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Thèse

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On birational transformations and automorphisms of some Hyperkähler manifolds

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Résumé

L'objectif de cette thèse est d'étudier le groupe d'automorphismes de certaines classes de variétés hyperkähleriennes.

La première partie de ce travail de thèse porte sur les doubles *EPW* sextiques, une famille de variétés hyperkähleriennes qui, dans le cas général, sont équivalentes par déformation au schéma de Hilbert de deux points sur une surface *K3*. Notamment nous avons utilisé le lien que ces variétés ont avec les variétés de Gushel-Mukai, qui sont des variétés de Fano dans une Grassmannienne si leur dimension est plus grande que deux, des surface *K3* si la dimension est deux. Nous avons donné des bornes sur la structure d'un certain sous-groupe d'automorphismes d'une double *EPW* sextique sous condition d'existence d'un point fixe pour l'action du groupe. Si une surface *K3* peut être associée à une double *EPW* sextique X , nous avons donné des conditions explicites sur le groupe de Picard de la surface pour que X soit une variété hyperkählienne. Cela permet d'utiliser le théorème de Torelli pour une surface *K3* pour démontrer l'existence de quelques automorphismes sur la double *EPW* sextique. Toujours dans le cas d'existence d'une surface *K3* associée à X , nous avons amélioré la borne obtenue précédemment sur les automorphismes de X , en donnant un lien explicite avec le nombre de coniques sur la surface *K3*. Nous avons montré que la symplecticité d'un automorphisme sur X dépend de la symplecticité d'un automorphisme correspondant sur la surface *K3*.

Dans la deuxième partie de ce travail de thèse, en collaboration avec Alberto Cattaneo, nous avons étudié le groupe d'automorphismes birationnels sur le schéma de Hilbert des points sur une surface *K3* projective, dans le cas générique. Nous avons ensuite étudié les cas où il existe un modèle birationnel où ces automorphismes sont réguliers. Nous avons décrit de façon géométrique quelques involutions dont nous avons prouvé l'existence auparavant. Nous avons établi un isomorphisme entre une composante connexe de l'espace de modules des variétés de type $K3^{[n]}$ avec une polarisation, et l'espace de modules des variétés de même type avec une involution dont le rang de l'invariant est un.

Mots-clés : géométrie algébrique, variétés hyperkähleriennes, automorphismes, transformations birationnelles, surfaces *K3*, doubles *EPW* sextiques, variétés Gushel-Mukai, représentation géométrique.

Abstract

The aim of this thesis is to study the group of automorphisms of certain classes of hyperkähler manifolds.

The first part of this thesis deals with double *EPW* sextics, a family of hyperkähler manifolds which in the general case are equivalent by deformation to the Hilbert scheme of two points on a *K3* surface. In particular we use the relation that these manifolds have with Gushel-Mukai varieties, which are Fano varieties in a Grassmannian if their dimension is greater than two, *K3* surfaces if the dimension is two. We give bounds on the structure of a certain subgroup of automorphisms of a double *EPW* sextic under conditions of the existence of a fixed point for the action of the group. If some *K3* surface can be associated with a double *EPW* sextic X , we give explicit conditions on the Picard group of the surface so that X is a hyperkähler manifold. This allows us to use Torelli's theorem for *K3* surfaces to demonstrate the existence of some automorphisms on the double *EPW* sextic. Again in the case of the existence of a *K3* surface associated with X , we improve the bound obtained previously on the automorphisms of X , giving an explicit link with the number of conics on the *K3* surface. We show that the symplecticity of an automorphism on X depends on the symplecticity of a corresponding automorphism on the *K3* surface.

In the second part of this thesis, in collaboration with Alberto Cattaneo, we study the group of birational automorphisms on Hilbert scheme of points on a *K3* projective surface, in the generic case. Then we study the cases in which there is a birational model where these automorphisms are regular. We describe in a geometrical way some involutions whose existence we proved previously. We establish an isomorphism between a connected component of the moduli space of varieties of type $K3^{[m]}$ with a polarization, and the moduli space of varieties of the same type with an involution whose rank of the invariant is one.

Keywords : algebraic geometry, hyperkähler manifolds, automorphisms, birational transformations, *K3* surfaces, double *EPW* sextics, Gushel-Mukai varieties, geometric representation.

Ad Adelotta

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Antoine : Chaque prisonnier commence par ouvrir le tiroir correspondant à son numéro. Si son numéro n'y est pas, il ouvre le tiroir correspondant au numéro qui s'y trouve, etc., en espérant retourner à son tiroir de départ en moins de 50 essais. Facile.

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Introduction

Les variétés hyperkähleriennes, avec les tores complexes et les variétés Calabi-Yau, apparaissent dans la décomposition de Beauville-Bogomolov comme l'une des trois familles de blocs de construction des variétés compactes Kähériennes dont la première classe de Chern est triviale. Pour cette raison, et pour leur intérêt intrinsèque en tant que sujet d'étude, cette famille de variétés a suscité un grand intérêt ces dernières années.

D'ailleurs, la première classe connue d'exemples de variétés hyperkähleriennes apparaît déjà dans la classification Enriques Kodaira des surfaces compactes : il s'agit de la famille des surfaces $K3$, qui coïncident avec les variétés hyperkähleriennes en dimension deux.

Les variétés hyperkähleriennes de toutes dimensions partagent plusieurs propriétés avec les surfaces $K3$, dont la plus fondamentale est sûrement l'existence d'une structure naturelle de réseau pair intégral sur le deuxième groupe de cohomologie à coefficients entiers, donnée par la forme quadratique de Beauville-Bogomolov-Fujiki (BBF). Cette forme quadratique a une forte signification géométrique : pour des surfaces de $K3$, elle correspond à la forme d'intersection.

Dans ce sens, les variétés hyperkähleriennes peuvent être considérées comme une généralisation en la dimension supérieure des surfaces $K3$. Au cours des dernières années, de nombreuses tentatives ont été faites pour généraliser les résultats qui s'appliquent aux surfaces $K3$ pour leurs homologues de dimension supérieure. Dans cette thèse, nous nous concentrons principalement sur les groupes d'automorphismes des variétés hyperkähleriennes. Dans ce domaine, un exemple très important dans l'esprit de la généralisation ci-dessus est donné par les théorèmes de Torelli pour les variétés hyperkähleriennes, qui permettent de décrire les automorphismes du manifold à partir de certaines classes d'isométries du deuxième groupe de cohomologie. Il a été prouvé dans [105] que une isométrie du deuxième groupe de cohomologie d'une surface $K3$ est induite par un automorphisme si et seulement si elle fixe le cône Kählerien ; en dimension supérieure, une version plus technique a été prouvée par Huybrechts, Markman et Verbitsky, voir [77]. En particulier, les variétés hyperkähleriennes admettent des automorphismes birationnels non-biréguliers, alors que chaque automorphisme d'une surface de $K3$ est birégulier.

Le théorème de Torelli pour les surfaces $K3$ permet de formuler des problèmes sur les automorphismes des surfaces $K3$ en termes de théorie des réseaux. Les outils fondamentaux en ce sens ont été fournis par Nikulin dans [90]. Nikulin a en outre classifié les groupes abéliens finis agissant de manière symplectique sur des surfaces $K3$ (les automorphismes d'une variété hyperkählerienne sont symplectiques s'ils laissent invariant une 2-forme sur la variété, sinon ils sont

dits non-symplectiques).

Depuis lors, un grand nombre de résultats concernant les groupes agissant sur des surfaces $K3$ ont été prouvés ; en particulier, les groupes cycliques d'ordre premier, agissant de manière symplectique ou pas, ont été complètement classifiés, en termes d'action en cohomologie et de lieu fixe, voir [44] et [3]. Le groupe d'automorphismes des surfaces $K3$ projectives de rang Picard 1 s'avère être généré par une involution si et seulement si l'auto-intersection du diviseur qui génère le groupe de Picard de la surface $K3$ est 2 ; dans tous les autres cas, ce groupe est trivial. Lorsque l'auto-intersection est 2, l'involution est non-symplectique et la surface $K3$ est un revêtement double du plan projectif, ramifié sur une courbe sextique lisse. En général, les surfaces $K3$ dont le rang de Picard est inférieur à 8 n'admettent pas d'automorphismes symplectiques.

Au cours de cette thèse, nous nous concentrons sur la famille des variétés hyperkählériennes qui sont équivalentes par déformation aux schémas de Hilbert de n points sur une surface $K3$ projective. Celles-ci sont appelées variétés de type $K3^{[n]}$, le schéma de Hilbert de n points sur la surface S étant désigné par $S^{[n]}$. Lorsque $n \geq 2$, le deuxième groupe de cohomologie d'une variété de type $K3^{[n]}$ est isomorphe, en tant que réseau, à un réseau abstrait Λ_n de rang 23 dont la signature est $(3, 20)$. Ce réseau est bien connu et il ne dépend pas de la variété hyperkählérienne de type $K3^{[n]}$. Dans ce cadre, une étude minutieuse de la décomposition en chambre du cône positif permet d'étudier les groupes d'automorphismes à l'aide de la théorie des réseaux : ayant fixé une variété hyperkählérienne de type $K3^{[n]}$, la fermeture du cône généré par les diviseurs mobiles est la fermeture de l'union des tirés en arrière des cônes amples des modèles birationnels de la variété par le biais d'une isométrie induite par un morphisme birationnel. Lorsque S est une surface $K3$ projective dont le rang de Picard est un (ce qui est le cas très général dans l'espace des modules des surfaces $K3$ marquées), une description numérique du cône mobile et des chambres de type Kähler de $S^{[n]}$ est fournie par Bayer et Macrì dans [5]. Cela a permis à Boissière, An. Cattaneo, Nieper-Wiskirchen et Sarti pour $n = 2$, et ensuite à Al. Cattaneo pour $n \geq 3$, de décrire complètement le groupe d'automorphismes biréguliers du schéma de Hilbert de n points $S^{[n]}$, si S a rang Picard 1, voir [16] et [22]. Le groupe d'automorphismes est à nouveau soit trivial soit généré par une involution, et l'involution existe si et seulement s'il existe un diviseur ample dans $Pic(S^{[n]})$ dont le carré, par rapport à la forme BBF, est soit 2 soit $2(n - 1)$ (dans ce dernier cas, la divisibilité doit être $n - 1$) ; l'involution agit alors en cohomologie comme l'opposé de la réflexion par rapport au complément orthogonal de ce diviseur. Encore une fois l'involution est non-symplectique, car $S^{[n]}$ a rang de Picard 2 : pour les variétés de type $K3^{[n]}$ aussi, si le rang de Picard est inférieur à 8, tout automorphisme birégulier est non-symplectique, voir [82].

Plus en général, Camere, Al. Cattaneo et An. Cattaneo ont montré dans [21] qu'une involution non-symplectique sur une variété de type $K3^{[n]}$, agissant en cohomologie comme l'opposé de la réflexion par rapport au complément orthogonal d'un diviseur, existe si et seulement si la variété admet un diviseur ample dont le carré est $2d$ et la divisibilité γ , où $(d, \gamma) \in \{(1, 1), (1, 2), (n - 1, n - 1)\}$.

En ce qui concerne les automorphismes de variétés hyperkählériennes, un problème difficile consiste à trouver des constructions explicites pour les automorphismes, qu'ils soient biréguliers ou birationnels, car le théorème de Torelli ne donne pas un aperçu de la géométrie sous-jacente : quelques descriptions sont connues pour quelques familles de grande dimension, par exemple

l'involution de Beauville pour le cas birégulier, cfr. [8], et l'involution de O'Grady pour le cas birationnel, cfr. [94]. En effet, cela est l'un des sujets de cette thèse.

La première partie de la thèse débute sur ce point. Dans le Chapitres 1 et 2, nous introduisons les notions de base pour l'étude des variétés hyperkähleriennes, y compris un aperçu de la théorie des réseaux (pour tout réseau R , nous désignons par $O(R)$ le groupe de ses isométries) et de la théorie des équations de Pell. Nous introduisons également les espaces de modules que nous utilisons au cours de la thèse ; en particulier, nous présentons l'espace de modules des variétés de type $K3^{[n]}$ avec une polarisation de carré $2d$ et divisibilité γ , qui est dénoté par $\mathcal{M}_{2d,\gamma}^n$, suivant la notation de [25]. De plus, nous introduisons l'espace de modules des variétés de type $K3^{[n]}$ avec une involution agissant en cohomologie comme l'opposé de la réflexion par rapport au complément orthogonal d'un diviseur de carré $2d$ et divisibilité γ fixés. Nous le désignons par \mathcal{M}_M , où $M \subset \Lambda_n$ est un sous-réseau de rang 1 engendré par un élément de carré $2d$ et divisibilité γ . cet espace a été étudié par Joumaah dans [70] : nous résumons également les propriétés de base de cet espace de modules qui ont été obtenues par Joumaah.

Dans le Chapitre 3 nous mettons en relation les espaces de modules $\mathcal{M}_{2d,\gamma}^n$ et \mathcal{M}_M , où $M = \mathbb{Z}h$ avec $h \in \Lambda_n$ un élément de carré $2d$ et divisibilité γ . Nous appelons $\tilde{\Lambda}$ le réseau de Mukai : comme l'a prouvé Markman dans [77], chaque variété X de type $K3^{[n]}$ admet une orbite canonique $O(\Lambda)$ de plongements primitifs de $H^2(X, \mathbb{Z})$ dans $\tilde{\Lambda}$. Pour $h \in \Lambda_n$, nous avons désigné par $\mathcal{M}_{[h]} \subset \mathcal{M}_{2d,\gamma}^n$ l'espace des modules des variétés X avec une polarisation D de carré $2d$ et divisibilité γ , et telles qu'il existe une isométrie $\phi \in O(H^2(X, \mathbb{Z}))$ et un représentant du plongement canonique $\iota_X : H^2(X, \mathbb{Z}) \hookrightarrow \tilde{\Lambda}$ tel que $\iota_X(\phi(D)) = h$. Nous prouvons le résultat suivant, qui avait déjà été observé dans [15] pour $n = 2$.

Théorème 0.1. *Pour $(d, \gamma) \in \{(1, 1), (1, 2), (n-1, n-1)\}$, soit $\rho \in O(\Lambda_n)$ une involution telle que $(\Lambda_n)^\rho = \{x \in \Lambda_n \mid \rho(x) = x\}$ est généré par un élément primitif h de carré $2d$ et divisibilité γ . Alors \mathcal{M}_M est isomorphe à une composante connexe de $\mathcal{M}_{[h]} \subset \mathcal{M}_{2d,\gamma}^n$.*

Pour $n \geq 3$, une attention particulière est requise, car ces espaces de modules ne sont pas forcément connexes. Cette interprétation nous permet de décrire, à déformation près, les involutions birégulières de $S^{[n]}$, lorsque S est une surface $K3$ de rang de Picard 1 dont le groupe de Picard est engendré par un diviseur d'auto-intersection $2((n-1)k^2 + 1)$, pour k suffisamment grand. Ces involutions sont obtenues en déformant une combinaison d'involutions de Beauville sur le schéma de Hilbert de n points sur une surface $K3$ particulière dont le rang de Picard est deux, de sorte que le chemin de déformation ne passe que par des variétés hyperkähleriennes qui sont des schémas de Hilbert de points sur des surfaces $K3$. Cette partie du chapitre fait partie d'un travail récent, en collaboration avec Al. Cattaneo, voir [11].

Nous précisons cette description pour le cas quadridimensionnel, en généralisant la construction fournie dans [15] et en atteignant de nouveaux cas en déformant une combinaison d'une involution de Beauville et d'une involution de O'Grady.

Dans la deuxième partie de cette thèse, à partir du Chapitre 4, nous nous concentrons sur le cas de variétés de type $K3^{[2]}$ et nous introduisons une autre famille d'objets, les doubles Eisenbud-

Popescu-Walter (*EPW*) sextiques. Celles-ci se présentent sous la forme de revêtements doubles d'une famille particulière d'hypersurfaces sextiques en \mathbb{P}^5 , appelées *EPW* sextiques, chacune desquelles est obtenue à partir d'un sous-espace Lagrangien en $\bigwedge^3 V_6$. Lorsque ces variétés sont lisses, elles sont des variétés hyperkähleriennes de type $K3^{[2]}$. Les doubles *EPW* sextiques ont été introduites par O'Grady dans [96] et font l'objet d'une série d'articles, voir [96], [95], [97], [98], [99], [102].

Les doubles *EPW* sextiques constituent un sujet d'étude important, car elles constituent un sous-ensemble ouvert et dense à l'intérieur de $\mathcal{M}_{2d,1}^2$, l'espace de modules des variétés $\langle 2 \rangle$ -polarisées de type $K3^{[2]}$. Comme l'ont souligné Debarre et Macrì dans [32], l'élément très général à l'intérieur de $\mathcal{M}_{2d,1}^2$ est également une double *EPW* sextique chaque fois que $d \equiv -1 \pmod{4}$.

Au cours du Chapitre 4, nous proposons une sélection des propriétés des doubles *EPW* sextiques, avec un accent sur le rôle d'une autre famille de variétés, les variétés de Gushel-Mukai (*GM*) ordinaires. Les variétés *GM* ordinaires sont des intersections complètes, dans la Grassmannienne $G(2, \mathbb{C}^5) \subset \mathbb{P}(\bigwedge^2 V_5)$, d'un sous-espace linéaire et d'une hypersurface quadrique projective (pour une définition précise, voir 4.85). Les variétés *GM* ordinaires 2-dimensionnelles et lisses sont des surfaces $K3$, qui à leur tour sont générales dans l'espace de modules des surfaces $K3$ avec une polarisation dont l'auto-intersection est 10.

La relation entre les variétés de Gushel-Mukai et les sextiques *EPW* était claire depuis [96] pour le cas des variétés *GM* 2-dimensionnelles. O'Grady a ainsi prouvé, en utilisant de la théorie de la déformation, que la double *EPW* sextique générale est une variété hyperkählienne. De plus, dans [67], Iliev et Manivel ont montré un moyen d'obtenir une sextique *EPW* à partir d'une variété *GM* ordinaire générale de dimension 3, 4 ou 5. Pour finir, au cours de la série d'articles [27], [29], [30], [31], Debarre et Kuznetsov ont étudié systématiquement l'interaction entre les variétés de Gushel-Mukai et les *EPW* sextiques. En général, il existe une façon canonique d'associer à toute variété *GM* normale un sous-espace Lagrangien, d'où une (double) *EPW* sextique. L'inverse est vrai aussi : il existe une façon canonique d'associer une variété *GM* lisse à la donnée d'un sous-espace Lagrangien dans $\bigwedge^3 V_6$ et un point dans $(\mathbb{P}^5)^\vee$, à condition que le sous-espace Lagrangien ne contienne aucun vecteur décomposable.

Les variétés de Gushel-Mukai ont suscité beaucoup d'intérêt au cours de la période la plus récente, et leur relation avec les doubles *EPW* sextiques s'inscrit dans le discours plus général de la relation entre les variétés de Fano et les variétés hyperkähleriennes.

Tous les résultats de ce chapitre sont déjà connus, à l'exception de quelques considérations dans la Section 4.3.4. En outre, nous proposons une preuve alternative de la finitude du groupe d'automorphismes d'une *EPW* sextique sans vecteurs décomposables non nuls, voir Proposition 4.28.

Dans le Chapitre 5 nous utilisons la relation entre les *EPW* sextiques et les variétés *GM* pour étudier les groupes d'automorphismes des variétés dans les deux familles. Une variété *GM* est dite fortement lisse si elle est lisse et si elle est obtenue comme l'intersection d'une variété de Fano lisse et dimensionnellement transversale et d'une hypersurface quadrique. Notre premier objectif est de caractériser la lissité de la double *EPW* sextique associée à une variété *GM* fortement lisse de dimension 2. Nous obtenons les résultats suivants (voir Theorem 5.2 et Proposition 5.3).

Théorème 0.2. *Soit $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ une surface $K3$ fortement lisse. Soit $A(S)$ le sous-espace Lagrangien associé. On a*

$$|Y_{A(S)}^3 \cap \mathbb{P}(V_5)| = |\{\ell \subset S \mid \ell \text{ droite}\}|.$$

En particulier, si S ne contient pas de droites, alors $Y_{A(S)}^3 \cap \mathbb{P}(V_5) = \emptyset$. De plus, si S ne contient pas de pincesaux elliptiques de degré 5, la double EPW sextique $X_{A(S)}$ est une variété hyperkählérienne.

Dans la suite nous étudions les groupes d'automorphismes des EPW sextiques. Nous fournissons quelques bornes pour les automorphismes des EPW sextiques associées à une surface $K3$, et pour les EPW sextiques en général lorsqu'un groupe d'automorphismes fixe un point de la sextique. Ces résultats sont obtenus en étudiant le groupe d'automorphismes de quelques variétés GM ordinaires associées à l'hypersurface sextique. Notamment, des bornes pour le groupe d'automorphismes sont trouvées pour certaines familles de variétés GM décrites par Debarre, Iliev et Manivel dans [26]. Par ailleurs, par le biais du Théorème 0.2, nous pouvons induire des automorphismes sur une famille de doubles EPW sextiques et sur certaines variétés GM à partir d'une famille de surfaces $K3$. Nous concluons ce chapitre en étudiant la fibre du morphisme qui associe une double EPW sextique à une surface $K3$ $\langle 10 \rangle$ -polarisée générale.

Tous les résultats de ce chapitre sont originaux, à l'exception de ceux présentés dans la Section 5.3, qui proviennent du travail de O'Grady, voir [98].

Dans le Chapitre 6 nous revenons au cas des variétés de type $K3^{[n]}$ de toutes dimensions pour étudier le groupe de morphismes birationnels de $S^{[n]}$, où S est une surface $K3$ projective. Tout comme pour le cas régulier, le théorème de Torelli peut être utilisé pour fournir des conditions nécessaires et suffisantes pour que le schéma de Hilbert sur une surface $K3$ admette des morphismes birationnels ; ici les surfaces $K3$ ont toujours rang de Picard 1. Comme dans [22], la description numérique du cône mobile de $S^{[n]}$ donnée dans [5] permet de prouver qu'une involution existe lorsque $\text{Pic}(S^{[n]})$ admet un diviseur mobile dont le carré est soit 2 soit $2(n-1)$ (dans ce dernier cas, la divisibilité doit être $n-1$), comme on pouvait s'y attendre. Il est intéressant et surprenant de constater que des involutions birationnelles symplectiques peuvent apparaître dans ce cas. Celles-ci ne sont jamais birégulières d'après les résultats en [82]. Nous obtenons les conditions numériques suivantes, cfr. Theorem 6.3 : étant donné $\sigma \in \text{Bir } S^{[n]}$, nous appelons réseau invariant le sous-réseau fixé par l'action de σ en cohomologie, réseau coinvariant son complément orthogonal. Nous appelons réseau transcendant le complément orthogonal du groupe de Néron-Severi à l'intérieur de $H^2(S^{[n]}, \mathbb{Z})$.

