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Quasi-geostrophic vortices and their desingularization

Ludovic Godard-Cadillac

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Ludovic Godard-Cadillac. Quasi-geostrophic vortices and their desingularization. Analysis of PDEs [math.AP]. Sorbonne Université, 2020. English. NNT : 2020SORUS437 . tel-03168120v2

HAL Id: tel-03168120

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SORBONNE UNIVERSITÉ

LABORATOIRE JACQUES-LOUIS LIONS

MÉMOIRE DE THÈSE

**Les vortex quasi-géostrophiques
et leur désingularisation.**

Ludovic GODARD-CADILLAC

sous la direction de
Philippe GRAVEJAT et Didier SMETS



Lundi 28 septembre 2020

Résumé

Ce mémoire de thèse se situe dans le domaine des mathématiques de la mécanique des fluides. Il y est proposé une étude des équations quasi-géostrophiques surfaciques non visqueuses (SQG) dans une version générale. Ces équations ont été introduites par les physiciens dans le cadre de la modélisation de l'atmosphère terrestre pour effectuer des prévisions météorologiques et climatiques. Elles modélisent l'évolution d'un gaz stratifié dans un référentiel en rotation au voisinage de l'équilibre géostrophique. Cet équilibre correspond à la situation où le gradient de pression et l'accélération de Coriolis se compensent exactement. Les équations quasi-géostrophiques sont la perturbation de cet équilibre par les phénomènes thermodynamiques atmosphériques. Outre les aspects de modélisation physique, il s'agit d'équations ayant des liens structurels importants avec les équations d'Euler bi-dimensionnelles et tri-dimensionnelles écrites en vorticit .

1. La première partie de la thèse est consacrée aux problèmes de points-vortex pour les équations (SQG). Il s'agit de solutions pour (SQG) s'écrivant sous la forme d'une somme pondérée de masses de Dirac évoluant au cours du temps. Cette partie contient une extension de la théorie des points-vortex connue pour Euler bi-dimensionnel au cas quasi-géostrophique suivi d'une étude des collisions de vortex dans le cas d'Euler et dans le cas de (SQG).
2. La seconde partie de la thèse est une construction variationnelle de solutions spéciales à (SQG). Ce sont des solutions qui prennent la forme de N patches identiques (à rotation près) et dont les centres respectifs sont uniformément répartis sur un cercle. On obtient alors une structure ayant une symétrie d'ordre N . En outre, cette famille de solution est une désingularisation, une approximation, du système de points-vortex associé. Les désingularisations de systèmes de points-vortex sont un élément essentiel pour obtenir une dérivation rigoureuse du système de points-vortex à partir des équations aux dérivées partielles. Il s'agit d'un travail en collaboration avec mes directeurs de Thèse D. Smets et P. Gravejat.
3. La troisième partie est la construction de solutions C^∞ qui prennent la forme de deux zones de vorticit  identiques mais de signes opposés et qui sont immobiles dans un référentiel en translation. Il s'agit d'une extension à un travail antérieur de D. Smets et P. Gravejat.
4. La dernière partie porte sur la construction du réarrangement par tassement. Il s'agit d'un outil d'analyse fonctionnelle dont l'objectif est de décrire les zones de vorticit  des solutions d'équations de la mécanique des fluides du plan et plus g n ralement les ph nom nes de concentration non lin aires sous contraintes.   l'heure actuelle la construction est limit e   la dimension 1.

Abstract

This thesis is in the field of mathematics for fluid mechanics. There is proposed a study of quasi-geostrophic surface inviscid equations (SQG) in a general version. These equations were introduced by physicists as part of the modeling of the Earth's atmosphere for weather and climate forecasts. It models the evolution of a stratified gas in a rotating frame in the vicinity of the geostrophic equilibrium for which the pressure gradient and the Coriolis acceleration exactly offset each other. The perturbations of the geostrophic equilibrium leading to quasi-geostrophic motion come from thermodynamical atmospheric phenomena. In addition to the physical modeling aspects, there are important structural linkage with the two-dimensional and three-dimensional Euler equations written in vorticity and many properties and difficulties in common.

1. The first part of the thesis is devoted to vortices problems for equations (SQG). Vortex model refers to solutions for (SQG) written in the form of a weighted sum of Dirac masses evolving over time. This part contains an extension of the theory of points-vortices known for Euler two-dimensional to the quasi-geostrophic case followed by a study of vortices collisions in the case of Euler and in the case of (SQG).
2. The second part of the thesis is a variational construction of special solutions to (SQG). These are solutions that take the form of N identical patches up to rotation and uniformly distributed on circle. This leads to a solution with an N -fold symmetry. In addition, this family of solutions is a desingularization, an approximation, of the associated vortex system. Desingularizations of vortex systems are an essential element to obtain a rigorous derivation of the vortex system. The section is joint work with my PhD advisors D. Smets and P. Gravejat.
3. The third part is the construction of solutions \mathcal{C}^∞ which take the form of two zones of identical vorticities but of opposite signs and which are translating at constant speed. It is an extension of a previous work by D. Smets and P. Gravejat.
4. The last part deals with the construction of the rearrangement by tamping. It is a tool of functional analysis which objective is to describe the vorticity zones of the solutions of fluid mechanics equations on the plane and more generally the non-linear concentration phenomena under constraints. The construction is so far reduced to dimension 1.

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Chapitre 1

Introduction de la thèse

La mécanique des fluides est le domaine des mathématiques et de la physique consacré à l'étude du comportement mécanique des fluides, à savoir les liquides, les gaz et les plasmas. Les concepts développés servent également dans une moindre mesure à étudier les solides ductiles ou bien parfois des dynamiques de foule. La mécanique des fluides est contenue dans la mécanique des milieux continus qui modélise le comportement de la matière à l'aide d'une approximation de l'objet physique par un continuum. C'est-à-dire que le système est décrit mathématiquement par des fonctions continues régies par des équations aux dérivées partielles. La mécanique des fluides géophysiques est la branche de la mécanique des fluides qui étudie le comportement de l'atmosphère terrestre¹ afin d'effectuer des prévisions météorologiques et climatiques. En plus des équations issues directement de la mécanique des fluides, les modèles atmosphériques incluent en général des équations issues de la thermodynamique (dilatation thermique, condensation et évaporation, etc.) Dans les sections 1.1. et 1.2. où sont présentées respectivement la mécanique des fluides générale et la mécanique des fluides géophysiques, les notations de physique sont adoptées. Les illustrations utilisées dans ces deux sections sont des images personnelles ou issues de Wikipédia et sous licence libre.

1 Éléments de mécanique des fluides

Nous proposons dans cette section des considérations générales quant à la mécanique des fluides et la présentation des concepts centraux qui sont à la base de cette discipline. Bien entendu il ne s'agit pas d'une introduction exhaustive à la mécanique des fluides et l'accent sera mis surtout sur les équations qui font l'objet d'une étude dans le cadre de cette thèse : les équations d'Euler incompressibles et homogènes. Pour une introduction plus large aux équations d'Euler lire par exemple [6] ou bien [56]. Un lecteur familier des concepts centraux de la mécanique des fluides et des principaux théorèmes peut sauter cette section².

1.1 Équations de la mécanique des fluides

1.1.1 Du discret au continu

Dans le cadre de la modélisation de la mécanique des fluides (et plus généralement de la mécanique des milieux continus), on sépare généralement l'analyse en trois niveaux d'échelles tel qu'illustré en figure 1.1.

L'échelle moléculaire ($10^{-11} \sim 10^{-10}$ mètres environ) : Il s'agit de l'échelle à laquelle se trouvent les molécules constituant le fluide. On y trouve notamment l'information sur la distribution statistique des vitesses ainsi que les interactions entre les molécules entrant en collision (choc élastique, agitation thermique, interaction électro-magnétique ou quantique, réaction chimique, nucléaire, etc...)

L'échelle mésoscopique ($10^{-8} \sim 10^{-6}$ mètres par exemple) : Une fois l'échelle microscopique modélisée, le passage à l'échelle mésoscopique consiste à considérer des *particules de fluide* contenant un très grand nombre de molécules dont le comportement est décrit de manière statistique (par une densité de probabilité). On y définit également la pression p comme étant l'action de contact des molécules d'une

1. Les modèles étudiant la dynamique des océans sur de grandes échelles sont également modélisés par des fluides géophysiques.

2. Quoique se rafraichir la mémoire sur les bases est très loin d'être une mauvaise chose ; dixit l'auteur à qui la rédaction de cette section fit beaucoup de bien.

particule de fluide sur les molécules d'une particule voisine. Il s'agit de l'échelle à laquelle s'établissent les lois de conservation qui sont à la base des équations de la mécanique des fluides.

L'échelle macroscopique (10^{-5} mètres pour la microfluidique et jusqu'à l'échelle des nébuleuses interstellaires, soit $10^{15} - 10^{18}$ mètres) : La particule de fluide est réduite à un point qui évolue au cours du temps selon les lois d'interactions entre particules de fluides qui découlent de l'analyse mésoscopique. La description du fluide est alors un continuum, c'est-à-dire que le fluide est alors décrit par des champs scalaires, vectoriels ou tensoriels. Par exemple, $\vec{v}(t, \vec{x})$ est la vitesse au temps t de la particule de fluide qui est située au point x de l'espace à ce temps t .

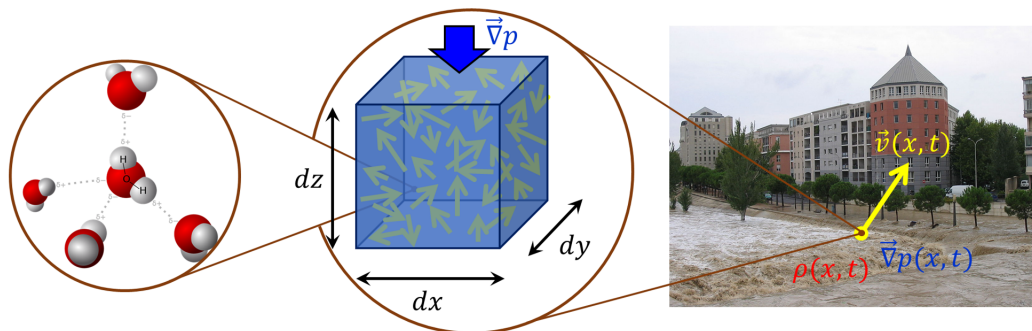


FIGURE 1.1 – Illustration de la modélisation du fluide en 3 échelles. De gauche à droite : échelle moléculaire (ici : des molécules d'eau en interaction mécanique), échelle mésoscopique (représentation de la distributions statistique des vitesses à l'intérieur de la particule de fluide), échelle macroscopique (ici : le Lez en crue à Montpellier le 6 septembre 2005) avec représentation de la pression \vec{p} , la vitesse \vec{v} et la masse volumique ρ en un point \vec{x} à un temps t .

La mécanique des fluides consiste donc pour les physiciens à extraire, à partir de cette représentation en 3 échelles, des équations aux dérivées partielles qui vont décrire le comportement macroscopique du fluide étudié. Le premier passage à la limite (microscopique vers mésoscopique) consiste à utiliser des méthodes permettant d'extraire une connaissance statistique simplifiée à partir d'un grand nombre d'occurrences (ici les molécules). Le second passage à la limite (mésoscopique vers macroscopique) consiste à remplacer les densités de probabilités dans une particule de fluide par la valeur moyenne et ainsi pouvoir réduire la particule de fluide à un point. Chacun de ces deux passages à la limite fait l'objet de nombreux travaux mathématiques.

1.1.2 Représentation eulérienne et lagrangienne

La représentation du fluide la plus naturelle est la représentation dite *lagrangienne* et qui consiste à suivre l'évolution d'une particule de fluide donnée au cours du temps $\vec{X}(t) \in \mathbb{R}^3$. Comme il s'agit d'une particule de fluide et que $\vec{v}(t, \vec{x})$ est la vitesse au temps t de la particule de fluide qui est située au point \vec{x} de l'espace à ce temps t , on en déduit que l'évolution de cette particule de fluide est donnée par

$$\frac{D}{Dt} \vec{X}(t) = \vec{v}(t, \vec{X}(t)). \quad (1.1)$$

L'opérateur D/Dt est souvent appelé la *dérivée lagrangienne* du fluide. On peut l'interpréter en disant que c'est la dérivée *ressentie* par une particule en mouvement. L'équation (1.1) peut paraître un peu tautologique à quelques égards puisque qu'elle dit simplement que la dérivée en temps de la position est la vitesse, c'est-à-dire la définition même de la notion de vitesse. Néanmoins, cette équation permet de mieux appréhender ce que signifie ce champ de vitesse \vec{v} et ce qu'est la *représentation eulérienne*. Cette représentation consiste non plus à suivre les particules de fluide mais à observer le fluide depuis une position \vec{x} fixée dans le temps. La variation d'une quantité est donnée par la *dérivée eulérienne* notée $\partial/\partial t$ et s'interprète en disant que c'est la dérivée *ressentie* par un observateur fixe.

Le lien entre les descriptions eulérienne et lagrangienne s'établit en suivant le schéma de preuve suivant. Pendant un intervalle de temps infinitésimal δt , une particule située en \vec{x} à l'instant t se sera déplacée en $\vec{x} + \vec{v} \delta t$ puisque elle va à la vitesse \vec{v} . La variation d'une grandeur scalaire quelconque ϕ s'écrit donc, grâce à un développement de Taylor :

$$\delta \phi = \phi(t + \delta t, \vec{x} + \vec{v} \delta t) - \phi(t, \vec{x}) = \left(\frac{\partial \phi}{\partial t} + \vec{v} \cdot \vec{\nabla} \phi \right) \delta t + \mathcal{O}(\delta t^2), \quad (1.2)$$

où l'opérateur de dérivation $\vec{\nabla}$ est le vecteur des dérivations spatiales eulériennes :

$$\vec{\nabla} := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right). \quad (1.3)$$

Par conséquent,

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \quad (1.4)$$

1.1.3 L'équation de conservation de la masse

L'équation de conservation de la masse est un cas particulier de loi de conservation en physique. Cette équation s'établit au niveau mésoscopique et modélise le fait que si le système physique est fermé alors la masse totale du fluide est constante. De manière générale, une loi de conservation d'une quantité scalaire quelconque $\phi(t, \vec{x})$ consiste à étudier la variation de ϕ dans une région d'échelle mésoscopique fixe donnée Ω . La variation de ϕ est alors égale à la quantité de ϕ entrante et sortante à travers la frontière de cette région plus le terme source de création ou de disparition de ϕ noté $F(t, \vec{x})$. C'est-à-dire :

$$\int_{\Omega} \frac{\partial \phi}{\partial t} dx = \int_{\partial \Omega} \phi \vec{v} \cdot \vec{dn}(\vec{x}) + \int_{\Omega} F dx, \quad (1.5)$$

où \vec{dn} est la normale extérieure au volume Ω au point \vec{x} . En utilisant la formule du flux-divergence, on obtient

$$\int_{\Omega} \frac{\partial \phi}{\partial t} dx + \int_{\Omega} \vec{\nabla} \cdot (\phi \vec{v}) dx = \int_{\Omega} F dx. \quad (1.6)$$

Ceci aboutit, comme Ω est quelconque, à

$$\frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot (\phi \vec{v}) = F. \quad (1.7)$$

Dans le cas où ϕ est la masse volumique du fluide notée ρ , il n'y a pas de terme source F puisque la masse totale du fluide est conservée³ et on obtient

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0. \quad (1.8)$$

1.1.4 L'équation de conservation de la quantité de mouvement

L'équation de conservation de la quantité de mouvement volumique $\rho \vec{v}$ est une loi de conservation et à ce titre s'écrit selon le modèle de (1.7). En vertu de la seconde loi de la mécanique de Newton [59], la variation de la quantité de mouvement est égale à la somme des forces extérieures appliquées. Sur une particule de fluide mésoscopique, les forces extérieures sont :

- La pression exercée par les particules de fluide voisines qui se modélise naturellement par la notion de pression du fluide p . Il s'agit de la force de contact exercée par une particule de fluide sur sa voisine au temps t . La force volumique qui en résulte macroscopiquement est $-\vec{\nabla} p$.
- Les forces de dissipation d'énergie liée au frottement visqueux du fluide. Ceci est modélisé par le tenseur des contraintes visqueuses $\underline{\tau}(t, \vec{x})$ dont le comportement varie d'un fluide à l'autre.
- Les interactions à distance (attraction gravitationnelle, champ électro-magnétique dans le cas des ferrofluides, etc...) qui s'écrivent sous la forme générale $\rho(t, \vec{x}) \vec{f}(t, \vec{x})$.

Ainsi, l'équation de conservation de la quantité de mouvement s'écrit sous la forme générale suivante

$$\frac{\partial(\rho \vec{v})}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v} \otimes \vec{v}) = -\vec{\nabla} p + \vec{\nabla} \cdot \underline{\tau} + \rho \vec{f}. \quad (1.9)$$

La masse volumique ρ s'exprime en $kg.m^{-3}$, la vitesse \vec{v} en $m.s^{-1}$, la pression et le tenseur des contraintes visqueuses en $Pa = kg.m^{-1}.s^{-2}$ et enfin les forces extérieures volumiques \vec{f} en $m.s^{-2}$. Les deux équations (1.8) et (1.9) sont les deux équations fondamentales de la mécanique des fluides. Elles sont généralement complétées par un bilan sur l'évolution de l'énergie locale, ce qui donne les *équations de Navier-Stokes* dans leur formulation la plus générale. Ces équations furent écrites pour la première fois et étudiées par Navier en 1823 [58] puis une dérivation plus rigoureuse fut donnée par Stokes en 1845 [78]. Notons

3. Bien que ceci ne concerne pas ce travail, remarquons que dans le cas des fluides radioactifs, la masse totale peut varier en vertu de la loi d'Einstein $E = mc^2$.

au passage les travaux pionniers de Cauchy [20] et Poisson [65]. Les équations de Navier-Stokes sont la généralisation des équations d'Euler incompressibles homogènes qui furent écrites en 1755 [39] par Euler et qui sont historiquement les premières équations aux dérivées partielles qui décrivent le comportement d'un fluide.

Les équations de Navier-Stokes sont alors complétées par des équations empiriques qui viennent modéliser la compressibilité du fluide et la viscosité $\underline{\tau}$.

1.2 Équations d'Euler incompressibles homogènes

Les équations de Navier-Stokes présentées précédemment ont été historiquement la généralisation du travail d'Euler présenté en 1757 et ont consisté en l'adjonction des deux phénomènes que sont la viscosité et la compressibilité du fluide. Néanmoins, vu que l'adjonction du terme de compressibilité peut se faire à partir des concepts légués par Euler, on considère généralement que les équations d'Euler consistent seulement à retirer le terme de viscosité et on distingue dans un deuxième temps entre Euler compressible et Euler incompressible. Cela s'explique également par le fait que l'ajout du terme de viscosité modifie considérablement la nature de l'équation puisqu'il s'agit d'un terme dissipatif. La notion d'homogénéité désigne quant à elle, dans le cas incompressible, le fait de considérer un fluide de masse volumique ρ constante dans l'espace et dans le temps.

1.2.1 Énoncé des équations

Les *équations d'Euler* sous leur forme générale consistent donc à négliger le terme dissipatif associé à la viscosité du fluide et à prendre $\underline{\tau} = 0$ dans (1.9). Cela signifie physiquement que les autres termes de l'équation de conservation de la quantité de mouvement sont grands devant la dissipation par viscosité. Les équations d'Euler sont dites *incompressibles* si on y ajoute la condition $\vec{\nabla} \cdot \vec{v} = 0$ qui correspond à l'idée qu'à l'intérieur d'un volume mésoscopique il y a autant de fluide qui entre que de fluide qui sort à chaque instant. Cette condition de divergence est l'utilisation de la loi de conservation (1.7) avec $\phi(t, \vec{x}) = 1$. Cela correspond localement à la préservation du volume du fluide. Les équations d'Euler sont dites *homogènes* si la masse volumique $\rho(t, \vec{x})$ est constante. Dans le cas des fluides incompressibles, il suffit que le fluide soit homogène au temps $t = 0$ pour le rester en tous temps. Les équations d'Euler incompressibles et homogènes s'écrivent alors

$$\begin{cases} \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\frac{1}{\rho_0} \vec{\nabla} p + \vec{f}, \\ \vec{\nabla} \cdot \vec{v} = 0. \end{cases} \quad (1.10)$$

Dans la suite de cette section, nous supposons que le terme de force extérieure \vec{f} est nul ; c'est-à-dire que le système est isolé. Dans ce cas, puisque les termes dissipatifs sont supprimés⁴, nous avons conservation de l'énergie globale du système⁵. Après redimensionnement de l'équation, on aboutit à

$$\begin{cases} \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} + \vec{\nabla} p = \vec{0}, \\ \vec{\nabla} \cdot \vec{v} = 0. \end{cases} \quad (1.11)$$

Par la suite et sauf mention contraire, les *équations d'Euler incompressibles et homogènes* seront simplement appelées *équations d'Euler*.

1.2.2 Formulation en terme de vorticit 

Un objet important à étudier pour comprendre les équations d'Euler (1.11) est la vorticit  $\vec{\omega} := \vec{\nabla} \wedge \vec{v}$. Cette quantité permet de comprendre et suivre l'évolution des phénomènes tourbillonnaires ou turbulents (voir illustration en figure 1.2). La vorticit  du fluide peut se comprendre au niveau de l'échelle mésoscopique comme  tant la rotation de la vitesse de la particule de fluide. En effet, par un d veloppement de Taylor,

$$\vec{v}(t, \vec{x} + \vec{h}) = \vec{v}(t, \vec{x}) + \underline{\epsilon}(\vec{v}) \cdot \vec{h} + \frac{1}{2} \vec{\omega} \wedge \vec{h} + \mathcal{O}(|\vec{h}|^2), \quad (1.12)$$

4. On peut se contenter de n gliger la composante dissipative de l'interaction \vec{f} et aboutir alors   une  quation ayant les m mes propri t s.

5. Ceci est vrai dans le cas de solutions de l' quation au sens classique. Pour des solutions moins r guli res, voir la section consacr e   la conjecture d'Onsager.

où $\underline{\epsilon}$ est le tenseur des déformations des vitesses défini par

$$\underline{\epsilon}(\vec{v}) := \frac{1}{2}(\nabla\vec{v} + (\nabla\vec{v})^t). \quad (1.13)$$



FIGURE 1.2 – Tourbillon généré par une aile d’avion au décollage mis en évidence par de la fumée colorée.

Ainsi, au premier ordre la vitesse évolue comme étant une translation, une rotation donnée par $\vec{\omega}$ et une déformation donnée par $\underline{\epsilon}$. L’équation d’Euler (1.11) en vorticité prend la forme suivante

$$\frac{\partial\vec{\omega}}{\partial t} + \vec{v} \cdot \vec{\nabla}\vec{\omega} = \vec{\omega} \cdot \vec{\nabla}\vec{v}. \quad (1.14)$$

L’équation (1.14), l’équation $\vec{\nabla} \wedge \vec{v} = \vec{\omega}$ et l’équation $\vec{\nabla} \cdot \vec{v} = 0$ forment un système fermé. Plus précisément, l’application $\vec{\omega} \mapsto \vec{v}$ est bien définie et celle-ci envoie $\mathcal{C}^{0,\alpha}$ dans $\mathcal{C}^{1,\alpha}$ (avec $0 < \alpha \leq 1$) et envoie H^s dans H^{s+1} [6].

1.2.3 Formulation hamiltonienne et théorème de Noether

Une solution de l’équation d’Euler (1.11) peut-être regardée comme étant une fonction qui évolue en suivant les extrémals d’une certaine fonctionnelle d’action \mathcal{A} sous la contrainte de préservation de la mesure par le flot (conséquence de $\vec{\nabla} \cdot \vec{v} = 0$). Voir [57] pour plus de détails sur cette reformulation. Ainsi, l’équation d’Euler peut être vue comme étant une équation hamiltonienne puisqu’on peut la voir comme étant une équation d’Euler-Lagrange, le lagrangien étant l’énergie cinétique du fluide et le multiplicateur de Lagrange associé va nous donner le terme de gradient de pression. La conséquence de cette formulation est que l’on peut appliquer le théorème de Noether [60] et ainsi extraire différents invariants pour le système. Cela permet d’obtenir la préservation de l’énergie (forces conservatives), la préservation de la *vorticité totale* :

$$\Omega := \int_{\mathbb{R}^3} \vec{\omega}(t, \vec{x}) \, dx,$$

la préservation du *moment cinétique total* (conséquence de l’invariance par translation)

$$M := \int_{\mathbb{R}^3} \vec{x} \wedge \vec{\omega}(t, \vec{x}) \, dx,$$

et la préservation du *moment angulaire total* (conséquence de l’invariance par rotation)

$$I := \int_{\mathbb{R}^3} |\vec{x}|^2 \vec{\omega}(t, \vec{x}) \, dx.$$

1.2.4 Principe de Bernoulli et théorème de Kelvin

Ces deux résultats sont plus anecdotiques dans le travail effectué dans cette thèse. Comme ils sont néanmoins des résultats fondamentaux et utilisés dans beaucoup de travaux sur les équations d’Euler, nous les mentionnons brièvement ici.

Théorème 1.1 (Théorème de la circulation, Kelvin [79], 1868). *Soit C un contour fermé dans le fluide paramétré par un vecteur tangent unitaire \vec{i} . Alors la circulation du fluide le long de ce contour est nul :*

$$\oint_C \vec{v} \cdot \vec{di} = 0. \quad (1.15)$$

Le principe de Bernoulli [11] de 1738 affirme que dans le flux d'un fluide homogène et incompressible soumis uniquement aux forces de pression et de pesanteur, une accélération se produit simultanément avec la diminution de la pression. Plus précisément, sur une même ligne de courant la quantité

$$\frac{|\vec{v}|^2}{2} + gz + \frac{p}{\rho}$$

est constante.

1.2.5 Théorème de Beale-Kato-Majda

Les résultats d'existence de solution sur un petit intervalle de temps sont connus depuis longtemps. La démonstration la plus ancienne connue est celle de Lichtenstein [6] basée sur un lemme de Gronwall de la forme

$$\frac{d}{dt}y \leq Cy^{\frac{3}{2}} \quad \implies \quad y(t) \leq \frac{y_0}{(1 - 2tC\sqrt{y_0})^2}. \quad (1.16)$$

Ceci permet de conclure si on choisit $y(t) = \|\vec{v}(t, \cdot)\|_{H^s}$ avec $s > \frac{5}{2}$ et une injection de Sobolev. Ce résultat a été significativement amélioré par le théorème suivant.

Théorème 1.2 (Existence pour Euler tri-dimensionnel, Beale-Kato-Majda [8], 1984,). *Soit \vec{v} une solution aux équations d'Euler (1.11) sur un intervalle de temps $[0, T]$ telle que*

$$\exists s > \frac{5}{3}, \quad \vec{v} \in \mathcal{C}_t^0([0, T[, H_x^s(\mathbb{R}^3)) \quad (1.17)$$

et telle que

$$\int_0^T \|\vec{\omega}(t, \cdot)\|_{L^\infty} dt < +\infty. \quad (1.18)$$

Alors il existe un $\delta > 0$ tel que la solution puisse être prolongée de manière unique sur l'intervalle de temps $[0, T + \delta]$

En plus de raffiner le résultat d'existence déjà précédemment connu, il nous donne une condition suffisante sur la vorticité pour que la solution existe et donc sur la façon dont la vorticité diverge si l'existence n'est que locale.

1.2.6 Le paradoxe de Scheffer-Shnirelman

La pertinence physique de la modélisation des fluides par les équations d'Euler a été remise en question par un article de V. Scheffer [72] au début des années 1990. Il y propose la construction de solutions à ces équations ayant la propriété d'être non nulles et à support compact en espace-temps. En d'autres termes, un fluide parfaitement modélisé par les équations d'Euler serait susceptible, alors que celui-ci est complètement immobile, de s'agiter spontanément et sans apport d'énergie extérieur, de manière locale, avant de retourner au repos. Il fut ensuite complété et précisé par A. Shnirelman [74]. Une construction de solutions paradoxales plus régulières et possédant une énergie finie a été proposée par De Lellis et Székelyhidi [32] [33]. Dans ces derniers articles, leurs solutions ont la propriété de s'écrire comme des limites de solutions \mathcal{C}^∞ . C'est-à-dire qu'elles sont topologiquement proches des solutions au sens classique. Ces solutions paradoxales restent néanmoins très irrégulières puisque qu'elles ne sont nulles part continues sur leur support et les fonctions approximantes font apparaître des structures multi-échelles nombreuses (fractalité).

1.2.7 Cascade d'énergie et conjecture d'Onsager

Le paradoxe de Scheffer-Shnirelman peut paraître anecdotique tant par la grande irrégularité des solutions que par son caractère contraire à l'intuition physique. Pourtant, l'idée selon laquelle une turbulence trop irrégulière peut remettre en cause la pertinence du modèle d'Euler n'est pas récente. L'idée

selon laquelle de fortes turbulences fassent intervenir des interactions multi-échelles date de la publication en 1826 (posthume) des observations de de Vinci [34]⁶ (voir figure 1.3). Il y décrit les phénomènes tourbillonnaires comme étant une évolutions de petits tourbillons pris dans des tourbillons plus gros et observe que ce sont les tourbillons de plus petite taille disparaissent le plus vite.



FIGURE 1.3 – Dessin et description de la turbulence par L. de Vinci vers 1508.

Ce phénomène s’appelle la cascade d’énergie et a été formalisé par Richardson [68] puis Kolmogorov [50, 51, 52, 53]. Leur objectif était de saisir l’idée que lorsque l’on arrive à des turbulences ayant lieu à petite échelle alors la viscosité n’est plus négligeable et des phénomènes dissipatifs apparaissent. En témoigne par exemple la conjecture d’Onsager formulée en 1949 qui affirme que pour la régularité $C^{0,\alpha}$ avec $\alpha < 1/3$ les solutions des équations d’Euler (1.11) échouent à préserver l’énergie [62]. Cette conjecture a été résolue par l’affirmative par Isett en 2018, soit presque 60 ans plus tard [47]. Les solutions de Isett donnent également une négation de l’unicité des solutions de Euler sous le niveau de régularité $1/3$. Pour une introduction plus complète aux problèmes de régularité dans Euler, voir [16]. La pertinence de ce facteur $1/3$ a également été mis en évidence expérimentalement [71].

Une autre réserve à avoir sur ces solutions peu régulières, en plus des interrogations sur leur pertinence physique, consiste à s’interroger sur la véracité de l’égalité (1.4). En effet, cette relation de passage de la représentation lagrangienne à la représentation eulérienne nécessite un flot suffisamment régulier pour être démontrée rigoureusement à partir des éléments donnés à la section 1.1.2.

1.3 Équations d’Euler incompressibles bi-dimensionnelles

1.3.1 Modélisation des écoulements du plan

Les équations d’Euler bi-dimensionnelles correspondent aux équations d’Euler (1.11) sauf que l’on considère seulement deux variables d’espace. Ceci est utilisé pour modéliser des écoulements non-visqueux planaires, c’est-à-dire soit un écoulement sur une faible épaisseur, soit tel que l’écoulement pour un certain $z = z_0$ reste dans le plan et est intépéndant de l’écoulement dans le plan $z = z_1 \neq z_0$ (voir figure 1.4).

1.3.2 Formulation en vorticité : une équation de transport

Dans le cas d’un écoulement dans le plan, la vorticité devient une quantité scalaire $\omega := \vec{\nabla}^\perp \cdot \vec{v}$, avec \perp désignant la rotation du plan d’angle $\pi/2$. La formulation en vorticité (1.14) pour les équations d’Euler bi-dimensionnelles sont alors de la forme suivante

$$\frac{\partial}{\partial t} \omega + \vec{v} \cdot \vec{\nabla} \omega = 0. \quad (1.19)$$

6. Contrairement à une légende tenace, les descriptions de la mécanique des fluides par Leonard de Vinci vers 1508 ne sont pas “tombées dans l’oubli car il était trop en avance sur son temps”. Ces travaux - plus artistiques que scientifiques - sont restés en réalité sous forme de manuscrits qui n’avaient pas été publiés et qui avaient été perdus. Ils ont été retrouvés dans la première moitié du XIXe siècle par un archiviste (Francesco Arconati) dans la *Biblioteca Ambrosiana* à Milan. Ils sont maintenant exposés au musée de la *Pinacoteca Ambrosiana* juste à côté de la bibliothèque.

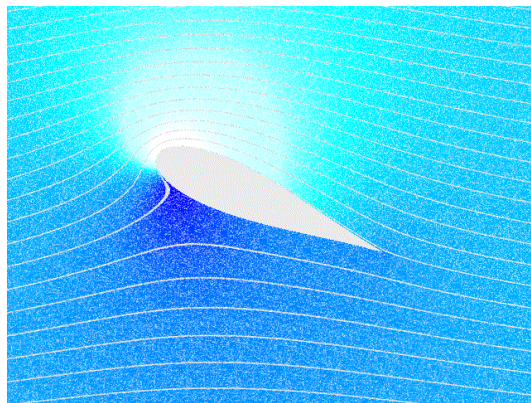


FIGURE 1.4 – Modélisation de l'écoulement de l'air autour d'une aile d'avion à l'aide des équations d'Euler bi-dimensionnelles.

Muni de $\omega = \nabla^\perp \cdot \vec{v}$ et de la condition d'incompressibilité $\vec{\nabla} \cdot \vec{v} = 0$, le système obtenu est fermé. On constate que l'équation (1.19) est une équation de transport de la vorticit   ω par la vitesse \vec{v} . Comme en outre $\vec{\nabla} \cdot \vec{v} = 0$, par le th  or  me de Liouville [4], il s'agit d'une   quation de transport qui pr  serve la mesure (on parle aussi de *rearrangement*). La cons  quence est la pr  servation de toutes les normes L^p de ω , pour $p \in [1, +\infty]$, au cours du temps pour une vitesse suffisamment r  guli  re. Le fait que la vitesse soit de divergence nulle permet de l'  crire    l'aide d'une fonction de courant ψ sous la forme $\vec{v} = \vec{\nabla}^\perp \psi$. Autrement dit, la vitesse s'  coule selon les lignes de niveau de la fonction ψ tel que repr  sent      la figure 1.4. On aboutit alors au syst  me suivant

$$\begin{cases} \frac{\partial}{\partial t} \omega + \vec{v} \cdot \vec{\nabla} \omega = 0, \\ \omega = (-\Delta) \psi, \\ \vec{v} = \vec{\nabla}^\perp \psi. \end{cases} \quad (1.20)$$

On peut ensuite   crire la loi de *Biot et Savart* qui permet de retrouver le vecteur vitesse \vec{v} connaissant la vorticit   ω :

$$\vec{v}(t, \vec{x}) = \vec{\nabla}^\perp (-\Delta)^{-1} \omega(t, \vec{x}) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \omega(t, \vec{y}) \frac{(\vec{y} - \vec{x})^\perp}{|\vec{y} - \vec{x}|^2} dy. \quad (1.21)$$

o   l'on a utilis   $-\frac{1}{2\pi} \log$ le noyau de Green du laplacien dans le plan.

1.3.3 Th  or  me de Yudovich

Le premier th  or  me d'existence globale des solutions classiques pour les   quations d'Euler bi-dimensionnelles a   t   donn   par Wolibner [84]. Ce r  sultat fut significativement am  lior   par Yudovich qui l'  tend    des solutions faibles et qui obtient un crit  re d'unicit  .

Th  or  me 1.3 (Existence pour Euler bi-dimensionnel, Yudovich [85], 1963). *On consid  re ω_0 donn  e initiale pour Euler bi-dimensionnel (1.19) avec \vec{v} d  finie par (1.21).*

- Si $\omega_0 \in L^\infty$ alors on a existence et unicit   de la solution en tous temps avec pr  servation des normes L^p pour tout p .
- Si $\omega_0 \in L^p$ avec $p \in]1, +\infty[$ alors on a seulement existence mais on a pr  servation de la norme L^p .

Une extension de ce r  sultat d'existence a   t   propos  e par Delort [35] et permet d'obtenir sous certaines conditions des solutions qui sont de r  gularit   n  gative. L'argument principal est une sym  trisation dans la formulation faible des   quations d'Euler bi-dimensionnelles en vorticit  .

Bien que les normes L^p soient pr  serv  es, il n'en est rien concernant les normes de Sobolev. Par exemple, dans [36] est construite    partir de [5] une solution r  guli  re aux   quations d'Euler bi-dimensionnelles dont la croissance de la norme $W^{1,\infty}$ au cours du temps se fait comme $\exp(\exp(t))$.

1.3.4 Modèle des points-vortex

Le modèle des points-vortex est un système d'équations différentielles sur des points du plan \mathbb{R}^2 qui approxime les situations où la vorticit e ω est concentr ee autour d'un nombre fini de points appel es vortex. En d'autres termes, la vorticit e peut  tre vue comme  tant une somme pond er ee de masses de Dirac :

$$\omega(t = 0, x) = \sum_{i=1}^N a_i \delta_{x_i}. \quad (1.22)$$

Les points x_i sont les positions respectives des vortex $a_i \delta_{x_i}$ et les coefficients a_i sont leur intensit e. L' equation (1.19) est une  equation de transport portant sur l' evolution de la vorticit e ω . Ainsi on s'attend   ce que la masse de Dirac situ ee en x_i soit laiss ee inchang ee mais d eplac ee par le flot. Formellement, si on utilise les  equations (1.20) avec comme donn ee initiale (1.22), nous obtenons que la vitesse s' ecrit

$$v(t = 0, x) = -\frac{1}{2\pi} \sum_{i=1}^N a_i \nabla_x^\perp \log(|x_i - x|), \quad (1.23)$$

avec $-\frac{1}{2\pi} \log$ le profil de la fonction de Green du laplacien $(-\Delta)$ dans le plan \mathbb{R}^2 . La vitesse v est singuliere au voisinage des points x_i et par cons equent l' equation (1.20) ne fait pas sens m eme en utilisant les formulations faibles aboutissant aux th eor emes d'existence de Yudovich et de Delort pr esent es   la section pr ecedente. N eanmoins, des consid erations relevant de la physique disent que si toute la vorticit e est concentr ee au voisinage d'un point alors celui-ci est pratiquement immobile. Certes, les particules de fluides vont tourner autour de ce point   tr es grande vitesse mais il s'agira d'un point immobile. Cet  l ement d'intuition physique se reformule math ematiquement en disant que le champ de vitesse v engendr e par un vortex $a_i \delta_{x_i}$ n'interagit pas avec ce m eme vortex $a_i \delta_{x_i}$. Autrement dit, le champ de vitesse avec lequel on travaille est

$$v(t = 0, x) = -\frac{1}{2\pi} \sum_{i=1}^N a_i \nabla_x^\perp \log(|x_i - x|) \mathbb{1}_{x \neq x_i} \quad (1.24)$$

o  $\mathbb{1}$ d esigne les fonctions indicatrices (rappelons que $0 \times \infty = 0$). Dans ce cas, en utilisant (1.24) dans (1.19), on obtient que l' equation diff erentielle qui r egit l' evolution de la position des vortex au cours du temps est :

$$\frac{d}{dt} x_i(t) = -\frac{1}{2\pi} \sum_{\substack{j=1 \\ j \neq i}}^N a_j \nabla^\perp \log(|x_i(t) - x_j(t)|). \quad (1.25)$$

Pour une  tude d etaill ee de ce syst eme d' equations diff erentielles et pour une d erivation rigoureuse de ce mod ele, lire [56,  4]. Puisque le syst eme des points-vortex ne rentre pas dans la th eorie de Yudovich ni de Delort, une d erivation rigoureuse du syst eme des points-vortex consiste alors en une approximation des masses de Dirac par des solutions de Yudovich assortie d'une  tude de l' energie lors du passage   la limite. Une telle approximation s'appelle une *d esingularisation* du syst eme des points-vortex [75].

1.3.5 Utilisation du mod ele des points-vortex

Le premier usage naturel des mod eles des points-vortex consiste   mod eliser la turbulence dans les cas simples. L'int er et  tant que l'int egration d' equations diff erentielles est parfois moins c oteuse que celle d'une EDP (si le nombre de vortex n'est pas trop  lev e).

D'un point de vue th eorique, le syst eme des points-vortex redonne les  equations d'Euler bi-dimensionnelles dans une limite de champs moyens (sous certaines hypoth eses). Dans [57], est  galement  tabli la possibilit e de retrouver les  equations de Navier-Stokes   l'aide des points-vortex dont le d eplacement serait stochastique, c'est- -dire que l'on ajoute un mouvement brownien qui mod elise la diffusion li ee aux frottements.

Un autre  l ement int eressant sur les syst emes de points-vortex est qu'ils servent   mod eliser l'analogie tri-dimensionnel des points-vortex,   savoir les filaments de vorticit e pour Euler tri-dimensionnelles. Cela correspond dans ce cas   une vorticit e qui est concentr ee non pas en des points mais sur des lignes. Le mod ele le plus  tudi e est le flot binormal (pour une pr esentation du flot binormal, lire par exemple [38]). Cependant, dans le cas o  les filaments ont une direction privil egi ee \vec{e}_z (pensons aux train ees derri ere les avions : les deux filaments de vorticit e laiss es par le passage de l'avion ont bien une direction privil egi ee

au sens où ils sont proches de la trajectoire de l'avion) alors on peut modéliser l'interaction de deux filaments dans un plan orthogonal à \vec{e}_z par des points-vortex. Dans la direction principale, l'interaction est modélisée à l'aide d'une équation de Schrödinger. Pour plus de détails sur ce modèle, lire [31].

2 Modéliser l'atmosphère terrestre

2.1 Mécanique des fluides géophysiques

La mécanique des fluides géophysiques est la discipline issue de la mécanique des fluides qui étudie l'évolution des différents mécanismes à l'œuvre dans le comportement de l'atmosphère terrestre et des mers et océans (voir illustration en figure 1.5). Pour une présentation générale de la mécanique des fluides géophysiques, voir les livres de Pedlow [64] et de Vallis [83] à partir desquels a été rédigée la présentation qui suit. Pour les équations quasi-géostrophiques de surface, lire aussi [54].

Le cas des équations quasi-géostrophiques concerne essentiellement la modélisation de l'atmosphère terrestre dans les régions basses de l'atmosphère. Il s'agit d'un modèle particulièrement pertinent pour la tropopause, la zone de transition entre la troposphère et la stratosphère et plus généralement pour les strates de fluides situées juste au dessus de la couche limite. Ce sont les principales équations utilisées pour effectuer des prédictions météorologiques. Il n'est pas possible en quelques pages de présenter de manière exhaustive la dérivation du modèle conduisant à ces équations. Il s'agit surtout d'expliquer les différents concepts issus de la physique et les modélisations qui permettent de décrire l'atmosphère terrestre. Une attention plus particulière sera donnée à ceux qui permettent d'aboutir aux équations quasi-géostrophiques. Le régime quasi-géostrophique est le régime d'écoulement où le fluide est stratifié (organisé en couches de densité variable et qui ne s'interpénètrent pas) en rotation rapide (la rotation de la terre est un terme dominant dans le bilan mécanique) et soumis à des phénomènes thermodynamiques (dilatation thermique pour l'interaction des strates de fluide les unes avec les autres).

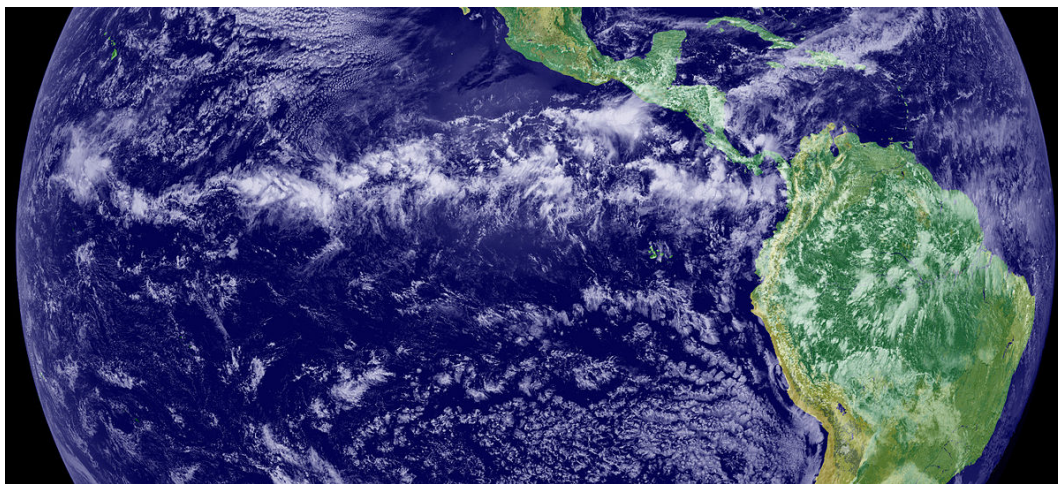


FIGURE 1.5 – Écoulements dans l'atmosphère terrestre vus depuis l'espace.

2.1.1 Utiliser les équations d'Euler ?

L'approche la plus naturelle pour modéliser l'atmosphère terrestre à partir des éléments de la mécanique des fluides présentés à la section précédente consisterait à considérer un fluide régi par les équations d'Euler tri-dimensionnelles (1.11) s'écoulant en dehors d'une boule ou bien entre deux sphères concentriques. La plus petite sphère modéliserait la surface de la terre, la plus grande la limite de l'atmosphère. Néanmoins, en plus des difficultés évoquées précédemment pour Euler tri-dimensionnel, des difficultés supplémentaires apparaissent quant aux interactions fluide-structure pour cette équation [7]. Une autre approche, en utilisant le fait que l'atmosphère est de très faible épaisseur consisterait à étudier un écoulement bi-dimensionnel sur la surface d'une sphère et ainsi réutiliser et adapter la théorie connue pour Euler bi-dimensionnel. Néanmoins, une telle approche ne prend en compte aucune des interactions spécifiques qui jouent un rôle crucial dans le comportement de l'atmosphère terrestre.

2.1.2 Rotation de la terre et mécanique en référentiel tournant

Le premier élément supplémentaire par rapport aux équations d'Euler est la rotation de la terre. En effet, dans le cas des écoulements de fluides à grande échelle, les effets inertiels engendrés par la rotation de la Terre deviennent non-négligeables, voire prépondérants. Si on note I le référentiel inertiel et R le référentiel en rotation selon le vecteur $\vec{\Omega}$ (avec la convention de Maxwell pour le sens de la rotation), l'évolution d'un vecteur \vec{B} dans chaque référentiel est relié par

$$\left(\frac{d\vec{B}}{dt}\right)_I = \left(\frac{d\vec{B}}{dt}\right)_R + 2\vec{\Omega} \wedge \vec{B}. \quad (1.26)$$

Par conséquent, les accélérations dans les deux référentiels sont reliées par la formule du référentiel tournant

$$\left(\frac{d\vec{v}_I}{dt}\right)_I = \left(\frac{d\vec{v}_R}{dt}\right)_R + \vec{\Omega} \wedge (\vec{\Omega} \wedge \vec{r}) + \frac{d\vec{\Omega}}{dt} \wedge \vec{r} + 2\vec{\Omega} \wedge \vec{v}_R, \quad (1.27)$$

où $\vec{r} = r\vec{e}_r$ est le vecteur radial en coordonnées sphériques (centrées au centre de la terre). Le premier terme inertiel qui apparaît à droite de (1.27) est appelé *l'accélération centripète*. Ce terme tend à éjecter le fluide loin de l'axe de rotation. Le second terme est souvent appelé *accélération d'Euler* et rend compte des effets inertiels liés à la variation de l'axe et de la vitesse de la rotation. Dans notre cas celui-ci est nul⁷. Le dernier terme apparaissant dans (1.27) est *l'accélération de Coriolis* et il joue un rôle prépondérant dans la mécanique des fluides géophysiques. Faisons remarquer qu'il s'agit d'un terme qui est systématiquement orthogonal à la vitesse. A ce titre, il s'agit d'une force que ne *travaille pas*. Elle dévie les trajectoires sans faire accélérer ni ralentir les particules. La plupart des phénomènes mécaniques spécifiques à la dynamique atmosphérique à grande échelle s'expliquent en grande partie par la prépondérance du terme de Coriolis.



FIGURE 1.6 – Une dépression atmosphérique au large de l'Islande vue depuis l'espace. Au lieu de rejoindre le centre de la dépression en ligne droite, la particules de fluide sont déviées par l'effet Coriolis et forment cette spirale mise en évidence par la condensation.

2.1.3 Compressibilité, inhomogénéité, et stratification

Dans le cas de l'atmosphère terrestre, l'air est un gaz *compressible* et cette compressibilité n'est pas négligeable dans beaucoup de cas. La condition d'incompressibilité $\vec{\nabla} \cdot \vec{v} = 0$ centrale dans la modélisation des fluides chez Euler doit être remplacée par une modélisation physique de la compressibilité. De même, l'air est un gaz *inhomogène* au sens où la masse volumique ρ ne peut pas forcément être considérée comme constante. On observe par exemple empiriquement que les gaz atmosphériques tendent à se raréfier lorsque l'altitude augmente.

7. Certains éléments théoriques issus de la mécanique des fluides géophysiques sont utilisés pour étudier le mouvement des plaques tectoniques. Dans ce cas, puisqu'il s'agit d'échelles de temps géologiques, le terme issu de l'accélération d'Euler n'est pas négligeable.

Un fluide est dit *stratifié* lorsque les particules de fluide se déplacent horizontalement mais pas verticalement. Il n'y a donc pas d'échange de matière entre les couches de fluide. L'écoulement est alors purement planaire (ou sphérique) et les différentes strates de fluide interagissent les unes avec les autres uniquement par le biais de la poussée d'Archimède (et des frottements visqueux entre les couches de fluides s'ils ne sont pas négligeables).

2.1.4 Effets thermodynamiques et dilatation liée à la température

Un élément nouveau par rapport à la théorie standard de la mécanique des fluides et qui joue un rôle majeur dans le cas des fluides géophysiques est la thermodynamique et la dilatation thermique. Dans le cadre de cette présentation des équations quasi-géostrophiques, l'influence de la thermodynamique dans notre modèle sera détaillée en un second temps. Nous rappelons néanmoins ici quelques concepts centraux de la thermodynamique tel que le *premier principe* de la thermodynamique (référentiel inertiel) :

$$\rho \frac{de}{dt} = -\rho p \frac{d}{dt} \left(\frac{1}{\rho} \right) + k\Delta T + \chi + \rho Q, \quad (1.28)$$

avec e l'énergie interne massique en $J kg^{-1} = m^2 s^{-2}$, T la température en Kelvin K , Q le taux de chauffage massique en $m^2 s^{-3}$ et χ le chauffage par frottement visqueux en $kg m^{-1} s^{-3}$. Le *second principe* de la thermodynamique concerne la variation d'entropie

$$T \frac{ds}{dt} = \frac{de}{dt} + p \frac{d}{dt} \left(\frac{1}{\rho} \right) \quad (1.29)$$

et affirme que dans un système thermiquement isolé, celle-ci ne peut qu'augmenter. L'entropie peut être interprétée comme étant la mesure du degré de désordre d'un système au niveau microscopique. Elle mesure donc l'irréversibilité du phénomène étudié. L'entropie se mesure en Joules par Kelvin et donc l'entropie massique s apparaissant dans (1.29) se mesure en $m^2 \cdot s^{-2} \cdot K^{-1}$.

La dilatation thermique est le phénomène de thermodynamique consistant en l'expansion à pression constante du volume d'un corps occasionné par son réchauffement. Dans le cadre de la modélisation de l'atmosphère, cela se modélise à l'aide de la *loi des gaz parfaits* :

$$P = \rho(c_p - c_v)T, \quad (1.30)$$

où c_p est la capacité thermique isobare massique de l'air et c_v sa capacité thermique isochore. On pose en général $RM = c_p - c_v$ où M est la masse molaire de l'air et R est une constante universelle appelée constante des gaz parfaits.

2.1.5 Interactions avec les continents et les océans

Dans des modèles plus élaborés que celui que nous étudions ici, des éléments liés à l'interaction de l'atmosphère avec le sol terrestre ou les océans sont pris en compte. Citons par exemple les phénomènes de frottement visqueux (interaction fluide-structure) avec formation de couches limites, les échanges thermiques entre le sol et l'atmosphère ou bien encore l'évaporation de l'océan.

2.1.6 Autres phénomènes

Dans des modèles plus élaborés, on peut également intégrer

- l'énergie libérée ou absorbée lors des changements d'états (condensation, formation des flocons de neige)
- les effets électro-magnétiques (trou dans la couche d'ozone, aurores boréales)
- les réactions chimiques
- la salinité (dans le cas de la modélisation de l'océan)
- etc...

2.2 Approximations et modélisation : vers le modèle géostrophique

L'approximation *géostrophique* consiste à supposer que le fluide est dans une disposition telle que le gradient de pression et l'accélération de Coriolis se compensent exactement. Dans cette section nous présentons différentes approximations et modélisations classiques qui constituent plus ou moins directement le cadre de l'approximation géostrophique.

2.2.1 Approximation de la surface terrestre par un plan

Une approximation souvent utilisée consiste à approximer la surface de la terre au voisinage d'un point par un plan (voir figure 1.7). Pour étudier l'évolution de l'atmosphère sur une échelle qui est celle de la France, ceci est une très bonne approximation. L'intérêt est que l'on peut alors utiliser les coordonnées cartésiennes et non plus les coordonnées sphériques qui sont plus difficiles à manipuler.

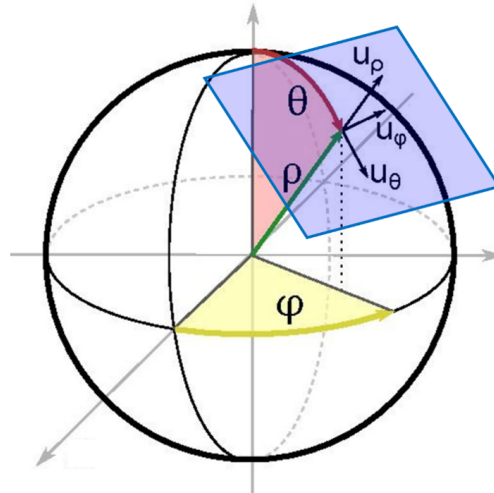


FIGURE 1.7 – Système de coordonnées sphériques et approximation locale par un plan (représenté en bleu).

La coordonnée verticale ze_z coïncide alors dans ce cas avec la coordonnée radiale re_r au dessus du point choisi pour faire l'approximation. Dans le cadre de cette approximation, l'écoulement du fluide se fait dans le demi-espace de \mathbb{R}^3 associé à la condition $z \geq 0$. Afin de prendre en compte la courbure de la terre, certaines approximations intègrent un terme supplémentaire appelé β -correction. Cela permet d'utiliser l'approximation planaire pour étudier l'atmosphère et prédire la météo à une échelle comme celle de l'Europe. Pour des échelles plus importantes, on utilise soit des coordonnées polaires, soit des recouvrements d'atlas.

2.2.2 Attraction gravitationnelle et force inertielle

L'équation de conservation de la quantité de mouvement (1.9) admet comme terme d'interaction à distance l'attraction gravitationnelle $-g\rho e_r$ où g est l'accélération de la pesanteur terrestre et vaut environ $9,8m.s^{-2}$ à la surface du globe. Quoique l'attraction gravitationnelle diminue avec l'altitude, l'atmosphère terrestre est suffisamment mince pour pouvoir négliger cet effet. Un autre terme qui peut s'écrire sous la forme d'une interaction potentielle (qui s'écrit comme le gradient d'une énergie) est l'accélération inertielle centripète. Néanmoins, dans la plupart des cas celle-ci est petite devant l'accélération gravitationnelle et elle est alors négligée (entre les tropiques, pour des prévisions d'une semaine et plus on observe qu'elle a une légère influence).

2.2.3 Nombre de Rossby et influence de l'accélération de Coriolis

Le nombre de Rossby mesure l'influence de l'accélération de Coriolis, dernier terme de (1.27). On note L l'échelle de longueur horizontale caractéristique des phénomènes étudiés (par exemple la distance moyenne entre deux extrema de pression), U l'échelle de vitesse horizontale caractéristique et Ω la vitesse angulaire de la rotation de la terre. Alors le nombre de Rossby est :

$$R := \frac{U}{L\Omega}. \quad (1.31)$$

Si le nombre de Rossby est petit, cela signifie que l'accélération de Coriolis est un terme dominant dans la dynamique. Ceci constitue un élément central des approximations géostrophique et quasi-géostrophique. Par exemple, dans le cas de l'atmosphère terrestre, nous avons L de l'ordre de $10^6 \sim 10^7 m$ et U de l'ordre de $10^1 \sim 10^2 m.s^{-1}$. La vitesse angulaire de rotation de la terre est $\Omega = 7.3 \times 10^{-5} \simeq 10^{-6} s^{-1}$. L'ordre de grandeur du nombre de Rossby est donc $10^{-1} \sim 10^{-3}$. Il s'agit donc bien d'une quantité petite mais qui est loin d'être négligeable, d'où la nécessité de renforcer par la suite le modèle géostrophique.

2.2.4 Nombre de Burger et stratification du fluide

Le *nombre de Burger* est le nombre qui va mesurer la stratification du fluide. Rappelons ici qu'un fluide est dit stratifié lorsque l'écoulement des particules de fluide est purement planaire et qu'il n'y a pas d'échange de particules de fluide verticalement. C'est à dire que la dérivée lagrangienne D/Dt de la vitesse est nulle dans la direction verticale. Les différentes strates de fluides interagissent seulement par le biais de la poussée d'Archimède. En conséquence, pour qu'une stratification soit stable sur une certaine échelle de temps il faut que la masse volumique ρ du fluide au sein d'une même couche soit pratiquement constante ; la variation caractéristique horizontale de ρ doit être petite par rapport à la variation caractéristique verticale. On compare cette caractéristique à la situation de l'équilibre hydrostatique (immobilité) qui implique en particulier que

$$\frac{\partial p}{\partial z} = -\rho g. \quad (1.32)$$

La variation caractéristique verticale de la masse volumique est $\delta\rho/\rho$. Si on note D l'échelle caractéristique de déplacement vertical, et si on suppose que ρ est constante sur chaque strate, on définit le nombre de Burger par

$$S := g \frac{\delta\rho}{\rho} \frac{D}{\Omega^2 L^2}. \quad (1.33)$$

Ce nombre mesure la stratification du fluide. Lorsque celui-ci est petit, on peut approximer le flot par un écoulement purement horizontal sans échange de matière entre les strates. L'interaction entre deux strates étant alors réduit à la poussée d'Archimède et celle-ci vient faire varier la valeur de ρ à l'intérieur d'une strate. L'ordre de grandeur du nombre de Burger est très variable selon les situations.

2.2.5 Nombre de Rossby pour un écoulement stratifié

Dans le cas d'un écoulement stratifié, la composante verticale de la vitesse est nulle. Dans ce cas, l'influence de l'accélération de Coriolis va dépendre de la latitude⁸. En effet, le plan dans lequel on travaille sera plus ou moins incliné selon que l'on soit entre les tropiques ou vers les pôles. Si ce plan est perpendiculaire à l'axe de rotation de la terre (aux pôles) alors le produit vectoriel $\vec{\Omega} \wedge \vec{v}_R$ sera toujours maximal tandis qu'à l'équateur il peut aller jusqu'à valoir 0 si \vec{v}_R est aligné dans la direction des méridiens. Le nombre de Rossby pour les écoulements stratifiés est alors défini de la manière suivante

$$R_s := \frac{U}{L\Omega \sin(\vartheta_0)}, \quad (1.34)$$

où ϑ_0 est la latitude à laquelle on travaille. Comme les approximations quasi-géostrophiques s'obtiennent dans le régime des nombres de Rossby petits, elles ne sont donc pas utilisées pour prédire la météo entre les tropiques.

2.2.6 Nombre d'Ekman et influence de la viscosité

Le nombre d'Ekman est le nombre qui contrôle si la viscosité du fluide peut être considérée comme négligeable. Le nombre d'Ekman donne le rapport entre les forces de viscosité et la force de Coriolis et s'écrit donc de la manière suivante

$$E := \frac{R_s}{Re} = \frac{\nu}{L^2 \Omega \sin(\vartheta_0)}, \quad (1.35)$$

où $Re := UL/\nu$ est le nombre de Reynolds défini à partir de la viscosité cinématique ν . Lorsque le nombre d'Ekman est petit, les effets de viscosité peuvent être négligés. Ceci est le cas de l'atmosphère terrestre, sauf à proximité du sol où l'on peut voir se former des phénomènes de couches limites (ils sont en général étudiés séparément).

2.2.7 Théorème de Taylor-Proudman

Le théorème de Taylor-Proudman est un résultat de mécanique des fluides pour les référentiels en rotation. Il s'agit d'un phénomène contre-intuitif car absent pour les référentiels inertiels. Le théorème peut s'énoncer de la manière suivante : on suppose que le fluide est incompressible, homogène et non visqueux. On se place dans le régime des nombres de Rossby faibles (domination de l'effet Coriolis). Alors dans ce cas,

$$\vec{\Omega} \cdot \vec{\nabla} \vec{v} = \vec{0}. \quad (1.36)$$

8. la latitude ϑ vaut $\pi/2 - \theta$ où θ est la deuxième coordonnée sphérique.

Cela signifie que \vec{v} est une constante dans la direction de $\vec{\Omega}$. Une conséquence par exemple est que s’il y a un obstacle immobile dans une certaine région \mathcal{V} (une chaîne de montagnes par exemple) alors le fluide est immobile dans toute la colonne de fluide contenue dans le cylindre $\mathcal{V} + \vec{\Omega}\mathbb{R}$. Cette propriété s’observe aisément en laboratoire en utilisant de l’eau comme fluide incompressible non visqueux et en faisant tourner le récipient à bonne allure jusqu’à ce que le nombre de Rossby soit petit (voir figure 1.8). Bien que ce phénomène s’observe dans certains cas pour la dynamique atmosphérique, c’est loin d’être la généralité. Ce résultat nous renseigne donc sur l’importance joué par la stratification et par la compressibilité de l’air dans l’étude des phénomènes atmosphériques.



FIGURE 1.8 – Mise en évidence de l’effet Taylor-Proudman par une équipe du laboratoire de géoscience de l’UCLA. L’obstacle se situe au fond de l’eau et ne fait que 10% de la hauteur d’eau totale. Et pourtant l’eau est immobile dans toute la colonne située au dessus.

2.2.8 Équilibre géostrophique et vent thermique

Après toutes ces considérations et ces éléments de modélisation, nous pouvons aborder la notion d’*équilibre géostrophique*. On travaille dans l’approximation planaire, on suppose que le fluide est rigoureusement stratifié (ρ ne dépend que de l’altitude et pas d’échange de particule de fluide entre les strates), et que les nombres de Rossby et Ekman sont faibles.

La première hypothèse de l’équilibre géostrophique consiste à supposer que la pression reste constante pendant le mouvement. Supposons que le mouvement de l’air dans l’atmosphère est immobile à l’instant $t = 0$. Ce qui va faire se mouvoir le fluide à partir de cet état est exclusivement le gradient de pression puisque la vitesse étant nulle, l’accélération de Coriolis l’est aussi. Le fluide va donc se déplacer depuis les régions de forte pression vers celles de basse pression. Néanmoins, lorsque leur vitesse devient non nulle, la force de Coriolis fait dévier les particules de fluide, vers la droite dans l’hémisphère nord, et vers la gauche dans l’hémisphère sud (vu du dessus). Plus la vitesse de l’air augmente, plus la force de Coriolis augmente en proportion. La déviation engendrée par le terme de Coriolis en est donc d’autant accentuée et ce jusqu’à ce qu’elle atteigne une valeur égale et opposée à celle de la force engendrée par le gradient de pression. On obtient donc un écoulement de l’air qui se fait parallèlement aux lignes isobares (les lignes de niveau de la pression, lignes le long desquelles la pression est constante) et à une vitesse égale au gradient de pression en chaque point. Cet équilibre est l’*équilibre géostrophique*. Les observations montrent que l’écoulement des masses d’air atmosphérique en dehors des tropiques est presque toujours en “quasi-équilibre” géostrophique⁹. L’intérêt du concept d’équilibre géostrophique est qu’il permet de connaître très simplement le sens et la vitesse du vent pour chaque strate par la simple connaissance de la pression atmosphérique. La description est très simple : le vent tourne le long des lignes isobares (voir figure 1.9). Néanmoins, cette description n’est pas adaptée pour des descriptions de plus de quelques heures puisque qu’elle fait l’hypothèse que la pression reste constante au cours du temps. La grande faiblesse de cette description est que malheureusement elle ne fournit aucune information sur la façon dont va évoluer la pression et donc les isobares au cours du temps. L’approximation géostrophique devient donc de plus en plus mauvaise au fur et à mesure que le temps s’écoule et que le profil réel de la pression évolue. On parle dans ce cas de dégénérescence géostrophique.

9. Nous n’utilisons pas ici le terme “quasi-géostrophique” qui est un concept bien défini, voir la section suivante.

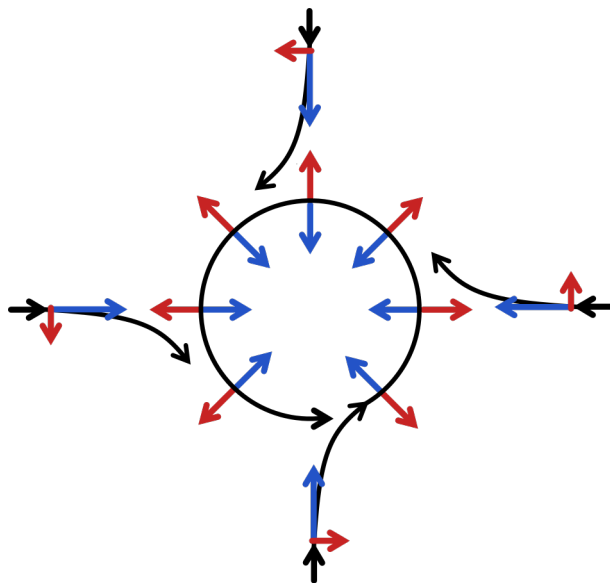


FIGURE 1.9 – Illustration montrant la déviation des vents pour donner une circulation autour d’une dépression le long des lignes de niveau de la pression. La force de gradient de pression est en bleu, celle de Coriolis en rouge et le déplacement en noir. L’équilibre géostrophique est réalisé dans la région centrale.

2.3 Influence de la thermodynamique et équations quasi-géostrophiques

L’équilibre géostrophique est un outil intellectuel puissant pour rendre compte d’une partie des phénomènes apparaissant dans les fluides géophysiques et particulièrement le sens du vent. Néanmoins, il échoue à comprendre comment évolue le système dès que l’on considère des échelles de temps de quelques heures ou davantage. Pour comprendre l’évolution de cet équilibre géostrophique, il nous faut relaxer diverses approximations faites précédemment et particulièrement la stratification. Le deuxième élément qui manque dans l’équilibre géostrophique est la thermodynamique, qu’il nous faut rajouter ici. On parle alors de système quasi-géostrophique, au sens où il s’agit d’un équilibre géostrophique qui évolue avec le temps par l’intermédiaire de la thermodynamique.

2.3.1 Ondes de gravité et fréquence de Brunt-Väisälä

L’idée ici consiste à relâcher l’hypothèse de stratification en autorisant de petits déplacements verticaux pour les particules de fluide. Ces déplacements sont couplés avec des considérations thermodynamiques. On considère une particule de fluide initialement à l’équilibre mécanique et thermique située à une certaine altitude z_0 . On perturbe alors cette particule dans la direction verticale jusqu’à une altitude $z_0 + \delta z$. On suppose que ce déplacement est suffisamment rapide pour qu’il puisse être considéré comme adiabatique, mais suffisamment lent pour que la pression de la particule de fluide puisse s’adapter continuellement à la pression de l’altitude à laquelle elle se trouve. Une fois en $z_0 + \delta z$, puisque le mouvement est adiabatique, la particule de fluide considérée va avoir une masse volumique ρ plus élevée que celle des autres particules de fluides de la couche $z_0 + \delta z$ et va alors être animée par un mouvement descendant. En effet, par la loi des gaz parfaits, si sa température est plus basse à pression identique alors sa masse volumique ρ est plus grande et la particule de fluide va redescendre en vertu du principe d’Archimède. Le mouvement ainsi créé, en l’absence de terme dissipatif, est un mouvement d’oscillation de la particule entre les strates $z_0 - \delta z$ et $z_0 + \delta z$. Ce phénomène d’oscillations au voisinage de l’équilibre entre attraction gravitationnelle et poussée d’Archimède s’appelle *ondes de gravité*¹⁰. Lorsque ce phénomène est observé empiriquement, le déclenchement des oscillations (la perturbation initiale) est le plus souvent lié à la présence d’un obstacle au sol de la taille d’une grosse colline ou d’une montagne (voir figure 1.10). La fréquence de ces oscillations est appelée *fréquence de Brunt-Väisälä* et est notée N .

10. Ne pas confondre avec les ondes gravitationnelles présentes en relativité générale.

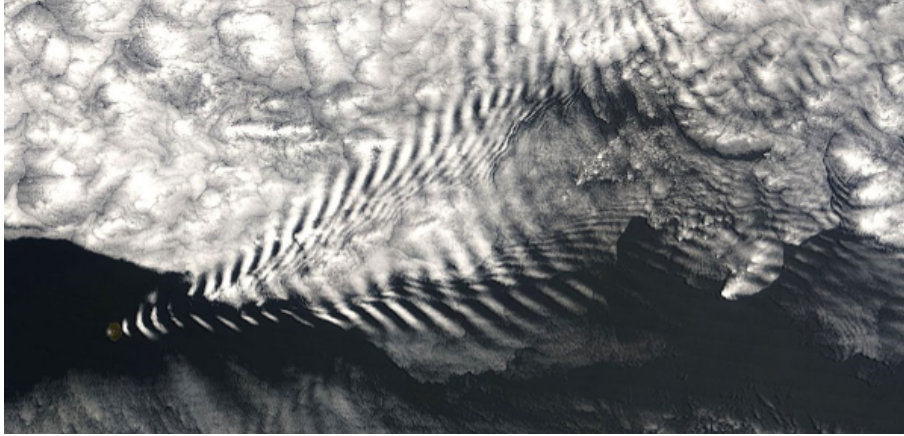


FIGURE 1.10 – Motif nuageux formé par les ondes de gravité en aval de l'Île Amsterdam, une île volcanique de l'Océan Indien.

2.3.2 Notion de température potentielle

La *température potentielle*, parfois appelé *scalaire actif*, d'un fluide quelconque est la température que ce fluide aurait s'il était détendu ou bien comprimé adiabatiquement (sans transferts thermiques avec l'extérieur) jusqu'à atteindre une pression de référence p_0 . Dans le cas de l'étude de l'atmosphère terrestre, on choisit généralement $p_0 = 1000$ hPa. A l'aide de la loi des gaz parfaits (1.30) et des principes de la thermodynamique (1.28)(1.29), on peut montrer que la température potentielle s'écrit

$$\theta = T \left(\frac{p_0}{p} \right)^{\frac{R}{c_p}}. \quad (1.37)$$

On peut également, en utilisant de nouveau la loi des gaz parfaits, écrire la densité du fluide en fonction de la température potentielle.

$$\rho = \frac{p_0}{\theta R} \left(\frac{p}{p_0} \right)^{\frac{c_v}{c_p}}. \quad (1.38)$$

A partir de l'équation ci-dessus, il est alors possible d'exprimer la fréquence de Brunt-Väisälä à l'aide de la température potentielle. Le raisonnement conduit à

$$N^2 = \frac{g}{\theta} \frac{\partial \theta}{\partial z}, \quad (1.39)$$

où g est l'accélération de la pesanteur terrestre. Remarquons que N^2 peut éventuellement être une quantité négative. Dans ce cas-là, les ondes de gravité ont une fréquence imaginaire pure. Cela correspond à une instabilité : deux couches de fluides superposés telles que la couche du dessous soit moins dense que la couche du dessus. Ces situations existent mais elles sont exclues dans le cadre de ce modèle car on reste au voisinage de situations d'équilibre.

L'intérêt remarquable de ce concept de température potentielle est que dans le cadre des ondes de gravité cette quantité est conservée. En effet, ces oscillations se font à des vitesses telles que les transferts thermiques n'ont pas le temps de se faire avant que la particule de fluide ne redescende ou remonte vers la strate initiale. La température de la particule de fluide change à cause de la dilatation ou compression imprimée sur la particule de fluide lors des changements d'altitudes, mais la température potentielle est inchangée.

2.3.3 Les équations quasi-géostrophiques de surface

Les équations quasi-géostrophiques, par rapport au cas géostrophique, consistent à remplacer la contrainte de stratification (absence de déplacement vertical) par la contrainte moins rigide de préservation de la température potentielle. Plus précisément, la théorie montre que la seule connaissance de la stratification de la fréquence de Brunt-Väisälä nous donne suffisamment d'information sur la façon dont les strates de fluide échangent de la matière entre elles mais également comment la thermodynamique fait évoluer l'équilibre géostrophique. En combinant une telle modification avec les différentes approximations de mécanique des fluides présentées précédemment, on aboutit aux équations quasi-géostrophiques

non-visqueuses. Une idée importante de cette dernière étape de la construction consiste à dire que la seule connaissance de la température potentielle à la plus basse strate de fluide est nécessaire car la température potentielle des autres strates se déduit à l'aide de (1.39). Un autre élément est l'utilisation d'une fonction de courant, notée ψ , avec des arguments similaires à ceux utilisés pour les équations d'Euler incompressibles et des manipulations standards sur les fonctions de courant. Les équations obtenues sont alors

$$\begin{cases} \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial}{\partial z} \left(\frac{f_0^2}{N^2} \frac{\partial \psi}{\partial z} \right) = 0, \\ \vec{v} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right), \\ \frac{\theta}{\theta_0} = \frac{f_0}{g} \frac{\partial \psi}{\partial z} \Big|_{z=0}, \\ \frac{\partial \theta}{\partial t} + \vec{v} \cdot \vec{\nabla} \theta = 0, \end{cases} \quad (1.40)$$

où θ_0 désigne une température potentielle de référence reliée à p_0 et où $f_0 = 2\Omega \sin \vartheta_0$ est le paramètre de Coriolis¹¹. Précisons que les deux dernières équations de (1.40), celles qui font intervenir le scalaire actif θ , ne sont posées que dans le plan $z = 0$. Concernant le détail du raisonnement qui permet d'obtenir ces équations à partir de ces éléments de modélisations issus de la mécanique des fluides et de la thermodynamique, nous revoyons aux ouvrages de référence [64] et [83]. Pour étudier ces équations dans la pratique, la fréquence de Brunt-Väisälä $N(z)$ est connue empiriquement et dépend de nombreux facteurs qui caractérisent la masse d'air étudiée. Une fois le type de masse d'air identifié, on peut "résoudre" les équations quasi-géostrophiques (1.40) et on retrouve le comportement mécanique du fluide donné par la vitesse \vec{v} . Concernant les données thermodynamiques du fluide, on connaît la température potentielle θ uniquement en $z = 0$. Pour en déduire la température potentielle pour tout z , on utilise (1.39). Un élément à signaler concernant les ondes de gravité est que leur amplitude croît en sens inverse de leur fréquence N . Ainsi, la limite où la fréquence de $N(z)$ vaut 0 pour tout $z \geq 0$ peut être assimilée au cas physique où il n'y a pas de stratification. On retrouve par exemple le théorème de Taylor-Proudman¹². D'un autre côté, la limite N valant $+\infty$ correspond à une stratification rigoureuse avec absence d'échanges de particules de fluides entre strates. Notons aussi que dans de nombreuses modélisations physiques, la fréquence de Brunt-Väisälä est considérée comme étant constante. La fonction ψ est alors harmonique.

2.3.4 Prédire la météo à l'aide des équations quasi-géostrophiques

Pour prédire la météo avec les équations quasi-géostrophiques, il est en réalité nécessaire de considérer plusieurs masses d'air de caractéristiques différentes (en général de températures potentielles moyennes différentes) chacune évoluant selon les équations quasi-géostrophiques. L'interface entre deux masses d'air est appelé un front météorologique (voir figure 1.11). Les phénomènes se produisant un niveau d'un front sont des phénomènes faisant intervenir de manière directe la variable verticale et où donc le fluide n'est plus stratifié. Il s'y passe notamment l'essentiel des phénomènes de formations des nuages.

Les météorologues distinguent en général 4 principaux types de fronts.

- **Front chaud** : la masse d'air la plus chaude pousse la masse d'air la plus froide et tend à passer par dessus avec formation de nuages nimbostratus (précipitations faibles sur de grandes étendues). Il est représenté en rouge sur les cartes météorologiques.
- **Front froid** : La masse d'air la plus froide pousse la masse d'air la plus chaude et tend à passer par en dessous avec formation de nuages cumulus (précipitation importantes sur de faibles étendues). Il est représenté en bleu sur les cartes météorologiques.
- **Front stationnaire** : Les masses d'air chaud et froid se poussent l'une l'autre le long d'une ligne (en général peu longue) à l'équilibre. C'est au niveau d'un front stationnaire que se forment les dépressions. Il est représenté en rouge et bleu sur les cartes météorologiques.
- **Front oculus** : Un front froid a rattrapé un front chaud¹³. La masse d'air chaud est pincée entre deux masses froides. C'est une zone d'instabilité météorologique complexe à analyser et qui peuvent générer des orages et des précipitations importantes. Il sont représentés en violet sur les cartes météorologiques.

11. Ce paramètre contrôle l'influence de l'accélération de Coriolis en fonction de la latitude ϑ_0 . Dans le cas où on travaille avec une approximation planaire sans terme correctif prenant en compte la courbure de la terre, ce terme est une constante.

12. Si on prend formellement $N = 0$ dans (1.40) alors la première équation implique que $\partial_z \psi = 0$. On dérive ensuite la deuxième équation par rapport à z et on utilise cette dernière identité pour conclure que $\partial_z v = 0$.

13. Il arrive également, plus rarement, qu'un front chaud rattrape un front froid. On parle alors d'oculus chaud.

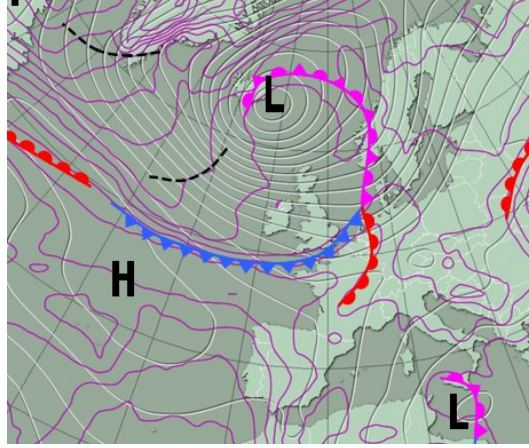


FIGURE 1.11 – Exemple de carte météorologique. On observe au centre la formation d'un front oculus (violet) au niveau de la dépression lorsque le front froid (bleu) a rattrapé le front chaud (rouge).

Ainsi nous avons rassemblé l'essentiel des concepts physiques amenant aux équations quasi géostrophiques et expliqué les grandes lignes de la modélisation la plus basique qui permet de prédire la météo à partir du modèle quasi géostrophique complété par les fronts météorologiques.

3 Etat de l'art sur les équations quasi-géostrophiques surfaciques non visqueuses (SQG)

Maintenant que nous avons une bonne compréhension de l'intérêt de ces équations et des éléments de modélisation physique qui permettent d'y aboutir, nous pouvons nous consacrer à son analyse mathématique.

3.1 Présentation des équations

Nous allons partir de la formulation des équations quasi-géostrophiques (SQG) donnée par les physiciens (1.40). Dans beaucoup de situations, il est possible de supposer que le profil de la fréquence de Brunt-Väisälä $N(z)$ est une fonction constante. Quitte à dilater l'axe vertical d'un facteur constant, on peut se ramener alors au cas où $\frac{f_0}{N^2(z)} = 1$. Dans cette situation, les équations (1.40) s'écrivent

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0, \quad (1.41)$$

$$\theta = \frac{\partial \psi}{\partial z} \Big|_{z=0}, \quad (1.42)$$

$$v = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x} \right), \quad (1.43)$$

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0. \quad (1.44)$$

La deuxième équation dit que le scalaire actif θ en $z = 0$ se déduit de la fonction de courant ψ à l'aide d'un opérateur de trace. Dans une telle situation, il est bien connu par la théorie de la trace et l'analyse standard des opérateurs *Dirichlet to Neumann* que l'on peut remplacer les équations (1.41) et (1.42) ci-dessus par une seule équation dans le plan $z = 0$ donnée par

$$\theta = (-\Delta)^{\frac{1}{2}} \psi. \quad (1.45)$$

On s'est donc ramené à un système de 3 équations (1.43)(1.44)(1.45) dont les 3 inconnues définies dans le plan sont le scalaire actif $\theta : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$, la fonction de courant $\psi : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ et la vitesse du fluide $v : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$. Ce système est généralement appelé "*Équations quasi géostrophiques surfaciques*" par les mathématiciens. Néanmoins, dans le cadre de ce travail, elle sera plutôt désignée par '*Équations*

quasi géostrophiques surfaciques d'exposant $\frac{1}{2}$ " et abrégée en (SQG- $\frac{1}{2}$) en référence à l'exposant $\frac{1}{2}$ sur le laplacien fractionnaire.

Dans le cas général, la fréquence de Brunt-Väisälä N dépend de l'altitude et il n'est pas toujours possible d'appliquer des opérateurs de trace explicites dans le cas d'une fonction N quelconque. Néanmoins, il est souhaitable d'enrichir la théorie à des profils de fréquences de Brunt-Väisälä plus généraux afin de pouvoir saisir une plus large variété de comportements. A défaut de pouvoir considérer une fonction $z \mapsto N(z)$ quelconque, on étudie le cas où ce profil est une fonction puissance avec un exposant proche de 0, correspondant au cas constant. Ce type d'approche est courante en physique afin de disposer d'une large variété de situations sur lesquelles travailler à défaut de les avoir toutes. Dans le cas présent, cette réduction est largement suffisante pour effectuer la plupart des études qualitatives et quantitatives sur les phénomènes physiques étudiés. On suppose donc l'existence d'un η tel que dans les équations (1.40) la quantité $f_0/N^2(z)$ soit remplacée par z^η . Dans ce cas, on va effectuer un changement de variable sur l'axe des z consistant à poser la fonction φ définie à partir de ψ par

$$\psi(x, y, z) = \varphi(x, y, z^{1-\eta}) \quad (1.46)$$

Autrement dit, on effectue le changement de variable consistant à remplacer z par $z^{1-\eta}$. La fonction φ va alors vérifier l'équation

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + z^\alpha \frac{\partial^2 \varphi}{\partial z^2} = 0 \quad (1.47)$$

avec $\alpha = -2\frac{\eta}{1-\eta}$. Le cas où η vaut 0 correspond à une fréquence de Brunt-Väisälä constante. Nous nous intéressons à des situations proches du cas $\eta = 0$ et la théorie mathématique présentée ci-dessous nous permet de considérer les cas où $\eta \in]-1, 1[$. Dans ce cas, on écrit $\eta = 1 - 2s$ avec $s \in]0, 1[$. Pour de tels choix d'exposants, la théorie de Caffarelli-Silvestre [18] permet de généraliser le raisonnement menant à (1.45) et ainsi obtenir à la place le laplacien fractionnaire d'ordre s . Autrement dit, si on remplace l'équation (1.41) par une équation de la forme (1.47), alors cela équivaut, d'après [18], à remplacer l'équation (1.45) par

$$\theta = (-\Delta)^s \psi. \quad (1.48)$$

Ce système (1.43)(1.44)(1.48) sera désigné par la suite comme étant les "*équations quasi-géostrophiques surfaciques*", notées (SQG). Rappelons ici à toutes fins utiles la définition du laplacien fractionnaire pour $u : \mathbb{R}^n \rightarrow \mathbb{R}$,

$$(-\Delta)^s u(x) := p.v. C_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \quad (1.49)$$

où $C_{n,s}$ est une constante de renormalisation et *p.v.* désigne la valeur principale de la singularité du noyau $x \mapsto 1/|x|^{n+2s}$. Il est également possible de le définir en Fourier en tant qu'opérateur pseudo-différentiel

$$\widehat{(-\Delta)^s u}(\xi) = |\xi|^{2s} \widehat{u}. \quad (1.50)$$

Remarquons alors que cette formulation nous permet de concevoir le laplacien fractionnaire comme étant un opérateur de convolution. Une telle formulation est également pertinente pour définir l'opérateur réciproque $(-\Delta)^{-s}$. On obtient alors la fonction de Green du laplacien fractionnaire qui dans \mathbb{R}^n s'écrit

$$G_s(x) := \frac{\widetilde{C}_{n,s}}{|x|^{n-2s}}, \quad (1.51)$$

où $\widetilde{C}_{n,s}$ est une constante de normalisation. Dans le cas des équations d'Euler bi-dimensionnelles, nous avons écrit une loi de Biot et Savart (1.21). Dans le cas des équations quasi-géostrophiques, celle-ci prend la forme suivante

$$v = \nabla^\perp (-\Delta)^{-s} \theta = \nabla^\perp G_s \star \theta. \quad (1.52)$$

Nous ne disposons pas d'un opérateur de trace général pour tous les profils de fréquence de Brunt-Väisälä $N(z)$. Néanmoins, une possibilité pour comprendre des situations assez générales pour des équations prenant cette forme consiste à remplacer (1.48) par une convolution radiale quelconque

$$\psi = G \star \theta \quad (1.53)$$

où G est une fonction radiale. Le système d'équations (1.43)(1.44)(1.53) constitue les "*Équations quasi-géostrophiques surfaciques généralisées*", notées (g-SQG). La très grande généralité pour (g-SQG) n'est qu'apparente car on retrouve une structure identique quel que soit le choix de G :

$$\begin{cases} \frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \\ v = \nabla^\perp G \star \theta. \end{cases} \quad (\text{g-SQG})$$

La vitesse v transporte le scalaire actif θ (première équation) et le scalaire actif fait tourner le fluide autour de lui (deuxième équation). La régularité des solutions à ces équations dépend fortement de la singularité du noyau G en 0.

3.2 Liens avec les équations d'Euler

Les équations quasi-géostrophiques (1.43)(1.44)(1.48) ont des liens mathématiques importants avec les équation d'Euler bi-dimensionnelles mises sous forme vorticité (1.20). En effet, les équations d'Euler bi-dimensionnelles en vorticité ont la même forme que (SQG) avec comme seule différence que le laplacien fractionnaire est un laplacien entier dans le cas d'Euler. Comme nous le développerons plus longuement par la suite et à plusieurs reprises, cette formulation très proche permet d'établir de nombreuses propriétés communes entre ces deux équations. Cette proximité est souvent le point de départ des réflexions sur (SQG). Ainsi, les équations quasi-géostrophiques peuvent être regardées comme étant un ensemble d'équations qui interpolent les équations d'Euler bi-dimensionnelles (cas $s \rightarrow 1^-$) avec le cas stationnaire (lorsque $s \rightarrow 0^+$) [28]. En effet, il est possible sous divers rapport de concevoir le laplacien comme étant la limite des laplaciens fractionnaires [37]. Le passage à la limite $s \rightarrow 0^+$ est plus formel. Si formellement on prend $s = 0$ dans (1.48), cela signifie $\theta = \psi$. Alors (1.43) et (1.44) deviennent

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \quad \text{et} \quad v = \nabla^\perp \theta. \quad (1.54)$$

En observant à présent que $\nabla^\perp \theta \cdot \nabla \theta = 0$, on obtient bien $\frac{\partial \theta}{\partial t} = 0$ et donc $\frac{\partial v}{\partial t} = 0$. De manière plus générale et pour compléter cette analogie, on doit observer que les équations d'Euler bi-dimensionnelles sont en réalité incluses dans la formulation généralisée des équations quasi-géostrophiques (g-SQG). Il suffit pour cela de prendre dans ce cas la fonction G apparaissant dans (1.53) égale au noyau de Green du laplacien dans le plan. Commentons les deux équations de (g-SQG). La première équation est une équation de transport de θ par le fluide évoluant à la vitesse v . Comme v s'écrit comme le gradient tourné d'une certaine quantité, il suit que $\operatorname{div} v = 0$ (conséquence du théorème de Schwarz). Donc, par le théorème de Liouville [4], il s'agit d'une équation de transport incompressible car elle préserve la mesure. La deuxième équation nous dit que les particules de fluide tournent autour des zones de grande vorticité¹⁴. Autrement dit, les particules de fluide tournent en cercle autour du centre des tourbillons et la vitesse de cette rotation se calcule à partir de ∇G . Signalons également que l'équation garde une structure hamiltonienne et à ce titre le théorème de Noether reste valable dans ce contexte.

Si la ressemblance avec les équations d'Euler bi-dimensionnelles est frappante, la proximité avec Euler tri-dimensionnel dans le cas du laplacien $\frac{1}{2}$ mérite d'être soulignée. Effectivement, le noyau de Green du laplacien dans l'espace et du laplacien $\frac{1}{2}$ dans le plan sont de la même forme : l'inverse de la norme euclidienne. En outre, $\nabla^\perp \theta$ satisfait

$$\left(\frac{\partial}{\partial t} + v \cdot \nabla \right) (\nabla^\perp \theta) = \nabla v \cdot \nabla^\perp \theta. \quad (1.55)$$

La proximité avec (1.14) est patente. Ces deux éléments combinés permettent d'établir de nombreux liens entre $\nabla^\perp \theta$ pour (SQG- $\frac{1}{2}$) et ω pour Euler tri-dimensionnel, particulièrement dans la reconstruction du vecteur vitesse v . Lire [25] pour davantage de détails.

3.3 Existence et unicité des solutions

Le premier résultat d'existence et d'unicité a été donné par Constantin, Majda et Tabak en 1994 [25] pour (SQG- $\frac{1}{2}$).

Théorème 3.1 (Existence locale et unicité pour (SQG- $\frac{1}{2}$), Constantin-Majda-Tabak [25], 1994). *Soit $\theta_0 \in H^k(\mathbb{R}^2)$ avec $k \geq 3$ une donnée initiale pour (1.43)(1.44)(1.45). Alors ces équations admettent une unique solution $C_t^0([0, T[, H_x^k(\mathbb{R}^2))$ pour un certain $T > 0$. De plus, T est le temps maximal d'existence de la solution si et seulement si*

$$\int_0^t \|\nabla \theta\|_{L^\infty}(s) ds \longrightarrow +\infty \quad \text{as } t \rightarrow T^-. \quad (1.56)$$

14. L'analogie entre la vorticité ω des équations d'Euler bi-dimensionnelles et le scalaire actif θ pour les équations quasi-géostrophiques est telle que les mathématiciens emploient souvent le mot "vorticité" pour l'ensemble des phénomènes se décrivant par des équations de la forme (g-SQG)

La formulation de ce théorème est très proche du théorème de Beale-Kato-Majda (théorème 1.2). L'argumentaire consiste précisément à utiliser la proximité entre les équations (SQG- $\frac{1}{2}$) et Euler tri-dimensionnel. Une bonne partie des éléments de la preuve du théorème 3.1 reprend ceux du théorème 1.2. Les équations quasi-géostrophiques (1.43)(1.44)(1.48) étant d'autant plus singulières que l'exposant du laplacien est proche de 0, ce théorème implique l'existence et l'unicité locale des solutions régulières pour (SQG) avec $s \geq \frac{1}{2}$. Notons par ailleurs que l'argumentaire développé par Yudovich dans le cas Euler bi-dimensionnel (théorème 1.3) ne peut quant à lui être repris pour obtenir existence et unicité pour des solutions faibles de (SQG). L'existence globale en temps de solutions faibles pour (SQG) a été obtenue par [67] dans l'espace $L^2(\mathbb{R}^2)$. Ces résultats d'existence ont été améliorés significativement en 2012 dans [21]. Dans cet article sont démontrées l'existence locale des solutions régulières pour tout $s \in]0, 1[$, l'existence locale des solutions patches¹⁵ et l'existence globale de solutions faibles dans l'espace des fonctions $L_t^\infty(\mathbb{R}_+, L_x^2(\mathbb{T}^2))$ à moyenne nulle. A notre connaissance, ces résultats n'ont pas connu d'améliorations depuis. Ce même article [21] étudie également les équations quasi-géostrophiques dans le cas où un terme dissipatif est ajouté [26]. Ce terme prenant la forme d'un laplacien fractionnaire est ajouté dans l'équation de transport (1.44) afin de modéliser des effets dissipatifs liés à la viscosité. Voir également [30].

En 2016, Buckmaster, Shkoller et Vicol [15] réutilisent les techniques développées pour les solutions faibles des équations d'Euler tri-dimensionnelles (voir sections 1.2.6 et 1.2.7) pour obtenir des contre-exemples à l'unicité pour les solutions faibles de (SQG). Cette non-unicité s'étend au cas dissipatif. La question de savoir s'il est possible de discriminer entre ces solutions faibles (par des méthodes de solutions de viscosité par exemple) est ouverte. L'existence de solutions aux équations quasi-géostrophiques qui soient non triviales, globales en temps et de régularité C^∞ a quant à elle été donnée par [19] puis [42].

3.4 Formation de front de discontinuité

Ainsi que nous l'avons évoqué à la section 2.3.4, l'étude des équations quasi-géostrophiques par les physiciens est complétée par l'étude de l'évolution de fronts météorologiques. Ceci correspond sur le plan mathématique à un scalaire actif θ qui présenterait une discontinuité, ou bien un fort gradient, le long d'une ligne en mouvement. Ce type d'étude a été introduit pour l'étude des patches pour Euler bi-dimensionnel [86] et peut s'étendre, au moins dans ses principes à l'étude de patch et des fronts de discontinuités pour (SQG). Il est à noter cependant que les démonstrations de nombreux théorèmes connus pour les discontinuités dans Euler bi-dimensionnel en vorticit e ne s'étendent pas au cas (SQG). C'est le cas par exemple de la démonstration du théorème de Chemin [22] qui affirme que si la frontière d'un patch est régulière alors le flot eulérien préserve cette régularité. La question de savoir si ce résultat s'étend à (SQG) est ouverte. L'étude de l'évolution des fronts de discontinuité va de pair avec l'étude de la formation de ces fronts dans les équations. Diverses simulations numériques complétées par des analyses mathématiques attestent de formations spontanées au cours du temps de fronts quasi-géostrophiques [25] [24]. Pour des fronts qui sont des "presques discontinuités" (zones de fort gradient), lire [27].

L'étude numérique de (SQG) fait apparaître des structures comparables à la cascade de Kolmogorov évoquée en section 1.2.7. Ceci est particulièrement visible dans [45] où dans les régions où le gradient de θ devient grand on voit apparaître des zones de turbulences multi-échelles. Pour un résumé des principaux phénomènes observés numériquement dans (SQG), lire [54].

3.5 Vortex et vorticit e pour (SQG)

L'autre phénomène important à étudier est la proximité des équations quasi-géostrophiques avec Euler bi-dimensionnel écrit en vorticit e. En particulier, de nombreuses descriptions connues pour la vorticit e du plan s'étendent moyennant diverses transformations du scalaire actif θ des équations quasi-géostrophiques. Le parall e entre (SQG) et Euler bi-dimensionnel est exploité en ce sens dans [70] où sont écrites les équations d'évolution pour le patch chez (SQG) et est effectué le début de l'analyse dans le cas régulier. Le théorème d'existence locale pour le patch dans (SQG) le plus abouti sera donné par [40] où l'existence du patch est établie localement en temps pour des exposants $s \in [\frac{1}{2}, 1[$ et l'unicité pour $s > \frac{1}{2}$. De m eme, un résultat connu pour Euler bi-dimensionnel est l'existence de patch de vorticit e simplement connexe et possédant une sym etrie d'ordre m [48], c'est-à-dire des patch de vorticit e ressemblant à des polygones réguliers. Un tel résultat a été étendu par [44] au cas (SQG). Pour $m = 6$ on retrouve le fameux hexagone de Saturne (voir figure 1.12). Dans [46] le cas de deux patches est étudié. De m eme que pour Euler bi-dimensionnel, si les deux patches sont identiques et symétriques l'un de l'autre alors ils tournent

15. Un patch est une solution à une équation d'évolution qui prend la forme d'une fonction indicatrice.

l'un autour de l'autre à vitesse constante. Dans la configuration inverse où ils sont de signe opposé, ils se traduisent à vitesse constante orthogonale à la droite reliant leur centre respectif. Une version C^∞ de cette dernière situation a été donnée par [42].

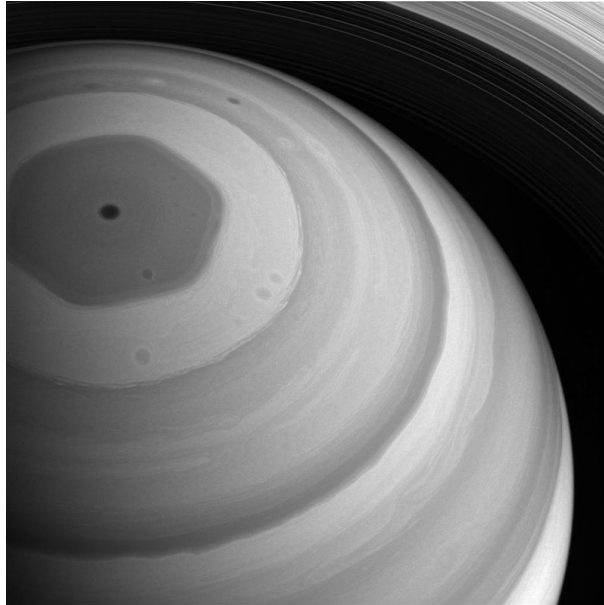


FIGURE 1.12 – L’hexagone de Saturne pris en photo par la sonde Cassini en septembre 2016. Cet hexagone est un gigantesque tourbillon situé au pôle nord de la planète et mesurant 3 fois le diamètre de la terre.

4 Plan de la thèse

L’objectif principal de ces travaux consiste à poursuivre et développer l’étude des liens mathématiques entre les équations quasi-géostrophiques et les équations d’Euler bi-dimensionnelles. Cette thèse est organisée en quatre parties.

4.1 Collision de vortex pour les modèles de Euler et quasi-géostrophique

4.1.1 Modèle de point-vortex pour (SQG)

Le chapitre 2 de cette thèse se propose de développer les liens mathématiques entre les équations de Euler bi-dimensionnel et les équations quasi-géostrophiques en proposant l’extension du modèle des points-vortex connu et étudié pour Euler bi-dimensionnel au cas (SQG). Le modèle des points-vortex consiste à chercher des solutions pour lesquelles ϑ s’écrit sous la forme d’une somme pondérée de masses de Dirac évoluant au cours du temps :

$$\vartheta = \sum_{i=1}^N a_i \delta_{x_i(t)}.$$

L’évolution de la position de la masse de Dirac se fait en vertu de l’équation de transport $\partial_t \vartheta + v \cdot \nabla \vartheta = 0$. En supposant dans le modèle qu’une masse de Dirac n’interagit pas avec elle-même, nous obtenons le système d’équations différentielles

$$\frac{d}{dt} x_i(t) = \sum_{j \neq i}^N a_j \nabla^\perp G(|x_i(t) - x_j(t)|). \quad (1.57)$$

Si on choisit G comme étant le profil radial de la fonction de Green du Laplacien, on retrouve le système des point-vortex pour Euler bi-dimensionnel. L’extension au cas quasi-géostrophique est donc naturelle et consiste à choisir pour G le profil radial du noyau de Green du Laplacien fractionnaire. Nous pouvons également considérer le cas général où G est une fonction de \mathbb{R}_+^* régulière quelconque. Quel que soit le

choix de la fonction G , le système de vortex garde une structure hamiltonienne car ces équations peuvent se réécrire

$$a_i \frac{d}{dt} x_i(t) = \nabla_{x_i}^\perp H(X) \quad \text{avec} \quad H(X) = \sum_{i \neq j} a_i a_j G(|x_i - x_j|). \quad (1.58)$$

En conséquence, les résultats standards de la mécanique hamiltonienne s'appliquent [4].

En plus d'une partie de présentation de cette généralisation du modèle des points-vortex suivi d'une étude du cas où le système est constitué de seulement deux vortex, nous avons démontré les quatre théorèmes qui suivent. Ces théorèmes s'inscrivent dans le cadre plus général de l'étude des collisions de vortex, à savoir les situations où le vecteur vitesse diverge et où donc le système différentiel n'est plus défini.

4.1.2 Théorème de la borne uniforme

Dans leur important travail sur les point-vortex eulérien, Marchioro et Pulvirenti [56, §4] se sont plus particulièrement intéressés aux systèmes pour lesquels les intensités a_i des vortex vérifient l'hypothèse de *non-neutralité des clusters* :

$$\forall A \subseteq \{1 \dots N\} \quad \text{tel que} \quad A \neq \emptyset, \quad \sum_{i \in A} a_i \neq 0. \quad (1.59)$$

Dans le cadre de cette hypothèse ils y établissent un résultat de borne uniforme sur les trajectoires. L'idée est que sous une telle hypothèse, le centre de vorticité de chaque cluster est bien défini. Or le centre de vorticité est un invariant du mouvement et on s'attend à ce que les vortex d'un cluster donné tournent autour de leur centre (éventuellement à très grande vitesse) et donc ne seront pas éjectés au loin. En reprenant les idées de leur preuve, nous avons pu améliorer ce résultat et obtenir :

Théorème 4.1 (Borne uniforme sur les trajectoires, L.G-C., 2020). *Soit $J : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ telle que $J(r) \rightarrow 0$ pour $r \rightarrow +\infty$. On suppose que les profils G satisfont l'hypothèse de décroissance suivante :*

$$\forall r \geq 1, \quad \left| \frac{dG}{dr}(r) \right| \leq J(r). \quad (1.60)$$

On considère également l'hypothèse de *non-neutralité des clusters* (1.59). Soit $T > 0$.

Alors il existe une constante C telle que pour tout profil G satisfaisant (1.60) et pour tout donnée initiale $X \in \mathbb{R}^{2N}$ qui n'aboutit pas à une collision sur l'intervalle de temps $[0, T)$, on a

$$\sup_{t \in [0, T)} |X - S_G^t X| \leq C, \quad (1.61)$$

où S_G^t est le flot de l'équation (1.58) associé au noyau dont le profil radial est G . Cette constante C dépend de N , des intensités a_i , du temps final T et du choix de la fonction J .

Contrairement à l'intuition, la borne ne dépend pas de la singularité du noyau G en 0 alors que c'est précisément cette singularité qui nous renseigne sur la divergence de la norme du vecteur vitesse au voisinage des collisions.

4.1.3 Improbabilité des collisions pour les points-vortex

Un autre aspect qui intervient dans le cadre de l'étude des collision est la question de la probabilité qu'une collision se produise dans le cadre d'une dynamique hamiltonienne. Rappelons que la dynamique hamiltonienne préserve la mesure (Théorème de Liouville) et donc un ensemble "*petit*" (par exemple l'ensemble des collisions) reste petit au cours du temps. Nous pouvons formuler la conjecture suivante :

Conjecture 1. *L'ensemble des données initiales $X \in \mathbb{R}^{2N}$ qui aboutissent à une collision au bout d'un certain temps T_X est un ensemble de mesure de Lebesgue \mathcal{L}^{2N} égale à 0.*

Cette conjecture est à ce jour toujours ouverte. Elle a été notamment étudiée dans [56, §4] où elle a été résolue par la positive sous l'hypothèse d'une borne uniforme sur les trajectoires (hypothèse en lien avec le théorème précédent). Nous avons pu reprendre leurs arguments et les améliorer pour obtenir un résultat un peu plus général :

Théorème 4.2 (Improbabilité des collisions de vortex pour Euler et (SQG), L.G-C., 2020). *Considérons le modèle des points-vortex (1.58) avec le noyau G_s , la fonction de Green du Laplacien ou du laplacien fractionnaire.*

Alors l'ensemble des données initiales $X \in \mathbb{R}^{2N}$ qui aboutissent à une collision en un temps $T_X > 0$ et telles que

$$\limsup_{t \rightarrow T_X^-} \max_{i \neq j} |x_i(t) - x_j(t)| < +\infty \quad (1.62)$$

est un ensemble de mesure nulle.

Ce théorème ne clôt pas la conjecture car nous avons recours à l'hypothèse (1.62) mais améliore néanmoins le résultat précédemment connu sous 3 aspects. D'abord, la borne présente dans (1.62) n'est plus une borne uniforme (on a remplacé une hypothèse de type “ $\leq \text{constante}$ ” par “ $< +\infty$ ”). Ensuite, cette nouvelle hypothèse nécessite que les vortex restent bornés les uns par rapport aux autres (mais peuvent éventuellement diverger collectivement) alors qu'une hypothèse de borne absolue (distance à 0 bornée) était nécessaire dans la formulation précédente. Enfin, notre démonstration permet d'étendre le résultat au cas quasi-géostrophique.

4.1.4 Convergence des vortex eulériens sous l'hypothèse de non-neutralité des clusters

Lorsque les vortex vont jusqu'à une collision à un certain temps $T_X > 0$, leur vitesse diverge et par conséquent tout comportement pathologique devient *a priori* possible. Seule la continuité sur $[0, T_X)$ est assurée. Nous avons pu prouver que, sous l'hypothèse de non-neutralité des clusters et dans le cas spécifique des vortex eulériens, les trajectoires sont en réalité convergentes.

Théorème 4.3 (Convergence pour les vortex Eulérien, L.G-C., 2020). *On travaille avec le système des points vortex sous l'hypothèse de non-neutralité des clusters (1.59) muni du noyau G_1 correspondant au noyau de Green du laplacien dans le plan. Soit $X \in \mathbb{R}^{2N}$ une donnée initiale aboutissant à une collision au temps T_X . Alors pour tout $i = 1 \dots N$, il existe $x_i^* \in \mathbb{R}^2$ tel que*

$$x_i(t) \longrightarrow x_i^* \quad \text{as } t \rightarrow T_X^-. \quad (1.63)$$

La question de savoir si ce résultat de convergence s'étend aux vortex quasi-géostrophiques est une question ouverte.

4.1.5 Collisions Mono and multi-échelles pour les points-vortex eulériens

L'idée derrière la notion de mono-échelle consiste à comparer la dynamique des points-vortex eulériens, pour lesquels la vitesse se comporte comme l'inverse des distances, avec le système jouet unidimensionnel $\dot{x} = -\frac{\alpha}{x}$. Pour le système jouet, la solution s'écrit $x(t) = \sqrt{2\alpha(T-t)}$ et “entre en collision” avec 0 au temps T .

Théorème 4.4 (Collisions mono et multi-échelles pour les vortex eulériens, L.G-C., 2020). *On travaille avec le système des points vortex sous l'hypothèse de non-neutralité des clusters (1.59) muni du noyau G_1 correspondant au noyau de Green du laplacien dans le plan. Soit $X \in \mathbb{R}^{2N}$ une donnée initiale aboutissant à une collision au temps T_X . On suppose qu'il existe des clusters $P_l \subseteq \{1 \dots N\}$ avec $l = 1 \dots L$ deux-à-deux disjoints tels que les hypothèses de collision mono-échelle suivantes soient vérifiées pour tout $k \neq l$:*

- $\max_{i,j \in P_l} |x_k(t) - x_j(t)| \longrightarrow 0^+ \quad \text{lorsque } t \rightarrow T_X^-,$
- $\min_{i \in P_k} \min_{j \in P_l} |x_k(t) - x_j(t)| \geq c > 0,$
- $\max_{i,j \in P_l} |x_i(t) - x_j(t)| \leq C \min_{\substack{i,j \in P_l \\ i \neq j}} |x_i(t) - x_j(t)|.$

(i) *Alors chaque cluster de collision P_l satisfait la condition algébrique des clusters équilibrés :*

$$\left(\sum_{i \in P_l} a_i \right)^2 = \sum_{i \in P_l} a_i^2.$$

(ii) *Pour tout $i = 1 \dots N$, il existe $x_i^* \in \mathbb{R}^2$ tel que pour t suffisamment proche de T_X ,*

$$|x_i(t) - x_i^*| \leq C \sqrt{T_X - t}. \quad (1.64)$$

(iii) *Dans la situations analogue avec des collisions multi-échelles, il est possible d'établir la condition algébrique (4.4) dans sa version généralisée (Voir chapitre 2 pour les détails).*

Le point (i) du théorème est un résultat standard dans le cas des vortex pour Ginzburg-Landau [73, 13, 12] et est largement utilisé pour étudier les collisions de vortex dans ce cadre. Les vortex pour Ginzburg-Landau partagent avec les points-vortex pour Euler le même hamiltonien :

$$H(x) := \sum_{i \neq j} a_i a_j \log(|x_i - x_j|). \quad (1.65)$$

La différence est, entre-autres choses, que pour Ginzburg-Landau on considère le flot-gradient. Le comportement au niveau des collision reste assez proche. L'idée de la preuve dans les grandes lignes est que, si on a $|x_i - x_j| \simeq \varepsilon$, alors la préservation du hamiltonien implique que $\sum_{i \neq j} a_i a_j = 0$. Cette relation algébrique est équivalente à $(\sum_i a_i)^2 = \sum_i a_i^2$. On raisonne alors cluster par cluster pour obtenir la formulation donnée par le théorème ci-dessus. Pour le multi-échelles, si une distance $|x_i - x_j|$ est de l'ordre de ε alors elle va contribuer à la relation algébrique avec le terme $a_i a_j$ car $\log(\varepsilon)$ diverge pour $\varepsilon \rightarrow 0^+$. Si en revanche une distance est de l'ordre de ε^2 alors son logarithme est de l'ordre de $2\log(\varepsilon)$ et par conséquent elle contribue à la relation algébrique avec le terme $2a_i a_j$. Une formalisation générale de cette idée donne la relation algébrique des collisions multi-échelles.

4.2 Désingularisation du collier de vortex pour (SQG)

4.2.1 Présentation du problème

La question de la désingularisation du système de points-vortex est nettement plus délicate que pour Euler bi-dimensionnel car nous ne disposons pas d'un équivalent du théorème de Yudovich (théorème 1.3). Cependant, dans le cas du collier de vortex il est possible de procéder à une désingularisation à l'aide de solutions particulières des équations (SQG). Le collier de vortex est le système de formé de N points-vortex equi-répartis sur un cercle (c'est-à-dire qu'ils forment les sommets d'un polygone régulier d'ordre N) et qui tournent à vitesse constante autour du centre de ce cercle. Dans le cas où $N = 2$, on retrouve le système formé d'une paire de vortex identiques en co-rotation. La désingularisation d'un tel système prend la forme de N patches identiques (à rotation près) qui sont très concentrés autour de leur centre respectifs, et ces centres sont equi-répartis sur un cercle et tournent à vitesse constante autour du centre de ce cercle.

Le travail présenté dans cette section a été réalisée en collaboration avec mes directeurs de thèse Philippe Gravejat et Didier Smets.

Redonnons ici les équations quasi-géostrophiques surfaciques non visqueuses :

$$\begin{cases} \frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \\ \theta = (-\Delta)^s \psi, \\ v = \nabla^\perp \psi, \end{cases} \quad (1.66)$$

pour $s \in]0, 1[$. Les inconnues sont la vitesse $v : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$, le scalaire actif $\theta : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ et la fonction de courant $\psi : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$.

Dans le cadre des équations de Euler bi-dimensionnelles, la désingularisation du collier de vortex a été réalisée par B. Turkington [82] à partir de ses travaux sur la paire de vortex en translation [80, 81]. Les constructions présentées ici s'inspirent de ses travaux bien que de nombreux arguments valables pour Euler ne sont pas utilisables ici pour des raisons qui viennent à la fois de régularité insuffisante des solutions et du caractère non-local du laplacien fractionnaire.

4.2.2 Énoncé du théorème

Puisque l'on cherche des vortex en rotation constante, l'approche la plus naturelle consiste à chercher des solutions qui sont stationnaires dans un certain référentiel en rotation. Pour cela, si α est une vitesse angulaire, on cherche des solutions qui se mettent sous la forme

$$\theta(t, x) =: \omega(R_{\alpha t} x) \quad (1.67)$$

avec $\omega : \mathbb{R}^2 \rightarrow \mathbb{R}$ et où R_ϕ désigne la rotation du plan d'angle ϕ . A partir de (1.66) on obtient sur ω l'équation suivante

$$\nabla \omega(x) \cdot \nabla^\perp \left(G_s \star \omega(x) + \frac{\alpha}{2} |x|^2 - \mu \right) = 0, \quad (1.68)$$

où μ est une constante d'intégration et où G_s est la fonction de Green du laplacien fractionnaire dans le plan définie par (1.49). Nous cherchons des solutions qui prennent la forme de patches :

$$\omega_\lambda(x) = \lambda \mathbb{1}_S(x), \quad (1.69)$$

pour $\lambda > 0$ un paramètre et $S \subseteq \mathbb{R}^2$ ouvert compact. En particulier, $\omega_\lambda \in L^p(\mathbb{R}^2)$ pour tout $p \in [1, +\infty]$ mais nous ne savons pas si cette fonction a une meilleure régularité. Pour un patch compact dont la frontière est lipschitzienne, sa régularité est $\dot{W}^{\frac{1}{p}-\delta, p}$ pour tout $\delta > 0$ et avec $p \in]1, +\infty[$. Dans la construction de Turkington pour le cas eulérien on obtient une frontière \mathcal{C}^1 par un argument faisant intervenir le théorème des fonctions implicites. Les équations quasi-géostrophiques ne sont pas assez régulières pour réutiliser cet argument. Ainsi, on a seulement $\omega_\lambda \in L^p$ et par conséquent, par la théorie standard pour les opérateurs elliptiques fractionnaires, $G_s \star \omega_\lambda \in \dot{W}^{2s, p}$. La formulation faible de (1.68) s'écrit (avec φ une fonction test)

$$\int_{\mathbb{R}^2} \omega(x) \nabla \varphi(x) \cdot \nabla^\perp \left(k_s \star \omega(x) + \frac{\alpha}{2} |x|^2 - \mu \right) dx = 0. \quad (1.70)$$

Le régularité est telle que cette formulation faible n'est *a priori* pas définie si $s < \frac{1}{2}$. Néanmoins, dans le cas particulier de patches, il est tout de même possible de donner un sens à une notion de solution faible étendue à des s petits.

Pour obtenir la désingularisation du collier de vortex, il est naturel d'imposer que les N patches aient la même intensité λ et que leur centre soit situé aux sommets respectifs d'un polygone régulier à N sommets. On impose également la condition de symétrie d'ordre N

$$\omega(x) = \omega\left(R_{\frac{2\pi}{N}} x\right), \quad (1.71)$$

pour tout $x \in \mathbb{R}^2$. Cela nous permet en particulier de restreindre l'analyse mathématique au secteur d'angle

$$\mathfrak{S}_N := \left\{ (r \cos(\theta), r \sin(\theta)) \in \mathbb{R}^2 : -\frac{\pi}{N} < \theta < \frac{\pi}{N} \right\},$$

puis de compléter par symétrie. En outre, nous allons utiliser les propriétés du réarrangement de Steiner pour la variable angulaire. Nous effectuons cette symétrisation de Steiner par rapport à $\theta = 0$ dans le secteur d'angle \mathfrak{S}_N . Cela se définit pour une fonction $\theta \mapsto \omega(r, \theta)$ comme étant l'unique fonction paire telle que

$$\omega^\sharp(r, \theta) > \nu \quad \text{si et seulement si} \quad |\theta| < \frac{1}{2} \text{meas} \left\{ \theta' \in \left] -\frac{\pi}{2}, \frac{\pi}{2} \right[: \omega(r, \theta') > \nu \right\},$$

pour tout $r > 0$ et $\nu > 0$, et pour $-\pi/N < \theta < \pi/N$. Une fonction ω est Steiner-symétrique angulaire lorsque

$$\omega(r, \theta) = \omega^\sharp(r, \theta), \quad (1.72)$$

où (r, θ) , avec $r > 0$ et $-\pi/N < \theta < \pi/N$, désignent les coordonnées polaires. Cette condition de symétrie (1.72) est importante dans la construction variationnelle des N patches. On note $L_{\text{sym}}^p(\mathbb{R}^2)$ le sous ensemble des fonctions de $L^p(\mathbb{R}^2)$, qui satisfont (1.71) et (1.72). Le théorème démontré est le suivant.

Théorème 4.5 (Collier de vortex quasi-geostrophique, L. G.-C., P. Gravejat, D. Smets, (2019)). *Soit λ un réel positif suffisamment grand. Alors il existe une vitesse angulaire α_λ et une fonction de vorticité $\omega_\lambda \in L_{\text{sym}}^\infty(\mathbb{R}^2)$, qui satisfont les propriétés suivantes*

(i) *La vorticité ω_λ est formé de patches de vorticité d'intensités λ au sens où il existe μ_λ tel que*

$$\omega_\lambda = \lambda \mathbb{1}_{\{\psi_\lambda > 0\}}, \quad (1.73)$$

avec

$$\psi_\lambda(x) := G_s \star \omega_\lambda + \frac{\alpha_\lambda}{2} |x|^2 - \mu_\lambda, \quad (1.74)$$

où G_s est la fonction de Green du laplacien fractionnaire dans le plan.

(ii) *Si $1/2 \leq s < 1$, alors ω_λ est un solution faible de (1.68), au sens donné par (1.70), pour les paramètres α_λ et μ_λ .*

(iii) *Soit $S_\lambda := \{\psi_\lambda > 0\}$ le support de ω_λ . Il existe R , indépendant de λ tel que*

$$S_\lambda \cap \mathfrak{S}_N \subseteq B\left((1, 0), \frac{R}{\sqrt{\lambda}}\right).$$

en particulier S_λ a au moins N composantes connexes.

(iv) À la limite $\lambda \rightarrow \infty$, on a

$$\omega_\lambda \longrightarrow \sum_{n=0}^{N-1} \delta_{R_{\frac{2n\pi}{N}}(1,0)},$$

au sens faible des mesures. Pour $1/2 \leq s < 1$ on peut calculer la limite de la vitesse angulaire.

$$\alpha_\lambda \longrightarrow \sum_{n=1}^{N-1} \frac{C_s(1-s)}{|(1,0) - R_{\frac{2n\pi}{N}}(1,0)|^{2(1-s)}},$$

où C_s est la constante de renormalisation intervenant dans la définition de la fonction de Green du laplacien fractionnaire (1.49). Enfin, lorsque $s \rightarrow 1^-$, la limite obtenue ci-dessus converge vers la vitesse angulaire du collier de vortex eulérien.

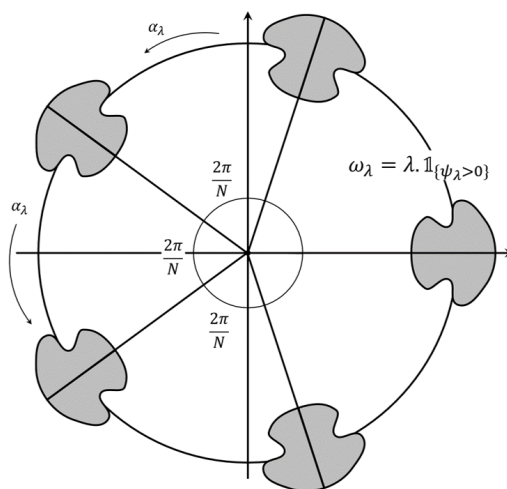


FIGURE 1.13 – Illustration des patches de vortex construits pour le théorème 4.5. Il s’agit d’une représentation de la fonction ω_λ dans le cas $N = 5$.

4.2.3 Principales idées de la démonstration

Nous allons présenter les grandes lignes de la démonstration de ce théorème d’existence. Le début de la démonstration s’inspire des idées développées par Arnold [4] et consiste à réécrire le problème sous la forme d’une maximisation sous contraintes. On définit d’abord l’énergie fractionnaire du fluide :

$$E_s(\omega) := \frac{1}{2} \int_{\mathbb{R}^2} \omega(x) G_s \star \omega(x) dx, \quad (1.75)$$

sa circulation,

$$C(\omega) := \int_{\mathbb{R}^2} \omega(x) dx, \quad (1.76)$$

et sa quantité de mouvement

$$L(\omega) := \int_{\mathbb{R}^2} |x|^2 \omega(x) dx. \quad (1.77)$$

Ainsi l’équation (1.68) est l’équation d’Euler-Lagrange associée à

$$\sup \left\{ E_s(\omega) : \omega : \mathbb{R}^2 \rightarrow \mathbb{R}_+ \text{ s.t. } C(\omega) = L(\omega) = N \right\}. \quad (1.78)$$

Les nombres α et μ sont alors les multiplicateurs de Lagrange associés respectivement à $L(\omega) = N$ et $C(\omega) = N$. L’énergie E_s est définie sur $\dot{H}^{-s}(\mathbb{R}^2)$, la circulation C sur $L^1(\mathbb{R}^2)$ et la quantité de mouvement L sur l’espace de Lebesgue à poids $L^1(\mathbb{R}^2, |x|^2 dx)$.

Pour éviter d'avoir à travailler avec cet espace fonctionnel compliqué, l'idée développée par Turkington [82] consiste à travailler dans un secteur d'angle réduit et dans une couronne :

$$S := \left\{ (r \cos(\theta), r \sin(\theta)) : a_0 \leq r \leq a_1 \text{ and } -\frac{\pi}{2N} \leq \theta \leq \frac{\pi}{2N} \right\},$$

pour $0 < a_0 < 1 < a_1$. On complète ensuite S par la symétrie d'ordre N .

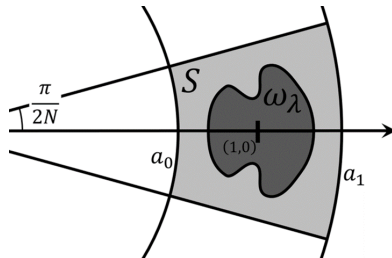


FIGURE 1.14 – Illustration du domaine de travail S .

Cette restriction permet de réduire le problème à des espaces fonctionnels plus standards, de travailler sur un compact indépendant des paramètres et de bien séparer les uns des autres les différents patches approximant chaque vortex. L'intuition derrière cette restriction est que l'on s'attend à ce que ω_λ , dans la limite $\lambda \rightarrow +\infty$, ait son support qui soit concentré dans N boules centrées respectivement aux N sommets $x_n = (\cos(2n\pi/N), \sin(2n\pi/N))$. L'idée consiste alors à résoudre le problème en restreignant l'analyse aux fonction à support dans S , puis, dans un deuxième temps, on démontre que la solution ω_λ vérifie

$$\inf \left\{ |x - y| : x \in \text{supp}(\omega_\lambda) \text{ et } y \in \partial S \right\} > 0. \quad (1.79)$$

Cette condition implique que si on prolonge la solution ω_λ par 0 en dehors de S on obtient bien une solution au problème initial car alors les formulations faibles du problème initial et du problème restreint aux fonctions à support dans S coïncident.

La construction de la solution dans S se fait en deux étapes. Dans un premier temps, on travaille avec l'énergie pénalisée

$$E_{s,p}(\omega) := E_s(\omega) - \frac{\lambda N}{p} \int_S \left(\frac{\omega}{\lambda} \right)^p. \quad (1.80)$$

Le problème de maximisation sous contraintes dans S associé à l'énergie pénalisée donne des solutions qui s'écrivent sous la forme suivante :

$$\omega_{\lambda,p} = \lambda (\psi_{\lambda,p})_+^{\frac{1}{p-1}}, \quad (1.81)$$

et

$$\psi_{\lambda,p} := K_s \omega_{\lambda,p} + \frac{\alpha_{\lambda,p}}{2} |\cdot|^2 - \mu_{\lambda,p}, \quad (1.82)$$

où l'indice $+$ désigne la partie positive définie par $x_+ := \max(x, 0)$. Dans un deuxième temps, la solution patch est obtenue en passant à la limite L^∞ faible- \star dans S pour la variable p après s'être assuré que les multiplicateurs de Lagrange $\alpha_{\lambda,p}$ et $\mu_{\lambda,p}$ restaient bornés. On démontre que l'on converge vers des patches en utilisant le fait que la fonction $x \in \mathbb{R} \mapsto x_+^a$ converge ponctuellement lorsque $a \rightarrow 0^+$ vers la fonction $x \mapsto \mathbb{1}_{\mathbb{R}_+}(x)$.

Pour conclure le théorème d'existence, c'est-à-dire établir (1.79), il est suffisant de démontrer le point (iii) du théorème dans S et de choisir ensuite λ assez grand. Pour cela on utilise la définition de la fonction de Green du laplacien fractionnaire (1.49) pour réécrire l'énergie comme suit :

$$E_s = \frac{C_{2,s}}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\omega(x) \omega(y)}{|x - y|^{2(1-s)}} dx dy. \quad (1.83)$$

La fonction G_s est grande lorsque x est proche de y . Ainsi, lorsque l'on maximise l'énergie, on favorise les situations où ω est concentré autour de quelques points. C'est cette idée qui guide toute la démonstration du point (iii) du théorème. Les principaux arguments sont des inégalités de réarrangements couplées avec les propriétés de l'équation établie dans S .

Une présentation plus précise de ce problème et la démonstration détaillée du théorème sont données au chapitre 3 de cette thèse.

4.3 La paire de vortex \mathcal{C}^∞ en translation

Un autre système de points-vortex souvent étudié est celui constitué de deux vortex d'intensités opposées. Ils sont alors animés d'un mouvement de translation rectiligne uniforme dans la direction de leur médiatrice. Ceci se produit aussi bien pour Euler bi-dimensionnel que pour (SQG). Dans l'article [42] est étudié ce système pour (SQG- $\frac{1}{2}$) où le scalaire actif est \mathcal{C}^∞ . La désingularisation n'y est en revanche pas traitée. Le travail présenté dans cette troisième partie de la thèse consiste à étendre ce résultat au laplacien fractionnaire d'exposant quelconque (SQG).

Puisque l'on étudie un système où le scalaire actif est animé d'un mouvement de translation rectiligne uniforme, il est plus aisé de travailler dans un référentiel en translation à vitesse c donnée et d'y chercher des solutions stationnaires. Le cas d'une paire de vortex en translation correspond à deux vortex d'intensités opposées, par exemple une masse de Dirac d'intensité $+1$ initialement au point $(1, 0)$ et une masse de Dirac d'intensité -1 au point $(-1, 0)$. La recherche de solutions \mathcal{C}^∞ en translation constante se fait par analogie avec ce qui se passe pour les deux vortex en translation. Cela correspond à une fonction θ impaire par rapport à la première variable. Les fonctions H^s satisfaisant cette condition de symétrie forment l'ensemble H_{sym}^s . L'idée de la construction consiste en une approche variationnelle. Il s'agit d'une recherche d'un point-col en dimension infinie à l'aide d'une variété de Nehari. Un contrôle sur la géométrie du support de θ donne alors la compacité des suites minimisantes. En utilisant le réarrangement de Steiner et l'inégalité de réarrangement de Riesz, on peut montrer que la solution est symétrique-décroissante pour la deuxième variable.

Soit c et k strictement positifs. Soit $f : \mathbb{R} \rightarrow \mathbb{R}$ de classe \mathcal{C}^∞ telle que

- $f|_{\mathbb{R}_-} = 0$ et $f|_{\mathbb{R}_+^*} > 0$,
- $\exists \nu \in]1, \frac{1+s}{1-s}[$, $\forall \xi \geq 0$, $f(\xi) \leq C\xi^\nu$.
- $\exists \mu \in]1, \nu[$, $\forall \xi$, $\mu f(\xi) \leq \xi f'(\xi)$.

On pose $F(\xi) := \int_0^\xi f(\xi') d\xi'$ et on considère la fonctionnelle d'énergie suivante

$$E(\Psi) := \frac{1}{2} \int_{\mathbb{R}^2} \Psi(-\Delta)^s \Psi - \int_{\mathbb{R}_+ \times \mathbb{R}} F(\Psi - cx - k) + \int_{\mathbb{R}_- \times \mathbb{R}} F(\Psi + cx - k). \quad (1.85)$$

La variété de Nehari associée à l'énergie E est

$$\mathcal{N} := \left\{ \Psi \in H_{sym}^s \setminus \{0\} : E'(\Psi)(\Psi) = 0 \right\}. \quad (1.86)$$

Le théorème démontré, à partir de [42], est le suivant.

Théorème 4.6 (Existence d'ondes voyageantes \mathcal{C}^∞ solutions aux équations quasi-géostrophiques, L. G.-C., 2017). *L'énergie E admet un minimiseur $\Psi = 0$ sur \mathcal{N} . En conséquence il existe une solution régulière non-triviale θ aux équations quasi-géostrophiques (1.66) qui prend la forme suivante : pour tout $x, y \in \mathbb{R}^2$,*

$$\theta(x, y, t) = \Theta(x, y - ct) \quad \text{et} \quad \psi(x, y, t) = \Psi(x, y - ct). \quad (1.87)$$

Les fonctions Θ et Ψ sont reliés par la relation

$$\Theta(x, y) = f\left(\Psi(x, y) - cx - k\right), \quad (1.88)$$

pour tout $y \in \mathbb{R}$ et pour tout $x \geq 0$. La fonction Θ satisfait en outre les relations de symétrie $\Theta(x, y) = -\Theta(-x, y) = \Theta(x, -y)$, pour tout x, y .

Enfin, la restriction de Θ aux $x \geq 0$ est positive, à support compact et décroissante par rapport à la variable y lorsque $y \geq 0$.

Ce théorème est présenté en détail et démontré au chapitre 4 de cette thèse.

4.4 Le tassement, un réarrangement un dimension 1

Dans le cadre de ces deux travaux précédents, une des principales difficultés à laquelle nous sommes confrontés est celle de décrire la géométrie des zones de vortécité des solutions étudiées. Nous ne parvenons pas par exemple à montrer la connexité du support du scalaire actif pour la solution obtenue à la section précédente. C'est dans ce contexte qu'a été développé le réarrangement par tassement. Ce réarrangement n'est à l'heure actuelle défini qu'en dimension 1. L'objectif est d'obtenir un réarrangement qui partage de nombreuses propriétés communes avec le fameux réarrangement décroissant de Schwarz mais qui, contrairement à celui de Schwarz, préserve la condition au bord de Dirichlet en 0.

4.4.1 Généralités sur les réarrangements

Soit Ω un domaine de \mathbb{R}^d . On note $\{\varphi \geq \nu\}$ l'ensemble de sur-niveau de la fonction $\varphi : \Omega \rightarrow \mathbb{R}_+$ pour $\nu \geq 0$ défini par $\{x \in \Omega : \varphi(x) \geq \nu\}$ (autrement dit : l'image réciproque de $[\nu, +\infty[$ par f). Une application T qui envoie $\mathfrak{M}_+(\Omega)$, l'ensemble des fonctions mesurables positives, dans lui-même est appelé un *réarrangement* lorsque

$$\forall \nu \geq 0, \quad \mathcal{L}^d\{\varphi \geq \nu\} = \mathcal{L}^d\{T\varphi \geq \nu\}, \quad (1.89)$$

avec \mathcal{L}^d la mesure de Lebesgue de \mathbb{R}^d . Rappelons ici la *représentation par tranches* pour une fonction positive mesurable [55],

$$\varphi(x) = \int_0^{\varphi(x)} d\nu = \int_0^{+\infty} \mathbb{1}_{\{\varphi \geq \nu\}}(x) d\nu, \quad (1.90)$$

où $\mathbb{1}_A$ est la fonction indicatrice de l'ensemble A . Cette représentation implique que pour tout $f \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ croissante telle que $f(0) = 0$,

$$\int_{\Omega} f \circ \varphi(x) dx = \int_0^{+\infty} f'(\nu) \text{meas}(\{\varphi \geq \nu\}) d\nu. \quad (1.91)$$

Dans l'égalité ci-dessus, il est possible en réalité de baisser la régularité de f jusqu'aux fonctions $BV(\mathbb{R}_+)$ et dans ce cas la dérivée f' est une mesure (pas nécessairement positive). Cette égalité implique en particulier que pour $p \in [1, +\infty[$,

$$\|\varphi\|_{L^p}^p = p \int_0^{+\infty} \nu^{p-1} \text{meas}(\{\varphi \geq \nu\}) d\nu. \quad (1.92)$$

Par conséquent, une propriété importante pour un réarrangement T est qu'il préserve toutes les normes L^p pour $p \in [1, +\infty]$ (le cas $p = +\infty$ se démontre séparément) et plus généralement toutes les intégrales de φ de la forme $\int f \circ \varphi$.

En guise d'illustrations, des exemples de réarrangements élémentaires pour $\Omega = \mathbb{R}^d$ peuvent être fabriqués de la manière suivante. Soit $v \in \mathbb{R}^d$ et $M \in \mathcal{M}_d(\mathbb{R})$ une matrice telle que $\det(M) = \pm 1$. Si pour $\varphi \in \mathfrak{M}_+(\mathbb{R}^d)$ on pose

$$T\varphi(x) := \varphi(Mx + v) \quad (1.93)$$

alors $T : \mathfrak{M}_+(\mathbb{R}^d) \rightarrow \mathfrak{M}_+(\mathbb{R}^d)$ est un réarrangement.

4.4.2 Présentation du réarrangement décroissant de Schwarz

Le *réarrangement décroissant de Schwarz* est défini comme étant l'unique réarrangement de $\Omega = \mathbb{R}_+$ qui envoie $\mathfrak{M}_+(\mathbb{R}_+)$ dans l'ensemble des fonctions décroissantes. Pour $\varphi \in \mathfrak{M}_+(\mathbb{R}_+)$, on note φ^* son réarrangé décroissant. Dans le cas des fonctions constantes par morceaux, si on les voit comme étant des histogrammes alors le réarrangement décroissant de Schwarz correspond au tri des valeurs prises par l'histogramme de la plus élevée à la plus basse. Il est également possible de le définir par le biais des ensembles de sur-niveau comme suit

$$\{\varphi^* \geq \nu\} = [0; \mathcal{L}^1\{\varphi \geq \nu\}]. \quad (1.94)$$

Une version symétrisée sur \mathbb{R} existe également et s'appelle selon les auteurs le *réarrangement symétrique-décroissant* ou bien le *réarrangement de Steiner*. Il est défini par

$$\{\varphi^\# \geq \nu\} = \left[-\frac{1}{2}\mathcal{L}^1\{\varphi \geq \nu\}; \frac{1}{2}\mathcal{L}^1\{\varphi \geq \nu\} \right]. \quad (1.95)$$

Voici présentées brièvement les trois inégalités fondamentales pour les réarrangements de Schwarz et Steiner.

Théorème 4.7 (Inégalité de réarrangement de Hardy-Littlewood [43]). *Soit φ et ψ deux fonctions mesurables positives définies sur \mathbb{R}_+ . Alors*

$$\int_0^\infty \varphi \psi \leq \int_0^\infty \varphi^* \psi^*.$$

Cette inégalité est également vérifiée pour le réarrangement de Steiner. Ceci implique en particulier que le réarrangement décroissant de Schwarz en tant qu'opérateur de $L^p_+(\mathbb{R}_+)$ à valeurs dans lui-même est un opérateur continu pour la norme L^p , avec $p \in [1, +\infty]$.

Théorème 4.8 (Inégalité de Pólya–Szegő [66], 1951). *Si on suppose que $\varphi \in \dot{W}_+^{1,p}$ alors il en est de même pour φ^* et*

$$\int_{\mathbb{R}_+} |\nabla(\varphi^*)|^p \leq \int_{\mathbb{R}_+} |\nabla\varphi|^p.$$

La notation gradient a été utilisée car cette inégalité reste vraie pour des fonctions définies sur $\mathbb{R}_+ \times \Omega$ (le réarrangement de Schwarz étant alors effectué par rapport à la première variable). Cette inégalité est également vraie pour le réarrangement de Steiner. Le cas $p = 2$ dans l’inégalité de Pólya–Szegő est un cas particulièrement intéressant puisqu’il correspond à l’énergie associée à un opérateur de Laplace. L’opérateur associé à l’énergie définie pour p quelconque est souvent appelé p -laplacien.

Théorème 4.9 (Inégalité de réarrangement de Riesz [69], 1930). *Soit φ, ψ et $\chi : \mathbb{R} \rightarrow \mathbb{R}_+$. Alors,*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) \psi(y) \chi(x-y) dx dy \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi^\sharp(x) \psi^\sharp(y) \chi^\sharp(x-y) dx dy.$$

Cette inégalité peut être considéré sous un certain rapport comme étant une généralisation des deux précédentes. En effet, si on choisit formellement la fonction χ égale à la masse de Dirac en 0 on retrouve l’inégalité de Hardy–Littlewood ci-dessus. De plus, une conséquence importante de cette inégalité est que pour tout $s \in (0, 1)$ nous avons

$$|\varphi^\sharp|_{H^s} \leq |\varphi|_{H^s} \quad (1.96)$$

où les semi-normes H^s sont définis par

$$|\varphi|_{H^s}^2 := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\varphi(x) - \varphi(y)|^2}{|x-y|^{2(1-s)}} dx dy.$$

La preuve se fait en développant le carré au numérateur puis en appliquant l’inégalité de Riesz. En raisonnant de manière similaire à l’aide de la fonction de Green du laplacien, on peut établir la même chose pour les semi-normes \dot{H}^1 . La généralisation aux semi-normes $\dot{W}^{s,p}$ est un peu plus difficile et se fait à l’aide de la formule de la co-aire.

4.4.3 Inégalité de réarrangement de Schwarz

Une propriété importante du réarrangement décroissant de Schwarz est qu’il “déplace la masse de la fonction vers la gauche”. Plus précisément,

Lemme 4.1 (Inégalité de réarrangement de Schwarz).

$$\forall x \in \mathbb{R}_+, \quad \text{meas} \left([0, x] \cap \{\varphi \geq \nu\} \right) \leq \text{meas} \left([0, x] \cap \{\varphi^* \geq \nu\} \right).$$

Dans le cas particulier du réarrangement décroissant de Schwarz, cette inégalité est une conséquence de l’inégalité de Hardy–Littlewood (théorème 4.7). Néanmoins, dans le cas général, ces deux inégalités sont indépendantes au sens où il existe des réarrangements vérifiant l’une mais pas l’autre ou bien ni l’une ni l’autre. Cette inégalité est le point de départ des réflexions ayant mené à la construction du réarrangement par tassement. L’intérêt principal de cette inégalité est que nous pouvons en déduire de nombreuses inégalités intégrales. On peut par exemple établir que le réarrangement de Schwarz diminue les moments et plus généralement, pour tout α et $\beta > 0$,

$$\int_0^{+\infty} x^\alpha |\varphi^*(x)|^\beta dx \leq \int_0^{+\infty} x^\alpha |\varphi(x)|^\beta dx. \quad (1.97)$$

Une autre inégalité qui découle de l’inégalité de réarrangement de Schwarz est

$$\int_0^\infty F(\varphi(x) - cx^\alpha - k) dx \leq \int_0^\infty F(\varphi^*(x) - cx^\alpha - k) \quad (1.98)$$

pour $\alpha > 0$, $c, k \geq 0$ et $F : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ croissante. L’intérêt de cette dernière inégalité dans le cas $\alpha = 1$ est qu’elle correspond exactement au type d’inégalité nécessaire pour effectuer des estimées d’énergie dans le cadre du travail présenté à la section 4.3. Le cas $\alpha = 2$ dans l’inégalité ci-dessus est également pertinent pour pouvoir obtenir des inégalités d’énergie dans le cadre du travail réalisé à la section 4.2. L’inégalité de réarrangement de Schwarz combiné avec l’inégalité de Pólya–Szegő (ou Riesz pour le laplacien fractionnaire) est donc un outil puissant pour effectuer des estimations d’énergies dans le cadre de problèmes variationnels. Le réarrangement de Schwarz peut être étendu à des fonctions de plusieurs variables sous certaines conditions et préserver ses bonnes propriétés. Néanmoins, cet outil ne peut être utilisé en l’état dans le cadre des travaux présentés aux sections 4.2 et 4.3 à cause d’une objection majeure : le réarrangement de Schwarz ne préserve pas la condition au bord de Dirichlet en 0.

4.4.4 Présentation du réarrangement par tassement

L’objectif général qui préside ce travail consiste donc à créer des réarrangements qui partagent avec le réarrangement de Schwarz des équivalents des inégalités intégrales présentés à la section 4.4.2 et de l’inégalité de réarrangement de Schwarz (lemme 4.1) mais qui en plus stabilisent les conditions aux bords de Dirichlet. Le tassement qui est présenté ici doit être considéré comme étant une contribution à un questionnement mathématique plus large sur les rapports entre réarrangements et conditions aux bords. L’objectif à terme dans le cadre des travaux présentés aux sections précédentes serait d’utiliser un réarrangement pour obtenir des descriptions géométriques de la zone de vorticit ,   savoir le support de la fonction de vorticit  not e θ pour SQG et ω pour Euler bi-dimensionnel. Plus g n ralement un d veloppement de la th orie du r arrangement et des outils intellectuels associ s permettrait d’obtenir des descriptions pr cises des fonctions solutions   de nombreuses  quations.

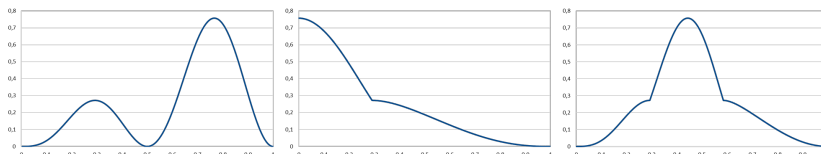


FIGURE 1.15 – Illustration pour le tassement. De gauche   droite : Le graphe de la fonction $x \mapsto x \cdot \sin^2(2\pi x) \cdot \mathbb{1}_{[0,1]}(x)$, le graphe du r arrangement de Schwarz de cette fonction et le graphe du r arrangement par tassement.

Concernant la construction du r arrangement par tassement, le r sultat obtenu est le suivant :

Th or me 4.10 (Existence du r arrangement par tassement, L.G-C., 2018). *Il existe un r arrangement d fini pour les fonctions mesurables positives de \mathbb{R}_+ qui*

- satisfait l’in galit  de P lya–Szeg  (th or me 4.8),
- satisfait l’in galit  de r arrangement de Schwarz (lemme 4.1),
- donne des fonctions unimodales (croissante sur un intervalle $[0, x_0[$ puis d croissante sur $]x_0, +\infty[$, avec $x_0 \in [0, +\infty[$),
- pr serve la condition au bord de Dirichlet en 0 pour les fonctions absolument continues.

En outre, l’exemple d’un tel r arrangement est donn  de mani re constructive, et a  t  baptis  le “r arrangement par tassement” ou plus simplement le “tassement”.

Les deux premi res conditions ci-dessus sont les m mes que pour le r arrangement de Schwarz. La troisi me consiste   remplacer la propri t  de d croissance dans le cas de Schwarz par une propri t  d’unimodalit  (la fonction a “une seule bosse”) car la condition de d croissance emp che de pr server la condition au bord de Dirichlet. Malgr  cette transformation, on retrouve l’id e selon laquelle le r arrangement “simplifie” la fonction et fait dispara tre les oscillations. La derni re condition porte sur la stabilisation des conditions de Dirichlet en 0 (pas seulement Dirichlet homog ne : quelle que soit la valeur, le tassement la pr serve d s lors que la fonction est absolument continue). Pour une illustration du r arrangement par tassement, voir figure 1.15

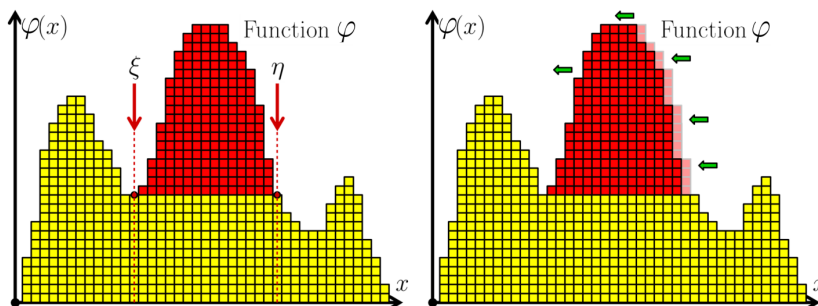


FIGURE 1.16 – Illustration de la construction du tassement pour les fonctions constantes par morceaux : L’algorithme de tassement en action.

Il existe trois d finitions  quivalentes du r arrangement par tassement. Nous en esquissons une ici qui est la d finition algorithmique. On consid re pour cela des fonctions  l mentaires, constantes par

morceaux, et on les regarde comme étant des piles de carrés superposés. L'algorithme déplace ces cubes vers la gauche de telle façon que les "creux" de la fonction (les endroits où se trouvent les minima locaux) soient bouchés au fur et à mesure de l'algorithme. Cependant, l'algorithme ne pousse pas les lignes de cubes "jusqu'au bout" car alors on perd la condition de Dirichlet mais s'arrête dès que les "creux" sont bouchés. Voir figure 1.16 pour une illustration. Un passage à la limite bien choisi permet alors d'avoir une définition pour toutes les fonctions mesurables positives de \mathbb{R}_+ . Cette définition dans une version rigoureuse ainsi que deux autres définitions possibles sont données aux chapitre 5 de cette thèse, chapitre consacré à la construction du réarrangement par tassement et à l'étude de ses principales propriétés.

————— ∼ * ∼ —————



FIGURE 1.17 – Jean Louis Théodore GÉRICAULT, Apologie de la Mécanique des Fluides (Le Radeau de La Méduse), 1819, Musée du Louvre.

Chapitre 2

Collisions of vortices for Euler and Quasi-Geostrophic models

Abstract

This article provides an extension to inviscid surface quasi-geostrophic equation (SQG) of the standard theory for the point-vortex model associated to the surface Euler equation. In the case of an inviscid fluid with planar motion, the point-vortex model gives account of the cases where the vorticity profile is sharply concentrated around some points and approximated by Dirac masses. Using the links between (SQG) and Euler equations, we define here the analogous of the point vortex model for (SQG) and extend some aspects of the standard theory to (SQG). We provide in a second part an improvement of a standard result on the improbability of collisions of Euler vortices and this new result is also extended to (SQG). In a third part, we prove a convergence theorem for the point-vortex Euler system under the standard *non-neutral clusters* hypothesis. In a last part, a particular class of collisions of Euler vortices called *multi-scale collisions* is studied.

1 The point-vortex system

1.1 Inviscid surface quasi-geostrophic equation

We are interested in a model of point-vortices to study the inviscid surface quasi-geostrophic equation given by

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, \\ v = \nabla^\perp (-\Delta)^{-s} \omega, \end{cases} \quad (\text{SQG})$$

where $v : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the velocity of the fluid and $\omega : \mathbb{R}_+ \times \mathbb{R}^2 \rightarrow \mathbb{R}$ is called the active scalar. The notation \perp refers to the counterclockwise rotation of angle $\frac{\pi}{2}$.

Typically, the quasi-geostrophic equation gives account of the dynamics in a rotating frame of the potential temperature for a stratified fluid subject to Brünt-Väisälä oscillations. This is a standard model for the study of geophysical fluids and it is intensively used for weather forecast and climatology. For more details about the physical modeling leading to these equations, see [64] or [83]. Mathematically, the quasi-geostrophic equation has many properties in common with the bi-dimensional Euler equations written in terms of vorticity :

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0, \\ v = \nabla^\perp (-\Delta)^{-1} \psi. \end{cases} \quad (\text{Eu})$$

The Euler equation in dimension 2 can be seen as a particular case of the quasi-geostrophic equation where the parameter s would have been chosen equal to 1. Elements of standard theory for Euler equations can be found here : [6]. Local well-posedness of classical solutions for (SQG) was established in [25], where are also studied the several analogies with the Euler equations of dimensions 2 and 3. Solutions with arbitrary Sobolev growth were constructed in [49] in a periodic setting. So far and contrarily to the two dimensional Euler equations, establishing global well-posedness of classical solutions for (SQG), or alternatively describing their finite time singularities, is not known. Note also that the global existence of weak solutions in $L^2(\mathbb{R}^2)$ was established by [67], but below a certain regularity threshold, these weak

solutions show dissipative behaviors and non-uniqueness. The arguments appearing in these cases are close to the problems related to the Onsager conjecture. See [15] for an example of two weak solutions to (SQG) with same initial datum. Exhibiting global smooth solutions or patch solutions is a challenging issue as there is no equivalent of the Yudovitch theorem [85]. A first example was recently provided in [19] by developing a bifurcation argument from a specific radially symmetric function. The variational construction of an alternative example in the form of a smooth traveling-wave solution was completed in [42]. Corotating patch solutions with two patches were provided by [46] with bifurcation argument and this construction was extended using variational argument by [41] to N patches forming an N -fold symmetrical solution.

The present work aims at developing the understanding of the links between the quasi-geostrophic equation and the 2-dimensional Euler equation through the study of the point-vortex model. In the case of the Euler equation in dimension 2, the points-vortex model is a system of differential equations for points on \mathbb{R}^2 that approximates situations where the vorticity ω is highly concentrated around several points. In such a situation, it is more convenient to see the vorticity as being a pondered sum of Dirac masses. This model is widely studied in fluid mechanics and a good presentation of the main results on this system can be found at [56, §4], completed by [57]. The objective of this work is to define (SQG) point-vortices and to study this system. In a first section, the (SQG) point-vortex model is presented and some standard result for Euler point-vortices are extended to (SQG). In a second part, a standard result on the fact that the measure of the initial datum leading to collisions is 0 [56, §4] is improved and this new result is also extended to (SQG). We provide in a third part a convergence theorem for the point-vortex Euler system under the standard *non-neutral clusters* hypothesis. In a last part, a particular class of collisions of vortices called *mono-scale collisions* is studied. The notion of *multi-scale collision* is also sketched with a study of some properties.

1.2 Presentation of the point-vortex model

1.2.1 Vorticity as Dirac masses

The point-vortex model on the plane \mathbb{R}^2 consists in the supposition that at time $t = 0$ the vorticity can write as a pondered sum of Dirac masses :

$$\omega(t = 0, x) = \sum_{i=1}^N a_i \delta_{x_i}. \quad (2.1)$$

The points x_i are the respective position of the vortices $a_i \delta_{x_i}$ and the coefficients a_i are their intensity. The first equation in (SQG) or (Eu) is a transport equation on the vorticity ω . Then, it is expected that the Dirac masses initially located at x_i be left unchanged but transported by the flow. Formally, if we use the evolution equations, (SQG) or (Eu) with initial datum (2.1), we obtain that the speed writes

$$v(t = 0, x) = \sum_{i=1}^N a_i \nabla_x^\perp G_s(|x_i - x|), \quad (2.2)$$

where G_s is the profile of the Green function of the fractionnal Laplace operator $(-\Delta)^s$ in the plane \mathbb{R}^2 . Parameter s is chosen to be in $(0, 1]$. These profiles $\mathbb{R}_+^* \rightarrow \mathbb{R}$ are given by the following formulas :

$$G_1(r) := \frac{1}{2\pi} \log\left(\frac{1}{r}\right) \quad \text{and} \quad G_s(r) := \frac{\Gamma(1-s)}{2^{2s}\pi\Gamma(s)} \frac{1}{r^{2(1-s)}}, \quad (2.3)$$

with Γ the Gamma function. In order to get rid of the problems of definition arising from the singularity of the speed at (2.2), the idea consists in claiming that since the vorticity is concentrated in one point then it does not interact with itself but only with the vorticity that is at a positive distance. This aspect of the modeling consists roughly speaking in replacing the kernel profile $G_s : \mathbb{R}_+ \rightarrow \mathbb{R}$ by $G_s \times \mathbb{1}_{\mathbb{R}_+^*}$ with the convention that $+\infty \times 0 = 0$. In this case, using (2.2), we get that the differential equation describing the evolution of the position of the vortices is given by

$$\frac{d}{dt} x_i(t) = \sum_{\substack{j=1 \\ j \neq i}}^N a_j \nabla^\perp G(|x_i(t) - x_j(t)|). \quad (2.4)$$

In the work [56, §4], the authors only consider the case $s = 1$ with kernel profile G_1 corresponding to the Euler equation in dimension 2. We are going to generalize some of their results to more general kernel profiles $G : \mathbb{R}_+^* \rightarrow \mathbb{R}$ that include the quasi-geostrophic case. For all the following, the function G is of regularity

$$G \in \mathcal{C}_{loc}^{1,1}(\mathbb{R}_+^*) \cap \mathcal{C}^{1,1}([1, +\infty[). \quad (2.5)$$

By the Cauchy-Lipschitz theorem, it is possible to conclude that the point-vortex system (2.4) is well-posed as long as there are no vortex collisions. In other words, as long as the system stays far from the singularity of the kernel G in 0^+ . We add some other hypothesis on the function G later on. These hypothesis later added are always satisfied by the Green functions of the fractional laplacian. It is at this condition that this work can be understood as an extension of [56, §4] to quasi-geostrophic equation.

1.2.2 Hamiltonian dynamic

The first element to point-out on this dynamic is that it is an hamiltonian dynamic. Indeed, if we define

$$\begin{aligned} H : \quad \mathbb{R}^{2N} &\longrightarrow \mathbb{R}, \\ X = (x_1, \dots, x_N) &\longmapsto \sum_{i \neq j} a_i a_j G(|x_i - x_j|), \end{aligned} \quad (2.6)$$

then Equation (2.4) can be rewritten under the following form

$$a_i \frac{d}{dt} x_i(t) = \nabla_{x_i}^\perp H(X). \quad (2.7)$$

The first consequence is that the hamiltonian H is conserved by the flow S^t of Equation (2.4).

Lemma 1.1 (Conservation of the hamiltonian). *Let $X \in \mathbb{R}^{2N}$. Then,*

$$\frac{d}{dt} H(S^t X) = 0. \quad (2.8)$$

Another consequence of the hamiltonian property is the Liouville theorem that ensures the preservation of the Lebesgue measure by the flow S^t .

Lemma 1.2 (Liouville theorem). *Let $V_0 \subseteq \mathbb{R}^{2N}$ measurable. Then*

$$\frac{d}{dt} \mathcal{L}^{2N}(S^t V_0) = 0, \quad (2.9)$$

where \mathcal{L}^{2N} denotes the Lebesgue measure on \mathbb{R}^{2N} .

For the proof of the Liouville Theorem, we refer to the one given by Arnold in [4, §3]. With the hamiltonian formulation comes the Noether theorem that provides the quantities left invariant by the flow knowing the geometrical invariants of the hamiltonian H . The invariance of H with respect to the translations of the plane implies the conservation of the vorticity vector.

Lemma 1.3 (Conservation of the vorticity vector). *Let $X \in \mathbb{R}^{2N}$. The vorticity vector is defined by*

$$M(X) := \sum_{i=1}^N a_i x_i. \quad (2.10)$$

Then,

$$\frac{d}{dt} M(S^t X) = 0. \quad (2.11)$$

In the case where the system is non-neutral, meaning that $\sum_i a_i \neq 0$, this lemma implies the preservation of the *center of vorticity* of the system defined by

$$B(X) := \left(\sum_{i=1}^N a_i \right)^{-1} \sum_{i=1}^N a_i x_i = \left(\sum_{i=1}^N a_i \right)^{-1} M(X). \quad (2.12)$$

Concerning the invariance of the hamiltonian by the rotations of the plane, it implies the conservation of the moment of inertia.

Lemma 1.4 (Conservation of the moment of inertia.). *Let $X \in \mathbb{R}^{2N}$. The moment of inertia is defined by*

$$I(X) := \sum_{i=1}^N a_i |x_i|^2. \quad (2.13)$$

Then

$$\frac{d}{dt} I(S^t X) = 0. \quad (2.14)$$

1.2.3 Solutions in the case of a pair of vortices

The conservation of H , M and I allows us to determine explicitly all the solutions to the point-vortex equations (2.4) in the particular case where $N = 2$, which corresponds to a pair of vortices. The hamiltonian writes then as a function of the distance between the two vortices. Since the hamiltonian is conserved, this implies that the distance r_0 between the two vortices remains constant. The speed at which every vortex moves is also constant and proportional to $|dG/dr|$ evaluated at r_0 and to the intensity a_i of the other vortex.

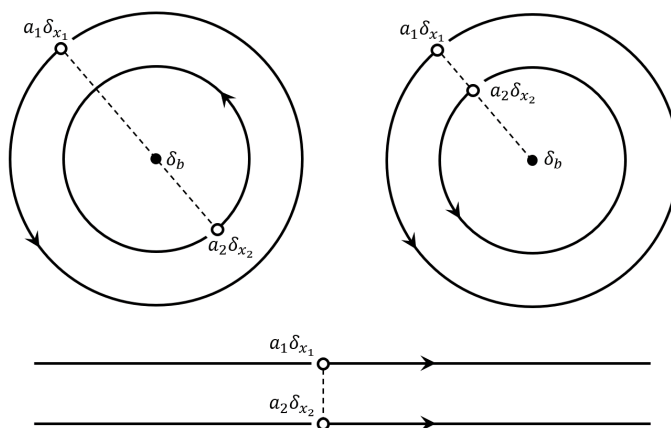


FIGURE 2.1 – The different solutions of the point-vortex problem in the case of 2 vortices $a_1 \delta_{x_1}$ and $a_2 \delta_{x_2}$. The center of vorticity, that is immobile, is represented by δ_b . Left : the case where a_1 and a_2 have the same sign. Right : the case where a_1 and a_2 have opposite sign but $a_i + a_j \neq 0$. Bottom : the case $a_i + a_j = 0$.

In the case where $a_i + a_j \neq 0$, the center of vorticity is well-defined and left invariant by the dynamic. This one is at a fixed distance to respectively x_1 and x_2 , and therefore we can deduce that the trajectories are circles. The sense in which a vortex x_i moves along its circle is given by the sign of a_i and by the sign of the derivative of G in r_0 . On the contrary, if $a_1 + a_2 = 0$, the center of vorticity is no longer defined (or is at infinity). The vortices are translating at same speed along two parallel lines. See Figure 2.1 for an illustration.

1.2.4 An example of collision

To study the well-posedness of this system of differential equations, the natural tool is the Cauchy-Lipschitz theorem. Nevertheless, the term at the right-hand side of (2.4) may be not locally Lipschitz because this term may blow up if the system comes close to a collision situation, meaning $x_i = x_j$ for some $i \neq j$. By regularity assumption on G , the system is well-posed if and only if no collision occur. Unfortunately, there exist initial conditions leading to a collision of vortices in finite time and therefore the system in this case is not well-defined for all time. For the cases where G is the Green kernel of the Laplace operator (2.3), which is $-\log$, a construction of such a collision is provided by [56, §4]. We refer to it for more details about this construction and give here the main ideas (see Figure 2.2 for an illustration).

At time $t = 0$, the three vortices x_1 , x_2 and x_3 are at positions respectively $(-1, 0)$, and $(1, 0)$ and at $(1, \sqrt{2})$. Their respective intensities are $a_1 = 2$, $a_2 = 2$ and $a_3 = -1$. This configuration leads to a collision in finite time and the three vortices collapse on their center of vorticity B . This collision is *auto-similar* in the sense that in this particular configuration the angles between the the vertices of

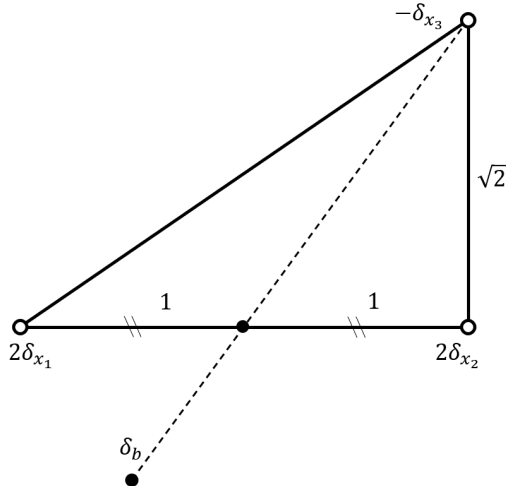


FIGURE 2.2 – Initial datum of the point-vortex model (2.4) with $G = -\frac{1}{2\pi}\log$ leading to a collision in finite time.

the triangle are preserved by the flow. This triangle at time t is, up to a rotation-dilatation, equal to the initial one. Moreover, the distances $|x_i(t) - x_j(t)|$ for $i \neq j$ behaves like $t \mapsto C_{ij}\sqrt{T-t}$, where $T > 0$ is the time of collision. This means that these distances behave like the uni-dimensional system defined by $dx/dt = -\alpha/x$ for $\alpha > 0$. This scale of convergence in square root, since $G = -\log$, is conform to the natural intuition. This intuition is that, seen the form of system (2.4), the collisions of vortices in the case where $dG/dr = r^{-a-1}$ with $a \leq 0$ have an auto-similar behavior and the distance between vortices behaves like the uni-dimensional system given by $dx/dt = -x^{-a-1}$. Nevertheless, to our knowledge there exist no study of the different types of collisions existing for the point-vortex system.

2 Main results

2.1 The point-vortex system in a general framework

2.1.1 Non-neutral clusters hypothesis

The specific case where the point-vortex system is associated to the Euler equations corresponds to the Green function of the laplacian :

$$G_1(r) = \frac{1}{2\pi}\log\left(\frac{1}{r}\right), \tag{2.15}$$

and this particular case is studied in [56, §4]. For instance the authors provide the following well-posedness result.

Theorem 2.1 (Marchioro-Pulvirenti existence theorem, [56, §4]). *Define the kernel G by (2.15) and assume that the intensities satisfy*

$$\forall A \subseteq \{1\dots N\} \text{ s.t. } A \neq \emptyset, \quad \sum_{i \in A} a_i \neq 0. \tag{2.16}$$

Then for almost every initial datum $X_0 \in \mathbb{R}^{2N}$, the evolution problem (2.4) admits a unique global solution.

A vortex system such that the sum of all the intensities a_i is equal to 0 is called in [56, §4] a “neutral system”. Nevertheless, they give no name to Hypothesis (2.16) that they extracted and studied. We suggest that it should be called “non-neutral clusters hypothesis since it exactly means that any cluster of vortices is non-neutral. We state later in section 2.2 that this existence theorem remains true for (SQG) as a consequence of a slightly more general theorem that both weakens hypothesis (2.16) and consider the fractional laplacian.

2.1.2 Uniform bound theorem

The main reason why the authors use the non-neutral clusters hypothesis (2.16) for their proof is the fact that the center of vorticity

$$B(X) := \left(\sum_{i=1}^N a_i \right)^{-1} \sum_{i=1}^N a_i x_i = \left(\sum_{i=1}^N a_i \right)^{-1} M(X) \quad (2.17)$$

is well defined, not only for the whole system but also for any subset of vortices. Seen the properties of the centers of vorticity presented in the previous section, it can be said at an intuitive level that a cluster of vortices is expected to “*turn around its center of vorticity*”. If all the subsets of vortices have their relative center of vorticity well-defined then it is possible to claim that the vortices will not diverge in finite time. More precisely, we prove a much stronger result which is a bound on the trajectories that is uniform with respect to the initial datum $X \in \mathbb{R}^{2N}$ but also with respect to the singularity of the kernel profile G near 0. Nevertheless, the bound is not uniform with respect to the behavior of the kernel profile G at $+\infty$.

Theorem 2.2 (Uniform bound on the trajectories). *Let $J : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $J(r) \rightarrow 0$ as $r \rightarrow +\infty$. We suppose that the studied profiles G satisfy the following hypothesis.*

$$\forall r \geq 1, \quad \left| \frac{dG}{dr}(r) \right| \leq J(r). \quad (2.18)$$

We also consider a system satisfying the non-neutrality of clusters hypothesis (2.16). Let $T > 0$. Then there exist a constant C such that for all profile G satisfying (2.18) and for all initial datum $X \in \mathbb{R}^{2N}$ that are not leading to collision on $[0, T)$,

$$\sup_{t \in [0, T)} |X - S_G^t X| \leq C, \quad (2.19)$$

where S_G^t is the flow associated to (2.4) with the kernel profile equal to G . This constant C depends on N , on the intensities a_i , on the final time T and on the choice of the function J .

The proof of this theorem is reminiscent from Theorem 2.1 in [56, §4], where was proved an analogous weaker result. This theorem is for instance a good tool when one tries to drop the singularity of a kernel profile G near 0. Indeed, if G is replaced by a smooth approximation G_ε that coincides with G on $[\varepsilon, +\infty)$ then the bound given by Theorem 2.2 is independent of ε . This idea is for instance the first step of the proof of the existence result in Theorem 2.1. We will not make use of this idea for the proof of our existence result later on. Nevertheless, Theorem 2.2 is very interesting in itself because it provides a very good bound on the behaviors of vortices under the non-neutral clusters hypothesis. In [56, §4], this theorem is established only for the Euler case. Thus, we extended their theorem to a general case where the same result holds no matter what is the singularity of the kernel G in 0^+ . This theorem may seem quite surprising at first sight since the singularity of the kernel in 0 governs the situations where the speed of the vortices become very high. Then, for a boundedness result we could expect hypothesis on this singularity at 0 of the kernel G and not only on its behavior at infinity. This is why this result must be understood as a quantitative result on the intuition that a cluster of vortices is expected to “*turn around its center of vorticity*” and not leave whatever its speed is.

Remark that in this theorem the non-neutral cluster hypothesis (2.16) is essential. Indeed, the simple situation of a pair of vortices with intensities $+1$ and -1 give raise to a motion of translation along two parallel lines and the speed at which these vortices go blows up as the initial distance between both goes to 0^+ . The trajectories are also bounded in finite time in this situation but, unlike the conclusion of Theorem 2.2, the bound is not uniform and depends on the initial conditions and on the singularity of the kernel. The existence of initial datum for which the vortices are unbounded in finite time is an open problem.

2.1.3 The case of intensities a_i all positive

Another consequence of Theorem 2.2 is that it provides a possible improvement of Theorem 2.1 in the case where the intensities a_i are all positive.

Lemma 2.1. *We make on G the two following hypothesis*

$$\bullet \quad |G(r)| \longrightarrow +\infty \quad \text{as } r \rightarrow 0. \quad (2.20)$$

$$\bullet \quad \left| \frac{dG}{dr}(r) \right| \longrightarrow 0 \quad \text{as } r \rightarrow +\infty. \quad (2.21)$$

Suppose now that the a_i are all positive. Let $X \in \mathbb{R}^{2N}$ an initial datum for the point-vortex model (2.4). We make the supposition that this initial datum is such that $i \neq j$ on a $x_i \neq x_j$. Then there is no collision of vortices at any time.

Contrary to Theorem 2.1, this result is true for all initial datum and not only for almost every. This is a consequence of the following energetical argument.

Démonstration. Let $X \in \mathbb{R}^{2N}$ be an initial datum for the point-vortex system (2.4) such that for all $i \neq j$ we have $x_i \neq x_j$. Suppose by the absurd that there exist an increasing sequence of time (t_n) converging towards some T_X and a set of couples $\mathcal{P} \subseteq \{(i, j) : i, j = 1, \dots, N\}$ such that

$$\forall (i, j) \in \mathcal{P}, i \neq j, \quad |x_i(t_n) - x_j(t_n)| \longrightarrow 0^+ \quad \text{for } n \rightarrow +\infty \quad (2.22)$$

and

$$\forall (i, j) \notin \mathcal{P}, i \neq j, \quad |x_i(t_n) - x_j(t_n)| \geq c, \quad (2.23)$$

with $c > 0$ a constant independent of n . Since the a_i are all positive then the non-neutral cluster hypothesis is satisfied and then Theorem 2.2 combined with (2.21) and (2.23) give

$$\sum_{i \neq j} a_i a_j G(|x_i(t_n) - x_j(t_n)|) \geq \sum_{\substack{(i,j) \in \mathcal{P} \\ i \neq j}} a_i a_j G(|x_i(t_n) - x_j(t_n)|) + \left(\min_{r \in [c, C]} G(r) \right) \sum_{\substack{(i,j) \notin \mathcal{P} \\ i \neq j}} a_i a_j. \quad (2.24)$$

Therefore and since the a_i are all positive, the inequality above with (2.20) and (2.22) implies that

$$\sum_{i \neq j} a_i a_j G(|x_i(t_n) - x_j(t_n)|) \longrightarrow +\infty \quad \text{pour } n \rightarrow +\infty. \quad (2.25)$$

This is in contradiction with the preservation of the energy stated at Lemma 1.1. \square

2.2 Improbability of collisions result for point-vortex

First, we define the set of initial datum that leads to a collision of vortices.

Definition 2.3 (Set of collisions). The set of initial datum such that two or more vortices collide in finite time is

$$\mathfrak{C} := \left\{ X \in \mathbb{R}^{2N} : \exists T_X \in \mathbb{R}_+, \liminf_{t \rightarrow T_X^-} \min_{i \neq j} |x_i(t) - x_j(t)| = 0 \right\}. \quad (2.26)$$

The set \mathfrak{C} is called the *set of collisions* and T_X is the time of collision associated to the initial datum $X \in \mathfrak{C}$.

Concerning the Marchioro-Pulvirenti result given by Theorem 2.1, the most natural question consists in removing the non-neutral clusters hypothesis (2.16). This leads to the following conjecture.

Conjecture 2. *The set of collisions \mathfrak{C} as a Lebesgue measure 0.*

It is not yet possible to prove such a conjecture. The main difficulty lays in the understanding of the situations where some vortices collide in such a way that they go to infinity in finite time. Nevertheless, we are able to improve Theorem 2.1 by weakening hypothesis (2.16) but we still need to restrict the work to systems of vortices that stay bounded with respect to one another.

Theorem 2.4 (improbability of collisions for Euler and (SQG) vortices). *Consider the point-vortex problem (2.4) with kernel profile G_s , the kernel associated to the Green function of the laplacian or the fractional laplacian on the plane (2.3). Then, the set of initial data for (2.4) leading to collisions and such that*

$$\limsup_{t \rightarrow T_X^-} \max_{i \neq j} |x_i(t) - x_j(t)| < +\infty \quad (2.27)$$

has a Lebesgue measure equal to 0.

This theorem is slightly more general than Theorem 2.1 for two reasons. First, it is true both for Euler and quasi-geostrophic point-vortex models. Another aspect is hypothesis (2.16) in Theorem 2.1 that implies a uniform bound on the trajectories of the vortices as stated in Theorem 2.2. Under this point of view, hypothesis (2.27) is quite weaker. Nevertheless, it is not possible to directly check from the initial datum if this hypothesis is satisfied, unlike for Theorem 2.1. The question whether it is possible to get rid of Hypothesis (2.27) but obtain the same conclusion is the open question by Conjecture 2. It is not clear for instance whether it is possible to decompose a complicated vortex system into different clusters which individually satisfy (2.27) on a *not-too-short* period of time.

2.3 Convergence result for Euler point-vortices with non-neutral clusters hypothesis

The systems of vortices that are studied with more details in [56, §4] are the systems for which the non-neutral clusters hypothesis (2.16) holds. Concerning this kind of systems, in the Euler case, we are also able to write a convergence result. When vortices come to collision, their speed may become infinite as a consequence of the kernel profile singularity in 0^+ . But if their speed blows up, then any pathological behavior near the time of collision T_X is *a priori* possible. Only the continuity of the trajectories on $[0, T_X)$ is ensured. In the situation of the non-neutral clusters hypothesis, Theorem 2.2 implies that the trajectories remain bounded. We proved here after that the trajectories are actually convergent in the Euler case.

Theorem 2.5 (Convergence for Euler vortices under non-neutral clusters hypothesis). *Consider the point-vortex model under hypothesis (2.16) with a kernel profile G_1 corresponding to the Green function of the laplacian (2.3). Let $X \in \mathfrak{C}$ be an initial datum leading to a collision at time T_X . Then, for all $i = 1 \dots N$, there exists an $x_i^* \in \mathbb{R}^2$ such that*

$$x_i(t) \longrightarrow x_i^* \quad \text{as } t \rightarrow T_X^-. \quad (2.28)$$

The first step of the proof consists in the following idea by Didier Smets. For a fixed value of $t \in [0, T_X)$ define the measure

$$P_t := \sum_{i=1}^N a_i \delta_{x_i(t)}. \quad (2.29)$$

Then, testing against a test function ϕ and using the point-vortex equation give that this measure converges in the sense of distributions as $t \rightarrow T_X$. In a second time, it is possible to prove that this convergence is actually stronger and obtain (2.28).

The proof of this result is specific to the Euler case and we do not know yet whether the conclusion extends to the (SQG) case.

2.4 Mono and multi-scales collisions for Euler point-vortices

The idea behind this notion of mono-scale collisions consists in comparing the Euler point-vortex model, where the speed behaves like the inverse of the distances with the uni-dimensional differential equation given by $\dot{x} = -\frac{\alpha}{x}$. In this situation, the solutions are $x(t) = \sqrt{2\alpha(T-t)}$. Such a solution “collides” with 0 at time T . It is natural to expect that the distances $|x_i(t) - x_j(t)|$ are also expected to behave in time like the square-root function or at least some power function. In the Euler case it is possible to study with more details the collisions in some special cases where the collision shows a nice multi-scale structure.

Definition 2.6 (multi-scale collision). Let $X \in \mathfrak{C}$ be an initial datum for the point-vortex problem (2.4) leading to a collision and let $T_X \geq 0$ be the time of collision. This initial datum is said to show a *multi-scale collision* if satisfying the following properties.

There exists a finite collection of functions $(f_m)_{m=0}^M$ with $f_m : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$, such that $f_0(z) = 1$ and

$$\forall m = 1 \dots M, \quad \frac{f_m(z)}{f_{m-1}(z)} \longrightarrow 0. \quad \text{as } z \rightarrow 0^+, \quad (2.30)$$

and there exists constants $C > c > 0$ such that for all $i \neq j$ there exists $m = 1 \dots M$,

$$c f_m(T_X - t) \leq |x_i(t) - x_j(t)| \leq C f_m(T_X - t). \quad (2.31)$$

Section 2. Main results

The fact that $f_0(z) = 1$ says that we restrict the analysis to situations where we know *a priori* that the trajectories are bounded. With (2.30) it implies in particular

$$\forall m = 1 \dots M, \quad f_m(z) \longrightarrow 0. \quad \text{as } z \rightarrow 0^+. \quad (2.32)$$

The functions f_m are called the *scaling functions* of the collision as they give the description of how the different collisions occur and at which scale. The minimal integer M such that (2.32), (2.30) and (2.31) hold is the number of scalings involved in the collisions. The set of $X \in \mathfrak{C}$ which number of scalings is M is noted \mathfrak{C}_M . The set \mathfrak{C}_1 is called the set of *mono-scale collisions*. The example of collision for the Euler equation that has been presented in Section 1.2.4 is an example of mono-scale collision with $f_1(z) = \sqrt{z}$. The set of collisions $\mathfrak{C}^* := \mathfrak{C} \setminus (\bigcup_{M \in \mathbb{N}^*} \mathfrak{C}_M)$ is called the set of *special collisions* and corresponds to collisions that are too complicated or degenerated and which study is excluded here. A good illustration of these scaling functions are for instance the power functions

$$f_m : z \in \mathbb{R}_+^* \mapsto z^{b_m}$$

with $b_{m+1} > b_m$. We now define $\mathfrak{C}_0 := \mathbb{R}^{2N} \setminus \mathfrak{C}$, the set where there are no collisions. Indeed, roughly speaking the case $M = 0$ can be interpreted formally as being an absence of collision.

We need to make an extra assumption for technical issues.

$$\forall m = 1 \dots M, \quad \exists \sigma_m \in (1, +\infty), \quad \frac{\log(f_m(z))}{\log(f_{m-1}(z))} \longrightarrow \sigma_m \quad \text{as } z \rightarrow 0^+. \quad (2.33)$$

Concerning this extra technical assumption, it comes from the fact that if a function f is negligible with respect to a function g , it is not necessarily the case for $\log(f)$ with respect to $\log(g)$, unlike the case of power functions. Such an hypothesis allows us to use the properties of the \log to make energy estimates. Nevertheless, this technical hypothesis should not be considered as very restrictive. For instance, it is always satisfied in the case of power function $f_m(z) = z^{b_m}$ with $b_{m+1} > b_m$. Energy estimates for a collision situation is a strong tool to obtain algebraic conditions on the intensities a_i , otherwise the term $G_1(|x_i - x_j|)$ would make the energy blow up as $|x_i - x_j| \rightarrow 0^+$. We aim at writing this algebraic condition, which formulation depends on the type of collision and for that purpose we have to track what happens at every scale of the collision.

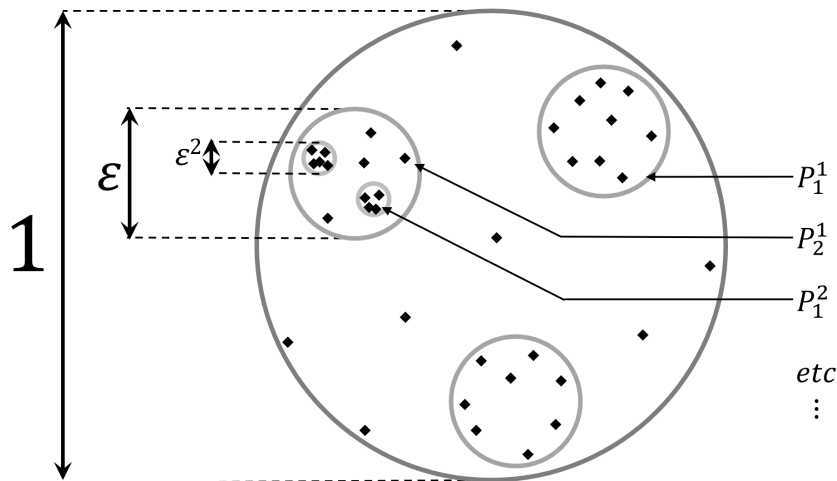


FIGURE 2.3 – Illustration for Lemma 2.2. The system of vortices is separated into clusters at every scale involved in the collision.

Lemma 2.2 (Separation of the system into clusters at every scale). *Let $X \in \mathfrak{C}$ such that (2.30) and (2.31) are satisfied. Consider the scaling functions $(f_m)_{m=1}^M$ associated to X . Then, there exist constants $C > 0$ and $\delta > 0$ such that for all $m = 1 \dots M$ there exists a family $\{P_l^m\}_{l=1}^{L_m}$ of non-empty subsets of $\{1, \dots, N\}$ two by two disjoint and that verify the two following conditions.*

1. For all $l = 1 \dots L_m$,

$$\max_{i, j \in P_l^m} |x_i(t) - x_j(t)| \leq C f_m(T_X - t). \quad (2.34)$$

2. For all $l = 1 \dots L_m$, for all $i \in P_l^m$ and $j \notin P_l^m$,

$$|x_i(t) - x_j(t)| \geq \delta f_{m-1}(T_X - t). \quad \forall t \in [0, T_X]. \quad (2.35)$$

The constants depend on the initial datum X and on the a_i .

The clusters associated to the first scale of collisions f_m with $m = 1$, which are the sets P_l^1 for $l = 1 \dots L_1$, are called the *clusters of collision*. Indeed, their respective size vanishes as a consequence of (2.34) but the distance between two remains bounded below by a positive constant, consequence of (2.35) with $f_0(z) = 1$. For an illustration of Lemma 2.2, see Figure 2.3. With such a description of the clusters at every scale of a multi-scale collision given by Lemma 2.2, we are able to state the following theorem.

Theorem 2.7 (Mono and multi-scales collisions for Euler vortices). *Consider the point-vortex problem (2.4) associated to the Green function of the laplacian (2.3). Let $X \in \mathfrak{C}_M$ a multi-scale collision with M scales as in Definition 2.6. Consider also the technical hypothesis (2.33) where are defined the coefficients σ_m .*

Then the following results hold.

(i) For all cluster of collision $l = 1 \dots L_1$,

$$\left(\sum_{i \in P_l^1} a_i \right)^2 + \sum_{m=2}^M \alpha_m \sum_{k=1}^{L_m} \left(\sum_{i \in P_k^m \cap P_l^1} a_i \right)^2 = \sum_{i \in P_l^1} a_i^2 + \sum_{m=2}^M \alpha_m \sum_{k=1}^{L_m} \sum_{i \in P_k^m \cap P_l^1} a_i^2. \quad (2.36)$$

where

$$\alpha_m := (\sigma_m - 1) \prod_{q=2}^{m-1} \sigma_q > 0. \quad (2.37)$$

(ii) In the mono-scale case $M = 1$, the algebraic collision condition given by Equation (2.36) gives back the condition of balanced clusters [73],

$$\left(\sum_{i \in P_l^1} a_i \right)^2 = \sum_{i \in P_l^1} a_i^2. \quad (2.38)$$

(iii) Still in the mono-scale case $M = 1$, For all $i = 1 \dots N$, there exists $x_i^* \in \mathbb{R}^2$ such that for t close enough to T_X ,

$$|x_i(t) - x_i^*| \leq C \sqrt{T_X - t}. \quad (2.39)$$

In the mono-scale situation, the main interest of Formula (2.38) is the fact that since $a_i \neq 0$ then necessarily we have the non-neutrality of the clusters of collision

$$\sum_{i \in P_l^1} a_i \neq 0. \quad (2.40)$$

Indeed, suppose that $\sum_{i \in P_l^1} a_i = 0$. Then, the algebraic condition (2.38) implies $\sum_{i \in P_l^1} a_i^2 = 0$. This is incompatible with the fact that $a_i \neq 0$ for all i . This formula is intensively used to study collisions for Ginzburg-Landau vortices, which have several common points with Euler vortices. See [73, 13, 12] for more details.

The possibility to extend Theorem 2.7 to quasi-geostrophic equations is not known (unlike for Theorem 2.5). Another open question concerning (2.66) and (2.39) is whether the converse inequality is true. In other words, is the scale of convergence actually $f_1(z) = \sqrt{z}$ in the general case or is it only an upper bound?

The open problems from these results are whether there exist multi-scale collisions ($\mathfrak{C}_M \neq \emptyset$ when $M > 1$?) and whether there exist special solutions ($\mathfrak{C}^* \neq \emptyset$?) or do the point-vortex model (2.4) only shows mono-scale collisions? And if these kinds of collisions exist, is it possible to estimate their Lebesgue measure or, which is better, their Hausdorff dimension? Another important question concerns the converse of Theorem 2.7. Given an algebraic condition of the form (2.36) for given values of σ_m , does the associated multi-scale collision exists for some initial datum?

3 Sketch of the proof for Theorem 2.4

3.1 The modified system

Let i be fixed in $\{1 \dots N\}$. The modified system consists in studying the evolution of $y_{ij} := x_i - x_j$ for $j \in \{1 \dots N\} \setminus \{i\}$. The point of this modification, since i is fixed, is that it decreases the dimension of the studied system because the vector $Y_i = (y_{ij})_{j \neq i}$ belongs to $\mathbb{R}^{2(N-1)}$. This modification can also be interpreted in a way as quotienting the space by the translations of the plane. The initial problem (2.4) implies that

$$\frac{d}{dt}(x_i - x_j)(t) = \sum_{k \neq i} a_k \nabla^\perp G_s(|x_i - x_k|) - \sum_{k \neq j} a_k \nabla^\perp G_s(|x_j - x_k|). \quad (2.41)$$

Therefore, the evolution of y_{ij} is given by

$$\frac{d}{dt} y_{ij} = (a_i + a_j) \nabla^\perp G_s(|y_{ij}|) + \sum_{k \neq i, j} a_k \left(\nabla^\perp G_s(|y_{ik}|) + \nabla^\perp G_s(|y_{ij} - y_{ik}|) \right). \quad (2.42)$$

Concerning Hypothesis (2.27), it can be rewritten in terms of y_{ij} as follows

$$\forall i = 1 \dots N, \quad \sup_{t \in [0, T]} \max_{j=1 \dots N} |y_{ij}(t)| < +\infty. \quad (2.43)$$

The main interest of this new system is that Theorem 2.4 can be reformulated using only the y_{ij} .

Lemma 3.1 (Reformulation of Theorem 2.4). *Denote by $Y_i(t) := (y_{ij}(t))_{j \neq i}$ the solution to (2.42) at time t with initial datum $Y_i \in \mathbb{R}^{2(N-1)}$. Suppose that for all $i \in \{1 \dots N\}$ and for all $T > 0$ and $\rho > 0$ the set*

$$\left\{ \begin{aligned} Y_i := (y_{ij})_{j \neq i} \in \mathbb{R}^{2(N-1)} : \exists T_X \in [0, T], \quad \liminf_{t \rightarrow T_X^-} \min_{j \neq i} |y_{ij}(t)| = 0 \\ \text{and} \quad \limsup_{t \rightarrow T_X^-} \max_{j=1 \dots N} |y_{ij}(t)| \leq \rho. \end{aligned} \right\} \quad (2.44)$$

has its Lebesgue measure $\mathcal{L}^{2(N-1)}$ equal to 0. Then in this case the conclusion of Theorem 2.4 holds.

The rest of the work consists then in studying this system (2.42) and to establish that (2.44) does have measure 0. This modified dynamics has many properties in common with the original one. In particular, it is proved in this section that this new dynamics still satisfies the Liouville property. Define the function $\mathcal{H}_{ij} : \mathbb{R}^{2(N-1)} \rightarrow \mathbb{R}$ by

$$\mathcal{H}_{ij} \left[(y_{ik})_{k \neq i} \right] := (a_i + a_j) G_s(|y_{ij}|) + y_{ij} \cdot \sum_{k \neq i, j} a_k \nabla G_s(|y_{ik}|) + \sum_{k \neq i, j} a_k G(|y_{ij} - y_{ik}|). \quad (2.45)$$

Combining (2.42) and (2.45),

$$\frac{d}{dt} y_{ij} = \nabla_{y_{ij}}^\perp \mathcal{H}_{ij} \left[(y_{ik})_{k \neq i} \right]. \quad (2.46)$$

Equation (2.46) says that, in a certain sense, the dynamics of the vector $Y_i := (y_{ij})_{j \neq i}$ shows an Hamiltonian structure. It is not an Hamiltonian system because the function \mathcal{H}_{ij} does depend on j but there is still a structure with an operator ∇^\perp . Therefore, the Schwartz theorem gives

$$\operatorname{div}_{y_{ij}} \left(\frac{d}{dt} y_{ij} \right) = \operatorname{div}_{y_{ij}} \left(\nabla_{y_{ij}}^\perp \mathcal{H}_{ij} \left[(y_{ik})_{k \neq i} \right] \right) = 0. \quad (2.47)$$

Since by (2.47) the velocity is divergent-free, a Liouville theorem holds for this dynamics.

Lemma 3.2 (Liouville theorem). *The Lebesgue measure on the space $\mathbb{R}^{2(N-1)}$ given by*

$$\prod_{j \neq i} dy_{ij} \quad (2.48)$$

is preserved by the flow \mathfrak{S}_i^t associated to (2.46).

A detailed proof of the Liouville theorem in a more general setting can be found at [4, §3]. The main argument for the proof of the Liouville theorem is that for a dynamical system $\dot{x} = f(x)$, and for $\Omega \subset \mathbb{R}^d$ an open set, by a change of variables and a Taylor expansion :

$$\frac{d}{dt} \mathcal{L}^d(S^t \Omega) = \int_{\Omega} \operatorname{div} f(x) dx. \quad (2.49)$$

where S^t is the flow of the equation.

3.2 Regularization of the kernel G_s

From the kernel G_s given at (2.15), define now the regularized profiles $G_{s,\varepsilon}$ for $\varepsilon > 0$. The objective here is to drop the singularity of the kernel G_s near 0^+ . We ask $G_{s,\varepsilon}$ to be \mathcal{C}^1 on \mathbb{R}_+ (until the boundary) and to verify

$$\bullet \quad G_{s,\varepsilon}(q) = G_s(q) \quad \text{when} \quad \varepsilon \leq q, \quad (2.50)$$

$$\bullet \quad |G_{s,\varepsilon}(q)| \leq |G_s(q)| \quad \text{for all } q \in \mathbb{R}_+, \quad (2.51)$$

$$\bullet \quad \left| \frac{d}{dq} G_{s,\varepsilon}(q) \right| \leq \left| \frac{d}{dq} G_s(\varepsilon) \right| \quad \text{for all } q \in \mathbb{R}_+, \quad (2.52)$$

$$\bullet \quad |G_{s,\varepsilon}(q)| \leq 2|G_s(\varepsilon)| \quad \text{for all } q \in \mathbb{R}_+. \quad (2.53)$$

Since $G_{s,\varepsilon}$ is of class \mathcal{C}^1 on \mathbb{R}_+ , the dynamic defined by (2.4) for the kernel $G_{s,\varepsilon}$ is well-defined and is Hamiltonian. In the sequel we denote $\mathfrak{S}_{i,\varepsilon}^t$ the flow at time t associated to the evolution equation (2.42) with the kernel profile G_s replaced by the regularization $G_{s,\varepsilon}$. The motion induced by the kernel profile $G_{s,\varepsilon}$ coincides with the original motion provided that the distances between the vortices remain higher than ε . Theorem 2.4 can be reformulated as follows.

Lemma 3.3. *Suppose that for all $T > 0$ and for all $\rho > 0$*

$$\begin{aligned} \mathcal{L}^{2(N-1)} \left\{ Y_i = (y_{ij})_{j \neq i} \in \mathbb{R}^{2(N-1)} : \max_{j \neq i} \sup_{t \in [0, T]} |y_{ij}^\varepsilon(t)| \leq \rho \right. \\ \left. \text{and } \min_{j \neq i} \inf_{t \in [0, T]} |y_{ij}^\varepsilon(t)| \leq \varepsilon \right\} \xrightarrow{\varepsilon \rightarrow 0^+} 0. \end{aligned} \quad (2.54)$$

Then for all $i \in \{1 \dots N\}$, the set (2.44) has Lebesgue measure 0.

Combined with Lemma 3.1, this lemma gives Theorem 2.4 provided that the convergence (2.54) holds. The rest of the proof is then devoted to prove that this convergence is true.

3.3 Estimate the collisions

Lemma 3.4. *Let $i \in \{1 \dots N\}$, $\varepsilon > 0$ and $\rho > 0$. Then*

$$\begin{aligned} \mathcal{L}^{2(N-1)} \left\{ Y_i = (y_{ij})_{j \neq i} \in \mathbb{R}^{2(N-1)} : \max_{j \neq i} \sup_{t \in [0, T]} |y_{ij}^\varepsilon(t)| \leq \rho \right. \\ \left. \text{and } \min_{j \neq i} \inf_{t \in [0, T]} |y_{ij}^\varepsilon(t)| \leq \varepsilon \right\} \leq C \begin{cases} \varepsilon & \text{if } s > 0.5, \\ \varepsilon \log(1/\varepsilon) & \text{if } s = 0.5, \\ \varepsilon^{2s} & \text{if } s < 0.5. \end{cases} \end{aligned} \quad (2.55)$$

where the constant C only depends on N , $|a_i|$, T and ρ .

The proof of this lemma is reminiscent from the proof of Theorem 2.1 with some new arguments. The main idea is to rely on a Bienaymé-Tchebycheff inequality applied to a well-chosen function. This last estimate (2.55) with Lemma 3.3 concludes the proof of Theorem 2.4. \square

4 Sketch of the proof for Theorem 2.5

The proof is separated into two parts. The first part consists in considering Dirac masses located on the vortices and to prove that this converges in the distributional sense at $t \rightarrow T_X$, the time of collision. The second part uses this distributional convergence and Hypothesis (2.16) to conclude that the vortices converge on \mathbb{R}^2 . The proofs of the different lemmas are provided in section 6.2.

4.1 The vortices are Dirac masses

Lemma 4.1 (Convergence of the Dirac measures). *Let $X \in \mathbb{R}^{2N}$. From the vortices $x_i(t)$ evolving according to the differential equations (2.4) with initial datum X , define the distribution*

$$P_X(t) := \sum_{i=1}^N a_i \delta_{x_i(t)}, \quad (2.56)$$

where δ_x denotes the Dirac mass at point x . Suppose now that the evolution problem associated to the initial datum X is well defined on an interval $[0, T[$ for some $T > 0$. Then, there exist $X^* \in \mathbb{R}^{2N}$ and $b \in \{0, 1\}^N$ such that

$$\sum_{i=1}^N a_i \delta_{x_i(t)} \longrightarrow \sum_{i=1}^N a_i b_i \delta_{x_i^*} \quad \text{in the weak sense of measure as } t \rightarrow T^-. \quad (2.57)$$

Moreover, if $b_i = 0$ then $\sup_{t \in [0, T[} |x_i(t)| = +\infty$.

This lemma makes no use of the non-neutral clusters hypothesis (2.16). In the particular case where this hypothesis is satisfied, theorem 2.2 implies that the vortices stay bounded on bounded intervals of time. Thus, in this particular case all the coefficients b_i given by this lemma worth 1.

Corollary 4.1. *Under Hypothesis (2.16), Lemma 4.1 implies that*

$$\sum_{i=1}^N a_i \delta_{x_i(t)} \longrightarrow \sum_{i=1}^N a_i \delta_{x_i^*} \quad \text{in the weak sense of measure as } t \rightarrow T^-. \quad (2.58)$$

4.2 Convergence of the vortices

The idea here is to exploit more precisely Hypothesis (2.16) to obtain that a given vortex $x_i(t)$ can only converge as $t \rightarrow T^-$. The first lemma below tells us that a given vortex can only have $(x_i^*)_{i=1}^N$ as adherence points.

Lemma 4.2. *Consider the $X^* \in \mathbb{R}^{2N}$ given by Lemma 4.1. Let $z \in \mathbb{R}^2$ such that for all $i = 1 \dots N$, $z \neq x_i^*$. Then, for all $i = 1 \dots N$,*

$$\liminf_{t \rightarrow T^-} |x_i(t) - z| > 0. \quad (2.59)$$

Indeed, if one takes some test function φ with support localized near z then as a consequence of Corollary 4.1, the quantity $\langle \sum_{i=1}^N a_i \delta_{x_i(t)}, \varphi \rangle$ vanishes as $t \rightarrow T^-$. The non-neutral clusters hypothesis (2.16) gives the conclusion. The convergence result is the following.

Lemma 4.3. *Under Hypothesis (2.16), for any initial datum X belonging to the set of collisions \mathfrak{C} , the positions of the vortices $x_i(t)$ are converging in \mathbb{R}^2 as $t \rightarrow T^-$.*

5 Sketch of the proof for Theorem 2.7

5.1 Energy estimate of the clusters of collision

The proof starts by recalling the multi-scale clusterization lemma (Lemma 2.2). We underline the particular role of the first scale of collision $m = 1$. We call the associated clusters P_l^1 the *clusters of collision* since their respective size vanishes as a consequence of (2.34) but the distance between two remains bounded below by a positive constant, consequence of (2.35) with $f_0(z) = 1$. Now that the dynamics of the vortices is separated into clusters, it is possible to study the vorticity energy associated to a given cluster.

Lemma 5.1 (The energy of a cluster). *Let $X \in \mathfrak{C}$ an initial datum for the point-vortex problem associated to the Green function G_1 defined by (2.30) and (2.31) are satisfied at least for $m = 1$ (clusters of collisions well defined). Consider the number $\delta > 0$ and the subsets $\{P_l^1\}_{l=1}^{L_1}$ given by Lemma 2.2.*

Then, for all $l = 1 \dots L_1$,

$$\left| \frac{d}{dt} \sum_{\substack{i, j \in P_l^1 \\ i \neq j}} a_i a_j G_1 |x_i(t) - x_j(t)| \right| \leq \frac{C}{\delta^2} \quad (2.60)$$

where C depends only on the intensities a_i .

The proof uses arguments close to the ones that give the preservation of the Hamiltonian by the motion (Lemma 1.1). A Taylor expansion as $|x_i(t) - x_j(t)| \rightarrow 0^+$ leads to the conclusive estimate.

5.2 Algebraic condition associated to a multi-scale convergence

It is now possible to write the algebraic condition on the intensities that comes from the boundedness of the energy given by Lemma 5.1.

Lemma 5.2. *(Algebraic condition associated to a multi-scale collision) Let $X \in \mathfrak{C}$ such that (2.30) and (2.31) are satisfied. Consider the scaling functions $(f_m)_{m=0}^M$ associated to X . Let $\{P_l^m\}_{l=1}^{L_m}$ be the subsets of $\{1, \dots, N\}$ given by Lemma 2.2. Consider the extra technical hypothesis (2.33) where in particular the coefficients σ_m are defined.*

Then, for all cluster of collision $l = 1 \dots L_1$,

$$\left(\sum_{i \in P_l^1} a_i \right)^2 + \sum_{m=2}^M \alpha_m \sum_{k=1}^{L_m} \left(\sum_{i \in P_k^m \cap P_l^1} a_i \right)^2 = \sum_{i \in P_l^1} a_i^2 + \sum_{m=2}^M \alpha_m \sum_{k=1}^{L_m} \sum_{i \in P_k^m \cap P_l^1} a_i^2. \quad (2.61)$$

where

$$\alpha_m := \left(\sigma_m - 1 \right) \prod_{q=2}^{m-1} \sigma_q > 0. \quad (2.62)$$

In the mono-scale case $M = 1$, the second term appearing in both sides of (2.61) becomes an empty sum. Thus, the algebraic collision condition given by Equation (2.36) gives the condition of *balanced clusters*,

$$\left(\sum_{i \in P_l^1} a_i \right)^2 = \sum_{i \in P_l^1} a_i^2. \quad (2.63)$$

The main interest of the balanced clusters formula is that it implies the non-neutrality of the clusters of collision :

$$\sum_{i \in P_l^1} a_i \neq 0. \quad (2.64)$$

Indeed, suppose that $\sum_{i \in P_l^1} a_i = 0$. Then, the algebraic condition (2.38) implies $\sum_{i \in P_l^1} a_i^2 = 0$. This is incompatible with the fact that $a_i \neq 0$ for all i . The algebraic condition (2.36) can be seen as the multi-scale generalization of (2.38). In the multi-scale situation, the obtained result is weaker.

$$\exists m \in \{1, \dots, M\}, \quad \exists q \in \{1, \dots, L_m\}, \quad \sum_{i \in P_q^m \cap P_l^1} a_i \neq 0. \quad (2.65)$$

Indeed, since $a_i \neq 0$ and $P_l^1 \neq \emptyset$, the term in the right-hand side of (2.36) is positive. But the coefficients α_m are also positive and thus equality (2.36), which implies (2.65). Unfortunately and unlike the mono-scale case, it is not possible to conclude that $\sum_{i \in P_l^1} a_i \neq 0$, because the energy interactions between the different scales of the point-vortex are not known. Therefore, it is not possible so far to obtain a convergence result concerning multi-scale collisions from the tools we have although we can reasonably conjecture it.

5.3 Sketch of the proof for point (iii)

The proof for point (iii) relies on the following Lemma.

Lemma 5.3 (estimate on the distances). *Let $X \in \mathfrak{C}$ an initial value leading to a collision that satisfies (2.30) and (2.31) with $M = 1$ (mono-scale collision). Then there exists C that depends only on the a_i such that, when t is close enough to T_X ,*

$$\forall i, j \in P_l^1, \quad |x_i(t) - x_j(t)| \leq C \sqrt{T_X - t}. \quad (2.66)$$

Recall now that a cluster of collision is non-neutral as stated at (2.40). The fact that the intensities do not sum to 0 inside a cluster of collision P_l^1 implies that the center of vorticity (2.17) of a given cluster is well defined. Concerning the vorticity vector (2.10), we have the following Lemma.

Lemma 5.4 (Lipschitz vorticity vector for a cluster). *Let $X \in \mathfrak{C}$ such that (2.30) and (2.31) are satisfied. Consider the $\delta > 0$ and the subsets $\{P_l^1\}_{l=1}^L$ given by Lemma 2.2. Then, for all $l = 1 \dots L$,*

$$\left| \frac{d}{dt} \sum_{i \in P_l^1} a_i x_i(t) \right| \leq \frac{C}{\delta}, \quad (2.67)$$

where C depends only on the intensities a_i .

The quantity $\sum_{i \in P_l^1} a_i x_i$ is the vorticity vector of the cluster P_l^1 . The proof of this lemma relies on the same arguments that gives the preservation of the vorticity vector (2.10) of the whole system. The consequence is that the center of vorticity of a cluster $t \mapsto B(P_l^1)(t)$ has a limit as $t \rightarrow T_X$. We now observe that for $i \in P_l^1$

$$\left| B(P_l^1) - x_i \right| = \left| \frac{\sum_{j \in P_l^1} a_j x_j}{\sum_{j \in P_l^1} a_j} - x_i \right| = \left| \frac{\sum_{j \in P_l^1} a_j (x_j - x_i)}{\sum_{j \in P_l^1} a_j} \right| \leq \frac{\sum_{j \in P_l^1} |a_j|}{\left| \sum_{j \in P_l^1} a_j \right|} \max_{k, l \in P_l^1} |x_k - x_l|. \quad (2.68)$$

Combining now (2.68) with the fact that $t \mapsto B(P_l^1)(t)$ is lipschitzian (Lemma 5.4) and with the estimate (2.66) gives the announced convergence (2.39). \square

6 Proofs of the main lemmas and theorems

6.1 Proof of Theorem 2.2

The proof is done with a recursive argument on the number of vortices. Denote $x_i^{G,N}(t)$ for $i = 1 \dots N$ the solution to the N point-vortices problem (2.4) that interact through the rotating potential which profile is given by G . We add to this system the action of an exterior field F .

$$\frac{d}{dt} x_i^N = \sum_{\substack{j=1 \\ j \neq i}}^N a_j \nabla^\perp G(|x_i^N(t) - x_j^N(t)|) + F_i(x_i^N(t), t). \quad (2.69)$$

This field is supposed to model the interaction with distant vortices. When vortices are far enough, it is possible to remove them from the system and replace them by this smooth external field. This idea permits to do the recursive argument. Suppose that

$$|F_i(x, t)| \leq \lambda \quad \text{where } \lambda > 0 \text{ is fixed.} \quad (2.70)$$

The recursive hypothesis consists in supposing that for all $k \leq N$ there exists a constant C_k that satisfies

$$\sup_{G \text{ s.t. (2.18)}} \max_{X \in \mathbb{R}^{2k}} \sup_{t \in [0, T]} |S_{G,k}^t X - X| \leq C_k, \quad (2.71)$$

where $S_{G,k}^t$ designates the flow of the equation (2.69) with k vortices and associated to the profile G . We are going to prove that there exist a constant C_{N+1} such that

$$\sup_{G \text{ s.t. (2.18)}} \max_{X \in \mathbb{R}^{2(N+1)}} \sup_{t \in [0, T]} |S_{G,N}^t X - X| \leq C_{N+1}, \quad (2.72)$$

Suppose by the absurd that “ $C_{N+1} = +\infty$ ”. More precisely, suppose that for S chosen arbitrarily large, there exist a profile G , a time $t^* \in [0, T]$ and an index j such that

$$|x_j^{G,N+1}(t^*) - x_j| = S. \quad (2.73)$$

Define

$$a = \max_{i=1 \dots N+1} |a_i|, \quad (2.74)$$

$$A = \min_{\substack{\mathcal{P} \subseteq \{1, \dots, N+1\} \\ \mathcal{P} \neq \emptyset}} \left| \sum_{i \in \mathcal{P}} a_i \right|, \quad (2.75)$$

$$M_{N+1}^G(t) := \sum_{i=1}^{N+1} a_i x_i^{G,N+1}(t). \quad (2.76)$$

Using Lemma 1.3, we get that the vorticity vector varies only under the action of the external field F . This external field is bounded by λ as stated in (2.70) and then

$$|M_{N+1}^G(t) - M_{N+1}^G(0)| = \left| \sum_{i=1}^{N+1} a_i \int_0^t F(x_i^{G,N+1}(t'), t') dt' \right| \leq (N+1) \lambda a T =: b. \quad (2.77)$$

On the other hand,

$$|M_{N+1}^G(t^*) - M_{N+1}^G(0)| = \left| \sum_{i=1}^{N+1} a_i (x_j^{G,N+1}(t^*) - x_j) + \sum_{i=1}^{N+1} a_i (z_i^{G,N+1}(t^*) - z_i^{G,N+1}) \right| \quad (2.78)$$

with $z_i := x_i^{G,N+1} - x_j^{G,N+1}$ and where t^* and j are given by (2.73). Combining (2.73), (2.77) and (2.78) with a triangular inequality

$$S \left| \sum_{i=1}^{N+1} a_i \right| \leq b + \left| \sum_{i=1}^{N+1} a_i (z_i^{G,N+1}(t^*) - z_i^{G,N+1}) \right|. \quad (2.79)$$

As a consequence and since $|\sum_{i=1}^{N+1} a_i| \geq A$,

$$SA - b \leq a(N+1) \max_{i=1 \dots N+1} |z_i^{G,N+1}(t^*) - z_i^{G,N+1}|. \quad (2.80)$$

We note k the index for which is realised the maximum above and then we obtain

$$|z_k^{G,N+1}(t^*) - z_k^{G,N+1}| \geq \frac{SA - b}{a(N+1)}. \quad (2.81)$$

The first consequence is the existence of a time \hat{t} for which the distance between the particles of indices respectively j and k is bigger than

$$\frac{SA - b}{2a(N+1)}.$$

At such a time, the particles are divided into two clusters separated by a distance d that is at least

$$\frac{SA - b}{2a(N+1)^2}.$$

Indeed, the least favorable case consists in the situation where all the remaining vortices forms a rectilinear chain made of $N+1-2$ points spaced with intervals of same length and that link the vortex of index j to the vortex of index k .

The non-neutral clusters hypothesis given by (2.16) implies that $A > 0$. Therefore, if S is chosen very large then so is d . Using now (2.18), we get that the interaction between these two clusters is bounded by

$$(N+1) a J \left(\frac{SA - b}{2a(N+1)^2} \right). \quad (2.82)$$

Yet, the function J vanishes at infinity by hypothesis. Thus, we can choose S so big that the expression above is smaller than the λ defined at (2.70). This means that we can apply the recursive hypothesis (2.71) to each of these clusters which are made of a number of vortices equal or lower than $N+1$. At time \hat{t} , the vortices of one cluster evolve under the action of the other vortices of the cluster and under the action of an exterior field which strength is bounded by λ . The recursive hypothesis provides a constant C_N such that on an interval of time $[0, T]$ the vortices do not move on a distance larger than C_N . We now choose the parameter S much bigger in such a way that the distance d between the clusters cannot be filled during the interval of time $[0, T]$. This gives a contradiction because in this configuration the displacement of the vortices is uniformly bounded by C_N whereas it is supposed that at least one moves on a distance S and S can be chosen strictly larger than C_N . \square

6.2 Proofs of the lemmas for Theorem 2.5

6.2.1 Proof of Lemma 4.1

Let $P_X(t)$ be the distribution defined by (2.56), supposed well-defined for all $t \in [0, T[$. Let $\varphi \in \mathcal{D}(\mathbb{R}^2)$. Then,

$$\frac{d}{dt} \langle P_X(t), \varphi \rangle_{\mathcal{D}'\mathcal{D}} = \frac{d}{dt} \sum_{i=1}^N a_i \varphi(x_i(t)) = \sum_{i=1}^N a_i \nabla \varphi(x_i(t)) \cdot \frac{d}{dt} x_i(t). \quad (2.83)$$

The equations of motion (2.4) give

$$\begin{aligned} \frac{d}{dt} \langle P_X(t), \varphi \rangle_{\mathcal{D}'\mathcal{D}} &= \sum_{i \neq j} a_i a_j \nabla \varphi(x_i(t)) \cdot \frac{(x_j(t) - x_i(t))^\perp}{|x_j(t) - x_i(t)|^2} \\ &= \frac{1}{2} \sum_{i \neq j} a_i a_j \left(\nabla \varphi(x_i(t)) - \nabla \varphi(x_j(t)) \right) \cdot \frac{(x_j(t) - x_i(t))^\perp}{|x_j(t) - x_i(t)|^2}. \end{aligned} \quad (2.84)$$

Therefore,

$$\left| \frac{d}{dt} \langle P_X(t), \varphi \rangle_{\mathcal{D}'\mathcal{D}} \right| \leq \frac{1}{2} \sum_{i \neq j} |a_i a_j| \|\nabla^2 \varphi\|_\infty. \quad (2.85)$$

Then, $t \mapsto \langle P_X(t), \varphi \rangle_{\mathcal{D}'\mathcal{D}}$ is Lipschitz and converges as $t \rightarrow T^-$. Since this holds for any $\varphi \in \mathcal{D}(\mathbb{R}^2)$, this implies that $P_X(t)$ converges in the sense of distributions towards some $P_X \in \mathcal{D}'(\mathbb{R}^2)$ as $t \rightarrow T^-$. There remains to prove that P_X is actually a measure that takes the form given by (2.57). Consider now an increasing sequence (t_n) converging towards T^- . Remark first that it is always possible, up to an omitted extraction of the sequence, to reduce the problem to

$$\text{either } x_i(t_n) \rightarrow x_i^* \quad \text{or} \quad |x_i(t_n)| \rightarrow +\infty \quad (2.86)$$

for some $X^* \in \mathbb{R}^{2N}$. Indeed, if $|x_i(t_n)| \rightarrow +\infty$ is not satisfied then there exists an extraction such that $x_i(t_n)$ stays bounded. But if it stays bounded then another extraction makes this sequence converge towards some x_i^* . Repeating this process for i from 1 to N gives (2.86). Now that (2.86) holds, define

$$b_i = \begin{cases} 0 & \text{if } |x_i(t_n)| \rightarrow +\infty, \\ 1 & \text{either.} \end{cases} \quad (2.87)$$

Therefore it holds

$$\sum_{i=1}^N a_i \delta_{x_i(t_n)} \rightarrow \sum_{i=1}^N a_i b_i \delta_{x_i^*} \quad \text{as } n \rightarrow +\infty. \quad (2.88)$$

By uniqueness of the limit, it is possible to identify

$$P_X = \sum_{i=1}^N a_i b_i \delta_{x_i^*}. \quad (2.89)$$

The fact that the convergence of $P_X(t)$ towards P_X in \mathcal{D}' is actually a convergence in the weak sense of measure comes from the fact that the measure $P_X(t)$ is bounded by $\sum_i |a_i|$ for all t . \square

6.2.2 Proof of Lemma 4.2

Consider the $X^* \in \mathbb{R}^{2N}$ given by Lemma 4.1. Let $z \in \mathbb{R}^2$ such that for all $i = 1 \dots N$, $z \neq x_i^*$. Suppose by the absurd that there exists $A \subseteq \{1 \dots N\}$ with $A \neq \emptyset$ such that for all $i \in A$,

$$\liminf_{t \rightarrow T^-} |x_i(t) - z| = 0. \quad (2.90)$$

This set A can be chosen such that for all $i \notin A$,

$$\liminf_{t \rightarrow T^-} |x_i(t) - z| > 0. \quad (2.91)$$

Define

$$\begin{aligned} d_z^1 &:= \min \{|x_i^* - z| : i = 1 \dots N\} > 0, \\ d_z^2 &:= \min \{\liminf_{t \rightarrow T^-} |x_i(t) - z| : i \notin A\} > 0, \\ d_z^* &:= \min \{d_z^1, d_z^2\} > 0, \end{aligned} \quad (2.92)$$

where by convention, for the definition of d_z^2 , the minimum of the empty set is $+\infty$. Let φ be a \mathcal{C}^∞ function supported on the ball $\mathcal{B}(z, d_z^*/2)$ and equal to 1 on the ball $\mathcal{B}(z, d_z^*/4)$. As a consequence of Corollary 4.1 and by definition of d_z^* ,

$$\left\langle \sum_{i=1}^N a_i \delta_{x_i(t)}, \varphi \right\rangle_{\mathcal{D}'\mathcal{D}} \rightarrow \left\langle \sum_{i=1}^N a_i \delta_{x_i^*}, \varphi \right\rangle_{\mathcal{D}'\mathcal{D}} = 0 \quad \text{as } t \rightarrow T^-. \quad (2.93)$$

As a consequence of (2.90) and (2.91), there exists an increasing sequence $(t_n)_{n \in \mathbb{N}}$ converging towards T^- such that for all $n \in \mathbb{N}$,

$$\forall i \in A, \quad x_i(t_n) \in \mathcal{B}\left(z, \frac{d_z^*}{4}\right) \quad \text{and} \quad \forall i \notin A, \quad x_i(t_n) \notin \mathcal{B}\left(z, \frac{d_z^*}{2}\right). \quad (2.94)$$

With the definition of φ and the definition of d_z^* given at (2.92), holds for all $n \in \mathbb{N}$,

$$\left\langle \sum_{i=1}^N a_i \delta_{x_i(t_n)}, \varphi \right\rangle_{\mathcal{D}', \mathcal{D}} = \sum_{i \in A} a_i. \quad (2.95)$$

As a consequence of the non-neutral clusters hypothesis (2.16), Equations (2.93) and (2.95) are in contradiction. \square

6.2.3 Proof of Lemma 4.3

Suppose that a given vortex $x_i(t)$ has two adherence points. By Lemma 4.2, these two points must be some x_i^* . For instance, x_j^* and x_k^* with j and k such that $x_j^* \neq x_k^*$. Define the smallest distance between x_j^* and any other possible adherence point

$$r_j^* := \min \left\{ r > 0 : \exists l = 1 \dots N, |x_j^* - x_l^*| = r \right\}. \quad (2.96)$$

Consider then the circle

$$\mathcal{S} := \left\{ x \in \mathbb{R}^2 : |x - x_j^*(t)| = \frac{1}{2} r_j^* \right\}. \quad (2.97)$$

Since x_j^* is inside the ball of radius $r_j^*/2$ and x_k^* is outside, since these two points are adherence points for the dynamics of $x_i(t)$ as $t \rightarrow T^-$ and since the trajectories are continuous, there exist an increasing sequence of time $(t_n)_{n \in \mathbb{N}}$ converging towards T^- such that

$$\forall n \in \mathbb{N}, \quad x_i(t_n) \in \mathcal{S}. \quad (2.98)$$

By compactness of \mathcal{S} , it can be supposed that, up to an extraction, $x_i(t_n) \rightarrow x^* \in \mathcal{S}$ as $n \rightarrow \infty$. By definition of r_j^* , for all $l = 1 \dots N$, $x_l^* \notin \mathcal{S}$. These two facts together are in contradiction with Lemma 4.2. \square

6.3 Proofs of the lemmas for Theorem 2.4

6.3.1 Proof of Lemma 3.1

First, Theorem 2.4 can be formulated as follows

$$\mathcal{L}^{2N} \left\{ X \in \mathbb{R}^{2N} : \exists T_X \in \mathbb{R}_+, \quad \liminf_{t \rightarrow T_X^-} \min_{i \neq j} |x_i(t) - x_j(t)| = 0 \right. \\ \left. \text{and} \quad \limsup_{t \rightarrow T_X^-} \max_{i,j} |x_i(t) - x_j(t)| < +\infty. \right\} = 0, \quad (2.99)$$

where \mathcal{L}^d refers to the Lebesgue measure of dimension d . It is possible to reduce the problem to bounded intervals of time and bounded regions of space by rewriting (2.99) as follows.

$$\mathcal{L}^{2N} \left(\bigcup_{T=1}^{+\infty} \bigcup_{\rho=1}^{+\infty} \left\{ X \in \mathbb{R}^{2N} : \exists T_X \in [0, T], \quad \liminf_{t \rightarrow T_X^-} \min_{i \neq j} |x_i(t) - x_j(t)| = 0 \right. \right. \\ \left. \left. \text{and} \quad \limsup_{t \rightarrow T_X^-} \max_{i,j} |x_i(t) - x_j(t)| \leq \rho. \right\} \right) = 0. \quad (2.100)$$

Since the reunion above is a countable reunion, it is enough to prove that for all $T > 0$ and $\rho > 0$,

$$\mathcal{L}^{2N} \left\{ X \in \mathbb{R}^{2N} : \exists T_X \in [0, T], \quad \liminf_{t \rightarrow T_X^-} \min_{i \neq j} |x_i(t) - x_j(t)| = 0 \right. \\ \left. \text{and} \quad \limsup_{t \rightarrow T_X^-} \max_{i,j} |x_i(t) - x_j(t)| \leq \rho. \right\} = 0. \quad (2.101)$$

Now, let $T > 0$ and $\rho > 0$. For i fixed in $\{1 \dots N\}$ denote by \mathcal{T}_i the isomorphism that gives the position of the point vortices $(x_k)_{k=1}^N$ knowing the position of x_i and knowing the differences $(y_{ij})_{j \neq i}$. In other words define,

$$\mathcal{T}_i : \mathbb{R}^2 \times \mathbb{R}^{2(N-1)} \rightarrow \mathbb{R}^{2N} \\ (x, Y) \mapsto (x + y_{i1}, \dots, x + y_{i(i-1)}, x, x + y_{i(i+1)}, \dots, x + y_{iN}). \quad (2.102)$$

Thus, the following inclusion holds

$$\left\{ X \in \mathbb{R}^{2N} : \exists T_X \in [0, T], \liminf_{t \rightarrow T_X^-} \min_{i=1 \dots N} \min_{\substack{j=1 \dots N \\ j \neq i}} |x_i(t) - x_j(t)| = 0 \right. \\ \left. \text{and } \limsup_{t \rightarrow T_X^-} \max_{i=1 \dots N} \max_{j=1 \dots N} |x_i(t) - x_j(t)| \leq \rho. \right\} \\ = \bigcap_{i=1}^N \left\{ X \in \mathbb{R}^{2N} : \exists T_X \in [0, T], \liminf_{t \rightarrow T_X^-} \min_{\substack{j=1 \dots N \\ j \neq i}} |x_i(t) - x_j(t)| = 0 \right. \\ \left. \text{and } \limsup_{t \rightarrow T_X^-} \max_{j=1 \dots N} |x_i(t) - x_j(t)| \leq \rho. \right\} \quad (2.103) \\ \subseteq \bigcap_{i=1}^N \mathcal{T}_i \left[\mathbb{R}^2 \times \left\{ Y_i := (y_{ij})_{j \neq i} \in \mathbb{R}^{2(N-1)} : \exists T_X \in [0, T], \liminf_{t \rightarrow T_X^-} \min_{\substack{j \neq i}} |y_{ij}(t)| = 0 \right. \right. \\ \left. \left. \text{and } \limsup_{t \rightarrow T_X^-} \max_{j=1 \dots N} |y_{ij}(t)| \leq \rho. \right\} \right].$$

Using hypothesis (2.44) and the Fubini theorem,

$$\mathcal{L}^{2N} \left(\mathbb{R}^2 \times \left\{ Y_i := (y_{ij})_{j \neq i} \in \mathbb{R}^{2(N-1)} : \exists T_X \in [0, T], \liminf_{t \rightarrow T_X^-} \min_{\substack{j \neq i}} |y_{ij}(t)| = 0 \right. \right. \\ \left. \left. \text{and } \limsup_{t \rightarrow T_X^-} \max_{j=1 \dots N} |y_{ij}(t)| \leq \rho. \right\} \right) = 0. \quad (2.104)$$

Since \mathcal{T}_i is a linear map, it is absolutely continuous and therefore maps any sets of Lebesgue measure 0 into sets of Lebesgue measure 0. Therefore, combining this fact with (2.103) and (2.104) gives (2.101). \square

6.3.2 Proof of Lemma 3.3

Let $T > 0$, $\rho > 0$ and $\varepsilon > 0$. For $i \neq j$, we define the set of initial datum such that occurs an ε -collision between the two vortices x_i and x_j

$$\Gamma_{ij}^{\varepsilon, \rho} := \left\{ Y_i = (y_{il})_{l \neq i} \in \mathbb{R}^{2(N-1)} : \exists t \in [0, T], |y_{ij}(t)| \leq \varepsilon \right. \\ \left. \text{and } \forall l \neq i, \sup_{\tau \in [0, t]} |y_{il}(\tau)| \leq \rho \right\}. \quad (2.105)$$

We also define the time at which occurs the ε -collision. Let $Y_i \in \bigcup_{j \neq i} \Gamma_{ij}^{\varepsilon, \rho}$,

$$T_{Y_i}^\varepsilon := \inf \left\{ t \in [0, T] : \min_{\substack{j \neq i}} |y_{ij}(t)| \leq \varepsilon \right\}. \quad (2.106)$$

We are also interested in the situations where other collisions occur, far from x_i . This corresponds to the ε -collisions of vector $y_{jk} := x_j - x_k = y_{ij} - y_{ik}$. Let $k \neq i, j$, define

$$\Gamma_{ijk}^{\varepsilon, \rho} := \left\{ Y_i \in \Gamma_{ij}^{\varepsilon, \rho} : \exists t < T_{Y_i}^\varepsilon, |y_{ij}(t) - y_{ik}(t)| \leq \varepsilon \right\}. \quad (2.107)$$

The fact that $\Gamma_{ijk}^{\varepsilon, \rho}$ gives information on whether another ε -collision occurs far from x_i with x_j before the expected ε -collision between x_i and x_j implies the following inclusion.

$$\Gamma_{ijk}^{\varepsilon, \rho} \subseteq \Gamma_{kj}^{\varepsilon, 2\rho} \setminus \Gamma_{kji}^{\varepsilon, 2\rho}. \quad (2.108)$$

Another inclusion is

$$\Gamma_{ijk}^{\varepsilon, \rho} \subseteq \Gamma_{ijk}^{\varepsilon, 2\rho}. \quad (2.109)$$

These two definitions (2.105) and (2.107) study the ε -collision on the exact system with kernel G_s . We need the same definitions with the regularized kernels $G_{s, \varepsilon}$.

$$\begin{aligned} \widehat{\Gamma}_{ij}^{\varepsilon, \rho} &:= \left\{ Y_i = (y_{il})_{l \neq i} \in \mathbb{R}^{2(N-1)} : \exists t \in [0, T], \quad |y_{ij}^\varepsilon(t)| \leq \varepsilon \right. \\ &\quad \left. \text{and } \forall l \neq i, \sup_{\tau \in [0, t]} |y_{il}^\varepsilon(\tau)| \leq \rho \right\}, \\ \widehat{T}_{Y_i}^\varepsilon &:= \inf \left\{ t \in [0, T] : \min_{j \neq i} |y_{ij}^\varepsilon(t)| \leq \varepsilon \right\}, \\ \widehat{\Gamma}_{ijk}^{\varepsilon, \rho} &:= \left\{ Y_i \in \widehat{\Gamma}_{ij}^{\varepsilon, \rho} : \exists t < \widehat{T}_{Y_i}^\varepsilon, \quad |y_{ij}^\varepsilon(t) - y_{ik}^\varepsilon(t)| \leq \varepsilon \right\}. \end{aligned} \quad (2.110)$$

Remark now that as long as the quantities $|y_{ij}|$ and $|y_{ij} - y_{ik}|$ remain higher than ε for all $j \neq i$ and $k \neq i, j$, then the dynamics of y_{ij} and y_{ij}^ε coincide as a consequence of (2.50). This property can be rephrased using the sets defined at (2.105), (2.107) and (2.110) as follows.

$$\Gamma_{ij}^{\varepsilon, \rho} \setminus \left(\bigcup_{k \neq i, j} \Gamma_{ijk}^{\varepsilon, \rho} \right) = \widehat{\Gamma}_{ij}^{\varepsilon, \rho} \setminus \left(\bigcup_{k \neq i, j} \widehat{\Gamma}_{ijk}^{\varepsilon, \rho} \right). \quad (2.111)$$

The hypothesis of Lemma 3.3 are that for all $\rho > 0$, and for all $i \neq j$,

$$\mathcal{L}^{2N} \left(\widehat{\Gamma}_{ij}^{\varepsilon, \rho} \right) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+. \quad (2.112)$$

Concerning the conclusion, it is enough to prove that for all $\rho > 0$, and for all $i \neq j$,

$$\mathcal{L}^{2N} \left(\Gamma_{ij}^{\varepsilon, \rho} \right) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0^+, \quad (2.113)$$

because

$$\begin{aligned} \left\{ Y_i := (y_{ij})_{j \neq i} \in \mathbb{R}^{2(N-1)} : \exists T_X \in [0, T], \quad \liminf_{t \rightarrow T_X^-} \min_{j \neq i} |y_{ij}(t)| = 0 \right. \\ \left. \text{and } \limsup_{t \rightarrow T_X^-} \max_{j=1 \dots N} |y_{ij}(t)| \leq \rho \right\} = \bigcap_{n=1}^{+\infty} \bigcup_{j \neq i} \Gamma_{ij}^{\frac{1}{n}, \rho}. \end{aligned} \quad (2.114)$$

The fact that the convergences (2.112) imply the convergences (2.113) is given by the following computations. First (2.108) gives.

$$\Gamma_{ij}^{\varepsilon, \rho} = \left[\Gamma_{ij}^{\varepsilon, \rho} \setminus \left(\bigcup_{k \neq i, j} \Gamma_{ijk}^{\varepsilon, \rho} \right) \right] \cup \left(\bigcup_{k \neq i, j} \Gamma_{ijk}^{\varepsilon, \rho} \right) \subseteq \left[\Gamma_{ij}^{\varepsilon, \rho} \setminus \left(\bigcup_{k \neq i, j} \Gamma_{ijk}^{\varepsilon, \rho} \right) \right] \cup \left(\bigcup_{k \neq i, j} \Gamma_{kj}^{\varepsilon, 2\rho} \setminus \Gamma_{kji}^{\varepsilon, 2\rho} \right). \quad (2.115)$$

Remark that it is possible to do the same computation with the remaining term on the very right using again (2.108) and this gives

$$\begin{aligned} \Gamma_{kj}^{\varepsilon, 2\rho} \setminus \Gamma_{kji}^{\varepsilon, 2\rho} &= \left[\Gamma_{kj}^{\varepsilon, 2\rho} \setminus \left(\bigcup_{l \neq k, j} \Gamma_{kjl}^{\varepsilon, 2\rho} \right) \right] \cup \left(\bigcup_{l \neq i, j, k} \Gamma_{kjl}^{\varepsilon, 2\rho} \right) \\ &\subseteq \left[\Gamma_{kj}^{\varepsilon, 2\rho} \setminus \left(\bigcup_{l \neq k, j} \Gamma_{kjl}^{\varepsilon, 2\rho} \right) \right] \cup \left(\bigcup_{l \neq i, j, k} \Gamma_{lj}^{\varepsilon, 4\rho} \setminus \Gamma_{ljk}^{\varepsilon, 4\rho} \right). \end{aligned} \quad (2.116)$$

Doing the same computation as above recursively until the residual term is empty and using (2.109) transforms (2.115) into

$$\Gamma_{ij}^{\varepsilon, \rho} \subseteq \bigcup_{k \neq j} \left[\Gamma_{kj}^{\varepsilon, 2^N \rho} \setminus \left(\bigcup_{l \neq j, k} \Gamma_{kjl}^{\varepsilon, 2^N \rho} \right) \right]. \quad (2.117)$$

Using now (2.111), we finally get

$$\Gamma_{ij}^{\varepsilon, \rho} \subseteq \bigcup_{k \neq j} \left[\widehat{\Gamma}_{kj}^{\varepsilon, 2^N \rho} \setminus \left(\bigcup_{l \neq j, k} \widehat{\Gamma}_{kjl}^{\varepsilon, 2^N \rho} \right) \right] \subseteq \bigcup_{k \neq j} \widehat{\Gamma}_{kj}^{\varepsilon, 2^N \rho}. \quad (2.118)$$

Thus the convergences (2.112) imply the convergences (2.113) and the lemma is proved. \square

6.3.3 Proof of Lemma 3.4

Let $i \in \{1 \dots N\}$, $\varepsilon > 0$ and $\rho > 0$.

Step 1. Define

$$B_i^{\varepsilon, \rho} := \left\{ Y_i = (y_{ij})_{j \neq i} \in \mathbb{R}^{2(N-1)} : \max_{j \neq i} \sup_{t \in [0, T]} |y_{ij}^\varepsilon(t)| \leq \rho \right\}. \quad (2.119)$$

Let $a > 0$. We define a kernel L_a by

$$L_a(q) := q^{-2-a}. \quad (2.120)$$

and we associate to the kernel L_a its ε -regularization $L_{a, \varepsilon}$ as defined by (2.50)-(2.53). From this we define the function

$$\Phi(Y_i) := \sum_{j \neq i} L_{a, \varepsilon}(|y_{ij}|). \quad (2.121)$$

This function is all the most high valued as the system is close to collision. Denote by $\mathfrak{S}_{i, \varepsilon}^t$ the flow of the modified system (2.42) with the regularized kernel $G_{s, \varepsilon}$. This gives

$$\begin{aligned} \frac{d}{dt} \Phi(\mathfrak{S}_{i, \varepsilon}^t Y_i) &= \sum_{j \neq i} \nabla L_{a, \varepsilon}(|y_{ij}^\varepsilon(t)|) \cdot \frac{d}{dt} y_{ij}^\varepsilon(t), \\ &= \sum_{j \neq i} \nabla L_{a, \varepsilon}(|y_{ij}^\varepsilon(t)|) \cdot \left[(a_i + a_j) \nabla^\perp G_{s, \varepsilon}(|y_{ij}^\varepsilon|) + \sum_{k \neq i, j} a_k \left(\nabla^\perp G_{s, \varepsilon}(|y_{ik}^\varepsilon|) + \nabla^\perp G_{s, \varepsilon}(|y_{ij}^\varepsilon - y_{ik}^\varepsilon|) \right) \right], \\ &= \sum_{j \neq i} \sum_{k \neq i, j} a_k \nabla L_{a, \varepsilon}(|y_{ij}^\varepsilon(t)|) \cdot \left(\nabla^\perp G_{s, \varepsilon}(|y_{ik}^\varepsilon|) + \nabla^\perp G_{s, \varepsilon}(|y_{ij}^\varepsilon - y_{ik}^\varepsilon|) \right), \end{aligned} \quad (2.122)$$

where for the last equality we used the identity $\nabla f \cdot \nabla^\perp g = 0$ that holds for f and g two radial functions. Thus,

$$\left| \frac{d}{dt} \Phi(\mathfrak{S}_{i, \varepsilon}^t Y_i) \right| \leq \Psi(\mathfrak{S}_{i, \varepsilon}^t Y_i), \quad (2.123)$$

where

$$\Psi(Y_i) := \sum_{j \neq i} \sum_{k \neq i, j} a_k |\nabla L_{a, \varepsilon}(|y_{ij}|)| \left(|\nabla G_{s, \varepsilon}(|y_{ik}|)| + |\nabla G_{s, \varepsilon}(|y_{ij} - y_{ik}|)| \right). \quad (2.124)$$

We now observe, recalling the definition of G_s given at (2.3), that the ε -regularization (2.50)-(2.53) implies, by a direct computation using polar coordinates,

$$\int_{\mathcal{B}(0, \rho)} |\nabla G_{s, \varepsilon}(|y|)| dy \leq C \begin{cases} 1 & \text{if } s > 0.5, \\ \log(1/\varepsilon) & \text{if } s = 0.5, \\ \varepsilon^{2s-1} & \text{if } s < 0.5, \end{cases} \quad (2.125)$$

where $\mathcal{B}(0, \rho)$ is the euclidean ball on \mathbb{R}^2 . The constant C depends on ρ and s . Similarly with the definition of the kernel L_a at (2.120) and since $a > 0$,

$$\int_{\mathcal{B}(0, \rho)} L_{a, \varepsilon}(|y|) dy \leq C \varepsilon^{-a} \quad \text{and} \quad \int_{\mathcal{B}(0, \rho)} |\nabla L_{a, \varepsilon}(|y|)| dy \leq C \varepsilon^{-1-a}. \quad (2.126)$$

Therefore using (2.126) with the definition of Φ gives

$$\int_{\mathcal{B}(0, \rho)^{N-1}} \Phi(Y_i) dY_i \leq C \varepsilon^{-a} \quad \text{and} \quad \int_{\mathcal{B}(0, \rho)^{N-1}} \Psi(Y_i) dY_i \leq C \varepsilon^{-1-a}. \quad (2.127)$$

Similarly, the definition of Ψ given at (2.124) gives

$$\begin{aligned} &\int_{\mathcal{B}(0, \rho)^{N-1}} \Psi(Y_i) dY_i \\ &= \int_{\mathcal{B}(0, \rho)^{N-1}} \left[\sum_{j \neq i} \sum_{k \neq i, j} a_k |\nabla L_{a, \varepsilon}(|y_{ij}|)| \left(|\nabla G_{s, \varepsilon}(|y_{ik}|)| + |\nabla G_{s, \varepsilon}(|y_{ij} - y_{ik}|)| \right) \right] \prod_{\substack{l=1 \\ l \neq i}}^N dy_{il} \\ &= 2 \left(\sum_{j \neq i} \sum_{k \neq i, j} a_k \right) \left(\int_{\mathcal{B}(0, \rho)} dy \right)^{N-3} \left(\int_{\mathcal{B}(0, \rho)} |\nabla L_{a, \varepsilon}(|y|)| dy \right) \left(\int_{\mathcal{B}(0, \rho)} |\nabla G_{s, \varepsilon}(|y|)| dy \right), \end{aligned} \quad (2.128)$$

and then, using (2.125) and (2.126),

$$\int_{\mathcal{B}(0,\rho)^{N-1}} \Psi(Y_i) dY_i \leq C\varepsilon^{-2-a} \begin{cases} \varepsilon & \text{if } s > 0.5, \\ \varepsilon \log(1/\varepsilon) & \text{if } s = 0.5, \\ \varepsilon^{2s} & \text{if } s < 0.5. \end{cases} \quad (2.129)$$

Step 2. It is now possible to integrate Φ along the flow on the set (2.119). We obtain

$$\int_{B_i^{\varepsilon,\rho}} \sup_{t \in [0,T]} \Phi(\mathfrak{S}_{i,\varepsilon}^t Y_i) dY_i \leq \int_{B_i^{\varepsilon,\rho}} \Phi(Y_i) dY_i + \int_{B_i^{\varepsilon,\rho}} \int_0^T \left| \frac{d}{dt} \Phi(\mathfrak{S}_{i,\varepsilon}^t Y_i) \right| dt dY_i \quad (2.130)$$

Using (2.123) in the estimate above gives

$$\int_{B_i^{\varepsilon,\rho}} \sup_{t \in [0,T]} \Phi(\mathfrak{S}_{i,\varepsilon}^t Y_i) dY_i \leq \int_{B_i^{\varepsilon,\rho}} \Phi(Y_i) dY_i + \int_{B_i^{\varepsilon,\rho}} \int_0^T \Psi(\mathfrak{S}_{i,\varepsilon}^t Y_i) dt dY_i. \quad (2.131)$$

Using the Fubini theorem in (2.131) and the Liouville theorem 3.2 leads to

$$\int_{B_i^{\varepsilon,\rho}} \sup_{t \in [0,T]} \Phi(\mathfrak{S}_{i,\varepsilon}^t Y_i) dY_i \leq \int_{B_i^{\varepsilon,\rho}} \Phi(Y_i) dY_i + \int_0^T \int_{\mathfrak{S}_{i,\varepsilon}^t B_i^{\varepsilon,\rho}} \Psi(Y_i) dY_i dt. \quad (2.132)$$

By definition of $B_i^{\varepsilon,\rho}$, we have for all $t \in [0, T]$, $\mathfrak{S}_{i,\varepsilon}^t B_i^{\varepsilon,\rho} \subseteq \mathcal{B}(0, \rho)^{N-1}$. Thus, the estimate above becomes

$$\begin{aligned} \int_{B_i^{\varepsilon,\rho}} \sup_{t \in [0,T]} \Phi(\mathfrak{S}_{i,\varepsilon}^t Y_i) dY_i &\leq \int_{B_i^{\varepsilon,\rho}} \Phi(Y_i) dY_i + \int_0^T \int_{\mathcal{B}(0,\rho)^{N-1}} \Psi(Y_i) dY_i dt. \\ &\leq C\varepsilon^{-a} + CT\varepsilon^{-2-a} \begin{cases} \varepsilon & \text{if } s > 0.5, \\ \varepsilon \log(1/\varepsilon) & \text{if } s = 0.5, \\ \varepsilon^{2s} & \text{if } s < 0.5. \end{cases}, \end{aligned} \quad (2.133)$$

where for the last estimate we used (2.127) and (2.129).

Step 3. By definition of the function Φ , there exists a constant $c > 0$ such that,

$$\begin{aligned} \left\{ Y_i = (y_{ij})_{j \neq i} \in B_i^{\varepsilon,\rho} : \min_{j \neq i} \inf_{t \in [0,T]} |y_{ij}^\varepsilon(t)| \leq \varepsilon \right\} \\ \subseteq \left\{ Y_i = (y_{ij})_{j \neq i} \in B_i^{\varepsilon,\rho} : \sup_{t \in [0,T]} \Phi(\mathfrak{S}_{i,\varepsilon}^t Y_i) \geq c\varepsilon^{-2-a} \right\}. \end{aligned} \quad (2.134)$$

Combining this inclusion with the Bienaymé-Tchebycheff inequality gives

$$\mathcal{L}^{2(N-1)} \left\{ Y_i = (y_{ij})_{j \neq i} \in B_i^{\varepsilon,\rho} : \min_{j \neq i} \inf_{t \in [0,T]} |y_{ij}^\varepsilon(t)| \leq \varepsilon \right\} \leq \frac{\varepsilon^{2+a}}{c} \int_{B_i^{\varepsilon,\rho}} \sup_{t \in [0,T]} \Phi(\mathfrak{S}_{i,\varepsilon}^t Y_i) dY_i. \quad (2.135)$$

Using now (2.133) in (2.135),

$$\mathcal{L}^{2(N-1)} \left\{ Y_i = (y_{ij})_{j \neq i} \in B_i^{\varepsilon,\rho} : \min_{j \neq i} \inf_{t \in [0,T]} |y_{ij}^\varepsilon(t)| \leq \varepsilon \right\} \leq C \begin{cases} \varepsilon & \text{if } s > 0.5, \\ \varepsilon \log(1/\varepsilon) & \text{if } s = 0.5, \\ \varepsilon^{2s} & \text{if } s < 0.5. \end{cases}, \quad (2.136)$$

where C is a constant that depends on T , ρ , N and on the a_i . The lemma is proved. \square

6.4 Proofs of the lemmas for Theorem 2.7

6.4.1 Proof of Lemma 2.2

Let $m \in \{1, \dots, M\}$. The clusters P_l^m are built the following way. Let $i \in \{1, \dots, N\}$ and consider

$$\left\{ j \in \{1, \dots, N\} : \exists C > 0, \quad |x_i(t) - x_j(t)| \leq C f_m(T_X - t) \right\}, \quad (2.137)$$

the set of vortices that collapse with vortex x_i at a scale lower or equal to the the scale associated to f_m . If this set is reduced to $\{i\}$ then just drop vortex x_i . Otherwise, the set (2.137) defines the cluster

P_1^m . Do this process recursively with the remaining vortices and this defines the clusterization $\{P_l^m\}_{l=1}^{L_m}$. Property (2.34) follows from this construction.

Concerning Property (2.35), let $i \in P_l^m$ and $j \notin P_l^m$. As a consequence of the construction given by (2.137), there exist a function $\gamma : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ that is unbounded near 0 and such that

$$\gamma(T_X - t) f_m(T_X - t) \leq |x_i(t) - x_j(t)|. \quad (2.138)$$

Using now (2.31), there exists constants $C > c > 0$ and a $q \in \{1, \dots, M\}$ such that

$$c f_q(T_X - t) \leq |x_i(t) - x_j(t)| \leq C f_q(T_X - t). \quad (2.139)$$

Condition (2.30) with (2.138) implies that $q \leq m - 1$. Using (2.30) with (2.139) knowing that $q \leq m - 1$ gives condition (2.35). \square

6.4.2 Proof of Lemma 5.1

Let $l = 1 \dots L$. The derivative in time of the energy of a cluster gives

$$\frac{d}{dt} \sum_{\substack{i,j \in P_l^1 \\ i \neq j}} a_i a_j \log(|x_i(t) - x_j(t)|) = \sum_{\substack{i,j \in P_l^1 \\ i \neq j}} a_i a_j \nabla \log(|x_i(t) - x_j(t)|) \cdot \frac{d}{dt}(x_i(t) - x_j(t)). \quad (2.140)$$

Using the equations of motion (2.4) and the identity $\nabla f \cdot \nabla^\perp f = 0$ gives

$$\begin{aligned} \frac{d}{dt} \sum_{\substack{i,j \in P_l^1 \\ i \neq j}} a_i a_j \log(|x_i(t) - x_j(t)|) &= \sum_{\substack{i,j \in P_l^1 \\ i \neq j}} \sum_{k \neq i,j} a_i a_j a_k \nabla \log(|x_i(t) - x_j(t)|) \\ &\quad \cdot \left(\nabla^\perp \log(|x_i(t) - x_k(t)|) - \nabla^\perp \log(|x_j(t) - x_k(t)|) \right). \end{aligned} \quad (2.141)$$

Observing that, since $U \cdot V^\perp = -U^\perp \cdot V$,

$$\sum_{\substack{i,j \in P_l^1 \\ i \neq j}} \sum_{\substack{k \in P_l^1 \\ k \neq i,j}} a_i a_j a_k \nabla \log(|x_i(t) - x_j(t)|) \cdot \nabla^\perp \log(|x_i(t) - x_k(t)|) = 0 \quad (2.142)$$

leads to

$$\begin{aligned} \frac{d}{dt} \sum_{\substack{i,j \in P_l^1 \\ i \neq j}} a_i a_j \log(|x_i(t) - x_j(t)|) &= \sum_{\substack{i,j \in P_l^1 \\ i \neq j}} \sum_{k \notin P_l^1} a_i a_j a_k \nabla \log(|x_i(t) - x_j(t)|) \\ &\quad \cdot \left(\nabla^\perp \log(|x_i(t) - x_k(t)|) - \nabla^\perp \log(|x_j(t) - x_k(t)|) \right). \end{aligned} \quad (2.143)$$

Moreover,

$$|\nabla \log(|x|)| \leq \frac{C}{|x|} \quad (2.144)$$

$$|\nabla^2 \log(|x|)| \leq \frac{C}{|x|^2} \quad (2.145)$$

$$|\nabla^3 \log(|x|)| \leq \frac{C}{|x|^3}. \quad (2.146)$$

Then, using Estimate (2.146), the following Taylor expansion as $|x_k| \rightarrow +\infty$, or equivalently $x_i \rightarrow x_j$, is satisfied,

$$\begin{aligned} \nabla^\perp \log(|x_i - x_k|) &= \nabla^\perp \log(|x_i - x_j + x_j - x_k|) \\ &= \nabla^\perp \log(|x_j - x_k|) + \nabla \nabla^\perp \log(|x_j - x_k|)(x_i - x_j) + \mathcal{O}_{|x_i - x_j| \rightarrow 0^+} \left(\frac{|x_i - x_j|^2}{|x_j - x_k|^3} \right) \end{aligned} \quad (2.147)$$

Using this expansion in (2.143) gives

$$\begin{aligned} \frac{d}{dt} \sum_{\substack{i,j \in P_l^1 \\ i \neq j}} a_i a_j \log(|x_i(t) - x_j(t)|) &= \sum_{\substack{i,j \in P_l^1 \\ i \neq j}} \sum_{k \notin P_l^1} a_i a_j a_k \nabla \log(|x_i(t) - x_j(t)|) \\ &\cdot \left(\nabla \nabla^\perp \log(|x_j(t) - x_k(t)|) (x_i(t) - x_j(t)) \right) + \mathcal{O}_{|x_i - x_j| \rightarrow 0^+} \left(\frac{|x_i - x_j|^2}{|x_j - x_k|^3} \right). \end{aligned} \quad (2.148)$$

Using now (2.144),

$$\left| \nabla \log(|x_i - x_j|) \right| \leq \frac{C}{|x_i - x_j|} \quad (2.149)$$

whereas (2.145) gives

$$\left| \nabla \nabla^\perp \log(|x_j - x_k|) (x_i - x_j) \right| \leq C \frac{|x_i - x_j|}{\delta^2}, \quad (2.150)$$

with δ defined by Lemma 2.2. Thus, combining (2.148), (2.149) and (2.150),

$$\left| \frac{d}{dt} \sum_{\substack{i,j \in P_l^1 \\ i \neq j}} a_i a_j \log(|x_i(t) - x_j(t)|) \right| \leq \frac{C}{\delta^2}. \quad (2.151)$$

□

6.4.3 Proof of Lemma 5.2

Let $X \in \mathfrak{C}$ such that (2.30) and (2.31) are satisfied. Consider for the Euler case the technical hypothesis (2.33) where the coefficients σ_m are defined. Consider also the scaling functions $(f_m)_{m=0}^M$ associated to X . Let $\{P_l^m\}_{l=1}^{L_m}$ be the subsets of $\{1, \dots, N\}$ given by Lemma 2.2. Since the energy of every single cluster is bounded as stated in Lemma 5.1, it is possible to do the proof in the particular case where there is a single cluster of collision ($L_1 = 1$). Indeed, the only property of the equation used to extract the algebraic condition at collision is the bound of the Hamiltonian of a cluster given by Lemma 5.1.

• Step 1 : Convergence to 0 of the rescaled energy.

Suppose that $L_1 = 1$. The energy of this cluster can be computed as follows, using the fact that the sets $(P_l^m)_{l=1}^{L_m}$ are two by two disjoint.

$$\begin{aligned} &\sum_{\substack{(i,j) \in P_1^1 \times P_1^1 \\ i \neq j}} a_i a_j \log(|x_i - x_j|) \\ &= \sum_{\substack{(i,j) \in P_1^1 \times P_1^1 \setminus \left(\bigcup_{l=1}^{L_2} P_l^2 \times P_l^2 \right) \\ i \neq j}} a_i a_j \log(|x_i - x_j|) + \sum_{l=1}^{L_2} \sum_{\substack{(i,j) \in P_l^2 \times P_l^2 \\ i \neq j}} a_i a_j \log(|x_i - x_j|). \end{aligned} \quad (2.152)$$

Observe now that it is possible to do the same work on the diagonal terms

$$\sum_{\substack{(i,j) \in P_l^2 \times P_l^2 \\ i \neq j}} a_i a_j \log(|x_i - x_j|)$$

that appear in the last sum of (2.152) using the sets P_l^3 . Therefore, the iteration of (2.152) for $m = 2 \dots M$ gives (with $L_1 = 1$),

$$\begin{aligned} &\sum_{\substack{(i,j) \in P_1^1 \times P_1^1 \\ i \neq j}} a_i a_j \log(|x_i - x_j|) \\ &= \sum_{m=2}^M \sum_{n=1}^{m-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in \left(\bigcap_{q=1}^{m-1} P_{l_q}^q \times P_{l_q}^q \right) \setminus \left(\bigcup_{q=1}^L P_q^m \times P_q^m \right) \\ i \neq j}} a_i a_j \log(|x_i - x_j|) \\ &+ \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in \left(\bigcap_{q=1}^M P_{l_q}^q \times P_{l_q}^q \right) \\ i \neq j}} a_i a_j \log(|x_i - x_j|), \end{aligned} \quad (2.153)$$

Thus,

$$\begin{aligned}
 & \sum_{\substack{(i,j) \in P_1^1 \times P_1^1 \\ i \neq j}} a_i a_j \log(|x_i - x_j|) \\
 = & \left(\sum_{m=2}^M \sum_{n=1}^{m-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\bigcap_{q=1}^{m-1} P_{l_q}^q \times P_{l_q}^q) \setminus (\bigcup_{q=1}^{L_m} P_q^m \times P_q^m) \\ i \neq j}} a_i a_j \frac{\log(|x_i - x_j|)}{\log(\min_{i \neq j} |x_i - x_j|)} \right. \\
 & \left. + \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\bigcap_{q=1}^M P_{l_q}^q \times P_{l_q}^q) \\ i \neq j}} a_i a_j \frac{\log(|x_i - x_j|)}{\log(\min_{i \neq j} |x_i - x_j|)} \right) \log\left(\min_{i \neq j} |x_i - x_j|\right). \tag{2.154}
 \end{aligned}$$

Since $X \in \mathfrak{C}$ leads to a collision, then $\liminf_{t \rightarrow T_X} \min_{i \neq j} |x_i(t) - x_j(t)| = 0$. The multi-scale Hypothesis (2.30) and (2.31) imply that this \liminf is actually a \lim . Nevertheless, the energy of a cluster is estimated in Lemma 5.1, and then it cannot diverge. Therefore, we conclude that the term between the large parentheses in (2.154) vanishes :

$$\begin{aligned}
 & \sum_{m=2}^M \sum_{n=1}^{m-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\bigcap_{q=1}^{m-1} P_{l_q}^q \times P_{l_q}^q) \setminus (\bigcup_{q=1}^{L_m} P_q^m \times P_q^m) \\ i \neq j}} a_i a_j \frac{\log(|x_i - x_j|)}{\log(\min_{i \neq j} |x_i - x_j|)} \\
 & + \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\bigcap_{q=1}^M P_{l_q}^q \times P_{l_q}^q) \\ i \neq j}} a_i a_j \frac{\log(|x_i - x_j|)}{\log(\min_{i \neq j} |x_i - x_j|)} \longrightarrow 0 \quad \text{as } t \rightarrow T_X. \tag{2.155}
 \end{aligned}$$

• **Step 2 : Writing the scaling relations.**

We now use Lemma 2.2 with the multi-scale Hypothesis (2.30) and (2.31) to conclude the following. For all $m = 2 \dots M$, for all $(i, j) \in (\bigcap_{q=1}^{m-1} P_{l_q}^q \times P_{l_q}^q) \setminus (\bigcup_{q=1}^{L_m} P_q^m \times P_q^m)$,

$$\exists C > c > 0, \quad c f_m(T_X - t) \leq |x_i(t) - x_j(t)| \leq C f_m(T_X - t). \tag{2.156}$$

And for all $(i, j) \in (\bigcap_{q=1}^M P_{l_q}^q \times P_{l_q}^q)$,

$$\exists C > c > 0, \quad c f_M(T_X - t) \leq |x_i(t) - x_j(t)| \leq C f_M(T_X - t). \tag{2.157}$$

In particular

$$\exists C > c > 0, \quad c f_M(T_X - t) \leq \min_{i \neq j} |x_i(t) - x_j(t)| \leq C f_M(T_X - t). \tag{2.158}$$

• **Step 3 : Obtaining the algebraic condition in the Euler case.**

Combining (2.156) and (2.158) with Hypothesis (2.33) gives that for all $m = 2 \dots M$, for all $(i, j) \in (\bigcap_{q=1}^{m-1} P_{l_q}^q \times P_{l_q}^q) \setminus (\bigcup_{l_m=1}^{L_m} P_{l_m}^m \times P_{l_m}^m)$,

$$\begin{aligned}
 & \frac{\log(|x_i(t) - x_j(t)|)}{\log(\min_{i \neq j} |x_i(t) - x_j(t)|)} \\
 = & \frac{\log(|x_i(t) - x_j(t)|)}{\log(f_{m-1}(T_X - t))} \times \frac{\log(f_M(T_X - t))}{\log(\min_{i \neq j} |x_i - x_j|)} \times \prod_{q=m}^M \frac{\log(f_{q-1}(T_X - t))}{\log(f_q(T_X - t))} \longrightarrow \prod_{q=m+1}^M \frac{1}{\sigma_q}. \tag{2.159}
 \end{aligned}$$

Similarly, (2.157) and (2.158) with Hypothesis (2.33) give for all $m = 2 \dots M$, for all $(i, j) \in (\bigcap_{q=1}^M P_{l_q}^q \times P_{l_q}^q)$,

$$\frac{\log(|x_i(t) - x_j(t)|)}{\log(\min_{i \neq j} |x_i(t) - x_j(t)|)} \longrightarrow 1 \quad \text{as } t \rightarrow T_X. \tag{2.160}$$

The limits (2.159) and (2.160) with (2.155) give the following algebraic condition.

$$\begin{aligned}
 & \sum_{m=2}^M \left(\prod_{q=m}^M \frac{1}{\sigma_q} \right) \sum_{n=1}^{m-1} \sum_{l_n=1}^{L_n} \sum_{(i,j) \in \left(\bigcap_{q=1}^{m-1} P_{l_q}^q \times P_{l_q}^q \right) \setminus \left(\bigcup_{q=1}^L P_q^m \times P_q^m \right)}_{i \neq j} a_i a_j \\
 & + \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{(i,j) \in \left(\bigcap_{q=1}^M P_{l_q}^q \times P_{l_q}^q \right)}_{i \neq j} a_i a_j = 0.
 \end{aligned} \tag{2.161}$$

Multiplying both sides with $\prod_{p=2}^M \sigma_p$ gives

$$\begin{aligned}
 & \sum_{m=2}^M \left(\prod_{p=2}^{m-1} \sigma_p \right) \sum_{n=1}^{m-1} \sum_{l_n=1}^{L_n} \sum_{(i,j) \in \left(\bigcap_{q=1}^{m-1} P_{l_q}^q \times P_{l_q}^q \right) \setminus \left(\bigcup_{q=1}^L P_q^m \times P_q^m \right)}_{i \neq j} a_i a_j \\
 & + \left(\prod_{p=2}^M \sigma_p \right) \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{(i,j) \in \left(\bigcap_{q=1}^M P_{l_q}^q \times P_{l_q}^q \right)}_{i \neq j} a_i a_j = 0.
 \end{aligned} \tag{2.162}$$

The first term of the left-hand side above can be written as follows by splitting the main summation whether $m = M$ or not.

$$\begin{aligned}
 & \sum_{m=2}^M \left(\prod_{p=2}^{m-1} \sigma_p \right) \sum_{n=1}^{m-1} \sum_{l_n=1}^{L_n} \sum_{(i,j) \in \left(\bigcap_{q=1}^{m-1} P_{l_q}^q \times P_{l_q}^q \right) \setminus \left(\bigcup_{q=1}^L P_q^m \times P_q^m \right)}_{i \neq j} a_i a_j \\
 & = \sum_{m=2}^{M-1} \left(\prod_{p=2}^{m-1} \sigma_p \right) \sum_{n=1}^{m-1} \sum_{l_n=1}^{L_n} \sum_{(i,j) \in \left(\bigcap_{q=1}^{m-1} P_{l_q}^q \times P_{l_q}^q \right) \setminus \left(\bigcup_{q=1}^L P_q^m \times P_q^m \right)}_{i \neq j} a_i a_j \\
 & + \left(\prod_{p=2}^{M-1} \sigma_p \right) \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{(i,j) \in \left(\bigcap_{q=1}^{M-1} P_{l_q}^q \times P_{l_q}^q \right) \setminus \left(\bigcup_{q=1}^L P_q^M \times P_q^M \right)}_{i \neq j} a_i a_j.
 \end{aligned} \tag{2.163}$$

Concerning the second term of the left-hand side of (2.162),

$$\begin{aligned}
 & \left(\prod_{p=2}^M \sigma_p \right) \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{(i,j) \in \left(\bigcap_{q=1}^M P_{l_q}^q \times P_{l_q}^q \right)}_{i \neq j} a_i a_j \\
 & = \left(\prod_{p=2}^{M-1} \sigma_p \right) \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{(i,j) \in \left(\bigcap_{q=1}^M P_{l_q}^q \times P_{l_q}^q \right)}_{i \neq j} a_i a_j \\
 & + \left(\prod_{p=2}^M \sigma_p - \prod_{p=2}^{M-1} \sigma_p \right) \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{(i,j) \in \left(\bigcap_{q=1}^M P_{l_q}^q \times P_{l_q}^q \right)}_{i \neq j} a_i a_j.
 \end{aligned} \tag{2.164}$$

Section 6. Proofs of the main lemmas and theorems

Therefore, combining (2.163) and (2.164),

$$\begin{aligned} & \sum_{m=2}^M \left(\prod_{p=2}^m \sigma_p \right) \sum_{n=1}^{m-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\cap_{q=1}^{m-1} P_{l_q}^q \times P_{l_q}^q) \setminus (\cup_{l_m=1}^{L_m} P_{l_m}^m \times P_{l_m}^m) \\ i \neq j}} a_i a_j \\ & + \left(\prod_{p=2}^M \sigma_p \right) \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\cap_{q=1}^M P_{l_q}^q \times P_{l_q}^q) \\ i \neq j}} a_i a_j \end{aligned} \quad (2.165)$$

$$\begin{aligned} & = \sum_{m=2}^{M-1} \left(\prod_{p=2}^{m-1} \sigma_p \right) \sum_{n=1}^{m-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\cap_{q=1}^{m-1} P_{l_q}^q \times P_{l_q}^q) \setminus (\cup_{q=1}^{L_m} P_q^m \times P_q^m) \\ i \neq j}} a_i a_j \\ & + \left(\prod_{p=2}^{M-1} \sigma_p \right) \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\cap_{q=1}^{M-1} P_{l_q}^q \times P_{l_q}^q) \setminus (\cup_{q=1}^{L_M} P_q^M \times P_q^M) \\ i \neq j}} a_i a_j \\ & + \left(\prod_{p=2}^{M-1} \sigma_p \right) \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\cap_{q=1}^M P_{l_q}^q \times P_{l_q}^q) \\ i \neq j}} a_i a_j \\ & + \left(\prod_{p=2}^M \sigma_p - \prod_{p=2}^{M-1} \sigma_p \right) \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\cap_{q=1}^M P_{l_q}^q \times P_{l_q}^q) \\ i \neq j}} a_i a_j \end{aligned} \quad (2.166)$$

$$\begin{aligned} & = \sum_{m=2}^{M-1} \left(\prod_{p=2}^m \sigma_p \right) \sum_{n=1}^{m-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\cap_{q=1}^{m-1} P_{l_q}^q \times P_{l_q}^q) \setminus (\cup_{l_m=1}^{L_m} P_{l_m}^m \times P_{l_m}^m) \\ i \neq j}} a_i a_j \\ & + \left(\prod_{p=2}^{M-1} \sigma_p \right) \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\cap_{q=1}^{M-1} P_{l_q}^q \times P_{l_q}^q) \\ i \neq j}} a_i a_j \\ & + \left(\prod_{p=2}^M \sigma_p - \prod_{p=2}^{M-1} \sigma_p \right) \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\cap_{q=1}^M P_{l_q}^q \times P_{l_q}^q) \\ i \neq j}} a_i a_j, \end{aligned} \quad (2.167)$$

where for the last equality we used

$$\begin{aligned} & \left(\prod_{p=2}^{M-1} \sigma_p \right) \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\cap_{q=1}^{M-1} P_{l_q}^q \times P_{l_q}^q) \setminus (\cup_{q=1}^{L_M} P_q^M \times P_q^M) \\ i \neq j}} a_i a_j \\ & + \left(\prod_{p=2}^{M-1} \sigma_p \right) \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\cap_{q=1}^M P_{l_q}^q \times P_{l_q}^q) \\ i \neq j}} a_i a_j \\ & = \left(\prod_{p=2}^{M-1} \sigma_p \right) \sum_{n=1}^{M-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in (\cap_{q=1}^{M-1} P_{l_q}^q \times P_{l_q}^q) \\ i \neq j}} a_i a_j. \end{aligned} \quad (2.168)$$

The equality above is a direct consequence of the set equality

$$\begin{aligned} & \bigcup_{n=1}^{M-1} \bigcup_{l_n=1}^{L_n} \left[\left(\bigcap_{q=1}^{M-1} P_{l_q}^q \times P_{l_q}^q \right) \setminus \left(\bigcup_{q=1}^{L_M} P_q^M \times P_q^M \right) \right] \cup \left[\bigcup_{n=1}^M \bigcup_{l_n=1}^{L_n} \left(\bigcap_{q=1}^M P_{l_q}^q \times P_{l_q}^q \right) \right] \\ &= \bigcup_{n=1}^{M-1} \bigcup_{l_n=1}^{L_n} \left(\bigcap_{q=1}^{M-1} P_{l_q}^q \times P_{l_q}^q \right), \end{aligned} \quad (2.169)$$

where the reunions above are two by two disjoint. Equality (2.169) must be understood as follows. The first term of the left-hand side contains all the pairs of indices (i, j) associated to the scale $M - 1$ but not the pairs of indices associated to the scale M . The second term of the left-hand side contains all the pairs of indices (i, j) associated to the scale M . Then the reunion of both does give the right-hand side of (2.169) which is made of the pairs of indices (i, j) associated to a scale larger or equal to $M - 1$.

We observe that the two first terms appearing in (2.167) are the same terms as the ones in (2.165) but with M replaced by $M - 1$. Therefore, applying iteratively the process described by (2.164) and (2.165)-(2.167) to the left-hand side term of the algebraic condition (2.162) leads to

$$\sum_{\substack{(i,j) \in P_1^1 \times P_1^1 \\ i \neq j}} a_i a_j + \sum_{m=2}^M \left(\prod_{p=2}^m \sigma_p - \prod_{p=2}^{m-1} \sigma_p \right) \sum_{n=1}^{m-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in \left(\bigcap_{q=1}^m P_{l_q}^q \times P_{l_q}^q \right) \\ i \neq j}} a_i a_j = 0. \quad (2.170)$$

We now use the definition of α_m given at (2.37) and we are obtain

$$\sum_{\substack{(i,j) \in P_1^1 \times P_1^1 \\ i \neq j}} a_i a_j + \sum_{m=2}^M \alpha_m \sum_{n=1}^{m-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in \left(\bigcap_{q=1}^m P_{l_q}^q \times P_{l_q}^q \right) \\ i \neq j}} a_i a_j = 0. \quad (2.171)$$

Remark that (2.30) and (2.31) with the definition of the $(P_l^m)_{l=1}^{L_m}$ (given at Lemma 2.2) imply that for all $m = 2 \dots M$ and for all $l = 1 \dots L_m$,

$$\exists q \in \{1, \dots, L_{m-1}\}, \quad P_l^m \subseteq P_q^{m-1} \quad (2.172)$$

and this q is unique because the $(P_k^m)_{k=1}^{L_m}$ are two by two disjoint. Then,

$$\bigcup_{n=1}^M \bigcup_{l_n=1}^{L_n} \bigcap_{q=1}^M P_{l_q}^q \times P_{l_q}^q = \bigcup_{m=1}^{L_M} P_m^M \times P_m^M \quad (2.173)$$

and the reunions are made of two by two disjoint sets. Thus,

$$\sum_{n=1}^{m-1} \sum_{l_n=1}^{L_n} \sum_{\substack{(i,j) \in \left(\bigcap_{q=1}^m P_{l_q}^q \times P_{l_q}^q \right) \\ i \neq j}} a_i a_j = \sum_{l_m=1}^{L_m} \sum_{\substack{(i,j) \in \left(P_{l_m}^m \times P_{l_m}^m \right) \\ i \neq j}} a_i a_j. \quad (2.174)$$

Using (2.174) in (2.171) gives

$$\sum_{\substack{(i,j) \in P_1^1 \times P_1^1 \\ i \neq j}} a_i a_j + \sum_{m=2}^M \alpha_m \sum_{l=1}^{L_m} \sum_{\substack{(i,j) \in \left(P_l^m \times P_l^m \right) \\ i \neq j}} a_i a_j = 0. \quad (2.175)$$

This last equality is equivalent to (2.36) by expanding the squares. \square

6.4.4 Proof of Lemma 5.3

The proof start by this sub-lemma :

Lemma 6.1 (Mono-scale collision reformulated). *Let $X \in \mathfrak{C}$ a collision point. This X satisfies (2.30) and (2.31) with $M = 1$ (mono-scale) if and only if there exist a constant $C > 0$ and $\{P_l^1\}_{l=1}^L$ subsets of $\{1, \dots, N\}$ 2-by-2 disjoint such that*

$$\max_{\substack{i,j \in P_l^1 \\ i \neq j}} |x_i(t) - x_j(t)| \leq C \min_{\substack{i,j \in P_l^1 \\ i \neq j}} |x_i(t) - x_j(t)|, \quad (2.176)$$

Proof of this equivalence

1. Suppose that X satisfies (2.30) and (2.31). Then the sets $\{P_l^1\}_{l=1}^L$ given by Lemma 2.2 and the fact that $M = 1$ gives (2.176).
2. Conversely, define

$$f_1(t) := \min_{l=1\dots L} \min_{\substack{i,j \in P_l^1 \\ i \neq j}} |x_i(t) - x_j(t)|. \quad (2.177)$$

Since X is a collision point, this definition with (2.176) and $M = 1$ give (2.30) and (2.31). \square

Define now

$$\zeta_0(t) := \min_{\substack{i,j \in P_l^1 \\ i \neq j}} |x_i(t) - x_j(t)|^2 \quad \text{and} \quad \xi_0(t) := \max_{\substack{i,j \in P_l^1 \\ i \neq j}} |x_i(t) - x_j(t)|^2. \quad (2.178)$$

Denote by i_0, j_0 the two indices for which the minimum in (2.178) is achieved. By the equations of motions,

$$\frac{d}{dt} \zeta_0(t) = 2(x_{i_0}(t) - x_{j_0}(t)) \cdot \left(\sum_{k \neq i_0} a_k \nabla^\perp \log(|x_{i_0}(t) - x_k(t)|) - \sum_{k \neq j_0} a_k \nabla^\perp \log(|x_{j_0}(t) - x_k(t)|) \right). \quad (2.179)$$

The distances that appears in the right-hand side above are larger than $|x_{i_0}(t) - x_{j_0}(t)|$ when t is close enough to T_X . Indeed, either x_k belongs to the cluster of collision P_l^1 and this is simply the definition of the min, or x_k belongs to another cluster and then the distance stays larger than $\delta > 0$ during all the dynamics. Thus,

$$\left| \frac{d}{dt} \zeta_0(t) \right| = 2 \leq 4 \sum_{i=1}^N |a_i| \quad (2.180)$$

Thus, using Lemma 6.1,

$$|\zeta_0(t)| \leq Ct \quad \text{and} \quad |\xi_0(t)| \leq Ct. \quad (2.181)$$

This is equivalent to

$$\forall i, j \in P_l^1, \quad |x_i(t) - x_j(t)| \leq C \sqrt{T_X - t}. \quad (2.182)$$

\square

6.4.5 Proof of Lemma 5.4

The vorticity vector (2.10) of the cluster P_l^1 is

$$M(P_l^1) := \sum_{i \in P_l^1} a_i x_i. \quad (2.183)$$

Using the equations of motion (2.4), the computation of the derivative in time of $M(P_l^1)$ gives,

$$\frac{d}{dt} M(P_l^1)(t) = \sum_{i \in P_l^1} \sum_{\substack{j \in P_l^1 \\ j \neq i}} a_i a_j \nabla^\perp \log(|x_i - x_j|) + \sum_{i \in P_l^1} \sum_{j \in P_l^{1c}} a_i a_j \nabla^\perp \log(|x_i - x_j|). \quad (2.184)$$

Since $\nabla^\perp \log(|x_i - x_j|) = -\nabla^\perp \log(|x_j - x_i|)$, the first sum in (2.184) is 0 and therefore,

$$\frac{d}{dt} M(P_l^1)(t) = \sum_{i \in P_l^1} \sum_{j \in P_l^{1c}} a_i a_j \nabla^\perp \log(|x_i - x_j|). \quad (2.185)$$

Then, (2.185) can be estimated by

$$\left| \frac{d}{dt} M(P_l^1)(t) \right| \leq \max_{i,j} |a_i a_j| \sum_{i \in P_l^1} \sum_{j \notin P_l^1} \frac{1}{|x_i - x_j|}. \quad (2.186)$$

Using now $|x_i - x_j| \geq \delta$,

$$\left| \frac{d}{dt} M(P_l^1)(t) \right| \leq \frac{N^2 \max_{i,j} |a_i a_j|}{\delta}. \quad (2.187)$$

\square

Chapitre 3

Co-rotating vortices with N fold symmetry for the inviscid surface quasi-geostrophic equations

Joint work with Philippe GRAVEJAT and Didier SMETS.

Abstract

We provide a variational construction of special solutions to the generalized surface quasi-geostrophic equations. These solutions take the form of N vortex patches with N -fold symmetry, which are steady in a uniformly rotating frame. Moreover, we investigate their asymptotic properties when the size of the corresponding patches vanishes. In this limit, we prove these solutions to be a desingularization of N Dirac masses with same intensity, located on the N vertices of a regular polygon and turning around each other at a constant angular velocity.

1 Introduction

In this paper, we consider the generalized versions of the surface quasi-geostrophic equation given by the system

$$\begin{cases} \partial_t \vartheta + v \cdot \nabla \vartheta = 0, \\ \vartheta = (-\Delta)^s \psi, \\ v = \nabla^\perp \psi, \end{cases} \quad (\text{gSQG})$$

for $0 < s < 1$. For $s = 1/2$, the surface quasi-geostrophic equation is a standard model in the context of geophysical fluid mechanics. Typically, it gives account of the dynamics of the potential temperature $\vartheta : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}$ in a rapidly rotating stratified fluid with uniform potential vorticity (see [45] and the references therein). The function ψ is the stream function, while the map v is the flow velocity. The family of nonlinear transport equations in (gSQG) interpolates between the surface quasi-geostrophic equation and two classical equations : the two-dimensional Euler equation in the limit $s \rightarrow 1$ and the stationary equation $\partial_t \vartheta = 0$ when $s \rightarrow 0$. On a mathematical point of view, the surface quasi-geostrophic equation has attracted a lot of interest due to its analogy with the three-dimensional Euler equation (see [25] and the references therein). Local well-posedness of classical solutions was established in [25], but solutions with arbitrary Sobolev growth were constructed in [49] (in a periodic setting). To our knowledge, establishing global well-posedness of classical solutions, or alternatively describing their finite time singularities, remains an open problem. Note also that the global existence of weak solutions in $L^2(\mathbb{R}^2)$ was shown in [67], but their non-uniqueness below a certain regularity threshold was highlighted in [15]. Concerning special solutions, all the radially symmetric functions are stationary solutions to (gSQG). Exhibiting other global smooth solutions is a challenging issue. A first example was recently provided in [19] by developing a bifurcation argument from a specific radially symmetric function. The variational construction of an alternative example in the form of a smooth traveling-wave solution was completed in [42]. These latter results are the starting point of the following analysis.

It is well-known that the two-dimensional Euler equation exhibits point vortex solutions (see [56] and the references therein), which take the form

$$\omega(x, t) = \sum_{n=1}^N a_n \delta_{x=x_n(t)}.$$

In this formula, the notation ω stands for the vorticity of the ideal fluid under consideration, whereas a_n and $x_n(t)$ denote the intensity and position at time t of the vortex n . When the number N of vortices reduces to 2, two special configurations are of interest. The first one corresponds to the case $a_1 + a_2 = 0$ for which the two vortices translate at a constant velocity. When $a_1 \neq -a_2$ instead, the two vortices turn around each other at a constant angular velocity. This latter configuration can be generalized to an arbitrary choice of the number $N \geq 3$, for which N vortices, with same intensity and located on the N vertices of a regular polygon, co-rotate at a constant angular velocity. Point vortices are singular solutions to the Euler equation. On finite time intervals, it is possible to justify that they are weak solutions by regularizing them (see e.g. [56]). An interesting issue is to ask for the existence of regularized solutions, which approximate the point vortices for any time. This desingularization issue was answered positively for numerous vortex configurations of the Euler equation (see e.g. [75] and the references therein). In particular, the construction of smooth traveling-wave solutions corresponding to a vortex pair in translation was achieved in [61] (see also [10, 17]). In parallel, less regular approximations called vortex patches were constructed both for the vortex pair in translation [80, 81] and for the ones in rotation [82].

The notion of point vortex solutions extends to the context of the generalized surface quasi-geostrophic equations. This was the starting point of the constructions in [42], which exhibited a smooth traveling-wave solution to (gSQG) corresponding to a vortex pair in translation. In this direction, a natural question is to ask for a similar construction in the case of a vortex pair in rotation. For $1/2 < s < 1$, this question was answered positively in [46], where a pair of vortex patches was shown to exist by a perturbative argument. Our main goal in this paper is to revisit and extend this latter result by providing a variational construction of N co-rotating vortex patches with N -fold symmetry in the spirit of [82]. Additionally, we also prove these patches to be a desingularization of N point vortices in rotation.

Before stating our main result, we first explicit the equation satisfied by a solution to (gSQG), which remains steady in a uniformly rotating frame. We introduce an angular velocity α and we look for solutions under the form

$$\vartheta(x, t) = \omega(R_{\alpha t}x), \quad (3.1)$$

where R_ϕ stands for the rotation with angle ϕ . Here, we make the choice to use the notation ω for the profile of the solution in the rotating frame in reference to the vorticity formulation of the two-dimensional Euler equation. Note also that the ansatz in (3.1) gives account of a clockwise rotation when α is positive. In view of the equation for the velocity v in (gSQG), we next set

$$u = \nabla^\perp(k_s \star \omega). \quad (3.2)$$

In this identity, the function k_s is the fundamental solution of the operator $(-\Delta)^s$ on \mathbb{R}^2 . This solution is explicitly given by the formula

$$k_s(x) = \frac{C_s}{|x|^{2(1-s)}}, \quad \text{with } C_s := \frac{\Gamma(1-s)}{2^{2s}\pi\Gamma(s)}. \quad (3.3)$$

Combining (3.1) and (3.2) provides

$$v(x, t) = R_{-\alpha t}u(R_{\alpha t}x),$$

so that the first equation in (gSQG) reduces to

$$\alpha x^\perp \cdot \nabla \omega(x) + \nabla^\perp(k_s \star \omega)(x) \cdot \nabla \omega(x) = 0.$$

Hence, a solution to (gSQG), which is steady in a uniformly rotating frame, satisfies the stationary equation

$$\nabla \omega(x) \cdot \nabla^\perp\left(k_s \star \omega(x) + \frac{\alpha}{2}|x|^2 - \mu\right) = 0, \quad (3.4)$$

where μ is a constant of integration. The weak formulation of (3.4) writes as

$$\int_{\mathbb{R}^2} \omega(x) \nabla \varphi(x) \cdot \nabla^\perp\left(k_s \star \omega(x) + \frac{\alpha}{2}|x|^2 - \mu\right) dx = 0, \quad (3.5)$$

Section 1. Introduction

for any function $\varphi \in C_c^\infty(\mathbb{R}^2)$. The vortex patch solutions, which we seek for, own a priori no more regularity than being in the Lebesgue spaces $L^p(\mathbb{R}^2)$ for any $1 \leq p \leq \infty$. Due to standard elliptic theory, the corresponding functions $k_s \star \omega$ belong to the Sobolev spaces $W^{2s,p}(\mathbb{R}^2)$ for any $1 < p < \infty$. For $s < 1/2$, this regularity is not enough to give a rigorous sense to the weak formulation in (3.5). In the sequel, we will explain how to reformulate the notion of vortex patch solution in order to give it a meaningful sense even for $0 < s < 1/2$.

Recall here that a vortex patch ω writes as

$$\omega = \lambda \mathbb{1}_S.$$

In this formula, the notation $\mathbb{1}_S$ refers to the characteristic function of the support S of the patch. The positive number λ is its intensity. Since we aim at desingularizing N point vortices with same intensity, it is natural to demand that the N patches own the same intensity λ . Since the point vortices are located on the N vertices of a regular polygon, we also ask for the symmetry condition

$$\omega(x) = \omega\left(R_{\frac{2\pi}{N}}x\right), \quad (3.6)$$

for any $x \in \mathbb{R}^2$. In the sequel, this symmetry property is used in order to restrict the construction to only one vortex patch inside the angular sector

$$\mathfrak{S}_N := \left\{ (r \cos(\theta), r \sin(\theta)) \in \mathbb{R}^2 : -\frac{\pi}{N} < \theta < \frac{\pi}{N} \right\}. \quad (3.7)$$

Moreover, we also demand this vortex patch to be angular Steiner symmetric with respect to the angle $\theta = 0$. Recall that this condition is defined as

$$\omega(r, \theta) = \omega^\sharp(r, \theta), \quad (3.8)$$

where the notation (r, θ) , with $r > 0$ and $-\pi/N < \theta < \pi/N$, stands for the usual polar coordinates. In this definition, the angular Steiner symmetrization ω^\sharp of the function ω for the variable θ is the unique even function such that

$$\omega^\sharp(r, \theta) > \nu \quad \text{if and only if} \quad |\theta| < \frac{1}{2} \text{meas} \left\{ \theta' \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) : \omega(r, \theta') > \nu \right\},$$

for any positive numbers r and ν , and any $-\pi/N < \theta < \pi/N$. The symmetry condition in (3.8) is helpful in our variational construction of N vortex patches. In Appendix 4 below, we collect some useful properties of the angular Steiner symmetrization.

In the sequel, we denote by $L_{\text{sym}}^p(\mathbb{R}^2)$ the subset of functions in $L^p(\mathbb{R}^2)$, which satisfy both (3.6) and (3.8). With this notation at hand, we are now in position to state our main result.

Theorem 1.1. *Let λ be a large enough positive number. There exists an angular velocity α_λ and a function $\omega_\lambda \in L_{\text{sym}}^\infty(\mathbb{R}^2)$, which satisfy the following properties.*

(i) *The vorticity ω_λ is a vortex patch with intensity λ , in the sense that there exists a number μ_λ such that*

$$\omega_\lambda = \lambda \mathbb{1}_{\{\psi_\lambda > 0\}}, \quad (3.9)$$

with

$$\psi_\lambda(x) := \frac{\Gamma(1-s)}{2^{2s}\pi\Gamma(s)} \int_{\mathbb{R}^2} \frac{\omega_\lambda(x')}{|x-x'|^{2(1-s)}} dx' + \frac{\alpha_\lambda}{2}|x|^2 - \mu_\lambda. \quad (3.10)$$

(ii) *When $1/2 \leq s < 1$, the function ω_λ is a weak solution (in the sense of (3.5)) to (3.4) for the parameters α_λ and μ_λ .*

(iii) *Let $S_\lambda := \{\psi_\lambda > 0\}$ be the support of ω_λ . There exists a positive number R , independent of λ large enough, such that*

$$S_\lambda \cap \mathfrak{S}_N \subseteq B\left((1,0), \frac{R}{\sqrt{\lambda}}\right).$$

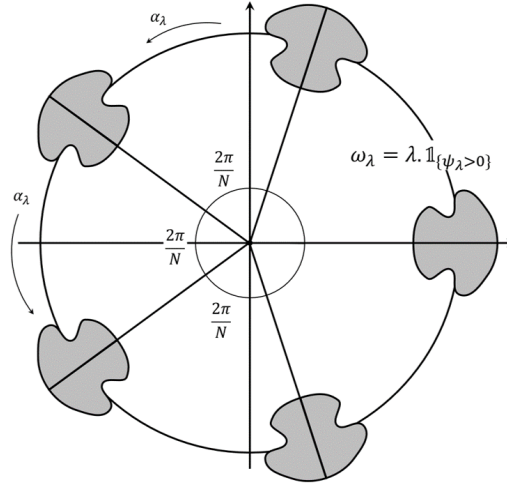
In particular, the support S_λ has at least N connected components.

(iv) *In the limit $\lambda \rightarrow \infty$,*

$$\omega_\lambda \longrightarrow \sum_{n=0}^{N-1} \delta_{R_{\frac{2n\pi}{N}}(1,0)},$$

in the weak sense of measures. If $\frac{1}{2} \leq s < 1$ then the limit of the angular speed is known :

$$\alpha_\lambda \longrightarrow \sum_{n=1}^{N-1} \frac{C_s(1-s)}{|(1,0) - R_{\frac{2n\pi}{N}}(1,0)|^{2(1-s)}}.$$


 FIGURE 3.1 – Illustration of the vortex patch solution ω_λ in the case $N = 5$.

2 Outline of the proof of Theorem 1.1

2.1 Construction of the co-rotating vortex patches

The proof follows the general approach introduced by Arnold in [4]. We rewrite the problem of finding a solution to (3.4) as a maximization problem under constraint. For that purpose, we consider the energy $E_s(\omega)$ of the fluid

$$E_s(\omega) := \frac{1}{2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} k_s(x - x') \omega(x) \omega(x') dx dx', \quad (3.11)$$

its circulation

$$C(\omega) := \int_{\mathbb{R}^2} \omega(x) dx, \quad (3.12)$$

and its impulse

$$L(\omega) := \int_{\mathbb{R}^2} |x|^2 \omega(x) dx. \quad (3.13)$$

Equation (3.4) then appears as the Euler-Lagrange equation corresponding to the maximization problem under constraint

$$\sup \left\{ E_s(\omega) : \omega : \mathbb{R}^2 \rightarrow \mathbb{R}_+ \text{ s.t. } C(\omega) = L(\omega) = N \right\}, \quad (3.14)$$

the numbers α and μ being determined as the Lagrange multipliers of this problem. Here, we impose the circulation C and the impulse L to be equal to N in order that it is eventually equal to 1 for each vortex patch.

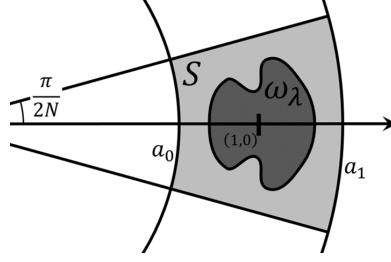
2.1.1 Reduction of the maximization problem to a compact setting

The energy E_s is well-defined on $\dot{H}^{-s}(\mathbb{R}^2)$, while the circulation C and the impulse L make sense in the (weighted) Lebesgue spaces $L^1(\mathbb{R}^2)$, respectively $L^1(\mathbb{R}^2, |x|^2 dx)$. This non-trivial functional framework is a first obstacle to solve this problem. A second one originates in the non-compact nature of the setting. In order to by-pass these difficulties, we follow the arguments developed by Turkington in [82]. We consider the section of annulus

$$S := \left\{ (r \cos(\theta), r \sin(\theta)) : a_0 \leq r \leq a_1 \text{ and } -\frac{\pi}{2N} \leq \theta \leq \frac{\pi}{2N} \right\},$$

for given numbers $0 < a_0 < 1 < a_1$.

The introduction of this set is motivated by the fact that, in the limit $\lambda \rightarrow \infty$, the vortex patches are supported into N small balls centered at the N vertices $x_n = (\cos(2n\pi/N), \sin(2n\pi/N))$ of a regular polygon. Taking advantage of the symmetry property (3.6), we can expect that solving the maximization problem for symmetric functions, whose restriction to \mathfrak{S}_N is supported in S , will eventually provide a solution in the whole space.


 FIGURE 3.2 – Illustration of the domain S .

More precisely, we now fix a positive vortex strength λ and we introduce the function set

$$X_\lambda := \left\{ \omega \in L^\infty(\mathbb{R}^2) : \omega \circ R_{\frac{2\pi}{N}} = \omega \text{ a.e.}, \text{ and } \omega = 0 \text{ a.e. on } \mathfrak{S}_N \setminus S \right\}, \quad (3.15)$$

where

$$L^\infty(\mathbb{R}^2) := \left\{ \omega \in L^\infty(\mathbb{R}^2) : 0 \leq \omega \leq \lambda \text{ a.e.} \right\}.$$

The choice of this uniform framework is motivated by the fact that the vortex patches are non-negative bounded functions, with maximum equal to λ . Finally, we consider the maximization problem under constraint

$$\mathcal{E}_\lambda := \sup \left\{ E_s(\omega) : \omega \in X_\lambda \text{ s.t. } C(\omega) = L(\omega) = N \right\}. \quad (3.16)$$

In this definition, we observe that the energy E_s may be rewritten as

$$E_s(\omega) = \frac{N}{2} \int_S \int_S \kappa_s(r, r', \theta - \theta') \omega(r, \theta) \omega(r', \theta') r' dr' d\theta' r dr d\theta, \quad (3.17)$$

due to the symmetry and support properties of the function ω . Here, the kernel κ_s is equal to

$$\kappa_s(r, r', \xi) := \sum_{n=0}^{N-1} \frac{C_s}{\left(r^2 + r'^2 - 2rr' \cos\left(\xi - \frac{2n\pi}{N}\right) \right)^{1-s}}. \quad (3.18)$$

For further use, let us also define the function

$$K_s \omega(r, \theta) := \int_S \kappa_s(r, r', \theta - \theta') \omega(r', \theta') r' dr' d\theta',$$

as well as the functional

$$V_s(\omega_1, \omega_2) := \int_S \int_S \kappa_s(r, r', \theta - \theta') \omega_1(r, \theta) \omega_2(r', \theta') r' dr' d\theta' r dr d\theta,$$

so that the energy E_s may be expressed as

$$E_s(\omega) = \frac{N}{2} V_s(\omega, \omega). \quad (3.19)$$

Coming back to the maximization problem (3.16), we first check the existence of functions $\omega \in X_\lambda$ such that $C(\omega) = L(\omega) = N$ when λ is large enough. Setting $\varepsilon = 1/\sqrt{\pi\lambda}$, we can find a number $1 - \varepsilon \leq r_\varepsilon \leq 1 + \varepsilon$ such that the function ϖ_λ given by

$$\varpi_\lambda := \sum_{n=0}^{N-1} \lambda \mathbb{1}_{B\left(R_{\frac{2n\pi}{N}}(r_\varepsilon, 0), \varepsilon\right)}, \quad (3.20)$$

satisfies the constraints $C(\varpi_\lambda) = L(\varpi_\lambda) = N$. Due to the choice $a_0 < 1 < a_1$, this function also belongs to X_λ for λ large enough, that is for ε small enough compared to $1 - a_0$ and $a_1 - 1$.

Invoking the boundedness of the set S and the Sobolev embedding theorem of $L^\infty(S)$ into $H^{-s}(S)$ then guarantees that the maximization problem (3.16) is well-defined for λ large enough. Moreover, we can prove

Proposition 2.1. *Let $0 < s < 1$ and assume that λ is a large enough positive number. There exists an angular Steiner symmetric function $\omega_\lambda \in X_\lambda$ such that*

$$E_s(\omega_\lambda) = \mathcal{E}_\lambda.$$

Moreover, there exist two numbers α_λ and μ_λ such that

$$\omega_\lambda = \lambda \mathbb{1}_{\{\psi_\lambda > 0\}}, \quad (3.21)$$

where we have set

$$\psi_\lambda := K_s \omega_\lambda + \frac{\alpha_\lambda}{2} |\cdot|^2 - \mu_\lambda. \quad (3.22)$$

When $1/2 \leq s < 1$, the function ψ_λ belongs to $W^{1,q}(S)$ for any real number $q > 1$, and it satisfies the weak equation

$$\int_S \omega_\lambda(x) \nabla^\perp \psi_\lambda(x) \cdot \nabla \varphi(x) dx = 0, \quad (3.23)$$

for any function $\varphi \in C_c^\infty(S)$.

2.1.2 The penalized problems

Our proof of Proposition 2.1 does not rely on a direct variational argument for which it is not so direct to derive (3.21) and (3.23). Instead, we introduce the penalized functionals

$$E_{\lambda,p}(\omega) := E_s(\omega) - \frac{\lambda N}{p} \int_S \left(\frac{\omega}{\lambda} \right)^p, \quad (3.24)$$

for any number $p > 1/s$, as well as the corresponding maximization problems

$$\mathcal{E}_{\lambda,p} := \sup \left\{ E_{\lambda,p}(\omega) : \omega \in L_+^p(S) \text{ s.t. } C(\omega) = L(\omega) = 1 \right\}. \quad (3.25)$$

Due to the boundedness of the set S , we observe that

$$\sup_{(r,\theta) \in S} \int_S \kappa_s(r, r', \theta - \theta')^q dr' d\theta' < \infty,$$

when $1 \leq q < 1/(1-s)$. Hence it follows from the Hölder inequality that both the terms in the right-hand side of (3.24) are well-defined when the vorticity ω lies in $L^p(S)$ with $p > 1/s$. In this case, the quantities $C(\omega)$ and $L(\omega)$ are also well-defined, so that the maximization problem (3.25) makes sense. Moreover, we can solve this problem as follows.

Lemma 2.1. *Let $0 < s < 1$, $\lambda > 0$ and $p > 1/s$. Denote by p' the Hölder conjugate of p defined by $1/p + 1/p' = 1$. There exists a function $\omega_{\lambda,p} \in L_+^p(S)$ such that*

$$E_{\lambda,p}(\omega_{\lambda,p}) = \max \left\{ E_{\lambda,p}(\omega) : \omega \in L_+^p(S) \text{ s.t. } C(\omega) = L(\omega) = 1 \right\}. \quad (3.26)$$

The function $\omega_{\lambda,p}$ is angular Steiner symmetric and bounded. Moreover, there exist two numbers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ such that the function $\omega_{\lambda,p}$ can be written as

$$\omega_{\lambda,p} = \lambda (\psi_{\lambda,p})_+^{p'-1}, \quad (3.27)$$

with

$$\psi_{\lambda,p} := K_s \omega_{\lambda,p} + \frac{\alpha_{\lambda,p}}{2} |\cdot|^2 - \mu_{\lambda,p}. \quad (3.28)$$

The proof of Lemma 2.1 is standard. Existence follows from bounding the L^p -norm of the minimizing sequences corresponding to (3.25) and from applying standard weak compactness results. The angular Steiner symmetry of the function $\omega_{\lambda,p}$ is a consequence of Lemmas 4.1 and 4.2. Equations (3.27) and (3.28) are no more than the Euler-Lagrange equations of the maximization problem (3.25). In particular, the numbers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ are interpreted as the Lagrange multipliers of the problem. We refer to Subsection 3.1.1 for more details.

2.1.3 Bounds on the maximizers $\omega_{\lambda,p}$ and the Lagrange multipliers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$

Our goal is now to construct solutions of the maximization problems (3.16) as limits when $p \rightarrow \infty$ of the functions $\omega_{\lambda,p}$ of Lemma 2.1. Before passing to this limit, we establish that the Lagrange multipliers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ are bounded uniformly with respect to p .

Lemma 2.2. *Let $0 < s < 1$ and $\lambda > 0$. There exists a positive number $p_0 > 1/s$ such that there exists a positive number C_λ independent of p for which the Lagrange multipliers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ in Lemma 2.1 satisfy*

$$|\alpha_{\lambda,p}| + |\mu_{\lambda,p}| \leq C_\lambda,$$

for any $p \geq p_0$.

This lemma follows from a uniform bound on the function $K_s \omega_{\lambda,p}$. As a consequence of the integrability properties of the kernel K_s , there exists a positive number C_λ independent of p such that

$$\|K_s \omega_{\lambda,p}\|_{L^\infty} \leq C_\lambda.$$

With this inequality at hand, we can control the support of the function $\psi_{\lambda,p}$ and then of the function $\omega_{\lambda,p}$. In case the Lagrange multipliers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ are not bounded independently of p , this control is enough to establish a contradiction with the constraints $C(\omega_{\lambda,p}) = L(\omega_{\lambda,p}) = 1$.

We next apply standard regularity estimates to the Euler-Lagrange equations (3.27) and (3.28) in order to bound the functions $\omega_{\lambda,p}$ and $\psi_{\lambda,p}$ uniformly with respect to p . When $1/2 \leq s < 1$, the functions $\omega_{\lambda,p}$ and $\psi_{\lambda,p}$ own sufficient smoothness so as to satisfy the weak formulation of (3.4) in (3.23). As a matter of fact, this weak formulation follows from the collinearity of the gradients $\nabla \omega_{\lambda,p}$ and $\nabla \psi_{\lambda,p}$, which in turn is a consequence of (3.27) provided that $\omega_{\lambda,p}$ and $\psi_{\lambda,p}$ are smooth enough. More precisely, we show

Lemma 2.3. *Let $0 < s < 1$, $\lambda > 0$ and $p \geq p_0$, where p_0 is defined in Lemma 2.2. Consider the solution $\omega_{\lambda,p}$ to the maximization problem (3.26) constructed in Lemma 2.1, and set*

$$\psi_{\lambda,p} = K_s \omega_{\lambda,p} + \frac{\alpha_{\lambda,p}}{2} |\cdot|^2 - \mu_{\lambda,p},$$

where $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ are the corresponding Lagrange multipliers. There exists a positive number C_λ independent of p such that

$$\|\omega_{\lambda,p}\|_{L^r} \leq C_\lambda, \tag{3.29}$$

for any $1 \leq r \leq \infty$, while there exist positive numbers $C_{\lambda,r}(R)$, not depending on p , such that

$$\|\psi_{\lambda,p}\|_{W^{2s,r}(B(0,R))} \leq C_{\lambda,r}(R), \tag{3.30}$$

for any $1 < r < \infty$ and any positive number R . In particular, when $1/2 \leq s < 1$, the functions $\omega_{\lambda,p}$ and $\psi_{\lambda,p}$ satisfy the weak equation

$$\int_S \omega_{\lambda,p}(x) \nabla^\perp \psi_{\lambda,p}(x) \cdot \nabla \varphi(x) dx = 0,$$

for any function $\varphi \in C_c^\infty(S)$.

2.1.4 Convergence in the limit $p \rightarrow \infty$ towards vortex patch solutions

With Lemmas 2.2 and 2.3 at hand, we are in position to complete the proof of Proposition 2.1. Since the Lagrange multipliers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$, as well as the functions $\omega_{\lambda,p}$ and $\psi_{\lambda,p}$, are uniformly bounded with respect to p , we can invoke a compactness argument in order to take the limit $p \rightarrow \infty$. This eventually provides a limit function $\omega_\lambda \in X_\lambda$, which maximizes the energy E_s under the constraints $C(\omega_\lambda) = L(\omega_\lambda) = 1$. We next derive from the following lemma that this function is a vortex patch.

Lemma 2.4 (Bathtub principle for Riesz integrals). *Let μ and ν be two positive numbers. Given a positive number η , set*

$$\mathcal{G}_\eta(\mathbb{R}) := \left\{ g \in L^\infty(\mathbb{R}, [0, 1]) : \int_{\mathbb{R}} g \leq \mu \right\},$$

and consider a function $f \in L^1(\mathbb{R})$, which is even and non-increasing on \mathbb{R}_+ . Then the maximization problem

$$\mathcal{I}_{\mu,\nu} := \sup_{(g,h) \in \mathcal{G}_\mu(\mathbb{R}) \times \mathcal{G}_\nu(\mathbb{R})} \int_{\mathbb{R}} \int_{\mathbb{R}} f(x-y) g(x) h(y) dx dy, \tag{3.31}$$

owns a solution, which is given by the functions

$$g = \mathbb{1}_{[-\frac{\mu}{2}, \frac{\mu}{2}]} \quad \text{and} \quad h = \mathbb{1}_{[-\frac{\nu}{2}, \frac{\nu}{2}]}.$$

Moreover, when the function f is decreasing on \mathbb{R}_+ , this solution is unique (up to a translation).

The proof of this lemma consists in applying the bathtub principle (see e.g. [55, Theorem 1.14]) to the double integral in the right-hand side of (3.31). Double integrals taking this form are usually called Riesz integrals in reference to the Riesz rearrangement inequality [69]. For fixed numbers r and r' , the integrals depending on the variables θ and θ' in the expression (3.17) of the energy E_s are exactly of this form. In particular, it is enough to apply Lemma 2.4 to them in order to prove that the functions ω_λ are the characteristic functions of measurable sets Ω_λ .

The proof of Proposition 2.1 then reduces to establish that these sets are equal to the upper level sets $\{\psi_\lambda > 0\}$. In the limit $p \rightarrow \infty$, we deduce from (3.27) that

$$\{\psi_\lambda > 0\} \subset \Omega_\lambda \subset \{\psi_\lambda \geq 0\}.$$

It then results from the monotonicity properties of the kernel κ_s given by Lemma 4.3 that the functions $\theta \mapsto \psi_\lambda(r, \theta)$ are decreasing on $(0, \pi/2N)$ for any fixed number $a_0 < r < a_1$. As a consequence, the level sets $\{\psi_\lambda > 0\}$ have measure zero, which eventually gives (3.21).

Note also that the weak equation (3.23) follows as in the proof of Lemma 2.3 from the fact that ψ_λ belongs to the Sobolev spaces $W^{1,r}(S)$ for any $1 < r < \infty$ when $1/2 \leq s < 1$. We refer to Subsection 3.1.5 for more details.

2.2 Description of the vortex patch support

Proposition 2.1 eventually provides a solution to the initial problem provided that the solution satisfies (3.32).

Corollary 2.1. *Assume that the maximizer ω_λ to (3.16) given by Proposition 2.1 satisfies*

$$\text{dist}(\text{supp } \omega, \mathfrak{S}_N \setminus S) > 0. \tag{3.32}$$

Then the function ω_λ is a weak solution to (3.4) in the sense of (3.5).

Indeed, condition (3.32) implies that the weak formulations of this equation on S and on \mathbb{H} coincide. In this case, the reformulated weak formulation of the problem on S is equal to the weak formulation on \mathbb{R}^2 using the N -fold symmetry condition (3.6). The main idea to obtain (3.32) consists in explaining that the maximization of the energy E_s under the additional constraint $0 \leq \omega \leq \lambda$ gives a solution that is “close” to ϖ_λ when λ is large enough. This additional constraint is enough to guarantee that the maximization problem of the energy E_s subject to the constraints $C(\omega) = 1$ and $L(\omega) = 1$ makes sense. Provided that λ is large enough, Proposition 2.1 below establishes the existence of an absolute maximizer $\omega_\lambda \in L^\infty_\lambda(S)$ to the previous maximization problem and that it has the form of a “patch” meaning that ω takes only the two values 0 and λ . We construct this maximizer by solving first a penalized maximization problem corresponding to smooth solutions. We then take the limit in which the penalization disappears. This argument provides a so-called patch solution ω_λ , since it takes the special form

$$\omega_\lambda = \lambda \mathbb{1}_{\Omega_\lambda},$$

with

$$\Omega_\lambda := \{x \in S : \psi_\lambda(x) > 0\}.$$

Here, the function ψ_λ is given by

$$\psi_\lambda := K_s \omega_\lambda - \mu_\lambda + \frac{\alpha_\lambda}{2} |\cdot|^2, \tag{3.33}$$

and it corresponds to the stream function of the induced flow. The numbers α_λ and μ_λ are the Lagrange multipliers of the constrained maximization problem. We next show that this solution in S defines a solution in \mathbb{R}^2 . This property follows from checking Condition (3.32). We finally complete the description of the solution ω_λ by providing some of its asymptotic properties in the limit $\lambda \rightarrow \infty$. More precisely we prove that the solution ω_λ converges towards a Dirac mass centered at point $(1, 0)$. This correspond to a solution of the point vortex problem made up with N Dirac vortices of same intensity turning around one-another at angular speed α .

Section 2. Outline of the proof of Theorem 1.1

The main purpose of this section consists in proving that (3.32) holds provided that λ is large enough. Such a result would imply that the continuation by 0 of the solution out of S (with the N -fold symmetry condition) provides a solution of the initial problem. The distance condition (3.32) is obtained for λ large enough as a consequence of the following lemma.

Lemma 2.5. *Let ω_λ solution of the problem (3.16). There exist $R > 1$ such that for ε small enough*

$$\int_{\mathcal{B}((1,0),R\varepsilon)} \omega_\lambda(x) dx = 1. \quad (3.34)$$

2.2.1 Energy estimates and rearrangements

We start this work with several integral estimates using the fact that ω_λ maximizes the energy E_s and rearrangement inequalities. The energy defined by (3.11), in a context of localization of the problem on the compact S can be separated into two using the symmetry ansatz as follows. First, separate the interaction of the patch with himself from the interaction with other patches.

$$E_s(\omega_\lambda) = 2C_s \int_{S \times S} \frac{\omega_\lambda(x)\omega_\lambda(x')}{|x-x'|^{2(1-s)}} dx dx' + 2C_s \sum_{n=1}^{N-1} \int_{S \times R_{\frac{2n\pi}{N}}S} \frac{\omega_\lambda(x)\omega_\lambda(x')}{|x-x'|^{2(1-s)}} dx dx'. \quad (3.35)$$

Since the distance between S and $R_{\frac{2n\pi}{N}}(S)$ is positive and bounded below by a constant (depends only on a_0), the energy term corresponding to the second term above remains bounded as $\lambda \rightarrow \infty$. Thus, the behavior of $E_s(\omega_\lambda)$ as $\lambda \rightarrow \infty$ at the first order is given by the integral on $S \times S$. The following lemma provides an estimate of this term.

Lemma 2.6. *Let ω_λ be a solution to (3.16). There exists a positive number C , depending only on s , a_0 and a_1 , such that*

$$\mathcal{I}_s \leq \int_S \int_S \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} \omega_\lambda(x) \omega_\lambda(x') dx dx' \leq \mathcal{I}_s + C\varepsilon^{2(1-s)}, \quad (3.36)$$

where we have set

$$\mathcal{I}_s := \frac{1}{\pi^2} \int_{\mathcal{B}(0,1)} \int_{\mathcal{B}(0,1)} \frac{dx dx'}{|x-x'|^{2(1-s)}}. \quad (3.37)$$

As a consequence, the studied energy term behaves like $\mathcal{I}_s \varepsilon^{2(s-1)}$ as $\varepsilon \rightarrow 0^+$. The proof of this lemma relies on the maximality of ω_λ and rearrangement arguments. We also provide an estimate on the behavior of the constant \mathcal{I}_s with respect to the parameter s .

Lemma 2.7. *The constant \mathcal{I}_s defined by (3.37) satisfies*

$$\frac{1}{s(s+1)(2s+1)} \leq \mathcal{I}_s \leq \frac{1}{s}.$$

Démonstration. The first inequality follows from the inclusion $\mathcal{B}(0,1-|x|) \subseteq \mathcal{B}(x,1)$. More precisely, we compute

$$\mathcal{I}_s = \frac{1}{\pi^2} \int_{\mathcal{B}(0,1)} \int_{\mathcal{B}(x,1)} \frac{dy}{|y|^{2(1-s)}} dx \geq \frac{1}{\pi^2} \int_{\mathcal{B}(0,1)} \int_{\mathcal{B}(0,1-|x|)} \frac{dy}{|y|^{2(1-s)}} dx = \frac{1}{s(s+1)(2s+1)}.$$

For the second inequality, we write

$$\mathcal{I}_s = \frac{1}{\pi^2} \int_{\mathcal{B}(0,1)} \int_{\mathcal{B}(x,1)} \frac{dy}{|y|^{2(1-s)}} dx \leq \frac{1}{\pi^2} \int_{\mathcal{B}(0,1)} \int_{\mathcal{B}(0,1)} \frac{dy}{|y|^{2(1-s)}} dx = \frac{1}{s}.$$

□

The aim of this section is to obtain the concentration result enunciated at Lemma 2.5. The first step to this result consists in proving that at least a small part of ω_λ concentrates somewhere. This first concentration result is given by the following lemma.

Lemma 2.8. *Assume that the function ω satisfies (3.9) and (3.36). Given a positive number R , define the number η_R^* as*

$$(\eta_R^*)^s = s\mathcal{I}_s - \frac{s}{R^{2(1-s)}}.$$

Then there exists x_λ^* , independent of R , such that

$$\text{meas} \left(\mathcal{B}(x_\lambda^*, R\varepsilon) \cap \text{supp } \omega_\lambda \right) \geq \pi \eta_R^* \varepsilon^2. \quad (3.38)$$

Moreover, if ω_λ is angular Steiner symmetric, then the point x_λ^* can be chosen of the form $x_\lambda^* = (r_\lambda^*, 0)$.

The proof of this lemma relies on a rearrangement argument. We remark that when R goes to $+\infty$ then η_R converges towards $\eta_\infty := (s\mathcal{I}_s)^{\frac{1}{s}}$. Using Lemma 2.7, we obtain $\eta_\infty \geq e^{-3} \approx 0.05$. The guiding idea to improve the concentration result given by Lemma 2.8 is to use the energy estimate given by Lemma 2.6. For that purpose, the three following lemmas provide estimates the concentration of ω with respect to the energy integrals. By concentration of ω we mean the set of quantities $\eta(X) := \int_X \omega(x) dx \in [0, 1]$, where X is any measurable subset of S . Although this problem is non-local, these three lemmas tell us in a sense that if the energy is high near a set X then so is the concentration $\eta(X)$.

Lemma 2.9. *Assume that the function ω satisfies (3.9) and (3.36). Given a subset X of S , we have*

$$\int_X \int_X \left(\frac{\varepsilon}{|x - x'|} \right)^{2(1-s)} \omega(x) \omega(x') dx dx' \leq \mathcal{I}_s \eta(X)^{1+s}.$$

Démonstration. We infer from the Riesz rearrangement inequality [69] that

$$\int_X \int_X \left(\frac{\varepsilon}{|x - x'|} \right)^{2(1-s)} \omega(x) \omega(x') dx dx' \leq \lambda^2 \int_X \int_{\mathcal{B}(0, \varepsilon \sqrt{\eta(X)})} \left(\frac{\varepsilon}{|y|} \right)^{2(1-s)} dy dx = \mathcal{I}_s \eta(X)^{1+s}.$$

□

Lemma 2.10. *Assume that the function ω satisfies (3.9) and (3.36), and consider two measurable subsets X and Y of S whose distance $d(X, Y)$ is positive. Then, we have*

$$\int_X \int_Y \left(\frac{\varepsilon}{|x - x'|} \right)^{2(1-s)} \omega(x) \omega(x') dx dx' \leq \left(\frac{\varepsilon}{d(X, Y)} \right)^{2(1-s)} \eta(X) \eta(Y).$$

Démonstration. This is a direct consequence of the inequality

$$|x - x'| \geq d(X, Y),$$

which holds for any $x \in X$ and $x' \in Y$.

□

Lemma 2.11. *Assume that the function ω satisfies (3.9) and (3.36). Given two measurable subsets X and Y of S , we have*

$$\int_X \int_Y \left(\frac{\varepsilon}{|x - x'|} \right)^{2(1-s)} \omega(x) \omega(x') dx dx' \leq \frac{\eta(X) \eta(Y)^s}{s}. \quad (3.39)$$

Démonstration. Since the map $x \mapsto |x|^{2(s-1)}$ is radially decreasing, we can rearrange the function ω so as to write

$$\begin{aligned} \int_X \omega(x') \int_Y \left(\frac{\varepsilon}{|x - x'|} \right)^{2(1-s)} \omega(x) dx dx' &\leq \lambda \int_X \omega(x') \int_{\mathcal{B}(x, \varepsilon \sqrt{\eta(Y)})} \left(\frac{\varepsilon}{|x - x'|} \right)^{2(1-s)} dx dx' \\ &= \lambda \eta(X) \int_{\mathcal{B}(0, \varepsilon \sqrt{\eta(Y)})} \left(\frac{\varepsilon}{|y|} \right)^{2(1-s)} dy. \end{aligned}$$

Estimate (3.39) then follows from the formula $\lambda \pi \varepsilon^2 = 1$.

□

2.2.2 First decay estimate on the vortex patch solution

With the energy estimate provided by Lemma 2.6 and the three lemmas 2.9, 2.11 and 2.10 the concentration result enunciated at Lemma 2.8 can be improved. More precisely, we provide in this section at Lemma 2.13 a decay estimate on the solution ω_λ . To start with, we write the following enumeration result consequence of the fact that $\int \omega = 1$.

Lemma 2.12. *Assume that the function ω satisfies (3.9) and (3.36). Given two positive numbers ρ and γ , there exists an integer $1 \leq m \leq \rho^\gamma$ such that*

$$\int_{\mathcal{B}(x_\lambda^*, (m+1)\rho\varepsilon)} \omega(x) dx \leq \int_{\mathcal{B}(x_\lambda^*, m\rho\varepsilon)} \omega(x) dx + \frac{1}{\rho^\gamma}.$$

Démonstration. We assume for the sake of a contradiction that

$$\int_{\mathcal{B}(x_\lambda^*, (m+1)\rho\varepsilon) \setminus \mathcal{B}(x_\lambda^*, m\rho\varepsilon)} \omega_\lambda(x) dx > \frac{1}{\rho^\gamma},$$

for any integer $1 \leq m \leq \rho^\gamma$. If we sum this inequality above the index m , we get

$$\int_{\mathcal{B}(x_\lambda^*, (\rho^\gamma+1)\rho\varepsilon)} \omega_\lambda(x) dx > 1,$$

and we obtain a contradiction with the constraint $C(\omega_\lambda) = 1$ due to the non-negativeness of the function ω_λ . \square

The combination of this enumeration result and the lemmas given in the previous sections leads to the announced decay estimate.

Lemma 2.13 (Decay estimate on the solution). *Assume that the function ω satisfies (3.9) and (3.36). Given any number $R \geq \mathcal{R}_s$, we have*

$$\int_{\mathcal{B}(x_\lambda^*, R\varepsilon)^c} \omega(x) dx \leq \frac{\mathcal{Q}_s}{R^{\gamma_s}},$$

where we have set

$$\mathcal{Q}_s := \frac{1 + \frac{2(1+s)}{s\mathcal{I}_s}}{1 - 2^{-s}}, \quad \mathcal{R}_s := \left(\frac{\mathcal{Q}_s + s}{s\mathcal{I}_s} \right)^{1 + \frac{1}{2(1-s)}} \quad \text{and} \quad \gamma_s := \frac{1}{1 + \frac{1}{2(1-s)}}. \quad (3.40)$$

2.2.3 The condition on the support

With this decay estimate at hand, we can exploit that fact that ω_λ satisfies equations to obtain the final concentration result enunciated at lemma 2.5. As we saw before, the solution ω_λ tends to concentrate weakly around x_λ^* , where by “weakly” we refer to the decay estimate. Since we want to prove that ω_λ concentrate around $(1, 0)$, we first have to study the behavior of x_λ^* as $\lambda \rightarrow \infty$.

Lemma 2.14. *The point x_λ^* given by Lemma 2.8 satisfies*

$$x_\lambda^* \rightarrow (1, 0),$$

in the limit $\lambda \rightarrow \infty$.

The proof of this fact relies on the decay estimate given by lemma 2.13 that would contradict the constraints $C(\omega_\lambda) = L(\omega_\lambda) = 1$ if x_λ^* is far from $(1, 0)$ when λ is large enough. On the other hand, the fact that ω_λ solves the maximization problem (3.16) permits to improve the decay estimate given by lemma 2.13 and obtain that ω is exactly zero out of a ball of radius $R\varepsilon$.

Lemma 2.15. *Let ω_λ be the maximizer of (3.16) given by Proposition 2.1. There exists a number $R > 1$ such that*

$$\int_{\mathcal{B}(x_\lambda^*, R\varepsilon)} \omega_\lambda(x) dx = 1,$$

for ε small enough.

We can combine the two previous lemmas in order to obtain

Lemma 2.16. *There exists a positive number C such that the point x_λ^* given by Lemma 2.8 satisfies*

$$|x_\lambda^* - (1, 0)| \leq C\varepsilon,$$

for ε small enough.

Démonstration. Otherwise Lemma 2.15 would be in contradiction with the two constraints $C(\omega_\lambda) = L(\omega_\lambda) = 1$ when ε is small enough. \square

As a consequence, we can replace x_λ^* by the point $(1, 0)$ in Lemma 2.15, and this substitution is enough to complete the proof of Lemma 2.5.

2.3 Desingularization of a point vortex pair

Now that we proved that Equation (gSQG) admits a solution that takes the form of a pair of positive symmetric patches of height λ turning around each other at speed α_λ , we study the limit as $\lambda \rightarrow \infty$. We prove in the sequel that this family of solutions indexed by parameter λ provide a desingularization of the symmetric pair of point vortex. For that purpose we have to establish first that the solution ω_λ converges in the sens of measure on \mathbb{H} towards the Dirac mass centered at $(1, 0)$ and second that α_λ the angular speed of rotation in the inertial frame of the two vortices converges as $\lambda \rightarrow \infty$. Here is the first convergence result.

Lemma 2.17. *The solution ω_λ converges on S towards $\delta_{(1,0)}$, the Dirac mass centered at $(1, 0)$, in the sense of measure as $\lambda \rightarrow +\infty$.*

Démonstration. This is straightforward from $\int \omega = 1$ and Lemma 2.5. \square

Concerning the limit of α_λ as $\lambda \rightarrow \infty$, it can be computed exactly using the weak formulation (3.5).

Lemma 2.18. *Suppose that $1/2 \leq s < 1$. The Lagrange multiplier α_λ that corresponds to the angular speed at which the vortex patches are turning around each other satisfies*

$$\alpha_\lambda \longrightarrow \sum_{n=1}^{N-1} \frac{C_s(1-s)}{|(1,0) - R_{\frac{2n\pi}{N}}(1,0)|^{2(1-s)}}, \quad (3.41)$$

as $\lambda \rightarrow +\infty$.

In addition to this desingularization result, we can provide an estimate of how μ_λ blows up as $\lambda \rightarrow \infty$ using again the equations (3.9) and (3.10).

Lemma 2.19. *There exist two positive numbers c and C , depending only on s , such that the Lagrange multipliers μ_λ satisfy*

$$c\lambda^{1-s} \leq \mu_\lambda \leq C\lambda^{1-s}, \quad (3.42)$$

for λ large enough.

In the case of the Euler equations corresponding to the case $s = 1$, the divergence speed of parameter μ_λ with respect to λ is $\log(\lambda)$ (the proof of this fact is provided in [82]). The case of a logarithm divergences does appears to be the limit as $s \rightarrow 1^-$. These three asymptotic properties allows us to emphasize once more the paternity between the Euler equations and (gSQG) because even in the limit of very irregular solutions (solutions that take the form of Dirac masses) we obtain the same phenomena with different scalings.

Moreover, if we compute the α limit given by (3.41) with respect to s in the particular case $N = 2$ we obtain, using the asymptotics of the Γ function,

$$\frac{(1-s)\Gamma(1-s)}{4\pi\Gamma(s)} \rightarrow \frac{1}{4\pi},$$

as $s \rightarrow 1^-$. This does correspond to the speed of rotation of the pair of Dirac vortices on the unit circle evolving according to the Euler equations.

3 Details of the proofs

3.1 Proofs for Section 2.1

3.1.1 Proof of Lemma 2.1

We first establish the existence of the maximizing vorticity $\omega_{\lambda,p}$. Recall that the kernel κ_s is defined so as to satisfy

$$\kappa_s(r, r', \theta - \theta') = \sum_{n=0}^{N-1} \frac{C_s}{\left(r^2 + r'^2 - 2rr' \cos\left(\theta - \theta' - \frac{2n\pi}{N}\right)\right)^{1-s}}. \quad (3.43)$$

for $x = (r \cos(\theta), r \sin(\theta))$ and $x' = (r' \cos(\theta'), r' \sin(\theta'))$. Hence, we derive from the boundedness of S that

$$\sup_{(r,\theta) \in S} \int_S \kappa_s(r, r', \theta - \theta')^p r' dr' d\theta' < \infty, \quad (3.44)$$

under the condition $p > 1/s$. Given any function $\omega \in L^p(S)$, we infer from the Hölder inequality that

$$\|K_s \omega\|_{L^\infty} \leq C \|\omega\|_{L^p}, \quad (3.45)$$

where the number C only depends on a_0 , a_1 and s . The energy E_s is then controlled by

$$E_s(\omega) \leq \frac{CN}{2} \|\omega\|_{L^1} \|\omega\|_{L^p} = \frac{CN}{2} \|\omega\|_{L^p}, \quad (3.46)$$

under the conditions $\omega \geq 0$ and $C(\omega) = 1$. We next apply the Young inequality in order to obtain

$$E_s(\omega) \leq \left(\frac{C\lambda^{\frac{1}{p'}} N^{\frac{1}{p'}}$$

Here, the constant $K_{\lambda,p} := C^{p'} \lambda N / (2^{p'} p')$ is uniformly bounded with respect to $p \rightarrow \infty$, and it controls the penalized energy as

$$E_{\lambda,p}(\omega) \leq K_{\lambda,p}. \quad (3.47)$$

In particular, the value of the supremum $\mathcal{E}_{\lambda,p}$ in (3.25) is finite.

In another direction, we also deduce from (3.46) that

$$\frac{\lambda^{1-p} N}{p} \|\omega\|_{L^p}^p \leq \frac{CN}{2} \|\omega\|_{L^p} - E_{\lambda,p}(\omega) \leq \frac{C^{p'} \lambda N}{2p'} + \frac{\lambda^{1-p} N}{2p} \|\omega\|_{L^p}^p - E_{\lambda,p}(\varpi_\lambda),$$

when $E_{\lambda,p}(\varpi_\lambda) \leq E_{\lambda,p}(\omega)$. Here, the notation ϖ_λ refers as before to the function in (3.20). We next obtain

$$\|\omega\|_{L^p}^p \leq (p-1) \lambda^p C^{p'} - \frac{2p\lambda^{p-1}}{N} E_{\lambda,p}(\varpi_\lambda), \quad (3.48)$$

so that there exists a maximizing sequence $(\omega_j)_{j \in \mathbb{N}}$ for (3.25), which is bounded in $L^p(S)$. Hence, there exists a function $\omega_{\lambda,p} \in L^p(S)$ such that, up to a subsequence, $(\omega_j)_{j \in \mathbb{N}}$ weakly tends to $\omega_{\lambda,p}$ in $L^p(S)$. This convergence is enough to guarantee that $\omega_{\lambda,p} \geq 0$ a.e., as well as the identities $C(\omega_{\lambda,p}) = 1$ and $L(\omega_{\lambda,p}) = 1$. Moreover,

$$\mathcal{E}_{\lambda,p} = \limsup_{j \rightarrow \infty} E_{\lambda,p}(\omega_j) \leq E_{\lambda,p}(\omega_{\lambda,p}). \quad (3.49)$$

Indeed, estimate (3.44) implies that $\kappa_s \in L^{p'}(S^2)$, and the functions $(x, y) \mapsto \omega_j(x) \omega_j(y)$ weakly tend to the function $(x, y) \mapsto \omega_{\lambda,p}(x) \omega_{\lambda,p}(y)$ in $L^p(S^2)$ as $j \rightarrow \infty$. Therefore, the quantities $E_s(\omega_j)$ converge to $E_s(\omega_{\lambda,p})$, and inequality (3.49) follows from the weak lower semi-continuity of the L^p -norm. Finally, this inequality guarantees that the limit vorticity $\omega_{\lambda,p}$ solves the maximization problem (3.25). Moreover, we can invoke Lemmas 4.1 and 4.2 in order to claim that the vorticity $\omega_{\lambda,p}$ is angular Steiner symmetric.

We now turn to the proof of (3.27) and (3.28). These two equations correspond to the Euler-Lagrange equations of the maximization problem (3.25). In order to write them, we observe that, given any measurable subset X of S with positive measure, the non-vanishing linear forms C and L are linearly independent on $L^\infty(X)$. Otherwise, there exists a number $a \neq 0$ such that

$$\int_X (a - |x|^2) \varphi(x) dx = 0,$$

for any $\varphi \in L^\infty(X)$. In particular, the function $x \mapsto a - |x|^2$ vanishes almost everywhere on X , which is only possible when a is positive and X is a subset of the circle of center $(0, 0)$ and radius \sqrt{a} . Since this circle is of measure 0, this contradicts the fact that X has positive measure. As a consequence of the linear independence of C and L on $L^\infty(X)$, we conclude that the linear mapping $\varphi \mapsto (C(\varphi), L(\varphi))$ is onto \mathbb{R}^2 .

Consider next a positive number δ such that $X_\delta = \{\omega_{\lambda,p} \geq \delta\}$ has positive measure. Note that all the numbers δ small enough satisfy this assumption, otherwise the vorticity $\omega_{\lambda,p}$ vanishes, which contradicts the condition $C(\omega_{\lambda,p}) = 1$. Taking into account the previous arguments, we derive the existence of two functions φ_1 and φ_2 in $L^\infty(X_\delta)$ with $C(\varphi_1) = L(\varphi_2) = 1$ and $C(\varphi_2) = L(\varphi_1) = 0$. In particular, given any function $\zeta \in L^\infty(S)$ such that $\zeta \geq 0$ on $S \setminus X_\delta$, we can introduce the function

$$\omega_h = \omega_{\lambda,p} + h(\zeta - C(\zeta)\varphi_1 - L(\zeta)\varphi_2),$$

for any positive number h . When this number is small enough, we check that this function belongs to $L^p_+(S)$, with $C(\omega_h) = L(\omega_h) = 1$. The maximizing nature of the vorticity ω then gives

$$0 \geq \lim_{h \rightarrow 0^+} \frac{d}{dh} (E_{\lambda,p}(\omega_h)) = E'_{\lambda,p}(\omega_{\lambda,p}) \zeta - C(\zeta) E'_{\lambda,p}(\omega_{\lambda,p}) \varphi_1 - L(\zeta) E'_{\lambda,p}(\omega_{\lambda,p}) \varphi_2.$$

In this expression, the differential $E'_{\lambda,p}(\omega)$ is given by

$$E'_{\lambda,p}(\omega) \zeta = N \int_S \zeta \left(K_s \omega - \left(\frac{\omega}{\lambda} \right)^{\frac{1}{p'-1}} \right).$$

Setting $\mu_{\lambda,p} = E'_{\lambda,p}(\omega_{\lambda,p}) \varphi_1 / N$, $\alpha_{\lambda,p} = -2E'_{\lambda,p}(\omega_{\lambda,p}) \varphi_2 / N$ and $\psi_{\lambda,p}(x) := K_s \omega_{\lambda,p}(x) + \alpha_{\lambda,p} |x|^2 / 2 - \mu_{\lambda,p}$ as in (3.28), we obtain

$$\int_S \zeta \left(\psi_{\lambda,p} - \left(\frac{\omega_{\lambda,p}}{\lambda} \right)^{\frac{1}{p'-1}} \right) \leq 0.$$

Going back to the condition $\zeta \geq 0$ on $S \setminus X_\delta$, we infer that

$$\psi_{\lambda,p} = \left(\frac{\omega_{\lambda,p}}{\lambda} \right)^{\frac{1}{p'-1}} \text{ on } X_\delta, \quad \text{and} \quad \psi_{\lambda,p} \leq \left(\frac{\delta}{\lambda} \right)^{\frac{1}{p'-1}} \text{ on } S \setminus X_\delta.$$

Equation (3.27) follows by taking the limit $\delta \rightarrow 0$.

Finally, it follows from (3.45) that the function $\psi_{\lambda,p}$ is bounded on S . In view of (3.27), so is the vorticity $\omega_{\lambda,p}$. This completes the proof of Lemma 2.1. \square

3.1.2 Proof of Lemma 2.2

The proof relies on a uniform bound on the function $K_s \omega_{\lambda,p}$, which is derived from estimates (3.45) and (3.48). Indeed, we deduce from the definition of the function ϖ_λ that

$$E_{\lambda,p}(\varpi_\lambda) = E_s(\varpi_\lambda) - \frac{N}{p}.$$

Since $p > 1/s$, this gives

$$|E_{\lambda,p}(\varpi_\lambda)| \leq C_\lambda,$$

where C_λ denotes, here as in the sequel, a positive number not depending on p and possibly changing from line to line. Hence, we infer from (3.45) and (3.48) that

$$\|K_s \omega_{\lambda,p}\|_{L^\infty} \leq C_\lambda. \quad (3.50)$$

As a consequence of its definition, the function $\psi_{\lambda,p}$ then satisfies

$$\frac{\alpha_{\lambda,p}}{2} r^2 - \mu_{\lambda,p} - C_\lambda \leq \psi_{\lambda,p}(r, \theta) \leq \frac{\alpha_{\lambda,p}}{2} r^2 - \mu_{\lambda,p} + C_\lambda. \quad (3.51)$$

If the Lagrange multipliers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ were not bounded independently of p , these estimates would provide a contradiction with the constraints $C(\omega_{\lambda,p}) = L(\omega_{\lambda,p}) = 1$.

More precisely, let us now assume that

$$\mu_{\lambda,p} \rightarrow +\infty, \quad (3.52)$$

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as $p \rightarrow \infty$. In this case, we can suppose that $\mu_{\lambda,p} > C_\lambda$ for p large enough. It then follows from inequalities (3.51) that $\alpha_{\lambda,p} > 0$. Otherwise, the function $\psi_{\lambda,p}$ is non-positive, which eventually contradicts the constraint $C(\omega_{\lambda,p}) = 1$. Consider next the number

$$r_0(\psi_{\lambda,p}) := \inf_{-\pi/4 \leq \theta \leq \pi/4} \left(\inf \text{ess} \left(\text{supp}(r \mapsto (\psi_{\lambda,p})_+(r, \theta)) \right) \right).$$

Since the function $(\psi_{\lambda,p})_+$ is angular Steiner symmetric, this infimum is achieved for $\theta = 0$. Observe that the inequality $f \geq g$ implies that $r_0(f) \leq r_0(g)$. Therefore, we derive from (3.51) that

$$r_0(\omega_{\lambda,p}) = r_0(\psi_{\lambda,p}) \geq r_0\left(\frac{\alpha_{\lambda,p}}{2} |\cdot|^2 - \mu_{\lambda,p} + C_\lambda\right) = \sqrt{\frac{2}{\alpha_{\lambda,p}}(\mu_{\lambda,p} - C_\lambda)} := r_0^*.$$

Similarly, we obtain

$$r_0(\omega_{\lambda,p}) = r_0(\psi_{\lambda,p}) \leq r_0\left(\frac{\alpha_{\lambda,p}}{2} |\cdot|^2 - \mu_{\lambda,p} - C_\lambda\right) = \sqrt{\frac{2}{\alpha_{\lambda,p}}(\mu_{\lambda,p} + C_\lambda)} := r_1^*,$$

and we notice that

$$\frac{r_0^*}{r_1^*} = \sqrt{\frac{\mu_{\lambda,p} - C_\lambda}{\mu_{\lambda,p} + C_\lambda}} \rightarrow 1. \quad (3.53)$$

when $p \rightarrow \infty$. Given a positive number δ such that $1 + \delta < a_1$, we now face two possibilities.

- **Case 1 :** $\limsup_{p \rightarrow \infty} r_1^* \geq 1 + \delta$.

It then follows from (3.53) that

$$\limsup_{p \rightarrow \infty} r_0^* \geq 1 + \delta,$$

and we obtain the following contradiction

$$1 = \int_S |x|^2 \omega_{\lambda,p}(x) dx \geq r_0(\omega_{\lambda,p})^2 \int_S \omega_{\lambda,p} \geq (r_0^*)^2 \int_S \omega_{\lambda,p} = (r_0^*)^2 > 1, \quad (3.54)$$

with the constraints $C(\omega_{\lambda,p}) = L(\omega_{\lambda,p}) = 1$ for a number p large enough.

- **Case 2 :** $\limsup_{p \rightarrow \infty} r_1^* \leq 1 + \delta$.

In this case, we have $r_1^* \leq 1 + \delta$ for any number p large enough. Combining the constraint $C(\omega_{\lambda,p}) = 1$ with (3.51), we get

$$\int_{r_1^*}^{a_1} \left(\frac{\alpha_{\lambda,p}}{2} r^2 - \mu_{\lambda,p} - C_\lambda \right)^{p'-1} r dr \leq \frac{1}{\lambda\pi}. \quad (3.55)$$

The computation of the integral leads to

$$\frac{\alpha_{\lambda,p}}{2} a_1^2 - \mu_{\lambda,p} - C_\lambda \leq \left(\frac{p' \alpha_{\lambda,p}}{\lambda\pi} \right)^{\frac{1}{p'}}.$$

Assuming that δ is small enough and applying the Young inequality to the right-hand side of this inequality give

$$\frac{\alpha_{\lambda,p}}{2} a_1^2 - \mu_{\lambda,p} - C_\lambda \leq \frac{\delta \alpha_{\lambda,p}}{2} + \frac{1}{p} \left(\frac{2}{\delta \lambda \pi} \right)^{\frac{p}{p'}},$$

and then

$$(a_1^2 - \delta) \frac{\alpha_{\lambda,p}}{2} \leq \frac{1}{p} \left(\frac{2}{\delta \lambda \pi} \right)^{\frac{p}{p'}} + \mu_{\lambda,p} + C_\lambda.$$

In view of the definition of r_0^* , we are led to

$$(r_0^*)^2 \geq \frac{(a_1^2 - \delta)(\mu_{\lambda,p} - C_\lambda)}{\mu_{\lambda,p} + C_\lambda + \frac{1}{p} \left(\frac{2}{\delta \lambda \pi} \right)^{p-1}} \xrightarrow{\mu_{\lambda,p} \rightarrow \infty} a_1^2 - \delta.$$

Since δ is arbitrary small and $a_1 > 1$, we finally get a contradiction by arguing as for (3.54).

We conclude that assumption (3.52) does not hold, and we can argue similarly in order to prove that $\mu_{\lambda,p}$ does not converge towards $-\infty$ when $p \rightarrow \infty$. This shows that the numbers $\mu_{\lambda,p}$ remain bounded in this limit, and so do the numbers $\alpha_{\lambda,p}$ due to the constraint $C(\omega_{\lambda,p}) = 1$. \square

3.1.3 Proof of Lemma 2.3

The proof relies on standard regularity theory. In view of the definition of the function $\psi_{\lambda,p}$, we can derive from (3.50) and Lemma 2.2 that

$$\|\psi_{\lambda,p}\|_{L^\infty} \leq C_\lambda, \quad (3.56)$$

where C_λ denotes, here as in the sequel, a positive number not depending on p and possibly changing from line to line. In view of (3.27), we obtain

$$\|\omega_{\lambda,p}\|_{L^\infty} \leq \lambda C_\lambda^{\frac{1}{p-1}}.$$

Since $p \geq p_0$, this is enough to get

$$\|\omega_{\lambda,p}\|_{L^\infty} \leq C_\lambda,$$

and to deduce (3.29) from the constraint $C(\omega_{\lambda,p}) = 1$ and the Hölder inequality. As a consequence of (3.56), we also observe that

$$\|\psi_{\lambda,p}\|_{L^r(B(0,R))} \leq C_\lambda R^{\frac{2}{r}}, \quad (3.57)$$

for any $1 \leq r \leq \infty$ and any positive number R .

For $1 < r < \infty$, we next recall the existence of a positive number $A_{s,r}$, depending only on s and r , such that

$$\|K_s f\|_{\dot{W}^{2s,r}(\mathbb{R}^2)} \leq A_{s,r} \|f\|_{L^r(\mathbb{R}^2)}, \quad (3.58)$$

for any function $f \in L^r(\mathbb{R}^2)$. Applying this inequality to the function $f = \omega_{\lambda,p} \mathbb{1}_{\text{supp}(\omega_{\lambda,p})}$, we obtain

$$\|K_s \omega_{\lambda,p}\|_{\dot{W}^{2s,r}(\mathbb{R}^2)} \leq A_{s,r} \|\omega_{\lambda,p}\|_{L^r(S)},$$

and (3.30) results from (3.57) and Lemma 2.2.

When $1/2 \leq s < 1$, this guarantees that the function $\psi_{\lambda,p}$ lies in $W^{1,r}(S)$ for any $1 < r < \infty$, so that its positive part $(\psi_{\lambda,p})_+$ is also in these spaces. Hence, we can compute

$$\nabla^\perp \left((\psi_{\lambda,p})_+^{p'} \right) = p' (\psi_{\lambda,p})_+^{p'-1} \nabla^\perp \psi_{\lambda,p} = \frac{p'}{\lambda} \omega_{\lambda,p} \nabla^\perp \psi_{\lambda,p},$$

and this identity holds for integrable functions. Given a function $\varphi \in \mathcal{C}_c^\infty(S)$, we conclude that

$$\int_S \omega_{\lambda,p}(x) \nabla^\perp \psi_{\lambda,p}(x) \cdot \nabla \varphi(x) dx = \int_S \nabla^\perp \left((\psi_{\lambda,p})_+^{p'} \right)(x) \cdot \nabla \varphi(x) dx = 0,$$

by integrating by parts. \square

3.1.4 Proof of Lemma 2.4

Invoking the Riesz rearrangement inequality, we reduce the analysis of the maximization problem (3.31) to the situation where both the functions g and h are even and non-increasing on \mathbb{R}_+ . In this case, since the function f is even and non-increasing on \mathbb{R}_+ , so is the function $f \star g$. In particular, the sets $\{f \star g > t\}$ have finite measure for any positive number t . Therefore, we can apply the bathtub principle in [55, Theorem 1.14] in order to show that the function $h_\mu = \mathbb{1}_{[-\frac{\mu}{2}, \frac{\mu}{2}]}$ is a solution to the problem

$$\mathcal{J}_\mu := \max_{h \in \mathcal{G}_\mu(\mathbb{R})} \int_{\mathbb{R}} f \star g(y) h(y) dy.$$

Moreover, when f is decreasing on \mathbb{R}_+ , so is the function $f \star g$, and the bathtub principle guarantees the uniqueness of the solution h_μ . We finally complete the proof of Lemma 2.4 by observing that the functions g and h play symmetric roles in the maximization problem (3.31). \square

3.1.5 Proof of Proposition 2.1

Let $p > 1/s$. Consider a solution $\omega_{\lambda,p}$ to the maximization problem (3.26) and the corresponding Lagrange multipliers $\alpha_{\lambda,p}$ and $\mu_{\lambda,p}$ given by Lemma 2.1. Up to an omitted subsequence, we first derive from Lemma 2.2 the existence of two numbers α_λ and μ_λ such that

$$\alpha_{\lambda,p} \rightarrow \alpha_\lambda \quad \text{and} \quad \mu_{\lambda,p} \rightarrow \mu_\lambda, \quad (3.59)$$

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when $p \rightarrow \infty$. In view of Lemma 2.3, we can invoke the Sobolev embedding theorem in order to exhibit a continuous function $\psi_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that, up to a further subsequence,

$$\psi_{\lambda,p} \rightarrow \psi_\lambda \text{ in } L^\infty(B(0, R)), \quad (3.60)$$

as $p \rightarrow \infty$, for any positive number R . In particular, this convergence provides

$$\limsup_{p \rightarrow \infty} \|\omega_{\lambda,p}\|_{L^\infty(S)} \leq \lambda \lim_{p \rightarrow \infty} \|(\psi_{\lambda,p})_+\|_{L^\infty(S)}^{p'-1} \leq \lambda. \quad (3.61)$$

Going back to Lemma 2.3, we can apply the Banach-Alaoglu theorem to construct a non-negative function $\omega_\lambda \in L^\infty(S)$ such that, up to a further subsequence,

$$\omega_{\lambda,p} \overset{*}{\rightharpoonup} \omega_\lambda \text{ in } L^\infty(S), \quad (3.62)$$

when $p \rightarrow \infty$. In view of (3.61), the function ω_λ lies in $L^\infty_\lambda(S)$ and we can extend it as a function in X_λ by performing a N -fold symmetrization. Moreover, we infer from (3.59), (3.60) and (3.62) that the function ψ_λ is given by (3.22).

We now check that this function solves the maximization problem (3.16). Since the maps $x \mapsto 1$ and $x \mapsto |x|^2$ are locally integrable, we derive from the constraints $C(\omega_{\lambda,p}) = L(\omega_{\lambda,p}) = 1$ and (3.62) that $C(\omega_\lambda) = L(\omega_\lambda) = N$. Consider next a function $\omega \in X_\lambda$ such that $C(\omega) = L(\omega) = N$. It follows from Lemma 2.1 and (3.61) that

$$E_s(\omega) \leq E_s(\omega_{\lambda,p}) + \frac{\lambda N}{p} \int_S \left(\frac{\omega}{\lambda}\right)^p \leq E_s(\omega_{\lambda,p}) + \frac{\lambda N |S|}{p}.$$

Invoking the compact embedding of $L^\infty(S)$ into $\dot{H}^{-s}(S)$, we can extract a further subsequence for which

$$E_s(\omega_{\lambda,p}) \rightarrow E_s(\omega_\lambda),$$

when $p \rightarrow \infty$. Hence, we are led to

$$E_s(\omega) \leq E_s(\omega_\lambda),$$

by taking the limit $p \rightarrow \infty$ in the previous inequality. Therefore, the function ω_λ solves the maximization problem (3.16).

At this point, we explain why the function ω_λ is a vortex patch. Given any function $\omega \in L^\infty_\lambda(S)$, we define the function $\widehat{\omega}$ as being the only angular Steiner symmetric function of $L^\infty_\lambda(S)$, with values into the pair $\{0, \lambda\}$, such that

$$\int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \omega(r, \theta) d\theta = \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \widehat{\omega}(r, \theta) d\theta,$$

for almost any $a_0 < r < a_1$. Using the decreasing nature of the map $\xi \mapsto \kappa_s(r, r', x, i)$ given by Lemma 4.3, we deduce from Lemma 2.4 that

$$\begin{aligned} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} k_s(r, r', \theta - \theta') \omega_1(r, \theta) \omega_2(r', \theta') d\theta d\theta' \\ \leq \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} k_s(r, r', \theta - \theta') \widehat{\omega}_1(r, \theta) \widehat{\omega}_2(r', \theta') d\theta d\theta', \end{aligned}$$

for any functions $(\omega_1, \omega_2) \in L^\infty_\lambda(S)^2$ and almost any $(r, r') \in (\mathbb{R}_+)^2$. Moreover, this inequality is an equality if and only if $\omega_1(r, \cdot) = \widehat{\omega}_1(r, \cdot)$ and $\omega_2(r', \cdot) = \widehat{\omega}_2(r', \cdot)$ for almost any $-\pi/(2N) \leq \theta \leq \pi/(2N)$. Therefore, we have

$$V_s(\omega_1, \omega_2) \leq V_s(\widehat{\omega}_1, \widehat{\omega}_2),$$

so that by (3.19),

$$E_s(\omega) \leq E_s(\widehat{\omega}),$$

with equality if and only if $\omega = \widehat{\omega}$. This guarantees that the function $\omega_\lambda = \widehat{\omega}_\lambda$ is angular Steiner symmetric, and up to the multiplicative factor λ , equal to the characteristic function of a measurable subset Ω_λ of S .

Since $\omega_{\lambda,p} = (\psi_{\lambda,p})_+^{p'-1}$, we derive from (3.60) that this set satisfies

$$\{\psi_\lambda > 0\} \subseteq \Omega_\lambda \subseteq \{\psi_\lambda \geq 0\}.$$

In order to conclude that this set matches (up to a set of measure 0) with the support of the function $(\psi_\lambda)_+$, we are reduced to show that the set $\{\psi_\lambda = 0\}$ has zero measure. Let $a_0 < r < a_1$ and $0 < \theta_1, \theta_2 < \pi/4$, with $\theta_1 < \theta_2$. In view of definition (3.22), we compute

$$\begin{aligned} \psi_\lambda(r, \theta_1) - \psi_\lambda(r, \theta_2) &= K_s \omega_\lambda(r, \theta_1) - K_s \omega_\lambda(r, \theta_2) \\ &= \int_{a_0}^{a_1} \int_0^{\frac{\pi}{2N}} \left(k_s(r, r', \theta' - \theta_1) - k_s(r, r', \theta' - \theta_2) \right) \omega_\lambda(r', \theta') r' dr' d\theta, \end{aligned}$$

and this double integral is positive due to the fact that the map $\xi \mapsto k_s(r, r', \xi)$ is decreasing on $(0, \pi/2N)$ as a consequence of Lemma 4.3 below. By the Fubini theorem, we conclude that the measure of the set $\{\psi_\lambda = 0\}$ is equal to 0, which eventually provides (3.21).

We finally turn to the proof of (3.23). When $1/2 \leq s < 1$, Lemma 2.3 guarantees that the functions $\psi_{\lambda,p}$ are uniformly bounded in $W^{1,r}(B(0,R))$ with respect to $p \rightarrow \infty$ for any exponent $1 < r < \infty$ and any positive number R . As a consequence, we can assume that the function ψ_λ lies in $W_{\text{loc}}^{1,r}(\mathbb{R}^2)$. In particular, we are allowed to write

$$\nabla^\perp((\psi_{\lambda,p})_+) = \mathbb{1}_{\{\psi_\lambda > 0\}} \nabla^\perp \psi_{\lambda,p},$$

so that, by integrating by parts,

$$\int_S \omega_\lambda(x) \nabla^\perp \psi_\lambda(x) \cdot \nabla \varphi(x) dx = \lambda \int_S \nabla^\perp (\psi_\lambda)_+(x) \cdot \nabla \varphi(x) dx = 0,$$

for any function $\varphi \in C_c^\infty(S)$. This completes the proof of Proposition 2.1. \square

3.2 Proofs for Section 2.2

3.2.1 Proof of Lemma 2.6

On the one hand, since ω_λ is the maximizer of the energy E_s ,

$$E_s(\varpi_\lambda) \leq E_s(\omega_\lambda), \quad (3.63)$$

where ϖ_λ is defined in (3.20). Then,

$$E_s(\varpi_\lambda) \geq C_s \int_S \int_S \frac{\lambda \mathbb{1}_{\mathcal{B}((r_\varepsilon, 0), \varepsilon)}(x) \lambda \mathbb{1}_{\mathcal{B}((r_\varepsilon, 0), \varepsilon)}(x')}{|x - x'|^{2(1-s)}} dx dx' = \frac{C_s}{\varepsilon^{2(1-s)}} \mathcal{I}_s.$$

Together with (3.63), this gives the lower bound in (3.36). Concerning the upper bound,

$$E_s(\omega_\lambda) \leq C_s \int_S \int_S \frac{\omega_\lambda(x) \omega_\lambda(x')}{|x - x'|^{2(1-s)}} dx dx' + C.$$

Using the Riesz rearrangement inequality,

$$E_s(\omega_\lambda) \leq C_s \int_S \int_S \frac{\lambda \mathbb{1}_{\mathcal{B}(0, \varepsilon)}(x) \lambda \mathbb{1}_{\mathcal{B}(0, \varepsilon)}(x')}{|x - x'|^{2(1-s)}} dx dx' + C = \frac{C_s}{\varepsilon^{2(1-s)}} \mathcal{I}_s + C.$$

The lemma is proved. \square

3.2.2 Proof of Lemma 2.8

Given any positive numbers η and R , set

$$\Upsilon_\eta^R := \left\{ x \in \text{supp } \omega : \text{meas}(\mathcal{B}(x, R\varepsilon) \cap \text{supp } \omega) \geq \pi \eta \varepsilon^2 \right\}. \quad (3.64)$$

Assume for the sake of a contradiction that this set is empty. First, we split the double integral in (3.36) as

$$\begin{aligned} \int_S \int_S \left(\frac{\varepsilon}{|x - x'|} \right)^{2(1-s)} \omega(x) \omega(x') dx dx' &= \int_S \int_{\mathcal{B}(x, R\varepsilon)} \left(\frac{\varepsilon}{|x - x'|} \right)^{2(1-s)} \omega(x) \omega(x') dx' dx \\ &\quad + \int_S \int_{\mathcal{B}(x, R\varepsilon)^c} \left(\frac{\varepsilon}{|x - x'|} \right)^{2(1-s)} \omega(x) \omega(x') dx' dx. \end{aligned}$$

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If the set Υ_η^R is supposed to be empty, then a rearrangement argument with respect to the variable x' provides

$$\int_S \int_{\mathcal{B}(x, R\varepsilon)} \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} \omega(x)\omega(x') dx dx' < \lambda \int_S \int_{\mathcal{B}(x, \sqrt{\eta\varepsilon})} \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} \omega(x) dx dx' = \frac{\eta^s}{s}.$$

On the other hand, we check that

$$\int_S \int_{\mathcal{B}(x, R\varepsilon)^c} \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} \omega(x)\omega(x') dx dx' \leq \frac{1}{R^{2(1-s)}},$$

so that, by (3.36), we get

$$\mathcal{I}_s < \frac{\eta^s}{s} + \frac{1}{R^{2(1-s)}}.$$

This gives the desired contradiction when $\eta = \eta_R^*$.

The point $x_\lambda^* \in \Upsilon_\eta^R$ can then be chosen independently of R due to the inclusion

$$\Omega_\eta^{R_2} \subseteq \Omega_\eta^{R_1},$$

when $R_2 \geq R_1$. Moreover, if we assume the function ω to be angular Steiner symmetric, then replacing $x_\lambda^* := (r_\lambda^*, \theta_\lambda^*)$ by $(r_\lambda^*, 0)$ in (3.38) cannot decrease the measure of the considered set. As a conclusion, we can suppose in this case that $\theta_\lambda^* = 0$. This ends the proof of Lemma 2.8. \square

3.2.3 Proof of Lemma 2.13

Let $m \in \mathbb{N}^*$. We first decompose the double integral in Lemma 2.6 as

$$\begin{aligned} \mathcal{I}_s &\leq \int_{\mathcal{B}(x_\lambda^*, m\rho\varepsilon)} \int_{\mathcal{B}(x_\lambda^*, m\rho\varepsilon)} \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} \omega(x)\omega(x') dx dx' \\ &\quad + \int_{\mathcal{B}(x_\lambda^*, (m+1)\rho\varepsilon) \setminus \mathcal{B}(x_\lambda^*, m\rho\varepsilon)} \int_{\mathcal{B}(x_\lambda^*, (m+1)\rho\varepsilon) \setminus \mathcal{B}(x_\lambda^*, m\rho\varepsilon)} \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} \omega(x)\omega(x') dx dx' \\ &\quad + \int_{\mathcal{B}(x_\lambda^*, (m+1)\rho\varepsilon)^c} \int_{\mathcal{B}(x_\lambda^*, (m+1)\rho\varepsilon)^c} \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} \omega(x)\omega(x') dx dx' \\ &\quad + 2 \int_{\mathcal{B}(x_\lambda^*, m\rho\varepsilon) \cup \mathcal{B}(x_\lambda^*, (m+1)\rho\varepsilon)^c} \int_{\mathcal{B}(x_\lambda^*, (m+1)\rho\varepsilon) \setminus \mathcal{B}(x_\lambda^*, m\rho\varepsilon)} \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} \omega(x)\omega(x') dx dx' \\ &\quad + 2 \int_{\mathcal{B}(x_\lambda^*, m\rho\varepsilon)} \int_{\mathcal{B}(x_\lambda^*, (m+1)\rho\varepsilon)^c} \left(\frac{\varepsilon}{|x-x'|} \right)^{2(1-s)} \omega(x)\omega(x') dx dx'. \end{aligned} \tag{3.65}$$

In order to simplify the notations, we next set

$$\eta_0 := \int_{\mathcal{B}(x_\lambda^*, m\rho\varepsilon)} \omega(x) dx, \quad \eta_1 := \int_{\mathcal{B}(x_\lambda^*, (m+1)\rho\varepsilon) \setminus \mathcal{B}(x_\lambda^*, m\rho\varepsilon)} \omega(x) dx \text{ and } \eta_2 := \int_{\mathcal{B}(x_\lambda^*, (m+1)\rho\varepsilon)^c} \omega(x) dx.$$

We now apply the estimates in Lemmas 2.9, 2.10 and 2.11 to (3.65) in order to get

$$1 \leq \eta_0^{1+s} + \eta_1^{1+s} + \eta_2^{1+s} + \frac{2}{\mathcal{I}_s} \left(\frac{\eta_1(\eta_0 + \eta_2)^s}{s} + \frac{\eta_0\eta_2}{\rho^{2(1-s)}} \right). \tag{3.66}$$

Since η_0, η_1, η_2 and $\eta_0 + \eta_2$ are less than 1, this gives

$$1 \leq \eta_0^{s+1} + \eta_2^{s+1} + \left(1 + \frac{2}{s\mathcal{I}_s} \right) \eta_1 + \frac{2}{\mathcal{I}_s \rho^{2(1-s)}}. \tag{3.67}$$

We now estimate η_1 by using Lemma 2.12, with $\gamma = 2(1-s)$. In particular, we fix the value of the integer m in (3.65) as the one provided by Lemma 2.12. With these choices, we infer from (3.67) that

$$1 \leq \eta_0^{s+1} + \eta_2^{s+1} + \left(1 + \frac{2}{s\mathcal{I}_s} \right) \frac{1}{\rho^{2(1-s)}} + \frac{2}{\mathcal{I}_s \rho^{2(1-s)}}. \tag{3.68}$$

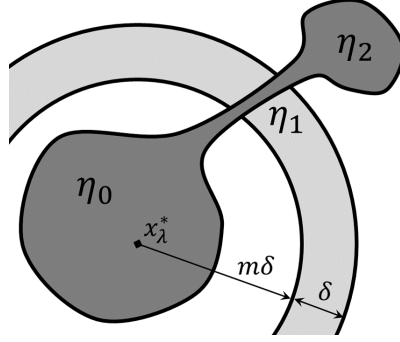


FIGURE 3.3 – Illustration for the proof of Lemma 2.13

Since $\eta_0 + \eta_1 + \eta_2 = 1$, we are led to

$$1 - \eta_0^{s+1} - (1 - \eta_0)^{s+1} \leq \left(1 + \frac{2(1+s)}{s\mathcal{I}_s}\right) \frac{1}{\rho^{2(1-s)}}.$$

We now use the concavity of the function $t \mapsto 1 - t^{s+1} - (1-t)^{s+1}$ on the segment $[0, 1]$ in order to obtain

$$1 - t^{s+1} - (1-t)^{s+1} \geq (1 - 2^{-s}) \left(\frac{1}{2} - \left|t - \frac{1}{2}\right|\right),$$

so that we are reduced to the alternative

$$\eta_0 \leq \frac{\mathcal{Q}_s}{\rho^{2(1-s)}} \quad \text{or} \quad 1 - \eta_0 \leq \frac{\mathcal{Q}_s}{\rho^{2(1-s)}}. \quad (3.69)$$

In view of Lemma 2.8, we know that

$$\eta_0 \geq s\mathcal{I}_s - \frac{s}{m\rho^{2(1-s)}} \geq s\mathcal{I}_s - \frac{s}{\rho^{2(1-s)}}.$$

Therefore, when $s\mathcal{I}_s\rho^{2(1-s)} \geq \mathcal{Q}_s + s$, the alternative in (3.69) reduces to

$$1 - \eta_0 \leq \frac{\mathcal{Q}_s}{\rho^{2(1-s)}}.$$

Due to the constraint $C(\omega) = 1$, this leads to

$$\int_{\mathcal{B}(x_\lambda^*, m\rho\varepsilon)^c} \omega(x) dx = 1 - \eta_0 \leq \frac{\mathcal{Q}_s}{\rho^{2(1-s)}}.$$

It then follows from Lemma 2.12 that $m \leq \rho^{2(1-s)}$, so that

$$\int_{\mathcal{B}(x_\lambda^*, \rho^{2(1-s)+1}\varepsilon)^c} \omega(x) dx \leq \frac{\mathcal{Q}_s}{\rho^{2(1-s)}}.$$

We finally complete the proof of Lemma 2.13 by setting $R := \rho^{2(1-s)+1}$. □

3.2.4 Proof of Lemma 2.14

Let R be a positive number. We start the proof with the following lower and upper bounds

$$\begin{aligned} \left(\inf_{x \in \mathcal{B}(x_\lambda^*, R\varepsilon)} |x|^2\right) \int_{\mathcal{B}(x_\lambda^*, R\varepsilon)} \omega(x) dx &\leq \int_S |x|^2 \omega(x) dx \\ &\leq \left(\sup_{x \in \mathcal{B}(x_\lambda^*, R\varepsilon)} |x|^2\right) \int_{\mathcal{B}(x_\lambda^*, R\varepsilon)} \omega(x) dx + a_1^2 \int_{\mathcal{B}(x_\lambda^*, R\varepsilon)^c} \omega(x) dx. \end{aligned}$$

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According to Lemma 2.8, the point x_λ^* takes the form $(r_\lambda^*, 0)$. Therefore, we obtain

$$\begin{aligned} \left(r_\lambda^* - \frac{R^2}{\pi\lambda}\right) \int_{\mathcal{B}(x_\lambda^*, R\varepsilon)} \omega(x) dx &\leq \int_S |x|^2 \omega(x) dx \\ &\leq \left(r_\lambda^* + \frac{R^2}{\pi\lambda}\right) \int_{\mathcal{B}(x_\lambda^*, R\varepsilon)} \omega(x) dx + a_1^2 \int_{\mathcal{B}(x_\lambda^*, R\varepsilon)^c} \omega(x) dx. \end{aligned}$$

We now assume that $R > \mathcal{R}_s$ and use Lemma 2.13. In view of the $C(\omega) = L(\omega) = 1$, we are led to

$$\left(r_\lambda^* - \frac{R^2}{\pi\lambda}\right) \left(1 - \frac{\mathcal{Q}_s}{R^{\gamma_s}}\right) \leq 1 \leq \left(r_\lambda^* + \frac{R^2}{\pi\lambda}\right) + a_1^2 \frac{\mathcal{Q}_s}{R^{\gamma_s}}.$$

In the limit $\lambda \rightarrow \infty$, this gives

$$\left(1 - \frac{\mathcal{Q}_s}{R^{\gamma_s}}\right) \limsup_{\lambda \rightarrow \infty} r_\lambda^* \leq 1 \leq a_1^2 \frac{\mathcal{Q}_s}{R^{\gamma_s}} + \liminf_{\lambda \rightarrow \infty} r_\lambda^*.$$

The convergence in Lemma 2.14 then follows in the limit $R \rightarrow \infty$. This concludes the proof of this lemma. \square

3.2.5 Proof of Lemma 2.15

Let $R > 0$. When $x \in \mathcal{B}(x_\lambda^*, R\varepsilon)$, we can write

$$K_s \omega_\lambda(x) \geq \frac{C_s}{(2R\varepsilon)^{2(1-s)}} \int_{\mathcal{B}(x_\lambda^*, R\varepsilon)} \omega_\lambda(z) dz - C. \quad (3.70)$$

On the contrary, if $x \in \mathcal{B}(x_\lambda^*, 2R\varepsilon)^c$, then

$$K_s \omega_\lambda(x) \leq C_s \int_S \frac{\omega_\lambda(z)}{|x-z|^{2(1-s)}} dz + C \leq C_s \int_{\mathcal{B}(x_\lambda^*, R\varepsilon)^c} \frac{\omega_\lambda(z)}{|x-z|^{2(1-s)}} dz + \frac{C_s}{(R\varepsilon)^{2(1-s)}} + C,$$

and a rearrangement argument provides

$$K_s \omega_\lambda(x) \leq \frac{C_s}{s\varepsilon^{2(1-s)}} \left(\int_{\mathcal{B}(x_\lambda^*, R\varepsilon)^c} \omega_\lambda(z) dz \right)^s + \frac{C_s}{(R\varepsilon)^{2(1-s)}} + C. \quad (3.71)$$

Consider now $x \in S$ such that $\psi_\lambda(x) \geq 0$ and $y \in S$ such that $\psi_\lambda(y) \leq 0$. We then have

$$0 \geq \psi_\lambda(y) - \psi_\lambda(x) = K_s \omega_\lambda(y) - K_s \omega_\lambda(x) + \frac{\alpha}{2} (|y|^2 - |x|^2). \quad (3.72)$$

We next assume for the sake of a contradiction the existence of a point $x \in \mathcal{B}(x_\lambda^*, 2R\varepsilon)^c$ such that $\psi_\lambda(x) \geq 0$. When ε is small enough, we can invoke Lemma 2.14 in order to find $y_1 \in S$ such that $\psi_\lambda(y_1) < 0$ and the two following conditions are satisfied

$$\text{sign}(\alpha) = \text{sign}(|y_1|^2 - |x|^2) \quad \text{and} \quad ||y_1|^2 - |x|^2| \geq \frac{a_1 - a_0}{4}. \quad (3.73)$$

Combining equations (3.72) and (3.73) with (3.71) leads to the estimate

$$\frac{a_1 - a_0}{8} |\alpha| \leq \frac{C_s}{s\varepsilon^{2(1-s)}} \left(\int_{\mathcal{B}(x_\lambda^*, R\varepsilon)^c} \omega_\lambda(z) dz \right)^s + \frac{C_s}{(R\varepsilon)^{2(1-s)}} + C. \quad (3.74)$$

Given a positive number R_1 , consider now $y_2 \in \mathcal{B}(x_\lambda^*, R_1\varepsilon)$ such that $\psi_\lambda(y_2) < 0$. Inequalities (3.70), (3.71) and (3.72) provide

$$\begin{aligned} \frac{C_s}{(2R_1\varepsilon)^{2(1-s)}} \int_{\mathcal{B}(x_\lambda^*, R_1\varepsilon)} \omega_\lambda(x) dx &\leq \frac{C_s}{s\varepsilon^{2(1-s)}} \left(\int_{\mathcal{B}(x_\lambda^*, R\varepsilon)^c} \omega_\lambda(x) dx \right)^s \\ &\quad + \frac{C_s}{(R\varepsilon)^{2(1-s)}} + \frac{|\alpha|}{2} (a_1 - a_0) + C, \end{aligned}$$

so that we deduce from (3.74) that

$$\frac{C_s}{(2R_1)^{2(1-s)}} \int_{\mathcal{B}(x_\lambda^*, R_1 \varepsilon)} \omega_\lambda(x) dx \leq \frac{5C_s}{s} \left(\int_{\mathcal{B}(x_\lambda^*, R\varepsilon)^c} \omega_\lambda(x) dx \right)^s + \frac{5C_s}{R^{2(1-s)}} + C\varepsilon^{2(1-s)}.$$

We next assume that R and R_1 are larger than the number \mathcal{R}_s in (3.40) and we apply Lemma 2.13 in order to get

$$\frac{C_s}{(2R_1)^{2(1-s)}} \left(1 - \frac{Q_s}{R_1^s}\right) \leq \frac{5C_s}{s} \left(\frac{Q_s}{R^s}\right)^s + \frac{5C_s}{R^{2(1-s)}} + C\varepsilon^{2(1-s)}.$$

Finally, we check that we can choose R and R_1 such that the above inequality is false when ε is small enough. This ends the proof of Lemma 2.15. \square

3.3 Proofs for Section 2.3

3.3.1 Proof of Lemma 2.18

We start from the weak formulation (3.5) that we rewrite using a Delort symmetrization [35] :

$$\alpha \int_{\mathbb{R}^2} \omega(x) \nabla \varphi(x) \cdot x^\perp dx = C_s(1-s) \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \omega(x) \omega(x') \left(\nabla \varphi(x) - \nabla \varphi(x') \right) \cdot \frac{(x-x')^\perp}{|x-x'|^{4-2s}} dx' dx \quad (3.75)$$

Let $y = (1, 0) \in \mathbb{R}^2$ the position of the vortex Dirac mass in S . We choose a function $\varphi \in \mathcal{C}_{sym}^\infty(S)$ and such that

$$\varphi(x) = y^\perp \cdot x,$$

when $x \in \text{supp } \omega_\lambda \cap S$. Using the fact that φ satisfies the same N fold symmetry condition that ω provides

$$\begin{aligned} & \alpha_\lambda \int_S \omega_\lambda(x) y^\perp \cdot x^\perp dx \\ &= C_s(s-1) \sum_{n=1}^{N-1} \int_S \int_S \omega_\lambda(x) \omega_\lambda(x') (y^\perp - R \frac{2n\pi}{N} y^\perp) \cdot \frac{(x - R \frac{2n\pi}{N} x')^\perp}{|x - R \frac{2n\pi}{N} x'|^{4-2s}} dx dx', \end{aligned} \quad (3.76)$$

where we dropped the term in the sum above corresponding to $n = 0$ because in this case $y^\perp - R \frac{2n\pi}{N} y^\perp = 0$. We now rely on the convergence of ω_λ on S towards the Dirac mass centered at $(1, 0) = y$ as $\lambda \rightarrow \infty$ in order to obtain

$$\int_S \omega_\lambda(x) y^\perp \cdot x^\perp dx \longrightarrow y^\perp \cdot y^\perp = 1,$$

when $\lambda \rightarrow \infty$. This convergence also implies the following limit

$$\begin{aligned} & \sum_{n=1}^{N-1} \int_S \int_S \omega_\lambda(x) \omega_\lambda(x') (y^\perp - R \frac{2n\pi}{N} y^\perp) \cdot \frac{(x - R \frac{2n\pi}{N} x')^\perp}{|x - R \frac{2n\pi}{N} x'|^{4-2s}} dx dx' \\ & \longrightarrow \sum_{n=1}^{N-1} (y^\perp - R \frac{2n\pi}{N} y^\perp) \cdot \frac{(y - R \frac{2n\pi}{N} y)^\perp}{|y - R \frac{2n\pi}{N} y|^{4-2s}} = \sum_{n=1}^{N-1} \frac{1}{|y - R \frac{2n\pi}{N} y|^{2(1-s)}}. \end{aligned} \quad (3.77)$$

From these two convergences and (3.76) we conclude that

$$\alpha_\lambda \longrightarrow \sum_{n=1}^{N-1} \frac{C_s(1-s)}{|y - R \frac{2n\pi}{N} y|^{2(1-s)}}. \quad (3.78)$$

This completes the proof of Lemma 2.18. \square

3.3.2 Proof of Lemma 2.19

Concerning the lower estimate, we choose $y \in \mathcal{B}((1, 0), R\varepsilon)$ such that $\omega(y) = 0$, where the number R is defined in Lemma 2.15. We then have $\psi(y) \leq 0$, and we deduce from (3.22) that

$$K_s \omega_\lambda(x) + \frac{\alpha_\lambda}{2} |y|^2 \leq \mu. \quad (3.79)$$

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On the one hand, it follows from Lemma 2.18 that the numbers $\frac{\alpha\lambda}{2}|y|^2$ are bounded as $\lambda \rightarrow \infty$. On the other hand, we infer from Lemma 2.15 and the condition $C(\omega_\lambda) = 1$ that

$$K_s \omega_\lambda(x) \geq \int_{\mathcal{B}((1,0), R\varepsilon)} \frac{\omega_\lambda(y')}{|y - y'|^{2(1-s)}} dy' - C \geq \frac{1}{(2R\varepsilon)^{2(1-s)}} - C.$$

Pulling these two facts back in (3.79) gives the lower estimate in (3.42).

For the upper estimate, we consider a point $y \in \mathcal{B}((1,0), R\varepsilon)$ such that $\omega(y) = \lambda$. Then $\psi(x) \geq 0$ and (3.22) gives

$$\mu \leq K_s \omega_\lambda(x) + \frac{\alpha\lambda}{2}|y|^2. \quad (3.80)$$

The numbers $\frac{\alpha\lambda}{2}|y|^2$ still remain bounded as $\lambda \rightarrow \infty$, whereas by rearrangement,

$$K_s \omega_\lambda(x) \leq \int_{\mathcal{B}((1,0), R\varepsilon)} \frac{\omega_\lambda(x')}{|x - x'|^{2(1-s)}} dx' + C \leq \lambda \int \frac{\mathbb{1}_{\mathcal{B}((1,0), \varepsilon)}(x')}{|x - x'|^{2(1-s)}} dx' + C = \frac{1}{s\varepsilon^{2(1-s)}} + C.$$

This gives the upper bound in (3.42) and completes the proof of Lemma 2.19. \square

4 The angular Steiner symmetrization

Our previous construction of co-rotating vortices with N -fold symmetry relies several times on a reduction on each fold to angular Steiner symmetric functions. This reduction is made possible by the fact that the restriction to each fold of these co-rotating vortices are indeed angular Steiner symmetric. In this appendix, we collect the properties of this symmetrization, which we have used for our construction, and we give their proofs. Consider a non-negative measurable function ω defined on the angular sector

$$\mathfrak{S}_N := \left\{ x = (r \cos(\theta), r \sin(\theta)) \in \mathbb{R}^2 : -\frac{\pi}{N} < \theta < \frac{\pi}{N} \right\}.$$

Its angular Steiner symmetrization ω^\sharp is defined as the unique even function for the variable θ such that

$$\omega^\sharp(r, \theta) > \nu \quad \text{if and only if} \quad |\theta| < \frac{1}{2} \text{meas} \left\{ \theta' \in \left(-\frac{\pi}{N}, \frac{\pi}{N} \right) : \omega(r, \theta') > \nu \right\},$$

for any positive numbers r and ν , and any angle $-\pi/N < \theta < \pi/N$. This function is well-defined and measurable on \mathfrak{S}_N . Moreover, the function ω is said to be angular Steiner symmetric if and only if $\omega^\sharp = \omega$ almost everywhere. In view of its definition, the angular Steiner symmetrization satisfies

$$\text{meas} \left\{ \theta \in \left(-\frac{\pi}{N}, \frac{\pi}{N} \right) : \omega^\sharp(r, \theta) > \nu \right\} = \text{meas} \left\{ \theta \in \left(-\frac{\pi}{N}, \frac{\pi}{N} \right) : \omega(r, \theta) > \nu \right\},$$

for any positive numbers r and ν , which means that it is a rearrangement with respect to the angular variable θ . As a consequence of the layer-cake representation of non-negative measurable functions, the angular Steiner symmetrization maps the set

$$L_+^p(\mathfrak{S}_N) := \{ \omega \in L^p(\mathfrak{S}_N) \text{ s.t. } \omega \geq 0 \text{ a.e.} \},$$

into itself. Moreover, the circulation C and the impulse L are invariant under this symmetrization, when they make sense.

Lemma 4.1. *Let $\omega \in L_+^1(\mathfrak{S}_N)$. We have*

$$C(\omega^\sharp) = C(\omega).$$

Moreover, if the impulse $L(\omega)$ is finite, then

$$L(\omega^\sharp) = L(\omega).$$

Démonstration. This is a direct consequence of the layer-cake formula

$$\omega(r, \theta) = \int_0^{+\infty} \mathbb{1}_{\{\omega(r, \cdot) > \nu\}}(\theta) d\nu,$$

the Fubini theorem and the definition of the angular Steiner symmetrization. \square

In contrast, the energy E is not conserved by the angular Steiner symmetrization, but it is increased by this transformation. The proof of this claim relies on the Riesz rearrangement inequality for the angular Steiner symmetrization.

Lemma 4.2. *Let $\omega \in L_+^\infty(S)$. We have*

$$E(\omega) \leq E(\omega^\sharp).$$

Démonstration. Recall first that the quantity $E(\omega)$ is defined by

$$E(\omega) = \int_{a_0}^{a_1} r \, dr \int_{a_0}^{a_1} r' \, dr' \left(\int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \kappa(r, r', \theta - \theta') \omega(r, \theta) \omega(r', \theta') \, d\theta \, d\theta' \right). \quad (3.81)$$

In this expression, the kernel κ is equal to

$$\kappa(r, r', \xi) = \sum_{n=0}^{N-1} \frac{NC_s}{(r^2 + (r')^2 - 2rr' \cos(\xi - \frac{2\pi n}{N}))^{1-s}},$$

for $a_0 < r \neq r' < a_1$ and $-\pi/N < \xi < \pi/N$. This function is non-negative and even with respect to the variable ξ due to the identity

$$\cos\left(-\xi - \frac{2\pi n}{N}\right) = \cos\left(\xi - \frac{2\pi(N-n)}{N}\right),$$

which holds for $-\pi/N < \xi < \pi/N$ and $1 \leq n \leq N-1$.

Moreover, given two fixed numbers $a_0 < r \neq r' < a_1$, the map $\xi \mapsto \kappa(r, r', \xi)$ is smooth on $(-\pi/N, \pi/N)$, and its derivative is given by

$$\frac{\partial \kappa}{\partial \xi}(r, r', \xi) = -2C_s(1-s)rr' \sum_{n=0}^{N-1} \frac{\sin\left(\xi - \frac{2\pi n}{N}\right)}{(r^2 + (r')^2 - 2rr' \cos(\xi - \frac{2\pi n}{N}))^{2-s}}.$$

We claim that this quantity is negative on $(0, \pi/N)$, while it is positive on $(-\pi/N, 0)$. More precisely, we have

Lemma 4.3. *Let $N \geq 2$, $0 \leq s \leq 1$ and $0 < \varrho < 1$. Consider the function $\phi : [-1, 1] \rightarrow \mathbb{R}$ defined by*

$$\phi(x) := \frac{1}{(1 - \varrho x)^{2-s}}, \quad (3.82)$$

for any number $-1 \leq x \leq 1$, and set

$$\Phi(\xi) = \sum_{n=0}^{N-1} \sin\left(\xi - \frac{2\pi n}{N}\right) \phi\left(\cos\left(\xi - \frac{2\pi n}{N}\right)\right),$$

for any number $\xi \in \mathbb{R}$. There exists a continuous, positive and $2\pi/N$ -periodic function $F : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\Phi(\xi) = \sin(N\xi) F(\xi), \quad (3.83)$$

for any number $\xi \in \mathbb{R}$.

Our previous claim directly follows from applying formula (3.83) with $\varrho = 2rr'/(r^2 + (r')^2) \in (0, 1)$. For sake of clarity, we postpone the proof of Lemma 4.3 and now complete the proof of Lemma 4.2.

Coming back to (3.81), we fix two positive numbers $r \neq r'$ and extend the functions $\theta \mapsto \omega(r, \theta)$ and $\theta \mapsto \omega(r', \theta)$ to \mathbb{R} by letting them equal to 0 outside the interval $(-\pi/2N, \pi/2N)$. Similarly, we extend the map $\theta \mapsto \kappa(r, r', \theta)$ to \mathbb{R} by letting it equal to 0 outside the interval $(-\pi/N, \pi/N)$. We then have

$$\int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \kappa(r, r', \theta - \theta') \omega(r, \theta) \omega(r', \theta') \, d\theta \, d\theta' = \int_{\mathbb{R}} \int_{\mathbb{R}} \kappa(r, r', \theta - \theta') \omega(r, \theta) \omega(r', \theta') \, d\theta \, d\theta',$$

and we can apply the Riesz rearrangement inequality (see e.g. [55, Lemma 3.6]) in order to obtain

$$\int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \kappa(r, r', \theta - \theta') \omega(r, \theta) \omega(r', \theta') \, d\theta \, d\theta' \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \kappa^\sharp(r, r', \theta - \theta') \omega^\sharp(r, \theta) \omega^\sharp(r', \theta') \, d\theta \, d\theta'.$$

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Since the map $\xi \mapsto \kappa(r, r', \xi)$ is even and non-increasing on \mathbb{R}_+ , we have

$$\kappa^\sharp(r, r', \theta - \theta') = \kappa(r, r', \theta - \theta'),$$

for any $(\theta, \theta') \in \mathbb{R}^2$. Moreover, since the functions $\theta \mapsto \omega(r, \theta)$ and $\theta \mapsto \omega(r', \theta)$ are supported in $(-\pi/2N, \pi/2N)$, so are the functions $\theta \mapsto \omega^\sharp(r, \theta)$ and $\theta \mapsto \omega^\sharp(r', \theta)$, and this property leads to

$$\int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \kappa(r, r', \theta - \theta') \omega(r, \theta) \omega(r', \theta') d\theta d\theta' \leq \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \int_{-\frac{\pi}{2N}}^{\frac{\pi}{2N}} \kappa(r, r', \theta - \theta') \omega^\sharp(r, \theta) \omega^\sharp(r', \theta') d\theta d\theta'.$$

Lemma 4.2 finally follows from introducing this inequality into (3.81). \square

We now provide the proof of Lemma 4.3.

Proof of Lemma 4.3. For sake of simplicity, we set

$$\xi_n = \xi - \frac{2\pi n}{N},$$

for any number $\xi \in \mathbb{R}$ and any integer $0 \leq n \leq N-1$. With this notation at hand, the function Φ rewrites as

$$\Phi(\xi) = \sum_{n=0}^{N-1} \sin(\xi_n) \phi(\cos(\xi_n)),$$

and we can express it as in (3.83) by developing the following inductive argument.

The first step, as the subsequent ones, relies on a Taylor expansion of the function ϕ . We define the maps

$$T_p[f](x) := \int_0^1 f^{(p)}(sx)(1-s)^{p-1} ds$$

for any integer $p \geq 1$, any number $-1 \leq x \leq 1$ and any function $f \in C^\infty([-1, 1])$. These maps allow to write the remainder term in the Taylor formula as

$$f(x) = \sum_{k=0}^{p-1} \frac{f^{(k)}(0)}{k!} x^k + \frac{x^p}{(p-1)!} T_p[f](x).$$

Since the function ϕ is smooth on \mathbb{R} , we infer from this formula that

$$\begin{aligned} \Phi(\xi) &= \sum_{k=0}^{N-2} \frac{\phi^{(k)}(0)}{k!} \sum_{n=0}^{N-1} \sin(\xi_n) \cos(\xi_n)^k \\ &\quad + \frac{1}{(N-2)!} \sum_{n=0}^{N-1} \sin(\xi_n) \cos(\xi_n)^{N-1} T_{N-1}[\phi](\cos(\xi_n)). \end{aligned} \tag{3.84}$$

We next claim that the trigonometric sums $\sum_{n=0}^{N-1} \sin(\xi_n) \cos(\xi_n)^k$ are equal to 0 when $0 \leq k \leq N-2$. For further use, we more generally set

$$\sigma_{k,\ell}(\xi) := \sum_{n=0}^{N-1} \sin(\ell\xi_n) \cos(\xi_n)^k, \tag{3.85}$$

for any integer $(k, \ell) \in \mathbb{N}^2$, and we check that

$$\sigma_{k,\ell}(\xi) = 0, \tag{3.86}$$

provided that $0 \leq k + \ell \leq N-1$. Indeed, the Euler formula for the cosine function and the binomial theorem give

$$\sigma_{k,\ell}(\xi) = \frac{1}{2^k} \sum_{n=0}^{N-1} \sin(\ell\xi_n) \sum_{j=0}^k \binom{k}{j} e^{i(k-2j)\xi_n}.$$

Since the sum $\sigma_{k,\ell}(\xi)$ is real-valued, this expression reduces to

$$\sigma_{k,\ell}(\xi) = \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} \sum_{n=0}^{N-1} \sin(\ell\xi_n) \cos((k-2j)\xi_n),$$

so that

$$\sigma_{k,\ell}(\xi) = \frac{1}{2^{k+1}} \sum_{j=0}^k \binom{k}{j} \sum_{n=0}^{N-1} \left(\sin((k+\ell-2j)\xi_n) + \sin((2j+\ell-k)\xi_n) \right). \quad (3.87)$$

We rewrite this formula as

$$\sigma_{k,\ell}(\xi) = \frac{1}{2^{k+1}} \sum_{j=0}^k \binom{k}{j} \Im \left(e^{i(k+\ell-2j)\xi} \sum_{n=0}^{N-1} \left(e^{\frac{2\pi i(2j-k-\ell)}{N}n} \right)^n + e^{i(2j+\ell-k)\xi} \sum_{n=0}^{N-1} \left(e^{\frac{2\pi i(k-\ell-2j)}{N}n} \right)^n \right). \quad (3.88)$$

Recall that

$$\sum_{n=0}^{N-1} \left(e^{\frac{2\pi i p}{N}n} \right)^n = \begin{cases} N & \text{if } p \text{ is a multiple of } N, \\ 0 & \text{elsewhere,} \end{cases}$$

and observe that the integers $2j - \ell - k$ and $k - \ell - 2j$ lie in the interval $[-(N-1), N-1]$ due to the constraints $0 \leq k + \ell \leq N - 1$ and $0 \leq j \leq k$. As a consequence, the sums with respect to n in (3.88) are equal to 0 except if $2j - \ell - k = 0$ for the first sum, respectively $k - \ell - 2j = 0$ for the second one. In these two cases, the quantity in the imaginary part is real-valued, so that the quantity $\sigma_{k,\ell}(\xi)$ is indeed equal to 0.

