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# Essais en théorie de l'appariement et ses applications

Julien Combe

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Ecole des Hautes Etudes en Sciences Sociales

**THESE**

Pour l'obtention du grade de docteur en Sciences Économiques  
de l'École des Hautes Études en Sciences Sociales

*Discipline: Sciences Economiques - Spécialité: Analyse et Politique Economiques*

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Julien Combe

**Essais en théorie de l'appariement  
et ses applications.**

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Ecole des Hautes Etudes en Sciences Sociales

## PhD Thesis

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*Prepared and defended at the Paris School of Economics on October 24, 2017 by:*

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## Essays in matching theory and its applications

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# Résumé

Cette thèse étudie l'affectation centralisée des enseignants aux écoles et un nouveau modèle d'appariement inspirée par cette dernière.

Dans le premier chapitre, nous développons un modèle théorique de réaffectation afin d'étudier le problème de réaffecter des enseignants titulaires enseignant au sein d'un établissement et demandant une mutation. Le problème est similaire à celui d'affecter des élèves dans des écoles. Dans ce cas, l'algorithme à Acceptation Différée a été identifié comme étant le seul algorithme qui: i) est stable, ii) efficace et qui iii) incite les élèves à soumettre sincèrement leurs préférences. La différence principale avec le problème d'affecter des élèves aux écoles est que les enseignants ont déjà une position initiale au sein d'un établissement. On doit donc prendre en compte une contrainte additionnelle, la Rationalité Individuelle (RI): un enseignant doit être affecté dans un établissement qu'il préfère faiblement à son établissement d'origine. Pour prendre en compte cette contrainte, une modification de l'algorithme à Acceptation Différée a été identifiée dans la littérature académique et utilisé en pratique pour affecter les enseignants aux écoles en France. Nous montrons que cet algorithme modifié souffre d'un important défaut: il n'est pas efficace au sens fort. Il est en effet possible de réaffecter les enseignants aux écoles de telle sorte que: i) les enseignants obtiennent une école qu'ils préfèrent et ii) les écoles obtiennent des enseignants mieux classés. Partant de ce constat, nous identifions la classe de tous les algorithmes, les algorithmes Block-Exchange (BE), qui ne souffrent pas de ce défaut. Parmi eux, nous montrons qu'il en existe un unique qui incite les enseignants à soumettre leurs préférences sincèrement: le Teacher Optimal Block-Exchange algorithm (TO-BE). En utilisant un modèle de marché large, nous montrons théoriquement que ces algorithmes ont de meilleures performances en termes de mouvement et de bien-être des enseignants que l'algorithme actuel. Nous utilisons ensuite une base de données sur l'affectation des enseignants aux écoles du secondaire en France en 2013 pour quantifier les gains possibles que nos algorithmes peuvent apporter.

Dans un cadre de réaffectation pur sans enseignant néotitulaire et places vacantes, nous montrons qu'il est possible de plus que double le nombre d'enseignants obtenant une nouvelle affectation.

Dans le second chapitre, nous concevons un algorithme pratique, inspiré de nos résultats du chapitre précédent, pour la procédure française d'affectation des enseignants du secondaire. Plus globalement, cette conception a également pour but de fournir un outil face à deux problèmes importants communs aux pays de l'OCDE: i) le manque d'attractivité de la profession enseignants et ii) les importantes inégalités de réussites des élèves issus de milieux sociaux différents. Nous considérons l'ensemble du marché français composé des enseignants titulaires demandant une réaffectation, les enseignants sans affectation initiale et des places vacantes. Améliorer la mobilité des enseignants permet de leur donner de meilleures perspectives de carrière ce qui peut potentiellement attirer plus d'entrants dans la profession. Mais cette mobilité accrue peut entraîner l'affectation de plus d'enseignants peu expérimentés au sein d'académies déjà très défavorisées, affectant in fine la réussite des élèves au sein de celles-ci. Nous proposons un algorithme flexible qui permet de mieux contrôler le mouvement et la distribution des enseignants au sein de régions, notamment celles très désavantagées. En utilisant les données françaises d'affectation de 2013, nous simulons plusieurs scénarios contre factuels et montrons que notre algorithme peut prendre en compte plusieurs objectifs de politique publique.

Dans un troisième chapitre, nous proposons un nouveau modèle d'appariement inspiré par l'affectation des enseignants et des élèves aux écoles. Deux types d'agents doivent être affectés ensemble à des objets. Une application possible est l'affectation de managers et travailleurs à des projets. Ce modèle est un hybride entre le modèle classique d'affectation avec deux types où des hommes (ou élèves) et femmes (ou écoles) ont des préférences sur l'un et l'autre et doivent être appariées ensemble et un modèle d'allocation de logements ou des logements sans préférences doivent être affectés à des agents ayant des préférences sur ces logements. Nous explorons théoriquement les questions classiques de la littérature d'appariement en définissant un concept de stabilité: par exemple il ne doit pas exister de manager (ou travailleur) qui préférerait renvoyer son partenaire et demander à un autre travailleur (ou manager) de le rejoindre au sein du projet qui lui a été affecté. Nous montrons que ce concept de stabilité peut ne pas exister. Cependant, cette impossibilité dépend fortement de la possibilité pour un manager et pour un travailleur de pouvoir renvoyer son partenaire assigné et garder son projet. Cela pose la question suivante: qui possède, entre un manager et un

travailleur, le projet ? Nous introduisons donc une notion de structure de propriété qui définit, pour chaque triplet de manager-travailleur-projet possible, le propriétaire du projet au sein de ce triplet. On peut donc ensuite considérer une notion de stabilité où uniquement les propriétaires peuvent renvoyer leur partenaire et demander à un autre de les rejoindre au sein de leur projet. Nous montrons que si l'on donne toujours la propriété au même type d'agent, par exemple les managers possèdent toujours les projets, alors l'existence d'appariements stables est rétablie. Ce résultat peut fournir une explication sur la raison pour laquelle on tend à n'observer que ce type de structure en pratique. Il souligne également une structure similaire avec celle des appariements stables dans le modèle standard d'affectation avec deux types. Nous explorons ce lien en définissant des concepts de coeur et montrons que, contrairement au cadre classique, ils peuvent être vides.

**Mots clés:** Appariement, Conception de marchés, Affectation des enseignants.



# Summary

This thesis studies the centralized assignment of teachers to schools and a new matching framework inspired by it.

In the first chapter, we develop a theoretical model of reassignment to study the problem of reassigning tenured teachers who already have a position and are willing to move to another school. The problem is similar to the one of assigning students to schools. In this case, the well known Deferred Acceptance algorithm has been identified as the only algorithm that: i) is stable ii) efficient and iii) gives incentives to students to report their true preferences. The main difference with the problem of assigning students to schools is that teachers have an initial assignment. One has to consider an additional constraint, Individual Rationality (IR): a teacher must receive a school that he weakly prefers to his initial one. To incorporate this constraint, a modification of the Deferred Acceptance algorithm has been identified in the academic literature and used in practice to assign teachers to schools in France. We show that this modified algorithm has a serious drawback: it is not efficient in a strong sense. Indeed, it is possible to reassign teachers to schools such that both: i) teachers obtain a school that they prefer and ii) schools are assigned teachers that they rank higher. Thus, we identify the class of all algorithms, the Block-Exchange (BE) algorithms, that do not suffer from this drawback. Among them, we show that there is a unique one that gives good incentives to teachers to report their true preferences, the Teacher Optimal Block-Exchange algorithm (TO-BE). In using a large market setting, we theoretically show that these algorithms perform better in terms of movement and welfare for teachers than the currently used one. We then use a dataset on the assignment of teachers to schools in France in 2013 to quantify the possible gains that can bring our algorithms. In a reassignment setting with no newly tenured teachers or empty seats, we show that we can more than double the number of teachers obtaining a new assignment.

In the second chapter, we aim to design a practical algorithm, inspired by our findings in the



previous chapter, for the French assignment system of teachers to schools. More generally, this design also aims to provide a tool about two important issues common to OECD countries: i) the lack of attractiveness of the teaching profession and ii) the high achievement inequality between students from different social backgrounds. We consider the complete French market composed of tenured teachers looking for a reassignment, newly tenured teachers with no initial assignment and empty positions. In improving the mobility of teachers, one can give them better career perspectives and so potentially attract more teachers into the profession. But in doing so, it can also hurt deprived regions in assigning more teachers with low experience to them and ultimately the students from these regions. We propose a flexible algorithm that allows to better control the movement and distribution of teachers across regions, especially deprived ones. Using the data of the French assignment of teachers in 2013, we simulate several counterfactuals and show that our algorithm can accommodate a wide range of policy objectives.

In the third chapter, we propose a new matching setting inspired by the assignment of teachers and students to schools. Two types of agents have to be assigned together to objects. One possible application is the assignment of managers and workers to projects. This model is a hybrid between the classical two-sided matching market where men (or students) and women (or schools) have preferences over each others and have to be matched together; and the allocation problem about assigning houses with no preferences to agents having preferences over them. We theoretically explore the classical questions of the matching literature in defining a concept of stability: for instance no manager (or worker) can prefer to dismiss his partner and ask another worker (or manager) to join his assigned project. We show that such concept may fail to exist. However, this impossibility strongly relies on the ability for both managers and workers to dismiss their assigned partner and keep their project. It asks the question: who owns, between a manager and a worker, their assigned project ? Thus, we introduce the notion of ownership structure that defines, for each manager-worker-object triplet, who owns the object. Then one can define a stability notion where only owners can dismiss their partner and ask another one to join their project. We show that if one always gives the ownership to the same type of agents, e.g. managers are always the owners of the projects, then one can restore the existence of stable matchings. This result can give an explanation about why we tend to always observe such type of ownerships in practice. It also highlights a similar structure than the one of stable matchings in the standard two-sided matching

framework. We additionally explore this link in defining core notions and show that, contrary to the standard setting, they can all be empty.

**Keywords:** Matching, Market Design, Teacher Assignment



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# Introduction

## Introduction Générale

L'étude des problèmes d'appariement<sup>2</sup> a pris une place croissante en Sciences Economiques au cours des 20 dernières années, aboutissant en 2012 à l'attribution du Prix de la Banque de Suède en l'honneur d'Alfred Nobel en Sciences Economiques à Alvin Roth et Lloyd Shapley. Une part importante de ce succès vient sûrement du lien étroit qu'il existe entre la théorie et les applications pratiques qui ont menées à nombre de recommandations politiques réussies. Cette thèse essaie de suivre cette tradition en motivant son analyse par le problème d'affectation des enseignants avec une application au cas français. Dans le premier chapitre, notre but est de développer un cadre théorique afin d'étudier des problèmes de réaffectation, une classe de problèmes d'appariement à laquelle appartient l'affectation des enseignants. Ce cadre nous permet d'identifier une classe importante de mécanismes à utiliser dans ces problèmes tout en fournissant des prédictions théoriques et de premières évaluations empiriques des améliorations possibles. Dans un second chapitre, appuyés par nos prédictions théoriques précédentes, nous tentons de proposer la conception pratique d'une procédure<sup>3</sup> alternative à celle utilisée actuellement en France; en prenant en compte les spécificités concernant les inégalités entre académies et les différents objectifs de politiques publiques. Dans le dernier chapitre, inspirés par le lien qu'il existe entre l'affectation des enseignants et des élèves au sein des écoles, nous explorons théoriquement un nouveau modèle d'appariement intermédiaire où deux types d'agents doivent être affectés à un ensemble d'objets.

Dans cette introduction, nous commençons par rappeler la littérature sur les problèmes d'appariement. Nous nous concentrons sur les deux grands types de modèles qui sont pertinents pour les travaux

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<sup>2</sup>Dans ce qui suit nous utiliserons indifféremment: apparier, affecter et allouer ainsi que les noms correspondants. Le verbe apparier sera utilisé dans un sens plus général que juste "mettre par paires" et se référera à l'action d'affecter des agents entre eux.

<sup>3</sup>Dans ce qui suit, procédure, mécanisme ou algorithme seront utilisés indifféremment.



de cette thèse tout en illustrant quelques exemples de réformes politiques au fil des années. Nous introduisons ensuite les travaux réalisés dans le cadre de cette thèse en soulignant leurs connections avec la littérature exposée.

## Problèmes d'appariement: présentation

### Pourquoi étudier les problèmes d'appariement ?

Il existe un nombre important de définitions de ce qu'est la Science Économique. L'économiste français Edmond Malinvaud fournit la suivante<sup>4</sup>:

*L'économie est la science qui étudie comment des ressources rares sont employées pour la satisfaction des besoins des hommes vivant en société. Elle s'intéresse d'une part aux opérations essentielles que sont la production, la distribution et la consommation des biens, d'autre part aux institutions et aux activités ayant pour objet de faciliter ces opérations.*

Cette dernière met en lumière deux points importants. Une partie de la Science Économique s'intéresse à la distribution des ressources rares. La question est dès lors de décider de la façon dont quelqu'un peut allouer, par exemple, un bien à une multitude de personnes (agents) qui le désireraient sachant qu'à la fin, un seul d'entre eux peut l'obtenir. Comment résoudre ce conflit de préférences ? Comment résoudre ce problème basique de congestion ? Une autre partie de la Science Économique s'intéresse quant à elle aux institutions aux travers desquelles la distribution est réalisée. Au cours de l'Histoire, une institution particulière a émergé: le marché, où les personnes peuvent échanger des biens et services afin de satisfaire leurs besoins. Ces marchés peuvent prendre une multitude de formes. Par exemple, l'échange peut se faire via un système de troc. Cependant, la plupart des marchés actuels étudiés par les économistes prennent la forme d'un échange de biens et services contre de la monnaie via un système de prix. Afin de résoudre le problème de congestion susnommé, les prix s'ajustent. Si plus d'agents demandent un bien qu'il y en a de disponible, alors le prix augmente et vice-versa. Les économistes ont rapidement commencé à se concentrer sur l'étude de ce type de marchés avec un système de prix en étudiant leurs propriétés. En effet, Malinvaud, au sein du même écrit, reporta:

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<sup>4</sup>Tirée de: Leçons de théorie microéconomique, Dunod, 4th edition, 1982.

*Dans sa double recherche explicative et normative, notre science a été amenée à attribuer un rôle central aux prix qui président aux échanges de biens entre agents. Ces prix reflètent pour les individus, de manière plus ou moins exacte, la rareté sociale des produits qu’il achète et qu’il vend.*

Les économistes ont identifié le rôle central joué par les prix dans la gestion des échanges sans que ces derniers soient les seules solutions possibles. Revenons en effet en arrière et demandons nous: pour certains types de ressources, peut-on ou veut-on utiliser la monnaie et un système de prix comme moyen d’allocation de cette ressource ? Par exemple, de nombreux patients avec un défaut rénal ont besoin d’une transplantation et il y a généralement plus de patients que de reins disponibles. C’est donc un problème de congestion: comment peut-on allouer un nombre limité de ressources, ici des reins, à des agents qui en ont besoin sachant qu’il y a plus d’agents que de reins disponibles ? Une réponse naturelle suite aux écrits ci-dessus serait de définir un système de prix pour des reins et d’utiliser de la monnaie. C’est ce qu’a choisi l’Iran depuis 1988: les donneurs sont payés afin de donner leurs reins. Par exemple, [Becker and Elias \(2007\)](#) discutent de la possibilité d’introduire aux Etats-Unis ce système de prix en évaluant le potentiel prix d’équilibre pour les reins en prenant en compte plusieurs facteurs tels que la valeur monétaire statistique d’une vie, le risque de mortalité ou encore la qualité de vie et la valeur du temps pour les individus<sup>5</sup>. Cependant, la plupart des pays interdisent par la loi l’usage de transferts monétaires afin d’allouer des organes. Comme mentionné par [Roth \(2007\)](#), l’Article 21 du Protocole Additionnel à la Convention des Droits de l’Homme et de la Biomedecine datant de 2002 sur la Transplantation d’Organes et de Tissus d’Origine Humaine stipule que<sup>6</sup>

*Le corps humain et ses parties ne doivent pas, en tant que tels, amener à des gains monétaires.*

La loi Nationale de Transplantation d’Organes<sup>7</sup> de 1984 aux Etats-Unis contient un texte similaire. Il existe un sentiment général et répandu que la monnaie n’est pas un instrument approprié pour traiter des échanges d’organes. [Roth \(2007\)](#) appela ce sentiment *répugnance*<sup>8</sup>. Ce concept de répugnance

<sup>5</sup>Des articles plus anciens discutent également de cette possibilité pour les dons d’organes depuis des donneurs décédés, le lecteur intéressé est invité à se référer à l’article mentionné pour plus de détails sur cette littérature.

<sup>6</sup>“*The human body and its parts shall not, as such, give rise to financial gain*”, (Article 21 of the Council of Europe’s (2002) Additional Protocol to the Convention on Human Rights and Biomedicine, on Transplantation of Organs and Tissues of Human Origin)

<sup>7</sup>*National Organ Transplant Act*

<sup>8</sup>*Repugnance* en anglais qui renvoie à la notion de dégoût moral.

est bien entendu différent et changeant entre localisations géographiques et au fil du temps. Il peut être influencé par une multitude de facteurs culturels, psychologiques, religieux...etc. Par exemple, en mettant de côté le marché noir, les pays diffèrent dans leur législation sur la consommation de certains stupéfiants ou encore sur la prostitution. Roth (2007) mentionna également la consommation de viande de cheval qui est interdite par exemple en Californie à l'encontre d'autres régions du globe. Au fil du temps, des transactions considérées comme répugnantes peuvent ne plus l'être à présent. Il mentionna notamment, p.3, l'exemple bien connu suivant<sup>9</sup>:

*Le prêt d'argent avec intérêts était considéré à une époque comme répugnant mais ne l'est plus (à l'exception importante de la loi islamique qui est vastement interprétée comme l'interdisant).*

L'utilisation des ajustement de prix pour résoudre le problème de congestion de certains problèmes d'allocation peut également être considérée en tant que telle comme opposée à l'objectif de politique publique fixé. L'éducation publique et généralisée est considérée dans beaucoup de pays comme un droit fondamental et comme objectif important de politique publique grâce aux externalités positives qu'elle génère pour la société. Si plusieurs candidats souhaitent être affectés à une école, l'augmentation des frais de scolarité, c.a.d. le prix de l'école, n'est pas considérée comme un outil politique utile afin de résoudre la congestion. En effet, cela pourrait exclure certains candidats issus de familles à revenus plus modestes qui ne pourraient pas payer ces frais. Un autre exemple important est celui de l'allocation des logements sociaux. L'objectif politique de fournir des logements sociaux est de permettre à des familles relativement plus pauvres d'accéder à un logement dans des localités où le marché immobilier, au travers des ajustement des prix, les exclue de facto. La motivation étant que la diversité sociale crée des externalités positives pour la société dans son ensemble. La motivation des politiques de logements sociaux exclue par nature l'utilisation du loyer, c.a.d. du prix, comme un moyen d'ajustement afin de résoudre ce problème d'allocation des logements aux familles<sup>10</sup>.

Il est important de débattre sur la qualité répugnante ou non de certaines transactions monétaires.

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<sup>9</sup>“Lending money for interest was once widely repugnant, and no longer is, (with the important exception that Islamic law is commonly interpreted as prohibiting it).”

<sup>10</sup>Il est important ici de mentionner que des courants économiques, dans une tradition de l'école de Chicago, mentionneraient que la distortion ou la non utilisation du prix serait une erreur et entacherait l'allocation optimale. Ils préconiseraient plutôt un système de bourses directement aux familles concernées. Bien entendu, il se pose le problème du coût d'un tel financement. Comme nous le verrons, nous allons vite mettre de côté le débat sur l'utilisation ou non d'un système d'ajustement des prix.

Par exemple, [Leider and Roth \(2010\)](#) menèrent une enquête auprès d'un échantillon représentatif d'américains afin d'identifier les opinions sur l'utilisation d'un marché de reins avec prix. Comme nous le mentionnerons plus tard, décider ou non d'utiliser les salaires afin de résoudre le problème de congestion dans l'allocation des enseignants est un débat important de politique publique. Cependant, certains économistes ont choisi, comme mentionné dans [Roth \(2007\)](#), de s'abstraire des raisons poussant telles ou telles transactions à être considérées comme répugnantes et ont considéré la notion de répugnance des transactions monétaires sur certains marchés comme une contrainte. Une fois que l'on accepte cette contrainte, le but est de trouver d'autres méthodes afin de résoudre le problème fondamental de congestion, c.a.d. de *concevoir le marché*<sup>11</sup>. Cette considération fondamentale est la motivation de base des économistes s'intéressant aux problèmes d'appariement et à la conception de marchés: ils souhaitent développer des théories et solutions pratiques pour résoudre ces problèmes de congestion tout en prenant en compte les objectifs de politiques publiques des marchés pour lesquels les transactions monétaires ne sont pas possibles ou souhaitées. Les théories et solutions proposées sont intrinsèquement liées aux marchés concrets que l'on souhaite concevoir. Plusieurs problèmes pratiques peuvent appartenir à la même classe de problèmes d'allocation et peuvent utiliser les mêmes solutions que celles qui ont été développées pour cette classe. Les objectifs pour l'allocation des élèves au sein des écoles vont être fondamentalement différents de ceux utilisés pour allouer des reins à des patients qui vont être à leur tour différents de ceux de l'allocation des logements sociaux. Tout ces problèmes ont vu leurs propres solutions émerger. Cette thèse va s'atteler à montrer que le problème de réallouer des enseignants à des écoles est conceptuellement différent de celui d'allouer des élèves à des écoles. Ce problème va partager des similarités avec ce dernier tout comme avec celui de réallouer des logements sociaux entre des occupants. Dans la suite de cette introduction, nous allons exposer les résultats de la littérature en se concentrant sur ces deux types de problèmes<sup>12</sup>.

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<sup>11</sup>Le terme anglais utilisé dans la littérature est *market design*.

<sup>12</sup>De par la nature de cette thèse, nous choisissons volontairement de ne pas traiter les résultats concernant une autre application de la littérature d'appariement: celle de l'allocation des reins, qui a également eu des impacts politiques importants. Une différence notable est que la prise en compte des incitations des patients à transmettre leurs caractéristiques est de moindre importance. Le lecteur intéressé peut facilement trouver des références sur le sujet. Une excellente introduction à ce type de problèmes est faite dans [Sonmez and Unver \(2013\)](#). Pour une présentation brève des problématiques récentes, le lecteur peut également se référer à [Ashlagi and Roth \(2012\)](#).

## Marchés d'appariement avec deux types

Le problème d'allouer des élèves au sein d'écoles, connu comme le *college admission problem*, a été introduit dans l'article fondateur de Gale and Shapley (1962). Ce problème est simple: des élèves doivent être affectés dans des écoles, chaque école ayant un nombre limité de places. Chaque élève a un ordre de préférences sur les écoles et chaque école a également un ordre de préférences sur les élèves. Il est supposé que ces préférences sont strictes de telle façon que les élèves et écoles ne peuvent être indifférentes. De façon plus abstraite, nous faisons face à un problème où *deux types* d'agents, ici élèves et écoles, ont des préférences les uns sur les autres. Ils commencent par ne pas être affectés et voudraient l'être ensemble. Pour un décideur politique, la question est de décider comment utiliser l'information des listes de préférences de chacun des types d'agents afin de décider d'un appariement. Une première étape est de décider des bonnes propriétés qu'un appariement doit satisfaire vis à vis de ces informations que sont les préférences. Une première propriété intuitive, que les économistes appellent *rationalité individuelle* (RI) ou *contrainte de participation*, est que si un élève (resp. une école), préfère rester seul que d'être affecté à une certaine école (resp. élève) alors il ne faut pas les affecter ensemble. Pour les économistes, une autre propriété largement utilisée est celle d'*optimalité de Pareto*. Dans ce contexte, les deux types d'agents ont des préférences que l'on devrait utiliser pour décider d'une allocation. Une allocation est dite Pareto-optimale (ou Pareto-efficace ou des fois juste efficace) si on ne peut pas réaffecter les élèves aux écoles de sorte que tous les élèves et écoles soient indifférents ou préfèrent strictement leur nouvelle allocation et certains strictement<sup>13</sup>. Bien que la Pareto-efficacité soit une propriété attrayante, il existe une multitude d'allocations différentes qui sont Pareto-efficaces et certaines d'entre elles peuvent ne pas être très attractives par rapport à d'autres critères. Dans ce contexte, Gale et Shapley ont proposé un concept naturel additionnel: celui de *stabilité*. Une allocation est dite stable si elle est individuellement rationnelle et qu'il n'existe pas un élève et une école qui ne sont pas affectés ensemble mais préféreraient l'être. Si une telle paire existe, elle est appelée une *paire bloquante*. Un appariement est donc stable s'il est individuellement rationnel et qu'il n'y a pas de paire bloquante<sup>14</sup>. Cette propriété intuitive peut s'interpréter de deux façons:

- D'un point de vue normatif, elle peut être vue comme un critère de *justice*, de *justesse*. En

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<sup>13</sup>Dans ce contexte avec préférences strictes, être indifférent est équivalent à garder la même allocation.

<sup>14</sup>On peut prouver que tout matching stable doit être Pareto-efficace au sens dans lequel nous l'avons défini.

effet, une allocation non stable verrait un élève se faire refuser l'entrée au sein d'une école qu'il préfère et cette école le préférerait en retour à l'un des élèves qui lui aura été accepté au sein de celle-ci. Le premier élève pourrait donc "envier" ce dernier au sein de cette école qu'il préfère et cette envie serait justifiée puisque l'école le préfère également. Cela pourrait donc créer ce qui est appelé une *envie justifiée* pour le premier élève envers le second.

- D'un point de vue positif, une allocation non stable ne pourrait pas perdurer très longtemps. En effet, si un élève et une école préféreraient être affectés ensemble alors l'école renverrait son élève le moins préféré et demanderait au premier élève de la rejoindre. Roth (1991) a en effet étudié différents marchés d'appariement de jeunes docteurs et chirurgiens au Royaume-Uni et nota que les procédures menant à un appariement stable étaient toujours utilisées alors que la plupart de celles menant à des appariement non stables ont été abandonnées.

Ayant défini ce qu'est un appariement stable, la prochaine question naturelle est de se demander si un tel appariement peut toujours être obtenu peu importe les préférences des élèves et des écoles. Et si oui, *comment* peut-on en trouver un ? Gale et Shapley donnèrent, avec une preuve élégante, une réponse affirmative à la première question et une procédure pratique pour la deuxième. Ils proposèrent un algorithme, appelé *l'algorithme de Gale et Shapley* ou également *l'algorithme à Acceptation Différée* (AD) qui, pour toutes préférences fournies, retourne un appariement stable. Ce dernier fonctionne comme suit:

**Etape Préliminaire.** Le décideur politique choisit les élèves ou les écoles comme étant *les proposants* pour la suite de l'algorithme. Dans ce qui suit, nous supposons que les élèves seront les proposants<sup>15</sup>.

**Etape 1.** Chaque élève postule à son école qu'il a classée première. Les écoles acceptent *temporairement* tous les élèves qui postulent. Si une école reçoit plus de postulants qu'elle a de places disponibles, elle n'accepte que ceux qui sont les mieux classés dans ses préférences jusqu'à remplir son nombre de places et rejette les autres.

**Etape k.** Tous élèves rejetés à l'étape  $k - 1$  postulent à leur école préférée parmi celles qui ne les ont pas encore rejetés aux étapes précédentes. Une école considère tous les élèves qu'elle a

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<sup>15</sup>Il suffit d'intervertir élèves et écoles dans ce qui suit pour obtenir l'autre version où les écoles proposent. Si les écoles proposent, alors on considère les élèves comme ayant une "capacité de 1" dans la suite, puisqu'ils ne peuvent être affectés qu'à une seule école. Une école ayant  $q$  places doit proposer à ses  $q$  meilleurs élèves classés.

temporairement acceptés à l'étape précédente ainsi que les nouveaux postulants: s'il y en a plus que son nombre de places disponibles, elle sélectionne encore une fois les mieux classés au sein de ses préférences et rejette les autres<sup>16</sup>.

Ce processus continue jusqu'à ce qu'il n'y ait plus aucun rejet. L'allocation définie à cette dernière étape sera la définitive. Comme il y a un nombre fini d'écoles et d'élèves et, qu'à chaque étape, au moins un élève descend au sein de sa liste de préférences en postulant à une nouvelle école, cet algorithme s'arrête après un nombre fini d'étapes. Comme à chaque étape, un élève n'est rejeté d'une école que si un autre préféré par l'école est accepté, on peut facilement montrer que l'allocation finale est stable. Une autre propriété intéressante de cet algorithme est qu'il est "rapide". En effet, s'il y a  $n$  élèves et  $m$  écoles, il faut au plus  $n \times m$  étapes pour que l'algorithme s'arrête. En Science de l'Informatique, plus précisément en théorie de la complexité, ce type d'algorithmes est appelé *polynomial* c.a.d. que son temps d'exécution est une fonction polynomiale de la taille de ses entrées, ici la taille des listes de préférences des élèves et écoles. Puisque les problèmes d'appariement s'intéressent à des problématiques d'allocation concrètes, avoir des algorithmes pratiques et efficaces est une condition importante<sup>17</sup>. Dans leur article, Gale et Shapley prouvèrent également la propriété importante suivante: si les élèves sont les proposants, alors l'allocation stable obtenue est celle préférée par tous les élèves et la moins préférée par toutes les écoles de toutes les allocations stables possibles. Cette allocation est appelée *l'Allocation Stable Optimale pour les Eleves*. On ne peut donc pas trouver un autre appariement stable que tous les élèves préféreraient faiblement (et certains strictement) à celui obtenu par l'algorithme AD où les élèves sont les proposants. Le résultat est évidemment symétrique si les écoles proposent, l'appariement obtenu étant appelé *l'Allocation Stable Optimale pour les Ecoles*, de telle sorte qu'il existe un arbitrage entre favoriser les élèves ou les écoles.

Dans les années 80, une importante littérature s'est développée afin d'étudier les allocations stables, l'algorithme AD et leurs propriétés. Notre objectif n'est pas de fournir une liste exhaustive de l'ensemble de ces résultats, le lecteur intéressé est invité à se référer à l'excellent manuel de [Roth and Sotomayor \(1990\)](#) qui expose formellement l'ensemble de ces résultats. Cependant, nous allons

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<sup>16</sup>On peut noter qu'un étudiant accepté à l'étape précédente peut être rejeté à cette étape. C'est pourquoi on parle de décision d'acceptation temporaire et qui ne sera définitive qu'à la dernière étape de l'algorithme.

<sup>17</sup>La littérature d'appariement est également importante en Science de l'Informatique, initiée par Donald Knuth. Il y a par exemple nombre de résultats dans la branche de la théorie de la complexité reliés aux problématiques d'appariement. Un nombre croissant de travaux sont à l'intersection de la Science Economique et la Science de l'Informatique. Le Chapitre 2 présentera un résultat lié à la théorie de la complexité.

en mentionner un essentiel: les incitations. Inspiré par une solide tradition de théorie des jeux en Science Economique, Roth (1982) a étudié les propriétés incitatives de l'algorithme AD: est ce qu'un élève ou une école pourrait fournir une fausse liste de préférences afin d'obtenir une allocation qu'il ou elle préfère à celui qu'il/elle obtiendrait en étant sincère dans sa liste ? Son premier résultat n'est pas très encourageant:

**Roth (1982), Théorème 3.** Il n'existe aucune procédure retournant systématiquement une allocation stable et pour laquelle fournir ses vraies préférences est une stratégie dominante.

Cela implique que l'algorithme AD peut être manipulé. Le concept de stratégie dominante est standard en théorie des jeux: il requiert que peu importe ce que les autres agents (élèves et écoles) fournissent comme préférences, un élève ou une école ne peut obtenir une allocation qu'il/elle préfère strictement en soumettant une fausse liste de préférences. Une procédure qui ne souffre pas de ce type de manipulation est appelée *non manipulable*. Cependant, le résultat suivant est plus positif:

**Roth (1982), Théorème 5.** Si les élèves sont les proposants sous l'algorithme AD, alors ce dernier est non manipulable par les élèves<sup>18</sup>.

En utilisant le premier théorème, nous savons que sous l'algorithme AD où les élèves proposent, les écoles peuvent éventuellement manipuler les préférences qu'elles fournissent. Cependant, le deuxième théorème montre qu'aucun élève ne pourra le faire. Une version plus forte du deuxième théorème a également été prouvée indépendamment par Dubins and Freedman (1981): aucun groupe d'élèves ne peut conjointement soumettre de fausses préférences et tous obtenir une allocations qu'ils préfèrent strictement<sup>19</sup>. Ce résultat est encourageant. En effet, généralement en pratique, les écoles ne sont pas stratégiques. Leurs préférences au sein de la procédure d'affectation sont souvent déterminées par la loi (dans ce cas, nous parlons de *priorités*) et ne sont pas stratégiquement fournies à l'algorithme contrairement aux listes de préférences des élèves. La non-manipulabilité est une propriété importante pour un mécanisme. C'est probablement celle que les économistes,

<sup>18</sup>La version symétrique pour les écoles est plus subtile. Dans le cas où chaque école n'a qu'une seule place, appelé environnement *un-pour-un*, le résultat symétrique est trivialement vrai puisqu'il n'y a plus de différence conceptuelle entre élèves et écoles. Cependant, Roth (1985) montra que dans un environnement *plusieurs-pour-un* où les écoles peuvent avoir plusieurs places, la proposition symétrique pour AD ne tient plus. La principale différence est que dans ce cadre, une école se doit maintenant de comparer des ensembles d'élèves à partir de sa liste de préférences qui elle ne classe que les élèves individuels entre eux.

<sup>19</sup>Dans la littérature, cette notion est appelée *non manipulabilité par groupe*.



et les théoriciens des jeux, ont introduite comme étant une propriété importante, voir essentielle, contrairement à la littérature de Science de l'Informatique ou de Recherche Opérationnelle.

Roth (1984) fût le premier à étudier un marché concret d'appariement: celui affectant les internes et résidents de médecine aux hôpitaux aux Etats-Unis. Ce type d'affectation fait parti de la classe de problèmes analysée par Gale et Shapley. Il y a en effet deux types d'agents, médecins et hôpitaux, qui ont des préférences les uns sur les autres et qui ne sont pas initialement affectés les uns aux autres mais souhaiteraient l'être. Il commença par fournir une description institutionnelle et historique de ce marché d'appariement, en soulignant les objectifs, critiques et problèmes rencontrés au cours de son histoire, notamment le passage d'un processus décentralisé très critiqué à un algorithme centralisé. Il fournit une description de l'algorithme adopté en 1953, l'algorithme *National Intern Matching Program* (NIMP) et il en analysa ses propriétés. Ce dernier s'avéra être équivalent à l'algorithme AD où les hopitaux proposent. En utilisant les propriétés déjà étudiées, il nota plusieurs conséquences de cette équivalence:

- L'appariement obtenu, avec des préférences sincères, est l'appariement stable le moins préféré pour les médecins. Cela pourrait expliquer pourquoi de multiples plaintes ont émergé au fil des années par ces derniers, arguant que l'algorithme "favorise injustement les hôpitaux au détriment des étudiants en médecine".
- L'algorithme NIMP n'est pas non manipulable par les médecins: cela peut créer des problèmes stratégiques dans la soumission des préférences par ces derniers.

Une première recommandation naturelle, en utilisant les résultats déjà mentionnés, a été de conseiller le passage à l'algorithme AD où les docteurs proposent. Roth mentionna qu'une inquiétude des institutionnels concernait la distribution des médecins au sein des différents hôpitaux. Certains hôpitaux, principalement en région rurale, étaient fortement sous demandés par les médecins et la crainte fût que le passage à un algorithme alternatif favorisant ces derniers aurait pu aggraver la situation. Motivé par cette préoccupation pratique, Roth prouva le résultat important suivant, appelé le *Théorème de l'Hôpital Rural*:

**Roth (1984a), Théorème 9.** Quand les préférences sont strictes, l'ensemble des positions d'un hôpital assignées à des médecins est le même sous n'importe quel appariement stable,

tout comme l'ensemble des médecins qui reçoivent une affectation.

Ce théorème a une conséquence importante: si la stabilité est considérée comme une propriété essentielle, alors les inégalités d'affectation régionales entre hôpitaux en termes de nombre de docteurs affectés ne peuvent être résolues en utilisant simplement une procédure d'appariement stable différente<sup>20</sup>. Une autre préoccupation d'ordre pratique qu'il mentionna fût la présence de couples de docteurs, tout deux demandant une affectation. Cela introduit une nouveauté: les préférences des membres du couple sont liées. En effet, ils ne veulent pas se retrouver affectés loin l'un de l'autre. Conceptuellement, le problème était nouveau: un couple peut être vu comme une entité unique ayant des préférences sur les paires d'affectations possibles qui représentent l'affectation des deux membres du couple. En introduisant cela, on peut redéfinir la notion de stabilité: un couple peut bloquer si ses membres peuvent être affectés à deux positions que le couple préfère conjointement et les hôpitaux recevant les membres du couple les préfèrent à leur affectation. Avec cette nouvelle considération pratique, il prouva le résultat fondateur suivant:

**Roth (1984a), Théorème 10.** Dans un marché d'appariement où certains agents sont des couples, l'ensemble des appariement stables peut être vide.

Ce théorème a eu un impact important au sein de la littérature. Une multitude de travaux futurs, en Science Economique ou en Science de l'Informatique, s'intéressèrent aux propriétés et possibilités des marchés d'appariement avec la présence de couples. Puisque le Chapitre 2 de cette thèse introduit la littérature d'appariement en présence de couples, nous ne le ferons pas ici. Cependant, nous recommandons au lecteur intéressé l'excellente revue interdisciplinaire de [Biró and Klijn \(2013\)](#). Deux résultats importants ont été montrés: le problème de décider de l'existence d'un appariement stable en présence de couples devient, en termes d'algorithmie, "difficile" comme montré par [Ronn \(1990\)](#)<sup>21</sup> et il n'y a pas de mécanisme qui peut trouver un appariement stable s'il existe tout en étant non manipulable comme montré par [Biró and Klijn \(2013\)](#). A la lumière de l'analyse précédente de Roth, le *Board of Directors of the National Resident Matching Program (NRMP)*<sup>22</sup> decida en

<sup>20</sup>Plus tard, [Roth \(1986\)](#) prouva un résultat plus fort: les hôpitaux ne voyant pas toutes leurs positions affectées au sein d'un appariement stable seront affectés aux mêmes docteurs sous n'importe quel appariement stable. On ne peut donc pas changer le nombre ou l'identité des docteurs affectés à des hôpitaux sous demandés, on peut seulement influencer l'identité des docteurs affectés à des hôpitaux dont toutes les positions sont affectées.

<sup>21</sup>En vocabulaire de théorie de la complexité, ce problème appartient à la classe des problèmes dits *NP-difficiles* (*NP-hard*).

<sup>22</sup>Le nouvel acronyme donné plus tard au système précédent mentionné comme NIMP.

1995 de réformer le programme d'allocation des docteurs. [Roth and Peranson \(1999\)](#) décrivent l'algorithme final adopté, qui fût élaboré sur les bases des travaux théoriques évoqués ainsi qu'avec une approche d'ingénierie afin de prendre en compte des contraintes pratiques spécifiques à ce marché. Une citation importante de l'article, p.749, est la suivante<sup>23</sup>

*Le présent papier, en plus de présenter quelques résultats nouveaux, a également pour objectif de passer une étape supplémentaire dans la direction d'une littérature d'ingénierie, en décrivant comment les faits furent appris et comment ils ont impactés les décisions de conception.*

Une contrainte pratique importante fût, comme mentionné, la présence de couples. L'algorithme final, inspiré de la théorie de l'appariement et élaboré empiriquement, eu de bonnes performances pratiques. Par exemple, l'existence d'appariement stable même avec la présence de couples n'était pas un problème en pratique. Plus tard, cette observation empirique, et une autre également faite sur le programme d'affectation des psychologues, motiva une recherche théorique sur la question. [Kojima, Pathak and Roth \(2013\)](#) développèrent un modèle d'analyse de grand marché (ou marché large), c.a.d. avec beaucoup d'agents, pour montrer que l'existence d'un appariement stable sous l'algorithme NMRP, même en présence de couples, est très probable quand le marché est suffisamment grand. [Ashlagi, Braverman and Hassidim \(2011\)](#), avec un modèle de marché large différent, montrèrent également qu'une notion de stabilité moins restrictive que la précédente existe également avec une forte probabilité dans un marché large. De plus, ils montrèrent que le nouvel algorithme qu'ils proposent donne de bonnes incitations aux agents, dans un marché large, à être sincères en fournissant leurs listes de préférences<sup>24</sup>.

Une autre réforme concrète dans la classe des problèmes d'appariement avec deux types fût celle des procédures de *choix d'école* pour affecter des élèves à des écoles. [Abdulkadiroglu and Sonmez](#)

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<sup>23</sup>"The present paper then, in addition to presenting some new results, is intended to take a step in the direction of an engineering literature as well, by describing how those facts were learned and how they impacted design decisions."

<sup>24</sup>L'utilisation d'une approche en marchés larges a une importance croissante au sein de la littérature des problèmes d'appariement. Elle fût utilisée dans les travaux fondateurs de [Immorlica and Mahdian \(2005\)](#) et [Kojima and Pathak \(2008\)](#) qui montrèrent qu'avec une structure particulière de marché large, les écoles n'ont pas d'incitations à mentir sur leurs listes de préférences même sous l'algorithme AD où les élèves proposent. Plus tard, des modèles alternatifs de marchés larges ont été développés. [Abdulkadiroglu, Che and Yasuda \(2015a\)](#) et [Azevedo and Leshno \(2016\)](#) développèrent un modèle de continuum d'agents afin de prouver qu'il n'existe qu'un seul appariement stable dans un marché large. [Lee \(2014\)](#) utilisa une approche en termes de graphes aléatoires pour montrer que les élèves et les écoles peuvent obtenir un appariement au sein de leur liste de préférences dans le plus haut quantile si les préférences sont indépendantes et distribuées uniformément. Récemment, [Che and Tercieux \(2015b,a\)](#) utilisèrent les mêmes techniques de graphes aléatoire pour comparer les propriétés d'efficacité et de stabilité dans des marchés larges.

(2003) décrivent un algorithme répandu dans ce domaine, qui fût plus tard appelé le *mécanisme de Boston*, et utilisé à Boston, Columbus, Minneapolis et Seattle. Ils montrèrent que cet algorithme a un défaut important: il est manipulable par les élèves, leur créant ainsi des incitations à être stratégiques en fournissant leurs listes de vœux. Ils recommandèrent naturellement le passage à l'algorithme AD élèves-proposant pour remédier à ce problème. Comme mentionné précédemment, il y a une différence conceptuelle avec le problème standard de *college admission*: les écoles ne sont pas stratégiques et leurs préférences sont plutôt considérées comme des priorités fixées par la loi. Afin d'évaluer les performances en termes d'efficacité, elles peuvent donc être omises, le rang de l'affectation que les écoles reçoivent en soit importe peu en dehors du seul critère de stabilité. En considérant la Pareto-efficacité comme un critère important d'un appariement, seuls les élèves peuvent être considérés comme importants. Un appariement est donc considéré comme *efficace* si on ne peut pas réaffecter les élèves aux écoles de telle sorte que les élèves obtiennent une école qu'ils préfèrent<sup>25</sup>. Cette modification de la définition de Pareto-efficacité a une conséquence importante: les appariements stables peuvent ne plus être Pareto-efficaces avec cette nouvelle définition. En démarrant d'un appariement stable, il peut être possible de réaffecter les élèves afin qu'ils obtiennent une école qu'ils préfèrent, mais ce changement créerait des paires bloquantes. En construisant une procédure de choix d'école, on fait face à un premier arbitrage: veut-on respecter la Pareto-efficacité ou la stabilité ? Abdulkadiroglu and Sonmez (2003) proposèrent un algorithme efficace et non manipulable, inspiré des problèmes d'allocation de logements que nous décrirons en présentant les modèles d'allocation de logements, et appelé l'algorithme *Top Trading Cycles* (TTC). Ce dernier s'exécute ainsi:

**Etape 1.** Tous les élèves pointent vers l'école qu'ils classent en première (si un élève trouve les écoles inacceptables, alors il est non affecté et retiré de la procédure). Chaque école pointe vers l'élève classé en premier au sein de ses priorités. En utilisant les pointages ainsi définis, il existera un cycle<sup>26</sup>, affecter les élèves au sein de ce cycle à l'école vers laquelle ils pointent. Supprimer ces élèves et diminuer d'un la capacité des écoles impliquées dans le cycle. Si une

<sup>25</sup>La différence avec la notion précédente est que l'on requiert uniquement que les élèves soient mieux et non les élèves et écoles. Dans ce nouveau cadre, il est donc possible qu'une école reçoive un élève qu'elle classe plus bas suite à la réaffectation.

<sup>26</sup>Nous sommes sûr qu'il y en aura un. C'est une propriété classique en théorie des graphes: un graphe pour lequel chaque noeud possède une unique flèche de sortie a au moins un cycle. De plus, s'il y a plusieurs cycles, ces derniers sont disjoints et ne comportent pas les mêmes noeuds, l'ordre dans lequel l'algorithme les choisit n'affectera donc pas l'appariement final obtenu.

école finit avec une capacité nulle, elle est supprimée définitivement du processus. Passer à l'étape suivante.

**Etape k.** Tous les élèves restants à cette étape pointent vers l'école qu'ils classent la plus haute dans leur liste de préférences parmi celles restantes à cette étape, c.a.d qui ont encore une capacité positive. Chaque école restante pointe vers l'étudiant classé le plus élevé dans sa liste de priorité parmi ceux restants. Il y aura un cycle en utilisant ces comportements de pointage. L'implémenter en affectant les élèves au sein de ce dernier à l'école vers laquelle ils pointent. Diminuer d'un les capacités des écoles impliquées dans ce cycle. Supprimer les écoles qui finissent avec une capacité nulle et passer à l'étape suivante.

Ces étapes continuent jusqu'à ce que tous les élèves soient affectés. Ils prouvèrent que cet algorithme est non manipulable par les élèves et Pareto-efficace. Un décideur politique, en décidant de la procédure d'affectation de choix d'école se retrouve donc face à deux candidats potentiels: l'algorithme AD s'il privilégie la stabilité ou l'algorithme TTC s'il privilégie la Pareto-efficacité. [Abdulkadiroglu et al. \(2005\)](#) conseillèrent ces deux possibilités à la ville de Boston afin de remplacer leur algorithme qui était manipulable. [Abdulkadiroglu et al. \(2006\)](#) étudièrent plus précisément ce mécanisme de Boston avec une approche empirique. Ils soulignèrent que les parents comprenaient bien les potentielles manipulations de listes de préférences. Dans le guide 2004 du *Boston Public School*, il était notamment donné une recommandation claire en ce sens<sup>27</sup>

*Pour une meilleure chance pour votre "premier choix": considérez une école moins populaire. Demandez au Centre de Ressources des Familles pour des informations sur les école sous demandées.*

Ils montrèrent qu'au sein des listes fournies par les familles, on observait une chute claire d'écoles, considérées comme populaires, classées secondes dans les listes de préférences. Cette discontinuité dans les listes était révélatrice de la non sincérité des parents: classer second une école populaire sous la procédure de Boston n'était pas une bonne stratégie car ils prenaient le risque de perdre des opportunités dans d'autres écoles et donc préféreraient ne pas classer en deuxième une école populaire. Cela a une implication importante en termes d'évaluation des politiques publiques: il n'est

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<sup>27</sup>"For a better chance of your "first choice" school: consider choosing less popular schools. Ask Family Resource Center staff for information on "underchosen" schools."

pas possible d'utiliser les listes de préférences fournies par les familles afin d'évaluer la performance de la procédure. En effet, le comité de Boston mentionnait que l'algorithme affectait 90% des familles à l'école qu'elles classaient en première mais cette analyse est fautive car l'école classée en première n'était, à cause des manipulations, probablement pas leur vrai premier choix. Le dernier argument qui convaincu les décideurs politiques d'initier une réforme de la procédure fût que les familles qui ne réussissaient pas à être stratégiques et fournissaient leurs vraies préférences étaient pénalisées et la plupart finissaient sans aucune affectation. De plus, ces dernières étaient majoritairement des familles issus de milieux sociaux défavorisés et à faibles revenus. L'adoption d'un mécanisme non manipulable, comme les décideurs politiques ont mentionné à l'époque, permettrait *d'aplanir le champs stratégique* (*level the playing field*) entre les familles et donc d'être moins inégalitaire. Cette première analyse et réforme de politique publique réussie mena à nombre important de recommandations et un grand nombre de recherches que nous ne détaillerons pas ici. Plusieurs villes comme Boston, New York City, Chicago, Denver, Washington DC parmi d'autres adoptèrent l'algorithme AD élèves-proposant. La Nouvelle-Orléans choisit l'algorithme TTC mais changea récemment pour AD. En France, [Hiller and Tercieux \(2014\)](#) montrèrent que l'algorithme Affelnet actuellement utilisé pour affecter les élèves aux lycées est équivalent à l'algorithme AD écoles-proposant.

## Problèmes d'allocation de logements

Une autre classe importante de problèmes d'appariement concerne l'allocation d'objets. Contrairement au cas précédent avec deux types, seulement un type d'agents a des préférences et l'autre est juste considéré comme des objets qu'il faut allouer, sans préférences ou priorités particulières. La principale application de cette classe de problèmes est celle des logements sociaux. Trois types de problèmes ont été identifiés au sein de la littérature:

1. L'allocation de logements sans occupants.
2. L'allocation de logements avec uniquement des occupants.
3. L'allocation de logements avec des occupants existants.<sup>28</sup>

Le premier cas, introduit par [Hylland and Zeckhauser \(1979\)](#) étudie le plus basique des problèmes d'allocation: un nombre fini d'objets doit être alloué à des agents ayant un ordre de préférences

<sup>28</sup>Cette dernière est aussi appelée *marché de logement* (*housing market*).

sur chacun des types d'objets présents. Il peut y avoir plusieurs copies d'un même type d'objets mais toujours en nombre limité afin de faire face à un problème de congestion. La Pareto-efficacité est une propriété naturelle pour une allocation: il ne faudrait pas pouvoir réallouer les objets de telle sorte que tous des agents préféreraient la nouvelle allocation. Un mécanisme standard dans la littérature est celui du *Dictateur en Série* (DS)<sup>29</sup>. Il est simple et intuitif:

**Etape Préliminaire.** On fixe un classement des agents.

**Etape 1.** Le premier agent classé choisit son objet favori et l'obtient. Ils sont tous deux supprimés du marché (s'il y a plusieurs copies de cet objet, on ne supprime qu'une seule copie).

**Etape k.** Le k-ième agent classé choisit son objet préféré parmi ceux restants à cette étape et l'obtient. Ils sont tous deux supprimés du marché<sup>30</sup>.

Le processus continue jusqu'à ce que tous les agents ont pu choisir. Ce mécanisme est non manipulable donc les agents n'ont aucune incitation à mentir sur leurs listes de préférences. Bien entendu, le choix de l'ordre de classement à l'étape préliminaire est crucial et le premier agent classé à un avantage certain<sup>31</sup>. Dans l'allocation des logements sociaux, le délai d'attente est un critère naturel: l'agent ayant attendu le plus longtemps choisit en premier et ainsi de suite. S'il n'existe aucun critère pour sélectionner ce classement, afin d'assurer une certaine équité du processus, on peut choisir aléatoirement le classement, le mécanisme est alors appelé *Dictateur en Série Aléatoire* (DSA)<sup>32</sup>. Avec cette approche aléatoire, on peut étudier les propriétés probabilistes d'un algorithme, le problème devient alors un problème *d'allocation aléatoire*<sup>33</sup> où il faut choisir des distributions de probabilités sur les allocations possibles. Si les agents ont des préférences cardinales, en plus de leurs ordres de préférences, c.a.d. attachent une valeur numérique aux objets qu'ils peuvent obtenir, il est alors possible d'évaluer différentes allocations aléatoires avec le critère d'utilité espérée. Une importante littérature s'est concentrée sur les allocations aléatoires. Par exemple, [Abdulkadiroglu and Sonmez \(1998\)](#) montrèrent que les mécanismes DSA et TTC, où les occupants sont tirés aléatoirement, sont stochastiquement équivalents. [Bogomolnaia and Moulin \(2001\)](#) proposèrent

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<sup>29</sup> *Serial Dictatorship*.

<sup>30</sup> Dans le cas où il n'y aurait plus d'objets disponibles, ou que ceux restants ne sont pas acceptables pour l'agent, ce dernier n'en reçoit aucun.

<sup>31</sup> Il est important que ce choix de classement se fasse en amont ou indépendamment des listes de préférences fournies, sinon la procédure risque d'être manipulable.

<sup>32</sup> *Random Serial Dictatorship*.

<sup>33</sup> *Probabilistic assignment*.

une notion d'efficacité *ex ante*, *l'efficacité ordinale*, et proposèrent un mécanisme, *le Probabilistic Serial* pour caractériser l'ensemble des mécanismes probabilistes menant à une allocation aléatoire ordinalement efficace<sup>34</sup>. Le mécanisme DS est simple et facile à mettre en place. Il a été répertorié dans plusieurs marchés d'appariement concrets. Les universités l'utilisent beaucoup pour l'allocation de leurs étudiants aux logements de leur campus. [Abdulkadirouglu and Sonmez \(1999\)](#) citèrent par exemple Carnegie-Mellon, Duke, Harvard, Northwestern et Pennsylvania.

[Shapley and Scarf \(1974\)](#) introduisirent le second type de problèmes d'allocation de logements où un groupe d'occupants aimerait échanger leurs logements actuels<sup>35</sup>. Ici, les occupants ont un droit sur leur logement initial et peuvent le garder dans le cas où ils ne seraient pas satisfaits de la nouvelle allocation. Cette contrainte est, comme avant, appelée *Rationalité Individuelle* (RI): si un occupant obtient un nouveau logement, il doit le préférer à son logement initial<sup>36</sup>. Cette contrainte pour un appariement peut s'ajouter au critère de Pareto-efficacité mentionné précédemment. Avec l'existence d'une allocation initiale, on peut également se référer à une propriété plus forte que la simple Pareto-efficacité: la notion de *coeur* (*core*). Dans ce contexte avec des occupants initiaux pour les logements, un appariement est dans le coeur s'il n'y a pas de groupe d'occupants tel que ces derniers pourraient échanger leurs logements initiaux et préférer cet échange à l'appariement considéré<sup>37</sup>. L'interprétation est que si l'on utilise un mécanisme qui ne donne pas un appariement dans le coeur, alors un groupe d'agents préférerait ne pas participer à ce dernier et organiser eux-mêmes leur échange. On dit alors qu'un tel groupe *bloque* l'allocation ou encore qu'il *dévie*. [Shapley and Scarf](#)

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<sup>34</sup>Nous n'avons pas pour but ici de donner une revue complète de la littérature sur les allocations aléatoires. [Hylland and Zeckhauser \(1979\)](#) proposèrent par exemple une approche similaire à un marché monétaire, *l'Équilibre Compétitif avec Revenus Égaux* (*Competitive Equilibrium with Equal Income*), où chaque agent est doté d'un faux revenu et les prix permettent d'acheter des parts de probabilités. [Budish et al. \(2013\)](#) étendirent cette approche pour construire des allocations aléatoires plus complexes sous contraintes où des groupes d'objets peuvent avoir des quotas supérieurs et inférieurs d'allocation.

<sup>35</sup>Cet échange se fait sans transferts monétaires. On peut penser à des réallocations de logements publics ou de chambres universitaires.

<sup>36</sup>Il y a un parallèle conceptuel clair entre le cadre précédent et celui-ci. Dans les deux cas, la rationalité individuelle impose qu'on doit donner à chaque agent une meilleure allocation que celle qu'il obtiendrait dans le cas du *statu quo*. Dans le cadre d'appariement avec deux types, les agents commençaient sans aucune allocation, leur *statu quo* était donc de rester seuls. Dans ce nouveau cadre, le *statu quo* est décidé par l'allocation initiale. C'est également pour cela que l'on parle parfois de contrainte de participation: si un agent était sûr d'obtenir une allocation moins bonne qu'à sa situation de *statu quo* et qu'il serait libre de participer ou non, il choisirait de ne pas participer et rester dans son logement initial.

<sup>37</sup>Dans le cadre des problèmes d'appariement avec deux types mentionné dans la section précédente, il est également possible de définir une notion de coeur. Dans ce cas, sans allocation initiale, les agents du groupe échangent leurs affectations obtenues sous l'appariement considéré et non, comme ici, leur logement initial. On peut voir cette notion comme une généralisation du concept de paire bloquante définie dans le concept de stabilité. [Roth \(1982\)](#) prouva que l'ensemble des appariements stables est exactement égale à l'ensemble des allocations du coeur, donc il n'y a pas de perte de généralité à se concentrer sur les déviations par paires plutôt que par groupe.



(1974) prouvèrent l'existence d'appariements dans le coeur. Gale proposa également un algorithme qui trouve systématiquement un appariement dans le coeur, l'algorithme *Top Trading Cycles*:

**Etape 1.** Tous les agents pointent vers l'agent occupant initialement leur logement favori. Si un agent n'a aucun logement favori à l'exception du sien, il pointe vers lui même, formant un cycle avec lui même. Puisqu'il y aura au moins un cycle, affecter chaque agent de ce cycle au logement de l'agent vers lequel il pointe. Supprimer l'agent et les logements du cycle de la procédure.

**Etape k.** Les agents restants à cette étape pointent vers celui qui occupe leur logement favori restant. Si un agent n'a pas de logement favori en dehors du sien, il pointe vers lui même. Puisqu'il y aura au moins un cycle, les agents de celui-ci sont affectés au logement de l'agent vers lequel ils pointent. Supprimer ces agents et les logements présents dans ce cycle.

Ce processus continue jusqu'à ce que tous les agents sont affectés par un cycle. Roth and Postlewaite (1977) prouvèrent que l'appariement trouvé par l'algorithme TTC est l'unique allocation du coeur. On peut également montrer que cet algorithme est non manipulable. En fait, Ma (1995) montra que c'est le seul algorithme qui retourne un appariement RI, Pareto-efficace et qui est non manipulable donc si ces trois propriétés sont essentielles, on est forcé d'utiliser un algorithme qui sera équivalent à TTC<sup>38</sup>.

En ce qui concerne la dernière classe de problèmes, introduite par Abdulkadirouglu and Sonmez (1999), elle considère un environnement où des logements doivent être affectés. Certains d'entre eux ont un occupant initial qui souhaite obtenir un nouveau logement, d'autres logements sont vides et certains agents n'ont pas de logement initial mais souhaiteraient en obtenir un. Ils définirent un mécanisme non manipulable, RI et Pareto-efficace appelé *You-Request-My-House-I-Get-Your-Turn* (YRMH-IGYT)<sup>39</sup>:

**Etape préliminaire.** On définit un classement des agents.

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<sup>38</sup>Dans le premier contexte d'allocation d'objets, on aimerait obtenir un résultat similaire déclarant DS comme étant le seul mécanisme Pareto-efficace et non manipulable. Malheureusement, des hypothèses supplémentaires sont nécessaires. Svensson (1999) prouva qu'il était le seul algorithme non manipulable, Pareto-efficace et neutre. Papai (2000) prouva qu'une généralisation de l'algorithme TTC dans ce contexte sans occupants initiaux est le seul mécanisme non manipulable par groupe, Pareto-efficace et autre axiome appelé *reallocation-proof*. Le lecteur intéressé est invité à se référer aux articles mentionnés pour les définitions formelles des axiomes supplémentaires.

<sup>39</sup>Qu'on peut traduire par "Vous demandez mon logement j'obtiens votre tour".

**Etape 1.** L'agent classé en premier pointe vers son logement favori. Si ce dernier est vide, alors il y est affecté, est supprimé de la procédure et on passe à l'étape suivante. Si le logement est occupé par un occupant initial alors ce dernier pointe également vers son logement favori. Encore une fois, s'il est vide, alors toute cette chaîne est implémentée en affectant les agents au logement vers lequel ils pointent. S'il est occupé, ce processus de pointage continu jusqu'à: i) qu'il atteigne un logement vide ou ii) qu'il cycle. Dans les deux cas, la chaîne, ou le cycle, formée est implémentée en affectant les agents au logement vers lequel ils pointent, ces derniers sont supprimés et on passe à l'étape suivante.

**Etape k.** Elle est similaire à l'Etape 1 mais en utilisant uniquement les agents et logements restants à cette étape.

Ce processus continue jusqu'à ce que tous les agents reçoivent une affectation. On peut le voir comme une généralisation des algorithmes TTC et de DS. En effet, s'il n'y a que des logements occupés et aucun agent sans logement initial, on revient à l'algorithme TTC. S'il n'y a que des logements vides et des agents sans logement initial, on revient à DS. [Sonmez and Unver \(2010\)](#) ont caractérisé l'algorithme YRMH-IGYT comme étant le seul mécanisme Pareto-efficace, RI, non manipulable ainsi que deux axiomes supplémentaires: *neutralité faible* et *consistance*. Dans ce contexte avec des occupants initiaux et des logements vides, un autre algorithme, identifié pour la première fois par [Guillen and Kesten \(2012\)](#) pour l'allocation des étudiants aux logements de campus au MIT peut être utilisé: c'est une modification de l'algorithme AD. Des priorités sont définies pour chaque logement avec la seule contrainte étant que son occupant initial, s'il y en a un, doit être classé premier. L'algorithme AD où les agents proposent est ensuite utilisé pour affecter ces derniers aux logements. Par construction, comme chaque occupant initial est classé premier dans son logement, il est assuré d'obtenir une affectation qu'il préfère faiblement à son logement initial: ce mécanisme est donc RI. Comme il s'agit de l'algorithme AD où les agents proposent, on sait qu'il est non manipulable par ces derniers. Il ne sera en revanche pas Pareto-efficace si on utilise uniquement les préférences des agents pour évaluer ce critère, comme mentionné dans la section précédente.

## **Cette thèse: étude du problème d'affectation des enseignants en France et d'un nouveau problème d'appariement.**

Cette thèse est inspirée par un marché concret d'appariement: celui qui consiste à affecter des enseignants dans des écoles. De nombreux pays utilisent une procédure centralisée pour affecter leurs enseignants. La France est l'une d'entre eux tout comme par exemple l'Italie, le Mexique, le Pérou, la Turquie, l'Uruguay ou encore le Portugal. La question de savoir si oui ou non nous devons utiliser un système centralisé d'affectation contre un système décentralisé comme pour un marché du travail classique est importante. Plusieurs pays, comme le Royaume-Uni ou les Etats-Unis, ont choisi par exemple le deuxième choix. Les syndicats d'enseignants français sont par exemple très attachés au concept *d'école républicaine* et au concept de traitement égal. Chaque enseignant doit être traité identiquement et personne ne doit être discriminé en termes de salaires en dehors des critères de base comme l'ancienneté par exemple. On pourrait donc voir cette contrainte comme une contrainte de répugnance mentionnée plus haut. Notre but ici est donc de prendre cette contrainte comme donnée et non de discuter de la pertinence ou non de celle-ci et du cas d'un système décentralisé. Ce dernier, même dans la littérature académique, est encore sous un débat important en termes de comparaisons avec un système centralisé<sup>40</sup>. Notre analyse peut cependant être vue comme prenant part à ce débat. En effet, nous montrons que dans un contexte de réallocation, l'algorithme actuellement identifié dans la littérature académique et utilisé en pratique peut être grandement amélioré. Dans un sens, nous montrons que les résultats possibles d'un système centralisé, dans ce cadre, sont grandement sous évalués.

L'affectation des enseignants aux écoles peut être vue comme étant à l'intersection des environnements que nous avons décrits dans les sections précédentes. Elle partage des similarités avec les modèles d'appariement avec deux types: même si les écoles ne sont pas stratégiques, nous argumentons qu'il y a un sens dans lequel les priorités incorporent des objectifs politiques importants

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<sup>40</sup>Par exemple pour le problème NRMP d'allocation des docteurs, certains avancent qu'un système décentralisé permettrait de mieux répartir les médecins tout en fournissant des salaires plus élevés. Cette critique est importante et récurrente. [Agarwal \(2015\)](#) traita cette question en estimant l'impact potentiel d'un passage à un système décentralisé avec des ajustements de salaires. Il estima que: "A cause de la volonté des résidents en médecine de payer pour des programmes désirables et des contraintes de capacité de ces derniers, les salaires dans n'importe quel équilibre compétitif demeureront, en moyenne, au moins 23 000\$ en dessous du produit marginal du travail", concluant ainsi que le système centralisé ne causait pas de salaires trop bas. L'utilisation des variations de salaire dans le cadre de l'affectation des enseignants est encore sous débat au sein de la littérature académique. Les travaux empiriques existants peuvent même avoir des conclusions opposées. Nous invitons le lecteur à se référer aux discussions et citations des Chapitres 1 et 2 sur la question.

et qu'elles ne peuvent être complètement ignorées pour évaluer l'efficacité des affectations obtenues. Dans un problème de *college admission* de Gale et Shapley, la prise en compte des préférences des deux types d'agents en évaluant les propriétés d'efficacité est plus pertinente dans notre contexte. Afin de la différencier de l'autre notion de Pareto-efficacité introduite dans le problème d'allocation de logements où les problèmes de choix d'école par exemple, nous l'appellerons *2-Pareto efficacité* (au contraire de la deuxième qui est mentionnée comme *1-Pareto efficacité*). Une fois cela défini, on pourrait penser que, dans ce contexte, utiliser l'algorithme AD où les enseignants proposent produirait un appariement stable, 2-Pareto efficace et que ce dernier serait non manipulable par les enseignants. Cette approche serait valide si l'on se concentrait uniquement sur l'affectation des *enseignants néotitulaires* qui entrent tout juste dans la profession et recherchent leur toute première affectation. Cependant, il existe un second type d'enseignants: *les titulaires*. Ces derniers enseignent déjà au sein d'un établissement et souhaiteraient une mutation au sein d'un autre. Sur les données 2013 du mouvement inter-académique utilisées dans cette thèse, ils représentent 63% des enseignants. Dans ce cas, tout comme dans le cas du problème d'allocation de logements avec des occupants initiaux, une contrainte importante doit être prise en compte: la *rationalité individuelle* (RI). En effet, pour ces enseignants titulaires, nous aimerions leur assurer le *droit à rester dans leur établissement d'origine* de sorte que, s'ils obtiennent une nouvelle affectation, elle doit être dans une école qu'ils préfèrent. Le but est donc d'assurer la contrainte RI tout en prenant en compte les préférences des enseignants et des écoles. Un premier résultat théorique important est qu'il est impossible d'assurer en même temps une affectation RI et stable. Si la rationalité individuelle est une propriété obligatoire alors on peut facilement construire des exemples où il est obligatoire d'avoir des paires bloquantes ou, dans le sens inverse, où tout appariement stable n'est pas RI. L'algorithme TTC proposé par Shapley et Scarf dans le cadre de l'allocation de logements avec occupants initiaux n'est pas très adapté: même s'il assure RI, il ne prend pas du tout en compte le côté des écoles. Comme évoqué dans la section précédente, une variation de l'algorithme AD a été identifiée par [Guillen and Kesten \(2012\)](#). Si les enseignants sont classés en haut des priorités de leur école initiale, la contrainte RI est assurée tout en respectant une certaine notion de stabilité, qui permet de relativement mieux prendre en compte les classements des écoles. Dans ce contexte, il y a une différence fondamentale avec la version précédemment décrite dans le problème d'allocation de logements. Les priorités des écoles sont initialement fixées et données par la loi. Il se peut donc qu'un

enseignant titulaire initialement affecté à une école n'ait pas la plus haute priorité au sein de celle-ci. Dans ce cas, la procédure consiste à modifier artificiellement la priorité de l'école en question afin de classer ce dernier en premier. Une fois cette étape faite, l'algorithme AD enseignants-proposant est utilisé sur ces priorités modifiées. Nous appelons cette procédure AD\*. C'est celle qui est utilisée en France actuellement pour affecter les enseignants du second degré néotitulaires et titulaires. Ses propriétés peuvent être déduites de la première étude de [Guillen and Kesten \(2012\)](#) qui ont décrit l'affectation des logements de campus au MIT et de [Pereyra \(2013\)](#) qui étudia la version présentée ici également dans le cas de l'affectation des enseignants. Ce dernier se concentra sur une définition plus faible de la stabilité où un enseignant ne peut avoir qu'un seul type d'envie envers un autre: les enseignants préférant une école à leur affectation et qui ont une priorité supérieure à un enseignant de cette école qui y était initialement affecté et qui n'en a pas bougé. Il appela ce type d'envie une *envie inappropriée*<sup>41</sup>. L'autre type, *l'envie justifiée*<sup>42</sup>, est celle qui voit un enseignant préférer une école en ayant une plus haute priorité au sein de celle-ci qu'un autre enseignant qui n'était pas initialement au sein de cette école mais y a été affecté par l'algorithme. Comme mentionné, il est impossible d'assurer en même temps une allocation RI et stable où les deux types d'envies mentionnés ne se produisent pas. Pereyra imposa donc que seul le premier type, les envies inappropriées, peut se produire mais que le deuxième type ne doit pas. Il montra que l'algorithme AD\* permet d'obtenir un appariement de ce type et qu'en plus, il minimise le nombre d'envies inappropriées. [Compte and Jehiel \(2008\)](#), en étudiant également un problème de réaffectation, montrèrent que si l'on veut interdire les envies justifiées, alors AD\* mène au plus grand mouvement possible parmi les algorithmes qui n'ont également pas d'envies justifiées.

En France, le mécanisme AD\* est celui qui est utilisé pour affecter en même temps les enseignants titulaires et néotitulaires aux écoles. Dans le Chapitre 2, nous donnons une description précise du contexte institutionnel et la description formelle de l'algorithme utilisé. Le point important est qu'en France, cette affectation est faite en deux étapes. La première concerne l'affectation inter-académique où les enseignants souhaitant changer d'affectation vers une autre académie soumettent une liste de vœux sur potentiellement les 31 académies françaises. Une fois que cette affectation est décidée en utilisant l'algorithme AD\*, les enseignants soumettent une liste de vœux sur les écoles cette fois à

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<sup>41</sup> *Inappropriate claim.*

<sup>42</sup> *justified claim.*

l'intérieur de la région où ils ont été affectés à la première étape<sup>43</sup>.

La plupart des systèmes éducatifs des pays de l'OCDE font face à deux problèmes majeurs:

1. un manque d'attractivité de la profession enseignante.
2. de fortes inégalités de réussite entre étudiants issus de milieux sociaux différents.

La France ne fait pas exception. Concernant le premier problème, la [Direction de l'Evaluation de la Prospective et de la Performance \(2015\)](#) fait part, qu'en 2014, 24% des postes ouverts au concours par le gouvernement sont restés vacants à cause d'un manque de candidats ou à la pauvre qualité de ceux postulant. De plus, en 2014, seulement 43.9% des enseignants du second degré demandant une mutation l'ont obtenu via la procédure d'affectation actuelle. En ce qui concerne le deuxième problème, le rapport [Organisation for Economic Co-operation and Development \(2014\)](#) mentionna que la France est l'un des pays des l'OCDE où l'origine sociale des élèves a le plus haut pouvoir prédictif sur leur performance à l'école. L'affectation des enseignants aux écoles est à l'intersection de ces deux problématiques.<sup>44</sup> Tout d'abord, cette affectation détermine la possibilité des enseignants de pouvoir gérer leur mobilité géographique au sein de leur carrière, influençant de potentiels candidats dans leur choix de devenir enseignants. Ensuite, elle détermine également la distribution géographique des enseignants dans les différentes écoles, influençant in fine les performances des étudiants. En effet, il est par exemple connu ([Chetty, Friedman and Rockoff, 2014](#); [Rockoff, 2004](#)) que les enseignants dans leur première année d'expérience ont une capacité moindre à faire progresser leurs élèves comparé aux enseignants plus expérimentés. La distribution des enseignants néotitulaires et des titulaires avec uniquement une ou deux années d'expérience peut donc devenir une variable importante de politique éducative. En France, selon la [Direction de l'Evaluation de la Prospective et de la Performance \(2014\)](#), les deux académies de Créteil et Versailles sont celles qui combinent la plus grande part d'élèves affectés à des écoles d'éducation prioritaire et le plus petit ratio d'enseignants de plus de 50 ans sur ceux de moins de 30 ans. La procédure utilisée afin de (re)affecter les enseignants est donc un outil clé. Un arbitrage simple émerge. Afin de satisfaire au mieux les enseignants faisant une demande de mobilité, on pourrait tenter de construire une procédure d'affectation cherchant à maximiser le nombre de demandes

<sup>43</sup>Pour une description détaillée, le lecteur intéressé peut se référer au Chapitre 2 de cette thèse.

<sup>44</sup>Pour des exemples au sein d'autres pays, quelque soit leur procédure d'affectation, le lecteur peut se référer à l'introduction du Chapitre 2.

satisfaites. Cependant, en faisant cela, on pourrait créer d'importants départs au sein des académies défavorisées et, in fine, être forcé d'affecter des enseignants ayant relativement moins d'expérience à ces académies, impactant négativement la réussite des élèves au sein de celles-ci. En France, chaque année, environ 50% des demandes de mobilité enseignantes proviennent d'enseignants affectés au sein des académies de Créteil et Versailles qui sont, comme mentionné, des académies avec des taux de réussite des élèves relativement plus faibles. Cette thèse, en utilisant la longue tradition d'analyse et de conception de marchés mentionnée dans les sections précédentes, va montrer que l'algorithme utilisé pour affecter les enseignants peut être un outil puissant de gestion des problèmes mentionnés.

## Chapitre 1: Fondations théoriques et premières évaluations empiriques

Dans un contexte standard d'appariement avec deux types, nous savons que l'algorithme AD élèves-proposant est l'unique mécanisme stable, 2-Pareto efficace et non manipulable par les élèves. Dans ce chapitre, nous montrons que dans le contexte de réaffectation des enseignants, l'algorithme AD\*, identifié dans la littérature et utilisé en France pour l'affectation des enseignants, souffre d'un point négatif important: il n'est pas 2-Pareto efficace. On peut donc réallouer les enseignants aux écoles de telle sorte que ces derniers obtiennent une école qu'ils préfèrent et les écoles un enseignant plus haut classé. Le point marquant est qu'en faisant cela, on peut améliorer l'appariement en termes d'efficacité mais également en termes de *justice* puisque l'ensemble des paires bloquantes se réduit. L'exemple stylisé illustrant cela est simple: un enseignant initialement affecté à une école très impopulaire mais qui a une priorité très élevée dans toutes les écoles peut, sous AD\*, empêcher d'autres enseignants d'échanger leurs positions même si cet échange peut être bénéfique pour les enseignants et les écoles. Ce point a une implication importante pour le cas du marché français: il peut expliquer les plaintes de la part des enseignants mentionnées ci-dessus concernant le faible taux de satisfaction des requêtes de mutation. Cette mobilité faible n'est donc pas uniquement due à une congestion structurelle du marché due à la combinaison de préférences corrélées des enseignants et du faible nombre de voeux effectués au sein des listes de préférences. Similaire à l'analyse de Roth (1984) concernant les plaintes des docteurs, notre analyse de ce marché d'appariement nous permet d'identifier le problème pour le cas des enseignants français comme étant intrinsèque aux règles d'allocation utilisées via l'algorithme. Le désavantage de AD\* est qu'il ne permet pas de gérer correctement les échanges de positions possibles entre enseignants titulaires. En pratique, la situation

de l'exemple stylisé mentionné est courante. En effet, les enseignants affectés au sein d'établissements d'éducation prioritaire reçoivent un important bonus de priorités après 5 ans d'enseignement dans ces établissements. Cela crée effectivement la situation évoquée: des enseignants avec priorités élevées dans des écoles très peu populaires.

Partant de ce constat, nous nous restreignons à l'analyse d'un modèle simple de réaffectation pure dit *un-pour-un*, où il n'y a qu'une seule place par école, ces places sont toutes occupées par des enseignants titulaires et il n'y a pas d'enseignants néotitulaires ni de places vacantes. Nous identifions une classe de procédures, les algorithmes *Block Exchange* (BE), qui, contrairement à  $AD^*$ , sont 2-Pareto efficaces et ne peuvent être améliorés en termes d'efficacité et justice. Cependant, parmi ces algorithmes, un unique est non manipulable par les enseignants: nous l'appelons l'algorithme *Teacher Optimal Block Exchange* (TO-BE). Comme pour un cadre de choix d'école, si les écoles ne sont pas considérées comme des entités importantes dans l'évaluation de l'efficacité et qu'on ne considère que le rang de l'affectation obtenue par les enseignants; on peut identifier une deuxième classe d'algorithmes, les algorithmes *1-Sided Block Exchange*. Ces derniers améliorent uniquement le rang obtenu par les enseignants sous la seule contrainte de réduire l'ensemble des paires bloquantes. Cependant, nous montrons que bien que ces algorithmes tendent à favoriser les enseignants, il n'en existe aucun qui soit non manipulable par ces derniers. Pour une évaluation théorique plus poussée des différences entre  $AD^*$  et les algorithmes BE, nous adoptons une approche de marchés larges avec graphes aléatoires similaire à celle utilisée par [Che and Tercieux \(2015b\)](#). Nous pouvons montrer que quand il y a beaucoup d'enseignants et d'écoles<sup>45</sup>:

- Pour tous les algorithmes BE, le pourcentage d'enseignants bougeant de leur école initiale peut être rendu arbitrairement proche de 100% alors que ce n'est pas le cas sous  $AD^*$ .
- Il existe un algorithme BE qui donne à tous les enseignants (resp. écoles) une école (resp. enseignant) qu'ils classent arbitrairement haut(e).
- L'algorithme TO-BE donne à tous les enseignants une école qu'ils classent arbitrairement haute mais la proposition inverse pour les écoles n'est pas vraie<sup>46</sup>.

<sup>45</sup>Nous renvoyons le lecteur aux résultats correspondants au sein du Chapitre 1 pour les énoncés formels.

<sup>46</sup>Formellement, pour les deux dernières propriétés, le rang obtenu est de l'ordre  $o(n)$  c.a.d. que le ratio du rang sur la taille du marché tend vers zéro quand la taille du marché tend vers l'infini. Avec un vocabulaire plus statistique, cela veut dire que le rang obtenu se trouve dans un quantile arbitrairement haut si la taille du marché est suffisante.



Afin de quantifier les améliorations de ces algorithmes par rapport à  $AD^*$ , nous utilisons les données françaises de l'affectation inter-académique en 2013. Afin de rester proche de notre modèle théorique, nous nous restreignons uniquement au sous échantillon des enseignants titulaires en supprimant l'ensemble des néotitulaires, c.a.d. sans affectation initiale, et nous supprimons tous les postes inoccupées au sein des académies. Les résultats sont frappant. Premièrement, sur 10 579 enseignants dans notre échantillon, seuls 564 obtiennent une nouvelle affectation sous  $AD^*$ . Parmi l'ensemble des enseignants, 75% d'entre eux ne classent qu'une seule académie en dehors de leur académie d'affectation. Combiné à la forte corrélation existant dans les préférences, par exemple les académies du sud ou de l'ouest sont très demandées, on pourrait s'attendre à une faible marge d'amélioration. Cependant, sous l'algorithme TO-BE, 1573 enseignants obtiennent une nouvelle affectation, plus du double donc<sup>47</sup>. Même si ces résultats sont importants pour quantifier les effets théoriques mis en avant, ils ne fournissent pas, en soi, une analyse contre factuelle du marché français puisque nous nous sommes restreints à un sous échantillon en supprimant les néotitulaires et les positions vacantes au sein des académies.

## Chapitre 2: Conception pratique et contre factuels

Dans ce chapitre, notre but est de fournir un algorithme pratique et utilisable pour le marché français en prenant en compte ses spécificités et afin de fournir de vrais résultats contre factuels. Comme pour [Roth and Peranson \(1999\)](#) quand ils ont conçu la nouvelle procédure NMRP, nous prenons ici une approche plus d'ingénierie soutenue par nos résultats théoriques du premier chapitre. Dans une première partie, nous étendons l'algorithme TO-BE à un cadre avec places vacantes au sein des académies et enseignants néotitulaires, sans affectation initiale. Cette extension a un parallèle clair avec l'algorithme YRMH-IGYT proposé par [Abdulkadirouglu and Sonmez \(1999\)](#) dans le contexte d'allocation de logements avec occupants initiaux, agents sans logements et logements vides. Notre généralisation de TO-BE peut être vue comme une généralisation de l'algorithme YRMH-IGYT. Nous montrons théoriquement que l'algorithme garde ses bonnes propriétés d'efficacité et de non manipulabilité.

En utilisant les données françaises complètes sur l'affectation inter-académique de 2013, 3991

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<sup>47</sup>Nous avons calculé le mouvement maximal structurellement possible en imposant uniquement la contrainte de RI. Il est de l'ordre de 2000 enseignants. Bien entendu, cet algorithme serait manipulable mais nous donne un ordre de grandeur sur l'amélioration maximale possible.

enseignants titulaires obtiennent une nouvelle affectation sous AD\* contre environ 3880 sous notre généralisation de TO-BE. Pour comprendre cette performance moindre sur ces données complètes, il faut analyser en détails le marché français. Comme évoqué, Créteil et Versailles sont deux académies très peu attractives et, dans une moindre mesure, une troisième peut être ajoutée: Amiens. Les deux premières concentrent 48.4% des enseignants titulaires demandant une réaffectation et la dernière, Amiens, 6.2%. Donc ces trois académies concentrent à elles trois plus de la moitié des demandes de mobilité. Ces académies attirent peu d'enseignants, notamment titulaires et tendent à concentrer une grande proportion d'élèves en difficulté. Dans un contexte sans aucune place vacante, comme dans le Chapitre 1, une très grande proportion d'enseignants au sein de ces trois académies restaient à leur allocation initiale sous AD\* ou TO-BE car peu d'enseignants titulaires sont prêts à échanger leurs postes avec eux. Avec des places vides, ces enseignants ont la possibilité de bouger. Cependant, l'algorithme TO-BE, qui prend en compte le rang obtenu par les écoles, ne les autorise pas à partir sans être remplacés par un enseignant avec une priorité plus élevée. En particulier, on ne peut affecter un néotitulaire pour les remplacer alors que AD\* autorise ce genre de remplacements. De par la taille de ces trois académies, cet effet négatif en termes de mouvement pour TO-BE compense le fait qu'il permet plus d'échanges d'enseignants entre des académies plus populaires que AD\*, amenant in fine à moins de mouvement total. En retenant plus d'enseignants titulaires dans ces trois académies désavantagées, l'effet mécanique de TO-BE est d'y augmenter l'expérience des enseignants affectés comparé à AD\* puisque ce dernier les remplace principalement par des néotitulaires. Cet effet peut être vu comme bénéfique pour les élèves au sein de ces régions. Comme mentionné, il existe un arbitrage entre satisfaire les requêtes de mobilité des enseignants et diminuer les inégalités d'affectation des enseignants expérimentés. Cependant, retenir trop d'enseignants au sein de ces académies peut ne pas être un objectif désirable du point de vue du décideur politique. Motivés par cette idée, nous proposons de relâcher la contrainte de l'algorithme TO-BE en imposant seulement que *certain*s enseignants dans *certain*es académies (appelées *académies cibles*) doivent être remplacés par des enseignants à priorité plus élevée. Cela permet plus de flexibilité dans la conception de l'algorithme afin d'avoir une meilleure adéquation avec les différents objectifs politiques. En variant le nombre d'enseignants et d'académies cibles, on peut aller de l'algorithme TO-BE originel à celui YRMH-IGYT.

En testant différents scénarios de contre factuels avec nos données, nous montrons par exemple

qu'on peut augmenter la mobilité des enseignants titulaires de 44.9% comparé à AD\* tout en maintenant la même sortie dans les trois académies désavantagées de Créteil, Versailles et Amiens<sup>48</sup>. L'augmentation de mobilité dans les autres académies est de 79.2%. Cela permet d'améliorer la proportion d'enseignants relativement plus expérimentés pour les trois régions désavantagées, réduisant les inégalités d'affectation. Nous fournissons plusieurs résultats selon les scénarios adoptés pour montrer la flexibilité de notre approche qui peut amener à une augmentation, stagnation ou diminution des sorties au sein des trois académies désavantagées.

De la même façon que Roth (1984), nous explorons les effets de la présence de couples d'enseignants dans un problème de réaffectation. Tout comme dans le cadre standard d'appariement avec deux types, un certain nombre de résultats d'impossibilité émergent. Si les couples sont initialement affectés au sein de la même académie, veulent uniquement finir affectés dans une académie identique et qu'il n'y a pas d'enseignants seuls (ne faisant pas parti d'un couple), alors on peut définir un algorithme 2-Pareto efficace, RI et non manipulable qui s'exécute en temps polynomial. Cependant, si l'une des hypothèses est relâchée, on perd l'existence d'un mécanisme non manipulable. Nous montrons également que dans le cadre d'un marché avec couples et enseignants seuls, le problème de trouver un appariement RI différent de l'allocation initiale est NP-difficile.

Enfin, nous explorons un modèle alternatif de réaffectation des enseignants afin de mieux contrôler le taux de mobilité au sein des académies. Plutôt que de s'appuyer sur un classement complet des enseignants pour ses priorités, chaque académie à un objectif plus simple, comme par exemple décroître le nombre de "jeunes"enseignants qui y sont affectés. Dans un premier modèle très simple avec uniquement des enseignants "jeunes"et "vieux", nous exhibons une classe d'algorithmes, les algorithmes *Type Exchange* (TE), qui permettent de trouver les appariements qui respectent les objectifs des académies et qui ne peuvent pas être améliorés en termes de préférences des enseignants par un autre appariement respectant également les objectifs des académies. Cette approche alternative offre des pistes intéressantes pour de futures recherches.

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<sup>48</sup>Pour les néotitulaires, des différences existent. N'ayant pas d'affectation initiale, il se peut qu'il finissent non affectés. Pour empêcher cela, leurs listes de préférences sont complétées pour contenir (presque) toutes les académies. On peut dès lors compter le nombre de ces enseignants affectés à un voeu qui leur a été ajouté ou encore regarder la distribution des rangs des voeux obtenus. Le lecteur est invité à se référer au Chapitre 2 pour les résultats obtenus.

### Chapitre 3: un nouveau cadre d'appariement

Ce chapitre explore une nouvelle classe de problèmes d'appariement. Il a été inspiré par l'interdépendance qui peut exister entre l'allocation des enseignants aux écoles et celle des élèves aux écoles. Enseignants et élèves vont tout deux être affectés au même ensemble d'écoles. Pour qu'un enseignant puisse définir ses préférences sur les écoles, leurs caractéristiques comme la position géographique, leur budget ou leur infrastructures sont prises en compte mais ces préférences peuvent aussi être influencées par les types d'élèves qui seront affectés à ces écoles. L'idée inverse pour les étudiants est également vraie: ils peuvent également prendre en compte la qualité des enseignants affectés à leur école potentielle. En prenant une approche plus abstraite que la seule application des enseignants et élèves, on définit un modèle d'appariement où deux types d'individus doivent être affectés à un ensemble commun d'objets. Chaque individu d'un type a des préférences sur les paires d'objets et d'individus de l'autre type auxquelles il peut être affecté. Ce modèle est à l'intersection d'un modèle classique d'appariement avec deux types où deux ensembles d'agents, par exemple hommes et femmes, doivent être appariés ensemble; et un modèle d'allocation de logement où ces paires d'agents doivent également être affectées à des objets, par exemple des logements. Dans ce chapitre, nous utilisons les terminologies hommes, femmes et logements pour souligner cette intersection. En effet, dans leur article fondateur, [Gale and Shapley \(1962\)](#) utilisèrent la terminologie hommes et femmes pour les appariements avec deux types. [Shapley and Scarf \(1974\)](#) utilisèrent la terminologie logements pour les objets dans leur problème de réallocation de logements. Le problème est donc d'affecter au sein de chaque logement, un homme et une femme. Une application possible est l'affectation de managers et travailleurs à des projets au sein d'une organisation, une administration publique par exemple.

En suivant les questions traditionnelles de la littérature sur les problèmes d'appariement, on peut se demander si une généralisation du concept d'appariement stable existe dans ce contexte ? La stabilité requiert qu'une fois les couples d'hommes et de femmes sont affectés à leur logement, il n'existe aucun homme (resp. femme) qui préférerait inviter une autre femme (resp. homme) dans son logement actuel de façon à ce que les deux préfèrent strictement cette situation à leurs paires logement-partenaire qui leurs ont été affectées. Sans surprise, nous fournissons un contre exemple où un tel matching stable peut ne pas exister. Cependant, en réfléchissant au problème, la notion de stabilité mentionnée permet à n'importe quel agent de "prendre à son partenaire" le logement qui lui

est affecté tout en demandant à un autre partenaire de le rejoindre. Dans beaucoup d'applications, cela n'est pas toujours possible pour tout type d'agent. Dans le cas des managers, travailleurs et projets, il n'est pas possible pour un travailleur de renvoyer son manager, seul ce dernier peut le faire avec un travailleur. Il y a donc un sens dans lequel il existe une *structure de propriété sur les projets*. Dans notre cadre plus abstrait, nous nous référons aux projets en utilisant le terme "logements". Nous introduisons donc une *structure de propriété* qui donne pour chaque affectation d'un homme et d'une femme à un logement, le "propriétaire" du logement. Nous pouvons ensuite définir une notion de *stabilité par rapport à une structure de propriété* qui impose que seuls les propriétaires peuvent demander à un autre agent de les rejoindre dans le logement qui leur est affecté. Pour une structure de propriété arbitraire, on peut facilement montrer qu'un appariement stable par rapport à cette structure peut ne pas exister. Cependant, nous montrons qu'ils peuvent exister pour des structures naturelles: celles qui donnent toujours la propriété d'un logement à un agent du même type, c.a.d. toujours une femme ou toujours un homme. Nous les appelons les *structures de propriété pour un type*<sup>49</sup>. La preuve se base sur l'existence d'appariements stables dans un contexte standard d'appariement avec deux types comme montré par [Gale and Shapley \(1962\)](#). Cependant, nous montrons que la structure de nos appariements stables diffère. En effet, on peut définir une notion de coeur par rapport à une structure de propriété: un groupe d'agents peut essayer de réallouer leurs affectations entre eux pour tous préférer cette réaffectation mais ils doivent inclure les propriétaires des logements qu'ils utilisent pour cette réaffectation. Dans un cadre standard d'appariement avec deux types, les appariements stables sont équivalents aux appariements du coeur. Dans notre contexte avec des structures de propriété à un type, les appariements Pareto-efficaces et stables peuvent être disjoints, impliquant que le coeur peut être vide.

Afin d'explorer les liens de cette nouvelle classe de problèmes d'appariement avec la classe des problèmes d'allocation de logements définie auparavant, nous introduisons une affectation initiale des agents aux logements. On peut alors définir une notion de coeur dans ce contexte similaire à celle, déjà mentionnée ci-dessus, par [Roth and Postlewaite \(1977\)](#). La principale différence réside dans le fait que les groupes d'agents tentant de dévier ne peuvent pas échanger les logements qui leurs sont affectés mais uniquement utiliser leurs logements initiaux. Pour une allocation initiale qui affecte les logements aux agents du même type, c.a.d. toujours femmes ou toujours hommes, alors nous

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<sup>49</sup> *One sided ownership structures.*

pouvons également montrer que cette notion de coeur peut être vide. Cependant, de façon similaire au résultat précédent, on peut montrer qu'on peut toujours trouver un appariement qui n'est pas bloqué par des groupes de deux agents, composés d'un homme et d'une femme. Contrairement à la notion de coeur précédente, on peut trouver un appariement Pareto-efficace qui ne peut pas être bloqué par une coalition formée d'un homme et d'une femme.

Ce qui ressort de cette analyse est que malgré de nombreux résultats d'impossibilité, les structures de propriétés pour un type semblent être celles qui permettent toujours l'existence minimale de l'absence de paires bloquantes, que cela soit dans un contexte similaire à l'appariement avec deux types, où dans un contexte plus proche du marché de logements avec une allocation initiale. Si l'absence de blocage par paires est la condition minimale pour un système pour perdurer, alors ce résultat peut justifier pourquoi, en pratique, nous observons principalement de telles structures de propriétés. Les notions de coeur, contrairement au cadre standard, peuvent être vides. Cependant, le contexte de marché de logement avec une allocation initiale permet, contrairement au premier contexte plus proche de l'appariement avec deux types, de concilier l'absence de blocage par des paires d'agents avec la Pareto-efficacité.

## General Introduction

Over the past 20 years, matching has been a growing field in economics, culminating with the award of the 2012 Nobel Memorial Prize in Economic Sciences to Alvin Roth and Lloyd Shapley. An important determinant of its success is the existence of a strong link between the theory and its applications, which led to the adoption of several successful policies. This thesis aims at following this tradition in motivating its market design analysis with the assignment of teachers to schools with an emphasis on the French system. In the first chapter, our goal is to develop a theoretical framework to study the class of problems related to reassignment, to which teacher assignment belongs. This framework allows us to identify an important class of mechanisms that can be used in such setting, and to make theoretical predictions and preliminary empirical assessments of their possible improvements. In the second chapter, we rely on the theoretical findings from the previous analysis to practically redesign the French assignment procedure,<sup>50</sup> taking into account its specificities such as the problem of regional inequality and its established policy objectives. We provide a set of possible designs depending on different policy goals, and we validate them through the simulation of counterfactual scenarios. Finally, given the interdependence between the problem of teachers' and students' assignment to schools, in the last chapter we theoretically explore a new hybrid matching model where two sets of agents have to be matched to a common set of objects.

In this introduction, we start with a review of the matching literature where we focus on the two types of matching models that are relevant for this thesis, and we illustrate them with some policy designs that have been practically implemented over the years. Finally, we introduce the analysis done in the three chapters of this thesis while highlighting its connections to the literature.

### Matching: an overview

#### Why do we need matching ?

There are many different definitions of what economics is about. The French economist Edmond Malinvaud gave the following one:<sup>51</sup>

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<sup>50</sup>In what follows, we will indifferently use mechanism, procedure or algorithm.

<sup>51</sup>“*L'économie est la science qui étudie comment des ressources rares sont employées pour la satisfaction des besoins des hommes vivant en société. Elle s'intéresse d'une part aux opérations essentielles que sont la production, la distribution et la consommation des biens, d'autre part aux institutions et aux activités ayant pour objet de faciliter ces opérations.*”(Leçons de théorie microéconomique, Dunod, 4th edition, 1982)

*Economics is the science that studies how scarce resources are employed to satisfy the needs of human beings living in society. It is interested, on the one hand, in the essential operations, which are production, distribution and consumption of goods and, on the other hand, in the institutions and activities, which goal it is to facilitate such operations.*

This definition highlights two crucial points. First, that an important part of economics is interested in the distribution of scarce resources. The question is to decide how to allocate a good when there are several agents who want it, knowing that in the end only one of them will be able to use it. Economics thus studies the way to solve such conflict of preferences and the congestion that arises from them. The second point is the focus on the institutions that are in charge of the distribution of resources. In the course of History, an important one emerged: the market. The latter allows individuals to exchange goods and services to match their needs. Markets can take many forms. For instance, people can exchange their scarce resources through barter. However, most markets extensively studied by economists take the form of exchanges of goods against money through a system of prices. To solve the congestion problem, prices adjust: if there are more agents demanding a good than copies available, then the price increases and vice versa. Economists started to focus on markets regulated by prices, and evaluated their properties. Malinvaud, wrote in the same text: <sup>52</sup>

*In its double explanatory and normative research, our science attributed a central role to prices that govern exchanges of goods between agents. Such prices reflect for individuals, more or less precisely, the social scarcity of the goods that one buys or sells.*

Economists have thus identified the central role of prices in governing exchanges, but it is important to recall that there are other ways to regulate markets. Let us take a step back and ask for some specific kind of resources: can we and do we want to use prices as a mean to allocate them? For example, many patients with a kidney disease need a transplant and there are more patients than available kidneys. This is a typical congestion problem, raising the question of how to allocate a limited amount of resources, here kidneys, to agents who need it, knowing that there are more agents than available kidneys. One possible answer is to define a price for kidneys. This is what Iran chose

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<sup>52</sup>“Dans sa double recherche explicative et normative, notre science a été amenée à attribuer un rôle central aux prix qui président aux échanges de biens entre agents. Ces prix reflètent pour les individus, de manière plus ou moins exacte, la rareté sociale des produits qu’il achète et qu’il vend.”, (Leçons de théorie microéconomique, Dunod, 4th edition, 1982)



in 1988, by having donors paid to give their kidneys. This possibility was also discussed for the US context by [Becker and Elias \(2007\)](#) ), who estimated a potential equilibrium price for kidneys taking into account the statistical value of life, the risk of death, the quality of life and the value of time.<sup>53</sup> Most countries however prohibit the use of monetary transfers to allocate organs. As mentioned in [Roth \(2007\)](#), Article 21 of the Council of Europe's (2002) Additional Protocol to the Convention on Human Rights and Biomedicine, on Transplantation of Organs and Tissues of Human Origin states:

*The human body and its parts shall not, as such, give rise to financial gain*

In the U.S, the National Organ Transplant Act of 1984 contains a similar statement. There is a widespread feeling that money is not appropriate to deal with the exchange of organs. [Roth \(2007\)](#) called this feeling *repugnance*. The concept of repugnance is, of course, different across regions and across time. It can be influenced by many factors such as culture, psychology, religion and so on. For instance, black market aside, countries differ in their legislation over drug consumption or prostitution. Another example presented by [Roth \(2007\)](#) is the one of horsemeat consumption, which is forbidden in California but is commercialized without concerns in many other places. Over time, transactions that used to be considered repugnant might also cease to be seen as such. [Roth \(2007\)](#), p.3, mentioned the following well known example:

*Lending money for interest was once widely repugnant, and no longer is, (with the important exception that Islamic law is commonly interpreted as prohibiting it).*

The use of prices to solve congestion problems may also be directly at odds with the policy objective that drives the allocation motive. For example, free public education is considered in many countries as a fundamental right and an important policy objective, given the externalities it can bring. Therefore, in a case where many applicants would like to be assigned to a popular school, increasing tuition fees is rarely considered as a potential policy tool to solve the congestion, since it would exclude applicants from lower income families who cannot afford to pay more. Another common example is the one of social housing. Since the goal of social housing is to allow poorer families to live in places where the market price de facto excludes them, an adjustment through prices would be directly in contradiction with the policy objective. The motivation behind such goal is that social

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<sup>53</sup>Earlier papers discussed this possibility for organs from cadavers, the interested reader may refer to the cited papers for more details.

diversity creates positive externalities that benefit the society as a whole, and should therefore be encouraged.<sup>54</sup>

It is important to debate whether some transactions should or not be considered as repugnant. For instance, [Leider and Roth \(2010\)](#) surveyed a representative sample of Americans to identify opinions about the potential introduction of a market with prices for kidneys. In our teacher assignment application, as we describe later, deciding whether to use wages to solve for congestion in the regional assignment is an important policy debate. However, as mentioned in [Roth \(2007\)](#), some economists have taken the repugnance of monetary transactions in some markets as a given constraint, abstracting from the reasons motivating it. Once one accepts the existence of such constraint, one has to find alternative means of solving the congestion problem i.e. to *design the market*. This fundamental consideration constitutes the main motivation of matching economists: developing theories and practical solutions to solve congestion problems and to achieve policy goals for markets where the use of monetary transfers is not allowed. The theories and solutions developed are intrinsically linked to the specific markets they aim to design. However, the practical problems encountered in several markets can belong to the same class, and thus the solutions developed for this class can all be applied. The goal of allocating students to schools is fundamentally different from the one of assigning kidneys to patients, which is in turn different from assigning social housing. All of these problems have therefore seen their own solutions emerge. As this thesis will show, the problem of (re)assigning teachers to schools is conceptually different from the one of assigning students to schools. It will share similarities between the latter problem and the one of reassigning social housing between tenants. In the following section, we review important results in the literature focusing on the two classes of the assignment of students and the one of social housing.<sup>55</sup>

## Two-sided matching markets

The problem of assigning students to colleges, known as the *college admission problem*, has been introduced by the seminal article of [Gale and Shapley \(1962\)](#). The problem is simple: students

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<sup>54</sup>It is important to highlight that some economic schools of thought, in the tradition of the Chicago School, would argue that the distortion or the no-use of prices would be a mistake that would distort economic efficiency. According to this thinking, a more relevant policy would be to allocate grants to the families. As we will mention, we will put aside and not enter into this debate on the use or not of price adjustments.

<sup>55</sup>Because of this thesis focus, we voluntarily omit an important literature concerning kidney exchange problems that has important policy implications. A key difference is that incentives are relatively less of a concern concerning reports of patient characteristics. The interested reader can easily find references. An excellent handbook overview is [Sonmez and Unver \(2013\)](#). For a short presentation, one can also refer to [Ashlagi and Roth \(2012\)](#).

have to be assigned to colleges, each college having a limited number of seats. Each student has a preference order over the colleges and each college has a preference order over the students. The authors assume that the preferences are strict so that a student (resp. a college) cannot be indifferent between two colleges (resp. students). Let us look at the abstract class of problems we are facing: there are two sides, students and colleges, which have preferences over each other. They start unassigned and would like to be matched together. For a policy maker, the question is to decide how to use the information contained in the preferences of each side to decide an appropriate matching. To do so, one has to establish what desirable properties a matching should have. A first very intuitive requirement, that economists refer to as *individual rationality* or *participation constraint*, is that if a student (resp. a college) prefers to stay unmatched than being matched to a certain college (resp. student), then one should not match them together. Among economists, another widely used requirement is the one of *Pareto-optimality*. A matching is Pareto-optimal (or Pareto-efficient, or sometimes efficient for short) if one cannot reassign students to colleges such that all students and colleges are at least indifferent and some of them strictly prefer their new assignment.<sup>56</sup> While Pareto-efficiency is an appealing requirement, there exist many different matchings that are Pareto-efficient and some of them might not be very desirable. Given the latter, Gale and Shapley proposed an additional requirement, which they called *stability*. A matching is stable if it is individually rational and if there is no pair of student and college which is not matched together where it would prefer to be. If such pair exists, it is called a blocking pair. Therefore, a stable matching is an individually rational matching with no *blocking pairs*.<sup>57</sup> This intuitive requirement can be interpreted in two ways:

- From a normative point of view, it can be seen as a criterion of justice, or *fairness*.<sup>58</sup> Indeed, a non-stable matching will have a student who has been denied entry at a college that he prefers and where he is preferred over one of the students accepted. The former student will thus have a *justified-envy* towards the one that took a spot for which he was less preferred than him.
- From a positive point of view, a non-stable matching cannot be sustainable on the long term. Indeed, if a student and a school prefer to be matched together rather than their current

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<sup>56</sup>In this setting with strict preferences, being indifferent means that one keeps the same assignment.

<sup>57</sup>One can actually prove that any stable matching must be Pareto-efficient according to the aforementioned definition.

<sup>58</sup>Stable matchings are also referred to as fair matchings.

situation, then the college will find a way to expel one of its least preferred students to create a free spot for his preferred one. Roth (1991) studied seven different assignment markets for new physicians and surgeons in the United Kingdom, and showed that stable matches were still in use while most of the unstable ones were quickly abandoned.

Having defined what a stable matching is, the question that naturally arises is whether it is always possible to find one, no matter the form of preferences expressed by students and colleges. Gale and Shapley provided an elegant proof of its existence, and introduced a practical procedure for finding it. They proposed an algorithm, now called the *Gale-Shapley algorithm* or the *Deferred-Acceptance* (DA) algorithm, which returns a stable matching for each possible set of preferences reported. It can be described as follows:

**Preliminary step.** The policy maker chooses which side, students or colleges, is the *proposing side*. In what follows, we will assume that students propose.<sup>59</sup>

**Step 1.** Each student applies to his first ranked college and colleges *tentatively* accept all the students applying to them. If a college receives more applicants than its available seats, it only accepts the best ranked ones up to its capacity and rejects the others.

**Step k.** All the students rejected at Step  $k - 1$  apply to their next favorite college in the preference list. Colleges consider all the students tentatively accepted in the previous steps and the new applicants, and if a college has more applicants than its capacity, it again selects the top ranked ones and rejects the others.<sup>60</sup>

The process continues until there are no rejections anymore. The matching defined in the last step becomes the final one. Since there are a finite number of colleges and students and at each step at least one student applies to his next ranked college, this algorithm stops after a finite number of steps. Since at each step, a student is rejected only if a preferred one applies to the college, the resulting matching can easily be proved to be stable. Another remarkable property is that this algorithm is “fast”: if there are  $n$  students and  $m$  colleges, it requires at most  $n \times m$  steps. In computer science, and more precisely in complexity theory, this type of algorithms is referred to as *polynomial*, since

<sup>59</sup>One just has to change students into schools in the following to obtain the other version. If colleges propose, we consider, in what follows, that students have a “capacity of one” since they can only be matched with one college. At each step, a college with  $q$  seats will propose to its  $q$  best ranked students who have not rejected it yet.

<sup>60</sup>Note that student accepted at a previous step can still be rejected at this step. That is why the acceptance decision is tentative and will only be definitive at the final step of the algorithm.

its execution time is a polynomial function of the size of its inputs, i.e. the length of the preferences of students and colleges. Given that we are dealing with the design of real markets, having practical and efficient algorithms is an important requirement for their policy applications.<sup>61</sup> In their article, Gale and Shapley also proved the following important property: if students are the proposing side, then the obtained stable matching is, among all the possible stable matchings, the most preferred one for students and the least preferred one for colleges. It is therefore called the *Student Optimal Stable Matching*. This means that one cannot find a stable matching that all the students weakly prefer (and some strictly) to the one found by the algorithm under which they propose. The result is symmetric if colleges are the proposing side, so that the resulting matching is the *College Optimal Stable Matching*. This property creates a trade-off in choosing which side to favor.

In the eighties, an important literature developed to study stable matchings, the DA algorithm and their properties. Our goal here is not to give an exhaustive list of all the results, for which we refer the reader to the excellent textbook of Roth and Sotomayor (1990) that formally exposes them. Rather, we will focus on one important aspect: incentives. Inspired by a solid tradition of game theory in economics, Roth (1982) focused on the incentive properties of the DA algorithm, asking whether it is possible for any student or college to misreport his preferences in order to obtain a better match than the one he would get while being truthful. He finds the following result:

**Roth (1982), Theorem 3.** There exists no procedure leading to a stable matching for which truth telling is a dominant strategy.

The concept of dominant strategy is standard in game theory and in this case it requires that, no matter what the other agents report, one cannot have a strictly better match by misreporting his preferences. This result thus implies that the DA algorithm can be manipulated. A procedure that does not suffer from incentives to manipulate is called *strategy-proof*. However, Roth (1982) also finds a more encouraging result:

**Roth (1982), Theorem 5.** If students propose, the Deferred-Acceptance mechanism is strategy-proof for the students.<sup>62</sup>

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<sup>61</sup>The matching literature is also important in computer science and was initiated by Donald Knuth. There are, for instance, numerous results in complexity theory related to matching. An increasing amount of works intersects both fields. Chapter 2 will have a result related to complexity theory.

<sup>62</sup>The symmetric version for colleges is trickier. In the case where each college only has one seat, called the *one-to-one* setting, the symmetric result trivially holds since there is no conceptual difference between students and

Since the first Theorem states that the algorithm is not strategy-proof overall and the second one states that it is for students, then in a DA where students propose, colleges will have an incentive to manipulate. A stronger version of Theorem 5 was independently proved by [Dubins and Freedman \(1981\)](#): no group of students can jointly misreport their preferences such that they all obtain a strictly better match.<sup>63</sup> This result is encouraging since, in practice, secondary education colleges are mostly non-strategic entities: their “preferences” are determined by law in the form of a list of criteria. Consequently, strategy-proofness is a relevant concept only from the student side. To make this distinction, the literature referred to them as schools rather than colleges and called their preferences *priorities*. Strategy-proofness has become an important property for mechanism design among economists and game theorists, opposing them to the literature in computer science or operation research, where this feature is not considered as central.

[Roth \(1984\)](#) provided the first analysis of a practical matching market: the one that allocates medical interns and residents to hospitals in the U.S. This market falls in the same class of matching problems analyzed by Gale and Shapley. Here there are also two-sides, interns and hospitals, that have preferences over each other, are initially unassigned and who need to be matched together. After describing the institutional and historical characteristics of the market, including its issues and the criticism it faced, he provided a description of the algorithm adopted in 1953, the National Intern Matching Program (NIMP) algorithm, and analyzed its properties. The latter turned out to be equivalent to a DA hospital-proposing algorithm. Given the aforementioned results, he noted several implications:

- The resulting matching is the least preferred stable matching for interns. This could explain why over the years there have been some complaints from the latter, which claimed that the algorithm “unfairly favors hospitals at the expense of students”.
- The NIMP algorithm is not strategy-proof for interns, creating possible manipulation issues.

Based on these results, the natural policy recommendation that arises is to switch towards a DA doctor-proposing algorithm. However, [Roth \(1982\)](#) reported the existence of a practical concern about the distribution of interns and residents within hospitals. Some hospitals, mostly in rural

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schools. However, [Roth \(1985\)](#) showed that in a *many-to-one* setting where colleges have more than one seat, the symmetric result for the college-proposing DA does not hold. The main technical difference lies in the fact that now, a college has to compare sets of assigned students using its preferences that rank only individual students.

<sup>63</sup>In the literature, this notion is referred to as *group strategy-proofness*.

areas, are under-demanded by doctors, and switching to an alternative algorithm favoring the doctor side may contribute to worsen the problem. Motivated by this concern, he studied the issue under the two possible DA mechanisms, and proved the following important result, usually referred to as the *Rural Hospital Theorem*:

**Roth (1984a), Theorem 9.** When all preferences are strict, the set of hospital positions filled is the same at every stable outcome, as is the set of doctors who are assigned positions.

This has an important implication: the problem of regional inequality in the number of doctors assigned to hospitals cannot be overcome by switching to a different stable matching, and if one believes that stability is an essential property of the assignment algorithm, then the solution to this problem needs to be found elsewhere.<sup>64</sup> More importantly, it implies that switching to the DA doctor-proposing algorithm will not worsen the inequality in the distribution of doctors across rural and urban hospitals.

Another important practical concern that he mentioned is the presence of couples of doctors requesting an assignment. This introduces the novelty of correlated preferences among individuals, who wish to be matched close to each other. Conceptually, this problem was new, since a couple can be considered as a unique entity having preferences over pairs of possible assignments, representing the assignment of the two members. Having introduced this, one can redefine the concept of stability as follows: a couple blocks if its members can be re-matched to two positions that they jointly prefer and if the hospitals receiving them prefer this outcome over their current match. Concerned with this practical consideration, he proved the following seminal theorem:

**Roth (1984a), Theorem 10.** In a market in which some agents are couples, the set of stable outcomes may be empty.

This theorem had an important impact on the literature. Many future works, both in economics and computer science, studied the properties and possibilities of matching markets in the presence of couples. Since the Chapter 2 of this thesis already introduces the literature on couples, we will not go into the details here. However, we recommend to the interested reader the excellent interdisciplinary

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<sup>64</sup>Later on, Roth (1986) proved a stronger version: hospitals that do not fill all their seats under some stable matching will be matched to the exact same candidates under all stable matchings. So one cannot change the number or identity of doctors matched to underdemanded hospitals but can only influence the identity of matched doctors in hospitals filling all their positions.

survey of [Biró and Klijn \(2013\)](#), which describes two important results in this context: First, proving the existence of a stable matching becomes computationally “hard” ([Ronn, 1990](#)).<sup>65</sup> Second, whenever a stable matching exists, there is no strategy-proof mechanism that can generate it ([Biró and Klijn, 2013](#)).