Théorème 0.3. *Soit S une surface $K3$ projective dont le groupe de Picard est engendré par L dont l'auto-intersection est $2t$, et soit $n \geq 2$.*

Si $t \geq 2$ il existe un automorphisme birationnel non trivial $\sigma : S^{[n]} \dashrightarrow S^{[n]}$ si et seulement si $t(n-1)$ n'est pas un carré et la solution minimale $(X, Y) = (z, w)$ de l'équation de Pell $X^2 - t(n-1)Y^2 = 1$ avec $z \equiv \pm 1 \pmod{n-1}$ satisfait $w \equiv 0 \pmod{2}$ et $(z, z) \equiv (j, k) \in \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}} \times \frac{\mathbb{Z}}{2t\mathbb{Z}}$ avec $(j, k) \in \{(1, 1), (1, -1), (-1, -1)\}$. Si tel est le cas, $\text{Bir}(S^{[n]}) = \langle \sigma \rangle \cong \mathbb{Z}/2\mathbb{Z}$ et

- si $(j, k) = (1, -1)$ alors σ est non-symplectique avec réseau invariant $T_\sigma \cong \langle 2 \rangle$;

- si $(j, k) = (-1, -1)$, alors σ est non-symplectique avec réseau invariant $T_\sigma \cong \langle 2(n-1) \rangle$;
- si $(j, k) = (1, 1)$, alors σ est symplectique avec réseau coinvariant isomorphe à $\langle -2(n-1) \rangle$, et le réseau invariant contient le réseau transcendant.

Si $t = 1$, soit $(X, Y) = (a, b)$ la solution entière de $(n-1)X^2 - Y^2 = -1$ avec le plus petits $a, b > 0$. Si $n-1$ est un carré ou $b \equiv \pm 1 \pmod{n-1}$, alors $\text{Bir}(S^{[n]}) = \text{Aut}(S^{[n]}) \cong \mathbb{Z}/2\mathbb{Z}$. Autrement $n \geq 9$, $\text{Bir}(S^{[n]}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ et $\text{Aut}(S^{[n]}) \cong \mathbb{Z}/2\mathbb{Z}$.

Le Théorème 0.3 généralise le résultat donné par Debarre dans [25] pour $n = 2$. Nous fournissons, pour chaque cas dans le Théorème 0.3, des familles de surfaces $K3$ polarisées admettant l'involution décrite.

L'involution σ est birégulière si et seulement si l'intérieur du cône mobile de $S^{[n]}$ contient une seule chambre ; lorsque ce n'est pas le cas, nous donnons des conditions pour vérifier si $S^{[n]}$ admet un modèle birationnel hyperkählérien sur lequel σ est birégulière ; notamment, nous vérifions que les involutions birationnelles symplectiques restent strictement birationnelles sur tous les modèles birationnels.

Nous produisons une analyse approfondie du cas $n = 3$, en vérifiant que le nombre de chambres dans le cône mobile de $S^{[3]}$ (c'est-à-dire le nombre de modèles birationnels hyperkählérien distincts) est soit un, deux, trois ou cinq. Pour $n = 2$, on sait que le nombre de chambres dans le cône mobile de $S^{[2]}$ est au plus trois.

Nous montrons une généralisation du Théorème 0.3 qui permet d'étudier les morphismes birationnels entre les schémas de Hilbert de points sur deux surfaces $K3$ distinctes dont le rang de Picard est 1. Pour finir, nous présentons quelques nouveaux exemples géométriques d'involutions birationnelles pour des schémas de Hilbert de points sur des surfaces quartiques.

Les résultats présentés dans ce dernier chapitre font partie d'un récent travail en collaboration avec Al. Cattaneo, voir [11].

Introduction

Hyperkähler manifolds, together with complex tori and Calabi-Yau varieties, appear in the Beauville-Bogomolov decomposition as one of the three families of building blocks for compact Kähler manifolds with trivial first Chern class. For this reason, and for their intrinsic interest as a matter of study, this family of manifolds has attracted great interest in recent years.

On the other hand, the first known class of examples of hyperkähler manifolds appears already in the Enriques Kodaira classification of compact surfaces: it is the family of $K3$ surfaces, which coincide with hyperkähler manifolds in dimension two.

Hyperkähler manifolds of any dimension share several properties with $K3$ surfaces, of which the fundamental one is surely the existence of a natural structure of even integral lattice on the second cohomology group with integer coefficients, given by the Beauville-Bogomolov-Fujiki (BBF) quadratic form. This quadratic form has a strong geometry meaning: for $K3$ surfaces it corresponds to the intersection form.

In this sense, hyperkähler manifolds can be seen as a generalization in higher dimension of $K3$ surfaces. In the last years, many attempts have been made to generalise results which hold for $K3$ surfaces for their higher dimensional counterparts. In this thesis we focus mostly on automorphism groups of hyperkähler manifolds. In this area, a very important example in the spirit of generalization above are the Torelli theorems for hyperkähler manifolds, which allow to describe automorphisms of the manifold from certain classes of isometries of the second cohomology group. It has been proved in [105] that isometries of the second cohomology group of a $K3$ surface are induced by some automorphism if and only if they fix the Kähler cone; in higher dimension a more technical version has been proved by Huybrechts, Markman and Verbitsky, see [77]. In particular, while every automorphism of a $K3$ surface is biregular, hyperkähler manifolds admit birational non-biregular automorphisms.

The Torelli theorem for $K3$ surfaces allows to formulate problems about automorphisms on $K3$ surfaces in terms of lattice theory. The fundamental tools in this sense have been provided by Nikulin in [90], who moreover classified finite abelian groups acting symplectically on $K3$ surfaces (automorphisms on a hyperkähler manifold are symplectic if they leave invariant a 2-form on the manifold, non-symplectic otherwise).

Since then, a great amount of results about groups acting on $K3$ surfaces has been proved; in particular, cyclic groups of prime order acting both symplectically and non-symplectically are completely classified, in terms of action in cohomology and fixed locus, see [44] and [3]. The

automorphism group of projective $K3$ surfaces of Picard rank 1 turns out to be generated by an involution if and only if the self-intersection of the divisor spanning the Picard group of the $K3$ surface is 2; in all the other cases it is trivial. When the self-intersection is 2 the involution is non-symplectic and the $K3$ surface is the double cover of the projective plane ramified on a smooth sextic curve. It is a general fact that $K3$ surfaces whose Picard rank is less than 8 cannot admit symplectic automorphisms.

In this thesis we focus on the family of hyperkähler manifolds which are deformation equivalent to Hilbert schemes of n points on a projective $K3$ surface. They are called manifolds of $K3^{[n]}$ -type, the Hilbert scheme of n points on the surface S being denoted by $S^{[n]}$. When $n \geq 2$ the second cohomology group of a manifold of $K3^{[n]}$ -type is isomorphic, as a lattice, to an abstract lattice Λ_n of rank 23 and signature $(3, 20)$. This lattice is well-known and it does not depend on the hyperkähler manifold of $K3^{[n]}$ -type.

In this setting, to study automorphism groups from the lattice-theoretic point of view, a careful study of the chamber decomposition of the positive cone is needed: fixed a hyperkähler manifold of $K3^{[n]}$ -type, the closure of the cone spanned by movable divisors on it is obtained as the closure of the union of the pullbacks, through birational maps, of the ample cones of the birational models of the manifold. When S is a projective $K3$ surface whose Picard rank is one (which is the very general case in the moduli space of marked $K3$ surfaces), a numerical description of the movable cone and Kähler-type chambers of $S^{[n]}$ is provided by Bayer and Macrì in [5]. This allowed Boissière, An. Cattaneo, Nieper-Wisskirchen and Sarti for $n = 2$, and then Al. Cattaneo for $n \geq 3$, to completely describe the group of biregular automorphisms of the Hilbert scheme of n points $S^{[n]}$, when S has Picard rank 1, see [16] and [22]. The automorphism group is again trivial or generated by an involution, and the involution exists if and only if there exists an ample divisor in $\text{Pic}(S^{[n]})$ which has either square 2 or square $2(n - 1)$ and divisibility $n - 1$, with respect to the BBF form; the involution then acts in cohomology as the opposite of the reflection with respect to the orthogonal complement of this divisor. The involution is non-symplectic in this case too, as $S^{[n]}$ has Picard rank 2: also for manifolds of $K3^{[n]}$ -type, if their Picard rank is less than 8 they cannot admit biregular symplectic automorphisms, see [82].

More in general, Camere, Al. Cattaneo and An. Cattaneo showed in [21] that a non-symplectic involution on a manifold of $K3^{[n]}$ -type, acting in cohomology as the opposite of the reflection with respect to the orthogonal complement of a divisor, exists if and only if the manifold admits an ample divisor with square $2d$ and divisibility γ , where $(d, \gamma) \in \{(1, 1), (1, 2), (n - 1, n - 1)\}$.

When dealing with automorphisms of hyperkähler manifolds, a challenging problem is to find explicit constructions for automorphisms, whether biregular or birational, since the Torelli theorem does not give an insight of the subjacent geometry: a few descriptions are known for some families of big dimension, for example Beauville's involution for the biregular case, cfr. [8], and O'Grady's involution for the birational case, cfr. [94]. This is one of the topics of this thesis.

The first part of the thesis has this starting point. In Chapter 1 and Chapter 2 we introduce the basic notions for the study of hyperkähler manifolds, including an insight of lattice theory (for any lattice R , we denote by $O(R)$ the group of its isometries) and of the theory of Pell's equations.

We also introduce the moduli spaces that we use in the thesis; in particular we present the moduli space of manifolds of $K3^{[n]}$ -type with a polarization of square $2d$ and divisibility γ , which is denoted by $\mathcal{M}_{2d,\gamma}^n$, following the notation of [25]. Moreover we introduce the moduli space of manifolds of $K3^{[n]}$ -type with an involution acting in cohomology as the opposite of the reflection with respect to the orthogonal complement of a divisor with fixed square $2d$ and divisibility γ . We denote it by \mathcal{M}_M , where $M \subset \Lambda_n$ is a rank-1 sublattice spanned by an element of square $2d$ and divisibility γ . It has been studied by Joumaah in [70]: we also summarize the basic properties of this moduli space that were obtained by Joumaah.

In Chapter 3 we relate the moduli spaces $\mathcal{M}_{2d,\gamma}^n$ and \mathcal{M}_M , where $M = \mathbb{Z}h$ with $h \in \Lambda_n$ a square- $2d$ element with divisibility γ . We call $\tilde{\Lambda}$ the Mukai lattice: as proved by Markman in [77], every manifold X of $K3^{[n]}$ -type admits a canonical $O(\tilde{\Lambda})$ -orbit of primitive embeddings of $H^2(X, \mathbb{Z})$ in $\tilde{\Lambda}$. For $h \in \Lambda_n$, we denote by $\mathcal{M}_{[h]} \subset \mathcal{M}_{2d,\gamma}^n$ the moduli space of manifolds X with a polarization D with square $2d$ and divisibility γ , and such that there exists an isometry $\phi \in O(H^2(X, \mathbb{Z}))$ and a representative of the canonical embedding $\iota_X : H^2(X, \mathbb{Z}) \hookrightarrow \tilde{\Lambda}$ such that $\iota_X(\phi(D)) = h$. We prove the following, which was already pointed out in [15] for $n = 2$.

Theorem 0.4. *For $(d, \gamma) \in \{(1, 1), (1, 2), (n-1, n-1)\}$, let $\rho \in O(\Lambda_n)$ be an involution such that $(\Lambda_n)^\rho = \{x \in \Lambda_n \mid \rho(x) = x\}$ is generated by a primitive element h of square $2d$ and divisibility γ . Then \mathcal{M}_M is isomorphic to a connected component of $\mathcal{M}_{[h]} \subset \mathcal{M}_{2d,\gamma}^n$.*

For $n \geq 3$ this result requires some additional care, since we deal with moduli spaces which may be non-connected. This interpretation allows us to describe, up to deformation, biregular involutions of $S^{[n]}$, when S is a $K3$ surface of Picard rank 1 whose Picard group is spanned by a divisor of self-intersection $2((n-1)k^2 + 1)$ for some sufficiently large k . These involutions are obtained by deforming a combination of Beauville's involutions on the Hilbert scheme of n points of a specific $K3$ surface of Picard rank two, in such a way that the deformation path goes only through hyperkähler manifolds which are still Hilbert schemes of points on $K3$ surfaces. This part of the chapter is part of a recent joint work with Al. Cattaneo, see [11].

We precise this description in the four-dimensional case, generalizing the construction provided in [15] and reaching new cases by deforming a combination of a Beauville's involution and an O'Grady involution.

In the second part of this thesis, from Chapter 4, we focus on the four-dimensional case and we introduce another family of objects, which are double Eisenbud-Popescu-Walter (*EPW*) sextics. They arise as double covers of a particular family of sextic fourfolds in \mathbb{P}^5 , each obtained from a Lagrangian subspace in $\bigwedge^3 V_6$, called *EPW* sextics. When these fourfolds are smooth they are hyperkähler manifolds of $K3^{[2]}$ -type. Double *EPW* sextics have been introduced by O'Grady in [96] and are the subject of a series of papers, see ([95], [96], [97], [98], [99], [102]).

They are an important matter of study, since they fill a dense open subset inside $\mathcal{M}_{2,1}^2$, the moduli space of $\langle 2 \rangle$ -polarized manifolds of $K3^{[2]}$ -type. As Debarre and Macrì pointed out in [32], also the very general element inside $\mathcal{M}_{2d,1}^2$ is a double *EPW* sextic whenever $d \equiv -1 \pmod{4}$.

In Chapter 4 we propose a summary of some of the main properties of double *EPW* sextics, with

a stress on the role of another family of varieties, which are ordinary Gushel-Mukai (GM) varieties. Ordinary GM varieties are complete intersections, inside the Grassmannian $G(2, \mathbb{C}^5) \subset \mathbb{P}(\wedge^2 \mathbb{C}^5)$, of a linear subspace and a projective quadric hypersurface (for a precise definition see 4.85). When 2-dimensional and smooth, they are $K3$ surfaces, which in turns are general in the moduli space of $K3$ surfaces with a polarization of self-intersection 10.

The relation between Gushel-Mukai varieties and EPW sextics was clear since [96] for the case of 2-dimensional GM varieties. In this way O'Grady proved by a deformation argument that the general double EPW sextic is a hyperkähler manifold. Moreover, in [67], Iliev and Manivel showed a way to obtain a EPW sextic from a general ordinary GM variety of dimension 3, 4 or 5. Finally, in the series of papers [27], [29], [30], [31], Debarre and Kuznetsov studied systematically the interplay between Gushel-Mukai varieties and EPW sextics. In general, there is a canonical way to associate to any normal GM variety a Lagrangian subspace, hence a (double) EPW sextic. Vice versa, there is a canonical way to associate a smooth GM variety to the data of a Lagrangian subspace in $\wedge^3 V_6$ and a point in $(\mathbb{P}^5)^\vee$, under the condition that the Lagrangian subspace does not contain any decomposable vector.

Gushel-Mukai varieties have attracted much interest in the most recent period, and their relation with double EPW sextics is a subject in the wide topic of the relation between Fano varieties and hyperkähler manifolds.

All the results in this chapter are already known, except some considerations in Section 4.3.4. Moreover we propose an alternative proof of the finiteness of the automorphism group of an EPW sextic with no non-zero decomposable vector, see Proposition 4.28.

In Chapter 5 we use the relation between EPW sextics and GM varieties to study the automorphism groups of varieties on both families. A GM variety is strongly smooth if it is smooth and it is obtained as the intersection of a smooth and dimensionally transverse Fano variety and a quadric hypersurface. Our first aim is to characterize smoothness of the double EPW sextic associated to a strongly smooth GM variety of dimension 2. We obtain the following (see Theorem 5.2 and Proposition 5.3).

Theorem 0.5. *Let $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ be a strongly smooth $K3$ surface. Let $A(S)$ be the associated Lagrangian subspace. Then*

$$|Y_{A(S)}^3 \cap \mathbb{P}(V_5)| = |\{\ell \subset S \mid \ell \text{ line}\}|.$$

In particular, if S contains no line then $Y_{A(S)}^3 \cap \mathbb{P}(V_5) = \emptyset$. If moreover S contains no quintic elliptic pencils, the double EPW sextic $X_{A(S)}$ is a hyperkähler manifold.

Then we study automorphism groups of EPW sextics. We provide some bounds for automorphisms of EPW sextics associated to some $K3$ surface, and more in general for EPW sextics when a group of automorphisms fixes a point of the sextic. These results are obtained by a study of the automorphism group of some ordinary GM varieties associated to the sextic hypersurface. Namely, bounds on the automorphism group are found for some families of GM varieties described by Debarre, Iliev and Manivel in [26]. On the other hand, by means of Theorem 0.5, we can induce automorphisms on a family of double EPW sextics and on some GM varieties from a family of $K3$

surfaces. Finally, we study the fiber of the map which associates to a general $\langle 10 \rangle$ -polarized $K3$ surface its double EPW sextic.

All the results in this chapter are original, except for the ones presented in Section 5.3, which come from O'Grady's work [98].

In Chapter 6 we return to the case of manifolds of $K3^{[n]}$ -type of any dimension to study the group of birational morphisms of $S^{[n]}$, where S is a projective $K3$ surface. Also for birational morphisms, the Torelli theorem can be used to provide necessary and sufficient conditions on Hilbert schemes of $K3$ surfaces of Picard rank 1. As in [22], the numerical description of the movable cone of $S^{[n]}$, given in [5], allows to prove that an involution exists when $\text{Pic}(S^{[n]})$ admits a movable divisor which has either square 2 or square $2(n-1)$ and divisibility $n-1$, as we could expect. It is interesting and surprising to remark that symplectic birational involutions may appear in this case. These are surely not biregular by results in [82]. We obtain the following numerical conditions, cfr. Theorem 6.3: given $\sigma \in \text{Bir } S^{[n]}$, we call invariant lattice the sublattice fixed by the action of σ in cohomology, the coinvariant lattice its orthogonal complement and we call transcendental lattice the orthogonal complement of the Néron-Severi group inside $H^2(S^{[n]}, \mathbb{Z})$.

Theorem 0.6. *Let S be a projective $K3$ surface whose Picard group is spanned by L with self-intersection $2t$, and let $n \geq 2$.*

If $t \geq 2$, there exists a non-trivial birational automorphism $\sigma : S^{[n]} \dashrightarrow S^{[n]}$ if and only if $t(n-1)$ is not a square and the minimal solution $(X, Y) = (z, w)$ of the Pell's equation $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$ satisfies $w \equiv 0 \pmod{2}$ and $(z, z) \equiv (j, k) \in \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}} \times \frac{\mathbb{Z}}{2i\mathbb{Z}}$ with $(j, k) \in \{(1, 1), (1, -1), (-1, -1)\}$. If so, $\text{Bir}(S^{[n]}) = \langle \sigma \rangle \cong \mathbb{Z}/2\mathbb{Z}$ and

- *if $(j, k) = (1, -1)$, then σ is non-symplectic with invariant lattice $T_\sigma \cong \langle 2 \rangle$;*
- *if $(j, k) = (-1, -1)$, then σ is non-symplectic with invariant lattice $T_\sigma \cong \langle 2(n-1) \rangle$;*
- *if $(j, k) = (1, 1)$, then σ is symplectic with coinvariant lattice isomorphic to $\langle -2(n-1) \rangle$, and the invariant lattice contains the transcendental lattice.*

If $t = 1$, let $(X, Y) = (a, b)$ be the integer solution of $(n-1)X^2 - Y^2 = -1$ with smallest $a, b > 0$. If $n-1$ is a square or $b \equiv \pm 1 \pmod{n-1}$, then $\text{Bir}(S^{[n]}) = \text{Aut}(S^{[n]}) \cong \mathbb{Z}/2\mathbb{Z}$. Otherwise $n \geq 9$, $\text{Bir}(S^{[n]}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\text{Aut}(S^{[n]}) \cong \mathbb{Z}/2\mathbb{Z}$.

Theorem 0.6 generalizes the result given by Debarre in [25] for $n = 2$. We provide, for every case in Theorem 0.6, families of polarized $K3$ surfaces admitting the involution described.

The involution σ is biregular if and only if there is only one chamber inside the movable cone of $S^{[n]}$; when this is not the case, we give conditions to check whether $S^{[n]}$ admits a birational model on which σ is biregular; in particular, we verify that symplectic birational involutions remain strictly birational on all birational hyperkähler models.

We provide an in-depth analysis of the case $n = 3$, checking that the number of chambers in the movable cone of $S^{[3]}$ (i.e. the number of distinct hyperkähler birational models) is either one, two, three or five. For $n = 2$ it is known that the number of chambers in the movable cone of $S^{[2]}$ is at most three.

We show how Theorem 0.6 can be generalized to study birational maps between Hilbert schemes of points on two distinct $K3$ surfaces of Picard rank 1. Finally, we present new geometrical examples of birational involutions for Hilbert schemes of points on quartic surfaces.

The results in this chapter are part of a recent joint work with Al. Cattaneo, see [11].

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Chapter 1

General facts

1.1 Introduction and notations

In this chapter we recall some basis results, which will be useful in the thesis. In the first section we present some notions of the theory of Pell's equations, a particular type of diophantine equation. They are useful when dealing with rank-2 lattices, as they describe the conditions for having a vector of a fixed square; we will use them mostly in Chapter 3 and Chapter 6.

In the second section we give an insight of the theory of integral lattices. The second cohomology group of a hyperkähler manifold admits a natural structure of lattice, and a lot of properties of the manifold can be read off this structure. For this reason, lattice theory appears in numerous fundamental results about hyperkähler manifolds: see, for instance, the construction of moduli spaces for hyperkähler manifolds in Section 2.3, or the chamber decomposition of the positive cone of a hyperkähler manifold and its relation with the group of birational morphisms of the manifold in Section 2.4.

In the third section we provide the construction of double covers for normal varieties. This construction is very classical, but we found much less references for the case of non-étale covers whose branch locus has codimension bigger than one. This last case is involved in the construction of double *EPW* sextics, a locally complete family of hyperkähler manifolds which we will study in detail in this thesis. The construction of these manifolds is described for the first time by O'Grady in [96]; it will be presented in Section 4.3.1. This section takes up the introduction of [28].

All the results in this chapter are already known, except for Lemma 1.6, a technical result which appears in a joint work with Al. Cattaneo, see Lemma 2.1 of

On birational transformations of Hilbert schemes of points on K3 surfaces,
<https://arxiv.org/abs/2005.05187>, submitted.

We fix some general notations, which we will keep for the whole thesis.

Let V be a finite-dimensional complex vector space. Given $\mu \in GL(V)$, we denote by μ also its

class in $\mathrm{PGL}(V)$.

Consider the vector space $\bigwedge^k V$; non-zero decomposable vectors are the classes of elements in the form $[v_1 \wedge \dots \wedge v_k] \in \mathbb{P}(\bigwedge^k V)$ or, alternatively, the points of the Grassmannian $G(k, V)$, seen as an embedded variety through the Plücker embedding.

Let X be a topological space. We say that a property holds for $x \in X$ general if the condition is satisfied by all the points inside an open subset of X . We say that a property holds for $x \in X$ *very* general if the condition is satisfied by all the points in the complement of a countable union of closed subspaces inside X .

A scheme is integral if it is reduced and irreducible. For us, a variety is a separated integral scheme of finite-type over \mathbb{C} , not necessarily projective. A surface is defined as a 2-dimensional variety. Let \mathbb{P}^n be the complex projective space, we denote by $Z(f) \subset \mathbb{P}^n$ the vanishing of some homogeneous equation $f \in \mathbb{C}[x_0, \dots, x_n]$.

Let X be a variety: a polarization on X is the choice of a primitive ample (not only big and nef) line bundle on it. Given a line bundle L on X , we denote by $\phi_{|L|} : X \dashrightarrow \mathbb{P}(H^0(X, L)^\vee)$ the rational map associated to it.

The singular locus of X will be always denoted by $\mathrm{Sing}(X)$.

We will consider the same definition of a *dimensionally transverse intersection* as Debarre and Kuznetsov in [27].

Definition 1.1. *Given two closed subschemes $X_1, X_2 \subset \mathbb{P}(W)$, we say that the intersection $X_1 \cap X_2$ is dimensionally transverse if $\mathrm{Tor}_{>0}(\mathcal{O}_{X_1}, \mathcal{O}_{X_2}) = 0$.*

Remark 1.2. Consider X_1 and X_2 which are both Cohen–Macaulay. Then being a dimensionally transverse intersection is equivalent to have

$$\mathrm{codim}_x(X_1 \cap X_2) = \mathrm{codim}_x(X_1) + \mathrm{codim}_x(X_2)$$

for any closed point $x \in X_1 \cap X_2$ (where the codimensions are in $\mathbb{P}(W)$). When $X_2 \subset \mathbb{P}(W)$ is a hypersurface, the intersection $X_1 \cap X_2$ is dimensionally transverse if and only if no associated point of X_1 is contained in X_2 .

1.2 Pell's equations

Definition 1.3. *A generalized Pell's equation is a quadratic diophantine equation*

$$X^2 - rY^2 = m \tag{1.1}$$

in the unknowns $X, Y \in \mathbb{Z}$, for $r \in \mathbb{N}$ and $m \in \mathbb{Z} \setminus \{0\}$. If $m = 1$, the equation is called standard Pell's equation.

Even when it is not specified, we will always refer to integer solutions when talking about Pell's equations. If the equation (1.1) is solvable and r is not a square, the (infinite) set of solutions is divided into equivalence classes: two solutions (X, Y) and (X', Y') are said to be equivalent if

$$\frac{XX' - rYY'}{m} \in \mathbb{Z}, \quad \frac{XY' - X'Y}{m} \in \mathbb{Z}.$$

If the equation is standard, then all solutions are equivalent.

Inside any equivalence class of solutions, the *fundamental* solution (X, Y) is the one with smallest non-negative Y , if there is a single solution with this property. Otherwise, the smallest non-negative value of Y is realized by two *conjugate* solutions (X, Y) , $(-X, Y)$: in this case, the fundamental solution of the class will be (X, Y) with $X > 0$. If (X, Y) is a fundamental solution, all other solutions (X', Y') in the same equivalence class are of the form

$$\begin{cases} X' = aX + rbY \\ Y' = bX + aY \end{cases} \quad (1.2)$$

where (a, b) is a solution of the standard Pell's equation $a^2 - rb^2 = 1$. In particular, for a standard Pell's equation every solution can be found by recursion from a fundamental solution; for example the second solution obtained from (X, Y) is $(X^2 + rY^2, 2XY) = (2X^2 - 1, 2XY)$. Note that, for a standard Pell's equation, the solution (X', Y') obtained by (1.2) from (X, Y) always has $X' > X$.

Definition 1.4. A solution (X, Y) of (1.1) is called *positive* if $X > 0$, $Y > 0$. The *minimal solution* of the equation is the positive solution with smallest X .

In particular, the minimal solution is one of the fundamental solutions. We also use the expression “minimal solution with a property P ” to denote the positive solution with smallest X among those which satisfy the property P .

Let $(X, Y) = (z, w)$ be the minimal solution of the equation $X^2 - rY^2 = 1$. The half-open interval $[(\sqrt{m}, 0), (z\sqrt{m}, w\sqrt{m})]$ on the hyperbola $X^2 - rY^2 = m$ contains exactly one solution for each equivalence class. The solutions in this interval are all the solutions (X, Y) of (1.1) such that $X > 0$ and $0 \leq \frac{Y}{X} < \frac{w}{z}$. Moreover, if (X, Y) is a fundamental solution of (1.1) and $m > 0$, then:

$$0 < |X| \leq \sqrt{\frac{(z+1)m}{2}}, \quad 0 \leq Y \leq w\sqrt{\frac{m}{2(z+1)}}. \quad (1.3)$$

Debarre stated the following useful lemma, see [25, Lemma A.2].

Lemma 1.5. Let e_1 and e_2 be positive integers. Assume that for some $\epsilon \in \{-1, 1\}$ the equation $e_1X^2 - e_2Y^2 = \epsilon$ is solvable and let (a_ϵ, b_ϵ) be its minimal solution. Then e_1e_2 is not a square and the minimal solution of the equation $X^2 - e_1e_2Y^2 = 1$ is $(e_1a_\epsilon^2 + e_2b_\epsilon^2, 2a_\epsilonb_\epsilon) = (2e_1a_\epsilon^2 - \epsilon, 2a_\epsilonb_\epsilon)$, unless $e_1 = \epsilon = 1$ or $e_2 = -\epsilon = 1$.

In Chapter 6 we will often make use of the following technical result, that we prove.

Lemma 1.6. For $t \geq 1$ and $n \geq 2$ such that $t(n-1)$ is not a square, let (z, w) be the minimal solution of $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$. If $w \equiv 0 \pmod{2}$, then

- (i) $z \equiv 1 \pmod{2(n-1)}$ and $z \equiv 1 \pmod{2t}$ if and only if (z, w) is not the minimal solution of $X^2 - t(n-1)Y^2 = 1$.
- (ii) $z \equiv -1 \pmod{2(n-1)}$ and $z \equiv -1 \pmod{2t}$ if and only if the equation $X^2 - t(n-1)Y^2 = -1$ has integral solutions.
- (iii) if $z \equiv 1 \pmod{2(n-1)}$ and $z \equiv -1 \pmod{2t}$ then the equation $(n-1)X^2 - tY^2 = -1$ has integral solutions; if $t \geq 2$ the converse also holds.
- (iv) if $z \equiv -1 \pmod{2(n-1)}$ and $z \equiv 1 \pmod{2t}$ then the equation $(n-1)X^2 - tY^2 = 1$ has integral solutions; if $n \geq 3$ the converse also holds.

Proof. Let $w = 2m$ for some $m \in \mathbb{N}$.

- (i) – (ii) Write $z = 2(n-1)p \pm 1 = 2tq \pm 1$ for $p, q \in \mathbb{N}$. Then $p((n-1)p \pm 1) = tm^2$ and $(n-1)p = tq$, hence $r = \frac{p}{t} \in \mathbb{N}$. We have $r((n-1)rt \pm 1) = m^2$ and it follows that there exist $s, u \in \mathbb{N}$ such that $r = s^2$, $(n-1)tr \pm 1 = u^2$, $m = su$, hence $u^2 - t(n-1)s^2 = \pm 1$. If the sign is +, notice that $u < z$, therefore (z, w) is not the minimal solution of $X^2 - t(n-1)Y^2 = 1$. Conversely, assume first that $X^2 - t(n-1)Y^2 = -1$ has integer solutions and let (a, b) be the minimal one. Then, by Lemma 1.5 the minimal solution of $X^2 - t(n-1)Y^2 = 1$ is $(z, w) = (2t(n-1)b^2 - 1, 2ab)$, which satisfies $z \equiv -1 \pmod{2(n-1)}$ and $z \equiv -1 \pmod{2t}$. Similarly, if the minimal solution (u, s) of $X^2 - t(n-1)Y^2 = 1$ does not satisfy $u \equiv \pm 1 \pmod{n-1}$, then by (1.2) we have $(z, w) = (2t(n-1)s^2 + 1, 2us)$, hence $z \equiv 1 \pmod{2(n-1)}$ and $z \equiv 1 \pmod{2t}$.
- (iii) – (iv) Write $z = 2(n-1)p \pm 1 = 2tq \mp 1$ for $p, q \in \mathbb{N}$. Then $p((n-1)p \pm 1) = tm^2$ and $(n-1)p \pm 1 = tq$, hence $pq = m^2$. Since $\gcd(p, q) = 1$, there exist $s, u \in \mathbb{N}$ such that $p = s^2$, $q = u^2$ and $m = su$, thus $(n-1)s^2 - tu^2 = \mp 1$. Vice versa, let (a, b) be the solution of $(n-1)X^2 - tY^2 = \pm 1$ with smallest $a > 0$. By Lemma 1.5, the assumption $t \geq 2$ (if the sign is $-$) or $n \geq 3$ (if the sign is $+$) implies that the minimal solution of $X^2 - t(n-1)Y^2 = 1$ is $(z, w) = (2(n-1)a^2 \mp 1, 2ab)$, which satisfies $z \equiv \mp 1 \pmod{2(n-1)}$ and $z \equiv \pm 1 \pmod{2t}$. \square

Consider a generalized Pell's equation in the form $X^2 - rY^2 = m$ and let

$$\sqrt{r} = r_0 + \frac{1}{r_1 + \frac{1}{r_2 + \dots}}$$

be the continued fraction of \sqrt{r} . The continued fraction of a square root is always periodic by Lagrange's theorem: we will note it as $[r_0; \overline{r_1, \dots, r_k}]$. For the following results and details on the theory of Pell's equations, see [23, Chapter XXXIII, Section 16].

Theorem 1.7. *If $r > m^2$ and m is square-free then for every solution (X, Y) of $X^2 - rY^2 = m$ the quotient X/Y is a convergent to \sqrt{r} .*

As an immediate consequence we have the following.

Corollary 1.8. *If $r > m^2$ then for every solution (X, Y) of $X^2 - rY^2 = m$ the quotient X/Y is a convergent to \sqrt{r} whenever $\gcd(X, Y) = 1$.*

Moreover, fixed $r \in \mathbb{N}$ which is not a square, the evaluation of $X^2 - rY^2$ on the pairs (p, q) for p/q a convergent of \sqrt{r} is periodic with the same period of the continued fraction of \sqrt{r} . So the set of square-free integers $m < \sqrt{r}$ such that (1.1) admits a solution can be found by evaluating $X^2 - rY^2$ on the first k convergents of \sqrt{r} , where k is the period of the continued fraction of \sqrt{r} ; the whole set of $m < \sqrt{r}$ such that (1.1) admits a solution is then obtained by multiplication by squares.

1.3 Lattices

1.3.1 Definition of lattices

Lattice theory is a fundamental tool for the study of automorphisms and birational transformations of hyperkähler manifolds: this is a consequence of Theorem 2.3. Here we list the basic results that will be useful later; for a more complete treatment see for example [64, Chapter 14] or [90].

Definition 1.9. *An integral lattice, or simply a lattice, is a free \mathbb{Z} -module Λ of finite rank, endowed with a non-degenerate symmetric integral bilinear form $(-, -) : \Lambda \times \Lambda \rightarrow \mathbb{Z}$. For $x \in \Lambda$, we will also denote by x^2 the integer (x, x) , and we will call it the square of x .*

Consider a lattice $(\Lambda, (-, -))$.

- The *rank* of $(\Lambda, (-, -))$ is the rank of the \mathbb{Z} -module Λ .
- Denote by $\Lambda_{\mathbb{R}}$ the real extension of Λ . The *signature* (n_+, n_-) of $(\Lambda, (-, -))$ is the signature of the real extension of $(-, -)$.
- $(\Lambda, (-, -))$ is *hyperbolic* if its signature is $(1, \rho - 1)$, with $\rho \geq 2$.
- $(\Lambda, (-, -))$ is *positive (resp. negative) definite* if its signature is $(\rho, 0)$ (resp. $(0, \rho)$).

When there will be no ambiguity, we will denote by Λ the pair $(\Lambda, (-, -))$.

Since Λ is free, we can fix a basis $\{x_1, \dots, x_n\}$ for it; the associated *Gram matrix* is the symmetric matrix

$$\begin{bmatrix} (x_1, x_1) & \dots & (x_1, x_n) \\ \vdots & \ddots & \vdots \\ (x_n, x_1) & \dots & (x_n, x_n) \end{bmatrix}.$$

The *discriminant* of Λ , denoted by $\text{disc}(\Lambda)$, is the determinant of the Gram matrix associated to a basis of Λ ; it does not depend on the choice of the basis.

Definition 1.10. *We say that Λ is even if $x^2 \in 2\mathbb{Z}$ for every $x \in \Lambda$.*

Equivalently, we can say that Λ is even if the Gram matrix associated to one (hence any) basis of Λ has only even integers on the diagonal.

We give some examples of lattices. All of them are defined up to isomorphism, see Definition 1.19 below.

- For $n \in \mathbb{Z}$, $n \neq 0$, we denote by $\langle n \rangle$ the rank-1 lattice such that $x^2 = n$ for a generator x of $\langle n \rangle$.
- The *hyperbolic plane* U is the rank-2 lattice such that there exists a basis of U whose associated Gram matrix is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

- We denote by E_8 the rank-8 lattice such that there exists a basis of E_8 whose associated Gram matrix is

$$\begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & -1 \\ & & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & \\ & -1 & & & & & & 2 \end{bmatrix}.$$

- For any lattice Λ , with bilinear form $(-, -)$, and for any integer $n \neq 0$, the lattice $\Lambda(n)$ has Λ as subjacent \mathbb{Z} -module and $(x, y)_n = n(x, y)$ for any $x, y \in \Lambda$ as bilinear form.
- For Λ_1 and Λ_2 lattices, the lattice $\Lambda_1 \oplus \Lambda_2$ has the direct sum $\Lambda_1 \oplus \Lambda_2$ as subjacent \mathbb{Z} -module and, as bilinear form, $(x_1 + x_2, y_1 + y_2)_{\Lambda_1 \oplus \Lambda_2} = (x_1, y_1)_{\Lambda_1} + (x_2, y_2)_{\Lambda_2}$ for $x_1, y_1 \in \Lambda_1$ and $x_2, y_2 \in \Lambda_2$.

We will often make use of the negative definite lattice $E_8(-1)$. An alternative definition of U and E_8 is presented in Remark 1.15.

We say that a lattice is *non-divisible* if it is not of the form $\Lambda(n)$, for some lattice Λ and integer n .

We say that $x \in \Lambda$ is *primitive* if there is no $\tilde{x} \in \Lambda$ such that $x = k\tilde{x}$ for some integer $k \geq 2$, or equivalently if $\Lambda/\mathbb{Z}x$ is a free \mathbb{Z} -module.

Definition 1.11. *Given a lattice Λ , and $x \in \Lambda$, the divisibility of x in Λ is the positive integer $\text{div}_\Lambda(x)$ such that the image of the morphism*

$$\begin{aligned} (x, -) : \Lambda &\rightarrow \mathbb{Z} \\ \ell &\mapsto (x, \ell) \end{aligned}$$

is generated by $\text{div}_\Lambda(x)$.

Clearly, if $\text{div}_\Lambda(x)$ is prime then x is primitive, while the converse is not necessarily true.

We denote by $(-, -)$ also the \mathbb{Q} -extension of the symmetric form on $\Lambda_{\mathbb{Q}}$. The dual of the \mathbb{Z} -module Λ is $\text{Hom}(\Lambda, \mathbb{Z})$: we see it as a submodule of $\Lambda_{\mathbb{R}}$ i.e. $\Lambda^\vee = \{\ell \in \Lambda_{\mathbb{R}} \mid (\ell, x) \in \mathbb{Z} \text{ for all } x \in \Lambda\}$.

Definition 1.12. *The dual lattice of Λ is the free \mathbb{Z} -module Λ^\vee endowed with the symmetric form $(-, -)$. The discriminant group of Λ is $A_\Lambda = \Lambda^\vee/\Lambda$.*

As for Λ , we refer to Λ^\vee also for the pair $(\Lambda^\vee, (-, -))$. The rank of Λ^\vee is the same as Λ , so A_Λ is a torsion group; its cardinality is $|A_\Lambda| = |\text{disc}(\Lambda)|$.

Remark 1.13. The dual lattice Λ^\vee is not necessarily a lattice with the definition given above.

Definition 1.14. *The lattice Λ is unimodular if $\Lambda^\vee = \Lambda$ i.e. its discriminant group is trivial.*

A simple computation shows that both U and E_8 are unimodular lattices, while $\langle n \rangle$ is not for $n \neq 1$, as $A_{\langle n \rangle} = \mathbb{Z}/n\mathbb{Z}$. Moreover $\text{disc}(\Lambda(n)) = n^{\text{rk}(\Lambda)} \text{disc}(\Lambda)$. In particular $E_8(-1)$ is unimodular too. For Λ, Λ' lattices, the discriminant group of $\Lambda \oplus \Lambda'$ is $A_{\Lambda \oplus \Lambda'} = A_\Lambda \oplus A_{\Lambda'}$.

Remark 1.15. We can define U as the unique even unimodular hyperbolic lattice of rank 2, up to isomorphism (see Definition 1.19). We can define E_8 as the unique even unimodular positive definite lattice of rank 8, up to isomorphism.

Two natural notions arise with the definition of a lattice and its dual.

Definition 1.16. *An overlattice of Λ is a lattice $R \subseteq \Lambda_{\mathbb{Q}}$ which contains Λ . In this case we write $\Lambda \subseteq R$.*

Clearly Λ, Λ^\vee and any overlattice R have the same rank, so Λ^\vee/Λ and R/Λ are torsion groups; since $R/\Lambda \subseteq \Lambda^\vee/\Lambda = A_\Lambda$, we have that the cardinality of R/Λ divides $\text{disc}(\Lambda)$. In particular, if Λ is unimodular it admits no non-trivial overlattice.

We say that an overlattice $\Lambda \subseteq R$ is *primitive* if Λ is generated by primitive elements of R . An element $x \in \Lambda$ is primitive, as an element, if and only if $\mathbb{Z}x$ is primitive as a lattice.

Definition 1.17. *A sublattice of Λ is a (free) submodule of Λ endowed with the restriction of the bilinear form of Λ , if the restriction is again non-degenerate.*

We say that a sublattice $T \subset \Lambda$ is *primitive* if Λ/T is a free module.

For $T \subset \Lambda$ a sublattice of Λ , the orthogonal complement of T is $T^\perp = \{x \in \Lambda \mid (x, t) = 0 \text{ for every } t \in T\}$; it is always a primitive sublattice of Λ and $T \oplus T^\perp \subset \Lambda$ is an overlattice of Λ . Hence the rank of T^\perp is $\text{rk}(\Lambda) - \text{rk}(T)$ and its signature is $(n_+ - t_+, n_- - t_-)$ where (n_+, n_-) and (t_+, t_-) are the signatures of, respectively, Λ and T .

Remark 1.18. For any primitive overlattice $\Lambda \subseteq R$ the sublattice Λ is primitive in R if and only if $\Lambda = R$.

1.3.2 Isometries of lattices

Definition 1.19. Let Λ, Λ' be two lattices. An isometry between Λ and Λ' is a morphism of groups $\phi : \Lambda' \rightarrow \Lambda$ such that $(\phi(x), \phi(y)) = (x, y)$ for every $x, y \in \Lambda'$. We denote by $O(\Lambda)$ the group of isometries of Λ into itself.

Remark 1.20. The image of Λ' via ϕ is a sublattice of Λ . In particular ϕ is always injective. For this reason, isometries of lattices are also called *embeddings*.

An embedding $\phi : \Lambda' \rightarrow \Lambda$ is *primitive* if $\phi(\Lambda') \subset \Lambda$ is a primitive sublattice of Λ . Two primitive embeddings $\phi_1 : \Lambda' \rightarrow \Lambda$ and $\phi_2 : \Lambda' \rightarrow \Lambda$ are isometric if there exists $\lambda \in O(\Lambda)$ such that $\lambda \circ \phi_1 = \phi_2$ and $\lambda|_{\phi_1(\Lambda')} = \text{id}_{\phi_1(\Lambda')}$. We stress the difference between sublattices of Λ and lattices admitting an embedding in Λ : the reason is that the same lattice can admit different non-isometric embeddings into a fixed lattice.

Remark 1.21. We consider an embedding $\phi : \Lambda' \rightarrow \Lambda$ and we call R the saturated lattice of $\phi(\Lambda')$ in Λ i.e. the smallest primitive sublattice of Λ containing $\phi(\Lambda')$. We have $R = (\phi(\Lambda')^\perp)^\perp$, in particular $\phi(\Lambda') \subset R$ is an overlattice of $\phi(\Lambda') \cong \Lambda'$.

Clearly ϕ is a primitive embedding if and only if $R \cong \Lambda'$. This means, for example, that every embedding $\Lambda' \rightarrow \Lambda$ is primitive if Λ' is unimodular.

We focus on a particular family of isometries. For more details about it, see for example [51, Section 3].

Definition 1.22. Let Λ be an even lattice. For $D \in \Lambda$ with $D^2 \neq 0$, we call R_D the associated reflection

$$R_D : \Lambda_{\mathbb{Q}} \rightarrow \Lambda_{\mathbb{Q}}$$

$$v \mapsto v - 2 \frac{(v, D)}{D^2} \cdot D$$

This map restricts to a well-defined isometry of Λ if and only if the divisibility of D is either D^2 or $D^2/2$.

The reflection R_D acts as the identity on D^\perp and $R_D(D) = -D$; the whole action of R_D is then determined.

Remark 1.23. We take an element $D \in \Lambda$ such that R_D restricts to an isometry of Λ , and we consider $\alpha \in O(\Lambda)$. If α does not send D to $-D$, then α and R_D commutes if and only if $\alpha(D) = D$. The condition on the image of D through α is very natural in our setting.

The following definition is fundamental when the lattices on which we are working are some cohomology groups of Kähler varieties. This happens often when dealing with hyperkähler manifolds, cfr. Theorem 2.3 and more in general Section 2.3.1.

Definition 1.24. Consider two lattices Λ_1, Λ_2 such that their complex extension admits a Hodge structure of weight n , $(\Lambda_h)_{\mathbb{C}} = \bigoplus_{p+q=n} H_h^{p,q}$ for $h = 1, 2$. Let $\phi : \Lambda_1 \rightarrow \Lambda_2$ be an isometry. We say that ϕ is a Hodge isometry if the complex extension of ϕ is an isomorphism of Hodge structures.

The following definitions will be useful in Section 2.3.2 and, consequently, in Section 3.2.

Definition 1.25. A lattice with a distinguished vector is a pair (Λ, ℓ) where Λ is a lattice and $\ell \in \Lambda$ is a primitive vector. Two lattices with a distinguished vector (Λ_1, ℓ_1) and (Λ_2, ℓ_2) are isomorphic if there exists an isometry between Λ_1 and Λ_2 sending ℓ_1 to ℓ_2 .

1.3.3 The discriminant group

Definition 1.26. Let A be a finite abelian group. A finite bilinear form is a symmetric bilinear form $b : A \times A \rightarrow \mathbb{Q}/\mathbb{Z}$. A finite quadratic form is a morphism $q : A \rightarrow \mathbb{Q}/2\mathbb{Z}$ such that

- $q(ka) = k^2q(a)$ for $a \in A, k \in \mathbb{Z}$,
- $q(a + a') - q(a) - q(a') = 2b(a, a')$ where $b : A \times A \rightarrow \mathbb{Q}/\mathbb{Z}$ is a finite bilinear form (called the bilinear form associated to q). An isometry of A is an isomorphism of groups $\phi : A \rightarrow A$ such that $q(\phi(a)) = q(a)$ for every $a \in A$; we denote by $O(A)$ the group of isometries of A .

A finite bilinear form $b : A \times A \rightarrow \mathbb{Q}/\mathbb{Z}$ is said to be *non-degenerate* if for every non-zero $a \in A$ there exists $x \in A$ such that $b(a, x) \neq 0$; a finite quadratic form is said to be non-degenerate if the associated bilinear form is non-degenerate.

For $H \subset A$, the orthogonal complement $H^\perp \subset A$ is the orthogonal complement with respect to the associated bilinear form. Moreover H is *isotropic* if $q|_H = 0$ or equivalently if $H \subseteq H^\perp$.

Let Λ be a lattice; the \mathbb{Q} -extension of the bilinear form on it endows the dual lattice Λ^\vee with a symmetric form $(-, -) : \Lambda^\vee \times \Lambda^\vee \rightarrow \mathbb{Q}$. For $x \in \Lambda^\vee$, we denote by $[x]$ the class of x in the discriminant group A_Λ . We can then define a non-degenerate finite bilinear form

$$b_\Lambda : A_\Lambda \times A_\Lambda \rightarrow \mathbb{Q}/\mathbb{Z} \tag{1.4}$$

$$([x], [y]) \mapsto [(x, y)]. \tag{1.5}$$

Since the bilinear form on Λ is integral, the form b_Λ is well-defined. Moreover, when Λ is even, we can define a finite quadratic form

$$q_\Lambda : A_\Lambda \rightarrow \mathbb{Q}/2\mathbb{Z} \tag{1.6}$$

$$[x] \mapsto [(x, x)]; \tag{1.7}$$

by definition, the associated finite form is b_Λ , hence q_Λ is non-degenerate. We will always consider the discriminant group endowed with this quadratic form. If A_Λ is cyclic of order m , we write $A_\Lambda \cong \frac{\mathbb{Z}}{m\mathbb{Z}}(\alpha)$ if q_Λ takes value $\alpha \in \mathbb{Q}/2\mathbb{Z}$ on a generator of A_Λ .

Remark 1.27. When Λ is unimodular A_Λ is trivial. The discriminant group of $\langle n \rangle$ is $A_{\langle n \rangle} = \frac{\mathbb{Z}}{n\mathbb{Z}}(1/n)$. The isomorphism $A_{\Lambda_1 \oplus \Lambda_2} \cong A_{\Lambda_1} \oplus A_{\Lambda_2}$ holds for the bilinear form as well.

Let $\phi : \Lambda \rightarrow \Lambda$ be an isometry and consider the induced morphism $(\phi^{-1})^\vee : \Lambda^\vee \rightarrow \Lambda^\vee$. Since it sends Λ to Λ it descends to an isomorphism of the discriminant group A_Λ .

Definition 1.28. For $\phi \in O(\Lambda)$ we denote by $\bar{\phi}$ the isometry induced by ϕ on A_Λ . We call $\tilde{O}(\Lambda)$ the kernel of the morphism of groups sending ϕ to $\bar{\phi}$ i.e. the group of isometries acting as the identity on the discriminant group.

Orbits of elements in Λ with respect to the action of $\tilde{O}(\Lambda)$ have a strong geometric meaning, see Section 3.2. Moreover, we will need the following classical result, see for example [32, Theorem 2.1].

Theorem 1.29 (Eichler's criterion). *Let Λ be a lattice containing two orthogonal copies of U . Then the $\tilde{O}(\Lambda)$ -orbit of a primitive vector $\ell \in \Lambda$ is determined by two invariants: its square ℓ^2 and the class $[\ell^*]$, where $\ell^* = \ell / \text{div}_\Lambda(\ell)$, in the discriminant group A_Λ .*

We presented in Remark 1.21 a relation between embeddings of a lattice Λ in some other lattice and overlattices of Λ . The overlattices of Λ can be described looking at the discriminant group A_Λ .

Proposition 1.30. *There is a bijection between the set of overlattices of Λ and the set of the subgroups of A_Λ which are isotropic.*

Proof. See [90, Proposition 1.4.1, Item a)]. Call $\pi : \Lambda^\vee \rightarrow \Lambda^\vee / \Lambda = A_\Lambda$ the natural projection. The bijection is given by the function sending $\Lambda \subseteq R$ to $H_R = \pi(R)$: a priori it induces a bijection between the set of groups R with $\Lambda \subset R \subset \Lambda^\vee$ and subgroups of A_Λ , but $\pi(R)$ is isotropic if and only if $b_{A_\Lambda}(x, y) \in \mathbb{Z}$ for every $x, y \in R$, which is equivalent to R being a lattice. \square

Remark 1.31. Finite quadratic forms over a finite abelian group have been completely classified by Nikulin, see [90, Theorem 1.8.1]. In this thesis we will use this classification to compute the overlattices in some particular cases. For this purpose we will use some Sage packages, see [112]: in particular we make use of `sage.modules.free_quadratic_module_integer_symmetric`, to define lattices, and `sage.modules.torsion_quadratic_module` to obtain an explicit description by a matrix of the quadratic form on A_Λ . Both packages have been developed by Simon Brandhorst. Then isotropic subgroups of A_Λ can be found by a direct calculation.

The following result has been proved by Nikulin in [90, Theorem 1.14.4]: here we present the weaker form in [64, Chapter 14, Theorem 1.12]. The *length* $l(A_\Lambda)$ is the minimal number of generators for A_Λ .

Theorem 1.32. *Let Λ be an even, unimodular lattice of signature (n_+, n_-) and Λ' be an even lattice of signature (m_+, m_-) . If $m_\pm < n_\pm$ and*

$$l(\Lambda') + 2 \leq \text{rk}(\Lambda) - \text{rk}(\Lambda') \tag{1.8}$$

then there exists a primitive embedding $\Lambda' \rightarrow \Lambda$, which is unique up to automorphisms of Λ .

Clearly $l(A_\Lambda) \leq \text{rk}(\Lambda)$. As an immediate consequence, condition (1.8) holds whenever $2(\text{rk}(\Lambda') + 1) \leq \text{rk}(\Lambda)$.

We put together Proposition 1.30 and Theorem 1.32 for the particular case of $\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$; this is the $K3$ lattice, see Definition 2.18. Consider an even hyperbolic lattice Λ' whose rank is less or equal to 10: by Theorem 1.32 not only Λ' admits, up to isometry, a unique primitive embedding into Λ_{K3} , but the same holds for every overlattice of Λ' . As an example, take a rank-2 lattice Λ' whose Gram matrix with respect to a basis $\{L, D\}$ is

$$\begin{bmatrix} 10 & 2 \\ 2 & 0 \end{bmatrix}.$$

A computer algebra calculation, see Remark 1.31, shows that the quadratic form on $A_{\Lambda'}$ is $\frac{\mathbb{Z}}{2\mathbb{Z}}(1/2) \oplus \frac{\mathbb{Z}}{2\mathbb{Z}}(3/2)$; there is one non-trivial isotropic lattice of $A_{\Lambda'}$, corresponding by Proposition 1.30 to the lattice $\mathbb{Z}L + \mathbb{Z}\frac{D}{2} \cong U$. Then Λ' admits two different embeddings in Λ_{K3} : a primitive embedding and a non-primitive embedding as a sublattice of U . Note, however, that in both cases the image of L is primitive.

We end the section with a result by Nikulin, about isometries of sublattices extending to the whole lattice. Let $T \subset \Lambda$ be a primitive sublattice; the sublattice $T \oplus T^\perp$ has maximal rank in Λ , so $T \oplus T^\perp \subseteq \Lambda$ is an overlattice. As above, we call $\pi : (T \oplus T^\perp)^\vee \rightarrow A_{T \oplus T^\perp}$ the natural quotient morphism, and H_Λ the isotropic group in $A_{T \oplus T^\perp}$ associated to Λ . We call $p_T : A_{T \oplus T^\perp} \rightarrow A_T$ and $p_{T^\perp} : A_{T \oplus T^\perp} \rightarrow A_{T^\perp}$ the projections, and we define $M_T = p_T(H_\Lambda)$ and $M_{T^\perp} = p_{T^\perp}(H_\Lambda)$. Since T is primitive in Λ , the restrictions of the projections to H_Λ are isomorphisms. Following Nikulin, we define $\gamma = p_{T^\perp} \circ (p_T^{-1})|_{M_T} : M_T \rightarrow M_{T^\perp}$.

Remark 1.33. The map γ is an anti-isometry i.e. $q_T(x) = -q_{T^\perp}(\gamma(x))$ for every $x \in M_T$.

Theorem 1.34. *Let Λ be an even lattice with a primitive sublattice $T \subset \Lambda$. For every isometry $\varphi \in O(T)$, there exists an isometry $\phi \in O(\Lambda)$ such that $\phi|_T = \varphi$ if and only if there exists an isometry $\psi \in O(T^\perp)$ such that $\bar{\psi} \circ \gamma = \gamma \circ \bar{\varphi}$.*

Proof. See [90, Corollary 1.5.2] for $S = S_1$. □

1.4 Double covers

Coverings will be fundamental in the definition of double *EPW* sextics. Recall that a morphism $f : \tilde{S} \rightarrow S$ between normal varieties is a *double cover* if it is finite of degree 2. In this case there is an involution $\tau \in \text{Aut}(\tilde{S})$ such that $S \cong \tilde{S}/\langle \tau \rangle$: we will call it *covering involution*. The *branch locus* of the double cover f is the subscheme of S of points where the cardinality of the fiber of f drops. The *ramification locus* of f is the subscheme of \tilde{S} of points whose image via f is contained in the branch locus. The scheme-theoretic structure of them will be made precise later.

We say that two double coverings $f_1 : \tilde{S}_1 \rightarrow S_1$ and $f_2 : \tilde{S}_2 \rightarrow S_2$ are isomorphic if there exists a

commutative diagram

$$\begin{array}{ccc} \tilde{S}_1 & \xrightarrow{\tilde{\phi}} & \tilde{S}_2 \\ \downarrow f_1 & & \downarrow f_2 \\ S_1 & \xrightarrow{\phi} & S_2 \end{array}$$

where $\tilde{\phi}$ and ϕ are biholomorphisms. This is equivalent to ask that there is some biholomorphism $\tilde{\phi} : \tilde{S}_1 \rightarrow \tilde{S}_2$ that commutes with the covering involutions τ_1 and τ_2 , that is $\tilde{\phi} \circ \tau_1 = \tau_2 \circ \tilde{\phi}$. On the one hand such a morphism would descend to a morphism $S_1 \cong \tilde{S}_1 / \langle \tau_1 \rangle \xrightarrow{\phi} \tilde{S}_2 / \langle \tau_2 \rangle \cong S_2$. On the other hand, given the diagram above, for every $s \in \tilde{S}_1$ we have $f_2 \tilde{\phi} \tau_1(s) = f_2 \tau_2 \tilde{\phi}(s)$. Since $f_2 : \tilde{S}_2 \rightarrow S_2$ is a double cover, we have $\tilde{\phi} \tau_1(s) = \tau_2 \tilde{\phi}(s)$ for every $s \in \tilde{S}_1$.

Remark 1.35. Let τ be the covering involution associated to $f : \tilde{S} \rightarrow S$, and let $H^*(\tilde{S}, \mathbb{Q})^\tau$ be the subgroup of classes in $H^*(\tilde{S}, \mathbb{Q})$ which are invariant with respect to the action in cohomology of τ . Then, see [18, Chapter 3, Corollary 2.3], we have

$$H^*(\tilde{S}, \mathbb{Q})^\tau = f^*(H^*(S, \mathbb{Q})).$$

Let $f : \tilde{S} \rightarrow S$ be a double cover, with covering involution τ . For every group of automorphisms $\tilde{G} \subseteq \text{Aut}(\tilde{S})$, if every element of \tilde{G} commutes with τ there is a natural homomorphism $\tilde{G} \rightarrow \text{Aut}(S)$: the image of $\tilde{g} \in \tilde{G}$ is the automorphism g such that $g(f(x)) = f(\tilde{g}(x))$.

Definition 1.36. Let $G \subseteq \text{Aut}(S)$ be a subgroup of automorphisms of S . We say that G lifts to \tilde{S} if there exists a group of automorphisms $\tilde{G} \subseteq \text{Aut}(\tilde{S})$ which commutes with the covering involution τ and the corresponding homomorphism $\tilde{G} \rightarrow \text{Aut}(S)$ is an isomorphism onto the group G .

We cite some classical definitions and results; we refer to [28, Section 2] for notations.

Definition 1.37. Let S be a normal variety, $D \in \text{Cl}(S)$ a Weil divisor on S . The Cartier locus $U \subseteq S$ of D is the maximal open subset on which D is Cartier.

Let S be a normal variety. Since on the smooth locus of S every Weil divisor is Cartier, for every Weil divisor the complement of the Cartier locus is contained in the singular locus of S and has thus codimension at least 2.

Consider now a coherent sheaf \mathcal{R} on S . We say that \mathcal{R} has rank r if there is a dense open subset of S where the restriction of \mathcal{R} is locally free of rank r ; the *locally free locus* of \mathcal{R} is the maximal open subscheme on which \mathcal{R} is locally free. In the following we will call dual sheaf the sheaf $\mathcal{R}^\vee = \mathcal{H}om(\mathcal{R}, \mathcal{O}_S)$. There is a natural evaluation morphism of sheaves between \mathcal{R} and $\mathcal{R}^{\vee\vee} = (\mathcal{R}^\vee)^\vee$: for any open $U \subseteq S$ the associated morphism of modules is

$$ev_U : \mathcal{R}(U) \rightarrow \mathcal{H}om(\mathcal{H}om(\mathcal{R}, \mathcal{O}_S))(U) = \mathcal{H}om(\mathcal{H}om(\mathcal{R}|_U, \mathcal{O}_U), \mathcal{O}_U)$$

such that for $r \in \mathcal{R}(U)$ and any open $V \subseteq U$, for $f \in \mathcal{H}om(\mathcal{R}|_V, \mathcal{O}_V)$ a morphism of sheaves, we have $(ev_U(r))_V(f) = f_V(r|_V) \in \mathcal{O}(V)$.

Definition 1.38. Consider a coherent sheaf \mathcal{R} on a normal variety S . We say that \mathcal{R} is a reflexive sheaf if the evaluation morphism $\mathcal{R} \rightarrow \mathcal{R}^{\vee\vee}$ is an isomorphism. Moreover, we say that \mathcal{R} is self-dual if $\mathcal{R} \cong \mathcal{R}^\vee$.

The dual sheaf \mathcal{R}^\vee of a coherent sheaf \mathcal{R} is reflexive, see [57, Corollary 1.2]; for other results about reflexive sheaves, cfr. [57]. There is a canonical way to associate a rank-one reflexive sheaf $\mathcal{O}_S(D)$ to a Weil divisor D , and vice versa. Let $D \in \text{Cl}(S)$ be the linear equivalence class of a prime divisor, and call U its Cartier locus. Then $d = D|_U$ is the class of a Cartier divisor on U : we call \mathcal{I}_d the associated ideal sheaf. The sheaf associated to D is then the only coherent sheaf $\mathcal{O}_S(D)$ whose restriction to U is isomorphic to \mathcal{I}_d^\vee , the dual of the ideal sheaf of d ; the sheaf obtained in this way is reflexive. The general case follows by additivity. Furthermore, consider a rank-one reflexive sheaf \mathcal{R} , and call U its locally free locus. Since U has codimension at least 2 in S , the groups of classes of Cartier divisors $\text{Cl}(S)$ and $\text{Cl}(U)$ are isomorphic through the restriction morphism. The sheaf $\mathcal{R}|_U$ is locally free on U , so there is a Cartier divisor D' on U whose associated ideal sheaf is $\mathcal{R}|_U^\vee$: the Weil divisor associated to \mathcal{R} is then the only divisor D whose restriction to U is D' . Note that the locally free locus of $\mathcal{O}_S(D)$ is the Cartier locus of D .

On rank-one reflexive sheaves, the operation $D_1 + D_2$ correspond to $(\mathcal{O}_S(D_1) \otimes \mathcal{O}_S(D_2))^\vee$ so that the inverse of $\mathcal{O}_S(D)$ is $(\mathcal{O}_S(D))^\vee$. Since, by additivity, $\mathcal{O}_S(D)^\vee \cong \mathcal{O}_S(-D)$, the sheaf $\mathcal{O}_S(D)$ is self-dual if and only if D is 2-torsion, that means that $2D = 0$ modulo linear equivalence. In this case we say that D is a two-torsion class, and we have $\mathcal{O}_S(D)^{\vee\vee} \cong \mathcal{O}_S$.

The following result is much more known for double covers coming from a Cartier divisor. For our purpose we will need this generalised statement.

Theorem 1.39. *Let S be a normal variety, and D a two-torsion Weil divisor class on S . There is a normal variety \tilde{S} and a double cover $f : \tilde{S} \rightarrow S$ that satisfy the following properties:*

- a) *there is an isomorphism $f_*\mathcal{O}_{\tilde{S}} \cong \mathcal{O}_S \oplus \mathcal{O}_S(D)$;*
- b) *the morphism f is étale over the Cartier locus of D .*

If moreover all invertible functions on S are squares, the double cover is unique up to isomorphism.

Proof. See [28, Proposition 2.5]. □

The key point in the construction above is to provide a structure of \mathcal{O}_S -algebra to $\mathcal{O}_S \oplus \mathcal{O}_S(D)$. It is sufficient to observe that there is an isomorphism

$$\mathbf{m} : \mathcal{O}_S(D) \otimes \mathcal{O}_S(D) \xrightarrow{ev} (\mathcal{O}_S(D) \otimes \mathcal{O}_S(D))^{\vee\vee} \xrightarrow{\sim} \mathcal{O}_S$$

which is symmetric with respect to the permutation of factors (the first morphism in the definition of \mathbf{m} is the evaluation, while the existence of the second one is granted by the fact that D is two-torsion). So the structure of \mathcal{O}_S -algebra is given by the inclusion in the first component $\mathcal{O}_S \rightarrow \mathcal{O}_S \oplus \mathcal{O}_S(D)$. We call \mathbf{p} the product on \mathcal{O}_S : the product on $\mathcal{O}_S \oplus \mathcal{O}_S(D)$ is induced by the map

$$\begin{array}{c} ((\mathcal{O}_S \otimes \mathcal{O}_S) \oplus (\mathcal{O}_S(D) \otimes \mathcal{O}_S(D))) \oplus (\mathcal{O}_S \oplus \mathcal{O}_S(D)) \oplus (\mathcal{O}_S \oplus \mathcal{O}_S(D)) \\ \downarrow (\mathbf{p} + \mathbf{m}) \oplus (\text{id} + \text{id}) \\ \mathcal{O}_S \oplus \mathcal{O}_S(D), \end{array}$$

where the upper side is isomorphic to the tensor product of $\mathcal{O}_S \oplus \mathcal{O}_S(D)$, by distributivity of tensor product and permutation of terms.

The \mathcal{O}_S -module $\mathcal{O}_S \oplus \mathcal{O}_S(D)$ is coherent, hence for any open affine cover \mathcal{U} of S there exists, for every $U \in \mathcal{U}$, an $\mathcal{O}(U)$ -module M_U such that $\tilde{M}_U \cong (\mathcal{O}_S \oplus \mathcal{O}_S(D))|_U$ as \mathcal{O}_U -modules.

The covering space \tilde{S} is then defined as $\tilde{S} = \text{Spec}_S(\mathcal{O}_S \oplus \mathcal{O}_S(D))$, in the sense that locally we define $\tilde{U} = \text{Spec}(\tilde{M}_U)$ and \tilde{S} is obtained by glueing them. The covering map $f : \tilde{S} \rightarrow S$ is induced locally by the inclusion $\mathcal{O}_U \rightarrow \tilde{M}_U \cong \mathcal{O}_U \oplus \mathcal{O}_U(D)$ and is finite since $\mathcal{O}_{\tilde{S}}$ is a coherent \mathcal{O}_S -module. There is a natural involution τ on \tilde{S} that acts as the identity on \mathcal{O}_S and as -1 on $\mathcal{O}_S(D)$. Since the invariant subsheaf of τ in $\mathcal{O}_S \oplus \mathcal{O}_S(D)$ is \mathcal{O}_S , the involution τ commutes with f and $\tilde{S}/\langle \tau \rangle \cong S$.

As for the uniqueness of the cover, in general the structure of \mathcal{O}_S -algebra of the sheaf $\mathcal{O}_S \oplus \mathcal{O}_S(D)$ depends on \mathbf{m} , and while the first map in the definition of \mathbf{m} is set to be the evaluation morphism, the choice of the second one yields, a priori, non-isomorphic structures of \mathcal{O}_S -algebra. However, two morphisms

$$(\mathcal{O}_S(D) \otimes \mathcal{O}_S(D))^{\vee\vee} \xrightarrow{\sim} \mathcal{O}_S$$

that differ by the square of an invertible function yield to isomorphic double covers: if every function on S satisfies this property, then uniqueness is granted.

Remark 1.40. Since the covering map $f : \tilde{S} \rightarrow S$ is finite, the pullback of an ample line bundle through f is again ample. In particular, if S is projective then \tilde{S} is projective too.

Lemma 1.41. *Consider S , \tilde{S} and D as in Theorem 1.39: if S is integral, and D is not linearly equivalent to zero, then \tilde{S} is integral too.*

Proof. See [28, Lemma 2.7]. □

Call U the Cartier locus of D : on U the sheaf $\mathcal{O}_S(D)$ is invertible and the map \mathbf{m} is an isomorphism. The *branch locus* $B(f)$ of f is the subscheme of S whose ideal sheaf is the image of the map $\mathbf{m} : \mathcal{O}_S(D) \otimes \mathcal{O}_S(D) \rightarrow \mathcal{O}_S$, so the branch locus is contained in the complement of U , which is contained in $\text{Sing}(S)$ hence it has codimension at least 2. The inclusion $B(f) \hookrightarrow S$ induces a projection $\pi : \mathcal{O}_S \rightarrow \mathcal{O}_{B(f)}$, and thus another surjective morphism $\pi + 0 : \mathcal{O}_S \oplus \mathcal{O}_S(D) \rightarrow \mathcal{O}_{B(f)}$. This morphism of sheaves produces a morphism $\iota : B(f) \rightarrow \tilde{S}$ whose image is called *the ramification locus* of f . Note that, while the composition $f \circ \iota$ is the identity on $B(f)$, the scheme-theoretic preimage through f of $B(f)$ is a non-reduced scheme, see [28, Lemma 2.9].

Remark 1.42. Even if \mathbf{m} induces an isomorphism between $\mathcal{O}_S(D) \otimes \mathcal{O}_S(D)$ and \mathcal{O}_S over U , this isomorphism does not extend on the whole S , since a priori $\mathcal{O}_S(D) \otimes \mathcal{O}_S(D)$ is not reflexive. We know that the ramification locus $B(f)$ is contained in the Cartier locus of D , so everytime $B(f)$ is not empty the Weil divisor D is not a Cartier divisor, and in particular $\mathcal{O}_S(D)$ is not trivial.

There is a converse of Theorem 1.39, for a proof of the result below see [28, Lemma 2.6].

Lemma 1.43. *Consider S and \tilde{S} normal varieties, and assume that $f : \tilde{S} \rightarrow S$ is a double cover which is étale over the nonsingular locus of S . There exists a 2-torsion Weil divisor D on S such that $f_*\mathcal{O}_{\tilde{S}} \cong \mathcal{O}_S \oplus \mathcal{O}_S(D)$. If f is étale everywhere, D is a Cartier divisor.*

Chapter 2

Hyperkähler manifolds

2.1 Introduction

We introduce here the main objects of study of this thesis, which are hyperkähler manifolds. Aside from being an interesting matter of study in themselves, hyperkähler manifolds occur in the Beauville-Bogomolov decomposition, hence they are a natural object of study when dealing with manifolds whose first Chen class is trivial.

In dimension two these manifolds are well-known, as they coincide with $K3$ surfaces, which have been studied from the 19th-century. The first examples in higher dimension were provided by Fujiki in dimension 4, then by Beauville in every even dimension, see [9]; hyperkähler manifolds do not exist in odd dimension.

In the first section we introduce hyperkähler manifolds and we present the known examples. A fundamental result is Theorem 2.3, which relates the study of hyperkähler manifolds with lattice theory. Many properties of hyperkähler manifolds are preserved under deformation; in this chapter we present some general results on hyperkähler manifolds, but our final object of interest are the manifolds that are equivalent by deformation to the Hilbert scheme of n points on a projective $K3$ surface, see Definition 2.19. Manifolds of this type and of dimension $2n$ are called *manifolds of $K3^{[n]}$ -type*.

Several different moduli spaces can be constructed for hyperkähler manifolds. In the second section we focus on some of them, see for example Theorem 2.42 and Definition 2.69. Every moduli space which we will present arises with a *period map*, i.e. a map from the moduli space to a homogeneous domain, or a quotient of a homogeneous domain by an arithmetic group. A turning point for the study of hyperkähler manifolds of $K3^{[n]}$ -type is the Global Torelli theorem for polarized hyperkähler manifolds of $K3^{[n]}$ -type; it states that the period map, when restricted to an irreducible component of the moduli space of polarized manifolds, defined in Definition 2.69, is an algebraic map and is an open embedding. This has been proved by Markman in [77]. The number of irreducible components of this moduli spaces has been computed by Apostolov in

[2]: we explain how to count them and how to recognize the component on which a particular polarized manifold lies. This will be useful later in Section 3.2.

In the third section we focus on automorphisms of hyperkähler manifolds of $K3^{[n]}$ -type; we will consider both birational and biregular automorphisms. A crucial point is Corollary 2.109, another Torelli-like theorem, which allows to study automorphisms through their action on the second cohomology group, seen as a lattice. The chamber decomposition of the positive cone of a hyperkähler manifold will also play an important role with respect to automorphisms; we will give an insight of this in Section 2.4.1. Section 2.4.5 is a summary of results on involutions that act in cohomology as the inverse of a reflection with respect to a divisor. All this tools will be necessary for the results of Chapter 6.

In Section 2.4.4 we introduce the theory of deformations of automorphisms, with a special focus on involutions which act non-trivially on a symplectic 2-form on the manifold. This will allow us to use deformation theory to give a geometric description of some involutions in Chapter 3.

Most of the results in this chapter are for manifolds of $K3^{[n]}$ -type for every $n \geq 1$. In Section 2.2.3, Section 2.3.3 and Section 2.4.3 we present some results specific to $K3$ surfaces; in particular, in Definition 2.33 we present $\langle 10 \rangle$ -polarized Brill-Noether general $K3$ surfaces, which will play a fundamental role in Chapter 5.

All results in this chapter are already known. We refer mostly to [9], [77], [70] for the general results, and to [64] for $K3$ surfaces.

2.2 An introduction to hyperkähler manifolds

2.2.1 Definition of hyperkähler manifolds

Let X be a manifold. We say that a holomorphic 2-form on X is symplectic if it has everywhere maximal rank.

Definition 2.1. *A hyperkähler manifold is a compact Kähler manifold X which is simply-connected and such that $H^0(X, \Omega_X^2) = \mathbb{C}\omega$, where ω is a symplectic 2-form on X .*

Let X be a hyperkähler manifold. The existence of a symplectic 2-form on X has strong consequences: the very first one is that X has even complex dimension, since for every $x \in X$ the skew-symmetric form ω_x on the tangent space $T_x X$ has maximal rank, and the rank of a skew-symmetric form is always even.

Again, call K_X the canonical bundle of X and let $2n$ be the dimension of X . Since ω^n is nowhere vanishing on X , the canonical bundle is trivial, $K_X \cong \mathcal{O}_X$; in particular the first Chern class of X is trivial. As $\omega : TX \times TX \rightarrow \mathbb{C} \otimes \mathcal{O}_X$ has maximal rank, the tangent bundle TX is isomorphic to Ω_X^1 .

Hyperkähler manifolds have a special place among manifolds whose first Chern class is zero. Indeed, they arise naturally as building blocks of manifolds with this property, as was proved by Beauville.

Theorem 2.2 (Beauville-Bogomolov decomposition). *Let M be a compact Kähler manifold such that $c_1(M)_{\mathbb{R}} = 0$. Then there exists a finite étale covering \tilde{M} of M in the form*

$$\tilde{M} = T \times \prod_i C_i \times \prod_j X_j,$$

where T is a complex torus, C_i is a Calabi-Yau manifold for every i and X_j is a hyperkähler manifold for every j .

Proof. See [9, Theorem 2]. □

Some general results can be given for every hyperkähler manifold, especially on the cohomology level. Let X be a hyperkähler manifold. We already know that $h^{2,0}(X) = 1$, and moreover, for $\dim(X) = 2n$,

$$H^{k,0}(X) = \begin{cases} 1 & \text{if } k \text{ even, } k \leq 2n \\ 0 & \text{if } k \text{ odd,} \end{cases}$$

see [9, Section 4, Proposition 3, Item *ii*]. In particular the Picard group $\text{Pic}(X)$ and the Néron-Severi group $NS(X)$ are isomorphic: via this identification, we will sometimes see $\text{Pic}(X)$ as a subgroup of $H^2(X, \mathbb{Z})$.

The second cohomology group plays an important role in the study of hyperkähler manifolds. The related Hodge numbers for X are

$$1 \quad h^{1,1}(X) \quad 1$$

and, by the Universal Coefficient theorem for cohomology, the group $H^2(X, \mathbb{Z})$ has no torsion, hence it is free, since X is simply-connected. Actually, we can say more.

Theorem 2.3. *Let X be a hyperkähler manifold of dimension $2n$ and whose second Betti number is $b_2(X)$. There exists a unique non-degenerate symmetric integral and non-divisible bilinear form $(-, -)$ that endows $H^2(X, \mathbb{Z})$ with the structure of a lattice of signature $(3, b_2(X) - 3)$ with the following property: there exists a positive rational number c_X such that the equality*

$$\int_X \alpha^{2n} = c_X (\alpha, \alpha)^n$$

holds for every $\alpha \in H^2(X, \mathbb{Z})$.

Moreover, let ω be a symplectic 2-form on X , then

$$(\omega, \omega) = 0 \text{ and } (\omega, \bar{\omega}) > 0.$$

Proof. See [9, Section 8, Theorem 5, Item *a*) – *c*]. □

Remark 2.4. In the case of a hyperkähler manifold X whose Betti number is $b_2(X) = 6$, we have to require that the square of every Kähler class on X is strictly positive. If $b_2(X) \neq 6$ this condition is automatically satisfied: since we will never deal with hyperkähler manifolds whose second Betti number is 6, we does not specify the request on Kähler classes and we will always consider positivity on Kähler classes to be satisfied.

Definition 2.5. We call Beauville-Bogomolov-Fujiki form (BBF) the bilinear form in Theorem 2.3. When not otherwise specified, for a hyperkähler manifold X , a bilinear form $(-, -)$ on $H^2(X, \mathbb{Z})$ will always be intended as the Beauville-Bogomolov-Fujiki form; every time we will mention the structure of a lattice on $H^2(X, \mathbb{Z})$ we will refer to the one provided by in Theorem 2.3. We will call c_X the Fujiki constant for X .

Since the orthogonal complement of $H^{2,0}(X)$ is $H^{1,1}(X)$, the Néron-Severi group can be seen as $\text{NS}(X) = \omega^\perp \cap H^2(X, \mathbb{Z})$. For X projective, the Néron-Severi group is then a primitive sublattice of $H^2(X, \mathbb{Z})$ whose signature is $(1, b_2(X) - 3)$. We will refer as *Néron-Severi lattice* to the Néron-Severi group endowed with the restriction of the BBF form, as *Picard lattice* to the Picard group with the lattice structure induced by the natural automorphism with $\text{NS}(X)$. Given two divisors D_1, D_2 on X , we will write (D_1, D_2) for the product of the corresponding line bundles, $(D_1, D_2) = (\mathcal{O}_X(D_1), \mathcal{O}_X(D_2))$.

Remark 2.6. The BBF form and the Fujiki constant are deformation invariant, so for X, Y deformation equivalent hyperkähler manifolds we have an isomorphism of lattices $H^2(X, \mathbb{Z}) \cong H^2(Y, \mathbb{Z})$.

Definition 2.7. We say that a line bundle $L \in \text{Pic}(X)$ is primitive if it is primitive as an element of the lattice $\text{Pic}(X)$.

The Kleiman projectivity criterion for surfaces generalizes on hyperkähler manifolds, as Huybrechts showed in the following result.

Theorem 2.8. Let X be a hyperkähler manifold. Then X is projective if and only if there exists a line bundle $D \in \text{Pic}(X)$ such that $D^2 > 0$.

Proof. See [62, Theorem 3.11]. □

Definition 2.9. Let X be a projective hyperkähler manifold. A $\langle k \rangle$ -polarization on X is the choice of a primitive ample line bundle $L \in \text{Pic}(X)$ such that $L^2 = k$. By definition of the BBF form, the integer k is always strictly positive and even, since an ample class is also Kähler.

As a final note, the morphism induced by a birational map respects the BBF form.

Proposition 2.10. Let $f : X_1 \dashrightarrow X_2$ be a birational map between hyperkähler manifolds. Then f restricts to a biregular map $f|_{U_1} : U_1 \rightarrow U_2$, $U_i \subseteq X_i$ open of codimension at least two for $i = 1, 2$, and the map

$$f_* : H^2(X_1, \mathbb{Z}) \rightarrow H^2(X_2, \mathbb{Z})$$

induced through the inclusion of U_i in X_i , is a Hodge isometry. Moreover f_* does not depend on the choice of U_i .

Proof. See [91, Proposition 1.6.2]. □

This last result is very important, as the lattice structure on the second cohomology group is a fundamental tool for the study of morphisms between hyperkähler manifolds.

Definition 2.11. Given $f : X_1 \dashrightarrow X_2$ a birational map between hyperkähler manifolds, we denote $f^* = (f^{-1})_* : H^2(X_2, \mathbb{Z}) \rightarrow H^2(X_1, \mathbb{Z})$.

2.2.2 Some examples of hyperkähler manifolds

Every example of hyperkähler manifold gives rise to a whole family of hyperkähler manifolds, due to the following results.

Theorem 2.12. *Let X be a hyperkähler manifold. There exists a universal deformation $\mathcal{X} \rightarrow \text{Def}(X)$ with smooth $\text{Def}(X)$.*

Proof. See [62, Remark 1.12]. □

The family $\mathcal{X} \rightarrow \text{Def}(X)$ is based in X , in the sense that the fiber over 0 is isomorphic to X . The universal deformation is the germ of an analytic space, whose Zariski tangent space in $0 \in \text{Def}(X)$ is naturally isomorphic to $H^1(X, TX) \cong H^1(X, \Omega_X)$; since $\text{Def}(X)$ is smooth, this implies that $\text{Def}(X)$ has dimension $h^{1,1}(X)$.

Proposition 2.13. *Let $\mathcal{X} \rightarrow B$ be a smooth and proper family over an analytic base B and assume that the fiber X_0 over a point $0 \in B$ is a hyperkähler manifold. Then, for every $b \in B$ such that X_b is Kähler, X_b is a hyperkähler manifold too.*

Proof. See [9, Proposition 9 and next Remark]. □

Corollary 2.14. *Every fiber of the universal deformation $\mathcal{X} \rightarrow \text{Def}(X)$ is a hyperkähler manifold.*

We are now ready to present some examples of hyperkähler manifolds. We will say that two hyperkähler manifolds are *of the same deformation type* if they are deformation equivalent.

Recall that hyperkähler manifolds appear only in even dimension. The first example of hyperkähler manifold is classical.

Definition 2.15. *A K3 surface is a smooth compact surface which is simply-connected, and whose canonical bundle is trivial.*

In dimension two the definition of a hyperkähler manifold is equivalent to that of a K3 surface (every K3 surface is a Kähler manifold, see [110]). Many properties of hyperkähler manifolds are generalizations of properties of K3 surfaces, and many conjectures about hyperkähler manifolds are based on similar conjectures on K3 surfaces.

Remark 2.16. By Proposition 2.13, deforming a K3 surface results again in a K3 surface: a surface is Kähler if and only if its second Betti number is even and a deformation of the complex structure preserves the underlying topology, so a deformation is always Kähler. Conversely, every two K3 surfaces are deformation equivalent, see [64, Chapter 7, Theorem 1.1].

We compute now $h^{1,1}(S)$ for any K3 surface S ; the other Hodge numbers come directly from the structure of hyperkähler manifold. By triviality of the canonical bundle, the Noether's formula reads $e(S) = 12\chi(\mathcal{O}_S)$, where $e(S)$ is the topological Euler characteristic of S and $\chi(\mathcal{O}_S)$ is the holomorphic Euler characteristic, $\chi(\mathcal{O}_S) = h^{0,2} - h^{0,1} + 1 = 2$. Hence we have $e(S) = 24$ and second

Betti number $b_2(S) = e(S) - 2 = 22$. Since $H^0(S, \Omega_X^2)$ is generated by a symplectic form, we have moreover $h^{1,1}(S) = 20$. The lattice $H^2(S, \mathbb{Z})$ has rank 22 and in this case the BBF form is the cup product.

Theorem 2.17. *The second cohomology group with the BBF form is isomorphic, as an abstract lattice, to the unimodular lattice $U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$.*

Proof. See [64, Chapter 14 example 1.4]. □

Definition 2.18. *We call K3 lattice the abstract rank-22 unimodular lattice $\Lambda_{K3} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$.*

Examples of hyperkähler manifolds in higher dimension were given for the first time by Fujiki in dimension 4, and then generalized to every even dimension by Beauville in [9]; most of the results we present in the rest of this chapter come from this last paper. We present the family in which we are mostly interested.

Let S be a K3 surface; in our setting, we can reduce to the case of S projective.

Definition 2.19. *The Hilbert scheme of n points on S is the Douady space parametrizing zero-dimensional subschemes (Z, \mathcal{O}_Z) of length n (i.e. $\dim \mathcal{O}_Z = n$) on the surface S . We denote it by $S^{[n]}$. We will sometimes call the Hilbert square of S the Hilbert scheme $S^{[2]}$ of 2 points on S .*

Following [9, Section 6], we denote by

- $S^{(n)}$ the variety of 0-cycles of degree n , seen as the quotient of S^n by the symmetric group on n elements;
- $\epsilon : S^{[n]} \rightarrow S^{(n)}$ the natural map that associates to every finite scheme the associated 0-cycle (*Hilbert-Chow morphism*);
- $D \subset S^{(n)}$ the diagonal, that is the locus of cycles in the form $p_1 + \dots + p_n$ such that there exists $i \neq j$ with $p_i = p_j$.

By the Fogarty theorem, see for example [13, Theorem 3.1], the Hilbert scheme $S^{[n]}$ is smooth, connected and $2n$ -dimensional. Moreover, the singularity locus of $S^{(n)}$ is the diagonal D , the Hilbert-Chow morphism ϵ is a birational map that is a desingularization of $S^{(n)}$ and $E = \epsilon^{-1}(D)$ is an irreducible divisor on $S^{[n]}$. This also means that a general point on $S^{[n]}$ can be thought of as an n -uple of non-ordered distinct points $p_1 + \dots + p_n$ on S . Those results come from [40], [41] and [65] and were summarized by Beauville at Items a) – d) at the beginning of [9, Section 6].

Theorem 2.20. *For any projective K3 surface S and for any $n \geq 2$, the Hilbert scheme of n points on S is a hyperkähler manifold.*

Proof. See [9, Section 6, Theorem 3]. □

Fix again a projective $K3$ surface S and $n \geq 2$. There is a natural primitive embedding of lattices

$$i : H^2(S, \mathbb{Z}) \rightarrow H^2(S^{[n]}, \mathbb{Z})$$

such that $H^2(S^{[n]}, \mathbb{Z}) = i(H^2(S, \mathbb{Z})) \oplus \mathbb{Z}\delta$ as lattices, with $2\delta = E$, cfr. [9, Proposition 6 and next remark]. Moreover $\delta^2 = -2(n-1)$, so in particular $H^2(S^{[n]}, \mathbb{Z}) \cong U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1) \rangle$. As for the Picard group of the Hilbert scheme of n points $S^{[n]}$, the inclusion of lattices i induces an identification

$$\text{Pic}(S^{[n]}) = i(\text{Pic}(S)) \oplus \mathbb{Z}\delta.$$

In particular the Picard rank of some Hilbert scheme of n points on a $K3$ surface S is always at least two.

Remark 2.21. By Theorem 2.8, if S is projective then $S^{[n]}$ is projective too. However, for any line bundle $L \in \text{Pic}(S)$ the corresponding line bundle $i(L)$ on $S^{[n]}$ is never ample for $n \geq 2$, since it always has product zero with the class of the exceptional divisor E , that is $(i(L), E) = 0$.

Definition 2.22. For $n \geq 2$, we call $K3^{[n]}$ -lattice the abstract lattice $\Lambda_{K3^{[n]}} = U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1) \rangle$. Given a $K3$ surface S , for any $L \in \text{Pic}(S)$ we will denote by L_n the line bundle $i(L) \in \text{Pic}(S^{[n]})$.

Definition 2.23. Let X be a hyperkähler manifold of dimension $2n$. If X is deformation equivalent to $S^{[n]}$ for some (hence any) $K3$ surface S , we say that X is of $K3^{[n]}$ -type.

Remark 2.24. Since $h^{1,1}(S^{[n]}) = h^{1,1}(S) + 1$ for any $n \geq 2$, by Theorem 2.12 the space $\text{Def}(S)$ can be seen as a divisor in $\text{Def}(S^{[n]})$, representing hyperkähler manifolds of $K3^{[n]}$ -type that are Hilbert schemes of a small deformation of S . So the very general hyperkähler manifold of $K3^{[n]}$ -type is not isomorphic to the Hilbert scheme of n points on some $K3$ surface, see [9, Theorem 6]. A geometric description for some class of these manifolds will be given in Chapter 4.

Lemma 2.25. For any hyperkähler manifold X of $K3^{[n]}$ -type the Fujiki constant is $c_X = \frac{(2n)!}{n!2^n}$.

Proof. See [9, Section 9]. □

Let S be a projective $K3$ surface. The full cohomology group $H^*(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus H^2(S, \mathbb{Z}) \oplus H^4(S, \mathbb{Z})$ is a rank-24 free \mathbb{Z} -module. It admits a structure of lattice which is coherent with the BBF form on the second cohomology group: the bilinear form, called the *Mukai pairing*, is given by

$$\begin{aligned} H^*(S, \mathbb{Z}) \times H^*(S, \mathbb{Z}) &\rightarrow \mathbb{Z} \\ ((r, L, s), (r', L', s')) &\mapsto (L, L') - rs' - r's. \end{aligned}$$

The rank-2 free \mathbb{Z} -module generated by the 0-th and 4-th cohomology groups is isomorphic to the unimodular lattice U , so that $H^*(S, \mathbb{Z})$ is isomorphic to $U^{\oplus 4} \oplus E_8(-1)^{\oplus 2}$ as a lattice.

For $n \geq 2$, consider the vector $v \in H^*(S, \mathbb{Z})$, $v = (1, 0, 1-n)$: its orthogonal complement in $H^*(S, \mathbb{Z})$ is $H^2(S, \mathbb{Z}) \oplus \mathbb{Z}(1, 0, n-1) \cong H^2(S, \mathbb{Z}) \oplus \langle -2(n-1) \rangle$. Hence there is an isometry of lattices

$$\begin{aligned} \theta : v^\perp &\rightarrow H^2(S^{[n]}, \mathbb{Z}) \\ (0, L, 0) &\mapsto -L_n \\ (1, 0, n-1) &\mapsto -\delta. \end{aligned}$$

Definition 2.26. *The Mukai lattice $\tilde{\Lambda}$ is the unimodular lattice given by the \mathbb{Z} -module $H^*(S, \mathbb{Z})$ endowed with the Mukai pairing. For every $n \geq 2$, the isometry of lattices θ defined above is called Mukai's homomorphism.*

The algebraic Mukai lattice is the sublattice $H_{alg}^(S, \mathbb{Z}) = H^0(S, \mathbb{Z}) \oplus \text{NS}(S) \oplus H^4(S, \mathbb{Z}) \subset H^*(S, \mathbb{Z})$.*

Note that the lattice structure on $H^*(S, \mathbb{Z})$, unlike the definition of the algebraic Mukai lattice, does not depend on the choice of a particular $K3$ surface S , so $\tilde{\Lambda}$ is well-defined even when no $K3$ surface is explicitly considered. For every Hilbert scheme of points $S^{[n]}$, with S a projective $K3$ surface, we have then a natural primitive isometric embedding of $H^2(S^{[n]}, \mathbb{Z})$ in $\tilde{\Lambda}$, given by θ^{-1} . Actually Markman showed in [77] that a generalized version of this holds.

Proposition 2.27. *A hyperkähler manifold X of $K3^{[n]}$ -type, $n \geq 2$, comes with a natural choice of an $O(\tilde{\Lambda})$ -orbit $[\iota_X]$ of primitive isometric embeddings of $H^2(X, \mathbb{Z})$ in the Mukai lattice $\tilde{\Lambda}$.*

Proof. See [77, Corollary 9.5, Item 1)]. □

Markman found a characterization of the group of monodromy operators for manifolds of $K3^{[n]}$ -type using Proposition 2.27, see [77, Corollary 9.5, Item 1)]; following Apostolov, [2], we will utilize these embeddings to treat irreducible components of the moduli spaces of polarized manifolds of $K3^{[n]}$ -type.

Remark 2.28. In the particular case of the Hilbert scheme of n points on a $K3$ surface $S^{[n]}$, the inverse of Mukai's homomorphism $\theta^{-1} : H^2(S^{[n]}, \mathbb{Z}) \rightarrow v^\perp \subset \tilde{\Lambda}$ is a representative of $[\iota_{S^{[n]}}]$. This is [77, Example 9.6], once we observe that the Hilbert scheme of n points on a $K3$ surface S can be seen as the moduli space of rank-one torsion free sheaves on S whose first Chern class is trivial and second Chern class is n , for details see for example [109, Example above Theorem 3].

The examples we gave, $K3$ surfaces and hyperkähler manifolds of $K3^{[n]}$ -type for $n \geq 2$, provide a list of examples of hyperkähler manifolds for every even dimension. For completeness, we cite here the other known families of hyperkähler manifolds. Every example carries its own deformation family as above.

- *Generalized Kummer manifolds*, [9, Section 7] - They appear in every even dimension. Given a generalized Kummer manifold X , if $\dim X = 2$ then X is a Kummer surface, i.e. a $K3$ surface obtained by blowing up along the singularities the quotient of a torus by $\{\pm 1\}$. For $\dim X > 2$ the second Betti number of X is $b_2(X) = 7$, in particular X is **not** of $K3^{[n]}$ -type. We say that a hyperkähler manifold is of *Kummer-type* if it is deformation equivalent to some generalized Kummer manifold.

- *O'Grady's example in dimension 6*, [93] - This example appears only in dimension 6. A manifold X in this family has second Betti number $b_2(X) = 8$, in particular it is neither of $K3^{[3]}$ -type nor of Kummer-type.
- *O'Grady's example in dimension 10*, [92] - This example appears only in dimension 10. A manifold X in this family has second Betti number $b_2(X) = 24$, in particular it is neither of $K3^{[5]}$ -type nor of Kummer-type.

In this thesis every hyperkähler manifold is of $K3^{[n]}$ -type for some integer $n \geq 2$, or a $K3$ surface.

2.2.3 About $K3$ surfaces

Here we list some results specific to $K3$ surfaces, that we will utilise later.

When talking about curves we will refer to one-dimensional \mathbf{C} -schemes separated of finite-type which are reduced, hence allowing schemes with more than an irreducible component.

We fix a $K3$ surface S .

Definition 2.29. *A (-2) -class Γ on S is a line bundle $\Gamma \in \text{Pic}(S)$ with $\Gamma^2 = -2$. A (-2) -curve Γ on S is an irreducible curve with $\Gamma^2 = -2$.*

Note that every (-2) -curve is a (-2) -class, but the converse is not always true: consider for example $\Gamma_1 + \Gamma_2$, where Γ_1 and Γ_2 are two (-2) -curves which intersect in one point.

Recall that we say that a line bundle is *big and nef* if it is nef and it has positive square; in this case, by the Kodaira-Ramanujam theorem, see [64, Theorem 1.8], we have

$$H^i(X, L) = 0 \text{ for } i > 0.$$

For $L \in \text{Pic}(S)$ a line bundle on S , the Riemann-Roch theorem reads

$$\chi(S, L) = h^0(S, L) - h^1(S, L) + h^0(S, -L) = \frac{1}{2}L^2 + 2.$$

This has some immediate consequences.

- For $L \in \text{Pic}(S)$ big and nef with $L^2 = 2t$ we have $h^0(S, L) = t + 2$, so that the associated map is of the form $\phi_{|L|} : S \dashrightarrow \mathbb{P}^{t+1}$.
- For a curve C on S , we have $C^2 = 2p_a(C) - 2$, $p_a(C)$ being the arithmetic genus of C ; in particular $C^2 \geq -2$ and C is a (-2) -curve if and only if C is a smooth rational curve on S , arithmetic and geometric genus being equal if and only if C is smooth.
- Given a (-2) -class Γ on S , the Riemann-Roch theorem gives $h^0(S, \Gamma) + h^0(S, -\Gamma) \geq 1$, so one between $|\pm \Gamma|$ is non-empty.

We will see that the (-2) -curves have a fundamental importance in the study of linear systems on a $K3$ surface.

By definition, any nef line bundle on S has non-negative product with an effective (-2) -class on S . There is a converse of this statement.

Lemma 2.30. *Let S be a $K3$ surface and $L \in \text{Pic}(S)$ a line bundle with $L^2 \geq 0$. If $(L, \Gamma) \geq 0$ for all (-2) -curves Γ on S then L is nef, unless there exists no such Γ : in this case either L or $-L$ is nef.*

Proof. See [64, Corollary 1.5]. □

Theorem 2.31. *Let S be a $K3$ surface, and $L \in \text{Pic}(S)$ a big and nef line bundle on S . Then L is very ample if and only if $L^2 \geq 4$ and there is no effective line bundle $D \in \text{Pic}(S)$ such that one of the followings holds. Either*

- $D^2 = -2$ and $(L, D) = 0$, or
- $D^2 = 0$ and $(L, D) = 1, 2$, or
- $L^2 = 8$, $L = 2D$.

Proof. This is [73, Theorem 1.1] in the case $k = 1$, since 1-very ampleness and very ampleness are equivalent. The case $D^2 = -2$ with $(L, D) < 0$ is ruled out by nefness of D . The case $D^2 = 0$ with $(L, D) = 0$ cannot happen, since under these hypothesis D is trivial, as an application of the Hodge index theorem. □

For the result above we cite [73]; see also [108] and [107], which provide furthermore a study of the hyperelliptic case (there exists $D \in \text{Pic}(S)$ with $D^2 = 0$, $(L, D) = 2$) and the non-ample case (there exists $D \in \text{Pic}(S)$ with $D^2 = -2$, $(L, D) = 0$).

Given a very ample line bundle L on a $K3$ surface S with $L^2 = 2t$, the embedding $\phi_{|L|} : S \rightarrow \mathbb{P}^{t+1}$ sends S to a smooth surface of degree $2t$. The simplest possible case is for $t = 2$: in this case $\phi_{|L|}(S)$ is a quartic hypersurface in \mathbb{P}^3 . Conversely, every smooth quartic hypersurface in \mathbb{P}^3 is a $\langle 4 \rangle$ -polarized $K3$ surface, see [7, Example VIII.8]. This is the first case of $K3$ surface ever studied. Again, as we can see in [7, Example VIII.8], also for $t = 3, 4$ the embedded $K3$ surface is a complete intersection, respectively of a quadric and a cubic and of three quadrics; these are the only $K3$ surfaces arising as complete intersections, in [7, Proposition 7.5] the author proves that for any other case a complete intersection is either rational or of general-type.

For S a $K3$ surface, and L a $\langle 2 \rangle$ -polarization on it, L cannot be very ample by Theorem 2.31. However, the map $\phi_{|L|}$ is well understood in this case: for any $\langle 2 \rangle$ -polarized $K3$ surface (S, L) , the map $\phi_{|L|} : S \rightarrow \mathbb{P}^2$ is a double covering of the plane whose ramification locus is a smooth sextic curve in \mathbb{P}^2 , cfr. [7, Proposition VIII.13].

Remark 2.32. It is natural to ask if something similar happens for $\langle 2 \rangle$ -polarized hyperkähler manifolds of $K3^{[n]}$ -type, with $n \geq 2$. This is the reasoning that leads to the description of *EPW* sextics, that are one of the main objects of this thesis, see Chapter 4: as we will see, for (X, D)

general with X of $K3^{[n]}$ -type and $D^2 = 2$, the map $\phi_{|D|} : X \rightarrow \mathbb{P}^5$ factorizes as $X \xrightarrow{2:1} Y \hookrightarrow \mathbb{P}^5$, where Y is a special type of a sextic hypersurface and the branch locus of the first morphism is a smooth surface. An important difference is that in this case the construction holds true only for (X, D) general in the moduli space of such pairs, that we will present in Section 2.3.2.

We will be particularly interested in the geometric description of $K3$ surfaces which admit a polarization of degree 10. As we said, these are not complete intersections inside \mathbb{P}^6 ; however, for the general case they are *dimensionally transverse intersections*, see Definition 1.1. In particular they can be thought of as complete intersections inside a Grassmannian instead of a projective space.

We will need the following definition.

Definition 2.33. *A polarized $K3$ surface (S, L) is said to be Brill—Noether general if*

$$h^0(S, D) \cdot h^0(S, L \otimes D^{-1}) < h^0(S, L)$$

for all line bundles D on S which are neither trivial nor isomorphic to L .

Brill-Noether-general $K3$ surfaces are an interesting matter of study in their own, however here we will not study them in detail.

The importance for us of Brill-Noether general $K3$ surfaces stems from the following result, due to Mukai.

Theorem 2.34. *Let (S, L) be a $\langle 10 \rangle$ -polarized $K3$ surface. Then (S, L) is Brill-Noether general if and only if $\phi_{|L|} : S \rightarrow \mathbb{P}^6$ is a closed embedding and $\phi_{|L|}(S)$ is a smooth dimensionally transverse intersection of a linear section of the Grassmannian $G(2, \mathbb{C}^5) \cap \mathbb{P}^6 \subset \mathbb{P}^9$ and a quadric hypersurface in \mathbb{P}^6 .*

Conversely, every smooth transverse intersection obtained in this way is an embedded $K3$ surface on which the restriction of the class of $\mathcal{O}_{\mathbb{P}^6}(1)$ has degree 10.

Proof. See [72, Theorem 10.3]. This is in fact a result by Mukai, which can be found in [84]. \square

The theorem above implies that (S, L) can be seen, in this case, as a 2-dimensional smooth ordinary Gushel-Mukai variety, that we will describe in detail in Section 4.5.

The definition of Brill-Noether general $K3$ surfaces suggests that polarized surfaces that satisfy this property fill an open dense subset inside the moduli space of $\langle 10 \rangle$ -polarized $K3$ surfaces; Greer, Li and Tian gave in [47, Lemma 1.7] some necessary and sufficient conditions on the Picard lattice of a $K3$ surface to be Brill-Noether general, hence an open condition on elements of \mathcal{K}_{10} . We present this result in Proposition 2.83. So we can consider the general $\langle 10 \rangle$ -polarized $K3$ surface as a Brill-Noether general $K3$ surface too.

2.3 Moduli spaces, period maps and Torelli theorems

2.3.1 Marked hyperkähler manifolds and monodromy operators

One important property of $K3$ surfaces is that a Torelli theorem holds for them: we enunciate it in Theorem 2.56. This is the first theorem of this kind arising for hyperkähler manifolds, and a

strong motivation to trace similar results in higher dimension. We will see that such generalizations exist, although their formulation is far from trivial: a naïve formulation for Torelli theorem in higher dimension is proved to be false by Namikawa, in [87].

We fix an abstract lattice Λ , and we will consider a hyperkähler manifold X whose cohomology group $H^2(X, \mathbb{Z})$ is isomorphic to Λ as a lattice. For our purpose we will be interested in the case of Λ as the unimodular lattice Λ_{K3} , for the case of $K3$ surfaces, and as $\Lambda_{K3^{[n]}}$ for manifolds of $K3^{[n]}$ -type with $n \geq 2$.

Definition 2.35. *A marking on X is an isometry $\eta : H^2(X, \mathbb{Z}) \rightarrow \Lambda$. A marked hyperkähler manifold is a pair (X, η) , where X is a hyperkähler manifold together with a marking $\eta : H^2(X, \mathbb{Z}) \rightarrow \Lambda$ on X . Two marked hyperkähler manifolds (X_1, η_1) and (X_2, η_2) are isomorphic if there exists an isomorphism $f : X_1 \rightarrow X_2$ such that $\eta_2 = \eta_1 \circ f^*$.*

Let (X, η) be a marked hyperkähler manifold, and consider again the universal deformation $\pi : \mathcal{X} \rightarrow \text{Def}(X)$. We can extend the marking η to the family \mathcal{X} , in the sense that there exists a family of markings $(F_b : \mathcal{X}_b \rightarrow \Lambda)_{b \in B}$ such that $F_0 = \eta$, see [74, Theorem 2.4]. Then we can define a local period map

$$\begin{aligned} \mathcal{P} : \text{Def}(X) &\rightarrow \mathbb{P}(\Lambda) \\ b &\mapsto [F_b(H^{2,0}(\mathcal{X}_b))]. \end{aligned}$$

Clearly $\mathcal{P}(0) = [\eta(H^{2,0}(X))]$.

Definition 2.36. *We call period domain the analytic subvariety*

$$\Omega = \{[x] \in \mathbb{P}(\Lambda \otimes \mathbb{C}) \mid x^2 = 0, (x, \bar{x}) > 0\}.$$

It is an open subvariety on a quadric hypersurface in $\mathbb{P}(\Lambda)$, which is isomorphic to \mathbb{P}^{21} in the case of $K3$ surface, to \mathbb{P}^{22} for manifolds of $K3^{[n]}$ -type with $n \geq 2$.

Remark 2.37. The image of \mathcal{P} is contained in Ω by the properties of BBF form, see Theorem 2.3. We call *period point* of (X, η) the image $\mathcal{P}(0)$.

Theorem 2.38 (Local Torelli theorem). *Let (X, η) be a marked hyperkähler manifold; the local period map $\mathcal{P} : \text{Def}(X) \rightarrow \Omega$ is a local isomorphism.*

Proof. See [9, Section 8 Theorem 5, Item b)]. □

Now we move to the global case, for which we will need the definition of parallel transport operator. These isometries play a key role in the theory of hyperkähler manifolds.

Definition 2.39. *Let X_1, X_2 be hyperkähler manifolds. An isomorphism $f : H^2(X_1, \mathbb{Z}) \rightarrow H^2(X_2, \mathbb{Z})$ is said to be a parallel transport operator if there exist*

- a smooth and proper family $\pi : \mathcal{X} \rightarrow B$ of hyperkähler manifolds over an analytic base B ,

- $b_i \in B$, for $i = 1, 2$, such that there exists $\psi_i : X_i \xrightarrow{\sim} \mathcal{X}_{b_i}$ isomorphisms.
- a continuous path $\gamma : [0, 1] \rightarrow B$ with $\gamma(0) = b_1$, $\gamma(1) = b_2$

such that f is induced, through the isomorphisms ψ_i , by the parallel transport in the local system $R^2\pi_*\mathbb{Z}$ along γ .

Given a hyperkähler manifold X , a monodromy operator is a parallel transport operator $f : H^2(X, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$.

Every parallel transport operator is a lattice isometry. Moreover, it is easy to see that, given two families of deformation, they can be glued to obtain a third family of deformation, hence the composition of two parallel transport operators is a parallel transport operator; in particular monodromy operators form a subgroup inside $O(H^2(X, \mathbb{Z}))$, which we call *monodromy group* and we denote by $\text{Mon}^2(X)$.

Definition 2.40. A Hodge monodromy operator is a monodromy operator in $O(H^2(X, \mathbb{Z}))$ which is a Hodge isometry; we denote by $\text{Mon}_{\text{Hdg}}^2(X)$ the subgroup of such operators.

Definition 2.41. Let Λ be the abstract lattice isomorphic to the second cohomology group of some hyperkähler manifold of $K3^{[n]}$ -type for $n \geq 1$. We denote by $\text{Mon}(\Lambda)$ the subgroup of $O(\Lambda)$ given by

$$\eta^{-1} \circ \text{Mon}^2(X) \circ \eta$$

where (X, η) is a marked hyperkähler manifold of $K3^{[n]}$ -type.

A different choice of (X, η) could lead to define $\text{Mon}^2(X)$ as a different subgroup of $O(\Lambda_{K3^{[n]}})$, conjugate to the first one. However the group is well defined whenever the group of monodromy operators is normal inside $O(H^2(X, \mathbb{Z}))$, which is the case for the $K3^{[n]}$ -type, as Markman verified, see Corollary 2.52.

If the group of monodromy operators is not normal for some deformation type of hyperkähler manifolds, then the definition of $\text{Mon}(\Lambda)$ depends on the connected component of (X, η) inside the moduli space of marked hyperkähler manifolds of this deformation type, which we define now.

The coarse moduli space of marked hyperkähler manifolds, of the same deformation type as X , can be defined as the set of marked manifolds (Y, η) , where Y is deformation equivalent to X , modulo isomorphism of marked manifold. We call it \mathcal{M}_Λ . By the local Torelli theorem it can be endowed with a structure of complex manifold using universal deformations to produce local charts.

Theorem 2.42. The moduli space \mathcal{M}_Λ of marked hyperkähler manifolds which are deformation equivalent to X is a smooth non-Hausdorff complex manifold.

Proof. See [63, Section 3.1]. □

Looking at $\mathcal{X} \rightarrow \text{Def}(X)$, we check that \mathcal{M}_Λ has dimension 20 for $K3$ surfaces, 21 for the $K3^{[n]}$ -type if $n \geq 2$.

Remark 2.43. Every two marked hyperkähler manifolds (X_1, η_1) and (X_2, η_2) are deformation equivalent if and only if they are in the same connected component of \mathcal{M}_Λ . In particular, they are in the same connected component if and only if $\eta_2^{-1} \circ \eta_1$ is a parallel transport operator, see [77, Lemma 7.5]. This also implies that the number of connected components of \mathcal{M}_Λ is equal to $[O(H^2(X, \mathbb{Z})) : \text{Mon}^2(X)]$.

Patching together local period maps we obtain a *global period map*

$$\mathcal{P} : \mathcal{M}_\Lambda \rightarrow \Omega.$$

Theorem 2.44 (Surjectivity of the period map). *Let \mathcal{M}_Λ^0 be a connected component of \mathcal{M}_Λ . Then the restriction to \mathcal{M}_Λ^0 of the period map is surjective.*

Proof. See [62, Theorem 8.1]. □

Surjectivity of the period map admits this useful corollary. We will treat separately the case of $K3$ surfaces, $n = 1$, in Corollary 2.45.

Corollary 2.45. *Let $R \subset \Lambda_{K3^{[n]}}$ for $n \geq 2$ be an even hyperbolic lattice such that $\text{rk}(R) \leq 20$. Then there exists a marked hyperkähler manifold (X, η) of $K3^{[n]}$ -type whose Néron-Severi lattice is isomorphic to R and such that $\eta(\text{NS}(X)) = R$.*

Proof. See [14, Proposition 5.3] for the case $n = 2$, the general case for $n \geq 2$ then works in the same way without changes. □

Note that this is a result of existence, and a priori it does not give any hint on the geometric structure of the produced manifold: the statement follows from the fact that the period point allows to individuate the Néron-Severi lattice of an associated $K3$ surface by orthogonality.

The global Torelli theorem generalizes for every deformation type of hyperkähler manifolds in the following way. The two parts of the statement were proved respectively by Huybrechts and Verbitski.

Theorem 2.46 (Global Torelli theorem for marked varieties). *For every two inseparable points (X_1, η_1) and (X_2, η_2) in \mathcal{M}_Λ , the hyperkähler manifolds X_1 and X_2 are bimeromorphic.*

Let \mathcal{M}_Λ^0 be a connected component of \mathcal{M}_Λ , and call \mathcal{P}_0 the restriction to \mathcal{M}_Λ^0 of the period map: for every $p \in \Omega_\Lambda$, the fiber $\mathcal{P}_0^{-1}(p)$ consists of pairwise inseparable points.

Proof. See [62, Theorem 4.3], for the first part, and [111, Theorem 1.16], for the second part. □

A Hodge-theoretic version of the Global Torelli theorem has been stated by Markman in [77].

Theorem 2.47 (Hodge-theoretic Torelli theorem). *Let X and Y be two hyperkähler manifolds which are deformation equivalent. Then X and Y are bimeromorphic if and only if there exists a parallel transport operator $f : H^2(X, \mathbb{Z}) \rightarrow H^2(Y, \mathbb{Z})$ which is a Hodge isometry.*

Moreover, there exists a biregular morphism $\tilde{f} : Y \rightarrow X$ such that $f = \tilde{f}^$ if and only if f maps a Kähler class on X to a Kähler class on Y .*

Proof. See [77, Theorem 1.3]. □

This last result implies that automorphisms on a hyperkähler manifold can be studied using lattice theory. For this, we will need a characterization of the group of monodromy operators inside the group of isometries of the second cohomology group of the manifold. The *positive cone* \mathcal{C}_X of X is the connected component of

$$\{x \in H^{1,1}(X, \mathbb{R}) \mid x^2 > 0\}$$

containing a Kähler class. A more complete description of its structure will be presented in Section 2.4.1.

Let X be a hyperkähler manifold with a symplectic form ω . Consider some $h \in \mathcal{C}_X$: the three-dimensional real vector space $V_X^h = \mathbb{R}\Im(\omega) \oplus \mathbb{R}\Re(\omega) \oplus \mathbb{R}h$ admits an oriented basis $\{\Im(\omega), \Re(\omega), h\}$.

Definition 2.48. *An isometry $f : H^2(X_1, \mathbb{Z}) \rightarrow H^2(X_2, \mathbb{Z})$ is orientation-preserving if for some $h \in \mathcal{C}_{X_1}$ the induced morphism $V_{X_1}^h \rightarrow V_{X_2}^{f(h)}$ is orientation preserving. We call $O^+(H^2(X, \mathbb{Z})) < O(H^2(X, \mathbb{Z}))$ the index-two subgroup of isometries of $H^2(X, \mathbb{Z})$ that preserves orientation.*

Remark 2.49. The definition of orientation-preserving isometries does not depend on the choice of the vector h belonging to the positive cone in the definition. Any isometry that preserves Hodge decomposition is orientation preserving if and only if its complex extension sends the positive cone of X_1 to the positive cone of X_2 , since the symplectic form on X_1 is always sent to some complex multiple of the symplectic form on X_2 .

In particular, an isometry of $H^2(X, \mathbb{Z})$ is orientation-preserving if it fixes the positive cone \mathcal{C}_X of X .

Every parallel transport operator is orientation-preserving. From now on we concentrate on the $K3^{[n]}$ -type case and on monodromy operators: in this case an important result by Markman, [77, Theorem 9.1], gives a class of generators of the group of monodromy operators. Recall that we define in Definition 1.22 the reflection associated to a vector $u \in \Lambda$; note that if $u^2 = \pm 2$ the reflection always results in an isometry of Λ . We denote by ρ_u the isometry $\pm R_u$, with $u \in \Lambda$ such that $u^2 = \mp 2$.

Theorem 2.50. *The monodromy group $\text{Mon}^2(X)$ is generated by the elements in the form $\pm R_u$, with $u \in H^2(X, \mathbb{Z})$ such that $u^2 = \mp 2$.*

Proof. See [77, Theorem 9.1]. □

Looking at the action of such generators on the discriminant group of $H^2(X, \mathbb{Z})$, Markman obtains the following result. We denote by $A_{H^2(X, \mathbb{Z})}$ the discriminant group of the lattice $H^2(X, \mathbb{Z})$, which we defined in Definition 1.12.

Proposition 2.51. *Let X be a hyperkähler manifold of $K3^{[n]}$ -type. An orientation-preserving isometry f on $H^2(X, \mathbb{Z})$ is a monodromy operator if and only if $\bar{f} = \pm \text{id} \in O(A_{H^2(X, \mathbb{Z})})$.*

Proof. See [77, Lemma 9.2]. □

Corollary 2.52. *The subgroup $\text{Mon}^2(X) < O(H^2(X, \mathbb{Z}))$ is normal.*

Remark 2.53. The number of distinct primes dividing $n - 1$ will be denoted by $\rho(n - 1)$. As $A_{H^2(X, \mathbb{Z})}$ is cyclic and isomorphic to $\mathbb{Z}/(2(n - 1))\mathbb{Z}$, the Proposition 2.51 implies that $\text{Mon}^2(X)$ has index $2^{\rho(n-1)-1}$ in $O^+(H^2(X, \mathbb{Z}))$: in particular, the moduli space \mathcal{M}_Λ admits a finite number of connected component. Since $[O(H^2(X, \mathbb{Z})) : O^+(H^2(X, \mathbb{Z}))] = 2$, there are always at least two connected components.

For any even lattice L with $\text{rk}(L) = k$, we give a criterion for an isometry $f \in O(L)$ to be orientation-preserving. This will be useful in view of the characterization of monodromy operators given in Proposition 2.51.

By the Cartan-Dieudonné theorem, any isometry of $L_{\mathbb{R}}$ decomposes as the composition of at most k reflections R_{u_i} , with $u_i \in L_{\mathbb{R}}$. The *real spinor norm* of L is the group homomorphism $\text{sn}_{\mathbb{R}}^L : O(L) \rightarrow \mathbb{R}^* / (\mathbb{R}^*)^2 \cong \{\pm 1\}$ defined on $f \in O(L)$ as

$$\text{sn}_{\mathbb{R}}^L(f) = \left(-\frac{(u_1, u_1)}{2} \right) \cdots \left(-\frac{(u_r, u_r)}{2} \right) \pmod{(\mathbb{R}^*)^2}$$

where $f_{\mathbb{R}} = R_{u_1} \circ \dots \circ R_{u_r}$ is the factorization of $f_{\mathbb{R}} \in O(L_{\mathbb{R}})$ with respect to reflections defined by elements $u_i \in L$. In particular, if the signature of L is (l_+, l_-) , after diagonalizing the bilinear form of L over \mathbb{R} we check that $\text{sn}_{\mathbb{R}}^L(-\text{id}) = (-1)^{l_+}$.

Remark 2.54. An isometry $f \in O(L)$ is orientation-preserving if and only if $\text{sn}_{\mathbb{R}}^L(f) = 1$.

Remark 2.55. For $K3$ surfaces, or if $n - 1$ is the power of a prime number, then $[O^+(H^2(X, \mathbb{Z})) : \text{Mon}^2(X)] = 1$, so \mathcal{M}_Λ has two connected components and every orientation preserving isometry is a monodromy operator: in this setting Torelli theorem is much simpler to apply.

We specialize some results for the case of $K3$ surfaces. Any birational map between $K3$ surfaces is biregular (we present a proof which relies on chamber decomposition of the positive cone in Lemma 2.104): by this, and by Remark 2.55, we can rewrite Hodge-theoretic Torelli theorem as follows.

Theorem 2.56 (Global Torelli theorem for $K3$ surfaces). *Two $K3$ surfaces S and Σ are isomorphic if and only if there exists a Hodge isometry $H^2(S, \mathbb{Z}) \rightarrow H^2(\Sigma, \mathbb{Z})$.*

Moreover, for any Hodge isometry $f : H^2(S, \mathbb{Z}) \rightarrow H^2(\Sigma, \mathbb{Z})$, there exists an isomorphism $\tilde{f} : \Sigma \rightarrow S$ such that $\tilde{f}^ = f$ if and only if f sends a Kähler class on S to a Kähler class on Σ . If such \tilde{f} exists it is unique.*

Proof. For the original proof see [19]. Without the unicity, this is Theorem 2.47 with Remark 2.55, once we observe that if the Hodge isometry $f : H^2(S, \mathbb{Z}) \rightarrow H^2(\Sigma, \mathbb{Z})$ in the first part of the statement does not preserve orientation, then $-f$ does. □

A result similar to Corollary 2.45 holds in the case of $K3$ surfaces, once we request the rank of the sublattice $R \subset \Lambda_{K3}$ to be at most 19 instead of 20, since $h^{1,1}(S, \mathbb{R})$ has rank 20 instead of 21 in this case. Actually, we present a result which is a direct consequence of the analogue of Corollary 2.45 for $K3$ surfaces and Theorem 1.32.

Proposition 2.57. *For every even hyperbolic lattice R with rank less than 10, there exists a projective $K3$ surface whose Picard lattice is isomorphic to R .*

Proof. See [64, Chapter 14, Corollary 3.1]. □

Note that in this case we know that the manifold we obtain is a $K3$ surface, since any element in the moduli space of deformations of $K3$ surfaces is again a $K3$ surface.

Riemann-Roch theorem, together with Corollary 2.45, implies that for every $t \geq 2$ there exists a $K3$ surface $S \subset \mathbb{P}^{t+1}$ of degree $2t$: consider a $K3$ surface whose Picard lattice is isomorphic to $\langle 2t \rangle$. Ampleness is granted by Theorem 2.31.

2.3.2 Polarized hyperkähler manifolds

A moduli space for polarized hyperkähler manifolds can be constructed too, although some attention is needed. Given a polarized hyperkähler manifold (X, D) , we will denote by $\text{Aut}_D(X) \leq \text{Aut}(X)$ the subgroup of automorphisms of X fixing the class of the polarization D .

Definition 2.58. *Two hyperkähler manifolds X_1, X_2 of dimension $2n$ are numerically equivalent if there exists an isomorphism $f : H^2(X_1, \mathbb{Z}) \xrightarrow{\sim} H^2(X_2, \mathbb{Z})$ such that $\int_{X_1} \alpha^{2n} = \int_{X_2} f(\alpha)^{2n}$ for every $\alpha \in H^2(X_1, \mathbb{Z})$. Given a hyperkähler manifold X , we call numerical type of X the pair $(H^2(X, \mathbb{Z}), c_X)$, where the second cohomology group is intended as a lattice and c_X is the Fujiki constant of X .*

We will start by dealing with moduli space of numerically equivalent hyperkähler manifolds. As usual, we consider the cohomology groups with the BBF form. Clearly hyperkähler manifolds of equal dimension and same numerical type are numerically equivalent, while the converse is a priori not necessarily true. However, O’Grady proved the following.

Lemma 2.59. *Consider two hyperkähler manifolds X_1, X_2 of dimension $2n$, with n odd or $b_2(X_i) \neq 6$ for $i = 1, 2$. Then X_1 and X_2 are numerically equivalent if and only if they have same numerical type, up to isomorphism of lattices.*

Proof. See [97, Section 2.1]. □

Remark 2.60. In the case of a hyperkähler manifold of dimension $2n$ whose Betti number is $b_2(X) = 6$ and n is even, Lemma 2.59 is weaker and states that two numerically equivalent hyperkähler manifolds have the same numerical type *up to a sign*. However, as we already said in Remark 2.4, this case will never occur in this thesis.

Fix two integers $n, d \geq 1$ and a numerical type $\mathbf{N} = (\Lambda, c)$. For every element $h \in \Lambda$ such that $h^2 = 2d$, we denote by Λ_h the orthogonal to h in Λ , and by Ω_h the domain

$$\Omega_h = \{[x] \in \mathbb{P}(\Lambda_h \otimes \mathbb{C}) \mid x^2 = 0, (x, \bar{x}) > 0\};$$

the group of isometries $O(\Lambda, h) = \{\theta \in O(\Lambda) \mid \theta(h) = h\}$ can be seen, then, as a subgroup of $O(\Lambda_h)$.

Theorem 2.61. *There exists a smooth quasi-projective coarse moduli space $\mathcal{M}_{\mathbf{N}, 2d}^n$ parametrising $\langle 2d \rangle$ -polarised manifolds of dimension $2n$ and numerical type \mathbf{N} . The choice of an irreducible component \mathcal{M} of $\mathcal{M}_{\mathbf{N}, 2d}^n$ determines*

- a primitive vector $h \in \Lambda$ with $h^2 = 2d$ (up to isometry of Λ),
- a connected component Υ of $\Omega_h/O(\Lambda, h)$

such that there exists a period map

$$\mathcal{M} \rightarrow \Upsilon$$

which is a dominant and finite morphism of quasi-projective varieties.

Proof. See [50, Theorem 1.5]. □

As a first consequence, for $K3$ surfaces the moduli space has dimension 19 for $K3$ surfaces, and 20 for manifolds of $K3^{[n]}$ -type with $n \geq 2$.

Definition 2.62. *Take two elements $(X_1, D_1), (X_2, D_2)$ in $\mathcal{M}_{\mathbf{N}, 2d}^n$ which have the same deformation type, and whose second cohomology group is isomorphic to the abstract lattice Λ . For $i = 1, 2$, call h_i the primitive vector of Λ associated to the irreducible component of $\mathcal{M}_{\mathbf{N}, 2d}^n$ in which (X_i, D_i) lies. We say that $(X_1, D_1), (X_2, D_2)$ have the same polarization type if h_1 and h_2 are in the same $O(\Lambda)$ -orbit. We call polarization type any choice $[h]$ of an $O(\Lambda)$ -orbit of vectors in Λ .*

The moduli space of hyperkähler manifolds of $K3^{[n]}$ -type with a given polarization type $[h]$, that we denote by $\mathcal{M}_{[h]}$, is the union of irreducible components of $\mathcal{M}_{\mathbf{N}, 2d}^n$ which contain manifolds of $K3^{[n]}$ -type, and whose period map lands in $\Omega_h/O(\Lambda, h)$. In this setting we have then a period map $\mathcal{M}_{[h]} \rightarrow \Omega_h/O(\Lambda, h)$.

Now we fix a deformation type, and thus a numerical type; in particular we work with manifolds of $K3^{[n]}$ -type, with $n \geq 1$.

For a fixed polarization type, the period maps from all components point to the same $\Omega_h/O(\Lambda, h)$. In general the choice of d , does not induce a polarization type. We have, however, some sufficient conditions.

Lemma 2.63. *Consider n, d, γ positive integers with $n \geq 2$ and $\gamma \mid \gcd(d, 2(n-1))$ such that $\gcd(\frac{2(n-1)}{\gamma}, \frac{2d}{\gamma}, \gamma) = 1$. Then there exists at most one $O(\Lambda_{K3^{[n]}})$ -orbit of vectors $h \in \Lambda_{K3^{[n]}}$.*

Proof. See [50, Corollary 3.7]. □

Lemma 2.64. *Consider n, d positive integers with $n \geq 2$. Then there exists a vector $h \in \Lambda_{K3^{[n]}}$, with $h^2 = 2d$ and with divisibility 2, if and only if $n+d \equiv 1 \pmod{4}$; in this case the $O(\Lambda_{K3^{[n]}})$ -orbit of vectors is unique.*

Proof. See [50, Example 3.10]. □

The two lemmas above provide a list of situations in which the polarization type is unique, for example by Lemma 2.63 this holds when h has divisibility 1. Conversely, for every $d \geq 1$ there always exists some $h \in \Lambda$ such that $h^2 = 2d$ with divisibility 1, since Λ always contains a copy of U .

Remark 2.65. If $n = 2$, so that we are dealing with moduli spaces of manifolds of $K3^{[2]}$ -type, the divisibility of a vector in $\Lambda_{K3^{[2]}}$ is either 1 or 2, hence the polarization type is unique for any γ, d in this case.

Even for fixed polarization type, the moduli space is not necessarily irreducible: we will present some results in this sens, due to Apostolov, at the end of this section.

A polarized version of the Torelli theorem exists for manifolds of $K3^{[n]}$ -type for $n \geq 1$. For this we need to quotient the period domain with respect to a smaller group.

Definition 2.66. *A polarized parallel transport operator between (X_1, D_1) and (X_2, D_2) is a parallel transport operator $f : H^2(X_1, \mathbb{Z}) \rightarrow H^2(X_2, \mathbb{Z})$ such that $f(D_1) = D_2$.*

Remark 2.67. The equivalence between this definition and the definition 1.1 in [77] is guaranteed by [77, Corollary 7.4]. Two elements in $\mathcal{M}_{[h]}$ are in the same irreducible component if and only if there is a polarized parallel transport operator between them, see [2, Proposition 1.6 and Theorem 2.1].

The definition of polarized parallel transport operators, along with the characterization in Proposition 2.51, motivates the definition of

$$\hat{O}(\Lambda, h) = \{\theta \in O(\Lambda) \mid \theta(h) = h, \bar{\theta} = \pm \text{id} \in O(A_\Lambda)\}.$$

The quotient $\mathcal{P}_{[h]} = \Omega_h / \hat{O}(\Lambda, h)$ is irreducible and quasi-projective. We are now ready to enunciate the following.

Theorem 2.68 (Global Torelli theorem for polarized hyperkähler manifolds of $K3^{[n]}$ -type). *Fix a polarization type $[h]$ for manifolds of $K3^{[n]}$ -type. For any irreducible component \mathcal{M} of the moduli space $\mathcal{M}_{[h]}$ of polarized manifolds with polarization type $[h]$, the restriction to \mathcal{M} of the period map*

$$\mathcal{M} \rightarrow \mathcal{P}_{[h]}$$

is an algebraic map which is an open embedding.

Proof. See [77, Theorem 1.10]. □

Definition 2.69. We denote by $\mathcal{M}_{2d,\gamma}^n$ the moduli space of classes of $\langle 2d \rangle$ -polarized manifolds of $K3^{[n]}$ -type (X, D) such that the divisibility of D is γ .

The moduli space $\mathcal{M}_{2d,\gamma}^n$ is the union of the irreducible components of $\mathcal{M}_{\mathbf{N},2d}^n$ which contain manifolds of $K3^{[n]}$ -type, and such that the corresponding period map sends the component to some $\Omega_h/O(\Lambda, h)$ with $\text{div}_\Lambda(h) = d$. If (X_1, D_1) lies in $\mathcal{M}_{2d,\gamma}^n$ and (X_2, D_2) has the same polarization type as (X_1, D_1) , then it lies in $\mathcal{M}_{2d,\gamma}^n$ too; the converse is not necessarily true.

Definition 2.70. Every moduli space of polarized manifolds with a given polarization type $\mathcal{M}_{[h]}$ in $\mathcal{M}_{2d,\gamma}^n$ has its own period map to $\mathcal{P}_{[h]}$; if n, d, γ determine a unique polarization type, we call $\mathcal{P}_{2d,\gamma}^n$ the associated period domain.

Now we consider a polarization type $[h]$ with prescribed degree and positivity; we have $\mathcal{M}_{[h]} \subseteq \mathcal{M}_{2d,\gamma}^n$ the associated moduli space and $P : \mathcal{M}_{[h]} \rightarrow \mathcal{P}_{[h]}$ its period map. Take a primitive hyperbolic rank-two sublattice $K \subset \Lambda$ containing h , so that $\mathcal{K} = \mathbb{P}(K^\perp \otimes \mathbb{C}) \cap \Omega_h$ is an hypersurface of Ω_h . We call a *Heegner divisor* the image \mathcal{D}_K of \mathcal{K} in $\mathcal{P}_{[h]}$. Points on \mathcal{D}_K correspond, via the period map, to isomorphism classes of $\langle 2d \rangle$ -polarized hyperkählers of $K3^{[n]}$ -type whose Picard lattice contains a primitive copy of K as a sublattice. We set

$$\mathcal{D}_e = \bigcup_{\text{disc}(K^\perp) = -e} \mathcal{D}_K,$$

and

$$\mathcal{D} = \bigcup_{e \in \mathbb{N}} \mathcal{D}_e.$$

We call *Noether-Lefschetz locus* the set of points, in a moduli space, that corresponds to varieties whose Picard rank is bigger than one. We write $\mathcal{C}_{2d,\gamma}^n$ for the Noether-Lefschetz locus of $\mathcal{M}_{2d,\gamma}^n$. The restriction of the Noether-Lefschetz locus to $\mathcal{M}_{[h]}$ can be seen as the inverse image, through the period map P , of \mathcal{D} : if we set $\mathcal{C}_e = P^{-1}(\mathcal{D}_e)$, we have $\mathcal{C}_{2d,\gamma}^n \cap \mathcal{M}_{[h]} = \bigcup_{e \in \mathbb{N}} \mathcal{C}_e$.

Remark 2.71. The Noether-Lefschetz locus is the countable union, on every component of $\mathcal{M}_{2d,\gamma}^n$ of the divisors described above; each of them is called a *Noether-Lefschetz divisor*. The very general element in $\mathcal{M}_{2d,\gamma}^n$, outside the Noether-Lefschetz locus, has Picard rank one, and its Picard group is generated by the polarization.

Remark 2.72. Fix a polarization type, hence a period map $P : \mathcal{M}_{[h]} \rightarrow \mathcal{P}_{[h]}$. Since the Picard lattice of the very general element in \mathcal{C}_e has discriminant $-e$, if $e \neq f$ then $\mathcal{C}_e = \mathcal{C}_f$ if and only if they are empty. The Noether-Lefschetz locus is thus a union of countably many divisors (one can prove that, for example, for $\mathcal{M}_{2,1}^n$ the Noether-Lefschetz divisor \mathcal{C}_{2t} is never empty for t satisfying conditions in Theorem 2.161, see Proposition 3.5). However, the divisor \mathcal{D}_e is a finite union of algebraic hypersurfaces for any fixed e , see [25, Lemma 3.23].

In contrast with the case of $K3$ surfaces, which we will treat in Section 2.3.3, for $n \geq 2$ the moduli space in Definition 2.69 is not necessarily connected. The number of connected components

of $\mathcal{M}_{2d,\gamma}^n$ is computed by Apostolov in [2]; we will shortly present his result here. Here we talk about connected components to follow the notation of Apostolov, in this situation they coincide with irreducible components as the moduli space is smooth.

Fix n, d, γ positive integers with $n \geq 2$ and $\gamma \mid \gcd(d, 2(n-1))$. We call again $\tilde{\Lambda}$ the Mukai lattice; by Proposition 2.27, every hyperkähler manifold X of type $K3^{[n]}$ comes with a natural choice of an $O(\tilde{\Lambda})$ -orbit of primitive embeddings of lattices $H^2(X, \mathbb{Z}) \rightarrow \tilde{\Lambda}$. We fix an embedding ι_X in this orbit. Following [2], given $(X, D) \in \mathcal{M}_{2d,\gamma}^n$ we call $T(X, D)$ the rank-two positive definite lattice obtained by the saturation, in $\tilde{\Lambda}$, of the lattice generated by $\iota_X(D)$ and $\iota_X(H^2(X, \mathbb{Z}))^\perp$.

We will need the following property of parallel transport operators, by Markman.

Theorem 2.73. *Given $(X_1, D_1), (X_2, D_2) \in \mathcal{M}_{2d,\gamma}^n$, an isometry $f : H^2(X_1, \mathbb{Z}) \rightarrow H^2(X_2, \mathbb{Z})$ is a parallel transport operator if and only if it is orientation preserving and there exists $\lambda \in O(\tilde{\Lambda})$ such that*

$$\lambda \circ \iota_{X_1} = \iota_{X_2} \circ f.$$

Proof. See [77, Theorem 9.8]. □

The last result allows to give the following characterization, in terms of the associated rank-two lattice, of manifolds which are deformation equivalent as polarized varieties.

Proposition 2.74. [2, Proposition 1.6] *Consider $(X_1, D_1), (X_2, D_2) \in \mathcal{M}_{2d,\gamma}^n$: a polarized parallel transport operator between them exists (so that they are in the same connected component of $\mathcal{M}_{2d,\gamma}^n$) if and only if $(T(X_1, D_1), \iota_{X_1}(D_1))$ and $(T(X_2, D_2), \iota_{X_2}(D_2))$ are isomorphic as lattices with a distinguished vector.*

Proof. (Sketch) One implication is clear, since if there is a polarized transport operator $f : H^2(X_1, \mathbb{Z}) \rightarrow H^2(X_2, \mathbb{Z})$ then λ as in Theorem 2.73 realizes a polarized isomorphism between the rank-two lattices, as $\lambda(\iota_{X_1}(D_1)) = \iota_{X_2}(f(D_1)) = \iota_{X_2}(D_2)$ and $\lambda(\iota_{X_1}(H^2(X_1, \mathbb{Z}))^\perp) = \iota_{X_2}(H^2(X_2, \mathbb{Z}))^\perp$.

On the other hand, given an isomorphism between the polarized rank-two lattices, it extends to an isometry $\lambda \in O(\tilde{\Lambda})$ by unimodularity of $\tilde{\Lambda}$. Moreover, the isometry λ restricts to an isometry between $H^2(X_1, \mathbb{Z})$ and $H^2(X_2, \mathbb{Z})$ since their orthogonal complements are sent one to the other by construction of λ : one of the generators of the rank-two lattices is always in the orthogonal complement of the image of the second cohomology group. So we have an isometry $f : H^2(X_1, \mathbb{Z}) \rightarrow H^2(X_2, \mathbb{Z})$ such that $\lambda \circ \iota_{X_1} = \iota_{X_2} \circ f$; either f or $-R_u \circ f$ is orientation preserving, where $u \in H^2(X_2, \mathbb{Z})$ is a square-two vector which is orthogonal to D_2 . So by Theorem 2.73 a monodromy operator exists, and it is also polarized since the one between f and $-R_u \circ f$ which is orientation preserving sends D_1 to D_2 . □

We denote by $\Sigma_{2d,\gamma}^n$ the set of isomorphism classes of lattices with a distinguished vector (T, h) , where T is a rank-two, even positive definite lattice of discriminant $\frac{4d(n-1)}{\gamma^2}$, $h \in T$ has square $2d$ and a generator of h^\perp in T has square $2(n-1)$. By Proposition 2.74, we can associate a lattice with a distinguished vector in $\Sigma_{2d,\gamma}^n$ to any connected component of $\mathcal{M}_{2d,\gamma}^n$, simply by taking $(T(X, D), \iota_X(D))$ for any (X, D) belonging to this component. Actually, something more can be said.

Theorem 2.75. *Call μ_n the set of connected components of $\mathcal{M}_{2d,\gamma}^n$. The map $\mu_n \rightarrow \Sigma_{2d,\gamma}^n$, that sends a component \mathcal{M} to the lattice with a distinguished vector associated to one element of \mathcal{M} , is bijective.*

Proof. See [2, Corollary 2.4]. □

Apostolov made an explicite computation of all the possible cardinalities of $\Sigma_{2d,\gamma}^n$, which leads to an explicit formula for the number of connected components of $\mathcal{M}_{2d,\gamma}^n$, by the result above.

Proposition 2.76. [2, Proposition 3.1] *Set $g = \gcd(2d, 2n - 2)/\gamma$, $\tilde{n} = (2n - 2)/\gcd(2d, 2n - 2)$, $\tilde{d} = 2d/\gcd(2d, 2n - 2)$, $w = \gcd(g, \gamma)$, $g_1 = g/w$, $\gamma_1 = \gamma/w$. Call $w_+(t_1)$ the product of all powers of primes dividing (w, γ_1) and $w_-(\gamma_1)$ such that $w = w_+(\gamma_1)w_-(\gamma_1)$. For any integer x , we denote by $\varphi(x)$ the Euler function evaluated at x , and $\rho(x)$ the number of distinct prime divisors of x . Then:*

- $|\Sigma_{2d,\gamma}^n| = w_+(\gamma_1)\varphi(w_-(\gamma_1))2^{\rho(\gamma_1)-1}$ if $\gamma > 2$ and one of the following sets of conditions hold: g_1 is even, $\gcd(\tilde{d}, \gamma_1) = \gcd(\tilde{n}, \gamma_1) = 1$ and the residue class $-\tilde{d}/\tilde{n}$ is a quadratic residue modulo γ_1 ; **OR** g_1, γ_1 , and \tilde{d} are odd, $\gcd(\tilde{d}, \gamma_1) = \gcd(\tilde{n}, 2\gamma_1) = 1$ and $-\tilde{d}/\tilde{n}$ is a quadratic residue modulo $2\gamma_1$; **OR** g_1, γ_1 , and w are odd, \tilde{d} is even, $\gcd(\tilde{d}, \gamma_1) = \gcd(\tilde{n}, 2\gamma_1) = 1$ and $-\tilde{d}/(4\tilde{n})$ is a quadratic residue modulo γ_1 ;
- $|\Sigma_{2d,\gamma}^n| = w_+(\gamma_1)\varphi(w_-(\gamma_1))2^{\rho(\gamma_1/2)-1}$ if $\gamma > 2$, g_1 is odd, γ_1 is even, $\gcd(\tilde{d}, \gamma_1) = \gcd(\tilde{n}, 2\gamma_1) = 1$ and $-\tilde{d}/\tilde{n}$ is a quadratic residue modulo $2\gamma_1$;
- $|\Sigma_{2d,\gamma}^n| = 1$ if $\gamma \leq 2$ and one of the following sets of conditions hold: g_1 is even, $\gcd(\tilde{d}, \gamma_1) = \gcd(\tilde{n}, \gamma_1) = 1$ and the residue class $-\tilde{d}/\tilde{n}$ is a quadratic residue modulo γ_1 ; **OR** g_1, γ_1 , and \tilde{d} are odd, $\gcd(\tilde{d}, \gamma_1) = \gcd(\tilde{n}, \gamma_1) = 1$ and $-\tilde{d}/\tilde{n}$ is a quadratic residue modulo $2\gamma_1$; **OR** g_1, γ_1 , and w are odd, \tilde{d} is even, $\gcd(\tilde{d}, \gamma_1) = \gcd(\tilde{n}, 2\gamma_1) = 1$ and $-\tilde{d}/(4\tilde{n})$ is a quadratic residue modulo γ_1 ; **OR** g_1 is odd, γ_1 is even, $\gcd(\tilde{d}, \gamma_1) = \gcd(\tilde{n}, 2\gamma_1) = 1$ and $-\tilde{d}/\tilde{n}$ is a quadratic residue modulo $2\gamma_1$;
- $|\Sigma_{2d,\gamma}^n| = 0$ otherwise.

Proof. (sketch) The computation is made by counting the number of possible overlattices of fixed discriminant of the lattice $\langle 2d \rangle \oplus \langle 2(n - 1) \rangle$. Some calculations are from [50, Proposition 3.6], that computes the number of $\tilde{O}(\Lambda_{K3^{[n]}})$ -orbits of vectors with square $2d$ and divisibility γ in $\Lambda_{K3^{[n]}}$. □

Remark 2.77. As a consequence, the number of connected components of $\mathcal{M}_{2d,\gamma}^n$ is finite.

Finally, note that, for any polarization type, the period map in Theorem 2.68 is not surjective. We are interested in two particular cases: $\langle 2t \rangle$ -polarized $K3$ surfaces and $\langle 2 \rangle$ -polarized manifolds of $K3^{[2]}$ -type (so that the divisibility is necessarily 1, see Lemma 2.64). By Proposition 2.76, in both cases the moduli space of manifolds with a polarization of such square and divisibility is connected; in particular in every case the polarization type is unique, so there is a period map whose domain is the whole moduli space. The Noether-Lefschetz locus $\mathcal{C}_{2d,\gamma}^n$ is then the inverse image of \mathcal{D} through the period map.

Remark 2.78. Whenever \mathcal{D}_e is non-empty then e is even. In the case of the period domain $\mathcal{P}_{2,1}^2$, a divisor \mathcal{D}_e is non-empty if and only if $e \equiv 0, 2, 4 \pmod{8}$, and in this case it has two irreducible component if $e \equiv 2 \pmod{8}$, it is irreducible otherwise (see [32, Proposition 4.1]).

Explicit descriptions for the image of the period map in these cases are given.

- For $K3$ surfaces, call $h(t)$ the polarization type associated to t . By Proposition 2.106 and Lemma 2.107, that we present in Section 2.3.3, the complement of the image is the quotient through $O(\Lambda_{K3}, h(t))$ of $\cup_{\Gamma \in \Delta_{h(t)}} \Gamma^\perp$, with

$$\Delta_{h(t)} = \{\Gamma \in \Lambda_{K3} \mid \Gamma^2 = -2, (\Gamma, h) = 0\}.$$

- Consider now the moduli space $\mathcal{M}_{2,1}^2$. Debarre and Macrì proved the following result.

Theorem 2.79. *The image of the period map*

$$P : \mathcal{M}_{2,1}^2 \rightarrow \mathcal{P}_{2,1}^2$$

is exactly the complement of the union of finitely many Heegner divisors. More precisely, those divisors are

- *the hypersurface \mathcal{D}_2 (which has two irreducible components),*
- *the hypersurface \mathcal{D}_8 (which has one irreducible component),*
- *one (out of two) irreducible component of the hypersurface \mathcal{D}_{10} .*

Proof. This is a special case of [32, Theorem 6.1]. □

In Section 4.3.3 we will consider again the irreducible components of \mathcal{D}_{10} and we will call them \mathcal{D}'_{10} and \mathcal{D}''_{10} . In particular, we will call \mathcal{D}''_{10} the one that is not contained in the image of the period map.

2.3.3 Polarized $K3$ surfaces

We treat the case of $K3$ surfaces separately, since it will occur multiple times. In this case, since Λ_{K3} is unimodular, by Eichler's criterion the degree $2t$ of a polarization determines automatically the polarization type $h(t)$ for every $t \geq 1$ and the divisibility of $h(t)$ is always one.

Definition 2.80. *We denote by \mathcal{K}_{2t} the moduli space of $\langle 2t \rangle$ -polarized $K3$ surfaces; clearly $\mathcal{K}_{2t} = \mathcal{M}_{[h(t)]} = \mathcal{M}_{2t,1}^1$. The divisor $D_{x,y} \subset \mathcal{K}_{2t}$ is the inverse image, through the period map, of the Heegner divisor \mathcal{D}_K , where K is a sublattice of Λ whose Gram matrix is*

$$\begin{bmatrix} 2t & x \\ x & y \end{bmatrix}.$$

In other words, $D_{x,y}$ corresponds to classes of polarized $K3$ surfaces (S, L) for which there exists $B \in \text{Pic}(S)$ with $(L, B) = x$, $B^2 = y$ and $\mathbb{Z}L + \mathbb{Z}B \subseteq \text{Pic}(S)$ is a primitive lattice.

Remark 2.81. For every $t \geq 1$ the moduli space \mathcal{K}_{2t} is irreducible, see [49, Theorem 2.10]. Hence, in this case, Theorem 2.68 simply reads "the period map $\mathcal{K}_{2t} \rightarrow \mathcal{P}_{[h(t)]}$ is an open embedding".

In this setting, Theorem 2.31 can be rewritten as follows.

Lemma 2.82. *For $(S, L) \in \mathcal{K}_{2t}$, the primitive ample divisor L is very ample if and only if $(S, L) \notin D_{h,0}$ with $h = 1, 2$.*

Proof. It is sufficient to note that, for any $(S, L) \in \mathcal{K}_{2t}$ such that there exists $D \in \text{Pic}(S)$ with $D^2 = 0$ and $(L, D) = h$, for $h = 1, 2$, the sublattice generated by L and D is either primitive, either a sublattice of U , see Section 1.3.3. \square

In Theorem 2.34 we gave a necessary and sufficient condition for a $\langle 10 \rangle$ -polarized $K3$ surface to be Brill-Noether general, in the following we see that this can be seen also as a divisorial condition on \mathcal{K}_{10} .

Proposition 2.83. *Let $(S, L) \in \mathcal{K}_{10}$ be a $\langle 10 \rangle$ -polarized $K3$ surface. Then (S, L) is Brill-Noether general if and only if $(S, L) \notin \mathcal{D}_{h,0}$ for $h \in \{1, 2, 3\}$.*

Proof. See [47, Lemma 1.7]. Greer, Li and Tian request also $(S, L) \notin D_{5,2}$ since they consider pairs (S, L) with L big and nef rather than ample. If $(S, L) \in D_{5,2}$ there is a (-2) -class in the orthogonal complement of L , so L cannot be ample by Proposition 2.106. \square

Note that, as we should expect, if (S, L) is Brill-Noether general then L is very ample by Lemma 2.82. The missing case is $(S, L) \in D_{3,0}$: if this happens we say that S is *trigonal*, see for example [108, Section 7.7] or [107, Section 4]. However, in this thesis we deal only with Brill-Noether general $K3$ surfaces.

2.3.4 Involutions of moduli spaces

We are in the same setting of Section 2.3.2; we focus to the case of manifolds of $K3^{[2]}$ -type, so that $\gamma \in \{1, 2\}$: by Remark 2.65, in these cases the polarization type is uniquely determined, and we have a period map

$$\mathcal{M}_{2d,\gamma}^2 \rightarrow \mathcal{P}_{2d,\gamma}^2.$$

If $\gamma = 1$ the moduli space $\mathcal{M}_{2d,1}^2$ is never empty, while by Lemma 2.64 we know that $\mathcal{M}_{2d,2}^2$ is not empty if and only if $d \equiv -1 \pmod{4}$. We call $\rho(d)$ the number of distinct primes dividing d .

Proposition 2.84. *If d is odd, the space $\mathcal{P}_{2d,\gamma}^2$ admits a generically free action by the group $(\mathbb{Z}/2\mathbb{Z})^\rho$, where*

- $\rho = \max(\rho(d), 1)$ if $d \equiv 1 \pmod{4}$ (in this case $\gamma = 1$),
- $\rho = \rho(d) - 1$ if $d \equiv -1 \pmod{4}$.

Proof. See [25, Proposition 3.1]. \square

By the Global Torelli theorem for polarized hyperkähler manifolds of $K3^{[n]}$ -type, this action induces, through the period map ρ , distinct birational involutions on the moduli space $\mathcal{M}_{2d,\gamma}^2$. In particular for $d \equiv 1 \pmod{4}$ there always exists a birational involution on $\mathcal{M}_{2d,\gamma}^2$.

For $d = 1$ we have a single non-trivial birational involution on $\mathcal{M}_{2,1}^2$: the case of $\langle 2 \rangle$ -polarized manifolds of $K3^{[2]}$ -type is particularly interesting, since it is the only case for which we have a geometrical description of the action. This description will be presented in Section 2.4.4, see in particular Remark 4.57.

Remark 2.85. When a divisor $\mathcal{D}_e \subset \mathcal{P}_{2,1}^2$ has two irreducible component (i.e. $e \equiv 2 \pmod{8}$), cfr. Remark 2.78), the involution exchanges the two irreducible components of it, see [32, Remark 4.2].

2.4 Automorphisms

2.4.1 Chamber decomposition

Here we focus on the action of a birational morphism in cohomology in the case of hyperkähler manifolds, and more in general on the structure of the second cohomology group. Markman pointed out that, again, monodromy operators are fundamental in this setting: a first hint is given by the following result, given by Huybrechts.

Proposition 2.86. *Let $f : X \dashrightarrow Y$ be a birational map between hyperkähler manifolds. Then $f^* : H^2(Y, \mathbb{Z}) \rightarrow H^2(X, \mathbb{Z})$ is a parallel transport operator.*

In particular, this means that the group of birational automorphisms of a hyperkähler manifold X acts on $H^2(X, \mathbb{Z})$ as a group of monodromy operators.

Let X be a projective hyperkähler manifold whose Néron-Severi group is $\text{NS}(X)$. We already defined the positive cone \mathcal{C}_X in Section 2.3.1.

Remark 2.87. For $x, y \in H^{1,1}(X, \mathbb{R})$ with $x \in \mathcal{C}_X$ and $y^2 > 0$, the element y belongs to the positive cone if and only if $(x, y) > 0$.

The decomposition into chambers of \mathcal{C}_X is elaborate: we give an insight of it, referring mostly to [77] and [1].

Definition 2.88. *We call prime exceptional divisor an effective divisor E which is reduced, irreducible and such that $E^2 < 0$, and we denote by \mathcal{P}_X the set of classes of such divisors.*

- The fundamental exceptional chamber of X is the cone

$$\mathcal{FE}_X = \{x \in \mathcal{C}_X \mid (x, E) > 0 \text{ for every } E \in \mathcal{P}_X\}.$$

- An exceptional chamber of the positive cone \mathcal{C}_X is a subset in the form $g(\mathcal{FE}_X)$ for some monodromy operator $g \in \text{Mon}_{Hdg}^2(X)$.

Clearly the fundamental exceptional chamber is an exceptional chamber of \mathcal{C}_X . By definition, an exceptional chamber is a cone too, and it is open since \mathcal{FE}_X is open. Two exceptional chambers coincide if their intersection is non-empty, see [77, Theorem 6.18, Item 2)]. A parallel transport operator that respects the Hodge decomposition sends an exceptional chamber to another, see [77, Lemma 5.12, Item 1)]; this holds in particular for monodromy operators and, by definition, the group $\text{Mon}_{Hdg}^2(X)$ acts transitively on the set of exceptional chambers.

Markman remarked that the action of a Hodge monodromy operator on the set of exceptional chambers allows to tell if the operator comes from a birational map of X to itself.

Proposition 2.89. *Given a monodromy operator $f \in \text{Mon}_{Hdg}^2(X)$, there exists a birational morphism \tilde{f} such that $\tilde{f}^* = f$ if and only if $f^*(\mathcal{FE}_X) = \mathcal{FE}_X$.*

Proof. See [77, Lemma 5.11, Item 6]. □

Given a Kähler class x on X , by definition of the BBF form x lies in the positive cone \mathcal{C}_X . Actually we can say more: since $(x, E) > 0$ for any effective class E by the Nakai-Moishezon criterion, the Kähler cone \mathcal{K}_X is contained in the fundamental exceptional chamber. Hence the fundamental exceptional chamber \mathcal{FE}_X can be defined also as the exceptional chamber containing a Kähler class.

Definition 2.90. *A monodromy birationally minimal (MBM) class is an element $s \in H^{1,1}(X, \mathbb{Q})$ with $s^2 < 0$ such that there exists a birational map $f : X \dashrightarrow Y$ between hyperkähler manifolds and a Hodge monodromy operator $g \in \text{Mon}_{Hdg}^2(X)$ such that $g(s)^\perp$ contains a face of $f^*(\mathcal{K}_Y)$.*

We denote by $\Delta(X)$ the set of primitive integral MBM classes. We call wall divisor any such class.

Definition 2.91. *The Kähler-type chambers of the positive cone \mathcal{C}_X are the connected components of*

$$\mathcal{C}_X - \bigcup_{s \in \Delta(X)} s^\perp.$$

The Kähler cone \mathcal{K}_X can be seen as the Kähler-type chamber that has positive product with every class in $\Delta(X)$.

By definition, Kähler-type chambers are open and they coincide if their intersection is non-empty. A parallel transport operator that respects the Hodge decomposition sends a Kähler-type chamber onto another Kähler-type chamber, see [77, Lemma 5.12, Item 2)]; .

Remark 2.92. Kähler-type chambers can be seen as the translation of the Kähler cone through a Hodge monodromy operator and a parallel transport operator induced by a birational map; the equivalence of this definition, which is [77, Definition 5.10], with the one we gave is granted by Amerik and Verbitski in [1, Theorem 6.2].

Consider the case of X a manifold of $K3^{[n]}$ -type whose second cohomology group is isomorphic, as a lattice, to $\Lambda_{K3^{[n]}}$; we need to define a class of elements of $\Lambda_{K3^{[n]}}$ that corresponds through markings to wall divisors, that we defined in Definition 2.90. To do it, we fix a connected component inside the moduli space \mathcal{M}_Λ .

Proposition 2.93. *Fixed a connected component $\mathcal{M}_{\Lambda_{K3^{[n]}}}^0$ in the moduli space of marked hyperkähler manifolds of $K3^{[n]}$ -type, there exists a subset $\Delta(\Lambda_{K3^{[n]}}) \subset \Lambda_{K3^{[n]}}$ with the following properties:*

- for any $(X, \eta) \in \mathcal{M}_{\Lambda_{K3^{[n]}}}^0$ we have $\Delta(X) = \eta^{-1}(\Delta(\Lambda_{K3^{[n]}})) \cap H^{1,1}(X, \mathbb{Z})$;
- the group $\text{Mon}(\Lambda_{K3^{[n]}})$ acts on $\Delta(\Lambda_{K3^{[n]}})$ with a finite number of orbits.

Let L be a sublattice of $\Lambda_{K3^{[n]}}$, we will denote $\Delta(L) = \Delta(\Lambda_{K3^{[n]}}) \cap L$. The elements in $\Delta(L)$ will be called *wall divisors* of L .

Now we introduce the definition of some convex cones inside $\text{NS}(X) \otimes \mathbb{R} \subseteq H^{1,1}(X, \mathbb{R})$; those two vector spaces are not necessarily equal. A *movable line bundle* is a line bundle on X that admits a positive multiple whose base locus has codimension at least 2. To motivate the first inclusion below see [25, Footnote 21].

Definition 2.94. *To simplify notation, we denote by $\text{Pos}(X)$ the intersection $\text{NS}(X)_{\mathbb{R}} \cap \mathcal{C}_X$.*

- The movable cone $\text{Mov}(X) \subseteq \overline{\text{Pos}(X)}$ is the convex cone generated by classes of movable line bundles on X .
- The nef cone $\text{Nef}(X) \subseteq \text{Mov}(X)$ is the closed convex cone generated by classes of nef line bundles on X .
- The ample cone $\mathcal{A}(X) \subseteq \text{Nef}(X)$ is the convex cone generated by classes of ample line bundles on X .

In particular, if X has Picard rank one all those cones are the same, since $\text{NS}(X)$ is generated by a single ample class in this situation. In general, the ample cone is open and it is the interior of the nef cone $\text{Nef}(X)$, see [76, Theorem 1.4.23].

As for \mathcal{C}_X in Definition 2.88, the cone $\text{Pos}(X)$ too admits a decomposition into chambers, induced by the decomposition of \mathcal{C}_X .

Proposition 2.95. *The interior of the movable cone $\text{Mov}(X)^0$ is the intersection $\text{NS}(X)_{\mathbb{R}} \cap \mathcal{FE}_X$. There is a one-to-one correspondence between the set of exceptional chambers in the positive cone \mathcal{C}_X and the set of the restrictions of the exceptional chambers to $\text{Pos}(X)$.*

Proof. See [77, Lemma 6.22]. □

Remark 2.96. Proposition 2.95, together with Proposition 2.89, implies that a monodromy operator is induced by a birational map if and only if it fixes the movable cone.

Also the decomposition in Kähler-type chambers of \mathcal{C}_X induces a decomposition on $\text{Pos}(X)$, so that the chambers are the connected components of

$$\text{Pos}(X) = \bigcup_{s \in \Delta(X)} s^{\perp}. \quad (2.1)$$

The ample cone is the intersection $\mathcal{K}_X \cap \text{Pos}(X)$, so it can be defined as the chamber of the movable cone containing an ample class. Moreover the movable cone $\text{Mov}(X)$ can be characterized as the connected component, in the decomposition of $\text{Pos}(X)$, containing an ample class.

Definition 2.97. We call exceptional (resp. Kähler-type) chambers of $\text{Pos}(X)$ the restrictions of the exceptional (resp. Kähler-type) chambers to $\text{Pos}(X)$. The chambers of the movable cone are then the Kähler-type chambers of $\text{Pos}(X)$ which lie inside $\text{Mov}(X)$. The walls of a chamber Ch are the subspaces of $\text{Pos}(X)$ which lie in the boundary of Ch and are a maximal open subset of a linear hyperspace, where maximality is taken with respect to inclusion.

Remark 2.98. By the definition of Kähler-type chambers given in Remark 2.92 and by Proposition 2.89, every Kähler-type chamber inside \mathcal{FE}_X corresponds to a birational model Y of X which is a hyperkähler manifold, and it is the image through a birational map $g : X \dashrightarrow Y$ of the Kähler cone of Y . Similarly, every Kähler-type chamber inside $\text{Mov}(X)$ corresponds to a birational model Y of X and is the image through a birational map $g : X \dashrightarrow Y$ of the ample cone of Y .

Finally the closure of the union of the Kähler-type chambers inside the movable cone is equal to $\overline{\text{Mov}(X)}$: this is the description of the closure of the exceptional chamber of X as the closure of the birational Kähler cone of X given in [77, