermore, the above polynomial minimizes the length of the corresponding polygonal walk in this case. However, if we multiply $\Phi_{18}(x)$ by $x^9 + 2x^8 + 2x^7 + 2x^6 + 3x^5 + 3x^4 + 2x^3 + 2x^2 + 2x + 1$ we obtain the polynomial

$$x^{15} + 2x^{14} + 2x^{13} + x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + 2x^2 + 2 + 1$$

with all coefficients positive. Multiplication by $x + 1$ yields a legitimate polynomial of degree 16 which determines a complex QCS of length 40 which omits one edge. This is the minimum length which uses 17 edges. Thus the minimum length may decrease as one allows more edges.

In the case $n = 30$, $\Phi_{30}(x) = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1$. Multiplication by $4x^{13} + 2x^{12} + 2x^{11} + 3x^{10} + 4x^9 + 5x^8 + 3x^7 + 3x^6 + 5x^5 + 4x^4 + 3x^3 + 2x^2 + 2x + 4$ yields the polynomial

$$4x^{21} + 6x^{20} + 4x^{19} + x^{18} + x^{17} + x^{16} + x^{15} + x^{14} + 2x^{13} + x^{12} + x^{11} + x^{10} + x^9 + 2x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + 4x^2 + 6x + 4.$$

Multiplication by $x + 1$ then gives a legitimate polynomial of degree 22 which determines a complex QCS with fundamental increment $y = e^{\pi i/15}$ of length 92 which omits 7 edges. Using computer analysis, this is the maximum number of edges that can be omitted for $n = 30$ and the above polynomial minimizes the length in this case.

Method used for finding the legitimate polynomials of smallest degree and minimum sum of coefficients:

We recall that if $p(x)$ is a defining polynomial of a complex algebraic QCS, having a root $y = e^{2im\pi/n}$, $m < n$, m and n relatively prime, so if we consider a corresponding legitimate sequence of increments $\{y^{s_0}, y^{s_1}, \dots, y^{s_{N-1}}\}$, then its elements are taken from the set $\{1, y, y^2, \dots, y^{n-1}\}$ such that $s_{j+1} = s_j \pm 1 \pmod{n}$, $j \in \mathbb{Z}/N\mathbb{Z}$. If we don't have a transition from y^{n-1} to y^0 in any sequence of increments corresponding to $p(x)$, then by Lemma 4.1.7 of §4.1, $p(x) = (x + 1)h(x)$ where $h(x)$ has *strictly positive integer coefficients*.

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When the coefficient of x^{n-1} is zero in $p(x)$, (that is when we don't use the edge y^{n-1} in any corresponding polygonal walk), then we don't have a transition from y^{n-1} to y^0 in any sequence of increments corresponding to $p(x)$. So, when looking for the polynomials $q(x)$ of Theorem 4.2.3, when the degree of $\Phi_n(x).q(x)$ is strictly less than $n - 2$, we require that $(x + 1)$ be a factor of $p(x)$, and the coefficients of $\Phi_n(x).q(x)$ all be strictly positive integers.

$n = 12$: $\Phi_{12}(x) = x^4 - x^2 + 1$.

If $q(x) = ax + b$, then

$$\Phi_{12}(x).q(x) = ax^5 + bx^4 - ax^3 - bx^2 + ax + b,$$

if $q(x) = ax^2 + bx + c$, then

$$\Phi_{12}(x).q(x) = ax^6 + bx^5 + (c - a)x^4 - bx^3 + (a - c)x^2 + bx + c,$$

if $q(x) = ax^3 + bx^2 + cx + d$, then

$$\Phi_{12}(x).q(x) = ax^7 + bx^6 + (c - a)x^5 + (d - b)x^4 + (a - c)x^3 + (b - d)x^2 + cx + d,$$

if $q(x) = ax^4 + bx^3 + cx^2 + dx + e$, then

$$\Phi_{12}(x).q(x) = ax^8 + bx^7 + (c - a)x^6 + (d - b)x^5 + (e - c + a)x^4 + (b - d)x^3 + (c - e)x^2 + dx + e,$$

and in each case, there are always two coefficients that can not be > 0 at the same time.

If $q(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$, then $\Phi_{12}(x).q(x)$ has as coefficients:

x^9	x^8	x^5	x^6	x^5	x^4	x^3	x^2	x	1
a	b	c	d	e	f				
		$-a$	$-b$	$-c$	$-d$	$-e$	$-f$		
			a	b	c	d	e	f	

So here, we need to look for conditions on a, b, c, d, e, f such that the 10 above coefficients of $\Phi_{12}(x).q(x)$ be strictly positive integers, that is we have the 10 following constraints:

$$\begin{aligned} a &\geq 1, & b &\geq 1, \\ -a + c &\geq 1, & -b + d &\geq 1, \\ a - c + e &\geq 1, & b - d + f &\geq 1, \\ c - e &\geq 1, & d - f &\geq 1, \\ e &\geq 1, & f &\geq 1. \end{aligned}$$

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The summation of the 10 coefficients of $\Phi_{12}(x).q(x)$ is $a + b + c + d + e + f$. We want to find the values of a, b, c, d, e, f minimizing the above sum and satisfying the 10 above inequalities. To this end, we create the following program using an implementation of the "Cassowary" algorithm [2] in "Rust":

Listing 4.1: Program that finds a, b, c, d, e, f , having minimal sum, under the 10 constraints.

```
1
2 extern crate cassowary; // ~0.3.0
3
4 use cassowary::strength::{REQUIRED, STRONG};
5 use cassowary::WeightedRelation::*;
6 use cassowary::{AddConstraintError, Expression, Solver, Variable};
7 use std::vec::Vec;
8
9 fn check_satisfiability(result: Result<(), AddConstraintError>) {
10     if result.is_err() {
11         eprintln!("Unsatisfiable constraints");
12         ::std::process::exit(1);
13     }
14 }
15
16 fn calculate_constraint_expression(variables: &Vec<Variable>,
17     i: usize) -> Expression {
18     let vars_count = variables.len();
19     let mut exp = Expression::new(vec![], 0.0);
20     if i < vars_count {
21         exp = exp + variables[i];
22     }
23     if i >= 2 && i < vars_count + 2 {
24         exp = exp - variables[i - 2];
25     }
26     if i >= 4 && i < vars_count + 4 {
27         exp = exp + variables[i - 4];
28     }
29     return exp;
30 }
31
32 fn main() {
33     //////////////////////////////////////
34     // Define the variables to be used //
35     //////////////////////////////////////
36
37     let var_names = ["a", "b", "c", "d", "e", "f"];
38
39     //////////////////////////////////////
40     // Create and add the constraints to the solver //
41     //////////////////////////////////////
```

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```

42
43     let vars_count = var_names.len();
44     let variables: Vec<Variable> =
45         var_names.iter().map(|_| Variable::new()).collect();
46     let mut solver = Solver::new();
47     let mut sum_exp = Expression::new(vec! [], 0.0);
48     let mut i = 0;
49     loop {
50         if i < vars_count {
51             sum_exp = sum_exp + variables[i];
52         }
53
54         let exp = calculate_constraint_expression(&variables, i);
55
56         if exp.terms.len() == 0 {
57             break;
58         }
59
60         check_satisfiability(
61             solver.add_constraint(exp | GE(REQUIRED) | 1.0)
62         );
63         i += 1;
64     }
65     check_satisfiability(
66         solver.add_constraint(sum_exp | EQ(STRONG) | 0.0)
67     );
68
69     //////////////////////////////////////
70     // Get the values from the solver and print them //
71     //////////////////////////////////////
72
73     for index in 0..vars_count {
74         println!(
75             "{} = {}",
76             var_names[index],
77             solver.get_value(variables[index])
78         );
79     }
80 }

```

Applying the above program, we get

$$a = 2, b = 2, c = 3, d = 3, e = 2, f = 2,$$

hence,

$$q(x) = 2x^5 + 2x^4 + 3x^3 + 3x^2 + 2x + 2.$$

Multiplying $q(x)$ by $\Phi_{12}(x).(x+1)$ we get a legitimate polynomial $p(x)$ of degree $n - 2 = 10$

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(so we don't use the edge y^{11}), it has the minimal length (since we minimized the sum $a + b + c + d + e + f$), and 5 is the smallest degree possible for $q(x)$ such that $p(x) = (x + 1) \cdot \Phi_{12}(x) \cdot q(x)$ is legitimate.

$n = 18$: $\Phi_{18}(x) = x^6 - x^3 + 1$.

As for the case $n = 12$, for all polynomials $q(x)$ of degree less than or equal 7, we can verify that $\Phi_{18}(x) \cdot q(x)$ can not admit all coefficients strictly positive.

For $q(x) = ax^8 + bx^7 + cx^6 + dx^5 + ex^4 + fx^3 + gx^2 + hx + i$, then $\Phi_{18}(x) \cdot q(x)$ has as coefficients:

$$\begin{array}{cccccccccccccccc}
 x^{14} & x^{13} & x^{12} & x^{11} & x^{10} & x^9 & x^8 & x^7 & x^6 & x^5 & x^4 & x^3 & x^2 & x & 1 \\
 \hline
 a & b & c & d & e & f & g & h & i & & & & & & \\
 & & & -a & -b & -c & -d & -e & -f & -g & -h & -i & & & \\
 & & & & & & a & b & c & d & e & f & g & h & i
 \end{array}$$

Requiring the coefficients of $\Phi_{18}(x) \cdot q(x)$ to be strictly positive integers, we have 15 constraints on the elements a, b, \dots, i . With the following suitable modifications applied on Listing 4.1,

```

1 // ...
2 fn calculate_constraint_expression(variables: &Vec<Variable>,
3                                 i: usize) -> Expression {
4     let vars_count = variables.len();
5     let mut exp = Expression::new(vec![], 0.0);
6     if i < vars_count {
7         exp = exp + variables[i];
8     }
9     if i >= 3 && i < vars_count + 3 {
10        exp = exp - variables[i - 3];
11    }
12    if i >= 6 && i < vars_count + 6 {
13        exp = exp + variables[i - 7];
14    }
15    return exp;
16 }
17
18 fn main() {
19     // ...
20     let var_names = ["a", "b", "c", "d", "e", "f", "g", "h", "i"];
21     // ...
22 }

```

we minimize the sum $a + b + \dots + i$, which is the sum of the 15 coefficients of $\Phi_{18}(x) \cdot q(x)$, under the required 15 constraints, and get the desired values for the coefficients of $q(x)$.

4.2. COMPLEX ALGEBRAIC QCS

We do quite the same to find the coefficients of $q(x) = ax^9 + bx^8 + cx^7 + dx^6 + ex^5 + fx^4 + gx^3 + hx^2 + ix + j$ of degree 9.

$$n = 30: \Phi_{30}(x) = x^8 + x^7 - x^5 - x^4 - x^3 + x + 1.$$

As for the two above cases, for all polynomials $q(x)$ of degree less than or equal 4, we can verify that $\Phi_{30}(x).q(x)$ can not admit all coefficients strictly positive.

However, when the degree of $q(x)$ is greater than 4, it is unreasonable to check by hand whether the coefficients of $\Phi_{30}(x).q(x)$ can all be strictly positive. For example, when the degree of $q(x)$ is 7, that is $q(x) = ax^7 + bx^6 + cx^5 + dx^4 + ex^3 + fx^2 + gx + h$, then $\Phi_{30}(x).q(x)$ has as coefficients:

x^{15}	x^{14}	x^{13}	x^{12}	x^{11}	x^{10}	x^9	x^8	x^7	x^6	x^5	x^4	x^3	x^2	x	1
a	b	c	d	e	f	g	h								
	a	b	c	d	e	f	g	h							
			$-a$	$-b$	$-c$	$-d$	$-e$	$-f$	$-g$	$-h$					
				$-a$	$-b$	$-c$	$-d$	$-e$	$-f$	$-g$	$-h$				
					$-a$	$-b$	$-c$	$-d$	$-e$	$-f$	$-g$	$-h$			
							a	b	c	d	e	f	g	h	
								a	b	c	d	e	f	g	h

so this time with the following modifications applied on Listing 4.1, requiring that the 16 coefficients of $\Phi_{30}(x).q(x)$ be strictly positive integers,

```

1 // ...
2 fn calculate_constraint_expression(variables: &Vec<Variable>,
3                                 i: usize) -> Expression {
4     let vars_count = variables.len();
5     let mut exp = Expression::new(vec![], 0.0);
6     if i < vars_count {
7         exp = exp + variables[i];
8     }
9     if i >= 1 && i < vars_count + 1 {
10        exp = exp + variables[i - 1];
11    }
12    if i >= 3 && i < vars_count + 3 {
13        exp = exp - variables[i - 3];
14    }
15    if i >= 4 && i < vars_count + 4 {
16        exp = exp - variables[i - 4];
17    }
18    if i >= 5 && i < vars_count + 5 {

```

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```
19     exp = exp - variables[i - 5];
20   }
21   if i >= 7 && i < vars_count + 7 {
22     exp = exp + variables[i - 7];
23   }
24   if i >= 8 && i < vars_count + 8 {
25     exp = exp + variables[i - 8];
26   }
27   return exp;
28 }
29
30 fn main() {
31   // ...
32   let var_names = ["a", "b", "c", "d", "e", "f", "g", "h"];
33   // ...
34 }
```

we get no solution ("Unsatisfiable constraints") when we ask to minimize $a + b + \dots + h$, which means that the coefficients of $\Phi_{30}(x).q(x)$ can not all be strictly positive integers. This remains the case for all degrees less than or equal 12.