Invoking (3.86) with $0 \leq k \leq N - 2$ and $\ell = 1$, we can simplify (3.84) as

$$\Phi(\xi) = \frac{1}{(N-2)!} \sum_{n=0}^{N-1} \sin(\xi_n) \cos(\xi_n)^{N-1} T_{N-1}[\phi](\cos(\xi_n)), \quad (3.89)$$

and put the focus on the trigonometric polynomial $\xi \mapsto \sin(\xi) \cos(\xi)^{N-1}$. As before, we more generally consider the functions $\xi \mapsto \sin(\ell\xi) \cos(\xi)^{N-\ell}$ for an integer $1 \leq \ell \leq N - 1$. Arguing as for the proof of (3.87) and using the identity $\sin(N\xi_n) = \sin(N\xi)$ for $0 \leq n \leq N - 1$, we observe that

$$\sin(\ell\xi_n) \cos(\xi_n)^{N-\ell} = \frac{1}{2^{N-\ell}} \left(\sin(N\xi) + \sum_{k=1}^{N-\ell} \binom{N-\ell}{k} \sin((N-2k)\xi_n) \right). \quad (3.90)$$

In this expression, the quantities $\sin((N-2k)\xi_n)$ can appear only once, with a positive multiplicative factor, or twice, with opposite multiplicative factors. For further use, we aim at eliminating this late behavior. For $2\ell \geq N$, this behavior is excluded, so that we now assume that $2\ell < N$. In this case, we use the change of indices $j = N - k$ for $N/2 < k \leq N - \ell$ in order to decompose the sum over k in (3.90) as

$$\begin{aligned} \sum_{k=1}^{N-\ell} \binom{N-\ell}{k} \sin((N-2k)\xi_n) &= \sum_{1 \leq k < \ell} \binom{N-\ell}{k} \sin((N-2k)\xi_n) \\ &\quad + \sum_{\ell \leq k < \frac{N}{2}} \left(\binom{N-\ell}{k} - \binom{N-\ell}{N-k} \right) \sin((N-2k)\xi_n). \end{aligned} \quad (3.91)$$

Note here that

$$\binom{N-\ell}{k} - \binom{N-\ell}{N-k} = \frac{(N-\ell)!}{k!(N-k)!} \left(\prod_{j=0}^{\ell-1} (N-k-j) - \prod_{j=0}^{\ell-1} (k-j) \right) > 0, \quad (3.92)$$

since $N - k > k$ when $\ell \leq k < N/2$. Hence, rewriting (3.90) as in (3.91) provides an expression in which each quantity $\sin((N-2k)\xi_n)$ appears only once, with a positive multiplicative factor.

Going back to (3.89), we can combine (3.90) and (3.91) for $\ell = 1$ in order to obtain

$$\begin{aligned} \Phi(\xi) &= \frac{\sin(N\xi)}{2^{N-1}(N-2)!} \sum_{n=0}^{N-1} T_{N-1}[\phi](\cos(\xi_n)) \\ &\quad + \frac{1}{2^{N-1}(N-2)!} \sum_{1 \leq k < \frac{N}{2}} \left(\binom{N-1}{k} - \binom{N-1}{N-k} \right) \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) T_{N-1}[\phi](\cos(\xi_n)). \end{aligned} \quad (3.93)$$

At this stage, we set

$$F^1(\xi) := \frac{1}{2^{N-1}(N-2)!} \sum_{n=0}^{N-1} T_{N-1}[\phi](\cos(\xi_n)), \quad (3.94)$$

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and

$$R_{N-2k}^1(x) = \frac{1}{2^{N-1}(N-2)!} \left(\binom{N-1}{k} - \binom{N-1}{N-k} \right) T_{N-1}[\phi](x), \quad (3.95)$$

for $1 \leq k < N/2$ and $-1 \leq x \leq 1$. With this notation at hand, we can write (3.93) as

$$\Phi(\xi) = \sin(N\xi)F^1(\xi) + \sum_{1 \leq k < \frac{N}{2}} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^1(\cos(\xi_n)), \quad (3.96)$$

and we observe that the sums

$$\sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^1(\cos(\xi_n))$$

have exactly the same form as the map Φ with the function ϕ being replaced by the functions R_{N-2k}^1 . As a consequence, we can expect that an inductive argument eventually provides an expression of the map Φ as in (3.83).

Before going into this inductive argument, we now check that the function F^1 is positive, while the remainder terms R_{N-2k}^1 are positive and absolutely monotone, which means that all their derivatives are positive. This claim originates into the two following properties.

First, in view of (3.82), the function ϕ is well-defined and smooth on $[-1, 1] \subset (-1/\varrho, 1/\varrho)$, and its successive derivatives are given by

$$\phi^{(p)}(x) = \frac{(2-s)(3-s)\dots(p+1-s)\varrho^p}{(1-\varrho x)^{p+2-s}} > 0,$$

for any $p \geq 1$ and any $x \in [-1, 1]$. Therefore, the function ϕ is absolutely monotone. Actually, it extends to an analytic function on the interval $(-1/\varrho, 1/\varrho)$ since it can be expanded as the power series

$$\phi(x) = \sum_{p=0}^{+\infty} \alpha_{p,s} \varrho^p x^p, \quad (3.97)$$

for $x \in (-1/\varrho, 1/\varrho)$. In this formula, the coefficients $\alpha_{p,s}$ are given by

$$\alpha_{0,s} := 1 \quad \text{and} \quad \alpha_{p,s} := \frac{(2-s)(3-s)\dots(p+1-s)}{p!} \text{ for } p \geq 1, \quad (3.98)$$

and all of them are positive.

Second, we remark that, if a function $f \in \mathcal{C}^\infty([-1, 1], \mathbb{R})$ is positive and absolutely monotone, so are the maps $T_p[f]$ for any $p \geq 1$. This is a direct consequence of the computation

$$T_p[f]^{(q)}(x) = \int_0^1 f^{(p+q)}(sx) s^q (1-s)^{p-1} ds > 0.$$

Hence, we can invoke (3.92), (3.94) and (3.95) to conclude that the function F^1 is positive, while the remainder terms R_{N-2k}^1 are positive and absolutely monotone.

We now go inside the inductive argument. Given an integer $p \geq 1$, we assume that we have constructed positive and smooth functions $(F^q)_{1 \leq q \leq p}$ on \mathbb{R} , and positive, smooth and absolutely monotone functions $(R_{N-2k}^q)_{1 \leq q \leq p, 1 \leq k < N/2}$ on $[-1, 1]$, such that

$$\Phi(\xi) = \sin(N\xi) \sum_{q=1}^p F^q(\xi) + \sum_{1 \leq k < \frac{N}{2}} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^p(\cos(\xi_n)), \quad (3.99)$$

for any $\xi \in \mathbb{R}$. In order to construct the function F^{p+1} and the remainders R_{N-2k}^{p+1} , we apply the same strategy as in the first inductive argument above. We fix an integer $1 \leq k < N/2$ and we invoke the Taylor formula so as to obtain

$$\begin{aligned} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^p(\cos(\xi_n)) &= \sum_{j=0}^{2k-1} \frac{(R_{N-2k}^p)^{(j)}(0)}{j!} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) \cos(\xi_n)^j \\ &+ \frac{1}{(2k-1)!} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) \cos(\xi_n)^{2k} T_{2k}[R_{N-2k}^p](\cos(\xi_n)). \end{aligned}$$

In view of (3.86), the double sum in the right-hand side of this identity is equal to 0, so that

$$\begin{aligned} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^p(\cos(\xi_n)) &= \\ &= \frac{1}{(2k-1)!} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) \cos(\xi_n)^{2k} T_{2k}[R_{N-2k}^p](\cos(\xi_n)). \end{aligned}$$

When $k \leq N/4$, we next use (3.90) in order to get

$$\begin{aligned} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^p(\cos(\xi_n)) &= \frac{\sin(N\xi)}{4^k(2k-1)!} \sum_{n=0}^{N-1} T_{2k}[R_{N-2k}^p](\cos(\xi_n)) \\ &+ \frac{1}{4^k(2k-1)!} \sum_{j=1}^{2k} \binom{2k}{j} \sum_{n=0}^{N-1} \sin((N-2j)\xi_n) T_{2k}[R_{N-2k}^p](\cos(\xi_n)). \end{aligned} \quad (3.100)$$

For $k > N/4$, we similarly invoke (3.91) so as to obtain

$$\begin{aligned} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^p(\cos(\xi_n)) &= \frac{\sin(N\xi)}{4^k(2k-1)!} \sum_{n=0}^{N-1} T_{2k}[R_{N-2k}^p](\cos(\xi_n)) \\ &+ \frac{1}{4^k(2k-1)!} \sum_{1 \leq j < N-2k} \binom{2k}{j} \sum_{n=0}^{N-1} \sin((N-2j)\xi_n) T_{2k}[R_{N-2k}^p](\cos(\xi_n)) \\ &+ \frac{1}{4^k(2k-1)!} \sum_{N-2k \leq j < \frac{N}{2}} \left(\binom{2k}{j} - \binom{2k}{N-j} \right) \sum_{n=0}^{N-1} \sin((N-2j)\xi_n) T_{2k}[R_{N-2k}^p](\cos(\xi_n)). \end{aligned}$$

Setting

$$F^{p+1}(\xi) := \sum_{1 \leq k < \frac{N}{2}} \frac{1}{4^k(2k-1)!} \sum_{n=0}^{N-1} T_{2k}[R_{N-2k}^p](\cos(\xi_n)), \quad (3.101)$$

as well as

$$\begin{aligned} R_{N-2k}^{p+1}(x) &= \sum_{\max\{1, \frac{k}{2}\} \leq j < \frac{N-k}{2}} \binom{2j}{k} \frac{T_{2j}[R_{N-2j}^p](x)}{4^j(2j-1)!} \\ &+ \sum_{\frac{N-k}{2} \leq j < \frac{N}{2}} \left(\binom{2j}{k} - \binom{2j}{N-k} \right) \frac{T_{2j}[R_{N-2j}^p](x)}{4^j(2j-1)!}, \end{aligned} \quad (3.102)$$

we conclude that

$$\Phi(\xi) = \sin(N\xi) \sum_{q=1}^{p+1} F^q(\xi) + \sum_{1 \leq k < \frac{N}{2}} \sum_{n=0}^{N-1} \sin((N-2k)\xi_n) R_{N-2k}^{p+1}(\cos(\xi_n)).$$

Moreover, we can prove as for the functions F^1 and R_{N-2k}^1 that the map F^{p+1} is positive and smooth on \mathbb{R} , while the maps R_{N-2k}^{p+1} are positive, smooth and absolutely monotone on $[-1, 1]$. This completes the inductive argument and establishes the validity of (3.99) for any $p \geq 1$.

We finally conclude the proof of Lemma 4.3 by showing that

$$\sum_{1 \leq k < N/2} |R_{N-2k}^p(x)| \rightarrow 0, \quad (3.103)$$

as $p \rightarrow +\infty$, uniformly with respect to $x \in [-1, 1]$. In order to establish this convergence, we use the analytic expansion of the function ϕ in (3.97). More generally, we consider an analytic function f on the interval $(-1/\rho, 1/\rho)$, which we expand as the power series

$$f(x) = \sum_{k=0}^{+\infty} a_k x^k,$$

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for $x \in (-1/\varrho, 1/\varrho)$. Given an integer $m \geq 1$ and a number $x \in [-1, 1]$, we compute

$$T_m[f](x) = \sum_{k=0}^{+\infty} a_{k+m}(k+m) \dots (k+1)x^k \int_0^1 s^k(1-s)^{m-1} ds.$$

Integrating by parts and arguing by induction provide

$$\int_0^1 s^k(1-s)^{m-1} ds = \frac{m-1}{k+1} \int_0^1 s^{k+1}(1-s)^{m-2} ds = \frac{(m-1)!}{(k+m) \dots (k+1)}.$$

Hence we obtain

$$T_m[f](x) = (m-1)! \tau_m f(x),$$

where the notation $\tau_m f$ refers to the analytic function f given by the power series

$$\tau_m f(x) = \sum_{k=0}^{+\infty} a_{k+m} x^k,$$

for any number $x \in (-1/\varrho, 1/\varrho)$.

With this expression at hand, we rewrite (3.102) as

$$R_{N-2k}^{p+1}(x) = \sum_{\frac{k}{2} \leq j < \frac{N}{2}} b_k^{2j} \tau_{2j} R_{N-2j}^p(x),$$

with

$$b_k^j = \begin{cases} \binom{j}{k} \frac{1}{2^j} & \text{if } k \leq j < N-k, \\ \left(\binom{j}{k} - \binom{j}{N-k} \right) \frac{1}{2^j} & \text{if } N-k \leq j < N. \end{cases}$$

Since

$$R_{N-2k}^1(x) = b_k^{N-1} \tau_{N-1} \phi(x),$$

by (3.95), a direct inductive argument provides

$$\begin{aligned} R_{N-2k}^{p+1}(x) = & \sum_{\max\{1, \frac{k}{2}\} \leq j_1 < \frac{N}{2}} b_k^{2j_1} \sum_{\max\{1, \frac{j_1}{2}\} \leq j_2 < \frac{N}{2}} b_{j_1}^{2j_2} \dots \\ & \dots \sum_{\max\{1, \frac{j_{p-1}}{2}\} \leq j_p < \frac{N}{2}} b_{j_{p-1}}^{2j_p} b_{j_p}^{N-1} \tau_{2j_1+2j_2+\dots+2j_p+N-1} \phi(x). \end{aligned} \quad (3.104)$$

In view of (3.97), we know that

$$\tau_m \phi(x) = \sum_{k=0}^{+\infty} \alpha_{k+m,s} \rho^{k+m} x^k.$$

Moreover, the radius of convergence of the power series $\sum_{k \geq 0} \alpha_{k,s} x^k$ is equal to 1. Therefore, given any number $0 < \sigma < 1$, there exists a positive number M_σ such that

$$\alpha_{k,s} \sigma^k \leq M_\sigma,$$

for any integer $k \in \mathbb{N}$. When $\tau < \sigma < 1$ and $-1 \leq x \leq 1$, we are led to the estimate

$$|\tau_m \phi(x)| \leq M_\sigma \frac{\tau^m}{\sigma^m} \sum_{k=0}^{+\infty} \frac{\rho^k}{\sigma^k} = \frac{M_\sigma \rho^m}{\sigma^{m-1}(\sigma - \rho)}.$$

Inserting this inequality into (3.104) gives

$$\begin{aligned} |R_{N-2k}^{p+1}(x)| \leq & \frac{M_\sigma \sigma}{\sigma - \rho} \sum_{\max\{1, \frac{k}{2}\} \leq j_1 < \frac{N}{2}} b_k^{2j_1} \sum_{\max\{1, \frac{j_1}{2}\} \leq j_2 < \frac{N}{2}} b_{j_1}^{2j_2} \dots \\ & \dots \sum_{\max\{1, \frac{j_{p-1}}{2}\} \leq j_p < \frac{N}{2}} b_{j_{p-1}}^{2j_p} b_{j_p}^{N-1} \left(\frac{\rho}{\sigma} \right)^{2j_1+2j_2+\dots+2j_p+N-1}. \end{aligned}$$

Hence, we deduce from the inequality $\rho < \sigma$ that

$$\begin{aligned}
 |R_{N-2k}^{p+1}(x)| &\leq \frac{M_\sigma \sigma}{\sigma - \rho} \left(\frac{\rho}{\sigma}\right)^{2p+N-1} \times \\
 &\times \sum_{\max\{1, \frac{k}{2}\} \leq j_1 < \frac{N}{2}} b_k^{2j_1} \sum_{\max\{1, \frac{j_1}{2}\} \leq j_2 < \frac{N}{2}} b_{j_1}^{2j_2} \dots \sum_{\max\{1, \frac{j_{p-1}}{2}\} \leq j_p < \frac{N}{2}} b_{j_{p-1}}^{2j_p} b_{j_p}^{N-1}.
 \end{aligned} \tag{3.105}$$

At this stage, we deduce from the discrete Fubini theorem

$$\begin{aligned}
 \sum_{1 \leq k \leq \frac{N}{2}} \sum_{\max\{1, \frac{k}{2}\} \leq j < \frac{N}{2}} b_k^{2j} \alpha_j &\leq \sum_{1 \leq k \leq \frac{N}{2}} \sum_{\max\{1, \frac{k}{2}\} \leq j < \frac{N}{2}} \binom{2j}{k} \frac{\alpha_j}{2^{2j}} \\
 &\leq \sum_{1 \leq j \leq \frac{N}{2}} \sum_{1 \leq k \leq 2j} \binom{2j}{k} \frac{\alpha_j}{2^{2j}} = \sum_{1 \leq j \leq \frac{N}{2}} \alpha_j,
 \end{aligned}$$

for any non-negative numbers $\alpha_1, \alpha_2, \dots$ and α_j . In view of (3.105), this yields

$$\sum_{1 \leq k < \frac{N}{2}} |R_{N-2k}^{p+1}(x)| \leq \frac{M_\sigma \sigma}{\sigma - \rho} \left(\frac{\rho}{\sigma}\right)^{2p+N-1} \sum_{1 \leq j < \frac{N}{2}} b_j^{N-1}.$$

Since

$$\sum_{1 \leq j < \frac{N}{2}} b_j^{N-1} \leq \frac{1}{2^{N-1}} \sum_{j=1}^{N-1} \binom{N-1}{j} \leq 1,$$

we conclude that

$$\sum_{1 \leq k < \frac{N}{2}} |R_{N-2k}^{p+1}(x)| \leq \frac{M_\sigma \sigma}{\sigma - \rho} \left(\frac{\rho}{\sigma}\right)^{2p+N-1},$$

which is enough to obtain the uniform convergence in (3.103).

Applying this convergence to (3.99) is enough to establish the identity

$$\Phi(\xi) = \sin(N\xi) F(\xi),$$

for any number $\xi \in \mathbb{R}$. The map F in this formula is defined as the function series

$$F(\xi) = \sum_{q=1}^{+\infty} F^q(\xi).$$

Since the convergence of this series is uniform on \mathbb{R} , and all the functions F^q are continuous and positive, so is the map F . Its $2\pi/N$ -periodicity then follows from its continuity and the $2\pi/N$ -periodicity of the functions Φ and $\xi \mapsto \sin(N\xi)$. This completes the proof of Lemma 4.3. \square

Chapitre 4

Smooth traveling-wave solutions to the generalized inviscid surface quasi-geostrophic equation

Abstract

In a recent article by Gravejat and Smets [42], it is built smooth solutions to the inviscid surface quasi-geostrophic equation that have the form of a traveling wave. In this article we work back on their construction to provide solution to a more general class of quasi-geostrophic equation where the half-laplacian is replaced by any fractional laplacian.

1 Presentation of the problem

1.1 The quasi-geostrophic equations

We consider the general transport equation for the vorticity of an incompressible fluid in dimension 2.

$$\frac{\partial \theta}{\partial t} + v \cdot \nabla \theta = 0, \quad (4.1)$$

where $\theta : \mathbb{R}^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is called the *active scalar* and $v : \mathbb{R}^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}^2$ is the *velocity* of the fluid. This equation tells that the active scalar is transported by the induced velocity. Since this velocity v is divergence free (incompressibility condition), it is convenient to relate v and θ through a stream function $\psi : \mathbb{R}^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}$. The generalized inviscid surface quasi-geostrophic equation corresponds to a stream function that verifies

$$v = \nabla^\perp \psi \quad \text{and} \quad (-\Delta)^s \psi = \theta, \quad (4.2)$$

with $s \in]0, 1[$ and \perp denotes the rotation in the plane of angle $\frac{\pi}{2}$. The equations (4.1)-(4.2) are the generalized inviscid surface quasi-geostrophic equations. In the particular case $s \rightarrow 1$ we obtain the well-known 2D Euler equation written in term of vorticity and stream function. Another important case is $s = \frac{1}{2}$ which correspond to the work made in [42] that we generalize here. This case is the standard surface quasi-geostrophic equation that first appeared as a limit model in the context of geophysical flows [64][83]. These equations are used to model a fluid in a rotation frame with stratified density and velocity and submitted to Brunt-Väisälä thermal oscillations. This models leads to (4.1)-(4.2) using the Cafferelli-Silverstre theory on fractional laplace operator [18]. The case of the exponent $s = \frac{1}{2}$ corresponds to the case of a Brunt-Väisälä frequency N that does not depend on the height. Other exponents for the fractional Laplace operator corresponds to different profiles for N . These equations has been intensely investigated since the work of Constantin, Majda and Tabak [25] on the case $s = \frac{1}{2}$ where they pointed out the mathematical links that arises between (SQG- $\frac{1}{2}$) and the Euler equation in dimension 3. Besides stationary solution, given by a radially symmetric rearrangement on the active scalar, the only two known examples of global smooth solutions where built by Castro, Córdoba and Gómez-Serrano [19] on the one hand and by Gravejat and Smets [42] on the other hand with two different techniques. The article of Castro, Córdoba and Gómez-Serrano also provides a wide bibliography related on SQG and its Cauchy problem. In this work we generalize the result and the construction provided by [42] to the more general

equations (4.1)-(4.2) with a fixed $s \in]0, 1[$. The idea consists in looking for solutions that have the form of traveling waves with a positive speed c in direction z . In short, solutions of the form

$$\theta(r, z, t) = \Theta(r, z - ct), \quad v(r, z, t) = V(r, z - ct), \quad \psi(r, z, t) = \Psi(r, z - ct). \quad (4.3)$$

We inject this form of solution in (4.1)-(4.2) and we get

$$\begin{aligned} 0 &= \frac{\partial}{\partial t}(\Theta(r, z - ct)) + V(r, z - ct) \cdot \nabla \Theta(r, z - ct) \\ &= -ce_z \cdot \nabla \Theta(r, z - ct) + V(r, z - ct) \cdot \nabla \Theta(r, z - ct) \\ &= -ce_z \cdot \nabla \Theta(r, z - ct) + \nabla^\perp \Psi(r, z - ct) \cdot \nabla \Theta(r, z - ct), \end{aligned} \quad (4.4)$$

where (e_r, e_z) denotes the canonical basis of \mathbb{R}^2 . This leads to the orthogonality condition

$$\left(\nabla \psi - ce_r \right)^\perp \cdot \nabla \Theta = 0, \quad (4.5)$$

with the remark that $e_r^\perp = e_z$. In other words, the two vectors $\nabla \Theta$ and $\nabla \Psi - ce_r$ must be collinear. Following an idea from Arnold [4], Condition (4.5) is immediately verified if Θ has the form

$$\Theta(r, z) = f\left(\Psi(r, z) - cr - k\right) \quad (4.6)$$

because in this case

$$\nabla \Theta(r, z) = f'\left(\Psi(r, z) - cr - k\right) \cdot \left(\nabla \Psi(r, z) - ce_r\right) \quad (4.7)$$

which gives (4.5). We now make consider the ansatz of a symmetry relatively to the z -axis that takes the form

$$\Psi(-r, z) = -\Psi(r, z). \quad (4.8)$$

This implies that $\Theta(-r, z) = -\Theta(r, z)$ and if we denote $V = (V_1, V_2)$ the two components of the velocity profile, then $V_1(-r, z) = -V_1(r, z)$ and $V_2(-r, z) = V_2(r, z)$. More precisely, we impose the following ansatz

$$\Theta(r, z) = \begin{cases} f(\Psi(r, z) - cr - k) & \text{if } r \geq 0, \\ -f(-\Psi(r, z) + cr - k) & \text{otherwise} \end{cases} \quad (4.9)$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function supported in \mathbb{R}_+ (to avoid a singularity at $x = 0$) with the condition $k > 0$. Using the stream equations (4.2) we obtain

$$(-\Delta)^s \Psi(r, z) = \begin{cases} f(\Psi(r, z) - cr - k) & \text{if } r \geq 0, \\ -f(-\Psi(r, z) + cr - k) & \text{otherwise} \end{cases} \quad (4.10)$$

1.2 Variational formulation

The studied equation is variational and its solutions are the critical points of

$$E(\Psi) := \frac{1}{2} \int_{\mathbb{R}^2} \Psi (-\Delta)^s \Psi - \int_{\mathbb{H}} F(\Psi - cr - k) + \int_{\mathbb{H}^c} F(\Psi + cr - k), \quad (4.11)$$

where $\mathbb{H} := \{x = (r, z) \in \mathbb{R}^2 : r \geq 0\}$ and $F(\xi) := \int_0^\xi f(\xi') d\xi'$. We are going to build a critical point of E using the technique of the Nehari manifold (defined later). For that purpose, since the choice of f is free, we are imposing on this function the following properties

$$\bullet \quad f \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}), \quad f|_{\mathbb{R}_-} = 0 \quad \text{and} \quad f|_{\mathbb{R}_+^*} > 0, \quad (4.12)$$

$$\bullet \quad \exists \nu \in \left] 1, \frac{1+s}{1-s} \right[, \quad \forall \xi \geq 0, \quad f(\xi) \leq C\xi^\nu, \quad (4.13)$$

$$\bullet \quad \exists \mu \in]1, \nu[, \quad \forall \xi \geq 0, \quad \mu f(\xi) \leq \xi f'(\xi). \quad (4.14)$$

This last hypothesis on the variations of f is equivalent to the hypothesis that the function

$$\xi \mapsto \frac{f(\xi)}{\xi^\mu}. \quad (4.15)$$

Section 1. Presentation of the problem

is non-decreasing on \mathbb{R}_+ . In particular and since $\mu > 1$,

$$\forall \xi_0 \geq 0, \quad \xi \in \mathbb{R}_+ \mapsto \frac{f(\xi)}{\xi + \xi_0} \quad (4.16)$$

is increasing and diverging at infinity. Examples of functions that satisfies these three hypothesis(4.12) (4.13) (4.14) are the functions

$$\xi \mapsto \xi^\beta e^{-\frac{1}{\xi}} \mathbb{1}_{\mathbb{R}_+}(\xi), \quad (4.17)$$

with $\beta \in [\mu, \nu]$. Given the hypothesis (4.12) and (4.13), the functional E is well-defined on the Hilbert space

$$X^s := L^{\frac{2}{1-s}} \cap \dot{H}^s(\mathbb{R}^2) \quad (4.18)$$

with the scalar product induced by \dot{H}^s given by

$$\langle \Phi, \Psi \rangle_{X^s} := P.V. \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{(\Phi(x) - \Phi(y))(\Psi(x) - \Psi(y))}{|x - y|^{2(1+s)}} dx dy \quad (4.19)$$

where $P.V.$ refers to the principal value of the singularity of the kernel $(x, y) \mapsto 1/|x - y|^{2(1+s)}$. For further work, we make use of the notations $x = (r_x, z_x)$ and $y = (r_y, z_y)$ to distinguish the coordinates of x and the coordinates of y . We recall here that the Gagliardo half-norms defining the spaces $\dot{W}^{s,p}$ are given in general by

$$|\Phi|_{W^{s,p}} := P.V. \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\Phi(x) - \Phi(y)|^p}{|x - y|^{d+sp}} dr dz. \quad (4.20)$$

For the rest of the work we refer E as being the “energy” of the problem although this energy does not correspond to a physical energy. We remark that it is invariant by the action of the group of symmetry generated by (4.8). We denote by X_{sym}^s the subspace of X^s made with the functions that are left invariant by the action of this symmetry group.

$$X_{sym}^s := \{\Psi \in X^s : \forall (r, z) \in \mathbb{R}^2, \Psi(-r, z) = -\Psi(r, z)\}. \quad (4.21)$$

It follows from the Palais principle of symmetric criticality [63] that any critical point of E on X^s actual belongs to X_{sym}^s . We can therefore restrict our investigations to the subspace X_{sym}^s , inside which the energy can be rewritten

$$E(\Psi) = \frac{1}{2} \|\Psi\|_{X^s}^2 - 2V(\Psi) \quad (4.22)$$

with

$$V(\Psi) := \int_{\mathbb{H}} F(\Psi - cr - k). \quad (4.23)$$

1.3 Nehari Manifold and presentation of the main result

The Nehari manifold associated to the energy E is defined by

$$\mathcal{N} = \{\Psi \in X_{sym}^s \setminus \{0\} : E'(\Psi)(\Psi) = 0\}, \quad (4.24)$$

so that $\Psi \in \mathcal{N}$ implies

$$\int_{\mathbb{R}^2} \Psi(-\Delta)^s \Psi - 2 \int_{\mathbb{H}} f(\Psi - cr - k) \Psi = 0. \quad (4.25)$$

It is proven after that the Nehari manifold \mathcal{N} is a sub-manifold of X_{sym}^s non empty, of regularity \mathcal{C}^1 without boundary. The main result of this article is the following theorem.

Theorem 1.1. *Let c and k positive. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ verifying (4.12), (4.13) and (4.14).*

Then the energy E admits a minimizer $\Psi \neq 0$ on \mathcal{N} . As a consequence there exist a non-trivial smooth solution Θ to the inviscid quasi-geostrophic equations (4.1)(4.1) which has the form

$$\Theta(r, z, t) = \Theta(r, z - ct) = f(\Psi(r, z - ct) - cr - k), \quad (4.26)$$

for all $(r, z) \in \mathbb{H}$ and that satisfies the symmetries $\Theta(r, z) = -\Theta(-r, z) = \Theta(r, -z)$, for all $(r, z) \in \mathbb{R}^2$. Moreover, The restriction of Θ to \mathbb{H} is non-negative, compactly supported and non-increasing relatively to the variable $|z|$.

2 Strategy of proof and main lemmas

We regroup in this section the main Lemmas involved in the proof of Theorem 1.1 and how they follow one another. The detailed proof of these different lemmas are provided in Section 3.

2.1 Properties of the Nehari Manifold and minimizing sequences

We are interested in the minimization problem

$$\alpha := \inf \{E(\Psi) : \Psi \in \mathcal{N}\}. \quad (4.27)$$

Since the function f worth 0 on \mathbb{R}_- then a given function Ψ cannot belong to \mathcal{N} if $\Psi \leq 0$ on \mathbb{H} . Indeed, this would imply that

$$\int_{\mathbb{H}} f(\Psi - cr - k)\Psi = 0 \quad (4.28)$$

and then $\|\Psi\|_{X^s}$. The only function on X_{sym}^s such that this quantity worth 0 is the null function which has been excluded from the definition of the Nehari manifold). We have the following description of the Nehari manifold.

Lemma 2.1. *The set \mathcal{N} is a C^1 non-empty sub-manifold of X_{sym}^s . For every $\Psi \in X_{sym}^s$ such that $\mathcal{L}^2(\text{supp}(\Psi_+) \cap \mathbb{H})$ is non zero¹, there exist a unique $t_\Psi > 0$ such that $t_\Psi \Psi \in \mathcal{N}$. The value of this t_Ψ is characterized by*

$$E(t_\Psi \Psi) = \max\{E(t\Psi)/t > 0\}. \quad (4.29)$$

Moreover, any local minimizer of E on \mathcal{N} is a smooth non-trivial solution of (4.10). We also have that

$$\beta := \inf \{\|\Psi\|_{X^s}^2 : \Psi \in \mathcal{N}\} > 0. \quad (4.30)$$

and for every $\Psi \in \mathcal{N}$,

$$\|\Psi\|_{X^s}^2 \leq \left(1 + \frac{1}{\mu}\right)E(\Psi). \quad (4.31)$$

Remark that this last assertion implies that α is positive. This proposition also implies that any minimizing sequence of E on \mathcal{N} is a bounded sequence.

Definition 2.1 (Polarization). We now define the *polarization* of a function $\Psi \in X^s$ by

$$\forall x = (r, z) \in \mathbb{R}^2, \quad \Psi^\dagger(X) := \begin{cases} \max\{\Psi(x), \Psi(\sigma(x))\} & \text{if } r > 0, \\ \min\{\Psi(x), \Psi(\sigma(x))\} & \text{if } r < 0, \end{cases} \quad (4.32)$$

where σ denotes the linear map $x = (r, z) \in \mathbb{R}^2 \mapsto (-r, z)$. In the particular case $\Psi \in X_{sym}^s$, we obtain $\Psi^\dagger_{\mathbb{H}} \geq 0$ and $\Psi^\dagger_{\mathbb{H}^c} \leq 0$. For more details about polarization, see for instance [76].

Lemma 2.2 (Polarization inequality). *For all $\Psi \in \mathcal{N}$,*

$$E(t_{\Psi^\dagger} \Psi^\dagger) \leq E(\Psi) \quad (4.33)$$

and this inequality is strict when $\Psi \neq \Psi^\dagger$.

Denote with a \dagger the image of a given set by the polarization. This lemma tells that if (Ψ_n) is a minimizing sequence for E on \mathcal{N} then so is $t_{\Psi^\dagger} \Psi^\dagger$ because by definition of $\Psi \mapsto t_\Psi$ the function $t_{\Psi^\dagger} \Psi^\dagger$ belongs to \mathcal{N} . Thus, the minimizer, if it exists, belongs to \mathcal{N}^\dagger . It is then possible to restrict the investigations to $X_{sym}^{s, \dagger}$.

Definition 2.2 (Steiner rearrangement). We define the *Steiner rearrangement* of $\Psi \in X_{sym}^{s, \dagger}$, noted Ψ^\sharp , as being the function of $X_{sym}^{s, \dagger}$ which super-level sets on \mathbb{H} are given for all $\nu > 0$ by

$$\{\Psi^\sharp \geq \nu\} := \bigcup_{r \in \mathbb{R}_+} \{r\} \times \left[-\frac{\zeta_\Psi(r)}{2}, +\frac{\zeta_\Psi(r)}{2}\right] \quad (4.34)$$

with

$$\zeta_\Psi(r) := L^1\{z \in \mathbb{R} : \Psi(r, z) \geq \nu\}. \quad (4.35)$$

We extend this definition on \mathbb{H}^c by symmetry to ensure that $\Psi^\sharp \in X_{sym}^{s, \dagger}$.

1. The notation \mathcal{L}^d refers to the d -dimensional Lebesgue measure, ie. the lebesgue measure on \mathbb{R}^d . The function $\Psi_+ := \max\{\Psi, 0\}$ is the positive part of Ψ .

Lemma 2.3 (Steiner inequality). *For all $\Psi \in \mathcal{N}^\dagger$,*

$$E(t_{\Psi^\sharp}\Psi^\sharp) \leq E(\Psi) \quad (4.36)$$

and the equality holds if and only if $\Psi = \Psi^\sharp$ up to a translation on the z axis.

Then, if (Ψ_n) is a minimizing sequence for E on \mathcal{N}^\dagger then so is $t_{\Psi_n^\sharp}\Psi_n^\sharp$. Thus, similarly as before it is possible to restrict the investigations to $X_{sym}^{s,\sharp}$.

2.2 Existence of the solution of the minimizaing problem

Let $(\Psi_n) \in \mathcal{N}^\dagger$ a minimizing sequence. We already know that such a sequence is bounded as a consequence of Lemma (2.1). To start with, we establish the following compactness result.

Lemma 2.4 (compactness). *Let c and k be positive. Define the map*

$$T : \Psi \in X^s \mapsto \begin{cases} (\Psi - cr - k)_+ & \text{on } \mathbb{H}, \\ -(\Psi - cr + k)_- & \text{on } \mathbb{H}^c. \end{cases} \quad (4.37)$$

Then T maps X_{sym}^s into himself and maps bounded sets into bounded sets. Moreover, the map $T \circ \sharp \circ \dagger$ is a compact map from X_{sym}^s into $L^p_{sym}(\mathbb{R}^2)$, with $1 \leq p < \frac{2}{1-s}$.

Up to an extraction we can suppose that the minimizing sequence $\Psi_n \rightarrow \Psi^*$ weakly in $X_{sym}^{s,\sharp}$ and that $(\Psi_n - cr - k)_+ \rightarrow (\Psi^\sharp - cr - k)_+$ strongly in $L^p(\mathbb{H})$ for all $p < \frac{2}{1-s}$.

Lemma 2.5 (convergence). *The convergence of Ψ_n towards Ψ^* in $X^{s,\sharp}$ is a strong convergence.*

This implies that Ψ^* is solution to the studied minimization problem.

2.3 Properties of the solution

We finally define Θ^* from Ψ^* according to formula (4.9). Since $T(\Psi^*) \in L^p(\mathbb{R}^2)$ for all $p \in [1, \frac{2}{1-s}]$ then $\Theta^* \in L^q$ for all $q \in [1, \frac{2}{\nu(1-s)}]$ as a consequence of (4.13). We have the following regularity result

Lemma 2.6 (regularity). *The functions Ψ^* and Θ^* are C^∞ .*

We can also establish a result on the decay of Ψ^* at infinity.

Lemma 2.7 (decay estimate). *There exists a constant $C > 0$ such that for all $r \in \mathbb{R}^2$,*

$$|\Psi^*(x)| \leq \frac{C}{1 + |x|^{2(1-s)}}, \quad (4.38)$$

With the positive cut-off level $k > 0$ appearing in the definition of T , this proposition implies in particular that Θ^* is compactly supported.

3 Proofs of the lemmas

3.1 Proofs of the lemmas of section 2.1

3.1.1 Proof of Lemma 2.1

Let $\Psi \in X_{sym}^s$ with $\mathcal{L}^2(\text{supp}(\Psi_+) \cap \mathbb{H}) \neq 0$. For any $t > 0$, we define

$$g(t) := \frac{E'(t\Psi)(t\Psi)}{t^2} = \frac{1}{2}\|\Psi\|_{X^s}^2 - 2t \int_{\mathbb{H}} f(t\Psi_+ - cr - k)\Psi_+. \quad (4.39)$$

We observe that the integral above can be rewritten

$$g(t) = \frac{1}{2}\|\Psi\|_{X^s}^2 - 2 \int_{\mathbb{H}} \frac{f(t\Psi_+(r, z) - cr - k)}{t\Psi_+(r, z) - cr - k + (cr + k)} (\Psi_+)^2(r, z) dr dz. \quad (4.40)$$

Since we have $\mathcal{L}^2(\text{supp}(\Psi_+) \cap \mathbb{H}) \neq 0$, our remark on the variations $\xi \mapsto f(\xi)/(\xi + \xi_0)$, consequence of (4.14), indicates that $t \mapsto g(t)$ is decreasing and $g(t) \rightarrow -\infty$ as $t \rightarrow +\infty$. Indeed, one has to apply this

property of f to (4.40) with $\xi = t\Psi + (r, z) - cr - k$ and $\xi_0 = cr + k$ and then integrate on \mathbb{H} against the non-negative weight $(\Psi_+)^2$. We use Hypothesis (4.13) to write on \mathbb{H}

$$0 \leq \frac{1}{t} f(t\Psi_+ - cr - k)\Psi_+ \leq \frac{1}{t} f(t\Psi_+)\Psi_+ \leq Ct^{\nu-1}(\Psi_+)^{\nu+1}. \quad (4.41)$$

Since $\nu > 1$,

$$g(t) \rightarrow \frac{1}{2}\|\Psi\|_{X^2}^2 > 0. \quad (4.42)$$

Since f is smooth then g is continuous, and then the function g admits a unique root on \mathbb{R}_+^* . The characterization (4.29) comes from the fact that

$$tg(t) = \frac{d}{dt}E(t\Psi). \quad (4.43)$$

The estimate (4.31) is obtained, for $\Psi \in \mathcal{N}$, as follows

$$\begin{aligned} E(\Psi) &= E(\Psi) - \frac{1}{\mu+1}E'(\Psi)(\Psi) \\ &= \frac{\mu}{2(\mu+1)}\|\Psi\|_{X^s}^2 + \frac{2}{\mu+1} \int_{\mathbb{H}} \left[f(\Psi - cr - k)\Psi(r, z) - (\mu+1)F(\Psi - cr - k) \right] dr dz \\ &\geq \frac{\mu}{2(\mu+1)}\|\Psi\|_{X^s}^2, \end{aligned} \quad (4.44)$$

The last inequality comes from the integration on $[0, r]$ of hypothesis (4.14) that gives $\mu F(t) \leq t f(t) - F(t)$. The fact that β is not zero is obtained using (4.13) and the Sobolev embedding

$$\|\Psi\|_{X^s}^2 = \int_{\mathbb{H}} f(\Psi_+ - cr - k)\Psi_+ \leq 4K \int_{\mathbb{H}} (\Psi_+)^{\frac{2}{1-s}} = 4K\|\Psi_+\|_{L^{\frac{2}{1-s}}}^{\frac{2}{1-s}} \leq C\|\Psi\|_{X^s}^{\frac{2}{1-s}}. \quad (4.45)$$

Concerning the regularity of \mathcal{N} , it is a consequence of the implicit functions theorem applied to $\Xi : (s, \Psi) \mapsto E'(s\Psi)(\Psi)$ defined on the open set $\mathbb{R}_+^* \times X_{sym}^s \setminus \{0\}$. The hypothesis of the theorem are verified because for $\Psi \in \mathcal{N}$ we have :

$$\partial_1 \Xi(1, \Psi) = t_{\Psi}^2 g'(t\Psi) < 0. \quad (4.46)$$

It remains to prove that any minimizer of E on \mathcal{N} is a critical point for E defined on the whole space. We first remark that a minimizer of E on \mathcal{N} is a minimizer of $\Psi \mapsto E(t\Psi\Psi)$ on X_{sym}^s . Then, using the definition of the Nehari manifold and the fact that we have $\Psi \in \mathcal{N}$ implies $t_{\Psi} = 1$, we conclude

$$\forall h \in X_{sym}^s, E'(\Psi)(h) = E'(t\Psi)(\Psi)[t'_{\Psi}(h)\Psi + t_{\Psi}h] = 0. \quad (4.47)$$

□

3.2 Proof of Lemma 2.2

We first recall that $\Psi \in \mathcal{N}$ implies that $\mathcal{L}^2(\text{supp}(\Psi_+) \cap \mathbb{H}) \neq 0$. Using the characterization (4.29) we get $E(\Psi) \geq E(t_{\Psi}\Psi)$. Using the fact that $\Psi(r, z) = -\Psi(-r, z)$, we conclude that here the polarization consists in switching the two values of $\Psi(r, z)$ and $\Psi(-r, z)$ if and only if we have $\Psi(r, z) \leq 0 \leq \Psi(-r, z)$. Therefore since F is zero on \mathbb{R}_- and is positive on \mathbb{R}_+ , we obtain

$$V(t_{\Psi}\Psi) \leq V(t_{\Psi}\Psi^{\dagger}). \quad (4.48)$$

To finish the proof of this lemma, we have to establish

$$\|\Psi^{\dagger}\|_{X^s} \leq \|\Psi\|_{X^s} \quad (4.49)$$

and that this inequality is strict if and only if $\Psi^{\dagger} \neq \Psi$. Actually the fact that the polarization decreases the $\dot{W}^{s,p}(\mathbb{R}^d)$ half-norms (4.20) is a general result so that we can establish it in the general case. By definition of the principal values, we have

$$\iint_{|x-y| \geq \varepsilon} \frac{|\Psi(x) - \Psi(y)|^p}{|x-y|^{d+sp}} dx dy \rightarrow |u|_{W^{s,p}}^p \quad \text{as } \varepsilon \rightarrow 0. \quad (4.50)$$

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We then establish the inequality for any fixed $\varepsilon > 0$. First, the integral is split as follows,

$$\begin{aligned} & \iint_{|x-y| \geq \varepsilon} \frac{|\Psi(x) - \Psi(y)|^p}{|x-y|^{d+sp}} dx dy \\ &= \iint_{\mathbb{H}^2 \setminus \{|x-y| < \varepsilon\}} \left(\frac{1}{|x-y|^{d+sp}} \left(|\Psi(x) - \Psi(y)|^p + |\Psi \circ \sigma(x) - \Psi \circ \sigma(y)|^p \right) \right. \\ & \quad \left. + \frac{1}{|x - \sigma(y)|^{d+sp}} \left(|\Psi(x) - \Psi \circ \sigma(y)|^p + |\Psi \circ \sigma(x) - \Psi(y)|^p \right) \right) dx dy \end{aligned} \quad (4.51)$$

Let $x, y \in \mathbb{H}$. Observe that

$$|x-y|^{d+sp} < |x-\sigma(y)|^{d+sp}. \quad (4.52)$$

Case 1 : $\Psi(x) \geq \Psi \circ \sigma(x)$ and $\Psi(y) \geq \Psi \circ \sigma(y)$. In this case, with the definition of the polarization, $\Psi(x) = \Psi^\dagger(x)$ and $\Psi(y) = \Psi^\dagger(y)$. Then when we integrate on the couples (x, y) that belongs to Case 1, the associated term in the integral (4.51) is not modified by the polarization.

Case 2 : $\Psi(x) \geq \Psi \circ \sigma(x)$ and $\Psi(y) < \Psi \circ \sigma(y)$. By computing its derivative, we obtain that the function

$$u_{\beta,s} : \alpha \in \mathbb{R} \mapsto |\alpha + \beta|^p - |\alpha + s|^p \quad (4.53)$$

is non-decreasing when $\beta > s$. Indeed we have (with $p \geq 1$)

$$u'_{\beta,s}(\alpha) = p(\alpha + \beta)|\alpha + \beta|^{p-2} - p(\alpha + s)|\alpha + s|^{p-2} \quad (31) \quad (4.54)$$

which is non-negative because $y \mapsto y|y|^{p-2}$ is an non-decreasing function. We now use this property of $u_{\beta,s}$ with $\alpha_1 := \Psi(x) \geq \Psi \circ \sigma(x) =: \alpha_2$ and with $\beta := -u(y) > s := -u \circ \sigma(y)$. We obtain

$$|\Psi(x) - \Psi(y)|^p + |\Psi \circ \sigma(x) - \Psi \circ \sigma(y)|^p > |\Psi \circ \sigma(x) - \Psi(y)|^p + |\Psi(x) - \Psi \circ \sigma(y)|^p. \quad (4.55)$$

If we combine this with (4.52) we get

$$\begin{aligned} & \frac{1}{|x-y|^{d+sp}} \left(|\Psi(x) - \Psi(y)|^p + |\Psi \circ \sigma(x) - \Psi \circ \sigma(y)|^p \right) \\ & \quad + \frac{1}{|x - \sigma(y)|^{d+sp}} \left(|\Psi \circ \sigma(x) - \Psi(y)|^p + |\Psi(x) - \Psi \circ \sigma(y)|^p \right) \\ & > \frac{1}{|x-y|^{d+sp}} \left(|\Psi \circ \sigma(x) - \Psi(y)|^p + |\Psi(x) - \Psi \circ \sigma(y)|^p \right) \\ & \quad + \frac{1}{|x - \sigma(y)|^{d+sp}} \left(|\Psi(x) - \Psi(y)|^p + |\Psi \circ \sigma(x) - \Psi \circ \sigma(y)|^p \right) \\ & = \frac{1}{|x-y|^{d+sp}} \left(|\Psi^\dagger(x) - \Psi^\dagger(y)|^p + |\Psi^\dagger \circ \sigma(x) - \Psi^\dagger \circ \sigma(y)|^p \right) \\ & \quad + \frac{1}{|x - \sigma(y)|^{d+sp}} \left(|\Psi^\dagger \circ \sigma(x) - \Psi^\dagger(y)|^p + |\Psi^\dagger(x) - \Psi^\dagger \circ \sigma(y)|^p \right). \end{aligned} \quad (4.56)$$

Case 3 : $\Psi(x) < \Psi \circ \sigma(x)$ and $\Psi(y) < \Psi \circ \sigma(y)$. In this case we have both $\Psi(x)$ and $\Psi(y)$ that are swapped with respectively $\Psi \circ \sigma(x)$ and $\Psi \circ \sigma(y)$. Then this case is the same as case 1 and the term associated to Case 3 in the integral (4.51) is not modified by the polarization.

Case 4 : $\Psi(x) < \Psi \circ \sigma(x)$ and $\Psi(y) \geq \Psi \circ \sigma(y)$. This case is the same as Case 2. Gathering these four cases we obtain that for any $\varepsilon > 0$,

$$\iint_{|x-y| \geq \varepsilon} \frac{|\Psi(x) - \Psi(y)|^p}{|x-y|^{d+sp}} dx dy \geq \iint_{|x-y| \geq \varepsilon} \frac{|\Psi^\dagger(x) - \Psi^\dagger(y)|^p}{|x-y|^{d+sp}} dx dy. \quad (4.57)$$

Concerning the cases of equality, we obtained from Cases 2 and 4 that if

$$\mathcal{L}^2 \left(\left\{ (x, y) \in \mathbb{H}^2 : \Psi(x) = \Psi^\dagger(x) \quad \text{and} \quad \Psi(y) \neq \Psi^\dagger(y) \right\} \cap |x-y| \geq \varepsilon \right) > 0 \quad (4.58)$$

then the inequality(4.57) is actually strict. We now observe that the above set is of measure zero for every $\varepsilon > 0$ if and only if we have either $\Psi = \Psi^\dagger$ or $\Psi = \Psi^\dagger \circ \sigma$. But this last case is not possible when $\Psi \in \mathcal{N}$ and then the only case of equality in our case is $\Psi = \Psi^\dagger$. \square

3.3 Proof of Lemma 2.3

Arguing similarly as the previous proof, we only have to prove that

$$\forall \Psi \in X_{sym}^{s,\dagger}, \quad E(\Psi^\sharp) \leq E(\Psi). \quad (4.59)$$

Since the Steiner rearrangement only involves rearrangements of the super-level sets perpendicularly to the r -axis, we get

$$V(\Psi^\sharp) = V(\Psi). \quad (4.60)$$

To conclude we have to establish that.

$$\|\Psi^\sharp\|_{X^s} \leq \|\Psi\|_{X^s}. \quad (4.61)$$

To start with, we suppose that Ψ is smooth and compactly supported. In this case

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\Psi(x) - \Psi(y)|^2}{(|x - y|^2 + \varepsilon^2)^{1+s}} dx dy \longrightarrow \|\Psi\|_{X^s}, \quad (4.62)$$

as $\varepsilon \rightarrow 0^+$. Since the considered functions are \mathcal{C}^∞ , it is possible to develop the square above and write

$$\begin{aligned} & \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{|\Psi(x) - \Psi(y)|^2}{(|x - y|^2 + \varepsilon^2)^{1+s}} dx dy \\ &= 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\Psi(x)^2}{(|x - y|^2 + \varepsilon^2)^{1+s}} dx dy - 2 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\Psi(x)\Psi(y)}{(|x - y|^2 + \varepsilon^2)^{1+s}} dx dy. \end{aligned} \quad (4.63)$$

The first integral in the right-hand side of the above inequality is not modified by rearrangement of the super-level sets of the function Ψ . Concerning the second integral, using the fact that $\Psi(r, z) = -\Psi(-r, z)$ we get

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Psi(r_x, z_x)\Psi(r_y, z_y)}{((r_x - r_y)^2 + (z_x - z_y)^2 + \varepsilon^2)^{1+s}} dz_x dz_y dr_x dr_y \\ &= 2 \int_0^{+\infty} \int_0^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Psi(r_x, z_x)\Psi(r_y, z_y)}{((r_x - r_y)^2 + (z_x - z_y)^2 + \varepsilon^2)^{1+s}} dz_x dz_y dr_x dr_y \\ &- 2 \int_0^{+\infty} \int_0^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{\Psi(r_x, z_x)\Psi(r_y, z_y)}{((r_x + r_y)^2 + (z_x - z_y)^2 + \varepsilon^2)^{1+s}} dz_x dz_y dr_x dr_y \end{aligned} \quad (4.64)$$

But the function

$$\Upsilon_{r_x, r_y} : u \longmapsto \frac{1}{((r_x - r_y)^2 + u^2 + \varepsilon^2)^{1+s}} - \frac{1}{((r_x + r_y)^2 + u^2 + \varepsilon^2)^{1+s}} \quad (4.65)$$

is non-negative and radially decreasing on \mathbb{R} . Moreover, for $r_x, r_y \geq 0$ the functions $z_x \mapsto \Psi(r_x, z_x)$ and $z_y \mapsto \Psi(r_y, z_y)$ are both non-negative \mathbb{R} . Thus, using the Riesz rearrangement inequality, we obtain

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi(r_x, z_x)\Psi(r_y, z_y)\Upsilon_{r_x, r_y}(z_x - z_y) dz_x dz_y \\ & \leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \Psi^\sharp(r_x, z_x)\Psi^\sharp(r_y, z_y)\Upsilon_{r_x, r_y}(z_x - z_y) dz_x dz_y. \end{aligned} \quad (4.66)$$

We now inject this inequality back into (4.64) and get

$$\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\Psi(x)\Psi(y)}{(|x - y|^2 + \varepsilon^2)^{1+s}} dx dy \leq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{\Psi^\sharp(x)\Psi^\sharp(y)}{(|x - y|^2 + \varepsilon^2)^{1+s}} dx dy. \quad (4.67)$$

We use this estimate in (4.63), we take the limit and we conclude by density of the smooth compactly supported functions. \square

Remark : It was not possible to use directly the Riesz rearrangement inequality to the second integral appearing in (4.63) because this inequality is only true for non-negative functions. \square

3.4 Proofs of the lemmas of section 2.2

3.4.1 Proof of Lemma 2.4

- **Step 1 : T maps $X_{sym}^{s,\dagger}$ into itself and maps bounded sets into bounded sets.**

First, if Ψ satisfies the symmetry property then so does $T(\Psi)$. Define the set²

$$\Omega(\Psi) := \{(r, z) \in \mathbb{H} : T(\Psi)(r, z) > 0\}, \quad (4.68)$$

where $T(\Psi)(r, z) := (\Psi(r, z) - cr - k)_+$. By definition of T and of Ω

$$\mathcal{L}^2(\Omega) = \int_{\Omega} 1 \leq \int_{\Omega} \left(\frac{\Psi(r, z)}{cr + k} \right)^{\frac{2}{1-s}} dr dz \leq \int_{\Omega} \left(\frac{\Psi(r, z)}{k} \right)^{\frac{2}{1-s}} dr dz \leq \frac{1}{k^{\frac{2}{1-s}}} \int_{\mathbb{H}} \Psi^{\frac{2}{1-s}}. \quad (4.69)$$

Using a Sobolev inequality above leads to

$$\mathcal{L}^2(\Omega) \leq \frac{C_s}{k^{\frac{2}{1-s}}} \|\Psi\|_{X^s}^{\frac{2}{1-s}}. \quad (4.70)$$

The computation of the double integral defining the \dot{H}^s half-norm 4.20 is done separating the integrals on \mathbb{R}^2 on two between Ω and Ω^c . On Ω^c the quantity $\Psi(r, z) - cr - k$ is non-positive and then $T(\Psi)(r, z) = 0$. Therefore,

$$\int_{\Omega^c} \int_{\Omega^c} \frac{|T(\Psi)(x) - T(\Psi)(y)|^2}{|x - y|^{2(1+s)}} dx dy = 0. \quad (4.71)$$

Concerning the integral on $\Omega \times \Omega$, using the notation $x = (r_x, z_x)$ and $y = (r_y, z_y)$,

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \frac{|T(\Psi)(x) - T(\Psi)(y)|^2}{|x - y|^{2(1+s)}} dx dy &= \int_{\Omega} \int_{\Omega} \frac{|(\Psi(x) - cr_x - k) - (\Psi(y) - cr_y - k)|^2}{|x - y|^{2(1+s)}} dr dz \\ &\leq \int_{\Omega} \int_{\Omega} \frac{|\Psi(x) - \Psi(y)|^2 + c^2|r_x - r_y|^2}{|x - y|^{2(1+s)}} dr dz. \end{aligned} \quad (4.72)$$

Denote with an $*$ the radially decreasing rearrangement and $R_{\Omega} > 0$ the radius such that

$$\mathcal{L}^2(\Omega) = \mathcal{L}^2(\mathcal{B}(0, R_{\Omega})). \quad (4.73)$$

To simplify the notations we simply note this ball $\mathcal{B}(\Omega)$. By the Riesz rearrangement inequality we have

$$\begin{aligned} \int_{\Omega} \int_{\Omega} \frac{|r_x - r_y|^2}{|x - y|^{2(1+s)}} dx dy &\leq \int_{\Omega} \int_{\Omega} \frac{dx dy}{|x - y|^{2s}} \leq \int_{\mathcal{B}(\Omega)} \int_{\mathcal{B}(\Omega)} \frac{dx dy}{|x - y|^{2s}} \\ &\leq \int_{\mathcal{B}(\Omega)} \int_{\mathcal{B}(\Omega)} \frac{dx}{|x|^{2s}} dy = \frac{\pi^s}{1-s} \mathcal{L}^2(\Omega)^{2-s}. \end{aligned} \quad (4.74)$$

Using now (4.70) we get

$$\int_{\Omega} \int_{\Omega} \frac{|r_x - r_y|^2}{|r - z|^{2(1+s)}} dr dz \leq C_s \|\Psi\|_{X^s}^{\frac{2}{1-s}}. \quad (4.75)$$

Concerning the last term,

$$\int_{\Omega} \int_{\Omega^c} \frac{|T(\Psi)(x) - T(\Psi)(y)|^2}{|x - y|^{2(1+s)}} dx dy = \int_{\Omega} |T(\Psi)(x)|^2 \int_{\Omega^c} \frac{dy}{|x - y|^{2(1+s)}} dx. \quad (4.76)$$

For all $x \in \Omega$ we define $\Lambda(x) := \frac{\Psi(x) - cr_x - k}{2c}$ and $\mathcal{O}_x := \{y \in \Omega^c : r_y \geq r_x + \Lambda(x)\}$. Then,

$$\int_{\mathcal{O}_x} \frac{dy}{|x - y|^{2(1+s)}} \leq \int_{\mathcal{O}_x} \frac{dy}{|r_x - r_y|^{2(1+s)}} \leq \int_{\mathcal{B}(x, \Lambda(x))^c} \frac{dy}{|x - y|^{2(1+s)}} = \frac{\pi^s}{\Lambda(x)^{2s}}. \quad (4.77)$$

Using the fact that $x \in \Omega$, that $y \in \Omega^c$ and that $y \in \mathcal{O}_x$ in this order, we obtain

$$\begin{aligned} 0 \leq T(\Psi)(x) &= \Psi(x) - cr_x - k \leq \Psi(x) - \Psi(y) + c(r_x - r_y) \\ &\leq \Psi(x) - \Psi(y) + \Psi(x) - \frac{cr_x - k}{2} = \Psi(x) - \Psi(y) + \frac{1}{2}T(\Psi)(x). \end{aligned} \quad (4.78)$$

2. The adherence of this set is the support of the function Θ . This corresponds physically speaking to the vorticity zone.

Therefore

$$|T(\Psi)(x)| \leq 2|\Psi(x) - \Psi(y)| \quad (4.79)$$

Combing (4.76), (4.77) and (4.79) leads to

$$\begin{aligned} & \int_{\Omega} \int_{\Omega^c} \frac{|T(\Psi)(x) - T(\Psi)(y)|^2}{|x - y|^{2(1+s)}} dx dy \\ & \leq 4 \int_{\Omega} \int_{\Omega^c \setminus \mathcal{O}_x} \frac{|\Psi(x) - \Psi(y)|^2}{|x - y|^{2(1+s)}} dx dy + \int_{\Omega} |T(\Psi)(x)|^2 \frac{\pi}{s\Lambda(x)^{2s}} dx \\ & \leq 4 \int_{\Omega} \int_{\Omega^c} \frac{|\Psi(x) - \Psi(y)|^2}{|x - y|^{2(1+s)}} dx dy + \frac{\pi}{s} \int_{\Omega} |T(\Psi)(x)|^{2(1-s)} dx. \end{aligned} \quad (4.80)$$

Now, to estimate the last term of the above inequality, we use the fact that $T(\Psi) \leq \Psi$ and then the Hölder inequality gives

$$\int_{\Omega} |T(\Psi)(x)|^{2(1-s)} dx \leq \int_{\Omega} |\Psi(x)|^{2(1-s)} dx \leq \mathcal{L}^2(\Omega)^{s(2-s)} \|\Psi\|_{L^{\frac{2}{1-s}}}^{2(1-s)}. \quad (4.81)$$

We continue this estimate using (4.70) and a Sobolev embedding,

$$\leq C \|\Psi\|_{X^s}^{2s(1+\frac{1}{1-s})} \|\Psi\|_{L^{\frac{2}{1-s}}}^{2(1-s)} \leq C \|\Psi\|_{X^s}^{\frac{2}{1-s}}. \quad (4.82)$$

Thus, gathering all these estimates we obtain.

$$\|T(\Psi)\|_{X^s}^2 := \int_{\Omega} \int_{\Omega^c} \frac{|T(\Psi)(x) - T(\Psi)(y)|^2}{|x - y|^{2(1+s)}} dx dy \leq C \|\Psi\|_{X^s}^2 \left(1 + \|\Psi\|_{X^s}^{\frac{2s}{1-s}}\right). \quad (4.83)$$

Therefore, T does map $X_{sym}^{s,\dagger}$ into itself and maps bounded subsets of $X_{sym}^{s,\dagger}$ into bounded subsets.

• **Step 2 : $T \circ \sharp \circ \dagger$ defined on X_{sym}^s is a compact operator for the L^p topology.** Set the convention that $\{|z| \geq R\}$ designates the set $\{(r, z) \in \mathbb{H} : |z| \geq R\}$. Let $\kappa > 0$ and $R \geq 0$. For all $r \in \Omega^\sharp$ we define

$$\mathcal{U}_x := \mathcal{B}(x, \kappa) \cap (\Omega^\sharp)^c. \quad (4.84)$$

Then,

$$\begin{aligned} \int_{\{|z| \geq R\}} |T(\Psi^\sharp)(x)|^2 dx &= \int_{\{|z| \geq R\}} \frac{1}{\mathcal{L}^2(\mathcal{U}_x)} \int_{\mathcal{U}_x} |T(\Psi^\sharp)(x)|^2 dy dx \\ &\leq \int_{\{|z| \geq R\}} \frac{1}{\mathcal{L}^2(\mathcal{U}_x)} \int_{\mathcal{U}_x} |T(\Psi^\sharp)(x) - T(\Psi^\sharp)(y)|^2 dy dx \\ &\leq \int_{\{|z| \geq R\}} \frac{\kappa^{2(1+s)}}{\mathcal{L}^2(\mathcal{U}_x)} \int_{\mathcal{U}_x} \frac{|T(\Psi^\sharp)(x) - T(\Psi^\sharp)(y)|^2}{|x - y|^{2(1+s)}} dy dx. \end{aligned} \quad (4.85)$$

Denote by $P_{\mathbb{R}}$ the projection on $\mathbb{R} \times 0$ (that is identified to \mathbb{R}). As a consequence of the Steiner symmetrization, with (4.69),

$$2R \mathcal{L}^1 \left(P_{\mathbb{R}} \left((\Omega^\sharp)^c \cap \{|z| \geq R\} \right) \right) \leq \mathcal{L}^2(\Omega^\sharp) \leq \frac{1}{k^{\frac{2}{1-s}}} \|\Psi^\sharp\|_{L^{\frac{2}{1-s}}}^{\frac{2}{1-s}}. \quad (4.86)$$

Since $|z_x| \geq R - \kappa$ then using again the Steiner symmetry of Ω^\sharp , gives that \mathcal{U}_x contains the ball $B(r, \kappa)$ minus the rectangle centered at x , of width

$$\mathcal{L}^1 \left(P_{\mathbb{R}} \left((\Omega^\sharp)^c \cap \{|z| \geq R - \kappa\} \right) \right)$$

height 2κ . Then, with (4.86),

$$\mathcal{L}^2(\mathcal{U}_x) \geq \pi\kappa^2 - \frac{\kappa}{(R - \kappa)k^{\frac{2}{1-s}}} \|\Psi^\sharp\|_{L^{\frac{2}{1-s}}}^{\frac{2}{1-s}}. \quad (4.87)$$

Section 3. Proofs of the lemmas

The choice of κ is free and then we choose to fix it equal to C/R with

$$C := \frac{4}{\pi k^{\frac{2}{1-s}}} \|\Psi^\sharp\|_{L^{\frac{2}{1-s}}}^{\frac{2}{1-s}}.$$

Since Choose now R such that $R \geq \sqrt{2C}$. Then in this case the inequality (4.87) becomes

$$\mathcal{L}^2(\mathcal{U}_x) \geq \frac{\pi C^2}{2R^2}. \quad (4.88)$$

Combining the estimate above with (4.85), leads to the following estimate

$$\int_{\{|z| \geq R\}} |T(\Psi^\sharp)(x)|^2 dx \leq \left(\frac{4}{\pi R}\right)^{2s} \left(\frac{\|\Psi^\sharp\|_{L^{\frac{2}{1-s}}}}{k}\right)^{\frac{4s}{1-s}} \|T(\Psi^\sharp)\|_{X^s}^2. \quad (4.89)$$

On the other hand, using the Hölder inequality,

$$\begin{aligned} \int_{\{r \geq R\}} |T(\Psi^\sharp)|^2 &= \int_{\{r \geq R\}} |T(\Psi^\sharp)|^2 \mathbb{1}_{(\Omega^\sharp)^c} \leq \left(\int_{\{r \geq R\}} |T(\Psi^\sharp)|^{\frac{2}{1-s}}\right)^{1-s} \left(\int_{\{r \geq R\}} \mathbb{1}_{(\Omega^\sharp)^c}\right)^s \\ &= cL^2\left((\Omega^\sharp)^c \cap \{r \geq R\}\right)^s \|\Psi^\sharp\|_{L^{\frac{2}{1-s}}}^2, \end{aligned} \quad (4.90)$$

where by convention $\{r \geq R\}$ designates the set $\{(r, z) \in \mathbb{H} : r \geq R\}$. Moreover

$$\begin{aligned} \mathcal{L}^2((\Omega^\sharp)^c \cap \{r \geq R\}) &= \int_{(\Omega^\sharp)^c \cap \{r \geq R\}} 1 \\ &\leq \int_{(\Omega^\sharp)^c \cap \{r \geq R\}} \left(\frac{\Psi^\sharp}{cr+k}\right)^{\frac{2}{1-s}} \leq \left(\frac{1}{cR+k}\right)^{\frac{2}{1-s}} \|\Psi^\sharp\|_{L^{\frac{2}{1-s}}}^{\frac{2}{1-s}}. \end{aligned} \quad (4.91)$$

Combining (4.90) and (4.91) leads to

$$\int_{\{r \geq R\}} |T(\Psi^\sharp)|^2 \leq \left(\frac{1}{cR+k}\right)^{\frac{2s}{1-s}} \|\Psi^\sharp\|_{L^{\frac{2}{1-s}}}^{\frac{2}{1-s}} \quad (4.92)$$

The two decay estimates (4.89) et (4.92) and the Rellich-Kondrachov compactness theorem (applied at the local level) give the result. \square

3.4.2 Proof of Lemma 2.5

It follows from the definition of \mathcal{N} and of Lemma 2.1 that

$$\int_{\mathbb{H}} f(\Psi_n - cr - k)\Psi_n = \frac{1}{2} \int_{\mathbb{R}} \Psi_n(-\Delta)^s \Psi_n = \frac{1}{2} \|\Psi_n\|_{X^s}^2 \geq \frac{\beta}{2} > 0. \quad (4.93)$$

By the previous lemma, up to a subsequence when $n \rightarrow +\infty$,

$$\int_{\mathbb{H}} f(\Psi_n - cr - k)\Psi_n \longrightarrow \int_{\mathbb{H}} f(\Psi^\star - cr - k)\Psi^\star \geq \frac{\beta}{2}. \quad (4.94)$$

In particular $(\Psi^\star - cr - k)_+ \not\equiv 0$ on \mathbb{H} . By proposition 2.1, there exists $t^\star > 0$ such that $t^\star \Psi^\star \in \mathcal{N}$. With the characterization of t^\star and since $\Psi_n \in \mathcal{N}$,

$$E(\Psi_n) = E(t_{\Psi_n} \Psi_n) \geq E(t^\star \Psi_n). \quad (4.95)$$

Thus,

$$\alpha = \lim_{n \rightarrow +\infty} E(\Psi_n) \geq \liminf_{n \rightarrow +\infty} E(t^\star \Psi_n) \geq E(t^\star \Psi^\star) \geq \alpha. \quad (4.96)$$

Therefore all these inequalities are equalities and $\|\Psi_n\|_{X^s}^2 \rightarrow \|\Psi^\star\|_{X^s}^2$. Since the space X^s is strictly convex, this gives that Ψ_n converges towards Ψ^\star strongly in X^s . \square

3.5 Proofs of the lemmas of section 2.3

3.5.1 Proof of Proposition 2.6

We already know that $T(\Psi^*) \in L^{1-\frac{2}{s}}(\mathbb{R}^2)$. Since the support of $T(\Psi^*)$ has finite measure, $T(\Psi^*) \in L^1(\mathbb{R}^2)$. Define Θ^* from Ψ^* using formula (4.26). Hypothesis (4.13) implies

$$\forall q \in \left[1, \frac{2}{\nu(1-s)}\right], \quad \Theta^* \in L^q(\mathbb{R}^2). \quad (4.97)$$

Define now the function $\tilde{\Psi}$ given by the following representation formula,

$$\tilde{\Psi}(x) = K_s \int_{\mathbb{R}^2} \frac{\Theta^*(y)}{|x-y|^{2(1-s)}} dy. \quad (4.98)$$

where K_s is some renormalization constant. It follows from the weighted inequalities for singular integrals [77, §5] that $\tilde{\Psi} \in \dot{W}^{2s,q}(\mathbb{R}^2)$, for all $q \in [1, \frac{2}{\nu(1-s)}]$. Moreover, by the Hardy-Littlewood-Sobolev convolution inequality, $\tilde{\Psi} \in L^q(\mathbb{R}^2), \forall q \in [\frac{1}{1-s}, \frac{2}{\nu-s(2+\nu)}]$. By standard interpolation, $\tilde{\Psi} \in X_{sym}^s$. Now, let $\varphi \in X_{sym}^s$ be a test function. Using the spectral properties of the Sobolev spaces [37] gives (up to multiplicative renormalization constants),

$$\begin{aligned} \langle \tilde{\Psi}, \varphi \rangle_{X^s} &= \int_{\mathbb{R}^2} |\xi|^{2s} \mathcal{F}[\tilde{\Psi}](\xi) \mathcal{F}[\varphi](\xi) d\xi \\ &= \int_{\mathbb{R}^2} |\xi|^{2s} \mathcal{F}\left[\Theta^* * \frac{1}{|\cdot|^{2(1-s)}}\right](\xi) \mathcal{F}[\varphi](\xi) d\xi \\ &= \int_{\mathbb{R}^2} \mathcal{F}[\Theta^*](\xi) \mathcal{F}[\varphi](\xi) d\xi = \langle \Theta^*, \varphi \rangle_{L^2}, \end{aligned} \quad (4.99)$$

where $\mathcal{F}[\cdot]$ designates the Fourier transform. Moreover, since Ψ^* is a critical point of E , then $\langle \Psi^*, \varphi \rangle_{X^s} = \langle \Theta^*, \varphi \rangle_{L^2}$, which implies $\tilde{\Psi} = \Psi^*$. The regularity known for Θ^* allows to conclude that Ψ^* is bounded and uniformly continuous. Seen the definitions, to conclude that Ψ^* is smooth by a bootstrap argument there remain to study possible discontinuities on $r = 0$. Nevertheless, it follows from the symmetry property of Ψ^* and its uniform continuity that $T(\Psi^*)$ worth 0 at a distance uniformly positive from $r = 0$, meaning on a strip $] -\delta, \delta[\times \mathbb{R}_+$. Therefore so is the case for Θ^* and then the smoothness of Ψ^* is proved. \square

3.5.2 Proof of Proposition 2.7

Let $r \in \mathbb{R}^2$ such that $|r| \geq 1$. We separate the integral (4.98) into two,

$$\Psi^*(x) = K_s \int_{|x-y| \leq \frac{|x|}{2}} \frac{\Theta^*(y)}{|x-y|^{2(1-s)}} dy + K_s \int_{|x-y| > \frac{|x|}{2}} \frac{\Theta^*(y)}{|x-y|^{2(1-s)}} dy \quad (4.100)$$

Concerning the first integral, we choose $\eta \in]1-s, \frac{1}{s+1}[$. This interval is non-empty and included in $]0, 1[$. We use the Hölder inequality and Hypothesis (4.13) and then we are led to

$$\int_{|x-y| \leq \frac{|x|}{2}} \frac{\Theta^*(y)}{|x-y|^{2(1-s)}} dy \leq \left(\int_{|\zeta| \leq \frac{|x|}{2}} \frac{d\zeta}{|\zeta|^{\frac{2(1-s)}{\eta}}} \right)^\eta \left(\int_{|x-y| \leq \frac{|x|}{2}} |T(\Psi^*)|^{\frac{\nu}{1-\eta}} \right)^{1-\eta}. \quad (4.101)$$

Using again the estimates (4.89) et (4.92),

$$\int_{|x-y| \leq \frac{|x|}{2}} |T(\Psi^*)|^2(y) dy \leq \frac{C}{(1+|x|)^{-2s}}. \quad (4.102)$$

Knowing that $\frac{\nu}{1-\eta} \geq 2$ the above estimate used in (4.101) leads to (the constant that depends only on η)

$$\int_{|x-y| \leq \frac{|x|}{2}} \frac{\Theta^*(y)}{|x-y|^{2(1-s)}} dy \leq C(\eta)(|x|^2 + 1)^{\eta(s+1)-1}. \quad (4.103)$$

The second integral in (4.100) can be estimated using directly the hypothesis on the function f ,

$$\int_{|x-y| > \frac{|x|}{2}} \frac{\Theta^*(y)}{|x-y|^{2(1-s)}} dy \leq \left(\frac{2}{|x|}\right)^{2(1-s)} \int_{\mathbb{R}^2} |T(\Psi^*)|^\nu \leq \frac{C}{|x|^{2(1-s)}}. \quad (4.104)$$

Section 3. Proofs of the lemmas

By choosing $\eta \in]1 - s, \frac{1}{s+1}[$ such that $\eta \geq \frac{s}{s+1}$, the estimates (4.103) et (4.104) give

$$\Psi^*(x) \leq \frac{C}{1 + |x|^{2(1-s)}}. \quad (4.105)$$

Chapitre 5

Tamped functions, A rearrangement in dimension 1

abstract We define a new rearrangement, called rearrangement by tamping, for non-negative measurable functions defined on \mathbb{R}_+ . This rearrangement has many properties in common with the well-known Schwarz non-increasing rearrangement such as the Pólya–Szegő inequality. Contrary to the Schwarz rearrangement, the tamping also preserves the homogeneous Dirichlet boundary condition of a function.

Introduction

The Schwarz non-increasing rearrangement is a powerful tool to establish symmetry properties of solutions to variational problems [3] [2] [55]. In this paper, we are interested in situations where the Schwarz rearrangement cannot be applied as such due to boundary constraints. Consider for example the simple minimization problem

$$\min \left\{ \int_0^1 |\varphi'(x)|^2 dx, \quad \varphi \in H_0^1(]0, 1[, \mathbb{R}), \quad \int_0^1 g(x) \varphi^2(x) dx = 1 \right\},$$

where $g : [0, 1] \rightarrow \mathbb{R}$ is positive and non-decreasing. Due to the constraint $\varphi(0) = 0$, the unique minimizer φ^* cannot be non-increasing. Since the weight $g(x)$ is decreasing, it is favorable to shift as much of the mass of φ^* towards the origin. However, this process is balanced by the Dirichlet energy which would otherwise blow-up, since $\varphi(0) = 0$, if φ^* is too much concentrated near the origin. For that reason, it appears natural to expect that φ^* is unimodal, meaning that there exists s^* in $(0, 1)$ such that φ^* is non-decreasing on $[0, s^*]$ and non-increasing on $[s^*, 1]$. Such a result still holds with a more general constraint of the form $\int_0^1 F(\varphi(x), x) dx = 1$, where $F : \mathbb{R}^2 \rightarrow \mathbb{R}_+$ is non-decreasing for each variables. As a matter of fact, any minimizer of the optimization problem is invariant by the rearrangement by tamping as a consequence of its four main properties. First, the rearrangement by tamping satisfies the Pólya–Szegő inequality, as stated in Theorem 3.2 and the Schwarz rearrangement inequality, stated at Property 1.2. It also maps the set of measurable non-negative functions into the set of unimodal functions. Finally, in addition to these three properties that it shares with the Schwarz non-increasing rearrangement, the tamping preserves the Dirichlet boundary condition of non-negative functions of $W_0^{1,p}(0, 1)$ or $W_0^{1,p}(\mathbb{R}_+)$. The main object of this work consists in defining the rearrangement by tamping and establishing its main properties.