In the light of Roth’s analysis on the couple’s problem, the Board of Directors of the National Resident Matching Program (NRMP)<sup>66</sup> decided in 1995 to redesign the allocation program. [Roth and Peranson \(1999\)](#) described the final algorithm, designed using the previously existing theories and some engineering approach to deal with practical concerns and constraints of this specific market. An important quote from the paper, p.749, is the following:

*The present paper then, in addition to presenting some new results, is intended to take a step in the direction of an engineering literature as well, by describing how those facts were learned and how they impacted design decisions.*

One of the important practical constraints was, as mentioned, the existence of couples. However, the final algorithm, inspired by matching theory, seemed to perform well in practice, despite the theoretical concerns. For instance, the existence of a stable matching with couples seemed not to be an issue in practice. This empirical observation, together with other similar examples, motivated the development of new theoretical research. [Kojima, Pathak and Roth \(2013\)](#) built a large market model to show that the existence of a stable matching under the NMRP algorithm, even with couples, is highly probable in large markets. [Ashlagi, Braverman and Hassidim \(2011\)](#), in a different large market setting, also showed that a weaker version of stability exists with high probability, even with couples. Moreover, they showed that the new algorithm they proposed gives agents good incentives, in the large, to report truthfully their preferences.<sup>67</sup>

Another important application of the two-sided market class of problems is the one of *school choice* procedures to assign students to schools. [Abdulkadiroglu and Sonmez \(2003\)](#) described

<sup>65</sup>In terms of complexity theory, the problem belongs to the class of NP-hard problems.

<sup>66</sup>The new name given to the previously called NIMP.

<sup>67</sup>The use of large markets has become more and more important in the matching literature. Starting with the seminal work of [Immorlica and Mahdian \(2005\)](#) and [Kojima and Pathak \(2008\)](#) who showed that, with a specific large market structure, the colleges do not have any incentive to misreport their rankings, even under the student-proposing DA algorithm. Later on, additional models of large markets have been developed. [Abdulkadiroglu, Che and Yasuda \(2015b\)](#) and [Azevedo and Leshno \(2016\)](#) provided a continuum model to prove that there is a unique stable matching in the large. [Lee \(2014\)](#) used a random graph approach to show that both students and schools can obtain their top quantile partners if preferences are independent and uniformly distributed. Recently, [Che and Tercieux \(2015b\)](#) or [Che and Tercieux \(2015a\)](#) used the same random graph approach to prove important results related to the comparison of Pareto-efficient and stable mechanisms.



a widely used algorithm in this context, called the *Boston mechanism*, that was used in Boston, Columbus, Minneapolis and Seattle. They showed that this mechanism had a serious drawback: it was not strategy-proof for students, creating incentives for them to misreport their preferences. The authors recommended the use of the student proposing DA mechanism to solve this issue. As we mentioned above, there is one conceptual difference between this problem and the standard college admission setting: schools are not strategic entities and their preferences are rather considered as a priority system. Consequently, when one considers Pareto-efficiency as a requirement for a matching, only students are considered welfare-relevant agents, and a matching is *efficient* if one cannot reassign students such that they would obtain schools that they weakly prefer, and some strictly.<sup>68</sup> This relaxation of the Pareto-efficiency definition has an important implication: stable matchings are not always Pareto-efficient anymore. In some cases of stable matchings, one could reassign students to make them better-off while, at the same time, creating blocking pairs. Therefore, in designing school choice procedures one may have to choose between Pareto-efficiency and stability. [Abdulkadiroglu and Sonmez \(2003\)](#) proposed a Pareto-efficient strategy-proof mechanism, inspired by the house allocation problem that we will describe below, and called the *Top Trading Cycles* (TTC) algorithm. It works as follows:

**Step 1.** Let all the students point to their first ranked school (if a student finds them all unacceptable then leave her unmatched and remove her from the process). Let all the schools point to their first ranked student. Using the pointing previously defined, there will be a cycle,<sup>69</sup> i.e. a student pointing to a school that points to another students until the pointing comes back to the first students. Assign the students in this cycle to the school they are pointing to. Remove these students and decrease the capacity of the schools involved in the exchange by one. If a school ends up with a null capacity, then remove it from the process.

**Step k.** Let all the remaining students point to their favorite school among those that still have positive capacity. Let each school point to its highest ranked student among the ones remaining. A new cycle will be generated. Assign the students in this cycle to the school they are pointing to and decrease by one the capacity of the schools concerned. Remove the schools

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<sup>68</sup>Note the difference with the previous definition: now we only require students to be better-off and not students and schools.

<sup>69</sup>It is a standard graph property: every graph with nodes all having a unique outgoing edge admits at least one cycle. There might be several cycles but those will be disjoint, i.e. they will include different nodes, so that the order in which one will implement them will not affect the final outcome of the algorithm.

reaching null capacity and move to the next step.

The process continues until all the students are assigned. The authors proved that this algorithm is strategy-proof for the students and Pareto-efficient, but it is not stable. Policy makers designing school choice procedures are thus left with two candidate algorithms: the DA algorithm if one cares about stability, and the TTC algorithm if one cares about Pareto-efficiency. [Abdulkadiroglu et al. \(2005\)](#) advocated for implementing one of these two propositions to replace the Boston mechanism that was not strategy-proof. In addition, [Abdulkadiroglu et al. \(2006\)](#) extensively studied the Boston mechanism from an empirical perspective, and highlighted that parents were well aware of the potential gains from manipulating the procedure by misreporting their true preferences. The Boston Public School guide of 2004 was for instance giving clear recommendations in this direction:

*For a better chance of your “first choice” school: consider choosing less popular schools.*

*Ask Family Resource Center staff for information on “underchosen” schools.*

The authors showed that in the reported preferences of families there was indeed a clear discontinuous drop in popular schools ranked second in the list. This discontinuity revealed the existence of manipulations, since ranking a popular school second under the used mechanism was not a good strategy, given that it will probably be full after the first step thus creating the risk to miss opportunities in other schools. Family understood this and strategically preferred not to rank popular schools in second place. This has an important implication for the evaluation of this mechanism: one cannot use family reported preferences to evaluate the performance of the procedure, since stated preferences are unlikely to be the true ones. The board claimed that the algorithm was successful since it was assigning 90% of families to their first ranked school, but this statement was flawed since for many families the first ranked school was probably not the truly preferred one but rather the best strategically. The final argument that convinced policy makers to redesign the procedure was that most of the families who were not strategic and reported their preferences truthfully were hurt by the algorithm and risked not to be assigned to any school. Moreover, these families were likely to be low income ones with less access to information on how to “game” the system. Adopting a strategy-proof mechanism was therefore presented by policy makers as a way to *level the playing field* between families, helping to decrease inequality. This first impactful analysis done by matching economists fostered numerous policy recommendations and a tremendous amount of additional research, which we will not detail here. Many cities such as Boston, New York City,

Chicago, Denver and Washington DC, among others, adopted the DA students proposing algorithm. New Orleans initially adopted the TTC algorithm but recently moved to DA. In France, [Hiller and Tercieux \(2014\)](#) showed that the currently used Affelnet procedure to assign students to high schools is in fact a school-proposing DA algorithm.

### House allocation problems

A second important class of problems deals with the allocation of objects. Contrary to the two-sided market setting, here only one side has preferences while the other is considered as an object to be allocated that does not possess any priorities or preferences. The most widely used application of this class is public housing assignment. Three types of problems arise:

1. House allocation with no tenants.
2. House allocation with tenants.
3. House allocation with existing tenants.<sup>70</sup>

The first one, introduced by [Hylland and Zeckhauser \(1979\)](#), studies the most basic allocation problem: how to assign a finite number of goods to agents having preferences over them. There can be several copies of the same good but the number has to be finite. Pareto-efficiency is a natural requirement for this kind of allocation, implying that one cannot reassign objects such that all the agents (weakly) prefer the new allocation. A standard mechanism used in this literature is the *Serial Dictatorship* (SD) mechanism, which is very intuitive and simple:

**Preliminary Step.** Fix an ordering of the agents.

**Step 1.** The first ranked agent chooses his favorite object and obtains it. They are both removed from the market (if there are several copies of the object, only one copy is deleted).

**Step k.** The k-th ordered agent chooses his favorite object among those remaining and obtains it. They are both removed from the market.<sup>71</sup>

The process continues until all the agents picked an object. This mechanism is strategy-proof so the agents do not have an incentive to lie on their reported preferences. Obviously, the choice of

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<sup>70</sup>Also referred to as *housing market*.

<sup>71</sup>In the case there is no remaining object, or those remaining are not acceptable for the agent, just do not allocate any object to this agent.

the order is crucial and higher ranked agents have a clear advantage. In the case of public housing assignment, seniority could be a suitable criterion to determine the priority order: the agent who has been waiting the longest picks first, and so on. If there is no defined criterion to discriminate, one can randomly pick the ordering of the agents to ensure a relative “fairness” of the process. The mechanism is then called *Random Serial Dictatorship* (RSD). In doing so, one can then study probabilistic properties of the algorithm, the problem becomes a *probabilistic assignment* where one has to set probability lotteries over the possible allocations. If agents have cardinal utility values on top of their ordinal rankings, i.e. they attach a numerical value to each object, they can evaluate probabilistic assignments using the expected utility criterion. Many research focused on probabilistic assignments. For instance, [Abdulkadiroglu and Sonmez \(1998\)](#) showed that that the RSD and the TTC are equivalent when owners are picked randomly. [Bogomolnaia and Moulin \(2001\)](#) introduced an ex ante notion of efficiency, *ordinal efficiency* and proposed an algorithm, the *Probabilistic Serial*, to characterize all the probabilistic mechanisms leading to ordinally efficient probabilistic assignments.<sup>72</sup> The SD mechanism is simple and easy to implement, and it has been reported in many practical allocation designs. For example, many universities use it for undergraduate housing allocation as is the case for Carnegie-Mellon, Duke, Harvard, Northwestern, and Pennsylvania ([Abdulkadiroglu and Sonmez, 1999](#)).

[Shapley and Scarf \(1974\)](#) introduced the second type of models where a group of tenants would like to exchange their current house.<sup>73</sup> Here, tenants have an initial right over their house and can keep it in the case they are not satisfied with the new assignment. This constraint is again called *Individual Rationality* (IR): if a tenant obtains a new house, he must prefer it to his initial one for the movement to take place.<sup>74</sup> This requirement is therefore added to the one of Pareto-efficiency. Furthermore, given the presence of an initial allocation of tenants to houses, one may also require a

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<sup>72</sup>Our goal is not to survey the complete and rich literature of probabilistic assignments. [Hylland and Zeckhauser \(1979\)](#) proposed a market approach, the *Competitive Equilibrium with Equal Income* where each agent is endowed with a fake budget and where prices allow to buy probability shares. [Budish et al. \(2013\)](#) extended this approach to design more complex constrained probabilistic assignments where, for instance, group of objects can have upper and lower assignment quotas.

<sup>73</sup>Since we are in a matching setting, this is done without any monetary transfer. One can consider the reallocation of public housing or dorms at universities for instance.

<sup>74</sup>There is a clear conceptual link between this IR notion and the one in the former two-sided matching context. In both cases, IR imposes that one must give to each agent, an assignment that he prefers to the one he would get under the *statu quo* matching. In two-sided matching, as highlighted, agents start being unassigned so that the *statu quo* matching is, for each agent, to stay unassigned. In this new setting, the *statu quo* matching is defined by the initial allocation of the houses. That is also why IR can also be referred to as *participation constraint*: if one gives an agent a worse assignment than his initial one, the latter will initially choose not to participate in the process.

stronger property than Pareto-efficiency called the *Core*. A matching is a *core matching* if no group of agents can exchange with each other their initial housing assignment and be better-off than under the algorithm proposed matching.<sup>75</sup> If instead one uses a mechanism returning non-core matchings, then some group of agents would prefer not to join it and organize their own exchanges. We say that such a group *is blocking* the matching or also that *he deviates*. [Shapley and Scarf \(1974\)](#) proved the existence of core matchings and Gale proposed an algorithm that always returns a core matching, the *Top Trading Cycles* algorithm:

**Step 1.** All agents point to the agent owning their favorite house. If an agent does not prefer any house over his own, he points to himself forming a (self) cycle. Since there will be at least one cycle, implement it in assigning agents to the house of the agent they are pointing to. Remove the agents and houses involved in the exchange.

**Step k.** The remaining agents point to the one owning their favorite remaining house. If an agent does not prefer any house over his own, he points to himself. Since there will be at least one cycle, implement it in assigning agents to the house of the agent they are pointing to. Remove the agents and houses involved in the exchange.

The process continues until every agent is matched in a cycle. [Roth and Postlewaite \(1977\)](#) proved that the matching found by the TTC algorithm is the unique core matching. Additionally, the TTC algorithm is strategy-proof and [Ma \(1995\)](#) proved that it is the only algorithm that is IR, Pareto-efficient and strategy-proof at the same time. Therefore, if one cares about these three properties above all, he must use an algorithm equivalent to TTC.<sup>76</sup>

Finally, the last model was introduced by [Abdulkadirouglu and Sonmez \(1999\)](#). They considered a setting where houses need to be assigned. Some of them are initially owned by a tenant while others are empty, and there are some initially unassigned agents looking for a house. They defined

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<sup>75</sup>One can also define a core definition in the previous setting of two-sided market. In that case, the deviating group of agents exchanges their assigned houses rather than, as here, their initial ones since they do not have any. In this context, one has to view this notion as the generalization of the notion of blocking pairs to group of agents. [Roth \(1982\)](#) proved that stable matchings coincide exactly with core matchings.

<sup>76</sup>In the context of the allocation of objects without tenants, one can hope to obtain a similar result with the SD mechanism as being the only Pareto-efficient and strategy-proof mechanism. [Svensson \(1999\)](#) proved that SD was the only strategy-proof, Pareto-efficient and neutral algorithm if the number of objects and agents is the same. [Papai \(2000\)](#) proved that a generalization of the TTC algorithm, to which SD belongs to, in the general setting, is the only group-strategy-proof, Pareto-efficient and reallocation-proof mechanism. The interested reader may refer to the papers for the definitions.

a strategy-proof and Pareto-efficient mechanism called *You-Request-My-House-I-Get-Your-Turn* (YRMH-IGYT):

**Preliminary Step.** Fix an ordering of all the agents.

**Step 1.** The first ranked agent points to his favorite house. If it is empty then he is assigned to it, they are removed and the algorithm moves to the next step. If it is occupied by an initial tenant, then the latter also points to his favorite house. Once again, if it is empty, then the chain is implemented; if it is occupied, the process continues until either: i) reaching an empty house or ii) cycling. In both cases the chain or cycle is implemented and the agents and houses are removed.

**Step k.** Same process as in Step 1 but only with the remaining agents and houses.

The process stops once all the agents are assigned. This algorithm can be seen as a generalization of the TTC and DS algorithms. Indeed, if there are only occupied houses and no agents without initial allocation, then it is equivalent to TTC. If there are only empty houses and agents without initial allocation, then it is equivalent to SD. [Sonmez and Unver \(2010\)](#) characterized the YRMH-IGYT algorithm as the only one being Pareto-efficient, IR, strategy-proof and respecting two additional axioms: weak neutrality and consistency. In the setting with tenants and possible empty houses, another algorithm can be used, which was first identified by [Guillen and Kesten \(2012\)](#) for the assignment of students to campus housing at MIT. One has first to define priorities for each house over the agents. If a house has an initial tenant, then it should rank him first in its priority list. Once the priorities are established, one can just use the standard agent proposing DA algorithm with this priority system. Since every tenant is ranked first in his initial house, he is ensured by construction to get a weakly better assignment, so that the mechanism is IR. Since it is a DA where agents propose, it is strategy-proof for the agents. However, as mentioned in the previous section, it is not going to be Pareto-efficient if one considers only agents as welfare relevant entities.

## **This thesis: the redesign of the French teacher assignment and a new matching problem**

This thesis is inspired by a practical market: assigning teachers to schools. Many countries chose to use centralized assignment procedures to assign their teachers. France is one of them but it is also

the case in Italy, Mexico, Peru, Turkey, Uruguay and Portugal, among others. An important question to address is whether one should prefer a centralized system of assignment or rather a decentralized one, as in a standard labor market. Many countries, like the U.K or the U.S for instance, chose the latter. On the other hand, French teacher unions are very attached to the concept of *Republican School* and to the value of equal treatment. They believe that every teacher should be treated the same way and that one should not discriminate in terms of wages. One could see this concern as the aforementioned *repugnance constraint*. Our goal is to take this constraint as given and not to discuss the relative performance of centralized versus decentralized systems, even though this debate is still very present in the literature.<sup>77</sup> Our analysis can however contribute to the discussion since we show that, in the reassignment context, the algorithm currently identified in the literature can be greatly improved. So, in a sense, we show that the potential outcomes of centralized assignment procedures were so far underestimated.

The assignment of teachers to schools can be seen as the intersection of the two standard frameworks that we described above: the two-sided market models and the housing allocation models. It shares similarities with the two-sided market models since, even though schools are not strategic agents, we show that their priorities reflect important policy goals that cannot be completely ignored. In our context, the college admission problem is thus the most relevant since it takes into account the preferences of both sides when performing welfare evaluations. To highlight the difference with the Pareto-efficiency concept in the house allocation setting, we call this notion 2-Pareto efficiency, which allows us to say that the DA teacher proposing algorithm produces a stable, 2-Pareto efficient matching and is strategy-proof for the teachers. This algorithm can be directly applied to *newly tenured teachers* who are just entering the profession and are looking for a first assignment. However, in the case of *tenured teachers* that are currently located in a school and would like to be *reassigned*, the additional constraint of *individual rationality* (IR) emerges, similarly to the house allocation problem with tenants. Indeed, one would like to grant these teachers *the right to stay at their initial school*, so that if they are reassigned it must be to a school they prefer. Our goal is thus to ensure

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<sup>77</sup>For instance, for the NRMP allocation, one could argue that a decentralized system would better allocate doctors while leading to potential higher wages. It has indeed been an important debate and criticism. Agarwal (2015) treated this question and estimated the impact of a potential decentralized system on wages. He estimated that “*Due to residents’ willingness to pay for desirable programs and capacity constraints, salaries in any competitive equilibrium would remain, on average, at least \$23,000 below the marginal product of labor.*” concluding that the centralized system was not causing a lowering of wages. There is still an ongoing debate on whether the use of salary changes is a better tool to incentivize teachers. The existing empirical results can even be contradictory. The reader can refer to the discussions and citations of the Chapters 1 and 2.

IR while also taking into account both sides of the market. In our 2013 French dataset used in this thesis, tenured teachers represent about 63% of the teachers.

A first important change with respect to the previous theoretical literature is that it is impossible to always ensure both IR and stability. If individual rationality is a compulsory requirement, then one has to accept the possibility of blocking pairs. In this setting the standard Shapley-Scarf TTC proposed in the house allocation problem is not very convenient: even if it ensures IR, it completely omits the schools' side. As mentioned above, [Guillen and Kesten \(2012\)](#) identified DA as a suitable mechanism for these types of problems. If teachers are initially ranked at the top of their initial school, then the algorithm ensures IR and some notion of stability while taking into account schools' rankings. In our context, there is however a slight difference with respect to the house allocation version: as in the school choice problem, priorities of schools are initially fixed and usually given by the law such that a tenured teacher may not have the highest priority in his current school. In this case, one has to artificially modify the priority of schools in order to rank their currently assigned teachers first. Once this process is done for all schools, the DA algorithm is run over the modified priorities. This procedure, which we call the DA\* mechanism, corresponds to the mechanism currently used in France to assign teachers. Its properties have been studied in the literature: [Guillen and Kesten \(2012\)](#) described it in the context of student housing assignment at MIT and [Pereyra \(2013\)](#) studied the specific case of teacher assignment that interests us. He focused on a relaxed definition of stability where one can only allow one form of justified envy: the one directly conflicting with the IR requirement. He allows the existence of teachers that would prefer to be assigned to a different school where they have higher priority, only if the teacher with lower priority in that school was initially assigned there and has not moved from it. Such envy is called an *inappropriate claim*, in the sense that they are not valid and are disregarded. Conversely, *the justified claims* are those consisting of a teacher preferring a school in which he has higher priority and where the assigned lower priority teacher was not there before. As mentioned above, if we want to ensure the IR constraint, then we have to accept some blocking pairs. Pereyra required that only the first type, the ones coming from inappropriate claims, should be allowed. He showed that the DA\* algorithm achieves IR with the minimum number of them. [Compte and Jehiel \(2008\)](#) also studied the reassignment problem and showed that, if one wants to forbid justified claims, then DA\* achieves the highest possible movement among the algorithms forbidding justified claims. In



France, the DA\* algorithm assigns both newly tenured and tenured teachers at the same time. In Chapter 2, we give a precise description of the institutional context and of the detailed French procedure. The important aspect is that this procedure is done in two steps. The first one is the between-regions assignment, where teachers willing to move to another region in France are asked to submit a preference list over the 31 administrative areas. Once this assignment is done using the DA\* algorithm, teachers are asked to submit a list of schools in the region they were assigned during the first step.<sup>78</sup>

France's education system, as in most OECD countries, faces two problems:

1. a lack of attractiveness of the teaching profession.
2. high inequalities in achievement between students from different social background.

Concerning the first issue, the [Direction de l'Evaluation de la Prospective et de la Performance \(2015\)](#) mentioned that, in 2014, 24% of the positions opened by the government remained vacant because of a lack of applicants and the poor quality of those applying. Moreover, in 2014, only 43.9% of the secondary school teachers asking for a reassignment obtained one under the current procedure. Concerning the second point, the [Organisation for Economic Co-operation and Development \(2014\)](#) reports that France is the one of the OECD countries where students' social background has the highest predictive power on their performance at school. The assignment of teachers to schools is at the intersection of these two problems. First, it determines the ability of teachers to manage their geographic mobility over their career, influencing the attractiveness of the teaching profession. Second, it determines how teachers are distributed across the different schools, influencing in fine the achievement of students.<sup>79</sup> For instance, it is well known ([Chetty, Friedman and Rockoff, 2014](#); [Rockoff, 2004](#)) that teachers in their first year of experience have lower ability to make their students progress compared to more experienced ones. Hence, the distribution of newly tenured teachers and tenured teachers with only one or two years of experience is an important policy question. In France, according to the [Direction de l'Evaluation de la Prospective et de la Performance \(2014\)](#), the two regions of Créteil and Versailles have among the highest shares of students enrolled in deprived schools and the smallest ratio of teachers aged more than 50 over teachers aged less than 30. Here a basic trade-off emerges: one can choose to maximize the number of satisfied teacher

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<sup>78</sup>For precise details, we refer the reader to the Chapter 2 of this thesis.

<sup>79</sup>For examples in other countries than France, the reader can refer to the introduction of the Chapter 2.

reassignment requests. However, in doing so, there is a risk of generating important movements of teachers away from the deprived regions and, in fine, be forced to match less experienced teachers to these regions, negatively impacting the achievement of students that are already disadvantaged across many other dimensions. In France, around 50% of the reassignment requests every year come from teachers assigned to the two regions of Créteil and Versailles, which also have the lowest performing students. This thesis will show that the algorithm assigning teachers to schools can be a powerful tool to address these issues, and that the long tradition of market design analysis described in this introduction can provide insightful solutions to these problems.

## Chapter 1: theoretical foundations and first empirical assessments

In the standard two-sided matching setting, we know that the student-proposing DA algorithm is the unique stable, 2-Pareto-efficient and strategy-proof (for students) mechanism. In this Chapter we show that, in the context of teacher reassignment, the aforementioned  $DA^*$  mechanism has a serious drawback: it is not 2-Pareto efficient. One can reassign teachers to schools such that teachers obtain a school that they prefer and schools obtain teachers that they rank higher. The striking point is that, in doing so, one can improve the assignment in terms of welfare but also in terms of *fairness* since the set of blocking pairs shrinks. It is simple to describe a stylized example: a teacher initially matched to a very unpopular school who has a high priority in all schools can, under  $DA^*$ , prevent other teachers from exchanging their initial positions, even if such exchange would make everyone better off. This has a clear practical implication for the French market: it can explain the recurrent complaints from teachers about the low number of mobility requests satisfied. This low mobility is not only due to a structural congestion arising from correlated preferences concentrated in very few highly demanded regions. As in Roth (1984) on the doctors' complaints, our market design analysis allows us to identify that part of the low movement problem comes in fact from the design of the algorithm itself. The drawback of  $DA^*$  is that it does not correctly manages possible exchanges of positions between tenured teachers. In practice, the situation presented in the stylized example arises frequently: since teachers assigned to disadvantaged schools receive an important priority bonus after 5 years of teaching in such schools, this precisely results in highly ranked teachers located in unpopular schools that was previously described.

Starting from this observation, we restrict ourselves to a simple *one-to-one pure reassignment*

*problem*, where there are only tenured teachers with an initial assignment, and each school has only one position initially occupied by a teacher. We identify a class of mechanisms, the *Block-Exchange* (BE) algorithms that, contrary to  $DA^*$ , are 2-Pareto efficient and maximize welfare and fairness. However, among all these mechanisms, we show that there is a unique one that is strategy-proof for the teachers: we called it the *Teacher Optimal Block Exchange* (TO-BE) algorithm. Similarly to the school choice approach, if one considers that schools are not welfare relevant entities and only cares about blocking pairs and the ranking that teachers obtain, we identify a second class of mechanisms, the *1-Sided Block Exchange* (1S-BE) algorithms, that only improve the teachers' rankings under the constraint of shrinking the set of blocking pairs. However, we show that even if these algorithms tend to favor the teachers, there is no strategy-proof one among them. To better assess the theoretical differences between  $DA^*$  and BE algorithms, we take a large market approach similar to [Che and Tercieux \(2015b\)](#) and we show that in presence of many teachers and schools:<sup>80</sup>

- In all the BE algorithms, the percentage of teachers moving from their initial school is close to 100% while it is not the case under  $DA^*$ .
- There is a BE algorithm that gives all teachers and schools a matching that they rank arbitrarily high.
- The TO-BE algorithm gives all teachers a school that they rank arbitrarily high but the reverse for the schools is not true.<sup>81</sup>

To assess the improvements of the identified algorithms over  $DA^*$ , we use French data from the between-regions assignment of 2013. To stay close to our theoretical framework, we restrict the data to the subsample of the tenured teachers by deleting all newly tenured teachers who do not have any initial assignment and all the available empty positions in the regions. The results are striking. First, over the 10 579 teachers in our sample, only 564 move from their initial assignment under  $DA^*$  and among all the teachers, 75% of them rank only one region in addition to their initial one. If we combine this fact with the high correlation among preferences, showing for instance that the regions in the south and in the west are very demanded; one can expect little scope for

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<sup>80</sup>We refer the reader to the corresponding results in Chapter 1 for the formal statements.

<sup>81</sup>Formally, in the two last claims, the rank obtained is of order  $o(n)$  i.e. the ratio of their rank over the size of the market tends to zero when the size of the market tends to infinity. With a statistical terminology, it means that they obtain a rank of arbitrarily high quantile if the market is sufficiently large.

improvement. However, under TO-BE, 1573 teachers would move from their initial region, which is more than the double.<sup>82</sup> Finally, while these numbers are important to highlight the relevance of our theoretical results, it is important to caveat that they do not provide a counterfactual analysis for the entire French market, since we omitted newly tenured teachers and empty positions. A more careful consideration of these aspects is studied in Chapter 2.

## Chapter 2: practical design and counter factials

In this chapter, our goal is to provide a practical algorithm for the French market, taking into account its specificities and running proper counterfactual scenarios for alternative policies. Similarly to Roth and Peranson (1999) design of the new NMRP procedure, we take a more engineering approach backed by our theoretical findings of Chapter 1. In a first part, we extend the TO-BE algorithm to a setting with empty seats and newly tenured teachers. This extension has a clear parallel with the YRMH-IGYT algorithm proposed by Abdulkadirouglu and Sonmez (1999) for the case of tenants, since they also had initially unassigned agents and empty houses. Our generalized TO-BE can be seen as a generalization of the YRMH-IGYT algorithm, and we show theoretically that it keeps the desirable properties of efficiency and strategy-proofness.

From the complete French dataset of between-regions assignment of teachers in 2013, we see that 3991 tenured teachers move from their initial region under  $DA^*$ , while only around 3880 would move under the generalized TO-BE. To understand the lower performance of TO-BE in the complete market, one has to analyze the market specificities in further detail. As previously mentioned, Créteil, Versailles and to a lesser extent Amiens are three very unattractive regions. The first two concentrate 48.4% of the tenured teachers asking for a reassignment while the last one counts 6.2% of them, so that the three regions account for more than half of the total mobility requests. In addition of not being attractive for teachers, they also tend to concentrate a high proportion of low achieving students. In the absence of empty seats, as in the framework of the Chapter 1, a very high proportion of teachers from these three regions would stay at their initial assignment under both  $DA^*$  and TO-BE, since very few teachers would prefer to exchange their position with them. With the availability of empty positions instead, these teachers have more possibilities to leave from these three regions. However, the TO-BE algorithm, by taking into account the regions' welfare in

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<sup>82</sup>We calculated that the maximal structural possible movement using only the IR constraint is around 2000 teachers. Obviously, in practice, such an algorithm would not be strategy-proof.

addition to the teachers' one, does not allow one to leave without being replaced by a teacher having a higher priority. In particular, one cannot assign newly tenured teachers to replace them under TO-BE, while this is possible under  $DA^*$ . Due to the large size of these three regions, this negative effect of TO-BE compensates for the fact that it allows more exchanges of teachers between more popular regions, leading to a lower global movement of teachers. By retaining more teachers in these disadvantaged regions, it also has the mechanical effect of increasing the experience of the teachers in these regions compared to  $DA^*$ , since the latter replace most teachers by newly tenured ones. This effect is likely to be beneficial for the students of these regions. As mentioned above, there is a trade-off between satisfying the mobility requests of teachers and decreasing the inequality in number of experienced teachers between regions. However, retaining too many teachers might not be a desirable objective for the policy maker. Motivated by this idea, we propose to relax the constraint of the TO-BE algorithm to allow *some* teachers in *some* regions (called *targeted regions*) to be replaced by teachers with higher priority. This allows more flexibility in the design of the algorithm to better match the different objectives of the policy maker. In varying the number of teachers and targeted regions, one can go from the generalized TO-BE algorithm to the YRMH-IGYT algorithm.

Testing different counterfactual scenarios using our data, we show that one can, for instance, increase the mobility of tenured teachers by 44.9% compared to  $DA^*$  while keeping the same turnover in the three deprived regions of Amiens, Créteil and Versailles.<sup>83</sup> The increase of mobility in other regions is 79.2%. It also allows to slightly improve the proportion of experienced teachers in the three unpopular regions, reducing the assignment inequality. We provide a range of results across different scenarios to show the flexibility of the approach, which can lead to higher, same or lower mobility in the three disadvantaged regions.

Following Roth (1984), we further explore the effect of introducing couples of teachers in the reassignment problem. As in the standard two-sided matching setting, a lot of impossibility results arise. If couples are initially assigned to the same region, they want to move together to the same new regions, and there are no single teachers, then one can find a 2-Pareto efficient, individually rational and strategy-proof mechanism that runs in polynomial time. However, as soon as one of the hypotheses is relaxed, a strategy-proof mechanism does not anymore exist. We also show that in a

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<sup>83</sup>Since newly tenured teachers do not have any initial allocation, they can end up being unassigned. To avoid it, the ministry completes the preference lists of these teachers for them to rank (almost) all the regions. One can then count the number of newly tenured teachers assigned to a region that he did not initially rank or also compare their rank distribution under each algorithm. For the results, the reader can refer to the dedicated section in Chapter 2.

context with both couples and singles, the problem of finding an IR matching different from the initial allocation is NP-hard.

Last, we explore an alternative model of teacher reassignment to better control the turnover rate in some regions. Rather than relying on a complete priority ranking over the teachers, each region has a simple objective like, for instance, to decrease the number of “young” teachers. In a first toy model with only “young” and “old” teachers, we provide an algorithm characterizing the matchings that respect the regions’ goals and that cannot be improved in terms of teachers’ welfare by another matching respecting such goals. This alternative approach offers interesting lines for future research.

### **Chapter 3: a new matching setting**

This chapter explores a new matching setting inspired by the interdependence that can exist between the assignment of teachers to schools and the one of students to schools. Both teachers and students are matched to the same set of schools. When teachers evaluate their preferences over schools they consider characteristics such as geographic position, budget, and quality of the infrastructure, but they are also influenced by the type of students that will be assigned to the school. The reverse is true for students : they also care, among other things, about the quality of the teachers that will be assigned to their potential school. We build a matching model abstracting from the teachers and students’ motivations, where two sets of individuals are assigned to a common set of objects. An individual has preferences over the objects and over the other individuals he may be matched to. This model is at the intersection between a standard two-sided matching setting, where two sets of individuals are matched together, and the housing allocation setting, since the pairs have also to be assigned to an object. In this chapter, we use the terminology of men, women and houses to highlight this intersection. Indeed, [Gale and Shapley \(1962\)](#) used the terminology of men and women to introduce their two-sided matching framework. Later, [Shapley and Scarf \(1974\)](#) used the term houses to refer to the objects in their housing market problem. The problem is then to assign to each house a man and a woman. One of the possible applications that we have in mind is to assign managers and workers together to projects.

Following the traditional questions in the matching literature, one may ask whether a generalization of stable matchings always exists in this setting. Stability would require that once couples of men and women are assigned to houses, there is no man (resp. woman) who would prefer to invite

another woman (resp. man) in his (resp. her) assigned house and the two would strictly prefer it to their assigned house-partner pair. Unsurprisingly, we can provide a counter example where no such stable matching exists. However, when thinking about the problem, this stability notion allows any agent to take his assigned house from his partner and ask another one to join. However, in some settings not everyone can do it. In the case of managers, workers and projects, for example, it is not possible for a worker to dismiss his manager, while the opposite can happen. In a sense, there is a specific *ownership over the projects*, which in our abstract case are called houses. We therefore introduce the notion of *ownership structure*, which defines a house owner for each matching of a woman, a man and a house. We then define a notion of *stability with respect to the ownership structure* that requires that only owners can ask another agent to join them in their assigned house. For an arbitrary ownership structure, it can be easily seen that a stable matching can fail to exist. However, we show that they exist for a type of structure that is commonly observed in practice: the one that always assign ownership of houses to the same type of agents, i.e. only women or only men. We call them *one-sided ownerships*. The proof relies on the existence of stable matchings in a standard two-sided market setting as proved by [Gale and Shapley \(1962\)](#). However, we show that the structure of those matchings is different. Indeed, one can also define a core notion with respect to a given ownership: no group of agents can rematch among each others in houses and be strictly better-off but they have to include the owners of the houses they exchange. While in the standard two-sided settings stable matchings are equivalent to core matchings, here we show that with one-sided ownerships, Pareto-efficient and stable matchings can be disjoint, leading to an empty core.

To further explore the intersection with the housing market setting, we define an initial allocation of agents to houses. In this setting, one can define a core similar to the one in [Roth and Postlewaite \(1977\)](#). The main difference is that agents do not block using their newly assigned houses but rather with their initial ones. For an initial allocation of houses to agents of the same type, i.e. only women or only men, we also show that this notion of core can be empty. However, similarly to the result with one-sided ownerships, we show that one can find an allocation that is not blocked by coalitions of size two i.e. formed by one man and one woman. However, contrary to the previous notion, one can find, in this housing market setting, a Pareto-efficient matching not blocked by any coalition of size two.

This analysis highlights that, even with several impossibility results, structures of a specific type (one sided) seem to be a requirement for the existence of a matching that is not blocked by a pair of agents. If pairwise blocking is a minimal requirement for a system to be sustainable, this result can provide an explanation about why we rarely observe ownership structures that are not one sided. Contrary to the standard framework, core notions can be empty. However, in the setting of housing market with an initial allocation, contrary to the setting closer to the two-sided market one, the existence of Pareto-efficient matching not blocked by a pair of agents is guaranteed.





# Chapter 1

## The Design of Teacher Assignment: Theory and Evidence <sup>1</sup>

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## Abstract

The reassignment of teachers to schools is a central issue in education policies. In several countries, this assignment is managed by a central administration that faces a key constraint: ensuring that teachers obtain an assignment that they weakly prefer to their current position. The deferred acceptance mechanism (DA) proposed by [Gale and Shapley \(1962\)](#) fails to satisfy this constraint. As a solution, a variation on this mechanism has been proposed in the literature and used in practice—for example, in the assignment of French teachers to schools. We show that this mechanism fails to be efficient in a strong sense: after such an assignment, it remains possible to reassign teachers in a way that makes both teachers and schools better-off. In addition, this reassignment increases fairness by reducing the set of blocking pairs. To address this weakness, we characterize a class of mechanisms that cannot be improved upon in terms of both efficiency and fairness. Our main theoretical finding shows that this class contains an essentially unique strategy-proof mechanism. We refine these results in two ways. First, we consider a random environment in which preferences and schools' rankings are drawn randomly from a rich class of distributions and show that when the market becomes large, our mechanism performs quantitatively better than the modified DA in terms of utilitarian efficiency and the number of blocking pairs. Second, we use a rich dataset on teachers' applications for transfers in France to empirically assess the extent of potential gains associated with the adoption of our mechanisms. These empirical results confirm both the poor performance of the modified DA mechanism and the significant improvement that our alternative mechanisms deliver in terms of both efficiency and fairness.

JEL Classification Numbers: C70, D47, D61, D63.

Keywords: Two-sided matching markets, Teacher Assignment, Fairness, Efficiency.

## 1.1 Introduction

The reassignment of teachers to schools is a central problem in education policies. The literature shows that teachers are a key determinant of student achievement (Chetty, Friedman and Rockoff, 2014; Rockoff, 2004). The distribution of teachers across schools can therefore have a major impact on achievement gaps between students from different ethnic or social backgrounds. Growing concerns that disadvantaged students may have less access to effective teachers have given rise to policies intended to better distribute effective teachers across schools.<sup>2</sup> However, such policies must be implemented with caution, as they might have unexpected effects on teachers' satisfaction and, ultimately, on the attractiveness of the teaching profession.<sup>3</sup> This leaves us with a central question: how can we design an assignment procedure for teachers that would take into account both sides of the market: teachers' preferences and schools' demands?

In many countries, the labor market for teachers is highly regulated by a central administration, and the assignment of teachers to schools is often managed centrally.<sup>4</sup> This is the case, for example, in France, Italy (Barbieri, Cipollone and Sestito, 2007), Mexico (Pereyra, 2013), Turkey (Dur and Kesten, 2014), Uruguay (Vegas, Urquiola and Cerdàn-Infantes, 2006) and Portugal. In such systems, teachers submit ranked lists of preferences over schools, and each school has a ranking over teachers. The criteria used to rank teachers are diverse, ranging from teachers' performance to standardized tests, teachers' experience and geographical distance from a partner.<sup>5</sup> The teacher assignment problem is similar to the college admission problem, as defined in Gale and Shapley (1962). In the latter problem, students have to be assigned to colleges and a *matching mechanism* is a mapping from both students' preferences and schools' rankings over students<sup>6</sup> into possible assignments of students. Typically, one requires three simple desirable properties of this mechanism. First, the assignment

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<sup>2</sup>Recent initiatives in the U.S. are intended to measure teacher effectiveness and ensure that disadvantaged students have equal access to effective teachers. These policies (for instance, Race to the Top, the Teacher Incentive Fund, and the flexibility policy of the Elementary and Secondary Education Act) allow states to waive a number of provisions in exchange for a commitment to key reform principles. One could also cite "Teach for America," which recruits and trains teachers to teach for at least two years in a low-income community. In the U.K., "Teach First" provides outstanding training for new teachers.

<sup>3</sup>Two of the most important issues facing the teaching profession are the increasing shortage of qualified teachers (Corcoran, Evans and Schwab, 1994) and the difficulty of retaining new teachers in the profession (Boyd et al., 2005).

<sup>4</sup>We take a standard matching without transfers approach. For a discussion on the use of monetary transfers, the reader can refer to the dedicated part at the end of the introduction.

<sup>5</sup>Standardized tests are used, for instance, in Turkey and Mexico, whereas teachers' experience and geographical distance for partners are used, for instance, in France.

<sup>6</sup>Criteria used to rank students can also be quite diverse. Typically, they can depend on student characteristics such as geographic distance from the school, academic performance, and socioeconomic status

produced should be *efficient* in the sense that one cannot find alternative assignments whereby all students and schools would be weakly, and some strictly, better-off.<sup>7</sup> Second, the assignment should be *fair*; i.e., there should be no student who is refused by a school while other students with lower rankings at that school are accepted (otherwise, we use standard terminology and say that this student and school form a *blocking pair*). Finally, one requires *strategy-proofness on the student side*, i.e., that students have incentives to report their preferences truthfully.<sup>8</sup> One of the main results in the literature is that these three properties, taken together, specify a unique solution: the deferred acceptance mechanism in which students are proposing (Gale and Shapley, 1962) – DA, for short. Thus, any other mechanism will violate one of these three properties, and this is certainly an important reason that this mechanism is extensively used to assign students.<sup>9</sup> A first objective of this paper is to demonstrate that this mechanism – or variations of it – will not perform well at assigning teachers to schools. In a second step, we identify a unique (new) mechanism that satisfies a very similar set of desirable properties.

To understand why one cannot simply use the DA mechanism, we note that there is an important difference between the teacher assignment and college admission problems: when one wants to assign teachers to schools, many teachers already have a position and are willing to be reassigned. In practice, tenured teachers have the right to keep their initial position if they wish, and thus, the administration has to offer them a position that they weakly prefer to the school to which they are currently assigned. In other words, the assignment of teachers must be *individually rational*. While fairness has emerged as an important normative criterion in the matching literature, in (the realistic) context in which there are tenured teachers willing to be reassigned, all fair matchings may violate the individual rationality constraint.<sup>10</sup> Thus, the prominent mechanism in the college admission problem (DA) fails to be individually rational. While no mechanism satisfies the three aforementioned properties together with individual rationality, one can hope to construct mechanisms that accommodate these properties in the best possible way. To achieve this goal, one approach

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<sup>7</sup>Note that, here, one has to adopt an “as if” approach and assume that schools’ ranking over teachers can be interpreted as schools’ preferences. The basic idea is that these rankings reflect normative criteria that this “as if” approach can take into account. A precise discussion of this approach is deferred to the end of the introduction (see “Two-sided efficiency with priorities”).

<sup>8</sup>Indeed, generally, students are the only strategic entities, and schools’ orderings are given by law.

<sup>9</sup>It is used in many real assignment problems, for instance, for the assignment of students to high schools in Boston, Hong Kong, New Orleans, and New York City.

<sup>10</sup>For cases with only first-year teachers (without an initially assigned position), the problem is formally equivalent to a two-sided matching problem as studied extensively since the seminal contribution by Gale and Shapley (1962).

is to use a variation on the DA mechanism to make it individually rational. Such mechanism has been identified in the literature and used in practice, for example, in the assignment of teachers to schools in France or the assignment of on-campus housing at MIT (Guillen and Kesten, 2012). This mechanism consists of, first, artificially modifying the school’s ordering over teachers, such that all teachers initially assigned to a school are moved to the top of that school’s ranking. In a second step, we run the DA mechanism using the modified priorities. While, by construction, this modified version of DA is individually rational, we show that one loses an important property of DA: it fails to be efficient, i.e., one can reassign teachers in such a way that *both* teachers and schools are better-off (according to the school’s true ranking over teachers).<sup>11</sup> In addition, this Pareto improvement can be achieved while simultaneously improving fairness (i.e., ensuring that the set of blocking pairs shrinks). Our first goal is therefore to design mechanisms that do not suffer from the same limitations as this modified version of DA: we say that a matching is *two-sided maximal* if (1) it Pareto dominates the initial assignment and (2) cannot be improved upon in terms of (2i) efficiency and (2ii) fairness. This requirement is actually quite weak, and it is easily shown that two-sided maximal matchings correspond to assignments that are both Pareto efficient and individually rational on both sides of the market.<sup>12</sup>

To characterize two-sided maximal matchings, we provide an algorithm called the block exchange (BE) algorithm. The idea is simple: if two teachers block one another’s schools, we allow these teachers to exchange their schools. Obviously, larger exchanges involving many teachers are possible. We can identify these exchanges by finding cycles in an appropriately defined directed graph. The outcome of the BE algorithm depends on the order in which we select the cycles. However, irrespective of how one selects cycles, one obtains a two-sided maximal matching, and conversely, starting from a two-sided maximal matching, one can find a way to select cycles in the BE algorithm that will eventually yield this matching. While this result is a useful first step, we obtain a plethora of different possible mechanisms depending on how we select cycles. However, we show that imposing incentive properties dramatically shrinks the set of possible mechanisms. Our main theoretical result states that there is a unique way to select cycles that makes this algorithm strategy-proof. We provide a

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<sup>11</sup>Under (standard) DA, it is well known that one can reassign teachers and make all of them better-off, some strictly. However, this will be done at the expense of schools given that (standard) DA is in the Core and, hence, efficient. Here, in stark contrast to standard DA, we show that, under the modified DA, both teachers and schools can be made better-off.

<sup>12</sup>In particular, we can dispense with (2ii) in the definition of two-sided maximality.

mechanism, called the teacher-optimal BE algorithm, which produces two-sided maximal matchings and is the unique strategy-proof mechanism with this property. Interestingly, this mechanism can be characterized using a simple modification of the standard top trading cycle (Shapley and Scarf, 1974) – TTC, for short: in a first step, one modifies teachers’ preferences such that no teacher ranks as acceptable a school that considers him unacceptable. Once teachers’ preferences have been modified, we run TTC.

Our characterization of the teacher assignment problem is related to the college admission problem in two ways. First, we see the above result as the counterpart of the characterization result of the college admission problem. While DA is the unique mechanism that is efficient, fair and strategy-proof in the college admission problem, the teacher-optimal BE algorithm is the unique strategy-proof mechanism that is two-sided maximal (and thus cannot be improved upon in terms of efficiency or fairness). Second, while, a priori, we did not wish to favor one side of the market – the BE algorithm treats schools and teachers symmetrically – once we impose incentive constraints, we ultimately favor the teacher side.<sup>13</sup> Thus, imposing strategy-proofness for teachers has an important cost on the school side. This also shows that the teacher assignment problem has a similar “structure” to the college admission problem, in which, among the set of all fair mechanisms, only the student-optimal mechanism is strategy-proof. We show that, among the set of two-sided maximal mechanisms, only the teacher-optimal mechanism is strategy-proof.

We also provide additional theoretical results in two respects. First, we consider a case in which only teachers are welfare-relevant entities. In this context, we provide a similar characterization to that obtained with the BE algorithm. Here again, we identify a large class of mechanisms. However, while this approach obviously favors teachers, we show – contrary to our main theoretical result – that no mechanism in this class is strategy-proof.<sup>14</sup> Second, we simulate matchings in a large market approach in which preferences and schools’ rankings are drawn randomly from a rich class of distributions.<sup>15</sup> In this context, we show that when the market becomes large, our mechanisms

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<sup>13</sup>Schools’ preferences are only taken into account by ensuring that schools are not assigned an unacceptable teacher.

<sup>14</sup>Dur, Gitmez and Yilmaz (2015) independently characterize the same class of mechanisms as we obtain, in which only teachers are welfare-relevant entities. They consider the allocation given by DA in a school choice environment. For each school, the authors define a set of teachers who are allowed to form a blocking pair with that school. They characterize the allocations that are (one-sided) efficient under this constraint and Pareto dominate the DA assignment. Our class of mechanisms starts from an arbitrary initial assignment, while given their motivation, they are interested only in the class improving on DA, and thus, they begin from the DA allocation. Our main message concerns the non-existence of a strategy-proof selection in our class of mechanisms. When starting from an arbitrary exogenous initial assignment (which excludes starting from the DA allocation), this result is non-trivial.

<sup>15</sup>These markets can involve a large number of agents. For instance, in France, each year, approximately 65,000

“perform quantitatively better” than the modified DA in terms of utilitarian efficiency and the number of blocking pairs. We also identify the cost that the adoption of the unique strategy-proof mechanism could have in terms of utilitarian outcomes and number of blocking pairs compared to a first-best approach whereby one could select any two-sided maximal mechanism. Our arguments build on techniques from random graph theory as in [Lee \(2014\)](#), [Che and Tercieux \(2015a\)](#) and [Che and Tercieux \(2015b\)](#).

Finally, we use a nationwide labor market to empirically estimate the magnitude of gains and trade-offs in a real teacher assignment problem. France, like several other countries, has a highly centralized labor market; the 400,000 teachers in the public sector are civil servants, and hence, their recruitment, assignment to schools, and salary scale are entirely managed by the central administration. We use data on the assignment of 10,500 teachers to regions in 2013. Exploiting the straightforward incentives of the modified version of DA currently employed, we use preferences reported by teachers to run counterfactuals and quantify the performance of our mechanisms. We confirm that the modified version of DA (DA\* for short) can be improved upon in terms of both welfare (on both sides of the market) and blocking pairs. Of the 49 markets (i.e., disciplines) for which we ran DA\*, 30 could be simultaneously improved in these two dimensions. Importantly, these 30 fields represent nearly the entire market we are analyzing, as they contain 94.2% of the teachers. In addition, the gains obtained by our alternative mechanisms are significant in the two dimensions. Compared to DA\*, the number of teachers moving from their initial assignment is more than doubled under our mechanisms. Additionally, under our mechanisms, the distribution of ranks of teachers (over schools they obtain) stochastically dominates that of DA\*. Regarding fairness, the number of teachers who are not blocking with any school increases by 36%. Finally, the percentage of schools having all of their positions improved in terms of the priority of the assigned teachers (compared to the initial assignment) doubles, from 12.7% under DA\* to 25.6% under our proposed mechanisms. These figures are essentially the same for the unique strategy-proof mechanism mentioned above, which makes it particularly appealing and a natural candidate for practical implementation.

**Two-sided efficiency.** Our notion of efficiency treats both teachers and schools as welfare-relevant entities (two-sided efficiency). At first glance, this might be surprising, as schools’ rankings over teachers (hereafter, priorities) are given by law. However, there are clear social objectives

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tenured teachers ask for an assignment. In Turkey, 8,850 positions were filled by new teachers in 2009 ([Dur and Kesten, 2014](#)).



motivating the criteria used to define teacher priorities. This makes it relevant to adopt an approach “as if” schools’ priority rankings were schools’ preferences. For instance, teacher priorities at schools are primarily determined by their experience, notably in disadvantaged schools, which reflects the administration’s efforts to assign more-experienced teachers to disadvantaged students.<sup>16</sup> If we were to consider only teachers as welfare-relevant entities, teachers could exchange their positions while decreasing the number of experienced teachers in disadvantaged schools. From an administrative perspective, it would be difficult to consider this a Pareto improvement, as disadvantaged students would be harmed by such a reassignment. Hence, a meaningful requirement is to allow for exchanges of positions across teachers only when they are not done at the expense of the experience of teachers in disadvantaged schools. This is exactly what the “as if” approach gives us when we require a Pareto improvement over the initial assignment. In practice, several other criteria are used to determine teacher priorities. Again, these may reflect broader social objectives. For instance, “spousal reunification ” and “children reunification” give a priority bonus to teachers at schools close to where their partner or children live. Again, one can easily see the social objective motivating these priorities. In this context, a meaningful requirement is to allow for exchanges of positions across teachers that are not done at the expense of the experience of teachers in (possibly disadvantaged) schools except potentially when it can allow a teacher to join his/her family. Here again, this is what the “as if” approach provides.

**Monetary transfers.** Our analysis focuses on a centralized reassignment procedure that does not use any monetary transfer. A first reason is practical: as mentioned, many countries use such system. In many of them, including France, the salary scale is publicly fixed and its modification is limited.<sup>17</sup> Second, there is an increasing debate in the literature on the comparison between centralized systems with no monetary transfers and decentralized ones. Both seem to have pros and cons and there is still no definitive answer about which one would be more appropriate in this context. For instance, [Agarwal and Somaini \(2014\)](#) reported that switching to a decentralized,

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<sup>16</sup>It is well known that early career teachers benefit from additional years of experience ([Rockoff, 2004](#)). However, it is important to emphasize that it is not the only factor of students’ achievements and that there is an important debate on the literature about the value-added of teachers and which characteristics are important for students’ success. We refer the reader to the survey of [Jackson, Rockoff and Staiger \(2014\)](#) for a discussion about these issues. Here, we chose a more pragmatic approach in taken as given the priority score system that has been chosen by the ministry. Any additional characteristic that may be of interests can be accommodated in the design of such scores. The main message of our paper would remain the same.

<sup>17</sup>In France, there is also a strong reluctance from teachers’ unions to differentiate salaries based on other criteria than experience. One can see this constraint as a *repugnance constraint* on the use of monetary transfers as defined by [Roth \(2007\)](#).

market based, system of allocation of interns to hospitals in the U.S would result in a decrease of the average wage. Concerning the attrition of teachers, [Stinebrickner \(1998\)](#) showed, using U.S longitudinal surveys, that the first spell in teaching is more responsive to wages than working conditions. However, [Imazeki \(2005\)](#) and [Ondrich, Pas and Yinger \(2008\)](#) found that large salary increases are needed to reduce the attrition of teachers. Salaries of public teachers being already an important part of public expenditures, budget balance is an important constraint so that the use of salary changes might not be feasible. Conversely, [Hanushek, Kain and Rivkin \(1999\)](#) found, using panel data in Texas, that teachers' mobility is more affected by characteristics of the students than by salaries. In the 2007 report on the Follow-Up Survey in the U.S ([Marvel et al., 2007](#)), 38% of public school teachers who moved from their position rated the opportunity for a better teaching assignment as very important or extremely important as opposed to only 16.5% rating better salary or benefits. In France, [Prost \(2013\)](#) found that teachers tend to move from a school with a higher proportion of less able students, students from minority groups and/or students from economically disadvantaged backgrounds. Those not working in the region where they were born are also more likely to move. She found that the salary bonus policy implemented in France to attract teachers in disadvantaged schools was not enough to retain teachers in these schools. Overall, there is still an important need for further evidences for both decentralized and centralized systems. Our work contributes to this debate in showing that the mobility in centralized systems of assignment was underestimated. Indeed, the main mechanism that has been identified in the literature and used in practice can be greatly improved.

**Related literature.** Our theoretical setup in this paper covers two standard models in matching theory. The first is the *college admission problem* as defined by [Gale and Shapley \(1962\)](#). In this context, schools have preferences that are taken into account regarding both efficiency considerations and fairness issues. Second, our model also embeds the *house allocation problem* as developed by [Shapley and Scarf \(1974\)](#). In this framework, individuals own a house and are willing to exchange their initial assignments. Hence, in this problem, only one side of the market has preferences. Among the goals in the case is to ensure that all individuals eventually obtain an assignment that they weakly prefer to their initial assignment. This problem is very similar to ours, but in our context, we wish to take into account the school side in a way that is similar to the approach in the college admission problem. While covering important applications, this “mixed ”model has only been

studied by a small number of authors. [Guillen and Kesten \(2012\)](#) is one of the exceptions and notes that the modified version of the DA mechanism is used for the allocation of on-campus housing at MIT. [Compte and Jehiel \(2008\)](#) and [Pereyra \(2013\)](#) provide results on the properties of this mechanism. They note that fairness and individual rationality are not compatible. They propose a weakening of the notion of blocking pairs and show that the modified version of DA “maximizes fairness” under their weakening. By contrast, our work retains the standard definition of blocking pairs and addresses notions of maximal fairness using the usual definition. More important, our theoretical and empirical results highlight the high cost that maximizing their notion of fairness can have in terms of efficiency and the traditional notion of fairness.

Our work is also related to [Dur and Unver \(2015\)](#). They introduce a matching model to study the “tuition-exchange market” (which includes applications such as tuition exchanges or temporary worker exchanges). In their environment, agents initially attached to an institution are allowed to spend some time at another institution. They require that the flow of exports from an institution be equal to the flow of imports to it (a mechanism satisfying this assumption is said to be balanced). They show that, in this environment, a unique mechanism is balanced efficient (i.e., Pareto efficient within mechanisms that are balanced), student group–strategy-proof, acceptable, individually rational, and respects internal priorities. Their notion of efficiency is, like ours, two-sided, and the mechanism they characterize is a version of TTC, similar to our teacher-optimal BE. However, our applications and theirs are quite different. In general, we do not wish to impose any balancedness condition. For instance, returning to one of our motivations, if a position becomes available in a disadvantaged school, the designer is likely to be willing to allow an effective teacher to go there, despite that this may create an imbalance for both the disadvantaged school and the school to which he is initially matched. More generally, many mechanisms used in practice (including that used in France) allow teachers to join a school when a position becomes available which suggests that the balancedness condition is not well suited to our environment.<sup>18</sup> More important, the main focus of our paper is on the conflict between efficiency and fairness. We show that DA\* can be improved upon in these two dimensions while our alternative mechanism cannot. This is orthogonal to their main points.

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<sup>18</sup>[Dur and Unver \(2015\)](#) report that some tuition-exchange programs care about balancedness only over a moving time window (they extend some of their results to this richer environment). So under tuition-exchange programs, some imbalancedness can be allowed for a short period of time provided that balancedness is (approximately) satisfied over a longer time span.

## 1.2 Teacher Assignment to Schools in France

France, like several other countries, has a highly centralized labor market for teachers.<sup>19</sup> Since the 400,000 public school teachers are civil servants, the French Ministry of Education is responsible for their recruitment, assignment to schools, and salary scale and collects extensive data on these activities.<sup>20</sup> This gives us the opportunity to use a nationwide labor market to compare the performance of different assignment algorithms.

Prior to assignment, the central administration defines priorities over teachers using a point system, which takes into account three legal priorities: spousal reunification, disability, and having a position in a disadvantaged or violent school. Additional teacher characteristics are also accounted for to compute the score including total seniority in teaching, seniority in the current school, time away from the spouse and/or children. This score determines schools' rankings or preferences. In this section, we will use the terms "priorities" and "preferences" interchangeably (for the motivation on this terminology, see the end of the introduction). The point system is defined by the central administration and well known by all teachers wishing to change schools.<sup>21</sup>

The French Ministry of Education divides French territory into 31 administrative regions, which are called *académies* (see the map in Appendix 1.8.1). We will refer to them as *regions* hereafter. Since 1999, the matching process has taken place in two successive phases. First, during a *region assignment* phase, newly tenured teachers and teachers who wish to move to another region submit an ordered list of them. A matching mechanism (described in the next section) is used to match teachers to regions, using priorities defined by the point system. This phase is managed by the central administration. Then, during the second phase, each region proceeds to the *school assignment*. In each region, teachers matched to it after the first phase and teachers who already have a position in it but wish to change schools within this region report their preferences over the schools in the region. The same mechanism as in Phase 1 is used to complete the matching using priorities defined

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<sup>19</sup>Italy (Barbieri, Cipollone and Sestito, 2007), Portugal, Mexico (Pereyra, 2013), Turkey (Dur and Kesten, 2014), and Uruguay (Vegas, Urquiola and Cerdàn-Infantes, 2006)

<sup>20</sup>Anyone who wishes to become a teacher has to pass a competitive examination, organized once per year by the Ministry of Education. Those who pass the public-sector exam will become civil servants, and thus, their salary is completely regulated by a detailed pay scale. Neither schools nor teachers can influence salary or promotions. All teachers having the same number of years of experience and having passed the same exam earn the same salary. Further details on the recruitment and assignment process are available on the Matching in Practice website: <http://www.matching-in-practice.eu/matching-practices-of-teachers-to-schools-france/>

<sup>21</sup>An official list of criteria used to compute the point system is available on the government website [http://cache.media.education.gouv.fr/file/42/84/6/annexeI-493\\_365846.pdf](http://cache.media.education.gouv.fr/file/42/84/6/annexeI-493_365846.pdf)

by a similar point system as in Phase 1. The only difference is that teachers are limited in the number of schools that they can rank during this phase, as discussed in the empirical section (Section 1.5). If a teacher is assigned to new region during Phase 1, then he is considered as not having any initial position during the Phase 2. So it is possible that he ends up being unassigned at the end of this phase. In that case, he becomes a substitute teacher affected to a geographical area inside the region that tries, whenever possible, to cover the area of his first ranked school. During the year, the teacher will replace some absent colleagues in the area whenever it is necessary. After this year, he is forced to participate again at least to the school assignment phase inside this region.

In 2013, just over 25,000 teachers applied in Phase 1, and approximately 65,000 submitted a list to be assigned a new school within a region (i.e., in Phase 2). These figures include all newly tenured teachers, who have never been assigned a position, and tenured teachers who request a transfer. In practice, the assignment process is decomposed into as many markets as subjects taught. There are 107 different subjects taught, which have different numbers of teachers. Some are large such as Sport (approximately 2,500 teachers), Contemporary Literature (approximately 2,000 teachers), Mathematics (approximately 2,000 teachers), while others are smaller such as Thermal Engineering (approximately 60 teachers) or Beauticians (approximately 15 teachers), with a wide range of subjects in between. As a teacher teaches only one subject, and positions are specific to a subject, the markets can be considered independent from one subject to another.<sup>22</sup>

A lack of mobility has emerged as a concern for the Ministry. In 2013, of the 17,000 tenured teachers requesting a new assignment, only 40.9% were satisfied. Similarly, 29% of the teachers asking to move closer to their family did not obtain a new assignment, many of them for several consecutive years. Due to the important lack of mobility, the [Mediator of the French Ministry of Education \(2015\)](#), responsible for resolving conflicts between the Ministry and teachers, receives approximately 700 complaints from primary and secondary school teachers every year related to assignment issues. He states that “the assignment algorithm opens doors to difficult personal situations that can, eventually, tarnish the quality and the investment of human resources”. An additional concern has been raised by the Mediator: every year, many teachers who do not obtain their desired region decide to resign or request a year off. This leaves some students without teachers and regularly requires regions to hire last-minute replacement staff who are not trained to teach. In

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<sup>22</sup>In practice, couples from different fields can submit joint applications, which connects the fields. However, we eliminated all couples from our sample. Details are provided in Appendix 1.8.8.

the least attractive schools, labeled “priority education”, 30% of the teachers do not have a teaching certification, versus 7.6% in other schools. From that perspective, one of the objectives of this paper is to show that using an alternative mechanism can significantly reduce the current lack of mobility.

### 1.3 Basic Definitions and Motivation

Consider a problem in which a finite set of teachers  $T$  has to be assigned to a finite set  $S$  of schools. For now, we restrict our attention to a one-to-one setting, i.e., an environment in which each school has a single seat (see Appendix 1.8.7 for the treatment of the many-to-one case). Each teacher  $t$  has a strict preference relation  $\succ_t$  over the set of schools and being unmatched (being unmatched is denoted by  $\emptyset$ ). Similarly, each school  $s$  has a strict preference relation  $\succ_s$  over teachers and being unmatched.<sup>23</sup> For any teacher  $t$ , we write  $s \succeq_t s'$  if and only if  $s \succ_t s'$  or  $s = s'$ . For any school  $s$ , we define  $\succeq_s$  in a similar way. For simplicity, we assume that all teachers and schools prefer to be matched rather than being unmatched. A **matching**  $\mu$  is a mapping from  $T \cup S$  into  $T \cup S \cup \{\emptyset\}$  such that (i) for each  $t \in T$ ,  $\mu(t) \in S \cup \{\emptyset\}$  and for each  $s \in S$ ,  $\mu(s) \in T \cup \{\emptyset\}$  and (ii)  $\mu(t) = s$  iff  $\mu(s) = t$ . That is, a matching simply specifies the school to which each teacher is assigned or that a teacher is unmatched. It also specifies the teachers assigned to each school, if any. We will also sometimes use the term assignment instead of matching. Thus far, our environment is not different from the college admission problem (Gale and Shapley, 1962). However, in a teacher assignment problem, there is an additional component: teachers have an initial assignment. Let us denote the corresponding matching by  $\mu_0$ . For now, we assume that  $\mu_0(t) \neq \emptyset$  for each teacher  $t$  and  $\mu_0(s) \neq \emptyset$  for each school  $s$ .<sup>24</sup> Hence, initially, all teachers are assigned a school (there is no incoming flow of teachers) and there is no available seat at schools (there is no outgoing flow of teachers). We define a **teacher allocation problem** as a triplet  $[T, S, \succ]$  where  $\succ := (\succ_a)_{a \in S \cup T}$ .

We will be interested in different efficiency and fairness criteria, depending on whether we regard both teachers and schools or only teachers as welfare-relevant entities. First, we say that a matching  $\mu$  is **two-sided individually rational** (2-IR) if for each teacher  $t$ ,  $\mu(t)$  is acceptable to  $t$ , i.e.,  $\mu(t) \succeq_t \mu_0(t)$  and, in addition, for each school  $s$ ,  $\mu(s)$  is acceptable to  $s$ , i.e.,  $\mu(s) \succeq_s \mu_0(s)$ . Similarly, a matching is **one-sided individually rational** (1-IR) if each teacher finds his assignment

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<sup>23</sup>Our results easily extend to the case of weak preferences for schools.

<sup>24</sup>This implies that  $\mu_0$  defines a bijection from  $T$  to  $S$  and thus  $|T| = |S|$ .

acceptable. We say that a matching  $\mu$  2-Pareto dominates (resp. 1-Pareto dominates) another matching  $\mu'$  if all teachers and schools (resp. teachers) are weakly better-off – and some strictly better off – under  $\mu$  than under  $\mu'$ . A matching is **two-sided Pareto-efficient** (2-PE) if there is no other matching that 2-Pareto dominates it. Similarly, we define **one-sided Pareto-efficient** (1-PE) matchings as assignments for which no alternative matching exists that 1-Pareto dominates it. We say that under matching  $\mu$ , a teacher  $t$  has justified envy for teacher  $t'$  if  $t$  prefers the assignment of  $t'$ , i.e.,  $\mu(t') =: s$ , to his own assignment  $\mu(t)$  and  $s$  prefers  $t$  to its assignment. Using the standard terminology from the literature, we say that  $(t, s)$  **blocks** matching  $\mu$ . A matching  $\mu$  is **stable** if there is no pair  $(t, s)$  blocking  $\mu$ . We will sometimes say that a matching  $\mu$  dominates another matching  $\mu'$  in terms of stability if the set of blocking pairs of  $\mu$  is included in that of  $\mu'$ .

Finally, a **matching mechanism** is a function  $\varphi$  that maps problems into matchings. We abuse notation and write  $\varphi(\succ)$  for the matching obtained in problem  $[T, S, \succ]$ . We also write  $\varphi_t(\succ)$  for the school that teacher  $t$  obtains under matching  $\varphi(\succ)$ . It is 2-IR/1-IR/1-PE/2-PE/stable if for each problem, it systematically selects a matching that is 2-IR/1-IR/1-PE/2-PE/stable.<sup>25</sup>

One of the standard matching mechanisms is deferred acceptance (DA for short), proposed by [Gale and Shapley \(1962\)](#). Because we discuss a closely related mechanism, we first recall the definition of DA.

- **Step 1.** Each teacher  $t$  applies to his most preferred school. Each school tentatively accepts its most preferred teacher among the offers it receives and rejects all other offers.

In general,

- **Step  $k \geq 1$ .** Each teacher  $t$  who was rejected at step  $k - 1$  applies to his most preferred school among those to which he has not yet applied. Each school tentatively accepts its most favorite teacher among the new offers in the current step and the applicant tentatively selected from the previous step (if any), and it rejects all other offers.

The following proposition is well known.

**Proposition 1.3.1** ([Gale and Shapley, 1962](#)). *DA is a stable and 2-PE mechanism.*

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<sup>25</sup>We focus on the 2-IR condition since it simply requires that a 2-IR matching must 2-Pareto dominates the initial allocation. Moreover, we will see in the empirical analysis that such notion better fulfills the policy maker objectives.

While DA is stable and 2-PE, it fails to be 1-IR (and thus 2-IR). This is unavoidable: in general, there is a conflict between individual rationality and stability. The basic intuition is that imposing 1-IR on a mechanism yields situations in which some teacher  $t$  may be able to keep his initial assignment  $\mu_0(t) =: s$  while school  $s$  may perfectly prefer other teachers to  $t$ . These other teachers may rank  $s$  at the top of their preference relation and hence block with school  $s$ . We summarize this discussion in the following observation.<sup>26</sup>

**Proposition 1.3.2.** *There is no mechanism that is both 1-IR and stable. Hence, DA is not 1-IR.*

Thus, there is a fundamental trade-off between 1-IR and stability, and one may wish to find a mechanism that restores individual rationality while retaining “to the greatest extent possible” the other desirable properties of DA, such as its stability and its 2-Pareto efficiency. To do so, one approach – followed by the literature (see, for instance, [Pereyra, 2013](#) or [Compte and Jehiel, 2008](#)) and used in practice – consists in modifying artificially the schools’ preferences such that each teacher  $t$  is ranked at the top of the (modified) ranking of the school he is initially assigned to, namely,  $\mu_0(t)$ . Other than this modification, the schools’ preference relations remain unchanged.<sup>27</sup> Once this is done, one runs DA as defined above using schools’ modified preferences. We denote the corresponding mechanism  $DA^*$ . By construction, this is a 1-IR mechanism. This mechanism is used in practice in several situations. For instance, it is used for the assignment of on-campus housing at MIT ([Guillen and Kesten, 2012](#)). It is also used in France for the assignment of teachers to schools. Specifically, a *school-proposing deferred acceptance* mechanism is run using the modified priorities and the reported preferences. Then, *Stable Improvement Cycles* are executed as defined in [Erdil and Ergin \(2008\)](#). Using Theorem 1 in [Erdil and Ergin \(2008\)](#), we know that this process yields to the outcome of the *teacher-proposing deferred acceptance* mechanism according to the modified priorities.<sup>28</sup> Hence, the mechanism used to assign French teachers to public schools is equivalent to  $DA^*$ .

By construction, this mechanism is 1-IR, hence, by Proposition 1.3.2, we know that it is not

<sup>26</sup>This is highlighted in [Compte and Jehiel \(2008\)](#) and [Pereyra \(2013\)](#).

<sup>27</sup>Formally, for each school  $s$ , a new preference relation  $\succ'_s$  is defined such that  $\mu_0(s) \succ'_s t'$  for each  $t' \neq \mu_0(s)$ , and for each  $t, t'$  distinct from the school’s initial assignment  $\mu_0(s)$ , we have  $t \succ'_s t'$  if and only if  $t \succ_s t'$ .

<sup>28</sup>In the French system, teachers’ priorities at schools can be coarse. Hence, in practice, the algorithm starts by breaking ties (using teachers’ birth dates). Once ties are broken, school-proposing deferred acceptance is run using the modified priorities *with no ties* and the reported preferences. From this outcome, stable improvement cycles are run, again using the modified (strict) priorities. Thus, the outcome is equivalent to the teacher-proposing deferred acceptance with the same tie-breaking rule, which in turn may be Pareto dominated by a teacher-optimal stable mechanism. Our mechanisms and results can be easily extended to an environment with coarse priorities.



stable. However, is there a sense in which the violation of stability is minimal? What about efficiency: Is  $DA^*$  2-PE? Furthermore, if the answers to those questions are negative, can we find ways to improve upon this? The following example will illustrate one important disadvantage of  $DA^*$  to which we will return both in our theoretical analysis and in our empirical assessment.

**Example 1.** *We consider a simple environment with  $n$  teachers and  $n$  schools with a 1-IR initial assignment  $\mu_0$ . Let us assume that some teacher  $t^*$  is initially assigned to school  $s^*$  (i.e.,  $\mu_0(t^*) = s^*$ ) and is ranked first by all schools. In addition, school  $s^*$  is ranked at the bottom of each teacher's preference relation – including  $t^*$ ; hence  $t^*$  is willing to move. We claim that under these assumptions, no teacher will move from his initial assignment if we use  $DA^*$  to assign teachers. To see this, note first that  $t^*$  does not move from his initial assignment. Indeed, because  $DA^*$  is 1-IR, if  $t^*$  were to move, then some teacher  $t$  would have to take the seat at school  $s^*$  (or be unmatched), but since  $s^*$  is the worst school for every teacher (and  $\mu_0$  is 1-IR), this assignment would violate the individual rationality condition for teacher  $t$ , a contradiction. Note that this implies that under the  $DA^*$  algorithm,  $t^*$  applies to every school  $s$  (but is eventually rejected). Now, to see that no teacher other than  $t^*$  moves, assume on the contrary that  $t \neq t^*$  is assigned a school  $s \neq \mu_0(t)$ . As mentioned above, at some step of the  $DA^*$  algorithm,  $t^*$  applies to  $s$ . Since  $t^*$  is ranked above  $t$  in the preference relation of school  $s$  (recall that  $s \neq \mu_0(t)$ ),  $t$  cannot eventually be matched to school  $s$ , a contradiction.*

*To recap, under our assumptions, no teacher moves from his initial assignment. Since the initial assignment can perform very poorly in terms of basic criteria such as stability or 2-Pareto efficiency, we can easily imagine the existence of alternative matchings that would make both teachers and schools better-off and hence also shrink the set of blocking pairs.*

The driving force in this example is the existence of a teacher ranked at the top of each school's ranking and who is initially assigned the worst school. This is, of course, a stylized example, and one can easily imagine less extreme examples in which a similar phenomenon would occur. The basic idea is that for  $DA^*$  to perform poorly, it is enough to have one teacher (a single one is enough) being assigned an unpopular school and who himself has a fairly high ranking for a relatively large fraction of the schools. Our theoretical analysis and our empirical assessment will give a sense in which the described phenomenon is far from being a peculiarity.

**Remark 1.** *Contrary to what we have in the example, in practice, there are open seats at schools.*

One may argue that high-priority teachers like  $t^*$  will succeed in obtaining available seats at schools they desire and, hence, that the above phenomenon would be considerably weakened. However, in an environment in which teachers' preferences tend to be similar (i.e., are positively correlated), there will be competition to access good schools. These good schools have a limited number of seats available, and one may easily imagine that once these open seats are filled by some of the high-priority teachers, a similar phenomenon as in the example could occur among the remaining teachers. We ran simulations in a rich environment (allowing for correlation in teachers' and schools' preferences, available seats, new comers and positive assortment in the initial assignment) that confirm this intuition. The results are reported in Appendix 1.8.2.

The above example identifies a weakness of DA\*: it can be improved upon in terms of both efficiency (on both sides) as well as in terms of the set of blocking pairs (i.e., we can shrink its set of blocking pairs). Thus, we are interested in mechanisms/matchings that do not have this type of disadvantage. We also wish to retain the elementary property that our mechanism/matching improves on the initial assignment. This suggests the following definitions.

**Definition 1.** A matching  $\mu$  is **two-sided maximal** if  $\mu$  is 2-IR<sup>29</sup> and there is no other matching  $\mu'$  such that (1) all teachers and schools are weakly better-off and some strictly better-off and (2) the set of blocking pairs under  $\mu'$  is a subset of that under  $\mu$ .

This notion treats both schools and teachers as welfare-relevant entities. As we argued above, one may also ignore the school side. In such a case, we obtain the following natural counterpart.

**Definition 2.** A matching is **one-sided maximal** if  $\mu$  is 1-IR and there is no other matching  $\mu'$  such that (1) all teachers are weakly better-off and some strictly better-off and (2) the set of blocking pairs under  $\mu'$  is a subset of that under  $\mu$ .

Consistent with our previous notions, we say that a mechanism is two-sided (resp. one-sided) maximal if it systematically selects a two-sided (resp. one-sided) maximal matching.

Let us note that if there is a matching  $\mu'$  under which all teachers and schools are weakly better-off and some strictly better-off than under a matching  $\mu$ , then the set of blocking pairs under  $\mu'$  is a subset of that under  $\mu$ . Thus, in the definition of two-sided maximality, requirement (2) can be dropped.<sup>30</sup> This yields the following straightforward equivalent definition.

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<sup>29</sup>Recall that the motivation for imposing 2-IR is to ensure that our assignments Pareto-dominate the initial assignment.

<sup>30</sup>However, one can easily check that (2) in the definition of one-sided maximality cannot be dropped.

**Proposition 1.3.3.** *A matching  $\mu$  is **two-sided maximal** if and only if  $\mu$  is 2-IR and 2-PE.*

Given Example 1 above, we have the following straightforward proposition.

**Proposition 1.3.4.** *DA\* is not two-sided maximal and hence not one-sided maximal. (Thus, DA\* is not 2-PE.)*

Given the above weaknesses of DA\*, the obvious goal henceforth is to identify the class of mechanisms characterizing two-sided and one-sided maximality and to study the properties of those mechanisms. The aim of the next section is to do so.

## 1.4 Theoretical Analysis

For each notion of maximality defined above (Definitions 1 and 2), the following two sections identify a class of mechanisms that characterizes it. Once the characterization results are proved, we analyze the properties of the mechanisms in that class. While the class of mechanisms can be very large (as illustrated by Proposition 1.3.3), imposing standard additional conditions reduces drastically the set of candidate mechanisms. In particular, a striking outcome of this analysis is that, once the standard strategy-proofness notion is imposed, a unique two-sided maximal mechanism is shown to survive. In addition, while one may expect that giving more weight to teachers (as opposed to schools) as in one-sided maximal mechanisms may help in terms of incentive properties, another conceptually interesting outcome of our analysis is that no one-sided maximal mechanism is strategy-proof.

### 1.4.1 Two-sided maximality

In the next section, we define a class of mechanisms that characterizes the set of two-sided maximal mechanisms. The mechanism will sequentially “clear” cycles of an appropriately constructed directed graph in the spirit of Gale’s top trading cycle, originally introduced in Shapley and Scarf (1974) and later studied by Abdulkadiroglu and Sonmez (2003).

#### 1.4.1.1 The Block Exchange Algorithm

The basic idea behind the mechanisms we define is the following: starting from the initial assignment, if a teacher  $t$  blocks with the school initially assigned to  $t'$  and  $t'$  also blocks with the

school initially assigned to  $t$ , then we allow  $t$  and  $t'$  to “trade” their initial assignments. This is a pairwise exchange between  $t$  and  $t'$ , but one may of course imagine three-way exchanges or even larger exchanges. Once such an exchange has been made, we obtain a new matching, and we can again search for possible trades. More precisely, our class of mechanisms is induced by the following algorithm, named the Block Exchange (BE, for short):

- **Step 0** : set  $\mu(0) := \mu_0$ .
- **Step  $k \geq 1$**  : Given  $\mu(k-1)$ , let the teachers and their assignments stand for the vertices of a directed graph, where for each pair of nodes  $(t, s)$  and  $(t', s')$ , there is an edge  $(t, s) \rightarrow (t', s')$  if and only if teacher  $t$  blocks  $\mu(k-1)$  with school  $s'$ . If there is no cycle, then return  $\mu(k-1)$  as the outcome of the algorithm. Otherwise, select a cycle in this directed graph. For each edge  $(t, s) \rightarrow (t', s')$  in the cycle, assign teacher  $t$  to school  $s'$ . Let  $\mu(k)$  be the matching so obtained. Go to step  $k+1$ .

It is easy to verify that this algorithm converges in (finite and) polynomial time.<sup>31</sup> In the above description, we do not specify how the algorithm should select the cycle of the directed graph. Therefore, one may think of the above description as defining a class of mechanisms, where a mechanism is determined only after we fully specify how to act when confronted with multiple cycles. One can imagine these selections to be random or dependent on earlier selections. In general, for each profile of preferences for teachers and schools  $\succ$ , a possible outcome of BE is a matching that can be obtained by using an appropriate selection of cycles in the above procedure. Hence, we consider the following correspondence  $BE : \succ \rightrightarrows \mu$  where  $BE(\succ)$  stands for the set of all possible outcomes of BE. A selection of the BE algorithm is a mapping  $\varphi : \succ \mapsto \mu$  s.t.  $\varphi(\succ) \in BE(\succ)$ . Obviously, each selection  $\varphi$  of  $BE$  defines a mechanism.

As mentioned above, our class of mechanisms shares some similarities with Gale’s TTC. However, there are two important differences. The first, and the most minor, is that a teacher in a node can point to several nodes and thus, implicitly, to several schools. This is why, contrary to TTC, we have an issue regarding selection of cycles and why our algorithm does not define a unique mechanism.

<sup>31</sup>To see that this algorithm converges in a finite number of steps, observe that whenever we carry out a cycle, at least one teacher is strictly better-off. Hence, in the worst case, one needs  $(n-1)n$  steps for this algorithm to end. Because finding a cycle in a directed graph can be solved in polynomial time, the algorithm converges in polynomial time.

However, as we will see in the next result, this is necessary for our characterization. Second, and certainly more important, our algorithm takes into account the welfare on both sides of the market. Indeed, a teacher in a node  $(t, s)$  can point to a school in  $(t', s')$  only if  $s'$  agrees (i.e.,  $s'$  prefers  $t$  to its assignment  $t'$ ). This is what ensures, contrary to TTC, that each time we carry out a cycle, *both* teachers and schools become better-off. This has the desirable implication that each time a cycle is cleared, the set of blocking pairs is shrunked.

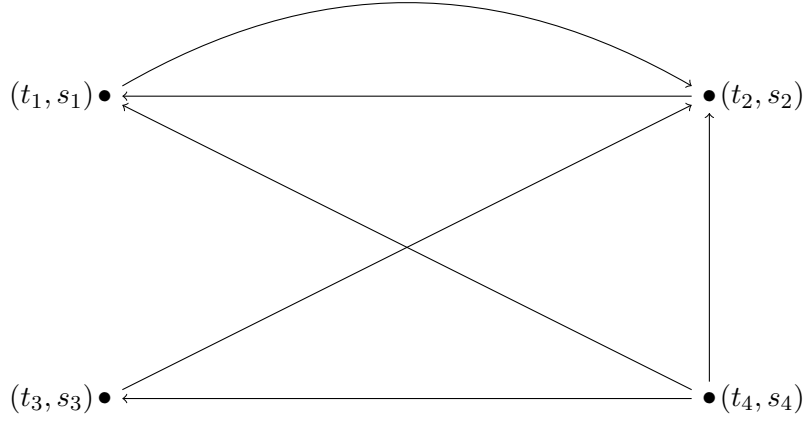
The BE algorithm starts from the initial assignment and then improves on it in terms of the welfare of teachers and schools. More generally, one could start from any matching, obtained by running another mechanism  $\varphi$ . Doing so will guarantee that the (modified) BE algorithm will select a matching that dominates that of  $\varphi$  in terms of the welfare of both teachers and schools. This modification of the BE algorithm that takes the “composition” of BE and  $\varphi$  will be denoted  $\text{BE} \circ \varphi$ . Given our starting point that  $\text{DA}^*$  performs poorly in terms of welfare of teachers and schools, we will be particularly interested in  $\text{BE} \circ \text{DA}^*$ .

The next example illustrates how the BE algorithm works.

**Example 2.** *There are 4 teachers  $t_1, \dots, t_4$  and 4 schools  $s_1, \dots, s_4$  with one seat each. The initial matching  $\mu_0$  is such that for  $k = 1, \dots, 4$ ,  $\mu_0(t_k) = s_k$ . Preferences are the following:*

$$\begin{array}{ll}
 \succ_{t_1}: & s_2 \quad s_3 \quad s_1 \quad s_4 & \succ_{s_1}: & t_4 \quad t_2 \quad t_1 \quad t_3 \\
 \succ_{t_2}: & s_3 \quad s_1 \quad s_2 \quad s_4 & \succ_{s_2}: & t_4 \quad t_3 \quad t_1 \quad t_2 \\
 \succ_{t_3}: & s_1 \quad s_2 \quad s_3 \quad s_4 & \succ_{s_3}: & t_4 \quad t_3 \quad t_2 \quad t_1 \\
 \succ_{t_4}: & s_1 \quad s_2 \quad s_3 \quad s_4 & \succ_{s_4}: & t_4 \quad t_1 \quad t_2 \quad t_3
 \end{array}$$

*This example has a similar feature to Example 1:  $t_4$  is the best teacher and is matched to the worst school. Thus, we know that in that case,  $\text{DA}^*$  coincides with the initial assignment. We have six blocking pairs:  $(t_1, s_2)$ ,  $(t_2, s_1)$ ,  $(t_3, s_2)$  and  $(t_4, s_k)$  for  $k = 1, 2, 3$ . The graph of BE is then the following:*



The only cycle in this graph is  $(t_1, s_1) \rightleftharpoons (t_2, s_2)$ , and it can be verified that once implemented, there are no cycles left in the new matching, and thus, the matching of BE is given by

$$BE = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ s_2 & s_1 & s_3 & s_4 \end{pmatrix}$$

There are now 4 blocking pairs:  $(t_3, s_2)$  and  $(t_4, s_k)$  for  $k = 1, 2, 3$ , and teachers  $t_1$  and  $t_2$  as well as schools  $s_1$  and  $s_2$  are better-off.

We now turn to our characterization result.

**Theorem 1.** *Fix a preference profile. The set of possible outcomes of the BE algorithm coincides with the set of two-sided maximal matchings.*

Before proving the above statement, we prove the following simple lemma.

**Lemma 1.** *Assume that  $\mu'$  2-Pareto dominates  $\mu$ . Starting from  $\mu(0) = \mu$ , there is a collection of disjoint cycles in the directed graph associated with the BE algorithm that, once carried out, yields matching  $\mu'$ .*

*Proof.* Consider the directed graph where teachers and their assignments under  $\mu$  stand for the vertices and for each pair of nodes  $(t, s)$  and  $(t', s')$ , there is an edge  $(t, s) \rightarrow (t', s')$  if and only if teacher  $t$  is assigned to  $s'$  under  $\mu'$ . By the definition of matchings, this directed graph has at least one cycle and cycles are disjoint. Note that because  $\mu'$  2-Pareto dominates  $\mu$ , in this graph,  $(t, s) \rightarrow (t', s')$  if and only if teacher  $t$  blocks  $\mu$  with school  $s'$ . Hence, the graph we built is a

subgraph of the directed graph associated with the BE algorithm starting from  $\mu$ . By construction, we have a collection of disjoint cycles in this directed graph that, once carried out, yields matching  $\mu'$ , as was to be shown.  $\square$

We are now in a position to complete the proof of Theorem 1.

*Proof of Theorem 1.* If  $\mu$  is an outcome of BE, then it must be two-sided maximal. Indeed, if this were not the case, then by the above lemma, there would exist a cycle in the directed graph associated with the BE algorithm starting from  $\mu$ , which contradicts our assumption that  $\mu$  is an outcome of the BE algorithm. Now, if  $\mu$  is two-sided maximal, it 2-Pareto dominates the initial assignment  $\mu_0$ . Hence, appealing again to the above lemma, there is a collection of disjoint cycles in the directed graph associated with the BE algorithm starting from  $\mu_0$  that, once carried out, yields the assignment  $\mu$ . Clearly, once  $\mu$  is achieved by the BE algorithm, there are no more cycles in the associated graph.  $\square$

While this result provides a simple and computationally easy procedure to find two-sided maximal matchings, the class of mechanisms defined by this algorithm is huge. Indeed, appealing to Proposition 1.3.3, this corresponds to the whole class of mechanisms that are both 2-PE and 2-IR. As we will see, by imposing the standard requirement of strategy-proofness, a unique mechanism will remain. The next section will state and prove this result and identify this mechanism.

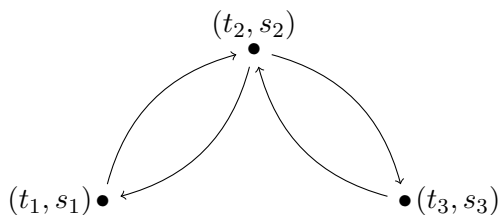
#### 1.4.1.2 Incentives under Block Exchanges

First, recall that a mechanism  $\varphi$  is **strategy-proof** if, for each profile of preferences  $\succ$  and teacher  $t$ ,  $\varphi_t(\succ) \succeq_t \varphi_t(\succ'_t, \succ_{-t})$  for any possible report  $\succ'_t$  of teacher  $t$ .<sup>32</sup> The following example shows that some selections of the BE algorithm are not strategy-proof.

**Example 3.** Consider an environment with three teachers  $\{t_1, t_2, t_3\}$  and three schools  $\{s_1, s_2, s_3\}$ . For each  $i = 1, 2, 3$ , we assume that teacher  $t_i$  is initially assigned to school  $s_i$ . Teacher  $t_1$ 's most preferred school is  $s_2$ , and he ranks his initial school  $s_1$  second. Teacher  $t_2$  ranks  $s_1$  first, followed by  $s_3$ . Teacher  $t_3$  ranks  $s_2$  first and his initial assignment  $s_3$  second. Finally, we assume that each teacher is ranked in last position by the school to which he is initially assigned. We obtain the following graph for the BE algorithm.

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<sup>32</sup>Using standard notation,  $\succ_{-t}$  denotes the vector of preference relations  $(\succ_{t'})_{t' \neq t}$ .



There are two possible cycles that overlap at  $(t_2, s_2)$ . Consider a selection of the BE algorithm that picks cycle  $(t_2, s_2) \rightleftharpoons (t_3, s_3)$ . In that case, the algorithm ends at the end of step 1, and teacher  $t_2$  is eventually matched to school  $s_3$ , his second most preferred school. However, if teacher  $t_2$  lies and claims that he ranks  $s_3$  below his initial assignment, the directed graph associated with the BE algorithm has a single cycle  $(t_1, s_1) \rightleftharpoons (t_2, s_2)$ . In that case, the unique selection of the BE algorithm assigns  $t_2$  to his most preferred school  $s_1$ . Hence,  $t_2$  has a profitable deviation under the selection of the BE algorithm considered here.

While this example is simple, one important objection for practical market design purposes is that the manipulation requires that teachers have fairly precise information regarding the preferences in the market (i.e., of the other teachers and of schools). While this is true for many mechanisms, there is a sense in which – in some realistic instances – some selections of the BE (or associated) algorithm can be manipulated without requiring a considerable amount of information on both preferences in the market and the details of the mechanism. A simple instance of this phenomenon can be illustrated for BE $\circ$ DA\*. Indeed, under this mechanism, a teacher who would initially be assigned to a popular school that dislikes him can use the following strategy: report his most preferred school sincerely and then rank the school to which he is initially assigned in second place (even though this may not match his true preferences). If the teacher does not receive his first choice under DA\*, he will certainly receive his initial assignment under DA\*. Given that this school is popular and dislikes him, the teacher is likely to be part of a cycle involving his most preferred school under the BE algorithm. Hence, at an intuitive level, this mechanism can be manipulated by teachers who may only have coarse information on preferences in the market.

In the following, we define a mechanism that is a selection of the BE algorithm and is strategy-proof. More surprisingly, we will prove further in the text that this is the unique selection satisfying this property. Before providing the definition of the mechanism, we need an additional piece of notation. Given a matching  $\mu$  and a set of schools  $S' \subseteq S$ , we let  $\text{Opp}(t, \mu, S') := \{s \in S' \mid t \succeq_s \mu(s)\}$



be the opportunity set of teacher  $t$  among schools in  $S'$ . Note that for each teacher  $t$ , if  $\mu_0(t) \in S'$ , then  $\text{Opp}(t, \mu_0, S') \neq \emptyset$  since  $\mu_0(t) \in \text{Opp}(t, \mu_0, S')$ .

- **Step 0** : Set  $\mu(0) = \mu_0$ ,  $T(0) := T$  and  $S(0) := S$ .
- **Step  $k \geq 1$**  : Given  $T(k-1)$  and  $S(k-1)$ , let the teachers in  $T(k-1)$  and their assignments stand for the vertices of a directed graph, where for each pair of nodes  $(t, s)$  and  $(t', s')$ , there is an edge  $(t, s) \rightarrow (t', s')$  if and only if teacher  $t$  ranks school  $s'$  first in his opportunity set  $\text{Opp}(t, \mu(k-1), S(k-1)) = \text{Opp}(t, \mu_0, S(k-1))$ . The directed graph so obtained is a directed graph with out-degree one<sup>33</sup> and, as such, has at least one cycle and cycles are pairwise disjoint. For each edge  $(t, s) \rightarrow (t', s')$  in a cycle, assign teacher  $t$  to school  $s'$ . Let  $\mu(k)$  be the assignment obtained and  $T(k)$  (resp.,  $S(k)$ ) be the set of teachers (resp., schools) who are not part of any cycle at the current step. If  $T(k)$  is empty, then return  $\mu(k)$  as the outcome of the algorithm. Otherwise, go to step  $k+1$ .

As will become clear, our mechanism has a tight relationship with the TTC mechanism. Recall that TTC operates in the same manner as the above mechanism except that the pointing behavior does not refer to the opportunity set: an edge  $(t, s) \rightarrow (t', s')$  is added if and only if teacher  $t$  ranks school  $s'$  first within the set of all remaining schools (i.e., at step  $k$ , those are the schools in  $S(k-1)$ ). We will make use of the following straightforward equivalence result.

**Lemma 2.** *Fix a preference profile  $\succ$ . TO-BE( $\succ$ ) is equal to TTC( $\succ'$ ) where for each teacher  $t$ , the preference relation  $\succ'_t$  ranks schools outside his opportunity set  $\text{Opp}(t, \mu_0, S)$  below his initial assignment.*

From this simple lemma, we obtain the following proposition.

**Theorem 2.** *TO-BE is strategy-proof and is a selection of the BE algorithm.*

*Proof.* Given that an agent's report has no impact on his opportunity sets, Lemma 2 above (together with the well-known fact that TTC is strategy-proof) implies that TO-BE is strategy-proof. Now,

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<sup>33</sup>Since, by construction, if  $t$  is not yet eliminated from the algorithm (i.e., he is in  $T(k-1)$ ), the school to which  $t$  is initially assigned is also not eliminated. Hence,  $\mu_0(t) \in S(k-1)$ . As we have already noted, this implies that  $\text{Opp}(t, \mu_0, S(k-1))$  is non-empty. Now, because teachers have strict preferences, there is a unique most preferred school for  $t$  in  $\text{Opp}(t, \mu_0, S(k-1))$ .

we show that TO-BE is a selection of BE. Appealing to Theorem 1, it is enough to show that TO-BE is a two-sided maximal mechanism. If this were not to be the case, it would mean that for some preference profile  $\succ$ , starting from TO-BE( $\succ$ ), there would be a cycle in the directed graph associated with the BE algorithm. It is easily verified that this cycle would still be present if the preferences of teachers were to be modified such that any school outside the opportunity set of a teacher  $t$  (i.e., outside  $\text{Opp}(t, \mu_0, S)$ ) is ranked below his initial assignment. In this modified problem, by Lemma 2, TO-BE is equivalent to TTC. However, if we carry out the cycle starting from TO-BE, we obtain a matching that, for teachers, Pareto dominates TO-BE and hence TTC. This contradicts the well-known fact that TTC is 1-PE.  $\square$

Say that a selection  $\varphi$  of the BE algorithm is teacher-optimal if there is no selection of BE that 1-Pareto dominates  $\varphi$ . The following result justifies the terminology used thus far: TO-BE is indeed teacher-optimal.

**Proposition 1.4.1.** *Take any mechanism  $\varphi$  that is 2-IR. TO-BE is not 1-Pareto dominated by  $\varphi$ .*

*Proof.* Proceed by contradiction and assume that TO-BE is 1-Pareto dominated by  $\varphi$  at preference profile  $\succ$ . Since  $\varphi$  is 2-IR,  $\varphi(t) \in \text{Opp}(t, \mu_0, S)$ . Hence, TO-BE is still 1-Pareto dominated by  $\varphi$  at the modified preference profile where each teacher  $t$  ranks schools outside his opportunity set  $\text{Opp}(t, \mu_0, S)$  below his initial assignment. By Lemma 2, this implies that at the modified preference profile, TTC is 1-Pareto dominated by  $\varphi$ , which is not possible given that TTC is 1-PE.  $\square$

**Corollary 1.** *TO-BE is a teacher-optimal selection of BE.*

We now turn to the most striking result of this section. Apart from TO-BE, no selection of the BE algorithm is strategy-proof.

**Theorem 3.** *TO-BE is the unique selection of the BE algorithm that is strategy-proof.*

*Proof.* The proof is relegated to Appendix 1.8.3.  $\square$

While the formal details of the argument are provided in the appendix, let us give a sketch of the proof for this result.

As is well-known, in a Shapley-Scarf economy (in which schools are replaced by objects with no preferences but are initially owned by the other side of the market), TTC is the unique element of

the core [Shapley and Scarf, 1974](#) and [Roth and Postlewaite, 1977](#)). Because TO-BE is related to TTC, there is a sense in which it can be related to some notion of the core. This notion is used in the course of the argument for [Theorem 3](#). Define the two-sided notion of the core as the set of matchings  $\mu$  s.t. there is no (two-sided blocking) coalition  $B \subseteq T$  for which there is a matching  $\nu$  s.t. for each  $t \in B$ ,  $\nu(t)$  is a school to which a teacher in  $B$  is initially matched and for all  $t \in B : \nu(t) \succeq_t \mu(t)$  and, for  $s := \nu(t)$ ,  $t \succeq_s \mu_0(s)$  with a strict equality for some teacher (or school). Given a profile of preferences, it is easily verified that a matching is in the two-sided core if and only if it is in the (standard) core when preferences are modified in such a way that each teacher  $t$  ranks schools outside his opportunity set  $\text{Opp}(t, \mu_0, S)$  below his initial assignment. Thus, appealing to the results mentioned above (i.e., [Shapley and Scarf, 1974](#) and [Roth and Postlewaite, 1977](#)), we conclude that the two-sided core is a singleton and – given [Lemma 2](#) – coincides with TO-BE.

Now, to provide intuition for [Theorem 3](#), let us consider a selection  $\varphi$  of BE that is strategy-proof. Toward a contradiction, assume that  $\varphi$  and TO-BE differ at  $\succ$ . We first prove a useful technical result: there exists a teacher  $t$  s.t.  $\text{TO-BE}_t(\succ) \succ_t \varphi_t(\succ) \succ_t \mu_0(t)$ . That there is a teacher who strictly prefers the assignment of TO-BE to that of  $\varphi$  is straightforward given that TO-BE is teacher-optimal ([Proposition 1](#)). The non-trivial part consists in showing that this same teacher also *strictly* prefers the assignment of  $\varphi$  to that of  $\mu_0$ . If this were not the case, then among all teachers who strictly prefer TO-BE to  $\varphi$ , the assignment they would obtain with  $\varphi$  would coincide with the initial assignment. Hence, if we denote by  $B$  the complement set of teachers, namely, those who weakly prefer the assignment given by  $\varphi$  to that given by TO-BE, we know that the assignment they obtain under  $\varphi$  corresponds to the initial assignment of some other teacher in  $B$ . Given that  $\varphi$  is 2-IR, this is very close to showing that  $B$  is a two-sided blocking coalition. To show that  $B$  is indeed a two-sided blocking coalition, we need to find a teacher in  $B$  who actually strictly prefers  $\varphi$  to TO-BE. Our argument shows that if this were not the case, then this would contradict that  $\varphi$  is 2-PE (and thus a selection of BE).

Now, given the above technical point, the proof proceeds as follows. Given the profile  $\succ$ , we consider modified preferences  $\succ'_t$  for teachers who only rank as acceptable their school under  $\text{TO-BE}(\succ)$ . Given that this is the unique acceptable assignment for each teacher, the technical lemma implies that  $\text{TO-BE}(\succ')$  must be equal to  $\varphi(\succ')$ . We consider a sequence of unilateral deviations of teachers reporting  $\succ_t$  instead of  $\succ'_t$ , which ultimately returns us to  $\succ$  and along which the equality

between TO-BE and  $\varphi$  is maintained. To give an idea of why the equality is maintained along the sequence of unilateral deviations, let us assume that, starting from  $\succ'$ ,  $t$  reports  $\succ_t$  instead of  $\succ'_t$ . If under  $(\succ_t, \succ'_{-t})$ ,  $\varphi$  and TO-BE select different outcomes, then again by the technical lemma, we know that  $\text{TO-BE}_t(\succ_t, \succ'_{-t}) \succ_t \varphi_t(\succ_t, \succ'_{-t}) \succ_t \mu_0(t)$ .<sup>34</sup> By definition, TO-BE is not affected by  $t$ 's deviation, but then because TO-BE and  $\varphi$  coincide at  $\succ'$ , we have  $\text{TO-BE}(\succ_t, \succ'_{-t}) = \varphi(\succ')$ , which, by the previous argument, is strictly preferred to  $\varphi_t(\succ_t, \succ'_{-t})$  at  $\succ_t$ . Thus, at  $(\succ_t, \succ'_{-t})$ ,  $t$  can claim that his preferences are  $\succ'_t$  and be made better-off, which contradicts the strategy-proofness of  $\varphi$ .

Hence, for a unilateral deviation by teacher  $t$ ,  $\varphi$  and TO-BE must remain equal. Proceeding inductively in this way, we can show that after a sequence of unilateral deviations from  $\succ'_t$  to  $\succ_t$  by each teacher, the equality between TO-BE and  $\varphi$  is maintained, and hence, TO-BE and  $\varphi$  coincide at  $\succ$ .

Before closing this section, we discuss our approach relative to that of [Ma \(1994\)](#). Ma shows that in the Shapley-Scarf economy, the unique mechanism that is 1-IR, 1-PE and strategy-proof is TTC. Intuitively, our result applies to richer environments in which schools have non-trivial preferences that are taken into account when determining welfare. This suggests that our result is a generalization of Ma's. Indeed, to see this, note that in the specific situation in which each school ranks its initial assignment at the bottom of its ranking, TO-BE and TTC coincide. In this context, 1-IR and 2-IR are obviously equivalent. In addition, since 1-PE implies 2-PE, we obtain that the class of mechanisms considered by Ma is a subset of the selections of the BE algorithm. Applying [Theorem 3](#) to these selections yields Ma's result. While our argument builds upon that of Ma, there are a number of crucial differences. As mentioned above, even in the very specific environment in which each school ranks its initial assignment at the bottom of its preference relation, the BE algorithm contains many other mechanisms that include, in particular, all those that are 2-PE but not 1-PE and all 1-PE mechanisms that are "sensitive" to schools' preferences.<sup>35</sup> In addition, our result applies in general to settings in which schools' preferences are arbitrary and thus to many other types of mechanisms that are not even well defined in Ma's environment.

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<sup>34</sup>Obviously, this condition cannot hold for teachers other than  $t$  by construction of  $\succ'$ .

<sup>35</sup>That is, 1-PE mechanisms that select two different matchings for two different profiles of preferences where teachers' preferences remain unchanged.

### 1.4.2 One-sided maximality

We now turn to the characterization of one-sided maximality. As for two-sided maximality, we introduce a class of mechanisms with possible outcomes spanning the whole set of one-sided maximal matchings. With two-sided maximality, the underlying criteria targeted by the designer are the welfare of teachers and schools, as well as the set of blocking pairs. In contrast, with one-sided maximality, the designer only targets the welfare of teachers and the set of blocking pairs. The basic idea behind the mechanism of this section is as follows: under the BE algorithm, two teachers can exchange their assignments iff they both block with the school initially assigned to the other teacher. However, one can imagine a pair of teachers  $t$  and  $t'$  who each desire the school of the other teacher – say  $s$  and  $s'$ , respectively – and, while school  $s$  does not necessarily rank  $t'$  above  $t$ , it does rank first  $t'$  among the individuals who desire  $s$ .<sup>36</sup> If, similarly,  $s'$  ranks  $t$  first among the individuals who desire  $s'$ , then it is easily shown that an exchange between  $t$  and  $t'$  increases the welfare of teachers *and* shrinks the set of blocking pairs. Hence, based on a similar idea, we will weaken the definition of the pointing behavior in the directed graph defined in BE in such a way that – although schools may become worse-off – both teachers' welfare increases and the set of blocking pairs is shrunked each time we carry out a cycle. The following algorithm – named one-sided BE (1S-BE for short) – accomplishes this weakening, and Theorem 4 below provides a sense in which this is the best weakening one can hope to achieve.

- **Step 0** : set  $\mu(0) := \mu_0$ .
- **Step  $k \geq 1$**  : Given  $\mu(k-1)$ , let the teachers and their assignments stand for the vertices of a directed graph, where for each pair of nodes  $(t, s)$  and  $(t', s')$ , there is an edge  $(t, s) \rightarrow (t', s')$  if and only if either (1) teacher  $t$  blocks with school  $s'$  or (2)  $t$  desires  $s'$  and  $t$  is ranked first by  $s'$  among teachers who both desire  $s'$  and do not block with  $s'$ . If there is no cycle, then return  $\mu(k-1)$  as the outcome of the algorithm. Otherwise, select a cycle in this directed graph. For each edge  $(t, s) \rightarrow (t', s')$  in the cycle, assign teacher  $t$  to school  $s'$ . Let  $\mu(k)$  be the matching so obtained. Go to step  $k+1$ .

Here, again, it is easy to verify that this algorithm converges in (finite and) polynomial time. As for the BE algorithm, we do not specify how the algorithm should select the cycle of the directed

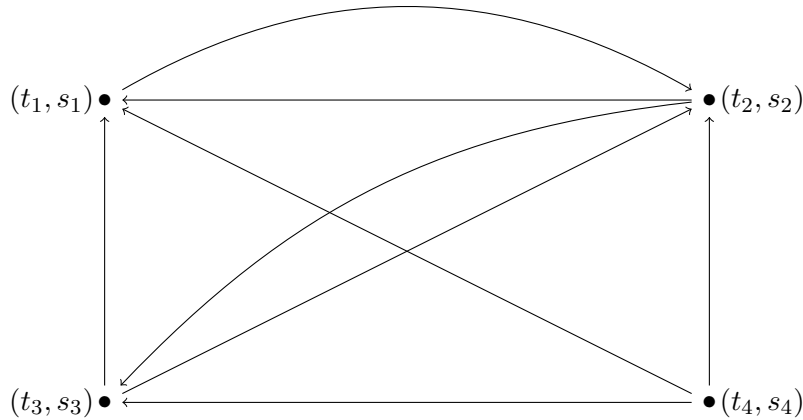
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<sup>36</sup>Henceforth, given a matching  $\mu$ , we say that  $t$  desires  $s$  if  $s \succ_t \mu(t)$ .

graph, and thus, this algorithm defines a class of mechanisms. Each mechanism in this class is a selection of the correspondence from preference profiles to matchings corresponding to the whole set of possible outcomes that can be achieved by the 1S-BE algorithm.

By construction, starting from  $\mu(k - 1)$ , the directed graph defined above is a supergraph of the directed graph that would have been built under the BE algorithm. Hence, there will be more cycles in our graph and more possibilities to improve teachers' welfare and to shrink the set of blocking pairs. This reflects the fact that we dropped the constraint that schools' welfare must increase along the algorithm, and thus, more can be achieved in terms of teachers' welfare and the set of blocking pairs. This is illustrated in the following example.

**Example 4.** Consider the same market as in Example 2. The graph of 1S-BE contains the edges of the graph of BE, but it also has two new additional edges. Indeed,  $t_1$  and  $t_2$  both desire  $s_3$  but do not block with it under  $\mu_0$ , and  $t_2$  is preferred to  $t_1$  at  $s_3$ , and thus, the node  $(t_2, s_2)$  can now point to  $(t_3, s_3)$ . Concerning  $t_3$ , he is the only one who desires  $s_1$  and does not block with it, and thus,  $(t_3, s_3)$  can point to  $(t_1, s_1)$ . Therefore the graph of 1S-BE is as follows:



Note that now there are two additional cycles:  $(t_1, s_1) \rightarrow (t_2, s_2) \rightarrow (t_3, s_3) \rightarrow (t_1, s_1)$  and  $(t_1, s_1) \leftrightarrow (t_2, s_2)$ . Having implemented the first cycle, it can be verified that there are no cycles left, and thus, the matching given by 1S-BE is<sup>37</sup>

$$\begin{pmatrix} t_1 & t_2 & t_3 & t_4 \\ s_2 & s_3 & s_1 & s_4 \end{pmatrix}$$

<sup>37</sup>Note that even if one wished to select one of the two other cycles, another cycle would lead to the same matching.

*Note that now there are only three blocking pairs:  $(t_4, s_k)$  for  $k = 1, 2, 3$ .*

Following the notions introduced for the BE algorithm, we will note  $1S\text{-BE} \circ \varphi$  for the “composition” of BE and of a mechanism  $\varphi$ . An outcome of such a (modified) 1S-BE algorithm selects matchings that dominate that of  $\varphi$  both in terms of teacher welfare and set of blocking pairs (but not necessarily in terms of school welfare); i.e., all teachers are weakly better off, and the set of blocking pairs is a subset of that of  $\varphi$ . Again, in the sequel, we will be particularly interested in starting the 1S-BE algorithm from the matching given by  $DA^*$ , i.e.,  $1S\text{-BE} \circ DA^*$ .

We now turn to our characterization result. We note that while the argument in the proof of Theorem 1 is simple, the proof of the characterization result below is non-trivial.

**Theorem 4.** *Fix a preference profile. The set of possible outcomes of the 1S-BE algorithm coincides with the set of one-sided maximal matchings.*

*Proof.* The proof is relegated to Appendix 1.8.4. □

Assume that matching  $\mu'$  dominates  $\mu$  in terms of teachers’ welfare and stability and consider the directed “exchange graph,” where teachers and their assignments under  $\mu$  stand for the vertices and for each pair of nodes  $(t, s)$  and  $(t', s')$ , there is an edge  $(t, s) \rightarrow (t', s')$  if and only if teacher  $t$  is assigned to  $s'$  under  $\mu'$ . If  $\mu'$  were to dominate  $\mu$  in terms of teachers’ *and schools’* welfare, as well as in terms of stability, then, as argued in the proof of Lemma 1, each “cycle of exchange” in this graph is actually a cycle of the graph associated with the BE algorithm. This is central to the characterization result in Theorem 1. In the present case, in which  $\mu'$  dominates  $\mu$  in terms of teachers’ welfare and blocking pairs (but not necessarily in terms of schools’ welfare), one may expect that these cycles of exchange would be cycles of the graph associated with the 1S-BE algorithm. This turns out not to be the case, and this is an important source of difficulty in the argument to prove Theorem 4. However, although cycles of exchange are not necessarily cycles of 1S-BE, we show that whenever there is a  $\mu'$  that dominates  $\mu$  in terms of teachers’ welfare and stability, there must exist a cycle in the graph (which may not be a cycle of exchange) of 1S-BE starting from  $\mu$ . With this existence, one direction of Theorem 4 can easily be proved. Indeed, given a matching  $\mu$  obtained with the 1S-BE algorithm, if, toward a contradiction, it is not one-sided maximal, then, by definition, there must exist a matching  $\mu'$  that 1-Pareto dominates  $\mu$ , such that its set of blocking pairs is a subset of that of  $\mu$ . However, in that case, we know that there must exist a cycle in the

graph associated with 1S-BE starting at  $\mu$ , which contradicts the fact that  $\mu$  is a matching obtained with the 1S-BE algorithm.

Here also, this result provides a computationally easy procedure to find one-sided maximal matchings. As for the BE algorithm, it is easy to construct selections of the 1S-BE algorithm that are not strategy-proof. In light of Theorem 3, an outstanding question naturally arises: is there any selection of the 1S-BE algorithm that is strategy-proof? While there is a unique selection of the BE algorithm that is strategy-proof, the next result provides a negative answer for the 1S-BE algorithm.

**Theorem 5.** *There is no selection of the 1S-BE algorithm that is strategy-proof.*

*Proof.* The proof is relegated to Appendix 1.8.5. □

This result highlights an important difference between the classes of two-sided and one-sided maximal mechanisms. One can understand the difference as follows. In contrast to the graph of BE, 1S-BE can have an edge  $(t, s) \rightarrow (t', s')$  if  $t$  desires  $s'$  and  $t$  is ranked first by  $s'$  among teachers who both desire  $s'$  and do not block with  $s'$ . Because of this condition, a teacher can modify the pointing behavior of others: indeed, if  $t$  is ranked first by  $s'$  among teachers who both desire  $s'$  and do not block with  $s'$ , then teacher  $t$  can change the set of outgoing edges of other teachers depending on whether he claims that he desires  $s'$ . In the course of the argument for Theorem 5, we rely on this additional feature. Indeed, we present an instance in which, for each possible selection of cycles under the 1S-BE algorithm, one teacher can profitably misreport his preferences. Two types of manipulations are used in that case: one is basic and consists in ranking as acceptable an unacceptable school to be able, once matched with it, to exchange it for a better one. However, for some selection of cycles, another manipulation is needed whereby a teacher ranks as unacceptable an acceptable school to expand the set of outgoing edges of other teachers. Again, this new type of manipulation is central to the argument in Theorem 5 and is not available under the BE algorithm.