Finally, when the degree of $q(x)$ is 13, that is

$$q(x) = ax^{13} + bx^{12} + cx^{11} + dx^{10} + ex^9 + fx^8 + gx^7 + hx^6 + ix^5 + jx^4 + kx^3 + lx^2 + mx + n,$$

then the coefficients of $\Phi_{30}(x).q(x)$ being strictly positive integers yields 21 constraints on a, b, \dots, n . Running the last program, but for n variables:

```
1 // ...
2
3 fn main() {
4   // ...
5   let var_names = [
6     "a", "b", "c", "d", "e", "f", "g", "h", "i", "j", "k", "l",
7     "m", "n",
8   ];
9   // ...
10 }
```

we get the desired values of a, b, \dots, n minimizing $a + b + \dots + n$ under the 21 constraints.

Complex QCS from algebraic increments which are not roots of unity: The smallest degree of a monic polynomial for which a unit modulus algebraic integer which is not a root of unity can occur is 4. This can be seen by first noting that the degree must be even (since the degree

4.2. COMPLEX ALGEBRAIC QCS

of the minimal polynomial of an algebraic number of unit modulus is even), and then that any unit modulus algebraic integer in a *quadratic* field extension must be a root of unity.

The palindromic polynomial

$$q(x) = x^4 + 3x^3 + 3x^2 + 3x + 1,$$

irreducible over the integers, has two conjugate complex roots a, \bar{a} and two real roots. Since for each root b , one must have $1/b$ a root, it follows that $\bar{a} = 1/a$ so that $|a| = 1$. However a cannot be an n 'th root of unity for any n , since if this were the case, $q(x)$ would divide $x^n - 1$ ($q(x)$ being monic and irreducible is the minimal polynomial of a over the integers). But this would mean that the two real roots are also roots of unity, which is not the case since 1 and -1 are not roots of $q(x)$. In fact the two complex roots are given approximately by $-0.191 \pm 0.982i$, and the two real roots, approximately by -2.154 and -0.464 . Although the coefficients of $q(x)$ do not yield a corresponding sequence of increments (see below), they are all strictly positive and so, using Lemma 4.1.7, we can multiply by $x + 1$ to obtain the polynomial

$$p(x) = (x + 1)(x^4 + 3x^3 + 3x^2 + 3x + 1) = x^5 + 4x^4 + 6x^3 + 6x^2 + 4x + 1$$

admitting, for example, the legitimate sequence of increments

$$y^0 y^1 y^2 y^3 y^4 y^5 y^4 y^3 y^4 y^3 y^2 y^3 y^2 y^1 y^2 y^1 y^2 y^1,$$

where y is one of the complex roots of $p(x)$. One then constructs a corresponding complex algebraic QCS.

Another example arises from the palindromic polynomial $x^4 + 2x^3 + 2x + 1$. This has two complex conjugate roots of modulus 1 (approx. $0.366 \pm 0.931i$) which are not roots of unity by the same reasoning as above. Indeed, there are two real roots given approx. by -2.297 and -0.435 . This time the polynomial does not have all coefficients strictly positive, however, multiplication by $x + 1$ yields $x^5 + 3x^4 + 2x^3 + 2x^2 + 3x + 1$ which, although not legitimate, does have all coefficients positive and once more multiplying by $x + 1$ yields the legitimate polynomial

$$p(x) = x^6 + 4x^5 + 5x^4 + 4x^3 + 5x^2 + 4x + 1.$$

Taking y to be one of the complex roots now yields a complex algebraic QCS, whose legitimate sequence of increments is

$$y^0 y^1 y^2 y^3 y^4 y^5 y^6 y^5 y^4 y^5 y^4 y^3 y^4 y^3 y^2 y^3 y^2 y^1 y^2 y^1 y^2 y^1$$

and the corresponding polygonal walk has approximate turning angle 68.5 degrees:

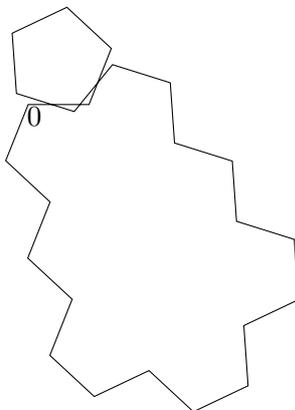


Figure 4.2: Complex algebraic QCS not arising from a root of unity

At last, we study switching from a real QCS with fundamental increment y to an algebraic complex QCS with fundamental increment $z = e^{2im\pi/n}$, $m < n$, m and n relatively prime.

Proposition 4.2.13. *Let X be a real QCS with $\gamma \neq 2$, with corresponding defining polynomial $p(x)$ and fundamental increment y , where y is a real root of $p(x)$. Then if $\Phi_n(x)$ divides $p(x)$, where $\Phi_n(x)$ is the n^{th} cyclotomic polynomial for some positive integer n , then there exists a complex algebraic QCS with fundamental increment $z = e^{2im\pi/n}$, $m < n$, m, n relatively prime, whose sequence of increments is the sequence of increments of X , up to replacement of y by $z = e^{2im\pi/n}$.*

Proof. As we saw in §4.1, any real QCS $X = (x_0, x_1, \dots, x_{N-1})$ with $\gamma \neq 2$ has as defining polynomial $p(x) = (x + 1)q(x)$ where $q(x)$ has strictly positive integer coefficients, and up to normalization, the corresponding sequence of increments $u = (u_0, u_1, \dots, u_{N-1})$ has the form $(y^{s_0}, y^{s_1}, \dots, y^{s_{N-1}})$, where y is a real root of $p(x)$, and $s_j = s_{j+1} \pm 1$, $j \in \mathbb{Z}/N\mathbb{Z}$. Clearly, if $\Phi_n(x)$ divides $p(x)$, then any primitive n^{th} root of unity is a root of $p(x)$, for it is a root of $\Phi_n(x)$. A primitive n^{th} root of unity has the form $z = e^{2im\pi/n}$ where $m < n$, m, n relatively prime, so replacing y by z in the sequence of increments u , we get a sequence $\underline{u} = (\underline{u}_0, \underline{u}_1, \dots, \underline{u}_{N-1})$ that has the form $(z^{s_0}, z^{s_1}, \dots, z^{s_{N-1}})$, where $s_j = s_{j+1} \pm 1 \pmod{n}$, $j \in \mathbb{Z}/N\mathbb{Z}$. Since z is a root of $p(x)$ then the sequence X' arising from the sequence of increments \underline{u} is cyclic. Hence X' is a complex algebraic QCS with legitimate sequence of increments \underline{u} . □

Example 4.2.14. Consider the polynomial

$$\begin{aligned} p(x) &= (x + 1) \cdot (x^2 - x + 1) \cdot (x^2 + 2x + 2) \cdot (x + 2) \\ &= x^6 + 4x^5 + 6x^4 + 5x^3 + 4x^2 + 6x + 4. \end{aligned}$$

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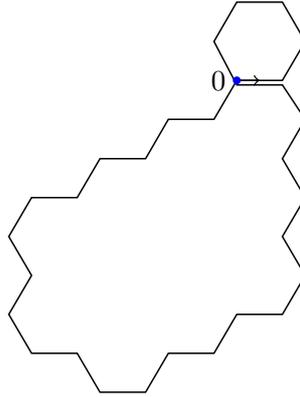
It admits $y = -2$ as a real root, and we can verify that $(x^2 - x + 1).(x^2 + 2x + 2).(x + 2)$ has strictly positive integer coefficients, hence by Theorem 4.1.3, $p(x)$ defines a real QCS with fundamental increment $y = -2$, and a corresponding sequence of increments is given by

$$y^0 y^1 y^2 y^3 y^4 y^5 y^6 y^5 y^4 y^5 y^4 y^5 y^4 y^3 y^4 y^3 y^4 y^3 y^2 y^3 y^2 y^1 y^2 y^1 y^0 y^1 y^0 y^1 y^0 y^1.$$

Since the cyclotomic polynomial $\Phi_6(x) = (x^2 - x + 1)$ divides $p(x)$, hence $z = e^{2i\pi/6}$ is a root of $p(x)$, and replacing y by z in the previous sequence of increments, and then z^6 by z^0 , we get a legitimate sequence of increments, namely the sequence

$$z^0 z^1 z^2 z^3 z^4 z^5 z^0 z^5 z^4 z^5 z^4 z^5 z^4 z^3 z^4 z^3 z^4 z^3 z^2 z^3 z^2 z^1 z^2 z^1 z^0 z^1 z^0 z^1 z^0 z^1,$$

corresponding to an algebraic complex QCS with fundamental increment z . The legitimate sequence of increments corresponds to the following walk with turning angle $\pi/3$:



The legitimate defining polynomial of this algebraic complex QCS is then given by

$$p(x) = 4x^5 + 6x^4 + 5x^3 + 4x^2 + 6x + 5.$$

Example 4.2.15. Another example is to take the polynomial

$$\begin{aligned} p(x) &= (x + 1)(x^2 + 1)(x^2 + 4x + 3) \\ &= x^5 + 5x^4 + 8x^3 + 3x^2 + 7x + 3, \end{aligned}$$

which admits $y = -3$ as a real root, and since $(x^2 + 1)(x^2 + 4x + 3)$ has strictly positive integer coefficients, then $p(x)$ defines a real QCS with fundamental increment $y = -3$. A corresponding sequence of increments is given by

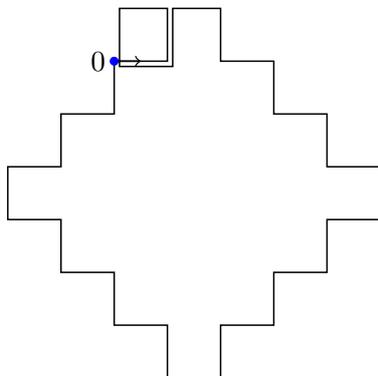
$$y^0 y^1 y^2 y^3 y^4 y^5 y^4 y^3 y^4 y^3 y^4 y^3 y^4 y^3 y^2 y^3 y^2 y^3 y^2 y^3 y^2 y^1 y^2 y^1 y^2 y^1 y^2 y^1 y^0 y^1 y^0 y^1.$$

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The cyclotomic polynomial $\Phi_4(x) = (x^2 + 1)$ divides $p(x)$, so $z = e^{2i\pi/4}$ is a root of $p(x)$, and replacing y by z in the previous sequence of increments, and then z^4 by z^0 , and z^5 by z^1 , we get the legitimate sequence of increments

$$z^0 z^1 z^2 z^3 z^0 z^1 z^0 z^3 z^0 z^3 z^0 z^3 z^2 z^3 z^2 z^3 z^2 z^3 z^2 z^1 z^2 z^1 z^2 z^1 z^0 z^1 z^0 z^1,$$

which corresponds to an algebraic complex QCS with fundamental increment z . The legitimate sequence of increment corresponds to the following walk in the plane with turning angle $\pi/2$:



The legitimate defining polynomial of the algebraic complex QCS is

$$p(x) = 8x^3 + 8x^2 + 8x + 8.$$

4.3 Combining QCS

On applying the normalization (4.3), cyclic sequences with common γ , can be combined to form new sequences. We will refer to this construction as *concatenation*. In this section, we show how this can be done for real QCS with $\gamma \neq 2$, and explain how concatenation is reflected in the defining polynomials. With suitable modifications, the same procedures apply to complex algebraic QCS. We begin with an example.