On the other hand, this rearrangement does *not* satisfy the Hardy–Littlewood nor the Riesz rearrangement inequalities. The continuity properties of the rearrangement by tamping are also somewhat weaker than their equivalent for the Schwarz non-increasing rearrangement [29][1]. Still, Theorem 3.1 states some form of continuity of the tamping process in $L^p(\mathbb{R}_+)$ ($1 \leq p < +\infty$). In our construction, this convergence property is essential for extending the properties established for the tamping, such as the Pólya–Szegő inequality, from elementary step functions to arbitrary functions in $L^p(\mathbb{R}_+)$. Compactness results for the rearrangement by tamping are also proved (Theorem 3.5 and its corollaries).

1 Presentation of the problem

In this section we provide a short survey on rearrangements with an emphasis on the properties that we are interested in. In the last subsection we explain that the classical Schwarz non-increasing rearrangement fails to preserve the Dirichlet boundary conditions and we present what are the properties that we require for our rearrangement. We denote by $\mathfrak{M}_+(\Omega)$ the set of non-negative measurable functions defined on a domain Ω , and by

$$\{\varphi \geq \nu\} := \{x \in \Omega : \varphi(x) \geq \nu\}$$

the *superlevel sets* of a function $\varphi \in \mathfrak{M}_+(\Omega)$. The sets $\{\varphi > \nu\}$ and $\{\varphi = \nu\}$ are defined similarly.

1.1 Layer-cake representation

Let Ω be an open subset of \mathbb{R}^d . We recall here the *layer-cake representation* of a non-negative measurable function [55],

$$\varphi(x) = \int_0^{\varphi(x)} d\nu = \int_0^{+\infty} \mathbb{1}_{\{\varphi \geq \nu\}}(x) d\nu, \quad (5.1)$$

where $\mathbb{1}_A$ refers to the indicator function of the set A . This layer-cake representation implies that for $f \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ non-decreasing such that $f(0) = 0$,

$$\int_{\Omega} f \circ \varphi(x) dx = \int_0^{+\infty} f'(\nu) \text{meas}(\{\varphi \geq \nu\}) d\nu. \quad (5.2)$$

In this equality, f can actually be chosen in $BV(\mathbb{R}_+)$ and in this case its derivative is a measure. In particular,

$$\|\varphi\|_{L^p}^p = p \int_0^{+\infty} \nu^{p-1} \text{meas}(\{\varphi \geq \nu\}) d\nu. \quad (5.3)$$

1.2 The Schwarz non-increasing rearrangement

Definition 1.1. Let φ be a non-negative measurable function. The function ψ is a *rearrangement* of the function φ if and only if for almost every ν we have

$$\text{meas}(\{\varphi \geq \nu\}) = \text{meas}(\{\psi \geq \nu\}).$$

As a consequence of the layer-cake representation (5.2), we have the following property.

Proposition 1.1. *Let φ be a non-negative measurable function and let ψ be a rearrangement of φ . The following equality holds (whenever these quantities are finite).*

$$\int_{\mathbb{R}} (f \circ \varphi)(s) ds = \int_{\mathbb{R}} (f \circ \psi)(s) ds, \quad (5.4)$$

for any BV -function $f : \mathbb{R} \rightarrow \mathbb{R}$. In particular,

$$\|\varphi\|_{L^p} = \|\psi\|_{L^p}. \quad (5.5)$$

Definition 1.2. Let φ be a non-negative measurable function defined on \mathbb{R}_+ . The *Schwarz non-increasing rearrangement* of φ , that we note φ^* , is defined as being the only rearrangement of φ that is a non-increasing function on \mathbb{R}_+ . Its superlevel sets are

$$\{\varphi^* \geq \nu\} = [0; \text{meas}\{\varphi \geq \nu\}].$$

The Schwarz non-increasing rearrangement can be interpreted as being the rearrangement which “shoves all the mass of the function until it reaches the origin”. The idea that this rearrangement “moves the mass of the function to the left” is embedded in the following important inequality.

Proposition 1.2 (Schwarz rearrangement inequality).

$$\forall x \in \mathbb{R}_+, \quad \text{meas}([0, x] \cap \{\varphi \geq \nu\}) \leq \text{meas}([0, x] \cap \{\varphi^* \geq \nu\}).$$

Section 1. Presentation of the problem

As a consequence of this inequality, the Schwarz non-increasing rearrangement also satisfies several other inequalities exploiting this idea that it shoves the mass down to the origin. For instance we can obtain inequalities for weighted L^p -norms when the weight is non-increasing. More generally we have the following two properties.

Proposition 1.3. *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a non-negative measurable function and let $g : \mathbb{R}_+ \rightarrow \mathbb{R}$ non-increasing. Let $\varphi \in \mathfrak{M}_+(\mathbb{R}_+)$. Then,*

$$\forall x \geq 0, \quad \int_0^x (f \circ \varphi)(s) g(s) ds \leq \int_0^x (f \circ \varphi^*)(s) g(s) ds. \quad (5.6)$$

Démonstration. Using twice the layer-cake representation we obtain

$$\begin{aligned} \int_0^x (f \circ \varphi)(s) g(s) ds &= \int_0^x \int_0^{+\infty} \mathbb{1}_{\{f \circ \varphi \geq \nu\}}(s) d\nu g(s) ds \\ &= \int_0^x \int_0^{+\infty} \int_0^{+\infty} \mathbb{1}_{\{f \circ \varphi \geq \nu\}}(s) \mathbb{1}_{\{g \geq \mu\}}(s) d\mu d\nu ds \\ &= \int_0^x \int_0^{+\infty} \int_0^{+\infty} \mathbb{1}_{\{f \circ \varphi \geq \nu\} \cap \{g \geq \mu\}}(s) d\mu d\nu ds. \end{aligned}$$

Since g is non-increasing, the set $\{g \geq \mu\}$ is a segment starting at 0. Noticing that $(f \circ \varphi)^* = f \circ \varphi^*$ we can conclude using the Schwarz rearrangement inequality. \square

Proposition 1.4. *Let $f : \mathbb{R} \rightarrow \mathbb{R}_+$ be non-decreasing BV-functions such that $\inf f = 0$ and let $g : \mathbb{R} \rightarrow \mathbb{R}$ be non-decreasing. Let $\varphi \in \mathfrak{M}_+(\mathbb{R}_+)$. Then,*

$$\forall x \geq 0, \quad \int_0^x f \circ (\varphi - g)(s) ds \leq \int_0^x f \circ (\varphi^* - g)(s) ds. \quad (5.7)$$

Démonstration. Since f is a BV-function, its weak derivative f' is a signed measure. Using the layer-cake representation, we have

$$\int_0^x f \circ (\varphi - g)(s) ds = \int_0^{+\infty} \text{meas} \left(\{\varphi - g \geq \nu\} \cap [0, x] \right) f'(\nu) d\nu. \quad (5.8)$$

We now observe that

$$\left\{ x \in \mathbb{R}_+ : \varphi(x) - g(x) \geq \nu \right\} = \bigcup_{\mu \geq \inf g} \left\{ x \in \mathbb{R}_+ : \varphi(x) \geq \nu + \mu \right\} \cap \left\{ x \in \mathbb{R}_+ : g(x) \leq \mu \right\}. \quad (5.9)$$

Since g is non-decreasing, the set $\{g \leq \mu\}$ is a segment starting at 0 and then we can conclude using the Schwarz rearrangement inequality that

$$\text{meas} \left(\{\varphi - g \geq \nu\} \cap [0, x] \right) \leq \text{meas} \left(\{\varphi^* - g \geq \nu\} \cap [0, x] \right). \quad (5.10)$$

Since f is a non-decreasing function, the measure f' is actually non-negative and then (5.8) with (5.9) gives (5.7). \square

Integral terms of the form $\int_0^x f(\varphi(s))g(s) ds$ like in Proposition 1.3 appear in the variational formulation of many reaction-diffusion equations. In these cases, φ models the density of the studied population while g can be understood as the hostility of the environment (see e.g. [9]). Integral terms of the form $\int_0^x f(\varphi(s) - g(s)) ds$ also appear in the variational formulation of some problems in fluid mechanics (see e.g. [10][61][42]).

1.3 Rearrangement inequalities

These are the three main rearrangement inequalities for the Schwarz and Steiner rearrangements. The proofs can be found respectively in [43], [66], [69].

Theorem 1.3 (Hardy-Littlewood rearrangement inequality). *Let φ and ψ be two non-negative measurable functions defined on \mathbb{R}_+ . Then,*

$$\int_0^\infty \varphi \psi \leq \int_0^\infty \varphi^* \psi^*.$$

It must be said that this inequality is not a consequence of the Schwarz rearrangement inequality 1.2 nor implies it. This inequality is also true for the Steiner rearrangement.

Theorem 1.4 (Pólya–Szegő inequality). *If we suppose that $\varphi \in \dot{W}_+^{1,p}$ then so is the function φ^* and*

$$\int_{\mathbb{R}_+} |\nabla(\varphi^*)|^p \leq \int_{\mathbb{R}_+} |\nabla\varphi|^p.$$

The corresponding inequality also holds in the case of the Steiner rearrangement. The case $p = 2$ in the Pólya–Szegő inequality is of particular interest since it involves the energy term associated to a Laplace operator.

Theorem 1.5 (Riesz Rearrangement inequality). *Let φ, ψ and $\chi : \mathbb{R} \rightarrow \mathbb{R}_+$. Then,*

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \varphi(x) \psi(y) \chi(x - y) dx dy \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi^\sharp(x) \psi^\sharp(y) \chi^\sharp(x - y) dx dy.$$

This inequality can be considered as being a generalization of the Hardy-Littlewood inequality 1.3 if we choose, at least formally, the function χ to be equal to the Dirac mass centered at 0. An important consequence of this inequality is the fact that for $s \in (0, 1)$ we have,

$$|\varphi^\sharp|_{H^s} \leq |\varphi|_{H^s}. \tag{5.11}$$

Here the H^s half-norms are defined by

$$|\varphi|_{H^s}^2 := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{2(1-s)}} dx dy.$$

1.4 Limitation of the Schwarz rearrangement, preserving Dirichlet boundary condition

1.4.1 The problem of tamping in dimension 1

However useful the Schwarz rearrangement may be, it does not work if we want to impose a Dirichlet boundary condition at 0 because such a condition is in general not respected after a Schwarz rearrangement. In the case of a Dirichlet boundary condition, the solution is expected to be unimodal instead of non-increasing, where by "unimodal" we mean non-decreasing on an interval $[0, s]$ and then non-increasing on $[s, +\infty)$ for a certain $s \in \mathbb{R}_+$. In short, we want to build a rearrangement that

- Satisfies a Pólya–Szegő inequality (Theorem 1.4),
- Satisfies a Schwarz rearrangement inequality (Property 1.2),
- Gives unimodal functions (non-decreasing then non-increasing),
- Preserves the Dirichlet boundary condition at 0.

The main observation that starts this work is the following intuitive explanation of why the Schwarz rearrangement verifies the Pólya–Szegő inequality. It can be said, roughly speaking, that when we apply the Schwarz rearrangement, the mass of the function is pushed all the way down to the origin and when this movement of mass is done, all the "hollows" of the function are filled up (we mean that all the strict local minima have disappeared after the Schwarz rearrangement). The idea that the Pólya–Szegő inequality is a consequence of the fact that the "hollows" are filled up, is the guiding idea of our construction. The spirit of the rearrangement by tamping is that the mass has to be moved in order to "fill all the hollows" but we must not move this mass "too far", because this implies the loss of the Dirichlet boundary condition.

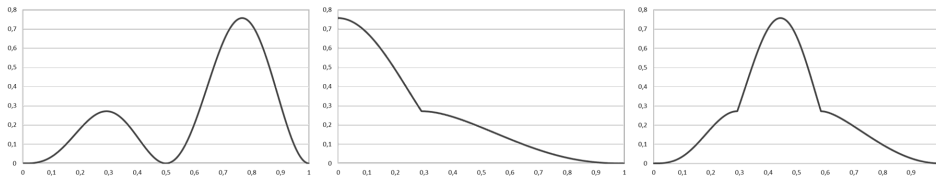


FIGURE 5.1 – From left to right : the graph of the function $x \mapsto x \cdot \sin^2(2\pi x) \cdot \mathbb{1}_{[0,1]}(x)$, the graph of the Schwarz rearrangement of this function and the graph of the rearrangement by tamping.

As an illustration, Figure 5.1 shows the graph of $x \mapsto x \cdot \sin^2(2\pi x) \cdot \mathbb{1}_{[0,1]}(x)$, the graph of the Schwarz non-increasing rearrangement of this function and the graph of the rearrangement by tamping that we are introducing in this article.

1.4.2 The problem of tamping in dimension 2

Although our construction is so far limited to functions of one variable, the problem of tamping in dimension 2 can still be enunciated. It consists in building a rearrangement for non-negative functions defined on $\mathbb{R}_+ \times \mathbb{R}$ that satisfies a 2D-analogous of the four conditions given at Section 1.4.1 for the 1D case.

- The extension of the Pólya–Szegő inequality in dimension two consists in replacing the derivative of the functions by a gradient.
- For a problem on $\mathbb{R}_+ \times \mathbb{R}$, the Schwarz rearrangement inequality becomes

$$\forall x \in \mathbb{R}_+, \quad \text{meas} \left([0, x] \times \mathbb{R} \cap \{\varphi \geq \nu\} \right) \leq \text{meas} \left([0, x] \times \mathbb{R} \cap \{\varphi^* \geq \nu\} \right),$$

where here *meas* refers to the Lebesgue measure on \mathbb{R}^2 .

- In dimension 2, the unimodal functions are the functions whose set of local maxima is a connected set (they are just “one bump”).
- On $\mathbb{R}_+ \times \mathbb{R}$, the Dirichlet boundary condition is set on $\{0\} \times \mathbb{R}$.

It must be said that the result obtained in dimension 1 for the problem of tamping cannot be immediately extended to the problem of tamping in dimension 2 because this rearrangement does not show good properties of tensorization.

2 Definition of the rearrangement by tamping

2.1 Definition of the tamping on voxel functions

At first we define the rearrangement by tamping on a special subclass of the piece-wise constant functions called the *voxel functions*. The general case for functions in $\mathfrak{M}_+(\mathbb{R}_+)$ will be treated later. The definition of the tamping rearrangement on voxel functions is given by an algorithm separated into two algorithms. We first define the *elementary tamping*. The general algorithm of tamping then consists in an iteration of the elementary tamping algorithm. Before all, we have to define what is a voxel function.

2.1.1 The voxel functions

Let $n \in \mathbb{N}^*$ and let $\Gamma : \{1, \dots, n\}^2 \rightarrow \{0, 1\}$ a function being non-increasing with respect to its second variable, in other words

$$\forall 1 \leq i, j, k \leq n, \quad \left. \begin{array}{l} k \leq j \\ \Gamma(i, j) = 1 \end{array} \right\} \implies \Gamma(i, k) = 1. \quad (5.12)$$

For convenience we extend the function Γ to \mathbb{N}^2 by setting its values to zero outside of $\{1, \dots, n\}^2$. We define the *voxels* (associated to Γ) as being the sets

$$a_\Gamma(i, j) := \begin{cases} \left[i - \frac{1}{2}, i + \frac{1}{2} \right] \times [j - 1, j] & \text{if } \Gamma(i, j) = 1, \\ \emptyset & \text{if } \Gamma(i, j) = 0, \end{cases} \quad (5.13)$$

and then the *voxel pile*

$$A_\Gamma := \bigcup_{i, j} a_\Gamma(i, j) \subset \mathbb{R}^2. \quad (5.14)$$

If we only consider the reunion on j then the obtained set is referred as being the *voxel column* at abscissa i and if on the contrary we consider the reunion on the index i then we get the *voxel line* at ordinate j . Condition (5.12) implies that the set A_Γ is the hypograph of a certain piece-wise constant function $\varphi_\Gamma : \mathbb{R} \rightarrow \mathbb{R}_+$. The set of piece-wise constant functions that can be defined in such a way is noted $\mathcal{E}_n(\mathbb{R})$, and its elements are called *voxel functions*.

2.1.2 The elementary tamping algorithm

For $\xi \in \mathbb{N}$ one defines the function :

$$\eta(\xi) := \min \{ \eta \in \mathbb{N}^* : \xi \leq \eta \text{ and } \varphi(\eta - 1) > \varphi(\xi) \geq \varphi(\eta) \}. \quad (5.15)$$

Roughly speaking, the *elementary tamping algorithm* consists in moving one step on the left the mountain that is between ξ and $\eta(\xi)$ (see Figure 5.2). This ξ will often be referred as being the “*pivot*” of the elementary tamping algorithm.

Definition 2.1. The *elementary tamping* associated to a given pivot ξ consists in modifying the function Γ following the algorithm :

for j from $\varphi(\xi) + 1$ to n do :
 for i from ξ to $\eta(\xi) - 1$ do :
 $\Gamma(i, j) \leftarrow \Gamma(i + 1, j)$

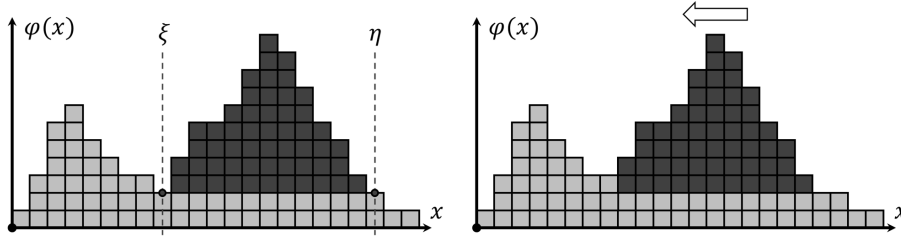


FIGURE 5.2 – The elementary tamping algorithm on a voxel function : the red voxels are slid one step to the left.

Once this definition is set, it is important to check that the new function Γ that we get after the elementary tamping still verifies Property (5.12). This is crucial because we want the function Γ to still define a function φ_Γ .

Lemma 2.1. *The elementary tamping preserves Property given by (5.12).*

Démonstration. To understand what happens during one step of the algorithm, we consider the hypograph of the function as being a collection of vertical strips of width 1 but of different height.

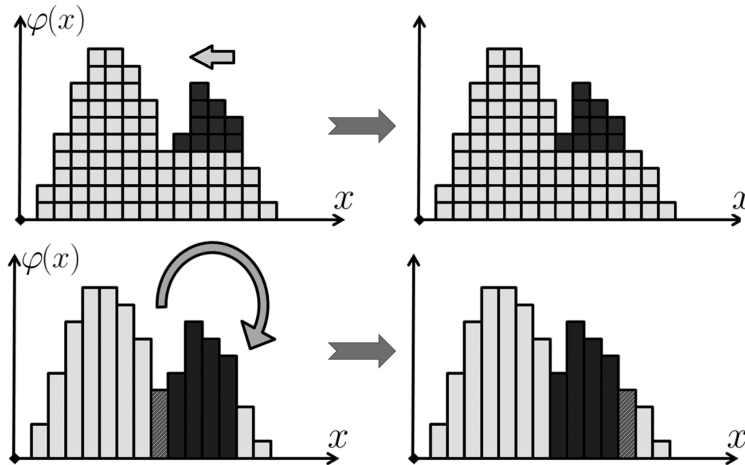


FIGURE 5.3 – The equivalence of the two algorithms. Top : Lebesguean point of view on the tamping. Bottom : Riemannian point of view on the tamping.

A step of the algorithm can be reformulated as the following (see Figure 5.3) :

1. The strip at ξ is removed. There is now a hollow (by hollow, we mean a missing strip) at ξ .
2. All the strips between $\xi + 1$ and η are translated one step to the left. Therefore, the hollow is now between η and $\eta + 1$.
3. The hollow is filled up with the strip that we removed at Step 1.

We can see that both algorithms are equivalent but with this version it is clear that Property (5.12) is preserved during the tamping process. □

We have now two equivalent algorithms to define the elementary step of the tamping algorithm. We call the first point of view the *Lebesguean* point of view and the second one is called the *Riemannian* point of view.

2.1.3 The tamping algorithm

Definition 2.2. Once the elementary tamping process is defined, the *tamping process*, also called *rearrangement by tamping*, is defined by the following algorithm :

1. Define the sets :

$$\underline{M} := \{x \in \mathbb{N} : \varphi(x) < \varphi(x+1)\}, \quad (5.16)$$

$$\underline{N} := \left\{x \in \mathbb{N} : \exists t \in \mathbb{N}, t < x, \forall s \in \left]t + \frac{1}{2}, x\right[, \varphi(t) > \varphi(s)\right\}. \quad (5.17)$$

2. If the set $\underline{N} \cap \underline{M}$ is empty, the algorithm ends.
3. Otherwise, choose a pivot $\xi \in \underline{N} \cap \underline{M}$.
4. Apply the elementary tamping associated to this pivot ξ .
5. Restart the algorithm at Step 1.

The result of this algorithm is illustrated at Figure 5.4. The set $\underline{N} \cap \underline{M}$ must be understood as the set of all the strict local minima of the function φ . Its definition comes from the fact that the classical definition for local minima makes no sense for piece-wise constant functions although this is the kind of concept we need. Similarly is defined the set $\overline{N} \cap \overline{M}$ of the strict local maxima for φ with

$$\overline{M} := \{x \in \mathbb{N}^* : \varphi(x-1) < \varphi(x)\}, \quad (5.18)$$

$$\overline{N} := \left\{x \in \mathbb{N}^* : \exists t \in \mathbb{N}, t > x, \forall s \in \left]x, t - \frac{1}{2}\right[, \varphi(t) < \varphi(s)\right\}. \quad (5.19)$$

Lemma 2.2. *This algorithm converges in a finite number of iterations.*

Démonstration. The non-negative integer $N_\Gamma := \sum_{i,j} i \Gamma(i,j)$ is decreasing at every iteration. □

Given a boolean function Γ and $\varphi_\Gamma \in \mathcal{E}_n(\mathbb{R})$, one notes Γ^\natural the boolean function given by the previous algorithm when initialized with the function Γ . This function is well defined because of the following uniqueness property.

Lemma 2.3. *The boolean function Γ^\natural does not depend on the choice of ξ at every iteration of the algorithm.*

Démonstration. This fact is a consequence of Proposition 2.1 that we prove later. □

The tamped-function, also called the rearrangement by tamping of φ_Γ , that we note $(\varphi_\Gamma)^\natural$, is defined by the natural definition

$$(\varphi_\Gamma)^\natural := \varphi_{\Gamma^\natural}. \quad (5.20)$$

This definition naturally extends to the functions of the set of the dilatations of the voxel functions

$$\widetilde{\mathcal{E}}_n(\mathbb{R}_+) := \left\{ \lambda \cdot \varphi \left(\frac{\cdot}{\mu} \right) : \varphi \in \mathcal{E}_n(\mathbb{R}_+), \lambda, \mu > 0 \right\}, \quad (5.21)$$

which is the set of the voxel functions made with voxels of size $\lambda \times \mu$. We set the definition

$$\left[\lambda \cdot \varphi \left(\frac{\cdot}{\mu} \right) \right]^\natural := \lambda \cdot \varphi^\natural \left(\frac{\cdot}{\mu} \right). \quad (5.22)$$

We also define

$$\widetilde{\mathcal{E}}(\mathbb{R}_+) := \bigcup_{n=1}^{\infty} \widetilde{\mathcal{E}}_n(\mathbb{R}_+).$$

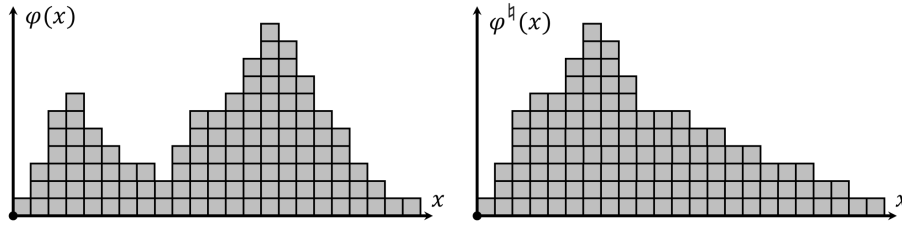


FIGURE 5.4 – The 1D tamping algorithm on a voxel function.

2.2 Definition of the tamping in $\mathfrak{M}_+(\mathbb{R}_+)$.

2.2.1 Problems of point-wise convergence for the tamping

The tamping is now defined on voxel functions. It is a well-known result that piece-wise constant functions are a dense subset of the set of measurable functions $\mathfrak{M}(\mathbb{R}_+)$ [55] for the point-wise convergence almost everywhere and it is always possible to approximate a piece-wise constant function by voxel functions of $\mathcal{E}_n(\mathbb{R})$. The main objective of this part is to define the tamping for any function in $\mathfrak{M}_+(\mathbb{R}_+)$ and the natural way to do it is to pass to the limit in which the size of the voxels tends to zero (and their number tends to $+\infty$). Nevertheless, such an approach cannot work because the tamping has unfortunately very bad continuity properties for the point-wise convergence. For instance if we consider the sequence

$$\varphi_n := \mathbb{1}_{[0,1]} + \mathbb{1}_{[n, n+1]},$$

then we have φ_n converging almost everywhere towards $\varphi := \mathbb{1}_{[0,1]}$ (which is a fixed point for the tamping) whereas φ_n^h is converging almost everywhere towards $\mathbb{1}_{[0,2]} \neq \varphi$.

A good way to pass through the difficulties evoked just before is to define the tamping by another approach but that coincides with the previous one on the sets $\mathcal{E}_n(\mathbb{R})$.

2.2.2 Definition of the hollows

To understand well the purpose of what follows, one must keep in mind that the main idea of the tamping is to “*move the cubes to the left in order to fill all the hollows*”. This key idea is the basis for the generalization of the tamping for all the functions in $\mathfrak{M}_+(\mathbb{R}_+)$. The idea behind the notion of *hollows* is to seize the lack of convexity of a given set (see Figure 5.5, Left). A natural definition for the *hollows* of a set A would be

$$\mathcal{H}(A) := (\text{conv } A) \setminus A. \tag{5.23}$$

Nevertheless, such a definition is not well adapted for the manipulation of sets in a context of measure theory and integration with respect to the Lebesgue measure. For instance the convex hull can be completely changed if we add only one point to the set A . The objective here is to define a notion of *hollows* that is the analog of the natural idea sketched by (5.23) but defined in such a way that $\mathcal{H}(A)$ remains unchanged if we modify the set A by a set of measure zero.

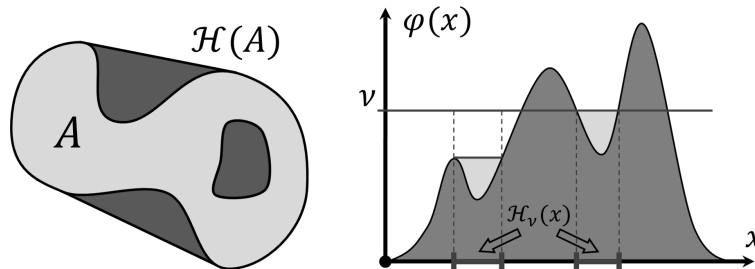


FIGURE 5.5 – Illustration of the hollows.

Left : the hollows of a set A . Right : the hollows at level ν of a function φ .

Definition 2.3. Let A be a measurable subset of \mathbb{R}_+ (for the Lebesgue measure). We define the *essential*

Section 2. Definition of the rearrangement by tamping

convex hull of the set A by

$$\text{conv ess } A := \bigcap_{\text{meas}(A\Delta B)=0} \text{conv } B,$$

where Δ is the symmetrical difference, $A\Delta B := (A \cup B) \setminus (A \cap B)$.

This definition allows us to define properly the notion of *hollow* that we adumbrated just before.

Definition 2.4. Let A be a measurable subset of \mathbb{R} . The *hollows* of the set A is the set defined by

$$\mathcal{H}(A) := \bigcup_{\text{meas}(A\Delta B)=0} (\text{conv } B \cap \text{conv ess } A) \setminus B.$$

This definition naturally extends to functions via the super-level sets (see Figure 5.5, Right).

Definition 2.5. Let $\varphi \in \mathfrak{M}_+(\mathbb{R}_+)$. The *hollows* of φ at level ν are the set

$$\mathcal{H}_\nu(\varphi) := \bigcup_{\lambda < \nu} \mathcal{H}(\{\varphi \geq \lambda\}).$$

2.2.3 A definition of the tamping

Here after is an equivalent description of the rearrangement by tamping. The main interest of this proposition is that the result above can be used to define the tamping for any function in $\mathfrak{M}_+(\mathbb{R}_+)$.

Proposition 2.1. Let $\varphi \in \widetilde{\mathcal{E}}_n(\mathbb{R})$. Let us define for all $\nu > 0$

$$x_\nu(\varphi) := \inf \text{ess } \{\varphi \geq \nu\} \tag{5.24}$$

and

$$y_\nu(\varphi) := x_\nu(\varphi) - \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi)] \right). \tag{5.25}$$

With these definitions we can describe the superlevel sets of the function φ^\natural as

$$\{\varphi^\natural \geq \nu\} = \left[y_\nu(\varphi), y_\nu(\varphi) + \text{meas}\{\varphi \geq \nu\} \right].$$

In other words,

$$y_\nu(\varphi) = x_\nu(\varphi^\natural).$$

Démonstration. It is enough to consider the case $\varphi \in \mathcal{E}_n(\mathbb{R})$. The fact that these superlevel sets are segments whose length is equal to $\text{meas}\{\varphi \geq \nu\}$ follows from the construction of the rearrangement by tamping.

The value of the infimum of these segments is determined using the algorithm. We can see that the notion of hollows for a function in \mathcal{E}_n exactly coincides with the number of times a given cube is slid one step on the left during the tamping algorithm. More precisely, such a cube in place (i, j) is slid on a distance which is the integer part of the quantity $\mathcal{H}_j(\varphi) \cap [0, i]$. Indeed this quantity is the size of the hollows at level j that we fill by moving cubes and a cube whose abscissa index is i is not concerned by the filling of the hollows that are positioned at indices bigger than i . Similarly, what happens at layers bigger than j does not interfere on what happens for the cube at (i, j) but only the levels below.

The conclusion of the demonstration then comes from the fact that, regarding the definition of $x_\nu(\varphi)$, we have

$$\mathcal{H}_\nu(\varphi) \cap [0, x_\nu(\varphi)] = \mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi)].$$

□

In the proposition above, the description of the super-level sets that we obtain remains well-defined for any function in $\mathfrak{M}_+(\mathbb{R}_+)$. From now, we can use the above proposition as the *definition of the tamping*.

Definition 2.6. Let $\varphi \in \mathfrak{M}_+(\mathbb{R}_+)$. We define the *tamping* of φ , noted φ^\natural , as being the function of $\mathfrak{M}_+(\mathbb{R}_+)$ which super-level sets are

$$\{\varphi^\natural \geq \nu\} = \left[y_\nu(\varphi), y_\nu(\varphi) + \text{meas}\{\varphi \geq \nu\} \right],$$

where y_ν is defined by (5.24) and (5.25).

To ensure that this defines a function, we have to check that $\mu \geq \nu$ implies

$$\left[y_\mu(\varphi), y_\mu(\varphi) + \text{meas}\{\varphi \geq \mu\} \right] \subseteq \left[y_\nu(\varphi), y_\nu(\varphi) + \text{meas}\{\varphi \geq \nu\} \right]. \quad (5.26)$$

Proposition 2.2. *The inclusion (5.26) is verified for any $\varphi \in \mathfrak{M}_+(\mathbb{R}_+)$.*

Démonstration. Let $\mu \geq \nu$. On the one hand we have

$$\begin{aligned} y_\mu(\varphi) - y_\nu(\varphi) &= \text{meas}\left([x_\nu(\varphi), x_\mu(\varphi)]\right) - \text{meas}\left(\mathcal{H}_\infty(\varphi) \cap [x_\nu(\varphi), x_\mu(\varphi)]\right) \\ &= \text{meas}\left(\mathcal{H}_\infty(\varphi)^c \cap [x_\nu(\varphi), x_\mu(\varphi)]\right) \geq 0. \end{aligned} \quad (5.27)$$

On the other hand,

$$\begin{aligned} \text{meas}\left(\mathcal{H}_\infty(\varphi)^c \cap [x_\nu(\varphi), x_\mu(\varphi)]\right) &\leq \text{meas}\left(\{\nu < \varphi\} \cap [x_\nu(\varphi), x_\mu(\varphi)]\right) \\ &= \text{meas}\left(\{\nu < \varphi \leq \mu\} \cap [x_\nu(\varphi), x_\mu(\varphi)]\right) \leq \text{meas}\left(\{\nu < \varphi \leq \mu\}\right). \end{aligned} \quad (5.28)$$

Combining (5.27) and (5.28) we get

$$y_\mu(\varphi) + \text{meas}\{\varphi \geq \mu\} \leq y_\nu(\varphi) + \text{meas}\{\varphi \geq \nu\}. \quad (5.29)$$

Equations (5.27) and (5.29) give (5.26). \square

Now that the tamping is well-defined, we can state the unimodality property, the Schwarz rearrangement inequality and the preservation of the Dirichlet boundary condition which are three of the four main properties that we required in Section 1.4.1.

Proposition 2.3 (Unimodality). *Let $\varphi \in \mathfrak{M}_+(\mathbb{R}_+)$. Then there exist $s \in \mathbb{R}_+$ such that φ^\natural is non-decreasing on $[0, s]$ and non-increasing on $[s, +\infty)$.*

Démonstration. This follows from the fact that the super-level sets of φ^\natural are segments. \square

Proposition 2.4 (Schwarz rearrangement inequality for the tamping). *Let $\varphi \in \mathfrak{M}_+(\mathbb{R}_+)$,*

$$\forall x \in \mathbb{R}_+, \quad \text{meas}\left([0, x] \cap \{\varphi \geq \nu\}\right) \leq \text{meas}\left([0, x] \cap \{\varphi^\natural \geq \nu\}\right).$$

Démonstration. This is a consequence of the fact that this rearrangement verifies $\forall \nu > 0$, $x_\nu(\varphi^\natural) \leq x_\nu(\varphi)$ and $\{\varphi^\natural \geq \nu\}$ is a segment. \square

Proposition 2.5 (Preservation of the Dirichlet boundary condition). *Let $\varphi \in \mathcal{C}^0(\mathbb{R}_+, \mathbb{R}_+)$ be absolutely continuous. Then, $\varphi(0) = \varphi^\natural(0)$.*

This fact is proved in section 4.

2.3 Best non-decreasing upper bound

In the first section devoted to the presentation of the problem, we emphasized the links that we want between the rearrangement by tamping and the Schwarz non-increasing rearrangement : the Schwarz inequality (Property 1.2) and the Pólya – Szegő inequality (Theorem 1.4). In this section we propose another equivalent way to define the tamping that directly involves the Schwarz non-increasing rearrangement (called here after the double Schwarz formula). Although we will not work with this second definition in the rest of this paper, it remains per se interesting because it tells more about the links between the tamping and the Schwarz rearrangement.

Definition 2.7. Let $\varphi \in \mathfrak{M}_+(\mathbb{R}_+)$. We define the *best non-decreasing upper bound* of φ as being

$$\varphi^\dagger(x) := \sup \text{ess} \left(\varphi \cdot \mathbb{1}_{[0, x]} \right). \quad (5.30)$$

Proposition 2.6. *The function φ^\dagger is the only function of*

$$\left\{ \psi \in \mathfrak{M}_+(\mathbb{R}_+) : \begin{array}{l} \varphi \leq \psi \text{ almost everywhere,} \\ \psi \text{ is non-decreasing,} \end{array} \right\} \quad (5.31)$$

that is minimal for the standard comparison of functions.

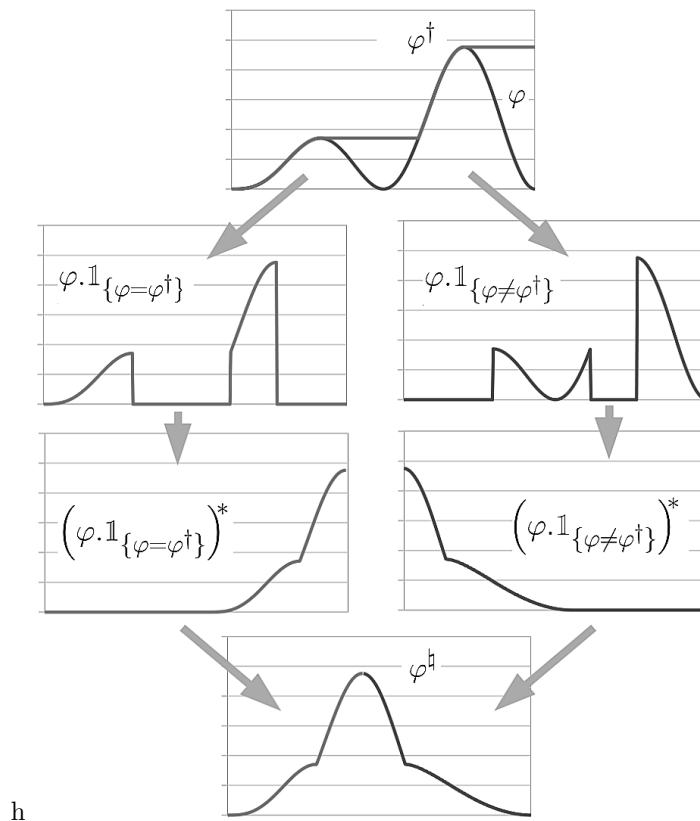


FIGURE 5.6 – Tamping process and Schwarz symmetrization. Tamping of the function $x \mapsto x \cdot \sin^2(2\pi x) \cdot \mathbb{1}_{[0,1]}(x)$ computed using the double Schwarz formula (5.34).

This proposition is proved in Section 4.

We define

$$s(\varphi) := \lim_{\nu \rightarrow (\sup \text{ess } \varphi)^-} \sup \text{ess } \{\varphi > \nu\}. \quad (5.32)$$

The value $s(\varphi)$ can be understood, roughly speaking, as being the supremum of the “argmax” of the function φ . It is not possible to define $s(\varphi)$ directly this way because φ is only measurable and defined almost everywhere. We now define

$$\sigma(\varphi) := s(\varphi) - \text{meas} \left(\{\varphi \neq \varphi^\dagger\} \cap [0, s(\varphi)] \right). \quad (5.33)$$

The purpose of this definition is the fact that we actually have $\sigma(\varphi) = s(\varphi^\natural)$ (this equality is given by the proof of the following lemma).

Lemma 2.4 (Double Schwarz formula). *Let $\varphi \in \mathfrak{M}_+(\mathbb{R}_+)$. Define $s(\varphi)$ and $\sigma(\varphi)$ respectively by (5.32) and (5.33). Suppose that $s(\varphi) < +\infty$. Then,*

$$\varphi^\natural(x) = \begin{cases} \left(\varphi \cdot \mathbb{1}_{\{\varphi = \varphi^\dagger\}} \right)^*(\sigma(\varphi) - x) & \text{if } x \leq \sigma(\varphi) \\ \left(\varphi \cdot \mathbb{1}_{\{\varphi \neq \varphi^\dagger\}} \right)^*(x - \sigma(\varphi)) & \text{otherwise,} \end{cases} \quad (5.34)$$

where the superscript $*$ refers to the Schwarz non-increasing rearrangement.

The proof of formula (5.34) is given in Section 4. If we note $\widehat{\varphi}$ the expression on the right hand side of (5.34), the main steps of the proof are the following.

1. Establish that $\{\varphi \neq \varphi^\dagger\} \cap [0, s(\varphi)] = \mathcal{H}_\infty(\varphi) \cap [0, s(\varphi)]$.
2. Conclude that $x_\nu(\varphi^\natural) = x_\nu(\widehat{\varphi})$.
3. Establish that $\text{meas}\{\varphi^\natural \geq \nu\} = \text{meas}\{\widehat{\varphi} \geq \nu\}$.

4. Conclude that $\{\varphi^\sharp \geq \nu\} = \{\widehat{\varphi} \geq \nu\}$ and then $\varphi^\sharp = \widehat{\varphi}$.

One interest of this formula is the fact that the notion of *best non-decreasing upper bound* also manages to seize the notion of *hollows* that we introduced at the previous section. This aspect is hidden in the formula but is more visible in the proof. The main idea behind this formula is the first point of the proof where we precise what we can obtain about the hollows of φ when we know φ^\dagger . Yet, the main aspect of Formula (5.34) that makes it interesting is the fact that the Schwarz non-increasing rearrangement appears in the computation of the tamping. With such a formula, we can reformulate and interpret the tamping as a double Schwarz non-increasing rearrangement. This idea is illustrated at Figure 5.6 where the tamping of the function $x \mapsto x \cdot \sin^2(2\pi x) \cdot \mathbb{1}_{[0,1]}(x)$ is proceeded using the Double Schwarz formula (5.34) (this is the process that we used to obtain the plots of Figure 5.1 in the section “Presentation of the problem”).

3 Main results about the tamping

3.1 Functional analysis

To deal with rearrangements, we provide in this section some tools of functional analysis for the manipulations of the super-level sets of functions in $L^p_+(\mathbb{R}_+)$. The aim is to obtain results about the links between convergence of the super-level sets and convergence in L^p .

3.1.1 Results on convergence of the super-level sets

Lemma 3.1. *Let Ω be a domain of \mathbb{R}^d and let $p \in [1, +\infty)$. Let φ, ψ in $L^p_+(\Omega)$. We have*

$$\|\varphi - \psi\|_{L^p}^p \leq p \int_0^\infty \nu^{p-1} \text{meas}(\{\varphi \geq \nu\} \Delta \{\psi \geq \nu\}) \, d\nu, \quad (5.35)$$

where Δ is the symmetrical difference between two sets : $A \Delta B := (A \cup B) \setminus (A \cap B)$. Equality holds if and only if $p = 1$ or if the support of the two functions are disjointed.

Note that the right-hand side of Inequality (5.35) is finite because we have

$$\begin{aligned} & p \int_0^\infty \nu^{p-1} \text{meas}(\{\varphi \geq \nu\} \Delta \{\psi \geq \nu\}) \, d\nu \\ & \leq p \int_0^\infty \nu^{p-1} (\text{meas}\{\varphi \geq \nu\} + \text{meas}\{\psi \geq \nu\}) \, d\nu \\ & = \|\varphi\|_{L^p}^p + \|\psi\|_{L^p}^p. \end{aligned}$$

The proof of Inequality (5.35) is given in Section 4 and it relies on an idea from [23].

Lemma 3.2. *Let Ω be a domain of \mathbb{R}^d and let $p \in [1, +\infty)$. Let φ and $(\varphi_n)_{n \in \mathbb{N}}$ in $L^p_+(\Omega)$. Then (φ_n) converges towards φ in L^p if and only if*

$$p \int_0^\infty \nu^{p-1} \text{meas}(\{\varphi \geq \nu\} \Delta \{\varphi_n \geq \nu\}) \, d\nu \xrightarrow{n \rightarrow \infty} 0. \quad (5.36)$$

The fact that the convergence in L^p follows from (5.36) is a consequence of Lemma 3.1. The converse is proved in Section 4.

Corollary 3.1. *Let Ω be a domain of \mathbb{R}^d and let $p \in [1, +\infty)$. Let $\varphi \in L^p_+(\Omega)$ and let $\varphi_n \rightarrow \varphi$ in L^p . Then, up to an omitted extraction of this sequence, we have for almost every $\nu \in \mathbb{R}_+$*

$$\text{meas}(\{\varphi \geq \nu\} \Delta \{\varphi_n \geq \nu\}) \xrightarrow{n \rightarrow \infty} 0. \quad (5.37)$$

Démonstration. Let $a > 0$. We have

$$p \int_a^\infty \nu^{p-1} \text{meas}(\{\varphi \geq \nu\} \Delta \{\varphi_n \geq \nu\}) \, d\nu \quad (5.38)$$

$$\geq p a^{p-1} \int_a^\infty \text{meas}(\{\varphi \geq \nu\} \Delta \{\varphi_n \geq \nu\}) \, d\nu, \quad (5.39)$$

and then $\nu \mapsto \text{meas}(\{\varphi \geq \nu\} \Delta \{\varphi_n \geq \nu\})$ belongs to $L^1(a, +\infty)$. But since (5.38) vanishes when $n \rightarrow \infty$ by Lemma 3.2 then so does (5.39). We conclude using a standard result [14] which states that the convergence in L^1 implies the convergence almost everywhere up to an extraction. Considering a sequence $a_k \rightarrow 0^+$ and successive extractions completes the proof. \square

3.1.2 Results on convergence in L^p

Lemma 3.3 (Weak convergence in L^p). *Let Ω be a domain of \mathbb{R}^d and let $p \in (1, +\infty)$. Let $\varphi \in L^p_+(\Omega)$ and let $\varphi_n \in L^p_+(\Omega)$ such that $\|\varphi_n\|_{L^p}$ is a bounded sequence and*

$$\text{meas} \left(\{\varphi_n \geq \nu\} \Delta \{\varphi \geq \nu\} \right) \xrightarrow{n \rightarrow +\infty} 0,$$

for almost every $\nu > 0$. Then (φ_n) weakly converges towards φ in L^p .

The proof of this lemma is done in Section 4. The two main ingredients of the proof are the inequality given at Lemma 3.1 and the Lebesgue dominated convergence theorem [55]. For the case $p = 1$, a counter-example is given by the sequence $\varphi_n := \frac{1}{n} \mathbb{1}_{[0,n]} \in L^1$ which satisfies all the hypothesis but which is not weakly converging in L^1 (integrate φ_n against a constant function for instance). Nevertheless, the case $p = 1$ turns out to be true for the strong version of this lemma.

Lemma 3.4 (Strong convergence in L^p). *Let Ω be a domain of \mathbb{R}^d and let $p \in [1, +\infty[$. Let $\varphi \in L^p_+(\Omega)$ and let $\varphi_n \in L^p_+(\Omega)$ such that*

$$\|\varphi_n\|_{L^p} \longrightarrow \|\varphi\|_{L^p} \tag{5.40}$$

and

$$\text{meas} \left(\{\varphi_n \geq \nu\} \Delta \{\varphi \geq \nu\} \right) \xrightarrow{n \rightarrow +\infty} 0, \tag{5.41}$$

for almost every $\nu > 0$. Then,

$$\|\varphi_n - \varphi\|_{L^p} \xrightarrow{n \rightarrow +\infty} 0.$$

By the Radon-Riesz lemma [14], a sequence (φ_n) weakly converging towards φ in L^p for $p \in (1, +\infty)$ is strongly converging if and only if the L^p norm of φ_n converges towards the L^p norm of φ . The proof in the case $p \neq 1$ therefore follows from Lemma 3.3 with the Radon-Riesz lemma. The case $p = 1$ is proved in Section 4. Note that Hypothesis (5.41) directly involves the super-level sets of the manipulated functions, while Hypothesis (5.40) is easy to verify using the layer-cake representation (5.3). This makes Lemma 3.5 be a good tool in the context of rearrangements.

This lemma is false in the case $p = +\infty$. Consider for instance the sequence $\varphi_n := \mathbb{1}_{[0,1+1/n]}$ as a counter-example.

3.2 Topological results in $L^p_+(\mathbb{R}_+)$.

3.2.1 Compactness result for the tamping in $L^p_+(\mathbb{R}_+)$

Before starting our result of convergence for the tamping, we provide some compactness results for the tamping in $L^p_+(\mathbb{R}_+)$.

Lemma 3.5 (Compactness of the super-level sets). *Let $p \in [1, +\infty)$ and let $(\varphi_n) \in L^p_+(\mathbb{R}_+)$ be a bounded sequence in L^p such that*

$$\exists \mu > 0, \quad \limsup_{n \rightarrow +\infty} x_\mu(\varphi_n) < +\infty. \tag{5.42}$$

Then there exists a function ψ such that, up to an omitted extraction and for almost every $\nu > 0$,

$$\text{meas} \left(\{\varphi_n^\sharp \geq \nu\} \Delta \{\psi \geq \nu\} \right) \xrightarrow{n \rightarrow +\infty} 0. \tag{5.43}$$

The proof of this lemma is provided in Section 4. Now, combining Lemma 3.3 and Lemma 3.5 we get the following corollary.

Corollary 3.2 (Weak compactness for the tamping). *Let $p \in (1, +\infty)$ and let $(\varphi_n) \in L^p_+(\mathbb{R}_+)$ be a bounded sequence such that*

$$\exists \mu > 0, \quad \limsup_{n \rightarrow +\infty} x_\nu(\varphi_n) < +\infty. \tag{5.44}$$

Then the sequence (φ_n^\sharp) is weakly compact in L^p .

Hypothesis (5.44) may look quite technical and little intuitive. Nevertheless, in the case of a sequence of functions (φ_n) whose support remains contained in a given compact K , this hypothesis is automatically verified if and only if (φ_n) does not converges to the null function. Since the case of the convergence towards the null function is easy to handle with, we can conclude that the following corollary holds.

Corollary 3.3 (Weak compactness for the tamping on compact support). *Let $p \in [1, +\infty)$ and let $(\varphi_n) \in L^p_+(\mathbb{R}_+)$ be a bounded sequence in L^p . We suppose that there exists $K \subseteq \mathbb{R}_+$ a compact such that*

$$\forall n \in \mathbb{N}, \quad \text{supp } \varphi_n \subseteq K. \tag{5.45}$$

Then the sequence (φ_n^\sharp) is weakly compact in L^p .

If we now go back to Lemma 3.5 and combine it with Lemma 3.4 and the fact that $\|\varphi_n\|_{L^p} = \|\varphi_n^\sharp\|_{L^p}$, we get the following result for strong compactness.

Corollary 3.4 (Strong compactness for the tamping). *Let $p \in [1, +\infty)$. Let $(\varphi_n) \in L^p_+(\mathbb{R}_+)$ such that φ_n converges in L^p towards a function φ . Then, up to an extraction, the sequence φ_n^\sharp converges strongly in L^p towards a function ψ that verifies*

$$\text{meas}\{\psi \geq \nu\} = \text{meas}\{\varphi \geq \nu\}. \tag{5.46}$$

Without more hypothesis on the sequence (φ_n) , we cannot obtain convergence of the sequence (φ_n^\sharp) towards the function φ^\sharp . We have for instance the counter-example (see Figure 5.7),

$$\begin{cases} \varphi(x) := \mathbb{1}_{[1,2]}(x), \\ \varphi_n(x) := \mathbb{1}_{[1,2]}(x) + \mathbb{1}_{[0, \frac{1}{n}]}(x). \end{cases}$$

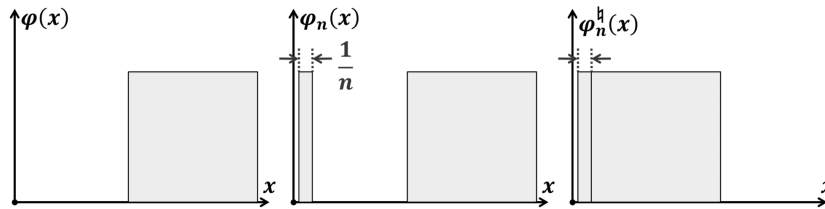


FIGURE 5.7 – Illustration of the first counter-example for the convergence.

The sequence (φ_n) converges towards φ in L^p whereas this is not the case for (φ_n^\sharp) which converges towards a different function in $L^p_+(\mathbb{R}_+)$.

Another interesting counter-example is the following (see Figure 5.8),

$$\begin{cases} \varphi(x) := \mathbb{1}_{[0,2]}(x) + 2 \cdot \mathbb{1}_{[2,3]}(x), \\ \varphi_n(x) := \mathbb{1}_{[0,1]}(x) + \left(1 - \frac{1}{n}\right) \mathbb{1}_{[1,2]}(x) + 2 \cdot \mathbb{1}_{[2,3]}(x). \end{cases}$$

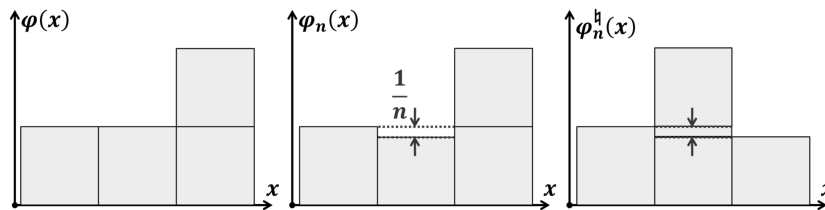


FIGURE 5.8 – Illustration of the second counter-example for the convergence.

Once again we choose a function φ that is already tamped and we exhibit a sequence φ_n that is converging towards the function φ but such that the sequence of tamped functions φ_n^\sharp is not converging towards the tamped function φ^\sharp ($= \varphi$ here).

The main common point between these two counter-examples that may seem quite different at first sight is that in both cases the approximating functions are in such a way that, roughly speaking, “*there is a new hollow that appears in front of the big bump*” and this hollow never vanishes as n grows. Although rough this observation may seem, this is the key idea to obtain a sufficient condition for convergence towards φ^\sharp .

3.2.2 A convergence result for the tamping

Theorem 3.1 (Convergence result). *Let $p \in [1, +\infty)$ and let $(\varphi_n)_{n \in \mathbb{N}}$ in $L^p_+(\mathbb{R}_+)$ converging towards φ . We suppose that we have the local hollows convergence condition, that is*

$$\forall a > 0, \quad \text{meas} \left((\mathcal{H}_\infty(\varphi_n) \Delta \mathcal{H}_\infty(\varphi)) \cap [0, a] \right) \xrightarrow[n \rightarrow +\infty]{} 0. \quad (5.47)$$

Then, up to an extraction we have

$$\|\varphi_n^\natural - \varphi^\natural\|_{L^p} \xrightarrow[n \rightarrow +\infty]{} 0. \quad (5.48)$$

Moreover, if for almost every $\nu \geq 0$

$$\text{meas} \left(\{\varphi_n \geq \nu\} \Delta \{\varphi \geq \nu\} \right) \xrightarrow[n \rightarrow +\infty]{} 0, \quad (5.49)$$

then it is also the case for (φ_n^\natural) and the full sequence converges in L^p towards φ^\natural .

Since it is possible to obtain Condition (5.49) from Corollary 3.1, it is enough to prove the last assertion. The proof is provided in Section 4. This condition (5.47) is actually not very surprising if we consider that the tamping is defined through the hollows of the function. If we want to converge towards the tamping of φ then it is natural to impose that the hollows are matching asymptotically.

We can now use this convergence theorem to obtain a convergence result on voxel functions.

Corollary 3.5. *Let $p \in [1, +\infty)$ and let $\varphi \in L^p_+(\mathbb{R}_+)$. There exist $\varphi_n \in \widetilde{\mathcal{E}}_n(\mathbb{R}_+)$ such that*

$$\|\varphi_n - \varphi\|_{L^p} \xrightarrow[n \rightarrow \infty]{} 0, \quad \text{and} \quad \|\varphi_n^\natural - \varphi^\natural\|_{L^p} \xrightarrow[n \rightarrow \infty]{} 0.$$

This sequence also verifies the convergence of the superlevel sets

$$\text{meas} \left(\{\varphi_n \geq \nu\} \Delta \{\varphi \geq \nu\} \right) \xrightarrow[n \rightarrow +\infty]{} 0, \quad (5.50)$$

$$\text{meas} \left(\{\varphi_n^\natural \geq \nu\} \Delta \{\varphi^\natural \geq \nu\} \right) \xrightarrow[n \rightarrow +\infty]{} 0, \quad (5.51)$$

for almost every $\nu > 0$.

Démonstration. We define the voxel function $\varphi_n \in \widetilde{\mathcal{E}}_n(\mathbb{R})$, following the construction (and the notations) of Section 2.1.1. We consider cubes $a(i, j)$ of vanishing size $\lambda_n \times \mu_n$. Since by definition of $\widetilde{\mathcal{E}}_n(\mathbb{R})$, the number of cubes is at most $n \times n$, to obtain the convergence in L^p we have to impose the following decay rates conditions,

$$n \cdot \lambda_n \xrightarrow[n \rightarrow \infty]{} +\infty \quad \text{and} \quad n \cdot \mu_n \xrightarrow[n \rightarrow \infty]{} +\infty. \quad (5.52)$$

Then we define the boolean function Γ by

$$\Gamma(i, j) := \begin{cases} 0 & \text{if } \text{meas}(a(i, j) \cap \text{hypo}(\varphi)) = 0, \\ 1 & \text{otherwise,} \end{cases} \quad (5.53)$$

where the *hypograph* of a function $\varphi \in \mathfrak{M}_+(\mathbb{R}_+)$ is given by

$$\text{hypo}(\varphi) := \{(x, y) \in \mathbb{R}_+ \times \mathbb{R} : \varphi(x) \leq y\}.$$

For every $a > 0$, by construction we have that the sequence $\{\varphi_n \geq \nu\} \cap [0, a]$ is non-increasing (for the inclusion) and $\mathcal{H}(\{\varphi_n \geq \nu\}) \cap [0, a]$ is non-decreasing when n is large enough (depending on ν and a). We also have $x_\nu(\varphi_n)$ converging towards $x_\nu(\varphi)$. The hypothesis of Theorem 3.1, including (5.49), are then verified and the conclusion follows. \square

3.3 Pólya–Szegő inequality for the tamping

We show in this section that it is possible to decrease the L^p norm of the derivative, also called the $W^{1,p}$ half-norm, using the rearrangement by tamping. In other words, we prove that the rearrangement by tamping satisfies a Pólya–Szegő inequality.

3.3.1 Piece-wise linear approximation

Since the tamping is constructed with functions in $\mathcal{E}_n(\mathbb{R})$, whose derivatives are not in L^p , these half-norms are computed by approximation. For a function $\varphi_n \in \mathcal{E}_n(\mathbb{R})$, we define its piece-wise linear and continuous approximation by :

$$(\Lambda\varphi_n)(x) := \sum_{i=0}^n \left[\varphi(i+1) \cdot (x-i) + \varphi(i) \cdot ((i+1)-x) \right] \mathbb{1}_{[i, i+1]}(x). \quad (5.54)$$

The operator Λ is a bijection between $\mathcal{E}_n(\mathbb{R})$ and $\Lambda\mathcal{E}_n(\mathbb{R})$. The operator Λ extends to $\widetilde{\mathcal{E}}_n(\mathbb{R})$ with a natural definition. The functions $\varphi_n \in \mathcal{E}_n(\mathbb{R})$ and $\Lambda\varphi_n$ coincide on the set \mathbb{Z} . When n goes to ∞ , the sequences (φ_n) and $(\Lambda\varphi_n)$ are such that if one converges then the other converges towards the same limit in L^p . Moreover, if we choose (φ_n) approximating a given function $\varphi \in W^{1,p}$, we can expect, using standard theory on piece-wise linear continuous approximation [23], that the sequence $(\Lambda\varphi_n)$ is converging $W^{1,p}$ towards φ .

3.3.2 Pólya–Szegő inequality for the tamping

We use the notation ∇ for the derivative of the functions, although we work in dimension 1, in order to make the formulas easier to read.

Theorem 3.2. *Let $\varphi \in W_+^{1,p}(\mathbb{R}_+)$. We have*

$$\int_{\mathcal{H}_\infty(\varphi)} |\nabla\varphi|^p \leq \int_{\mathbb{R}_+} |\nabla\varphi|^p - |\nabla\varphi^\natural|^p. \quad (5.55)$$

Or equivalently,

$$\int_{\mathbb{R}_+} |\nabla\varphi^\natural|^p \leq \int_{\mathbb{R}_+ \setminus \mathcal{H}_\infty(\varphi)} |\nabla\varphi|^p. \quad (5.56)$$

This theorem emphasizes the fact that we improve the L^p norm of the derivative by “filling the hollows” because the error term only involves the derivative of φ inside the hollows of φ . The proof of this theorem is provided in Section 4. It relies on the Riemannian point of view on the tamping on voxel functions φ_n that we defined in the proof of Lemma 2.1. Using this point of view, we can first estimate how does vary the L^p norm of the derivative of $\Lambda\varphi_n$ during one step of the elementary tamping algorithm. Then, we iterate the estimate given by one step of the tamping algorithm to obtain, for a well-chosen sequence $(\varphi_n) \in \widetilde{\mathcal{E}}_n(\mathbb{R})$, the inequality

$$\int_{\mathcal{H}_\infty(\varphi_n)} |\nabla\Lambda\varphi_n|^p \leq \int_{\mathbb{R}_+} |\nabla\Lambda\varphi_n|^p - |\nabla\Lambda\varphi_n^\natural|^p + o_{n \rightarrow \infty}(1), \quad (5.57)$$

and we can conclude by passing to the limit $n \rightarrow \infty$.

Corollary 3.6 (Pólya–Szegő inequality for the tamping). *Let $\varphi \in W_+^{1,p}(\mathbb{R}_+)$. We have*

$$\int_{\mathbb{R}_+} |\nabla\varphi^\natural|^p \leq \int_{\mathbb{R}_+} |\nabla\varphi|^p, \quad (5.58)$$

with equality if and only if $\varphi = \varphi^\natural$ almost everywhere.

This corollary is obtained directly from Theorem 3.2. The estimate (5.56) also gives the equality case above using the fact that $\varphi = \varphi^\natural$ is equivalent to $\mathcal{H}_\infty(\varphi) = \emptyset$ and the fact that $|\nabla\varphi|$ cannot worth identically 0 inside the hollows (in dimension 1 a $W^{1,p}$ function is Hölder continuous).

3.4 About a Riesz inequality for the tamping

Now that we have a Pólya–Szegő inequality for the tamping, one natural thing to expect is that we get a Riesz rearrangement inequality for the tamping similar to Theorem 1.5 because we can expect that the tamping also decreases the $W^{s,p}$ half-norms. Indeed, as we evoked in Section 1, this inequality gives that a rearrangement does not increase the H^s norms. We explain why this inequality is actually false in the case of the rearrangement by tamping and we discuss in this section some aspects about the tamping and H^s norms.

3.4.1 Riesz rearrangement inequality : A counter-example

Theorem 1.5 is false in the case of the tamping and we provide hereafter a counter-example. Such a result is not really surprising because we already know that the tamping does not verify the Hardy-Littlewood inequality (the cases presented at Figures 5.7 and 5.8 work as counter-examples since this inequality implies the continuity in L^2). As we explained at Subsection 1.3, the Riesz rearrangement inequality can be interpreted as a generalization of the Hardy-Littlewood inequality.

Here is a counter-example for the Riesz rearrangement inequality in the case of the tamping. Let $0 < a < b < c < d < e$ and let $t \geq 0$. If we make the supposition that $d \leq t \leq e - b$, then we have

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathbb{1}_{[0,e]}(x) \cdot \left(\mathbb{1}_{[a,b]} + \mathbb{1}_{[c,d]} \right)(y) \cdot \mathbb{1}_{[-t,t]}(x - y) \, dx \, dy \quad (5.59)$$

$$> \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathbb{1}_{[0,e]}(x) \cdot \mathbb{1}_{[a,b-c+d]}(y) \cdot \mathbb{1}_{[-t,t]}(x - y) \, dx \, dy. \quad (5.60)$$

This is a counter-example for the Riesz rearrangement inequality for the tamping because we have

$$\left(\mathbb{1}_{[a,b]} + \mathbb{1}_{[c,d]} \right)^\natural = \mathbb{1}_{[a,b-c+d]}. \quad (5.61)$$

The computation of the above inequality is provided in Section 4.

3.4.2 Decreasing the H^s half-norm : A counter-example

Working again on this counter-example, we can prove that the tamping sometimes fails to decrease the H^s half-norms defined by

$$|\varphi|_{H^s}^2 := \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{1+2s}} \, dx \, dy. \quad (5.62)$$

If we define $\psi := \mathbb{1}_{[a,b]} + \mathbb{1}_{[c,d]} + \mathbb{1}_{[0,e]}$, then a direct computation gives that, for $s \in]0, \frac{1}{2}[$,

$$\begin{aligned} \|\psi\|_{H^s}^2 &= \frac{1}{s \left(\frac{1}{2} - s \right)} \left[b^{1-2s} - a^{1-2s} + d^{1-2s} - c^{1-2s} + e^{1-2s} + (b - a)^{1-2s} \right. \\ &\quad - (c - a)^{1-2s} + (c - b)^{1-2s} + (d - a)^{1-2s} - (d - b)^{1-2s} + (d - c)^{1-2s} \\ &\quad \left. + (e - a)^{1-2s} - (e - b)^{1-2s} + (e - c)^{1-2s} - (e - d)^{1-2s} \right], \end{aligned}$$

whereas

$$\begin{aligned} \|\psi^\natural\|_{H^s}^2 &= \frac{1}{s \left(\frac{1}{2} - s \right)} \left[(b + d - c)^{1-2s} - a^{1-2s} + e^{1-2s} + (b + d - c - a)^{1-2s} \right. \\ &\quad \left. + (e - a)^{1-2s} - (e + c - d - b)^{1-2s} \right]. \end{aligned}$$

Using a computer, if we take $s = \frac{1}{4}$ and $a = 1, b = 2, c = 17, d = 32, e = 52$, we obtain that

$$\|\psi\|_{H^s}^2 \approx 124.07 \quad \text{and} \quad \|\psi^\natural\|_{H^s}^2 \approx 124.48,$$

which means that the rearrangement by tamping does not decrease the $H^{\frac{1}{4}}$ half norm in this case. We also find counter-examples for $s = 0.2, 0.3, 0.35, 0.4, \text{ and } 0.45$. This invites us to think that the tamping fails to decrease the H^s half-norms for all $s < \frac{1}{2}$ (but we have no systematic counter-example yet). Nevertheless, whether the tamping decreases the H^s half-norms or not when $s = \frac{1}{2}$ (or higher) remains unclear. The links between the problems of decreasing the $H^{\frac{1}{2}}$ half-norms in dimension 1 and decreasing the H^1 half-norms in dimension 2 are well known [18] and then such a result on the tamping for the $H^{\frac{1}{2}}$ half-norms would be an important step on the question whether it is possible to extend this work to dimension 2 or not.

4 Proofs of the main results

4.1 Proof of Property 2.5

Let $\varphi \in C^0(\mathbb{R}_+, \mathbb{R}_+)$ be absolutely continuous. First, since $\forall \nu \geq 0$, $x_\nu(\varphi^\natural) \leq x_\nu(\varphi)$, we deduce that

$$\{\nu \in \mathbb{R}_+ / x_\nu(\varphi^\natural) = 0\} \subseteq \{\nu \in \mathbb{R}_+ / x_\nu(\varphi) = 0\}.$$

Thus, $\varphi^\natural(0) \geq \varphi(0)$. Therefore, by definition of the tamping (Definition 2.6), we can conclude the proof if we obtain that

$$\forall \nu > \varphi(0), \quad \text{meas} \left(\mathcal{H}_\nu(\varphi) \cap [0, x_\nu(\varphi)] \right) < x_\nu(\varphi). \quad (5.63)$$

Indeed, in this case we have $x_\nu(\varphi^\natural) = x_\nu(\varphi) - \text{meas} \left(\mathcal{H}_\nu(\varphi) \cap [0, x_\nu(\varphi)] \right) > 0$ and then $\varphi^\natural(0) \leq \varphi(0)$. Let $\nu > \varphi(0)$. We first recall the Luzin property which states that the image of a set of Lebesgue measure zero by an absolutely continuous function is of Lebesgue measure zero. Here φ is absolutely continuous by hypothesis and

$$\varphi \left(\bigcup_{\mu \in [\varphi(0), \nu]} x_\mu(\varphi) \right) = [\varphi(0), \nu].$$

We infer that $\bigcup_{\mu \in [\varphi(0), \nu]} x_\mu(\varphi)$ has a positive measure. On the other hand, by definition of $\mathcal{H}_\lambda(\varphi)$ (Definition 2.5) we have,

$$\forall \mu < \nu, \quad x_\mu(\varphi) \notin \mathcal{H}_\nu(\varphi).$$

Finally, we observe that $\mu \mapsto x_\mu(\varphi)$ is nondecreasing. All these facts together give

$$\text{meas} \left(\mathcal{H}_\nu(\varphi) \cap [0, x_\nu(\varphi)] \right) \leq \text{meas}([0, x_\nu(\varphi)]) - \text{meas} \left(\bigcup_{\mu \in [\varphi(0), \nu]} x_\mu(\varphi) \right) < x_\nu(\varphi). \quad (5.64)$$

and hence (5.63) is proved. \square

4.2 Proof of Property 2.6

Let $\varphi^\dagger(x) := \sup \text{ess} (\varphi \mathbb{1}_{[0, x]})$. By definition we have

$$\varphi^\dagger \in \left\{ \psi \in \mathfrak{M}_+(\mathbb{R}_+) : \begin{array}{l} \varphi \leq \psi \text{ almost everywhere,} \\ \psi \text{ is non-decreasing.} \end{array} \right\}. \quad (5.65)$$

We want to explain why φ^\dagger is a minimizer of this set (for the usual comparison of functions) and is the unique one. Considering the uniqueness, suppose that ψ and χ verify the constraints. Then the function $x \mapsto \min\{\psi(x), \chi(x)\}$ also verifies the constraints and is strictly lower than at least one of the two unless $\psi = \chi$. For the existence, we have to explain why all the functions inside the considered set are higher than φ^\dagger . Suppose that we have a function ψ in the set (5.65) such that on some bounded set A of measure non zero

$$\sup \text{ess} (\psi \cdot \mathbb{1}_A) < \sup \text{ess} (\varphi \cdot \mathbb{1}_A). \quad (5.66)$$

We note $a := \sup \text{ess} A$.

— Case 1 : Suppose that for every $\varepsilon > 0$ holds

$$\sup \text{ess} (\varphi \cdot \mathbb{1}_{[0, a]}) = \sup \text{ess} (\varphi \cdot \mathbb{1}_{[a-\varepsilon, a]}). \quad (5.67)$$

In this case, combining (5.66) and (5.67), we obtain that there exists an $\varepsilon > 0$ small enough so that

$$\psi(x) < \varphi(x) \quad \text{for almost every } x \text{ in } A \cap [a - \varepsilon, a]. \quad (5.68)$$

Since $a = \sup \text{ess} A$, the set $A \cap [a - \varepsilon, a]$ is not of measure 0 and then equation (5.68) is in contradiction with the fact that $\psi \geq \varphi$.

— Case 2 : In the other case, there exists an $a_0 < a$ such that,

$$\forall \varepsilon \in \left] 0, \frac{a - a_0}{2} \right[, \quad \sup \text{ess} (\varphi \cdot \mathbb{1}_{[0, a]}) = \sup \text{ess} (\varphi \cdot \mathbb{1}_{[a_0 - \varepsilon, a_0 + \varepsilon]}). \quad (5.69)$$

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We use the fact that $A \subseteq [0, a]$ and (5.69) becomes

$$\sup \text{ess} (\varphi \cdot \mathbb{1}_A) \leq \sup \text{ess} (\varphi \cdot \mathbb{1}_{[a_0 - \varepsilon, a_0 + \varepsilon]}) \quad (5.70)$$

We now inject (5.66) in the equation above and we use $A \cap [a - \varepsilon, a] \subseteq A$. We obtain

$$\sup \text{ess} (\psi \cdot \mathbb{1}_{A \cap [a_0 + \varepsilon, a]}) < \sup \text{ess} (\varphi \cdot \mathbb{1}_{[a_0 - \varepsilon, a_0 + \varepsilon]}). \quad (5.71)$$

Finally, the fact that $\psi \geq \varphi$ with (5.71) gives

$$\sup \text{ess} (\psi \cdot \mathbb{1}_{A \cap [a_0 + \varepsilon, a]}) < \sup \text{ess} (\psi \cdot \mathbb{1}_{[a_0 - \varepsilon, a_0 + \varepsilon]}). \quad (5.72)$$

Since $\text{meas}(A \cap [a_0 + \varepsilon, a]) > 0$, the above inequality gives a contradiction with the fact that ψ is non-decreasing.

We conclude that φ^\dagger is the minimal function. \square

4.3 Proof of Lemma 2.4

We recall the formula (5.34) that we now prove,

$$\varphi^\natural := \begin{cases} (\varphi \cdot \mathbb{1}_{\{\varphi = \varphi^\dagger\}})^*(\sigma(\varphi) - x) & \text{if } x \leq \sigma(\varphi) \\ (\varphi \cdot \mathbb{1}_{\{\varphi \neq \varphi^\dagger\}})^*(x - \sigma(\varphi)) & \text{otherwise.} \end{cases} \quad (5.73)$$

Call $\widehat{\varphi}$ the function in the right-hand side of this equality.

Claim 1 : *Up to a set of measure 0, the following inclusion holds.*

$$\mathcal{H}_\infty(\varphi) \subseteq \{\varphi \neq \varphi^\dagger\}. \quad (5.74)$$

By definition of the hollows of the function φ (Definition 2.5), it is enough to prove that we have for all $\nu > 0$ the inclusion

$$\mathcal{H}(\{\varphi \geq \nu\}) \subseteq \{\varphi \neq \varphi^\dagger\}.$$

Let $X \subseteq \mathcal{H}(\{\varphi \geq \nu\})$ be a compact set. Since $X \subseteq \mathbb{R}$ the quantities

$$\inf \text{ess } X \quad \text{and} \quad \sup \text{ess } X$$

are well-defined. Since X is bounded and included in the hollows of the set $\{\varphi \geq \nu\}$, by definition of the hollows of a set (Definition 2.4) there exist two sets A and B of positive measure and included in $\{\varphi \geq \nu\}$ such that

$$\sup \text{ess } A \leq \inf \text{ess } X \leq \sup \text{ess } X \leq \inf \text{ess } B.$$

The definition of the hollows also gives

$$\mathcal{H}(\{\varphi \geq \nu\}) \subseteq \{\varphi \geq \nu\}^c = \{\varphi < \nu\}. \quad (5.75)$$

Since $\varphi \geq \nu$ on A , the fact that φ^\dagger is non-decreasing combined with (5.75) implies that $\varphi \neq \varphi^\dagger$ on X . Since X is any compact subset of $\mathcal{H}(\{\varphi \geq \nu\})$, Claim 1 is proved.

Claim 2 : *Up to a set of measure zero, the following inclusion holds.*

$$\{\varphi \neq \varphi^\dagger\} \cap [0, s(\varphi)] \subseteq \mathcal{H}_\infty(\varphi).$$

It is false in general that $\{\varphi \neq \varphi^\dagger\} \subseteq \mathcal{H}_\infty(\varphi)$ but this inclusion becomes true if we only consider what happens on $[0, s(\varphi)]$. Recall that $s(\varphi)$, defined by (5.32), must be understood as being the smallest element in $\text{argmax } \varphi$. We are going to prove that for any given $\delta > 0$ and $\varepsilon > 0$,

$$D_{\delta, \varepsilon} := \{\varphi \leq \varphi^\dagger - \delta\} \cap [0, s(\varphi) - \varepsilon] \subseteq \mathcal{H}_\infty(\varphi). \quad (5.76)$$

Let x in $D_{\delta,\varepsilon}$ and

$$E_{\delta,\varepsilon} := [s(\varphi) - \varepsilon, +\infty[\cap \left\{ \varphi \geq \sup \text{ess } \varphi - \frac{\delta}{2} \right\}. \quad (5.77)$$

By definition of $s(\varphi)$, the measure of $E_{\delta,\varepsilon}$ is positive. If $b \in E_{\delta,\varepsilon}$ then $b \geq x$ and

$$\varphi(x) \leq \varphi^\dagger(x) - \delta = \sup \text{ess } \varphi \cdot \mathbb{1}_{[0,x]} - \delta \leq \sup \text{ess } \varphi - \delta \quad (5.78)$$

$$\leq \varphi(b) - \frac{\delta}{2}. \quad (5.79)$$

Moreover, there exists a set $A_\delta \subseteq [0, x]$ of measure non zero such that for all $a \in A_\delta$,

$$\varphi(a) \geq \sup \text{ess } \left(\varphi \cdot \mathbb{1}_{[0,x]} \right) - \frac{\delta}{2} = \varphi^\dagger(x) - \frac{\delta}{2}. \quad (5.80)$$

Equation (5.80) and the fact that $\varphi(x) \leq \varphi^\dagger(x) - \delta$ give

$$\varphi(x) \leq \varphi(a) - \frac{\delta}{2}. \quad (5.81)$$

The upper bounds (5.79) and (5.81) with the definition of the hollows (Definition 2.4) give the inclusion (5.76). Since $\delta > 0$ and $\varepsilon > 0$ are arbitrary, the claim is proved.

Claim 3 : The essential infima of the super-level sets of φ^\natural and $\widehat{\varphi}$ coincide. In other words,

$$x_\nu(\varphi^\natural) = x_\nu(\widehat{\varphi}). \quad (5.82)$$

First, $\widehat{\varphi}$ is non-decreasing on $[0, \sigma(\varphi)]$ and non-increasing on $[\sigma(\varphi), +\infty[$ and therefore for almost every $\nu > 0$

$$x_\nu(\widehat{\varphi}) = x_\nu \left(x \in [0, \sigma(\varphi)] \mapsto \left(\varphi \cdot \mathbb{1}_{\{\varphi=\varphi^\dagger\}} \right)^* (\sigma(\varphi) - x) \right), \quad (5.83)$$

where $x_\nu(\varphi) := \inf \text{ess } \{\varphi \geq \nu\}$. We now remark that the hypothesis $s(\varphi) < +\infty$ implies that all the quantities that we manipulate here after are finite and then the subtractions are well-defined. Thus we can rewrite (5.83) as

$$x_\nu(\widehat{\varphi}) = \sigma(\varphi) - \sup \text{ess } \left\{ x \in \mathbb{R}_+ : \left(\varphi \cdot \mathbb{1}_{\{\varphi=\varphi^\dagger\}} \right)^* (x) \geq \nu \right\}. \quad (5.84)$$

Observing now that the super-level sets of a non-increasing function are intervals starting from the origin, (5.84) becomes

$$x_\nu(\widehat{\varphi}) = \sigma(\varphi) - \text{meas} \left\{ \left(\varphi \cdot \mathbb{1}_{\{\varphi=\varphi^\dagger\}} \right)^* \geq \nu \right\}. \quad (5.85)$$

We now use the fact that the Schwarz rearrangement preserves the measure of the super-level sets to obtain

$$x_\nu(\widehat{\varphi}) = \sigma(\varphi) - \text{meas} \left\{ \varphi \cdot \mathbb{1}_{\{\varphi=\varphi^\dagger\}} \geq \nu \right\}. \quad (5.86)$$

We observe that $\varphi < \nu$ on $[0, x_\nu(\varphi)[$ which implies that we have the set equality

$$\left\{ \varphi \cdot \mathbb{1}_{\{\varphi=\varphi^\dagger\}} \geq \nu \right\} = [x_\nu(\varphi), s(\varphi)] \setminus \{\varphi \neq \varphi^\dagger\}.$$

We replace this set equality in (5.86) and we obtain

$$\begin{aligned} x_\nu(\widehat{\varphi}) &= \sigma(\varphi) - \text{meas} \left([x_\nu(\varphi), s(\varphi)] \setminus \{\varphi \neq \varphi^\dagger\} \right) \\ &= \sigma(\varphi) - \left(s(\varphi) - x_\nu(\varphi) - \text{meas} \left(\{\varphi \neq \varphi^\dagger\} \cap [x_\nu(\varphi), s(\varphi)] \right) \right) \end{aligned}$$

Replacing $\sigma(\varphi)$ by its expression given at (5.33) leads to

$$\begin{aligned} x_\nu(\widehat{\varphi}) &= x_\nu(\varphi) + \text{meas} \left(\{\varphi \neq \varphi^\dagger\} \cap [x_\nu(\varphi), s(\varphi)] \right) - \text{meas} \left(\{\varphi \neq \varphi^\dagger\} \cap [0, s(\varphi)] \right) \\ &= x_\nu(\varphi) - \text{meas} \left(\{\varphi \neq \varphi^\dagger\} \cap [0, x_\nu(\varphi)] \right). \end{aligned}$$

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Since $x_\nu(\varphi) \leq s(\varphi)$, we can use the results of Claims 1 and 2 to get

$$x_\nu(\widehat{\varphi}) = x_\nu(\varphi) - \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi)] \right) = x_\nu(\varphi^\natural). \quad (5.87)$$

Claim 3 is proved.

Claim 4 : The measure of the super-level sets is the same

$$\text{meas}\{\varphi^\natural \geq \nu\} = \text{meas}\{\widehat{\varphi} \geq \nu\}. \quad (5.88)$$

The tamping is a rearrangement and so $\text{meas}\{\varphi^\natural \geq \nu\} = \text{meas}\{\varphi \geq \nu\}$. Therefore it is enough to prove that $\text{meas}\{\widehat{\varphi} \geq \nu\} = \text{meas}\{\varphi \geq \nu\}$. The fact that the Schwarz rearrangement preserves the measure of the super-level sets gives

$$\begin{aligned} \text{meas} \left(\{\widehat{\varphi} \geq \nu\} \right) &= \text{meas} \left(\{\widehat{\varphi} \geq \nu\} \cap [0, \sigma(\varphi)] \right) + \text{meas} \left(\{\widehat{\varphi} \geq \nu\} \cap [\sigma(\varphi), +\infty[\right) \\ &= \text{meas} \left(\left\{ \left(\varphi \cdot \mathbb{1}_{\{\varphi=\varphi^\dagger\}} \right)^* \geq \nu \right\} \right) + \text{meas} \left(\left\{ \left(\varphi \cdot \mathbb{1}_{\{\varphi \neq \varphi^\dagger\}} \right)^* \geq \nu \right\} \right) \\ &= \text{meas} \left(\left\{ \varphi \cdot \mathbb{1}_{\{\varphi=\varphi^\dagger\}} \geq \nu \right\} \right) + \text{meas} \left(\left\{ \varphi \cdot \mathbb{1}_{\{\varphi \neq \varphi^\dagger\}} \geq \nu \right\} \right) \\ &= \text{meas} \left(\{\varphi \geq \nu\} \right). \end{aligned}$$

Claim 4 is proved.

Conclusion of the proof

Let $\nu > 0$. By definition of the tamping (Definition 2.6) and by definition of $\widehat{\varphi}$, the two sets $\{\varphi^\natural \geq \nu\}$ and $\{\widehat{\varphi} \geq \nu\}$ are segments. By Claim 4 these two segments have the same length and by Claim 3 the inferior extremity of these two segments coincide. We infer that

$$\{\widehat{\varphi} \geq \nu\} = \{\varphi^\natural \geq \nu\}.$$

Therefore $\widehat{\varphi} = \varphi^\natural$ almost everywhere. □

4.4 Proof of Lemma 3.1

Let Ω be a domain of \mathbb{R}^d , let $p \in [1, +\infty[$ and let $\varphi, \psi \in L^p_+(\Omega)$. The idea behind the inequality stated in Lemma 3.1 is the observation that the set $\text{hypo}(\varphi) \Delta \text{hypo}(\psi)$ is a subset of $\Omega \times \mathbb{R}_+$ and thus it is possible to apply the Schwarz rearrangement to this set (for the last variable).

Since $p \geq 1$, the function $\nu \mapsto \nu^{p-1}$ is non-decreasing, for every $x \in \Omega$. Therefore, by the Schwarz rearrangement inequality (Property 1.2),

$$\int_0^\infty \nu^{p-1} \mathbb{1}_{[(\text{hypo}(\varphi) \Delta \text{hypo}(\psi))^*]}(x, \nu) \, d\nu \leq \int_0^\infty \nu^{p-1} \mathbb{1}_{[\text{hypo}(\varphi) \Delta \text{hypo}(\psi)]}(x, \nu) \, d\nu. \quad (5.89)$$

The inequality above is an equality if and only if $p = 1$ or if the support of the two functions ψ or ϕ are disjoint. On the other hand, by the layer-cake representation (5.2) and the Fubini theorem,

$$\|\varphi - \psi\|_{L^p}^p = p \int_0^\infty \nu^{p-1} \text{meas}\{|\varphi - \psi| \geq \nu\} \, d\nu \quad (5.90)$$

$$= p \int_\Omega \int_0^\infty \nu^{p-1} \mathbb{1}_{[\text{hypo}(|\varphi - \psi|)]}(x, \nu) \, d\nu \, dx, \quad (5.91)$$

where here *meas* refers to the d -dimensional Lebesgue measure (with $\Omega \subseteq \mathbb{R}^d$). Moreover,

$$\text{hypo}(|\varphi - \psi|) = \bigcup_{x \in \Omega} \{x\} \times [0, |\varphi(x) - \psi(x)|] = (\text{hypo}(\varphi) \Delta \text{hypo}(\psi))^*. \quad (5.92)$$

Combining (5.91) and (5.92) we get

$$\|\varphi - \psi\|_{L^p}^p = p \int_\Omega \int_0^\infty \nu^{p-1} \mathbb{1}_{[(\text{hypo}(\varphi) \Delta \text{hypo}(\psi))^*]}(x, \nu) \, d\nu \, dx. \quad (5.93)$$

With (5.89) the inequality above becomes

$$\|\varphi - \psi\|_{L^p}^p \leq p \int_{\Omega} \int_0^{\infty} \nu^{p-1} \mathbb{1}_{[\text{hypo}(\varphi) \Delta \text{hypo}(\psi)]}(x, \nu) \, d\nu \, dx. \quad (5.94)$$

We now make a small abuse of notation by identifying, for every ν , the set $\Omega \times \{\nu\}$ with Ω (and so we do with their subsets and for the computation of lebesgue measure) which makes the manipulations of integrals and measure theory much easier. With another use of the Fubini theorem, this gives,

$$\begin{aligned} & p \int_{\Omega} \int_0^{\infty} \nu^{p-1} \mathbb{1}_{[\text{hypo}(\varphi) \Delta \text{hypo}(\psi)]}(x, \nu) \, d\nu \, dx \\ &= p \int_0^{\infty} \int_{\Omega} \nu^{p-1} \mathbb{1}_{[(\text{hypo}(\varphi) \Delta \text{hypo}(\psi)) \cap (\Omega \times \{\nu\})]}(x) \, dx \, d\nu \\ &= p \int_0^{\infty} \nu^{p-1} \text{meas}_{\mathbb{R}^d} [(\text{hypo}(\varphi) \Delta \text{hypo}(\psi)) \cap (\Omega \times \{\nu\})] \, d\nu \\ &= p \int_0^{\infty} \nu^{p-1} \text{meas}_{\mathbb{R}^d} [(\text{hypo}(\varphi) \cap \Omega \times \{\nu\}) \Delta (\text{hypo}(\psi) \cap \Omega \times \{\nu\})] \, d\nu \\ &= p \int_0^{\infty} \nu^{p-1} \text{meas}_{\mathbb{R}^d} (\{\varphi \geq \nu\} \Delta \{\psi \geq \nu\}) \, d\nu. \end{aligned}$$

This with (5.94) gives the announced inequality. \square

4.5 Proof of Lemma 3.2

Let Ω be a domain of \mathbb{R}^d , let $p \in [1, +\infty[$ and let $\varphi, \psi \in L^p_+(\Omega)$.

Claim : If φ and ψ are bounded and compactly supported, then

$$\begin{aligned} & p \int_0^{\infty} \nu^{p-1} \text{meas} (\{\varphi \geq \nu\} \Delta \{\psi \geq \nu\}) \, d\nu \leq 2^{p-1} \|\varphi - \psi\|_{L^p}^p \\ & + p 2^{p-1} \left(\max(\|\varphi\|_{L^\infty}, \|\psi\|_{L^\infty}) \text{meas}(\text{supp} |\varphi - \psi|)^{\frac{1}{p}} \right)^{p-1} \|\varphi - \psi\|_{L^p}. \end{aligned} \quad (5.95)$$

First we observe that

$$\forall \nu \in \left[\min(\varphi(x), \psi(x)), \max(\varphi(x), \psi(x)) \right], \quad \nu \leq \max(\|\varphi\|_{L^\infty}, \|\psi\|_{L^\infty}).$$

Therefore for all $x \in \Omega$,

$$\int_{\min(\varphi(x), \psi(x))}^{\max(\varphi(x), \psi(x))} \nu^{p-1} \, d\nu \leq \int_{\max(\|\varphi\|_{L^\infty}, \|\psi\|_{L^\infty})}^{\max(\|\varphi\|_{L^\infty}, \|\psi\|_{L^\infty}) + (\max(\varphi(x), \psi(x)) - \min(\varphi(x), \psi(x)))} \nu^{p-1} \, d\nu. \quad (5.96)$$

We can rewrite this as follows,

$$\begin{aligned} & \int_0^{\infty} \nu^{p-1} \mathbb{1}_{[\text{hypo}(\varphi) \Delta \text{hypo}(\psi)]}(x, \nu) \, d\nu \\ & \leq \int_0^{\infty} \left(\nu + \max(\|\varphi\|_{L^\infty}, \|\psi\|_{L^\infty}) \right)^{p-1} \mathbb{1}_{[(\text{hypo}(\varphi) \Delta \text{hypo}(\psi))^*]}(x, \nu) \, d\nu, \end{aligned} \quad (5.97)$$

where $*$ refers to the Schwarz non-increasing rearrangement for the last variable. The above estimate can be understood as being the converse of (5.89). We now integrate for the x variable and the Fubini theorem to get

$$p \int_0^{\infty} \nu^{p-1} \text{meas} (\{\varphi \geq \nu\} \Delta \{\psi \geq \nu\}) \, d\nu \quad (5.98)$$

$$\leq p \int_0^{\infty} \left(\nu + \max(\|\varphi\|_{L^\infty}, \|\psi\|_{L^\infty}) \right)^{p-1} \cdot \text{meas} \left((\text{hypo}(\varphi) \Delta \text{hypo}(\psi))^* \cap (\mathbb{R} \times \{\nu\}) \right) d\nu. \quad (5.99)$$

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We then use in the estimate above the fact that $(a + b)^\alpha \leq 2^\alpha(a^\alpha + b^\alpha)$ (with $\alpha > 0$), the layer-cake representation of the L^p norms and Equality (5.92). This gives,

$$\begin{aligned} & p \int_0^\infty \nu^{p-1} \text{meas} \left(\{\varphi \geq \nu\} \Delta \{\psi \geq \nu\} \right) d\nu \\ & \leq 2^{p-1} \|\varphi - \psi\|_{L^p}^p + p 2^{p-1} \left(\max(\|\varphi\|_{L^\infty}, \|\psi\|_{L^\infty}) \right)^{p-1} \|\varphi - \psi\|_{L^1}. \end{aligned} \quad (5.100)$$

In the other hand, the Hölder inequality gives

$$\|\varphi - \psi\|_{L^1} \leq \left(\text{meas}(\text{supp} |\varphi - \psi|) \right)^{\frac{p-1}{p}} \|\varphi - \psi\|_{L^p}. \quad (5.101)$$

We then obtain (5.95) by combining (5.100) and (5.101).

Passing to the limit and conclusion of the proof

Consider now $\varphi \in L_+^p(\Omega)$ and a sequence $\varphi_n \in L_+^p(\Omega)$ converging towards φ in L^p . We note $\mathcal{B}(x, a)$ the open ball of center x and of radius a . If we apply the inequality given by the claim before to the functions $\varphi \cdot \mathbb{1}_{\mathcal{B}(0,b)} \cdot \mathbb{1}_{\{\varphi \leq a\}}$ and $\varphi_n \cdot \mathbb{1}_{\mathcal{B}(0,b)} \cdot \mathbb{1}_{\{\varphi \leq a\}}$, we obtain a constant C such that

$$p \int_0^a \nu^{p-1} \text{meas} \left((\{\varphi_n \geq \nu\} \Delta \{\varphi \geq \nu\}) \cap \mathcal{B}(0, b) \right) d\nu \quad (5.102)$$

$$\leq C \|\varphi_n - \varphi\|_{L^p}^p + C(a \cdot b^{\frac{1}{p}})^{p-1} \|\varphi - \varphi_n\|_{L^p}. \quad (5.103)$$

When a and b go to infinity, the term (5.102) converges towards

$$p \int_0^{+\infty} \nu^{p-1} \text{meas}(\{\varphi_n \geq \nu\} \Delta \{\varphi \geq \nu\}) d\nu < +\infty. \quad (5.104)$$

Let $\varepsilon > 0$. We fix a and b large enough such that the term (5.102) is at distance at most ε to the term (5.104). Now that a and b are fixed, we use the fact that (5.103) worthes less than an $\varepsilon > 0$ when n is chosen large enough. The lemma is proved. \square

4.6 Proof of Lemma 3.3

Let Ω be a domain of \mathbb{R}^d and let $p \in (1, +\infty)$. Let $\varphi \in L_+^p(\Omega)$ and let $\varphi_n \in L_+^p(\Omega)$ such that $M := \sup_n \|\varphi_n\|_{L^p} < +\infty$ and

$$\text{meas} \left(\{\varphi_n \geq \nu\} \Delta \{\varphi \geq \nu\} \right) \xrightarrow{n \rightarrow +\infty} 0,$$

for almost every $\nu > 0$. Let $g \in L^q(\Omega)$ where q is the Hölder conjugate of p defined by $\frac{1}{p} + \frac{1}{q} = 1$. We want to prove that

$$\int_{\Omega} g(x) (\varphi_n - \varphi)(x) dx \xrightarrow{n \rightarrow \infty} 0. \quad (5.105)$$

Claim 1 : The proof can be reduced to functions φ and φ_n with support in $\Omega \setminus \mathcal{B}(0, a)$ with a independent of n .

Let $a \in \mathbb{R}_+$. Using the Hölder inequality, we get that

$$\begin{aligned} & \int_{\Omega \setminus \mathcal{B}(0,a)} g(x) (\varphi_n - \varphi)(x) dx \\ & \leq \left(\int_{\Omega \setminus \mathcal{B}(0,a)} |g(x)|^q dx \right)^{\frac{1}{q}} \left(\int_{\Omega \setminus \mathcal{B}(0,a)} |(\varphi_n - \varphi)(x)|^p dx \right)^{\frac{1}{p}} \\ & \leq \left(\int_{\Omega \setminus \mathcal{B}(0,a)} |g(x)|^q dx \right)^{\frac{1}{q}} \left(\|\varphi_n\|_{L^p} + \|\varphi\|_{L^p} \right) \\ & \leq \left(\int_{\Omega \setminus \mathcal{B}(0,a)} |g(x)|^q dx \right)^{\frac{1}{q}} \left(M + \|\varphi\|_{L^p} \right). \end{aligned}$$

The above estimate does not depend on n and it is smaller than a fixed $\varepsilon > 0$ if a is chosen large enough because we have $q < \infty$. Then, in order to prove (5.105), it is enough to consider only uniformly compactly supported functions. Otherwise, it is always possible to separate the integral appearing in (5.105) into two - an integral on $\Omega \cap \mathcal{B}(0, a)$ and an integral on $\Omega \setminus \mathcal{B}(0, a)$ - and to prove that they are both smaller than any $\varepsilon > 0$ when the parameters a and n are well chosen.

Claim 2 : *The proof can be reduced to the case $\varphi \in L^\infty(\Omega)$.*

Let $b \in \mathbb{R}_+$. The reasoning is almost the same as the step before.

$$\begin{aligned} & \int_{\{\varphi \geq b\}} g(x) (\varphi_n - \varphi)(x) \, dx \\ & \leq \left(\int_{\{\varphi \geq b\}} |g(x)|^q \, dx \right)^{\frac{1}{q}} \left(\int_{\{\varphi \geq b\}} |(\varphi_n - \varphi)(x)|^p \, dx \right)^{\frac{1}{p}} \\ & \leq \left(\int_{\{\varphi \geq b\}} |g(x)|^q \, dx \right)^{\frac{1}{q}} \left(M + \|\varphi\|_{L^p} \right). \end{aligned}$$

The above estimate does not depend on n and it is smaller than a fixed $\varepsilon > 0$ if b is chosen large enough. Then, in order to prove (5.105), we can suppose that the function φ is essentially bounded.

Claim 3 : *The proof can be reduced to the case $g \in L^\infty$.*

The same reasoning as for Claim 2 with g instead of φ leads to the same estimate (mutatis mutandis).

Claim 4 : *The proof can be reduced to the case $\sup_{n \in \mathbb{N}} \|\varphi_n\|_{L^\infty} < +\infty$.*

This reasoning is also a variation of the two reasonings before. We have

$$\int_{\{\varphi_n \geq b\}} g(x) \cdot (\varphi_n - \varphi)(x) \, dx \leq \left(\int_{\{\varphi_n \geq b\}} |g(x)|^q \, dx \right)^{\frac{1}{q}} \left(M + \|\varphi\|_{L^p} \right) \quad (5.106)$$

For any sets A and B we always have $A \subseteq B \cup (A \Delta B)$. We use this fact in equation (5.106) and we obtain

$$\begin{aligned} & \int_{\{\varphi_n \geq b\}} g(x) \cdot (\varphi_n - \varphi)(x) \, dx \\ & \leq \left(M + \|\varphi\|_{L^p} \right) \left[\int_{\{\varphi \geq b\}} |g(x)|^q \, dx + \int_{\{\varphi \geq b\} \Delta \{\varphi_n \geq b\}} |g(x)|^q \, dx \right]^{\frac{1}{q}} \\ & \leq \left(M + \|\varphi\|_{L^p} \right) \left[\left(\int_{\{\varphi \geq b\}} |g(x)|^q \, dx \right)^{\frac{1}{q}} + \|g\|_{L^\infty} \cdot \left(\text{meas} \left(\{\varphi \geq b\} \Delta \{\varphi_n \geq b\} \right) \right)^{\frac{1}{q}} \right] \\ & = \left(M + \|\varphi\|_{L^p} \right) \|g\|_{L^\infty} \left(\text{meas} \left(\{\varphi \geq b\} \Delta \{\varphi_n \geq b\} \right) \right)^{\frac{1}{q}}, \end{aligned}$$

where we used Claim 2 that states $\{\varphi \geq b\}$ is of measure 0 if b is chosen large enough. Now we use the hypothesis

$$\text{meas} \left(\{\varphi \geq b\} \Delta \{\varphi_n \geq b\} \right) \xrightarrow{n \rightarrow \infty} 0,$$

and we conclude that the functions φ_n can be taken essentially bounded independently of n .

Conclusion of the proof.

It is now supposed that the functions we manipulate are all essentially bounded by the same bound $b \in \mathbb{R}_+$ and have a compact support contained in a ball $\mathcal{B}(0, a)$ with $a \in \mathbb{R}_+$ independent of n . If we combine the Hölder inequality and the inequality given by Lemma 3.1 we get

$$\begin{aligned} & \int_{\mathbb{R}} g(x) (\varphi_n - \varphi)(x) \, dx \\ & \leq \left(\int_{\mathbb{R}} |g(x)|^q \, dx \right)^{\frac{1}{q}} \left(p \int_0^\infty \nu^{p-1} \text{meas} \left(\{\varphi_n \geq \nu\} \Delta \{\varphi \geq \nu\} \right) \, d\nu \right)^{\frac{1}{p}}. \end{aligned}$$

We conclude the proof with the Lebesgue dominated convergence theorem. \square

4.7 Proof of Lemma 3.4 for $p = 1$

Let Ω be a domain of \mathbb{R}^d . Let $\varphi \in L^1_+(\Omega)$ and let $\varphi_n \in L^1_+(\Omega)$ such that

$$\|\varphi_n\|_{L^1} \longrightarrow \|\varphi\|_{L^1} \quad (5.107)$$

and

$$\text{meas} \left(\{\varphi_n \geq \nu\} \Delta \{\varphi \geq \nu\} \right) \xrightarrow{n \rightarrow +\infty} 0, \quad (5.108)$$

for almost every ν . We want to prove that

$$\|\varphi_n - \varphi\|_{L^1} \xrightarrow{n \rightarrow \infty} 0. \quad (5.109)$$

Let a and b in \mathbb{R}_+ . The inequality provided at Lemma 3.1 gives

$$\begin{aligned} & \int_{\mathcal{B}(0,a)} \left| \varphi \cdot \mathbb{1}_{\{\varphi \leq b\}} - \varphi_n \cdot \mathbb{1}_{\{\varphi_n \leq b\}} \right| (x) \, dx \\ & \leq \int_0^b \text{meas} \left((\{\varphi \geq \nu\} \Delta \{\varphi_n \geq \nu\}) \cap \mathcal{B}(0,a) \right) \, d\nu. \end{aligned}$$

Since the measure of the symmetrical difference of the super-level sets is vanishing when n grows, the Lebesgue dominated convergence theorem with the above estimate gives

$$\int_{\mathcal{B}(0,a)} \left| \varphi \cdot \mathbb{1}_{\{\varphi \leq b\}} - \varphi_n \cdot \mathbb{1}_{\{\varphi_n \leq b\}} \right| (x) \, dx \xrightarrow{n \rightarrow \infty} 0. \quad (5.110)$$

We now define

$$R_{a,b}(\varphi) := \varphi - \varphi \cdot \mathbb{1}_{\{\varphi \leq b\}} \cdot \mathbb{1}_{\mathcal{B}(0,a)}.$$

Since we work with L^1 norms, we have

$$\|R_{a,b}(\varphi_n)\|_{L^1} = \|\varphi_n\|_{L^1} - \|\varphi_n \cdot \mathbb{1}_{\{\varphi_n \leq b\}} \cdot \mathbb{1}_{\mathcal{B}(0,a)}\|_{L^1}.$$

We use the hypothesis $\|\varphi_n\|_{L^1} \longrightarrow \|\varphi\|_{L^1}$ and the convergence result (5.110) to obtain

$$\|R_{a,b}(\varphi_n)\|_{L^1} \xrightarrow{n \rightarrow \infty} \|\varphi\|_{L^1} - \|\varphi \cdot \mathbb{1}_{\{\varphi \leq b\}} \cdot \mathbb{1}_{\mathcal{B}(0,a)}\|_{L^1} = \|R_{a,b}(\varphi)\|_{L^1}. \quad (5.111)$$

We now write

$$\begin{aligned} \|\varphi - \varphi_n\|_{L^1} & \leq \|\varphi_n \cdot \mathbb{1}_{\{\varphi_n \leq b\}} \cdot \mathbb{1}_{\mathcal{B}(0,a)} - \varphi \cdot \mathbb{1}_{\{\varphi \leq b\}} \cdot \mathbb{1}_{\mathcal{B}(0,a)}\|_{L^1} \\ & \quad + \|R_{a,b}(\varphi_n)\|_{L^1} + \|R_{a,b}(\varphi)\|_{L^1}. \end{aligned}$$

The term $\|R_{a,b}(\varphi)\|_{L^1}$ can be made smaller than a given ε by choosing a and b large enough. By (5.111), the term $\|R_{a,b}(\varphi_n)\|_{L^1}$ can be made ε close to $\|R_{a,b}(\varphi)\|_{L^1}$ (and thus 2ε close to 0) by choosing n large enough. The first term of the above inequality is also smaller than a given ε when n is chosen large enough by the convergence result (5.110). Therefore,

$$\|\varphi - \varphi_n\|_{L^1} \xrightarrow{n \rightarrow \infty} 0.$$

\square

4.8 Proof of Lemma 3.5

Let $p \in [1, +\infty)$. Let $(\varphi_n) \in L^p_+(\mathbb{R}_+)$ be a bounded sequence such that

$$\exists \mu > 0, \quad \limsup_{n \rightarrow +\infty} x_\mu(\varphi_n) < +\infty, \quad (5.112)$$

where $x_\mu(\varphi_n)$ is the essential infimum of the super-level set at level μ of function φ_n . We want to prove that up to an omitted extraction there exists a function ψ such that for almost every $\nu > 0$

$$\text{meas} \left(\{\varphi_n^\sharp \geq \nu\} \Delta \{\psi \geq \nu\} \right) \xrightarrow{n \rightarrow +\infty} 0. \quad (5.113)$$

Claim 1 : For every $\nu > 0$ the sequence $x_\nu(\varphi_n^{\natural})$ is a bounded sequence

To prove this claim we use the hypothesis that

$$\exists \mu > 0, \quad \limsup_{n \rightarrow +\infty} x_\mu(\varphi_n) < +\infty. \quad (5.114)$$

Since $\nu \leq \mu \Rightarrow x_\nu(\varphi) \leq x_\mu(\varphi)$, the case $\nu \leq \mu$ is straight-forward. Concerning the case $\nu \geq \mu$, we first use the fact that, by inclusion of the super-level sets,

$$x_\nu(\varphi_n^{\natural}) \leq \sup \text{ess} \{\varphi_n^{\natural} \geq \mu\}. \quad (5.115)$$

Since the super-level sets of φ_n^{\natural} are segments and since $x_\mu(\varphi_n^{\natural})$ remains bounded, the quantity appearing at the right-hand side of (5.115) remains bounded as $n \rightarrow \infty$ if and only if the measure of $\{\varphi_n^{\natural} \geq \mu\}$ remains bounded as $n \rightarrow \infty$. Nevertheless, by hypothesis (φ_n) is bounded in L^p . Since the tamping is a rearrangement, we have for all $\lambda > 0$

$$\lambda^p \text{meas}\{\varphi_n^{\natural} \geq \lambda\} \leq \|\varphi_n^{\natural}\|_{L^p} = \|\varphi_n\|_{L^p} \leq \limsup_{n \rightarrow +\infty} \|\varphi_n\|_{L^p} < +\infty. \quad (5.116)$$

This implies in particular that $\text{meas}\{\varphi_n^{\natural} \geq \lambda\}$ is a bounded sequence. Therefore, the case $\lambda = \mu$ in (5.116) combined with (5.115) give the claim.

Claim 2 : There exist an extraction σ and a set $A \subseteq \mathbb{R}_+^*$ countable and dense such that there exists a decreasing family of segments $(T_\nu)_{\nu \in A}$ verifying the convergence property

$$\forall \nu \in A, \quad \text{meas} \left(\left\{ \varphi_{\sigma(n)}^{\natural} \geq \nu \right\} \Delta T_\nu \right) \xrightarrow[n \rightarrow +\infty]{} 0. \quad (5.117)$$

We first recall that $\{\varphi_n^{\natural} \geq \nu\}$ is also a segment. By Claim 1, the respective lower extremity $x_\nu(\varphi_n^{\natural})$ of the segments $\{\varphi_n^{\natural} \geq \nu\}$ remains bounded as $n \rightarrow +\infty$. Using again (5.116) we obtain that the length of these segments remains bounded as $n \rightarrow +\infty$ and therefore it is also the case for their upper extremity. By the Bolzano-Weierstrass theorem, we have for all $\nu > 0$ an extraction σ_ν such that the two extremities of the segment $\{\varphi_n^{\natural} \geq \nu\}$ converge as $n \rightarrow +\infty$. The claim follows from a classical diagonal extraction argument.

Conclusion of the proof.

To make the arguments easier to read, we now omit the extraction. The natural candidate for the limit is the function ψ defined by

$$\psi(x) := \sup \{ \nu \in A / x \in T_\nu \}. \quad (5.118)$$

This definition implies in particular that the super-level sets of ψ are segments. The result of Claim 2 gives (5.113) but for $\nu \in A$ which is only a countable set and then we cannot conclude yet. Observe first that the set of $\nu > 0$ such that the measure of $\{\psi = \nu\}$ is 0 is a countable set and since the announced result (5.113) holds for almost every $\nu > 0$ we can exclude this case. Now, let $\nu \in \mathbb{R}_+^* \setminus A$ such that $\text{meas}\{\psi = \nu\} = 0$. Proving that (5.113) holds for this choice ν concludes the proof. For that purpose we prove that the two extremities of the segment $\{\varphi_n^{\natural} \geq \nu\}$ are converging towards the corresponding extremities of the segment $\{\psi \geq \nu\}$.

We first claim that if $\nu \notin A$ is such that there exists $(\nu_k) \in A$ converging towards ν while $x_{\nu_k}(\psi)$ is not converging towards $x_\nu(\psi)$ then the measure of $\{\psi = \nu\}$ is not 0 (which is the excluded case). Indeed, suppose for instance that λ converges towards ν by upper values. By inclusion of the super-level sets, the sequence $x_\lambda(\psi)$ is non-increasing and then converges. Since the super-level sets of ψ are segments,

$$\forall x \in \left[x_\nu(\psi), \lim_{\lambda \rightarrow \nu^+} x_\lambda(\psi) \right], \quad \psi(x) = \nu. \quad (5.119)$$

The same reasoning works if λ converges towards ν by lower values. Since the set of ν such that $\text{meas}\{\psi = \nu\} \neq 0$ is countable and since we want to establish that (5.113) holds for almost every $\nu > 0$, we only have to concentrate on the case

$$x_\lambda(\psi) \xrightarrow[\lambda \rightarrow \nu]{} x_\nu(\psi), \quad \text{with } \lambda \in A. \quad (5.120)$$

Section 4. Proofs of the main results

We consider $\mu \leq \nu \leq \lambda$ with μ and λ in A . By inclusion of the super level-sets,

$$\liminf_{n \rightarrow +\infty} x_\mu(\varphi_n^\natural) \leq \liminf_{n \rightarrow +\infty} x_\nu(\varphi_n^\natural) \leq \limsup_{n \rightarrow +\infty} x_\nu(\varphi_n^\natural) \leq \limsup_{n \rightarrow +\infty} x_\lambda(\varphi_n^\natural). \quad (5.121)$$

Using now Claim 2. we are led to

$$x_\mu(\psi) \leq \liminf_{n \rightarrow +\infty} x_\nu(\varphi_n^\natural) \leq \limsup_{n \rightarrow +\infty} x_\nu(\varphi_n^\natural) \leq x_\lambda(\psi). \quad (5.122)$$

Making $\mu \rightarrow \nu^-$ and $\lambda \rightarrow \nu^+$ and using (5.120) we get

$$\lim_{n \rightarrow +\infty} x_\nu(\varphi_n^\natural) = x_\nu(\psi). \quad (5.123)$$

The same reasoning works for the upper extremity of the segment $\{\varphi_n^\natural \geq \nu\}$ and thus (5.113) holds. The theorem is proved. \square

4.9 Proof of Theorem 3.1

We recall that the definitions of x_ν and y_ν were given at Proposition 2.1. Since the super-level sets of φ that are of measure zero create no problem for the convergence of φ_n^\natural , we will make the assumption that, without loss of generality,

$$\forall \nu, \quad \text{meas}\{\varphi \geq \nu\} > 0.$$

This is equivalent to

$$\forall \nu, \quad x_\nu(\varphi) < \infty \quad \text{and} \quad \limsup_{n \rightarrow +\infty} x_\nu(\varphi_n) < \infty. \quad (5.124)$$

The strategy of the proof consists in proving that the condition $\varphi_n \rightarrow \varphi$ in L^p , the condition (5.49) on the convergence of the super-level sets and the *local hollows convergence condition*,

$$\forall a > 0, \quad \text{meas}\left(\left(\mathcal{H}_\infty(\varphi_n) \Delta \mathcal{H}_\infty(\varphi)\right) \cap [0, a]\right) \xrightarrow[n \rightarrow +\infty]{} 0, \quad (5.125)$$

together imply that for almost every $\nu \geq 0$

$$\text{meas}\left(\left\{\varphi_n^\natural \geq \nu\right\} \Delta \left\{\varphi^\natural \geq \nu\right\}\right) \xrightarrow[n \rightarrow +\infty]{} 0. \quad (5.126)$$

In this case the hypothesis of Lemma 3.4 are satisfied and we can conclude the proof. For that purpose, we prove that for almost every ν

$$y_\nu(\varphi_n) \xrightarrow[n \rightarrow +\infty]{} y_\nu(\varphi).$$

We start by defining the segment

$$X_\nu(\varphi_n, \varphi) := \left[\liminf_{n \rightarrow +\infty} x_\nu(\varphi_n), x_\nu(\varphi) \right].$$

The fact that the extremities of this segment are well-ordered, meaning that $\liminf_{n \rightarrow +\infty} x_\nu(\varphi_n) \leq x_\nu(\varphi)$, is a consequence of Claim 1 below.

Claim 1 : $\limsup_{n \rightarrow +\infty} x_\nu(\varphi_n) \in X_\nu(\varphi_n, \varphi)$.

By the absurd, suppose that $\limsup_{n \rightarrow +\infty} x_\nu(\varphi_n) > x_\nu(\varphi)$. The definition of $x_\nu(\varphi_n)$ gives

$$\{\varphi_n \leq \nu\} \cap [0, x_\nu(\varphi_n)] = [0, x_\nu(\varphi_n)]. \quad (5.127)$$

By (5.127) we obtain

$$\int_{[x_\nu(\varphi), x_\nu(\varphi_n)] \cap \{\varphi \geq \nu\}} |\varphi - \nu|^p \leq \int_{[x_\nu(\varphi), x_\nu(\varphi_n)] \cap \{\varphi \geq \nu\}} |\varphi - \varphi_n|^p \leq \int_{\mathbb{R}_+} |\varphi - \varphi_n|^p. \quad (5.128)$$

and therefore

$$\limsup_{n \rightarrow \infty} \int_{[x_\nu(\varphi), x_\nu(\varphi_n)] \cap \{\varphi \geq \nu\}} |\varphi - \nu|^p \leq \lim_{n \rightarrow \infty} \int_{\mathbb{R}_+} |\varphi - \varphi_n|^p = 0. \quad (5.129)$$

By definition of $x_\nu(\varphi)$ we have that for all $\varepsilon > 0$,

$$\text{meas} \left(\{\varphi > \nu\} \cap [x_\nu(\varphi), x_\nu(\varphi) + \varepsilon] \right) > 0. \quad (5.130)$$

The fact that $\limsup_{n \rightarrow +\infty} x_\nu(\varphi_n) > x_\nu(\varphi)$ with (5.130) gives

$$\limsup_{n \rightarrow \infty} \int_{[x_\nu(\varphi), x_\nu(\varphi_n)]} |\varphi - \nu|^p > 0. \quad (5.131)$$

Equations (5.129) and (5.131) are in contradiction.

Claim 2 : *We have $\varphi \leq \nu$ almost everywhere on $X_\nu(\varphi_n, \varphi)$.*

If we have $\varphi > \nu$ on a subset A of $X_\nu(\varphi_n, \varphi)$ whose measure is non zero then given the definition of x_ν and the fact that φ_n converges in L^p towards φ , holds

$$x_\nu(\varphi) \leq \inf \text{ess } A \quad \text{and} \quad \limsup_{n \rightarrow +\infty} x_\nu(\varphi_n) \leq \inf \text{ess } A.$$

This is in contradiction with the fact that, by definition of $X_\nu(\varphi_n, \varphi)$,

$$\inf \text{ess } A < \sup \text{ess } A \leq \max X_\nu(\varphi_n, \varphi)$$

because the maximum of $X_\nu(\varphi_n, \varphi)$ is $x_\nu(\varphi)$.

Claim 3 : *We have $\varphi \geq \nu$ almost everywhere on $X_\nu(\varphi_n, \varphi) \setminus \mathcal{H}_\infty(\varphi)$.*

Suppose that we have $\varphi \leq \nu - \varepsilon$ on a subset A of $X_\nu(\varphi_n, \varphi)$ with A of positive measure (for a certain $\varepsilon > 0$). Since by Claim 1 $\limsup_{n \rightarrow +\infty} x_\nu(\varphi_n) \leq x_\nu(\varphi)$, then

$$\liminf_{n \rightarrow +\infty} x_\nu(\varphi_n) \leq \inf \text{ess } A. \quad (5.132)$$

We assumed at (5.124) that $x_\nu(\varphi) < \infty$, then we know by L^p convergence that there exists a set B whose essential infimum is greater than $x_\nu(\varphi)$ and such that, for n large enough, $\varphi_n|_B \geq \nu - \varepsilon/2$. Such a set verifies

$$\sup \text{ess } A \leq \inf \text{ess } B. \quad (5.133)$$

Moreover, by definition of $x_\nu(\varphi_n)$, there exist sets C_n of positive measure and an extraction σ such that

$$\inf \text{ess } A \geq \liminf_{n \rightarrow +\infty} (\inf \text{ess } C_n) \quad (5.134)$$

and $\varphi_{\sigma(n)} \geq \nu$ on C_n . The extraction σ is an extraction such that $x_\nu(\varphi_{\sigma(n)}) \rightarrow \liminf_{n \rightarrow +\infty} x_\nu(\varphi_n)$. Such an extraction exists by property of the ‘‘liminf’’. Both (5.133) and (5.134) imply that

$$A \subseteq \left[\liminf_{n \rightarrow +\infty} (\inf \text{ess } C_n), \inf \text{ess } B \right] \setminus \left\{ \varphi_n \geq \nu - \frac{\varepsilon}{2} \right\}. \quad (5.135)$$

This inclusion is true for almost every element of A . Nevertheless, by definition of the *hollows* at (2.4), we have for all measurable set M

$$\text{conv ess } M \subseteq \text{conv } M \quad \text{and} \quad \text{conv ess } M \setminus M \subseteq \mathcal{H}(M),$$

Therefore, regarding the definitions of the sets B and C_n ,

$$[\inf \text{ess } C_n, \inf \text{ess } B] \setminus \left\{ \varphi_n \geq \nu - \frac{\varepsilon}{2} \right\} \subseteq \mathcal{H}_{\nu - \varepsilon/2}(\varphi_n) \subseteq \mathcal{H}_\infty(\varphi_n). \quad (5.136)$$

This inclusion is true for almost every element. Since we demanded that the sequence φ_n verifies the convergence of the hollows (5.125), we conclude that the inclusion (5.136) is preserved asymptotically, which means that

$$\left[\liminf_{n \rightarrow +\infty} (\inf \text{ess } C_n), \inf \text{ess } B \right] \setminus \left\{ \varphi \geq \nu - \frac{\varepsilon}{2} \right\} \subseteq \mathcal{H}_\infty(\varphi). \quad (5.137)$$

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We conclude from (5.135) and (5.137) that $A \subseteq \mathcal{H}_\infty(\varphi)$ for almost every element of A . This gives the conclusion.

Claim 4 : *If we have $\text{meas}\{\varphi = \nu\} = 0$ then, $X_\nu(\varphi, \varphi_n) \subseteq \mathcal{H}_\infty(\varphi)$.*

The inclusion is true up to a set of measure zero.

By Claim 2 we have, $\varphi|_{X_\nu(\varphi, \varphi_n)} \leq \nu$ almost everywhere on $X_\nu(\varphi, \varphi_n)$. But since $\text{meas}\{\varphi = \nu\} = 0$, we can conclude that $\varphi|_{X_\nu(\varphi, \varphi_n)} < \nu$ almost everywhere. Claim 3 gives the announced inclusion.

Claim 5 : *If we have $\text{meas}\{\varphi = \nu\} = 0$ then $y_\nu(\varphi_n) \xrightarrow{n \rightarrow +\infty} y_\nu(\varphi)$.*

We have :

$$\begin{aligned} y_\nu(\varphi_n) &= x_\nu(\varphi_n) - \text{meas} \left(\mathcal{H}_\infty(\varphi_n) \cap [0, x_\nu(\varphi_n)] \right) \\ &= y_\nu(\varphi) + x_\nu(\varphi_n) - x_\nu(\varphi) \\ &+ \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi)] \right) - \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi_n)] \right) \\ &+ \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi_n)] \right) - \text{meas} \left(\mathcal{H}_\infty(\varphi_n) \cap [0, x_\nu(\varphi_n)] \right). \end{aligned} \quad (5.138)$$

Thus,

$$|y_\nu(\varphi_n) - y_\nu(\varphi)| \leq \left| \left(x_\nu(\varphi_n) - x_\nu(\varphi) \right) \right. \quad (5.139)$$

$$\left. - \left(\text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi_n)] \right) - \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi)] \right) \right) \right| \quad (5.140)$$

$$+ \left| \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi_n)] \right) - \text{meas} \left(\mathcal{H}_\infty(\varphi_n) \cap [0, x_\nu(\varphi_n)] \right) \right| \quad (5.141)$$

We will explain why the two terms of the sum in the right hand side of the above inequality are vanishing when n goes to ∞ .

Concerning (5.141), we always have

$$\left| \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi_n)] \right) - \text{meas} \left(\mathcal{H}_\infty(\varphi_n) \cap [0, x_\nu(\varphi_n)] \right) \right| \quad (5.142)$$

$$\leq \text{meas} \left[\left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi_n)] \right) \Delta \left(\mathcal{H}_\infty(\varphi_n) \cap [0, x_\nu(\varphi_n)] \right) \right] \quad (5.143)$$

$$= \text{meas} \left[\left(\mathcal{H}_\infty(\varphi) \Delta \mathcal{H}_\infty(\varphi_n) \right) \cap [0, x_\nu(\varphi_n)] \right] \quad (5.144)$$

$$\leq \text{meas} \left(\mathcal{H}_\infty(\varphi) \Delta \mathcal{H}_\infty(\varphi_n) \right), \quad (5.145)$$

which is converging towards zero when n goes to infinity by hypothesis (5.125).

Concerning (5.139)-(5.140), as we explain here after, the result is a consequence of the properties established at the previous claims.

$$\text{--- Case 1 : } x_\nu(\varphi_n) \in \left[\liminf_{n \rightarrow +\infty} x_\nu(\varphi_n), \limsup_{n \rightarrow +\infty} x_\nu(\varphi_n) \right].$$

The result established at Claim 4 and the fact that $X_\nu(\varphi, \varphi_n)$ is a segment imply that

$$\mathcal{H}_\infty(\varphi) \cap \text{conv} \{x_\nu(\varphi_n), x_\nu(\varphi)\} = \text{conv} \{x_\nu(\varphi_n), x_\nu(\varphi)\}. \quad (5.146)$$

Thus,

$$\text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi)] \right) - \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi_n)] \right) = x_\nu(\varphi) - x_\nu(\varphi_n), \quad (5.147)$$

and then the studied term is 0.

— *Case 2* : $x_\nu(\varphi_n) \notin \left[\liminf_{n \rightarrow +\infty} x_\nu(\varphi_n), \limsup_{n \rightarrow +\infty} x_\nu(\varphi_n) \right]$

In this case, we cannot use directly the result of Claim 4 to get (5.147) but this equality remains true asymptotically if we apply this result not to $x_\nu(\varphi_n)$ but to its “liminf” and “limsup”. More precisely, if we suppose for instance that $x_\nu(\varphi_n) < \liminf_{n \rightarrow +\infty} x_\nu(\varphi_n)$, then we have

$$\text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi_n)] \right) \quad (5.148)$$

$$\begin{aligned} &\leq \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap \left[0, \liminf_{n \rightarrow +\infty} x_\nu(\varphi_n) \right] \right) + \text{meas} \left(\left[x_\nu(\varphi_n), \liminf_{n \rightarrow +\infty} x_\nu(\varphi_n) \right] \right) \\ &= \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap \left[0, \liminf_{n \rightarrow +\infty} x_\nu(\varphi_n) \right] \right) + \left(\liminf_{n \rightarrow +\infty} x_\nu(\varphi_n) - x_\nu(\varphi_n) \right). \end{aligned} \quad (5.149)$$

Equality (5.147) becomes

$$\begin{aligned} &\left| \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi_n)] \right) - \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi)] \right) - (x_\nu(\varphi_n) - x_\nu(\varphi)) \right| \\ &\leq \left| \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap \left[0, \liminf_{n \rightarrow +\infty} x_\nu(\varphi_n) \right] \right) \right. \\ &\quad \left. - \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi)] \right) - \left(\liminf_{n \rightarrow +\infty} x_\nu(\varphi_n) - x_\nu(\varphi) \right) \right| \\ &\quad + 2 \left(\liminf_{n \rightarrow +\infty} x_\nu(\varphi_n) - x_\nu(\varphi_n) \right). \end{aligned} \quad (5.150)$$

We apply the result of Claim 4 to the “lim inf” and we get that

$$\text{meas} \left(\mathcal{H}_\infty(\varphi) \cap \left[0, \liminf_{n \rightarrow +\infty} x_\nu(\varphi_n) \right] \right) - \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi)] \right) = \liminf_{n \rightarrow +\infty} x_\nu(\varphi_n) - x_\nu(\varphi). \quad (5.151)$$

Therefore

$$\begin{aligned} &\left| \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi_n)] \right) - \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi)] \right) - (x_\nu(\varphi_n) - x_\nu(\varphi)) \right| \\ &\leq 2 \cdot \left(\liminf_{n \rightarrow +\infty} x_\nu(\varphi_n) - x_\nu(\varphi_n) \right). \end{aligned} \quad (5.152)$$

If we do the same reasoning but we suppose on the contrary that $x_\nu(\varphi_n) > \limsup_{n \rightarrow +\infty} x_\nu(\varphi_n)$, we get

$$\begin{aligned} &\left| \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi_n)] \right) - \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi)] \right) - (x_\nu(\varphi_n) - x_\nu(\varphi)) \right| \\ &\leq 2 \cdot \left(x_\nu(\varphi_n) - \limsup_{n \rightarrow +\infty} x_\nu(\varphi_n) \right). \end{aligned} \quad (5.153)$$

We can thus conclude from (5.152) and (5.153) that we have this following estimate,

$$\begin{aligned} &\left| \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi_n)] \right) - \text{meas} \left(\mathcal{H}_\infty(\varphi) \cap [0, x_\nu(\varphi)] \right) - (x_\nu(\varphi_n) - x_\nu(\varphi)) \right| \\ &\leq 2 \text{dist} \left(x_\nu(\varphi_n), \left[\liminf_{n \rightarrow +\infty} x_\nu(\varphi_n), \limsup_{n \rightarrow +\infty} x_\nu(\varphi_n) \right] \right). \end{aligned} \quad (5.154)$$

This last term is converging towards zero when n goes to infinity.

Conclusion of the proof

The set of ν such that the convergence $\text{meas}\{\varphi_n \geq \nu\} \xrightarrow{n \rightarrow +\infty} \text{meas}\{\varphi \geq \nu\}$ does not happen is of measure zero by hypothesis (5.49). The set of ν such that $\text{meas}\{\varphi = \nu\} \neq 0$ is also of measure zero. Therefore we know that for almost every $\nu \geq 0$

$$\text{meas} \left(\{\varphi_n^\sharp \geq \nu\} \triangle \{\varphi^\sharp \geq \nu\} \right) \xrightarrow{n \rightarrow +\infty} 0. \quad (5.155)$$

We can conclude, using the crucial lemma 3.4, that we have

$$\varphi_n^{\natural} \xrightarrow[n \rightarrow +\infty]{} \varphi^{\natural} \quad \text{in } L^p.$$

□

Remark :

To understand the reason why we have to exclude the cases where $\text{meas}\{\varphi = \nu\} \neq 0$, one can meditate the following counter-example. Let $\alpha \in [0, 1]$ and

$$\varphi_n(x) := \left(1 - \frac{1}{n}\right) \mathbb{1}_{[0,\alpha]}(x) + \left(1 + \frac{1}{n}\right) \mathbb{1}_{[\alpha,1]}(x).$$

We have φ_n converging towards $\mathbb{1}_{[0,1]}$ and verifying the hypothesis of the theorem. Nevertheless, if $\nu = 1$, $y_\nu(\varphi_n)$ is converging towards α that can take any value between 0 and 1 while $y_\nu(\varphi) = 0$.

Another aspect to point out in the case $\text{meas}\{\varphi = \nu\} = 0$ is the fact that we do have $y_\nu(\varphi_n)$ converging towards $y_\nu(\varphi)$ (by our proof) even if we do not have $x_\nu(\varphi_n)$ converging towards $x_\nu(\varphi)$. An interesting illustration of this phenomenon is

$$\begin{cases} \varphi_n^1(x) := \left(1 + \frac{1}{n}\right) \sin(\pi x) \mathbb{1}_{[0,1]}(x) + 2 \cdot \mathbb{1}_{[1,2]}(x), \\ \varphi_n^2(x) := \left(1 - \frac{1}{n}\right) \sin(\pi x) \mathbb{1}_{[0,1]}(x) + 2 \cdot \mathbb{1}_{[1,2]}(x). \end{cases}$$

Both sequences verify the hypothesis of Theorem 3.1 and converge towards

$$\varphi(x) := \sin(\pi x) \mathbb{1}_{[0,1]}(x) + 2 \cdot \mathbb{1}_{[1,2]}(x).$$

Nevertheless, for $\nu = 1$ we have $\text{meas}\{\varphi = \nu\} = 0$ but

$$x_\nu(\varphi_n^1) \xrightarrow[n \rightarrow +\infty]{} \frac{1}{2} \quad \text{and} \quad x_\nu(\varphi_n^2) \xrightarrow[n \rightarrow +\infty]{} 1.$$

4.10 Proof of Theorem 3.2

Let $\varphi \in W_+^{1,p}(\mathbb{R}_+)$. We approximate this function by voxel functions $\varphi_n \in \widetilde{\mathcal{E}}_n$ using the construction given by Corollary 3.5. The theorem is first established on $\Lambda\varphi_n$, the piece-wise linear approximation of φ_n defined at (5.54) using the tamping algorithm. Then the general inequality is obtained by passing to the limit size of the voxels tending to 0. To study how vary the $W^{1,p}$ half-norms during the tamping algorithm, it is enough at first to consider the case of voxels that are squares of size 1 which corresponds to $\varphi_n \in \mathcal{E}_n(\mathbb{R})$. The case with voxels of other sizes is obtained by a rescale. If $\varphi_n \in \mathcal{E}_n(\mathbb{R})$,

$$|\Lambda\varphi_n|_{\dot{W}^{1,p}}^p = \sum_{i=0}^{+\infty} |\varphi_n(i+1) - \varphi_n(i)|^p. \tag{5.156}$$

Step 1 : The elementary tamping algorithm does not increase (5.156)

The equivalence of algorithms that we explained during the proof of Lemma 2.1 also tells us that the only quantities involved in $|\Lambda\varphi_n|_{\dot{W}^{1,p}}^p$ that are changing during one step of the algorithm (with ξ as pivot) are the two following :

$$\begin{aligned} & - \left| \varphi_n(\xi) - \varphi_n(\xi - 1) \right|^p + \left| \varphi_n(\xi + 1) - \varphi_n(\xi) \right|^p, \\ & - \left| \varphi_n(\eta + 1) - \varphi_n(\eta) \right|^p, \end{aligned}$$

and they are becoming respectively

$$\begin{aligned} & - \left| \varphi_n(\xi + 1) - \varphi_n(\xi - 1) \right|^p, \\ & - \left| \varphi_n(\eta + 1) - \varphi_n(\xi) \right|^p + \left| \varphi_n(\eta) - \varphi_n(\xi) \right|^p. \end{aligned}$$

The difference of the two L^p norms of the derivative (before and after one step of the tamping algorithm) is simply the difference of these quantities because all the other differences appearing in (5.156) remain

unchanged. Using the fact that we have $\varphi_n(\eta + 1) \leq \varphi_n(\xi) \leq \varphi_n(\eta)$, this difference can be bounded from below as follows.

$$\begin{aligned} & \left| \varphi_n(\xi) - \varphi_n(\xi - 1) \right|^p + \left| \varphi_n(\xi + 1) - \varphi_n(\xi) \right|^p + \left| \varphi_n(\eta + 1) - \varphi_n(\eta) \right|^p \\ & - \left| \varphi_n(\xi + 1) - \varphi_n(\xi - 1) \right|^p - \left| \varphi_n(\eta + 1) - \varphi_n(\xi) \right|^p - \left| \varphi_n(\eta) - \varphi_n(\xi) \right|^p \\ & \geq \left| \varphi_n(\xi) - \varphi_n(\xi - 1) \right|^p + \left| \varphi_n(\xi + 1) - \varphi_n(\xi) \right|^p - \left| \varphi_n(\xi + 1) - \varphi_n(\xi - 1) \right|^p \geq 0, \end{aligned} \quad (5.157)$$

where the non-negativeness comes from $\varphi_n(\xi - 1) \geq \varphi_n(\xi)$ and $\varphi_n(\xi + 1) \geq \varphi_n(\xi)$. The term on the left of (5.157) will be referred as the *residual* of the elementary tamping algorithm because it gives account of the variation of the L^p norm of the gradient during one step on the tamping algorithm.

Step 2 : Iteration of the estimate (5.157).

The objective of this step is to explain what becomes the residual in (5.157) when we iterate the tamping algorithm in order to obtain the residual of the full-tamping algorithm. We have $\varphi_n \in \mathcal{E}_n(\mathbb{R})$ and then the sets $\underline{N} \cap \underline{M}(\varphi_n)$ and $\overline{N} \cap \overline{M}(\varphi_n)$ are well-defined, see definitions at (5.16)-(5.19). In this section we will explain that the iteration of the estimate (5.157) leads to the inequality

$$\int_{\mathbb{R}_+} |\nabla \Lambda \varphi_n|^p - |\nabla \Lambda \varphi_n^\sharp|^p \quad (5.158)$$

$$\geq \sum_{\xi \in \mathbb{N}} \left| \varphi_n(\xi + 1) - \varphi_n(\xi) \right|^p \cdot \left(1 - \mathbb{1}_{\mathcal{H}_\infty^c}(\xi) \cdot \mathbb{1}_{\mathcal{H}_\infty^c}(\xi + 1) \right) \quad (5.159)$$

$$- \sum_{X \in \text{Co}(\mathcal{H}_\infty(\varphi_n))} \left| \varphi_n\left(\inf X - \frac{1}{2}\right) - \varphi_n\left(\sup X + \frac{1}{2}\right) \right|^p, \quad (5.160)$$

where $\text{Co}(A)$ designates the set of all the connected components of a given set A . The indication function $(1 - \mathbb{1}_{\mathcal{H}_\infty^c}(\xi) \cdot \mathbb{1}_{\mathcal{H}_\infty^c}(\xi + 1))$ is used to select only the terms such that ξ or $\xi + 1$ is inside a hollow. We must note that the term at (5.159) is almost the integral of $|\nabla \Lambda \varphi_n|^p$ on $\mathcal{H}_\infty(\varphi_n)$ and it will converge towards the integral of $|\nabla \Lambda \varphi|^p$ inside the hollows when n goes to infinity. To prove this estimate, it must first be noted that for any $\xi \in \mathcal{H}_\infty(\varphi_n) \cap \mathbb{N}$ there exists a step of the tamping algorithm in which the voxel column that was initially at ξ is the pivot of the elementary tamping algorithm. The two columns of voxels that will be just before and just after are the columns of voxels that initially were at the positions respectively

$$\underline{\xi} := \max \{x \in \mathbb{N}, x < \xi, : \varphi_n(x) \geq \varphi_n(\xi)\} \quad (5.161)$$

and

$$\overline{\xi} := \min \{x \in \mathbb{N}, x > \xi, : \varphi_n(x) > \varphi_n(\xi)\}. \quad (5.162)$$

These conditions are necessary because of the condition $\xi \in \underline{N} \cap \underline{M}$ that must be satisfied to make ξ a pivot. The fact that this will actually be these two columns that will be neighboring the pivot comes from the fact that, regarding the definitions, these two columns cannot satisfy the condition being inside $\underline{N} \cap \underline{M}$ before ξ is chosen as pivot (this would contradict either the maximality or the minimality). With this observation, we get that the iteration of (5.157) leads to

$$\int_{\mathbb{R}_+} |\nabla \Lambda \varphi_n|^p - |\nabla \Lambda \varphi_n^\sharp|^p \quad (5.163)$$

$$\geq \sum_{\xi \in \mathcal{H}_\infty(\varphi_n)} \left| \varphi_n(\overline{\xi}) - \varphi_n(\xi) \right|^p + \left| \varphi_n(\underline{\xi}) - \varphi_n(\xi) \right|^p - \left| \varphi_n(\overline{\xi}) - \varphi_n(\underline{\xi}) \right|^p. \quad (5.164)$$

In order to conclude, we need to explain what are the cancellations that occur in the sum just above and why these cancellations lead to (5.159)-(5.160). Suppose for instance that $\overline{\xi} \in \mathcal{H}_\infty(\varphi_n)$ and $\varphi_n(\underline{\xi}) \geq \varphi_n(\overline{\xi})$. Then by definition

$$(\overline{\overline{\xi}}) = \underline{\xi}.$$

Moreover, when it is the turn of the voxels column initially at $\overline{\xi}$ to be chosen as pivot for the tamping algorithm we get in the estimate of the residual (5.164) a term which is

$$+ \left| \varphi_n(\overline{\xi}) - \varphi_n\left(\overline{(\overline{\xi})}\right) \right|^p$$

Section 4. Proofs of the main results

and this term exactly cancels the term $|\varphi_n(\bar{\xi}) - \varphi_n(\underline{\xi})|^p$ that we get at the step where ξ is the pivot. This reasoning works the same if on the contrary we have $\underline{\xi} \in \mathcal{H}_\infty(\varphi_n)$ and $\varphi_n(\underline{\xi}) \leq \varphi_n(\bar{\xi})$. These cancellations occur whenever $\underline{\xi}$ or $\bar{\xi}$ are in the hollows $\mathcal{H}_\infty(\varphi_n)$. Therefore, the remaining terms that contribute with a positive sign are the ones corresponding to columns that were neighboring at the very beginning - which gives the term (5.159). The remaining terms that contribute with a negative sign are the terms of kind $|\varphi_n(\bar{\xi}) - \varphi_n(\underline{\xi})|^p$ that cannot be canceled because neither $\bar{\xi}$ nor $\underline{\xi}$ are in $\mathcal{H}_\infty(\varphi_n)$ - and this gives (5.160).

Step 3 : Estimate of the negative term (5.160) in the residual.

The last step of the proof consists in passing to the limit $n \rightarrow \infty$ while the size of the cubes $\lambda_n \times \mu_n$ (width \times height) is vanishing. Since by definition of $\widetilde{\mathcal{E}}_n(\mathbb{R})$, the number of cubes is at most $n \times n$, then to obtain the convergence in L^p we have to impose the following decay rates conditions,

$$n \cdot \lambda_n \xrightarrow{n \rightarrow \infty} +\infty \quad \text{and} \quad n \cdot \mu_n \xrightarrow{n \rightarrow \infty} +\infty. \quad (5.165)$$

Observe that $\Lambda\varphi_n$ is a variation of the standard piece-wise linear continuous approximations in $W^{1,p}$. To obtain that the convergence is actually in $W^{1,p}$ we have to impose a control on the derivative of $\Lambda\varphi_n$. For that purpose we impose this final condition

$$\frac{\mu_n}{\lambda_n} \xrightarrow{n \rightarrow \infty} 0. \quad (5.166)$$

The main difficulty of the next step is to explain why the term (5.160) is vanishing and for that purpose we need to work again on this term. If we rescale the estimate of the residual (5.159)-(5.160) to have cubes of size $\lambda_n \times \mu_n$ then this term becomes

$$\int_{\mathbb{R}_+} |\nabla \Lambda\varphi_n|^p - |\nabla \Lambda\varphi_n^h|^p \quad (5.167)$$

$$\geq \sum_{\xi \in \mathbb{N}} \left| \varphi_n(\lambda_n \cdot (\xi + 1)) - \varphi_n(\lambda_n \cdot \xi) \right|^p \cdot \left(1 - \mathbb{1}_{\mathcal{H}_\infty^c}(\lambda_n \cdot \xi) \cdot \mathbb{1}_{\mathcal{H}_\infty^c}(\lambda_n \cdot (\xi + 1)) \right) \quad (5.168)$$

$$- \sum_{X \in \text{Co}(\mathcal{H}_\infty(\varphi_n))} \left| \varphi_n\left(\inf X - \frac{\lambda_n}{2}\right) - \varphi_n\left(\sup X + \frac{\lambda_n}{2}\right) \right|^p, \quad (5.169)$$

The objective of this third step is to prove that if the function φ is actually smooth, compactly supported with $\#\text{Co}(\mathcal{H}_\infty(\varphi)) < +\infty$ then we have the following estimate

$$\sum_{X \in \text{Co}(\mathcal{H}_\infty(\varphi_n))} \left| \varphi_n\left(\inf X - \frac{\lambda_n}{2}\right) - \varphi_n\left(\sup X + \frac{\lambda_n}{2}\right) \right|^p \leq \#\text{Co}(\mathcal{H}_\infty(\varphi)) \lambda_n^p \|\nabla\varphi\|_{L^\infty}^p. \quad (5.170)$$

With the construction of piece-wise linear approximation sequences given by Corollary 3.5, it is a classical result of numerical analysis that

$$\|\nabla \Lambda\varphi_n\|_{L^\infty} \leq \|\nabla\varphi\|_{L^\infty}. \quad (5.171)$$

The above inequality comes from the mean value theorem. We also have

$$\#\text{Co}(\mathcal{H}_\infty(\varphi_n)) \leq \#\text{Co}(\mathcal{H}_\infty(\varphi)). \quad (5.172)$$

Then to obtain (5.170), it is enough to explain why for $X \in \text{Co}(\mathcal{H}_\infty(\varphi_n))$ we have

$$\left| \varphi_n\left(\inf X - \frac{\lambda_n}{2}\right) - \varphi_n\left(\sup X + \frac{\lambda_n}{2}\right) \right| \leq \lambda_n \|\nabla \Lambda\varphi_n\|_{L^\infty}. \quad (5.173)$$

First we point out that neither $\xi_0 := \inf X - \frac{\lambda_n}{2}$ nor $\xi_1 := \sup X + \frac{\lambda_n}{2}$ are in $\mathcal{H}_\infty(\varphi_n)$. But we have $\xi_0 + \lambda_n$ and $\xi_1 - \lambda_n$ that belong to $\mathcal{H}_\infty(\varphi_n)$. We use the mean value theorem to write that for any $x \in \lambda_n \mathbb{N}$ we have

$$\left| \varphi_n(x) - \varphi_n(x + \lambda_n) \right| \leq \lambda_n \|\nabla \Lambda\varphi_n\|_{L^\infty}. \quad (5.174)$$

Without restriction of generality since ξ_0 and ξ_1 have symmetrical roles, we can suppose that

$$\varphi_n(\xi_0) \leq \varphi_n(\xi_1). \quad (5.175)$$

In the other hand, the fact that ξ_0 and ξ_1 are the two extremities of a connected component of the hollows of φ_n gives

$$\varphi_n(\xi_0) \geq \varphi_n(\xi_1 - \lambda_n), \quad (5.176)$$

where we used the fact that $\xi_1 - \lambda_n$ does belong to the considered connected component of the hollows. Now, combining (5.174) and (5.176) we obtain (5.173).

Step 4 : Passing to the limit $n \rightarrow +\infty$.

Now let $\varphi \in W^{1,p}$. We note φ_n the approximation of φ with voxels of size $\lambda_n \times \mu_n$ constructed in the proof of Corollary 3.5. With the standard theory of approximation by piece-wise linear continuous functions (using the fact that the height of the voxels vanishes faster than their width), we have

$$\Lambda\varphi_n \xrightarrow[n \rightarrow \infty]{W^{1,p}} \varphi \quad \text{and} \quad \Lambda\varphi_n^{\natural} \xrightarrow[n \rightarrow \infty]{W^{1,p}} \varphi^{\natural}. \quad (5.177)$$

Nevertheless, it is not enough to conclude because we do not have a priori the fact that (5.160) is vanishing in the general case with the estimate (5.170) because both $\#\text{Co}(\mathcal{H}_\infty(\varphi))$ and $\|\nabla\varphi\|_{L^\infty}$ may be infinite. We now define the voxel approximation of $\Lambda\varphi_n$ that we note

$$\psi_{n,k} := \left(\Lambda\varphi_n \right)_k.$$

The estimate (5.170) obtained in the previous steps becomes

$$\sum_{X \in \text{Co}(\mathcal{H}_\infty(\psi_{n,k}))} \left| \psi_{n,k} \left(\inf X - \frac{\lambda_k}{2} \right) - \psi_{n,k} \left(\sup X + \frac{\lambda_k}{2} \right) \right|^p \quad (5.178)$$

$$\leq \#\text{Co}(\mathcal{H}_\infty(\varphi_n)) \lambda_k^p \|\nabla\varphi_n\|_{L^\infty}^p. \quad (5.179)$$

We set

$$k_n := \min \left\{ k \in \mathbb{N} : n \cdot \#\text{Co}(\mathcal{H}_\infty(\varphi_n)) \cdot \|\nabla\varphi_n\|_{L^\infty}^p \leq \frac{1}{\lambda_k^p} \right\}. \quad (5.180)$$

We have $\max\{n, k_n\} \rightarrow \infty$ when $n \rightarrow \infty$ and we define

$$\psi_n := \psi_{n, \max\{n, k_n\}}.$$

By construction we still have the convergences

$$\Lambda\psi_n \xrightarrow[n \rightarrow \infty]{W^{1,p}} \varphi \quad \text{and} \quad \Lambda\psi_n^{\natural} \xrightarrow[n \rightarrow \infty]{W^{1,p}} \varphi^{\natural}, \quad (5.181)$$

but the term (5.179) is also vanishing when n goes to infinity because with condition (5.180) this term is lower than $\frac{1}{n}$. The theorem is proved. \square

4.11 Proof of the counter-example for the Riesz inequality

We want to explain why the condition $d \leq t < e - b$, implies

$$\begin{aligned} & \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathbb{1}_{[0,e]}(x) \cdot \left(\mathbb{1}_{[a,b]} + \mathbb{1}_{[c,d]} \right)(y) \cdot \mathbb{1}_{[-t,t]}(x-y) \, dx \, dy \\ & > \int_{\mathbb{R}_+} \int_{\mathbb{R}_+} \mathbb{1}_{[0,e]}(x) \cdot \mathbb{1}_{[a,b-c+d]}(y) \cdot \mathbb{1}_{[-t,t]}(x-y) \, dx \, dy. \end{aligned}$$

Since we are working with indicator functions, the integrals that we intend to compare are equal to the area of the support of the two-variable product function which is a subset of \mathbb{R}^2 . This construction follows the one by Riesz for his rearrangement inequality [69] and is illustrated at Figure 5.9. We consider \mathbb{R}^2 - its canonical base is noted (e_1, e_2) - with three axis. The first axis is $e_1 \cdot \mathbb{R}$ that we also call the x -axis, the second one is $e_2 \cdot \mathbb{R}$ (also called the y -axis) and the last one is $(e_1 - e_2) \cdot \mathbb{R}$ (the $(x - y)$ -axis). We place the set $[0, e]$ on the x -axis, the set $[a, b] \cup [c, d]$ on the y -axis and the set $[-t, t]$ on the $(x - y)$ -axis and we consider the cartesian product of these sets with their relative orthogonal right line. The considered area is then the intersection of all these sets as illustrated at figure 5.9 (the considered area is in light-grey at the center).

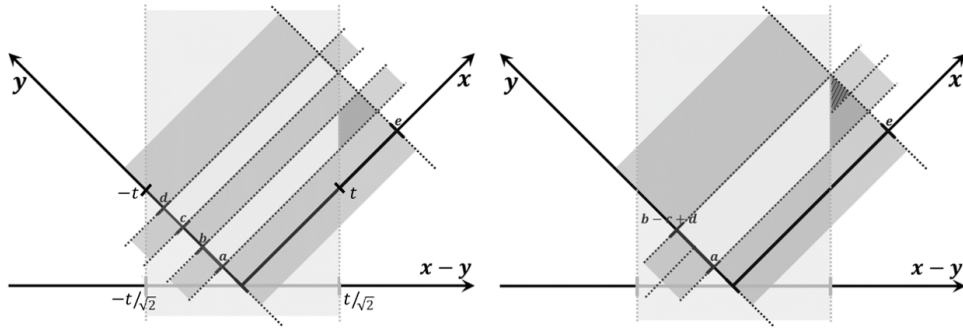


FIGURE 5.9 – Computation of the integrals as areas of the support of the integrand. The calculated area is the area in light grey at the center of the image. On the right we represented with hatches the area that is “lost” during the tamping process.

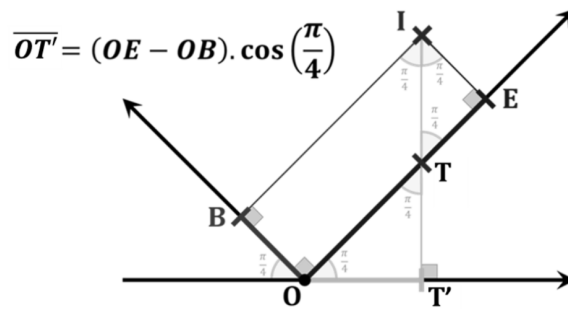


FIGURE 5.10 – Geometrical construction explaining the condition $t < e - b$. What happens at the equality case.

Why the condition $d \leq t < e - b$ makes the counter-example work is the following. First we want the point d on the y -axis to be inside the cylinder defined by the cartesian product of the segment $[-t, t]$ and its orthogonal right line the $(x + y)$ -axis. This point reaches the border of this cylinder when we have $t = d \cdot \cos(\pi/4)$, which gives the first inequality $d \leq t$. Indeed, if this inequality is not respected, then one may improve the considered area by sliding the segment $[c, d]$ along the y -axis in the decreasing sense. The other condition ensures that the area is decreased by the tamping. The area represented with hatches at Figure 5.9 (the area “lost” during the tamping) is positive. This area is positive when the point that is at the intersection of the right line orthogonal to the y -axis and passing by point b with the right line orthogonal to the x -axis passing by point e (we noted this point I at Figure 5.10) is outside the cylinder defined by the cartesian product of the segment $[-t, t]$ and its orthogonal right line the $(x + y)$ -axis. The reason why this corresponds to the condition $t < e - b$ is explained at Figure 5.10 where we drew the case of equality.

□

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