Before closing this section, we note that the 1S-BE algorithm shares some similarities with the stable improvement cycle (SIC) algorithm defined by Erdil and Ergin (2008). Indeed, the 1S-BE could be seen as a generalization of the SIC algorithm. To further discuss this relationship, let us recall the definition of the SIC algorithm.

- **Step 0** : set  $\mu(0) := \mu_0$ .



- **Step**  $k \geq 1$  : Given  $\mu(k-1)$ , let the teachers and their assignments stand for the vertices of a directed graph, where for each pair of nodes  $(t, s)$  and  $(t', s')$ , there is an edge  $(t, s) \rightarrow (t', s')$  if and only if  $t$  desires  $s'$  and  $t$  is ranked first by  $s'$  among teachers who desire  $s'$ . If there is no cycle, then return  $\mu(k-1)$  as the outcome of the algorithm. Otherwise, select a cycle in this directed graph. For each edge  $(t, s) \rightarrow (t', s')$  in the cycle, assign teacher  $t$  to school  $s'$ . Let  $\mu(k)$  be the matching so obtained. Go to step  $k+1$ .

The SIC algorithm has been constructed to improve on stable outcomes whenever an outcome is not teacher-optimal as is the case, for instance, with the outcome of teacher-proposing DA when schools have weak preferences. Now, note that when we start from a stable outcome, SIC and 1S-BE are the same. Obviously, in our environment, in which schools have strict preferences, if we start from the outcome of DA – which here is the teacher-optimal stable assignment – the SIC and the 1S-BE do not have any cycles in their associated directed graphs. More generally, if we were to weaken the assumption of strict preferences on the school side, the 1S-BE and the SIC algorithm – starting from DA – would yield the same set of possible outcomes. However, our mechanism goes much further in extending the properties of the SIC algorithm to cases in which the starting assignment is arbitrary. To illustrate why this is true and why we cannot make use of the SIC algorithm for our purposes, consider one of our initial motivations, which is to find ways to improve on the outcome of  $DA^*$ . Both  $BE \circ DA^*$  and  $1S-BE \circ DA^*$  succeed in doing so. However, the SIC algorithm (starting from the outcome of  $DA^*$ ) is of no help for this purpose. To see this, recall that under the SIC algorithm,  $(t, s) \rightarrow (t', s')$  iff  $t$  desires  $s'$  and  $t$  is ranked first by  $s'$  among teachers who desire  $s'$ . Since, given the individual rationality of  $DA^*$ , no teacher desires his initial assignment under the matching achieved by  $DA^*$ , the pointing behavior – and hence the directed graph – associated with SIC (starting from  $DA^*$ ) remains unchanged if we use the modified schools' preferences used to run  $DA^*$  as opposed to the true schools' preferences. However, under the modified preferences, by definition,  $DA^*$  yields the teacher-optimal stable matching. Hence, there cannot be any cycle in the graph associated with SIC (again, starting from  $DA^*$ ).

### 1.4.3 Large Markets

Let us summarize our findings thus far. We provided a stylized example in which  $DA^*$  performs poorly in terms of the set of teachers moving from their initial position. Due to this lack of movement,

we show that one can improve on this algorithm in terms of the welfare of both teachers and schools, as well as set of blocking pairs. We provide a whole class of mechanisms – characterized by the BE algorithm – that does not suffer from such flaws. While this seems to be an improvement over  $DA^*$ , this is still quite weak. As mentioned above, these results essentially show that  $DA^*$  is not on the Pareto frontier while our mechanisms are. These theoretical findings raise a new set of questions concerning both the magnitude of the under-performance of  $DA^*$  and the performance of the different selections of the BE algorithm. To answer these questions, we adopt a large-market approach in this section that allows us to quantify some aspects of the mechanisms’ performance when the market grows.

Specifically the aim of this section is to answer three questions. First, since lack of movement is an important weakness of  $DA^*$ , we may naturally wonder the following: *Is there more movement under all selections of the BE algorithm compared to  $DA^*$ ?* Second, while all selections of the BE algorithm are two-sided maximal, as noted in Proposition 1.3.3, how do the different selections of the BE algorithm compare in terms of the welfare of teachers and schools. In particular, we ask the following: *Based on standard welfare criteria, is there a best selection of BE?* As we will show, there is a meaningful sense in which a best selection of BE exists in terms of welfare on both sides of the market. From our analysis of incentives, we identified a natural candidate mechanism – the teacher-optimal BE – and one may wonder how it compares with the best selection of BE. In other words we wish to answer the following: *Is there a cost of strategy-proofness?* The following large-market analysis helps to answer these questions.

We assume that there are  $K$  tiers for the schools. More precisely, there is a partition  $\{S_k\}_{k=1}^K$  of  $S$  such that the utility of teacher  $t$  for school  $s \in S_k$  ( $k = 1, \dots, K$ ) is given by

$$U_t(s) = u_k + \xi_{ts}$$

where  $\xi_{ts} \sim U_{[0,1]}$ . We assume that  $u_1 > u_2 > \dots > u_K$ . For each  $k = 1, \dots, K$ , we denote by  $x_k$  the fraction of schools having common value  $u_k$  and further assume that  $x_k > 0$ .

This distribution of preferences in terms of tiers allows for some type of positive correlation in teachers’ preferences. Prior literature has highlighted positive correlation in preferences. Indeed, by studying teachers’ preferences for schools in the US, [Boyd et al. \(2013\)](#) find that teachers demonstrate

preferences for schools that are suburban and have a smaller proportion of students in poverty.<sup>38</sup> In addition, although this structure is special, we believe that the basic insights obtained under these distributions extend far beyond this class of distributions.

For schools' preferences, we assume that

$$V_s(t) = \eta_{ts}$$

where  $\eta_{ts} \sim U_{[0,1]}$ . The additive separability structure of our utilities and the specific uniform distribution employed are not essential to our argument.<sup>39</sup> In addition, we could assume that school's preferences are drawn in a similar way as students' preferences (allowing tiers); in that case our results would remain essentially the same. That schools' preferences are only based on an idiosyncratic shock is only to simplify the exposition.<sup>40</sup>

Finally, the initial assignment  $\mu_0$  is selected at random among all possible  $n!$  matchings, where  $n := |T| = |S|$ . A random environment is hence characterized by the number of tiers, their size and common values  $[K, \{x_k\}_{k=1}^K, \{u_k\}_{k=1}^K]$ . The maximum normalized sum of teachers' payoffs that can be achieved in this society is  $\bar{U}_T := \sum_{k=1}^K x_k (u_k + 1)$ , which is attained if all teachers are matched to schools with which they enjoy the highest possible idiosyncratic payoff. The maximum normalized sum of schools' payoffs that can be achieved in this society is  $\bar{V}_S := 1$ , which is attained if all schools are matched to teachers with which they enjoy the highest possible idiosyncratic payoff. Clearly, in our environment, in which preferences are drawn randomly, a mechanism can be seen as a random variable. In the sequel, we let  $\varphi(t)$  be the random assignment that teacher  $t$  obtains under mechanism  $\varphi$ .

In general, our mechanisms will fail to achieve the maximum sum of utilities on either side. However, a meaningful question is how often this phenomenon occurs when the market becomes large. The following concepts will help to answer this question. We say that a mechanism  $\varphi$

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<sup>38</sup>In France, in our data set, we also observe that some school regions are systematically preferred to others, as measured by the number of teachers ranking these regions first. Performing this exercise shows a clear pattern of tiers: whereas 43 and 57 teachers (out of 10,579) rank the regions of Amiens and Créteil first, respectively, more than 1,000 teachers rank the attractive regions of Paris, Bordeaux or Rennes as their first choice. The differences observed are likely related to differences across regions in the proportion of students from lower social backgrounds or minority students.

<sup>39</sup>We essentially need utilities to be continuous and increasing in both components and the distribution of the idiosyncratic shocks to have full support in a compact interval in  $\mathbb{R}$ .

<sup>40</sup>More precisely, the only issue when introducing a richer class of schools' preferences is that asymptotic stability and individual rationality become incompatible. However, if we ignore asymptotic stability, all of our results can be extended when allowing the richer class of preferences.

**asymptotically maximizes movement** if, for any random environment,

$$\frac{|\{t \in T \mid \varphi(t) \neq \mu_0(t)\}|}{|T|} \xrightarrow{p} 1.$$

A mechanism  $\varphi$  is **asymptotically teacher-efficient** if, for any random environment,

$$\frac{1}{|T|} \sum_{t \in T} U_t(\varphi(t)) \xrightarrow{p} \bar{U}_T.$$

Similarly,  $\varphi$  is **asymptotically school-efficient** if, for any random environment,

$$\frac{1}{|S|} \sum_{s \in S} V_s(\varphi(s)) \xrightarrow{p} \bar{V}_S.$$

Finally,  $\varphi$  is **asymptotically stable** if, for any random environment and any  $\varepsilon > 0$ ,

$$\frac{|\{(t, s) \in T \times S \mid U_t(s) > U(\varphi(t)) + \varepsilon \text{ and } V_s(t) > V(\varphi(t)) + \varepsilon\}|}{|T \times S|} \xrightarrow{p} 0.$$

The next three results provide some answers to the three questions posed at the beginning of this section. The proofs of these results are relegated to Appendix 1.8.6 (except for Theorem 6).

**Theorem 6.** *DA\* does not maximize movement, and hence, it is not asymptotically teacher-efficient, asymptotically school-efficient, or asymptotically stable.*

The basic idea behind the above theorem is very similar to the underlying argument in Example 1. Indeed, consider a random environment with two tiers of schools (i.e.,  $K = 2$ ) and where the second tier corresponds to “bad” schools (while the first corresponds to “good” schools). Formally, we assume that  $u_1 > u_2 + 1$ , and thus, irrespective of the idiosyncratic shocks, a school in tier 1 is always preferred to a school in tier 2. The intuition for the result is as follows. Fix any teacher  $t$  initially assigned to a school in the first tier. With non-vanishing probability, if  $t$  applies to a school in tier 1 other than his initial assignment, some teacher in the second tier will be preferred by that school. Hence, teacher  $t$  will be replaced by that teacher. This simple argument implies that – among teachers initially assigned to schools in tier 1 – the expected fraction of teachers staying at their initial assignment is bounded away from 0.

Specifically, for each  $k = 1, 2$ , let  $T_k$  denote the set of teachers who are initially assigned to

a school in  $S_k$ . Consider any teacher  $t \in T_1$ . Let  $E_t$  be the event that for each school  $s \in S_1$ , there is a teacher  $r \in T_2$  s.t.  $r$  is ranked above  $t$  (according to  $s$ 's preferences). Note that for a school  $s$ , the probability that  $t$  is ranked above each individual in  $T_2$  is the probability that  $\{t\} = \arg \max\{\eta_{ts}, \{\eta_{rs}\}_{r \in T_2}\}$ . Since  $\{\eta_{ts}, \{\eta_{rs}\}_{r \in T_2}\}$  is a collection of iid random variables, for each  $r \in T_2$ , by symmetry, the probability that the maximum is achieved by  $t$  must be the same as the probability that it is achieved by any  $r \in T_2$ . Hence, the probability of  $\{t\} = \arg \max\{\eta_{ts}, \{\eta_{rs}\}_{r \in T_2}\}$  must be  $\frac{1}{1+|T_2|}$ . We can now easily compute the probability of  $E_t$ :

$$\begin{aligned} \Pr(E_t) &= \left(1 - \frac{1}{|T_2| + 1}\right)^{|S_1|} = \left(\left(1 - \frac{1}{|T_2| + 1}\right)^{|T_2|}\right)^{|S_1|/|T_2|} \\ &\rightarrow \left(\frac{1}{e}\right)^{x_1/x_2} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Note that, using the same logic as in Example 1, whenever  $E_t$  realizes,  $t$  cannot move from his initial assignment. Indeed, if  $t$  applies to some school  $s$ , this must be to a school in  $S_1$ . However, by construction, each teacher  $t \in T_2$  applies to each school in  $S_1$ . In particular, the teacher in  $T_2$  being ranked above  $t$  by school  $s$  applies to  $s$ , showing that, eventually,  $t$  cannot be matched to  $s$  under DA\*. Thus, the expected fraction of individuals in  $T_1$  who do not move must be

$$\begin{aligned} \frac{1}{|T_1|} \mathbb{E} \left[ \sum_{t \in T_1} \mathbf{1}_{\{t \text{ does not move}\}} \right] &= \frac{1}{|T_1|} |T_1| \mathbb{E} \left[ \mathbf{1}_{\{t \text{ does not move}\}} \right] \\ &= \Pr\{t \text{ does not move}\} \\ &\geq \Pr(E_t). \end{aligned}$$

Thus, the lim inf of the expected fraction of teachers not moving is bounded away from 0. Note that the lower bound computed here can be improved. Indeed, for  $t$  not to move, one only needs that for each school  $s \in S_1$  that  $t$  finds acceptable, there is a teacher  $r \in T_2$  s.t.  $r$  is ranked above  $t$  (according to  $s$ 's true preferences). In general, simulations suggest that a much larger fraction of teachers are not moving. In addition, these simulations show that the assumption we made above that  $u_1 > u_2 + 1$  is not necessary and that the result seems to hold in much broader contexts.<sup>41</sup> Let us now think of the best possible outcome of the BE algorithm. While the way to implement this outcome may not be practical, we consider this a benchmark and want to compare this to what can

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<sup>41</sup>Available upon request.

typically be achieved by mechanisms that can be implemented easily such as TO-BE.

**Theorem 7.** *Each selection of BE asymptotically maximizes movement. There is a selection of BE that is asymptotically teacher-efficient, asymptotically school-efficient and asymptotically stable.*

The intuition for the first part of the result is basic. Indeed, assume, toward a contradiction, that the set of teachers not moving under some selection of BE is “large”. For each pair of teachers  $t$  and  $t'$  in that set, the probability that  $t$  blocks with the initial assignment (and hence the assignment under the given selection) of  $t'$  and that, vice versa,  $t'$  blocks with the initial assignment of  $t$  (and thus gains from the assignment under the given selection) is bounded away from 0. Hence, given our assumption that the set of teachers not moving is large, with high probability, there will be such a pair of teachers. In other words, there will be a cycle in the graph associated with BE when starting from the assignment given by the selection, which contradicts the definition of a selection.

As for the other part of the theorem, the intuition can be seen as follows: for a given tier  $k$ , for any school in  $S_k$  and any agent initially matched to such a school in  $S_k$ , the probability that they both enjoy high idiosyncratic payoffs when matched with one another is bounded away from 0. Thus, as the market grows, with probability approaching one, one can appeal to Erdős-Rényi’s result on the existence of a (perfect) matching within the set of individuals and schools. However, there are two difficulties here. First, one has to ensure the individual rationality of the matching so obtained, which we do by restricting the set of teachers and schools to those having idiosyncratic payoffs for their initial match bounded away from the upper bound. Second, in the sketch we just provided, we are implicitly assuming that the designer has access to an agent’s cardinal utilities. However, in this paper, we assume—as is usually the case in practice—that matching mechanisms map ordinal preferences into matchings, and a large part of the proof is devoted to addressing this issue.

Now, as we have already noted, while the BE algorithm treats teachers and schools symmetrically, TO-BE favors teachers at the expense of schools. Thus, it is natural to expect that TO-BE is asymptotically teacher-efficient. In addition, TO-BE only ensures that schools are assigned a teacher that they weakly prefer to their initial assignment. Hence, for each school, its assignment under TO-BE is a random draw within the set of teachers that it finds acceptable. Thus, given its idiosyncratic payoff for its initial assignment  $\eta_{\mu_0(s)s}$ , the expected payoff of a school  $s$  under TO-BE ( $s$ ) is  $\mathbb{E} \left[ \eta_{st} \mid \eta_{st} \geq \eta_{\mu_0(s)s} \right] = \frac{1}{2} \left( 1 + \eta_{\mu_0(s)s} \right)$ . Thus, the (unconditional) expected payoff of school  $s$  under TO-BE ( $s$ ) is  $\mathbb{E} \left[ \frac{1}{2} \left( 1 + \eta_{\mu_0(s)s} \right) \right] = \frac{3}{4}$ . Thus, TO-BE cannot be asymptotically school-efficient.

**Theorem 8.** *TO-BE is asymptotically teacher-efficient. Under TO-BE, the expected payoff of a school is  $\frac{3}{4}$ , and thus, TO-BE is neither asymptotically school-efficient nor asymptotically stable.*

To understand why TO-BE is asymptotically teacher-efficient, a heuristic is as follows: assume it is not. This implies that for some tier, there is a “large set” of teachers who are obtaining an idiosyncratic payoff bounded away from the upper bound. Now, for each pair of teachers  $t$  and  $t'$  in that set, intuitively, the probability that  $t$  blocks with the assignment of  $t'$  and that, vice versa, that  $t'$  blocks with the assignment of  $t$  is bounded away from 0. Hence, given our assumption that the set of teachers not moving is large, intuitively, with high probability, there will be such a pair of teachers. Thus, here again, there will be a cycle in the graph associated with BE when starting from the assignment given by the selection, which contradicts the definition of a selection. While this is an intuitive way of describing the result, there is a difficulty here. Indeed, if we fix the set of teachers who are obtaining an idiosyncratic payoff bounded away from the upper bound, this has some implications for the distribution of preferences. Hence, there is a conditioning issue, and the intuition provided above does not take this into account. This raises an important technical difficulty that we circumvent using random graph arguments in the spirit of those developed by [Lee \(2014\)](#) and, more particularly, [Che and Tercieux \(2015b\)](#).

From the above, we should expect several results from our data analysis. First,  $DA^*$  should rarely be two-sided maximal, particularly in markets with a large number of teachers. In addition, the BE algorithm and TO-BE should ensure more movement than  $DA^*$  and perform better in terms of teachers’ welfare. We will see that our data analysis largely confirms these findings. We should also expect TO-BE to perform less well than the BE algorithm: TO-BE may exhibit a loss in terms of schools’ welfare and blocking pairs relative to the BE algorithm. In terms of schools’ welfare and the set of blocking pairs, it is not clear a priori how to compare TO-BE and  $DA^*$ . Our data analysis will help further discriminate between these mechanisms. It is important to mention an important difference between our empirical analysis and our large market framework: in the latter we let both the number of schools and the number of teachers tend to infinity. In the following empirical analysis, we focus on the assignment of teachers to the 31 French administrative regions. This small number of regions is closer to a second type of large market analysis using continuum models like in [Abdulkadiroglu, Che and Yasuda \(2015b\)](#) or [Azevedo and Leshno \(2016\)](#); where the number of schools is fixed and where the number of agents is large. First, it is not clear how to

handle algorithms based on exchange graphs like TO-BE under a continuum model. Second, our empirical analysis can be seen as a complement to our large market framework. Which large market model is the best depends on the application that one considers. For instance, our large market analysis is closer to the second assignment phase of teachers to schools inside each region, which we chose to discard due to possible misreports of preferences during this phase as explained in the next section.

## 1.5 Empirical Analysis

The aim of this section is to assess our theoretical findings using a data set on the assignment of all public school teachers in France. We provide a brief presentation of the data set we use. Then, we run counterfactual scenarios for our mechanisms and measure the extent of the improvements they may yield, in terms of both school and teacher welfare, as well as in terms of fairness.

### 1.5.1 Data

We use several data sets related to teacher assignment in 2013. For both the first and the second phase of the assignment, these data sets contain four pieces of information: (1) teachers' reported preferences, (2) regions/schools' priorities, (3) each teacher's initial assignment (if any) and (4) regions/schools' vacant positions. This empirical study focuses on the first phase of the assignment, primarily because teachers have incentives to report their preferences<sup>42</sup> sincerely and the preferences we observe are more straightforward to interpret.<sup>43</sup> Several features of the assignment process support this position. First, since the mechanism at use is DA\* with no limit on the number of regions teachers can rank, it is a dominant strategy for all agents to be truthful. The fact that strategy-proof mechanisms generate reliable preference data for guiding policy is a common argument in their favor (see, e.g., [Abdulkadiroglu, Pathak and Roth, 2009](#), [Abdulkadiroglu, Agarwal and Pathak, 2015](#)).<sup>44</sup> Second, since the ministry manages the assignment process in partnership with teachers' trade unions, teachers are very well informed about the entire process, and trade unions

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<sup>42</sup>Given the agents' assessments over schools they may obtain in the second phase, agents have well-defined preferences over regions.

<sup>43</sup>As discussed in Appendix 1.8.8, preferences reported during the second phase of the assignment are more difficult to interpret because of both a binding constraint on the number of schools teachers can rank and the ability to rank larger geographic areas than a school (cities, for instance).

<sup>44</sup>This assumption is sometimes challenged, however. See, for instance, [Fack, Grenet and He \(2015\)](#).



have never advised teachers to strategically rank the school regions.<sup>45</sup> We therefore take for granted that agents' reported preferences in Phase 1 are the true preferences to run our counterfactuals.

The sample of teachers used for the analysis takes into account two restrictions. First, the sample is restricted to the 49 subjects that contain more than 10 teachers. Second, to match our theoretical framework, all initially non-matched teachers (newly tenured) and all empty seats in regions are suppressed. Hence, the initial assignment corresponds to a market in which each teacher is initially assigned to a region and each seat in each region is initially assigned a teacher. The final sample contains 10,579 teachers corresponding to 49 subjects ranging from 6 to 1,753 teachers. We end this section by providing two pieces of information on this market. It is important to keep in mind that, due to our above restrictions, the analysis presented will not be a counterfactual for the complete French assignment market. To do so, one would have to incorporate newly tenured teachers and empty seats and define a new version of TO-BE accordingly. We discuss this point in the conclusion. One can see the following results more as “data calibrated simulations”. Yet, this exercise offers interesting results to support our theoretical findings.

**Fact 0** (i) *Under the regular DA mechanism, there are at least 1,325 teachers for whom individual rationality is violated; i.e., they are assigned to a region that they consider worse than their initial region.*<sup>46</sup> (ii) *The individually rational mechanism that maximizes movement allows 2,257 teachers to move from their initial assignment.*<sup>47</sup>

This fact shows that the regular DA mechanism is indeed not individually rational in this market, and the violation of individual rationality is quite strong. The second point gives us a sense in which there is congestion on this market: if we focus only on individually rational matchings and attempt to ensure as much movement as possible, 21% of teachers will be able to move. We should bear in

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<sup>45</sup>The mobility process is the main source of teacher unionization in France. As a competition exists between trade unions, they have high incentives to provide detailed information and tailored help for teachers throughout the process. In practice, trade unions help teachers in identifying the criteria they can use to compute their priorities, they negotiate the number of positions offered in each region with the ministry, and they validate the mobility project submitted by the ministry.

<sup>46</sup>In our data set, for each teacher, we have the reported preferences only up to his initial region. Hence, we do not know how teachers rank regions below their initial assignment. However, one can show that when running DA on these truncated preferences, the number of unassigned teachers is a lower bound on the number of teachers for whom individual rationality is violated when running DA on the full preference lists.

<sup>47</sup>To find such an assignment, we build a bipartite graph with teachers on one side and schools on the other side. We consider the complete bipartite graph, where each edge will be associated with a weight. We assign weight  $\infty$  to edges  $(t, s)$ , where  $s$  is unacceptable to  $t$  (i.e., worse than his initial assignment). We assign weight 1 to the edge if  $t$  is initially matched to  $s$ . Finally, we assign weight 0 to all other edges (i.e., if  $t$  finds  $s$  strictly better than his initial assignment). The weight of a matching is defined as the sum of weights over all its edges. We use a standard algorithm to find a matching with minimal weight (see [Kuhn, 1955](#) and [Munkres, 1957](#)). It is easily verified that such a matching maximizes movement among all individually rational matchings.

mind this upper bound when considering the performance of our algorithms and the scope for their improvement.<sup>48</sup>

## 1.5.2 Results

### 1.5.2.1 Preliminaries: many-to-one

Before turning to the description of the results, we need to briefly discuss the generalization of our mechanisms to the many-to-one framework. A school/region can now be assigned several teachers, and starting with preferences over single teachers, we need to define schools' preferences over a bundle of teachers. We adopt a very conservative approach here that will only strengthen our main empirical findings. Consider a school/region with  $q$  positions to fill and two vectors of size  $q$ , say  $\mathbf{x} := (t_1, \dots, t_q)$  and  $\mathbf{y} := (t_1, \dots, t_q)$ . Let us assume that each of these vectors is ordered in such a way that for each  $k = 1, \dots, q - 1$ , the  $k$ th element of vector  $\mathbf{x}$  is preferred to its  $k + 1$ th element; we make analogous assumptions for vector  $\mathbf{y}$ . We say that  $\mathbf{x}$  is preferred by the school/region to  $\mathbf{y}$  if for each  $k = 1, \dots, q$ , the  $k$ th element of vector  $\mathbf{x}$  is (weakly) preferred to the  $k$ th element of vector  $\mathbf{y}$ .

With this definition in mind, all of our concepts (two-sided maximality or one-sided maximality) can be naturally extended. Mechanisms characterizing these notions can also be easily extended. Because these are the mechanisms we use to run our counterfactuals, we state them precisely in Appendix 1.8.7.

The aim of the following empirical analysis is to test our theoretical results. Therefore, we will focus on three main dimensions: teachers' welfare, regions' welfare and number of blocking pairs. Since BE and 1S-BE define a class of mechanisms, we randomly select outcomes within this class. For the TO-BE mechanisms parametrized by an ordering over teachers. We randomly select an ordering and hence randomly select an outcome in this class. We randomly draw selections for each mechanism ten times. In addition, there are indifferences in priorities of regions over teachers.<sup>49</sup> We use a single tie-breaking rule to break ties in regions' priorities. We draw randomly the tie-breaking rule ten times. The results reported in Tables 1 to 3 for BE, TO-BE and 1S-BE correspond to

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<sup>48</sup>The relatively small fraction of teachers able to move is explained primarily by the high proportion of teachers reporting short lists. Indeed, teachers rank on average 1.64 regions, and 75 % of teachers only ask for one region (beyond their initial region). Combined with correlation in preferences, this structurally restricts the possibility of movement in the market.

<sup>49</sup>Many young teachers use only one criterion – the number of years of experience – to compute their priorities, and thus, they have the same priority in a given region.

average outcomes over one hundred draws – ten random selections of tie-breaking rules times ten random selections of outcomes. The results for  $DA^*$  correspond to an average over ten iterations of tie-break only. The results for the BE algorithm and the 1S-BE are successively presented in the next section.<sup>50</sup>

### 1.5.2.2 Two-sided maximality: BE and TO-BE

#### How far is $DA^*$ from being two-sided maximal?

In the theoretical analysis, we noted an important flaw of  $DA^*$ : it can be improved in three main dimensions, namely, teachers' welfare, regions' welfare and fairness. In practice, a first simple way to test whether  $DA^*$  is two-sided maximal is to run the BE algorithm starting from the matching obtained by  $DA^*$  and then to observe whether the two matchings obtained with  $DA^*$  and  $BE \circ DA^*$  differ. If they differ, this means that some cycles exist in the graph associated with the BE algorithm starting from  $DA^*$ , and thus,  $DA^*$  is not two-sided efficient, which is a necessary condition for two-sided maximality. Fact 1 below illustrates that point in our data:

**Fact 1.**  *$DA^*$  is not two-sided maximal in 33 out of 49 subjects. These subjects represent 95.9% of the teachers.*<sup>51</sup>

This first fact suggests that the theoretical phenomenon we highlight is not rare. Based on this observation, we now estimate how far  $DA^*$  is from maximality in terms of the three criteria in which we are interested. Regarding teachers' welfare, two results reported in Table 1.1 are worthy of comment. First, on average,  $BE \circ DA^*$  more than doubles the number of teachers who are assigned to a new region relative to  $DA^*$ : 565 teachers move from their initial allocation under  $DA^*$  versus 1,488 under  $BE \circ DA^*$ . Second, the same table reports the cumulative distribution of the number of teachers who obtain school rank  $k$ . While we know from the theory that the distribution of the

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<sup>50</sup>As explained in Section 1.2, regions use multiple criteria to rank teachers (spousal reunification, disability, having a position in a disadvantaged or violent school, total seniority in teaching, seniority in the current school, time away from the spouse and/or children, etc). However, when running our alternative algorithms, we use only the seniority criteria (both total seniority in teaching and seniority in the current school) to determine a teacher's priority in his initial region. Indeed, the other criteria are supposed to help a teacher to leave his current region, so it would not make sense to use these criteria for the region to which he is currently assigned.

<sup>51</sup> Even if  $BE \circ DA^* = DA^*$ , it could be the case that  $DA^*$  is not two-sided maximal. Indeed, in our definition of two-sided maximality, we require that schools must not be harmed relative to the initial allocation. Since  $DA^*$  may harm schools, it can be two-sided Pareto efficient but still violate two-sided maximality. In 30 subjects,  $BE \circ DA^* \neq DA^*$ , and in 3 subjects,  $BE \circ DA^* = DA^*$ , but  $DA^*$  harms the welfare of at least one region relative to the initial assignment. Finally, we note that, if we restrict our attention to the 19 subjects with more than 100 teachers, in only one subject is  $DA^*$  two-sided maximal.

rank obtained by teachers under BE $\circ$ DA\* first-order stochastically dominates this same distribution under DA\*, the dominance is indeed significant.

There are several possible measures of regions' welfare. We focus below on one natural approach; however, we test the robustness of our results to the use of alternative approaches, which yield no significant differences in the results. Given a mechanism, we examine the improvement a region obtains (from the initial assignment) in terms of the number of positions improved. More precisely, given a region, we first take the initial assignment and sort it by decreasing order of priorities. We obtain a vector in which the first element/position is the teacher with the highest priority in that region at the initial assignment, the second element/position is the teacher with the second highest, and so forth. Call this vector  $\mathbf{x}$ . We perform the same operation for the assignment of this region with the mechanism under study. Let us call this vector  $\mathbf{y}$ . Finally, we say that a position  $k$  is assigned a teacher with higher (resp., lower) priority if the  $k$ th element of vector  $\mathbf{y}$  has a higher (resp., lower) priority than the  $k$ th element of vector  $\mathbf{x}$ . Based on this, we compute the percentage of net improvement in positions, i.e., the percentage of positions receiving a teacher with a higher priority minus the percentage of positions being assigned a teacher with a lower priority.

Table 1.3 reports, for the different mechanisms we run, the cumulative distribution of the percentage of net improvement in positions, i.e., for each percentage  $x$ , the proportion of regions having less than  $x$  percent of net improvement in positions. Again, we observe that the distribution under BE $\circ$ DA\* first-order stochastically dominates this same distribution under DA\*.

Finally, we compare the performance of DA\* and BE $\circ$ DA\* in terms of fairness. The first row of Table 1.2 reports that, on average, 2,496.5 teachers are not blocking under DA\* and 3,799.4 are not blocking under BE $\circ$ DA\*, which represents a 52.1% increase in the number of teachers who are not blocking with any region. More generally, we observe that fairness is significantly increased.<sup>52</sup>

Overall, these results show that DA\* fails to be two-sided maximal in a large number of cases, and the scope of improvement seems to be very large. To address this issue, a first natural candidate would be to run the BE algorithm from the assignment achieved by DA\*. However, as mentioned in our theoretical analysis, this mechanism is prone to easy manipulations. Alternatively, we focus our attention on both the BE algorithm that is run directly from the initial assignment (this is referred

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<sup>52</sup>The number of teachers who are part of a blocking pair is quite high. This is intuitive since the number of teachers moving is low and many teachers stay at their initial allocation, thus possibly creating envy. This can be seen as the cost of imposing the individual rationality constraint.

to as BE(Init) in our tables and graphs) and its strategy-proof selection: the TO-BE mechanism. In the next section, we evaluate the performance of these two mechanisms in terms of teachers' welfare, regions' welfare and the number of blocking pairs.

### Performance of BE and TO-BE

Before commenting on the results, it is worth briefly discussing the relevance of comparing BE and TO-BE to  $DA^*$ . We should bear in mind that, for an arbitrary outcome of the BE mechanism, its set of blocking pairs may differ from that of  $DA^*$ , and similarly, the outcome may not 2-Pareto dominate  $DA^*$ . However the comparison remains interesting for two reasons. First, we know from the above results that  $DA^*$  is far from being two-sided maximal, and thus, BE and TO-BE – which are two-sided maximal – can be expected to perform much better. Second, our theoretical results (Theorems 6, 7 and 8) suggest that BE and TO-BE perform better in large markets than  $DA^*$ .

We first focus our attention on the performance of BE and TO-BE in terms of teachers' welfare. Both mechanisms significantly improve the number of teachers moving relative to  $DA^*$ : on average, 565 teachers obtain a new assignment under  $DA^*$ , versus 1,461.5 under BE and 1,373 under TO-BE.

**Fact 2.** *The distribution of ranks obtained by teachers under TO-BE first-order stochastically dominates the distribution of BE, which dominates that of  $DA^*$ .*

Note, however, that there is no 2-Pareto domination between the matchings: some teachers may prefer their assignment under  $DA^*$  to the one that they obtain under BE or TO-BE.<sup>53</sup>

Regarding stability, BE and TO-BE also perform significantly better than  $DA^*$ . Table 1.2 shows that the average number of teachers not being part of a blocking pair increases from 2,496.5 under  $DA^*$  to 3,731.3 and 3,742.7 under BE and TO-BE, respectively.

**Fact 3.** *The distributions of the number of regions teachers can block with under  $DA^*$ , BE and TO-BE can be ranked stochastically:  $DA^*$  is dominated by BE, which is dominated by TO-BE.*<sup>54</sup>

Pereyra (2013) distinguished two types of envies:

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<sup>53</sup>On average, 258.5 teachers obtain a region they rank strictly higher under  $DA^*$  than under BE (254.5 under TO-BE). Conversely, 1,152.7 teachers strictly prefer their assignment under BE to that under  $DA^*$ , and the corresponding number is 1,096.7 under TO-BE.

<sup>54</sup>As discussed for teachers' welfare, it is worth noting that the set of blocking pairs of each matching may differ. Some teachers may block with a region under BE or TO-BE but not under  $DA^*$ .

- *Inappropriate envies*: a teacher prefers a region to his assignment and all the teachers less preferred to him by this region and assigned to it are teachers initially matched to it and who stayed at their position.
- *Justified envies*: a teacher who prefers a region and is preferred by this region to a teacher not initially assigned to it but who is matched to it under the given matching.

He imposed that one should not allow justified envies and showed that among all mechanisms having no justified envies,  $DA^*$  minimizes the set of inappropriate envies (in the setwise inclusion sense). Here, we do not distinguish between these two types of envies and we consider the standard definition of justified envies common in the literature. However, it is important to mention that the matching returned by TO-BE can create justified envies. The argument is similar to the one proving that the TTC algorithm is not stable: if one allows two teachers to exchange their positions, it can lead to an appropriate envy from a third teacher. TO-BE decreases the number of inappropriate envies but creates justified ones. Overall, as seen in the previous fact, the total number of blocking pairs decreases compared to  $DA^*$ . For further discussions about this trade-off, we refer the reader to the end of the Chapter 2.

Finally, comparing regions' welfare across mechanisms is of particular interest, as we know that  $DA^*$  can harm some regions, in contrast to BE and TO-BE. This is confirmed by Table 1.3. Under  $DA^*$ , 1.04% of the regions have at least 1% of their positions assigned a teacher with a lower priority than under the initial assignment. On the contrary, under BE and TO-BE, no region has a position for which the teacher assigned to it has a lower priority than the teacher initially assigned to that position.

**Fact 4.** *The distributions of the percentage of net improvement in positions can be stochastically ordered: the distribution of  $DA^*$  is dominated by that of TO-BE, which is dominated by that of BE.*

The lower performance of TO-BE compared to BE in terms of regions' welfare is in line with our theoretical predictions regarding the cost of the strategy-proofness imposed by TO-BE (Theorem 7 and 8).

As mentioned earlier, accounting for regions' preferences – through two-sided maximality – is a way to better consider policy maker's objectives because priorities incorporate several welfare relevant criteria. As described in Section 1.2, priorities are defined by a point system. Three main

criteria are used to rank a teacher: total seniority in teaching (experience), spousal reunification and whether the teacher has been teaching for several years in a disadvantaged school.<sup>55</sup> The analysis of regions' welfare presented above pools together all criteria, but we are also interested in the performance of the mechanisms for each criteria taken separately. In a sense, the points given for each criterion reflect the policy maker's trade-offs between the three criteria. For instance, we know that one of the main objective is to allocate more experienced teachers to disadvantaged regions. But policy makers would accept a slightly less experienced teacher in such region if he/she ends up being closer to his spouse. This is exactly what the ranking reflects. The lower priority in terms of experience is compensated by a higher priority in terms of spousal reunification. In Section 1.5.2.4, we decompose the welfare analysis for each criteria, and we take a finer look at the distribution of teachers across regions under each of the mechanisms.

Overall, these results suggest that BE and TO-BE perform much better than  $DA^*$  in terms of teachers' welfare, regions' welfare and fairness. The good performance of TO-BE is of particular interest due to its incentive properties. These results provide evidence that, although two-sided maximality is a strong requirement, our mechanisms can generate large improvements and distributions dominating those of  $DA^*$ .<sup>56</sup> The next section tests whether we can further improve upon  $DA^*$  by relaxing the constraint that no region should be harmed (relative to the initial assignment). To do so, we provide empirical evidence on the performance of 1S-BE, the one-sided maximal algorithm we defined in Section 1.4.2.

### 1.5.2.3 One-sided maximality: 1S-BE

As done previously for BE and TO-BE, we first aim to estimate how far  $DA^*$  is from being one-sided maximal. To do so, we compare the matching under  $DA^*$  to that under 1S-BE that we run from  $DA^*$ . For a large number of subjects, we have seen that  $DA^*$  is not two-sided maximal; thus, it is not one-sided maximal. Because the constraint on regions' welfare is relaxed under 1S-BE relative to BE, the improvements we have found for  $BE \circ DA^*$  in terms of teachers' welfare and blocking

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<sup>55</sup>Additional criteria are used such as disability, being a replacement teacher, or applying for a position after some years off teaching. Fewer teachers use these criteria, so we chose to focus on the three main ones. Time away from the spouse and/or children is used to determine the points attributed for the spousal reunification criterion. Disadvantaged schools are classified as such by the administration. After having taught 5 years in a disadvantaged school, teachers get additional points to move anywhere else.

<sup>56</sup>These results are all the more encouraging because they are obtained in a restrictive environment in which teachers rank a very limited number of regions. Even better results could be expected in environments in which agents have longer ranked lists.

pairs can be seen as a lower bound on the potential improvements under 1S-BE $\circ$ DA\*. Indeed, Table 1.1 reports that 1S-BE $\circ$ DA\* yields a threefold increase in the number of teachers moving relative to DA\* and increases this figure by 15% compared to BE $\circ$ DA\*. This suggests that there is still significant potential for improvement with respect to considering one-sided maximality.

**Fact 5.** *DA\* is not one-sided maximal in 31 subjects, and 95.3% of the teachers belong to these subjects. In one subject, DA\* is two-sided maximal but not one-sided maximal.*

We now turn to the results on 1S-BE starting from the initial allocation (referred to as 1S-BE(Init) in our tables and graphs) to compare its performance with that of the other mechanisms. Regarding teachers' welfare and fairness, the distributions of both the ranks obtained by teachers (Table 1.1) and the number of teachers blocking (Table 1.2) under 1S-BE stochastically dominate the distribution of all other algorithms mentioned previously: BE, TO-BE and DA\*.<sup>57</sup> Finally, regions' welfare is the key difference between two- and one-sided maximality. Even if improvements in teachers' welfare and fairness are large with 1S-BE, we know that this comes at the expense of the regions' welfare.

**Fact 6.** *Under 1S-BE, in 4.9 % of the regions, the percentage of positions filled by a teacher with a lower priority is higher than the percentage of positions filled by a teacher with a higher priority. This is in contrast with BE or TO-BE, under which, by definition, regions cannot be assigned teachers with a lower priority relative to the initial assignment.*

#### 1.5.2.4 Administration's objective

We motivated the two-sided efficiency notion by its better ability to fulfill the administration's objectives, as reflected by the three main criteria defining the priority system: i) experience in teaching, ii) spousal reunification and iii) years of teaching in a disadvantaged school. For instance, under the two-sided efficiency notion, a reassignment of teachers that, ceteris paribus, decreases the number of experienced teachers in disadvantaged schools would not meet the administration's objective to better distribute experienced teachers across schools. We end this empirical section by looking more closely at these three criteria.

The first criterion gives more points to more experienced teachers. Obviously, it is not possible to increase the experience of teachers in all regions. However, as some regions are more disadvantaged

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<sup>57</sup>As explained above, some teachers may prefer their match under DA\*; 195 teachers do so under 1S-BE, which is less than the corresponding figure under BE or TO-BE.



than others, one objective is to control the share of inexperienced teachers in these regions. To define disadvantaged regions, we follow a classification used by the French ministry of education. The regions of Créteil and Versailles are classified as disadvantaged regions as they have a high share of students enrolled in priority education, and a high share of students whose parents have no diploma. In addition, every year, around 50% of the teachers who ask for a re-assignment come from one of these two regions. The upper part of Table 1.4 reports, for both disadvantaged and non-disadvantaged regions, the percentage of teachers having only one or two years of experience under the current algorithm, and the difference under the alternative mechanisms we suggest.<sup>58</sup>

Under the current algorithm, 26.9% of the teachers have only one or two years of experience in non-disadvantaged regions. This rate goes up to 45.6% in disadvantaged regions. A first interesting result is that, under the TO-BE mechanism, this percentage of inexperienced teachers goes down by 0.67 percentage points in disadvantaged regions, while it goes up by 0.62 percentage points in non-disadvantaged regions. This shows that one of the objective of policy makers, which is to not increase the share of inexperienced teachers in deprived regions, is respected. It is also interesting to notice the difference between the TO-BE mechanism and 1S-BE. Given that 1S-BE does not require an exiting teacher to be replaced by a teacher whose priority is higher, we could expect this mechanism not to fulfill as well the administration's aim to not assign relatively inexperienced teachers in disadvantaged regions. This is what we see in Table 1.4. The reduction of the share of inexperienced teachers assigned to disadvantaged regions is smaller under 1S-BE than under TO-BE.

It is worth keeping in mind that most of the mobility gain happens in non-disadvantaged regions,<sup>59</sup> while mobility tends to remain constant or decrease in disadvantaged regions.<sup>60</sup> In order to cancel out these important differences in mobility, we look at two additional statistics: the percentage of inexperienced teachers among the ones *entering* each region, and the percentage of inexperienced teachers among the ones *leaving* each region. As expected, differences are much starker. Among teachers who leave disadvantaged regions, the share of inexperienced teachers is significantly higher

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<sup>58</sup>We distinguish between teachers having only one or two years of experience and more experienced teachers based on evidence that teachers in their first year of experience tend to perform less well than more experienced teachers (Chetty, Friedman and Rockoff, 2014; Rockoff, 2004).

<sup>59</sup>In Example 1, the disadvantaged school  $s^*$  is initially assigned to teacher  $t^*$ . Since  $t^*$  has the highest priority in all schools, he prevents all other teachers from moving from their initial school under  $DA^*$ , and stays himself in  $s^*$ . Under TO-BE, teacher  $t^*$  still stays in  $s^*$ , but the other teachers can exchange their positions.

<sup>60</sup>Mobility decreases in disadvantaged regions under the BE mechanism, and to a lesser extent under TO-BE, because of the requirement that any teacher leaving the region has to be replaced by a teacher with a higher priority. This requirement does not exist with  $DA^*$ , and can prevent some teachers from leaving their region if no teacher with a higher priority can replace them.

under TO-BE than under DA\*. This is consistent with the fact that, under TO-BE, any teacher leaving a region has to be replaced by a teacher with a higher priority. This makes it more difficult for experienced teachers to move, but not for inexperienced ones, so that the share of inexperienced teachers among exiting teachers increases. This higher priority rule under TO-BE also justifies why the share of inexperienced teachers among those who enter disadvantaged regions does not go up under TO-BE.

Finally, we look at performance for two additional criteria, which are spousal reunification and experience in disadvantaged schools. For these criteria, the objective is to help teachers to get closer to their spouse or leave a disadvantaged school. The lower part of Table 1.4 shows that the BE, TO-BE, and 1S-BE mechanisms significantly improve the number of teachers moving closer to their spouse. TO-BE performs slightly better than 1S-BE.<sup>61</sup> On the other hand, fewer teachers from disadvantaged schools move under TO-BE and 1S-BE than under DA\*.<sup>62</sup> This lower mobility is mainly due to the fact that TO-BE slightly reduces mobility for teachers in disadvantaged regions, and 48% of the teachers who wish to leave a disadvantaged school are located in the disadvantaged regions of Créteil and Versailles.

## 1.6 Concluding Remarks

**Efficiency vs. Fairness.** In the design of school choice allocation mechanisms, the notions of efficiency and fairness have received considerable attention. In this school choice context, it is known that these two goals are incompatible (see Roth, 1982, Abdulkadiroglu and Sonmez, 2003 and Abdulkadiroglu, Pathak and Roth, 2009). Efficient matching mechanisms, such as the top trading cycle, attain efficiency but fail to be fair. Fair mechanisms such as the DA algorithm do not guarantee efficiency. This trade-off between efficiency and stability is well studied. In particular, it is well understood how to attain one objective with the minimum possible sacrifice of the other goal. For instance, DA selects a fair matching that Pareto dominates all other fair mechanisms for the proposing side (Gale and Shapley, 1962). Similarly, there is a sense in which the top trading cycle mechanism (Abdulkadiroglu and Sonmez, 2003), which allows agents to sequentially trade their priorities, satisfies efficiency with a minimal amount of unfairness (Abdulkadiroglu et al., 2015).

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<sup>61</sup>The number of teachers who move closer to their spouse increases from 251.8 under the current algorithm to 513.5 under TO-BE, 547.6 under BE, and 482.5 under 1S-BE.

<sup>62</sup>110.3 teachers obtain a new assignment under DA\* compared to 83.4 under TO-BE and 106.8 under 1S-BE.

Hence, the choice for the designer often boils down to a choice between two mechanisms: the DA mechanism in which students propose or the top trading cycle mechanism.<sup>63</sup>

In contrast, our work shows that, in the teacher assignment problem, the individually rational version of DA identified in the literature can be improved in terms of both efficiency and fairness. In addition, a mechanism closely related to the top trading cycle mechanism (TO-BE) is identified as the right choice for a designer concerned about these two core notions. This contrast with the previous literature is striking and makes clear that the teacher assignment problem is a new environment with important differences from environments previously studied.

**Many-to-one environment.** In the theoretical part of this paper, we assumed a one-to-one environment. All our results extend to the many-to-one environment except for Theorem 3.<sup>64</sup> Indeed, as shown in our Appendix 1.8.7, there are several natural ways to extend the teacher-optimal BE to a many-to-one environment. Hence, the uniqueness result in Theorem 3 does not hold in this environment. However, in the many-to-one setting, we obtain a class of two-sided maximal mechanisms that are strategy-proof, which includes all versions of TO-BE defined in our appendix. As we mentioned in the data analysis, 75% of the participants rank only one additional region in addition to their initial one. In Section 1.8.9 of the Appendix, we show that in a many-to-one framework where teachers rank only one additional school above their initial one, TO-BE is the unique strategy-proof, IR and Two-sided maximal mechanism. In addition, the theoretical performance of TO-BE and BE (compared to DA\*) identified in the one-to-one environment can be observed in our empirical results where school regions have several available seats, confirming that the main message of our paper extends to the many-to-one environment.

**Vacant positions and newly recruited teachers.** To focus on improving the reallocation of teachers who initially have an assignment, we assumed away teachers who just graduated (new comers) and available seats at schools. In the Chapter 2, we complement the current approach by incorporating vacant positions and newly recruited teachers who do not have an initial assignment. Under current practice in France, one of the most important issues is the large outgoing flow of teachers faced by the least attractive regions (with, as a consequence, a large incoming flow of

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<sup>63</sup>See also Che and Tercieux (2015a) for additional perspectives on this topic.

<sup>64</sup>Theorem 1, 2 and 5 extend in a straightforward way. All results in the large market section (Section 1.4.3) are also fairly easy to extend to a many-to-one environment. The extension of Theorem 4 is non-trivial, and the proof is provided in the supplementary material section 1.8.10.

inexperienced teachers). Importantly, in our ongoing work, we observe that, under TO-BE, this flow is dramatically reduced. The reason is simple: to leave these regions, one has to be replaced by a teacher with higher priority/experience. As one would naturally expect, demand from experienced teachers for these regions is relatively low, making it more difficult for teachers to leave these regions. While a decrease in the outgoing flow mechanically increases the number of experienced teachers in these regions, this phenomenon may have several negative features. For instance, in the first place, this may discourage experienced teachers from applying to these regions. Moreover, if newcomers are eventually trapped in such regions, this may also make the teaching profession less attractive and decrease the overall quality of teachers. Overall, we sought to remain agnostic on how much the outgoing flow has to be reduced in these regions. Hence, in this ongoing work, we tailor the current mechanism to ensure that the outgoing flow can be targeted by the decision maker. In particular, one option is to leave this flow unchanged compared to the current flow (achieved by DA\*).<sup>65</sup> Even under such a conservative approach, the overall mobility of initially assigned teachers can still be significantly increased, by 44.9%.

**Centralized vs. decentralized systems.** This study has clear policy implications for countries using a centralized assignment system. However, we would like to stress that it also helps to envision what would be the impact of a transition from a decentralized to a centralized assignment system in other countries. In particular, we show that adopting the modified version of DA, rather than one of the alternative mechanisms we suggest, would largely under-estimate the performance of a centralized system (for instance, in terms of teacher mobility).

**Dynamics and Strategy-Proofness.** One can be concerned about the dynamic incentives that teachers may have under DA\* and TO-BE. Indeed, [Pereyra \(2013\)](#) defined a dynamic overlapping generation model where newly tenured teachers arrive without any initial assignment, are assigned to a school and then ask for a reassignment during a certain number of periods before exiting the market.<sup>66</sup> In this environment, if all newly tenured teachers entering at a given date are less preferred by all schools to the tenured teachers who already have an assignment, then he showed that DA\* is dynamically strategy-proof. So no teacher can ever misreport his preferences and obtain

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<sup>65</sup>Our conversation with agents from the Ministry of Education suggests that this is one of their most desired options.

<sup>66</sup>Their preferences are fixed over time and are assumed to be a linear order over the schools. Preferences of the schools can evolve over time.

a better-school at the current or some later date.<sup>67</sup> If the TO-BE algorithm is properly extended to account for newly tenured teachers and empty seats, then it is possible to show that, in the same setting as in [Pereyra \(2013\)](#), it is also dynamically strategy-proof. In our reassignment setting with no empty seats or newly tenured teachers, if one let initially assigned teachers arrive over time, neither DA\* nor TO-BE are dynamically strategy-proof.

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<sup>67</sup>In his setting, a teacher reports a preference list only once when entering the market.

## 1.7 Tables

Table 1.1 – WELFARE OF TEACHERS UNDER DIFFERENT MECHANISMS

Choice	Init	DA*	TO-BE	BE(Init)	BE(DA*)	1S-BE(Init)	1S-BE(DA*)
1	0	422.5 (2.4)	1163.0 (14.5)	1153.5 (17.9)	1169.9 (12.9)	1355.9 (13.5)	1347.8 (11.5)
2	7935	8084.2 (2.2)	8318.8 (9.1)	8287.2 (10.2)	8290.1 (6.8)	8386.1 (11.3)	8370.9 (7.8)
3	9125	9220.5 (0.5)	9361.5 (7.1)	9336.1 (9.9)	9341.7 (7.0)	9399.1 (8.4)	9390.8 (7.1)
4	9743	9796.1 (0.7)	9901.9 (6.2)	9882.7 (6.5)	9884.8 (5.8)	9929.0 (7.9)	9917.7 (6.2)
5	10038	10077.7 (0.5)	10150.3 (5.7)	10137.1 (5.3)	10140.1 (4.3)	10170.5 (5.6)	10162.7 (4.5)
6	10271	10297.0 (0.0)	10341.0 (4.7)	10328.7 (5.0)	10331.0 (3.7)	10354.8 (4.3)	10351.3 (3.7)
7	10366	10383.5 (0.5)	10418.8 (3.6)	10409.5 (3.9)	10408.7 (3.0)	10426.4 (3.5)	10422.7 (3.1)
8	10420	10432.5 (0.5)	10459.3 (3.7)	10450.6 (3.7)	10450.7 (2.7)	10463.5 (3.3)	10461.0 (2.9)
9	10461	10474.5 (0.5)	10493.7 (3.0)	10485.9 (3.4)	10487.5 (2.2)	10495.5 (2.9)	10494.6 (2.4)
>= 10	10579	10579 (0.0)	10579 (0.0)	10579 (0.0)	10579 (0.0)	10579 (0.0)	10579 (0.0)
Nb teachers moving	0	564.7	1373.0	1461.5	1488.2	1732.5	1709.7
Min	0	560.0	1333.0	1416.0	1456.0	1696.0	1677.0
Max	0	568.0	1408.0	1517.0	1513.0	1768.0	1739.0
SD	0	2.7	14.9	17.4	12.5	15.4	12.4

<sup>†</sup> Notes: This table presents the cumulative distribution of the number of teachers who obtain school rank  $k$  under their initial assignment in column 1, under DA\* in column 2, TO-BE in column 3, BE(Init) in column 4, BE(DA\*) in column 5, 1S-BE(Init) in column 6 and 1S-BE(DA\*) in column 7. The data come from the assignment process of French teachers to regions in 2013. Simulation standard errors are reported in parenthesis.

Table 1.2 – STABILITY OF THE MATCHINGS OBTAINED WITH DIFFERENT MECHANISMS

Nb regions	Init	DA*	TO-BE	BE(Init)	BE(DA*)	1S-BE(Init)	1S-BE(DA*)
0	1980	2496.5 (2.4)	3742.7 (56.8)	3731.3 (55.4)	3799.4 (49.9)	3940.5 (56.9)	4001.2 (54.9)
1	8722	8880.9 (2.0)	9246.9 (16.5)	9215.5 (17.7)	9234.8 (17.7)	9294.2 (17.9)	9306.1 (17.7)
2	9694	9787.8 (0.4)	10004.9 (13.2)	9983.4 (14.4)	9991.9 (12.0)	10026.4 (13.3)	10035.3 (13.3)
3	10096	10149.5 (0.5)	10299.1 (8.2)	10287.5 (9.3)	10292.7 (8.0)	10309.6 (7.9)	10312.3 (7.5)
4	10323	10360.2 (0.6)	10447.0 (5.4)	10438.9 (5.8)	10444.5 (5.3)	10457.6 (5.9)	10459.3 (4.8)
>= 5	10579	10579 (0.0)	10579 (0.0)	10579 (0.0)	10579 (0.0)	10579 (0.0)	10579 (0.0)
Nb of teachers blocking with at least one region							
Mean	.	8082.5	6836.3	6847.7	6779.6	6638.5	6577.8
Min	.	8078.0	6675.0	6703.0	6665.0	6527.0	6453.0
Max	.	8087.0	7003.0	6985.0	6917.0	6809.0	6701.0
SD	.	2.4	56.8	55.4	49.9	56.9	54.9

† Notes: The upper part of this table presents the cumulative distribution of the number of regions with which teachers are blocking. The data are from the assignment process of French teachers to regions in 2013. Column 1 reports the cumulative distribution of the number of regions with which teachers block under their initial assignment. The following columns report the cumulative distribution of the number of regions with which teachers block under DA\*, TO-BE, BE(Init), BE(DA\*), 1S-BE(Init) and 1S-BE(DA\*). Simulation standard errors are reported in parenthesis.



Table 1.3 – WELFARE OF REGIONS UNDER DIFFERENT MECHANISMS

Net percentage of positions	DA*	TO-BE	BE(Init)	BE(DA*)	1S-BE(Init)	1S-BE(DA*)
-100/-91%	0.18 (0.03)	0 (0)	0 (0)	0.18 (0.03)	0.80 (0.15)	0.71 (0.13)
-90/-71%	0.18 (0.0)	0 (0)	0 (0)	0.18 (0.0)	1.06 (0.21)	0.93 (0.15)
-70/-51%	0.31 (0.03)	0 (0)	0 (0)	0.31 (0.03)	1.74 (0.26)	1.41 (0.21)
-50/-31%	0.57 (0.03)	0 (0)	0 (0)	0.50 (0.04)	3.19 (0.33)	2.57 (0.25)
-30/-1%	1.03 (0.03)	0 (0)	0 (0)	0.93 (0.05)	4.86 (0.39)	4 (0.32)
0%	84.32 (0.06)	72.01 (0.40)	71.18 (0.42)	70.90 (0.31)	72.67 (0.47)	71.81 (0.40)
1/29%	87.95 (0.02)	75.62 (0.44)	74.59 (0.50)	74.73 (0.37)	76.56 (0.52)	76.02 (0.40)
30/49%	91.05 (0.04)	79.11 (0.43)	78.02 (0.55)	77.66 (0.38)	80.01 (0.45)	79.14 (0.46)
50/69%	94.84 (0.03)	85.20 (0.38)	84.31 (0.40)	84.03 (0.38)	85.76 (0.48)	84.94 (0.43)
70/89%	97.40 (0.03)	90.32 (0.36)	89.70 (0.41)	89.04 (0.33)	89.63 (0.43)	88.82 (0.37)
90/100%	100 (0.0)	100 (0.0)	100 (0.0)	100 (0.0)	100 (0.0)	100 (0.0)
% of regions with no priority change	83.22	72.01	71.18	69.83	67.05	67.12
SD	0.08	0.40	0.42	0.32	0.41	0.33

<sup>†</sup> Note: this table presents the cumulative percentage of regions having a net welfare improvement (relative to their initial assignment). For each of the 49 subjects\*31 regions, we compute the number of positions being assigned a teacher with a higher priority, from which we subtract the number of positions being assigned a teacher with a lower priority. Then, for each subject\*region, the net total is divided by the total number of positions to obtain the percentage of positions being improved in net terms. Finally, the total number of regions considered is divided by  $49 \times 31$  to obtain the average percentages of regions. For instance, on average, under the DA\*, 0.18% of the regions have between 91% and 100% of their seats assigned a teacher with a lower priority (in net terms). Simulation standard errors are reported in parenthesis.

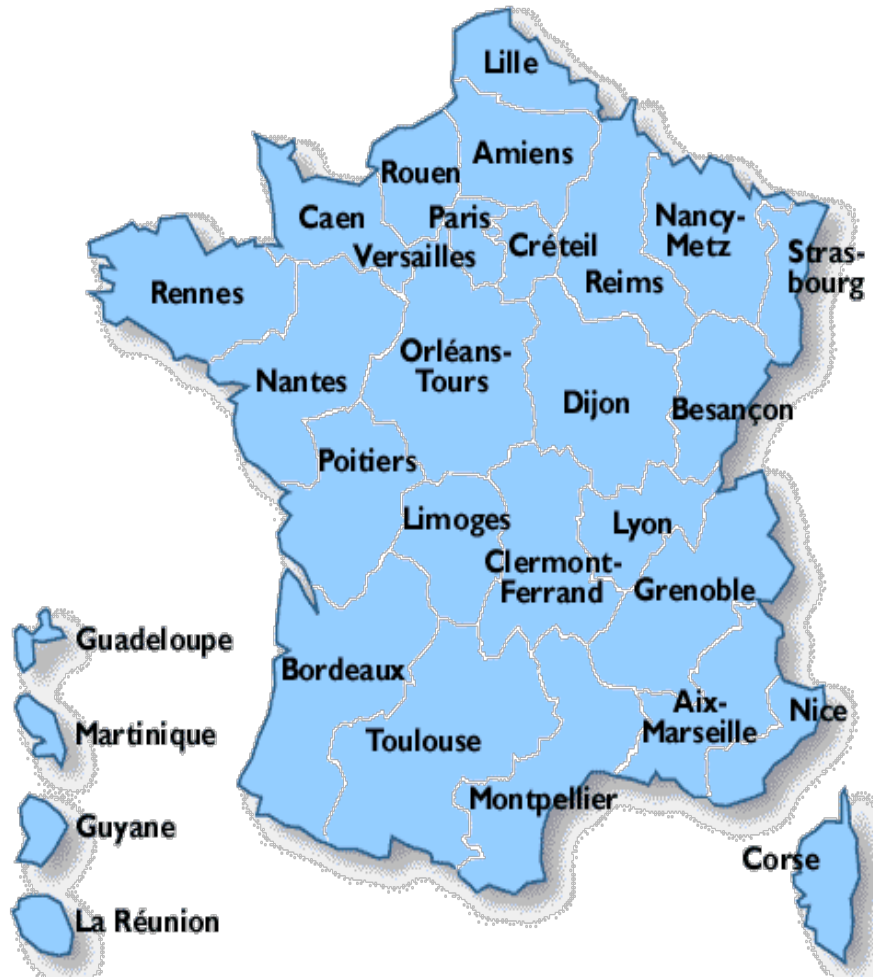
Table 1.4 – FULFILLMENT OF THE ADMINISTRATION’S OBJECTIVES

	DA*	TO-BE	BE(Init)	1S-BE(Init)
Share of inexperienced teachers				
Non-disadvantaged regions	26.88 (0.0)	27.50 (0.08)	27.66 (0.09)	27.42 (0.11)
Disadvantaged regions	45.66 (0.0)	44.99 (0.08)	44.82 (0.10)	45.07(0.11)
Share of inexperienced teachers (among incoming teachers)				
Non-disadvantaged regions	17.57 (0.07)	24.40 (0.65)	26.05 (0.74)	21.14 (0.61)
Disadvantaged regions	28.56 (0.08)	29.00 (1.34)	30.09 (1.19)	30.98 (0.98)
Share of inexperienced teachers (among exiting teachers)				
Non-disadvantaged regions	18.36 (0.07)	21.87 (0.56)	22.94 (0.63)	19.36 (0.51)
Disadvantaged regions	26.94 (0.07)	48.35 (2.78)	50.73 (2.61)	42.89 (2.68)
Nb spousal reunification	251.8 (1.48)	513.5 (10.11)	547.6 (12.01)	482.5 (10.55)
Nb teacher from disadvantaged school	110.3 (0.78)	83.4 (4.85)	90.4 (5.36)	106.8 (5.08)

† Note: The upper part of this table presents the share of teachers who have only one or two years of experience under the current algorithm (in column 1), TO-BE, BE, and 1S-BE. We successively reports statistics for all teachers, for incoming teachers only, and for exiting teachers. The last two rows of the table present statistics on the number of teachers who move closer to their spouse, and who leave a disadvantaged school. Simulation standard errors are reported in parenthesis.

## 1.8 Appendix

### 1.8.1 Map of French administrative regions



### 1.8.2 Simulations

In this section, we report a certain number of simulations. Our goal here is to argue that, in a realistic setting where a significant fraction of teachers are newcomers and where each school has a significant fraction of available seats, the effect identified in Example 1 can occur quite often. In order to do so, we consider the following environment. We have 600 teachers and 30 schools with 20 seats each; 300 teachers are newcomers, while the other half of teachers initially have an assignment. We assume that the total number of seats equals the total number of teachers so that half of the seats of each school are occupied and the other half are open seats.

We randomly draw the utility of teacher  $t$  for school  $s$  as follows:

$$U_t(s) = u_s + \xi_{st}$$

where  $u_s$  is the common value of school  $s$  drawn uniformly over the interval  $[0, a]$  and  $\xi_{st}$  is the idiosyncratic shock drawn uniformly over the interval  $[0, 1]$ . Parameter  $a$  measures the degree of correlation in teachers' preferences. For schools' preferences, we have a similar specification:

$$V_s(t) = v_t + \eta_{st}.$$

We assume that both common values ( $v_t$ ) and idiosyncratic shocks ( $\eta_{st}$ ) are drawn uniformly in  $[0, 1]$ .

Finally, given the dynamic nature of the teacher assignment process, teachers initially assigned good schools have a higher priority on average than teachers assigned poorer schools. In order to take this into account, we assume that the initial assignment is assortative: teachers with high common values are initially matched to schools with high common values. Formally, we assume that the 10 teachers  $t$  having a common value  $v_t$  within the 10 highest are matched to the school  $s$  with the highest common value  $u_s$ . Similarly, the next 10 teachers having a common value within the next 10 highest are matched to the school with the second highest common value, and so on.

We draw teachers' and schools' preferences 150 times. For each draw, we compute the outcome of  $DA^*$  and an outcome on the Pareto frontier, which Pareto-dominates  $DA^*$ .<sup>68</sup> The following table reports the number (averaged across iterations) of teachers staying at their initial assignment for both  $DA^*$  and the Pareto-dominating matching for several possible values of the correlation parameter  $a$ .<sup>69</sup>

In many instances,  $DA^*$  can be improved both in terms of efficiency and stability, as in Example 1. The intuition is essentially the same: within top schools (i.e., for instance, the schools corresponding to the ten highest common values), open seats are filled very quickly by teachers initially matched to these top schools and partly by teachers initially matched to poorer schools. Hence, at some

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<sup>68</sup>More precisely, in order to find the Pareto-dominating outcome, we run TO-BE starting from the allocation given by  $DA^*$ . See Section 1.4.1 for the definition of TO-BE.

<sup>69</sup>In our environment, teachers always prefer being matched to being unmatched. Given that the total number of seats equals the number of teachers, all teachers who are initially unmatched will end up being matched under  $DA^*$  and therefore under the Pareto-dominating matching.

Table 1.5 – NUMBER OF TEACHERS STAYING AT THEIR INITIAL ASSIGNMENT: MEAN AND STANDARD DEVIATION OVER 150 ITERATIONS.

	DA*	SD	TOBE(DA*)	SD
[0, 0.1]	22.49	4.76	22.22	4.64
[0, 0.5]	53.94	7.60	42.58	6.17
[0, 15]	185.22	7.29	147.73	11.26
[0, 30]	188.63	6.03	167.84	8.95
[0, 60]	188.37	6.70	177.58	8.73
[0, 100]	189.72	6.11	182.32	7.99
[0, 1000]	188.81	6.58	188.11	6.79

point, we return to the context of the example where teachers initially matched to good schools are willing to move to other good schools but cannot do so because of the large set of remaining tenured teachers – initially matched to poorer schools – among whom some teachers may have a higher priority at the schools they are targeting.

The simulations reveal that the scope of the improvement can potentially be large depending on the precise value of the correlation parameter  $a$  (it is maximized for intermediate values of the parameter).

### 1.8.3 Proof of Theorem 3

We want to prove the following proposition.

**Proposition 1.8.1.** *Let  $\varphi$  be any selection of BE. If  $\varphi \neq TO\text{-}BE$  then  $\varphi$  is not strategy-proof.*

**Lemma 3.** *Let  $\varphi$  be any selection of BE. Fix any profile of preferences  $\succ$  and assume that  $\varphi(\succ) \neq TO\text{-}BE(\succ)$ . Let  $x$  be the outcome of  $TO\text{-}BE(\succ)$  and let  $y$  be that of  $\varphi(\succ)$ . There exists  $t$  s.t.  $x(t) \succ_t y(t) \succ_t \mu_0(t)$ .*

*Proof.* Let  $T(x, y)$  be the set of teachers for which  $x(t) \succ_t y(t) \succeq_t \mu_0(t)$ . We know that  $x$  is not 1-Pareto-dominated by  $y$  (by Proposition 1.4.1) and since  $y$  is individually rational and  $x \neq y$ , we must have  $T(x, y) \neq \emptyset$ . Proceed by contradiction and assume that for all  $t \in T(x, y)$ , we have  $y(t) = \mu_0(t)$ . Let  $B := T \setminus T(x, y)$ . Note that for any  $t \in B$ ,  $y(t)$  is a school initially assigned to some teacher in  $B$ . In addition, by definition, for all  $t \in B$ ,  $y(t) \succeq_t x(t)$ . If there was no teacher  $t \in B$  for which  $y(t) \succ_t x(t)$ , then we would have the following situation:  $y$  would select the initial allocation for all  $t \in T(x, y)$  and would be identical to  $x$  for all  $t \notin T(x, y)$ . Given that  $x \neq y$ , we must have  $x(t) \neq y(t) = \mu_0(t)$  for some  $t \in T(x, y)$ . Since  $x$  is individually rational, we

have  $x(t) \succ_t y(t) = \mu_0(t)$  for those  $t \in T(x, y)$ . Hence  $x$  1-Pareto-dominates  $y$ . But all schools are also better-off under  $x$  rather than under  $y$ . Indeed, for each school  $s$  s.t.  $y(s) \notin T(x, y)$ ,  $y(s) = x(s)$  and for each school  $s$  s.t.  $y(s) \in T(x, y)$ , because  $x$  is individually rational on both sides,  $x(s) \succeq_s y(s) = \mu_0(s)$  with a strict inequality for  $s$  satisfying  $x(s) \neq y(s)$  (and such a  $s$  must exist since  $x \neq y$ ). Thus,  $x$  is individually rational on both sides and 2-Pareto-dominates  $y$ , which is not possible given that  $y$  is an outcome of BE.

To recap, we have that for any  $t \in B$ ,  $y(t)$  is a school initially assigned to some teacher in  $B$  and for all  $t \in B$ ,  $y(t) \succeq_t x(t)$  with a strict inequality for some  $t \in B$ . In addition, since  $y$  is the outcome of  $\varphi(\succ)$  and  $\varphi$  2-Pareto-dominates the initial allocation  $\mu_0$ , we must have that for all school  $s$ ,  $y(s) \succeq_s \mu_0(s)$ . Hence,  $B$  is a two-sided blocking coalition for  $x$ , which is a contradiction since  $x$  must be a point in the two-sided Core.  $\square$

**Proof of Proposition 1.8.1.** We start from a profile of preferences  $\succ$  under which  $\varphi(\succ) \neq \text{TO-BE}(\succ)$  which must exist by our assumption that  $\varphi \neq \text{TO-BE}$ . Given our profile of preferences  $\succ$ , we let the profile of preferences  $\succ'$  be defined as follows. For any  $t$ , any school  $s$  other than  $\text{TO-BE}(\succ)[t]$  are ranked as unacceptable for  $t$  under  $\succ'$ . We must have  $\text{TO-BE}(\succ) = \text{TO-BE}(\succ')$ . Now, we are in a position to prove the following lemma.

**Lemma 4.**  $\text{TO-BE}(\succ') = \varphi(\succ')$ .

*Proof.* Suppose  $x := \text{TO-BE}(\succ') \neq \varphi(\succ') =: y$ . By the above lemma, there exists  $t$  s.t.  $x(t) \succ'_t y(t) \succ'_t \mu_0(t)$  which yields a contradiction, by construction of  $\succ'_t$ .  $\square$

Note that TO-BE satisfies also the following property: for any profile of preferences  $\succ$ , for any teacher  $t$ ,  $\text{TO-BE}(\succ)(t) = \text{TO-BE}(\succ_{-t}, \succ'_t)(t)$ . This will be used in the following lemma.

**Lemma 5.** If  $\varphi$  is strategy-proof then  $\text{TO-BE}(\succ_Z, \succ'_{-Z}) = \varphi(\succ_Z, \succ'_{-Z})$  for any  $Z \subseteq T$ .

*Proof.* Assume  $\varphi$  is strategy-proof. The proof is by induction on the size of  $Z$ . For  $|Z| = 0$ , the result is given by the previous lemma. Now, the induction hypothesis is that  $\text{TO-BE}(\succ_Z, \succ'_{-Z}) = \varphi(\succ_Z, \succ'_{-Z})$  for any subset  $Z$  with  $|Z| = k$ . Proceed by contradiction and suppose that there is  $Z$  s.t.  $|Z| = k + 1$  for which  $x := \text{TO-BE}(\succ_Z, \succ'_{-Z}) \neq \varphi(\succ_Z, \succ'_{-Z}) =: y$ . By the first lemma above, there exists  $t$  s.t.  $\text{TO-BE}(\succ_Z, \succ'_{-Z})(t) \triangleright_t \varphi(\succ_Z, \succ'_{-Z})(t) \triangleright_t \mu_0(t)$  where  $\triangleright_t = \succ'_t$  if  $t \notin Z$  while  $\triangleright_t = \succ_t$  otherwise. If  $t \notin Z$ , then there is a straightforward contradiction since under  $\succ'_t$  there is a single school

which is ranked above  $\mu_0(t)$  for teacher  $t$ . Now, assume that  $t \in Z$ . By the property noticed just before the statement of the lemma, we must have  $\text{TO-BE}(\succ_{Z \setminus \{t\}}, \succ'_{-Z}, \succ'_t)(t) = \text{TO-BE}(\succ_Z, \succ'_{-Z})(t)$  and, by our induction hypothesis,  $\varphi(\succ_{Z \setminus \{t\}}, \succ'_{-Z}, \succ'_t)(t) = \text{TO-BE}(\succ_{Z \setminus \{t\}}, \succ'_{-Z}, \succ'_t)(t)$ . Thus, we obtain  $\varphi(\succ_{Z \setminus \{t\}}, \succ'_{-Z}, \succ'_t)(t) = \text{TO-BE}(\succ_{Z \setminus \{t\}}, \succ'_{-Z}, \succ'_t)(t) = \text{TO-BE}(\succ_Z, \succ'_{-Z})(t) \succ_t \varphi(\succ_Z, \succ'_{-Z})(t)$  which is a contradiction with the assumption that  $\varphi$  is strategy-proof (indeed, at  $(\succ_Z, \succ'_{-Z})$ , teacher  $t \in Z$  has an incentive to report  $\succ'_t$  instead of  $\succ_t$ ).  $\square$

Taking  $Z = T$  in the above lemma, given that  $\varphi(\succ) \neq \text{TO-BE}(\succ)$ , we obtain the following corollary which completes the proof of our proposition.

**Corollary 2.**  *$\varphi$  is not strategy-proof.*

■

#### 1.8.4 Proof of Theorem 4

In the sequel, we prove our characterization result of one-sided maximal matchings given in Theorem 4. Our proof is divided into two parts. We start by showing that any outcome of the 1S-BE algorithm is a one-sided maximal matching (Section 1.8.4.1):

**Proposition 1.8.2.** *If  $\mu$  is an outcome of the 1S-BE algorithm then  $\mu$  is one-sided maximal.*

Then, we move to the proof that any one-sided maximal matching corresponds to a possible outcome of the 1S-BE algorithm (Section 1.8.4.2):

**Proposition 1.8.3.** *If  $\mu$  is one-sided maximal then  $\mu$  is an outcome of the 1S-BE algorithm.*

##### 1.8.4.1 Proof of Proposition 1.8.2

Before moving to the proof we introduce a new notation. Given matching  $\mu$ , we denote  $\mathcal{B}_\mu$  for the set of blocking pairs of  $\mu$ .

In the sequel, we fix two matchings  $\mu$  and  $\mu'$  such that  $\mu'$  Pareto-dominates  $\mu$  for teachers and  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_\mu$ . We show below that starting from  $\mu$ , the graph associated to the 1S-BE algorithm must have a cycle. Hence, any outcome of 1S-BE must be one-sided maximal, as claimed in Proposition 1.8.2.

To give the intuition of each step of the proof, which uses a lot of graphical arguments, we will use an example to illustrate each part. This example involves 6 teachers,  $t_1, t_2, t_3, t_4, t_5, t_6$  and 6 schools  $s_1, s_2, s_3, s_4, s_5, s_6$ . In the example, matchings  $\mu$  and  $\mu'$  are as follows:

$$\mu = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \\ s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \end{pmatrix}$$

$$\mu' = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \\ s_2 & s_3 & s_4 & s_5 & s_6 & s_1 \end{pmatrix}$$

As in Lemma 1, we can exhibit “cycles of exchanges” which can be used to go from  $\mu$  to  $\mu'$  in the proposition.

In Lemma 1, these cycles of exchanges were actual cycles in the graph associated with BE. However, when considering the graph associated with 1S-BE, this is not the case anymore: the cycles of exchanges are not necessarily cycles of the graph associated with 1S-BE. Before moving to the first lemma, we note that all the nodes that are not part of cycles of exchanges are those where the teacher of that node has the same allocation between  $\mu$  and  $\mu'$ . In the following the “nodes of the cycles of exchanges” will be all the nodes  $(t, s)$  s.t  $\mu(t) \neq \mu'(t)$ . We will say that a node  $(t, s)$  1S-BE-points to another node  $(t', s')$  if  $(t, s)$  points toward  $(t', s')$  in the graph associated with the 1S-BE algorithm (starting from  $\mu$ ).

**Lemma 6.** *Fix a node  $(t, s)$  of the cycles of exchanges. Then:*

1. *either its predecessor according to the cycles of exchanges 1S-BE-points toward  $(t, s)$ ;*
2. *or there is a node  $(t', s')$  in the cycles of exchanges such that  $t'$  does not block with  $s$  under  $\mu$ ,  $s \succ_{t'} s'$  and  $t'$  has the highest priority among those who desire  $s$  but do not block with it under  $\mu$ . And so  $(t', s')$  1S-BE-points toward  $(t, s)$ .*

Before moving to the proof, let us illustrate this lemma in the example. Assume that all the nodes except  $(t_3, s_3)$  are 1S-BE-pointed by their predecessors in the cycle of exchanges. According to Lemma 6 there must be a node  $(t', s')$  in the cycle of exchanges which 1S-BE-points toward  $(t_3, s_3)$ . In the graph of Figure 1.1, this node is assumed to be  $(t_5, s_5)$ . The dashed edge from



$(t_2, s_2)$  to  $(t_3, s_3)$  is here to show that this is not an edge of the 1S-BE graph but it is only an edge corresponding to the cycle of exchanges.

*Proof.* Call  $(t'', s'')$  the predecessor of node  $(t, s)$  in the cycles of exchanges so that  $s'' := \mu(t'')$  and  $s := \mu'(t'')$ . Because  $\mu'$  Pareto-dominates  $\mu$  for teachers, we know that  $s \succ_{\mu'} s''$  so that  $t''$  desires  $s$  under  $\mu$ . Assume that  $(t'', s'')$  does not 1S-BE-point to  $(t, s)$ . This means that  $t''$  does not block with  $s$  under  $\mu$  and that there is another teacher  $t'$  who does not block with  $s$  and has the highest priority among those who desire  $s$  and do not block with it. Thus,  $(t', s')$  (where  $s' := \mu(t')$ ) 1S-BE points toward  $(t, s)$ . It remains to show that  $(t', s')$  is part of the cycles of exchanges. If it was not the case, it would mean that  $\mu(t') = \mu'(t') = s'$ . Let us recap. We have that  $t'$  does not block with  $s$  under  $\mu$ . In addition, by definition of  $t'$ , we must have that  $t' \succ_s t''$  (since  $t''$  does not block with  $s$  under  $\mu$  and desires  $s$ ). In addition,  $t'$  desires  $s$  under  $\mu$  and so  $\mu(t') = \mu'(t')$  implies that  $t'$  also desires  $s$  under  $\mu'$ . Hence, because  $t'' \in \mu'(s)$ , we obtain that  $t'$  blocks with  $s$  under  $\mu'$ . This contradicts our assumption that  $\mathcal{B}_{\mu'} = \mathcal{B}_{\mu}$ .  $\square$

Lemma 6 allows us to identify a subgraph  $(N', E')$  of the 1S-BE graph starting from  $\mu$  such that  $N'$  are the nodes of the cycles of exchanges and the set of edges  $E'$  is built as follows. We start from  $E' = \emptyset$  and we add the following edges: for each node  $(t, s)$  in the cycles of exchange, if its predecessor  $(\tilde{t}, \tilde{s})$  under the cycles of exchanges 1S-BE-points to  $(t, s)$  then  $((\tilde{t}, \tilde{s}), (t, s))$  is added to  $E'$ . If on the contrary,  $(\tilde{t}, \tilde{s})$  does not 1S-BE-points to  $(t, s)$ , then we pick the node  $(t', s')$  in the cycles of exchanges identified in the second condition of Lemma 6 which 1S-BE-points toward  $(t, s)$  and we add  $((t', s'), (t, s))$  to  $E'$ . Note that, by construction, each node in  $N'$  has a unique in-going edge in  $(N', E')$ . In the example, this subgraph  $(N', E')$  is given by the right graph of Figure 1.1 (the solid arrows). Note that this graph admits a cycle:  $(t_3, s_3) \rightarrow (t_4, s_4) \rightarrow (t_5, s_5) \rightarrow (t_3, s_3)$ . This is a simple property of digraphs with in-degree one:

**Lemma 7.** *Fix a finite digraph  $(N, E)$  such that each node has in-degree one. There exists a cycle in this graph.*

*Proof.* Fix a node  $n_1$  in the graph  $(N, E)$ . Because it has in-degree one, we can let  $n_2$  be the unique node pointing to  $n_1$ . Again from  $n_2$  we can let  $n_3$  be the unique node pointing to  $n_2$ . Because there is a finite number of nodes in the graph, this process must cycle at some point.  $\square$

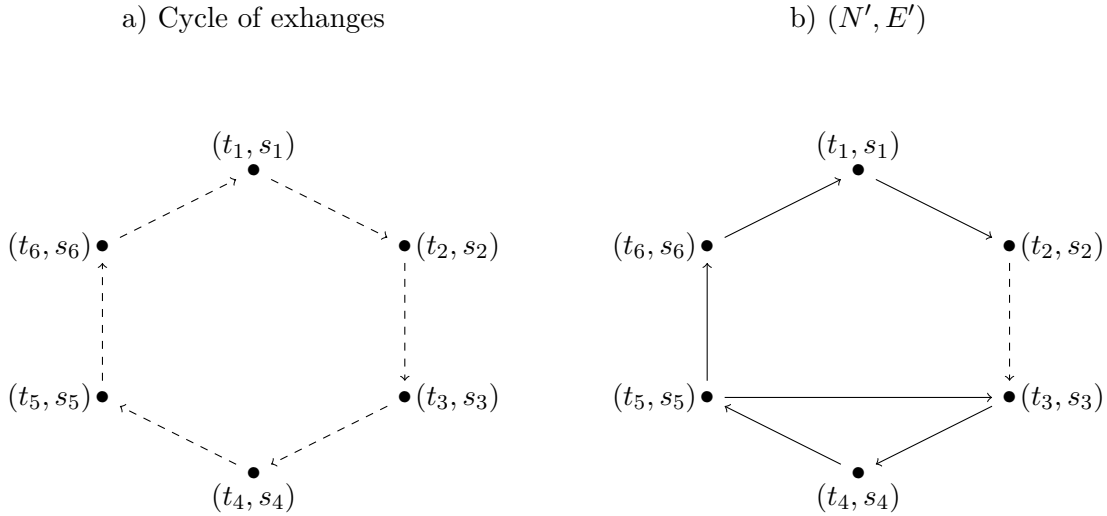
As the example illustrates, applying this lemma to  $(N', E')$  leads to the following corollary:

**Corollary 3.** *There is a cycle in the subgraph  $(N', E')$ .*

We are now in a position to prove Proposition 1.8.2.

*Completion of the proof of Proposition 1.8.2.* Let  $\mu$  be an outcome of the 1S-BE algorithm. Proceed by contradiction and assume that  $\mu$  is not one-sided maximal. Thus, there must exist a matching  $\mu'$  such that  $\mu'$  Pareto-dominates  $\mu$  for teachers and  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu}$ . Corollary 3 implies that there must be a cycle in the graph associated with 1S-BE starting from  $\mu$ , contradicting the fact that  $\mu$  is an outcome of 1S-BE.  $\square$

Figure 1.1 – CYCLES OF EXCHANGES AND  $(N', E')$ .



### 1.8.4.2 Proof of Proposition 1.8.3

In the sequel, we fix a one sided maximal matching  $\mu'$ . We let  $\mu$  be a matching such that  $\mu'$  Pareto-dominates for teachers  $\mu$  and satisfies  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu}$ . We claim that there is a cycle in the graph associated with 1S-BE starting from  $\mu$  which, once implemented, leads to a matching  $\tilde{\mu}$  such that  $\mu'$  Pareto-dominates  $\tilde{\mu}$  for teachers and satisfies  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\tilde{\mu}}$ . Note that this implies Proposition 1.8.3. Indeed, because, by definition,  $\mu'$  Pareto-dominates  $\mu_0$  and  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu_0}$ , we must have a cycle in the graph associated with 1S-BE starting from  $\mu_0$ , which once implemented, yields to a matching say  $\tilde{\mu}_1$  such that  $\mu'$  Pareto-dominates  $\tilde{\mu}_1$  for teachers and satisfies  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\tilde{\mu}_1}$ . Now, we can iterate the

reasoning and we get again that there is a cycle in the graph associated with 1S-BE starting from  $\tilde{\mu}_1$ , which, once implemented, yields to a matching say  $\tilde{\mu}_2$  such that  $\mu'$  Pareto-dominates  $\tilde{\mu}_2$  for teachers and satisfies  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\tilde{\mu}_2}$ . We can pursue this reasoning. At some point, because the environment is finite, we must reach matching  $\mu'$ , as was to be shown.

In the sequel, as in the proof of Proposition 1.8.2, we consider the digraph  $(N', E')$  as built in Section 1.8.4 after Lemma 6. Consider a cycle  $\tilde{C}$  in this graph (which exists by Lemma 7). Let  $\tilde{\mu}$  be the matching obtained once the cycle  $\tilde{C}$  is implemented. In the example introduced in Section 1.8.4, this matching would be:

$$\mu_1 = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \\ s_1 & s_2 & s_4 & s_5 & s_3 & s_6 \end{pmatrix}$$

We first show the following lemma:

**Lemma 8.**  $\mu'$  Pareto-dominates  $\tilde{\mu}$  for teachers.

*Proof.* Fix a teacher  $t$ . If the node  $(t, s)$  to which  $t$  belongs is not part of the cycles of exchanges, we know  $t$  does not move from  $\mu$  to  $\mu'$  and so  $(t, s)$  is not in the cycle  $\tilde{C}$ . Hence,  $\mu(t) = \tilde{\mu}(t) = \mu'(t)$ . So assume that  $(t, s)$  is part of the cycles of exchanges and let  $s := \mu(t)$  and  $s' := \mu'(t)$  with  $s \neq s'$ . There are three possible cases:

- *Case 1:*  $s = \tilde{\mu}(t) \neq s'$ . Because  $\mu'$  Pareto-dominates  $\mu$  for teachers, we have that  $\mu'(t) = s' \succeq_t \tilde{\mu}(t) = \mu(t) = s$ .
- *Case 2:*  $s \neq \tilde{\mu}(t) = s'$ . In such a case, we trivially have  $\mu'(t) \succeq_t \tilde{\mu}(t)$ .
- *Case 3:*  $s \neq \tilde{\mu}(t) := s_1 \neq s'$ . By construction of the graph  $(N', E')$  when we implement cycle  $\tilde{C}$ , we know that there is a unique edge  $((t, s), (t_1, s_1))$  in  $\tilde{C}$  and that  $(t, s)$  is not the predecessor of  $(t_1, s_1)$  under the cycles of exchanges, since otherwise,  $t$  would be matched to  $s'$  under  $\tilde{\mu}$  which is not the case by assumption. Hence, by construction of  $(N', E')$ , the predecessor of  $(t_1, s_1)$  under the cycles of exchanges, say  $(t'', s'')$ , does not 1S-BE point to  $(t_1, s_1)$  and, in addition,  $t$  does not block with  $s_1$  under  $\mu$ ,  $s_1 \succ_t s$  and  $t$  has the highest priority among those who desire  $s_1$  but do not block with it under  $\mu$  and 1S-BE-points to  $(t_1, s_1)$ . Because  $(t'', s'')$  does not 1S-BE point to  $(t_1, s_1)$ , we know that  $t''$  does not block with  $s_1$ . While because  $(t'', s'')$  points to  $(t_1, s_1)$  under the cycles of exchange, we must have that  $t''$  desires  $s_1$ . Thus, we conclude that  $t \succ_{s_1} t''$ .

Now, proceed by contradiction and assume that  $(\tilde{\mu}(t) =)_{s_1} \succ_t s' (= \mu'(t))$ . Because  $t'' \in \mu'(s_1)$  (recall that  $(t'', s'')$  is the predecessor of  $(t_1, s_1)$  under the cycles of exchange) and  $t \succ_{s_1} t''$ , we have that  $t$  blocks with  $s_1$  under  $\mu'$  i.e.  $(t, s_1) \in \mathcal{B}_{\mu'}$ . But, as already claimed,  $(t, s_1) \notin \mathcal{B}_{\mu}$ . This contradicts that  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu}$ . Thus, we must have  $\mu'(t) \succeq_t \tilde{\mu}(t)$ .<sup>70</sup>

So we have shown that  $\forall t, \mu'(t) \succeq_t \tilde{\mu}(t)$ . □

The following lemma completes the argument.<sup>71</sup>

**Lemma 9.** *We have that  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\tilde{\mu}}$*

*Proof.* The proof proceeds by contradiction. Assume that there is a teacher  $t$  and a school  $s$  s.t  $(t, s) \in \mathcal{B}_{\mu'}$  but  $(t, s) \notin \mathcal{B}_{\tilde{\mu}}$ . Note first that teacher  $t$  desires  $s$  under  $\tilde{\mu}$  because  $s \succ_t \mu'(t)$  (by  $(t, s) \in \mathcal{B}_{\mu'}$ ) and  $\mu'(t) \succeq_t \tilde{\mu}(t)$  (by Lemma 8). So because  $(t, s) \notin \mathcal{B}_{\tilde{\mu}}$  we must have that  $\tilde{t} \succ_s t$  for  $\tilde{t} := \tilde{\mu}(s)$ . Since  $(t, s) \in \mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu}$ , we know that  $t$  blocks with  $s$  under  $\mu$  and so, since we are in a one-to-one setting,  $\mu(\tilde{t}) \neq s$ . This implies that  $(\tilde{t}, \mu(\tilde{t}))$  is part of cycle  $\tilde{C}$ . Since  $(t, s) \in \mathcal{B}_{\mu'}$ , we also know that  $\mu'(\tilde{t}) \neq s$ . So to recap, we have that  $\mu(\tilde{t}) \neq \tilde{\mu}(\tilde{t}) \neq \mu'(\tilde{t})$ . But, this means that in the graph  $(N', E')$ ,  $(\tilde{t}, \mu(\tilde{t}))$  points to  $(s, \mu(s))$  while  $(\tilde{t}, \mu(\tilde{t}))$  is not the predecessor of  $(s, \mu(s))$  in the graph of exchanges. By construction of  $(N', E')$  this means that  $\tilde{t}$  does not block with  $s$  under  $\mu$  and has the highest priority among teachers who desire  $s$  under  $\mu$  and do not block with it under  $\mu$ . In particular, because  $\tilde{t}$  does not block with  $s$  under  $\mu$  (but desires it under  $\mu$ ) we must have  $\mu(s) \succ_s \tilde{t}$ . In addition, since  $(t, s) \in \mathcal{B}_{\mu}$ , we must have  $t \succ_s \mu(s) \succ_s \tilde{t}$ , contradicting that  $\tilde{t} \succ_s t$ . □

### 1.8.5 Proof of Theorem 5

In order to prove this result, we exhibit an instance where, irrespective of which (sequence of) cycle(s) one selects in the graphs associated with 1S-BE, one teacher will gain by misreporting his preferences. Assume that there are five teachers  $t_1, \dots, t_5$  and five schools  $s_1, \dots, s_5$ . Teachers' and

<sup>70</sup>Case 3 in Lemma 8 can be illustrated in the example. The node  $(t, s)$  would be  $(t_5, s_5)$  in the right graph of Figure 1.1.  $t_5$  is matched to  $s_3$  under  $\tilde{\mu}$  but is matched to  $s_6$  under  $\mu'$ . Under  $\tilde{C}$  (i.e.,  $(t_3, s_3) \rightarrow (t_4, s_4) \rightarrow (t_5, s_5) \rightarrow (t_3, s_3)$ ), node  $(t_5, s_5)$  points to  $(t_3, s_3)$  while  $(t_2, s_2)$  does not 1S-BE-point to  $(t_3, s_3)$ . Because  $(t_2, s_2)$  points to  $(t_3, s_3)$  in the cycle of exchanges, it means that  $t_2 \in \mu'(s_3)$  so that if  $t_5$  preferred  $s_3$  to his match under  $\mu'$ ,  $s_2$ , it would imply that  $t_5$  blocks with  $s_3$  under  $\mu'$  while he does not under  $\mu$  and so this would yield the contradiction.

<sup>71</sup>Note that up to here, all the arguments we provided can be applied to the many-to-one environment. However, the following Lemma explicitly uses the one-to-one environment and is not true anymore in many-to-one. However, we can use additional arguments to show that Proposition 1.8.3 goes through in the many-to-one setting. This is provided in Section 1.8.10 of the supplementary material.

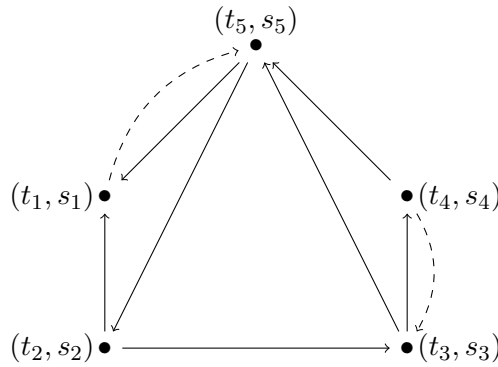
schools' preferences are given as follows:

$$\begin{array}{ll}
 \succ_{t_1}: s_5 & s_1 & \succ_{s_1}: t_5 & t_2 & t_1 \\
 \succ_{t_2}: s_1 & s_3 & s_2 & \succ_{s_2}: t_5 & t_2 \\
 \succ_{t_3}: s_4 & s_5 & s_3 & \succ_{s_3}: t_3 & t_2 & t_4 \\
 \succ_{t_4}: s_5 & s_3 & s_4 & \succ_{s_4}: t_3 & t_4 \\
 \succ_{t_5}: s_2 & s_1 & s_5 & \succ_{s_5}: t_4 & t_2 & t_5 & t_3 & t_1
 \end{array}$$

We let  $\succ := (\succ_{t_1}, \dots, \succ_{t_5})$ . The initial assignment is given by:

$$\mu_0 = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 & t_5 \\ s_1 & s_2 & s_3 & s_4 & s_5 \end{pmatrix}$$

Starting from the initial assignment, the solid arrows in the graph below correspond to the graph associated with 1S-BE.



We added dashed arrows from one node to another if the teacher in the origin of the arrow prefers the school in the pointed node. These arrows are not actual arrows of the graph associated with 1S-BE and so cannot be used to select a cycle. These arrows are just here to facilitate the understanding of the argument.

When  $\succ$  is submitted, there are two possible choices of cycles in the graph:

- A “large” cycle given by:  $(t_2, s_2) \rightarrow (t_3, s_3) \rightarrow (t_4, s_4) \rightarrow (t_5, s_5) \rightarrow (t_2, s_2)$ . Denote this cycle by  $\bar{C}$ .
- A “small” cycle given by:  $(t_2, s_2) \rightarrow (t_3, s_3) \rightarrow (t_5, s_5) \rightarrow (t_2, s_2)$ . Denote this cycle by  $\underline{C}$ .

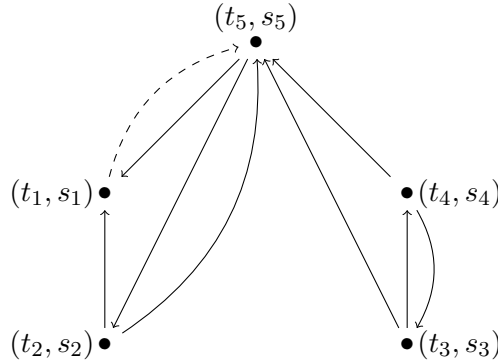
So we decompose the analysis for these two cases.

Case A: Under  $\succ$ ,  $\bar{C}$  is selected:

Once this cycle is cleared, there are no cycles left in the graph associated with 1S-BE and the final matching of 1S-BE is given by:

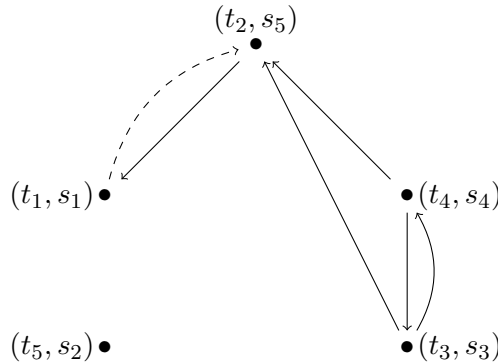
$$\bar{\mu} = \begin{pmatrix} t_1 & t_2 & t_3 & t_4 & t_5 \\ s_1 & s_3 & s_4 & s_5 & s_2 \end{pmatrix}$$

Now, assume that teacher  $t_2$  reports the following preference relation:  $\succ'_{t_2}: s_1, s_5, s_2$  while others report according to  $\succ$ . Under this profile, starting from the initial assignment, the graph associated with 1S-BE is:



Now, there are two possible choices of cycles.

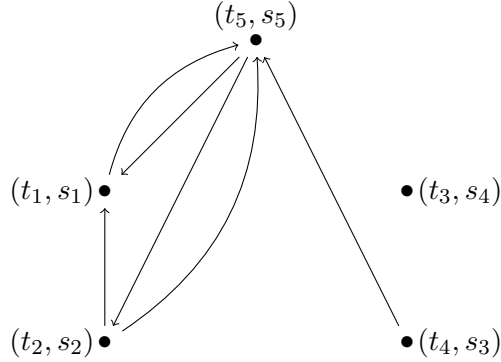
*Case A.1:* The cycle chosen is  $(t_2, s_2) \Leftrightarrow (t_5, s_5)$ . Once carried out, the graph associated with 1S-BE starting from the new matching is:



Clearly, there is a unique cycle  $(t_4, s_4) \Leftrightarrow (t_3, s_3)$ . Consider the new matching once this cycle is implemented. Teacher  $t_3$  obtains his most favorite school. Hence, in the graph associated with

1S-BE starting from the new matching, node  $(t_1, s_1)$  will now point to node  $(t_2, s_5)$ . In this graph, the only cycle is  $(t_2, s_5) \rightleftharpoons (t_1, s_1)$  and so  $t_2$  is eventually matched to school  $s_1$ . Hence,  $t_2$  obtains his most preferred school under  $\succ_{t_2}$  and so we exhibited a profitable misreport.

*Case A.2:* The cycle chosen is  $(t_4, s_4) \rightleftharpoons (t_3, s_3)$ . Once carried out, the graph associated with 1S-BE starting from the new matching is:



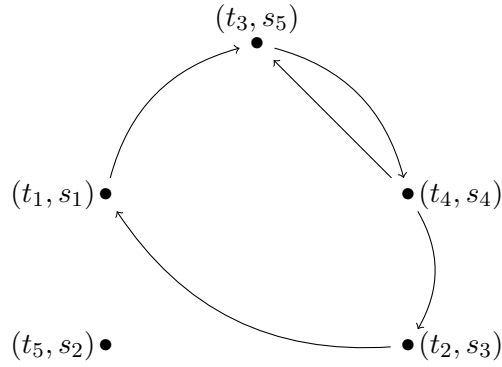
In this graph, there are three possible choices of cycles:

1.  $(t_2, s_2) \rightarrow (t_1, s_1) \rightarrow (t_5, s_5) \rightarrow (t_2, s_2)$ : in that case  $t_2$  is matched to  $s_1$  and so, again, we identified a profitable misreport.
2.  $(t_2, s_2) \rightleftharpoons (t_5, s_5)$ : Once cleared, the only cycle that is left is  $(t_1, s_1) \rightleftharpoons (t_2, s_5)$  and so  $t_2$  will be matched to  $s_1$  leading to a successful manipulation.
3.  $(t_1, s_1) \rightleftharpoons (t_5, s_5)$ : Once cleared, since  $t_5$  prefers  $s_2$  to  $s_1$  there is a unique cycle left which is  $(t_5, s_1) \rightleftharpoons (t_2, s_2)$ . Once again the manipulation of  $t_2$  is successful.

Thus, we have shown that when cycle  $\overline{C}$  is selected under the profile  $\succ$ , teacher  $t_2$  has a profitable misreport irrespective of the possible selections of cycles performed after  $t_2$ 's deviation. Let us now move to the other case.

*Case B: Under  $\succ$ ,  $\underline{C}$  is selected:*

Once this cycle is carried out, the graph associated with 1S-BE starting from the new matching is:



There are two possible choices of cycles.

*Case B.1:*  $(t_3, s_5) \rightleftharpoons (t_4, s_4)$  is chosen. Then the matching obtained is the same as the one obtained when we selected cycle  $\bar{C}$ . So we can come back to *Case A* and we know that  $t_2$  has a successful misreport.

*Case B.2:*  $(t_1, s_1) \rightarrow (t_3, s_5) \rightarrow (t_4, s_4) \rightarrow (t_2, s_3) \rightarrow (t_1, s_1)$  is chosen. In that case, each teacher but teacher  $t_4$  gets his most preferred school. Hence, there is no more cycles in the new graph associated with 1S-BE. In particular, teacher  $t_4$  gets matched to school  $s_3$ . Now, assume that  $t_4$  submits the following preferences:  $\succ'_{t_4} : s_5, s_4$ . The graph associated with 1S-BE starting from the initial assignment is the same as the one under truthful reports (note that, although these are not the arrows of the graph of 1S-BE, the dashed arrow from  $(t_4, s_4)$  disappears). So, again, we are left with a choice between cycle  $\bar{C}$  and  $\underline{C}$ .

1. If we carry out  $\underline{C}$ , the graph starting from the new matching will be given by the graph just above except that now  $(t_4, s_4)$  does not point to  $(t_2, s_3)$  anymore. Hence, we can only pick cycle  $(t_3, s_5) \rightleftharpoons (t_4, s_4)$  and so  $t_4$  obtains his best school and we identified a profitable misreport for teacher  $t_4$ .
2. If we select  $\bar{C}$ , we already know that we end up with matching  $\bar{\mu}$  as defined above. So, here again,  $t_4$  obtains his best school  $s_5$  and the manipulation is also a success.

To sum up, we have shown that for each possible selection of cycles under 1S-BE, there is a teacher who has a profitable misreport. Thus, no selection of the 1S-BE algorithm is strategy-proof, as was to be shown.



### 1.8.6 Proof of Theorem 7 and 8

#### 1.8.6.1 Preliminaries in random graph

In the sequel, we will exploit two standard results in random graph theory that are stated in this section. It is thus worth introducing the relevant model of random graph. A **graph**  $G(n)$  consists in  $n$  vertices,  $V$ , and edges  $E \subseteq V \times V$  across  $V$ . A **bipartite graph**  $G_b(n)$  consists of  $2n$  vertices  $V_1 \cup V_2$  (each of equal size) and edges  $E \subset V_1 \times V_2$  across  $V_1$  and  $V_2$  (with no possible edges within vertices in each side). Random (bipartite) graphs can be seen as random variables over the space of (bipartite) graphs. We will see two asymptotic properties of random graphs: one based on the notion of perfect matchings, the other on that of independent sets.

A **perfect matching** of  $G_b(n)$  is a subset  $E'$  of  $E$  such that each node in  $V_1 \cup V_2$  is contained in a single edge of  $E'$ .

**Lemma 10** (Erdős-Rényi). *Fix  $p \in (0, 1)$ . Consider a “random graph” which selects a graph  $G_b(n)$  with the following procedure. Each pair  $(v_1, v_2) \in V_1 \times V_2$  is linked by an edge with probability  $p$  independently (of edges created for all other pairs). The probability that there is a perfect matching in a realization of this random graph tends to 1 as  $n \rightarrow \infty$ .*

The second important technical result is about so called independent sets. An **independent set** of  $G(n)$  is  $\bar{V} \subseteq V$  such that for any  $(v_1, v_2) \in \bar{V} \times \bar{V}$ ,  $(v_1, v_2)$  is not in  $E$ .

**Lemma 11** (Grimmett and McDiarmid, 1975). *Fix  $p \in (0, 1)$ . Consider a “random graph” which selects a graph  $G(n)$  with the following procedure. Each pair  $(v_1, v_2) \in V \times V$  is linked by an edge with probability  $p$  independently (of edges created for all other pairs). Then,*

$$\Pr \left\{ \exists \text{ an independent set } \bar{V} \text{ such that } |\bar{V}| \geq \frac{2 \log n}{\log \frac{1}{1-p}} \right\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

#### 1.8.6.2 Proof of Theorem 7

In the sequel, we fix  $\mu_0$  and let  $T_k$  be  $\mu_0(S_k)$  where  $\mu_0$  is the initial allocation. We will prove the following result which implies the first part of Theorem 7.

**Proposition 1.8.4.** *Consider any selection  $\varphi$  of the BE-algorithm. Fix any  $k$ . Let  $\bar{T}_k := \{t \in T_k \mid \varphi(t) \neq \mu_0(t)\}$ . We have*

$$\frac{|\bar{T}_k|}{|T_k|} \xrightarrow{p} 1.$$

**Proof of Proposition 1.8.4.** Fix an arbitrary  $k$  and fix  $\varepsilon > 0$ . We define a random graph with  $\{(t, \mu_0(t))\}_{t \in T_k}$  as the set of vertices. An edge between  $(t, \mu_0(t))$  and  $(t', \mu_0(t'))$  is added if and only if  $\xi_{t\mu_0(t')} > 1 - \varepsilon$  and  $\xi_{t'\mu_0(t)} > 1 - \varepsilon$  and  $\eta_{t'\mu_0(t)} > 1 - \varepsilon$  and  $\eta_{t\mu_0(t')} > 1 - \varepsilon$ . Then, in the random graph, each edge between  $(t, \mu_0(t))$  and  $(t', \mu_0(t'))$  is added independently with probability  $\varepsilon^4 \in (0, 1)$ . Then, let  $\hat{T}_k := \{t \in T_k \mid \varphi(t) = \mu_0(t) \text{ and } U_t(\mu_0(t)) \leq u_k + 1 - \varepsilon \text{ and } V_{\mu_0(t)}(t) \leq 1 - \varepsilon\}$ . It must be that  $\{(t, \mu_0(t))\}_{t \in \hat{T}_k}$  is an independent set, or else if there is an edge  $(t, \mu_0(t)), (t', \mu_0(t'))$  where  $t, t' \in \hat{T}_k$  for some realization of the random graph, then

$$U_t(\mu_0(t')) > u_k + 1 - \varepsilon \geq U_t(\mu_0(t)) = U_t(\varphi(t)) \text{ and } V_{\mu_0(t)}(t) > 1 - \varepsilon \geq V_{\mu_0(t)}(t') = V_{\mu_0(t)}(\varphi(\mu_0(t')))$$

and similarly,

$$U_{t'}(\mu_0(t)) > u_k + 1 - \varepsilon \geq U_{t'}(\mu_0(t')) = U_{t'}(\varphi(t')) \text{ and } V_{\mu_0(t)}(t') > 1 - \varepsilon \geq V_{\mu_0(t)}(t) = V_{\mu_0(t)}(\varphi(\mu_0(t))).$$

Put in another way, both  $(t, \mu_0(t'))$  and  $(t', \mu_0(t))$  block  $\varphi$ . Since, by definition, under  $\varphi$ ,  $t$  is assigned  $\mu_0(t)$  and  $t'$  is assigned  $\mu_0(t')$ , this means that there are still cycles in the graph associated with BE when starting from the assignment given by  $\varphi$  which contradicts the fact that  $\varphi$  is a selection of BE.

Now, we can use Lemma 11 to get that  $\Pr \left\{ |\hat{T}_k| \geq \frac{2 \log(|T_k|)}{\log \frac{1}{1-p}} \right\} \rightarrow 0$  as  $n \rightarrow \infty$  and thus  $\frac{|\hat{T}_k|}{|T_k|} \xrightarrow{p} 0$  as  $n \rightarrow \infty$ . Setting  $\tilde{T}_k := \{t \in T_k \mid U_t(\mu_0(t)) \leq u_k + 1 - \varepsilon \text{ and } V_{\mu_0(t)}(t) \leq 1 - \varepsilon\}$ , we have

$$\frac{|\hat{T}_k|}{|T_k|} = \frac{|\bar{T}_k^c \cap \tilde{T}_k|}{|T_k|} = \frac{|\bar{T}_k^c \setminus \tilde{T}_k^c|}{|T_k|} \geq \frac{|\bar{T}_k^c|}{|T_k|} - \frac{|\tilde{T}_k^c|}{|T_k|}.$$

We know that for the left hand-side above :  $\frac{|\hat{T}_k|}{|T_k|} \xrightarrow{p} 0$  as  $n \rightarrow \infty$ . By the law of large numbers,  $\frac{|\tilde{T}_k^c|}{|T_k|} \xrightarrow{p} 1 - (1 - \varepsilon)^2$  which can be made arbitrarily close to 0 given that  $\varepsilon > 0$  is arbitrary. Hence, we obtain that  $\frac{|\bar{T}_k^c|}{|T_k|} \xrightarrow{p} 0$  as  $n \rightarrow \infty$ , as was to be proved.

Let us now move to the other part of Theorem 7. We have to show that there exists a selection of BE which is asymptotically teacher-efficient, asymptotically school-efficient and asymptotically stable. Note that in our environment asymptotic school-efficiency implies asymptotic stability. Hence, the following proposition is enough for this purpose.

**Proposition 1.8.5.** *There is a mechanism  $\varphi$  which is a selection of the BE algorithm such that for any  $k$  and any  $\varepsilon > 0$  we have*

$$\frac{|\bar{T}_k|}{|T_k|} \xrightarrow{p} 1 \text{ and } \frac{|\bar{S}_k|}{|S_k|} \xrightarrow{p} 1$$

where  $\bar{T}_k := \{t \in T_k \mid U_t(\varphi(t)) \geq u_k + 1 - \varepsilon\}$  and  $\bar{S}_k := \{s \in S_k \mid V_s(\varphi(s)) \geq 1 - \varepsilon\}$ .

**Proof of Proposition 1.8.5.** Fix  $\varepsilon > 0$ . We show that there exists a 2-IR mechanism  $\psi$  s.t. for each  $k = 1, \dots, K$ , it matches each teacher  $t \in T_k$  to a school in  $S_k$  and for each  $\delta > 0$ :

$$\Pr \left\{ \frac{|\{t \in T_k \mid \xi_{t\psi(t)} \geq 1 - \varepsilon\}|}{|T_k|} > 1 - \delta \right\} \rightarrow 1$$

and

$$\Pr \left\{ \frac{|\{s \in S_k \mid \eta_{\psi(s)s} \geq 1 - \varepsilon\}|}{|S_k|} > 1 - \delta \right\} \rightarrow 1$$

as  $n \rightarrow \infty$  where we recall that  $T_k := \mu_0(S_k)$ . This turns out to be enough for our purpose. Indeed, consider the matching mechanism given by  $\varphi := \text{BE} \circ \psi$  (i.e., the mechanism which runs BE on top of the assignment found by mechanism  $\psi$ ). Since  $\psi$  is 2-IR so is  $\varphi$ . Hence, by construction, this must be a selection of BE which satisfies

$$\frac{|\bar{T}_k|}{|T_k|} \xrightarrow{p} 1 \text{ and } \frac{|\bar{S}_k|}{|S_k|} \xrightarrow{p} 1$$

as  $n \rightarrow \infty$ .

Fix  $k = 1, \dots, K$ . Fix  $\varepsilon_0 \in (0, \varepsilon)$ . Further assume that  $\varepsilon_0$  is small enough so that  $(1 - \varepsilon_0)^2 > 1 - \delta$ . Consider the set of pairs  $(t, s) \in T_k \times S_k$  such that  $s = \mu_0(t)$  and either  $t$  ranks  $s$  within his  $\varepsilon_0|S_k|$  most favorite schools in  $S_k$  or  $s$  ranks  $t$  within his  $\varepsilon_0|T_k|$  most favorite teachers in  $T_k$ . We eliminate these pairs from  $T_k \times S_k$ . Observing that the remaining set is a product set we denote it by  $T_k^0 \times S_k^0$ . Note that for each pair  $(t, s) \in T_k \times S_k$  such that  $s = \mu_0(t)$ , there is a probability  $(1 - \varepsilon_0)^2$  that both  $t$  ranks  $s$  outside his  $\varepsilon_0|S_k|$  most favorite schools in  $S_k$  and  $s$  ranks  $t$  outside his  $\varepsilon_0|T_k|$  most favorite teachers in  $T_k$ . Let us call  $E_{ts}$  this event. For each such  $(t, s)$  where  $s = \mu_0(t)$  we denote  $\mathbf{1}_{ts}$  for the indicator function which takes value 1 if the event  $E_{ts}$  is true and 0 otherwise. Hence,  $|T_k^0| = \sum_{(t,s) \in T_k \times S_k: s = \mu_0(t)} \mathbf{1}_{ts}$ . Thus,  $|T_k^0| (= |S_k^0|)$  follows a Binomial distribution  $\text{Bin}(|T_k|, (1 - \varepsilon_0)^2)$ . By the law of large numbers,  $\frac{|T_k^0|}{|T_k|} \xrightarrow{p} (1 - \varepsilon_0)^2$  which by assumption is strictly greater than  $1 - \delta$ .

This proves that

$$\Pr \left\{ \frac{|T_k^0|}{|T_k|} \geq 1 - \delta \right\} \rightarrow 1$$

and

$$\Pr \left\{ \frac{|S_k^0|}{|S_k|} \geq 1 - \delta \right\} \rightarrow 1.$$

In the sequel, we condition w.r.t. a realization of the random set  $T_k^0 \times S_k^0$  assuming that both  $\frac{|T_k^0|}{|T_k|}$  and  $\frac{|S_k^0|}{|S_k|}$  are greater than  $1 - \delta$ . Now, fix  $\varepsilon'_0 > 0$  and note that conditional on this, each teacher  $t \in T_k^0$  draws randomly<sup>72</sup> in  $S_k^0$  his  $\varepsilon'_0 |S_k^0|$  most favorite schools in  $S_k^0$ . Similarly, each school  $s \in S_k^0$  draws randomly in  $T_k^0$  its  $\varepsilon'_0 |T_k^0|$  most favorite teachers in  $T_k^0$ . We build a random bipartite graph on  $T_k^0 \cup S_k^0$  where the edge  $(t, s) \in T_k^0 \times S_k^0$  is added if and only if  $t$  ranks  $s$  within his  $\varepsilon'_0 |S_k^0|$  most favorite schools in  $S_k^0$  and, similarly,  $s$  ranks  $t$  within its  $\varepsilon'_0 |T_k^0|$  most favorite teachers in  $T_k^0$ . This random bipartite graph can be seen as a mapping from the set of ordinal preferences into the set of bipartite graph  $G_b(|T_k^0|)$ . We denote this random graph by  $\tilde{G}_b$ . While Lemma 10 does not apply directly to this type of random graph, we will claim below that this random graph has a perfect matching, with probability approaching one as the market grows. Before stating and proving this result, we need the following lemma

**Lemma 12.** *With probability approaching one, for any teacher  $t \in T_k^0$ , any school  $s \in S_k^0$  with which  $\xi_{ts} \geq 1 - \frac{\varepsilon'_0}{2}$  must be within his  $\varepsilon'_0 |S_k^0|$  most favorite schools in  $S_k^0$ . Similarly, with probability approaching one, for any school  $s \in S_k^0$ , any teacher  $t \in T_k^0$ , with whom  $\eta_{ts} \geq 1 - \frac{\varepsilon'_0}{2}$  must be within his  $\varepsilon'_0 |S_k^0|$  most favorite teachers in  $T_k^0$ .*

*Proof.* We prove the first part of the statement, the other part follows the same argument. Fix  $t \in T_k^0$  and let  $E_t$  be the event that any school  $s \in S_k^0$  with which  $\xi_{ts} \geq 1 - \frac{\varepsilon'_0}{2}$  must be within his  $\varepsilon'_0 |S_k^0|$  most favorite schools in  $S_k^0$ . Let  $X_t := \sum_{s \in S_k^0} \mathbf{1}_{\left\{ \xi_{ts} \geq 1 - \frac{\varepsilon'_0}{2} \right\}}$  be the number of schools in  $S_k^0$  with which teacher  $t$  enjoys an idiosyncratic payoff greater than  $1 - \frac{\varepsilon'_0}{2}$ . Observe that  $X_t$  follows a Binomial distribution  $B(|S_k^0|, \frac{\varepsilon'_0}{2})$  (recall that  $\xi_{ts}$  follows a uniform distribution with support  $[0, 1]$ ) and that  $X_t \leq \varepsilon'_0 |S_k^0|$  implies that  $E_t$  is true. Hence, we have to prove that  $\Pr \{ \exists t \in T_k^0 : X_t > \varepsilon'_0 |S_k^0| \} \rightarrow 0$  as

<sup>72</sup>In the following, by randomly, we mean uniformly i.i.d.

$n \rightarrow \infty$ . In the sequel, we let  $Y_t$  be a Binomial distribution  $B(|S_k^0|, 1 - \frac{\varepsilon'_0}{2})$ , we have

$$\begin{aligned} \Pr \left\{ \exists t \in T_k^0 : X_t > \varepsilon'_0 |S_k^0| \right\} &\leq |T_k^0| \Pr \left\{ X_t > \varepsilon'_0 |S_k^0| \right\} \\ &= |T_k^0| \Pr \left\{ Y_t \leq (1 - \varepsilon'_0) |S_k^0| \right\} \\ &\leq |T_k^0| \exp \left\{ -2 |S_k^0| \left( \frac{\varepsilon'_0}{2} \right)^2 \right\} \rightarrow 0 \end{aligned}$$

as  $n \rightarrow \infty$ , where the first inequality is by the union bound and the last one uses Hoeffding inequality.

The limit result uses the fact that under our conditioning event,  $|T_k^0| = |S_k^0| \geq (1 - \delta) |S_k| \rightarrow \infty$ .  $\square$

We now move to our statement on the existence of a perfect matching in  $\tilde{G}_b$ .

**Lemma 13.** *With probability going to 1 as  $n \rightarrow \infty$ , the realization of  $\tilde{G}_b$  has a perfect matching.*

*Proof.* In our random environment, the state space, say  $\Omega$ , can be considered as the set of all possible profiles of idiosyncratic shocks for teachers and schools, i.e., the space of all  $\{\{\xi_{ts}\}_{ts}, \{\eta_{ts}\}_{ts}\}$ . We denote by  $\omega$  a typical element of that set. Let  $E$  be the event under which “For each  $(t, s) \in T_k^0 \times S_k^0$  :  $\xi_{ts} \geq 1 - \frac{\varepsilon'_0}{2}$  and  $\eta_{ts} \geq 1 - \frac{\varepsilon'_0}{2}$  imply that both  $t$  ranks  $s$  within his  $\varepsilon'_0 |S_k^0|$  most favorite schools in  $S_k^0$  and  $s$  ranks  $t$  within his  $\varepsilon'_0 |T_k^0|$  most favorite teachers in  $T_k^0$ ”. By Lemma 12,  $\Pr(E) \rightarrow 1$ . Now, let us build the following random graph on  $T_k^0 \cup S_k^0$  where this time the edge  $(t, s) \in T_k^0 \times S_k^0$  is added if and only if  $\xi_{ts} \geq 1 - \frac{\varepsilon'_0}{2}$  et  $\eta_{ts} \geq 1 - \frac{\varepsilon'_0}{2}$ . Let us call this graph  $\tilde{G}'_b$ . So this time,  $\tilde{G}'_b$  can be viewed as a mapping from the set of cardinal preferences into the set of bipartite graph  $G_b(|T_k^0|)$ . Let  $F$  be the event that the realization of  $\tilde{G}'_b$  has a perfect matching. By Lemma 10,  $\Pr(F) \rightarrow 1$ . By definition,  $E \cap F \subset \Omega$ . Let us consider the set of all possible profiles of teachers and schools’ ordinal preferences  $\succ$  induced by states  $E \cap F$  and let us denote this set by  $\mathcal{P}$ . Clearly,  $\Pr(\mathcal{P}) \geq \Pr(E \cap F) \rightarrow 1$ . Now, for each profile of preferences  $\succ$  in  $\mathcal{P}$ , let  $\tilde{G}_b(\succ)$  be the graph corresponding to  $\tilde{G}_b$  when  $\succ$  is the profile of realized preferences. We claim that for any  $\succ$  in  $\mathcal{P}$ ,  $\tilde{G}_b(\succ)$  has a perfect matching. Indeed, let  $\omega \in E \cap F$  be one state which induces  $\succ$  (this is well defined by construction of  $\mathcal{P}$ ). Because  $\omega \in F$ , the realization of  $\tilde{G}'_b$  at profile  $\omega$  has a perfect matching. In addition, because  $\omega \in E$ , the realization of  $\tilde{G}'_b$  at profile  $\omega$  is a subgraph of  $\tilde{G}_b(\succ)$ . We conclude that  $\tilde{G}_b(\succ)$  has a perfect matching. Combining this result with the observation that  $\Pr(\mathcal{P}) \rightarrow 1$ , we get

$$\Pr \left\{ \exists \text{ a perfect matching in } \tilde{G}_b \right\} \rightarrow 1$$

as  $n \rightarrow \infty$ , as claimed.  $\square$

Now, we build the mechanism  $\psi$  as follows. For each realization of ordinal preferences, (for each  $k = 1, \dots, K$ ) we build a graph on  $T_k^0 \cup S_k^0$  as defined above, i.e., where the edge  $(t, s) \in T_k^0 \times S_k^0$  is added if and only if  $t$  ranks  $s$  within his  $\varepsilon'_0 |S_k^0|$  most favorite schools in  $S_k^0$  and, similarly,  $s$  ranks  $t$  within its  $\varepsilon'_0 |T_k^0|$  most favorite teachers in  $T_k^0$ . If there is a perfect matching, then under  $\psi$ , teachers in  $T_k^0$  are matched according to this perfect matching while teachers in  $T_k \setminus T_k^0$  remain at their initial assignment. If there is no perfect matching then under  $\psi$ , all teachers in  $T_k$  remain at their initial assignment. Assuming that  $\varepsilon'_0 + \delta < \varepsilon_0$ , we get that the mechanism built in that way is 2-IR.<sup>73</sup> To see this, consider a teacher  $t$  who does not get matched to his initial school. This means that  $t$  is matched to a school  $s$  given by a perfect matching of the random bipartite graph. By construction, this means that  $t$  ranks  $s$  within his  $\varepsilon'_0 |S_k^0|$  most favorite schools in  $S_k^0$ . Hence, this means that  $s$  is within his  $\varepsilon'_0 |S_k^0| + \delta |S_k|$  most favorite schools in  $S_k$ . Since  $\varepsilon'_0 |S_k^0| + \delta |S_k| \leq (\varepsilon'_0 + \delta) |S_k| < \varepsilon_0 |S_k|$  and because  $t \in T_k^0$  implies that  $\mu_0(t)$  is not within  $t$ 's  $\varepsilon_0 |S_k|$  most favorite schools in  $S_k$ , we obtain that  $s$  is preferred by  $t$  to his initial assignment. Since a similar reasoning holds for schools, we obtain that  $\psi$  is 2-IR.

As we have shown, with probability approaching one, our bipartite graph actually has a perfect matching. Obviously, this perfect matching ensures that all teachers in  $T_k^0$  and all schools in  $S_k^0$  get matched to a partner within their  $\varepsilon'_0 |S_k^0|$  favorite. This holds for any realization of the random set  $T_k^0 \times S_k^0$  such that  $\frac{|T_k^0|}{|T_k|}$  and  $\frac{|S_k^0|}{|S_k|}$  are greater than  $1 - \delta$ . Thus, it holds conditional on the random sets  $\frac{|T_k^0|}{|T_k|}$  and  $\frac{|S_k^0|}{|S_k|}$  being greater than  $1 - \delta$ . Hence, this perfect matching ensures that all teachers in  $T_k^0$  and all schools in  $S_k^0$  get matched to a partner within their  $(\varepsilon'_0 + \delta) |S_k|$  favorite in  $S_k$  and  $T_k$  respectively. Hence, under our conditioning event that the random sets  $\frac{|T_k^0|}{|T_k|}$  and  $\frac{|S_k^0|}{|S_k|}$  are greater than  $1 - \delta$ ,

$$\Pr \left\{ \frac{|\{t \in T_k \mid \psi(t) \text{ is within the } (\varepsilon'_0 + \delta) |S_k| \text{ most favorite school in } S_k\}|}{|T_k|} > 1 - \delta \right\} \rightarrow 1$$

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<sup>73</sup>This is without loss of generality because if  $\Pr \left\{ \frac{|\{t \in T_k \mid \xi_{t\psi(t)} \geq 1 - \varepsilon\}|}{|T_k|} > 1 - \delta \right\} \rightarrow 1$  then,  $\Pr \left\{ \frac{|\{t \in T_k \mid \xi_{t\psi(t)} \geq 1 - \varepsilon\}|}{|T_k|} > 1 - \delta' \right\} \rightarrow 1$  for any  $\delta' > \delta$ .

and

$$\Pr \left\{ \frac{|\{s \in S_k \mid \psi(s) \text{ is within the } (\varepsilon'_0 + \delta) |S_k| \text{ most favorite teacher in } T_k\}|}{|S_k|} > 1 - \delta \right\} \rightarrow 1$$

Given that the conditioning event has a probability approaching 1 as  $n \rightarrow \infty$ , this is even true without conditioning.

Now, without loss of generality, let us assume that  $\delta$  is small enough so that  $\varepsilon'_0 + \delta < \varepsilon$ . It remains to show that these  $(\varepsilon'_0 + \delta) |S_k|$  favorite partners in  $S_k$  (resp.  $T_k$ ) yield an idiosyncratic payoff greater than  $1 - \varepsilon$ . The following lemma completes the argument.

**Lemma 14.** *With probability going to 1 as  $n \rightarrow \infty$ , the  $(\varepsilon'_0 + \delta) |S_k|$  most favorite schools of each teacher in  $T_k$  yield an idiosyncratic payoff higher than  $1 - \varepsilon$  and the  $(\varepsilon'_0 + \delta) |T_k|$  most favorite teachers of each school in  $S_k$  yield an idiosyncratic payoff higher than  $1 - \varepsilon$ .*

*Proof.* We show that with probability going to 1 as  $n \rightarrow \infty$ , the  $(\varepsilon'_0 + \delta) |S_k|$  most favorite schools of each teacher in  $T_k$  yield an idiosyncratic payoff higher than  $1 - \varepsilon$ . The other part of the statement is proved in the same way. For each  $t \in T_k$ , let  $Z_t$  be the number of schools  $s$  in  $S_k$  for which  $\xi_{ts} \geq 1 - \varepsilon$ . Note that if  $Z_t > (\varepsilon'_0 + \delta) |S_k|$  then  $t$ 's  $(\varepsilon'_0 + \delta) |S_k|$  first schools in  $S_k$  must yield an idiosyncratic payoff higher than  $1 - \varepsilon$ . Thus, it is enough to show that

$$\Pr\{\exists t \in T_k \text{ with } Z_t \leq (\varepsilon'_0 + \delta) |S_k|\} \rightarrow 0$$

as  $n \rightarrow \infty$ . Observe that  $Z_t$  follows a Binomial distribution  $B(|S_k|, \varepsilon)$  (recall that  $\xi_{ts}$  follows a uniform distribution with support  $[0, 1]$ ). Hence,

$$\begin{aligned} \Pr\{\exists t \in T_k \text{ with } Z_t \leq (\varepsilon'_0 + \delta) |S_k|\} &\leq \sum_{t \in T_k} \Pr\{Z_t \leq (\varepsilon'_0 + \delta) |S_k|\} \\ &= |T_k| \Pr\{Z_t \leq (\varepsilon'_0 + \delta) |S_k|\} \\ &\leq |T_k| \frac{1}{2} \exp \left( -2 \frac{(|S_k| \varepsilon - (\varepsilon'_0 + \delta) |S_k|)^2}{|S_k|} \right) \\ &= \frac{|T_k|}{2 \exp \left( 2 (\varepsilon - (\varepsilon'_0 + \delta))^2 |S_k| \right)} \rightarrow 0 \end{aligned}$$

where the first inequality is by the union bound while the second equality is by Hoeffding's inequality.  $\square$

### 1.8.6.3 Proof of Theorem 8

Recall that  $T_k$  stands for  $\mu_0(S_k)$  where  $\mu_0$  is the initial allocation. We will prove the following result.

**Proposition 1.8.6.** *Fix any  $k$  and any  $\varepsilon > 0$ . Let  $\bar{T}_k := \{t \in T_k \mid U_t(\text{TO-BE}(t)) \geq u_k + 1 - \varepsilon\}$ . We have*

$$\frac{|\bar{T}_k|}{|T_k|} \xrightarrow{p} 1.$$

**Proof of Proposition 1.8.6.** Recall that TO-BE is in the two-sided core. In particular, this implies that there is no pair of teachers  $t$  and  $t'$  so that  $\mu_0(t') \succeq_t \text{TO-BE}(t)$ ,  $\mu_0(t) \succeq_{t'} \text{TO-BE}(t')$  (with a strict preference for either  $t$  or  $t'$ ),  $t' \succeq_{\mu_0(t)} t$  and  $t \succeq_{\mu_0(t')} t'$ . Fix an arbitrary  $k$  and let  $E$  be the event that the fraction of schools  $s \in S_k$  s.t.  $\eta_{\mu_0(s)s} \leq 1 - \delta$  is greater than  $1 - 2\delta$  where  $\delta \in (0, 1)$ . By the law of large numbers, we have

$$\frac{1}{|S_k|} \sum_{s \in S_k} \mathbf{1}_{\{\eta_{\mu_0(s)s} \leq 1 - \delta\}} \xrightarrow{p} 1 - \delta.$$

Thus,  $\Pr(E) \rightarrow 1$ . Let  $T_k^0 := \{t \in T_k \mid \eta_{t\mu_0(t)} \leq 1 - \delta\}$ .

In the sequel, we condition on event  $E$  and we fix a realization of  $\{\eta_{\mu_0(s)s}\}_{s \in S}$  compatible with  $E$ . Observe that  $T_k^0$  is non-random once this has been fixed and note that conditional on these, individuals' preferences are still drawn according to the same distribution (as in the unconditional case) and for  $t \neq \mu_0(s)$ ,  $\eta_{ts}$  is also still drawn according to the same distribution. We further observe that, because that event  $E$  holds,  $\frac{|T_k^0|}{|T_k|} \geq 1 - 2\delta$  and hence  $|T_k^0|$  goes to infinity as  $n \rightarrow \infty$ . We define a random graph with  $\{(t, \mu_0(t))\}_{t \in T_k^0}$  as the set of vertices. An edge between  $(t, \mu_0(t))$  and  $(t', \mu_0(t'))$  is added if and only if  $\xi_{t\mu_0(t')} > 1 - \varepsilon$  and  $\xi_{t'\mu_0(t)} > 1 - \varepsilon$  and  $\eta_{t'\mu_0(t)} \geq \eta_{t\mu_0(t)}$  and  $\eta_{t\mu_0(t')} \geq \eta_{t'\mu_0(t')}$ . Then, in the random graph, each edge between  $(t, \mu_0(t))$  and  $(t', \mu_0(t'))$  is added independently with probability at least  $\varepsilon^2 \delta^2 \in (0, 1)$ . Now, let  $\bar{T}_k^0 := \{t \in T_k^0 \mid U_t(\text{TO-BE}(t)) \leq u_k + 1 - \varepsilon\}$ . It must be that  $\bar{T}_k^0$  is an independent set, or else if there is an edge  $(t, t') \in \bar{T}_k^0 \times \bar{T}_k^0$  for some realization of the random graph, then

$$U_t(\mu_0(t')) > u_k + 1 - \varepsilon \geq U_t(\text{TO-BE}(t)) \text{ and } U_{t'}(\mu_0(t)) > u_k + 1 - \varepsilon \geq U_{t'}(\text{TO-BE}(t')).$$

In addition,  $V_{\mu_0(t)}(t') = \eta_{t'\mu_0(t)} \geq \eta_{t\mu_0(t)} = V_{\mu_0(t)}(t)$  and  $V_{\mu_0(t')}(t) = \eta_{t\mu_0(t')} \geq \eta_{t'\mu_0(t')} = V_{\mu_0(t')}(t')$



and so TO-BE is blocked by a coalition of size two, a contradiction. Now, we can use Lemma 11 to get that  $\Pr \left\{ |\bar{T}_k^0| \geq \frac{2 \log(|T_k|)}{\log \frac{1}{1-p}} \right\} \rightarrow 0$  as  $n \rightarrow \infty$  and thus  $\frac{|\bar{T}_k^0|}{|T_k^0|} \xrightarrow{p} 0$  as  $n \rightarrow \infty$ . Now, since  $\bar{T}_k^c = \bar{T}_k^0 \cup \{t \in T_k \setminus T_k^0 \mid U_t(\text{TO-BE}(t)) \leq u_k + 1 - \varepsilon\}$  we must have

$$\frac{|\bar{T}_k^c|}{|T_k|} \leq \frac{|\bar{T}_k^0| + |T_k \setminus T_k^0|}{|T_k|} \leq \frac{|\bar{T}_k^0|}{|T_k|} + 2\delta$$

Hence, given that  $\frac{|\bar{T}_k^0|}{|T_k^0|} \xrightarrow{p} 0$ , we must have that with probability going to 1 as  $n$  goes to infinity,  $\frac{|\bar{T}_k^c|}{|T_k|} \leq 3\delta$  and so  $\frac{|\bar{T}_k|}{|T_k|} \geq 1 - 3\delta$ .

To recap, given event  $E$  and any realization of  $\{\eta_{\mu_0(s)s}\}_{s \in S}$ , we have  $\frac{|\bar{T}_k|}{|T_k|} \geq 1 - 3\delta$  with probability going to 1 as  $n \rightarrow \infty$ . Since the realization of  $\{\eta_{\mu_0(s)s}\}_{s \in S}$  is arbitrary, we obtain that given event  $E$ ,  $\frac{|\bar{T}_k|}{|T_k|} \geq 1 - 3\delta$  with probability going to 1 as  $n \rightarrow \infty$ . Since  $\Pr(E) \rightarrow 1$  as  $n \rightarrow \infty$ , we get that  $\frac{|\bar{T}_k|}{|T_k|} \geq 1 - 3\delta$  with probability going to 1 as  $n \rightarrow \infty$ . Since  $\delta > 0$  is arbitrary small, we obtain  $\frac{|\bar{T}_k|}{|T_k|} \xrightarrow{p} 1$  as  $n \rightarrow \infty$ , as claimed.

**Remark 2.** *The statement is related to that of (Che and Tercieux, 2015b, Theorem 1). However, since TO-BE is not Pareto-efficient, their proof/argument does not apply.*

**Remark 3.** *The argument relies on the fact that TO-BE is not blocked by any coalition of size 2. Hence, the result applies beyond the TO-BE mechanism and applies to any mechanism which cannot be blocked by any coalition of size 2.*

### 1.8.7 Many-to-one Extensions

We provide below the extensions of BE and 1S-BE to the many-to-one framework. So now, each school may have multiple seats. As before, we assume that all the teachers are initially matched to a school and that all seats are initially occupied by a teacher. As before let  $\mu_0$  be the initial matching.

#### The Block Exchange Algorithm

The main difference is that now, blocking with a school does not necessarily means that a teacher is preferred to a given matched one in this school. To keep the idea of not hurting any school, we have to allow a node to point to another one only if the teacher of the former is preferred to the teacher of the latter by the corresponding school.

- **Step 0** : set  $\mu(0) := \mu_0$ .

- **Step**  $k \geq 1$  : Given  $\mu(k-1)$ , let the teachers and their assignments stand for the vertices of a directed graph where for each pair of nodes  $(t, s)$  and  $(t', s')$ , there is an edge  $(t, s) \rightarrow (t', s')$  if and only if teacher  $t$  has a justified envy against teacher  $t'$  at  $s'$  i.e. he prefers  $s'$  to its match  $s$  and is preferred by  $s'$  to  $t'$ . If there is no cycle, then return  $\mu(k-1)$  as the outcome of the algorithm. Otherwise, select a cycle in this directed graph. For each edge  $(t, s) \rightarrow (t', s')$  in the cycle, assign teacher  $t$  to school  $s'$ . Let  $\mu(k)$  be the matching so obtained. Go to step  $k+1$ .

### The Teacher-Optimal Block Exchange Algorithm

In the following lines, we define a class of mechanisms which are all selections of the BE algorithm and are strategy-proof. They all reduce to the TO-BE mechanism (as defined in the main text) in the one-to-one environment.

Given a matching  $\mu$  and a set of school  $S' \subseteq S$ , we let

$\text{Opp}(t, \mu, S') := \{s \in S' \mid t \succeq_s t' \text{ for some } t' \in \mu(s)\}$  be the opportunity set of teacher  $t$  within schools in  $S'$ . Note that for each teacher  $t$ , if  $\mu_0(t) \in S'$ , then  $\text{Opp}(t, \mu_0, S') \neq \emptyset$  since  $\mu_0(t) \in \text{Opp}(t, \mu_0, S')$ .

Now, fix an ordering over teachers  $f : \{1, \dots, |T|\} \rightarrow T$  which will be the index for our class of mechanisms.

- **Step** 0 : Set  $\mu(0) = \mu_0$ ,  $T(0) := T$  and  $S(0) := S$ .
- **Step**  $k \geq 1$  : Given  $T(k-1)$  and  $S(k-1)$ , let the teachers in  $T(k-1)$  and their assignments stand for the vertices of a directed graph where for each pair of nodes  $(t, s)$  and  $(t', s')$ , there is an edge  $(t, s) \rightarrow (t', s')$  if and only if teacher  $t$  ranks school  $s'$  first in his opportunity set  $\text{Opp}(t, \mu(k-1), S(k-1)) = \text{Opp}(t, \mu_0, S(k-1))$ , teacher  $t'$  has a lower priority than teacher  $t$  at school  $s'$  and teacher  $t'$  has the lowest ordering according to  $f$  among all teachers forming a pair with school  $s'$  and having a lower priority than  $t$  at  $s$  (i.e.,  $f(t') \leq f(t'')$  for all  $t''$  such that  $\mu(k-1)(t'') = s'$  and  $t \succeq_s t''$ ). The directed graph so obtained is a directed graph with out-degree one and, as such, has at least one cycle and cycles are pairwise disjoint. For each edge  $(t, s) \rightarrow (t', s')$  in a cycle, assign teacher  $t$  to school  $s'$ . Let  $\mu(k)$  be the assignment obtained and  $T(k)$  be the set of teachers who are not part of any cycle at the current step. If  $T(k)$  is empty then return  $\mu(k)$  as the outcome of the algorithm. Otherwise, go to step  $k+1$ .

**Remark 4.** *This class of mechanisms is still tightly connected to the top trading cycle mechanism.*

To see this, fix an ordering  $f$  and assume first that each teacher ranks schools outside his opportunity set below his initial assignment. Now, for each teacher define preferences over pairs  $\{(t, \mu_0(t))\}_{t \in T}$  in the following way: for  $s \neq s'$ ,  $(t, s)$  is strictly preferred to  $(t', s')$  if and only if  $s$  is strictly preferred to  $s'$ ; in addition,  $(t, s)$  is strictly preferred to  $(t', s)$  if and only if  $f(t) < f(t')$ . We can consider this modified environment as a one-to-one environment where agents' preferences are strict. Top trading cycles is well-defined in this environment and coincides with the outcome of TO-BE (with the ordering  $f$ ) defined in the previous paragraph.

### The 1-Sided Block Exchange Algorithm

In order to keep the property that the graph associated with 1S-BE is a supergraph of that of BE, we build on the previous generalization of BE to define the extension of 1S-BE.

- **Step 0** : set  $\mu(0) := \mu_0$ .
- **Step  $k \geq 1$**  : Given  $\mu(k-1)$ , let the teachers and their assignments stand for the vertices of a directed graph where for each pair of nodes  $(t, s)$  and  $(t', s')$ , there is an edge  $(t, s) \rightarrow (t', s')$  if and only if either (1) teacher  $t$  has a justified envy toward  $t'$  at  $s'$ ; or (2)  $t$  desires  $s'$  and  $t$  is ranked higher by  $s'$  than each teacher who both desires  $s'$  and does not block with  $s'$ .<sup>74</sup> If there is no cycle, then return  $\mu(k-1)$  as the outcome of the algorithm. Otherwise, select a cycle in this directed graph. For each edge  $(t, s) \rightarrow (t', s')$  in the cycle, assign teacher  $t$  to school  $s'$ . Let  $\mu(k)$  be the matching so obtained. Go to step  $k+1$ .

#### 1.8.8 Empirical results

For the empirical part of the analysis, we decided not to focus on the second phase of the assignment because reported preferences seem to be less reliable. First, teachers are restricted to rank up to 20 schools and it is well known in the literature (Haeringer and Klijn, 2009) that such constraint gives rise to strategic reports.<sup>75</sup> Second, and perhaps more importantly, teachers can report “wide wishes” instead of reporting a precise school. A wide wish can be a geographic area

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<sup>74</sup>Note that here, teacher  $t$  may block with  $s'$  under condition (2). Thus, it is easy to see that if (1) is satisfied then (2) is satisfied as well. Hence, one could simplify the definition and suppress condition (1). We keep it just to have a parallel with the definition provided in the one-to-one environment.

<sup>75</sup>Teachers can rank up to 20 or 30 schools, depending on the region. In regions where they can rank a maximum of 20 schools, 10.79 % of the teachers rank 20 schools. In regions where teachers can rank up to 30 schools, the constraint is binding for less than 1 % of the teachers.

such as a city, a group of cities, a department or an entire region. For instance, instead of ranking a school within city  $x$ , a teacher can report the city  $x$  in his ranking. These wide wishes make it possible for teachers to cover all schools within a region – if they wish to – which would not be possible otherwise due to the limit of 20 schools that applies. Then, when a city is ranked, the designer is free to assign the teacher to any school within that city. This is a way for teachers to signal their strong preference to be in a city  $x$  (irrespective of where they eventually end up in that city). While this may make sense from a design point of view, it makes it more difficult for us to interpret the reported preferences.

As for the interpretation of the preferences reported during Phase 1, [Prost \(2013\)](#) provided a preliminary analysis of teachers preferences in France using the observations of teachers movements.<sup>76</sup> She noticed that the main determinant of the inter-regional mobility of teachers is the willingness to get closer to the region where they were born. This suggests a first weak evidence toward the possibility for teachers to have strong regional preferences.<sup>77</sup> We believe that the main message of our work, i.e.  $DA^*$  is not 2-Pareto efficient and that the scope of improvement is important, would remain if one refines the estimation of the preferences. As mentioned at the beginning of [Section 1.5](#), one can consider our analysis as simulations using reported preferences from a practical market.

25 067 teachers participate to the first phase of the assignment. We restrict the sample to the 49 subjects containing more than 10 teachers asking for a transfer : this restricts the sample to 20 808 teachers. We also remove from the sample all couples (1579 teachers) because of the specific treatment they receive in the assignment procedure.<sup>78</sup> Finally, only teachers who have an initial assignment are kept in the sample. The final sample contains 10579 teachers.

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<sup>76</sup>We also refer the reader to her description of the french assignment procedure in complement of ours.

<sup>77</sup>This estimation was done using data on the observed movements and not using the data of preferences during the assignment procedure. It uses a standard logistic regression to estimate the probability for a teacher to quit his position for a new one. A structural estimation using our data would be a natural next step. However, the dynamic component of the problem can require us to use new techniques than the ones developed in the matching literature. We leave it as a future research.

<sup>78</sup>Couples can jointly apply, in which case they have to submit two identical lists of regions to the central administration. A specific treatment is applied to the couple: one of the spouses will not be assigned a region if the other one does not get the same region. To achieve this, the central administration runs the algorithm once and checks the region obtained by each spouse. If they don't obtain the same region, one must have obtained a region that is ranked lower in their common ranking (for instance rank 5). In that case, the ministry would delete all regions ranked higher than rank 5 from their common list of preferences and run the algorithm again on the modified list. This process is repeated until both spouses obtain the same regions. If this does not happen, they stay in their initial region.

### 1.8.9 Many-to-one characterization of TO-BE when teachers rank only one additional school.

Consider a many-to-one environment where each school  $s$  has  $q_s \geq 1$  seat. We assume that each teacher finds acceptable at most one school beyond his initial assignment. Let us denote by  $\mathcal{P}$  the restricted domain of preferences/priorities. A mechanism in this context is a mapping from  $\mathcal{P}$  to matchings. In the sequel, we consider the correspondence given by all possible selections of TO-BE [to be defined].

**Theorem.** In the restricted domain  $\mathcal{P}$ , the set of two-sided maximal and strategy-proof mechanisms coincide with all selections of TO-BE.<sup>79</sup>

**Proposition.** Consider any two-sided maximal (and strategy-proof) mechanism  $\varphi$ . Fix any profile  $\succ$  which lies in  $\mathcal{P}$ . We must have  $\varphi(\succ) \in \text{TO-BE}(\succ)$ .

**Proof.** Consider the graph in the first step of TO-BE( $\succ$ ). We claim that there is a cycle of TO-BE( $\succ$ ) under which, any teacher who is part of the cycle gets assigned the object he points to under  $\varphi(\succ)$ . First, note that if there is a self-cycle in this step, i.e., a node  $(t, s)$  pointing to itself, then, by 2-IR of  $\varphi$ ,  $\varphi(\succ)$  must assign  $t$  to  $s$ . So let us assume that there is no self-cycle in the graph. Pick an arbitrary cycle denoted  $(t_1, s_1), (t_2, s_2), \dots, (t_K, s_K)$ .

We claim that in this cycle, there must be a node  $(t_k, s_k)$  such that  $\varphi(\succ)(t_k) \neq s_k$ . Indeed, if this was not the case, we would have that for all  $k = 1, \dots, K$ ,  $\varphi(\succ)(t_k) = s_k$ . But then  $\varphi(\succ)$  is not two-sided maximal since we can assign each  $t_k$  to  $s_{k+1}$ . Noticing that this gives a 2-IR assignment, the assignment we obtain 2-Pareto dominates  $\varphi(\succ)$ .

The following simple lemma shows that there must be a cycle of TO-BE( $\succ$ ) under which, any teacher who is part of the cycle gets assigned the object he points to under  $\varphi(\succ)$ .

**Lemma.** If there exists  $t_1$  such that  $\varphi(\succ)(t_1) \neq \mu_0(t_1)$  then there exists a cycle  $(t_1, s_1), (t_2, s_2), \dots, (t_K, s_K)$  in the graph such that  $\varphi(\succ)(t_k) = s_{k+1}$  for any  $k = 1, \dots, K$ .

**Proof.** Assume that there exists  $t_1$  such that  $\varphi(\succ)(t_1) \neq \mu_0(t_1)$ . Because  $\succ$  lies in  $\mathcal{P}$ ,  $\varphi(\succ)(t_1) =: s_2$  where  $s_2$  is  $t_1$ 's top choice. In addition, since, under  $\varphi(\succ)$ , one seat of school  $s_2$  is taken by  $t_1$ , there must be a teacher  $t_2$  such that  $\mu_0(t_2) = s_2$  and  $\varphi(\succ)(t_2) \neq s_2$ . In addition, because  $\varphi(\succ)$  is two-sided maximal, this teacher  $t_2$  can be chosen so that  $t_1 \succ_{s_2} t_2$  [il faut que je relise la defn en many-to-one...]. By definition of the graph,  $(t_1, s_1)$  points to  $(t_2, s_2)$ . Now, since

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<sup>79</sup>Two-sided maximality implies 1-IR which, under domain restriction  $\mathcal{P}$ , implies strategy-proofness.

$\varphi(\succ)(t_2) \neq s_2 = \mu_0(t_2)$ , we can iterate the reasoning to induce a path  $(t_1, s_1), (t_2, s_2), \dots$  in the graph such that  $\varphi(\succ)(t_k) = s_{k+1}$  for any  $k \geq 1$ . Since the graph is finite, this path will cycle at some point.  $\square$

Now, consider the new graph obtained after we removed the teachers who are part of this cycle and the seats they point to in the cycle. The exact same reasoning holds here. Hence, we can iterate the reasoning until we exhaust the market. We obtain a sequence of cycles selected in the graphs associated with TO-BE which, once implemented, yields the assignment given by  $\varphi(\succ)$ . This shows that  $\varphi(\succ) \in \text{TO-BE}(\succ)$ .  $\square$

### 1.8.10 Many-to-one extension of Theorem 4

In the sequel, we prove our characterization result of one-sided maximal matchings given in Theorem 4. Our proof is divided into two parts. We start by showing that any outcome of the 1S-BE algorithm is a one-sided maximal matching (Section 1.8.10.1):

**Proposition 1.8.7.** *If  $\mu$  is an outcome of the 1S-BE algorithm then  $\mu$  is one-sided maximal.*

Then, we move to the proof that any one-sided maximal matching corresponds to a possible outcome of the 1S-BE algorithm (Section 1.8.10.2):

**Proposition 1.8.8.** *If  $\mu$  is one-sided maximal then  $\mu$  is an outcome of the 1S-BE algorithm.*

#### 1.8.10.1 Proof of Proposition 1.8.7

Before moving to the proof we introduce a new notation. Given matching  $\mu$ , we denote  $\mathcal{B}_\mu$  for the set of blocking pairs of  $\mu$ .

In the sequel, we fix two matchings  $\mu$  and  $\mu'$  such that  $\mu'$  Pareto-dominates  $\mu$  for teachers and  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_\mu$ . We show below that starting from  $\mu$ , the graph associated to the 1S-BE algorithm must have a cycle. Hence, any outcome of 1S-BE must be one-sided maximal, as claimed in Proposition 1.8.7.

To give the intuition of each step of the proof, which uses a lot of graphical arguments, we will use an example to illustrate each part. This example involves 6 teachers,  $t_1, t'_1, t_2, t'_2, t_3, t_4$  and 4

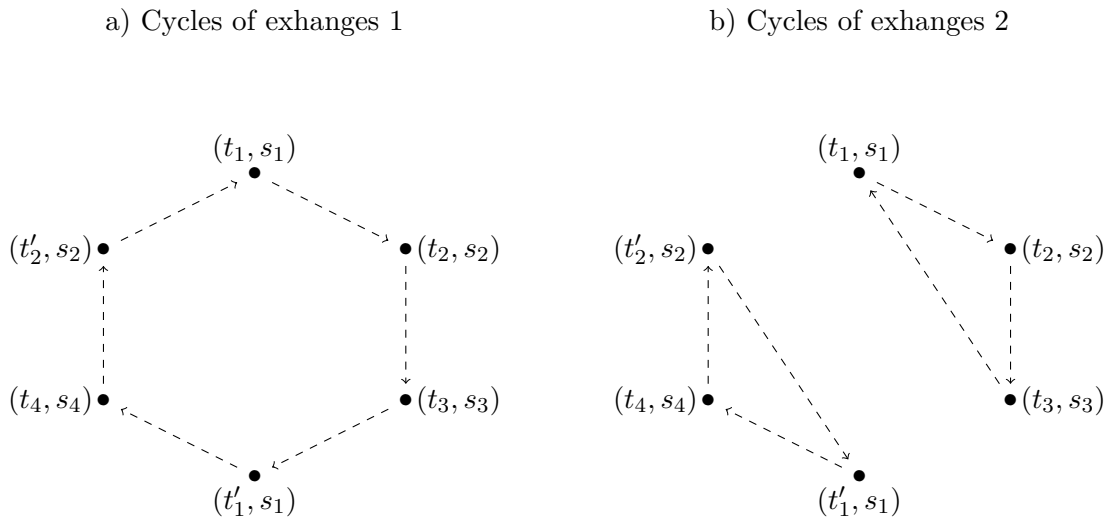
schools  $s_1, s_2, s_3, s_4$ . In the example, matchings  $\mu$  and  $\mu'$  are as follows:

$$\mu = \begin{pmatrix} t_1 & t'_1 & t_2 & t'_2 & t_3 & t_4 \\ s_1 & s_1 & s_2 & s_2 & s_3 & s_4 \end{pmatrix}$$

$$\mu' = \begin{pmatrix} t_1 & t'_1 & t_2 & t'_2 & t_3 & t_4 \\ s_2 & s_4 & s_3 & s_1 & s_1 & s_2 \end{pmatrix}$$

As in Lemma 1, we can exhibit “cycles of exchanges” which can be used to go from  $\mu$  to  $\mu'$  in the proposition. It is worth noting that in the many-to-one environment, these cycles of exchanges are not uniquely defined. Indeed, if for a given selection of cycles of exchanges, there are two nodes that involve the same school, then this cycle can be decomposed into two cycles of exchanges. Figure 1.2 illustrates this simple fact: in the left part of the figure, there is a cycle of exchanges starting from  $\mu$  which once implemented leads to  $\mu'$ . It is easy to see that we can decompose this cycle into two smaller cycles of exchanges, showed in the right part of the figure, that also lead to  $\mu'$  once implemented.

Figure 1.2 – TWO EQUIVALENT CYCLES OF EXCHANGES IN MANY-TO-ONE.



So for the rest of the proof, we fix (a collection of) cycles of exchanges which takes us from  $\mu$  to  $\mu'$  once implemented. To fix ideas, in the example, we consider the one on the left part of Figure 1.2. In Lemma 1, these cycles of exchanges were actual cycles in the graph associated with BE.

However, when considering the graph associated with 1S-BE, this is not the case anymore: the cycles of exchanges are not necessarily cycles of the graph associated with 1S-BE. Before moving to the first lemma, we note that all the nodes that are not part of cycles of exchanges are those where the teacher of that node has the same allocation between  $\mu$  and  $\mu'$ . In the following the “nodes of the cycles of exchanges” will be all the nodes  $(t, s)$  s.t  $\mu(t) \neq \mu'(t)$ . We will say that a node  $(t, s)$  1S-BE-points to another node  $(t', s')$  if  $(t, s)$  points toward  $(t', s')$  in the graph associated with the 1S-BE algorithm (starting from  $\mu$ ).

**Lemma 15.** *Fix a node  $(t, s)$  of the cycles of exchanges. Then:*

1. *either its predecessor according to the the cycles of exchanges 1S-BE-points toward  $(t, s)$ ;*
2. *or there is a node  $(t', s')$  in the cycles of exchanges that such that  $t'$  does not block with  $s$  under  $\mu$ ,  $s \succ_{t'} s'$  and  $t'$  has the highest priority among those who desire  $s$  but do not block with it under  $\mu$ . And so  $(t', s')$  1S-BE-points toward  $(t, s)$ .*

Before moving to the proof, let us illustrate this lemma in the example. Assume that all the nodes except  $(t_3, s_3)$  are 1S-BE-pointed by their predecessors in the cycle of exchanges. According to Lemma 15 there must be a node  $(t', s')$  in the cycle of exchanges which 1S-BE-points toward  $(t_3, s_3)$ . In the graph of Figure 1.3, this node is assumed to be  $(t_4, s_4)$ . The dashed edge from  $(t_2, s_2)$  to  $(t_3, s_3)$  is here to show that this is not an edge of the 1S-BE graph but it is only an edge corresponding to the cycle of exchanges.

*Proof.* Call  $(t'', s'')$  the predecessor of node  $(t, s)$  in the cycles of exchanges so that  $s'' := \mu(t'')$  and  $s := \mu'(t'')$ . Because  $\mu'$  Pareto-domination for teachers  $\mu$ , we know that  $s \succ_{t''} s''$  so that  $t''$  desires  $s$  under  $\mu$ . Assume that  $(t'', s'')$  does not 1S-BE-point to  $(t, s)$ . This means that  $t''$  does not block with  $s$  under  $\mu$  and that there is another teacher  $t'$  who does not block with  $s$  and has the highest priority among those who desire  $s$  and do not block with it. Thus,  $(t', s')$  (where  $s' := \mu(t')$ ) 1S-BE points toward  $(t, s)$ . It remains to show that  $(t', s')$  is part of the cycles of exchanges. If it was not the case, it would mean that  $\mu(t') = \mu'(t') = s'$ . Let us recap. We have that  $t'$  does not block with  $s$  under  $\mu$ . In addition, by definition of  $t'$ , we must have that  $t' \succ_s t''$  (since  $t''$  does not block with  $s$  under  $\mu$  and desires  $s$ ). In addition,  $t'$  desires  $s$  under  $\mu$  and so  $\mu(t') = \mu'(t')$  implies that  $t'$  also desires  $s$  under  $\mu'$ . Hence, because  $t'' \in \mu'(s)$ , we obtain that  $t'$  blocks with  $s$  under  $\mu'$ . This contradicts our assumption that  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu}$ . □



Lemma 15 allows us to identify a subgraph  $(N', E_1)$  of the 1S-BE graph starting from  $\mu$  such that  $N'$  are the nodes of the cycles of exchanges and the set of edges  $E_1$  is built as follows. We start from  $E_1 = \emptyset$  and we add the following edges: for each node  $(t, s)$  in the cycles of exchange, if its predecessor  $(\tilde{t}, \tilde{s})$  under the cycles of exchanges 1S-BE-points to  $(t, s)$  then  $((\tilde{t}, \tilde{s}), (t, s))$  is added to  $E_1$ . If on the contrary,  $(\tilde{t}, \tilde{s})$  does not 1S-BE-points to  $(t, s)$ , then we pick the node  $(t', s')$  in the cycles of exchanges identified in the second condition of Lemma 15 which 1S-BE-points toward  $(t, s)$  and we add  $((\tilde{t}, \tilde{s}), (t, s))$  to  $E_1$ . Note that, by construction, each node in  $N'$  has a unique in-going edge in  $(N', E_1)$ . In the example, this subgraph  $(N', E_1)$  is given by the right graph of Figure 1.3 (the solid arrows). Note that this graph admits a cycle:  $(t_3, s_3) \rightarrow (t'_1, s_1) \rightarrow (t_4, s_4) \rightarrow (t_3, s_3)$ . This is a simple property of digraphs with in-degree one:

**Lemma 16.** *Fix a finite digraph  $(N, E)$  such that each node has in-degree one. There exists a cycle in this graph.*

*Proof.* Fix a node  $n_1$  in the graph  $(N, E)$ . Because it has in-degree one, we can let  $n_2$  be the unique node pointing to  $n_1$ . Again from  $n_2$  we can let  $n_3$  be the unique node pointing to  $n_2$ . Because there is a finite number of nodes in the graph, this process must cycle at some point.  $\square$

As the example illustrates, applying this lemma to  $(N', E_1)$  leads to the following corollary:

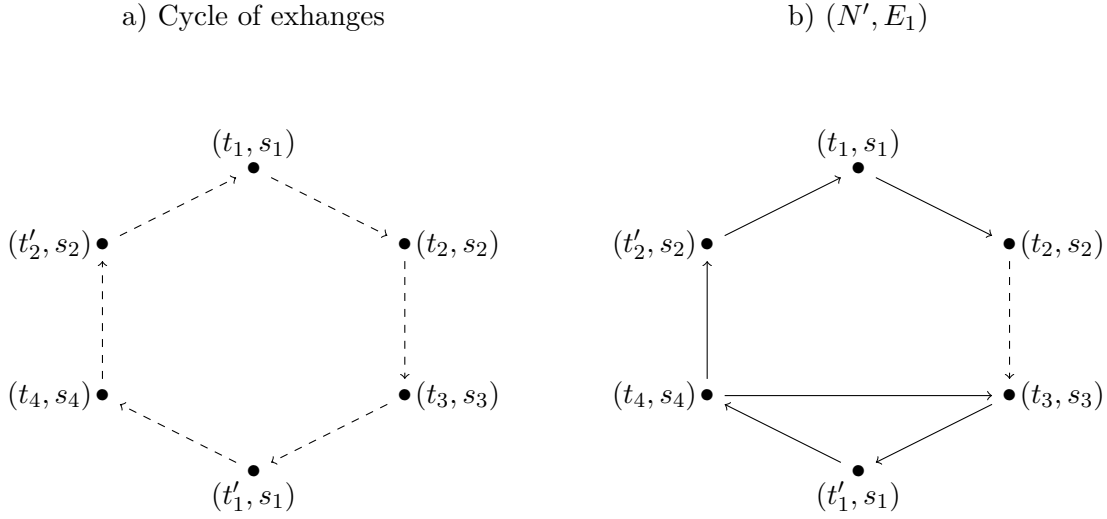
**Corollary 4.** *There is a cycle in the graph associated with 1S-BE starting from  $\mu$ .*

We are now in a position to prove Proposition 1.8.7.

*Completion of the proof of Proposition 1.8.7.* Let  $\mu$  be an outcome of the 1S-BE algorithm. Proceed by contradiction and assume that  $\mu$  is not one-sided maximal. Thus, there must exist a matching  $\mu'$  such that  $\mu'$  Pareto-dominates  $\mu$  for teachers and  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu}$ . Corollary 4 implies that there must be a cycle in the graph associated with 1S-BE starting from  $\mu$ , contradicting the fact that  $\mu$  is an outcome of 1S-BE.  $\square$

### 1.8.10.2 Proof of Proposition 1.8.8

In the sequel, we fix a one sided maximal matching  $\mu'$ . We let  $\mu$  be a matching such that  $\mu'$  Pareto-dominates for teachers  $\mu$  and satisfies  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu}$ . We claim that there is a cycle in the graph associated with 1S-BE starting from  $\mu$  which, once implemented, leads to a matching  $\tilde{\mu}$  such that  $\mu'$

Figure 1.3 – CYCLES OF EXCHANGES AND  $(N', E_1)$ .


Pareto-dominates  $\tilde{\mu}$  for teachers and satisfies  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\tilde{\mu}}$ . Note that this implies Proposition 1.8.8. Indeed, because, by definition,  $\mu'$  Pareto-dominates  $\mu_0$  and  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu_0}$ , we must have a cycle in the graph associated with 1S-BE starting from  $\mu_0$ , which once implemented, yields to a matching say  $\tilde{\mu}_1$  such that  $\mu'$  Pareto-dominates  $\tilde{\mu}_1$  for teachers and satisfies  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\tilde{\mu}_1}$ . Now, we can iterate the reasoning and we get again that there is a cycle in the graph associated with 1S-BE starting from  $\tilde{\mu}_1$ , which, once implemented, yields to a matching say  $\tilde{\mu}_2$  such that  $\mu'$  Pareto-dominates  $\tilde{\mu}_2$  for teachers and satisfies  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\tilde{\mu}_2}$ . We can pursue this reasoning, at some point, because the environment is finite, we must reach matching  $\mu'$ , as was to be shown.

We start by proving a lemma which will be useful in the subsequent arguments. Starting from any matching  $\mu$ , it states that – in the graph associated to the 1S-BE algorithm – if a node points to another node involving school  $s$  then it also points to all other nodes involving school  $s$ .

**Lemma 17.** *Let  $(t, s)$  be a node in the graph associated with the 1S-BE algorithm starting from a matching  $\mu$ . If  $(t, s)$  points to  $(t', s')$  then  $(t, s)$  points to  $(t'', s')$  for all  $t'' \in \mu(s')$ .*

*Proof.* Let us assume that  $(t, s)$  points to  $(t', s')$  and consider any node  $(t'', s')$ . Let us first consider the case where  $t \succ_{s'} t''$ . Given that  $(t, s)$  points to  $(t', s')$ , we must have that  $t$  desires  $s'$ . Hence,  $t$  has justified envy toward  $t''$  and so, by definition of 1S-BE,  $(t, s)$  must point to  $(t'', s')$ . Now, consider the other case in which  $t'' \succ_{s'} t$ . We have to check that  $t$  is preferred by  $s'$  to each teacher who desires  $s'$  and does not block with it. If  $t \succ_{s'} t'$ , then for any teacher  $\tilde{t}$  who desires  $s'$  and does

not block with  $s'$ , we have that  $t' \succ_{s'} \tilde{t}$  which implies  $t \succ_{s'} \tilde{t}$  so that  $t$  is preferred by  $s'$  to those who desire  $s'$  and do not block with it. Hence,  $(t, s)$  must point to  $(t', s')$ . Now, if  $t' \succ_{s'} t$ , because we know that  $(t, s)$  points to  $(t', s')$ ,  $t$  must be preferred by  $s'$  to those who desire  $s'$  and do not block with it so that  $(t, s)$  must also point to  $(t'', s')$ .  $\square$

In the sequel, as in the proof of Proposition 1.8.7, we fix (a collection of) cycles of exchanges which takes us from  $\mu$  to  $\mu'$  once implemented. We consider the digraph  $(N', E_1)$  as built in Section 1.8.10 after Lemma 15. Consider a cycle  $C_1$  in this graph (which exists by Lemma 16). Let  $\mu_1$  be the matching obtained once the cycle  $C_1$  is implemented. In the example introduced in Section 1.8.10, this matching would be:

$$\mu_1 = \begin{pmatrix} t_1 & t'_1 & t_2 & t'_2 & t_3 & t_4 \\ s_1 & s_4 & s_2 & s_2 & s_1 & s_3 \end{pmatrix}$$

We first show the following lemma.

**Lemma 18.**  $\mu'$  Pareto-dominates  $\mu_1$  for teachers.

*Proof.* Fix a teacher  $t$ . If the node  $(t, s)$  to which  $t$  belongs is not part of the cycles of exchanges, we know  $t$  does not move from  $\mu$  to  $\mu'$  and so  $(t, s)$  is not in the cycle  $C_1$ . Hence,  $\mu(t) = \mu_1(t) = \mu'(t)$ . So assume that  $(t, s)$  is part of the cycles of exchanges and let  $s := \mu(t)$  and  $s' := \mu'(t)$  with  $s \neq s'$ . There are three possible cases:

- *Case 1:*  $s = \mu_1(t) \neq s'$ . Because  $\mu'$  Pareto-dominates  $\mu$  for teachers, we have that  $\mu'(t) = s' \succeq_t \mu_1(t) = \mu(t) = s$ .
- *Case 2:*  $s \neq \mu_1(t) = s'$ . In such a case, we trivially have  $\mu'(t) \succeq_t \mu_1(t)$ .
- *Case 3:*  $s \neq \mu_1(t) := s_1 \neq s'$ . By construction of the graph  $(N', E_1)$  when we implement cycle  $C_1$ , we know that there is a unique edge  $((t, s), (t_1, s_1))$  in  $C_1$  and that  $(t, s)$  is not the predecessor of  $(t_1, s_1)$  under the cycles of exchanges, since otherwise,  $t$  would be matched to  $s'$  under  $\mu_1$  which is not the case by assumption. Hence, by construction of  $(N', E_1)$ , the predecessor of  $(t_1, s_1)$  under the cycles of exchanges, say  $(t'', s'')$ , does not 1S-BE point to  $(t_1, s_1)$  and, in addition,  $t$  does not block with  $s_1$  under  $\mu$ ,  $s_1 \succ_t s$  and  $t$  has the highest priority among those who desire  $s_1$  but do not block with it under  $\mu$  and 1S-BE-points to  $(t_1, s_1)$ .

Because  $(t'', s'')$  does not 1S-BE point to  $(t_1, s_1)$ , we know that  $t''$  does not block with  $s_1$ . While because  $(t'', s'')$  points to  $(t_1, s_1)$  under the cycles of exchange, we must have that  $t''$  desires  $s_1$ . Thus, we conclude that  $t \succ_{s_1} t''$ .

Now, proceed by contradiction and assume that  $(\mu_1(t) =)_{s_1} \succ_t s' (= \mu'(t))$ . Because  $t'' \in \mu'(s_1)$  (recall that  $(t'', s'')$  is the predecessor of  $(t_1, s_1)$  under the cycles of exchange) and  $t \succ_{s_1} t''$ , we have that  $t$  blocks with  $s_1$  under  $\mu'$  i.e.  $(t, s_1) \in \mathcal{B}_{\mu'}$ . But, as already claimed,  $(t, s_1) \notin \mathcal{B}_{\mu}$ . This contradicts that  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu}$ . Thus, we must have  $\mu'(t) \succeq_t \mu_1(t)$ .<sup>80</sup>

So we have shown that  $\forall t, \mu'(t) \succeq_t \mu_1(t)$ . □

If we were sure that it is always the case that  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu_1}$ , the proof would be completed. Unfortunately, even if this is true in the one-to-one environment, this may not be true in the many-to-one case. To give an intuition, assume that in the example we have  $t_1 \succ_{s_1} t_3 \succ_{s_1} t_2 \succ_{s_1} t'_2 \succ_{s_1} t'_1$  and  $s_1 \succ_{t_2} s_3 \succ_{t_2} s_2$ . So  $t_2$  blocks with  $s_1$  under both  $\mu$  and  $\mu'$ . But after implementing cycle  $C_1$ , we see that  $t_2$  does not block with  $s_1$  anymore. Indeed, the only teacher for whom  $t_2$  feel justified envy under  $\mu$  is  $t'_1$ . But  $t'_1$  is replaced by  $t_3$  once  $C_1$  is implemented and  $t_1$  has a higher priority than  $t_2$  at  $s_1$  (while  $t_1$  stays matched to  $s_1$ ). So we will show that if this arises, we can find another subgraph of the 1S-BE graph starting from  $\mu$ , call it  $(N', E_2)$ , still with in-degree one for each node so that there is a cycle  $C_2$  in this subgraph and the matching  $\mu_2$  obtained with this cycle keeps the blocking pair  $(t_2, s_1)$ .

**Lemma 19.** *Assume that there is  $(t_1, s_1) \in \mathcal{B}_{\mu'}$  but not in  $\mathcal{B}_{\mu_1}$ . Then there is a teacher  $t_1^*$  with  $s_1^* := \mu(t_1^*)$  s.t  $t_1 \succ_{s_1} t_1^*$  and  $(t_1^*, s_1^*)$  is part of the cycles of exchanges and points to all nodes of the form  $(t, s_1)$  in the graph associated with 1S-BE starting from  $\mu$ .*

Before moving to the proof, let us illustrate the lemma in the example. As explained above, the “problem” in the graph  $(N', E_1)$  (which yields to  $(t_2, s_1) \notin \mathcal{B}_{\mu_1}$ ) is that the node pointing to  $(t'_1, s_1)$  is  $(t_3, s_3)$  and that  $t_3 \succ_{s_1} t_2$ . But we have teacher  $t'_2$  who is less preferred by  $s_1$  than  $t_2$ , is not matched to  $s_1$  under  $\mu$  but is under  $\mu'$ . In addition, since we assumed that node  $(t'_2, s_2)$  is 1S-BE-pointing

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<sup>80</sup>Case 3 in Lemma 18 can be illustrated in the example. The node  $(t, s)$  would be  $(t_4, s_4)$  in the right graph of Figure 1.3.  $t_4$  is matched to  $s_3$  under  $\mu_1$  but is matched to  $s_2$  under  $\mu'$ . Under  $C_1$  (i.e.,  $(t_3, s_3) \rightarrow (t'_1, s_1) \rightarrow (t_4, s_4) \rightarrow (t_3, s_3)$ ), node  $(t_4, s_4)$  points to  $(t_3, s_3)$  while  $(t_2, s_2)$  does not 1S-BE-point to  $(t_3, s_3)$ . Because  $(t_2, s_2)$  points to  $(t_3, s_3)$  in the cycle of exchanges, it means that  $t_2 \in \mu'(s_3)$  so that if  $t_4$  preferred  $s_3$  to his match under  $\mu'$ ,  $s_2$ , it would imply that  $t_4$  blocks with  $s_3$  under  $\mu'$  while he does not under  $\mu$  and so this would yield the contradiction.

to  $(t_1, s_1)$  under  $\mu$ , we can use Lemma 17 to be sure that it is also pointing to  $(t'_1, s_1)$  so that  $t_1^*$  in the above lemma would be  $t'_2$  in the example. The argument in the proof below shows that this construction can be made in general.

*Proof.* Note first that because  $t_1$  desires  $s_1$  under  $\mu$  and  $\mu'$ ,  $t_1$  must also desire  $s_1$  under  $\mu_1$  because, by Lemma 18,  $\mu'(t_1) \succeq_{t_1} \mu_1(t_1)$ . Now, because  $t_1$  blocks with  $s_1$  under  $\mu$ , it means that there is  $t \in \mu(s_1)$  s.t  $t_1 \succ_{s_1} t$ . Fix one such teacher  $t$ . Since, by assumption,  $(t_1, s_1) \notin \mathcal{B}_{\mu_1}$ , it means that  $t$  is not matched to  $s_1$  under  $\mu_1$  and so, when implementing  $C_1$ ,  $t$  has been replaced by a teacher  $t'$  such that  $t' \succ_{s_1} t_1$  since  $t_1$  does not block with  $s_1$  under  $\mu_1$  but desires  $s_1$  under  $\mu_1$ . Since  $t_1$  blocks with  $s_1$  under  $\mu'$  it means that there is a teacher  $t'_1 \in \mu'(s_1)$  s.t  $t_1 \succ_{s_1} t'_1$ , let  $s'_1 := \mu(t'_1)$ . Note first that  $(t'_1, s'_1)$  is part of the cycles of exchanges. To see this, observe that if it was not the case then we would have that  $t'_1 \in \mu(s_1)$  but because  $(t_1, s_1)$  does not block  $\mu_1$ ,  $t'_1 \notin \mu_1(s_1)$ . But because  $\mu_1(t_1) \succ_{t_1} \mu(t_1) = s_1$  and, by Lemma 18,  $\mu'(t_1) \succeq_{t_1} \mu_1(t_1)$ , we conclude that  $t_1$  cannot be matched to  $s_1$  under  $\mu'$ , a contradiction. Hence, if the node  $(t'_1, s'_1)$  1S-BE-points to  $(t, s_1)$  then we can set  $t_1^* := t'_1$  and  $s_1^* := s'_1$  and the argument is complete using Lemma 17. So, now, consider the case where node  $(t'_1, s'_1)$  does not 1S-BE-point to  $(t, s_1)$ . We already know that  $(t'_1, s'_1)$  is part of the cycles of exchanges so let  $(\tilde{t}, s_1)$  be its successor under these cycles of exchanges ( $s_1$  has to be part of this node since  $t'_1 \in \mu'(s_1)$ ). If  $(t'_1, s'_1)$  was 1S-BE-pointing to  $(\tilde{t}, s_1)$  then by Lemma 17 it would also point to  $(t, s_1)$ , a contradiction. So node  $(t'_1, s'_1)$  does not 1S-BE-point to its successor under the cycles of exchanges, i.e.,  $(\tilde{t}, s_1)$ . Thus, we have that  $t'_1$  does not block with  $s_1$  under  $\mu$  (if he were to block with  $s_1$ ,  $(t'_1, s'_1)$  would be 1S-BE-pointing to some node which includes school  $s_1$  and so toward  $(\tilde{t}, s_1)$ , a contradiction) and, by condition 2 of Lemma 15, there is a teacher  $t''_1$ , whose node is part of the cycles of exchanges, who does not block with  $s_1$  under  $\mu$ , desires  $s_1$  and has the highest priority among those who do not block with  $s_1$  under  $\mu$  and desire it. In particular the node  $(t''_1, \mu(t''_1))$  1S-BE-points to  $(\tilde{t}, s_1)$  and so by Lemma 17 points also to  $(t, s_1)$ . Since  $t''_1$  does not block with  $s_1$  under  $\mu$  but that  $t_1$  does, it means that  $t_1 \succ_{s_1} t''_1$ , so we can set  $t_1^* := t''_1$  and  $s_1^* := \mu(t''_1)$ . Here again, we can use Lemma 17 to make sure that  $(t_1^*, s_1^*)$  indeed points to all the nodes  $(t, s_1)$  under the graph of 1S-BE.  $\square$

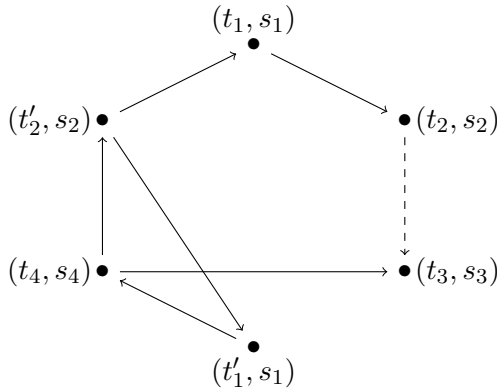
Coming back to our example, we can modify the graph  $(N', E_1)$  in deleting the edge  $((t_3, s_3), (t'_1, s_1))$  and replace it by  $((t'_2, s_2), (t'_1, s_1))$ . In doing so, we obtain a new subgraph of 1S-BE where each

node still has in-degree one and so still has a cycle. But, by construction of this new graph, the matching once the new cycle is implemented keeps  $(t_2, s_1)$  as a blocking pair. This is illustrated in the left graph in Figure 1.4 and the new cycle is now  $(t'_2, s_2) \rightarrow (t'_1, s_1) \rightarrow (t_4, s_4) \rightarrow (t'_2, s_2)$ . The general procedure is given below.

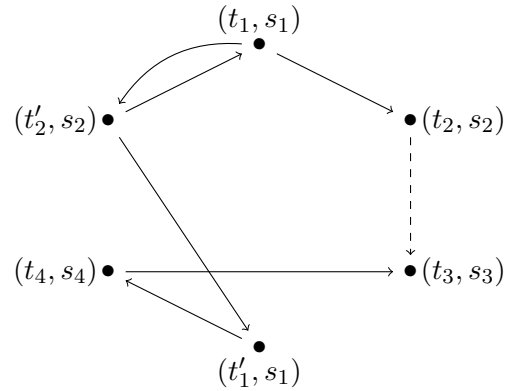
Let us assume that there is a node  $(t_1, s_1)$  such that it is in  $\mathcal{B}_{\mu'}$  but not in  $\mathcal{B}_{\mu_1}$ . Fix a teacher  $t \in \mu(s_1)$  such that  $t_1 \succ_{s_1} t$ . We know that  $t$  must leave  $s_1$  under  $\mu_1$  (because  $(t_1, s_1)$  does not block  $\mu_1$ ) and is replaced by a teacher  $t'$  such that  $t' \succ_{s_1} t_1$ . Since the teacher  $t'_1$  identified in Lemma 19 satisfies  $t_1 \succ_{s_1} t'_1$ , we have that  $t' \neq t'_1$ . So under the graph  $(N', E_1)$ , because  $t'$  replaces  $t$  when at  $s_1$  when we implement  $C_1$ , we must have that  $((t', \mu(t')), (t, s_1)) \in E_1$  and because each node has in-degree one,  $((t'_1, s'_1), (t, s_1)) \notin E_1$ . But that node is an edge in the graph of 1S-BE by construction. So define a new graph  $(N', E_2)$  where  $E_2$  corresponds to  $E_1$  where all edges of the form  $((t', \mu(t')), (t, s_1))$  with  $t_1 \succ_{s_1} t$  have been replaced by  $((t'_1, s'_1), (t, s_1))$  where  $(t'_1, s'_1)$  is as in Lemma 19. So  $(N', E_2)$  is still a subgraph of the 1S-BE graph starting from  $\mu$  and all the nodes in  $N'$  still have in-degree one so that using Lemma 16, we have a cycle  $C_2$ . We let  $\mu_2$  be the matching obtained once  $C_2$  is implemented.

Figure 1.4 – GRAPHS OF  $(N', E_2)$  AND  $(N', E_3)$ .

a)  $(N', E_2)$ .



b)  $(N', E_3)$ .



We obtain:

**Lemma 20.** *We have that:*

1.  $\mu'$  Pareto-dominates  $\mu_2$  for teachers.

2.  $(t_1, s_1) \in \mathcal{B}_{\mu_2}$ .

*Proof.* For part 1, fix a teacher  $t$  and let  $s := \mu(t)$ . Without loss, assume that  $(t, s)$  is part of the cycles of exchanges. First note that the only new edges (i.e. those in  $E_2 \setminus E_1$ ) are those of the form  $((t_1^*, s_1^*), (\tilde{t}, s_1))$  for  $\tilde{t} \in \mu(s_1)$  such that  $t_1 \succ_{s_1} \tilde{t}$ . So if the edge that matched  $t$  under  $C_2$  is an old one (i.e. belongs to  $E_1$ ), the same argument as in Lemma 18 can be used. So assume that the edge that matched  $t$  is of the form  $((t_1^*, s_1^*), (\tilde{t}, s_1))$ , so that  $t = t_1^*$  and  $\mu_2(t) = s_1$ . Using the same notations as in Lemma 19, there are two cases to consider:

- *Case 1:*  $t_1^* = t'_1$ . In that case, we know that  $t'_1 \in \mu'(s_1)$  and so trivially that  $s_1 = \mu'(t_1^*) \succeq_{t_1^*} \mu_2(t_1^*) = s_1$ .
- *Case 2:*  $t_1^* = t''_1$ . If  $\mu'(t''_1) = s_1$  then trivially,  $\mu'(t''_1) \succeq_{t''_1} \mu_2(t''_1) = s_1$ . So assume that  $\mu'(t''_1) \neq \mu_2(t''_1)$  and toward a contradiction that,  $\mu_2(t''_1) = s_1 \succ_{t''_1} \mu'(t''_1)$ . By the proof of Lemma 19, we know that  $t''_1$  does not block with  $s_1$  under  $\mu$  and since  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu}$ , he does not block with  $s_1$  under  $\mu'$  either. Again using the proof of Lemma 19, we know that because, by assumption,  $(t'_1, s'_1)$  does not 1S-BE-point to  $(t, s_1)$  and so  $t'_1$  does not block with  $s_1$  under  $\mu$ . In addition, since  $t'_1 \in \mu'(s_1)$ , we must have  $t'_1$  desires  $s_1$ . Thus, because, by construction of  $t''_1$ , teacher  $t''_1$  has the highest priority among those who do not block with  $s_1$  under  $\mu$  and desire  $s_1$ , we must have  $t''_1 \succ_{s_1} t'_1$ . Because  $t'_1 \in \mu'(s_1)$  and by assumption  $t''_1$  desires  $s_1$  under  $\mu'$ , we obtain that  $(t''_1, s_1) \in \mathcal{B}_{\mu'}$ , which yields a contradiction since, again by construction of  $t''_1$ , we must have  $(t''_1, s_1) \notin \mathcal{B}_{\mu}$ .

For part 2 assume that  $(t_1, s_1) \notin \mathcal{B}_{\mu_2}$ . Since  $(t_1, s_1) \in \mathcal{B}_{\mu'}$ , we have that  $s_1 \succ_{t_1} \mu'(t_1)$ . In addition, by Lemma 18,  $\mu'(t_1) \succeq_{t_1} \mu_2(t_1) \succeq_{t_1} \mu(t_1)$  and so  $s_1 \succ_{t_1} \mu_2(t_1)$ . Then because  $(t_1, s_1) \notin \mathcal{B}_{\mu_2}$ , we must have that all teachers  $t$  s.t  $t \in \mu(s_1)$  and  $t_1 \succ_{s_1} t$  are not matched to  $s_1$  anymore under  $\mu_2$ , i.e., once cycle  $C_2$  is implemented. But under  $(N', E_2)$  the only incoming edge for a node  $(t, s_1)$  with  $t_1 \succ_{s_1} t$  is  $((t_1^*, s_1^*), (t, s_1))$  and since  $t_1 \succ_{s_1} t_1^*$ , it contradicts that  $(t_1, s_1) \notin \mathcal{B}_{\mu_2}$  since  $t_1$  feels justified envy toward  $t_1^*$  under  $\mu_2$ .  $\square$

As for  $\mu_1$ , if we were sure that  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu_2}$ , the proof would be completed. However, as for  $\mu_1$ , it may not be the case. For instance, in the example, if we assume that  $t_2 \succ_{s_2} t_4 \succ_{s_2} t_3 \succ_{s_2} t_1 \succ_{s_2} t'_2$  and  $s_2 \succ_{t_3} s_1 \succ_{t_3} s_3$ , we have that  $(t_3, s_2) \in \mathcal{B}_{\mu'} \subset \mathcal{B}_{\mu}$ . Then, when we implement the cycle  $C_2$  given in

the left graph of Figure 1.4, we can see that we delete the blocking pair  $(t_3, s_2)$  and so  $(t_3, s_2) \notin \mathcal{B}_{\mu_2}$ . With this observation in mind, the idea now is to define a new graph, as we did when we constructed  $(N', E_2)$  from  $(N', E_1)$ , in order to be sure that this is a subgraph of 1S-BE and that it contains a cycle  $C_3$  which once implemented yields a matching which keeps the desired blocking pairs.

For the general case, assume that there is a pair  $(t_2, s_2)$  s.t  $(t_2, s_2) \in \mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu}$  but  $(t_2, s_2) \notin \mathcal{B}_{\mu_2}$ . In that case, we can apply exactly the same argument as in Lemma 19 and exhibit a teacher  $t_2^*$  such that  $t_2 \succ_{s_2} t_2^*$  and  $(t_2^*, s_2^*)$  1S-BE-points to all the nodes of the form  $(t, s_2)$  under the graph of 1S-BE starting at  $\mu$ . However, when  $s_2 = s_1$  if  $t_2^* \succ_{s_1} t_1^*$  then we reset  $t_2^*$  to be  $t_1^*$ . Then, we define a new graph  $(N', E_3)$  with  $E_3$  where  $E_3$  corresponds to  $E_2$  where all edges of the form  $((t', \mu(t')), (t, s'))$  with  $t_2 \succ_{s_2} t$  are replaced by  $((t_2^*, s_2^*), (t, s_1))$  where  $(t_2^*, s_2^*)$  as in the above paragraph. Here again  $(N', E_3)$  is indeed a subgraph of the graph associated with 1S-BE starting from  $\mu$  and each node still has in-degree one. Applying Lemma 16, we get the existence of a cycle  $C_3$  which, once implemented, yield to a matching  $\mu_3$ . In the example,  $t_2^*$  would be  $t_1$  and  $(N', E_3)$  is shown in the right graph of Figure 1.4.

It is easy to see that we can mimic the proof of Lemma 20 in order to obtain the following lemma.

**Lemma 21.** *We have that:*

1.  $\mu'$  Pareto-dominates  $\mu_3$  for teachers.
2.  $\{(t_1, s_1), (t_2, s_2)\} \subset \mathcal{B}_{\mu_3}$ .

In the example, the unique  $t_2^*$  is  $t_1$  and the graph of  $(N', E_3)$  is given in the right graph of Figure 1.4. In that case, the cycle  $C_3$  is  $(t_1, s_1) \rightleftharpoons (t_2', s_2)$ . Note that, once  $C_3$  is implemented, we indeed have  $\{(t_2, s_1), (t_3, s_2)\} \subset \mathcal{B}_{\mu_3}$  and  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu_3}$  so that we have found the desired matching.

Of course, in full generality, it is possible to have a pair  $(t_3, s_3)$  satisfying  $(t_3, s_3) \in \mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\mu}$  while  $(t_3, s_3) \notin \mathcal{B}_{\mu_3}$ . In order to prove the desired result – namely that there is a cycle in the graph associated with 1S-BE starting from  $\mu$  which, once implemented, leads to a matching  $\tilde{\mu}$  such that  $\mu'$  Pareto-dominates  $\tilde{\mu}$  for teachers and satisfies  $\mathcal{B}_{\mu'} \subseteq \mathcal{B}_{\tilde{\mu}}$  – we would continue to apply the same logic. Because we have a finite environment, at some point we must find a matching  $\tilde{\mu}$  with the desired property.





## Chapter 2

# A New Algorithm to Increase Teacher Mobility without Hurting Deprived Regions <sup>1</sup>

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## Abstract

The assignment of teachers to schools incorporates many important issues in education policies: attractiveness of the profession, geographic inequalities, achievements of students and so on. In many countries such assignment is centralized. We propose a generalization of the TO-BE algorithm proposed in [Combe, Tercieux and Terrier \(2016b\)](#) to the framework where there are newly tenured teachers without any initial assignment, tenured teachers asking for a reassignment and possible empty seats in schools. Such generalization keeps the good properties of efficiency and strategy-proofness. It also extends the so called You-Request-My-House-I-Get-Your-Turn (YRMH-IGYT) algorithm proposed by [Abdulkadirouglu and Sonmez \(2003\)](#) for the housing market framework with vacant houses and new applicants to the two-sided and many-to-one matching setting. Using French data on the assignment of teachers, we show the potential benefits in terms of movement brought by the algorithm and the finer controls that it offers in dealing with potentially deprived regions and geographic inequalities. Finally, we give some results on two important practical extensions: the introduction of couples and an alternative assignment approach to control inflows and outflows of teachers in each region in an alternative model.

JEL Classification Numbers: C70, D47, D61, D63, I24.

Keywords: Two-sided matching markets, Teacher Assignment, Fairness, Efficiency, Regional Inequalities, Market Design.

## 2.1 Introduction

In most countries, education systems face two issues: a lack of attractiveness of the teaching profession and a high achievement inequality, notably between students from different social backgrounds. The lack of attractiveness of the profession encompasses both the difficulty to recruit enough teachers (Schleicher, 2012; Corcoran, Evans and Schwab, 1994)<sup>2</sup> and to retain teachers in the profession.<sup>3</sup> The cited articles seem to support a widespread phenomenon both in Europe or in the U.S. In France, it is easy to measure the shortage of qualified teachers as their recruitment is managed centrally by the Ministry of Education. In 2014, 24% of the positions offered by the government remained vacant because of both a lack of applicants and the poor quality of those applying (Direction de l'Evaluation de la Prospective et de la Performance, 2015).

The second issue relates to achievement inequalities and is equally shared by most countries (Organisation for Economic Co-operation and Development, 2014). A student's socio-economic status appears to exert a powerful influence on learning outcomes. France is one of the OECD countries where the relationship between performance and socio-economic status is the highest. Although several factors can explain this poor performance, one seems particularly plausible: some regions gather both the most deprived students and the teachers with the least experience.<sup>4</sup> Knowing that teachers in their first year of experience tend to perform less well than more experienced teachers (Chetty, Friedman and Rockoff, 2014; Rockoff, 2004), the higher share on young teachers in deprived regions might contribute to France poor performance in terms of achievement inequalities.

The issue of teacher assignment to schools and regions stands at the crossroads between both attractiveness and inequality issues. Regarding the attractiveness of the teaching profession, teachers' geographic mobility prospect is a key factor. In the U.S, Boyd et al. (2005) showed that more qualified teachers are more likely to leave schools with low achieving students. Boyd et al. (2008) mentioned that almost a quarter of the entering public school teachers quit the profession within 3

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<sup>2</sup>In the PISA 2009 assessment, an average of close to 20% of 15-year-olds were enrolled in schools whose leaders reported that a lack of qualified mathematics or science teachers was hindering instruction in their schools. In Germany, Luxembourg or Turkey, over 40% of school leaders report that this is a problem. In Sweden, the educational system will lack teachers by 2020, around 20% of its labor force (Sweden, 2012).

<sup>3</sup>Ingersoll (2003) reported that the attrition rate of teachers is around 50-60% in the U.S. In Sweden, Lindqvist, Nordäng and Carlsson (2014) reported that around 16% of the graduated teachers work outside the educational system.

<sup>4</sup>Two of the most deprived regions (Créteil and Versailles) have (1) one of the highest share of students enrolled in "priority education" (Direction de l'Evaluation de la Prospective et de la Performance, 2014) and (2) the smallest ratio of the number of teachers aged more than 50 over the number of teachers aged less than 30. The term "priority education" refers to the diverse programs and labels used by the government to define deprived schools.

years and that this attrition has a negative impact on students' achievements. In the U.K, [Allen, Mian and Sims \(2016\)](#) reported that schools with lower income students are more likely to have teachers with less experience and a high turnover rate. In a recent French report on the attractiveness of the teaching profession, [Périer and Gurgand \(2016\)](#) reported that, among a sample of 423 people deciding not to pass the teaching qualification in France, 10% report that it is due to the evolution of careers and 2% explicitly mention the assignment process as a cause. These issues seem to be shared across many countries whether they have a centralized assignment system: France, Italy ([Barbieri, Cipollone and Sestito, 2007](#)), Mexico ([Pereyra, 2013](#)), Turkey ([Dur and Kesten, 2014](#)), Uruguay ([Vegas, Urquiola and Cerdàn-Infantes, 2006](#)) and Portugal, or decentralized ones: for instance, the United-States or the United Kingdom. Our work will focus on the systems where the assignment of teachers to schools is managed by the central administration.<sup>5</sup> In these centralized systems, teachers submit ranked lists of wishes to a public administration, each school ranks teachers, and the central administration obtains assignments by running an algorithm. In order to increase the attractiveness of the teaching profession, a spontaneous objective might be to satisfy as much as we can teachers' mobility requests. Unfortunately, these requests are currently poorly satisfied: in 2014 in France, only 43.9% of the secondary school tenured teachers asking for a new region were satisfied. For primary school teachers, this figure is even lower as less than 25% of the mobility requests are satisfied. A contribution of this paper is to show that a significant room for improvement exists for teacher mobility compared to the current algorithm. We explain how a different algorithm could significantly increase the number of teachers who obtain a new assignment.

The issue of teacher assignment to schools and regions is also highly related to students' achievement inequalities, as highlighted by a recent French report ([Inspection générale de l'administration de l'Education nationale et de la recherche, 2015](#)). In the U.S, [Ronfeldt, Loeb and Wyckoff \(2013\)](#) showed that the teachers' turnover has an important impact on students' achievement. In the

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<sup>5</sup>For decentralized systems, a more careful analysis is needed. First, our work can help in the comparison of both systems since it offers some alternative possibilities and results for the centralized ones. Second, the comparison between both systems is an important debate in the literature. For instance, [Agarwal and Somaini \(2014\)](#) reported that the use of a decentralized system for the allocation of interns would lead to lower wages. There are also conflicting evidences on the impact of wages on teachers' mobility and willingness to join or leave the profession. [Stinebrickner \(1998\)](#) showed, using U.S longitudinal surveys, that the first spell in teaching is more responsive to wages than working conditions. However, [Imazeki \(2005\)](#) and [Ondrich, Pas and Yinger \(2008\)](#) found that large salary increases are needed to reduce the attrition of teachers, which can be problematic since salaries of public teachers are already an important part of public expenditures. Conversely, [Hanushek, Kain and Rivkin \(1999\)](#) found, using panel data in Texas, that teachers' mobility is more affected by characteristics of the students than by salaries. Concerning the attrition of teachers, [Cooper and Alvarado \(2006\)](#) state that the U.S and the U.K have higher attrition rates (from 30 to 50% within the first three years of teaching) than for instance France, Germany or Portugal.

U.K, [Allen, Mian and Sims \(2016\)](#) reported that schools serving lower income students are more likely to have inexperienced teachers and a high turnover. The centralized process used to assign teachers to regions and then to schools is a powerful tool in policy makers' hands to homogenize the distribution of teachers across schools and regions, regulate teacher turnover, and better meet the specific needs of different schools. Yet, this tool is far from being fully exploited. Most importantly, the objective of better satisfying teachers' mobility requests and better distributing teachers across deprived schools might not be compatible. Increasing teachers' mobility can worsen the unequal distribution of teachers across schools. Indeed, in France, as the majority of teachers asking for a new assignment are initially assigned one of the three most deprived regions, better satisfying teacher mobility would significantly increase the outgoing mobility from these deprived regions. As a result, to ensure that no position remains vacant at the end of the mobility process, new teachers, most of them newly-tenured, would be assigned to these unattractive regions. Hence, every year, better satisfying teacher mobility would be done at the expense of the deprived regions, whose teaching workforce would have less experience. Based on that trade-off between teachers' satisfaction and regions equality, the main contribution of our paper is to suggest an algorithm which increases teacher mobility, while ensuring that deprived regions are not hurt by the mobility process.

The teacher assignment problem is similar to the college admission problem as defined in [Gale and Shapley \(1962\)](#). Yet, a key difference exists between the college admission problem and the assignment of teachers to schools: when one wants to assign teachers to schools, many of the teachers already have a position. Hence, an additional individual rationality constraint applies: tenured teachers must be able to keep their initial position if they do not obtain any of the schools they ranked. In [Combe, Tercieux and Terrier \(2016b\)](#), we showed that, to respect this constraint, the algorithm currently used by the French Ministry of Education to assign teachers to regions is a modified version of the Deferred Acceptance algorithm ([Gale and Shapley, 1962](#)). We highlight how this algorithm limits the overall mobility and suggest an alternative algorithm – the Teacher Optimal Block Exchange (TOBE) - which is strategy proof and two-sided maximal, meaning that (1) it Pareto-dominates the initial assignment for both teachers and schools and (2) it cannot be improved in terms of (2i) efficiency for both teachers and schools as well as (2ii) fairness. By using data on the assignment of teachers to public schools in France, we empirically estimate the magnitude of gains, and find that the number of teachers moving from their initial assignment is more than doubled

under our mechanisms.

In the current paper we enrich the previous analysis by adding two new features. Firstly, in [Combe, Tercieux and Terrier \(2016b\)](#), both the theoretical and the empirical analysis are done in a market that incorporates only teachers with an initial assignment. To better match the theoretical analysis of the paper<sup>6</sup> vacant positions and newly tenured teachers with no initial assignment have been ignored. In the current paper, we run simulations on the complete market by incorporating unassigned teachers and vacant positions, so that we almost perfectly replicate the assignments of the French Ministry of Education.<sup>7</sup> These new simulations are likely to be more policy-relevant. The second difference is perhaps the most important. While the previous paper characterizes a new class of mechanisms, and tests their performance, it does not take into account differences between regions in terms of attractiveness and experience of their teachers.<sup>8</sup> Yet, there is a great diversity in terms of attractiveness of the 31 French regions. Three regions are recognized for being particularly unattractive to teachers (called Créteil, Versailles and Amiens) whereas some other regions are very attractive. To illustrate this, the ratio of the number of tenured teachers asking to enter a region over the number of tenured teachers asking to leave the region varies from 15.5 in the most attractive region (Rennes) to 0.03 in the least attractive region of Créteil. Every year, the three least attractive regions concentrate about 50% of the teachers asking for a new assignment. One of the key contribution of this paper is to take into account the differences between regions in terms of attractiveness and experience of their teachers, with the objective to design an assignment mechanism which does not hurt the most deprived regions. This algorithm will be a modified version of the TOBE algorithm presented in [Combe, Tercieux and Terrier \(2016b\)](#).

The first part of this paper is theoretical and aims to present the algorithm currently used by

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<sup>6</sup>One of the contributions is to highlight the significant lack of mobility under the modified version of the Deferred-Acceptance because this algorithm does not allow the implementation of specific cycles of exchanges, contrary to the TOBE algorithm we suggest. TOBE allows teachers to exchange their assignments when a cycle of blocking pairs exists. As illustrated by a leading example, by reducing the number of blocking pairs, these cycles improve the welfare of both sides of the market (teachers and schools). A key point is that these specific cycles exist only for initially assigned teachers as no teacher would have a justified envy for a teacher who is initially unassigned. Hence, as the difference between the algorithm currently used by the French Ministry of Education and the TOBE algorithm stands in identifying cycles of matched teachers, it was natural for our analysis to consider only the sample of initially assigned teachers.

<sup>7</sup>Vacant positions mainly correspond to teachers who retire. In 2013, 9793 secondary school teachers retired in the public sector. The same year, about 7900 newly tenured teachers required a first assignment. As would be expected, the number of teachers retiring almost matches the number of newly tenured teachers asking for a first assignment.

<sup>8</sup>This omission makes sense in an environment where only assigned teachers are considered as one of the main difference between regions is the difference in their share of tenured versus newly-tenured teachers. This difference does not exist when only assigned teachers are considered.

the French Ministry of Education. Then, we present a generalized version of the TOBE algorithm (Combe, Tercieux and Terrier, 2016b) in a broader market where vacant positions and teachers with no initial assignment are incorporated. We show that the “generalized TOBE” keeps good properties, in particular the strategy-proofness and two-sided efficiency. As we will show, this generalization is tightly linked to the generalization of the well known Top Trading Cycles (TTC) algorithm proposed by Abdulkadirouglu and Sonmez (1999). In the case of a house allocation problem where one wants to incorporate vacant houses and agents without any initially, they proposed an algorithm that is efficient and strategy-proof. As we will show, our generalization of TOBE can be seen as a generalization of their algorithm that incorporates two additional features compared to their framework: i) a many-to-one environment where schools can have more than one position to allocate and ii) the existence of preferences on the schools’ side.

The second part of the analysis is empirical. We use French data on the 2013 assignment of teachers to regions to simulate different versions of the TOBE algorithm and compare their performance to the current algorithm for both attractive and unattractive regions. Firstly, we implement the standard version of TOBE. From the theoretical section, we know that TOBE is individually rational for schools. This implies that, in all regions, the position left by a departing teacher must be filled by an incoming teacher who is preferred by the school.<sup>9</sup> This is the “higher preference criterion”. Our simulations show that, with the standard TOBE which fully respects this criterion, mobility slightly decreases compared to the current situation (-2.48%). This is due to two effects that compensate each other: mobility significantly increases in attractive regions (+31.66%), but reduces in unattractive ones (-44.08%). As retaining too many teachers in unattractive regions might not be satisfying, we test several alternative options. A first option consists in relaxing the higher preference criterion in all regions, so that outgoing teachers do not need to be replaced by preferred teacher. This gives us a sense of the maximal improvement we could obtain and we find that the overall mobility could be increased by 56.98%. Yet, it might not be satisfying to significantly increase outgoing mobility from deprived regions since it can decrease the experience of the assigned teachers and ultimately hurt the students. Hence, the second option we test consists in relaxing the higher preference criterion in attractive regions only. This option increases the teacher mobility in these regions (by 79.43%), while ensuring that relatively deprived regions are not hurt in terms of

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<sup>9</sup>We will discuss later the meaning of the preferences of the schools in our setting.



experience of their incoming teachers. But this second option still significantly retains teachers in deprived regions. Hence, our last and most convincing option consists in relaxing the higher preference criterion in both attractive regions and for a certain share of the teachers in unattractive regions. Under this scenario, the number of teachers who obtain a new assignment increases by 79.47% in attractive regions, while remaining relatively constant by construction (+3.56%) in deprived regions. As it will be later detailed, these results depends on another specific parametrization of the TOBE algorithm concerning an exogenous order. We also provide in the Appendix additional results using a different ordering.

Taken together, our results show that a change in the centralized process used to assign teachers to regions in France could result in a significant improvement in the number of teachers obtaining a new region, without hurting the mobility in deprived regions. As different choices can be made regarding the identification of deprived regions, the choice of teachers to target in these regions and the parametrization of the order used, the high flexibility of our algorithm allows a fine tuning of teacher mobility in different regions. From a broader perspective, this paper highlights how centralized assignment processes can significantly impact education systems in two key dimensions. (1) By improving teachers' geographic mobility prospects, they can make the teaching profession more attractive. (2) By influencing the distribution of teachers across territories, they can increase teachers' experience in the most deprived regions and help reduce the persistent achievement gaps observed in most education systems.

## 2.2 Institutional context

**Teacher recruitment and assignment.** Teachers' certification and recruitment is highly centralized in France. Anyone who wishes to become a teacher has to pass a competitive examination. Those who succeed are allocated a teaching position by the ministry for a probation period of one year, at the end of which they get tenure or not. Once they get tenure, teachers in public schools become civil servants. The government manages both the first assignment of newly tenured teachers to a region, and the mobility between regions of tenured teachers who previously received an assignment but wish to change. This centralized assignment process is used for public school

teachers only. This research does not cover private school teachers.<sup>10</sup>

**Number of teachers, inflows and outflows of the profession.** In 2014, there are 700 000 secondary public school teachers in France. This number evolves every year due to both inflows and outflows of the profession. The outflows are mainly due to teachers retiring and leaving their position, while the inflows correspond to newly tenured teachers who have passed the recruitment exam and validated their probation year. As a result, every year, the central administration faces (1) some positions becoming vacant, (2) some newly tenured teachers wishing to get their first assignment and (3) a pool of tenured teachers who already have a position but wish to change. These three elements constitute the core of the yearly centralized mobility process managed by the French Ministry of Education.<sup>11</sup> In 2013, 9793 secondary school teachers retired in the public sector. The same year, about 25 100 teachers submitted a list to be assigned or reassigned to a region. Among them 17 200 are tenured teachers and 7900 are newly tenured teachers requiring a first assignment. Finally, it is worth noticing that the analysis done in this paper incorporates the three elements mentioned above: vacant positions, assigned teachers, and non-assigned teachers. This is one of the key difference compared to our previous paper where we had decided to run all simulations on assigned teachers only, hence omitting vacant positions and newly tenured teachers.

**A two-step assignment process.** Since 1999, teachers' assignment procedure takes place in two successive phases.<sup>12</sup> The first step consists in assigning teachers to one of the 31 French regions. It takes place between November and March for the following school year. This step is managed centrally by the government which runs the first algorithm that determines teachers' regional assignment. In a second step, teachers newly assigned to a region and teachers wishing to change schools within their region submit a list of ordered schools. This takes place between March and July (depending on the region). Since 1999, this stage is managed directly by administrations within the regions.<sup>13</sup> As explained in [Combe, Tercieux and Terrier \(2016b\)](#), our empirical analysis

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<sup>10</sup>The French education system is divided into public schools and private schools. Private schools make up 16% of the teachers. For them, the recruitment process is similar but public and private school teachers face completely different rules for their mobility – between regions and between schools. In private schools, teachers apply directly to schools – as would be the case in usual labor markets. Moreover, the certification to teach in private school requires to pass a different competitive exam than the one for public schools. The preparation of this exam being very time consuming, the number of teachers moving between public and private schools is small.

<sup>11</sup>This mobility process is called “Mouvement National à Gestion Déconcentrée” (MNGD)

<sup>12</sup>Before 1999, teachers' assignment to schools was managed centrally by running an algorithm once, which assigned teachers directly to schools. This highly centralized process was argued to be at odds with the regional nature of most demands: the majority of teachers asking for a transfer ranked schools within their current region. For them, the assignment could be managed directly by local authorities. To give more autonomy to regions in their teachers' assignments and reduce the workload at the level of the central administration, a two-step process has been introduced.

<sup>13</sup>Regions are in charge of running the second algorithm that determines the final assignment of teachers to schools.

focuses on the first step of the assignment – to regions - because of potential strategic issues in the reported preferences of teachers during the second phase of the assignment. Participation to the assignment mechanism is compulsory for all newly tenured teachers who have never been assigned a position. Participation is optional for any other tenured teacher wishing to change schools or regions. Students who have just passed the exam to become teachers are also assigned to a school for their probation year on the basis of a similar algorithm. This description does not cover this initial assignment.

**Regions’ preferences.** The central administration defines a ranking over teachers using a point system. It takes into account three legal priorities: spousal reunification, disability, and having a position in a disadvantaged or violent school. Several individual characteristics also enter the computed score: total seniority in teaching, seniority in the current school, time away from the spouse and/or children... The criteria considered to compute teachers’ scores are numerous but, in the end, most criteria are used by only a small number of teachers so that seniority is the main criterion (even though it is not the one that gives the most points). The criteria are commonly known and freely available online.<sup>14</sup> As discussed in [Combe, Tercieux and Terrier \(2016b\)](#), regions and schools are not strategic entities since the score of a teacher is fixed by law. However, the point system incorporates clear social objectives from the policy maker. As we discussed in [Combe, Tercieux and Terrier \(2016b\)](#):

*This makes it relevant to adopt an approach “as if” schools’ priority rankings were schools’ preferences. For instance, teacher priorities at schools are determined primarily by their experience, notably in disadvantaged schools, which reflects the administration’s efforts to assign more-experienced teachers to disadvantaged students. If we were to consider teachers as the only welfare-relevant entities, teachers could exchange their positions, and thereby decrease the number of experienced teachers in disadvantaged schools. From an administrative perspective, it would be difficult to consider this a Pareto improvement, as disadvantaged students would be harmed by these assignments. Hence, a meaningful requirement would be to allow for exchanges of positions across teachers only*

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Differences in the procedures between regions are limited. They use the same algorithm as the first step but can use different criteria to rank teachers. In practice, very few of them use different ones. In the case where a teacher has been assigned to a different region than his initial one at the first step and that, at the second step, he ends up being rejected by all the schools he ranked, he is temporarily assigned as a substitute teacher in this region.

<sup>14</sup> An official list of criteria used to compute the point system is available on the government website [http://cache.media.education.gouv.fr/file/42/84/6/annexeI-493\\_365846.pdf](http://cache.media.education.gouv.fr/file/42/84/6/annexeI-493_365846.pdf)

*when they do not negatively affect the experience of teachers in disadvantaged schools. This is exactly what the “as if” approach offers when we require a Pareto improvement over the initial assignment. As mentioned, several other criteria are used to determine teacher priorities that might also reflect broader social objectives. For instance, spousal reunification and children reunification give a priority bonus to teachers at schools close to where their spouses or children live. Again, one can easily see the social objective motivating these priorities. In this context, a meaningful requirement is to allow for exchanges of positions across teachers that are not at the expense of the experience of teachers in (possibly disadvantaged) schools, except when it can allow a teacher to join his/her family. Here again, this is what the “as if” approach provides.*

Thus in the following, we will refer to the ranking of a region or a school over teachers as “preferences of the region/school” and if a teacher has a higher ranking than another one we will say that the former is “preferred to” the latter.

## 2.3 The French Assignment Algorithm

In this section, we present the algorithm used to assign teachers to public schools in France, and show that this algorithm is equivalent to a modified version of the Deferred-Acceptance algorithm, where schools rank first the teachers they are initially assigned to.

### 2.3.1 The Model

A problem  $\mathcal{P}$  of teacher assignment to schools could be described as follows. There is a finite set  $T$  of teachers and a finite set  $S$  of schools. Each teacher  $t \in T$  has a preference order  $\succ_t$  over the schools and each school  $s \in S$  has a preference order  $\succ_s$  over the teachers.<sup>15</sup> Teachers could be either *initially assigned* or *initially unassigned*. Initially assigned teachers already have an initial assignment and would like to be reassigned to a new school. Initially unassigned teachers ask for an assignment and so do not have any initial one. As we will detail in the empirical section, the

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<sup>15</sup>Teachers are allowed to find some schools unacceptable. We will denote by  $\emptyset$  the *null school* i.e. the possibility to remain unmatched. However, as it is the case in practice, we assume that a initially assigned teacher ranks as acceptable his initial school. For schools, we assume that they can rank some of their initial teacher as “unacceptable”, meaning that this school accepts that this teacher leaves without being replaced and we will also use the notation  $\emptyset$  (this distinction will be useful later). However, we assume that all schools rank as acceptable teachers who are not initially matched to them.

French policy makers are used to another distinction. They distinguish between *tenured teachers* who got their civil servant position for at least more than a year and *newly tenured teachers* who are young teachers who have been just tenured and need their very first assignment. In practice, all initially assigned teachers are tenured ones and all newly tenured teachers are initially unassigned. However, there exist tenured teachers who do not have an initial assignment. They are teachers who did not teach for some years for various reasons (longer maternity leave, temporary work in the private sector...) but come back into the profession. In our dataset, they represent 18% of the initially unassigned teachers. The total number of positions in a school  $s$  will be denoted  $q_s$ , it could be decomposed in two types of positions: *Free Positions* (FP) and *Potentially Free Positions* (PFP). Free positions correspond to empty positions waiting to be filled by a teacher. They may have been created by the Ministry or may have been occupied by a former tenured teacher who retired. Potentially Free Positions are positions currently occupied by a tenured teacher who is asking a reassignment. If he/she obtains another position, the Potentially Free Positions will become a Free Position. We will denote respectively by  $q_s^f$  and  $q_s^p$  the number of FP and PFP of a school  $s$  so that  $q_s = q_s^f + q_s^p$ . A school cannot receive more than  $q_s$  teachers among those who participate to the assignment procedure i.e. teachers in  $T$ .

So far, we have defined schools' preferences as being on individual teachers, i.e. over  $T$ . Yet, as schools have multiple seats, we need to define comparisons over sets of teachers i.e.  $2^T$ . Fix a school  $s$  with  $q$  seats and two possible assignments for this school given by  $\mu = (t_1, \dots, t_q)$  and  $\mu' = (t'_1, \dots, t'_q)$ . We assume that teachers have been ordered in a decreasing order of preferences i.e.  $t_k \succ_s t_{k+1}$  and  $t'_k \succ_s t'_{k+1}$  for respectively  $\mu$  and  $\mu'$  (empty seats are ordered using  $\emptyset$ ). We say that a school  $s$  **prefers**  $\mu$  to  $\mu'$ , and write  $\mu \succ_s \mu'$ , if  $\forall k = 1, \dots, q: t_k \succeq_s t'_k$  and strictly for at least one.

A **matching** is described by a function  $\mu : T \mapsto S \cup \{\emptyset\}$  where  $\mu(t)$  is the school where teacher  $t$  is assigned under  $\mu$ . If he is unassigned, we let  $\mu(t) = \emptyset$ . A matching must respect the capacity constraints i.e.  $\forall s \in S, |\mu^{-1}(s)| \leq q_s$ . The **initial matching** of teachers will be denoted by  $\mu_0$ . Note that if teacher  $t$  is initially unassigned, then  $\mu_0(t) = \emptyset$  and for a initially assigned teacher,  $\mu_0(t) \in S$  is the initial school of teacher  $t$  at the beginning of the assignment procedure. A **matching mechanism** gives, for each possible teacher assignment problem, a given matching compatible with this problem. Formally, a mechanism  $\varphi$  is a function  $\varphi : \mathcal{P} \mapsto \mathcal{A}$  where  $\mathcal{A}$  is the set of all possible assignments.<sup>16</sup> In the following, we fix the set of teachers, the set of schools, the capacities, the

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<sup>16</sup>Formally here,  $\mathcal{A}$  is the set of all possible assignments for all possible problems. We assume that a mechanism is

preferences of the schools and the initial assignment. A mechanism will then be seen as a function which gives a matching  $\mu$  for each possible preference profile of the teachers  $\succ := (\succ_t)_{t \in T}$ .

A matching  $\mu$  is:

- **2-individually rational** (2-IR) if teachers and schools weakly prefer their assignment under  $\mu$  to the one under  $\mu_0$ . Formally,  $\forall t \in T, \mu(t) \succeq_t \mu_0(t)$  and  $\forall s \in S, \mu(s) \succeq_s \mu_0(s)$ .
- **2-Pareto efficient** (2-PE) if there is no other matching  $\mu'$  that Pareto dominates  $\mu$  for the teachers and schools. Formally, there is no  $\mu'$  s.t.  $\forall t \in T, \mu'(t) \succeq_t \mu(t)$  and  $\forall s \in S, \mu'(s) \succeq_s \mu(s)$  and strictly for some teacher or school.
- **1-Individually Rational** (1-IR) if only the teachers prefer their assignment under  $\mu$  than their one under  $\mu_0$ . Formally,  $\forall t \in T, \mu(t) \succeq_t \mu_0(t)$
- **1-Pareto Efficient** (1-PE) if there is no other matching  $\mu'$  that Pareto dominates  $\mu$  for the teachers. Formally, there is no  $\mu'$  s.t.  $\forall t \in T, \mu'(t) \succeq_t \mu(t)$  and strictly for some.

A mechanism  $\varphi$  is 2-PE/2-IR/1-PE/1-IR if it always gives a matching that is 2-PE/2-IR/1-PE/1-IR.

A mechanism  $\varphi$  is **strategy-proof** if for any possible preference profile  $\succ$ , any teacher  $t$  and any preferences  $\succ'_t$ , we have that<sup>17</sup>  $\varphi_t(\succ_t, \succ_{-t}) \succeq_t \varphi_t(\succ'_t, \succ_{-t})$ .

### 2.3.2 The French Assignment Mechanism

The French Assignment Mechanism (FAM) used to assign teachers to regions follows two steps. We will designate by respectively  $FAM_1$  and  $FAM_2$  the mechanisms used at each step. The final matching is the one given by  $FAM_2$ . Let  $\mu_1$  be the matching obtained at the end of  $FAM_1$ . At each step  $k$  of  $FAM_1$ , each school  $s$  has a counter of Free Positions  $q_s^f(k)$  and we let  $\mu_1(k)$  be the matching at that step. The steps of  $FAM_1$  are the following:

- **Step 0.**  $\forall s \in S$ , we let  $q_s^f(0) = q_s^f$  and  $\mu_1(0) = \mu_0$ .
- **Step  $k \geq 1$ .** If no school has Free Positions or if they all have proposed to all the teachers of their preference list, then let  $\mu_1(k-1)$  be the matching of  $FAM_1$ . If a school  $s$  still has Free Positions i.e.  $q_s^f(k-1) > 0$  and has not yet proposed to all the teachers of its list of

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restricted to choose a matching that is compatible with a given problem.

<sup>17</sup> $\varphi_t(\succ)$  is the school obtained by teacher  $t$  when the reported profile is  $\succ$ . For a preference profile  $\succ$ ,  $\succ_{-t}$  is the profile of all the teachers except teacher  $t$ .

preferences then let it propose to the next teacher it prefers among those it has not proposed yet. If a teacher receives several offers, he chooses the school he prefers between the offers he received and his school under  $\mu_1(k-1)$ . If he has not been assigned yet, he chooses the school he prefers. Let  $s$  be the school he chooses and  $s'$  his school under  $\mu_1(k-1)$ . If  $s \neq s'$ , we match this teacher to  $s$  under  $\mu_1(k)$  and we update the counter of free positions; i.e. if  $s' \neq \emptyset$ ,  $q_{s'}^f(k) = q_{s'}^f(k-1) + 1$  and  $q_s^f(k) = q_s^f(k-1) - 1$ . Then we go to step  $k+1$ .

FAM<sub>1</sub> is very similar to the standard school-proposing Deferred Acceptance (DA) algorithm in how it manages offers and rejections. The only difference comes from the presence of PFP which do not exist in the standard school-proposing DA. In the latter, no teacher is initially matched and schools make offers for all their vacant seats. In other words, the counter of free positions  $q_s^f(k)$  both counts FP and PFP in the standard schools proposing DA.

In our description of FAM<sub>1</sub>, we define the initial matching (at step 0) as being the initial assignment of teachers  $\mu_0$ . Under the school-proposing DA, it is equivalent to force the schools to first propose to their initially matched teachers. Indeed, under FAM<sub>1</sub>, a teacher is guaranteed to keep his initial position, unless he receives, during the algorithm, an assignment that he prefers. Therefore FAM<sub>1</sub> is individually rational. An equivalent matching would be obtained with an alternative solution: under the school-proposing DA, one can force schools to first offer their positions to their initially matched teachers. One has just to artificially modify the preference list of each school and put its initially matched teachers at the top of its list. It ensures that the school-proposing DA becomes individually rational. We call this procedure school-proposing DA\*. We summarize our discussion in the following proposition:

**Proposition 2.3.1.** *FAM<sub>1</sub> is equivalent to the school-proposing DA\*.*

Similarly, one can define the teacher-proposing DA\* where the same modification of the schools' preferences is done but where the teacher-proposing DA is used. Once  $\mu_1$  is obtained with FAM<sub>1</sub>, the procedure FAM<sub>2</sub> is used starting from this assignment. At each step  $k$  of FAM<sub>2</sub>, we note  $\mu_2(k)$  the matching obtained at this step:

- **Step 0.** Let  $\mu_2(0) = \mu_1$ .
- **Step  $k \geq 1$ .** Build a graph where the nodes are the teachers and the schools. For each school  $s$ , we consider the teacher  $t$  who is preferred by  $s$  among the set of teachers who:

- Are not matched with  $s$  under  $\mu_2(k-1)$ .
- Prefer  $s$  to their school under  $\mu_2(k-1)$ .

We let this teacher  $t$  point to  $s$ . Each school points to the teacher matched to it under  $\mu_2(k-1)$ . If there is no cycle in this graph, let  $\mu_2(k-1)$  be the final matching. Otherwise if there is a cycle, we implement it in matching teachers to the school they are pointing to and go to step  $k+1$ .

This mechanism is formally equivalent to the *Stable Improvement Cycles Algorithm* proposed by [Erdil and Ergin \(2008\)](#). Cycles previously defined are called *Stable Improvement Cycles* (SIC). A first proposition by [Erdil and Ergin \(2008\)](#) is that the matching obtained with this procedure is stable<sup>18</sup> and the next theorem is their main result:

**Theorem 9** ([Erdil and Ergin, 2008](#), Theorem 1). *If  $\mu$  is a stable matching and is Pareto-dominated by another stable matching  $\mu'$ , then there exists a SIC.*

Using the above theorem, we can now state the following proposition:

**Proposition 2.3.2.** *The FAM is equivalent to the teacher-proposing DA\*.*

*Proof.* Let  $M$  be the set of stable matchings with respect to the modified preferences of the schools i.e. where initially matched teachers at each school are moved at the top of the preference list of that school. Let  $\bar{\mu} \in M$  be the matching obtained by teacher-proposing DA\*. We know by [Gale and Shapley \(1962\)](#) that  $\forall \mu \in M$  s.t  $\mu \neq \bar{\mu}$ ,  $\bar{\mu}$  1-Pareto dominates  $\mu$ . According to Proposition 2.3.1, we know that the matching  $\mu_1$  obtained by FAM<sub>1</sub> is equal to the one obtained with school-proposing DA\* so  $\forall \mu \in M$ ,  $\mu$  1-Pareto dominates  $\mu_1$ . Suppose that the matching  $\mu_2$  obtained with FAM<sub>2</sub> is different from  $\bar{\mu}$ . In that case, we know by [Erdil and Ergin \(2008\)](#) that  $\mu_2 \in M$  so  $\bar{\mu}$  1-Pareto dominates  $\mu_2$ . So using Theorem 1 of [Erdil and Ergin \(2008\)](#), we know that there exists a SIC in the graph of FAM<sub>2</sub> starting at  $\mu_2$ , which contradicts the fact that  $\mu_2$  was the matching obtained by FAM<sub>2</sub>. □

In what follows, we refer to the teacher-proposing DA\* as simply DA\*.

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<sup>18</sup>In our environment, the stability concept is defined using the modified preferences of the schools under DA\*. As seen above, the matching may not be stable with respect to the true preferences of the schools.



## 2.4 Generalization of TOBE

In [Combe, Tercieux and Terrier \(2016b\)](#), we propose an alternative mechanism to DA\*: the *Teacher Optimal Block Exchange* algorithm (TOBE) in an environment where there are only tenured teachers and no Free Positions. This algorithm could be seen as a modified version of the *Top Trading Cycles* algorithm (TTC) proposed by [Shapley and Scarf \(1974\)](#) for house allocation problems with only existing tenants. Indeed, TOBE is equivalent to TTC where the preferences of the teachers are modified before running the algorithm. For each teacher, we delete from his preference list the schools with which he is not blocking the initial matching and then run TTC using these modified lists.<sup>19</sup> We showed that TOBE is 2-PE and strategy-proof. When testing its performance on the French data of teachers assignment (restricted to the initially assigned teachers), it performs significantly better than the FAM in terms of efficiency and fairness.

### 2.4.1 Preliminaries: generalization of TTC

In this section, we generalize TOBE to the case where there are initially unassigned teachers and Free Positions. Intuitively, this problem is tightly connected to the generalization of TTC where there are available houses and agents without any initial house. Hence, in the same way as we have defined TOBE as a TTC where the preferences of the teachers are modified before running the algorithm, we will present the generalization of TOBE to the complete market as the generalized TTC where the preferences of the teachers are modified before running the algorithm. The generalization of TTC has been proposed by [Abdulkadirouglu and Sonmez \(1999\)](#) who introduced the distinction between *available* and *occupied* houses in the description of the top trading cycles mechanism. Going further, they show that the outcome of the generalized TTC mechanism can be found by using the *You-Request-My-House-I-Get-Your-Turn* algorithm (YRMH-IGYT). As stated by the authors, this equivalence is particularly useful as the generalized TTC makes it easier to prove results, while the YRMH-IGYT can be easily coded on computers. Our generalization of TOBE can be seen as generalization of their algorithm. It allows to incorporate to additional features compared to their setting: i) a many-to-one environment where each school can have several positions and ii) the existence of preferences on the schools' side. Formally, when all the schools have only one seat

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<sup>19</sup>There is a slight change to operate due to the many-to-one environment since TTC is originally defined for a one-to-one setting.

and rank their initially assigned teachers at the bottom of their preferences, our environment is equivalent to theirs.