Example 4.3.1. As in Example 4.1.12, take the defining polynomial $(x + 1)(3x + 2)$ with root $y = -2/3$ leading to the cyclic sequence

$$(0, 9, 3, 12, 6, 10, 4, 8, 2, 6). \tag{4.12}$$

Normalize the sequence by dividing by 9 in order that the second term be 1:

$$(0, 1, \frac{1}{3}, \frac{4}{3}, \frac{2}{3}, \frac{10}{9}, \frac{4}{9}, \frac{8}{9}, \frac{2}{9}, \frac{2}{3}), \tag{4.13}$$

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and the corresponding sequence of increments of (4.13) is

$$(1, -\frac{2}{3}, 1, -\frac{2}{3}, \frac{4}{9}, -\frac{2}{3}, \frac{4}{9}, -\frac{2}{3}, \frac{4}{9}, -\frac{2}{3}) = (1, y, 1, y, y^2, y, y^2, y, y^2, y). \quad (4.14)$$

Now make another normalization of (4.12) by subtracting 6 and dividing by 4:

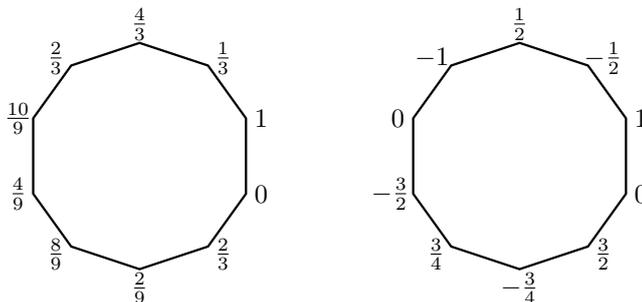
$$(-\frac{3}{2}, \frac{3}{4}, -\frac{3}{4}, \frac{3}{2}, 0, 1, -\frac{1}{2}, \frac{1}{2}, -1, 0), \quad (4.15)$$

and the corresponding sequence of increments of (4.15) is

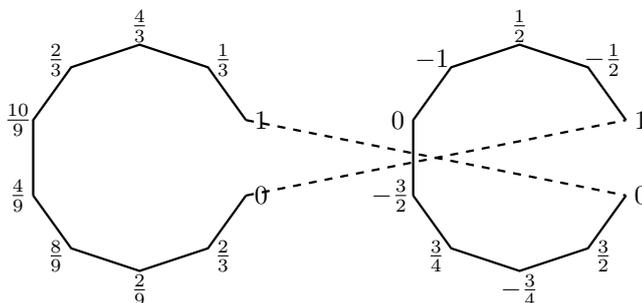
$$(\frac{9}{4}, -\frac{3}{2}, \frac{9}{4}, -\frac{3}{2}, 1, -\frac{3}{2}, 1, -\frac{3}{2}, 1, -\frac{3}{2}) = (\frac{1}{y^2}, \frac{1}{y}, \frac{1}{y^2}, \frac{1}{y}, 1, \frac{1}{y}, 1\frac{1}{y}, 1, \frac{1}{y}). \quad (4.16)$$

(Note that the sequence (4.15) has as defining polynomial $(x + 1)(2x + 3)$, the pair of $p(x)$, since the fundamental increment in its sequence of increments (4.16) is $\frac{1}{y}$.)

In so doing, we obtain the pair 0, 1 in a different location in (4.13) and (4.15). We can visualize the coefficients of the two sequences as labels on cyclic graphs:



Now switch edges as indicated to obtain a cyclic sequences of order 20:



$$(0, 1, -\frac{1}{2}, \frac{1}{2}, -1, 0, -\frac{3}{2}, \frac{3}{4}, -\frac{3}{4}, \frac{3}{2}, 0, 1, \frac{1}{3}, \frac{4}{3}, \frac{2}{3}, \frac{10}{9}, \frac{4}{9}, \frac{8}{9}, \frac{2}{9}, \frac{2}{3}) \quad (4.17)$$

We can now deduce the defining polynomial of the concatenated sequence. In fact, by concatenating the sequences (4.13) and (4.15), the sequence (4.17) has as corresponding sequence

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of increments:

$$(1, \frac{1}{y}, 1, \frac{1}{y}, 1, \frac{1}{y}, \frac{1}{y^2}, \frac{1}{y}, \frac{1}{y^2}, \frac{1}{y}, 1, y, 1, y, y^2, y, y^2, y, y^2, y). \quad (4.18)$$

By §4.1, some multiple of the corresponding sequence of increments yields a sequence of powers of one of the fundamental increments. So if we multiply (4.18) by y^2 we get the sequence

$$(y^2, y, y^2, y, y^2, y, 1, y, 1, y, y^2, y^3, y^2, y^3, y^4, y^3, y^4, y^3, y^4, y^3, y^4, y^3),$$

with corresponding defining polynomial

$$3x^4 + 5x^3 + 5x^2 + 5x + 2 = (x + 1)(x^2 + 1)(3x + 2) = (x^2 + 1)p(x).$$

Note that if the two equivalent sequences we chose, (4.13) and (4.15), came from two sequences of increments having the same fundamental increment y , then the corresponding polynomial of the concatenated sequence will be $2 \cdot p(x)$, that is $(x^0 + 1)p(x)$, since we won't have to multiply the corresponding sequence of increments by a power of y to get all its elements as positive powers of y .

An alternative construction is to combine two sequences of different lengths with common fundamental increment.

Example 4.3.2. Consider the two defining polynomials $p_1(x) = (x + 1)(x + 2)(2x + 1)$ and $p_2(x) = (x + 1)(x + 2)$ with common root $y = -2$:

$p_1(x) = 2x^3 + 7x^2 + 7x + 2$		$p_2(x) = x^2 + 3x + 2$
↓		↓
$(y^3, y^2, y^3, y^2, y, 1, y, 1, y, y^2, y, y^2, y, y^2, y, y^2)$	(increments)	$(y^2, y, 1, y, 1, y)$
↓		↓
$(-8, 4, -8, 4, -2, 1, -2, 1, -2, 4, -2, 4, -2, 4, -2, 4)$	$(y = -2)$	$(4, -2, 1, -2, 1, -2)$
↓		↓
$(0, -8, -4, -12, -8, -10, -9, -11, -10,$ $-12, -8, -10, -6, -8, -4, -6, -2, -4)$	(sequence)	$(0, 4, 2, 3, 1, 2)$
↓		↓
$(0, 4, 2, 6, 4, 5, \frac{9}{2}, \frac{11}{2}, 5, 6, 4, 5, 3, 4, 2, 3, 1, 2)$	$(\times -\frac{1}{2})$	

We now concatenate by placing one sequence after the other:

$$(0, 4, 2, 6, 4, 5, \frac{9}{2}, \frac{11}{2}, 5, 6, 4, 5, 3, 4, 2, 3, 1, 2, 0, 4, 2, 3, 1, 2)$$

with corresponding sequence of increments

$$(y^2, y, y^2, y, 1, \frac{1}{y}, 1, \frac{1}{y}, 1, y, 1, y, 1, y, 1, y, 1, y, y^2, y, 1, y, 1, y).$$

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Since by §4.1, some multiple of the corresponding sequence of increments yields a sequence of powers of one of the fundamental increments, then in this case, multiplication by -2 yields the sequence of increments

$$(y^3, y^2, y^3, y^2, y, 1, y, 1, y, y^2, y, y^2, y, y^2, y, y^2, y^3, y^2, y, y^2, y, y^2)$$

with defining polynomial $3x^3 + 10x^2 + 9x + 2 = (x+1)(x+2)(3x+1) = p_1(x) + x.p_2(x)$. The different ordering of the increments

$$(y^3, y^2, y^3, y^2, y, y^2, y, 1, y, 1, y, 1, y, 1, y, y^2, y, y^2, y, y^2, y, y^2, y, y^2),$$

which is obtained by placing the sequence $(y^2, y, 1, y, 1, y)$ after the fifth term of the sequence $(y^3, y^2, y^3, y^2, y, 1, y, 1, y, y^2, y, y^2, y, y^2, y, y^2)$, produces the polynomial $(x+1)(x+2)(2x+2) = p_1(x) + p_2(x)$.

We can recognize concatenation by the existence of two identical adjacent pairs as described by the following lemma, useful for the next proposition:

Lemma 4.3.3. *A real QCS X of the form $(x_0, x_1, \dots, x_k, a, b, y_0, y_1, \dots, y_\ell, a, b, z_0, z_1, \dots, z_m)$ is the concatenation of the two QCS $(a, b, y_0, y_1, \dots, y_\ell)$ and $(a, b, z_0, z_1, \dots, z_m, x_0, x_1, \dots, x_k)$.*

Proof. Since X has the form

$$(x_0, x_1, \dots, x_k, a, b, y_0, y_1, \dots, y_\ell, a, b, z_0, z_1, \dots, z_m),$$

with γ constant, say γ_c , then in particular we have

$$\frac{(a - x_k)^2 + (a - b)^2}{(a - x_k + a - b)^2} = \frac{(a - y_\ell)^2 + (a - b)^2}{(a - y_\ell + a - b)^2} = \gamma_c.$$

So the two sequences $(a, b, y_0, y_1, \dots, y_\ell)$ and $(a, b, z_0, z_1, \dots, z_m, x_0, x_1, \dots, x_k)$ have γ constant, ($= \gamma_c$), so are real QCS, and vice versa. \square

The above examples illustrate a more general property:

Proposition 4.3.4. *Let $p_1(x)$ and $p_2(x)$ be polynomials of degree $m - 1$ and $n - 1$ (resp.) of the form (4.4) of Theorem 4.1.3 which define real QCS with $\gamma \neq 2$, of orders M and N (resp.) deriving from a common fundamental increment y . Then there exists a real QCS of order $M + N$ with defining polynomial $p(x) = p_1(x) + x^k p_2(x)$, where $0 \leq k \leq m$, obtained by concatenation of two real QCS with defining polynomials $p_1(x)$ and $p_2(x)$ resp..*

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Proof. Let $(y^{s_0}, y^{s_1}, \dots, y^{s_{M-1}})$ and $(y^{t_0}, y^{t_1}, \dots, y^{t_{N-1}})$ be legitimate sequences of increments associated to real QCS deriving from $p_1(x)$ and $p_2(x)$ resp., where we suppose $s_0 = 0, s_1 = 1, \dots, s_{m-2} = m-2, s_{m-1} = m-1, s_m = m-2, \dots, s_{M-1} = 1$ and $t_0 = 0, t_1 = 1, \dots, t_{n-2} = n-2, t_{n-1} = n-1, t_n = n-2, \dots, t_{N-1} = 1$. That is, the powers are initially monotone increasing from 0 to $m-1$ (resp. $n-1$). This is always possible, see (4.19) of §4.4.

If $y = -1$, then the result is clear - we simply concatenate two oscillating sequences of the form $(0, 1, 0, 1, \dots, 0, 1)$.

A legitimate sequence of increments for $p(x)$ is given by

$$(y^{s_0}, y^{s_1}, \dots, y^{s_{k-1}}, y^{k+t_0}, y^{k+t_1}, \dots, y^{k+t_{N-1}}, y^{s_k}, y^{s_{k+1}}, \dots, y^{s_{M-1}})$$

with corresponding QCS:

$$\begin{aligned} (0, y^{s_0}, y^{s_0} + y^{s_1}, \dots, y^{s_0} + y^{s_1} + \dots + y^{s_{k-1}}, y^{s_0} + y^{s_1} + \dots + y^{s_{k-1}} + y^{k+t_0}, \\ y^{s_0} + y^{s_1} + \dots + y^{s_{k-1}} + y^{k+t_0} + y^{k+t_1}, \\ \dots, y^{s_0} + y^{s_1} + \dots + y^{s_{k-1}} + \underbrace{y^{k+t_0} + y^{k+t_1} + \dots + y^{k+t_{N-1}}}_0, \\ y^{s_0} + y^{s_1} + \dots + y^{s_{k-1}} + \underbrace{y^{k+t_0} + y^{k+t_1} + \dots + y^{k+t_{N-1}}}_0 + y^{s_k}, \dots) \end{aligned}$$

Since $(y^{t_0}, y^{t_1}, \dots, y^{t_{N-1}})$ is a legitimate sequence of increments associated to $p_2(x)$, we have $\sum_{j=0}^{N-1} y^{t_j} = 0 \Rightarrow \sum_{j=0}^{N-1} y^{k+t_j} = 0$. It follows that the pairs of successive terms

$$(y^{s_0} + y^{s_1} + \dots + y^{s_{k-1}}, y^{s_0} + y^{s_1} + \dots + y^{s_{k-1}} + y^{s_k})$$

and

$$\begin{aligned} (y^{s_0} + y^{s_1} + \dots + y^{s_{k-1}} + y^{k+t_0} + y^{k+t_1} + \dots + y^{k+t_{N-1}}, \\ y^{s_0} + y^{s_1} + \dots + y^{s_{k-1}} + y^{k+t_0} + y^{k+t_1} + \dots + y^{k+t_{N-1}} + y^{s_k}) \end{aligned}$$

coincide and by Lemma 4.3.3, the sequence can be obtained by concatenation of two sequences of orders M and N , with defining polynomials $p_1(x)$ and $p_2(x)$, resp.. \square

4.4 Eulerian digraphs

In order to better understand the correspondence between polynomials and QCS, notably the legitimate sequences of increments that can arise, it is useful to model the collection of increments with an Eulerian digraph.

4.4. EULERIAN DIGRAPHS

We recall that a digraph is a pair $D = (V, A)$ consisting of a (finite) set of vertices V and a set $A \subseteq V \times V$ – the (oriented) arcs. We allow now multiple arcs, loops and directed cycles of order 2. When $a \in A$ corresponds to an arc from x to y , we will write for more simplicity $a = xy$ instead of (x, y) .

A directed walk in a digraph $D = (V, A)$ is a sequence $v_0 a_1 v_1 a_2 \cdots a_k v_k$ where $v_j \in V$, $a_j \in A$ and $a_j = v_{j-1} v_j$, $1 \leq j \leq k$, with no arc repeated. A digraph is said to be *Eulerian* if it contains a closed spanning directed walk which traverses every arc of D . Eulerian digraphs are characterized by the following theorem [8].

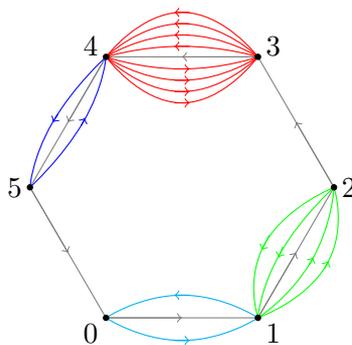
Theorem 4.4.1. *A digraph $D = (V, A)$ is Eulerian if and only if D is connected and for each of its vertices x , $d^-(x) = d^+(x)$.*

That is, a digraph is Eulerian if and only if it is a connected balanced digraph.

If the defining polynomial of a QCS has degree $n - 1$, then the digraphs we will use as a model will have vertices $\{0, 1, 2, \dots, n - 1\}$ and arcs only of the form $j(j - 1)$ or $(j - 1)j$ where j and $j - 1$ are taken modulo n . Call such a digraph a *1-step digraph*. We will view the vertices as arranged in cyclic order. For example, take

$$p(x) = (x + 1)(2x^4 + 4x^3 + x^2 + 2x + 2) = 2x^5 + 6x^4 + 5x^3 + 3x^2 + 4x + 2$$

(with real root -2). We construct a corresponding Eulerian digraph with 22 edges as follows.



For any vertex j , both the in-degree and the out-degree is equal to the coefficient of x^j . Call an *elementary closed directed walk between j and $j + 1$* , one of the form $ja(j + 1)bj$ ($a, b \in A$, $a \neq b$). For the above example, we have first constructed the cyclic digraph and then added elementary closed directed walks as necessary to correspond to the coefficients. As we see below there may be many non-isomorphic 1-step Eulerian digraphs associated to a given polynomial.

4.4. EULERIAN DIGRAPHS

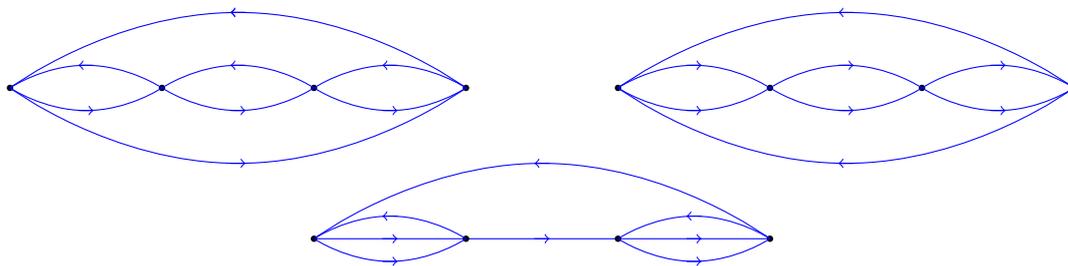
One can easily see that a digraph is Eulerian if and only if, after the removal of a closed directed walk, each of the connected components that remain are Eulerian, where we consider an isolated vertex as Eulerian. Thus we can recognize the above polynomial as being legitimate, if there is a corresponding associated 1-step Eulerian digraph obtained by removing elementary closed directed walks as follows:

$$\begin{aligned}
 p(x) = 2x^5 + 6x^4 + 5x^3 + 3x^2 + 4x + 2 &\rightarrow x^5 + 5x^4 + 5x^3 + 3x^2 + 4x + 2 \\
 &\rightarrow x^5 + x^4 + x^3 + 3x^2 + 4x + 2 \\
 &\rightarrow x^5 + x^4 + x^3 + x^2 + 2x + 2 \\
 &\rightarrow x^5 + x^4 + x^3 + x^2 + x + 1
 \end{aligned}$$

On the first step, we remove an elementary closed walk between 4 and 5; on the second step we remove $4 \times$ an elementary closed walk between 3 and 4; on the third step we remove $2 \times$ an elementary closed walk between 1 and 2; on the fourth step we remove an elementary closed walk between 0 and 1. The end polynomial now has corresponding 1-step Eulerian digraph, the cyclic digraph. The procedure of removing elementary closed walks is the analogue of reduction of §4.2. However, to check that a polynomial is not legitimate by this method means checking all possible reductions of all possible Eulerian digraphs.

For a balanced digraph of order n , its degree sequence is the sequence $(a_0, a_1, \dots, a_{n-1})$ where a_j is the in-degree (= out-degree) of vertex j . To such a digraph, we associate the polynomial $p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1 + a_0$.

Given the degree sequence for the defining polynomial of a QCS, an associated Eulerian 1-step digraph may not be unique. For example the polynomial $p(x) = 2 + 2x + 2x^2 + 2x^3$ has three realizations:



The top left-hand one contains elementary closed walks, whereas the top right-hand one contains no elementary closed walk. In this case the underlying multigraphs (the multigraph with the same vertex and edge set, but now with each edge undirected) are identical. However, the lower 1-step Eulerian digraph has underlying graph non-isomorphic to the top

4.4. EULERIAN DIGRAPHS

two. Corresponding legitimate sequences of increments are given by $1yy^2y^31y^3y^2y$ (top left), $1yy^2y^31yy^2y^3$ (top right), $1y1yy^2y^3y^2y^3$ (bottom).

We can exploit Eulerian digraphs to see that the polynomial

$$\begin{aligned} p(x) &= (x+1)(a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_1x + a_0) \\ &= a_{n-2}x^{n-1} + (a_{n-2} + a_{n-3})x^{n-2} + \cdots + (a_1 + a_0)x + a_0, \end{aligned}$$

of Theorem 4.1.3 is legitimate. Note however, that in the real case, we are not allowed to connect vertex $n-1$ with vertex 0.

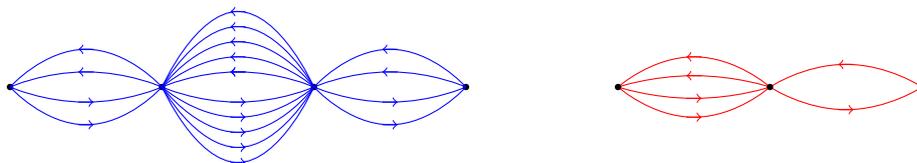
- Construct the directed edges $01, 12, \dots, (n-2)(n-1), (n-1)(n-2)$ (producing an elementary closed walk between $n-2$ and $n-1$).
- Construct $a_{n-2} - 1$ elementary closed walks between $n-2$ and $n-1$.
- Construct $(n-2)(n-3)$.
- Construct $a_{n-3} - 1$ elementary closed walks between $n-3$ and $n-2$.
- Construct $(n-3)(n-4)$.
- etc.
- Construct 10.
- Construct $a_0 - 1$ elementary closed walks between 0 and 1.

Note in particular that this shows it is always possible (in the notation of §4.1) to begin the sequence of increments

$$(1, y, y^2, \dots, y^{n-2}, y^{n-1}, y^{n-2}, \dots). \tag{4.19}$$

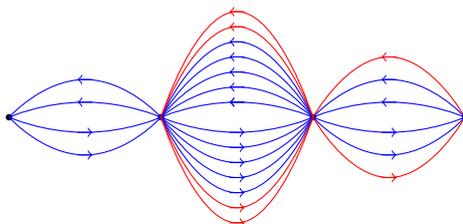
When enumerating all possible sequences of increments associated to a defining polynomial, we must consider all Eulerian 1-step digraphs associated to the polynomial (we don't distinguish between walks which take a different arc joining the same two vertices).

Concatenation of §4.3 can be represented in terms of corresponding digraphs. Consider the last example of §4.3, with $p_1(x) = 2x^3 + 7x^2 + 7x + 2$ and $p_2(x) = x^2 + 3x + 2$. Each of these has corresponding Eulerian digraphs given by the above algorithm as illustrated.

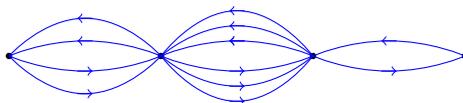


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When we concatenate the two sequences, we obtain a sequence with defining polynomial $3x^3 + 10x^2 + 9x + 2$, with (one of different possible) corresponding Eulerian digraph:



Similarly, we can reverse the process of concatenation by removing an Eulerian sub-digraph. For example, if, from the left-hand digraph corresponding to the polynomial $p(x) = 2x^3 + 7x^2 + 7x + 2$, we remove an Eulerian digraph corresponding to the right-hand digraph corresponding to the polynomial $p(x) = x^2 + 3x + 2$, we obtain the Eulerian digraph:



with corresponding polynomial $x^3 + 4x^2 + 5x + 2 = (x + 2)(x + 1)^2$. However, care needs to be taken, since we may lose the root -2 defining the QCS. If we remove another isomorphic copy of the same digraph, depending on how this is done, we arrive either at the digraph on the left with polynomial $x^2 + 3x + 2 = (x + 1)(x + 2)$ or the digraph on the right with polynomial $x^3 + 2x^2 + 2x + 1 = (x + 1)(x^2 + x + 1)$ whose only real root is -1 :



Note that, if the polynomial $p(x)$ has an associated 1-step Eulerian digraph $D = (V, A)$, then the polynomial $\tilde{p}(x) = x^{\deg p} p(1/x)$ has an associated 1-step Eulerian digraph $\tilde{D} = (\tilde{V}, \tilde{A})$ which is isomorphic to D . Here, isomorphism between digraphs means that the underlying multigraphs are isomorphic in a way which preserves the orientation of edges. In fact, if $p(x) = a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_1x + a_0$, then $\tilde{p}(x) = a_0x^{n-1} + a_1x^{n-2} + \dots + a_{n-2}x + a_{n-1}$. An isomorphism from D to \tilde{D} is given by mapping vertex j to $n - 1 - j$ ($0 \leq j \leq n - 1$) and mapping an edge $j(j + 1)$ to $(n - j - 1)(n - j - 2)$.

4.5 QCS and planar walks

Let us first review the construction of a QCS, both real and complex algebraic. Upon normalization (provided that $\gamma \neq 2$), the QCS can be put into the form $(0, 1, x_2, \dots)$. The sequence of increments is given

$$\left(1, y, \left\{ \begin{array}{c} y^2 \\ 1 \end{array} \right\}, \left\{ \begin{array}{c} y^3 \\ y \\ y^{-1} \end{array} \right\}, \dots \right)$$

where $y = x_2 - 1$. At each successive step, the increment y^s is multiplied either by y , or by y^{-1} . Thus, the normalized QCS has the form

$$\left(0, 1, 1 + y, 1 + y + \left\{ \begin{array}{c} y^2 \\ 1 \end{array} \right\}, \dots \right)$$

where y is the root of a polynomial whose coefficients are non-negative integers.

Let us now remove the requirement that the sequence be cyclic. Suppose that at each successive step, the increment y^s changes according to *either* $y^s \mapsto y^{s+1}$ or $y^s \mapsto y^{s-1}$, with equal probability $1/2$. This generates a random walk either along the real line, or in the complex plane according as to whether y is real or complex, respectively. Consider such a sequence with $y = e^{i\theta}$. Thus at each step we turn either right or left through an angle θ – we call this the *turning angle* of the walk. We are particularly interested in the cases $\theta = 2\pi/n$ when $n = 4$ or $n = 6$, for then the 2-step walk (the walk obtained by combining two successive steps) corresponds to standard walks on the square lattice, or triangular lattice, resp.

Consider a walk with an even number of steps, where at each step we are obliged to turn left or right with turning angle $\pi/2$ and where we take two steps at a time. We will refer to this as a *2-step walk with turning angle $\pi/2$* . We start at the origin $(0, 0)$ and set off in one of four directions, i.e. at the first step we arrive at one of $(0, \pm 1)$ or $(\pm 1, 0)$ with equal probability $1/4$.

Lemma 4.5.1. *After an even number of steps, we arrive at (k, ℓ) with $k + \ell \in 2\mathbb{Z}$.*

Proof. By induction on the number of steps $2t$. After two steps we arrive at one of $(\pm 1, \pm 1)$. Then each 2-step iteration replaces (k, ℓ) by $(k \pm 1, \ell \pm 1)$. \square

The *standard planar walk* is a walk on the integer lattice, for which, if one is at position (r, s) one moves to one of $(r \pm 1, s)$ or $(r, s \pm 1)$.

Lemma 4.5.2. *The 2-step planar walk with turning angle $\pi/2$ determines a standard planar*

4.5. QCS AND PLANAR WALKS

walk. Conversely, a standard planar walk corresponds to precisely two 2-step planar walks with turning angle $\pi/2$.

Proof. The walk is transformed into the standard walk by the mapping

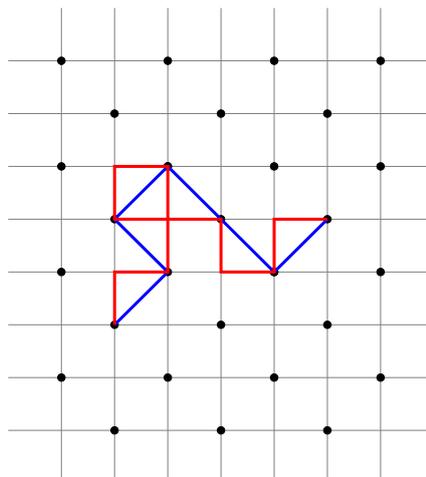
$$(k, \ell) \mapsto \left(\frac{k + \ell}{2}, \frac{\ell - k}{2} \right)$$

Note that, since $k + \ell \in 2\mathbb{Z}$, the right-hand side belongs to \mathbb{Z}^2 . If we set $r = (k + \ell)/2$ and $s = (\ell - k)/2$, then the possible outcomes of a 2-step walk (below left) map to the possible outcomes of the standard walk (below right):

$$\left. \begin{array}{l} (k + 1, \ell + 1) \mapsto (r + 1, s) \\ (k + 1, \ell - 1) \mapsto (r, s - 1) \\ (k - 1, \ell + 1) \mapsto (r, s + 1) \\ (k - 1, \ell - 1) \mapsto (r - 1, s) \end{array} \right\} \text{standard walk}$$

For each 2-step $(k, \ell) \mapsto (k \pm 1, \ell \pm 1)$ there are precisely two 1-step routes. For example, if $(k, \ell) \mapsto (k + 1, \ell - 1)$, then this is achieved by either $(k, \ell) \mapsto (k + 1, \ell) \mapsto (k + 1, \ell - 1)$ or $(k, \ell) \mapsto (k, \ell - 1) \mapsto (k + 1, \ell - 1)$. However, which of these two occurs is determined uniquely by the preceding step. Thus, given a standard walk, after an initial choice is made (of two possibilities), the 2-step walk is determined. \square

The diagram below gives a standard walk (blue) and one of the two possible corresponding 2-step walks (red).

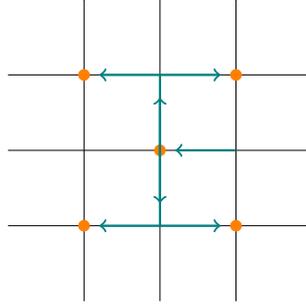


If we set $y = i$, then the 2-step walk is given by the following sequence of increments $(i, 1, i, -1, i, 1, -i, 1, -i, 1, i, 1)$. The alternative choice of 2-step walk is given by $(1, i, -1, i, 1, i, 1,$

4.5. QCS AND PLANAR WALKS

$-i, 1, -i, 1, i$). In terms of y , these are given by $(y, 1, y, y^2, y, 1, y^3, 1, y^3, 1, y, 1)$ and $(1, y, y^2, y, 1, y, 1, y^3, 1, y^3, 1, y)$, respectively. These correspond to different legitimate arrangements of the sequence of increments with defining polynomial $p(x) = 2x^3 + x^2 + 4x + 5 = (x + 1)(2x^2 - x + 5)$. Note that since the sequence is not cyclic, y is not a root of $p(x)$.

If we are in a particular position in a standard walk (at one of the orange nodes below), then in whatever direction we have arrived at that position in a 2-step walk, there is equal probability $1/4$ of arriving after two steps at one of the adjacent nodes in the standard walk, as illustrated in the diagram below, where we suppose we arrive along the horizontal arrow coming from the right.



Every standard planar walk of finite length determines a polynomial of the form $p(x) = b_3x^3 + b_2x^2 + b_1x + b_0$. This is obtained by choosing one of the two corresponding 2-step walks and setting b_k to be the cardinality of i^k in the sequence of increments. Then b_0 and b_2 correspond to horizontal increments and b_1 and b_3 to vertical increments. As for the example above, $x + 1$ must always be a factor of this polynomial, since each horizontal increment must be matched by a vertical increment. What polynomials $p(x)$ can arise from such a walk? In what follows, we will identify the lattice of the standard walk with the points (k, ℓ) in the plane with $k + \ell$ even.

Lemma 4.5.3. *Any polynomial $p(x)$ of the form $p(x) = (x + 1)(a_2x^2 + a_1x + a_0)$ with $a_0, a_2 \geq 0$ and $-a_1 \leq \min\{a_0, a_2\}$ determines a standard planar walk and conversely, each standard planar walk determines such a polynomial. The length of the standard planar walk is given by $a_0 + a_1 + a_2$. The walk is closed if and only if $p(x) = c(x + 1)(x^2 + 1)$ for some positive integer c . The end point of the walk is given by $p(i)$.*

Proof. Set $p(x) = b_3x^3 + b_2x^2 + b_1x + b_0 = (x + 1)(a_2x^2 + a_1x + a_0)$ where the a_j satisfy the conditions of the statement of the lemma. These conditions are equivalent to $b_k \geq 0$ for

$k = 0, \dots, 3$. Since $x + 1$ is a factor, $-b_3 + b_2 - b_1 + b_0 = 0$ so that the number of horizontal increments $b_0 + b_2$ (given by $y^0 = 1, y^2 = -1$ with $y = i$) is equal to the number of vertical increments $b_1 + b_3$. Clearly these can be ordered (non-uniquely in general) to give a 2-step walk determining a standard walk.

For the converse, given a standard planar walk, if b_k is the number of occurrences of y^k ($k = 0, 1, 2, 3, y = i$) as an increment in one of the two corresponding 2-step walks, then $p(x) = b_3x^3 + b_2x^2 + b_1x + b_0$ is a polynomial with the desired properties. The length of the standard planar walk is given by $(b_0 + b_1 + b_2 + b_3)/2 = a_0 + a_1 + a_2$.

If the walk is closed, i.e. it ends at its starting point, then $y = i$ is a root of $p(x)$. Since the coefficients of $p(x)$ are real, $-i$ must also be a root and $x^2 + 1$ is a factor. Thus $p(x)$ necessarily has the form $p(x) = c(x+1)(x^2+1)$ where c is a positive integer. Note that this is the same polynomial given in §4.2 for the case $n = 4$. More generally, $p(i) = -b_3i - b_2 + b_1i + b_0$ determines the end point of the walk. \square

We can be explicit about the coefficients of the defining polynomial as follows.

Theorem 4.5.4. *For a 2-step walk of turning angle $\pi/2$ of even length L from the origin to $k + i\ell$ ($k + \ell$ even), the defining polynomial is given by*

$$p(x) = (x + 1)(x - 1) \left(-\frac{\ell}{2}x - \frac{k}{2} \right) + \frac{L}{4}(x + 1)(x^2 + 1). \quad (4.20)$$

In particular it is uniquely defined by its length and its end point and as a consequence the (unordered) steps used to complete the walk are also uniquely defined.

Proof. Let $p(x) = b_3x^3 + b_2x^2 + b_1x + b_0$ be the defining polynomial of the walk. The length of the walk is given by $L = p(1) = b_3 + b_2 + b_1 + b_0$ and the end point by $p(i) = k + i\ell$, so that $k = b_0 - b_2$ and $\ell = b_1 - b_3$. Then together with $-b_3 + b_2 - b_1 + b_0 = 0$, we can solve for b_0, b_1, b_2, b_3 to obtain (4.20) \square

For a 2-step walk of odd length from the origin to the point $k + i\ell$, we can calculate the defining polynomial $\tilde{p}(x)$ of the walk to the preceding step (a walk of even length) as above. This could be one of the four possibilities: $(k - 1, \ell), (k + 1, \ell), (k, \ell - 1), (k, \ell + 1)$. Then we obtain $p(x)$ by adding on to $\tilde{p}(x)$, $1, x^2, x, x^3$, respectively. However, now the polynomial $p(x)$ depends upon the path.

Lemma 4.5.5. *For a given even length L , the number of 2-step paths from the origin to $k + i\ell$ is given by*

$$\frac{2(b_2 + b_0)!^2}{b_0!b_1!b_2!b_3!} = \frac{2(b_3 + b_1)!^2}{b_0!b_1!b_2!b_3!}$$

4.5. QCS AND PLANAR WALKS

where b_0, b_1, b_2, b_3 are the coefficients of $p(x) = b_3x^3 + b_2x^2 + b_1x + b_0$ given by (4.20). The number of standard paths of length $L/2$ from the origin to $k + il$ is half of this number.

Proof. Horizontal steps correspond to ± 1 and vertical steps to $\pm i$. These must occur alternately in the walk, i.e. we must either have horizontal - vertical - horizontal - ... , or vertical - horizontal - vertical - Otherwise, there is no restriction on the order in which we place $+1$ and -1 , similiary for $+i$ and $-i$. Thus the number of paths corresponds to the number of different orderings of $\{\underbrace{1, 1, \dots, 1}_{b_0}, \underbrace{-1, -1, \dots, -1}_{b_2}\}$ multiplied by the number of different orderings of $\{\underbrace{i, i, \dots, i}_{b_1}, \underbrace{-i, -i, \dots, -i}_{b_3}\}$. But the number of different orderings of $\{\underbrace{1, 1, \dots, 1}_{b_0}, \underbrace{-1, -1, \dots, -1}_{b_2}\}$ is given by the binomial coefficient

$$\binom{b_2 + b_0}{b_0} = \binom{b_2 + b_0}{b_2}$$

Similarly for the vertical steps. Finally, we can begin the walk with either a horizontal step or a vertical step, so the total number of paths is given by

$$2 \binom{b_2 + b_0}{b_0} \binom{b_3 + b_1}{b_1}$$

as required. This can be written differently using the identity $b_3 + b_1 = b_2 + b_0$. \square

If in the 2-step walk, the first step takes place with probability $1/4$ and successive steps with probability $1/2$, then each walk of length L occurs with probability

$$\frac{1}{4}2^{-L+1}.$$

Thus, for a given defining polynomial $p(x) = b_3x^3 + b_2x^2 + b_1x + b_0$, the probability that at least one walk defined by the polynomial occurs is given by

$$2 \binom{b_2 + b_0}{b_0} \binom{b_3 + b_1}{b_1} \times \frac{1}{4}2^{-L+1} = 4^{-L/2} \binom{b_2 + b_0}{b_0} \binom{b_3 + b_1}{b_1}.$$

Recalling that L is even, the latter expression gives the probability that one of the corresponding standard walks occurs.