Before moving to the generalization of TOBE, it is useful to present their generalization of the top trading cycles mechanism in a environment equivalent to theirs i.e. where each school has only one position.<sup>20</sup> We will also present the YRMH-IGYT algorithm that was shown to be equivalent to their generalization of TTC. Start with a set of teachers  $T$ , a set of schools  $S$  and an initial assignment  $\nu_0$  (with possibly non matched teachers/schools). Each teacher  $t \in T$  has a preference relation  $\succ_t$  over  $S \cup \{\emptyset\}$ . A school can have a vacant position or can be matched to a teacher. Fix an arbitrary exogenous ordering of the teachers  $f$ . For each school  $s$ , we define an order  $\succ_s$  as  $f$  but where its initially matched teacher under  $\nu_0$  (if any) is moved at the top of the  $f$  ranking. Then the generalized TTC corresponds to the following procedure:

- **Step 0.** Let  $T(0) = T$ ,  $S(0) = S$  and  $\nu(0) = \nu_0$ .
- **Step  $k \geq 1$ .** Build a directed graph where nodes are the teachers of  $T(k-1)$  and schools of  $S(k-1)$ . Every teacher points to the school he prefers in  $S(k-1)$ . If there is no acceptable school, we let him point to himself. Each school  $s \in S(k-1)$  points to the teacher of  $T(k-1)$  who is the highest ranked according to  $\succ_s$  (if  $\emptyset$  is ranked before all these teachers, we let this school point to itself<sup>21</sup>). Every node has an outdegree of 1 so there is at least one cycle. Let  $\nu(k)$  be the matching obtained after implementing all the cycles i.e. for a teacher  $t$  in a cycle  $\nu(k)$  matches him to the school he is pointing to. Let  $T(k)$  (resp.  $S(k)$ ) be  $T(k-1)$  (resp.  $S(k-1)$ ) where we remove all the agents (resp. schools) who were part of a cycle except schools that remain unmatched after having implemented the cycles. If one of this set of empty, then return  $\mu(k)$  as being the final matching otherwise go to step  $k+1$ .

They showed that the above algorithm is strategy-proof and (1-)Pareto-efficient. They also showed that the following algorithm, called *You Request My House I Get Your Turn* (YRMH-IGYT) is equivalent to the former one:<sup>22</sup>

- **Step 0.** Let  $T(0) = T$ ,  $S(0) = S$  and  $\nu(0) = \nu_0$ .

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<sup>20</sup>We will be using the terminology of schools and teachers but their original one was about houses and tenants.

<sup>21</sup>Note that, at this step, this cannot happen for schools with an initially assigned teacher since a school having an assigned teacher ranks it first and must point to him.

<sup>22</sup>Intuitively, a cycle involving an empty school in the generalized TTC will correspond to a chain in the YRMH-IGYT algorithm. Cycles of initially assigned teachers are mapped to cycles of initially assigned teachers below.

- **Step**  $k \geq 1$ . Pick the first ranked teacher in  $T(k-1)$  according to  $f$ , say  $t_1$ , and let him point to his favorite school in  $S(k-1)$ , say  $s_1$ , then:

- Case A. If  $\mu(k-1)(s_1) = \emptyset$ , match  $t_1$  to  $s_1$  and define  $\nu(k)$  accordingly. Note that if  $t_1$  was initially assigned a school, then this school will be empty under  $\nu(k)$ . Let  $T(k) = T(k-1) \setminus \{t_1\}$  and  $S(k) = S(k-1) \setminus \{s_1\}$ .

- Case B. If  $\mu(k-1)(s_1) \neq \emptyset$ , let  $t_2 := \mu(k-1)(s_1)$  and let him point to his favorite school in  $S(k-1)$ , say  $s_2$ , then either:

- \* Case A happens. In that case, match  $t_1$  and  $t_2$  to the school they are pointing to, define  $\nu(k)$  accordingly and update the sets  $T(k)$  and  $S(k)$  similarly than before.

- \* Case B happens. Then let  $t_3$  be  $s_2$ 's initial teacher and let him point to his favorite school in  $S(k-1)$ . If Case A happens, do has before. Otherwise, if Case B keeps happening, continue the chain of pointing as defined before in Case B until either: i) Case A happens, i.e. a teacher points to an empty school. In that case, match all the teachers of the chain to the school they are pointing to and update  $\nu(k)$ ,  $T(k)$  and  $S(k)$  accordingly. Or ii) the process cycles, i.e. a teacher ends up pointing to a school that was already assigned to a teacher of the ongoing chain. In that case, implement the cycle in matching all the teachers to the school they are pointing to and update  $\nu(k)$ ,  $T(k)$  and  $S(k)$  accordingly.

### 2.4.2 Many-to-one environment and two-sided efficiency

We will now present the generalization of TOBE. There are two key differences between the generalization of the top trading cycles mechanism presented above and the generalization of the TOBE algorithm we provide. Firstly, we are in a many-to one environment where schools have multiple positions, contrary to the generalized TTC presented in [Abdulkadirouglu and Sonmez \(1999\)](#) where each house can only have one tenant. Secondly and most importantly, when defining the TOBE algorithm, our two-sided efficiency notion considers both teachers and schools as welfare-relevant entities. In the generalized TTC, houses are not welfare-relevant entities. We explain in the two next paragraphs how the many-to-one environment and the two-sided efficiency criterion are incorporated in our generalization of the TOBE algorithm.

### 2.4.2.1 Many-to one environment

Regarding the many-to-one environment, we proceed in two steps. The idea is to first transform a problem where each school has several seats into a problem where they only have one seat. Based on this transformation, we modify both the initial matching and teachers' preferences to ensure that the final matching is 2-Pareto efficient. Intuitively, in this many-to-one environment, since each school might have several seats occupied by different teachers, when a teacher points to that school, we need to decide among these occupied seats the one he will point to and so with which teacher he is going to exchange. To do so, we introduce  $\succ_p$  as a total order over  $T \cup \{\emptyset\}$ , where the letter  $p$  stands for "pointing" order.

Formally, during the first step, define  $\tilde{S}$  as being the set of schools where each school in  $\tilde{S}$  has only one position. For each school  $s \in S$  with  $q_s$  positions, we create  $q_s$  *school-copies* of  $s$  in  $\tilde{S}$ . Each school-copy of  $s$  has the same preferences as  $s$  i.e.  $\tilde{\succ}_{\tilde{s}} := \succ_s$ .<sup>23</sup> We will note  $s(\tilde{s})$  the school in  $S$  where  $\tilde{s}$  is the school-copy. Once these duplications are done, we redefine the initial matching  $\tilde{\mu}_0 : T \mapsto \tilde{S}$ . Abusing a bit of notations, we will note  $\tilde{\mu}_0(\tilde{s})$  the teacher matched to the school-copy  $\tilde{s}$  (this is well defined since these school-copies only have one seat). For each school  $s$ , each teacher  $t$  initially matched to that school (s.t.  $\mu_0(t) = s$ ) is matched to a different school-copy of  $s$  in  $\tilde{S}$  (by construction, there are enough of them). For teachers, if  $\mu_0(t) = \emptyset$  then  $\tilde{\mu}_0(t) = \emptyset$ , meaning that if a teacher is initially unassigned, he remains unassigned with the transformation. For each teacher  $t$ , we define his transformed preferences  $\tilde{\succ}_t$  over  $\tilde{S} \cup \{\emptyset\}$  as  $\tilde{s} \tilde{\succ}_t \tilde{s}'$  if: (1)  $s(\tilde{s}) \succ_t s(\tilde{s}')$  or (2)  $s(\tilde{s}) = s(\tilde{s}')$  and  $\tilde{\mu}_0(\tilde{s}) \succ_p \tilde{\mu}_0(\tilde{s}')$ . In other words, since school-copies represent positions inside a school, for teachers' preferences, the ranking of two positions in two *different schools* remain the same and is defined by the original preferences  $\succ_t$  over the two schools. For two positions within the *same school*, we now use the pointing order  $\succ_p$  to rank them according to the teacher initially matched to these positions under  $\tilde{\mu}_0$ . We use the generalized TTC algorithm on this modified problem to obtain the final matching in a many-to-one environment.

### 2.4.2.2 Two-sided efficiency

Our two-sided efficiency notion considers both teachers and schools as welfare-relevant entities. Two modifications are required to ensure that both teachers' and schools' welfare are respected.

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<sup>23</sup>These objects are well defined since the set of teachers does not change.

Consider the school-copy environment as defined above where each school has only one position. Firstly, a teacher is prevented from ranking school-copy where he is less preferred than the initial teacher. Indeed, implementing a cycle between a teacher and a school where he/she is less preferred would hurt the school’s welfare. In the preliminary phase, for each school-copy, we identify the set of acceptable teachers. A teacher is acceptable for a position if he is preferred to the teacher who currently occupies that position. This can be defined as the “higher preference criterion”. Hence, for each teacher  $t$  and school-copy  $\tilde{s}$ , if  $\tilde{s}$  prefers its current teacher to  $t$ , then we declare  $t$  as unacceptable for  $\tilde{s}$ . In practice, we delete  $\tilde{s}$  from  $t$ ’s preferences.

As we will see in the empirical part, we will adopt both the above definition of the “higher preference criterion” and a more flexible approach. For instance, some schools could be identified as welfare-relevant entities – their assigned teacher must be preferred to the outgoing one – while for other schools, the assigned teacher could be less preferred to the outgoing one (or the outgoing teacher could not be replaced so that the position remains vacant). This flexibility allows to relax the “higher preference criterion” for some schools and will be particularly important in our simulations. The goal is to offer a more flexible approach to control the turnover of teachers in these targeted schools. As an extension in Section 2.7.2, we present an alternative approach to control this turnover.

The second modification aims to ensure that no teacher who is initially assigned will be re-assigned while leaving his initial position vacant, except if he is allowed to. Intuitively, this would violate the 2-IR criterion. To avoid this kind of situation, we define a set of matched teachers who cannot leave their position without being replaced.  $\succ_v$  is the pointing order of the vacant positions in schools. We modify these preferences in a way that prevents vacant positions from pointing to these teachers in the generalized TTC described before. In practice, in  $\succ_v$ , all matched teachers who are not allowed to be reassigned without being replaced are ordered below the vacant position:  $\emptyset \succ_v t$ . As a result, vacant positions will never point to these teachers.<sup>24</sup> More specifically, the pointing order of vacant positions  $\succ_v$  is defined as follows: above  $\emptyset$  are ranked all the newly-tenured teachers with no initial assignment ( $t$  s.t.  $\mu_0(t) = \emptyset$ ) and the tenured teachers who are allowed to leave their position without being replaced ( $t$  s.t.  $\emptyset \succ_{\mu_0(t)} t$ ). To order this set of teachers, we fix an exogenous total order,  $\succ_d$ , over  $T$ .<sup>25</sup> We rank the remaining teachers who cannot leave their current position empty below  $\emptyset$

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<sup>24</sup>Intuitively, this difference might affect the 1-PE of the resulting matching but here, we would like to focus on 2-PE. Abdulkadirouglu and Sonmez (1999)’s proof regarding strategy-proofness remains the same.

<sup>25</sup>The subscript corresponds to “drawing ”order. In the YRMH-IGYT analogue of our generalization of TOBE, this order is used to select the first teacher who will start to point to his favorite acceptable school.

according to  $\succ_d$ .

Again, the set of initially assigned teachers who cannot leave their position without being replaced can be used as a tool to change teacher distribution across regions. In the simulations we run, a more flexible approach of schools' welfare will be adopted where some tenured teachers will have the right to leave their initial school without being replaced by another teacher or by a preferred teacher. In  $\succ_v$ , these teachers will be ranked above the  $\emptyset$ .

### 2.4.2.3 Assignment Phase

Define the set of teachers  $t \in T$  with their respective preferences. The set of schools  $S$  is the set of school-copies  $\tilde{S}$ . The order  $\succ_v$  is used to let a vacant school-copy point to a teacher. After having incorporated the modifications related to the many-to-one environment and the two-sided efficiency criterion, we run the generalization of TTC and obtain an assignment  $\tilde{\mu}$  between teachers and school-copies. We can easily find a matching  $\mu$  of the original problem: if  $\tilde{\mu}(t) = \tilde{s}$  then  $\mu(t) = s(\tilde{s})$ . It is easy to check that this algorithm converges in a (finite and) polynomial time.<sup>26</sup>

### 2.4.3 From TTC to YRMH-IGYT

As mentioned by [Abdulkadirouglu and Sonmez \(1999\)](#), the outcome of the generalized TTC mechanism can be found by using the *You-Request-My-House-I-Get-Your-Turn* algorithm (YRMH-IGYT). To obtain the outcome of the generalization of TOBE presented above, we incorporate into YRMH-IGYT, the change related to the many-to-one environment and 2-sided efficiency. YRMH-IGYT is indeed easier to code than the generalized TTC.<sup>27</sup>

Running our alternative algorithms necessitates to define two orders: a drawing order  $\succ_d$  and a pointing order  $\succ_p$ . For all simulations performed in the empirical section, we use the same orders for all algorithms. In a many-to-one environment, within a school, the pointing order helps to select one teacher among several ones. In France, some teachers have the same number of points in all regions (that only incorporates their experience for instance), but some have points which vary depending on

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<sup>26</sup>To see that it converges in a finite number of steps, note that whenever we carry out a cycle, at least one teacher is removed from the set of available agents. Hence, in the worst case, one needs  $(n-1)n$  steps for this algorithm to end. Since finding a cycle in a directed graph can be solved in polynomial time, the algorithm converges in a polynomial-time.

<sup>27</sup>The latter is useful for the proofs since it clear highlights that the order with which the teachers are selected does not influence the existence of exchanges between initially assigned teachers. It is harder to see this property when using only the YRMH-IGYT version.

the region. For instance, some teachers get a higher score in the region where their partner/spouse or child is living, but not in other regions. For these teachers, considering the highest number of points is a means to move them up in the pointing order and encourage their mobility. The drawing is used to select the first teacher who is pointing. We have chosen to rank teachers in the following way: teachers initially matched to one of the attractive regions  $>$  teachers initially matched to one of the unattractive regions  $>$  teachers initially non-matched. Within each of the three categories, teachers are ordered according to the maximum score he has. In the hierarchy between the three categories, we have decided to order the teachers initially matched to one of the attractive regions higher than the teachers initially matched to one of the unattractive regions to avoid the latter to get assigned the vacant positions too quickly. As teachers from unattractive regions represent the majority of teachers (54.6%) and tend to have longer wish lists than other teachers, ranking them first in the drawing order might allow them to take the majority of vacant positions. This would impede the mobility of teachers from other regions.

In the Appendix, we provide results for alternative orders. Ranking the initially non-matched teachers at the bottom of the above order disadvantages them, it will be confirmed in our empirical results. It is possible to modify the order to correct this disadvantage. In Section 2.9.2, we keep the same structure as the above order except that we move a certain number of the best ranked teachers who are not initially matched at the top of the ordering. As expected, it leads to better results for the initially non-matched teachers while decreasing the overall movement compared to TOBE with the above order. We also provide alternative results in Section 2.9.3 using a more natural order: teachers are ranked according to their experience score. The same negative results for initially non-matched teachers will arise, we then provide results on two alternative orders ranking some of them at the top of the ordering while keeping the experience ranking. The final choice in front of these trade-offs will depend on the objectives of the policy maker. However it is important to keep in mind that the choice of the ordering, in addition to the one of targeted regions as we will present below, has an impact on the results.

## 2.5 Efficiency and strategy-proofness of the generalized TOBE

Before moving to the empirical simulations, let us state some theoretical results about the generalized TOBE we have defined:

**Proposition 2.5.1.** *The generalized TOBE is strategy-proof.*

*Proof.* A first point is that, in the Preliminary Phase, the matching  $\tilde{\mu}_0$  does not depend on the reported preference lists and the way to break ties between different school-copies of a same school does not depend on the reported lists. Then, the problem with school-copies defined in this phase can be seen as a house allocation problem as defined by [Abdulkadirouglu and Sonmez \(1999\)](#). So if a teacher  $t$  can manipulate TOBE in reporting  $\succ'_t$  instead of his true preferences  $\succ_t$ , then we could find a profile  $\tilde{\succ}'_t$  instead of  $\tilde{\succ}_t$  in the problem with school-copies where the teacher could manipulate the problem with school-copies. But then, it would mean that the YRMH-IGYT algorithm would be manipulable (the counter example would be the copy of this one where each school-copy would be considered as a house), which is not possible.  $\square$

**Proposition 2.5.2.** *The generalized TOBE is 2-PE.*

*Proof.* Fix a profile  $\succ$  of preferences for teachers and schools. Let  $\mu$  be a matching that weakly Pareto-dominates TOBE( $\succ$ ). We will show that necessarily  $\mu = \text{TOBE}(\succ)$ . We will use an induction on  $k \geq 1$  to show that each teacher assignment at the step  $k$  or less of the YRMH-IGYT in the Assignment Phase cannot obtain a strictly better assignment under  $\mu$  than under TOBE( $\succ$ ). Since preferences are strict and that  $\mu$  weakly Pareto-dominates TOBE( $\succ$ ), these teachers have to obtain the same assignment as under TOBE( $\succ$ ). If  $k$  is the last step then we obtain the result that  $\mu = \text{TOBE}(\succ)$ .

$k = 1$ . At this step, fix a teacher  $t$ . All teachers must point to their favorite school-copy (which corresponds to their favorite school under the original problem) among those available and not deleted from their preferences at the Preliminary Phase. The only way to give  $t$  a better school is to assign him to a school, say  $s$ , that has been declared as unacceptable at the Preliminary Phase (i.e. all the school-copies of  $s$  have been declared as unacceptable). But if it is the case, by definition of the Preliminary Phase, it would mean that  $\forall t'$  s.t.  $\mu_0(t') = s$ , we have that  $t' \succ_s t$ . Since TOBE is 2-IR, it means that for all teacher  $t''$  matched to  $s$ ,  $t'' \succ_s t$ . So it is not possible to reassign teachers of step 1 in order to weakly improve the schools.

$k - 1 \Rightarrow k$ : Suppose that the assumption holds for the teachers assigned up to step  $k - 1$ . All teachers assigned at step  $k$  are assigned to their favorite school-copy (also school in the original problem) among those left at this step or that haven't been declared as unacceptable at the



Preliminary Phase. To give teacher  $t$  a strictly better school we would have to (1) either give him a school that hasn't been deleted from his preference list at the Preliminary Phase or (2) either assign him to a school assigned to another teacher at a previous step. The case (1) could be excluded with the same argument as when  $k = 1$ . For case (2), there must exist a teacher assigned at a previous step who does not obtain the same allocation under TOBE, which is not possible by the inductive assumption.

We can then apply the inductive argument up to the last step of the procedure to obtain the desired result.  $\square$

## 2.6 Empirical simulations

The previous section showed that the TOBE algorithm has good properties in terms of efficiency and strategy-proofness. This section aims to assess the performance of this alternative algorithm by using a data set on the assignment of teachers to schools in France. We start with a short presentation of the data set. Then, we present results on the performance of the TOBE mechanism and detail some alternatives. The similarities and differences between this empirical analysis and the one in [Combe, Tercieux and Terrier \(2016b\)](#) will be fully detailed in the next section.

### 2.6.1 Dataset and descriptive statistics

We use several data sets related to the assignment of teachers to regions in 2013. These data sets contain four key information: (1) the reported preferences of teachers, (2) their score in each region that is used to define the ranking of each region,<sup>28</sup> (3) the initial assignment of each teacher (if any) and (4) the number of vacant positions in each the regions. In their list of preferences, teachers can report all regions if they wish to. Since the mechanism at use is DA\*, it is a dominant strategy for all agents to be truthful. Thus, we take for granted that agents' reported preferences in Phase 1 are the true preferences in order to run our counterfactuals. In addition, given the agents' assessments over the schools they may obtain in the second phase, agents have well defined preferences over regions. Our final sample incorporates all teachers who are teaching in one of the 49 subjects that contain more than 10 teachers. It is worth noticing that this sample is the same as the one used

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<sup>28</sup>If there is a tie, the ministry uses the birth date to break it: they give priority to the oldest teacher. We use the same tie break here.

in [Combe, Tercieux and Terrier \(2016b\)](#) with two important exceptions: in this paper we keep all teachers who are initially non-matched and all empty seats in regions. Hence, this sample is the same as the one used by the French Ministry of Education in 2013 to run their algorithm. Only one difference exists: for a matter of simplicity, we drop all couples from the sample because they benefit from a specific treatment in the assignment process.<sup>29</sup> In Section 2.7.1, we explore the effect of introducing couples in a reassignment setting like ours.

The final sample contains 19 229 teachers: 12 123 tenured teachers and 7106 newly tenured ones. Table 2.1 reports some descriptive statistics on the number of tenured teachers, newly tenured ones and vacant positions in the ten biggest subjects. Sport is the biggest one with 2487 teachers, followed by literature (2020 teachers) and mathematics (1948 teachers). As mentioned before, the number of vacant positions corresponds more or less to the number of teachers retiring every year. Finally, it is important to keep in mind that all newly tenured teachers are initially unassigned but the opposite is not true: not all tenured teachers are assigned: some are unassigned because they took some years off from teaching for instance. Table 2.1 reports that 12123 teachers are tenured in our sample. Among them, 10563 have an initial assignment (87.1%).

Finally, one of the objective of the current algorithm is to ensure that none of the initially unassigned teachers is left unassigned at the end of the process. Yet the ministry does not require that these teachers rank all regions. Instead it completes the lists of the unassigned teachers who have not ranked all regions. To complete the lists, the ministry uses the first region ranked by a teacher and completes the list step by step. The first region added is the unranked region which is geographically the closest to the first region ranked by this teacher. The second region added is the second unranked region which is geographically the closest to the first region ranked, and so on.<sup>30</sup> We use the same rule to complete the lists of all the unassigned teachers. Table 2.2 reports statistics about the length of the reported preference lists before the completion of the preferences for the initially unassigned teachers. We can note that newly tenured teachers tend to report longer

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<sup>29</sup>As explained in our previous paper, teachers can apply as a couple, in which case they have to submit the same list of regions to the central administration. They benefit from a specific treatment in the sense that one of the spouses will not be assigned a region if the other one does not get the same region. To achieve this, the central administration proceeds as follows: it runs the algorithm once and checks the region obtained by each spouse. If they don't obtain the same region, one must have obtained a region that is ranked lower in their common ranking (for instance rank 5). In that case, the ministry would delete all regions ranked higher than rank 5 from their common list of preferences and run the algorithm again on the modified list. This process is repeated until both spouses obtain the same regions. If this does not happen, they stay in their initial region.

<sup>30</sup>The following link describes the order considered to complete the list depending on the first region ordered by a teacher: [http://cache.media.education.gouv.fr/file/42/85/0/annexeIII-493\\_365850.pdf](http://cache.media.education.gouv.fr/file/42/85/0/annexeIII-493_365850.pdf)

Table 2.1 – STATISTICS ON TEACHERS FOR DIFFERENT SUBJECTS

Subject	All teachers (1)	Newly tenured (2)	Tenured (3)	Vacant positions (4)
All subjects	19229	7106	12123	8069
Sport	2487	623	1864	765
Literature	2020	784	1236	902
Mathematics	1948	1013	935	1112
English	1745	801	944	865
History-Geography	1489	657	832	746
Spanish	1193	324	869	364
Education	1117	361	756	416
Physics	1017	294	723	361
Pro: Literature - HG	688	215	473	233
Pro: Math - Sciences	513	206	307	231

Source: Dataset on public secondary school teacher assignment to region in 2013. The subjects in the first column corresponds to the subject taught by the teacher. In vocational high schools, some subjects are gathered in one course taught by one teacher (last two rows for instance). Newly tenured teachers ask for their first assignment, while tenured teachers are assigned a region, but wish to change.

lists than tenured ones. 50% of tenured teachers rank only two regions: their initial one and an additional one.

Table 2.2 – STATISTICS ON THE LENGTH OF THE PREFERENCE LISTS

	Nb	Mean	Median	SD	Min	Max
Tenured	12123	2.62	2	2.02	1	31
Newly Tenured	7106	8.09	5	7.32	1	31

Source: Dataset on public secondary school teacher assignment to region in 2013.

An important point for our analysis is the great diversity in terms of attractiveness of the 31 regions. Three regions are recognized for being particularly unattractive to teachers: two regions surrounding Paris – called Créteil and Versailles - and one region a bit further North – called Amiens. In stark opposition, some regions are very attractive. The attractiveness of regions depends on several factors. Geographic characteristics matter: regions in the South of France are usually more attractive than regions in the North. Another key determinant is the working conditions, perceived as more difficult in deprived regions gathering students from lower social background. Differences in attractiveness are easy to measure by computing the ratio of the number of tenured teachers asking to enter the region over the number of tenured teachers asking to leave the region. This statistic is presented in the first column of Table 2.3. Table 2.3 presents descriptive statistics for six of the thirty-one regions, presented by decreasing order of attractiveness. The ratio varies from 15.5 in the most attractive region (Rennes) to 0.03 in the least attractive region of Créteil. In addition,

column 2 shows that the three least attractive regions concentrate 50% of the tenured teachers asking for a new assignment. This statistic refers to 2013 but this pattern has been relatively stable over the past years. The outgoing mobility being high in these three regions, the positions need to be filled by new teachers. A direct consequence is that, every year, about 50% of the newly tenured teachers get their first assignment in one of these three regions. This high probability to be assigned one of the relatively deprived regions in the first place is one of the mentioned reasons for the lack of attractiveness of the teaching profession in France.<sup>31</sup> Finally, regarding the working conditions, columns 4 to 6 confirm that the least attractive regions gather an important share of students enrolled in “priority education” schools.<sup>32</sup> These students also tend to more likely have parents with no diploma and to have a lower success rate at the baccalaureate (the French diploma taken at the end of high school).

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<sup>31</sup>A recent French report by [Périer and Gurgand \(2016\)](#) reported that 2% of the students in a survey who chose not to join the teaching profession explicitly mention the assignment process as an important factor. Anyone wishing to become a teacher has to take a national exam in France, the most common of these exams being the CAPES. Every year, the ministry sets the number of teaching positions it opens which corresponds to the number of applicants who will be accepted at the exam. In 2014, for the CAPES exam, 24% of the positions offered by the government remained vacant because of both a lack of applicants and the poor quality of those applying.

<sup>32</sup>Priority education schools are schools designated by the administration as deprived schools with a low achievement rate for students, difficult working conditions for teachers and so on. They receive for instance additional budgets. Teachers assigned to these also receive a bonus of points used to compute the ranking of each region after some years of teaching.

Table 2.3 – STATISTICS ON REGIONS

<b>Region</b>	Nb tenured teachers asking to enter / leave the region <b>(1)</b>	% of teachers asking for a new assignment coming from each region <b>(2)</b>	Ratio of nb teachers aged more than 50 / less than 30 <b>(3)</b>	% students enrolled in "priority education" <b>(4)</b>	% students whose reference parent has no diploma <b>(5)</b>	% students obtaining their baccalaureate <b>(6)</b>
Rennes	15.55	0.5	8.10	7.9	14.18	91.54
Bordeaux	8.95	0.8	6.56	14.6	19.22	86.25
Toulouse	6.56	1.5	5.29	8.9	17.38	88.57
Paris	3.02	2.8	6.90	25.5	21.54	85.48
Aix-Marseilles	2.54	1.9	5.08	30.1	27.20	81.77
Grenoble	1.74	2.3	3.91	16.5	19.80	88.17
Amiens	0.08	6.2	1.89	23.9	27.71	82.41
Créteil	0.03	22.7	1.14	35.5	31.62	83.94
Versailles	0.05	25.7	1.62	24.9	21.88	87.92

Source: "Géographie de l'école", Ministère de l'éducation nationale, de l'enseignement supérieur et de la recherche, [Direction de l'Évaluation de la Prospective et de la Performance \(2014\)](#)

## 2.6.2 Performance of TOBE

The theoretical section shows that the TOBE algorithm has good properties in terms of efficiency, fairness and strategy-proofness. This section aims to assess the performance of this alternative algorithm by using the data presented in the previous section. It is important to keep in mind that, among the 12 123 tenured teachers, 1560 do not have an initial region. These teachers are tenured teachers who left the profession few years for various reasons (longer maternity leave, illness and so on). We could have splitted our analysis between assigned versus non-assigned teachers rather than tenured versus newly tenured but this distinction made more sense for the French policy makers with whom we discussed. Our results below will explicitly mention whether they are on tenured/newly tenured teachers or initially assigned/non-assigned ones.

The first set of results relates to the number of teachers who obtain a new assignment. Table 2.5 compares the number of assigned teachers obtaining a new assignment (region) under the current algorithm (DA\*) in column 1 and TOBE in column 2.<sup>33</sup> Results in Table 2.5 suggest that on average TOBE does not improve teachers' mobility compared to DA\*. The number of teachers obtaining a new region even decreases by 2.48%. This result might be surprising if one has in mind the significant improvement we found in [Combe, Tercieux and Terrier \(2016b\)](#). In this previous paper, we run all algorithms on a restricted market where vacant positions have been dropped as well as teachers without an initial assignment. We found that the number of teachers moving increases from 565 under DA\* to 1363 under TOBE, a significant improvement which stands in contrast with the results we obtain on the entire market.

The introduction of vacant positions easily justifies the lower performance of TOBE on the entire market. The key difference comes from the fact that the movement of teachers coming from deprived regions significantly increases under DA\* when we introduce vacant positions, whereas this movement does not increase under TOBE when vacant positions are introduced. The reason is the following: without vacant positions, under DA\* teachers in unattractive regions struggle to leave their region. Their mobility is all the more limited as they are higher ranked by regions, as we explained with a simple example in [Combe, Tercieux and Terrier \(2016b\)](#). The example highlights that teachers in unattractive regions face the same lack of mobility under TOBE as under DA\*, but TOBE significantly unlocks mobility for teachers in other regions. This is the core of the *positive*

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<sup>33</sup>Table 2.4 below also reports the mobility for tenured versus newly tenured teachers. In Table 2.11 in Section 2.9.1 of the Appendix, we provide the details of the movement in all regions under each algorithm.

*effect of TOBE in attractive regions compared to DA\*.*

However, when introducing vacant positions, the situation of teachers in unattractive regions evolves differently under DA\* and TOBE. Under DA\*, teachers in unattractive regions face new opportunities to take vacant positions in other regions so that their chance to quit their unattractive region significantly increases. This is all the more true as many teachers in unattractive regions tend to be preferred in other regions due to the points they have accumulated by teaching in a deprived school for several years.<sup>34</sup> This increased mobility under DA\* is only possible because DA\* does not take into account the welfare of the regions. In other words, under DA\*, teachers in deprived regions can leave their position without being replaced by a preferred teacher or without being replaced at all. This absence of consideration for regions' welfare justifies the significant increase in the outgoing mobility from unattractive regions. However, this is not true under TOBE which respects the 2-IR criterion: any teacher who quits a region has to be replaced by a preferred teacher. Intuitively, taking into account regions' welfare significantly reduces teachers' ability to leave unattractive regions because they need to be replaced by an incoming preferred teacher. These incoming teachers are very rare: few teachers ask to be assigned an unattractive region (compared to the number of teachers asking to leave) and among them, very few have enough points to be preferred by the region to replace an existing teacher.<sup>35</sup> This explains why introducing vacant positions, whatever their number, does not change much the probability that teachers in unattractive regions will be able to leave their region under TOBE. This paragraph presents the *negative effect of TOBE compared to DA\* in unattractive regions*: with vacant positions, DA\* allows teachers to quit deprived regions, whereas TOBE prevents them from quitting. Since teachers in the deprived regions represent about 50% of the teachers asking for a new region, this negative effect in deprived regions almost exactly compensates the aforementioned positive effect in attractive regions. This is why TOBE generates as much (or slightly less) less movement than DA\* when vacant positions are introduced in the market.

This explanation can be tested by comparing the mobility of teachers in attractive and unattractive regions under DA\* and TOBE. Table 2.5 presents teachers' mobility by region of initial assignment.

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<sup>34</sup>13.5% of the teachers in one of the three deprived regions benefit from points related to teaching in a deprived school, whereas only 3.9% of the teachers in other regions benefit from these points.

<sup>35</sup>To compute the points of a teacher in his initial region, we only take into account the points related to his experience. For instance, we do not add the points that they get for teacher in a "priority school". Indeed, this would reduce the number of teachers who would have a higher number of points and so that could point to them. Since this bonus of point was introduced to give them higher chances of leaving the deprived region, it makes sense to omit it when computing the ranking of a teacher in his initial region.

Column 1 reports the number of teachers who obtain a new region under DA\*, while column 2 reports the variation under TOBE compared to DA\*. As expected, in most attractive regions, the number of teachers moving is significantly higher under TOBE than under DA\*. The mobility increases by 31.66% in the attractive regions, but in stark contrast decreases by 44.08% in unattractive regions. More detailed statistics are provided for the three most attractive regions (Rennes, Bordeaux and Toulouse), and the three least attractive regions (Créteil, Versailles and Amiens), we also report in Table 2.11 of Section 2.9.1 of the Appendix all the results of movement for each region.

Table 2.4 – NUMBER OF TEACHERS OBTAINING A NEW ASSIGNMENT UNDER EACH ALGORITHM

	Nb	Current algorithm DA*	TOBE with all regions targeted	TOBE with no regions targeted	TOBE with deprived regions targeted (option 1)	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Newly Tenured	7106	6486	0.09%	-0.32%	-0.59%	-0.37%
Tenured	12123	5494	-2.24%	40.81%	15.07%	32.25%
Total	19229	11980	-0.98%	18.54%	6.59%	14.59%

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in columns 4 and 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.5 – COMPARISON OF THE NUMBER OF INITIALLY ASSIGNED TEACHERS OBTAINING A NEW ASSIGNMENT (REGION) UNDER THE CURRENT ALGORITHM (DA\*) AND THE ALTERNATIVE ALGORITHM WE SUGGEST (TOBE) - BY REGION OF INITIAL ASSIGNMENT -

	Current algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with deprived regions targeted (option 1)	TOBE with a proportion of teachers targeted in deprived regions (option 2)
	(1)	(2)	(3)	(4)	(5)
<b>All regions</b>	<b>3991</b>	<b>-2.48%</b>	<b>56.98%</b>	<b>21.55%</b>	<b>45.25%</b>
<b>All attractive regions</b>	<b>2192</b>	<b>31.66%</b>	<b>79.33%</b>	<b>79.43%</b>	<b>79.47%</b>
ex: Rennes	30	36.67%	40.0%	40.0%	40.0%
ex: Bordeaux	47	31.91%	46.81%	46.81%	46.81%
ex: Toulouse	73	63.01%	75.34%	75.34%	75.34%
<b>All unattractive regions</b>	<b>1799</b>	<b>-44.08%</b>	<b>29.74%</b>	<b>-48.97%</b>	<b>3.56%</b>
ex: Amiens	207	-17.39%	60.39%	-29.47%	2.9%
ex: Créteil	813	-61.13%	22.51%	-61.99%	2.58%
ex: Versailles	779	-33.38%	29.14%	-40.56%	4.75%

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in columns 4 and 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

The previous results show that TOBE can significantly improve the mobility of the teachers whose initial assignment is not in a deprived region. By construction, we know that TOBE is 2-IR.



This implies that, in all regions, the position left by a departing teacher must be filled by a preferred incoming. We call this criterion the “higher preference criterion”. As a teacher’s ranking mainly represents his/her experience, this criterion ensures that all regions are assigned incoming teachers that are more experienced than the departing ones. Yet, we might want to partly relax a this criterion for three reasons. Firstly, for a pragmatic one: the algorithm currently used by the French ministry of education does not respect the higher preference criterion. Replacing some teachers by less preferred ones does not seem to be perceived as problematic, so that by relaxing the higher preference criterion - completely or in some regions - we might better match the objectives and criteria of policy makers. Secondly, the previous results show that fully respecting this criterion can significantly reduce the mobility in deprived regions. We might want to better calibrate the mobility of teachers initially assigned in these deprived regions as reducing their overall mobility by 44.08% might have undesired effects (increased drop out from the profession, other teachers avoiding even more these deprived regions, being aware that it will be difficult to exit them). A fine tuning of the mobility in each region would be easy to achieve by relaxing the higher preference criterion in some regions but not others, or even by relaxing this criterion for some teachers in deprived regions, but not all. Both options will be tested in the next sections. Finally, the higher preference criterion might not be relevant in all regions, and a case-by-case approach might be more relevant. Indeed, prior to the mobility, regions significantly differ in terms of age and experience of their teachers, so that it might not be desirable to assign older teachers in all of them. In attractive regions, teachers are significantly older and more experienced than in less attractive regions. To see that, for different regions, we can compare the ratio of the number of teachers older than 50 over the number of teachers younger than 30. This ratio is higher than 16 in the region of Montpellier, close to 8 in Rennes, but lower than 2 in the relatively deprived regions of Créteil, Versailles or Amiens ([Direction de l’Evaluation de la Prospective et de la Performance, 2014](#)). This important inequality in the geographic distribution of young teachers could have an intuitive direct impact on policy makers in charge of assigning teachers: they might not want to respect the higher preference criterion - i.e assign more experienced teachers - in all regions, but rather use a case by case approach depending on the individual situation of each region.

We discussed in the previous section how the higher preference criterion can reduce teachers’ mobility. We will test several options in the next sections to solve this. A first option consists

Table 2.6 – COMPARISON OF THE NUMBER OF TENURED TEACHERS OBTAINING A NEW ASSIGNMENT (REGION) UNDER THE CURRENT ALGORITHM (DA\*) AND THE ALTERNATIVE ALGORITHM WE SUGGEST (TOBE) - BY SUBJECT -

	Current algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with deprived regions targeted (option 1)	TOBE with a proportion of teachers targeted in deprived regions (option 2)
	(1)	(2)	(3)	(4)	(5)
<b>All subjects</b>	<b>5494</b>	<b>-2.24%</b>	<b>40.81%</b>	<b>15.07%</b>	<b>32.25%</b>
Sport	705	-11.4%	29.7%	10.2%	24.3%
Literature	668	3.6%	34.3%	12.9%	25.0%
Mathematics	645	-4.8%	36.4%	2.3%	24.0%
English	512	7.4%	51.6%	19.1%	40.4%
History-Geography	437	2.1%	32.3%	13.7%	27.0%
Spanish	312	-11.9%	35.9%	9.3%	29.2%

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in columns 4 and 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

in relaxing the criterion in all regions.<sup>36</sup> This gives us a sense of the maximal improvement we could obtain, but it might not be satisfying to significantly increase outgoing mobility from deprived regions. Hence, as a second test, we relax the higher preference criterion in attractive regions only. This should intuitively increase teachers' mobility in these regions, while ensuring that relatively deprived regions are not hurt in terms of experience of their incoming teachers. But this second option significantly retains teachers in deprived regions which can in turn hurt the attractiveness of the profession. Hence, a last and convincing option consists in relaxing the higher preference criterion in attractive regions, and for a certain share of teachers in unattractive regions. We successively present these simulations in the next sections.

When discussing below the results for each option of TOBE, we will focus on the tenured teachers who are those for which the scope of improvement for TOBE is the most important. Remember that, for newly tenured teachers, their preference lists are completed in order for (most of) them to be sure to be assigned at the end of the assignment process.<sup>37</sup> We provide a detailed analysis for the newly tenured teachers in Section 2.6.8. Last, remember that all the results use the order for our generalized TOBE as defined in Section 2.4.3. As mentioned, we provide results on alternative

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<sup>36</sup>This option is equivalent to using a many-to-one version of the YRMH-IGYT algorithm.

<sup>37</sup>As it will be seen, it might still be the case that there are teachers unassigned at the end of the process. This is because the completion of the preference lists is not complete: the ministry does not incorporate the French overseas regions for a teacher who did not rank them. In practice, if a teacher is still unassigned at the end of the process, the ministry creates ex post additional positions depending on the regional needs.

orders in Sections 2.9.2 and 2.9.3 of the Appendix.

### 2.6.3 Modification of TOBE to account for the situation of deprived regions

As explained in the previous section, it might not be relevant to assign preferred teachers – mostly meaning with higher experience – in all regions. Based on this observation, we distinguish two types of regions. Firstly, “targeted” regions are relatively deprived regions that would benefit from being assigned more experienced teachers. For these regions, a first approach consists in requiring that any teacher living the region is replaced by a teacher whose experience is at least as high. The second type of regions corresponds to “non-targeted” regions. These are relatively attractive regions whose teaching workforce is relatively old . Based on this simple distinction, we modify the TOBE algorithm in the following way. For each teacher, we define the set of regions where he/she is “eligible”. A teacher is eligible in a region in two cases:

1. If the region is not targeted, all teacher are automatically eligible for that region.
2. If the region is targeted, a teacher will be eligible for that region only if he/she is preferred than at least one teacher initially assigned in that region who has not been reassigned yet through the algorithm.

This description of targeted regions can be incorporated in the TOBE mechanism we described in Section 2.3. Indeed, a non-targeted region would allow all its teachers to leave without necessarily replacing them. This is equivalent to declaring all the teachers matched to that region as “unacceptable” in the preferences of that region: any potential incoming teacher will be preferred, and will therefore be able to enter the region. It is worth noticing that if all regions are targeted, we obtain the TOBE mechanism presented earlier, that would be applied directly on the initial preferences. Reversely, if none of the regions are targeted, preferences of regions do not matter in the assignment and we obtain the YRMH-IGYT algorithm (Abdulkadirouglu and Sonmez, 1999) in the many-to-one environment we previously described. In the following, as mentioned before, we will consider four different targeting scenarios:

1. *TOBE with all regions targeted*: this version is equivalent to the original TOBE, all teachers in all regions can leave only if they are replaced by preferred teacher.

2. *TOBE with no region targeted*: all teachers can freely leave their region. This version is equivalent to the many-to-one YRMH-IGYT algorithm presented earlier.
3. *TOBE with deprived regions targeted (option 1)*: in the three deprived regions of Amiens, Créteil and Versailles, their teachers cannot leave without being replaced by a preferred teacher.
4. *TOBE with a proportion of teachers targeted in deprived regions (option 2)*: in the three deprived regions of Amiens, Créteil and Versailles we relax the higher priority criterion for only a proportion of them. For respectively the 65%, 40% and 45% most preferred teachers in Amiens, Créteil and Versailles, we relax the higher priority criterion i.e. they can leave their without being replaced by a preferred teacher. This calibration was chosen to replicate the movement of  $DA^*$  in these regions.

Before moving to the specific results on teachers' distribution and regional statistics under each algorithm, Table 2.7 gives the cumulative distribution of ranks obtained by tenured teachers. The cumulative distributions for each TOBE algorithm all assign more teachers than  $DA^*$  up to rank 8 but the latter does better for rank 9. So they tend to perform better than  $DA^*$  but there is no stochastic dominance relationship between each distribution for TOBE and the one of  $DA^*$ . This is mainly because all TOBE algorithms leave slightly more unassigned tenured teachers (with no initial region) than  $DA^*$ . Indeed, even is not reported here, the same distributions calculated on initially assigned teachers all stochastically dominate the one of  $DA^*$  for the same teachers.

#### **2.6.4 Performance of TOBE when no region is targeted**

To begin with, it is interesting to see what happens in the extreme situation where none of the regions are targeted. This is equivalent to running the YRMH-IGYT mechanism as defined by [Abdulkadirouglu and Sonmez \(1999\)](#), except that we are in a many-to-one environment. This mechanism does not take into account the welfare of the regions: when a cycle or a chain is implemented, an incoming teacher in a region is never required to be preferred to the exiting teacher. Intuitively, by removing all constraints on regions welfare, this allows more cycles to be implemented, so that this algorithm should be expected to have a higher movement than the versions where some regions are targeted. However, an intuitive drawback should be a significant increase in the exit mobility from deprived regions and no control on the experience of the incoming teachers in the

Table 2.7 – CUMULATIVE DISTRIBUTION OF THE RANKS FOR TENURED TEACHERS UNDER EACH ALGORITHM

	Current algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with deprived regions targeted (option 1)	TOBE with a proportion of teachers targeted in deprived regions (option 2)
	(1)	(2)	(3)	(4)	(5)
1	4219	4223	5953	4939	5643
2	9861	10067	10326	10192	10274
3	10745	10930	10956	10949	10932
4	11234	11393	11296	11376	11314
5	11482	11611	11501	11594	11516
6	11645	11756	11616	11730	11644
7	11764	11829	11728	11808	11745
8	11883	11894	11845	11889	11852
9	11961	11946	11906	11940	11908
$\geq 10$	12066	12052	12047	12047	12047
Unassigned	57	71	76	76	76

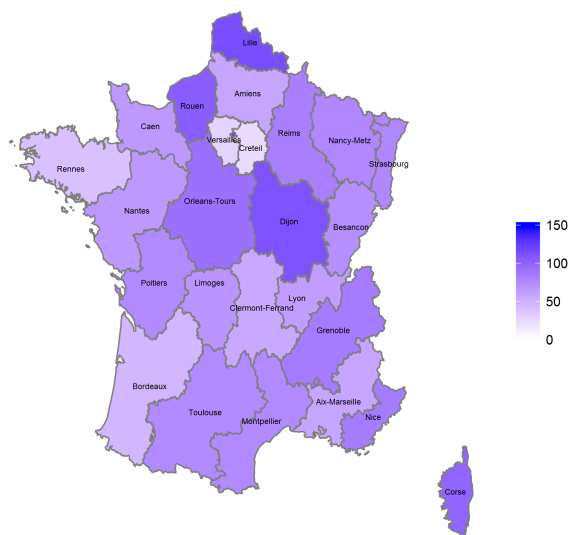
Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in columns 4 and 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A). The last line gives the number of unassigned tenured teachers among those who do not have an initial region.

deprived regions.

We run this algorithm on all subjects. The results are reported in Table 2.6, column 3. We find that the number of tenured teachers who obtain a new assignment can be increased by 40% compared to the number of teachers moving under the algorithm currently used by the French Ministry of Education. This significant increase is observed in almost all subjects. For instance, the mobility increases by 36% in maths, 34% in literature and 51% in English. In addition, the improvement is experienced by teachers originating from all regions as can be seen on Figure 2.1 and in Table 2.5. On average, the mobility increases by 79.33% in attractive regions, and by 29.74% in deprived regions.

At first sight, this significant improvement in teachers' mobility might be considered as a desirable outcome. Yet, as we mentioned earlier, increasing teachers' mobility can trigger important exit movements from the least attractive regions. These exits would need to be replaced by an equivalent incoming flow of teachers to ensure that none of the positions are left empty at the end of the mobility process. The drawback is that most of the incoming teachers in deprived regions are newly tenured teachers with very little experience in teaching. Hence, every year, an increased mobility would tend to increase the share of inexperienced teachers in the most deprived regions. In that

Figure 2.1 – % MOBILITY INCREASE COMPARED TO DA\*



perspective, any increase in teachers’ mobility that would be done to the detriment of the most deprived regions and students might be considered as problematic. Our simulations confirm that the number of teachers leaving the three most deprived regions would increase by 29.74%. As a consequence, our simulations show that the share of newly-tenured teachers – among the incoming teachers – would increase by 4.2% in Créteil and Versailles. Based on that observation, one of the contribution of this paper is to suggest a variant of this mechanism which, contrary to what happens here, does not hurt the deprived regions.

### 2.6.5 Performance of TOBE when some deprived regions are targeted (option 1)

As mentioned in Section 2.6.3, a simple modification of TOBE allows to account for the situation of deprived regions, and to avoid that too many teachers leave these regions every year. We identify three “targeted” regions that would benefit from being assigned more experienced teachers. In these regions, we require that any teacher who is leaving has to be replaced by a teacher whose experience is at least as high. The three regions we chose to target are Créteil, Versailles and Amiens. They are the three least attractive regions based on the ratio presented in Tables 2.3, column 1: the number of tenured teachers asking to enter over the number of tenured teachers asking to leave the region.

It is important to keep in mind that other criteria could be used and other regions could have been selected.<sup>38</sup> Based on these targeted regions, we run the modified version of TOBE and present the results in Table 2.6 and 2.5. We find that the mobility still increases significantly compared to the mobility observed with the current algorithm (+15.07% for tenured teachers, and +21.55% for those who were initially assigned a region). The new criterion requiring that any outgoing teacher has to be replaced by a more experienced incoming teachers implies that, in targeted regions, the ratio of the number of tenured teachers leaving over the number of tenured teachers entering gets closer to one. In addition, both the average age of the incoming teachers and the share of tenured teachers increase in these targeted regions. Based on these outcomes, the objective to ensure that our alternative algorithm does not hurt the deprived regions seems better satisfied. Yet a drawback emerges: targeting regions significantly reduces the number of teachers allowed to leave the region. As reported in Table 2.5, this number drops by 29.47% in Amiens, 61.99% in Créteil and 40.56% in Versailles. It is worth noticing that this drop in deprived regions is of similar magnitude to the one observed when all regions are targeted. But despite this similarity, the overall mobility is still much higher in the current option 1 than when all regions are targeted as the improvement for attractive regions is significantly higher (+79.43% with option 1 versus +31.66% when all regions are targeted). This mobility reduction in deprived regions could be perceived as problematic as it prevents teachers in these regions from moving and so can ultimately impact the attractiveness of the profession. A solution consists in targeting a proportion of teachers in deprived regions rather than targeting entire regions. We present this solution in the next section.

### **2.6.6 Performance of TOBE when some teachers in deprived regions are targeted (option 2)**

Based on the observation that targeting regions reduces teachers' mobility in these regions, we suggest an alternative which consists in identifying some “*non-target teachers*” in deprived regions that would not need to be replaced by a more experienced incoming teacher. In other words, this solution consists in relaxing the higher preference criterion for some teachers, who could be selected based on the preferences of the region. Relaxing this criterion is a means to achieve a fine tuning

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<sup>38</sup>For instance, to avoid boundary effects on some of the regions, one can think about a *smoothing targeting* scenario that tries to limit the exit of the teachers in other regions that are the next less popular ones after Créteil, Versailles and Amiens.

of the outgoing flows in deprived regions. This is particularly interesting if we want to keep the outgoing flows at the same level as they currently are for instance. This is obviously not the only target we could have, but it seems a relevant one in terms of policy recommendation. It allows to keep the mobility constant in deprived regions, while significantly improving the mobility in other regions. Hence, for each of the three deprived regions, we measure the percentage of teachers who need to be non-targeted to keep the outgoing number of teachers constant in deprived regions. By relaxing the higher preference criterion for 65% of the teachers in Créteil, 45% in Versailles and 40% in Amiens, the outgoing flows remain constant – or slightly increase - in these regions. Then, we simulate the assignment obtained with this new version of the targeting policy. The overall number of initially assigned teachers obtaining a new assignment is still significantly higher than under the current algorithm (+44.9% as reported in Table 2.5, column 5).<sup>39</sup> By definition, mobility remains close to constant in deprived regions (+2.90% in Amiens, +2.58% in Créteil and +4.75% in Versailles), but significantly increases in all other regions). For instance, the number of assigned teachers obtaining a new assignment increases by 40% in Rennes, 46.81% in Bordeaux and 75.34% in Toulouse. Hence, this option significantly increases the overall mobility, while ensuring that the regions of Créteil, Versailles and Amiens, which have been identified as deprived, do not suffer from a rise in the number of teachers leaving. In addition, as an important share of the tenured teachers leaving these regions are replaced by preferred incoming teachers, the age of the teachers assigned in deprived regions increases. It goes from 30.06 to 30.53 in Créteil, from 31.08 to 31.54 in Versailles and from 29.87 to 30.41 at Amiens.<sup>40</sup> And, as we have previously mentioned, we know from the literature that an additional year of experience at the beginning of a teacher’s career has a significant impact on his/her value-added. Finally, we show that other outcomes improve or remain constant in these three regions. The number of newly tenured teachers remains stable. The ratios between the number of tenured teachers entering and leaving the regions tend to improve, going from 3.18 to 2.79 in Créteil for instance (from 1.53 to 1.47 in Versailles and goes from 2.7 to 3 in Amiens).

### **2.6.7 Performance in terms of fairness**

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<sup>39</sup>For all tenured teachers, including those with no initial assignment, the improvement is of 32.25%.

<sup>40</sup>In the Section 2.9.1 of the Appendix, we report the densities of the age of the assigned teachers in the three deprived regions of Créteil, Versailles, Amiens and the three most attractive regions of Bordeaux, Rennes and Toulouse.



Table 2.8 – COMPARISON OF THE NUMBER OF TEACHERS BLOCKING WITH AT LEAST ONE REGION - DECOMPOSITION BY TYPE OF VIOLATION -

	Current algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with deprived regions targeted (option 1)	TOBE with a proportion of teachers targeted in deprived regions (option 2)
	(1)	(2)	(3)	(4)	(5)
<b>All blocking pairs</b>	<b>6542</b>	<b>8621</b>	<b>8501</b>	<b>8772</b>	<b>8579</b>
Tenured teachers	5094	6433	4646	5768	4965
Newly tenured teachers	1448	2188	3855	3004	3614
<b>Acceptable blocking pairs</b>	<b>6542</b>	<b>3408</b>	<b>3258</b>	<b>3007</b>	<b>3111</b>
Tenured teachers	5094	2628	1669	1993	1962
Newly tenured teachers	1448	780	1589	1014	1419
<b>Unacceptable blocking pairs</b>	-	<b>8178</b>	<b>8213</b>	<b>8464</b>	<b>8275</b>
Tenured teachers	-	6079	4387	5505	4702
Newly tenured teachers	-	2099	3826	2959	3573

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in columns 4 and 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

So far, the outcomes we have considered are related to the welfare of both teachers and regions. Another key criterion we haven't discussed yet relates to the fairness of the assignments we obtain. In other words, it is important to quantify situations where the priority of a teacher is not respected. Such situations exist when a teacher does not obtain a region, whereas other less preferred teachers are assigned that same region. In that case, the priority of the first teacher is not respected, and he/she would form what is called a blocking pair with the region. In this Section, for a matter of clarity, we distinguish two situations in which the priority of a teacher might be violated: violation due to another teacher *staying* in his region or *entering* the region. This distinction is equivalent to the concepts of inappropriate claims and justified claims in [Pereyra \(2013\)](#). Firstly, a highly preferred teacher might be rejected from a region whereas other less preferred teachers stay in that region because, through the algorithm, they have not obtained any of the regions they ranked. This violation of a teacher's priority is a direct consequence of the individual rationality constraint. The latter states that teachers can keep their initial assignment if they do not obtain any of the regions they ranked. The alternative algorithm we suggest, by significantly increasing the number of teachers obtaining a new assignment, proportionally reduces this first origin of violation of teachers' priorities. Indeed, as reported in [Table 2.8](#), the number of tenured teachers who experience a violation of their priority in at least one region because of another teacher staying to his/her initial assignment drops from 5094 to 1962 in the second option we suggest.

Secondly, a highly preferred teacher might be rejected from a region whereas other less preferred teachers are assigned, through the algorithm, to that region. This second situation of violation of a teacher's priority has its origin in the implementation of the cycles we suggest in the TOBE algorithm. By allowing some teachers to exchange their positions, a teacher part of a cycle might obtain a region A, while other preferred teachers don't. These other teachers might not be part of a cycle leading them to the region A because of a lack of teachers in the region A willing to be assigned their region. In our second option, 4702 teachers have their priority violated due to another less preferred teacher entering a region they do not obtain. Although the algorithm we suggest creates violations due to an entering teacher, two elements are worth noticing. Firstly, the number of violations due to an entering teacher created is partly compensated by the drop in the number of violations due to a staying teacher so that the increase in the overall number of blocking pairs remains limited (it goes from 6542 to 8579). Secondly, [Pereyra \(2013\)](#) and [Compte and Jehiel](#)

(2008) showed that if one does not allow any blocking pair due to entering teachers, then  $DA^*$  is the mechanism maximizing the movement and minimizing the number of blocking pairs due to teachers' staying among all the mechanisms with no blocking pairs due to entering teachers. Our results show that this requirement comes with an important cost: one has to accept a much lower mobility under  $DA^*$ . The question is then whether the benefit of having no blocking pairs due to entering teachers outperforms the cost of the low mobility. We hope that, at least, our empirical results will help to quantify the potential gains and losses of adopting one requirement or the other and will shed new lights into the debate.

Finally, we should mention that blocking pairs due to an entering teacher already exist in the current French assignment process, although they are justified by specific situations. For instance, the way couples are treated in the current algorithm creates such blocking pairs, which are accepted by couples as they are the only way for them to obtain the same region. Blocking pairs due to an entering teacher are also widespread in the assignment of primary school teachers, justified by the important lack of mobility if the algorithm was not allowing them. Indeed, they used a similar process as the one described for the French Assignment Mechanism in Section 2.3.2. They used to implement Stable Improvement Cycles at the second step of the procedure. However, due to several complaints because of an important lack of mobility, they chose to drop the stability requirement to define the cycles and so have used standard Pareto-improving cycles that just consider teachers' preferences. In doing so, they obviously create blocking pairs due to entering teachers.

### 2.6.8 Results for newly tenured teachers

Table 2.9 reports the cumulative distributions of the ranks for newly tenured teachers. A first observation is that the TOBE algorithms do worse than  $DA^*$  for newly tenured teachers. In our last proposed version in column (5), we only match 2560 of them to their first choice against 4044 under  $DA^*$ . The distribution of  $DA^*$  stochastically dominates the ones of columns (3), (4) and (5). These three TOBE algorithms also tend to leave slightly more newly tenured teachers unassigned. However, for the TOBE with all regions targeted in column (2), there is no stochastic dominance relationship:  $DA^*$  matches slightly more newly tenured teachers to their first choice but less for higher ranks, it also leaves slightly more unassigned newly tenured teachers.

As mentioned earlier, for teachers with no initial assignment, their preference lists are completed.

Table 2.9 – CUMULATIVE DISTRIBUTION OF THE RANKS FOR NEWLY TENURED TEACHERS UNDER EACH ALGORITHM

	Current algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with deprived regions targeted (option 1)	TOBE with a proportion of teachers targeted in deprived regions (option 2)
	(1)	(2)	(3)	(4)	(5)
1	4044	3955	2290	3224	2560
2	4804	4910	3286	4335	3628
3	5095	5231	3856	4807	4182
4	5285	5478	4315	5157	4614
5	5429	5662	4673	5419	4931
6	5560	5797	4956	5601	5187
7	5662	5892	5161	5729	5360
8	5803	6007	5434	5904	5607
9	5916	6092	5644	6011	5759
≥ 10	6491	6497	6467	6453	6465
Unassigned	615	609	639	653	641

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in columns 4 and 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

In Table 2.10, we report the number of tenured and newly tenured teachers assigned to a region that has been added on their preference list after the completion.<sup>41</sup> The column (1) reports the total number of teachers with no initial region, remember that there are 1560 of them among tenured teachers and all newly tenured do not have any initial region. Once again, except the TOBE with all regions targeted, the others tend to assign more teachers to an extended wish.

Table 2.10 – NUMBER OF TEACHERS ASSIGNED TO AN EXTENDED WISH

	Nb	Current algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with deprived regions targeted (option 1)	TOBE with a proportion of teachers targeted in deprived regions (option 2)
	(1)	(2)	(3)	(4)	(5)	(6)
Tenured	1560	639	554	1029	776	951
Newly Tenured	7106	603	522	1324	741	1111

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in columns 4 and 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

It seems that, under the TOBE algorithms, there is a competition between initially non-matched

<sup>41</sup>It reports the number of teachers assigned to a region that they did not rank in their reported list. However, it has to be interpreted with caution. Indeed, the teachers being aware of this completion can have anticipated it in their report and can still find acceptable these added regions.

teachers (both tenured and newly tenured) and the tenured teachers initially assigned to the three deprived regions of Créteil, Versailles and Amiens. Indeed, in both Table 2.9 and Table 2.10, algorithms in column (3) and (5) tend to better perform than the others.<sup>42</sup> In both of them, Créteil, Versailles and Amiens are targeted and, as one can see in Table 2.11 of the Appendix, the movement in these regions is drastically reduced compared to DA\*. As mentioned earlier, initially non-assigned teachers are disadvantaged compared to others since they are ranked at the bottom of the ordering of TOBE as defined in Section 2.4.3. For instance, it implies that they are always the last teachers who can take an empty seat in a region. In order to favor them, one can change the order of TOBE. In the Section 2.9.2 of the Appendix, we define a new order as follows:

- Rank first  $x$  % of the best initially non-assigned teachers with the highest maximum score.
- Then rank all teachers initially matched to one of the attractive regions.
- Then rank all teachers initially matched to one of the unattractive regions.
- Then rank the remaining initially non-assigned teachers.

Inside each group, teachers are ranked according to the maximum score they have. This order has the same structure as the previous one. The only difference is that it takes a proportion of the best initially non-assigned teachers and move them to the top of the ranking. Intuitively, this will allow these teachers to be the first to choose their assignment under TOBE. In the Appendix, we provide the results in terms of movement, distributions of ranks for tenured and newly tenured teachers as well as in terms of average age of the teachers in each region. We provide the results when  $x = 20\%$ ,  $x = 40\%$  and  $x = 100\%$ . The results support that there is a trade-off between the movement of teachers initially assigned to Créteil, Versailles and Amiens and the quality of the assignment of initially non-assigned teachers. Moreover, in the extreme case where all the initially non-assigned teachers are at the top of the ranking, we even match less of them in extended wishes than under DA\* and assign more to their first ranked region. However, this is done at the expense of the overall movement: +1.03% compared to DA\* with +2.15% for tenured teachers (and +3.61% for initially assigned teachers). Some can also argue that the structure of the orders used so far is artificial, especially when discriminating between teachers initially matched to attractive and unattractive

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<sup>42</sup>TOBE with all regions targeted even outperform DA\*. Since all the regions are targeted, initially assigned teachers cannot leave without being replaced by a preferred teacher. So initially non-matched teachers fully benefit from it and have almost no competition from initially assigned teachers for empty seats in the regions.

regions. In Section 2.9.3 of the Appendix, we consider another structure: teachers are only ordered according to their experience score. The additional movement for the TOBE algorithm with a proportion of teachers targeted in deprived regions (option 2) represents a 7% improvement compared to DA\* with a 14.83% improvement for tenured teachers. Once again, initially non-assigned teachers tend to be hurt: newly tenured teachers tend to have a lower experience than tenured ones, so most of them are still at the bottom of the order. We provide results for three additional orders that take the best  $x = 20\%, 40\%, 100\%$  of the initially non-assigned teachers and move them at the top of the order defined with the experience score. The results are similar to the previous ones.

Overall, our results suggest that a wide diversity of policy objectives can be achieved in carefully choosing the order and the targeting scenarios under TOBE. The final choice will depend on the objectives and the choices of the ministry and teacher unions.

## 2.7 Extensions

### 2.7.1 Reassignment of teachers in the presence of couples.

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The presence of couples in centralized labor markets is a standard problem in the matching literature. Roth (1984) was the first to introduce it for the assignment of medical interns in the U.S to hospitals. Since then, an important literature has studied such problem. The interested reader may refer to the excellent interdisciplinary survey of Biró and Klijn (2013). A couple is formally considered as an unique entity composed of two members. They have a preference ordering over the pairs of possible assignments that they could obtain. The literature has focused on the effect of introducing couples in a standard two-sided matching market. Roth (1984) has shown that in the presence of couples, stable matchings may fail to exist.<sup>44</sup> Other specific properties such that the existence of a one-sided optimal matching or the rural hospital theorem also fail in the presence of couples. Another interesting result for us from Biró and Klijn (2013) is that, with couples, there is no mechanism that is strategy-proof and finds a stable matching whenever such one exists. Finally, in terms of computational complexity, Romm (1990) proved the NP-hardness of the problem of finding a stable matching in the presence of couples when each hospital has only one position. Biró and McDermid (2014) showed that it is still NP-hard even if each couple always apply to positions at the same hospitals. In this section, we investigate the introduction of couples in our model of reassignment of teachers both in terms of existence results and computational complexity. In France for instance, in the complete dataset of 2013 we used, around 7.5% of the 25 196 teachers applying to the regional assignment phase apply for a couple assignment. In our former analysis, we chose to omit these couples. In practice, their treatment is particular and non standard, we refer the reader to the description done by Terrier (2014) of the French procedure on the Matching in Practice website.<sup>45</sup>

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<sup>43</sup>This section is based on an ongoing preliminary work of Biro, Combe and Tercieux.

<sup>44</sup>There are several definitions of stability in the presence of couples. The main issue is that, under responsive preferences for hospitals, it is not clear how the latter would consider the application of both members of a couples. For all the definitions, one can find an example such that stable matchings do not exist. Since stability cannot be guaranteed in our reassignment setting, we refer the reader to the survey of Biró and Klijn (2013).

<sup>45</sup><http://www.matching-in-practice.eu/matching-practices-of-teachers-to-schools-france/> For a brief description, couples are asked to report a common list over regions. They cannot rank pairs of regions that would allow them to express their preferences for one member of the couple being in one region and the other in another one. They then run the DA\* algorithm in considering each member of the couple as a single teacher with the list reported by the couple. If at the end, couple members are not matched together, they delete both regions from the common couple list and they run again the DA\* algorithm with the new list and iterate the procedure.

We consider a setting where there are now  $K$  **single** teachers:  $S := \{t_1, \dots, t_K\}$  and  $L$  **couples**:  $C := \{c_1, \dots, c_L\}$ . A couple is composed of two individual teachers:  $c_l = (m_l, f_l)$ . So in total there are  $N := K + 2 \times L$  teachers in the set  $T := S \cup (\bigcup_{l=1}^L \{m_l, f_l\})$ . With a slight abuse of notations, we will sometimes denote  $t$  has a generic teacher in  $T$  so that  $t$  can also be a member of a couple. To be closer to our former French application of assigning teachers to regions we will use the terminology “regions” rather than “schools”. So there are  $M$  **regions**:  $R = \{r_1, \dots, r_M\}$  and a generic region  $r$  has  $q_r$  positions for teachers. Each region  $r \in R$  has a strict preference order,  $\succ_r$  over individual teachers in  $T$ . Each single teacher  $t \in S$  has a strict preference order,  $\succ_t$ , over  $R$ . Each couple  $c = (m, f) \in C$  has strict preference order,  $\succ_c$  over pairs of possible assignments i.e. over  $R \times R$  where the first coordinate is the assignment of  $m$  and the second the one of  $f$ . Let  $\mathcal{P}_S$  and  $\mathcal{P}_C$  be the sets of all possible preferences for respectively singles and couples. We will sometimes restrict the possible domain, especially for couples and will denote  $\mathcal{D}_S \subset \mathcal{P}_S$  and  $\mathcal{D}_C \subset \mathcal{P}_C$  the considered domains. There is an initial assignment of teachers to regions  $\mu_0 : T \mapsto R$ . For simplicity, we assume that we are in a pure reassignment problem so that  $\sum_{r \in R} q_r = N$  and all teachers are initially matched so that  $\forall t \in T \mu_0(t) \neq \emptyset$ . With again a slight abuse of notations, we will sometimes refer to  $\mu_0(c_l) := (\mu_0(m_l), \mu_0(f_l))$  as the initial matching of couple  $c_l$ . We can redefine all the notions introduced in the paper with the presence of couples. A matching  $\mu : T \mapsto R$  is:

- **1-Individually Rational** (1-IR) if  $\forall c \in C, \mu(c) \succeq_c \mu_0(c)$  and  $\forall t \in S: \mu(t) \succeq_t \mu_0(t)$ .
- **2-Individually Rational** if it is 1-IR and  $\forall r \in R, \mu(r) \succeq_r \mu_0(r)$ .<sup>46</sup>
- **1-Pareto Efficient** (1-PE) if there is no other matching  $\mu'$  s.t.  $\forall c \in C, \mu'(c) \succeq_c \mu(c)$  and  $\forall t \in S, \mu'(t) \succeq_t \mu(t)$  and strictly for some  $c \in C$  or  $t \in S$ .
- **2-Pareto Efficient** (2-PE) if there is no other matching  $\mu'$  s.t.  $\forall c \in C, \mu'(c) \succeq_c \mu(c), \forall t \in S, \mu'(t) \succeq_t \mu(t)$  and  $\forall r \in R, \mu'(r) \succeq_r \mu(r)$  and strictly for at least one couple, single or region.
- **Two sided maximal** if it is 2-IR and 2-PE.

A mechanism  $\varphi$  maps admissible preference profiles of couples and singles to matchings. A mechanism is 1-IR (resp. 2-IR/1-PE/2-PE/ Two sided maximal) if it always returns a 1-IR (resp. 2-IR/1-PE/2-PE/ Two sided maximal) matching. A mechanism is **strategy-proof** on the domains  $\mathcal{D}_S$  and  $\mathcal{D}_C$  if:

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<sup>46</sup>For comparing sets of teachers for regions, we use the same definition as the one we used in Section 2.3.1.



1.  $\forall c \in C, \forall \succ_{-c}$  with profiles for singles and couples in the domains  $\mathcal{D}_S$  and  $\mathcal{D}_C, \forall \succ_c, \succ'_c \in \mathcal{D}_C$   
 $: \varphi(\succ_c, \succ_{-c})(c) \succeq_c \varphi(\succ'_c, \succ_{-c})(c);$
2.  $\forall t \in S, \forall \succ_{-t}$  with profiles for singles and couples in the domains  $\mathcal{D}_S$  and  $\mathcal{D}_C, \forall \succ_t, \succ'_t \in \mathcal{D}_S :$   
 $\varphi(\succ_t, \succ_{-t})(t) \succeq_t \varphi(\succ'_t, \succ_{-t})(t)$

Note that here, contrary to the current French procedure, a mechanism uses the preferences of couples over pairs of possible assignments. A mechanism is **polynomial** if it could be implemented using a polynomial time algorithm i.e. the time to solve any instance is a polynomial function of the number of teachers, couples and regions in the instance.<sup>47</sup> For some problems, there is no known polynomial algorithm to solve them or even check in polynomial time if a given solution is valid but there are known to be as “hard to solve” as any polynomial problem. These problems are called NP-hard problems.<sup>48</sup> Before moving to some results, we will describe an important domain of preferences for couples. In France, the distance between two regions is important. Since the main goal of couples is to stay together, one can reasonably assume that they would like to be assigned to the same region.<sup>49</sup> So let  $\mathcal{D}_C^r \subset \mathcal{P}_C$  be the domain of **regional preferences** for couples i.e.  $\succ_c \in \mathcal{D}_C^r$  if for two regions  $r \neq r'$  and  $\mu_0(c) \neq (r, r')$  then  $\mu_0(c) \succ_c (r, r')$ . So couples find unacceptable to be matched to different regions.<sup>50</sup>

Before moving to the first result, note that, contrary to the existence of stable matchings, the existence of a two-sided maximal matching does not fail once one introduces couples. Indeed, if the initial matching is not two-sided maximal, then there exists another one where all singles, couples and regions are weakly better-off and some strictly. If this new matching is not two-sided maximal then there exists another one where all singles, couples and regions are again weakly better-off and some strictly. We can iterate the argument and, by finiteness of the environment, it will lead to a

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<sup>47</sup>As we will mention later, it is also possible to fix the number of regions and to only let the number of teachers increase.

<sup>48</sup>We refer the reader to any standard textbook of computer science for a formal definition. By “as hard to solve”, we mean that any given instance of some polynomial problem could be reinterpreted (formally, reduced in polynomial time) as an instance of the studied problem. Over time, the literature has identified an important list of NP-hard problems. To prove that a problem is NP-hard, one has to show that one of the well known problems in the aforementioned list could be reduced to the studied problem. We will show in Proposition 2.7.4 such reduction.

<sup>49</sup>Obviously, one may argue that if both of them are assigned to different regions but next to the common boundary of the two, then they would prefer it. We rule out here this possibility.

<sup>50</sup>Since we will focus on IR mechanisms, we define acceptability of a match with respect to the initial allocation of a couple/teacher. Formally, this domain is the equivalent in our setting of “consistent” preferences as defined by [McDermid and Manlove \(2010\)](#). Remember that, in the standard two-sided setting, they proved the NP-hardness of finding a stable matching even when couples have consistent preferences.

two-sided maximal matching. Note that the previous argument is not constructive, being able to find such matching is the main challenge. We start with a simple positive result:

**Proposition 2.7.1.** *If there are no single teachers, i.e.  $K = 0$ , couples have regional preferences and  $\forall c = (m, f)$ ,  $\mu_0(m) = \mu_0(f)$ , then there exists a strategy-proof and two-sided maximal mechanism that is polynomial.*

*Proof.* Since there are no single teachers, we have  $N = 2 \times L$ . Since we are in a pure reassignment problem and that all couples are initially matched to the same region, then each region has an even number of seats so that  $q_r = 2 \times \tilde{q}_r \forall r \in R$ . We will consider a “couple one to one” environment where  $q_r = 2$  for all  $r$  so that there is only one couple per region.<sup>51</sup> So consider a teacher assignment problem as defined in Combe, Tercieux and Terrier (2016b) where there are  $M$  regions and where each region  $r$  has 2 seats. Couples are interpreted as individual teachers. Since preferences are regional, we can define, for each couple, a linear order over  $R$ , say  $\succ_c$  s.t.  $r \succ_c r' \Leftrightarrow (r, r) \succ_c (r', r')$ . We will use the TOBE algorithm as defined in Combe, Tercieux and Terrier (2016b) that is a strategy-proof and two-sided maximal mechanism that runs in polynomial time. To run it, we use the preferences of the couples over the regions as defined above. We just need to define how to use the preferences of the regions that are initially over individual teachers to compare couples. Note that, in the original one to one TOBE algorithm, teachers point to their favorite teacher-school pair among the ones in which the school is in their opportunity set i.e. the set of schools that strictly prefer them to their initial teachers. Here, say that the region  $r$ , initially matched with couple  $(m, f)$  is in the opportunity set of couple  $c' = (m', f')$  if either i)  $m' \succ_r m$  and  $f' \succ_r f$  or ii)  $m' \succ_r f$  and  $f' \succ_r m$ . This is to ensure that when couple  $(m', f')$  replaces couple  $(m, f)$  then region  $r$  is not hurt according to the definition we gave in Section 2.3.1. With this definition, one can run the TOBE algorithm and find a matching that is two-sided maximal. □

Unfortunately the next proposition shows that the positive result fails if one only drops the assumption that all couples are initially matched to the same region.<sup>52</sup>

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<sup>51</sup>The extension of the described procedure to the many to one case with several couples initially matched to a region is similar to the one used earlier: one just needs to define a pointing order over the couples.

<sup>52</sup>However, if one assumes that all couples prefer to be assigned together in the initial regions of its members and that it is always feasible in the region preferences to assigned the couples together in the same region. Then one can do this exchange and then use the same algorithm as in Proposition 2.7.1, this would be a polynomial, two-sided maximal and strategy-proof mechanism. However, if preferences of the regions can be arbitrary, it might not be feasible to initially assign couples together and so the following impossibility result would still hold.

**Proposition 2.7.2.** *If there are no single teachers, i.e.  $K = 0$ , that couples have regional preferences and that the initial matching is arbitrary, then there exists no strategy-proof and two-sided maximal mechanism.*

*Proof.* There are 4 couples:  $c_k := (m_k, f_k)$  for  $k = 1, \dots, 4$  and 3 regions  $r_k$ ,  $k = 1, 2, 3$ . Regions  $r_1$  and  $r_2$  have two seats each and region  $r_3$  has four seats. The initial matching is given by:

$$\mu_0 = \begin{pmatrix} m_1 & f_1 & m_2 & f_2 & m_3 & f_3 & m_4 & f_4 \\ r_1 & r_2 & r_1 & r_2 & r_3 & r_3 & r_3 & r_3 \end{pmatrix}$$

Preferences of the couples are as follows:

$$\begin{aligned} \succ_{c_1}: & (r_3, r_3) \quad (r_1, r_2) \\ \succ_{c_2}: & (r_3, r_3) \quad (r_1, r_2) \\ \succ_{c_3}: & (r_1, r_1) \quad (r_2, r_2) \quad (r_3, r_3) \\ \succ_{c_4}: & (r_1, r_1) \quad (r_2, r_2) \quad (r_3, r_3) \end{aligned}$$

Define preferences of the regions such that their initial teachers are ranked at the bottom so that any new matching would be IR for the regions. Note that under the profile  $\succ$  given by the above preferences, there are two possible two-sided matchings: either matching  $c_3$  to  $r_1$  and  $c_4$  to  $r_2$  or the reverse. If one chooses the former, then  $c_4$  could delete  $(r_2, r_2)$  from its profile (note that in doing so the resulting profile is still a regional profile) and the only possible choice is to match  $c_4$  to  $r_1$  and  $c_3$  to  $r_2$  so that the manipulation is successful. If one chooses the latter, the same manipulation occurs for  $c_3$  so that there is no strategy-proof and two-sided maximal mechanism.  $\square$

The next proposition shows that if one keeps regional preferences and the assumption that couples are initially matched in the same region but just adds single teachers, then the result also fails:

**Proposition 2.7.3.** *If  $K > 0$ , that couples have regional preferences and that  $\forall (m, f) \in C$ ,  $\mu_0(m) = \mu_0(f)$  then there exists no strategy-proof and two-sided maximal mechanism.*

*Proof.* There are 2 couples:  $c_k := (m_k, f_k)$  for  $k = 1, 2$ , two singles:  $t_1, t_2$  and 3 regions  $r_k$ ,  $k = 1, 2, 3$

with 2 seats each. The initial matching is given by:

$$\mu_0 = \begin{pmatrix} m_1 & f_1 & m_2 & f_2 & t_1 & t_2 \\ r_1 & r_1 & r_2 & r_2 & r_3 & r_3 \end{pmatrix}$$

Preferences are as follows:

$$\begin{aligned} \succ_{c_1}: & (r_3, r_3) \quad (r_1, r_1) \\ \succ_{c_2}: & (r_3, r_3) \quad (r_2, r_2) \\ \succ_{t_1}: & r_1 \quad r_2 \quad r_3 \\ \succ_{t_2}: & r_2 \quad r_1 \quad r_3 \end{aligned}$$

Define preferences of the regions such that their initial teachers are ranked at the bottom so that any new matching would be IR for the regions. Note that under the profile  $\succ$ , there are two possible two-sided maximal matchings: either couple  $c_1$  exchange seats with the two singles so that teachers  $t_1$  and  $t_2$  end up being matched to  $r_1$ . Or couple  $c_2$  exchanges seats with the two singles and the latter end up being matched to  $r_2$ . In the first case, let teacher  $t_2$  report the following profile:  $\succ'_{t_2}: r_2, r_3, r_1$ . In that case, doing the exchange with couple  $c_1$  would violate the IR constraint for this profile and so the only possible two-sided maximal matching is to match couple  $c_2$  to  $r_3$  and the two singles to  $r_2$ : a successful manipulation for  $t_2$ . In the second case, there is a similar deviation for  $t_1$  who could force the mechanism to match him to  $r_1$  in reporting the profile:  $\succ'_{t_1}: r_1, r_3, r_2$ .  $\square$

In terms of computational complexity, the next proposition shows the NP-hardness of the problem of finding a matching Pareto-dominating the initial allocation for teachers.

**Proposition 2.7.4.** *The problem of deciding whether there exists a matching that Pareto-dominates the initial allocation for teachers in a problem of reassignment of teachers with couples and singles is NP-hard. Even when couples have regional preferences and are all initially matched to the same region.*<sup>53</sup>

*Proof.* The reduction is from the problem of matching couples and singles to rooms (here we will use the term regions) with no preferences. In this problem, there are  $n$  twin rooms (here, regions with two seats) and  $2n$  agents, some of them forming couples. For each couple or single, there is

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<sup>53</sup>The following reduction cannot be used if one would like to prove a NP-hardness result with no singles but where couples are not initially matched in the same region. However, we conjecture that the NP-hardness would still hold.

a set of acceptable rooms. We can accommodate in a room either one couple or two singles. The problem is to decide whether we can allocate everyone to the rooms. [Biró and McDermid \(2014\)](#) showed that this problem is NP-hard.

We reduce the above problem to the problem of deciding Pareto-efficiency of an allocation of teachers with couples and singles. So start from a problem of matching  $k$  single teachers  $t_1, \dots, t_k$  and  $l$  couples  $c_1, \dots, c_l$  with  $n = k + 2l$  (w.l.o.g we assume  $n$  is even) to  $n$  regions with two seats each:  $r_1, \dots, r_n$ . For a teacher reassignment problem, these regions represent regions where the previous couples and singles are willing to move and so do not include their initial regions. All of the above regions are, in the teacher reassignment problem, occupied with couples who are willing to move, let us denote them by  $c'_1, \dots, c'_n$  where couple  $c'_i$  is initially matched with  $r_i$ . Each of these new couples, say  $c'_i$  is willing to move to exactly one other region  $r'_i$ . These regions  $r'_1, \dots, r'_n$  are initially occupied by singles,  $s'_1, \dots, s'_{2n}$  where  $s'_{2i-1}$  and  $s'_{2i}$  are initially assigned to  $r'_i$ . These singles are also willing to move to regions  $r''_1, \dots, r''_n$ .  $s'_1$  is willing to move to  $r''_1$ ,  $s'_2$  and  $s'_3$  are willing to move to  $r''_2$  and, so on,  $s'_{2i}$  and  $s'_{2i+1}$  are willing to move to  $r''_{i+1}$  and finally  $s'_{2n}$  is willing to move to  $r''_1$ . Now these regions are occupied with couples  $c^*_1, \dots, c^*_n$  where  $c^*_i$  is living in  $r''_i$ . They are also willing to move to regions  $r^*_1, \dots, r^*_n$  with  $r^*_i$  being the only acceptable region for the couple  $c^*_i$ . Finally, let us distribute the original singles  $t_1, \dots, t_k$  and couples  $c_1, \dots, c_l$  among regions  $r^*_1, \dots, r^*_n$  by assigning them to these regions arbitrarily so that in each region we have either two singles or one couple.

Now, we have to show that there is a Pareto-improvement for the instance if and only if there was a complete room allocation in the original instance.

( $\Leftarrow$ ) This direction is simple, if there is a complete room allocation, then we let the original agents  $t_1, \dots, t_k$  and  $c_1, \dots, c_l$  move to their acceptable regions according to the room allocation and we can complete the exchange by moving every other single and couple to their unique acceptable region.

( $\Rightarrow$ ) We have to show that if an agent is moving in the instance, then actually everybody must move and so there also exists a complete room allocation. Suppose now for a contradiction that some teachers are moving but not everybody. By the cyclical construction, it follows that the same number of agents must move from every set (the four sets of agents defined above).

Moreover, if  $s'_{2i-1}$  is moving, then  $s'_{2i}$  must also move since they are both living in  $r'_i$  that will be occupied by a couple. Let  $A$  denote the set of regions of the form  $r'_i$  and  $B$  the one of regions of the form  $r''_i$  that are involved in the exchange. If  $A$  is not complete then  $B$  must be larger, leading to a contradiction. Indeed, if  $r'_i$  is involved in the exchange then both  $s'_{2i-1}$  and  $s'_{2i}$  are moving and thus both  $r''_i$  and  $r''_{i+1}$  are involved in the exchange (modulo  $n$ ).

□

The above results suggest that introducing couples in the problem of reassignment of teachers leads to several negative results. In practice, one could use Integer Programming solvers to solve the above problems for reasonable market sizes. Obviously, Propositions 2.7.2 and 2.7.3 will imply that the resulting mechanism will not be strategy-proof. Evaluating with our data this IP-solver method and estimating the difference with the actual treatment of couples would be an interesting policy exercise.<sup>54</sup> Note that the NP-hardness result of Proposition 2.7.4 is proved for  $2n$  individuals and  $n$  regions so that the number of regions grows with the number of individuals. In practice, the number of regions is fixed. So the question of whether there exists a polynomial algorithm for a fixed number of regions but a growing number of agents is still open. We focused here on the first step of the French procedure i.e. the assignment of teachers to regions as described in the Institutional Context in Section 2.2. For this procedure, the domain of regional preferences seems reasonable. For the second step, that matches teachers to schools inside a given region, another domain should be considered.<sup>55</sup> The question of whether there exists some polynomial and strategy-proof mechanism for some well chosen domains is still open. Last, in the standard two-sided setting with couples, positive results have been found using large market approaches. Whether these techniques could be applied to the context of teacher reassignment problems is also an interesting future direction.

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<sup>54</sup>These techniques work well for relatively small instances. In France, the number of regions is 31 as we presented in our dataset. In our previous analysis, we considered each subject as an independent market so that the number of teachers involved in each market is reasonable (maximum 2500 teachers for Sport). However, if one introduces couples, the number of teachers in an instance of the problem could be much bigger. Indeed, it is likely that a couple is composed of two teachers from different subjects e.g. math and history so that these two subjects cannot be considered as disjoint markets. So if one has to merge all subjects together, the number of teachers in a given instance jumps to around 20 000 teachers.

<sup>55</sup>Informally, one may think that once a couple is assigned to the same region, they would like to be as close as possible to a city inside this region and as soon as one of the two members is closer to this city, it is acceptable for the couple. The idea is that the complexity of finding a feasible assignment with couples is driven by the fact that if one member of the couple could feasibly move in an exchange, then one has to check that the other one could also feasibly move in another exchange. The idea would be to break such interdependence in assuming that as soon as one finds a feasible exchange for one member of a couple then one could implement it without checking an exchange for the other member and it would be acceptable the the couple.

## 2.7.2 An alternative approach to the reassignment of teachers: the case of two types.

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Dur and Unver (2015) (DU) studied two-sided matchings via balanced exchanged and used tuition and worker exchanges as applications. They considered the setting where firms exchange *temporarily* their workers.<sup>57</sup> They proposed an algorithm called the Two-Sided Top Trading Cycle (2S-TTC) that ensures that, for each firm, imports and exports of workers are balanced, and further proved that 2S-TTC is the only mechanism that is balanced-efficient, worker strategy-proof, acceptable, individually rational, and satisfying another axiom called respecting internal priorities. In practice, for instance in our teacher assignment setting, as we discussed in previous sections, exact balancedness might not be a desirable goal and some imbalance tolerance might be acceptable in some regions. DU also defined the Two-Sided Tolerable Top-Trading Cycles mechanism (2S-TTTC) where for each school, there are lower and upper bounds imbalance tolerance between exports and imports of students. This mechanism is strategy-proof and produces a matching that is constrained Pareto-efficient so that no other matching also respecting the imbalance can Pareto-dominate it. In our setting, this approach has the advantage of controlling exactly the final imbalance of a region which is attractive from a policy perspective.

Although our approach in Combe, Tercieux and Terrier (2016b) and here pertains to the same two-sided environment with initial assignment, a key feature differentiates our analysis. We consider a richer set of schools' preferences. To capture the specificities of worker-exchange programs, Dur and Unver (2015) make assumptions about the preferences of workers and firms. They notably assume that firms do not have "strong" preferences over acceptable workers, i.e. they are all equally desirable for firms.<sup>58</sup> This assumption of coarse preferences over incoming agents might be plausible in the environment of temporary exchange of workers. Firms might only rank temporary workers as acceptable and unacceptable as the cost of not being matched with the best candidate is relatively limited since they will leave later on. However, when assignments are permanent, firms or schools are likely to have finer preferences over the set of applicants. Rather than being perceived as substitutable

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<sup>56</sup>This section is based on an ongoing preliminary work of Combe, Dur, Tercieux, Terrier and Unver.

<sup>57</sup>Temporary exchanges of teachers is a possible application.

<sup>58</sup>As stated by the authors, "each firm requires a set of specific skills, e.g., a mathematics teacher to replace their own mathematics teacher. Compatibility and ability to perform the task are the main preference criterion rather than a strict preference ranking. E.g., finding a good teacher with a specific degree is the first-order requirement, rather than finer details about the rankings of all good teachers."

applicants recruited to perform a teaching task, schools know that teachers' characteristics can vary widely, notably in terms of number of years of experience, number of different schools they have been working in, experience in teaching in disadvantaged schools, family characteristics, and so on. These characteristics are all used to define a schools' preferences, and some evidence shows that firms and schools do care about the characteristics of their employees. Boyd et al. (2013) find that schools show preferences for teachers with stronger academic achievement (e.g., attended a more selective college) and for teachers living in closer proximity to the school. Accounting for schools' fine preferences over teachers is particularly important when the distribution of experienced teachers differs across schools. Schools might want to maintain a balance in their teachers' characteristics and experience. If an experienced teacher wishes to leave, the school might want to replace her by an equally experienced teacher. If an inexperienced teacher wishes to leave, the school might care less about the experience of the incoming teacher. More formally, in their environment, under 2S-TTC, if a teacher  $t$  can replace another teacher  $t'$  who is initially matched with a school  $s$ , then teacher  $t$  could also replace any teacher  $t''$  who is also initially matched with  $s$ . In contrast, in our environment, teacher  $t$  might be more experienced than teacher  $t'$  but less experienced than teacher  $t''$  (so that  $t'' \succ_s t \succ_s t' \succ_s \emptyset$ ). School  $s$  would therefore accept teacher  $t$  only if she is replacing teacher  $t'$ , but it would not accept the exchange if she was replacing teacher  $t''$ . In practice, this type of preferences for schools arises quite frequently. In our dataset, for the ten biggest disciplines, on average 53,2% of the applicants have an "intermediate ranking", meaning that they are ranked strictly higher than the least preferred internal teacher, and strictly lower than the most preferred internal teacher. On average, 91.3% of the regions receive at least one application whose ranking is intermediate.<sup>59</sup> Incorporating schools' fine preferences has two important consequences.

First, with finer preferences, 2S-TTC is not two-sided maximal. In the above example, under 2S-TTC, teacher  $t$  could replace  $t''$ , which would violate individual rationality for school  $s$  since  $t$  is less preferred to  $t''$  in  $s$ .<sup>60</sup> In addition, due to the possibility to recruit teachers who are less preferred to those leaving, 2S-TTC could create new blocking pairs compared to those existing under

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<sup>59</sup>To compute this statistic, for every discipline\*region, we have defined the minimum and the maximum ranking of the internal teachers. Then, for every applicant teacher, we define her ranking as "intermediate" if it is strictly higher than the minimum and strictly lower than the maximum.

<sup>60</sup>One may avoid to violate the IR constraint for the schools in modifying the preferences of the schools before running 2S-TTC. In our example, one could instead use the modified preferences  $t'' \succ_s \emptyset$  i.e. truncating the preferences just after the first ranked initial teacher in  $s$ . But in doing so, 2S-TTC would not be 2-Pareto efficient since it could miss a potential profitable exchange between  $t$  and  $t'$ .



the initial matching.

Second, the characterization result in [Dur and Unver \(2015\)](#) relies on the axiom of the respect of internal priorities. This axiom states that, if a teacher  $t$  is initially matched to a school  $s$  moves from  $s$ , then adding a seat in  $s$  should not cause  $t$  to stay at his initial school, if that additional seat is occupied by an initial teacher  $t'$  who is less preferred by  $s$  to  $t$ .<sup>61</sup> However, in an environment with finer school preferences the next proposition shows that the characterization fails:

**Proposition 2.7.5.** *There is no mechanism that is two-sided maximal, strategy-proof and respects internal priorities.*

*Proof.* There are 4 teachers  $t_1, t'_1, t_2$  and  $t_3$ . And 3 schools:  $s_1, s_2$  and  $s_3$ . Preferences of the teachers are given by:

$$\begin{aligned} \succ_{t_1}: & s_2 \ s_1 \\ \succ_{t'_1}: & s_2 \ s_1 \\ \succ_{t_2}: & s_1 \ s_3 \ s_2 \\ \succ_{t_3}: & s_1 \ s_3 \end{aligned}$$

Preferences of the schools are:

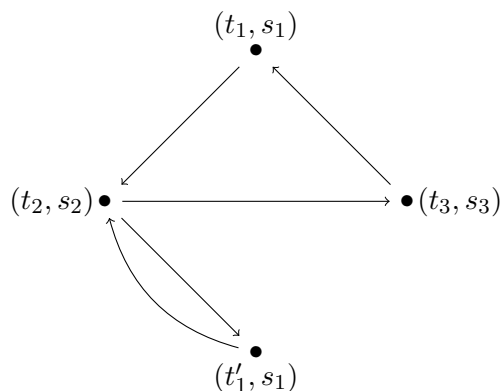
$$\begin{aligned} \succ_{s_1}: & t_3 \ t_1 \ t_2 \ t'_1 \\ \succ_{s_2}: & t_1 \ t'_1 \ t_2 \\ \succ_{s_3}: & t_2 \ t_3 \end{aligned}$$

The BE graph of the example is given in [Figure 2.2](#). Assume that  $t'_1$  is not on the market and that  $s_1$  has only one seat initially occupied by  $t_1$  and that  $t_2$  and  $t_3$  are respectively initially matched with  $s_2$  and  $s_3$ . The small two-way cycle in the above graph would not exist and the only two-sided maximal matching would match  $t_1$  in  $s_2$ ,  $t_2$  in  $s_3$  and  $t_3$  in  $s_1$ .  $t_1$  would be moving from  $s_1$ . Now, add  $t'_1$  in the market with an additional seat in  $c_1$ . According to the axiom of respecting internal priorities, since  $t'_1$  has a lower ranking than  $t_1$  in  $s_1$ , the existence of  $t'_1$  should not prevent  $t_1$  from moving from  $s_1$ . Therefore, one cannot implement the small two-way cycle in the graph. Only the large three-way cycle is left, and the only two-sided maximal matching is obtained by executing the same exchange as before the presence of  $t'_1$ . But in doing so,  $t_2$  has an obvious manipulation: in the

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<sup>61</sup>See [Dur and Unver \(2015\)](#) for a formal definition.

Figure 2.2 – BE GRAPH OF THE EXAMPLE.



presence of  $t'_1$ , he could claim that  $s_3$  is ranked below his initial school  $s_2$  so that the only two-sided maximal matching with this untruthful list from  $t_2$  is to make him switch his assignment with  $t'_1$ : a successful manipulation, contradicting strategy-proofness.  $\square$

Considering finer preferences of regions and controlling the exact imbalance of a given region are important tools to better manage the quality and the turnover of the pool of assigned teachers. However, [Dur and Unver \(2015\)](#) techniques to exactly control imbalance cannot be used when one consider finer preferences. Our approach considers a complete ordering over teachers for each region but we do not directly control the final imbalance of each region. By targeting teachers inside a given region we can influence it but the exact imbalance will also be influenced by reported preferences of teachers and those of schools so that one cannot predict it in only targeting teachers inside a region. So a natural next step would be to consider an in-between model where preferences are finer than those in [Dur and Unver \(2015\)](#) but coarser than ours and that one can still better control imbalance than in our approach. We provide here a first attempt with a simple model.

As we discussed in our analysis, an important policy issue is to control the proportion of teachers with relatively low experience in sensitive regions like Créteil and Versailles. Even if, in practice, newly tenured teachers, i.e. those without an initial assignment, are considered as teachers with low experience, we consider a simple pure reassignment problem with only tenured teachers i.e. with an initial assignment and no empty seats in each region. There is a finite set of teachers  $T$ . Each teacher  $t \in T$  has a type  $\tau(t) \in \mathcal{T}$ . In a general framework,  $\mathcal{T}$  is finite and could be multidimensional. Each dimension is a characteristic of a given teacher. For instance a teacher could have low experience,

medium experience or high experience. In addition, he could currently teach in a disadvantaged school or not. For simplicity, we assume here that there is only one dimension: a teacher  $t$  could be “Young” or “Old” so that  $\mathcal{T} = \{Y, O\}$ . With a slight abuse of notations,  $Y$  will denote the set of young teachers and  $O$  the set of old ones. There is a finite set of schools  $S$  and an initial matching  $\mu_0$ . Each teacher  $t$  has a strict preference order  $\succ_t$  over schools in  $S$ . For preferences of the schools, we do not assume that there is an ordering over teachers. However, the set of schools  $S$  is divided into three categories:  $S_-$ ,  $S_+$  and  $S_r$ . A matching  $\mu$  is said to be **feasible** if:

1. For  $s \in S_-$ :  $|\mu(s) \cap Y| \leq |\mu_0(s) \cap Y|$
2. For  $s \in S_+$ :  $|\mu(s) \cap Y| \geq |\mu_0(s) \cap Y|$

So for schools in  $S_-$ , the designer wants to weakly decrease their proportion of young teachers and for schools in  $S_+$ , he wants to increase it. Schools in  $S_r$ , the “rest”, are indifferent and accept every new matching. We say that a matching  $\mu$  is **maximal** if it is feasible and there is no other feasible matching  $\mu'$  that Pareto dominates it for the teachers. As before, a matching is **Individually Rational** (IR) if it Pareto-dominates the initial matching  $\mu_0$ . Note that here, preferences of schools are coarser than the approach we had before since only the number of young teachers matters for the schools. These preferences are also finer than those considered in [Dur and Unver \(2015\)](#). Indeed, fix a schools  $s \in S_-$ . The designer would like to weakly reduce the number of young teachers in this school. If no old teacher comes in then he does not want to realize an exchange that would increase its number of young teachers. But if one old teacher is matched to it, then the designer would be ready to let a young teacher leave. As in [Combe, Tercieux and Terrier \(2016b\)](#), we would like first to characterize the class of IR and maximal mechanisms in properly defining a class of mechanisms.

We are in a many-to-one environment,<sup>62</sup> so let us first redefine the problem into a one-to-one setting using similar techniques as the ones used in [Section 2.4.2.1](#). For each school  $s$ , create  $|\mu_0(s)|$  school-copies of  $s$ . And assign each school copy  $\tilde{s}$  a type  $\tau$  s.t. the number of school copies with type  $\tau$  is exactly  $|\mu_0(s) \cap \tau|$ . A general school copy will be denoted by  $(\tilde{s}, \tau)$ . For a school-copy  $\tilde{s}$  we will denote  $s(\tilde{s})$ , the school in the original problem from which  $\tilde{s}$  is a copy. Define an initial matching of school-copies  $\tilde{\mu}_0$  in assigning to a school copy  $(\tilde{s}, \tau)$  a teacher  $t$  s.t.  $\tau(t) = \tau$  and  $\mu_0(t) = s(\tilde{s})$ .

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<sup>62</sup>The one-to-one setting is not very interesting in this two-types environment. Indeed, one could redefine the TOBE algorithm as defined in [Combe, Tercieux and Terrier \(2016b\)](#) to have a strategy-proof and maximal mechanism.

By construction of the copies, this is well defined and one can easily transform a matching  $\tilde{\mu}$  of school-copies to teachers into a matching  $\mu$  in the original problem.

Now define the **Type-Echange algorithm** (TE) over the one-to-one transformation:

- *Step 1.* Build a directed graph  $(N, E)$  where the nodes are  $N := \bigcup_{t \in T} (\{(t, \tau(t))\} \cup \{(\tilde{\mu}_0(t), \tau(t))\})$ .

For each teacher  $t$  let his corresponding school-copy under  $\tilde{\mu}_0$  point to him,

i.e.  $[(\tilde{\mu}_0(t), \tau(t)), (t, \tau(t))] \in E$ . A node  $(t, \tau)$  points to a node  $(\tilde{s}, \tau')$  if 1)  $s(\tilde{s}) \succ_t \mu_0(t)$  and 2) either:

1.  $s \in S_r$
2.  $s \in S_-$  and either i)  $\tau = Y$  and  $\tau' = Y$  or ii)  $\tau = O$
3.  $s \in S_+$  and either i)  $\tau = O$  and  $\tau' = O$  or ii)  $\tau = Y$

If there is no cycle in the graph, let  $\tilde{\mu}_1 := \tilde{\mu}_0$  and stop the algorithm. Otherwise implement a cycle and let  $\tilde{\mu}_1$  be the resulting matching and go to Step 2.<sup>63</sup>

- *Step  $k \geq 2$ .* Build a directed graph  $(N, E)$  where the nodes are  $N := \bigcup_{t \in T} (\{(t, \tau(t))\} \cup \{(\tilde{\mu}_{k-1}(t), \tau(t))\})$ .

A node  $(t, \tau)$  points to  $(\tilde{s}, \tau')$  if  $s(\tilde{s}) \succ_t \mu_{k-1}(t)$  and if the same conditions as in Step 1 are satisfied. For a node  $(\tilde{s}, \tau)$ :

1. If  $\tau(\tilde{\mu}_{k-1}((\tilde{s}, \tau))) = \tau$ , then  $(\tilde{s}, \tau)$  points to  $\tilde{\mu}_{k-1}((\tilde{s}, \tau))$ .
2. If  $\tau(\tilde{\mu}_{k-1}((\tilde{s}, \tau))) \neq \tau$ , then  $(\tilde{s}, \tau)$  points to all nodes  $(t, \tau')$  s.t.  $s(\tilde{\mu}_{k-1}((t, \tau'))) = s(\tilde{s})$ .

If there is no cycle in the graph, let  $\tilde{\mu}_k := \tilde{\mu}_{k-1}$  and stop the algorithm. Otherwise implement a cycle. In this cycle however, since a node  $(\tilde{s}, \tau)$  matched with a different type, say  $(t, \tau')$ , could point to any teacher matched with  $s(\tilde{s})$ , say  $(t', \tau'')$ , one might end up unmatching  $(t, \tau')$  in implementing this cycle. If so, then match  $(t, \tau')$  to the seat that was assigned to  $(t', \tau'')$ . Let  $\tilde{\mu}_k$  be the resulting matching and go to Step  $k + 1$ .<sup>64</sup>

Note that, by finiteness of the environment, the algorithm must end in finite time. We will state a simple lemma:

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<sup>63</sup> $\tilde{\mu}_1$  is defined over the one-to-one transformation so that a teacher who has been matched is now matched to a particular school-copy that could have a different type than his. This will be important for the next steps.

<sup>64</sup>The condition on the schools' nodes is simple: if the seat has been taken by a teacher of the same type as the initial teacher who had that seat under  $\mu_0$ , then this seat points only to this teacher. If it has been taken by a teacher of a different type, then the seat points to all teachers currently matched to the corresponding school of the seat.

**Lemma 22.** *Every matching obtained after implementing a cycle of TE is feasible*

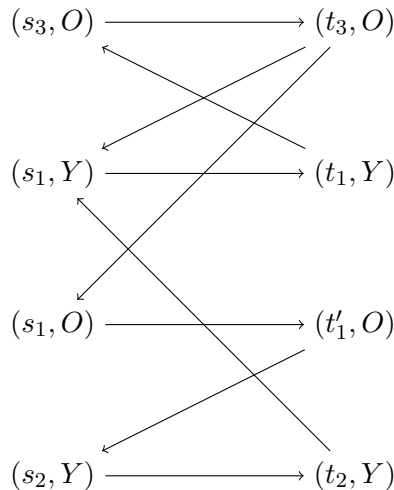
This lemma is simple to prove so we let the formal proof to the reader. Indeed, at each step, any school in  $S_-$  for instance can only receive weakly more old teachers since young teachers are only allowed to be assigned to seats of type  $Y$  and the number of these seats is exactly equal to the number of young teachers in this school under the initial matching. Before moving to the next proposition, let us give a simple example to illustrate the steps of the algorithm:

**Example 5.** *There are four teachers and 3 schools:  $t_1, t'_1$  are initially assigned to  $s_1$ ,  $t_2$  to  $s_2$  and  $t_3$  to  $s_3$ . We assume that  $s_1, s_2 \in S_-$  and  $s_3 \in S_r$ ,  $t_1, t_2 \in Y$  and  $t'_1, t_3 \in O$ . Preferences of the teachers are given by:*

$$\begin{aligned} \succ_{t_1}: & s_3 \ s_1 \\ \succ_{t'_1}: & s_2 \ s_1 \\ \succ_{t_2}: & s_1 \ s_2 \\ \succ_{t_3}: & s_1 \ s_3 \end{aligned}$$

*Schools  $s_2$  and  $s_3$  have only one seat and school  $s_1$  has to be transformed into two school-copies:  $(s_1, Y)$  and  $(s_1, O)$ . The first one is initially assigned to  $t_1$  and the second to  $t'_1$ . The graph of TE at the first step is given in Figure 2.3.*

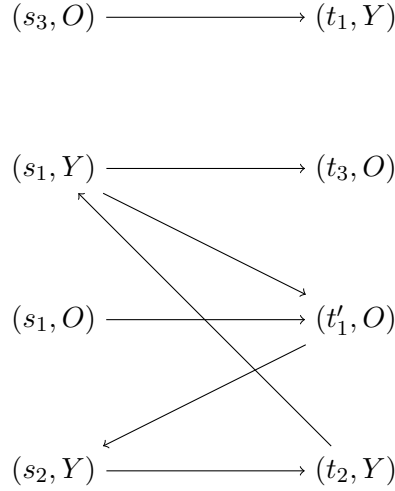
Figure 2.3 – GRAPH OF THE FIRST STEP OF TE IN EXAMPLE 5.



*There are two possible cycles: a long one that matches every teacher to his first choice. And a shorter one:  $(t_3, O) \rightarrow (s_1, Y) \rightarrow (t_1, Y) \rightarrow (s_3, O) \rightarrow (t_3, O)$ . To better illustrate some steps of the*

algorithm, let us pick the short one and implement it. The graph of the next step of the algorithm is given in Figure 2.4. Note that now, since the school copy  $(s_1, Y)$  is matched to  $(t_3, O)$ , it can

Figure 2.4 – GRAPH OF THE SECOND STEP OF TE IN EXAMPLE 5.



also point to  $(t'_1, O)$ . There is one cycle:  $(s_1, Y) \rightarrow (t'_1, O) \rightarrow (s_2, Y) \rightarrow (t_2, Y) \rightarrow (s_1, Y)$ . Once implemented, every teacher will be matched to his first ranked school and the algorithm will end. However, note that in doing so, one will match the school-copy  $(s_1, Y)$  to  $(t_2, Y)$  and this school-copy was assigned to  $(t_3, O)$  who was not part of the cycle. As noted in the description of the algorithm, one has to match  $(t_3, O)$  to the school-copy corresponding to the same school than  $(s_1, Y)$  and that was part of the cycle. Here, one has to match  $(t_3, O)$  to  $(s_1, Y)$ . Note that the corresponding matching in the many-to-one setting stays the same since  $(s_1, Y)$  and  $(s_1, O)$  both correspond to seats in school  $s_1$ .

As in the definition of the Block-Exchange algorithm in [Combe, Tercieux and Terrier \(2016b\)](#), we leave open the way in which the algorithm selects cycles so that this algorithm defines a class of mechanisms, each mechanism being pinned down by a rule to select cycles. So we will think of TE as a set of matchings that could be obtained in varying the possible selection of cycles.

**Proposition 2.7.6.** *The set of matchings that could be obtained with TE is equal to the set of feasible, IR, and maximal matchings.*

Before moving to the proof, let us prove a crucial Lemma. Let  $\mu$  be any feasible IR matching. Let  $\mu'$  be any feasible matching that Pareto dominates  $\mu$ . Let  $\tilde{\mu}$  be any school-copy matching that

represents  $\mu$  in the one-to-one transformation where the school-copies are defined with the initial matching  $\mu_0$ . Let  $G := (N, E)$  be the graph that is built at a step of TE where the school-copy matching is  $\tilde{\mu}$ . And let  $\tilde{T} := \{t : \mu(t) \neq \mu'(t)\}$  be the set of teachers who have a different school under  $\mu'$  and  $\mu$ . Then we have the following result:

**Lemma 23.**  $\forall t \in \tilde{T}, \exists(\tilde{s}, \tilde{\tau}) \in N$  and  $t' \neq t$  in  $\tilde{T}$  s.t:

1.  $s(\tilde{s}) = \mu'(t)$  and  $[(t, \tau(t)), (\tilde{s}, \tilde{\tau})] \in E$ .
2.  $\mu(t') = s(\tilde{s})$  and  $[(\tilde{s}, \tilde{\tau}), (t', \tau(t'))] \in E$ .

*Proof.* Fix  $t \in \tilde{T}$  and let  $s := \mu(t)$  and  $s' := \mu'(t)$ . Since  $t \in \tilde{T}$ , we have  $s \neq s'$  and  $s' \succ_t s$ . There are three cases to consider:

*Case 1:*  $s' \in S_r$ .

Since  $\mu'(t) = s'$  and  $\mu(t) \neq s'$ , there must be a teacher  $t'$  s.t  $\mu(t') = s'$  and  $\mu'(t') \neq s'$  so that  $t' \in \tilde{T}$ . Let  $\tilde{s}$  be the school-copy  $t'$  is assigned to under  $\tilde{\mu}$  so that  $s(\tilde{s}) = s'$  and let  $\tilde{\tau}$  be the type of school-copy  $\tilde{s}$ . Since  $s' \in S_r$ , and  $s' = s(\tilde{s}) \succ_t s$ , we have by construction of the graph  $G$  of TE that  $[(t, \tau(t)), (\tilde{s}, \tilde{\tau})] \in E$ . Since  $\tilde{s}$  is assigned to  $t'$  under  $\tilde{\mu}$ , we also have by construction of  $G$  that  $[(\tilde{s}, \tilde{\tau}), (t', \tau(t'))] \in E$ .

*Case 2:*  $s' \in S_-$  and  $|\mu(s') \cap Y| < |\mu_0(s') \cap Y|$ .

As in the previous case, we can find a teacher  $t'$  s.t  $\mu(t') = s'$  and  $\mu'(t') \neq s'$  so that  $t' \in \tilde{T}$ . Since  $|\mu(s') \cap Y| < |\mu_0(s') \cap Y|$ , there must be a school-copy  $\tilde{s}$  s.t  $s(\tilde{s}) = s'$  with type  $Y$  that is assigned under  $\tilde{\mu}$  to a teacher  $\tilde{t}$  with  $\tau(\tilde{t}) = O$ . So, by construction of  $G$ , since this school-copy is of type  $Y$  and is matched to a teacher of type  $O$ , it could point to any node  $(t'', \tau(t''))$  s.t  $\mu(t'') = s(\tilde{s}) = s'$ . In particular, we have that  $[(\tilde{s}, Y), (t', \tau(t'))] \in E$ . Moreover, by construction of  $G$ , since  $s(\tilde{s}) = s' \in S_-$ , then all teachers could point to this school-copy independently of their type. So in particular,  $[(t, \tau(t)), (\tilde{s}, Y)] \in E$ .

*Case 3:*  $s' \in S_-$  and  $|\mu(s') \cap Y| = |\mu_0(s') \cap Y|$ .

There are two subcases to consider:

*Case 3.1:*  $\exists t' \in \tilde{T}$  s.t  $\mu(t') = s'$  and  $\tau(t') = Y$ .

Since  $s' \in S_-$ , by construction of TE, teachers of type  $Y$  could only be matched to school-copies of type  $Y$ . So take the teacher  $t'$  aforementioned in Case 3.1. He is assigned under  $\tilde{\mu}$  to a school copy

$\tilde{s}$  of type  $Y$  s.t  $s(\tilde{s}) = s'$ . As in Case 2, by construction of  $G$ , we have that  $[(\tilde{s}, Y), (t', \tau(t'))] \in E$  and  $[(t, \tau(t)), (\tilde{s}, Y)] \in E$ .

*Case 3.2:*  $\forall t' \in \tilde{T}$  s.t  $\mu(t') = s', \tau(t') = O$ .

Since  $|\mu(s') \cap Y| = |\mu_0(s') \cap Y|$ , all the school-copies  $\tilde{s}$  of type  $Y$  s.t  $s(\tilde{s}) = s'$  are assigned under  $\tilde{\mu}$  to teachers of type  $Y$ . Fix  $t' \in \tilde{T}$  s.t  $\mu(t') = s'$ . By assumption,  $\tau(t') = O$  and by the previous remark, it must be the case that  $\tilde{\mu}(t') = (\tilde{s}, O)$  for some school-copy  $\tilde{s}$  s.t  $s(\tilde{s}) = s'$ . If  $\tau(t) = O$ , then we are done since by construction of the graph  $G$ ,  $(t, O)$  could point to  $(\tilde{s}, O)$  and so  $[(t, O), (\tilde{s}, O)] \in E$  and  $[(\tilde{s}, O), (t', O)] \in E$ .

Assume that  $\tau(t) = Y$ . Since by assumption, all the teachers leaving  $s'$  under  $\mu'$  are of type  $O$  and that  $|\mu(s') \cap Y| = |\mu_0(s') \cap Y|$ , then assigning  $t$ , who is of type  $Y$  to  $s'$  under  $\mu'$  would mean that  $|\mu'(s') \cap Y| > |\mu(s') \cap Y| = |\mu_0(s') \cap Y|$ , contradicting the feasibility of  $\mu'$ .