Example 4.5.6. There are eight 2-step walks of length 4 beginning and ending at the origin, each occuring with probability $2^{-3}/4$. Thus, to the corresponding polynomial $p(x) = x^3 + x^2 + x + 1$, we associate the probability $8 \times 2^{-3}/4 = 1/4$. On the other hand, there are four

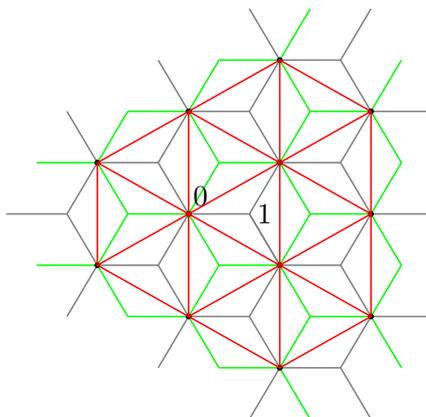
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standard walks of length 2 beginning and ending at the origin, each with probability 4^{-2} , thus the probability that one of these length-2 walks occurs is $4 \times 4^{-2} = 1/4$.

There are two 2-step walks of length 2 from the origin to the point $1 + i$, each with associated probability $1/4 \times 1/2 = 1/8$ and so we associate the probability value $1/4$ to the polynomial $p(x) = x + 1$. Equally, there is just one standard walk of length 1 from the origin to $1 + i$ with probability $1/4$ (recall, we identify the lattice of the standard walk with points $k + i\ell$ with $k + \ell$ even).

We now explore a duality between hexagonal walks and triangular walks given by algebraic complex QCS with exterior angle $2\pi/6 = \pi/3$. It turns out that a 2-step walk on one of two hexagonal lattices corresponds to a 1-step walk on the triangular lattice. Such lattices occur in the theory of random walks [20].

In the illustration below, the first step could be in one of three directions $1, y^2, y^4$ (grey), or one of three directions y, y^3, y^5 (green), where $y = e^{i\pi/3}$. In either case, the walk takes place entirely on either the grey hexagonal lattice or the green hexagonal lattice. After an even number of steps, the 2-step walk will arrive at one of the points $\frac{3}{2}k + i\frac{\sqrt{3}}{2}\ell$ for integers k, ℓ with $k + \ell$ even, that is, at a point on the triangular lattice (red).



In fact, we have:

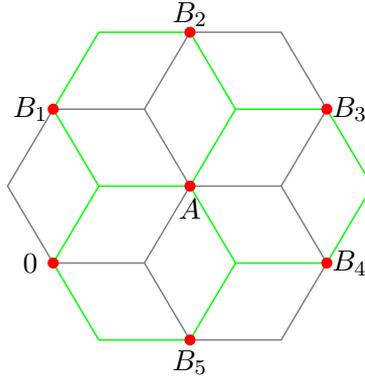
$$\begin{aligned}
 1 + y &= \frac{3}{2} + i\frac{\sqrt{3}}{2} \\
 y + y^2 &= \sqrt{3}i \\
 y^2 + y^3 &= -\frac{3}{2} + i\frac{\sqrt{3}}{2} \\
 y^3 + y^4 &= -\frac{3}{2} - i\frac{\sqrt{3}}{2} \\
 y^4 + y^5 &= -\sqrt{3}i \\
 y^5 + 1 &= \frac{3}{2} - i\frac{\sqrt{3}}{2}
 \end{aligned}$$

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So if we are at some point $x = \frac{3}{2}k + i\frac{\sqrt{3}}{2}l$ with k, l integers, $k+l$ even, (in particular, the point $(0, 0)$), then after two steps we will add to x one of the 6 values above and get $x' = \frac{3}{2}k' + i\frac{\sqrt{3}}{2}l'$, k', l' integers, $k' + l'$ even.

If the initial step is taken with probability $1/6$, then after two steps, the walk will arrive at one of $\frac{3}{2} + \frac{\sqrt{3}}{2}i, \sqrt{3}i, -\frac{3}{2} + \frac{\sqrt{3}}{2}i, -\frac{3}{2} - \frac{\sqrt{3}}{2}i, -\sqrt{3}i, \frac{3}{2} - \frac{\sqrt{3}}{2}i$ also with probability $1/6$, since there are two 2-step routes to arrive at each of the points on the red lattice each with probability $1/12$. This probability distribution agrees with the case of the standard walk on the triangular lattice.

However, unlike the case when $n = 4$, the triangular walk is no longer a Markovian process, that is, one which depends only on its present position and not on past positions. Furthermore, the relation between a walk on the triangular lattice and a 2-step walk with turning angle $\pi/3$ is more complicated. Let us examine this more closely.



If we arrive at A from the origin 0 via the grey route following successive increments y^0 and y , then there is a unique route to arrive at B_1, B_2, B_3, B_4 each with probability $1/4$. The nodes B_5 and 0 are inaccessible. On the other hand, if we arrive at A via the green route following successive increments y and y^0 , then there is a unique route to arrive at B_2, B_3, B_4, B_5 each with probability $1/4$. Now the nodes at B_1 and 0 are inaccessible. Thus, if we are engaged in a 2-step walk with turning angle $\pi/3$, the walk on the triangular lattice depends on the previous step and the route taken to arrive.

Given a 1-step walk on the triangular lattice which has arisen from a 2-step walk with turning angle $\pi/3$, depending on the walk, there may be either a unique corresponding 2-step walk or two such walks. For example, the walk $0AB_1$ is determined by a unique 2-step walk, whereas $0AB_2$ is determined by two: one on the grey lattice, one on the green lattice. In particular, if three successive steps on the triangular lattice lie on either a green, or grey

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hexagon, then the entire walk must have taken place on either the green hexagonal lattice, or the grey hexagonal lattice, respectively.

As for the case when $n = 4$, we can characterize the defining polynomials of 2-step walks with turning angle $\pi/3$.

Theorem 4.5.7. *Consider a 2-step walk with turning angle $\pi/3$ which begins at the origin and ends at the point $\frac{3}{2}k + i\frac{\sqrt{3}}{2}\ell$ with $k + \ell$ even. Let L be the length of the walk. Then the defining polynomial is given by*

$$p(x) = (x^3 + 1)(ax^2 + bx - a - b + \frac{L}{2}) - (x - 1)(x + 1) \left\{ \frac{k}{2}(x + 1) + \frac{\ell}{2}(x - 1) \right\} \quad (4.21)$$

where a and b are non-negative integers for which if $(k, \ell) \neq (0, 0)$, the following inequalities are necessary (but not sufficient) conditions:

$$\begin{aligned} L &\geq \max\{|k| + |\ell|, 2(a + b) + (k + \ell), 2(a + b) - (k - \ell)\} \\ a &\geq \max\{0, \frac{k - \ell}{2}\} \\ b &\geq \max\{0, -\frac{(k + \ell)}{2}\} \end{aligned}$$

and if $(k, \ell) = (0, 0)$, the inequalities $a, b > 0$ and $4 \max\{a, b\} < L < 4(a + b)$ are necessary and sufficient.

Proof. Let $p(x) = b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0$ be the defining polynomial of the walk. Then since each increment y^s is followed by either y^{s+1} or y^{s-1} , we must have $p(-1) = 0$; also $p(1) = L$ and $p(y) = \frac{3}{2}k + i\frac{\sqrt{3}}{2}\ell$ where $y = e^{i\pi/3}$. This yields the underdetermined system of equations

$$\begin{aligned} b_0 + b_1 + b_2 + b_3 + b_4 + b_5 &= L \\ b_0 - b_1 + b_2 - b_3 + b_4 - b_5 &= 0 \\ 2b_0 + b_1 - b_2 - 2b_3 - b_4 + b_5 &= 3k \\ b_1 + b_2 - b_4 - b_5 &= \ell \end{aligned}$$

Set $b_5 = a$ and $b_4 = b$ as arbitrary parameters. Then on solving the system, the defining polynomial is given by

$$\begin{aligned} p(x) = & ax^5 + bx^4 + \left(\frac{L}{2} - a - b - \frac{(k + \ell)}{2} \right) x^3 + \left(a - \frac{(k - \ell)}{2} \right) x^2 \\ & + \left(b + \frac{(k + \ell)}{2} \right) x + \frac{L}{2} - a - b + \frac{(k - \ell)}{2}, \end{aligned} \quad (4.22)$$

which yields (4.21).

Clearly the length L must be greater than or equal to the minimum length of a path joining the origin to $\frac{3}{2}k + i\frac{\sqrt{3}}{2}\ell$ which is given by $|k| + |\ell|$. In the case when $(k, \ell) \neq (0, 0)$, the other inequalities arise from the requirement that the coefficients of $p(x)$ must all be non-negative. If on the other hand $k = \ell = 0$ and the path is closed, then the condition on L is determined by the case $n = 6$ in §4.2. Recall, writing the polynomial $p(x) = (x^3 + 1)(ax^2 + bx + c)$, we require $a, b, c > 0$ and $\max\{a, b, c\} < a + b + c - \max\{a, b, c\}$, where $L = 2(a + b + c)$ in the notation of §4.2.

Suppose $\frac{L}{2} - a - b \geq \max\{a, b\}$, i.e. $L \geq 2(a + b) + 2\max\{a, b\}$. Then we require

$$\frac{L}{2} - a - b < a + b \quad \Leftrightarrow \quad L < 4(a + b).$$

On the other hand, if $\frac{L}{2} - a - b \leq \max\{a, b\}$, i.e. $L \leq 2(a + b) + 2\max\{a, b\}$, then we require

$$\max\{a, b\} < \frac{L}{2} - a - b + \min\{a, b\} \quad \Leftrightarrow \quad 4\max\{a, b\} < L.$$

However, for $a, b > 0$, we have

$$\left. \begin{array}{l} \text{either} \quad 2(a + b) + 2\max\{a, b\} \leq L < 4(a + b) \\ \text{or} \quad 4\max\{a, b\} < L \leq 2(a + b) + 2\max\{a, b\} \end{array} \right\} \Leftrightarrow 4\max\{a, b\} < L < 4(a + b)$$

as required. □

The inequalities for the case $(k, \ell) \neq (0, 0)$ in the above theorem are necessary conditions but not sufficient. For example, if we choose $k = 3$, $\ell = 5$, $L = 10$, $a = 0$, $b = 1$, then the inequalities are satisfied, however, the resulting polynomial given by $p(x) = x^4 + x^2 + 5x + 3$ is not legitimate.

Note that for the case $n = 6$, the defining polynomial may depend on the path. As the following example shows, this allows one to construct paths of the same length to the same point which make use of different (unordered) edges.

Example 4.5.8. In (4.22), set $k = 3$, $\ell = 5$ and $L = 12$. First choose $a = 1$ and $b = 0$ to give the defining polynomial

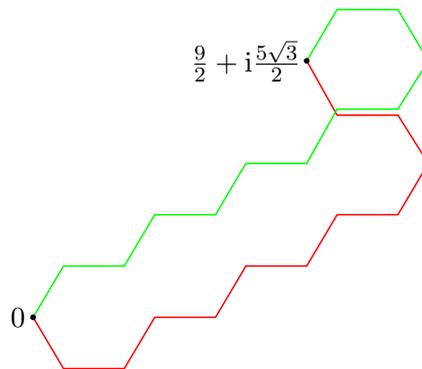
$$p(x) = x^5 + x^3 + 2x^2 + 4x + 4$$

with sequence of increments

$$(y^5, 1, y, 1, y, 1, y, 1, y, y^2, y^3, y^2)$$

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corresponding to the red path in the illustration below.



Second, choose $a = 0$ and $b = 1$ to give the defining polynomial

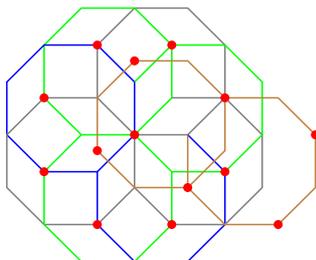
$$p(x) = x^4 + x^3 + x^2 + 5x + 4$$

with sequence of increments

$$(y, 1, y, 1, y, 1, y, 1, y, y^2, y^3, y^4)$$

corresponding to the green path in the illustration. Then the red path uses a different set of edges to the green path, for example it exploits the edge y^5 which is not used in the green path. Note that the two paths combine to yield a closed path of length 24. As affirmed by Proposition 4.2.12, this closed path requires each edge with its oppositely oriented counterpart.

We can proceed similarly with turning angle $2\pi/n$ for $n = 8, 10, \dots$, however in general, there are no longer convenient tilings of the plane which support the walks. We illustrate below part of the lattice for the case $n = 8$, where we see how the brown octagons begin to interfere with the tiling. Some points of the 2-step walk are illustrated as red nodes.



Chapter 5

Appendix

5.1 A non recursive algorithm for the path-function \mathcal{F}

We introduce another program, this time built using "R", to compute the values of the path-function \mathcal{F} defined in Chapter 3. What characterizes this program is that it is non recursive, but henceforth takes much more time for computing a desired value. This is done through three steps:

1. **Function "Zeroo".**

(a) **Explanation:**

This function eliminates the zeroes that can be found at the extremities of a tuple $\alpha \in \mathcal{N}$. If the zero isn't on the extremities, it sums its neighbors.

Example: for $\alpha = (0, 2, 1, 0, 3, 5, 0)$, we get the tuple $(2, 4, 5)$.

This function is used to reduce the number of components of a tuple α when some of them are zeroes, before using it in the principal algorithm.

(b) **Code:**

```
1 Zeroo=function(v)
2 {
3   while(v[1]==0) v=v[-1]
4   while(v[length(v)]==0) v=v[-length(v)]
5   wh=which(v %in% 0)
6   if(length(wh)>0)
7     {
8       for(k in 1:length(wh))
```

```
9      {
10     v=v[-wh[k]]
11     v[wh[k]]=v[wh[k]-1]+v[wh[k]]
12     v=v[-(wh[k]-1)]
13     }
14 }
15 return(v)
16 }
```

2. Function "Minus".

(a) Explanation:

This function generates new tuples from a given one, by subtracting the number 1 from each component of the initial tuple.

Example: $\alpha = (2, 4, 5)$ generates $(1, 4, 5)$ and $(2, 3, 5)$ and $(2, 4, 4)$.

This function is also used in the principal algorithm.

(b) Code:

```
1 Minus=function(vecteur)
2 {
3   vecteur=Zeroo(vecteur)
4   n=length(vecteur)
5   Matvecteur=matrix(0,n,n)
6   if(length(which(vecteur>0))!=0)
7   {
8     for(k in 1:n)
9     {
10      test=vecteur
11      test[k]=test[k]-1
12      Matvecteur[k,]=test
13    }
14    vecteur=Matvecteur
15  }
16  return(vecteur)
17 }
```

3. Function "PathsFunction".

(a) Description:

This function computes the desired values of the paths-function \mathcal{F} . It follows these steps:

Given a tuple $\alpha \in \mathbb{N}^n$.

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- It uses the function `Zeroo` to reduce the size of α if it contains zeroes so that it becomes ready for decomposition.
- It uses the function `Minus` to generate k tuples from α . These tuples are placed in a matrix of dimensions (k, n) , where n is the size (number of components) of α .
- For each of the generated tuples, it applies again the two functions `Zeroo` and `Minus`. The new tuples are placed in a new matrix of dimensions (k', n) where $k' = k_1 + k_2 + \dots + k_k$ and k_i is the number of tuples generated by the tuple i of the previous matrix.
- As long as the generated tuples contain numbers greater than 1, they undergo `Zeroo` and `Minus`.
- The algorithm finally converges to a matrix of big dimensions. This matrix only contains zeroes and ones. The number of lines of this matrix is equal to the desired value.

(b) Code:

```
1 PathsFunction=function(vecteur)
2 {
3   library(gtools)
4   vecteur=Zeroo(vecteur)
5   n=length(vecteur)
6   X=Minus(vecteur)
7   za=1
8   while(sum(X>1) | za==1 & length(X)>1)
9     {
10      za=0
11      to=apply(X,1,Zeroo)
12      v=lapply(to,Minus)
13      n <- max(sapply(v, nrow))
14
15      v=lapply(v, function (x)
16        rbind(x, matrix(, n-nrow(x), ncol(x))))
17
18      v=lapply(v, function (x)
19        cbind(x, matrix(, nrow(x), n-ncol(x))))
20
21      output <- do.call(rbind,t(lapply(v,matrix,ncol=n)))
22      output=output[apply(output, 1, function(y)
23        !all(is.na(y))),]
24      d=output
25      d[is.na(d)] <- 0
26      X=d
27    }
```

```

28 | X=data.frame(X)
29 | return(nrow(X))
30 | }

```

Now, we create the program "**PFallvalues**", using "PathsFunction", that computes all the possible values of $\mathcal{F}(a_1, \dots, a_s)$, $1 \leq s \leq p$, for $p = \sum_{i=1}^s a_i$:

(a) **Description:**

This function is a generalization of the previous one. It gives all the values corresponding to all the possible tuples of a given size and a given summation of the components (> 2), (and arranges them in increasing order).

Example: One may consider the number 3. The algorithm PFallvalues finds all the possible tuples of size 3 and whose components' summation is 3, which are: $V_1 = (3, 0, 0)$, $V_2 = (1, 2, 0)$, $V_3 = (2, 1, 0)$ and $V_4 = (1, 1, 1)$, and it applies the function PathsFunction on each one of these tuples to get:

	N_1	N_2	N_3	Result
V_1	3	0	0	1
V_2	1	2	0	3
V_3	2	1	0	3
V_4	1	1	1	5

(b) **Code:**

```

1 | PFallvalues=function(N)
2 | {
3 |   library(gtools)
4 |   library(plyr)
5 |   matriceALL=list(0,1000)
6 |   k=1
7 |   for(i in 2:N)
8 |   {
9 |     print(i)
10 |    matto=permutations(n=N,r=i,repeats.allowed=T)
11 |    matto=cbind(matto,apply(matto,1,sum))
12 |    m=ncol(matto)
13 |    matto=as.data.frame(matto)
14 |    matriceALL[[k]]=matto[which(matto[,m]==N),1:(m-1)]
15 |    k=k+1
16 |   }
17 |   #####

```

```

18 v=matriceALL
19 n <- max(sapply(v, nrow))
20 m <- max(sapply(v, ncol))
21
22 v=lapply(v, function (x)
23   rbind(x, matrix(, n-nrow(x), ncol(x))))
24
25 v=lapply(v, function (x)
26   cbind(x, matrix(, nrow(x), m-ncol(x))))
27
28 for(j in 1:length(v)) v[[j]]=as.matrix(v[[j]])
29
30 output <- do.call(rbind,t(lapply(v,matrix,ncol=m)))
31
32 output=output[apply(output, 1, function(y) !all(is.na(y))),]
33 d=output
34
35 d[is.na(d)] <- 0
36
37 matriceFIN=rbind( c(N,rep(0,(m-1))),d)
38 matriceFIN=matriceFIN[, colSums(matriceFIN != 0) > 0]
39 #####
40 #Application of PathsFunction
41 Results=NULL
42 for(i in 1:nrow(matriceFIN))
43 {
44   vecteur=matriceFIN[i,]
45   Results[i]=PathsFunction(vecteur)
46 }
47 #####
48 matriceResults=cbind(matriceFIN,Results)
49 colnames(matriceResults)[1:m]=paste("N",1:m,sep=" ")
50 #####
51 matriceResults=matriceResults[order(matriceResults[,m+1]),]
52
53 return(matriceResults)
54 }

```

5.2 On the defining polynomial of a complex algebraic QCS

As we saw in §4.2, the defining polynomial of a complex QCS with increment $y = e^{2m\pi i/n}$ (m, n relatively prime, $m < n$) exhibits certain symmetry properties: for $n \leq 11$, the corresponding walk in the plane must use all increments $\{y^0, y^1, \dots, y^{n-1}\}$, and when n is even, then for each occurrence of the increment y^k there is an occurrence of the increment $y^{\frac{n}{2}+k} = -y^k$.

5.2. ON THE DEFINING POLYNOMIAL OF A COMPLEX ALGEBRAIC QCS

We will take a different approach to prove this symmetry property, regardless the fact that the cyclotomic polynomial $\Phi_n(x)$ must be a factor of $p(x)$, and this is done involving linear algebra.

In other terms, we will prove that when $n \leq 11$, n even, the defining polynomial of a complex algebraic QCS with increment $y = e^{2\pi i/n}$ has the form:

$$\begin{aligned} p(x) &= \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{n-1} x^{n-1} \\ &= \alpha_0 + \alpha_1 x + \cdots + \alpha_{\frac{n}{2}-1} x^{\frac{n}{2}-1} \\ &\quad + \alpha_0 x^{\frac{n}{2}} + \alpha_1 x^{\frac{n}{2}+1} + \cdots + \alpha_{\frac{n}{2}-1} x^{n-1}, \end{aligned}$$

with all α_i strictly positive integers.

Let $p(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_{n-1} x^{n-1}$ be the defining polynomial associated to a complex algebraic QCS of order N with increment $y = e^{2\pi i/n}$. That is, y is a root of $p(x)$ since the QCS is cyclic, and the corresponding sequence of increments has the form $\{y^{s_0}, y^{s_1}, \dots, y^{s_{N-1}}\}$ where $s_{j+1} = s_j \pm 1 \pmod{n}$, $j \in \mathbb{Z}/N\mathbb{Z}$.

First, it is clear that when n is even, we have $y^k = -y^{\frac{n}{2}+k}$, $\forall k$, so we write:

$$\begin{aligned} p(y) &= \alpha_0 + \alpha_1 y + \alpha_2 y^2 + \cdots + \alpha_{n-1} y^{n-1} \\ &= \alpha_0 + \alpha_1 e^{\frac{2i\pi}{n}} + \alpha_2 e^{\frac{2i\pi(2)}{n}} + \cdots + \alpha_{n-1} e^{\frac{2i\pi(n-1)}{n}} \\ &= (\alpha_0 - \alpha_{\frac{n}{2}}) + (\alpha_1 - \alpha_{\frac{n}{2}+1}) e^{\frac{2i\pi}{n}} + (\alpha_2 - \alpha_{\frac{n}{2}+2}) e^{\frac{2i\pi(2)}{n}} + \cdots + (\alpha_{\frac{n}{2}-1} - \alpha_{n-1}) e^{\frac{2i\pi(\frac{n}{2}-1)}{n}} \\ &= \lambda_0 + \lambda_1 e^{\frac{2i\pi}{n}} + \lambda_2 e^{\frac{2i\pi(2)}{n}} + \cdots + \lambda_{\frac{n}{2}-1} e^{\frac{2i\pi(\frac{n}{2}-1)}{n}} \end{aligned}$$

Moreover, since y is a root of $p(x)$, then $p(y) = 0$, which imply that

$$\sum_{k=0}^{\frac{n}{2}-1} \lambda_k e^{\frac{2i\pi k}{n}} = 0,$$

As a result, we have:

$$\sum_{k=0}^{\frac{n}{2}-1} \lambda_k \sin\left(\frac{2\pi k}{n}\right) = 0, \tag{5.1}$$

$$\sum_{k=0}^{\frac{n}{2}-1} \lambda_k \cos\left(\frac{2\pi k}{n}\right) = 0. \tag{5.2}$$

Also, in the sequence of increments, successive terms differ by a power of one, so each occurrence of y^s must be followed by either y^{s-1} or y^{s+1} , cyclically. If we set $y = -1$, the

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sequence of increments has the form $(1, -1, \dots, 1, -1)$ or $(-1, 1, \dots, -1, 1)$ and it follows that $p(-1) = 0$, so that $x + 1$ is a factor of $p(x)$ (note that since n is even, the transition from y^{n-1} to $y^n = y^0$ is consistent with alternation from -1 to 1 when $y = -1$).

For $n = 2$, the result is trivial.

For $n = 4$: We have

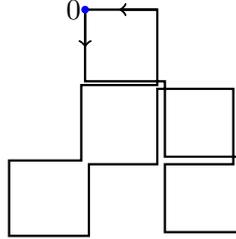
$$p(y) = \lambda_0 + \lambda_1 e^{i\frac{\pi}{2}} = 0,$$

so

$$\lambda_0 = \lambda_1 = 0$$

since the vectors $(1, 0)$ and $(0, 1)$ are linearly independent in \mathbb{R}^2 , and we get the desired form of the coefficients of $p(x)$.

Example: Consider the following walk with turning angle $\beta = \frac{\pi}{2}$,



The sequence of directions is

$$S = (-i, 1, -i, 1, -i, -1, i, 1, i, -1, -i, -1, -i, -1, i, 1, i, 1, i, -1),$$

and we have

$$p(x) = 5 + 5x + 5x^2 + 5x^3.$$

For $n = 6$: We have

$$p(y) = \lambda_0 + \lambda_1 e^{i\frac{\pi}{3}} + \lambda_2 e^{i\frac{2\pi}{3}} = 0,$$

so equation (5.1) implies that:

$$\frac{\sqrt{3}}{2} \lambda_1 + \frac{\sqrt{3}}{2} \lambda_2 = 0,$$

thus

$$\lambda_1 = -\lambda_2,$$

and equation (5.2) implies that:

$$\lambda_1 + \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 = 0,$$

and since $\lambda_1 = -\lambda_2$, we get

$$\lambda_0 = -\lambda_1.$$

Moreover, since -1 is a root of $p(x)$, so

$$p(-1) = \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 + \alpha_4 - \alpha_5 = \lambda_0 - \lambda_1 + \lambda_2 = 0,$$

and since $\lambda_1 = -\lambda_2 = -\lambda_0$, we get that

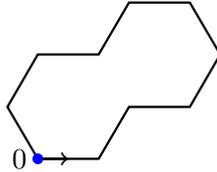
$$\lambda_0 = 0,$$

as a result,

$$\lambda_0 = \lambda_1 = \lambda_2 = 0,$$

which is required to be proved.

Example: Consider the following walk with turning angle $\beta = \frac{\pi}{3}$,



The sequence of directions is:

$$S = (1, e^{i\frac{\pi}{3}}, 1, e^{i\frac{\pi}{3}}, e^{i\frac{2\pi}{3}}, -1, -e^{i\frac{\pi}{3}}, -1, -e^{i\frac{\pi}{3}}, -e^{i\frac{2\pi}{3}}),$$

and the associated polynomial is:

$$p(x) = 2 + 2x + x^2 + 2x^3 + 2x^4 + x^5.$$

For $n = 8$: We have

$$p(y) = \lambda_0 + \lambda_1 e^{i\frac{\pi}{4}} + \lambda_2 e^{i\frac{\pi}{2}} + \lambda_3 e^{i\frac{3\pi}{4}} = 0.$$

equation (5.2) implies that

$$\lambda_0 + \frac{\sqrt{2}}{2}(\lambda_1 - \lambda_3) = 0,$$

and since $\frac{\sqrt{2}}{2}$ is irrational, then

$$\lambda_0 = 0,$$

and,

$$\lambda_1 - \lambda_3 = 0.$$

equation (5.1) implies that

$$\lambda_2 + \frac{\sqrt{2}}{2}(\lambda_1 + \lambda_3) = 0,$$

and also since $\frac{\sqrt{2}}{2}$ is irrational, then

$$\lambda_2 = 0,$$

and,

$$\lambda_1 + \lambda_3 = 0.$$

We finally get:

$$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = 0,$$

and we get our result.

For $n = 10$: We have

$$p(y) = \lambda_0 + \lambda_1 e^{i\frac{\pi}{5}} + \lambda_2 e^{i\frac{2\pi}{5}} + \lambda_3 e^{i\frac{3\pi}{5}} + \lambda_4 e^{i\frac{4\pi}{5}} = 0.$$

Note that

$$\cos \frac{\pi}{5} = \frac{\sqrt{5} + 1}{4},$$

so

$$\sin \frac{\pi}{5} = \sqrt{1 - \frac{6 + 2\sqrt{5}}{16}} = \frac{\sqrt{10 - 2\sqrt{5}}}{4}.$$

Moreover, $\sin \frac{2\pi}{5} = 2 \cos \frac{\pi}{5} \sin \frac{\pi}{5}$, and recall that $\sin \frac{\pi}{5} = \sin \frac{4\pi}{5}$, and $\sin \frac{2\pi}{5} = \sin \frac{3\pi}{5}$.

As a result, equation (5.2) implies that:

$$(\lambda_1 + \lambda_4) \sin \frac{\pi}{5} + (\lambda_2 + \lambda_3) 2 \sin \frac{\pi}{5} \cos \frac{\pi}{5} = 0,$$

which is equivalent to

$$(\lambda_1 + \lambda_4) + (\lambda_2 + \lambda_3) \frac{\sqrt{5} + 1}{2} = 0,$$

and since $\frac{\sqrt{5}+1}{2}$ is irrational, we get that

$$\lambda_1 + \lambda_4 = \lambda_2 + \lambda_3 = 0,$$

so we have

$$\lambda_1 = -\lambda_4,$$

and

$$\lambda_2 = -\lambda_3.$$

Now, recall that $\cos \frac{\pi}{5} = -\cos \frac{4\pi}{5}$, $\cos \frac{2\pi}{5} = -\cos \frac{3\pi}{5}$.

As a result, equation (5.1) implies:

$$\lambda_0 + (\lambda_1 - \lambda_4) \cos \frac{\pi}{5} + (\lambda_2 - \lambda_3) \cos \frac{2\pi}{5} = 0.$$

Since $\lambda_1 = -\lambda_4$ and $\lambda_2 = -\lambda_3$, then we get:

$$\lambda_0 + (2\lambda_1) \cos \frac{\pi}{5} + (2\lambda_2) \cos \frac{2\pi}{5} = 0.$$

Now, $\cos \frac{2\pi}{5} = 2 \cos^2 \frac{\pi}{5} - 1 = \frac{\sqrt{5}-1}{4}$, as a result we have now:

$$\lambda_0 + \frac{\sqrt{5}+1}{2} \lambda_1 + \frac{\sqrt{5}-1}{2} \lambda_2 = 0,$$

thus, since $\sqrt{5}$ is irrational, we get:

$$\left(\frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2\right) = 0$$

so

$$\lambda_1 = \lambda_2,$$

and

$$\left(\lambda_0 + \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2\right) = 0.$$

As $\lambda_1 = \lambda_2$, it implies that

$$\lambda_0 = -\lambda_1 = -\lambda_2.$$

On the other hand, since -1 is a root of $p(x)$, we have:

$$\begin{aligned} p(-1) &= 0 \\ \Leftrightarrow \alpha_0 - \alpha_1 + \alpha_2 - \alpha_3 + \alpha_4 - \alpha_5 + \alpha_6 - \alpha_7 + \alpha_8 - \alpha_9 &= 0 \\ \Leftrightarrow \lambda_0 - \lambda_1 + \lambda_2 - \lambda_3 + \lambda_4 &= 0, \end{aligned}$$

5.2. ON THE DEFINING POLYNOMIAL OF A COMPLEX ALGEBRAIC QCS

and since $\lambda_3 = -\lambda_2 = \lambda_0 = -\lambda_1 = \lambda_4$, we get:

$$\lambda_0 = 0,$$

as a result we have:

$$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 0,$$

and the result follows.

Obviously, after the symmetry was settled, it is clear that all α_i must be strictly positive, for if some $\alpha_k = 0$, then $\alpha_{k+\frac{n}{2}} = \alpha_k = 0$, and this will clearly disconnect the sequence of increments, which is impossible.

For $n \geq 12$, we can see that we don't have enough conditions to settle the symmetry of the coefficients, (which as we saw in §4.2, is no more true for $n \geq 12$):

For $n = 12$: We have

$$p(y) = \lambda_0 + \lambda_1 e^{i\frac{\pi}{6}} + \lambda_2 e^{i\frac{2\pi}{6}} + \lambda_3 e^{i\frac{3\pi}{6}} + \lambda_4 e^{i\frac{4\pi}{6}} + \lambda_5 e^{i\frac{5\pi}{6}} = 0.$$

equation (5.1) implies that:

$$\frac{1}{2}\lambda_1 + \frac{\sqrt{3}}{2}\lambda_2 + \frac{\sqrt{3}}{2}\lambda_4 + \frac{1}{2}\lambda_5 = 0,$$

which implies, as $\frac{\sqrt{3}}{2}$ is irrational, that

$$\lambda_3 = -\frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_5, \quad (a)$$

and

$$\lambda_4 = -\lambda_2. \quad (b)$$

equation (5.2) implies that:

$$\lambda_0 + \frac{\sqrt{3}}{2}\lambda_1 + \frac{1}{2}\lambda_2 + 2 - \frac{1}{2}\lambda_4 - \frac{\sqrt{3}}{2}\lambda_5 = 0,$$

which implies, also since $\frac{\sqrt{3}}{2}$ is irrational, that

$$\lambda_0 = \frac{1}{2}\lambda_4 - \frac{1}{2}\lambda_2, \quad (c)$$

and

$$\lambda_1 = \lambda_5. \quad (d)$$

Now combining (a) with (c) and (d) with (b), we get

$$\lambda_3 = -\lambda_1,$$

and

$$\lambda_0 = -\lambda_2.$$

We finally get:

$$\lambda_4 = \lambda_0 = -\lambda_2,$$

and

$$\lambda_5 = \lambda_1 = -\lambda_3,$$

which is a sub-determined system.

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Titre : Cycles Combinatoires et Géométriques

Mots clés : tournois, chemins et cycles Hamiltoniens, équation à différence quadratique, séquences cycliques quadratiques, polynômes cyclotomiques, marches dans le plan.

Résumé : Le travail de cette thèse se situe dans les domaines de la théorie combinatoire des graphes, la combinatoire algébrique et la géométrie discrète. D'un part, il concerne l'énumération des chemins et cycles Hamiltoniens de type donné dans un tournoi ; de l'autre part, il étudie des suites numériques vérifiant une équation à différence quadratique.

Parmi les résultats obtenus dans la première partie, on trouve : une égalité entre le nombre des chemins (resp. cycles) Hamiltoniens d'un type donné dans un tournoi et dans son complément; une expression du nombre de chemins Hamiltoniens d'un type donné pour un tournoi transitif en termes d'une fonction récursive F appelée « path-fonction »; la construction d'un algorithme pour le calcul de F . L'objet fondamental dans la deuxième partie est un graphe cyclique muni d'une solution d'une équation à différence quadratique.

Un paramètre de cette équation distingue les solutions réelles et les solutions complexes. Une correspondance entre les solutions réelles et une classe de polynômes à coefficients entiers positifs est établie. Pour compléter la correspondance, les digraphes Eulériens à un pas interviennent. Une solution complexe détermine une marche fermée dans le plan pour laquelle à chaque pas on tourne à gauche ou à droite par un angle constant (l'angle tournant). Cette fois-ci les polynômes cyclotomiques jouent un rôle important. La caractérisation des polynômes qui déterminent de telles suites est un problème qu'on surmonte afin d'élucider des propriétés géométriques de tels cycles polygonaux. Notamment, lorsque la marche exploite les côtés d'un polygone régulier avec angle extérieur $2\pi/n$, on trouve des phénomènes non anticipés lorsque $n \geq 12$.

Title : Combinatorial and Geometric Cycles

Keywords : tournaments, Hamiltonian paths and cycles, quadratic difference equation, quadratic cyclic sequences, cyclotomic polynomials, walks in the plane.

Abstract : The work in this thesis concerns the combinatorial theory of graphs, algebraic combinatorics and discrete geometry. On one side, it is about enumerating Hamiltonian paths and cycles of a given type in a tournament; On the other side, it studies numerical sequences verifying a quadratic difference equation.

Concerning the results of the first part, we find: an equality between the number of Hamiltonians paths (resp. cycles) of a given type, in a tournament and its complement; an expression of the number of Hamiltonian oriented paths of a given type in a transitive tournament in terms of a recursive function F called the « path-function »; and the construction of an algorithm to compute F . In the second part of the work, we study cyclic graphs altogether with a solution to a quadratic difference equation.

A parameter of this equation distinguishes real and complex sequences. A correspondence between real solutions and a class of polynomials with positive integer coefficients is established. To complete the correspondence, 1-step Eulerian digraphs interfere. A complex solution determines a closed planar walk in the plane, for which at each step we turn either left or right by a constant angle (the turning angle). This time, cyclotomic polynomials play a major role. Characterizing polynomials that determine such a solution is a problem that we study to the end of finding geometric properties of such polygonal cycles. When the walk exploits the sides of a regular polygon with exterior angle $2\pi/n$, we find unexpected phenomena when $n \geq 12$.