All the cases where  $s' \in S_+$  could be proved using the symmetric of our previous arguments. Having exhausted all the cases, we have proved the lemma.  $\square$

*Proof of Proposition 2.7.6. First Part.* Let  $\mu$  be a matching obtained at the end of TE. By construction of TE, it is easy to show that  $\mu$  is IR. Assume that  $\mu$  is not maximal. There exists a feasible matching  $\mu'$  that is feasible and Pareto-dominates  $\mu$ . Let  $\tilde{\mu}$  be the school-copy matching that corresponds to  $\mu$  obtained at the last step of TE when no cycle was found and let  $G := (N, E)$  be the graph of TE at this last step. The set  $\tilde{T}$  is defined as before. Start with a node  $n_1 := (t_1, \tau_1)$  in  $G$  s.t  $t_1$  is an arbitrary teacher in  $\tilde{T}$ . By Lemma 23, there exists two nodes of  $G$ ,  $n'_1 := (s_1, \tau_1)$  and another one  $n_2 := (t_2, \tau_2)$  s.t  $t_2 \in \tilde{T}$ ,  $t_2 \neq t_1$ ,  $n_1$  points to  $n'_1$  and  $n'_1$  points to  $n_2$  in  $G$ . Using again Lemma 23 with the node  $n_2$ , one can find two nodes of  $G$ ,  $n'_2 := (s_2, \tau_2)$  and  $n_3 := (t_3, \tau_3)$  s.t  $t_3 \neq t_2$ ,  $t_3 \in \tilde{T}$ ,  $n_2$  points to  $n'_2$  and  $n'_2$  points to  $n_3$  in  $G$ . Since the graph  $G$  is finite, this process has to cycle. But this contradicts that  $G$  was the graph of the last step of TE, since there should be no cycle left in the graph  $G$  of this step.

*Second part.* Fix a matching  $\mu'$  that is IR and maximal. Let  $\tilde{T} = \{t : \mu_0(t) \neq \mu'(t)\}$ . Let  $\tilde{\mu}$  be the school-copy matching equivalent of  $\mu_0$  that is defined in the one-to-one transformation preliminary phase of TE. Let  $\mu_1 := \tilde{\mu}$  and  $T_1 := \tilde{T}$  and  $G_1 := (N_1, E_1)$  be the graph of the first step of TE that starts at matching  $\mu_1$ . By using Lemma 23 and the same argument as in the previous part, one can find a cycle  $C_1$  in  $G_1$  s.t:



- All the nodes  $(t, \tau)$  in cycle  $C_1$  are s.t  $t \in T_1$ .
- If the edge  $[(t, \tau(t)), (\tilde{s}, \tilde{\tau})]$  is in cycle  $C_1$ , then  $s(\tilde{s}) = \mu'(t)$ .

So cycle  $C_1$  matches some teachers of  $\tilde{T}$  to the school they obtain under  $\mu'$ . All the teachers who are not part of cycle  $C_1$  stay at their initial school under  $\mu_0$ . So once implemented, cycle  $C_1$  leads to a matching  $\mu_2$ . If  $\mu_2 \neq \mu'$ , then  $\mu_2$  is feasible since it has been obtained by a cycle of TE. In addition,  $\mu'$  still Pareto-dominates  $\mu_2$ . Let  $T_2$  be the set  $T_1$  where one has removed all the teachers who have been matched with cycle  $C_1$ . It corresponds to the set  $T_2 = \{t \in T : \mu_2(t) = \mu_0(t) \neq \mu'(t)\}$ . Let  $G_2 := (N_2, E_2)$  be the graph of the next step of TE once one has implemented cycle  $C_1$ . In using again Lemma 23 and the same argument as in the previous part, one can find a cycle  $C_2$  in  $G_2$  s.t:

- All the nodes  $(t, \tau)$  in cycle  $C_2$  are s.t  $t \in T_2$ .
- If the edge  $[(t, \tau(t)), (\tilde{s}, \tilde{\tau})]$  is in cycle  $C_2$ , then  $s(\tilde{s}) = \mu'(t)$ .

Once  $C_2$  is implemented, it matches teachers of  $T_2$  to their schools under  $\mu'$  and leads to a matching  $\mu_3$  that is feasible and all teachers in  $T_2$  who are not part of cycle  $C_2$  stay to their initial school under  $\mu_0$ . If  $\mu_3 \neq \mu'$  then one can iterate the same above argument. Note that each time one implements a cycle in this procedure, it matches teachers of  $\tilde{T}$  to their schools under  $\mu'$  and leaves the other ones matched to their school under  $\mu_0$ . So by finiteness, this process will lead to the matching  $\mu'$ . Since  $\mu'$  is maximal, we know by the first step of the procedure that there will be no more cycle in the graph of TE so that the algorithm will indeed return  $\mu'$ .  $\square$

The TE algorithm completely characterizes the set of feasible, IR and maximal mechanisms. Following [Combe, Tercieux and Terrier \(2016b\)](#) approach, some selection of cycles can obviously lead to non strategy-proof mechanisms. So a next natural step would be to exhibit a selection of cycles leading to a strategy-proof mechanism. To start with, we considered a very simple setting with only two possible types. In this setting, decreasing the number of young teachers necessarily increases the number of old ones. In a setting with three types, say young, middle-aged and old teachers an increase/decrease targets for each type in each school, one has to redefine the TE algorithm in a non trivial way. The case of multidimensional types is also challenging but these offer interesting lines for future research.

## 2.8 Conclusion

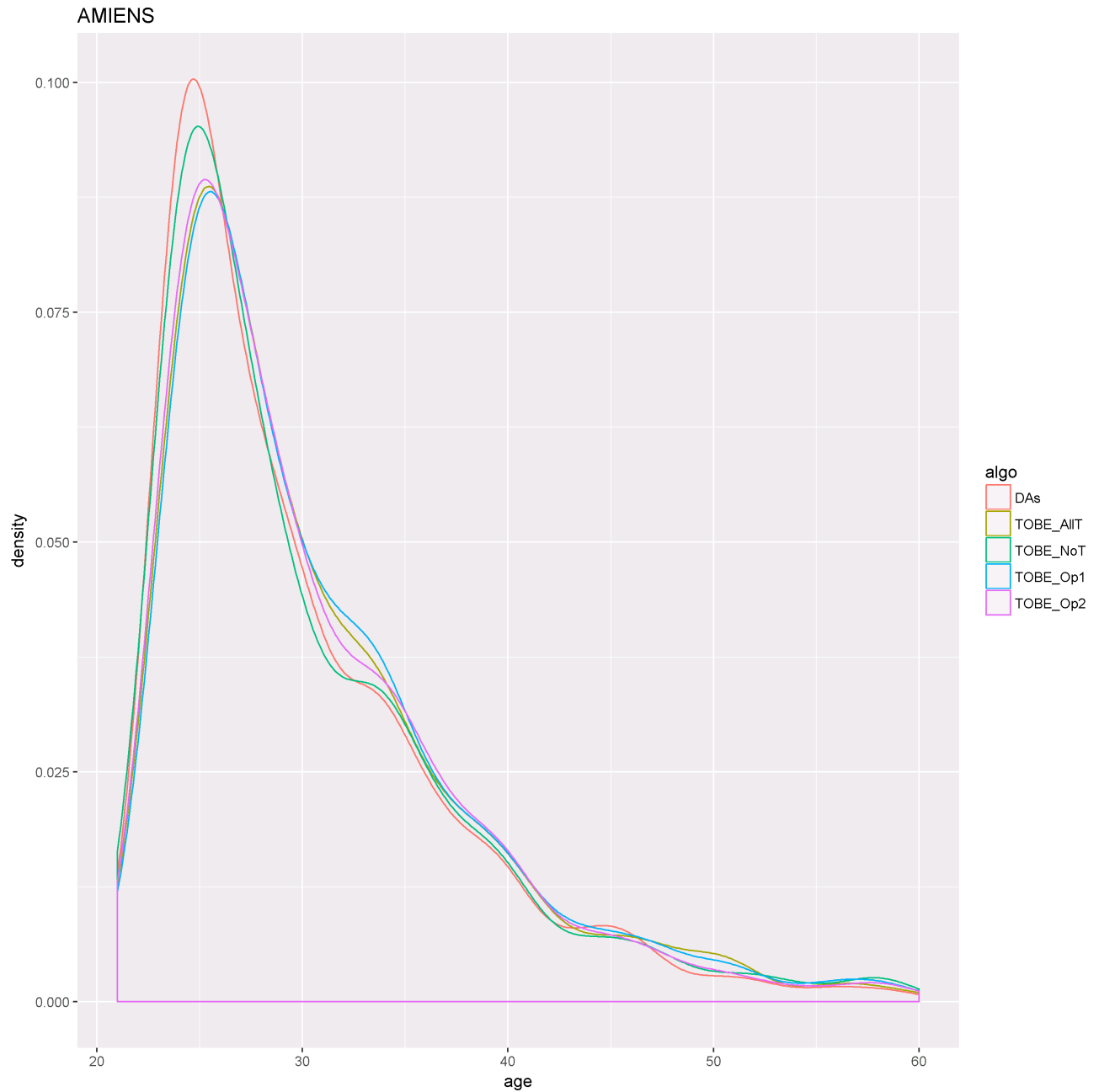
Taken together, our results show that a change in the centralized process used to assign teachers to regions in France could result in a significant improvement in the number of teachers obtaining a new region, without hurting the mobility in deprived regions. It is important to keep in mind that the results we present reflect some choices we have made for the simulations, in particular our choice to select the three regions of Créteil, Versailles and Amiens as deprived over the order of teachers chosen for TOBE. Other regions could obviously have been chosen in the option 1 we present and different proportions of teachers could have been selected in our option 2. Depending on the objectives and priorities of the policy makers, the high flexibility of our algorithm allows a fine tuning of teacher mobility in different regions. Hence, our algorithm could be used as a management tool which enables policy makers to simulate the impact of different assignment options and to test different human resources strategies in regions. It is also important to mention that the assignment of teachers is a dynamic problem. Hence the simulations made only highlight a gain a movement for a given period if one is switching from the current algorithm to TOBE. However, they do not reflect any stationary state for TOBE: a part of the gain of movement is due to the congestion that DA\* has created over the years due to its inefficiency. For instance, all the additional teachers obtaining a new assignment under TOBE are likely not to apply for a reassignment the next years. This would tend to decrease the number of teachers applying for a reassignment and decrease the movement of TOBE. An interesting exercise for future research would be to simulate a complete dynamic assignment for TOBE.

From a broader perspective, this paper highlights how centralized assignment processes can significantly impact education systems in two dimensions. Firstly, it can be used to improve teacher geographic mobility prospects. As such, it is a way to make the profession more attractive. Another lesson is that centralized assignment processes can be used to influence the distribution of teachers across territories. It makes it possible to increase teachers' experience in the most deprived regions, and to reduce it in others, notably in the most attractive ones. This could be a tool to reduce the persistent achievement gaps observed in most education systems.

## 2.9 Appendix

### 2.9.1 Additional results for Section 2.6

Figure 2.5 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF AMIENS



Note: DAs is DA\*, TOBE\_AllT is TOBE with all regions targeted, TOBE\_NoT is TOBE with no region targeted, TOBE\_Op1 is TOBE with the deprived regions of Créteil, Versailles and Amiens targeted and TOBE\_Op2 is the TOBE with a proportion of the teachers targeted in the deprived regions.

Table 2.11 – NUMBER OF TEACHERS MOVING UNDER EACH ALGORITHM - PER REGION OF INITIAL ASSIGNMENT

	Current algorithm DA*	TOBE with all regions targeted		TOBE with no regions targeted		TOBE with deprived regions targeted (option 1)		TOBE with a proportion of teachers targeted in deprived regions (option 2)	
	nb	nb	%	nb	%	nb	%	nb	%
All regions	3991	3892	-2.48%	6265	56.98%	4851	21.55%	5797	45.25%
Unassigned	7976	7968	-0.10%	7933	-0.54%	7919	-0.71%	7931	-0.56%
Paris	100	150	50.00%	222	122.00%	222	122.00%	222	122.00%
Aix	106	151	42.45%	168	58.49%	168	58.49%	168	58.49%
Besancon	37	50	35.14%	64	72.97%	65	75.68%	65	75.68%
Bordeaux	47	62	31.91%	69	46.81%	69	46.81%	69	46.81%
Caen	70	87	24.29%	116	65.71%	115	64.29%	116	65.71%
Clermont	34	46	35.29%	53	55.88%	53	55.88%	53	55.88%
Dijon	105	143	36.19%	223	112.38%	223	112.38%	223	112.38%
Grenoble	111	168	51.35%	206	85.59%	206	85.59%	206	85.59%
Lille	131	170	29.77%	283	116.03%	285	117.56%	285	117.56%
Lyon	92	139	51.09%	150	63.04%	150	63.04%	150	63.04%
Montpellier	69	115	66.67%	121	75.36%	121	75.36%	121	75.36%
Nancy	67	103	53.73%	119	77.61%	119	77.61%	119	77.61%
Poitiers	53	79	49.06%	93	75.47%	92	73.58%	93	75.47%
Rennes	30	41	36.67%	42	40.00%	42	40.00%	42	40.00%
Strasbourg	53	82	54.72%	95	79.25%	95	79.25%	95	79.25%
Toulouse	73	119	63.01%	128	75.34%	128	75.34%	128	75.34%
Nantes	85	124	45.88%	141	65.88%	141	65.88%	141	65.88%
Orleans	208	202	-2.88%	402	93.27%	402	93.27%	402	93.27%
Reims	98	91	-7.14%	180	83.67%	180	83.67%	180	83.67%
Amiens	207	171	-17.39%	332	60.39%	146	-29.47%	213	2.90%
Rouen	93	122	31.18%	191	105.38%	192	106.45%	191	105.38%
Limoges	39	40	2.56%	66	69.23%	66	69.23%	66	69.23%
Nice	81	138	70.37%	160	97.53%	161	98.77%	160	97.53%
Créteil	813	316	-61.13%	996	22.51%	309	-61.99%	834	2.58%
Versailles	779	519	-33.38%	1006	29.14%	463	-40.56%	816	4.75%
Corse	5	10	100.00%	10	100.00%	10	100.00%	10	100.00%
Reunion	63	89	41.27%	94	49.21%	94	49.21%	94	49.21%
Martinique	33	52	57.58%	57	72.73%	57	72.73%	57	72.73%
Guadeloupe	64	82	28.13%	102	59.38%	102	59.38%	102	59.38%
Guyane	89	85	-4.49%	131	47.19%	131	47.19%	131	47.19%
Mayotte	156	146	-6.41%	245	57.05%	244	56.41%	245	57.05%

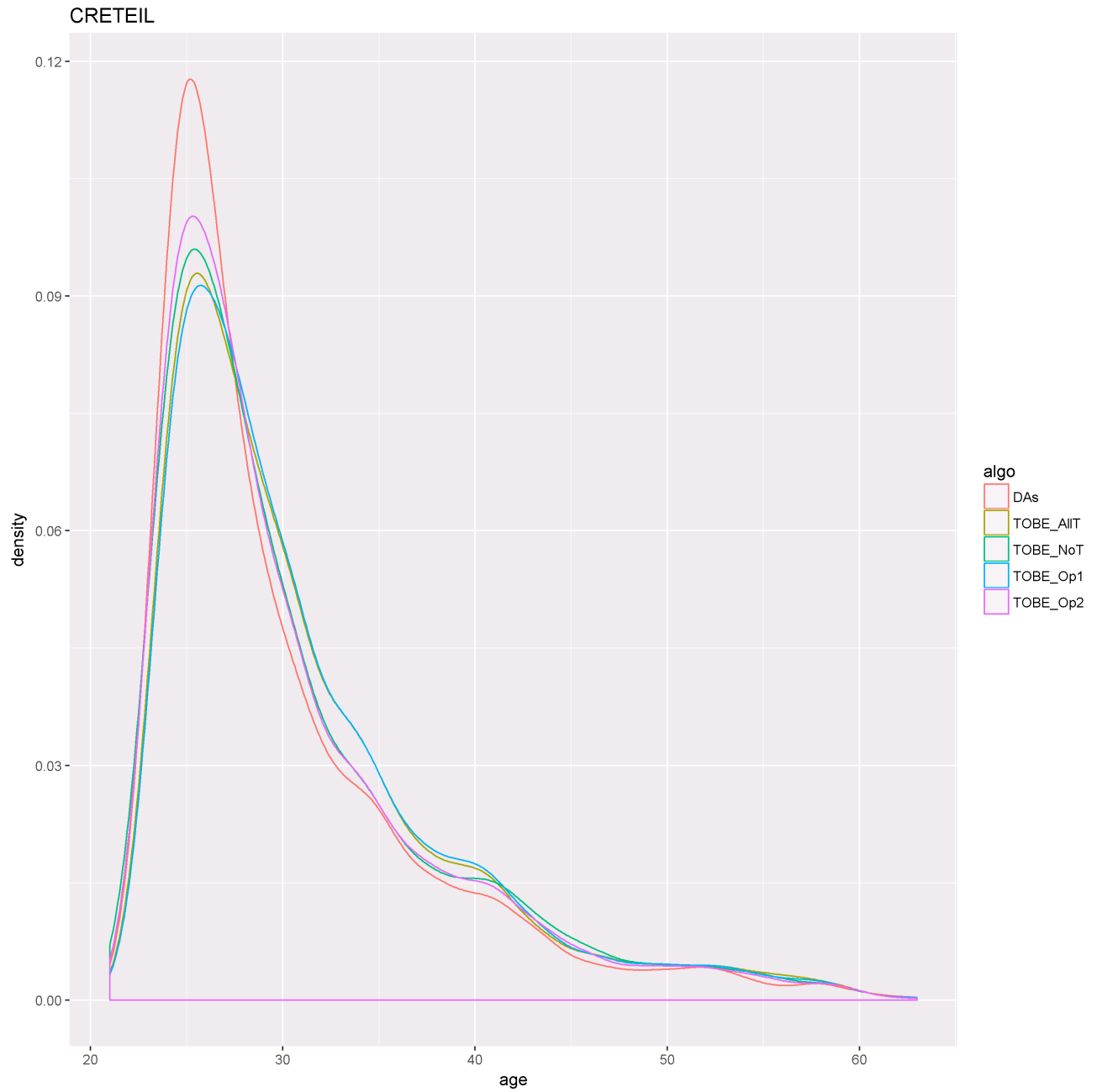
Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in columns 4 and 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A). The second column for each TOBE algorithm gives the percentage of improvement with respect to DA\*.

Table 2.12 – AVERAGE AGE OF ASSIGNED TEACHERS UNDER EACH ALGORITHM - PER REGION

	Current algorithm DA*	TOBE with all regions targeted	TOBE with no regions targeted	TOBE with deprived regions targeted (option 1)	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Unassigned	29.67	28.32	28.34	28.25	28.34
Paris	37.34	36.5	36.11	35.56	36.17
Aix	35.3	34.37	35	34.66	35.12
Besancon	32.95	32.49	33.67	33.33	33.59
Bordeaux	35.64	36.63	36.68	36.96	36.82
Caen	33.79	33.9	33.93	33.73	33.81
Clermont	34.33	34.11	34.48	34.47	34.42
Dijon	32.02	30.45	31.63	30.66	31.21
Grenoble	33.61	33.11	33.98	33.67	34.07
Lille	30.65	30.43	29.44	29.14	29.37
Lyon	33.14	32.71	32.37	33.31	32.57
Montpellier	36.83	36.84	36.99	37.39	37.36
Nancy	33.37	32.49	32.23	31.89	32.47
Poitiers	35.47	34.82	35.18	35.17	35.45
Rennes	35.98	36.23	36.21	36.49	36.26
Strasbourg	33.34	32.96	32.19	32.97	32.62
Toulouse	35.62	35.77	35.42	35.72	35.85
Nantes	34.76	34.42	34.48	35.27	35.04
Orleans	32.47	32.13	32.95	31.59	32.49
Reims	31.24	31.53	31	30.25	30.68
Amiens	29.87	30.57	30.12	30.73	30.41
Rouen	32.74	32.21	31.43	30.6	31.17
Limoges	34.84	36.46	35.87	36.88	36.6
Nice	35.84	35.15	35.79	35.89	35.93
Créteil	30.06	31.06	30.69	31.13	30.53
Versailles	31.08	31.26	31.63	31.52	31.54
Corse	40.53	36.97	38.47	38.9	38.73
Reunion	37.4	37.97	43.1	43.02	43.1
Martinique	39.72	39.42	38.73	38.73	38.64
Guadeloupe	39.23	39	39.56	39.58	39.35
Guyane	41.81	42.23	41.84	42.06	41.97
Mayotte	48.18	47.81	44.45	44.88	44.57

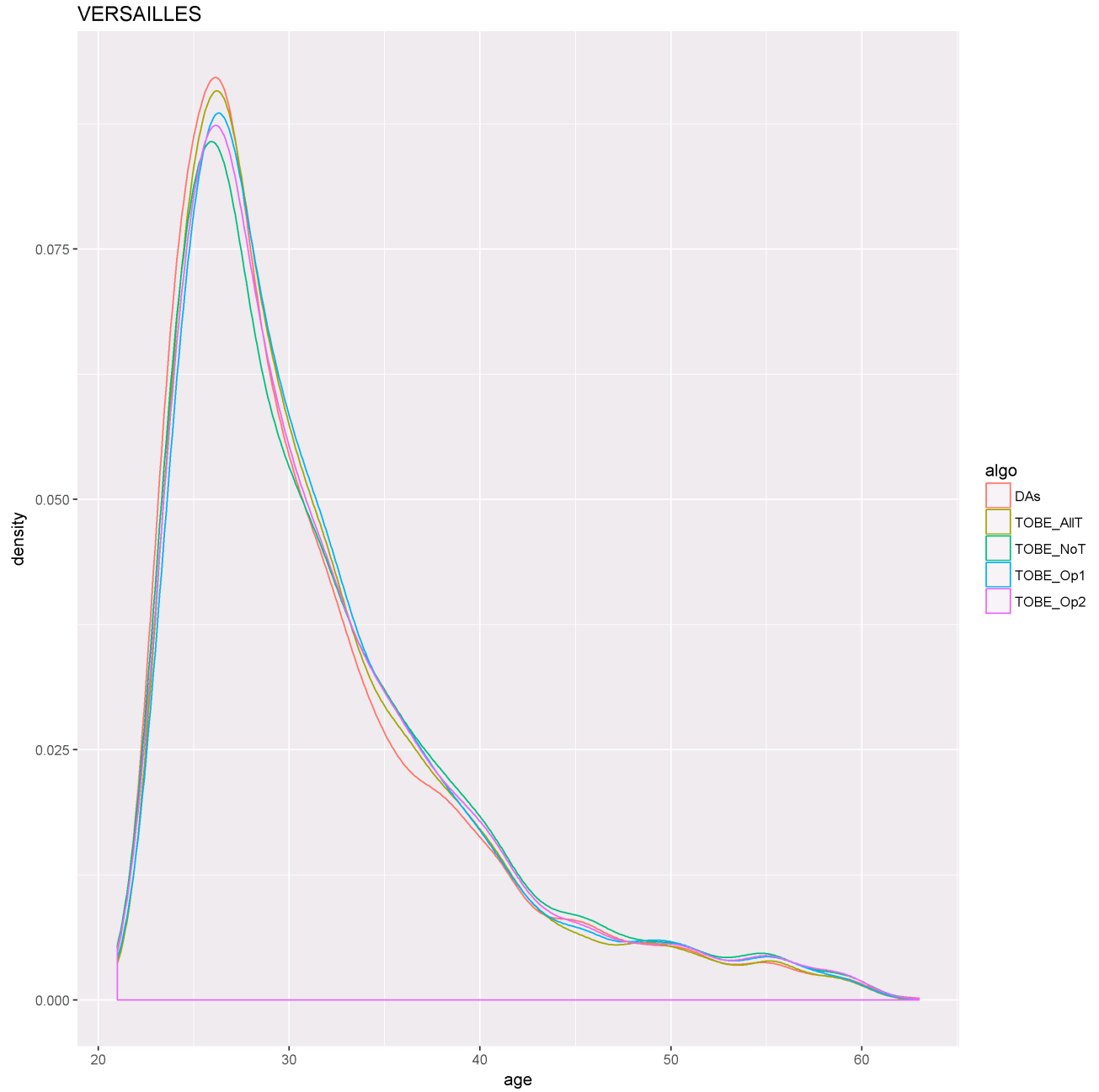
Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in columns 4 and 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A). The second column for each TOBE algorithm gives the percentage of improvement with respect to DA\*.

Figure 2.6 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF CRÉTÉIL



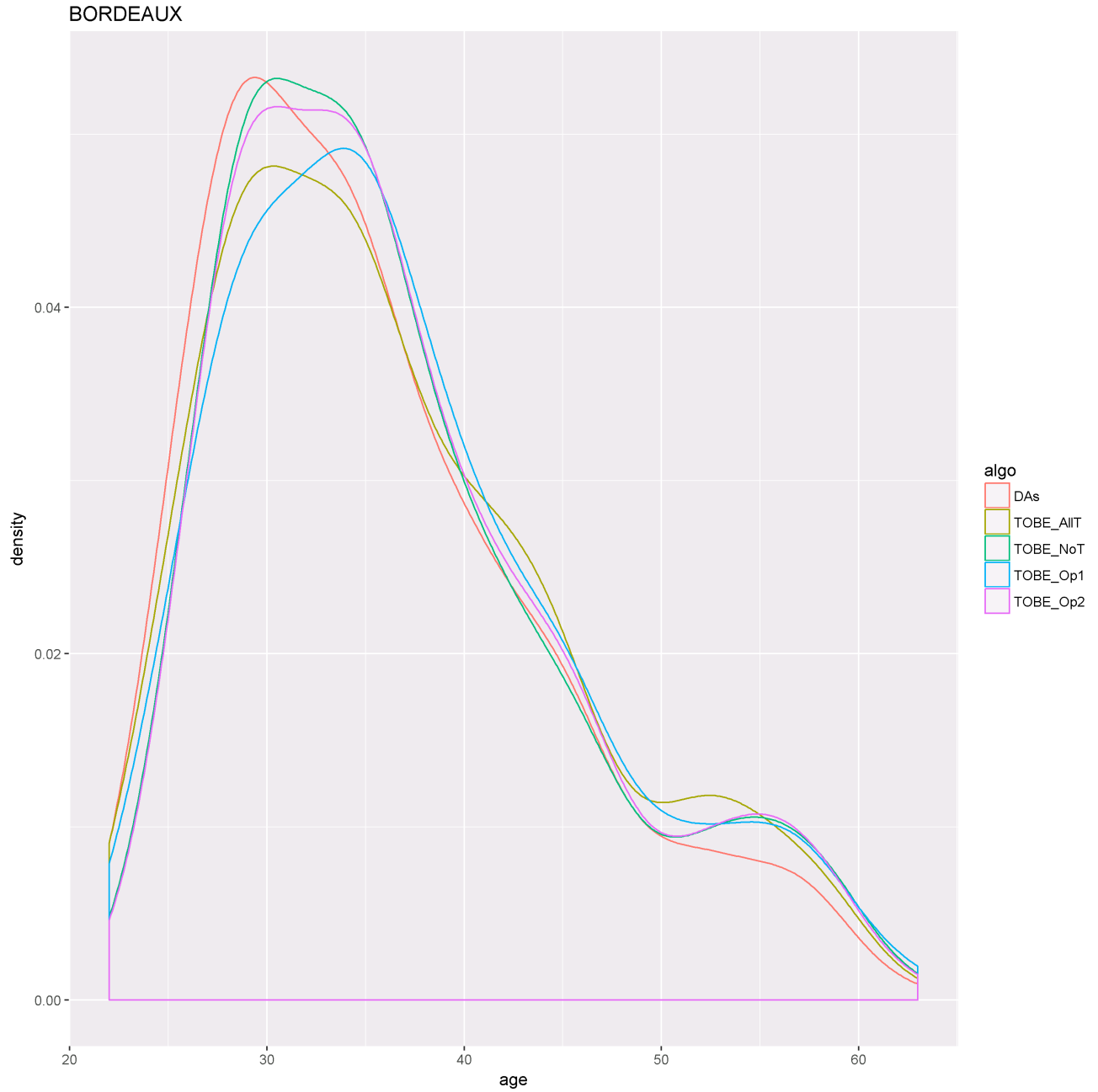
Note: DAs is DA\*, TOBE\_AllT is TOBE with all regions targeted, TOBE\_NoT is TOBE with no region targeted, TOBE\_Op1 is TOBE with the deprived regions of Créteil, Versailles and Amiens targeted and TOBE\_Op2 is the TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.7 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF VERSAILLES



Note: DAs is DA\*, TOBE\_AllT is TOBE with all regions targeted, TOBE\_NoT is TOBE with no region targeted, TOBE\_Op1 is TOBE with the deprived regions of Créteil, Versailles and Amiens targeted and TOBE\_Op2 is the TOBE with a proportion of the teachers targeted in the deprived regions.

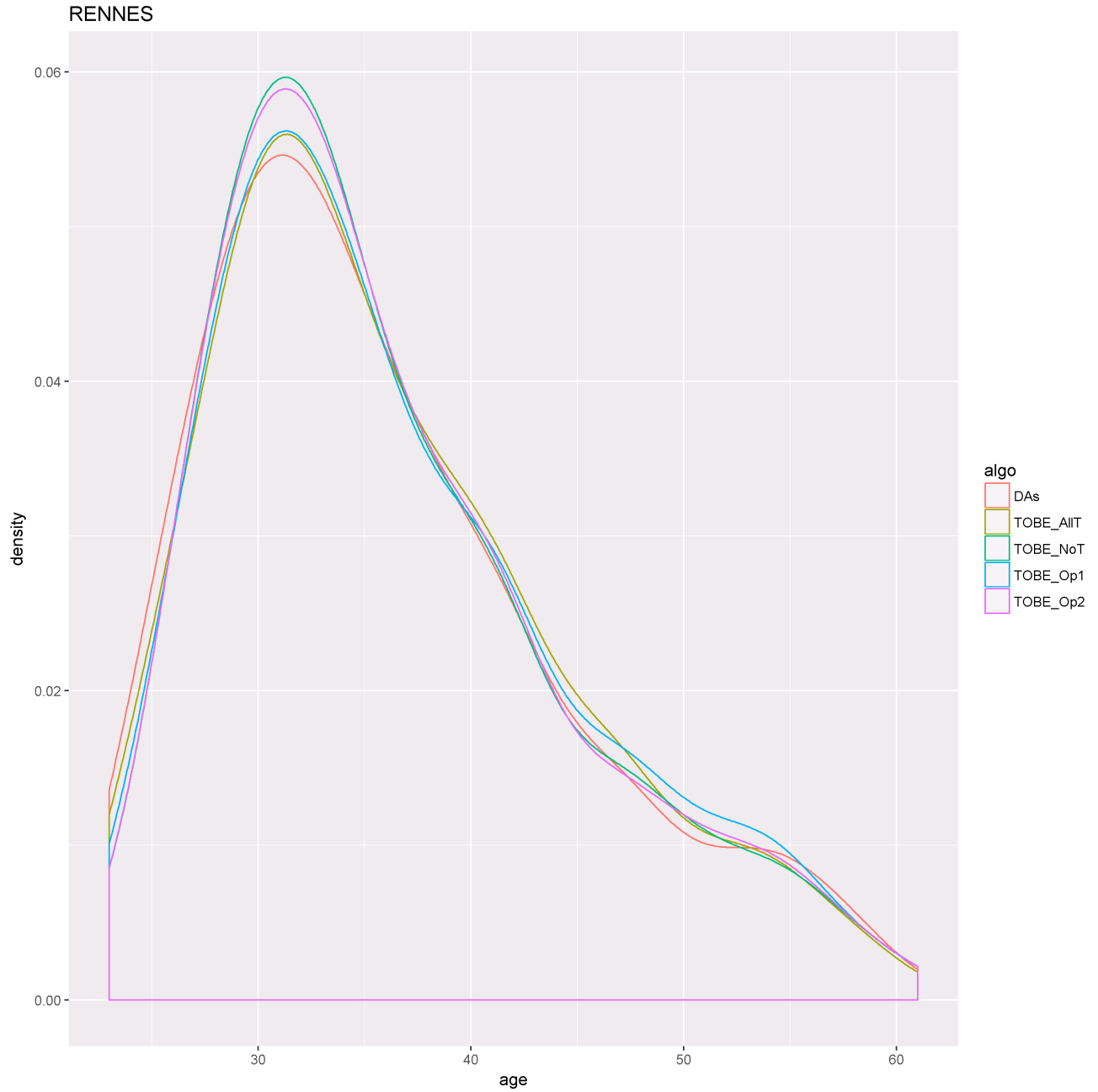
Figure 2.8 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF BORDEAUX



Note: DAs is DA\*, TOBE\_AllT is TOBE with all regions targeted, TOBE\_NoT is TOBE with no region targeted, TOBE\_Op1 is TOBE with the deprived regions of Créteil, Versailles and Amiens targeted and TOBE\_Op2 is the TOBE with a proportion of the teachers targeted in the deprived regions.

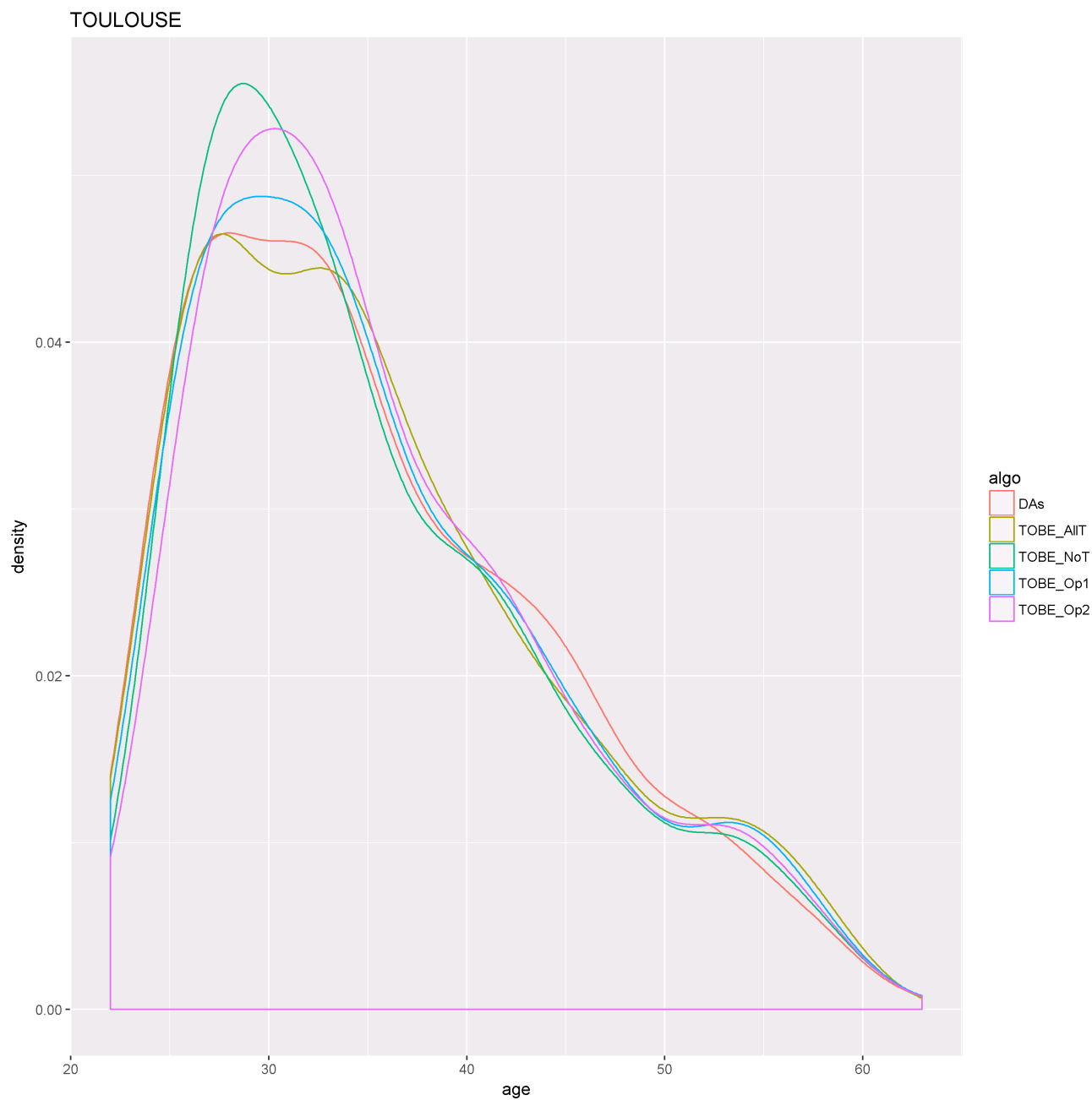


Figure 2.9 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF RENNES



Note: DAs is DA\*, TOBE\_AllT is TOBE with all regions targeted, TOBE\_NoT is TOBE with no region targeted, TOBE\_Op1 is TOBE with the deprived regions of Créteil, Versailles and Amiens targeted and TOBE\_Op2 is the TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.10 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF TOULOUSE



Note: DAs is DA\*, TOBE\_AllT is TOBE with all regions targeted, TOBE\_NoT is TOBE with no region targeted, TOBE\_Op1 is TOBE with the deprived regions of Créteil, Versailles and Amiens targeted and TOBE\_Op2 is the TOBE with a proportion of the teachers targeted in the deprived regions.

## 2.9.2 Additional results with improvement of non initially assigned teachers in the ordering of TOBE

In all the results, contrary to the tables in Section 2.6, we omit TOBE with deprived regions targeted (option 1) because only the last version with a proportion of teachers targeted (option 2) seemed to be relevant for policy makers. For the latter, we kept the same proportion of targeted teachers in Créteil, Versailles and Amiens that are respectively 45%, 65% and 40%.

In the following results, remember that the ordering used in TOBE is the following:

- $x$  % of the non initially assigned teachers with the highest maximum score, ordered according to their maximum score.
- All the teachers initially assigned to non deprived regions, ordered according to their maximum score.
- All the teachers initially assigned to a deprived region (Créteil, Versailles or Amiens), ordered according to their maximum score.
- All the  $(1 - x)$ % of the non initially assigned teachers left, ordered according to their maximum score.

In the next three sections, we present the results for  $x = 20\%$ ,  $x = 40\%$  and  $x = 100\%$ .

### 2.9.2.1 Results with 20% of initially non-assigned teachers improved in the ordering of TOBE

Table 2.13 – NUMBER OF TEACHERS OBTAINING A NEW ASSIGNMENT UNDER EACH ALGORITHM

	Nb	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Newly Tenured	7106	6486	0.09%	-0.12%	-0.12%
Tenured	12123	5494	-2.24%	25.50%	19.62%
Total	19229	11980	-0.98%	11.63%	8.93%

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.14 – NUMBER OF TEACHERS MOVING UNDER EACH ALGORITHM - PER REGION OF INITIAL ASSIGNMENT

	Current algorithm DA*	TOBE with all regions targeted		TOBE with no regions targeted		TOBE with a proportion of teachers targeted in deprived regions (option 2)	
	nb	nb	%	nb	%	nb	%
All regions	3991	3895	-2.41%	5425	35.93%	5102	27.84%
Unassigned	7976	7968	-0.10%	7948	-0.35%	7948	-0.35%
Paris	100	150	50.00%	191	91.00%	191	91.00%
Aix	106	151	42.45%	157	48.11%	157	48.11%
Besancon	37	51	37.84%	65	75.68%	65	75.68%
Bordeaux	47	62	31.91%	66	40.43%	66	40.43%
Caen	70	87	24.29%	98	40.00%	97	38.57%
Clermont	34	46	35.29%	51	50.00%	51	50.00%
Dijon	105	143	36.19%	201	91.43%	201	91.43%
Grenoble	111	168	51.35%	183	64.86%	183	64.86%
Lille	131	172	31.30%	234	78.63%	234	78.63%
Lyon	92	139	51.09%	141	53.26%	141	53.26%
Montpellier	69	115	66.67%	119	72.46%	119	72.46%
Nancy	67	103	53.73%	112	67.16%	112	67.16%
Poitiers	53	79	49.06%	86	62.26%	86	62.26%
Rennes	30	41	36.67%	42	40.00%	42	40.00%
Strasbourg	53	82	54.72%	87	64.15%	87	64.15%
Toulouse	73	119	63.01%	123	68.49%	123	68.49%
Nantes	85	124	45.88%	131	54.12%	132	55.29%
Orleans	208	202	-2.88%	328	57.69%	329	58.17%
Reims	98	91	-7.14%	154	57.14%	154	57.14%
Amiens	207	171	-17.39%	289	39.61%	195	-5.80%
Rouen	93	122	31.18%	164	76.34%	164	76.34%
Limoges	39	40	2.56%	61	56.41%	61	56.41%
Nice	81	138	70.37%	150	85.19%	150	85.19%
Créteil	813	316	-61.13%	790	-2.83%	677	-16.73%
Versailles	779	519	-33.38%	792	1.67%	675	-13.35%
Corse	5	10	100.00%	11	120.00%	11	120.00%
Reunion	63	89	41.27%	95	50.79%	95	50.79%
Martinique	33	52	57.58%	54	63.64%	54	63.64%
Guadeloupe	64	82	28.13%	99	54.69%	99	54.69%
Guyane	89	85	-4.49%	120	34.83%	120	34.83%
Mayotte	156	146	-6.41%	231	48.08%	231	48.08%

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in columns 4 and 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A). The second column for each TOBE algorithm gives the percentage of improvement with respect to DA\*.

Table 2.15 – CUMULATIVE DISTRIBUTIONS OF RANKS FOR TENURED TEACHERS UNDER EACH ALGORITHM

	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
1	4219	4223	5378	5147
2	9861	10067	10377	10307
3	10745	10930	11044	11007
4	11234	11393	11414	11410
5	11482	11611	11594	11604
6	11645	11756	11696	11721
7	11764	11829	11781	11792
8	11883	11894	11879	11879
9	11961	11946	11930	11929
$\geq 10$	12066	12052	12047	12046
Unassigned	57	71	76	77

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.16 – CUMULATIVE DISTRIBUTIONS OF RANKS FOR NEWLY TENURED TEACHERS UNDER EACH ALGORITHM

	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
1	4044	3955	2929	3103
2	4804	4910	3868	4086
3	5095	5231	4315	4535
4	5285	5478	4691	4906
5	5429	5662	4976	5178
6	5560	5797	5215	5375
7	5662	5892	5369	5525
8	5803	6007	5602	5731
9	5916	6092	5757	5865
$\geq 10$	6491	6497	6482	6483
Unassigned	615	609	624	623

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.17 – NUMBER OF TEACHERS ASSIGNED TO AN EXTENDED WISH

	Nb	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Tenured	1560	639	554	753	716
Newly Tenured	7106	603	522	998	852

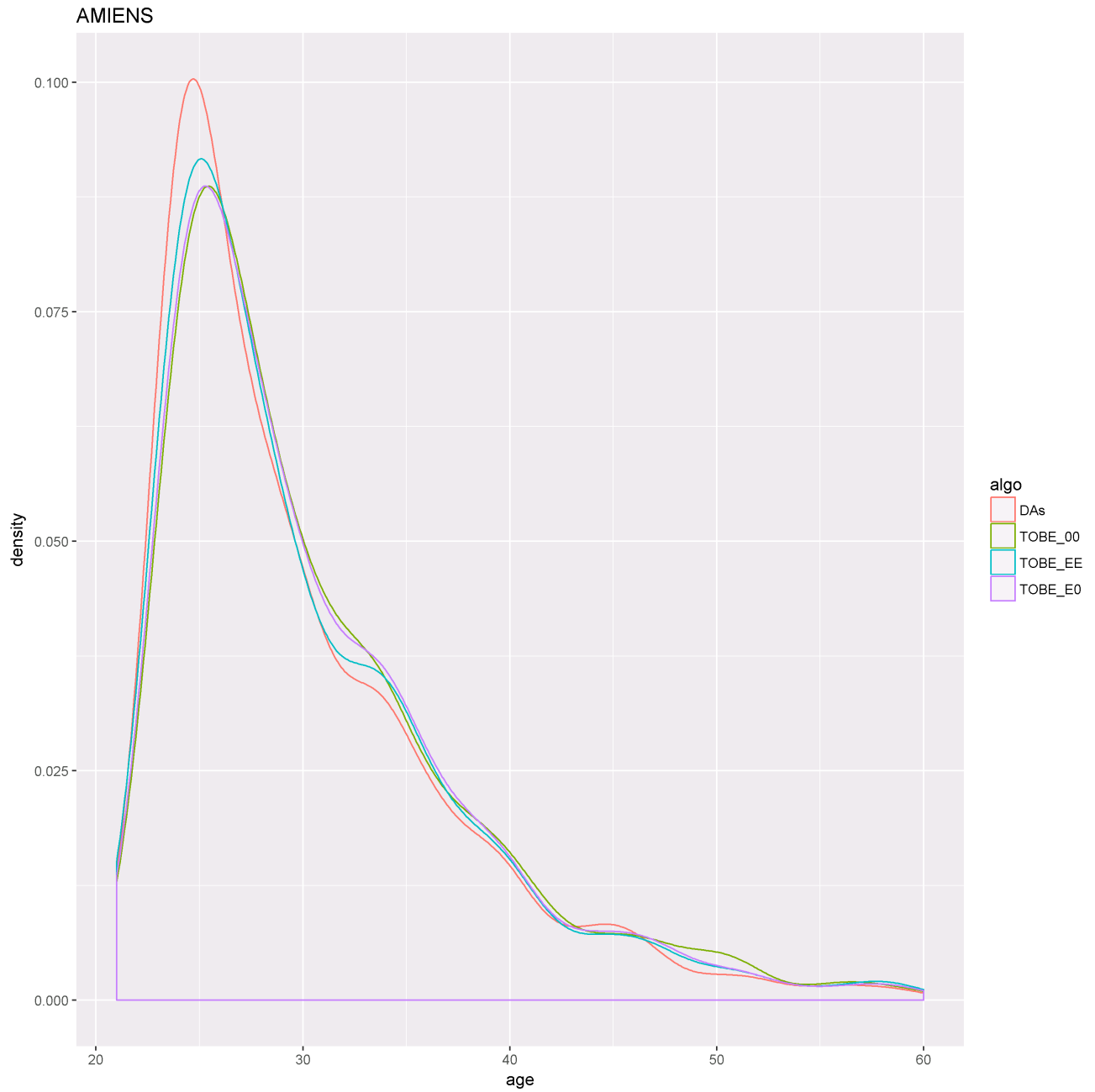
Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.18 – AVERAGE AGE OF ASSIGNED TEACHERS UNDER EACH ALGORITHM - PER REGION

	Current algorithm DA*	TOBE with all regions targeted	TOBE with no regions targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Unassigned	29.67	28.32	28.36	28.35
Paris	37.34	36.50	36.25	36.23
Aix	35.30	34.37	34.77	34.81
Besancon	32.95	32.49	33.48	33.37
Bordeaux	35.64	36.63	37.30	37.30
Caen	33.79	33.90	34.56	34.41
Clermont	34.33	34.11	34.89	34.72
Dijon	32.02	30.45	30.98	30.75
Grenoble	33.61	33.11	33.51	33.62
Lille	30.65	30.43	29.87	29.80
Lyon	33.14	32.71	33.05	33.11
Montpellier	36.83	36.84	37.56	37.80
Nancy	33.37	32.49	32.52	32.60
Poitiers	35.47	34.82	34.90	35.14
Rennes	35.98	36.23	36.23	36.46
Strasbourg	33.34	32.96	32.58	32.74
Toulouse	35.62	35.77	36.26	36.22
Nantes	34.76	34.42	35.14	35.25
Orleans	32.47	32.13	31.95	32.15
Reims	31.24	31.53	30.96	30.72
Amiens	29.87	30.57	30.21	30.40
Rouen	32.74	32.21	31.73	31.54
Limoges	34.84	36.46	35.94	36.33
Nice	35.84	35.14	36.27	36.34
Créteil	30.06	31.06	30.62	30.58
Versailles	31.08	31.26	31.46	31.37
Corse	40.53	36.97	37.10	37.13
Reunion	37.40	37.97	38.94	38.95
Martinique	39.72	39.42	39.12	39.12
Guadeloupe	39.23	39.00	39.17	38.97
Guyane	41.81	42.23	42.16	42.24
Mayotte	48.18	47.81	45.32	45.37

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 4 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

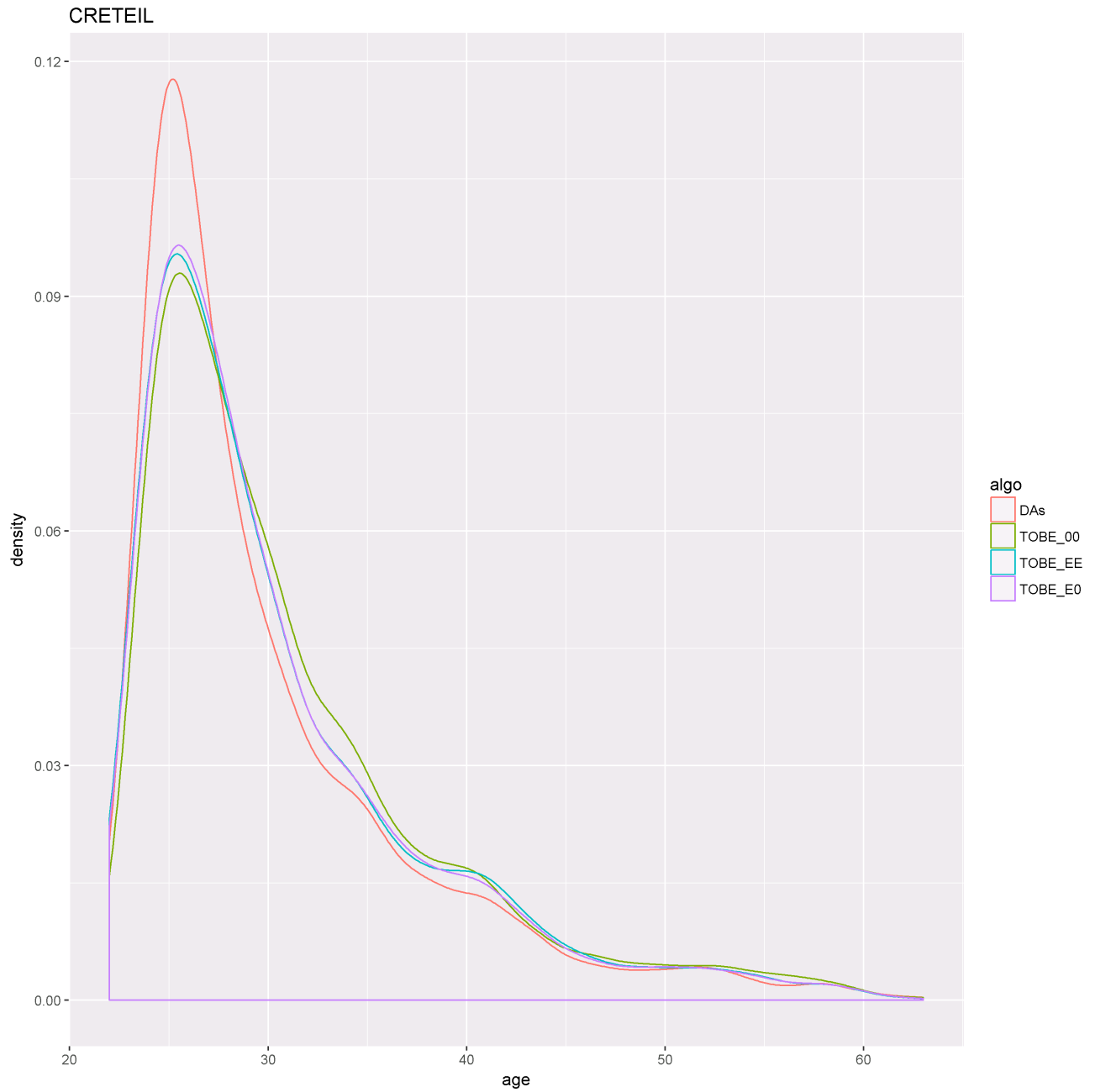
Figure 2.11 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF AMIENS



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

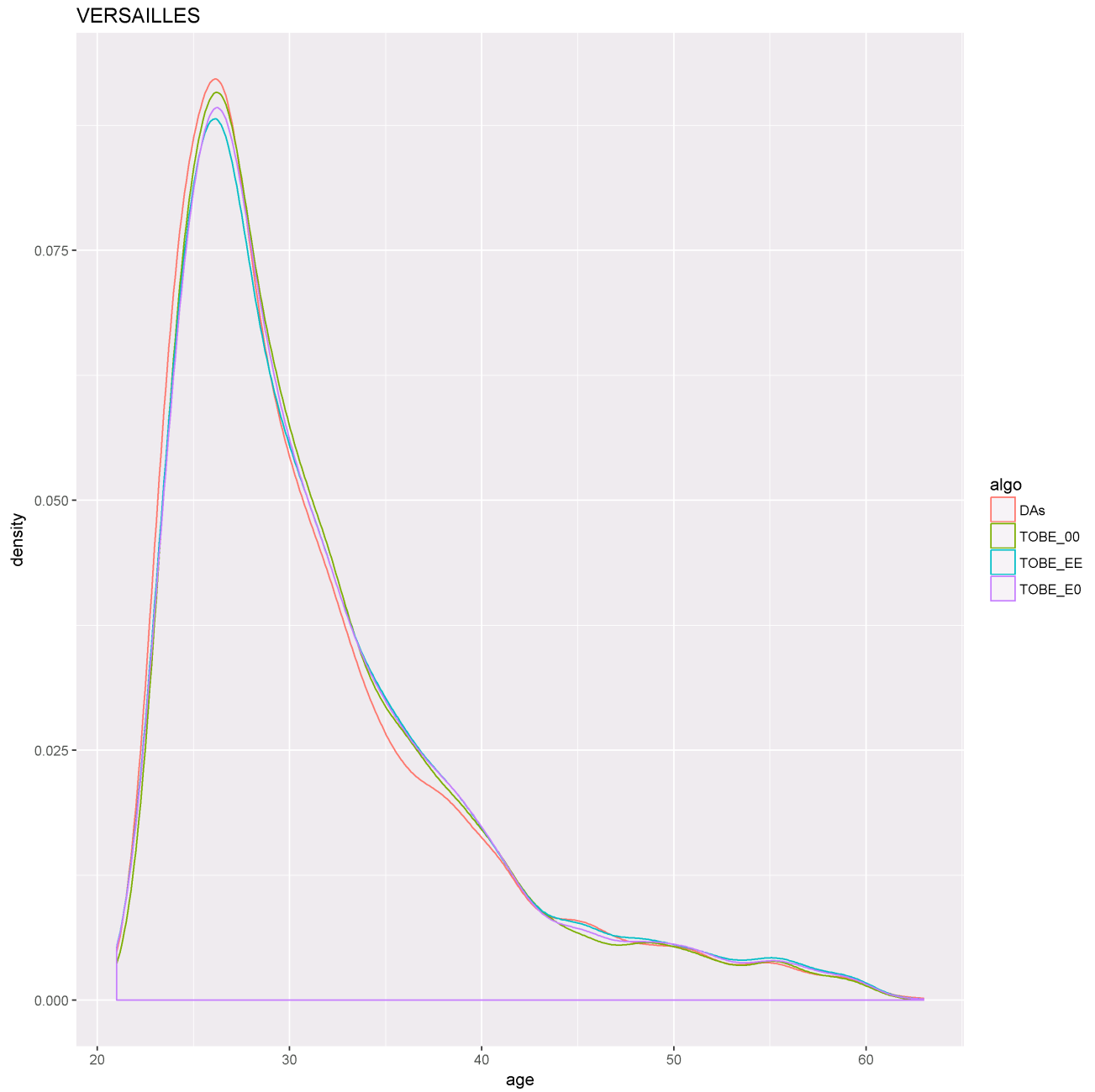


Figure 2.12 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF CRÉTÉIL



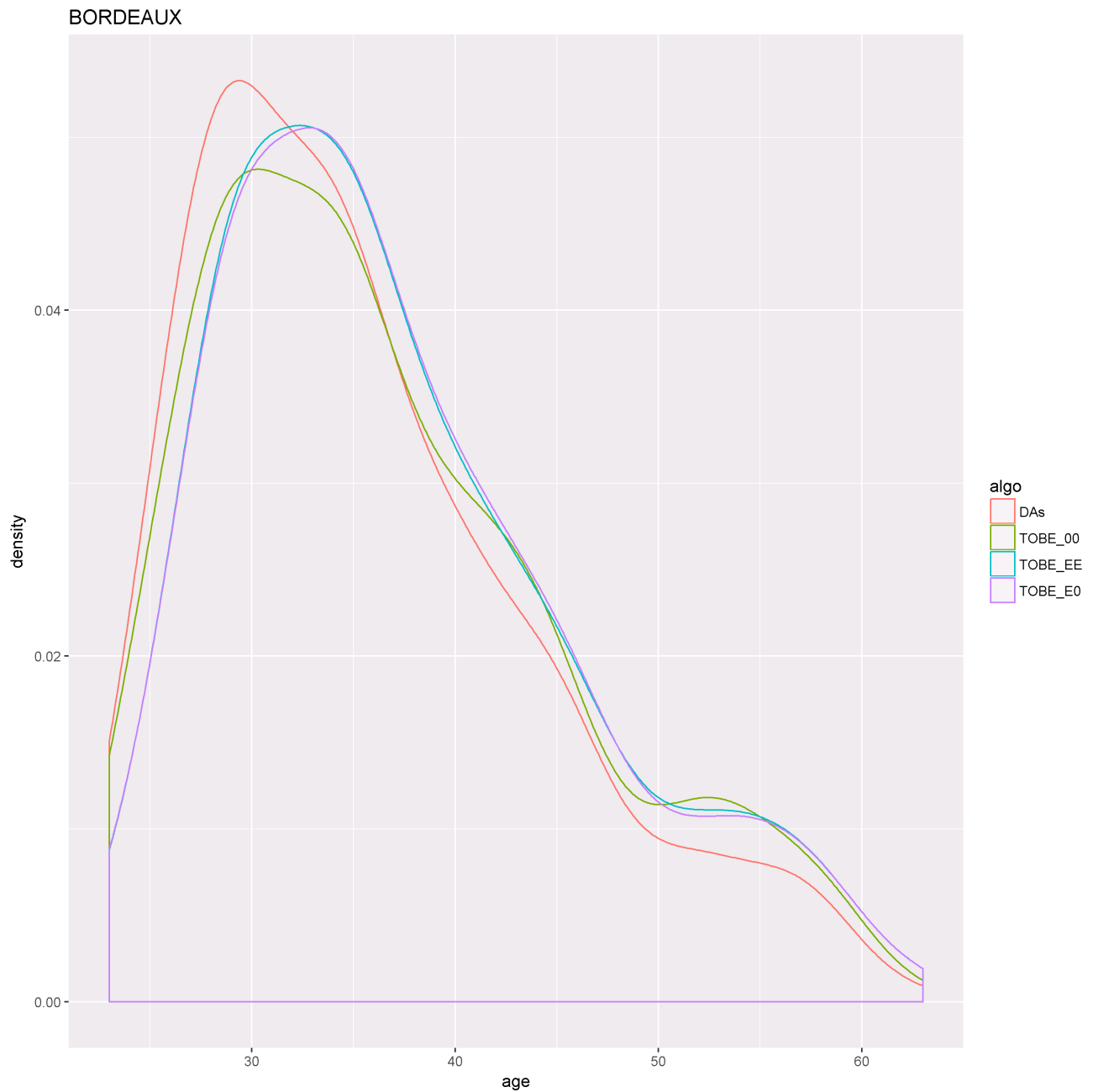
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.13 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF VERSAILLES



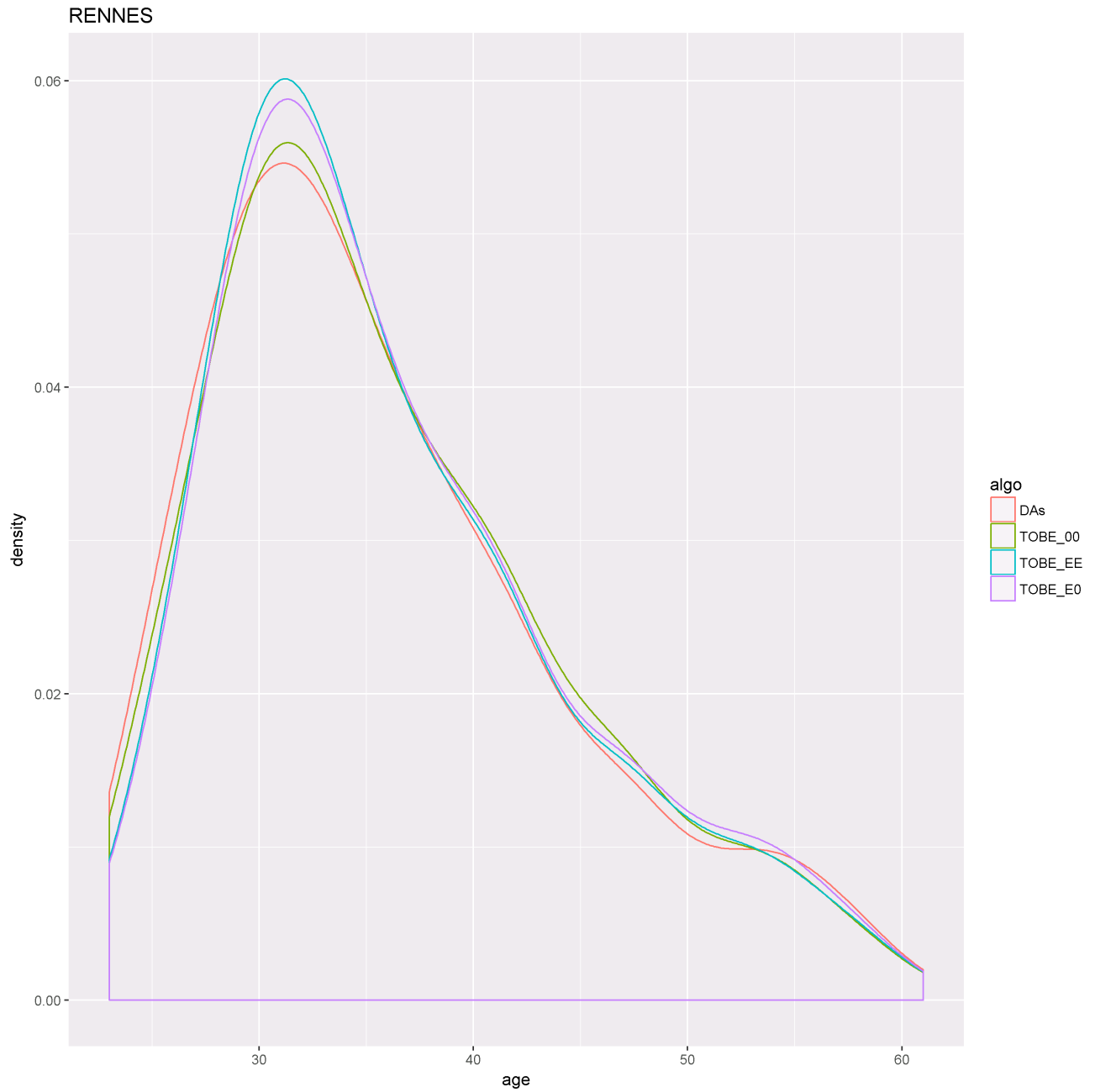
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.14 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF BORDEAUX



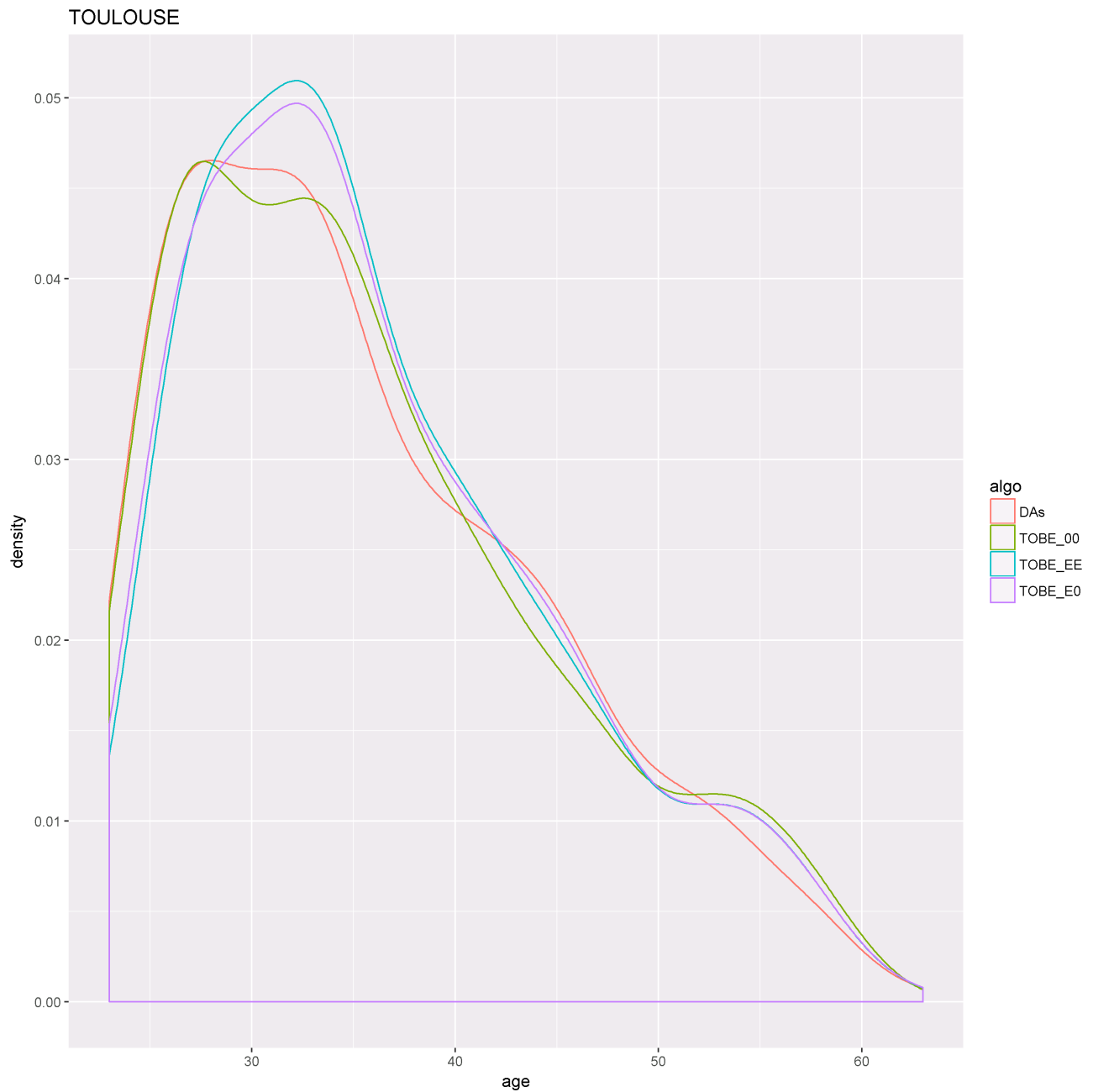
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.15 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF RENNES



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.16 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF TOULOUSE



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

### 2.9.2.2 Results with 40% of initially non-assigned teachers improved in the ordering of TOBE

Table 2.19 – NUMBER OF TEACHERS OBTAINING A NEW ASSIGNMENT UNDER EACH ALGORITHM

	Nb	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Newly Tenured	7106	6486	0.09	0.02	-0.05
Tenured	12123	5494	-2.24%	11.58%	8.52%
Total	19229	11980	-0.98%	5.32%	3.88%

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.20 – NUMBER OF TEACHERS MOVING UNDER EACH ALGORITHM - PER REGION OF INITIAL ASSIGNMENT

	Current algorithm DA*	TOBE with all regions targeted		TOBE with no regions targeted		TOBE with a proportion of teachers targeted in deprived regions (option 2)	
	nb	nb	%	nb	%	nb	%
All regions	3991	3895	-2.41%	4658	16.71%	4489	12.48%
Unassigned	7976	7968	-0.10%	7959	-0.21%	7956	-0.25%
Paris	100	150	50.00%	175	75.00%	175	75.00%
Aix	106	151	42.45%	157	48.11%	157	48.11%
Besancon	37	51	37.84%	62	67.57%	62	67.57%
Bordeaux	47	62	31.91%	67	42.55%	67	42.55%
Caen	70	87	24.29%	92	31.43%	91	30.00%
Clermont	34	46	35.29%	48	41.18%	48	41.18%
Dijon	105	143	36.19%	166	58.10%	166	58.10%
Grenoble	111	168	51.35%	165	48.65%	164	47.75%
Lille	131	172	31.30%	195	48.85%	194	48.09%
Lyon	92	139	51.09%	141	53.26%	141	53.26%
Montpellier	69	115	66.67%	119	72.46%	119	72.46%
Nancy	67	103	53.73%	105	56.72%	105	56.72%
Poitiers	53	79	49.06%	83	56.60%	83	56.60%
Rennes	30	41	36.67%	42	40.00%	42	40.00%
Strasbourg	53	82	54.72%	86	62.26%	86	62.26%
Toulouse	73	119	63.01%	123	68.49%	123	68.49%
Nantes	85	124	45.88%	130	52.94%	131	54.12%
Orleans	208	202	-2.88%	239	14.90%	239	14.90%
Reims	98	91	-7.14%	126	28.57%	126	28.57%
Amiens	207	171	-17.39%	233	12.56%	172	-16.91%
Rouen	93	122	31.18%	136	46.24%	136	46.24%
Limoges	39	40	2.56%	50	28.21%	50	28.21%
Nice	81	138	70.37%	145	79.01%	145	79.01%
Créteil	813	316	-61.13%	565	-30.50%	512	-37.02%
Versailles	779	519	-33.38%	621	-20.28%	568	-27.09%
Corse	5	10	100.00%	10	100.00%	10	100.00%
Reunion	63	89	41.27%	95	50.79%	95	50.79%
Martinique	33	52	57.58%	54	63.64%	54	63.64%
Guadeloupe	64	82	28.13%	93	45.31%	93	45.31%
Guyane	89	85	-4.49%	115	29.21%	115	29.21%
Mayotte	156	146	-6.41%	220	41.03%	220	41.03%

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in columns 4 and 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A). The second column for each TOBE algorithm gives the percentage of improvement with respect to DA\*.

Table 2.21 – CUMULATIVE DISTRIBUTIONS OF RANKS FOR TENURED TEACHERS UNDER EACH ALGORITHM

	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
1	4219	4223	4783	4665
2	9861	10067	10267	10214
3	10745	10930	11034	10998
4	11234	11393	11458	11434
5	11482	11611	11644	11632
6	11645	11756	11762	11759
7	11764	11829	11833	11829
8	11883	11894	11905	11907
9	11961	11946	11950	11951
$\geq 10$	12066	12052	12047	12048
Unassigned	57	71	76	75

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.22 – CUMULATIVE DISTRIBUTIONS OF RANKS FOR NEWLY TENURED TEACHERS UNDER EACH ALGORITHM

	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
1	4044	3955	3600	3691
2	4804	4910	4515	4624
3	5095	5231	4875	4979
4	5285	5478	5158	5252
5	5429	5662	5369	5464
6	5560	5797	5522	5610
7	5662	5892	5647	5723
8	5803	6007	5801	5863
9	5916	6092	5918	5974
$\geq 10$	6491	6497	6493	6489
Unassigned	615	609	613	617

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).



Table 2.23 – NUMBER OF TEACHERS ASSIGNED TO AN EXTENDED WISH

	Nb	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Tenured	1560	639	554	616	607
Newly Tenured	7106	603	522	683	621

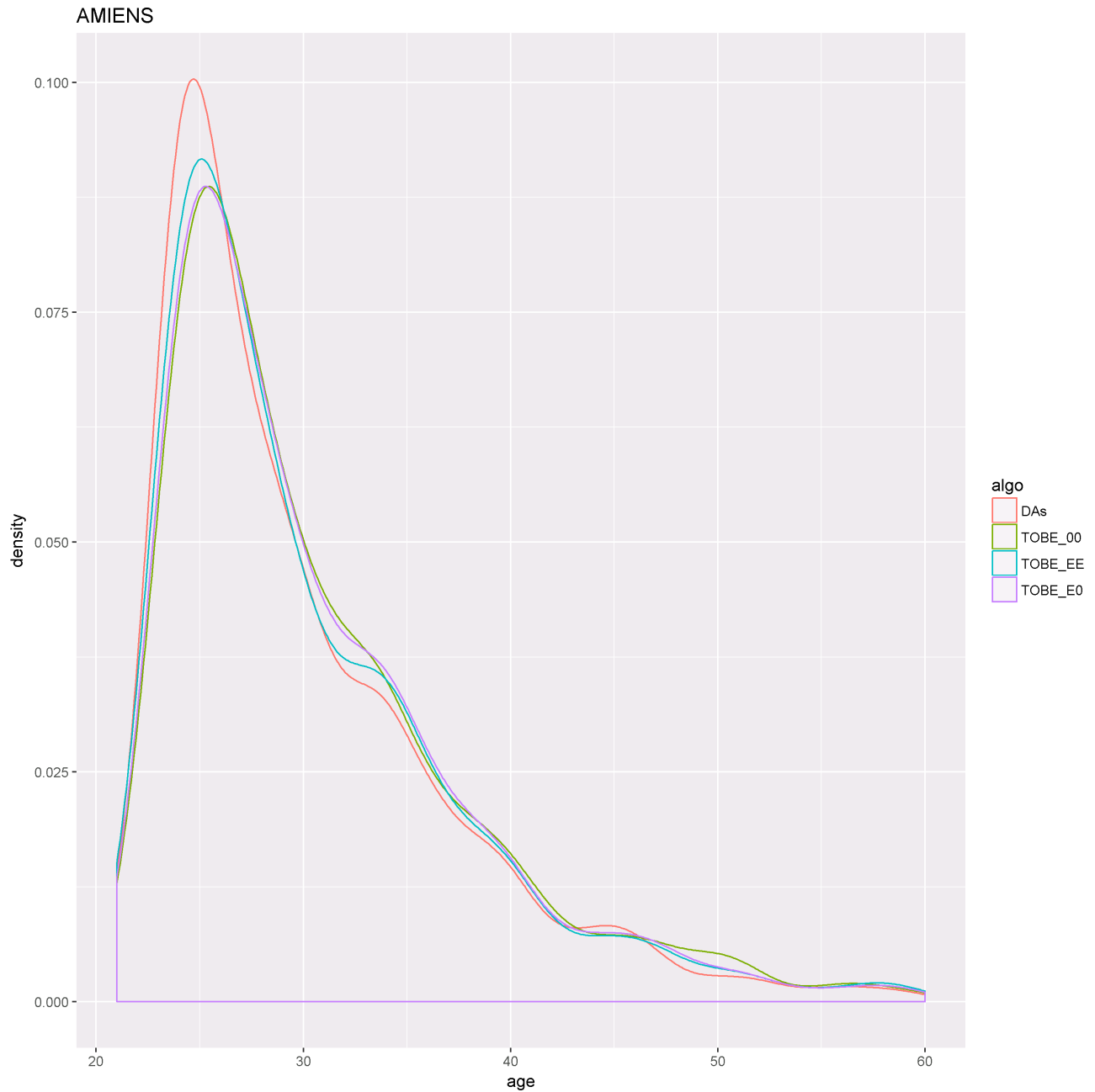
Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.24 – AVERAGE AGE OF ASSIGNED TEACHERS UNDER EACH ALGORITHM - PER REGION

	Current algorithm DA*	TOBE with all regions targeted	TOBE with no regions targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Unassigned	29.67	28.32	28.44	28.39
Paris	37.34	36.50	36.67	36.74
Aix	35.30	34.37	35.05	35.10
Besancon	32.95	32.49	33.16	33.10
Bordeaux	35.64	36.63	37.18	37.18
Caen	33.79	33.90	34.16	34.15
Clermont	34.33	34.11	34.83	35.01
Dijon	32.02	30.45	30.48	30.40
Grenoble	33.61	33.11	33.83	33.71
Lille	30.65	30.43	30.08	29.95
Lyon	33.14	32.71	32.73	32.72
Montpellier	36.83	36.84	37.26	37.28
Nancy	33.37	32.49	32.43	32.38
Poitiers	35.47	34.82	35.05	35.19
Rennes	35.98	36.23	36.34	36.36
Strasbourg	33.34	32.96	32.96	32.94
Toulouse	35.62	35.77	36.06	36.05
Nantes	34.76	34.42	34.73	34.78
Orleans	32.47	32.13	32.22	32.23
Reims	31.24	31.53	30.99	30.92
Amiens	29.87	30.57	30.31	30.55
Rouen	32.74	32.21	32.06	32.09
Limoges	34.84	36.46	35.84	35.77
Nice	35.84	35.14	35.82	35.87
Créteil	30.06	31.06	30.74	30.74
Versailles	31.08	31.26	31.28	31.23
Corse	40.53	36.97	37.13	37.47
Reunion	37.40	37.97	38.94	38.91
Martinique	39.72	39.42	38.87	38.87
Guadeloupe	39.23	39.00	38.71	38.98
Guyane	41.81	42.23	42.04	42.23
Mayotte	48.18	47.81	45.70	45.68

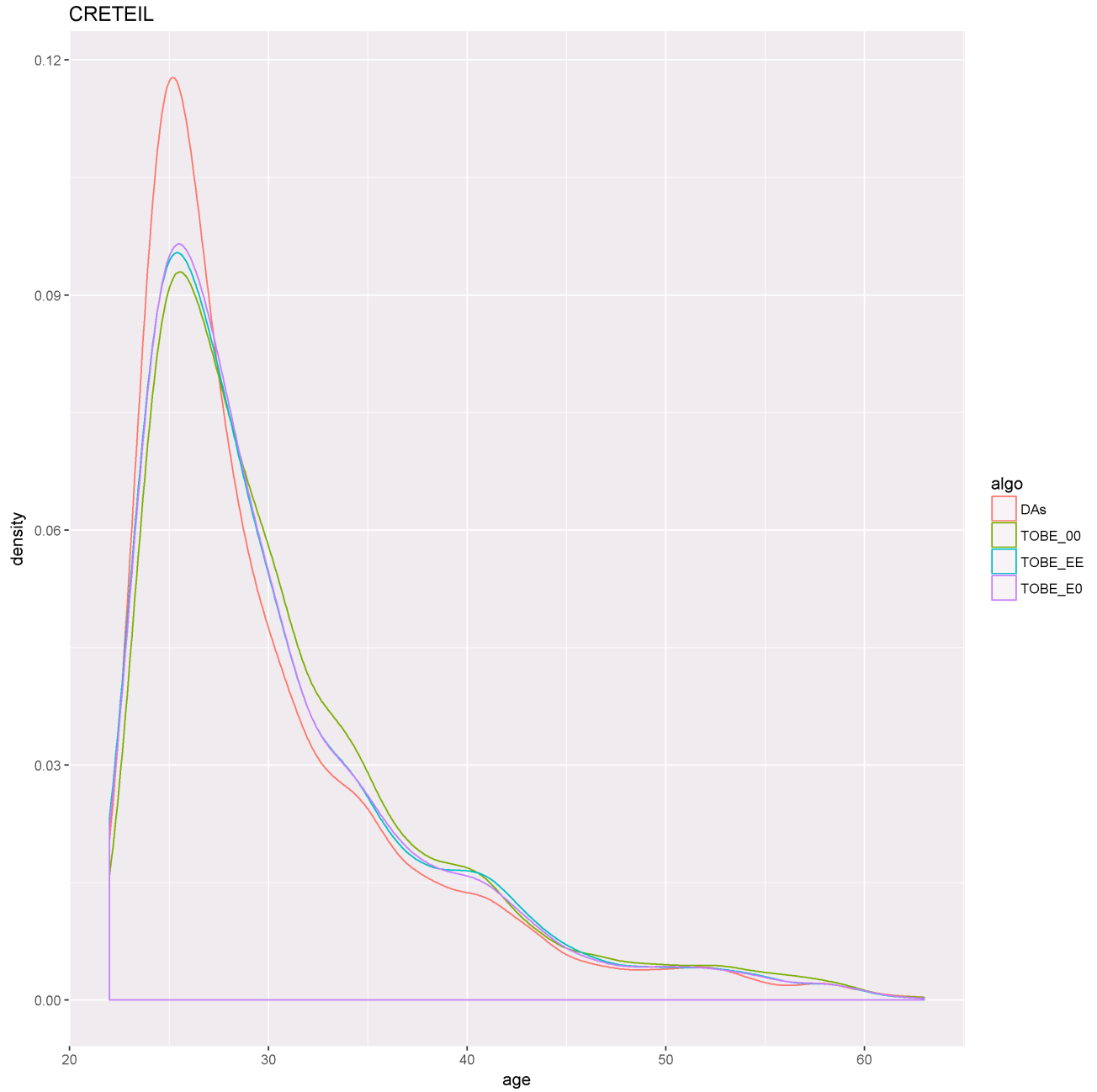
Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 4 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Figure 2.17 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF AMIENS



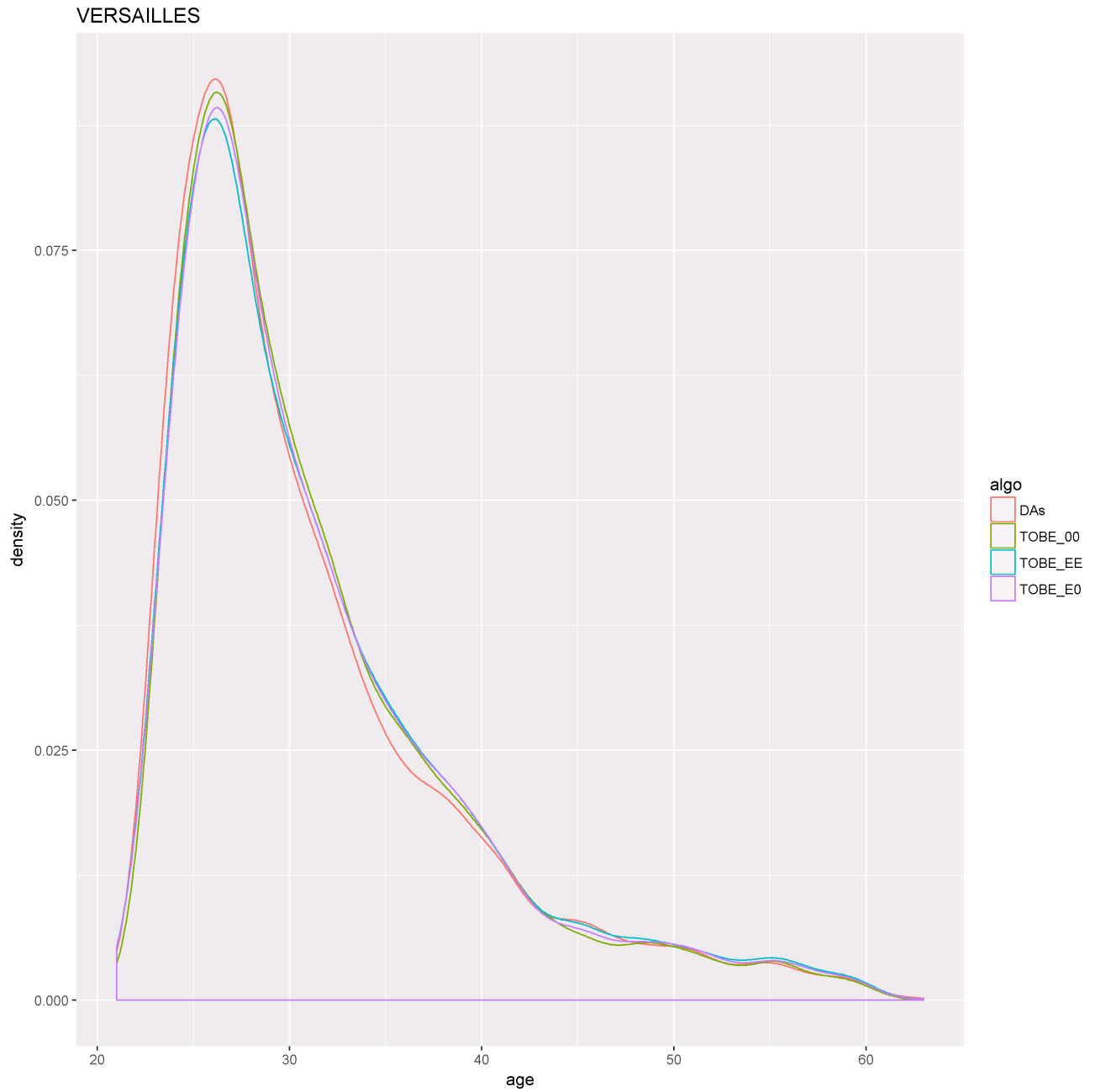
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.18 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF CRÉTÉIL



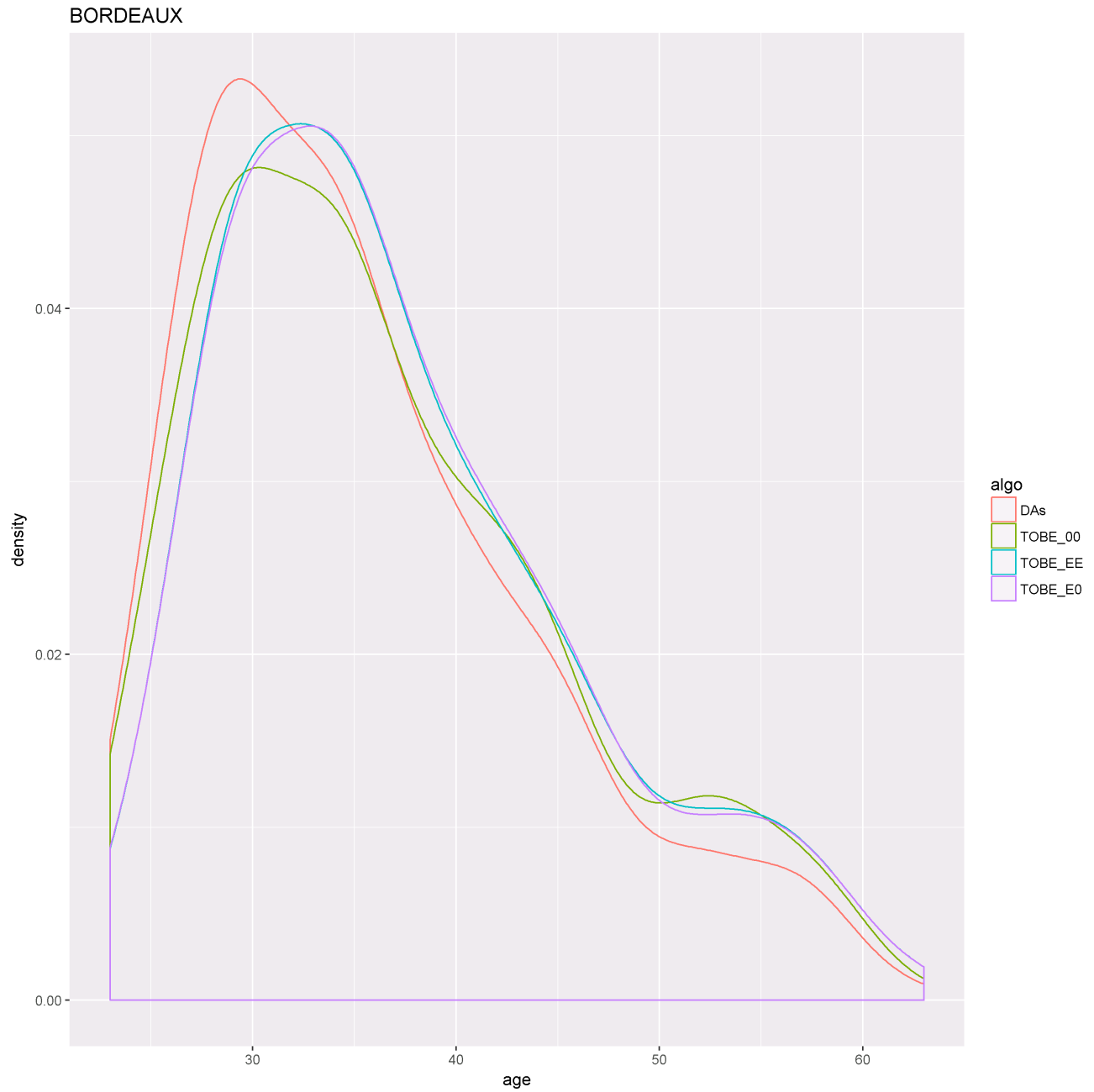
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.19 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF VERSAILLES



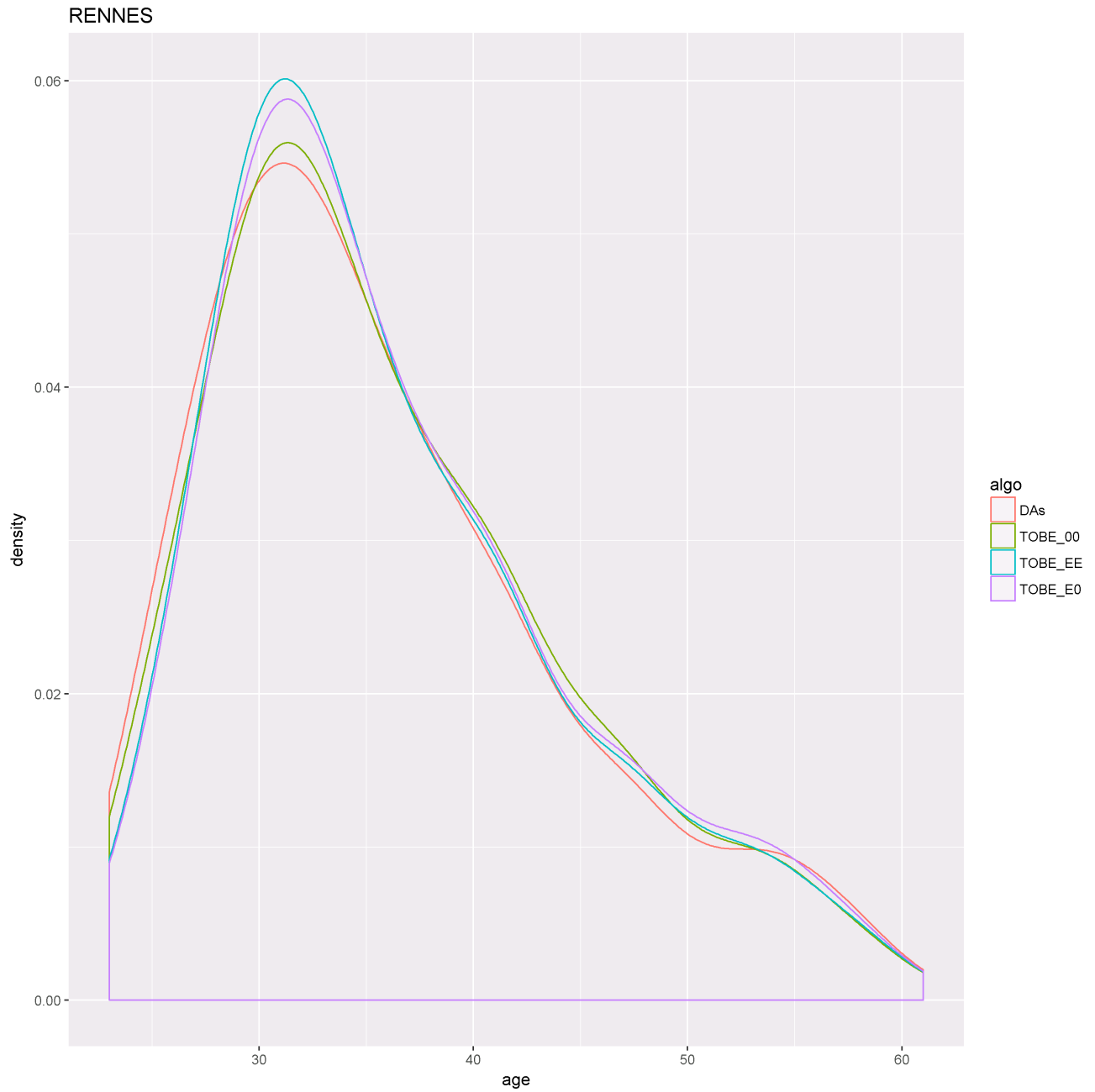
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.20 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF BORDEAUX



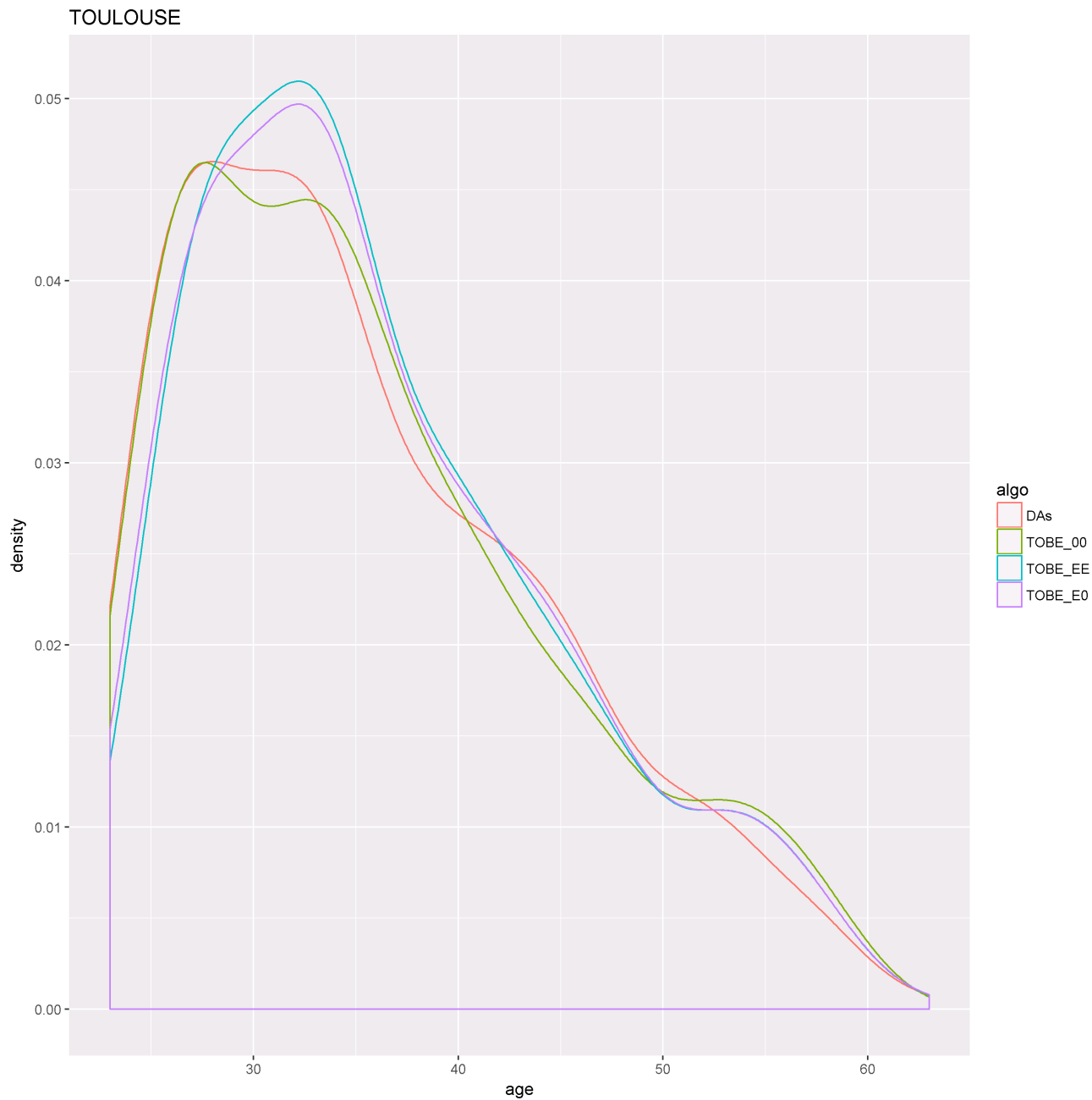
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.21 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF RENNES



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.22 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF TOULOUSE



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.



### 2.9.2.3 Results with 100% of initially non-assigned teachers improved in the ordering of TOBE

Table 2.25 – NUMBER OF TEACHERS OBTAINING A NEW ASSIGNMENT UNDER EACH ALGORITHM

	Nb	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Newly Tenured	7106	6486	0.09%	0.09%	0.08%
Tenured	12123	5494	-2.24%	3.02%	2.15%
Total	19229	11980	-0.98%	1.44%	1.03%

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.26 – NUMBER OF TEACHERS MOVING UNDER EACH ALGORITHM - PER REGION OF INITIAL ASSIGNMENT

	Current algorithm DA*	TOBE with all regions targeted		TOBE with no regions targeted		TOBE with a proportion of teachers targeted in deprived regions (option 2)	
	nb	nb	%	nb	%	nb	%
All regions	3991	3895	-2.41%	4183	4.81%	4135	3.61%
Unassigned	7976	7968	-0.10%	7969	-0.09%	7968	-0.10%
Paris	100	150	50.00%	161	61.00%	161	61.00%
Aix	106	151	42.45%	156	47.17%	156	47.17%
Besancon	37	51	37.84%	58	56.76%	58	56.76%
Bordeaux	47	62	31.91%	67	42.55%	67	42.55%
Caen	70	87	24.29%	89	27.14%	88	25.71%
Clermont	34	46	35.29%	48	41.18%	48	41.18%
Dijon	105	143	36.19%	146	39.05%	146	39.05%
Grenoble	111	168	51.35%	166	49.55%	165	48.65%
Lille	131	172	31.30%	185	41.22%	184	40.46%
Lyon	92	139	51.09%	141	53.26%	141	53.26%
Montpellier	69	115	66.67%	119	72.46%	119	72.46%
Nancy	67	103	53.73%	105	56.72%	105	56.72%
Poitiers	53	79	49.06%	83	56.60%	83	56.60%
Rennes	30	41	36.67%	42	40.00%	42	40.00%
Strasbourg	53	82	54.72%	84	58.49%	84	58.49%
Toulouse	73	119	63.01%	123	68.49%	123	68.49%
Nantes	85	124	45.88%	127	49.41%	128	50.59%
Orleans	208	202	-2.88%	200	-3.85%	203	-2.40%
Reims	98	91	-7.14%	91	-7.14%	91	-7.14%
Amiens	207	171	-17.39%	190	-8.21%	176	-14.98%
Rouen	93	122	31.18%	120	29.03%	120	29.03%
Limoges	39	40	2.56%	43	10.26%	43	10.26%
Nice	81	138	70.37%	143	76.54%	143	76.54%
Créteil	813	316	-61.13%	379	-53.38%	365	-55.10%
Versailles	779	519	-33.38%	565	-27.47%	545	-30.04%
Corse	5	10	100.00%	10	100.00%	10	100.00%
Reunion	63	89	41.27%	93	47.62%	93	47.62%
Martinique	33	52	57.58%	54	63.64%	54	63.64%
Guadeloupe	64	82	28.13%	87	35.94%	87	35.94%
Guyane	89	85	-4.49%	108	21.35%	108	21.35%
Mayotte	156	146	-6.41%	200	28.21%	199	27.56%

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 4 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A). The second column for each TOBE algorithm gives the percentage of improvement with respect to DA\*.

Table 2.27 – CUMULATIVE DISTRIBUTIONS OF RANKS FOR TENURED TEACHERS UNDER EACH ALGORITHM

	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
1	4219	4223	4454	4423
2	9861	10067	10134	10124
3	10745	10930	10979	10971
4	11234	11393	11415	11415
5	11482	11611	11619	11618
6	11645	11756	11760	11760
7	11764	11829	11826	11829
8	11883	11894	11903	11902
9	11961	11946	11953	11948
$\geq 10$	12066	12052	12052	12052
Unassigned	57	71	71	71

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.28 – CUMULATIVE DISTRIBUTIONS OF RANKS FOR NEWLY TENURED TEACHERS UNDER EACH ALGORITHM

	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
1	4044	3955	3889	3894
2	4804	4910	4854	4851
3	5095	5231	5178	5183
4	5285	5478	5433	5440
5	5429	5662	5616	5624
6	5560	5797	5752	5759
7	5662	5892	5854	5862
8	5803	6007	5967	5975
9	5916	6092	6066	6070
$\geq 10$	6491	6497	6498	6497
Unassigned	615	609	608	609

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.29 – NUMBER OF TEACHERS ASSIGNED TO AN EXTENDED WISH

	Nb	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Tenured	1560	639	554	571	568
Newly Tenured	7106	603	522	546	538

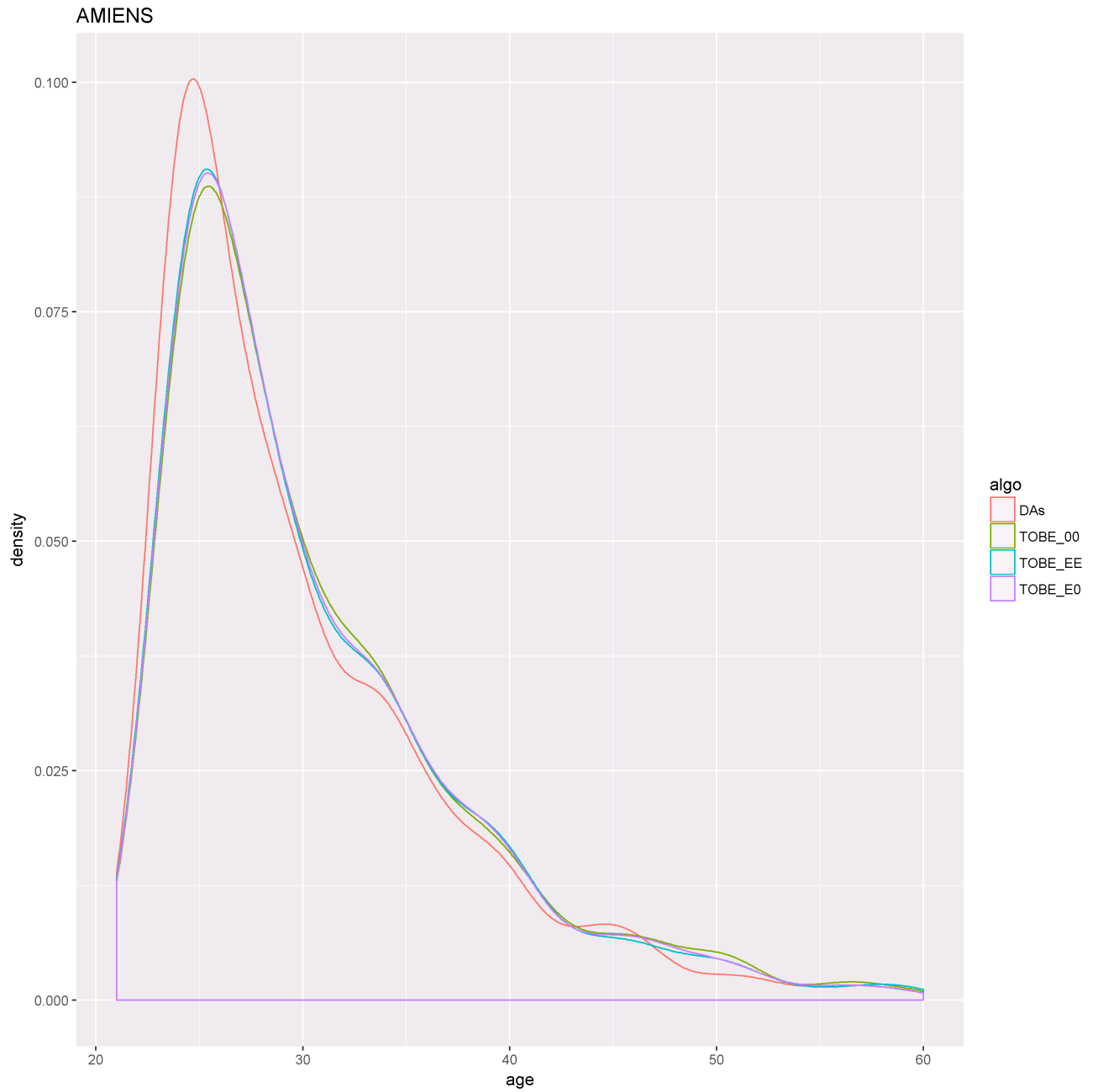
Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.30 – AVERAGE AGE OF ASSIGNED TEACHERS UNDER EACH ALGORITHM - PER REGION

	Current algorithm DA*	TOBE with all regions targeted	TOBE with no regions targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Unassigned	29.67	28.32	38.35	28.34
Paris	37.34	36.50	36.66	36.64
Aix	35.30	34.37	34.96	34.98
Besancon	32.95	32.49	33.18	33.20
Bordeaux	35.64	36.63	37.08	37.06
Caen	33.79	33.90	33.96	34.03
Clermont	34.33	34.11	34.53	34.53
Dijon	32.02	30.45	30.63	30.61
Grenoble	33.61	33.11	33.34	33.32
Lille	30.65	30.43	30.08	30.08
Lyon	33.14	32.71	32.68	32.67
Montpellier	36.83	36.84	37.28	37.28
Nancy	33.37	32.49	32.41	32.41
Poitiers	35.47	34.82	35.18	35.16
Rennes	35.98	36.23	36.35	36.35
Strasbourg	33.34	32.96	33.10	33.08
Toulouse	35.62	35.77	35.94	35.94
Nantes	34.76	34.42	34.82	34.73
Orleans	32.47	32.13	32.21	32.14
Reims	31.24	31.53	31.49	31.52
Amiens	29.87	30.57	30.40	30.43
Rouen	32.74	32.21	32.26	32.41
Limoges	34.84	36.46	35.70	35.70
Nice	35.84	35.14	35.63	35.63
Créteil	30.06	31.06	30.80	30.80
Versailles	31.08	31.26	31.16	31.18
Corse	40.53	36.97	37.20	37.20
Reunion	37.40	37.97	39.01	39.01
Martinique	39.72	39.42	38.87	38.87
Guadeloupe	39.23	39.00	39.36	39.32
Guyane	41.81	42.23	42.43	42.28
Mayotte	48.18	47.81	45.97	45.92

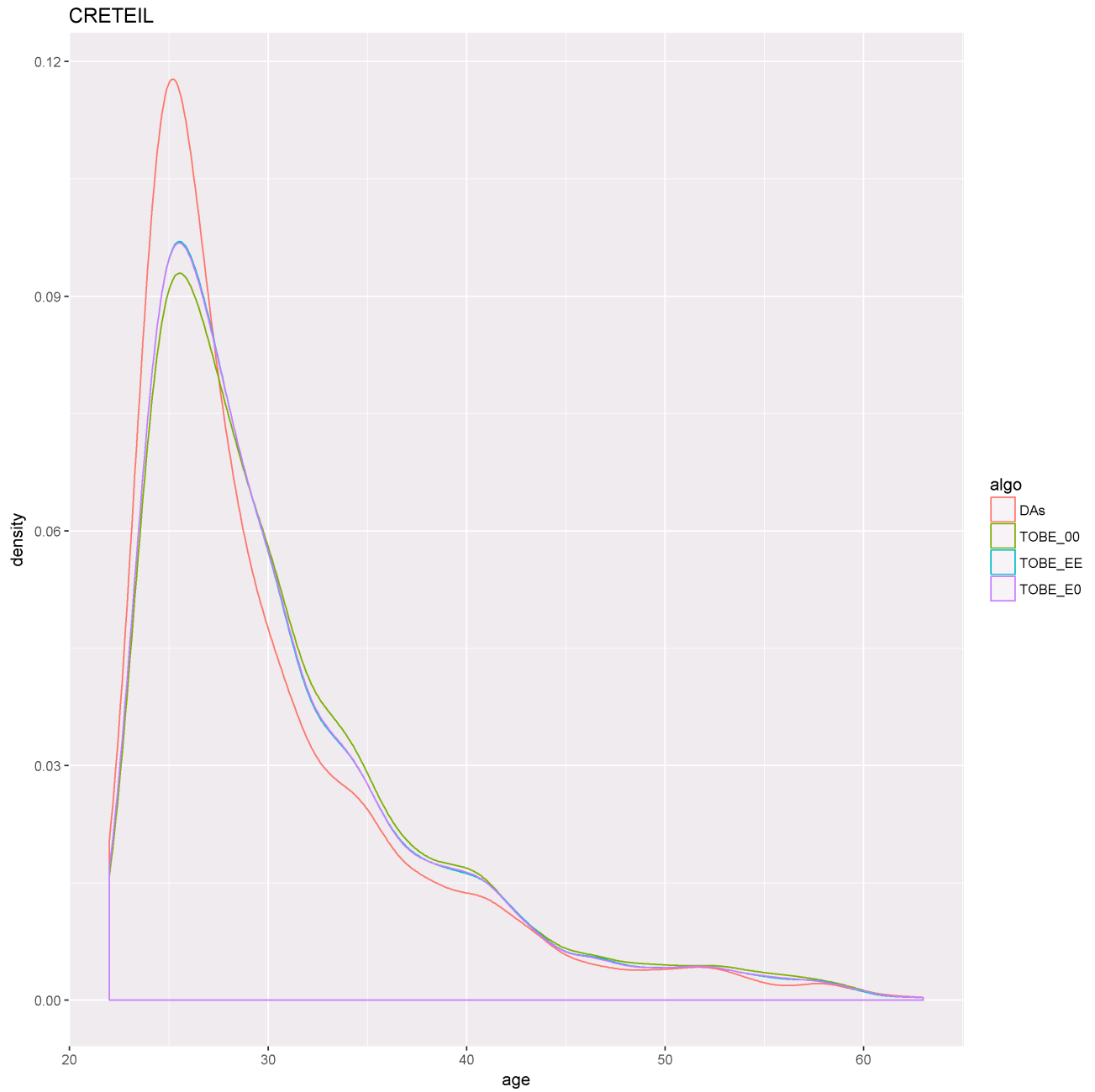
Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 4 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Figure 2.23 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF AMIENS



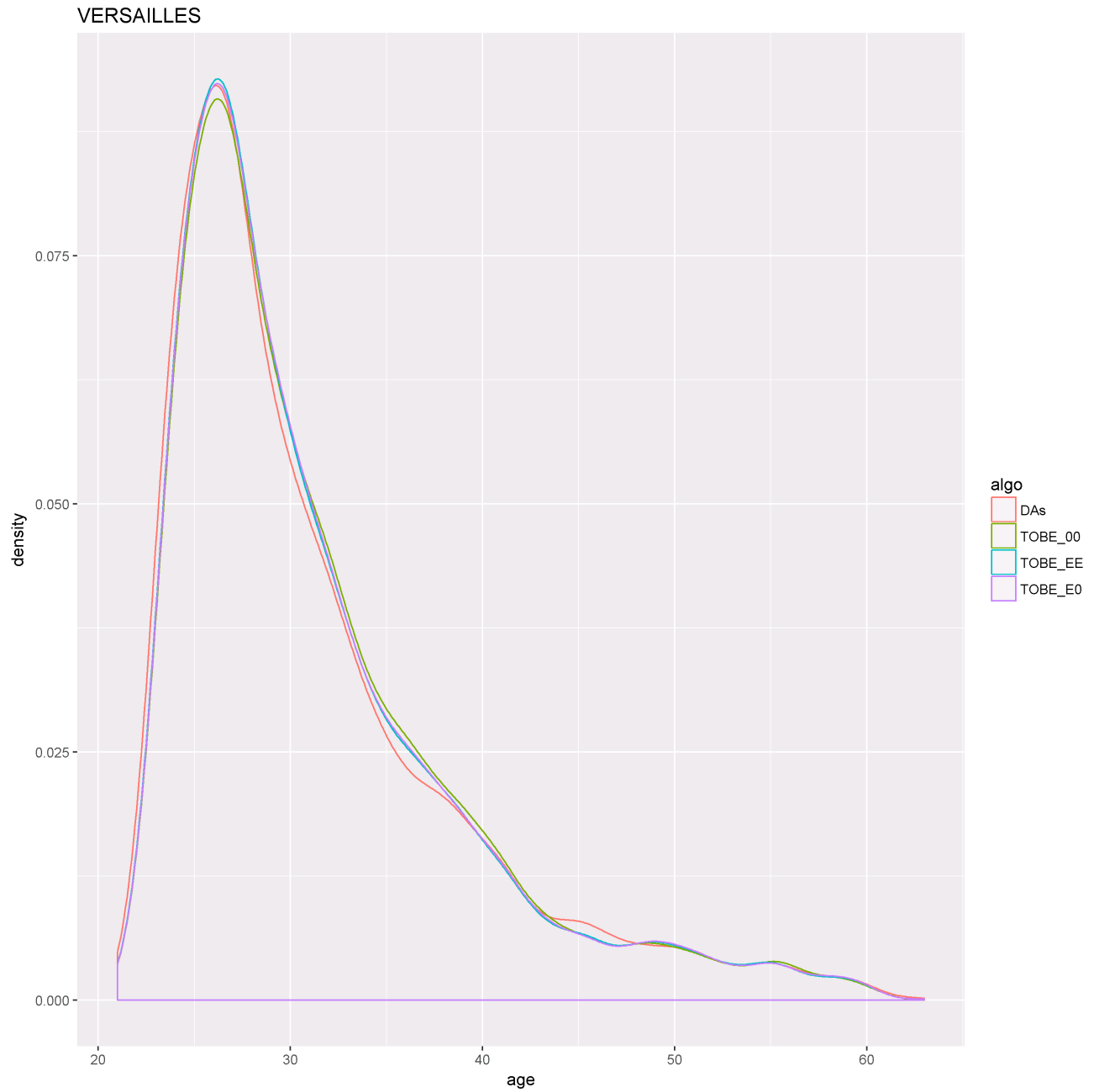
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.24 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF CRÉTÉIL



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

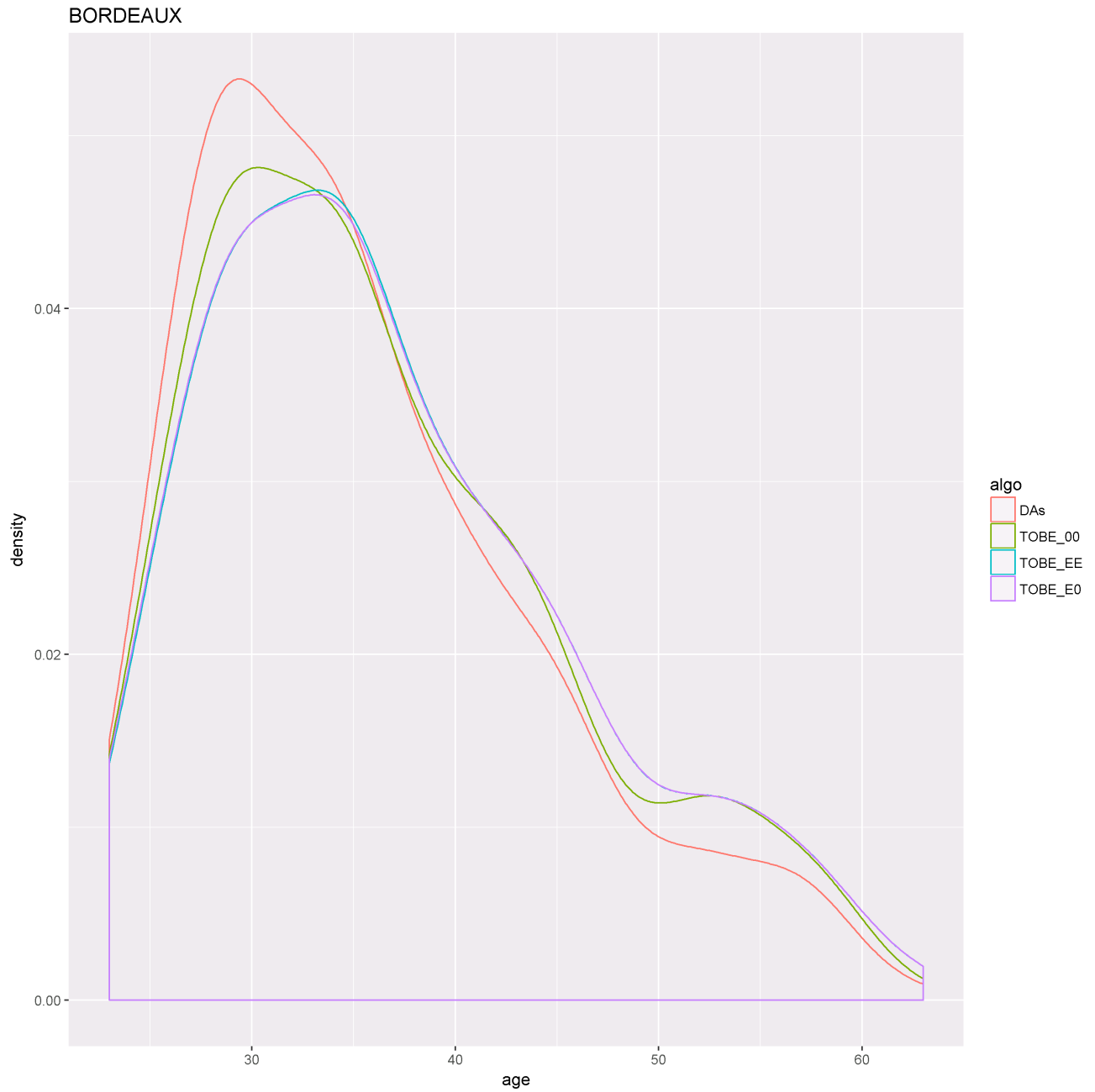
Figure 2.25 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF VERSAILLES



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

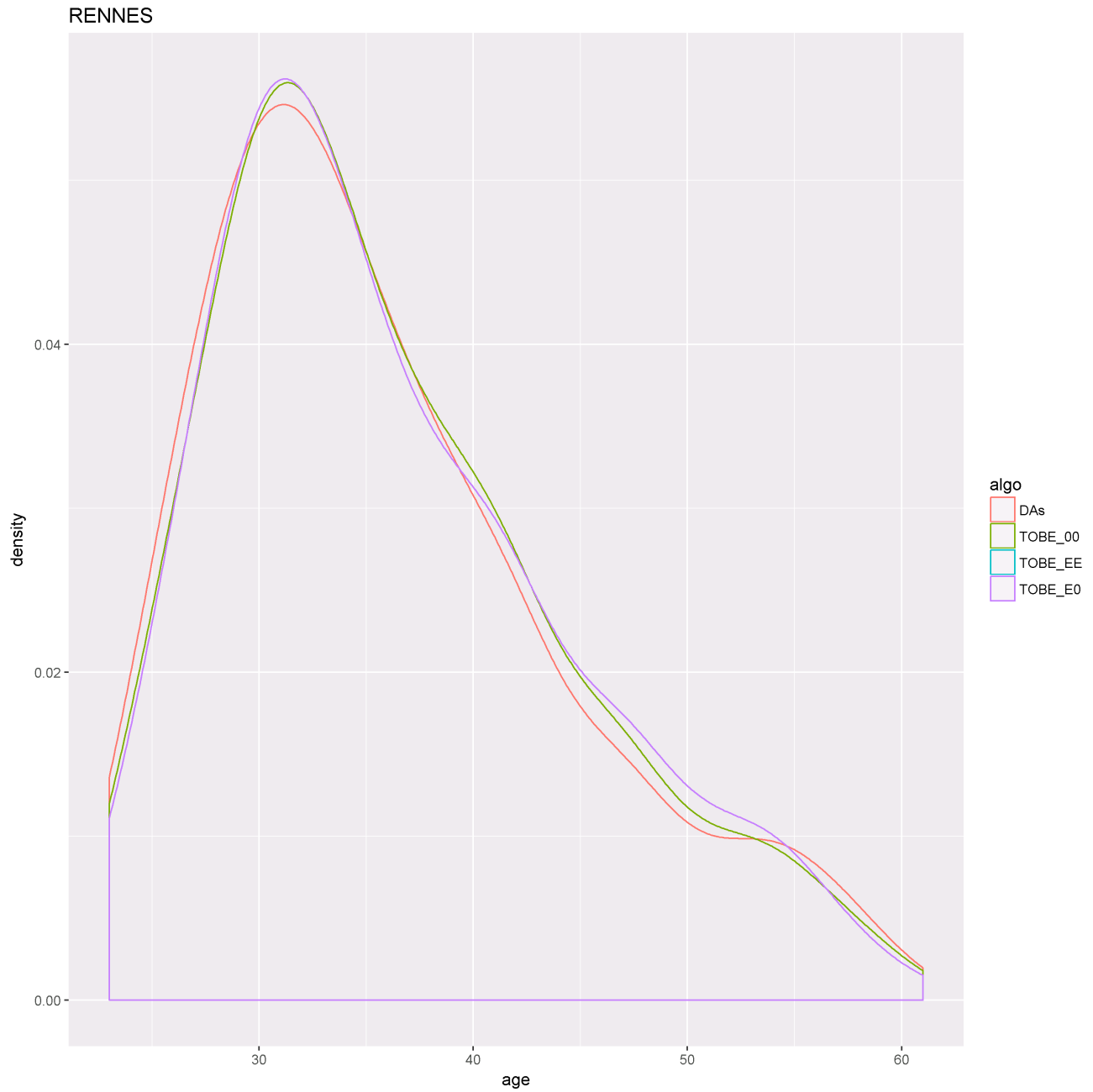


Figure 2.26 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF BORDEAUX



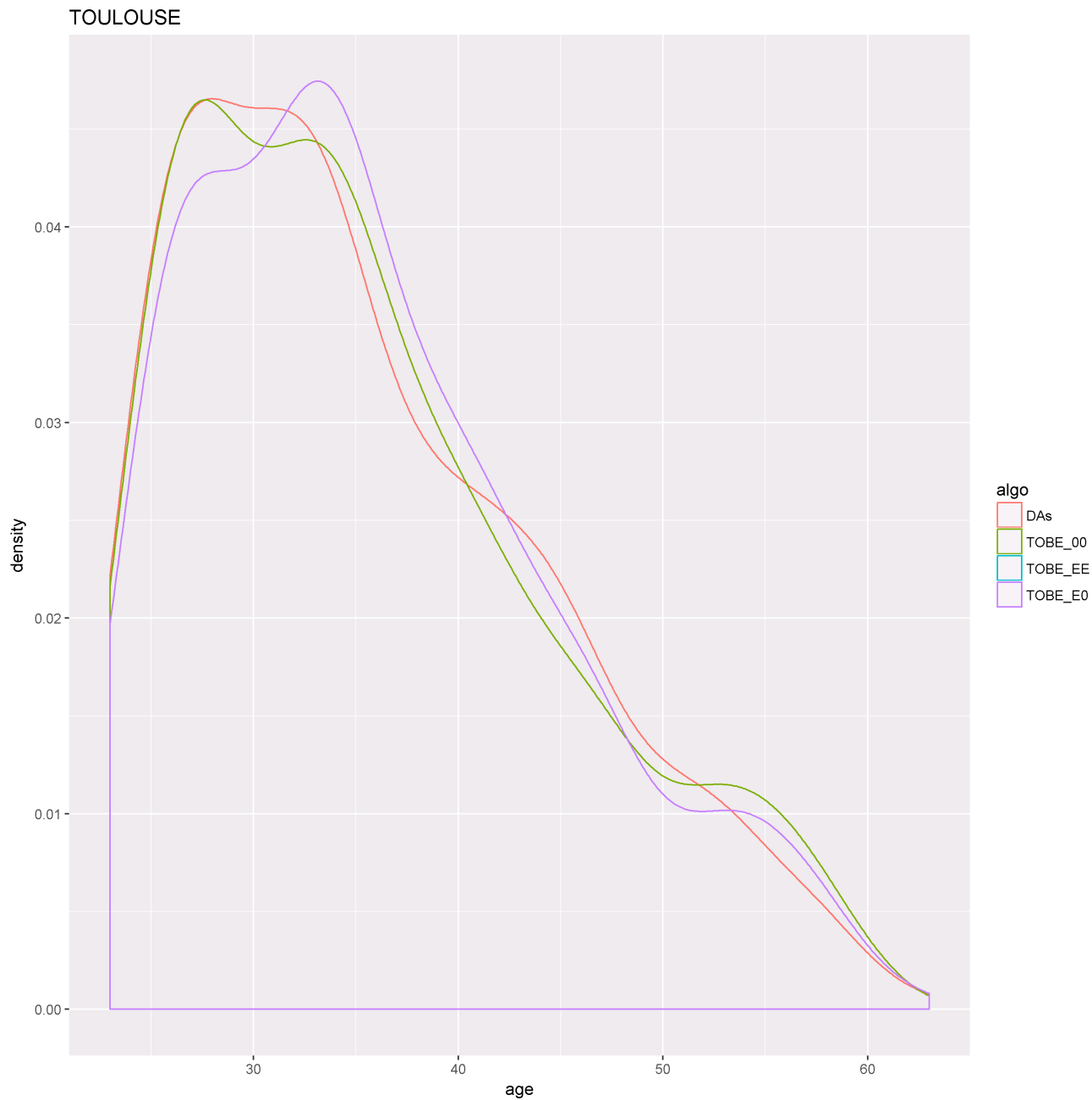
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.27 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF RENNES



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.28 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF TOULOUSE



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

### 2.9.3 Additional results using experience as the ordering of TOBE

In all the results, contrary to the tables in Section 2.6, we omit TOBE with deprived regions targeted (option 1) because only the last version with a proportion of teachers targeted (option 2) seemed to be relevant for policy makers. For the latter, we kept the same proportion of targeted teachers in Créteil, Versailles and Amiens that are respectively 45%, 65% and 40%.

In the following results, remember that the ordering used in TOBE is the following:

- $x$  % of the non initially assigned teachers with the highest experience, ordered according to their experience points.
- All the remaining teachers ordered according to their experience points.

In the next three sections, we present the results for  $x = 20\%$ ,  $x = 40\%$  and  $x = 100\%$ .

#### 2.9.3.1 Results with 20% of initially non-assigned teachers improved in the experience ordering of TOBE

Table 2.31 – NUMBER OF TEACHERS OBTAINING A NEW ASSIGNMENT UNDER EACH ALGORITHM

	Nb	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Newly Tenured	7106	6486	0.09%	0.09%	0.08%
Tenured	12123	5494	-2.24%	13.21%	12.43%
Total	19229	11980	-0.98%	6.11%	5.74%

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.32 – NUMBER OF TEACHERS MOVING UNDER EACH ALGORITHM - PER REGION OF INITIAL ASSIGNMENT

	Current algorithm DA*	TOBE with all regions targeted		TOBE with no regions targeted		TOBE with a proportion of teachers targeted in deprived regions (option 2)	
	nb	nb	%	nb	%	nb	%
All regions	3991	3895	-2.41%	4748	18.97%	4705	17.89%
Unassigned	7976	7968	-0.10%	7964	-0.15%	7963	-0.16%
Paris	100	150	50.00%	150	50.00%	150	50.00%
Aix	106	151	42.45%	155	46.23%	155	46.23%
Besancon	37	51	37.84%	62	67.57%	62	67.57%
Bordeaux	47	62	31.91%	65	38.30%	65	38.30%
Caen	70	87	24.29%	90	28.57%	89	27.14%
Clermont	34	46	35.29%	48	41.18%	48	41.18%
Dijon	105	143	36.19%	148	40.95%	149	41.90%
Grenoble	111	168	51.35%	167	50.45%	167	50.45%
Lille	131	172	31.30%	163	24.43%	164	25.19%
Lyon	92	139	51.09%	140	52.17%	140	52.17%
Montpellier	69	115	66.67%	119	72.46%	119	72.46%
Nancy	67	103	53.73%	102	52.24%	102	52.24%
Poitiers	53	79	49.06%	84	58.49%	84	58.49%
Rennes	30	41	36.67%	42	40.00%	42	40.00%
Strasbourg	53	82	54.72%	82	54.72%	82	54.72%
Toulouse	73	119	63.01%	122	67.12%	122	67.12%
Nantes	85	124	45.88%	125	47.06%	126	48.24%
Orleans	208	202	-2.88%	236	13.46%	236	13.46%
Reims	98	91	-7.14%	104	6.12%	104	6.12%
Amiens	207	171	-17.39%	220	6.28%	199	-3.86%
Rouen	93	122	31.18%	113	21.51%	113	21.51%
Limoges	39	40	2.56%	54	38.46%	54	38.46%
Nice	81	138	70.37%	146	80.25%	146	80.25%
Créteil	813	316	-61.13%	703	-13.53%	694	-14.64%
Versailles	779	519	-33.38%	744	-4.49%	728	-6.55%
Corse	5	10	100.00%	10	100.00%	10	100.00%
Reunion	63	89	41.27%	94	49.21%	94	49.21%
Martinique	33	52	57.58%	53	60.61%	53	60.61%
Guadeloupe	64	82	28.13%	90	40.63%	90	40.63%
Guyane	89	85	-4.49%	110	23.60%	110	23.60%
Mayotte	156	146	-6.41%	207	32.69%	208	33.33%

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 4 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A). The second column for each TOBE algorithm gives the percentage of improvement with respect to DA\*.

Table 2.33 – CUMULATIVE DISTRIBUTIONS OF RANKS FOR TENURED TEACHERS UNDER EACH ALGORITHM

	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
1	4219	4223	4864	4834
2	9861	10067	10187	10180
3	10745	10930	10932	10930
4	11234	11393	11363	11364
5	11482	11611	11575	11575
6	11645	11756	11706	11706
7	11764	11829	11791	11788
8	11883	11894	11885	11882
9	11961	11946	11942	11940
≥ 10	12066	12052	12049	12049
Unassigned	57	71	74	74

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.34 – CUMULATIVE DISTRIBUTIONS OF RANKS FOR NEWLY TENURED TEACHERS UNDER EACH ALGORITHM

	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
1	4044	3955	3359	3379
2	4804	4910	4346	4357
3	5095	5231	4749	4766
4	5285	5478	5076	5088
5	5429	5662	5297	5310
6	5560	5797	5473	5485
7	5662	5892	5606	5620
8	5803	6007	5760	5769
9	5916	6092	5878	5889
>= 10	6491	6497	6496	6495
Unassigned	615	609	610	611

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.35 – NUMBER OF TEACHERS ASSIGNED TO AN EXTENDED WISH

	Nb	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Tenured	1560	639	554	679	676
Newly Tenured	7106	603	522	743	736

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

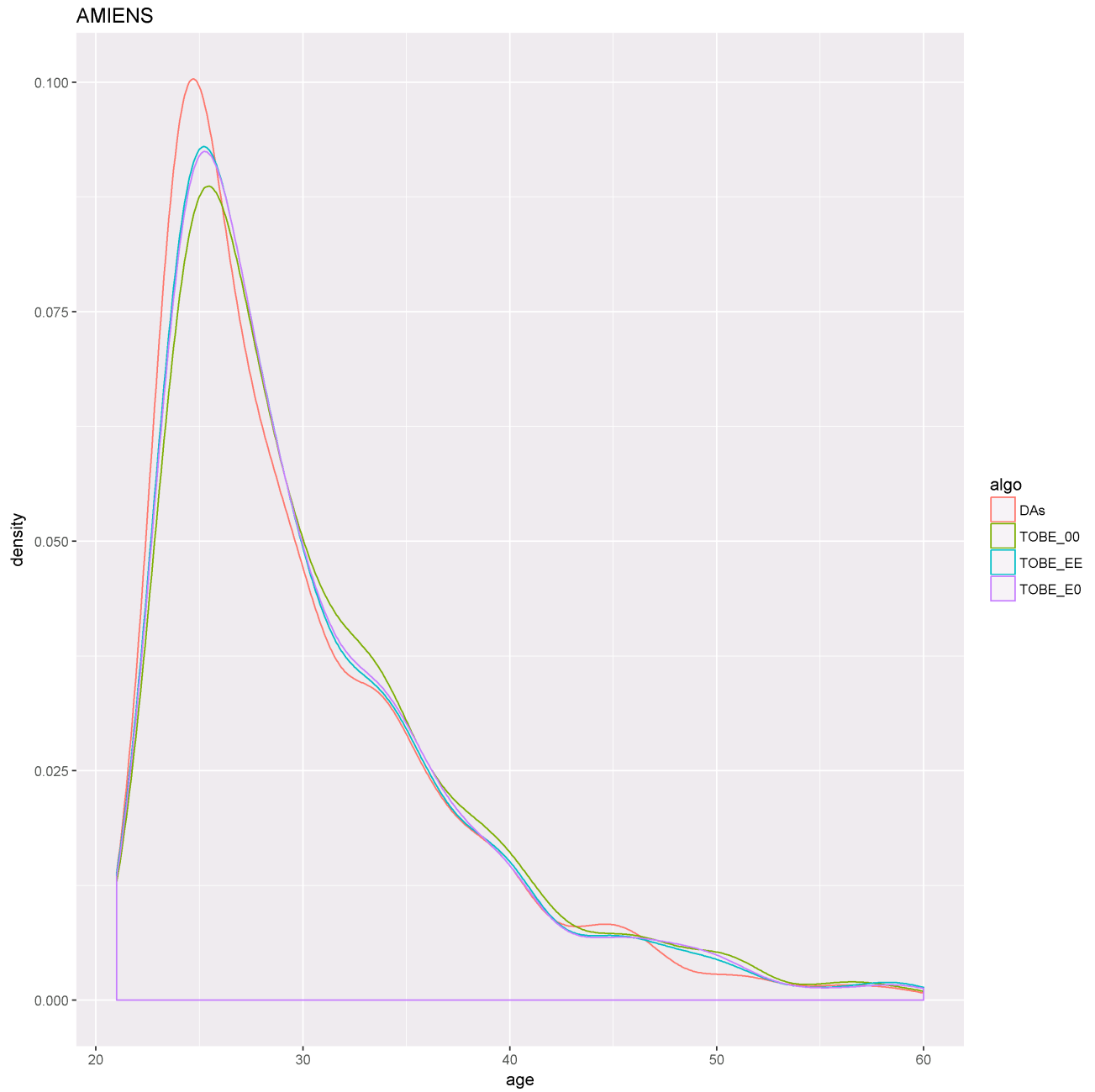
Table 2.36 – AVERAGE AGE OF ASSIGNED TEACHERS UNDER EACH ALGORITHM - PER REGION

	Current algorithm DA*	TOBE with all regions targeted	TOBE with no regions targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Unassigned	29.67	28.32	28.38	28.37
Paris	37.34	36.50	36.67	36.69
Aix	35.30	34.37	34.97	34.99
Besancon	32.95	32.49	33.46	33.44
Bordeaux	35.64	36.63	36.96	36.96
Caen	33.79	33.90	34.24	34.19
Clermont	34.33	34.11	34.69	34.78
Dijon	32.02	30.45	30.95	30.98
Grenoble	33.61	33.11	33.71	33.67
Lille	30.65	30.43	30.32	30.25
Lyon	33.14	32.71	32.93	32.87
Montpellier	36.83	36.84	37.33	37.34
Nancy	33.37	32.49	32.76	32.76
Poitiers	35.47	34.82	34.73	34.71
Rennes	35.98	36.23	36.41	36.41
Strasbourg	33.34	32.96	33.05	33.07
Toulouse	35.62	35.77	35.93	35.92
Nantes	34.76	34.42	35.08	35.05
Orleans	32.47	32.13	32.67	32.69
Reims	31.24	31.53	31.34	31.33
Amiens	29.87	30.57	30.22	30.27
Rouen	32.74	32.21	32.35	32.46
Limoges	34.84	36.46	35.98	35.98
Nice	35.84	35.14	35.99	36.02
Créteil	30.06	31.06	30.45	30.47
Versailles	31.08	31.26	31.15	31.13
Corse	40.53	36.97	37.93	37.97
Reunion	37.40	37.97	38.87	38.88
Martinique	39.72	39.42	39.02	39.02
Guadeloupe	39.23	39.00	39.22	39.22
Guyane	41.81	42.23	41.78	41.74
Mayotte	48.18	47.81	45.77	45.70

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 4 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

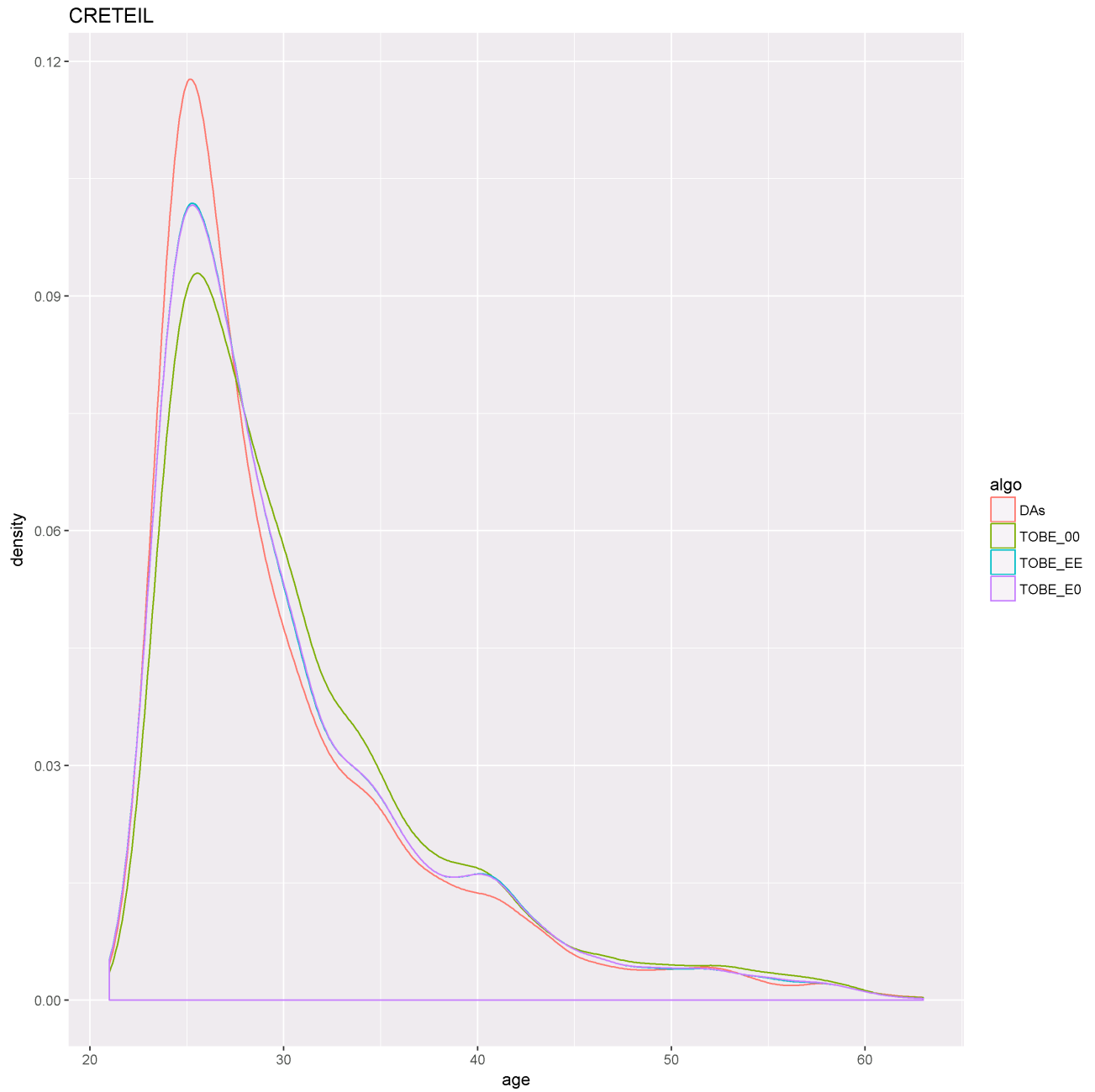


Figure 2.29 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF AMIENS



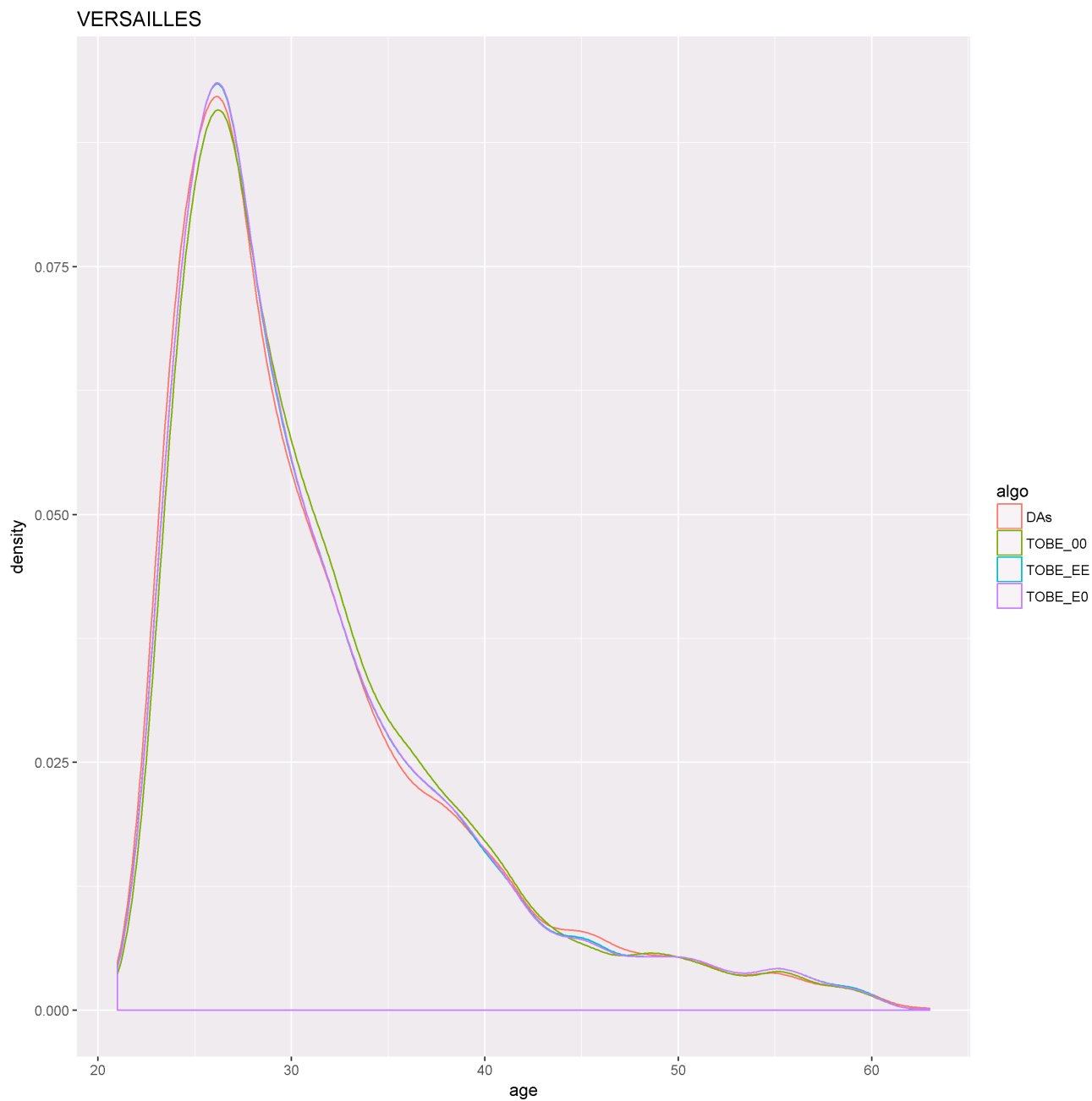
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.30 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF CRÉTÉIL



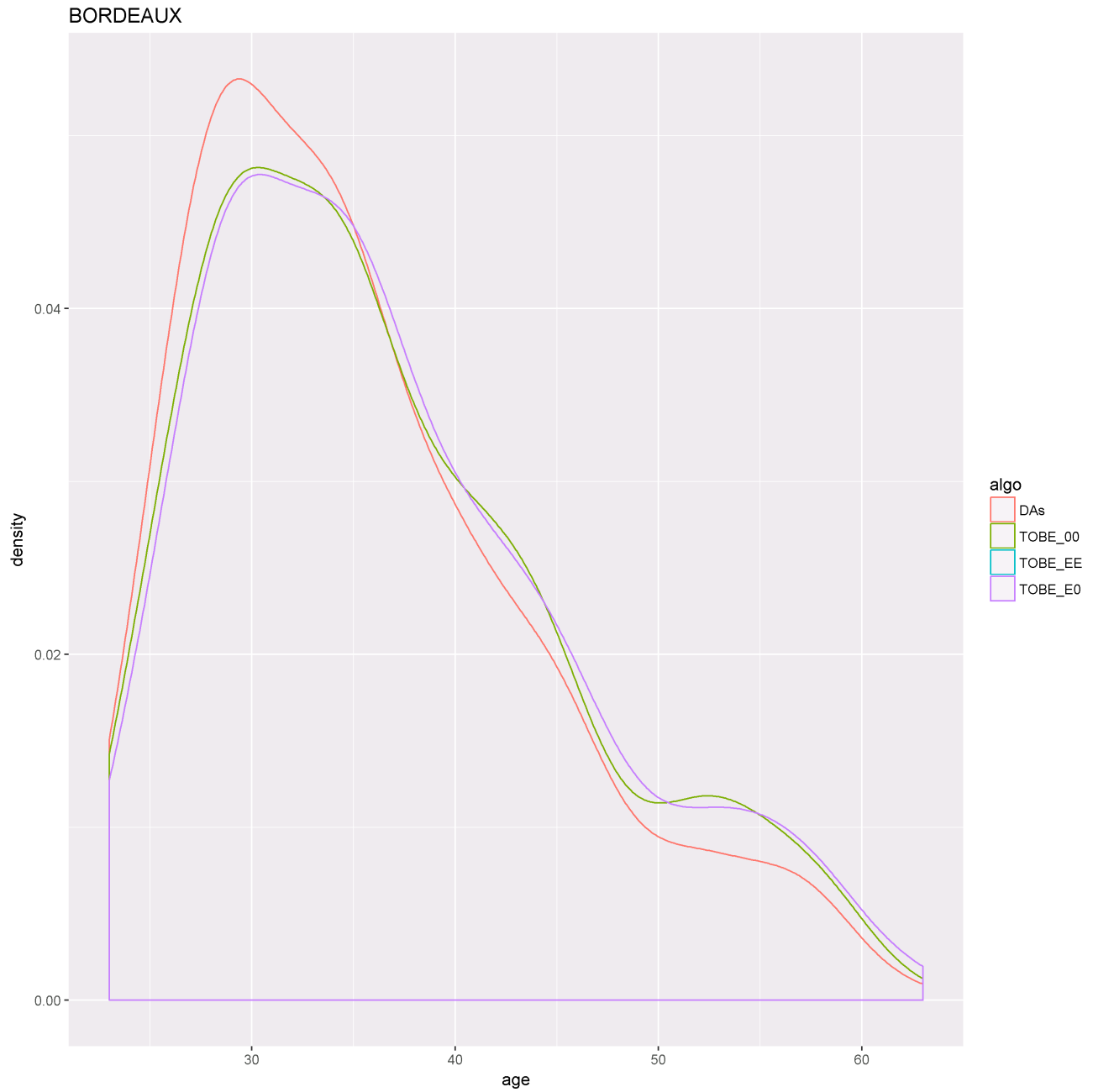
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.31 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF VERSAILLES



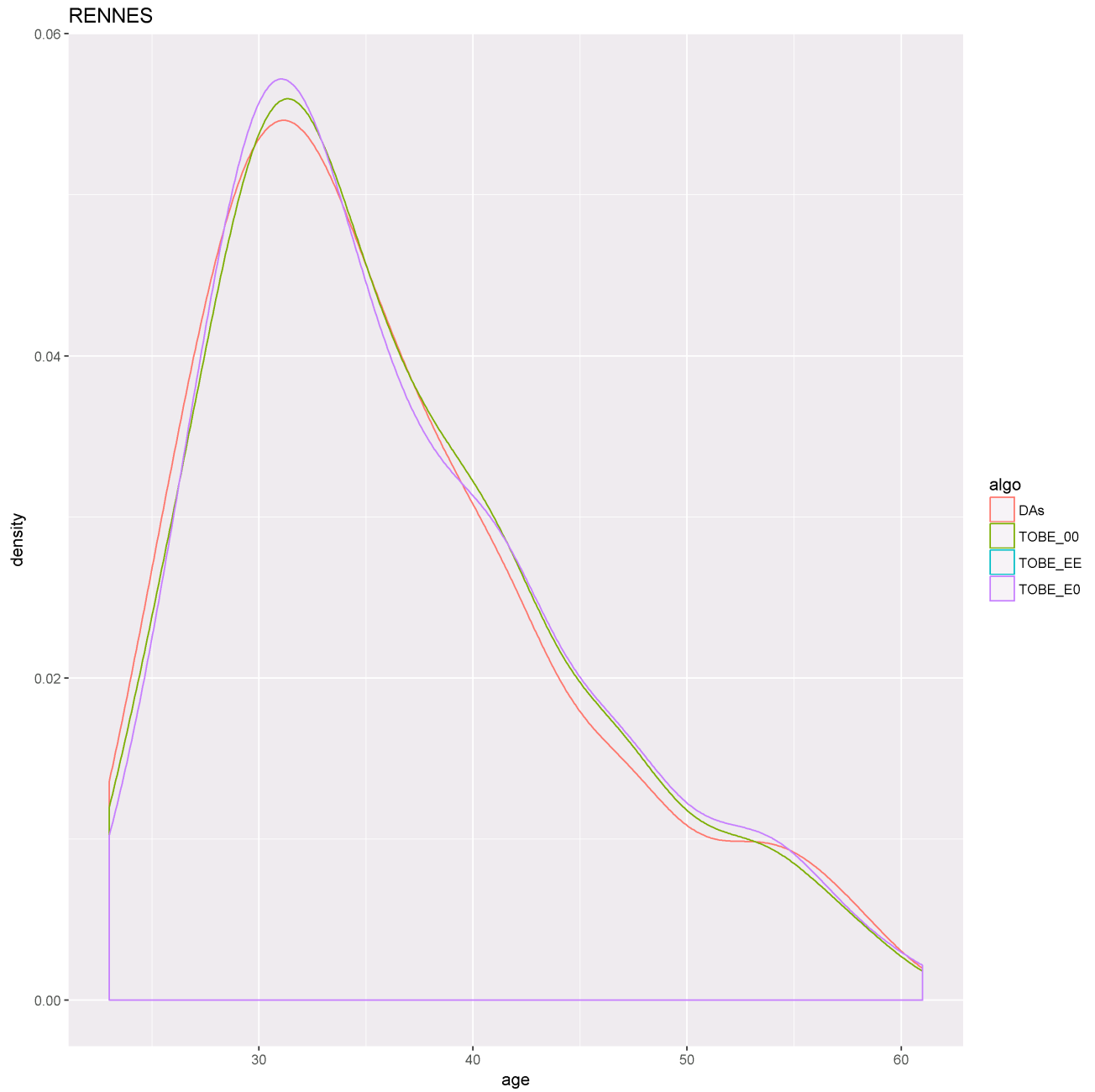
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.32 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF BORDEAUX



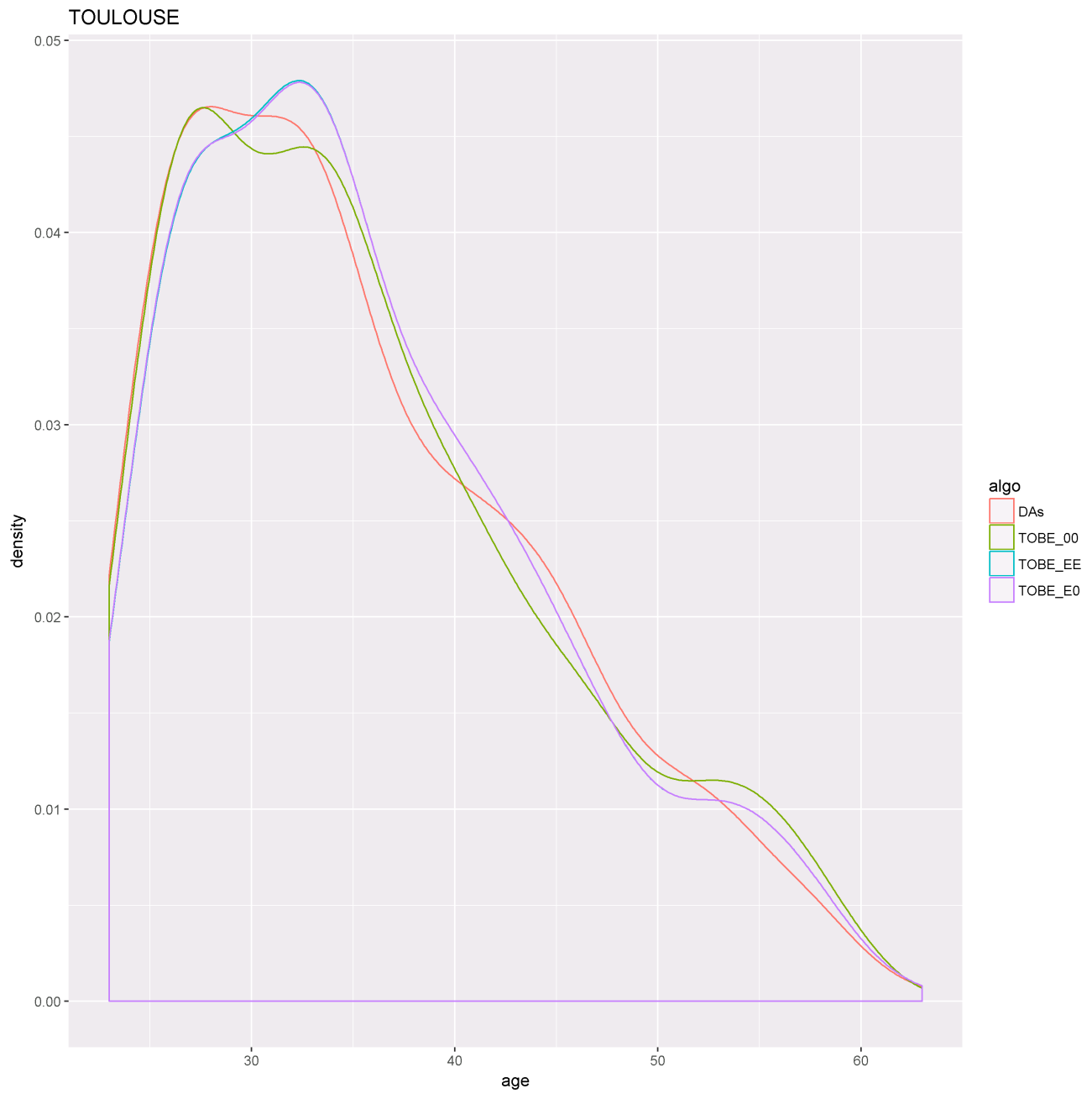
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.33 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF RENNES



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.34 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF TOULOUSE



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

### 2.9.3.2 Results with 40% of initially non-assigned teachers improved in the experience ordering of TOBE

Table 2.37 – NUMBER OF TEACHERS OBTAINING A NEW ASSIGNMENT UNDER EACH ALGORITHM

	Nb	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Newly Tenured	7106	6486	0.09%	0.08%	0.03%
Tenured	12123	5494	-2.24%	8.06%	7.08%
Total	19229	11980	-0.98%	3.74%	3.26%

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.38 – NUMBER OF TEACHERS MOVING UNDER EACH ALGORITHM - PER REGION OF INITIAL ASSIGNMENT

	Current algorithm DA*	TOBE with all regions targeted		TOBE with no regions targeted		TOBE with a proportion of teachers targeted in deprived regions (option 2)	
	nb	nb	%	nb	%	nb	%
All regions	3991	3895	-2.41%	4464	11.85%	4410	10.50%
Unassigned	7976	7968	-0.10%	7964	-0.15%	7961	-0.19%
Paris	100	150	50.00%	150	50.00%	150	50.00%
Aix	106	151	42.45%	157	48.11%	157	48.11%
Besancon	37	51	37.84%	60	62.16%	60	62.16%
Bordeaux	47	62	31.91%	67	42.55%	67	42.55%
Caen	70	87	24.29%	91	30.00%	90	28.57%
Clermont	34	46	35.29%	48	41.18%	48	41.18%
Dijon	105	143	36.19%	143	36.19%	143	36.19%
Grenoble	111	168	51.35%	163	46.85%	162	45.95%
Lille	131	172	31.30%	167	27.48%	166	26.72%
Lyon	92	139	51.09%	140	52.17%	140	52.17%
Montpellier	69	115	66.67%	119	72.46%	119	72.46%
Nancy	67	103	53.73%	103	53.73%	103	53.73%
Poitiers	53	79	49.06%	83	56.60%	83	56.60%
Rennes	30	41	36.67%	42	40.00%	42	40.00%
Strasbourg	53	82	54.72%	84	58.49%	84	58.49%
Toulouse	73	119	63.01%	123	68.49%	123	68.49%
Nantes	85	124	45.88%	128	50.59%	129	51.76%
Orleans	208	202	-2.88%	211	1.44%	211	1.44%
Reims	98	91	-7.14%	102	4.08%	102	4.08%
Amiens	207	171	-17.39%	200	-3.38%	180	-13.04%
Rouen	93	122	31.18%	120	29.03%	120	29.03%
Limoges	39	40	2.56%	45	15.38%	45	15.38%
Nice	81	138	70.37%	145	79.01%	145	79.01%
Créteil	813	316	-61.13%	589	-27.55%	571	-29.77%
Versailles	779	519	-33.38%	616	-20.92%	601	-22.85%
Corse	5	10	100.00%	10	100.00%	10	100.00%
Reunion	63	89	41.27%	95	50.79%	95	50.79%
Martinique	33	52	57.58%	54	63.64%	54	63.64%
Guadeloupe	64	82	28.13%	90	40.63%	90	40.63%
Guyane	89	85	-4.49%	111	24.72%	112	25.84%
Mayotte	156	146	-6.41%	208	33.33%	208	33.33%

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 4 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A). The second column for each TOBE algorithm gives the percentage of improvement with respect to DA\*.



Table 2.39 – CUMULATIVE DISTRIBUTIONS OF RANKS FOR TENURED TEACHERS UNDER EACH ALGORITHM

	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
1	4219	4223	4659	4613
2	9861	10067	10194	10178
3	10745	10930	10985	10976
4	11234	11393	11420	11415
5	11482	11611	11617	11617
6	11645	11756	11753	11753
7	11764	11829	11823	11824
8	11883	11894	11902	11906
9	11961	11946	11945	11949
$\geq 10$	12066	12052	12047	12049
Unassigned	57	71	74	74

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.40 – CUMULATIVE DISTRIBUTIONS OF RANKS FOR NEWLY TENURED TEACHERS UNDER EACH ALGORITHM

	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
1	4044	3955	3670	3692
2	4804	4910	4602	4629
3	5095	5231	4958	4988
4	5285	5478	5232	5254
5	5429	5662	5437	5455
6	5560	5797	5590	5604
7	5662	5892	5701	5712
8	5803	6007	5839	5844
9	5916	6092	5947	5957
$\geq 10$	6491	6497	6496	6493
Unassigned	615	609	610	613

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.41 – NUMBER OF TEACHERS ASSIGNED TO AN EXTENDED WISH

	Nb	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Tenured	1560	639	554	597	597
Newly Tenured	7106	603	522	639	627

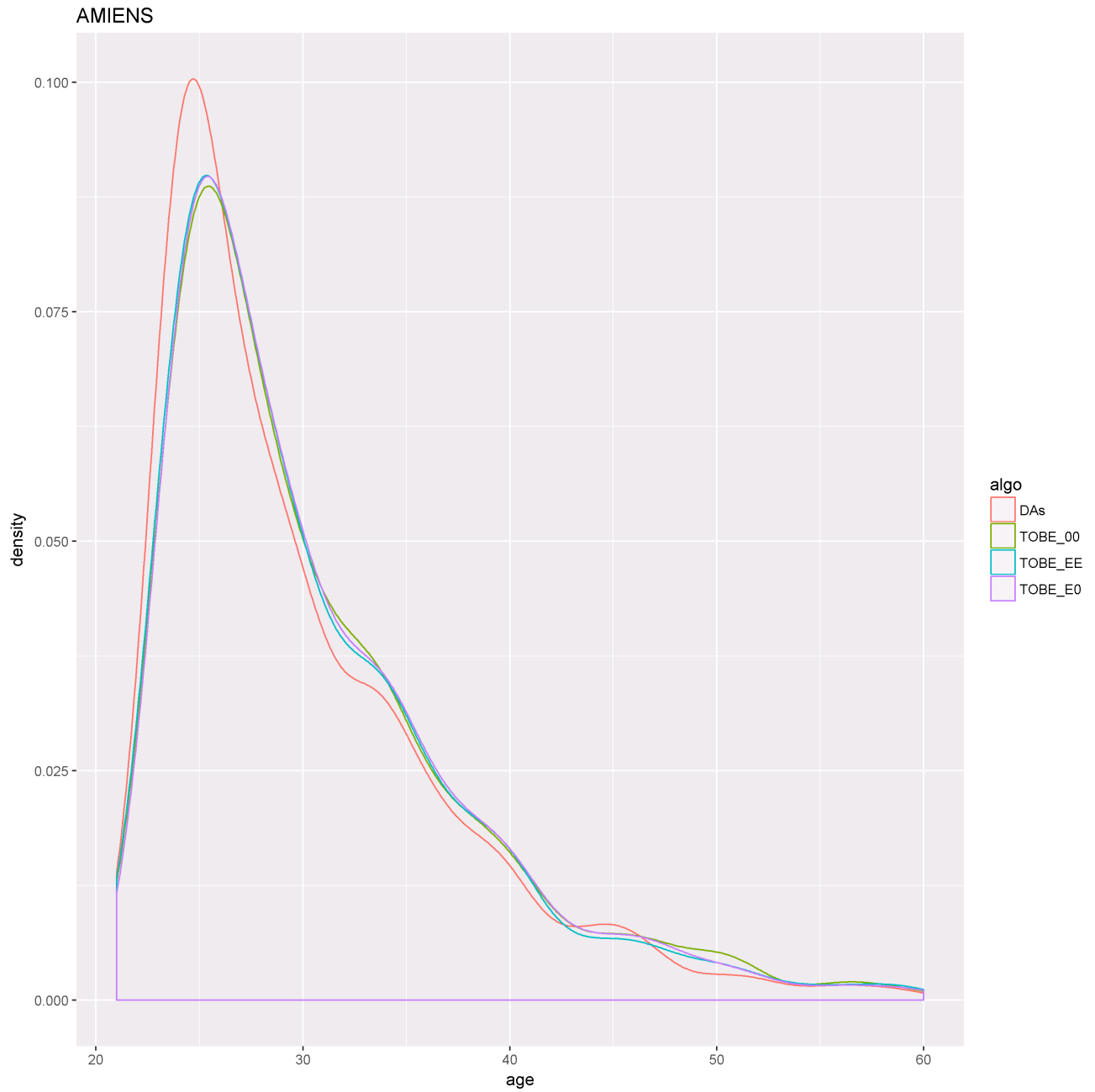
Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.42 – AVERAGE AGE OF ASSIGNED TEACHERS UNDER EACH ALGORITHM - PER REGION

	Current algorithm DA*	TOBE with all regions targeted	TOBE with no regions targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Unassigned	29.67	28.32	28.38	28.36
Paris	37.34	36.50	36.65	36.69
Aix	35.30	34.37	35.00	35.03
Besancon	32.95	32.49	33.24	33.24
Bordeaux	35.64	36.63	37.06	37.04
Caen	33.79	33.90	34.01	33.91
Clermont	34.33	34.11	34.59	34.60
Dijon	32.02	30.45	30.69	30.73
Grenoble	33.61	33.11	33.66	33.62
Lille	30.65	30.43	30.25	30.15
Lyon	33.14	32.71	32.77	32.69
Montpellier	36.83	36.84	37.19	37.20
Nancy	33.37	32.49	32.49	32.49
Poitiers	35.47	34.82	35.14	35.14
Rennes	35.98	36.23	36.34	36.34
Strasbourg	33.34	32.96	33.13	33.11
Toulouse	35.62	35.77	35.93	35.93
Nantes	34.76	34.42	34.81	34.81
Orleans	32.47	32.13	32.41	32.37
Reims	31.24	31.53	31.17	31.15
Amiens	29.87	30.57	30.38	30.50
Rouen	32.74	32.21	32.44	32.44
Limoges	34.84	36.46	35.87	35.87
Nice	35.84	35.14	35.71	35.69
Créteil	30.06	31.06	30.65	30.67
Versailles	31.08	31.26	31.16	31.18
Corse	40.53	36.97	37.17	37.10
Reunion	37.40	37.97	38.90	38.91
Martinique	39.72	39.42	38.87	38.87
Guadeloupe	39.23	39.00	39.41	39.41
Guyane	41.81	42.23	62.17	42.07
Mayotte	48.18	47.81	45.62	45.61

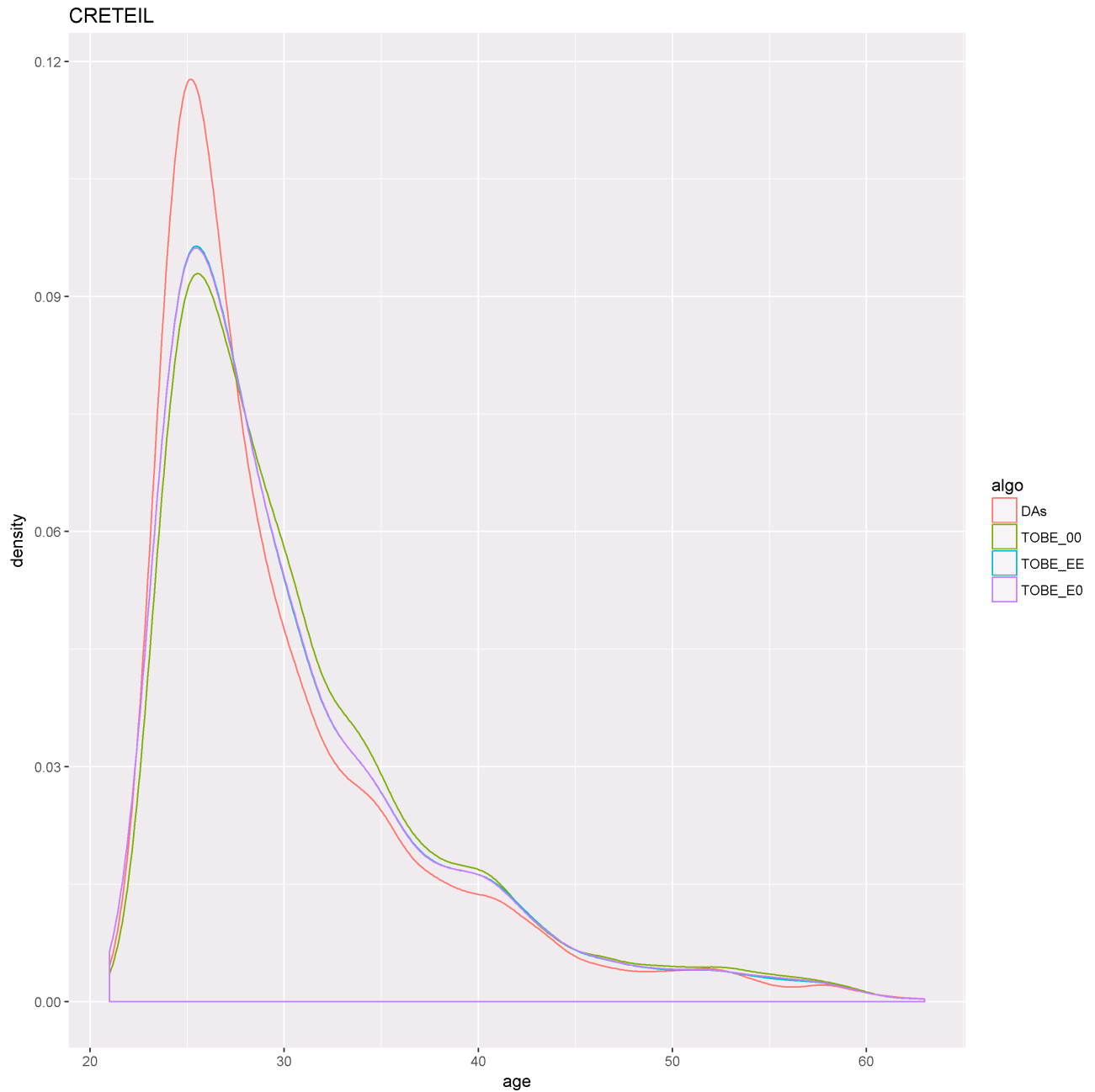
Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 4 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Figure 2.35 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF AMIENS



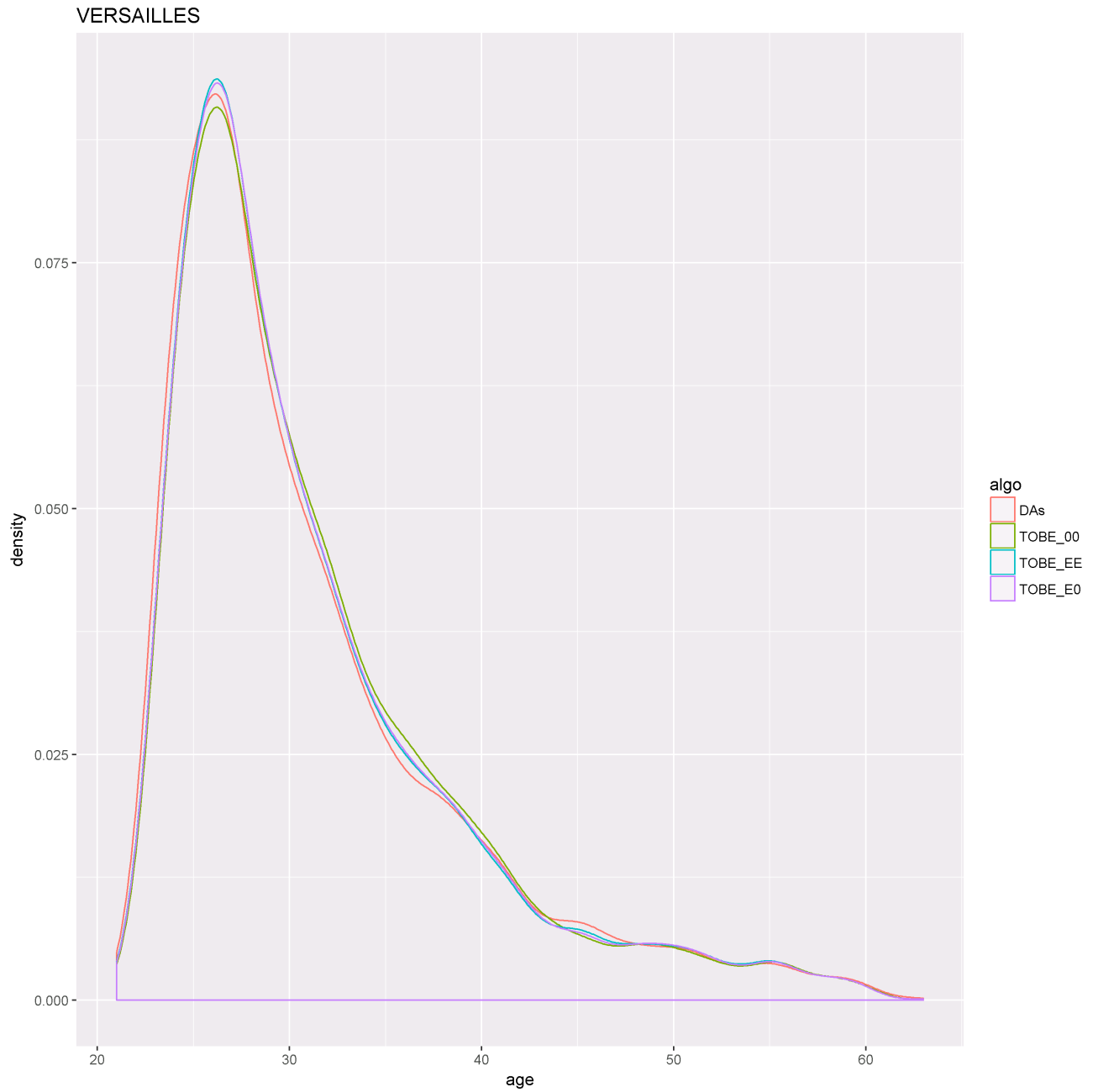
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.36 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF CRÉTÉIL



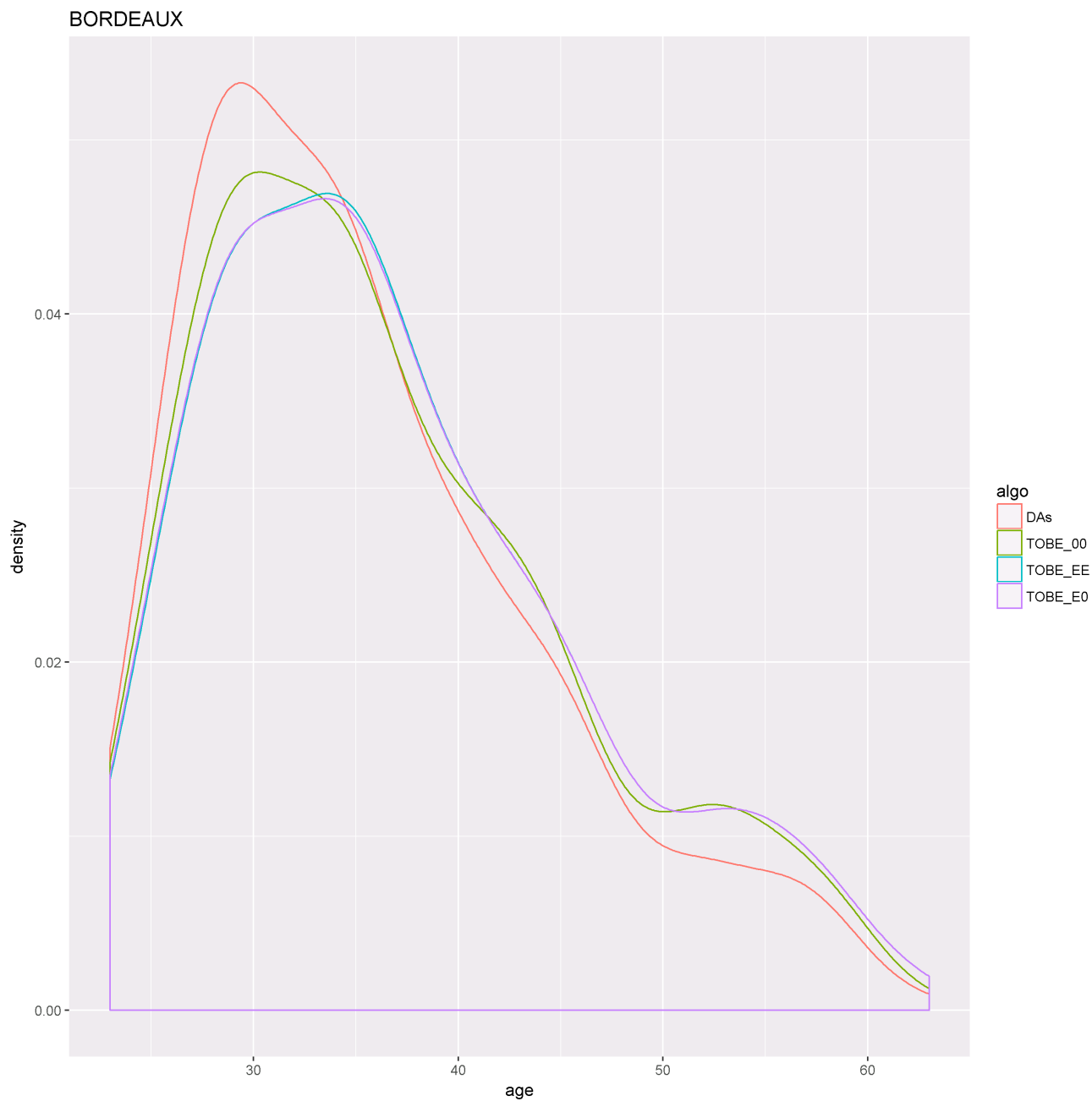
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.37 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF VERSAILLES



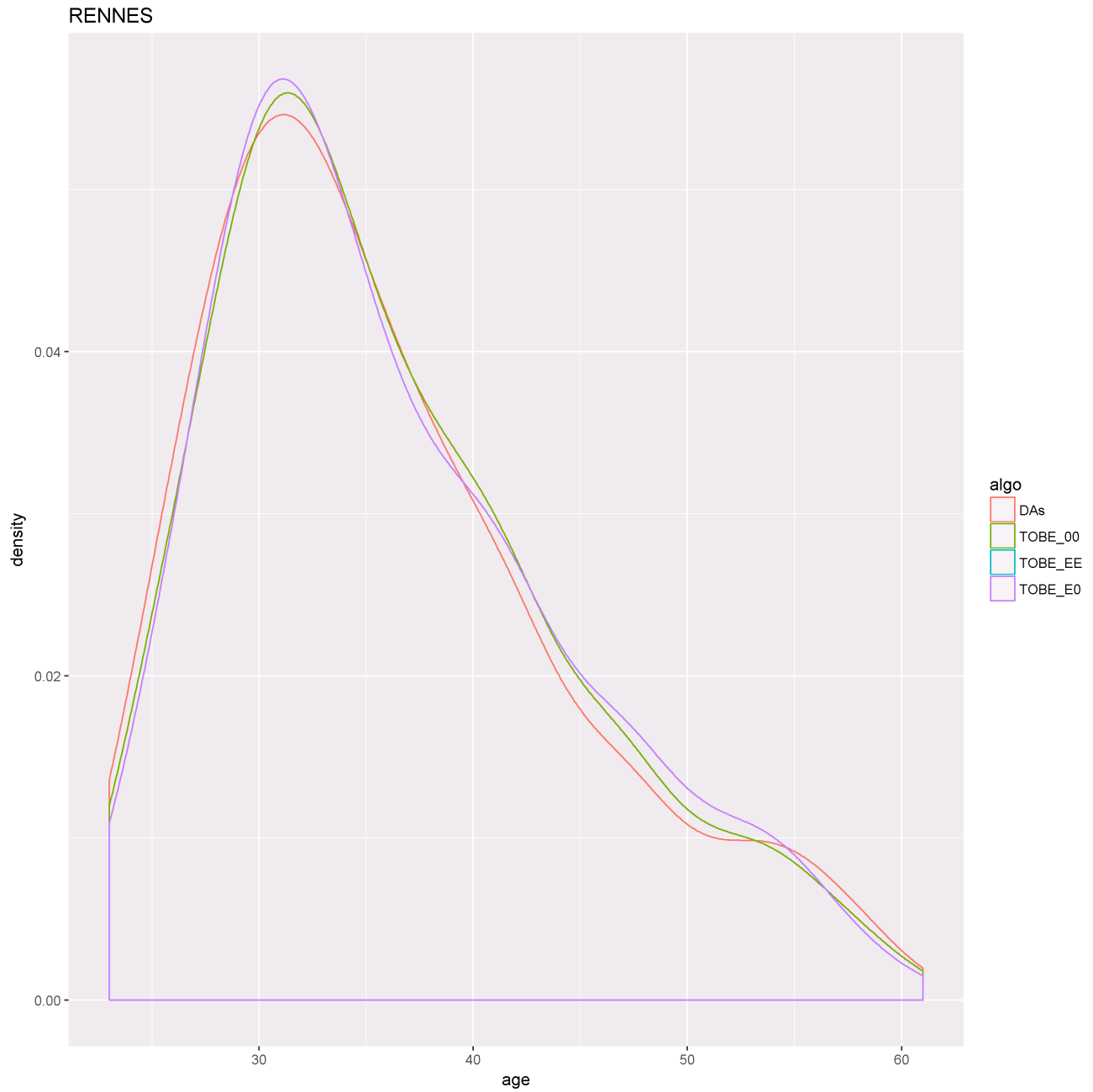
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.38 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF BORDEAUX



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

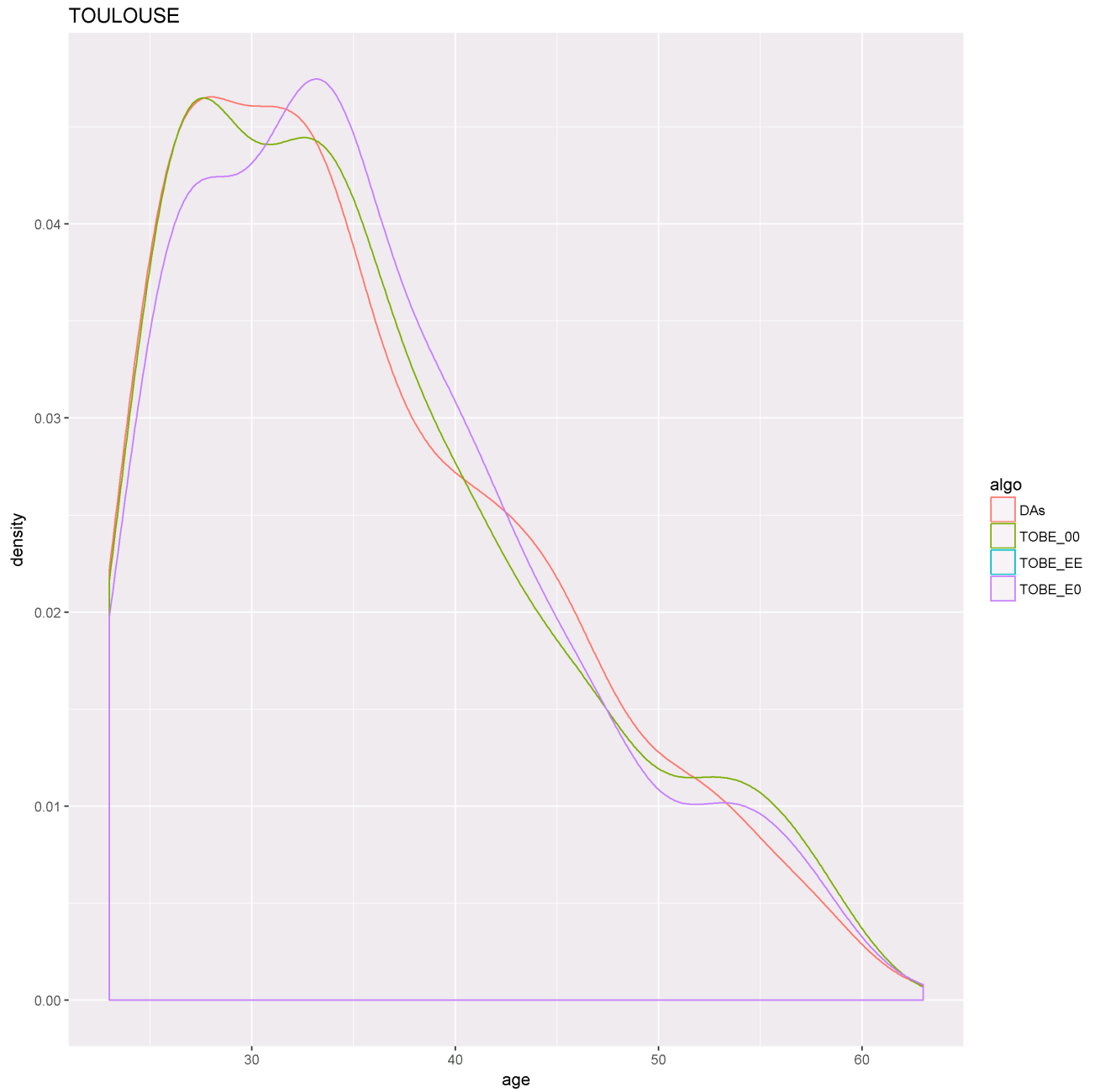
Figure 2.39 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF RENNES



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.



Figure 2.40 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF TOULOUSE



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

### 2.9.3.3 Results with 100% of initially non-assigned teachers improved in the experience ordering of TOBE

Table 2.43 – NUMBER OF TEACHERS OBTAINING A NEW ASSIGNMENT UNDER EACH ALGORITHM

	Nb	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Newly Tenured	7106	6486	0.09%	0.09%	0.08%
Tenured	12123	5494	-2.24%	3.02%	2.15%
Total	19229	11980	-0.98%	1.44%	1.03%

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.44 – NUMBER OF TEACHERS MOVING UNDER EACH ALGORITHM - PER REGION OF INITIAL ASSIGNMENT

	Current algorithm DA*	TOBE with all regions targeted		TOBE with no regions targeted		TOBE with a proportion of teachers targeted in deprived regions (option 2)	
	nb	nb	%	nb	%	nb	%
All regions	3991	3895	-2.41%	4183	4.81%	4135	3.61%
Unassigned	7976	7968	-0.10%	7969	-0.09%	7968	-0.10%
Paris	100	150	50.00%	161	61.00%	161	61.00%
Aix	106	151	42.45%	156	47.17%	156	47.17%
Besancon	37	51	37.84%	58	56.76%	58	56.76%
Bordeaux	47	62	31.91%	67	42.55%	67	42.55%
Caen	70	87	24.29%	89	27.14%	88	25.71%
Clermont	34	46	35.29%	48	41.18%	48	41.18%
Dijon	105	143	36.19%	146	39.05%	146	39.05%
Grenoble	111	168	51.35%	166	49.55%	165	48.65%
Lille	131	172	31.30%	185	41.22%	184	40.46%
Lyon	92	139	51.09%	141	53.26%	141	53.26%
Montpellier	69	115	66.67%	119	72.46%	119	72.46%
Nancy	67	103	53.73%	105	56.72%	105	56.72%
Poitiers	53	79	49.06%	83	56.60%	83	56.60%
Rennes	30	41	36.67%	42	40.00%	42	40.00%
Strasbourg	53	82	54.72%	84	58.49%	84	58.49%
Toulouse	73	119	63.01%	123	68.49%	123	68.49%
Nantes	85	124	45.88%	127	49.41%	128	50.59%
Orleans	208	202	-2.88%	200	-3.85%	203	-2.40%
Reims	98	91	-7.14%	91	-7.14%	91	-7.14%
Amiens	207	171	-17.39%	190	-8.21%	176	-14.98%
Rouen	93	122	31.18%	120	29.03%	120	29.03%
Limoges	39	40	2.56%	43	10.26%	43	10.26%
Nice	81	138	70.37%	143	76.54%	143	76.54%
Créteil	813	316	-61.13%	379	-53.38%	365	-55.10%
Versailles	779	519	-33.38%	565	-27.47%	545	-30.04%
Corse	5	10	100.00%	10	100.00%	10	100.00%
Reunion	63	89	41.27%	93	47.62%	93	47.62%
Martinique	33	52	57.58%	54	63.64%	54	63.64%
Guadeloupe	64	82	28.13%	87	35.94%	87	35.94%
Guyane	89	85	-4.49%	108	21.35%	108	21.35%
Mayotte	156	146	-6.41%	200	28.21%	199	27.56%

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 4 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A). The second column for each TOBE algorithm gives the percentage of improvement with respect to DA\*.

Table 2.45 – CUMULATIVE DISTRIBUTIONS OF RANKS FOR TENURED TEACHERS UNDER EACH ALGORITHM

	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
1	4219	4223	4454	4423
2	9861	10067	10134	10124
3	10745	10930	10973	10971
4	11234	11393	11415	11415
5	11482	11611	11619	11618
6	11645	11756	11760	11760
7	11764	11829	11826	11829
8	11883	11894	11903	11902
9	11961	11946	11953	11948
$\geq 10$	12066	12052	12052	12052
Unassigned	57	71	71	71

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.46 – CUMULATIVE DISTRIBUTIONS OF RANKS FOR NEWLY TENURED TEACHERS UNDER EACH ALGORITHM

	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
1	4044	3955	3889	3894
2	4804	4910	4854	4851
3	5095	5231	5178	5183
4	5285	5478	5433	5440
5	5429	5662	5616	5624
6	5560	5797	5752	5759
7	5662	5892	5854	5862
8	5803	6007	5967	5975
9	5916	6092	6066	6070
$\geq 10$	6491	6497	6498	6497
Unassigned	615	609	608	609

Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.47 – NUMBER OF TEACHERS ASSIGNED TO AN EXTENDED WISH

	Nb	Current Algorithm DA*	TOBE with all regions targeted	TOBE with no region targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Tenured	1560	639	554	571	568
Newly Tenured	7106	603	522	546	538

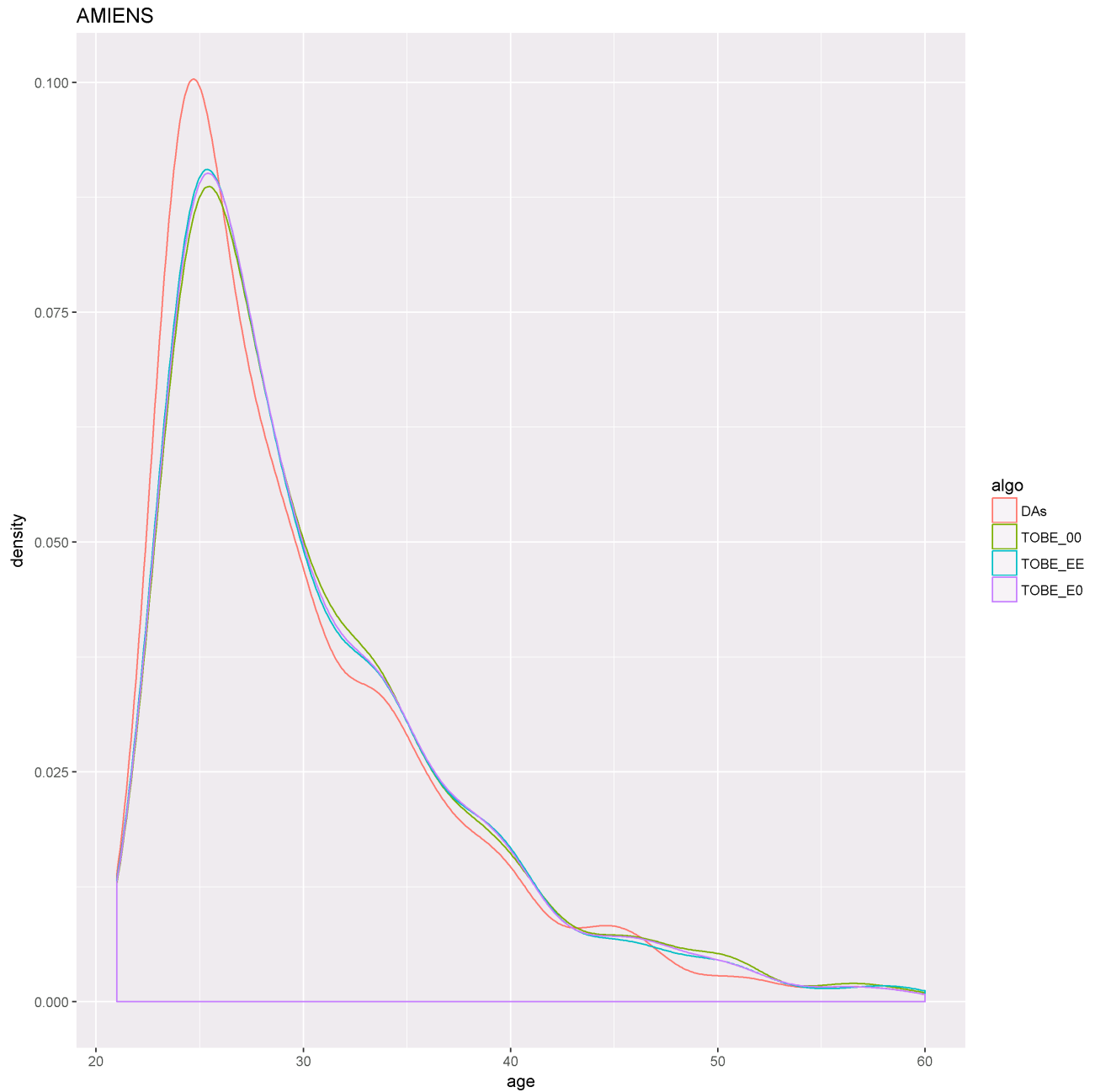
Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 5 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Table 2.48 – AVERAGE AGE OF ASSIGNED TEACHERS UNDER EACH ALGORITHM - PER REGION

	Current algorithm DA*	TOBE with all regions targeted	TOBE with no regions targeted	TOBE with a proportion of teachers targeted in deprived regions (option 2)
Unassigned	29.67	28.32	28.35	28.34
Paris	37.34	36.50	36.66	36.64
Aix	35.30	34.37	34.96	34.98
Besancon	32.95	32.49	33.18	33.20
Bordeaux	35.64	36.63	37.08	37.06
Caen	33.79	33.90	33.96	34.03
Clermont	34.33	34.11	34.53	34.53
Dijon	32.02	30.45	30.63	30.61
Grenoble	33.61	33.11	33.34	33.32
Lille	30.65	30.43	30.08	30.08
Lyon	33.14	32.71	32.68	32.67
Montpellier	36.83	36.84	37.28	37.28
Nancy	33.37	32.49	32.41	32.41
Poitiers	35.47	34.82	35.18	35.16
Rennes	35.98	36.23	36.35	36.35
Strasbourg	33.34	32.96	33.10	33.08
Toulouse	35.62	35.77	35.94	35.94
Nantes	34.76	34.42	34.82	34.73
Orleans	32.47	32.13	32.21	32.14
Reims	31.24	31.53	31.49	31.52
Amiens	29.87	30.57	30.40	30.43
Rouen	32.74	32.21	32.26	32.41
Limoges	34.84	36.46	35.70	35.70
Nice	35.84	35.14	35.63	35.63
Créteil	30.06	31.06	30.80	30.80
Versailles	31.08	31.26	31.16	31.18
Corse	40.53	36.97	37.20	37.20
Reunion	37.40	37.97	39.01	39.01
Martinique	39.72	39.42	38.87	38.87
Guadeloupe	39.23	39.00	39.36	39.32
Guyane	41.81	42.23	42.43	42.28
Mayotte	48.18	47.81	45.97	45.92

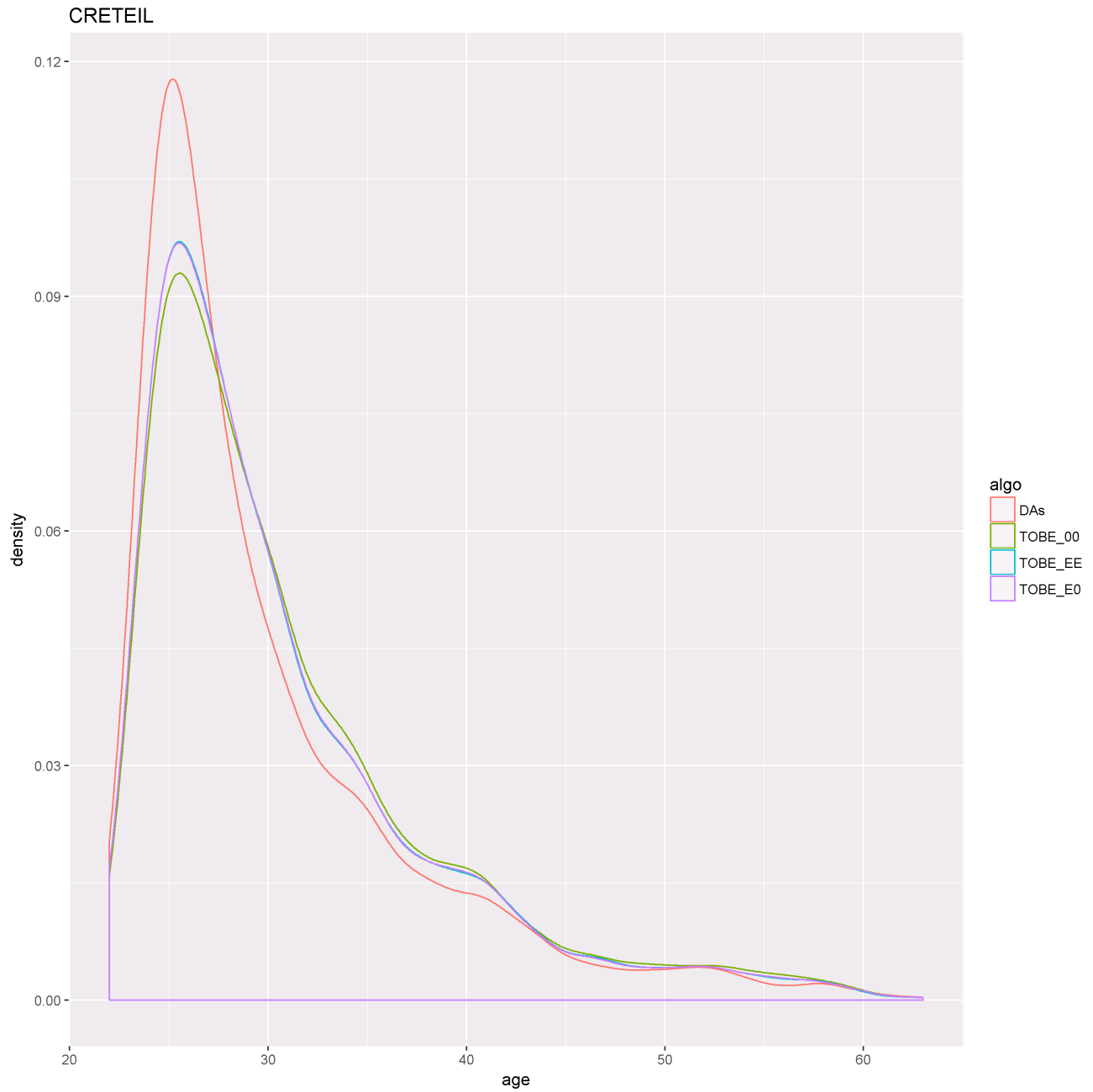
Source: Dataset on public secondary school teacher assignment to region in 2013. For results presented in column 4 of this table, we have chosen to select three regions as deprived: Créteil (C), Versailles (V) and Amiens (A).

Figure 2.41 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF AMIENS



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

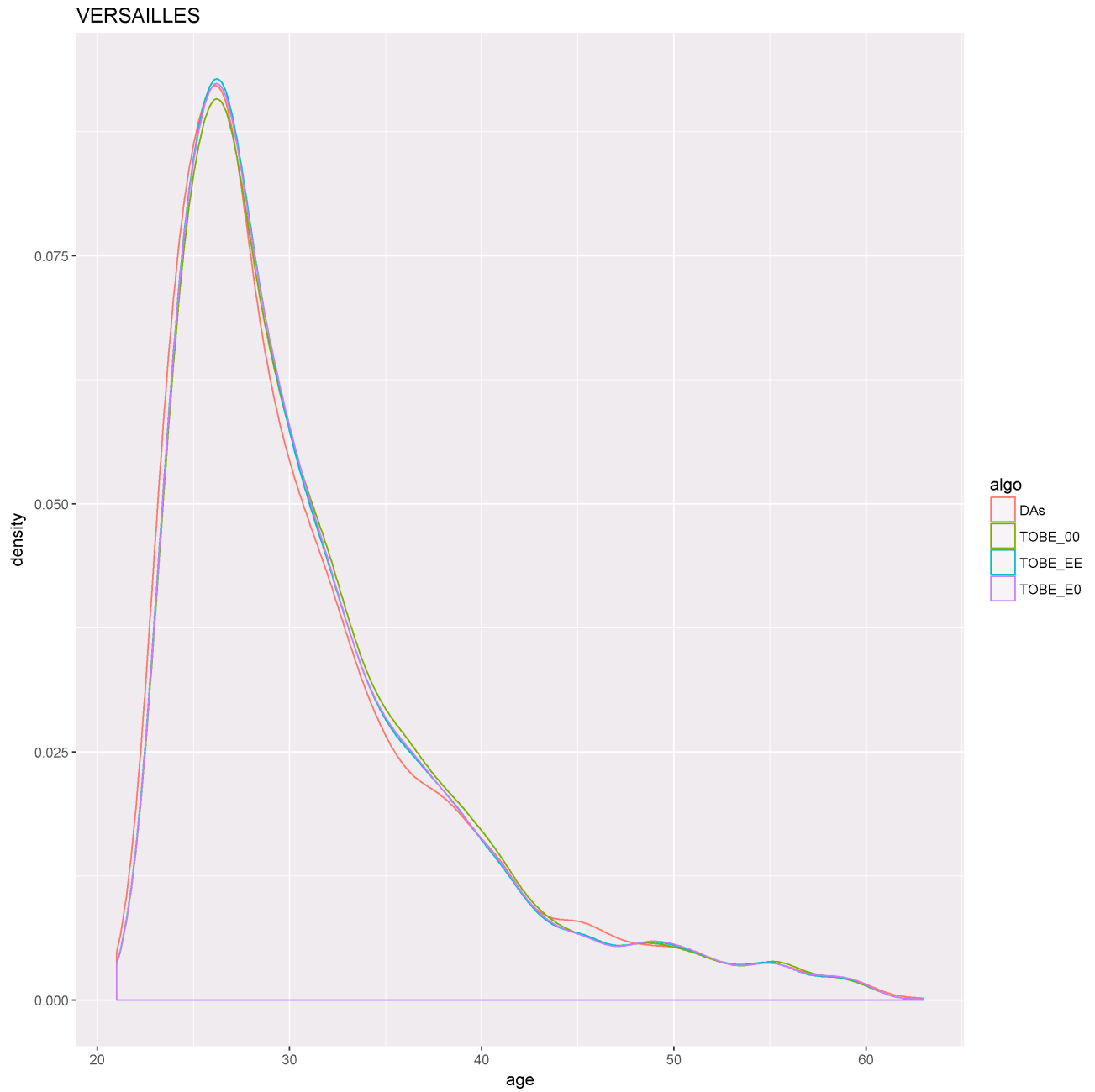
Figure 2.42 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF CRÉTÉIL



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

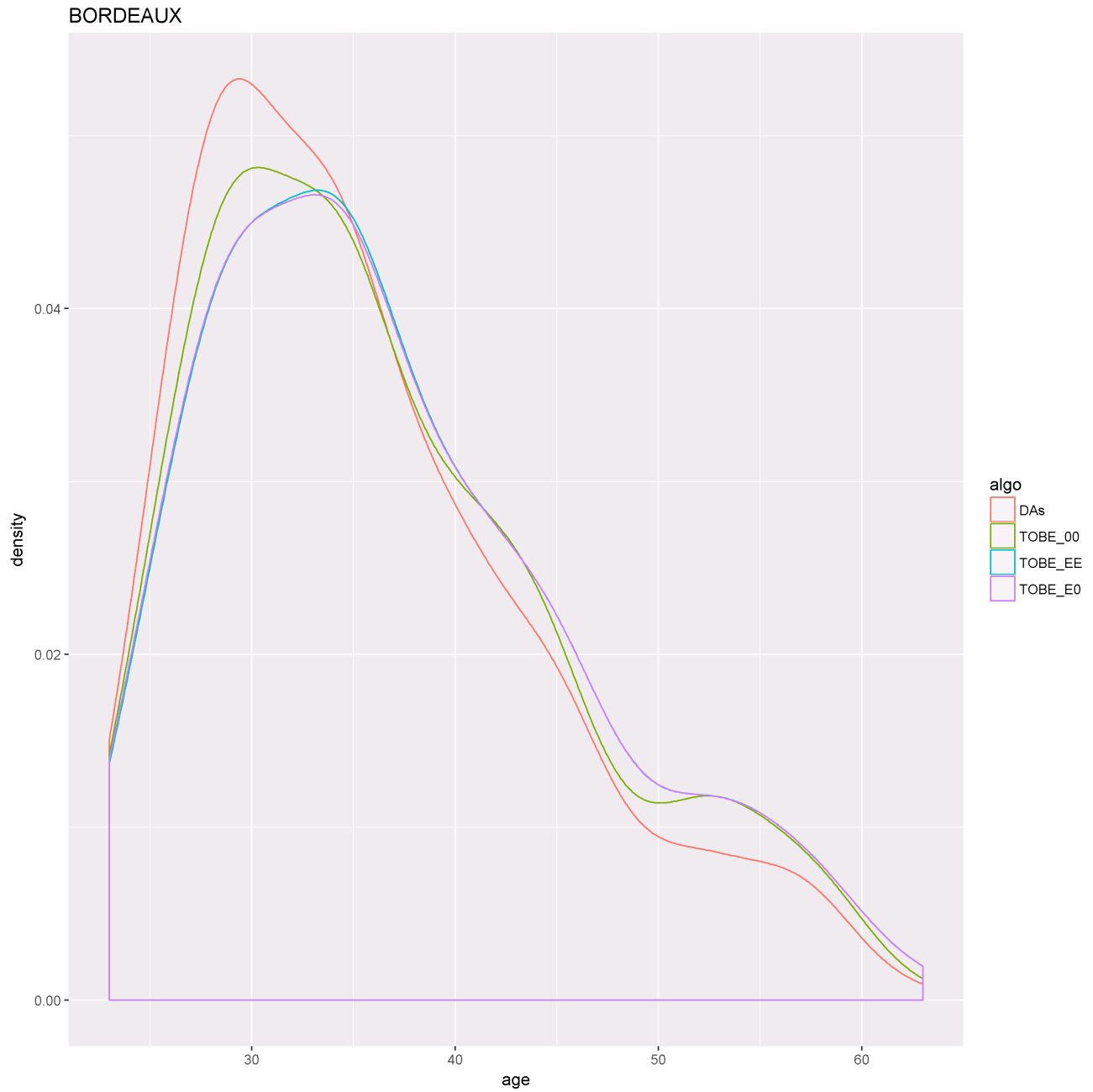


Figure 2.43 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF VERSAILLES



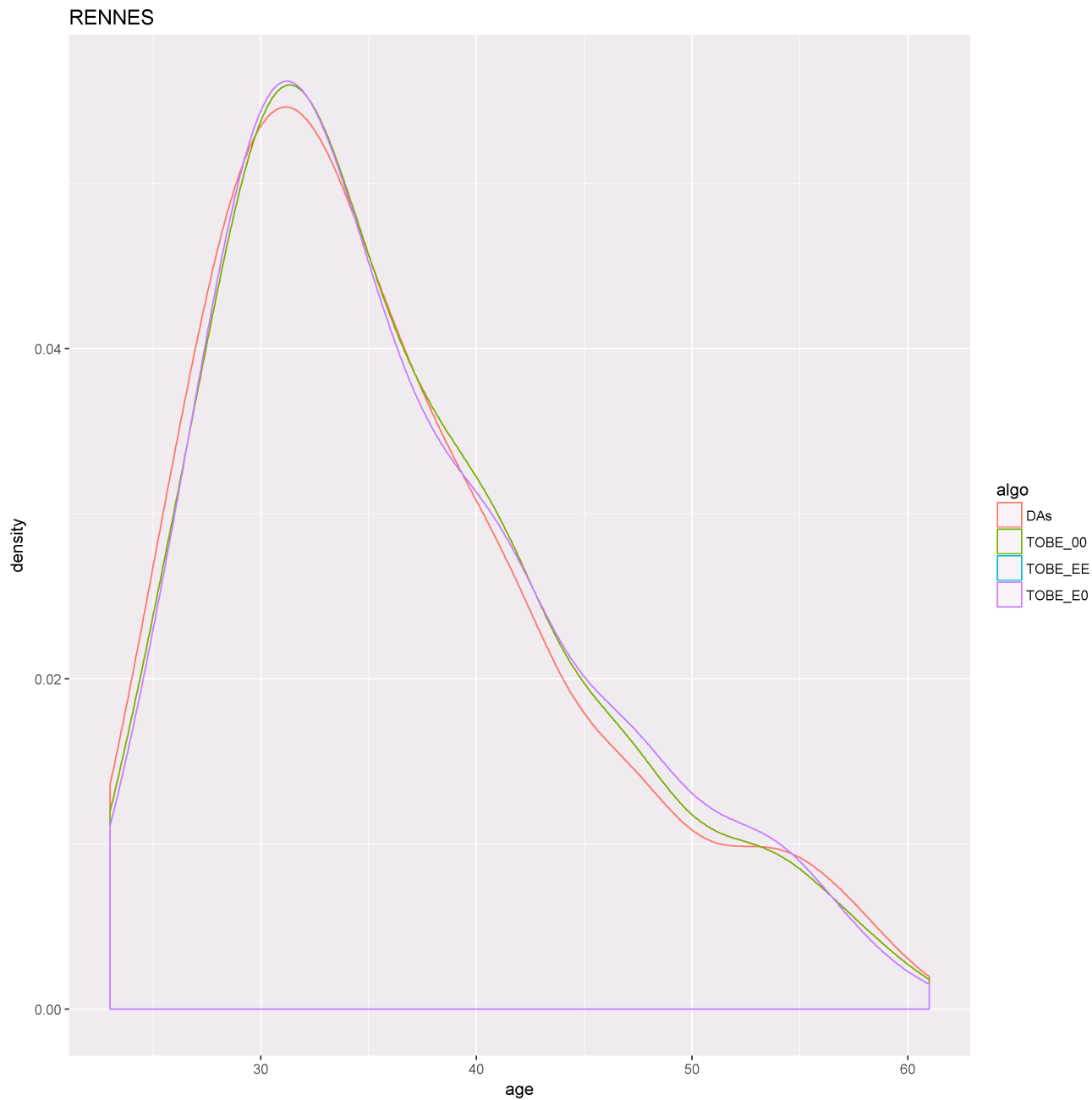
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.44 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF BORDEAUX



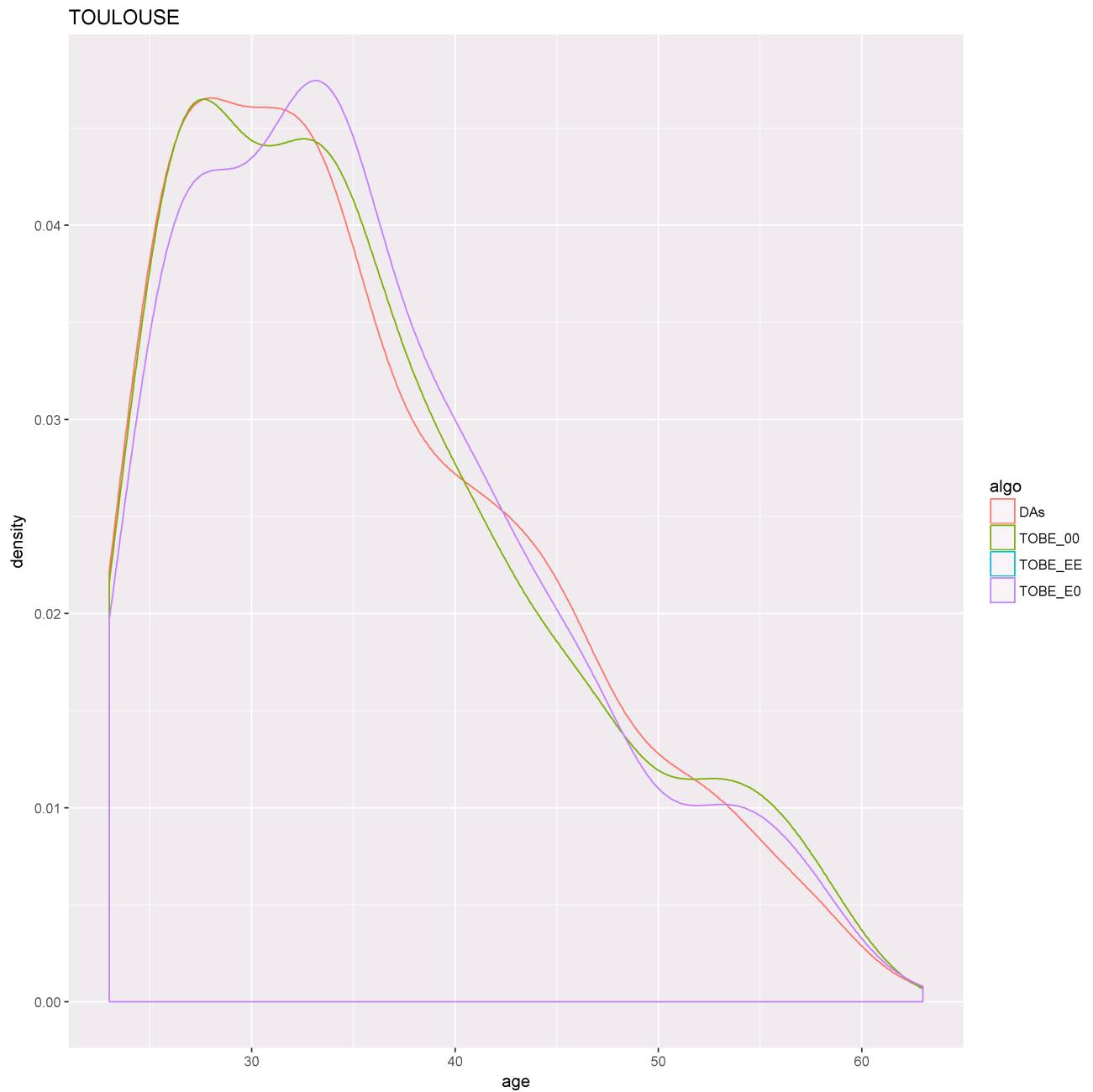
Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.45 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF RENNES



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.

Figure 2.46 – DISTRIBUTION OF THE AGE OF THE ASSIGNED TEACHERS UNDER EACH ALGORITHM - REGION OF TOULOUSE



Note: DAs is DA\*, TOBE\_00 is TOBE with all regions targeted, TOBE\_EE is TOBE with no region targeted and TOBE\_E0 is TOBE with a proportion of the teachers targeted in the deprived regions.



## Chapter 3

# Matching with Ownership <sup>1</sup>

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## Abstract

We consider a hybrid model at the intersection of the standard two-sided matching market as proposed by [Gale and Shapley \(1962\)](#) and a housing market as proposed by [Shapley and Scarf \(1974\)](#). Two sets of agents have to be matched in pairs to a common set of objects. Agents of one type have preferences that depend not only on the object they are matched to but also on the agent of the other type matched to this object. The crucial difference lies in the fact that the common side here is interpreted as an object and has no intrinsic preferences over the agents matched to it. In this context, stable matchings may fail to exist. However, we introduce a natural definition of ownership of the objects that determines which agent owns the object to which he is matched. In defining a stability notion whereby only owners can ask other agents to join them, we show that stable matchings exist for a very natural class of ownerships: ownership is given to agents of the same side. Even if this structure shares similarities with the classical two-sided matching framework, we present an important difference: stable matchings and Pareto-efficient matchings may be disjoint, thus implying that the core may be empty in this setting. To further investigate the link with the housing market literature, we introduce an initial allocation and define a core notion with respect to this initial allocation. We also show that, contrary to the standard setting, this core may be empty. However, we show that, in this housing market framework, there always exists a Pareto-efficient matching that is not blocked by any coalition of size two. In both settings, pairwise stability seems the only minimal requirement that one can ensure.

JEL Classification Numbers: C78, D47, D63.

Keywords: Three-sided matching, Ownership, Two-sided matching, Housing market, Core.

### 3.1 Introduction

Matching problems have received considerable attention over the past decade and have been used in several practical applications such as school choice (see [Abdulkadiroglu and Sonmez \(2003\)](#), for instance), kidney exchange ([Roth, Sonmez and Unver, 2004](#)) or public housing ([Shapley and Scarf, 1974](#)). In these problems, there are two main types of frameworks. The first, as introduced by [Gale and Shapley \(1962\)](#), is the two-sided market structure: two sets, men and women, have preferences over each other and have to be matched together. In this framework, an important question is whether there exists a stable matching, i.e., a matching such that there is no man and woman who would prefer to be matched together rather than with their current partner. One may also study other concepts such as Pareto-efficient matchings or core matchings, i.e., where no group of agents would like to rematch among each other. The second setting, introduced by [Shapley and Scarf \(1974\)](#), is the housing market structure where agents want to exchange their houses. It also has two sides (agents and houses), but only one has preferences. Moreover, each agent has an ownership right over his initial house. In this context, one may also define core matchings except that the deviating group of agents could only trade their initial houses and not the one they were assigned under a given matching.

However, one may think of other types of practical problems that cannot be described by a standard two-sided structure. For instance, in many public administrations, the salary scale is fixed by law, but one often has to match workers and managers to possible projects. Each project has to be assigned to a manager and to a certain number of workers. Even if managers have intrinsic preferences over the projects, the quality of and interest in working on a project also depend on the set of workers assigned to it. Workers also have similar preferences: an interesting project may not be valuable if a bad manager is assigned to it. This introduces complementarities into the preferences of the agents. Generally, once a manager is assigned to a project, he is free to choose his team of workers and to ask some to leave the project if needed. Hence, in this application, there is a natural *ownership structure* over the projects: the managers always own the project to which they are assigned. In some other contexts, ownership need not always be given to agents of the same type. In a school choice setting, heads of schools can have an important impact on the quality of their schools and thereby affect the school's desirability for parents. However, the head of the school also cares about the set of students who are assigned to her school. In some schools, the head is



responsible for deciding whether to accept or reject students. However, in other schools, such as charter schools in the U.S., parents are highly involved in the school's managerial decisions, meaning that recruitment or firing decisions concerning a badly managed school could be greatly influenced by parental satisfaction. We could capture this feature with an ownership structure over the schools.

Once again, such applications cannot be studied using the standard models in the literature. In this work, we introduce a new general framework: a hybrid model between a two-sided matching market and a housing market. There are two sides of individuals, for instance workers and managers, who have to be matched together and be assigned to objects, for instance projects. For each possible match, we introduce an *ownership structure* that defines for each object which of its matched individuals *owns* it. In most contexts, such ownership has a natural structure: managers are responsible for their teams and hence can decide whether to recruit a worker or to replace him with another. This structure corresponds to a *one-sided* ownership structure, where agents of one type always own the object (e.g., the managers always own the projects). In some other problems, this may not be the case, as in the aforementioned “heads of schools and students example”. Indeed, as we highlighted, some schools give more decision power to the head of the school, while in others, parents have greater influence on the managerial decisions. Our ownership structure allows for such flexibility. We can then define a natural notion of pairwise stability *with respect to the ownership structure*. In the standard notion of stability, every pair of agents who prefer to be matched together rather than with their assigned partner could decide to break their current assignment. With the introduction of the ownership, we impose that only owners can do so. We show that such matchings may fail to exist for general ownership structures but that they do exist for natural ones: for instance, one-sided ownerships. If one considers the pairwise stability as a minimum requirement for the sustainability of a system, then our result implies that one should mostly observe one-sided structures.

In the standard two-sided problem, it is well known that pairwise stable matchings are equivalent to core matchings. In contrast, we show that this is not the case for our notion of stable matchings, even for one-sided ownerships. We exhibit an instance in which the set of stable matchings and the set of Pareto-efficient matchings are disjoint.

Finally, we consider a setting closer to the housing market problem where certain agents initially own all the houses. [Shapley and Scarf \(1974\)](#) and [Roth and Postlewaite \(1977\)](#) defined a different core notion where a group of agents could only block a given matching by exchanging their initially

owned houses and not the one they are assigned to ex post. We show that in our environment, this notion of core could also be empty. However, we show that there always exists a Pareto-efficient matching that is not blocked by any coalition of size two.

The three-sided matching structure is not new in the literature. It was first introduced by [Knuth \(1976\)](#) with the matching of men, women and dogs. In such problems, all the sides have preferences over the pairs of the two other sides, and matchings are defined by a collection of triplets. Later, [Alkan \(1988\)](#) provided an example to prove the non-existence of stable matchings in such an environment. [Biró and McDermid \(2010\)](#) proved non-existence in the restricted set of cyclic preferences and proved the NP-completeness of the problem of deciding whether a stable matching exists in such an environment.<sup>2</sup> In the above impossibility results, all the sides need to have preferences over the pairs of the other sides since all the counterexamples need all the sides to be able to break their match to form a blocking triplet. In our model, however, projects cannot “unilaterally deviate” from their managers or workers; only agents can decide to leave.<sup>3</sup>

Another related literature addresses the formation of coalitions. Our model shares some features with the hedonic coalition literature initiated by [Dreze and Greenberg \(1980\)](#) and continued by [Bogomolnaia and Jackson \(2002\)](#). Works on partitioning games initiated by [Kaneko and Wooders \(1982\)](#) (and continued by [Kaneko and Wooders \(1986\)](#) or [Hammond, Kaneko and Wooders \(1989\)](#)) are also related to this literature. They ex ante restrict the possible admissible coalitions of an assignment game and ask whether core assignments exist. [Pápai \(2004\)](#) also studies a model where agents have hedonic preferences over coalitions but the set of admissible coalitions is restricted. Blocking coalitions can only be admissible ones, and she identifies a necessary and sufficient condition on the set of admissible coalitions to ensure the existence and uniqueness of core coalitions. Once again, objects cannot unilaterally deviate and do not have preferences over agents which is an important feature of our model and cannot be captured in her model. However, our ownership

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<sup>2</sup>[Danilov \(2003\)](#) proved the existence of stable triplet matchings in an environment where men only care about women and vice versa and dogs have preferences over couples of men and women. Later, [Eriksson, Sjöstrand and Strimling \(2006\)](#) proved a non-existence result in an environment with lexicographic preferences. Another open question arose in an environment with cyclic preferences where men care only about women, women only about dogs and dogs only about men. [Boros et al. \(2004\)](#) proved existence in environments with fewer than three agents, and [Biró and McDermid \(2010\)](#) proved general non-existence for more than three agents.

<sup>3</sup>Other works have focused on a multi-sided generalization of the [Shapley and Shubik \(1972\)](#) setting. In the latter paper, buyers want to buy a house from owners. Owners do not have intrinsic preferences over the identity of who is buying the house but care only about the monetary transfer they receive. [Quint \(1991\)](#) was the first to introduce an  $m$ -sided generalization of this setting. He gave a sufficient condition on the surplus generated by  $m$ -tuples to ensure the non-emptiness of the core in such a setting. Thereafter, several papers ([Marchi and Oviedo, 1997](#); [Stuart Jr, 1997](#); [Sherstyuk, 1999](#)) proposed other sufficient conditions. In contrast to those works, our model lacks transfers.

structure approach shares a similar idea of restricting the possible blocking coalitions.<sup>4</sup>

### 3.2 The Model

We consider a problem that consists of matching two finite sets  $M$  and  $W$  of individuals that we will call *men* and *women* to a set  $H$  of objects, which we will call *houses*.<sup>5</sup> For simplicity, we assume that the sets  $M$ ,  $W$  and  $H$  have the same cardinality  $n$ . Agents and objects will be ordered by indices:  $m_i, w_j, h_k$ , with  $i, j, k \in \{1, \dots, n\}$ , and a generic man, woman and house will be denoted  $m, w$  and  $h$ , respectively. For the ease of notation, we let  $I \in \{M, W\}$  and  $J \in \{M, W\} \setminus I$ , and a generic agent from these sets will be denoted  $i$  and  $j$  and called an  $I$ -agent and a  $J$ -agent, respectively. For simplicity, we assume a one-to-one setting such that each house  $h$  can be matched to at most one man and one woman. Objects are important in our model. Managers and workers can only work together if they are assigned to a project. A head of a school and a set of students have to interact in a school. In the following definitions, we always impose that agents have to be matched to a house, meaning that one cannot match a man and a woman together without assigning them to a house. If we restrict ourselves to the problem of matching men to houses (we will call it the “ $M$ -market”), we are back to a standard assignment problem (and similarly we will call the same problem for women a “ $W$ -market”). An  $M$ -matching is an injective map  $\mu_M : H \mapsto M \cup \{\emptyset\}$  that matches houses to men or leaves them unassigned. With a slight abuse of notation, we will write  $\mu_M(m) = h$  if  $m = \mu_M(h)$ . If  $m$  is not matched to any house, we use the standard notation  $\mu_M(m) = \emptyset$ . A  $W$ -matching  $\mu_W$  is defined analogously. The sets of such matchings will be denoted  $\mathcal{M}_M$  and  $\mathcal{M}_W$ , respectively. A **double matching**  $\mu$  is the combination of an  $M$ -matching and a  $W$ -matching. We sometimes refer to this as simply a “matching”. Thus, it can be seen as a function from  $H$  to  $(M \cup \{\emptyset\}) \times (W \cup \{\emptyset\})$ , where the projection on  $M$  (resp.  $W$ ) is an  $M$ -matching (resp.  $W$ -matching). Once again, to ease the notation, we sometimes see  $\mu(h)$  as a set and write  $a \in \mu(h)$  if agent  $a \in M \cup W$  is matched with  $h$  under  $\mu$ . Thus far, if preferences over houses were independent

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<sup>4</sup>One may also wonder if this could be interpreted as a many-to-many matching problem as studied in [Echenique and Oviedo \(2006\)](#) or [Hatfield and Kominers \(2009\)](#). In a one-to-one setting of their model, if one assumes complete indifference in the preferences of some agents (the objects in our model), then the primitives are equivalent. However, we do not take the same approach. The ownership structure, which is natural in our environment with objects, is a key feature of our model and does not appear in theirs. Moreover, assuming complete indifference is not neutral in their framework since it undermines most of the positive results.

<sup>5</sup>To emphasize the intersection with [Gale and Shapley \(1962\)](#) and [Shapley and Scarf \(1974\)](#), we chose to use the same terminology.

of who occupied them, the formulation would be equivalent to two independent assignment markets that could be studied separately. However, take the example of managers/workers and projects as presented in the introduction. Preferences over projects may depend on the project itself but also on the agents matched to it. For instance, some worker has specific skills relevant for a given project that make any manager prefer him to the other workers only when the worker is assigned to this specific project.<sup>6</sup> Thus, the preferences of an  $I$ -agent,  $i$ , could be seen as a linear order over  $((J \cup \{\emptyset\}) \times ((H \cup \{\emptyset\})).$  Allowing the preferences to depend simultaneously on the partner and the house allows us to capture some house-specific features and some possible complementarities between the house and the agents matched to it. Note that, thus far, one can be matched to a house  $h$  with no partner, i.e.,  $(\emptyset, h)$  or not be matched at all, i.e.,  $(\emptyset, \emptyset)$ . To ease notation, we will refer to this last case as simply  $\emptyset$ .<sup>7</sup> In the following, we will assume that an agent prefers to be matched alone in a house to not being assigned at all:

**Assumption 1.**  $\forall a \in A, \forall h \in H : (\emptyset, h) \succ_a \emptyset$

As a first natural requirement, a double matching  $\mu$  is said to be **Pareto-efficient** if there is no other double matching  $\mu'$  under which all men and women are weakly better off, and some strictly so. We also introduce a strong notion of stability to capture pairwise deviations. A double matching  $\mu = (\mu_M, \mu_W)$  is said to be **strongly stable** if there is no man or women who would prefer to be unmatched and there is no triplet  $(m, h, w)$  s.t.  $(m, w) \neq (\emptyset, \emptyset)$  and

1. Either:
  - (a)  $\mu_M(h) = m$  and  $m \neq \emptyset$ .
  - (b)  $\mu_W(h) = w$  and  $w \neq \emptyset$ .
  - (c)  $\mu_M(h) = \mu_W(h) = \emptyset$ .
2. if  $m \neq \emptyset, (w, h) \succ_m \mu(m)$ , if  $w \neq \emptyset, (m, h) \succ_w \mu(w)$ .

The interpretation is intuitive: a man and woman prefer to be together either in the house that was assigned to one of the two or in an empty house. If, for instance, woman  $w$  was assigned house  $h$

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<sup>6</sup>One may argue that in the workers/managers case, one could use transfers to correctly account for such complementarities. However, as emphasized in the introduction, in some contexts, this is not possible. Many public sectors, for instance, involve a standardized salary contract that does not depend on such details.

<sup>7</sup>As mentioned above, we assume that one cannot match a man and woman together without assigning them to a house. Therefore, we assume here that agents do not consider pairs of the form  $(m, \emptyset)$  or  $(w, \emptyset)$ .

with man  $m'$  under  $\mu$ , then she could make  $m'$  leave and ask  $m$  to join her in  $h$ . This is a quite strong requirement since we assume that an  $I$ -agent can break his relationship with the  $J$ -agent matched with him while keeping the house. This raises the point of the *ownership* of the house for a given matching. Indeed, since both agents are matched to a given house, it is not always possible that one could freely use it. If a manager is assigned the management of a project, one can imagine that he could freely choose his team and could invite new workers while asking others to leave. However, a worker cannot ask his manager to leave while asking another to join. Hence, there is a sense in which ownership restricts the admissible blocking pairs. As emphasized in the introduction, ownership structures do not always give ownership to agents of only one side, and one could consider more general structures. As opposed to [Shapley and Scarf \(1974\)](#), the ownership structure does not come from an initial allocation. It defines, for each house, its owners for each possible pairs of agents assigned to it. We postpone the [Shapley and Scarf \(1974\)](#) approach to [Section 3.4](#). An **ownership structure** is a function  $\theta : (M \cup \{\emptyset\}) \times (W \cup \{\emptyset\}) \times H \mapsto M \cup W$  that gives for each triplet  $(m, w, h) \in M \times W \times H$  the *owner*  $\theta(m, w, h) \in \{m, w\}$  of  $h$ . We impose that if there is only one agent, he is necessarily the owner. For instance, if the triplet is  $(h, \emptyset, w)$ , then the owner is necessarily  $w$ , i.e.,  $\theta(h, \emptyset, w) = w$ . Denote by  $\Theta$  the set of all possible ownership structures. Fix an ownership structure  $\theta$ ; we say that a double matching  $\mu = (\mu_M, \mu_W)$  is **stable with respect to  $\theta$**  if there is no man or woman who would prefer to be unmatched and there is no triplet  $(m, w, h)$  s.t.  $(m, w) \neq (\emptyset, \emptyset)$  and

1. Either:

(a)  $h = \mu_M(m)$  and  $\theta(m, \mu_W(h), h) = m$ .

(b)  $h = \mu_W(w)$  and  $\theta(\mu_M(h), w, h) = w$ .

(c)  $\mu_M(h) = \mu_W(h) = \emptyset$ .

2. if  $m \neq \emptyset$ ,  $(w, h) \succ_m \mu(m)$  and if  $w \neq \emptyset$ ,  $(m, h) \succ_w \mu(w)$ .

The interpretation is simple. Conditions (a) and (b) impose that if one of the blocking agents was assigned to house  $h$ , then he has to be the owner. Condition (c) considers the case of a couple moving to an empty house.

### 3.3 Main Results

A simple procedure could be provided to obtain a Pareto-efficient matching. Indeed, define a generalization of the well-known Serial Dictatorship mechanism: first order men. The first-ranked man, call him  $m_1$ , chooses his most preferred match among all the pairs  $W \times H$ ; call this pair  $(w_1, h_1)$ . Then, delete all pairs of the form  $(w_1, h)$  for some  $h$  and  $(w, h_1)$  for some  $w$  and let the second-ranked man  $m_2$  choose his preferred match among the remaining pairs. Continue this process until everyone is matched, and call  $\mu_{SD}$  the doubled matching induced by this procedure.<sup>8</sup> Clearly, this procedure leads to a Pareto-efficient matching. However, the matching proposed by the above generalization of the Serial Dictatorship mechanism completely omits the preferences of the women, and it is clear that the double matching  $\mu_{SD}$  may not be strongly stable.

However, the next proposition shows that a strongly stable matching does not always exist:

**Proposition 3.3.1.** *Strongly stable matchings may not exist, and there exist some ownership structures such that there is no stable matching with respect to them.*

*Proof.* Assume that there are two men, women and houses. Preferences are given as follows:

$m_1$	$m_2$	$w_1$	$w_2$
$(w_2, h_2)$	$(w_1, h_2)$	$(m_1, h_1)$	$(m_2, h_1)$
$(w_1, h_1)$	$(w_2, h_1)$	$(m_2, h_2)$	$(m_1, h_2)$
$(w_2, h_1)$	$(w_1, h_1)$	$(m_1, h_2)$	$(m_2, h_2)$
$(w_1, h_2)$	$(w_2, h_2)$	$(m_2, h_1)$	$(m_1, h_1)$
$(\emptyset, h_1)$	$(\emptyset, h_1)$	$(\emptyset, h_1)$	$(\emptyset, h_1)$
$(\emptyset, h_2)$	$(\emptyset, h_2)$	$(\emptyset, h_2)$	$(\emptyset, h_2)$

For the first part of the proposition, we will show that every possible matching is not strongly stable. First note that under a strongly stable matching, every agent has to be matched to a house. Indeed, assume that, for instance,  $w_1$  is unmatched. Since all agents prefer to be assigned to a house, even with no partner, then one of the two men, say  $m_1$ , has to be matched alone in a house, say  $h_1$ . However, preferences are such that he would always prefer to be in  $h_1$  with  $w_1$  than alone, and thus,

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<sup>8</sup>If one cares about individual rationality for the women, then one could do the following: if man  $m$  is choosing, then do not allow him to choose a pair  $(w, h)$  if woman  $w$  prefers to remain unmatched to being matched with  $(m, h)$ .

$(m_1, h_1, w_1)$  would form a blocking triplet. A similar argument can be used for all other possibilities for unassigned agents.

Next, we will show that every possible matching of agents to the houses has a blocking triplet:

- If we match  $m_1$  to  $h_1$  with  $w_1$  and  $m_2$  to  $h_2$  with  $w_2$ , then the latter would prefer to invite  $m_1$ , who prefers it to  $(w_1, h_1)$ .
- If we match  $m_2$  to  $h_1$  with  $w_1$  and  $m_1$  to  $h_2$  with  $w_2$ , then  $m_2$  would prefer to be in  $h_1$  with  $w_2$  who also prefers it to  $(w_1, h_2)$ .
- If we match  $m_2$  to  $h_1$  with  $w_2$  and  $m_1$  to  $h_2$  with  $w_1$ , then  $w_1$  would prefer to be in  $h_2$  with  $m_2$  who also prefers it to  $(w_2, h_1)$ .
- Finally, if we match  $m_1$  to  $h_1$  with  $w_2$  and  $m_2$  to  $h_2$  with  $w_1$ , then  $m_1$  would prefer to be in  $h_1$  with  $w_1$  who also prefers it to  $(w_2, h_2)$ .

For the second part of the proof, one can easily construct an ownership structure that would allow the above deviations. For instance, always give the ownership to men in house  $h_1$  and to women in house  $h_2$ . Thus, stable matching with respect to it would not exist.  $\square$

Naturally, we now face another question: does there exist a class of ownership structures under which there always exists a stable matching with respect to it? A first natural structure arising is the following: define a partition of the set of houses into two sets,  $H_w$  and  $H_m$ . Then, always give the ownership of a house in  $H_w$  to women and the one of a house in  $H_m$  to men. Unfortunately, the example given in the proof of Proposition 3.3.1 could be used to prove that this structure may not admit any matching stable with respect to it. Indeed, in this example, as highlighted in the proof, one can always give house  $h_2$  to women and house  $h_1$  to men and all deviations in the proof would be allowed by such an ownership structure. For an ownership structure  $\theta$ , call it a ***I*-ownership structure** for  $I \in \{H, W\}$  if there exists a matching  $\mu_I$  s.t. i)  $\forall h \in H, \mu_I(h) \in I$  and ii)  $\forall h \in H$  and  $\forall \mu_J \in \mathcal{M}_J: \theta(\mu_I(h), \mu_J(h), h) = \mu_I(h)$ . For instance, a *W*-ownership structure will always give the ownership of the houses to women for a certain allocation of women to houses. Among *I*-ownership structures, we will distinguish an important one: call  $\theta$  **the strong *I*-ownership structure** if for all agents  $i \in I, j \in J \cup \{\emptyset\}$  and houses  $h \in H$ , ownership is always given to agent  $i$ . The next proposition shows that existence is restored for *I*-ownerships structures:

**Proposition 3.3.2.** *Fix  $I \in \{M, W\}$ ; if  $\theta$  is an  $I$ -ownership structure, then there always exists a stable matching with respect to  $\theta$ .*

*Proof.* W.l.o.g., assume that  $\theta$  is an  $M$ -ownership structure. By definition, we know that there exists  $\mu_M$  such that men always obtain the ownership of their house under  $\mu_M$  irrespective of how we match women to the houses. First, match men to houses according to  $\mu_M$ . By point i) of the definition of an  $M$ -ownership, we know that all houses will be assigned to a man such that all men will be matched to a house. Thus, without loss, assume that  $\mu_M(h_k) = m_k$  for  $k = 1, \dots, n$ . Let  $p_k$  be the pair  $(h_k, m_k)$ . Once man  $m_k$  is matched to house  $h_k$ , he has a well-defined preference ordering over women. Let  $\succ_{p_k}$  be the ordering over women s.t.  $w \succ_{p_k} w' \Leftrightarrow (w, h_k) \succ_{m_k} (w', h_k)$ . Now, each woman  $w$  has a well-defined preference ordering over pairs of the form  $(h_k, m_k)$  for  $k = 1, \dots, n$  that is the restriction of her preferences  $\succ_w$  over  $H \times M$  to the set  $P := \{(h_k, m_k) : k = 1, \dots, n\} \subset H \times M$ , with a slight abuse of notation, write  $\succ_w$  for these preferences. Hence, one now has to match pairs of  $P$  to women of  $W$  where a pair  $p$  has preferences  $\succ_p$ , as defined above, over women in  $W$ , and a woman  $w$  has preferences  $\succ_w$  over pairs of  $P$ . We are then back to a standard two-sided matching problem as defined by [Gale and Shapley \(1962\)](#). Thus, we know that in this setting, there exists a matching  $\tilde{\mu}$  of the pairs in  $P$  to the women in  $W$  that is stable. There is no pair  $p$  and woman  $w$  who are not matched to each other under  $\tilde{\mu}$  but would strictly prefer to be so matched. Define  $\mu$  to be the double matching where men are matched according to  $\mu_M$  and  $\mu_W$  is defined as follows for women:  $\forall h_k, k = 1, \dots, n$  let  $\mu_W(h_k) = a \in (W \cup \{\emptyset\})$  iff  $\tilde{\mu}((h_k, m_k)) = a$ . First note that, by the definition of a one-sided ownership structure, all houses are occupied by a man, we have the same number of houses and men and that, by Assumption 1, all men find their matching acceptable even if they are alone in their assigned house. Thus, under  $\mu$ , there is no empty house. Now assume that  $\mu$  is not stable with respect to  $\theta_M$ . If  $\mu$  is not stable with respect to  $\theta$ , then there is a triplet  $(m, w, h)$  s.t.  $m$  and  $w$  both prefer to be matched together in  $h$  to their matching under  $\mu$ . There are three possible cases: (a)  $h$  was assigned to  $m$  under  $\mu$  and he was the owner, (b)  $h$  was assigned to  $w$  under  $\mu$  and she was the owner, and (c)  $h$  was not occupied under  $\mu$ . Since  $\theta$  is an  $M$ -ownership, case (b) is not possible. As mentioned above, there are no empty houses under  $\mu$ , and hence, case (c) is also ruled out. Therefore, it must be the case that  $\mu_M(h) = m$  and  $m$  was the owner of  $h$ . However, this would mean that, for the pair  $p := (h, m)$ ,  $w \succ_p \tilde{\mu}(p)$  and for woman  $w$ ,  $p \succ_w \tilde{\mu}(w)$ , contradicting the stability of  $\tilde{\mu}$  for the two-sided market formed by pairs in  $P$  and women in  $W$ .  $\square$



The one-sided structure of such ownerships seems an important property to ensure stability.<sup>9</sup> In light of the example in Proposition 3.3.1, one may wonder whether there are non one-sided ownership structures that would always allow the existence of stable matchings. This happens to be the case. Indeed, define  $\theta$  to be the ownership such that there is a unique  $m^* \in M$  and  $h^* \in H$  such that  $\forall w \in W$ ,  $\theta(m^*, w, h^*) = m$  and  $\theta$  always gives ownership to women otherwise. In that case, select the highest-ranked pair of the form  $(w, h^*)$  for  $m^*$ , call it  $(w^*, h^*)$ , and fix a matching  $\mu_W$  s.t.  $\mu_W(h^*) = w^*$ . Then, by Proposition 3.3.2, one can find a double matching  $\mu$  that is stable with respect to the strong  $W$ -ownership structure  $\theta'$  that always gives ownership to women. However, this matching also has to be stable with respect to  $\theta$ . Indeed, we know that, by construction, it is stable with respect to  $\theta'$ . Thus, if  $\mu$  was not stable with respect to  $\theta$ , the only possible blocking triplet would involve  $m^*$  in  $h^*$  such that  $m^*$  is matched with  $h^*$  under  $\mu$ . However, we know that, by the choice of  $w^*$ , for all other pairs of the form  $(w, h^*)$ , we have  $(w^*, h^*) \succ_m (w, h)$ , and this contradicts that  $m^*$  is currently blocking.

Let us focus now on the properties of the stable double matchings with respect to one-sided ownership structures. There is an obvious sense in which these stable double matchings share similar features with the standard stable ones in two-sided matching problems. Indeed the proof of Proposition 3.3.2 relies directly on the existence of stable matchings in the standard two-sided market problem. In our framework, one can define a natural generalization of the core. Fix an ownership structure  $\theta$ ; a matching  $\mu = (\mu_M, \mu_W)$  is in the  $\theta$ -core if there exists no **blocking coalition** of agents  $M' \cup W' \subseteq M \cup W$  and a matching  $\mu'$  s.t.

1. if  $\mu'(h') = (m', w') \in M \times W$ , then  $m' \in M' \Leftrightarrow w' \in W'$ .<sup>10</sup>
2. For  $m' \in M'$  and  $w' \in W'$ , if  $\mu'(h') = (m', w')$  then one of the following condition must hold:
  - (a) There is  $m \in M'$  s.t.  $h' = \mu_M(m)$  and  $\theta(m, \mu_W(h), h) = m$ .
  - (b) There is  $w \in W'$  s.t.  $h' = \mu_W(w)$  and  $\theta(\mu_M(h), w, h) = w$ .
  - (c)  $\mu_M(h') = \mu_W(h') = \emptyset$ .

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<sup>9</sup> Note that if  $\theta$  is the strong  $I$ -ownership structure, then one could use the procedure in the above proof to show a slightly stronger result: for every matching  $\mu_I$  of the  $I$ -agents to the houses, there exists a matching  $\mu_J$  of  $J$ -agents s.t. the resulting double matching  $\mu = (\mu_I, \mu_J)$  is stable with respect to  $\theta$ .

<sup>10</sup>This condition is only for men and women who are matched to each other under  $\mu'$  and are part of the coalition. For instance, if there is a man who was unassigned under  $\mu$  and who is matched alone to a house under  $\mu'$ , then this condition is trivially satisfied.

3. For all  $a \in M' \cup W'$ ,  $\mu'(a) \succeq_a \mu(a)$  and strictly for at least one agent.

The interpretation is standard: some group of men and women could be better-off rematching with one another and in possibly exchanging the houses of which some were the owners under  $\mu$ . If there is only one man and one woman, we return to the definition of a stable matching with respect to  $\theta$ . Note that two types of agents might be worse-off under  $\mu'$ : owners who have not changed houses but have seen their previous partner leave for a better match and previous partners of some owners who are part of the coalition. As in the standard core notion in two-sided markets, a core matching must be Pareto-efficient.<sup>11</sup> However, a stable double matching with respect to  $\theta$  need not be Pareto-efficient and thus may not be in the core. Specifically, there are instances in which the core could be empty while the set of stable matchings is not. This is one main difference from the standard two-sided market setting, where it is known that the set of pairwise stable matchings is equal to the core. In three-sided matching models as defined in Alkan (1988), it is also easy to see that stable matchings are in the core. Our departure is twofold: first, we consider a side composed of objects that do not have any preferences and hence cannot unilaterally break their matches. Second, we introduce an ownership structure that restricts the potential coalitions in forcing them to include the owners of the reassigned houses. We summarize these choices in the following proposition. Even if the statement is for general ownership structures, the proof explicitly uses the strong  $I$ -sided ownership.<sup>12</sup> Thus, even if stable matchings with respect to a one-sided ownership share similar features with stable matchings in standard two-sided matching markets, this is an important difference:

**Proposition 3.3.3.** *Fix an ownership structure  $\theta$ ; the sets of Pareto-efficient matchings and stable matchings with respect to  $\theta$  may be non-empty and disjoint. Thus, the  $\theta$ -core could be empty while stable matchings with respect to  $\theta$  exist.*

*Proof.* Assume that there are 3 men, 3 women and 3 houses. Preferences are given by the following

<sup>11</sup>Otherwise, using strict preferences, there would be a set of men and a set of women who would all change houses and be strictly better-off. Consider that set of men and women to be the blocking coalition.

<sup>12</sup>The proof uses a 3 men, 3 women, 3 houses structure. With only 2 men, 2 women and 2 houses and a strong  $I$ -sided ownership, one can find stable double matchings that are not Pareto-efficient. However, it can be shown that there always exists a stable double matching that is Pareto-efficient. To prove the impossibility, the minimal example involves a 3 agents structure.

table:

$m_1$	$m_2$	$m_3$	$w_1$	$w_2$	$w_3$
$(w_1, h_1)$	$(w_3, h_3)$	$(w_2, h_1)$	$(m_1, h_2)$	$(m_3, h_2)$	$(m_3, h_2)$
$(w_2, h_2)$	$(w_2, h_2)$	$(w_1, h_2)$	$(m_3, h_2)$	$(m_2, h_2)$	$(m_2, h_3)$
$(w_1, h_2)$	$(w_2, h_3)$	$(w_3, h_2)$	$(m_1, h_1)$	$(m_1, h_2)$	$(m_3, h_3)$
$(w_2, h_1)$	$(\emptyset, h_1)$	$(w_2, h_2)$	$(m_3, h_1)$	$(m_3, h_1)$	$(\emptyset, h_1)$
$(\emptyset, h_1)$	$(\emptyset, h_2)$	$(w_3, h_3)$	$(\emptyset, h_1)$	$(m_1, h_1)$	$(\emptyset, h_2)$
$(\emptyset, h_2)$	$(\emptyset, h_3)$	$(w_1, h_1)$	$(\emptyset, h_2)$	$(m_2, h_3)$	$(\emptyset, h_3)$
$(\emptyset, h_3)$		$(\emptyset, h_1)$	$(\emptyset, h_3)$	$(\emptyset, h_1)$	
		$(\emptyset, h_2)$		$(\emptyset, h_2)$	
		$(\emptyset, h_3)$		$(\emptyset, h_3)$	

Fix the ownership  $\theta$  as the strong  $M$ -ownership structure. This structure always assigns ownership to men. It trivially belongs to one-sided ownership structures such that, by Proposition 3.3.2, the set of stable matchings with respect to  $\theta$  is non-empty. Moreover, as mentioned in Footnote 9, for each possible assignment  $\mu_M$  of men to houses, there is a stable matching with respect to  $\theta$ . Since men always prefer to be assigned to a house even without any partner, then all men must be assigned a house. The intuition of the example is as follows: fix a double matching, every time one tries to implement the matching proposed by a blocking coalition, it creates additional blocking triplets or profitable Pareto-superior exchanges. To do so, one has to let the agents outside the coalition prefer less their new assignment. In trying to implement the matching of such newly created blocking coalitions, one will, at some point, come back to the initially fixed double matching. Obviously, one has to check all the possible double matchings for the impossibility to hold, that is what we are doing in what follows.

There are six ways to match men to houses. We will show that for each possible matching of women for each of these six cases, the double matching is either not Pareto-efficient or not stable with respect to  $\theta$ . For each possible  $\mu_M$ , we will present a table. The three first columns will represent the houses. The first line will restate the assignment  $\mu_M$  of men to houses, and each following line will give an assignment of women to houses. Since the matching of men is fixed in each case, each assignment of women determines a double matching. Next to each assignment of women, we will

present a possible blocking coalition. For ease of notation, each possible blocking coalition will be written using the relevant part of the blocking matching for the coalition. For instance, if a given matching is blocked by a coalition formed by a man  $m$  and a woman  $w$  who would prefer to be matched in  $h$ , we will say that the matching is “Blocked by  $(m, w, h)$ ”. If the coalition is formed by two couples  $(m, w)$  and  $(m', w')$  who would prefer to be assigned to houses  $h$  and  $h'$ , respectively, then we will say that the matching is “Blocked by  $(m, w, h)$  and  $(m', w', h')$ ”. Obviously, each of the stated parts of the blocking matching will respect the conditions (notably on ownership) given in the definition of a blocking coalition. Note that a Pareto-dominated assignment also admits a blocking coalition. In this example, all matchings blocked by a coalition of more than two agents will be Pareto-dominated by the matching used to block it.

Case 1.  $\mu_M(h_1) = m_1, \mu_M(h_2) = m_2, \mu_M(h_3) = m_3$ .

	$h_1$	$h_2$	$h_3$	
$\mu_M$	$m_1$	$m_2$	$m_3$	
$\mu_W$	$w_1$	$w_2$	$w_3$	$\Rightarrow$ Blocked by $(m_2, w_3, h_3), (m_3, w_2, h_2)$
	$w_2$	$\emptyset$	$w_3$	$\Rightarrow$ Blocked by $(m_2, w_2, h_2)$

For instance, note that here, one does not have to consider an assignment of women to houses such that only  $w_2$  is matched to  $h_2$  and the others remain unmatched. Indeed, the former could be obtained from the first assignment of women in the above table by letting  $w_1$  and  $w_3$  be unassigned. Thus, the matching would be blocked by exactly the same coalition using the same matching as in the table. A similar argument could be used for all other assignments of women that could be obtained from those above by just letting some women be unmatched. In all tables, we do not explicitly specify these matchings since they can be inferred using those we provide.

Case 2.  $\mu_M(h_1) = m_1, \mu_M(h_2) = m_3, \mu_M(h_3) = m_2$ .

	$h_1$	$h_2$	$h_3$	
$\mu_M$	$m_1$	$m_3$	$m_2$	
$\mu_W$	$w_1$	$w_2$	$w_3$	$\Rightarrow$ Blocked by $(m_3, w_3, h_2)$
	$w_1$	$w_3$	$w_2$	$\Rightarrow$ Blocked by $(m_3, w_1, h_2)$
	$w_2$	$w_1$	$w_3$	$\Rightarrow$ Blocked by $(m_1, w_1, h_2), (m_3, w_2, h_1)$
	$w_2$	$w_3$	$\emptyset$	$\Rightarrow$ Blocked by $(m_1, w_1, h_1)$

Case 3.  $\mu_M(h_1) = m_2, \mu_M(h_2) = m_1, \mu_M(h_3) = m_3$ .

	$h_1$	$h_2$	$h_3$	
$\mu_M$	$m_2$	$m_1$	$m_3$	
$\mu_W$	$\emptyset$	$w_2$	$w_3$	$\Rightarrow$ Pareto-dominated by $(m_1, w_1, h_1), (m_2, w_2, h_2), (m_3, w_3, h_3)$
	$\emptyset$	$w_1$	$w_3$	$\Rightarrow$ Blocked by $(m_1, w_2, h_2)$

Case 4.  $\mu_M(h_1) = m_2, \mu_M(h_2) = m_3, \mu_M(h_3) = m_1$ .

	$h_1$	$h_2$	$h_3$	
$\mu_M$	$m_2$	$m_3$	$m_1$	
$\mu_W$	$\emptyset$	$w_1$	$\emptyset$	$\Rightarrow$ Blocked by $(m_1, w_2, h_1), (m_2, w_3, h_3), (m_3, w_1, h_2)$
	$\emptyset$	$w_3$	$\emptyset$	$\Rightarrow$ Blocked by $(m_3, w_1, h_2)$
	$\emptyset$	$w_2$	$\emptyset$	$\Rightarrow$ Blocked by $(m_3, w_1, h_2)$

Case 5.  $\mu_M(h_1) = m_3, \mu_M(h_2) = m_1, \mu_M(h_3) = m_2$ .

	$h_1$	$h_2$	$h_3$	
$\mu_M$	$m_3$	$m_1$	$m_2$	
$\mu_W$	$w_1$	$w_2$	$w_3$	$\Rightarrow$ Blocked by $(m_1, w_1, h_1), (m_3, w_2, h_2)$
	$w_1$	$\emptyset$	$w_2$	$\Rightarrow$ Blocked by $(m_3, w_2, h_1)$
	$w_2$	$w_1$	$w_3$	$\Rightarrow$ Blocked by $(m_1, w_2, h_2)$
	$\emptyset$	$w_1$	$w_2$	$\Rightarrow$ Blocked by $(m_1, w_2, h_2)$
	$\emptyset$	$w_2$	$w_3$	$\Rightarrow$ Blocked by $(m_1, w_1, h_1), (m_2, w_3, h_3), (m_3, w_2, h_2)$
	$\emptyset$	$w_1$	$w_3$	$\Rightarrow$ Blocked by $(m_1, h_2, w_2)$

Case 6.  $\mu_M(h_1) = m_3, \mu_M(h_2) = m_2, \mu_M(h_3) = m_1$ .

	$h_1$	$h_2$	$h_3$	
$\mu_M$	$m_3$	$m_2$	$m_1$	
$\mu_W$	$w_1$	$w_2$	$\emptyset$	$\Rightarrow$ Blocked by $(m_1, w_1, h_1), (m_2, w_2, h_2), (m_3, w_3, h_3)$
	$w_2$	$\emptyset$	$\emptyset$	$\Rightarrow$ Blocked by $(m_2, w_2, h_2)$

Having exhausted all the cases, we conclude that the set of stable double matchings with respect to  $\theta$  and the set of Pareto-efficient matchings are disjoint. Since, as emphasized above, a matching in the  $\theta$ -core has to be stable with respect to  $\theta$  and Pareto-efficient, we conclude that the  $\theta$ -core is empty.  $\square$

The complementarities of the preferences over the possible agent-house pairs play an important role in this separation.<sup>13</sup> Indeed, it can easily be proved that the existence of a Pareto-efficient and stable double matching is restored for the strong  $I$ -ownership structure if we assume that  $I$ -agents have lexicographic preferences: they first value the house and then the women matched to it. Formally, for each  $I$ -agent  $i$  and his preference order  $\succ_i$  over  $(J \cup \{\emptyset\}) \times (H \cup \{\emptyset\})$ , there exists an order  $\tilde{\succ}_i$  over  $H$  s.t.  $\forall h, h' \in H, h \tilde{\succ}_i h' \Leftrightarrow (h, j) \succ_i (h', j') \forall j, j' \in J \cup \{\emptyset\}$ . A matching  $\mu_I$  of

<sup>13</sup>The emptiness of the core in the presence of externalities also exists in the auction literature (Jehiel and Moldovanu, 1996). Jeong (2017) showed the existence of weaker stability notions in this context. Contrary to them, our setting does not allow any monetary transfers, which creates important differences even in the standard framework.

$I$ -agents to houses is said to be Pareto-efficient with respect to preferences  $(\succsim_i)_{i \in I}$  if one cannot find another matching of  $I$ -agents to houses such that all  $I$ -agents, according to their preferences  $(\succsim_i)_i$ , prefer their newly assigned houses. In that case, one may match the  $I$ -agents to the houses in a Pareto-efficient manner with respect to  $(\succsim_i)_i$  and then select a stable matching to match the  $J$ -agents as in the proof of Proposition 3.3.2.<sup>14</sup> Since  $I$ -agents have been matched in a Pareto-efficient way and first value the houses, it can be shown that the obtained matching is necessarily Pareto-efficient. As mentioned above, the induced double matching is also stable with respect to  $\theta_M$ . Note however that this remark does not depend on any assumption on the preferences of the women, which may still exhibit complementarities.<sup>15</sup> In the following proposition, we also show that the obtained matching is in the  $\theta$ -core:

**Proposition 3.3.4.** *Let  $I \in \{M, W\}$ ; if preferences of  $I$ -agents are lexicographic and  $\theta$  is the strong  $I$ -ownership structure, then there exists a double matching that is in the  $\theta$ -core.*

*Proof.* W.l.o.g let  $\theta$  be the strong  $W$ -ownership such that women have lexicographic preferences. Match women to houses such that the resulting matching is Pareto-efficient with respect to preferences  $(\succsim_w)_w$ ; call this matching  $\mu_W$ . Since by Assumption 1, all women prefer to be matched to a house alone to not being matched at all and that there is the same number of houses as women, then all houses are occupied by a woman. As in the proof of Proposition 3.3.2, let  $P$  be the set of pairs  $p = (h, w)$  s.t.  $\mu_W(h) = w$ ,  $\succsim_p$  be the preferences of the pair  $p$  over the men and  $\succsim_m$  be the preferences of man  $m$  over the pairs. Thus, the market formed by pairs in  $P$  and men in  $M$  is a standard two-sided market as defined by Gale and Shapley (1962). We know that there exists an assignment of men to the pairs in  $P$  that is stable with respect to the aforementioned preferences. In a standard two-sided market, stable matchings are also in the core, and hence, there is no coalition formed by pairs in  $P$  and men in  $M$  that can block the matching in the two-sided matching setting, i.e., conditional on keeping  $\mu_W$  fixed in our setting. Therefore, if a coalition  $M' \cup W'$  blocks  $\mu$  with a matching  $\mu'$ , there must be one woman in  $W'$ , say  $w_1$ , who will be matched to a different house under  $\mu'$ , say  $h_1$ , than under  $\mu$ . Since she does not have the ownership of  $h_1$  under  $\mu$  and all houses are occupied, the woman, say  $w_2$ , matched to  $h_1$  under  $\mu$  must be part of the coalition and,

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<sup>14</sup>The Pareto-efficiency of the assignment of  $I$ -agents to houses is important; otherwise, one may still produce a stable matching with respect to  $\theta$  that is not Pareto-efficient.

<sup>15</sup>One can hope that a weaker notion would allow the existence, for instance, with additive preferences where the utility of a pair  $(w, h)$  for man  $m$  would be given by  $u(w) + v(h)$ . However, the problem becomes intractable. We have not been able to exhibit a counter example nor a proof of existence due to the combinatorial aspect of the problem.

hence, by strict preferences also be better-off, meaning that she must obtain a house  $h_2$  that, by lexicographic preferences, she prefers to  $h_1$ . Continuing the argument, one can exhibit a cycle of women exchanging their houses when going from matching  $\mu$  to  $\mu'$ . By lexicographic preferences, this contradicts the Pareto-efficiency of the matching  $\mu_W$  of women to houses with respect to preferences  $(\succsim_w)_w$ .  $\square$

### 3.4 An alternative notion of the core

In a housing market problem, as defined in [Shapley and Scarf \(1974\)](#), agents are initially assigned to a house and would like to exchange it. [Shapley and Scarf \(1974\)](#) and [Roth and Postlewaite \(1977\)](#) defined a notion of the core with respect to this initial assignment: a matching of agents to houses is in the core if no subset of agents can exchange their initial houses with one another while all being strictly better-off. Note the difference with respect to the core notion defined above: here, the set of agents blocking must exchange their initial house, not the one that they obtain under the blocked matching. One may wonder whether considering a housing market setting would allow a non-empty core as naturally defined in this context. This turns out not to be the case, as shown in [Proposition 3.4.1](#) below. Before turning to this question, let us define the core notion in this framework. An **initial ownership** is an injection  $\theta_0 : H \mapsto M \cup W$  where  $\theta_0(h)$  is interpreted as the agent initially owning house  $h$ . Note that we assume that all houses have an initial owner. As usual, we abuse notations and denote  $\theta_0(a)$  as the initial house, if any, of agent  $a \in M \cup W$  (if there is none, let  $\theta_0(a) = \emptyset$ ). An initial ownership  $\theta_0$  is an  **$I$ -initial ownership** if  $\forall h \in H, \theta_0(h) \in I$ . We will refer to the latter as a **one-sided initial ownership**. A double matching  $\mu$  is in the *housing market*  $\theta_0$ -core if there is no **blocking coalition** of agents  $M' \cup W' \subseteq M \cup W$  and a matching  $\mu'$  s.t. the following hold:

1. If  $\mu'(h') = (m', w') \in M \times W$  then  $m' \in M' \Leftrightarrow w' \in W'$ .<sup>16</sup>
2. For all  $a \in M' \cup W'$  s.t.  $a \in \mu'(h')$ , then  $h' \in \theta_0(M' \cup W')$ .
3.  $\forall a \in M' \cup W', \mu'(a) \succeq_a \mu(a)$  and strictly for some.

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<sup>16</sup>This condition is only for men and women who are matched to one another under  $\mu'$  and are part of the coalition. For instance, if there is a man who was unassigned under  $\mu$  and who is matched alone to his initial house under  $\mu'$ , then this condition is trivially satisfied.



The idea is standard: there is no subset of initial owners who could exchange their houses and be matched with other agents such that all of them are better-off. In a housing market setting, [Roth and Postlewaite \(1977\)](#) defined two different core notions: strict and weak. The former imposes that the preference relation in condition (3) of the above definition is strict for every member of the coalition. The weak version is similar to ours and allows indifference for some members of the coalition. Our next proposition shows that even for a  $I$ -initial ownership, this (weak) core notion may be empty:

**Proposition 3.4.1.** *Let  $\theta_0$  be an  $I$ -initial ownership. Then, the housing market  $\theta_0$ -core may be empty.*

*Proof.* There are 3 men, 3 women and 3 houses. The initial ownership is given by  $\theta_0(h_i) = m_i$  for  $i = 1, 2, 3$ . Preferences are as follows:

$m_1$	$m_2$	$m_3$	$w_1$	$w_2$	$w_3$
$(w_3, h_2)$	$(w_2, h_1)$	$(w_3, h_3)$	$(m_1, h_1)$	$(m_3, h_3)$	$(m_3, h_2)$
$(w_1, h_1)$	$(w_2, h_2)$	$(w_3, h_2)$	$(m_2, h_3)$	$(m_2, h_1)$	$(m_1, h_2)$
$(w_2, h_1)$	$(w_1, h_3)$	$(w_2, h_3)$	$(m_2, h_1)$	$(m_2, h_2)$	$(m_3, h_3)$
$(\emptyset, h_1)$	$(w_1, h_1)$	$(w_1, h_3)$	$(m_3, h_3)$	$(m_1, h_1)$	$(\emptyset, h_1)$
$(\emptyset, h_2)$	$(\emptyset, h_1)$	$(\emptyset, h_1)$	$(\emptyset, h_1)$	$(\emptyset, h_1)$	$(\emptyset, h_2)$
$(\emptyset, h_3)$	$(\emptyset, h_2)$	$(\emptyset, h_2)$	$(\emptyset, h_2)$	$(\emptyset, h_2)$	$(\emptyset, h_3)$
	$(\emptyset, h_3)$	$(\emptyset, h_3)$	$(\emptyset, h_3)$	$(\emptyset, h_3)$	

Assumption 1 implies that, with this core notion, a matching in the housing market  $\theta_0$ -core has to match all the men to all houses. Otherwise, the unmatched man could always block with his initial house. As in Proposition 3.3.3, we will check the six ways to assign men to houses. For each case, we will use the same table presentation and terminology as in Proposition 3.4.2 in listing the men matching to houses, the women matchings to houses and the same presentation of the blocking coalitions. The main difference will be that, now, men could block a matching by only using/exchanging their initial house and not the one that they are assigned to under the blocked matching.

*Case 1.*  $\mu_M(h_1) = m_1, \mu_M(h_2) = m_2, \mu_M(h_3) = m_3$ .

	$h_1$	$h_2$	$h_3$	
$\mu_M$	$m_1$	$m_2$	$m_3$	
	$w_1$	$w_2$	$w_3$	$\Rightarrow$ Blocked by $(m_1, w_3, h_2), (m_2, w_2, h_1)$
	$w_1$	$\emptyset$	$w_2$	$\Rightarrow$ Blocked by $(m_3, w_3, h_3)$
$\mu_W$	$\emptyset$	$w_2$	$w_1$	$\Rightarrow$ Blocked by $(m_3, w_3, h_3)$
	$w_2$	$\emptyset$	$w_1$	$\Rightarrow$ Blocked by $(m_3, w_3, h_3)$
	$w_2$	$\emptyset$	$w_3$	$\Rightarrow$ Blocked by $(m_1, w_1, h_1)$

For instance, note that here, one does not have to consider an assignment of women to houses such that only  $w_2$  is matched to  $h_2$  and the others remain unmatched. Indeed, the former could be obtained from the first assignment of women in the above table by letting  $w_1$  and  $w_3$  be unassigned. Therefore, the matching would be blocked by exactly the same coalition using the same matching as in the table. A similar argument could be used for all other assignments of women that could be obtained from those above by just letting some women be unmatched. In all the tables, we do not explicitly specify these matchings since they can be inferred using those we provide.

Case 2.  $\mu_M(h_1) = m_1, \mu_M(h_2) = m_3, \mu_M(h_3) = m_2$ .

	$h_1$	$h_2$	$h_3$	
$\mu_M$	$m_1$	$m_3$	$m_2$	
	$w_1$	$w_3$	$\emptyset$	$\Rightarrow$ Blocked by $(m_2, w_2, h_2)$
$\mu_W$	$w_2$	$w_3$	$w_1$	$\Rightarrow$ Blocked by $(m_1, w_1, h_1)$

Case 3.  $\mu_M(h_1) = m_2, \mu_M(h_2) = m_1, \mu_M(h_3) = m_3$ .

	$h_1$	$h_2$	$h_3$	
$\mu_M$	$m_2$	$m_1$	$m_3$	
$\mu_W$	$w_2$	$w_3$	$w_1$	$\Rightarrow$ Blocked by $(m_3, w_2, h_3)$
	$w_1$	$w_3$	$w_2$	$\Rightarrow$ Blocked by $(m_2, w_1, h_3), (m_3, w_3, h_2)$
	$w_1$	$\emptyset$	$w_3$	$\Rightarrow$ Blocked by $(m_1, w_1, h_1)$
	$w_2$	$\emptyset$	$w_3$	$\Rightarrow$ Blocked by $(m_1, w_1, h_1)$

Case 4.  $\mu_M(h_1) = m_2, \mu_M(h_2) = m_3, \mu_M(h_3) = m_1$ .

	$h_1$	$h_2$	$h_3$	
$\mu_M$	$m_2$	$m_3$	$m_1$	
$\mu_W$	$w_2$	$w_3$	$\emptyset$	$\Rightarrow$ Blocked by $(m_1, w_1, h_1)$
	$w_1$	$w_3$	$\emptyset$	$\Rightarrow$ Blocked by $(m_1, w_1, h_1)$

Case 5.  $\mu_M(h_1) = m_3, \mu_M(h_2) = m_1, \mu_M(h_3) = m_2$ .

	$h_1$	$h_2$	$h_3$	
$\mu_M$	$m_3$	$m_1$	$m_2$	
$\mu_W$	$\emptyset$	$w_3$	$w_1$	$\Rightarrow$ Blocked by $(m_3, w_2, h_3)$

Case 6.  $\mu_M(h_1) = m_3, \mu_M(h_2) = m_2, \mu_M(h_3) = m_1$ .

	$h_1$	$h_2$	$h_3$	
$\mu_M$	$m_3$	$m_2$	$m_1$	
$\mu_W$	$\emptyset$	$w_2$	$\emptyset$	$\Rightarrow (m_1, h_1, w_1)$ Blocked by

Having exhausted all the possible cases, we conclude that the housing market core is empty.  $\square$

Note that in the previous example, the matching  $\mu(h_1) = (m_1, w_1)$ ,  $\mu(h_2) = (m_2, w_2)$  and  $\mu(h_3) = (m_3, w_3)$  is Pareto-efficient and not blocked by any coalition of size two. As the next proposition shows, such a matching always exists for one-sided initial ownerships.<sup>17</sup> Since the previous counterexample involved coalitions of size four and smaller and because a coalition is always formed by an even number of agents, the result is tight.<sup>18</sup>

**Proposition 3.4.2.** *If the initial allocation is one-sided, there always exists a Pareto-efficient matching that is not blocked by any coalition  $M' \cup W'$ , of size two, i.e.,  $|M' \cup W'| = 2$ .*

*Proof.* Since the initial allocation is one-sided, w.l.o.g, assume that the women are initially matched to the houses s.t.  $\theta_0(h_i) = w_i$  for all  $i = 1, \dots, n$ . Fix this matching of women to the houses, and call it  $\mu_W$ . Since the number of houses and women is the same, there are no empty houses or unassigned women under  $\mu_W$ . As in the proof of Proposition 3.3.2, consider the two-sided market formed by a set  $P$  of pairs of the form  $p_i = (w_i, h_i)$ , where  $\mu_W(h_i) = \theta_0(h_i) = w_i$ ,  $\succ_{p_i}$  is the preference order of a pair  $p_i$  over the men in  $M$ , and  $\succ_m$  is the preference order of a man  $m$  over the pairs in  $P$ . In this two-sided market, as before, one can match men to the pairs s.t. the resulting matching is stable. The resulting double matching  $\mu$  will be composed of  $\mu_W$  and an assignment of men to houses,  $\mu_M$ . Therefore,  $\mu = (\mu_M, \mu_W)$  will not be blocked by any coalition of size two. Indeed, in such a coalition, there must be one man  $m$ , one woman  $w_i$  and matching  $\mu'$ . Since the coalition is of size two, by the definition of the housing market  $\theta_0$ -core, woman  $w_i$  cannot be matched to a house other than the one she is assigned to under  $\mu$ , i.e.,  $h_i$ . Indeed, under  $\mu$ , she is assigned to her initial house under  $\theta_0$ . Thus, if she were matched to another house, say  $h_j$  with  $j \neq i$ , by the definition of the housing market core, the coalition would have to include the initial owner of this house, woman  $w_j$ , and this would contradict there being only one woman in the blocking coalition. Thus,  $w_i$  has to remain in her initial house under the blocking matching  $\mu'$ . We therefore have the following:  $\mu'(w_i) = (m, h_i) \succ_{w_i} \mu(w_i)$  and  $(w_i, h_i) \succ_m \mu(m)$ . However, this would contradict the stability of the matching in the two-sided setting formed by pairs  $P$  and men in  $M$  since  $(h_i, w_i) \in P$  prefers being matched with man  $m$  and the latter prefers being matched to  $(h_i, w_i)$ .

<sup>17</sup>In the previous section, with the other core notion, we proved for such ownerships the existence of a matching not blocked by any coalition of size two. However, we showed in Proposition 3.3.3 that there could be no matching that is Pareto-efficient and not blocked by any coalition of size two.

<sup>18</sup>When one only imposes a coalition of size two in our setting, then there is no difference between the strict and weak notions as defined above. Indeed, if a coalition of size two formed by a man  $m$  and a woman  $w$  is blocking a matching  $\mu$  with the initial house  $h$  being held by one of them, then, in our definition, one of the two, say  $m$  must be strictly better-off. Thus, by strict preferences, this means that  $m$  and  $w$  were not matched together in  $h$  under  $\mu$ , and hence,  $w$  must also be strictly better-off.

Thus, the matching  $\mu$  is not blocked by any coalition of size two. Assume that it is not Pareto-efficient. There hence exists a double matching  $\mu' = (\mu'_M, \mu'_W)$  that Pareto dominates  $\mu$ . We will show that  $\mu'$  is also not blocked by any coalition of size two. If so, there exists a woman  $w_i$  and a man  $m$  who are blocking the matching  $\mu'$ . Using the same argument as above, it must be the case that they are blocking using the initial house of woman  $w_i$ , that is,  $h_i = \theta_0(w_i)$ . Therefore, by the definition of the blocking coalition and by strict preferences,

$$\begin{aligned} (m, h_i) \succ_{w_i} \mu'(w_i) \succeq_{w_i} \mu(w_i) \\ (w_i, h_i) \succ_m \mu'(m) \succeq_m \mu(m) \end{aligned}$$

The first preference relations come from the blocking coalition condition and strict preferences and the second from the assumption that  $\mu'$  Pareto-dominates  $\mu$ . However, this would mean that  $w_i$  and  $m$  were already a coalition of size two blocking  $\mu$ , a contradiction. Hence, to find a Pareto-efficient matching, one simply needs to iterate the procedure. If another matching  $\mu''$  Pareto-dominates  $\mu'$  then, using the same argument, it will not be blocked by any coalition of size two. Continuing the argument will, by the finiteness of the environment, lead to a Pareto-efficient matching that, by construction, will not be blocked by any coalition of size two.  $\square$

### 3.5 Conclusion

We introduced a new matching model in which two types of agents have to be matched to a set of objects. We defined a notion of ownership structure over the objects that specifies, for each object and its possible assignments of agents, the agent owning that object. Only owners are allowed to block a given matching and let other agents join them and their objects. For general ownership structures, we showed that stable matchings with respect to an ownership may fail to exist. However, assuming that all agents prefer to be matched to an object to being unassigned, we showed that stable matchings always exist for a simple class of one-sided ownership structures: those that assign the ownership of objects to agents of the same type. For one-sided ownerships, stable matchings share similarities with the standard two-sided matching setting. However, we showed that there is an important difference: they could be disjoint from Pareto-efficient matchings. Thus, in this context, a generalization of the core with respect to an ownership can be empty, which is not the case in

the standard two-sided market setting. We introduced an alternative ownership approach whereby each house is initially owned by one agent. We also showed that for initial ownerships where houses are owned by the same type of agents, an alternative notion of the core may fail to exist. However, one can always find a Pareto-efficient matching that is not blocked by any coalition of size two. In both settings, the one-sided structure of ownership is an important class to ensure the existence of matchings not blocked by pairs of agents. With initial ownership, one can in addition ensure the Pareto-efficiency of one of these matchings.

Our results show that the one-sided ownership structure is an important requirement for the existence of a pairwise stability notion. From a positive point of view, it implies that one should mostly observe one sided ownership structures in practice. As mentioned, the one-sided structures are not the only ones admitting a stable matching. A complete characterization would be needed to precisely identify the structures. However, the problem becomes combinatorial and intractable. Counter examples suggest that one cannot go much beyond the one-sided ownership structure.



# Conclusion

## Conclusion Générale

Cette thèse s’attèle à étudier le problème d’affectation des enseignants aux écoles. Nous avons montré que ce problème est l’intersection de deux grands types de problèmes au sein de la littérature d’appariement: l’appariement avec deux types et les problèmes d’allocation de logements. Dans le Chapitre 1, nous avons proposé un premier cadre théorique pour analyser les propriétés particulières de ce problème d’affectation et avons montré que la procédure actuellement identifiée dans la littérature et utilisée en pratique en France souffre d’un défaut important: il est possible de réaffecter les enseignants aux écoles de telle sorte qu’ils soient tous “mieux”. Nous avons étudié une classe de mécanismes qui ne souffre pas de ce défaut et avons identifié une unique procédure qui incite les enseignants à être sincères dans la soumission de leurs listes de préférences. Dans le Chapitre 2, nous avons exploré la conception pratique de cet algorithme pour le cas français d’affectation des enseignants du second degré en prenant en compte une multitude de contraintes pratiques pour ce marché. Nous avons proposé un algorithme pouvant être paramétrisé afin de s’adapter à un grand nombre d’objectifs de politiques régionales d’affectation des enseignants en termes de mouvement et de distribution de ces derniers. Plus généralement, cet algorithme peut s’avérer être un outil important de politique publique afin de répondre à deux problèmes centraux des systèmes éducatifs au sein des pays de l’OCDE: i) le manque d’attractivité de la profession enseignante et ii) les fortes inégalités de réussite entre élèves issus de différents milieux sociaux. Dans le Chapitre 3, nous avons développé une nouvelle classe de modèles d’appariement inspirée de l’affectation des enseignants et des élèves aux écoles. Des paires d’agents doivent être affectés ensemble à des objets, par exemple un manager et un travailleur à des projets. Chaque agent (manager ou travailleur) a des préférences sur les paires agent-objet auxquelles il peut être affecté. Nous avons exploré la question classique de



l'existence d'appariements stables dans ce nouveau modèle. Nous avons montré qu'une notion forte de stabilité, où un manager ou un travailleur peut librement renvoyer son partenaire et demander à un autre de rejoindre son projet, peut ne pas exister. Cependant, nous avons montré que lorsque l'on définit correctement une structure de propriété sur les projets, c.a.d. quel agent est autorisé à librement renvoyer son partenaire et demander à un autre de le rejoindre, alors on peut restaurer l'existence d'une notion de stabilité. La structure de propriété requise pour cette existence est naturelle: il faut toujours donner la propriété d'un projet au même type d'agents, c.a.d. toujours des managers ou toujours des travailleurs. Même si ces appariements stables ont des similarités avec ceux du cadre standard d'affectation avec deux types, nous avons montré que, contrairement au cadre classique, une notion naturelle de coeur peut être vide.

## Recherches futures

Les travaux développés au sein de cette thèse sont les premiers pas d'un agenda de recherche plus général sur l'affectation des enseignants aux écoles. Nous donnons ici une liste de potentielles recherches futures en plus de celles déjà mentionnées dans les conclusions des chapitres de cette thèse.

### Un modèle stochastique de réaffectation

L'analyse développée au sein des Chapitres 1 et 2 de cette thèse se concentrait sur des mécanismes dits *ex post* qui donnent une affectation finale des enseignants aux écoles. Cela nous a amené à considérer une contrainte de Rationalité Individuelle (RI) pour cette affectation finale: chaque enseignant doit toujours recevoir une affectation qu'il préfère faiblement à son affectation initiale. Dans la procédure d'affectation française, il n'est pas clair que cela soit pleinement le cas. En effet, comme nous l'avons décrit, la procédure d'affectation est divisée en deux étapes:

1. Une affectation inter-académique: les enseignants souhaitant changer d'académie d'affectation doivent soumettre leurs préférences sur les 31 académies françaises. La contrainte RI est imposée sur leur académie d'affectation initiale: ils sont sûrs d'être affectés à une académie qu'ils ont classée au dessus de leur académie initiale. Une fois l'affectation décidée par l'algorithme, elle est définitive.
2. Une affectation intra-académique: une fois l'affectation inter-académique de la première phase

décidée, les enseignants affectés à une académie donnée et ceux en poste au sein de cette académie et souhaitant changer d'établissement au sein de celle-ci soumettent leurs listes de préférences sur les établissements au sein de cette académie. Si un enseignant était en poste au sein d'un établissement de celle-ci, la contrainte RI lui assure d'être affecté à un établissement qu'il a classé au-dessus de son établissement initial. Cependant, un enseignant ayant été affecté à cette académie suite à la première phase est considéré comme n'ayant aucun établissement initial au sein de l'académie et peut donc se retrouver sans affectation à la fin de l'algorithme. Dans ce cas, le ministère l'affecte à un poste de remplaçant sur une zone géographique proche de sa première école classée. Ce dernier sera remplaçant au cours de l'année académique et participera encore à cette étape l'année suivante.

Un enseignant souhaitant changer d'académie fait donc face à un certain risque: celui de ne pas recevoir d'affectation au sein de sa nouvelle académie à la seconde étape. On peut modéliser ce risque comme une croyance que l'enseignant formule: il affecte une probabilité d'être affecté à tel ou tel établissement au sein de l'académie. Théoriquement, cela ouvre la possibilité de considérer des *mécanismes stochastiques* de réaffectation, c.a.d. qui retourneraient des probabilités d'obtenir tel ou tel appariement. Chaque enseignant posséderait une utilité cardinale pour chaque établissement et évaluerait une distribution de probabilités donnée en utilisant son utilité espérée. La contrainte RI serait alors *interim* et non *ex post*: pour une distribution de probabilités donnée, i.e. *une affectation probabiliste*, l'utilité espérée d'un enseignant doit être plus grande que l'utilité que lui procure son établissement initial. Cette approche implique qu'il est possible, pour certaines réalisations de la distribution de probabilités, qu'un enseignant se voit être affecté dans un établissement qui lui procure une plus faible utilité que son établissement initial. Ce dernier cas n'est pas autorisé par une contrainte RI *ex post*. Cette relaxation pourrait potentiellement permettre plus d'échanges et serait une étude théorique intéressante. Cependant, en pratique, il faudrait connaître ou demander les utilités cardinales de chaque enseignant, qui sont difficiles à estimer ou même à connaître pour les enseignants. De plus, il est connu dans la littérature de théorie des jeux que des contraintes *ex post*, comme celle considérée dans cette thèse, sont des solutions plus *robustes* quand il existe de l'incertitude sur les primitives du modèle (valeurs des utilités cardinales, information que possède chaque enseignant...etc)<sup>19</sup>.

<sup>19</sup>Voir par exemple la très bonne revue de littérature de [Kajii and Morris \(1997\)](#) et, pour la littérature de *mechanism design*, l'article de [Bergemann and Morris \(2005\)](#).

## Une caractérisation de notre généralisation de l'algorithme TO-BE

Dans le Chapitre 2, nous avons fourni une généralisation de l'algorithme TO-BE trouvé dans le Chapitre 1 afin de prendre en compte la présence d'enseignants néotitulaires sans affectation initiale et la présence de positions vacantes. Comme nous l'avons mentionné, ce problème est similaire à celui de généraliser l'algorithme TTC pour le problème d'allocation de logements avec locataires comme dans [Shapley and Scarf \(1974\)](#) à l'algorithme YRMH-IGYT proposé par [Abdulkadirouglu and Sonmez \(1999\)](#). En effet, ce dernier permet de généraliser l'algorithme d'échange de logements occupés entre locataires au cas où des agents sans logement initial et des logements vacants sont présents. Notre généralisation de TO-BE dans ce chapitre est une généralisation de l'algorithme YRMH-IGYT qui permet d'incorporer deux points additionnels: i) un environnement plusieurs-pour-un où les écoles ont plusieurs positions et ii) l'existence de "préférences" du côté des écoles. [Sonmez and Unver \(2010\)](#) ont fourni une caractérisation axiomatique de l'algorithme YRMH-IGYT.<sup>20</sup> Une caractérisation axiomatique similaire pour notre généralisation de TO-BE permettrait d'exhiber une classe plus importante de mécanismes.

## Un modèle de réaffectation dynamique

Tous les modèles présentés dans cette thèse prennent une approche statique de la réaffectation. L'aspect dynamique est pris comme une primitive du modèle via l'allocation initiale. Dans le cas de l'affectation inter-académique des enseignants en France, cette approche peut être justifiée. En effet, comme montré par [Prost \(2013\)](#), le principal déterminant de la mobilité inter-régionale des enseignants est la volonté de se rapprocher de leur région natale. Une fois qu'un enseignant rejoint cette dernière, il ne participera probablement plus au mouvement inter-académique<sup>21</sup>. Donc pour le problème de réaffectation pur étudié dans le Chapitre 1, l'approche statique peut être justifiée. Cependant, dans le Chapitre 2, il pourrait être intéressant de considérer un modèle dynamique simple: un enseignant doit d'abord postuler pour un premier poste en tant que néotitulaire puis peut postuler une deuxième fois dans sa carrière pour une réaffectation. Comme montré dans le Chapitre 2, un changement d'algorithme affecte la qualité de l'affectation des néotitulaires et celle

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<sup>20</sup>Les axiomes étant quelques peu techniques, nous renvoyons le lecteur à l'article cité pour plus de détails.

<sup>21</sup>Bien entendu, il peut toujours demander une réaffectation pour changer d'établissement au sein de cette académie. Il n'y a pas de statistiques ou rapports publics donnant le nombre moyen de fois qu'un enseignant participe au mouvement inter-académique dans sa carrière. Cependant, il serait possible de retrouver cette information en fusionnant l'ensemble des bases de données d'affectation avant 2013, qui est la base sur laquelle les travaux de cette thèse se sont appuyés.

des titulaires demandant une réaffectation. Les effets de long terme ne sont pas clairs et une analyse théorique complète serait utile. Récemment, la littérature académique en théorie de l'appariement a eu un intérêt croissant pour des modèles dynamiques. Notre but n'est pas de donner une revue complète de ces modèles, le lecteur intéressé peut se référer aux sections de revue de littérature des articles cités ci-après. Ils peuvent être divisés en deux types (les listes suivantes ne sont pas exhaustives):

1. Des modèles cherchant à étendre les notions de stabilité et d'efficacité à un contexte dynamique et à étudier leur existence: [Damiano and Lam \(2005\)](#), [Kurino \(2009\)](#), [Pereyra \(2013\)](#), [Doval \(2017\)](#).
2. Des modèles simplifiés de dynamique pour étudier les mécanismes optimaux: [Bloch and Cantala \(2013\)](#), [Bloch and Cantala \(2017\)](#), [Leshno \(2017\)](#), [Baccara, Lee and Yariv \(2015\)](#).

### Une estimation structurelle des préférences dynamiques des enseignants

La procédure d'affectation en France demande à chaque enseignant souhaitant une (re)affectation de soumettre une liste de préférences sur les académies (ou établissements). Cependant, au cours sa carrière, un enseignant se verra affecté à plusieurs établissements. Il est donc probable que ce dernier ait des préférences sur des *séquences d'affectations* c.a.d. l'ensemble des affectations possibles de sa carrière. Par exemple, un enseignant peut préférer un établissement A, par exemple car il est situé dans sa ville natale, à un établissement B lui même préféré à C. Cependant, il peut préférer être affecté à C une période pour ensuite rejoindre A, appelons cette affectation (C,A), plutôt que de rester deux périodes à B, appelons cette affectation (B,B). En effet, comme nous l'avons mentionné en évoquant la préférence qu'ont les enseignant à être plus proches de leur région natale, préférer (C,A) à (B,B) est très probable en pratique. Les néotitulaires demandent des établissements défavorisés afin d'obtenir un bonus de points qui leur permettra plus tard de rejoindre leur région natale. Cependant, recouvrir ces préférences dynamiques à partir des observations de listes statiques, par exemple préférer A à B à C, n'est pas aisé. Dans la littérature des problèmes d'appariement, il y a un nombre croissant de travaux sur l'estimation statique des préférences: voir par exemple [Fack, Grenet and He \(2015\)](#), [Abdulkadiroglu, Agarwal and Pathak \(2015\)](#) ou [Calsamiglia, Fu and Guell \(2016\)](#). Utiliser ces techniques sur un problème dynamique pourrait amener à ne pas correctement prendre en compte les possibles stratégies dynamiques et éventuellement biaiser les résultats. En

économétrie structurelle, il y a une littérature importante de modèles de choix dynamiques, voir par exemple la revue de littérature de [Aguirregabiria and Mira \(2010\)](#). Appliquer ces techniques dans un contexte d'appariement dynamique serait prometteur, notamment, voir notre discussion politique dans la section suivante, en termes de politiques publiques.

### **Un modèle avec entrée et sortie endogènes des enseignants.**

Comme évoqué dans le Chapitre 2, un problème important au sein des pays de l'OCDE est le manque d'attractivité de la profession enseignante. Un autre problème de politique éducative est le fort taux d'abandon des jeunes enseignants<sup>22</sup>. Pour prendre en compte ces effets, il faut étudier les déterminants des *flux d'entrées et de sorties* des enseignants. Bien entendu, comme montré par [Boyd et al. \(2005\)](#), les établissements dans lesquels les enseignants sont affectés (ou anticipent être affectés) sont importants dans leur décision de rejoindre ou de sortir de la profession. Par exemple, si l'on se fixe l'objectif de minimiser le nombre de néotitulaires affectés à un établissement défavorisé, cela peut réduire le taux d'abandon de ces derniers. Cependant, comme mentionné dans le Chapitre 2, augmenter la mobilité des enseignants titulaires, et donc leur donner de meilleures perspectives de carrière, peut induire une augmentation du nombre de personnes rejoignant la profession. Cependant, il n'est pas clair que ces deux effets peuvent être conjointement atteints. Si les enseignants ont des préférences relativement corrélées, augmenter la mobilité des enseignants titulaires peut se faire au détriment des néotitulaires et vice-versa. En effet, les différents résultats exposés dans le Chapitre 2 de cette thèse semblent confirmer cet arbitrage. Développer un modèle formel qui incorporerait les décisions d'entrées et de sorties pourrait aider à quantifier ces effets. Couplé avec le projet ci-dessus d'estimation des préférences pour les titulaires et néo-titulaires, cela pourrait se révéler être un outil important de politique publique.

### **Un modèle sur les interactions entre l'affectation des enseignants et des élèves aux écoles**

Ces dernières dizaines d'années, la littérature sur les problèmes d'appariement a mené à une multitude de réformes politiques des systèmes d'affectation des élèves aux écoles. L'algorithme AD a été utilisé dans un grand nombre de villes dans le monde et une importante partie de la littérature s'est intéressée aux moyens d'accomplir certains objectifs politiques via la conception des priorités des

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<sup>22</sup>Aux Etats-Unis, ce problème a été souligné par [Ingersoll \(2003\)](#) ou encore [Boyd et al. \(2005\)](#). En France, il n'existe pas de données disponibles ou rapports quantifiant ces taux d'abandon. Dans leur rapport, [Cooper and Alvarado \(2006\)](#) semblent souligner que c'est un problème moindre en Allemagne, France ou Portugal.

écoles. Un agenda de recherche important a été celui des politiques de *discrimination positive* afin de donner un meilleur accès aux bons établissements aux étudiants de milieux sociaux défavorisés.<sup>23</sup> D'autres se sont concentrés sur les moyens de mettre en œuvre des contraintes de diversité dans les appariements, cherchant une certaine balance entre différents types d'étudiants affectés au sein des établissements<sup>24</sup>. En France, [Hiller and Tercieux \(2014\)](#) ont montré que l'algorithme Affelnet utilisé pour affecter les élèves aux lycées est équivalent à l'algorithme AD où les écoles proposent. Pour définir les priorités au sein d'Affelnet, un système de points est utilisé, octroyant des points à un étudiant selon différents critères: zone géographique, présence d'un frère ou d'une soeur dans l'établissement, notes ou encore statut social. La dernier critère est en effet un bonus de points aux étudiants issus de milieux aux revenus modestes et qui reçoivent une bourse du CROUS. En pratique, ce bonus est tellement important que tous les élèves le recevant ont une priorité plus élevée que ceux ne le recevant pas. A Paris par exemple, ce bonus a eu un impact significatif sur la composition des cohortes d'entrées d'élèves au sein de certains établissements<sup>25</sup> au point que, récemment, un important lycée parisien, le lycée Turgot, s'en est plaint<sup>26</sup>. Un point important ici est que ces plaintes étaient également émises par le corps enseignant de l'établissement. Comme montré par [Prost \(2013\)](#), les enseignants au sein d'établissements avec une plus forte proportion d'élèves aux résultats plus faibles ou d'élèves issus de la minorité ou d'élèves aux origines sociales plus modestes ont tendance à plus quitter ces établissements. Si la composition des élèves au sein d'un établissement est un déterminant important dans les préférences des enseignants, alors il faut considérer conjointement l'analyse des problèmes d'affectation des enseignants et des élèves aux écoles afin d'en capturer tout les effets potentiels. Inversement, si la qualité d'une école dépend significativement des types d'enseignants qui y sont affectés, alors les préférences des élèves pour les établissements va également dépendre de la distribution des enseignants au sein de ces derniers. Une politique importante qui affecte la distribution des élèves au sein des établissements, comme par exemple une politique de discrimination positive, peut avoir des effets indirects importants sur la distribution des enseignants

<sup>23</sup>Notre but n'est pas de donner une revue détaillée des articles. Le lecteur peut se référer aux deux articles de [Kojima \(2012\)](#) et [Hafalir, Yenmez and Yildirim \(2013\)](#).

<sup>24</sup>Voir par exemple [Erdil and Kumano \(2012\)](#), [Kamada and Kojima \(2014\)](#) ou encore [Echenique and Yenmez \(2015\)](#) parmi d'autres. Récemment, [Ashlagi and Shi \(2015\)](#) ont montré comment concevoir des systèmes de priorité sous l'algorithme AD afin de prendre en compte une multitude d'objectifs politiques comme les couts de transport, la diversité, l'efficacité...etc.

<sup>25</sup>Pour les effets de cette politique, le lecteur peut se référer au rapport de [Fack, Grenet and Benhenda \(2014\)](#)

<sup>26</sup>Il y a une multitude d'articles français de presse mentionnant ce cas. Voir par exemple "Affelnet : Paris amende le système d'affectation des lycées", *Le Monde* 3/24/17 ([link](#)) ou encore "Lycées : à Paris, des parents d'élèves se mobilisent contre le logiciel Affelnet", *Libération* 8/7/17 ([link](#))

au sein des écoles, ce qui pourrait in fine avoir des effets de retour néfastes pour la politique initiale. Afin de pleinement considérer ces effets, il faudrait considérer un modèle complet prenant en compte cette interdépendance. Dans le Chapitre 3, nous avons analysé les effets de cette interdépendance sur l'existence des concepts de base comme la stabilité. Analyser les effets dans un modèle plus positif, où les enseignants et les élèves sont affectés chacun séparément par l'algorithme AD et doivent soumettre des listes ordonnées d'écoles, serait intéressant afin d'étudier les effets de long terme des certaines politiques. En soumettant leurs listes, les enseignants et étudiants doivent anticiper l'affectation finale afin de choisir comment classer les écoles: les préférences deviennent endogènes.

## Discussions pratiques de politiques publiques

L'analyse contre factuelle exposée dans le Chapitre 2 de cette thèse a aboutit à plusieurs discussions politiques et deux articles de vulgarisation additionnels: une note de l'Institut des Politiques Publiques (Combe, Tercieux and Terrier, 2016c) et une publication, en français, dans un journal du Ministère de l'Education Nationale (Combe, Tercieux and Terrier, 2016a).

Nous avons pu discuter et présenter ces travaux avec plusieurs institutions importantes pour l'affectation des enseignants:

### 1. Administrations:

- La direction de la DEPP<sup>27</sup>, le département de statistiques du Ministère de l'Education Nationale.
- Certains directeurs de la DGRH<sup>28</sup>, le département de ressources humaines du Ministère de l'Education Nationale, en charge de l'allocation inter-académique et qui nous ont fourni les données utilisées au sein de nos travaux.
- Le (ex) directeur de cabinet de Madame la Ministre de l'Education Nationale<sup>29</sup>, conjointement avec les directions de la DEPP et de la DGRH.

### 2. Les syndicats enseignants:

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<sup>27</sup> *Direction de l'Evaluation de la Prospective et de la Performance*

<sup>28</sup> *Direction Générale des Ressources Humaines*

<sup>29</sup> En juin 2016, sous Madame la Ministre Vallaud-Belkacem.

- Le SNES-FSU<sup>30</sup> le principal syndicat des enseignants du secondaire public qui regroupe environ 80% des enseignants syndiqués.
- Le SGEN<sup>31</sup> le deuxième plus grand syndicat pour les enseignants du secondaire public.

### 3. Think Tanks:

- Terra Nova: un think tank français, identifié à gauche du spectre politique.
- France Stratégie: un think tank gouvernemental directement attaché au du bureau du Premier Ministre.

Comment mentionné dans l'Introduction de cette thèse, le syndicat principal, le SNES-FSU, est très attaché au principe *d'égalité de traitement* des enseignants. Pour eux, le système de barème de points pour classer les enseignants dans les académies et établissements est un outil qui permet d'assurer ce traitement égalitaire et les enseignants ne doivent pas être traités différemment pour des critères en dehors de ceux définis au sein du barème. Ils ont émis deux commentaires importants:

1. L'existence possible sous TO-BE *d'envies justifiées*, comme définies par [Pereyra \(2013\)](#), c.a.d. l'existence d'enseignants qui préfèrent une académie sous l'affectation de TO-BE et qui ont un plus haut barème (priorité) qu'un autre enseignant qui n'était pas initialement affecté à cette académie mais l'a été sous TO-BE.
2. La possibilité d'avoir des académies cibles différentes entre les différentes matières, ce qui, selon eux, est une violation du principe d'égalité de traitement.<sup>32</sup>

Concernant le premier point, un de leur argument est que, sous  $DA^*$ , il est possible de calculer ce qu'ils appellent des *barres d'affectation* pour chaque académie. Une barre correspond à ce qui est appelé communément un *cutoff score* dans la littérature d'appariement, c.a.d. le plus petit nombre de points de priorité parmi les enseignants ayant été affectés par l'algorithme au sein de cette académie mais qui n'y étaient pas initialement affectés. S'il n'y a aucune envie justifiée, alors

<sup>30</sup>Syndicat National des Enseignements de Second degré - Fédération Syndical Unitaire

<sup>31</sup>Syndicat Général de l'Éducation Nationale

<sup>32</sup>Dans notre analyse du Chapitre 2, nous avons utilisé la même règle de ciblage des académies entre différentes matières. Cependant, en pratique, les différentes matières ne font pas face aux mêmes problèmes structurels de congestion. Par exemple, les Mathématiques sont une matière qui a beaucoup moins de néotitulaires que de besoins de postes contrairement à, par exemple, l'Histoire-Géographie ou la Littérature. Dans une mise en œuvre pratique, le ciblage devra probablement être différent entre ces différentes matières.



n'importe quel enseignant qui n'est pas affecté à une académie qu'il préfère aura moins de points de barème que la barre d'affectation de cette académie. Cela leur permet de facilement vérifier leur motif de rejet et de "faire confiance" à la procédure de l'administration centrale. Comme évoqué à la fin de la Section 6.7 du Chapitre 2, nous savons que si l'on s'impose de n'avoir aucune envie justifiée, alors DA\* permet d'obtenir le plus grand mouvement possible. Si les critiques récurrentes des enseignants concernant les faibles taux de mouvement sous DA\* sont suffisamment importantes, alors il faudrait considérer la possibilité d'autoriser l'existence d'envies justifiées. En un sens, nous revenons, comme en littérature sur le choix d'école, à un arbitrage entre efficacité, avec TO-BE, et une notion de stabilité, avec DA\*. Cependant, même si la plupart des décideurs publics ont opté pour l'algorithme DA dans le problème d'affectation des élèves aux écoles, il n'est pas clair que ce dernier soit l'algorithme le plus naturel dans le cas de la réaffectation des enseignants. En effet, avec l'existence d'une allocation initiale, l'idée d'échanges par cycles peut être mieux acceptée et comprise par les participants que sous un cadre standard de choix d'école avec des élèves. C'est ce qui s'est passé en France dans le cas de l'affectation des enseignants du primaire: à cause d'un taux de mouvement très bas sous DA\*, ils décidèrent de réaliser des cycles d'échanges avec l'algorithme TTC par dessus l'affectation obtenue par DA\*<sup>33</sup>. Comme l'algorithme TTC ignore complètement la notion de stabilité en réalisant ses échanges, cette procédure peut créer un nombre important d'envies justifiées. Il semble que le législateur et les différents acteurs ont bien compris ce débat. Par exemple, le deuxième syndicat enseignant, le SGEN, semble plus ouvert à l'existence d'envies justifiées. Le directeur de cabinet était également bien conscient des raisons du manque de mouvement comme illustré dans notre exemple stylisé du Chapitre 1. Il mentionna plusieurs discussions pour autoriser des échanges de postes ex post au sein des comités académiques mais ces échanges, créant des envies justifiées, étaient refusés par les syndicats. Une critique mentionnée par le SGEN sur l'interdiction totale d'envies justifiées est que cela peut créer des cas pratiques d'inégalités très importantes. Par exemple, un enseignant célibataire sans enfants et affecté au sein d'une école relativement défavorisée se verra, à cause de la forte contrainte qu'impose l'interdiction d'envies justifiées, rester à son poste pendant une période très importante sans pouvoir échanger (ils mentionnèrent des cas allant de 10 à 15 ans). De même, le Mediateur de l'Education Nationale ([Mediator of the French Ministry of Education, 2015](#)), en charge de concilier les conflits entre le ministère et les enseignants, reçoit

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<sup>33</sup>Cet algorithme est cependant manipulable.

environ 700 plaintes d'enseignants du primaire et secondaire chaque année concernant des problèmes d'affectation. Il déclare que *“fondé sur l'exploitation d'un barème, l'algorithme d'affectation laisse ouvertes des portes sur des situations humaines difficiles et qui peuvent, à terme, entacher la qualité et l'investissement de la ressource et l'acte éducatif”*. Dans le rapport cité, il souligne ce points en disant, p54: *“Le barème est une chose, le processus d'affectation en est une autre”*, et il mentionne la nécessité d'un débat approfondi sur le système entièrement basé sur le barème. Un article de presse mentionnant nos travaux<sup>34</sup> écrivit également qu'ils questionnent *“une lecture plus souple du barème”* afin de gagner en efficacité. Ce débat est important et le choix final relève d'un choix politique et public. Nous espérons que nos travaux ont aidé ce débat en quantifiant les possibles améliorations et arbitrages.

Concernant le second point de commentaire, il semble que leur notion d'égalité de traitement soit considérée ex ante: chaque enseignant doit être traité de façon égalitaire avant son affectation en utilisant seulement ses points de barème. Cependant, chaque matière peut être considérée comme un marché indépendant avec ses problématiques propres. Comme mentionné, un problème récurrent en Mathématiques est le manque important de nouveaux enseignants. Chaque année, un grand nombre de places au concours restent non affectées. Les néotitulaires étant peu nombreux dans cette matière, ils peuvent facilement obtenir une affectation dans une académie qu'ils souhaitent. Leur trajectoire de carrière est donc différente. En Histoire-Géographie par exemple, le scénario inverse se produit: il y a plus de candidats que de positions disponibles au concours et la plupart des néotitulaires se retrouvent affectés à une académie défavorisée. Toutes ces différences structurelles créent des inégalités d'affectation ex post, c.a.d. dans les types de carrières et les difficultés de travail. Notre point est que, pour obtenir une certaine égalité ex post des affectations, il est nécessaire de discriminer ex ante à travers, par exemple, l'outil de ciblage des régions au sein de TO-BE. Cependant, leur critique semble plus liée à la peur d'un potentiel système opaque, où les ciblage et objectifs peuvent changer chaque année et potentiellement après des essais au sein même du lancement de l'algorithme. Nous reconnaissons pleinement que la transparence d'un tel système est importante car un système opaque risque de nuire à la non manipulabilité de la procédure. Un système avec une règle officielle et contraignante d'options de ciblage basée sur des critères objectifs et vérifiables par tous, discutés et décidés par l'administration et les syndicats, serait une nécessité

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<sup>34</sup>TOUT EDUC, (8/4/2016): *“Mouvement des enseignants : pour plus d'efficacité, l'Ecole d'économie de Paris propose une lecture souple du barème ”*[\[link\]](#)

en cas de mise en oeuvre pratique d'un algorithme tel que TO-BE. Cela aiderait à la mise en place d'un système d'affectation transparent et digne de confiance pour les différents acteurs.

Sur un autre sujet, nous avons discuté avec Terra Nova de solutions afin d'augmenter la proportion d'enseignants titulaires entrant à Amiens, Créteil ou Versailles tout en conservant un taux de mobilité similaire à celui de l'algorithme DA\* actuel pour les enseignants demandant à sortir de ces académies. En effet, un nombre important de néotitulaires sont affectés au sein de ces trois académies ce qui, comme mentionné plus haut, peut être considéré comme un problème concernant la réussite des élèves. Une première observation est, qu'en utilisant les préférences soumises dans nos données de 2013, peu d'enseignants titulaires classent ces académies. Par exemple, sur les 12 123 enseignants titulaires du Chapitre 2, 159 classent Amiens dans leur liste, 260 Créteil et 371 Versailles<sup>35</sup>. Pour ces enseignants, le rang moyen d'Amiens, Créteil et Versailles est respectivement 5.53, 3.22 et 3.11. Il est intéressant de noter que le rang médian est de 2 pour les trois académies, donc la moitié de ces enseignants classent ces régions premier ou second<sup>36</sup>. On peut en déduire qu'avec les mêmes préférences soumises, un algorithme d'affectation différent peut difficilement augmenter la proportion de titulaires affectés dans ces académies puisque peu d'entre eux les classent au sein de leurs listes de préférences. Cela montre que cet objectif nécessite des outils qui vont au delà d'un simple changement d'algorithme: il faut inciter les enseignants titulaires à demander ces académies. Cela peut être réalisé via plusieurs outils, par exemple:

- Un bonus de salaire pour les enseignants affectés à des écoles d'éducation prioritaire. Il semble que, ces dernières années, le gouvernement français a mis en place ces bonus. Par exemple, depuis Septembre 2016, chaque enseignant au sein d'un établissement d'éducation prioritaire reçoit au moins 1734 euros et jusqu'à 2312 euros supplémentaires par an<sup>37</sup>. Récemment, Monsieur le Président Macron proposa dans son programme éducatif d'étendre ce bonus jusqu'à 3000 euros par an<sup>38</sup>.
- Une amélioration des conditions de travail au sein des écoles d'éducation prioritaire:

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<sup>35</sup>Parmi ces enseignants titulaires, respectivement 101, 145 et 217 sont des titulaires avec une affectation initiale. Pour les 7106 néotitulaires, 2219 classent Amiens, 2562 Créteil et 3155 Versailles.

<sup>36</sup>Pour les néotitulaires, les moyennes de rang pour Amiens, Créteil et Versailles sont de respectivement 8.52, 6.9 et 6.65 et les médianes respectivement 6, 3 et 3.

<sup>37</sup>Voir <http://www.education.gouv.fr/cid92275/education-prioritaire-un-nouveau-regime-indemnitare-en-faveur-des-personnels.html>

<sup>38</sup>Voir <https://en-marche.fr/emmanuel-macron/le-programme/education>

- La diminution des heures d’enseignement au sein de ces établissements. Cela a été mis en place en 2014<sup>39</sup>. Un problème important qu’une telle politique peut créer est le besoin de personnel enseignant supplémentaire que cela engendre au sein de ces établissements qui sont déjà très peu demandés. Cela crée le risque de faire appel à des enseignants temporaires ou vacataires ayant peu d’expérience.
- Augmenter le budget de ces écoles.

Bien que l’ensemble de ces outils soient importants, on sait peu de chose sur leur impact pratique sur les préférences des enseignants. Par exemple, comme mentionné dans l’introduction du Chapitre 1, [Hanushek, Kain and Rivkin \(1999\)](#) trouvèrent, avec des données sur le Texas, que la mobilité des enseignants était plus affectée par les caractéristiques des élèves que par les salaires. En France, [Prost \(2013\)](#) utilisa les données françaises sur les mouvements des enseignants du secondaire pour estimer les déterminants de la mobilité de ces derniers. Elle trouva que les incitations monétaires qui avaient été mises en place à l’époque n’étaient pas suffisantes pour retenir les enseignants des écoles défavorisées. Elle trouva que “les enseignants ont tendance à changer d’écoles quand ils travaillent dans des établissements avec une forte proportion d’élèves en difficultés scolaires, d’élèves de groupes minoritaires et/ou d’élèves issus de milieux sociaux défavorisés. De plus, les enseignants qui ne travaillent pas dans la région dans laquelle ils sont nés sont plus à même de bouger”. Récemment, [Benhenda and Grenet \(2016\)](#), en utilisant également les données françaises de mouvement, étudièrent l’impact d’un changement de barème de priorité pour les enseignants des écoles d’éducation prioritaire. En 2005, le nombre d’années d’enseignement nécessaires au sein d’un de ces établissements pour obtenir un bonus de barème a été étendu de 3 à 5 ans. Leurs résultats semblent montrer que cela a eu un effet positif sur le nombre d’années que ces enseignants passent au sein de ces établissements mais un effet négatif sur la qualité des enseignants entrant dans ces établissements. Tous ces résultats utilisent uniquement les données de mouvement et non les données de préférences soumises des enseignants. Des techniques récentes d’estimation de préférences dans la littérature d’appariement<sup>40</sup> offrent des outils intéressants pour estimer l’impact de ces politiques sur les préférences des enseignants et identifier précisément les meilleures politiques. Comme mentionné dans la section précédente, l’affectation des enseignants a une nature dynamique

<sup>39</sup>Voir [http://www.education.gouv.fr/pid25535/bulletin\\_officiel.html?cid\\_bo=87302](http://www.education.gouv.fr/pid25535/bulletin_officiel.html?cid_bo=87302)

<sup>40</sup>Voir [Fack, Grenet and He \(2015\)](#), [Abdulkadiroglu, Agarwal and Pathak \(2015\)](#) ou [Calsamiglia, Fu and Guell \(2016\)](#).

et pourrait nécessiter d'utiliser des techniques additionnelles que celles actuelles afin de précisément estimer leurs préférences. C'est une question technique ambitieuse et un outil nécessaire à développer pour les décideurs politiques.

Une explication possible discutée avec Terra Nova sur pourquoi les enseignants titulaires ne veulent pas demander les académies de Amiens, Créteil et Versailles est que leur affectation y serait définitive. Afin d'en sortir, ils doivent demander une nouvelle affectation qu'ils ne seraient pas sûr d'obtenir. Une possibilité que Terra Nova a proposée est de permettre des *échanges temporaires* de postes entre des néotitulaires affectés à l'une de ces académies et des titulaires volontaires qui seraient prêts à enseigner temporairement dans ces académies. La période de l'échange peut être déterminée à l'avance, par exemple 3 ans. L'avantage est, qu'après ces 3 ans, l'enseignant titulaire peut automatiquement revenir à son ancien poste pré-échange et que le néotitulaire doit revenir prendre son poste au sein de l'académie défavorisée. L'autre aspect positif est que le néotitulaire aura ainsi accumulé plus d'expérience et serait mieux préparé. Pour l'enseignant titulaire souhaitant faire l'échange, il est assuré de ne pas perdre son précédent poste. Comme mentionné précédemment, le nombre d'enseignants titulaires prêts à faire cet échange est inconnu et une estimation précise serait nécessaire. En termes pratiques, la mise en œuvre serait standard: on peut penser à une procédure post-affectation qui demanderait aux enseignants titulaires et néotitulaires désirant faire un échange de soumettre une liste de régions désirées. En utilisant un algorithme standard d'échange de type TTC, on peut trouver les échanges à réaliser. La principale différence serait, qu'après une période fixée, tous les enseignants de l'échange devront revenir à l'affectation initiale qu'ils avaient avant l'échange.

## General Conclusion

This thesis focused on the problem of assigning teachers to schools. We showed that this problem is at the intersection of two important classes in the matching literature: the two-sided matching problem and the house allocation one. In Chapter 1, we proposed a theoretical framework to study the assignment of teachers. We showed that the actual procedure identified in the matching literature and used in practice in France has an important drawback: it is possible to reassign teachers to schools such that both of them are “better-off”. We theoretically explored the class of mechanisms that does not suffer from this drawback and identified a unique one that gives incentives for the

teachers to report their true preferences. In Chapter 2, we explored the practical design of such algorithm for the French assignment of public secondary school teachers and accommodated several practical constraints of this specific market. We offered an algorithm that can be parametrized to incorporate a wide range of regional policy objectives in terms of movement of teachers and their distribution. More generally, this algorithm can be a useful policy tool to help mitigate two central problems in OECD countries: i) the lack of attractiveness of the teaching profession and ii) the important achievement inequality between students of different social backgrounds. In Chapter 3, we developed a new matching framework inspired by the assignment of teachers and students to schools. Pairs of agents have to be assigned together to objects, e.g. managers and workers have to be assigned to projects. Each agent (manager or worker) has preferences over the agent-object pairs he can be assigned to. We explored the standard question of the existence of a stable matching in this setting. We showed that a strong stability notion where a manager or a worker can freely dismiss his partner and ask another one to join his project may fail to exist. However, we showed that once one properly defines an ownership structure over the projects, i.e. which agent is able to dismiss his partner and ask another one to join then a stability notion exists. This notion has a natural structure: it always gives the ownership of the project to the same type of agents, e.g. always the managers. Even if these stable matchings share similarities with the ones in a standard two-sided matching setting, we showed that, contrary to the standard framework, a natural core notion can be empty.

## Future research

The works developed in this thesis are part of a more general research agenda about the assignment of teachers to schools. We enumerate here a number of future research in addition to those mentioned in the conclusion of each chapter of the thesis.

### A stochastic model of reassignment

The analysis exposed in the Chapters 1 and 2 of the thesis was focusing on *ex post* mechanisms that return an assignment of teachers to schools. In doing so, we imposed the Individual Rationality (IR) constraint on the final assignment: each teacher must always receive a school that he weakly prefers to the one he was initially matched. In the case of the French assignment procedure, it is not clear that it is the case. Indeed, as we described, the assignment process is divided into two phases:

1. An assignment between regions: the teachers willing to move from their region to another one are asked to report their preferences over the 31 French regions. The IR constraint is imposed on their initial region: they are sure to be assigned to a region that they weakly prefer to their initial one. Once the assignment is determined, it is final.
  
2. An assignment within regions: once the regional assignment of the first phase is given, teachers who were assigned to a region and those currently teaching inside this region but who would like to move to another school, are asked to report their preferences over the schools inside the region. If a teacher was currently teaching in a school, the IR constraint ensures him to get a school that he weakly prefers to his initial one. However, a teacher who has been assigned to the region at the first phase is considered as having no initial school, so that he can end up being unassigned at the end of the phase. In that case, the ministry assigns him to a *replacement zone*, that tries to cover the area of his first ranked school, where this teacher will spend a year as a substitute teacher. After this year, he has to participate again to the assignment phase within the region.

When applying to the first phase to move from his region, a teacher faces the risk of not being assigned at the second phase. One can model this risk as a belief of the teacher over the possible schools he might be assigned, i.e. probability distribution over the possible schools. Theoretically, one can consider *stochastic mechanisms* of reassignment, i.e. that return a probability distribution over the possible assignments. Each teacher has a cardinal utility for each school and evaluates a given probability distribution over assignments using an expected utility. The IR constraint can now be *interim* and not *ex post*: for a given probability distribution over assignments, called a *probabilistic assignment*, the expected utility of a teacher must be greater than his utility for his initial school. This approach implies that, for some realizations of the probability distribution, the teacher can possibly be assigned to a school with a lower utility than his initial one. This is not allowed with an *ex post IR constraint*. This relaxation of the IR constraint can potentially allow for more exchanges and would be an interesting theoretical exercise. However, in practice, one would have to know or ask the cardinal utility values of each teacher, a difficult exercise. Moreover, it is well known in the game theory literature that *ex post* constraints are *more robust solutions* when there exists some uncertainty about the primitives of the model (such as cardinal utility values, the

information that each teacher has...etc).<sup>41</sup>

### **A characterization of our generalized TO-BE algorithm**

In Chapter 2, we provided a generalization of the TO-BE algorithm found in Chapter 1 to accommodate vacant positions and newly tenured teachers without any initial assignment. As we mentioned, the problem is similar to the one of extending the TTC algorithm for the housing market as in [Shapley and Scarf \(1974\)](#) to the YRMH-IGYT algorithm proposed by [Abdulkadirouglu and Sonmez \(1999\)](#). Indeed, the latter extends the TTC algorithm used to exchange initially occupied houses between tenants to the case where there are initially non assigned agents looking for a house and vacant houses. Our generalization of TO-BE is a generalization of the YRMH-IGYT algorithm incorporating two additional features: i) a many-to-one framework where schools have several seats and ii) the existence of “preferences” for the schools. [Sonmez and Unver \(2010\)](#) provided a complete characterization of the YRMH-IGYT algorithm.<sup>42</sup> A complete axiomatic characterization for our generalized TO-BE algorithm would allow us to exhibit a broader class of mechanisms.

### **A dynamic model of reassignment**

All the models in this thesis took a static approach for the reassignment problem. The dynamic was considered as a primitive through the initial allocation. In the case of the French assignment phase between regions, this static approach can be justified. Indeed, as found by [Prost \(2013\)](#), the main determinant of the inter-regional mobility of teachers is their willingness to be closer to the region where they were born. So once a teacher is able to reach this region, he will probably never apply again to assignment phase between regions.<sup>43</sup> So in considering the pure reassignment problem of Chapter 1, the static approach can be justified. However, in Chapter 2, it would be interesting to investigate a simple dynamic model: a teacher has to apply as a newly tenured teacher for his first assignment and can apply for a reassignment only once during his career. As showed in Chapter 2, a change of algorithm affects both the quality of the match for newly tenured teachers and the one for tenured teachers asking for a reassignment. The long run effects are not fully clear and a complete theoretical analysis would be needed. Recently, there has been an increasing interest for dynamic

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<sup>41</sup>See for instance the excellent survey of [Kajji and Morris \(1997\)](#) and, for mechanism design, [Bergemann and Morris \(2005\)](#).

<sup>42</sup>The axioms being technical, the reader can refer to the cited article for a complete description.

<sup>43</sup>Obviously, he can be willing to move inside the region. There are no public statistics about the average number of times a teacher participates to the between-regions assignment phase. In this thesis, we focused on the 2013 assignment data. However, it can be possible to recover such information in merging all the datasets prior to 2013.



models in the matching literature. Our goal is not to do a complete review of these models, the interested reader can refer to the literature review sections of the next cited papers. They can be divided into two types:

1. Models trying to extend the stability or efficiency notions to a dynamic setting and that investigate their existence: [Damiano and Lam \(2005\)](#), [Kurino \(2009\)](#), [Pereyra \(2013\)](#), [Doval \(2017\)](#).
2. Simple dynamic models solving for dynamically optimal mechanisms: [Bloch and Cantala \(2013\)](#), [Bloch and Cantala \(2017\)](#), [Leshno \(2017\)](#), [Baccara, Lee and Yariv \(2015\)](#).

The idea would be to define a simple toy dynamic model similar to the ones in the papers of the second type. The main difference is that, in their models, once agents are matched, they definitely leave the market. It is not true in our context: the quality of one teacher's assignment today can affect his future willingness to apply for a reassignment. The dynamic stability notion explored by the papers of the first type seems to be less of a concern in practice. Participants of the market seem to be more concerned by the static notion of stability.<sup>44</sup> From a normative point of view, it seems, see our policy discussions below, that it is not a concern for policy makers or unions. From a positive point of view, it would be interesting to explore whether practical dynamic mechanisms tend to be abandoned if they are not dynamically stable. The French procedure, DA\*, is not dynamically stable but has been used for many years.

### **A structural estimation of dynamic preferences of teachers**

The French procedure asks each teacher willing to move to report a list of preferences over regions (or schools). However, in his career, a teacher will be matched to several schools. So he is likely to have preferences over *sequences of assignments* i.e. his possible complete career assignments. For instance, a teacher can prefer school A, say because it is located in his hometown, to school B to school C. However, he can still prefer to be matched to school C one period and then to school A, call this assignment (C,A) than being assigned the two periods to school B, call this assignment (B,B). Indeed, referring to our previous discussions on teachers' preference to be closer to their home-region, preferring (C,A) to (B,B) is likely in practice. Newly tenured teachers ask to join disadvantaged schools to obtain a bonus of points that will allow them to later join their home-region. However, it

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<sup>44</sup>With the notable exception of [Pereyra \(2013\)](#), see our discussion in Section 3.5 below.

is not clear how to recover this information in observing preference lists only over the schools i.e. A preferred to B preferred to C. In the matching literature, there has been an increasing number of works on the structural estimation of static preferences: see for instance [Fack, Grenet and He \(2015\)](#), [Abdulkadiroglu, Agarwal and Pathak \(2015\)](#) or [Calsamiglia, Fu and Guell \(2016\)](#). Directly applying those techniques on a dynamic problem, missing the possible dynamic strategies, could lead to biased estimates. In structural econometrics, there has been an important literature on dynamic discrete choice models, see for instance the survey of [Aguirregabiria and Mira \(2010\)](#). Applying such techniques in a dynamic matching context would be of interest, especially, see our policy discussions below, for policy making purposes.

### **A model with endogenous entry and exit of teachers**

As mentioned in Chapter 2, an important issue in OECD countries is the lack of attractiveness of the teaching profession. Another important concern for policy making is the high drop out rate of newly tenured teachers. To deal with these issues, one has to study the determinants of the *inflows* and *outflows* of teachers. Obviously, as showed in the U.S by [Ingersoll \(2003\)](#) or [Boyd et al. \(2005\)](#), the schools to which the teachers are assigned (or expect to be) matter in their decisions of entry or drop out.<sup>45</sup> For instance, aiming to minimize the number of newly tenured teachers assigned to a disadvantaged school can decrease the drop out of such teachers. As mentioned in Chapter 2, increasing the mobility of the tenured teachers, and so giving them better career perspectives, can lead to an increase of the number of people joining the profession. However, it is not clear that these two effects can be achieved together. Indeed, if there is correlation in the preferences of teachers, increasing the mobility of the tenured teachers will be done at the expense of the newly tenured ones and vice-versa. The different results exhibited in Chapter 2 of this thesis seem to confirm this trade-off. Developing a formal model that incorporates entry and exit decisions can help to quantify these effects. Coupled with the above project on the estimation of the preferences for newly tenured and tenured teachers, it can become an important tool and guidance for policy makers.

### **A model concerning the interactions between the assignments of teachers and students to schools**

Over the past decade, the matching literature led to many policy reforms concerning the design

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<sup>45</sup>In France, there is no available data or reports quantifying these drop outs. In their report, [Cooper and Alvarado \(2006\)](#) seem to claim that the dropout of teachers is much less of a concern in France, Germany or Portugal.

of centralized systems of assignment of students to schools. The DA algorithm has been widely implemented in many cities around the world and a part of the literature has focused on the ways to achieve certain policy goals through the careful design of schools' priorities. An important research agenda has focused on *affirmative action* policies that aim to give access to better schools to students from disadvantaged social backgrounds.<sup>46</sup> Another literature has focused on how to implement diversity constraints in a matching setting, seeking for a balance between different types of students.<sup>47</sup> In France, [Hiller and Tercieux \(2014\)](#) showed that the Affelnet algorithm used to assign students to high schools is equivalent to the DA school-proposing algorithm. To define priorities in Affelnet, a point system is used that grants points to students depending on several criteria: geography, whether he/she has a sibling in the school, grades and social status. The last aforementioned criterion is a bonus of points for all students from low income families that receive public grants called CROUS. Actually, this bonus of points is such that all the students receiving it end up having more points than those who do not. In Paris for instance, this bonus had a significant impact on the composition of the entering students in some high schools<sup>48</sup> to the point that, recently, an important Parisian high school, *lycée Turgot*, complained.<sup>49</sup> An important point for us is that these complaints also came from the teachers of this high school. As showed by [Prost \(2013\)](#), teachers in schools with higher proportions of low achieving students or students from minority groups or from disadvantaged social backgrounds have a higher probability to quit these schools. If teachers care about the composition of the students in the schools, then one has to jointly consider the assignments of teachers and students to schools in order to capture all the effects on both markets. On the other way around, if the quality of a school is significantly determined by the teachers assigned to it, then the preferences of the students over the schools also depend on the distribution of the teachers over the schools. An important policy affecting the distribution of the students in the schools, as for instance an important affirmative action policy, can have indirect impacts on the distribution of the teachers that can, in the long run, have potential counter effects on the initial policy. To fully consider all the effects, one

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<sup>46</sup>We do not aim to review the articles on the subject, the reader can refer to the two seminal papers of [Kojima \(2012\)](#) or [Hafalir, Yenmez and Yildirim \(2013\)](#).

<sup>47</sup>See for instance [Erdil and Kumano \(2012\)](#), [Kamada and Kojima \(2014\)](#) or [Echenique and Yenmez \(2015\)](#) among many others. Recently, [Ashlagi and Shi \(2015\)](#) also showed how to design priority systems under the DA algorithm to implement diverse policy objectives such as bus costs, diversity, efficiency and so on.

<sup>48</sup>For the detailed effects, we refer the reader to the report (in french) of [Fack, Grenet and Benhenda \(2014\)](#).

<sup>49</sup>There are many available articles in the French news. See for instance “Affelnet : Paris amende le système d'affectation des lycées”, *Le Monde* 3/24/17 ([link](#)) or “Lycées : à Paris, des parents d'élèves se mobilisent contre le logiciel Affelnet”, *Libération* 8/7/17 ([link](#))

has to develop a complete model taking into account the interdependence of the teachers and the students' assignments. In Chapter 3, we analyzed in a general matching framework the effects of such interdependence on the existence of standard concepts such as stability. Analyzing the effects in a positive model, where teachers and students are assigned using a standard DA algorithm and are asked to report an ordered list of schools would be of interest. It would allow to study the long run impact of some policies such as affirmative actions. In reporting their preferences, teachers and students would have to anticipate the resulting matching to choose how to rank schools: preferences would be endogenous.

## Practical policy discussions

The counterfactual analysis exposed in Chapter 2 of this thesis led to several policy discussions and two additional vulgarized papers: a Policy Brief from the Institut des Politiques Publiques (Combe, Tercieux and Terrier, 2016c) and a publication, in french, in a policy making journal from the French Ministry of Education (Combe, Tercieux and Terrier, 2016a).

We have been able to discuss and present our results with several important institutions relevant for the assignment of teachers in France:

### 1. Administrations:

- The board of directors of the DEPP,<sup>50</sup> the statistical department of the French Ministry of Education.
- Some directors of the DGRH,<sup>51</sup> the department of human resources of the French Ministry of Education, in charge of the assignment phase between regions for french teachers and who provided the data we used in our analysis in partnership with the DEPP.
- The (former) Chief of Staff of the French Minister of Education,<sup>52</sup> together with the heads of the aforementioned DEPP and DGRH.

### 2. Teacher unions:

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<sup>50</sup>*Direction de l'Evaluation de la Prospective et de la Performance*

<sup>51</sup>*Direction Générale des Ressources Humaines*

<sup>52</sup>In June 2016 under the Minister Vallaud-Belkacem.

- The SNES-FSU,<sup>53</sup> the main teacher union for the secondary public school teachers, regrouping roughly 80% of the union registered teachers.
- The SGEN,<sup>54</sup> the second biggest union for secondary public school teachers.

3. Think Tanks:

- Terra Nova: a French think tank, identified at the left of the French political scale.
- France Stratégie: a public think tank directly attached to the French Prime Minister.

As mentioned in the Introduction of the thesis, the main teacher union, the SNES-FSU, is really attached to what they call the *equal treatment of teachers*. For them, the point system used to rank teachers is a way to ensure such equal treatment and teachers should not be treated differently using criteria outside the ones defined by the point system. They expressed two important concerns:

1. The possible existence under TO-BE of *justified claims* as defined by [Pereyra \(2013\)](#), i.e. the existence of teachers who prefer a region to their TO-BE assignment and who have a higher priority than a teacher who is not initially assigned to this region but has been matched to it under TO-BE.
2. The possibility to have different targeted regions across different subjects that, according to them, is breaking the equal treatment principle.<sup>55</sup>

For the first concern, one of their arguments is that, under DA\*, they can calculate what they call *assignment bars* for each region. A bar corresponds to what is referred to as *a cutoff score* in the matching literature, i.e. the lowest score among the teachers assigned to a region by the algorithm but who were not initially matched to that region. If there is no justified claim, then any teacher not assigned to a region and who prefers it will have a lower priority score than the cutoff. It allows him to easily check his rejection motive and to “trust” the central administration procedure. As mentioned in Section 2.6.7 in Chapter 2, we know that if one does not want any justified claim, then DA\* achieves the highest possible movement. If the criticisms from teachers concerning the low

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<sup>53</sup>Syndicat National des Enseignements de Second degré - Fédération Syndicale Unitaire

<sup>54</sup>Syndicat Général de l'Éducation Nationale

<sup>55</sup>In our analysis in Chapter 2, we applied the same rule for targeting regions and teachers across subjects. However, in practice, different subjects do not face the same congestion problems. For instance, Mathematics is a subject that has much fewer entering newly tenured teachers than available seats, contrary to History-Geography or Literature. In a practical implementation, the targets will probably need to be different across subjects.

movement under  $DA^*$  are important enough, then one has to consider allowing justified claims. In a sense, we come back, as in the school choice literature, to a trade-off between efficiency, under TO-BE, and a notion of fairness, under  $DA^*$ . However, even if most of the policy makers chose the  $DA$  algorithm to assign students to schools, it is not clear that, in the context of the reassignment of teachers,  $DA^*$  is the most natural algorithm. Indeed, with the existence of an initial allocation, the idea of trading cycles can be better understood by participants than under the standard school choice framework. This is what actually happened in France for primary school teachers: due to a very low movement under  $DA^*$ , they decided to additionally run the TTC algorithm on top of the  $DA^*$  matching.<sup>56</sup> Since TTC completely ignores the fairness of the allocation, this procedure can potentially lead to an important number of justified claims. It seems that French policy makers and the different participants have well understood this debate. For instance, the second teacher union, the SGEN, is more open to consider justified claims. The Chief of Staff was well aware of the lack of movement illustrated in our stylized example of Chapter 1. He mentioned several discussions to allow ex post exchanges of positions during the regional committees but these exchanges, creating justified claims, were refused by the unions. One criticism mentioned by the SGEN about the current system, that forbids justified claims, is that it can create important inequalities. For instance, a single teacher with no children assigned to a relatively deprived school will potentially stay at his position for an important amount of time (they mentioned examples up to 10-15 years). Under TO-BE, such teacher would be more easily able to exchange his position. Along this line, the [Mediator of the French Ministry of Education \(2015\)](#), in charge of conciliating conflicts between the Ministry and teachers, receives around 700 complaints from primary and secondary school teachers, every year, related to assignment issues. He states that “*based on a score system, the assignment algorithm opens doors to difficult personal situations that can, eventually tarnish the quality and the investment of human resources*”. In the cited report, he highlighted this point in saying, p54: “*The point system is one thing, the assignment process another one*”, and mentioned the necessity to debate about the point system. An article mentioning our work<sup>57</sup> also wrote that it questions “*a softer reading of the point system*” to gain in efficiency. This debate is important and the final choice is a matter of public choice. We hope that our work helped the debate in quantifying the possible improvements

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<sup>56</sup>This procedure is not strategy-proof.

<sup>57</sup>TOUT EDUC, (4/8/2016): “*Mouvement des enseignants : pour plus d’efficacité, l’Ecole d’économie de Paris propose une lecture souple du barème*”[\[link\]](#)

and trade-offs.

For the second concern, it seems that their notion of equal treatment is *ex ante*: every teacher has to be treated equally prior to the assignment solely based on his priority. However, each subject can be considered as an independent market with its own issues. As mentioned, a recurrent problem in Mathematics is the lack of newly tenured teachers. Every year, an important number of positions remains unfilled, notably in Créteil and Versailles. The newly tenured teachers being few, they can easily be assigned to a region that they desire. Thus, their career trajectory is different than the one of the teachers teaching other subjects. In History-Geography for instance, the reverse occurs: there are more candidates than open recruitment positions and most of the newly tenured teachers end up being assigned to deprived regions. All of these structural differences create an *ex post* assignment inequality. Our point is that, to create an *ex post* assignment equality, one needs to discriminate *ex ante* through, for instance, the tool of targeting regions in TO-BE. However, their concern seems to be related to the fear of an opaque system, where targets and objectives can be changed yearly and potentially after trial and errors during the assignment process. We acknowledge that the transparency is important since opaque systems can hurt the strategy-proofness of the procedure. Our view is that an official rule to commit to targeting options according to objective criteria, discussed and decided by the administration and the unions, would help to establish a transparent and trusted assignment procedure.

On another topic, we discussed with Terra Nova solutions to increase the proportion of entering tenured teachers at Amiens, Créteil and Versailles while keeping the same exit rate for those asking to leave. As we mentioned in Chapter 2, doing so would improve the achievement of students in these regions without hurting the teachers. Indeed, an important number of newly tenured teachers are assigned to these three regions. A first observation is that, in the reported preferences in our 2013 data, very few tenured teachers rank these three regions. For instance, among the 12 123 tenured teachers in Chapter 2, 159 ranked Amiens, 260 Créteil and 371 Versailles.<sup>58</sup> For these teachers, the average rank of Amiens, Créteil and Versailles is respectively 5.53, 3.22 and 3.11. Interestingly, the median for the three regions is at 2 meaning that half of them rank these regions first or second.<sup>59</sup> The point is that, with the same reported preferences, the assignment algorithm can hardly increase

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<sup>58</sup>Among these tenured teachers, respectively 101, 145 and 217 are tenured teachers with an initial allocation. For the 7106 newly tenured teachers, 2219 rank Amiens, 2562 Créteil and 3155 Versailles.

<sup>59</sup>For newly tenured, the means for Amiens, Créteil and Versailles are respectively 8.52, 6.9 and 6.65 and the medians respectively 6, 3 and 3.

the proportion of tenured teachers in these regions since only few of the tenured teachers rank them. It shows that this goal requires policy tools that are beyond a simple change of algorithm: one has to give incentives to tenured teachers to rank these regions. This can be achieved using several tools, for instance:

- A salary bonus to those choosing to teach in disadvantaged schools. It seems that, recently, the French government has taken a step toward it. For instance since September 2016, each teacher in a disadvantaged school receives at least 1734 euros per year and up to 2312 euros per year on top of his salary.<sup>60</sup> President Macron, during his campaign, proposed in his education program to extend this bonus up to 3000 euros per year.<sup>61</sup>
- An improvement of the job quality in these disadvantaged schools:
  - To decrease the teaching load for the teachers in disadvantaged schools. This has been in place since 2014.<sup>62</sup> An important problem is that it creates a need to recruit more teachers in these regions that are already unpopular. So the risk is to fill the empty positions with temporary inexperienced teachers.
  - To increase the budget for these schools.

Even if all these tools are important, little is known about their practical impact on teachers' preferences. As mentioned in the introduction of Chapter 1, [Hanushek, Kain and Rivkin \(1999\)](#) found, using data from Texas, that teachers' mobility is more affected by students' characteristics than wages. In France, [Prost \(2013\)](#) used the data on French teachers' movements to estimate the determinants of their mobility. She found that the monetary incentives at that time were not sufficient enough to retain teachers in disadvantaged schools. She found that *“teachers tend to switch schools when they work in establishments with a high proportion of less able students, students from minority groups and/or students from economically disadvantaged backgrounds. In addition, teachers who do not work in the region where they were born are more likely to move”*. Recently, [Benhenda and Grenet \(2016\)](#), also using the French movement data, studied the impact of a change in the priority system for teachers in disadvantaged schools. In 2005, the number of years of teaching in a

<sup>60</sup>See <http://www.education.gouv.fr/cid92275/education-prioritaire-un-nouveau-regime-indemnitare-en-faveur-des-personnels.html>

<sup>61</sup>See <https://en-marche.fr/emmanuel-macron/le-programme/education>

<sup>62</sup>See [http://www.education.gouv.fr/pid25535/bulletin\\_officiel.html?cid\\_bo=87302](http://www.education.gouv.fr/pid25535/bulletin_officiel.html?cid_bo=87302)



disadvantaged school required to get a bonus of points in their priority score was extended from 3 to 5 years. Their first results seem to show that it had a positive effect on the number of years teachers stayed in a disadvantaged school but a negative effect on the quality of the teachers entering these schools. All these results only used the movement data and not the reported preferences of the teachers during the assignment procedure. Recent preference estimation techniques in matching<sup>63</sup> offer promising tools to estimate the impacts of the above policies on the teachers' preferences. As mentioned in the previous section of this conclusion, the assignment market for teachers is dynamic and so can require additional techniques than the actual ones to precisely identify the preferences of the teachers. It is a technically challenging question and a necessary tool to develop for policy making.

One possible explanation discussed with Terra Nova about why tenured teachers are not willing to move to Amiens, Créteil or Versailles can be that their assignment there would be definitive. To exit these regions, they would have to ask for a new assignment that they would not be sure to obtain. One possibility that Terra Nova proposed is to allow a *temporary exchange* of positions between a newly tenured teacher assigned to one of these regions and a tenured teacher who would be ready to temporarily teach in these regions. The period of the exchange can be determined in advance, say 3 years. The advantage is that, after these 3 years, the tenured teacher is entitled to automatically come back to his position before the exchange. The newly tenured teacher would also have to come back to teach in the disadvantaged region. The other positive aspect would be that this newly tenured teacher would have acquired experience and would be better prepared to teach. As for the tenured teacher willing to do the exchange, it ensures him not to lose his previous position. As mentioned before, it is not clear what would be the number of tenured teachers willing to do such exchange and a precise estimation would be necessary. Practically speaking, the implementation would be standard: one can think about an additional exchange procedure, after the main assignment is determined, that asks tenured teachers and newly tenured ones willing to exchange to report a list of desired regions. Using a standard exchange graph, one can find cycles of exchanges between teachers to determine the exchanges to implement. The only difference would be that, after a fixed period of time, all the teachers who exchanged would have to come back to their assignment before the exchange.

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<sup>63</sup>For instance, [Fack, Grenet and He \(2015\)](#), [Abdulkadiroglu, Agarwal and Pathak \(2015\)](#) or [Calsamiglia, Fu and Guell \(2016\)](#).

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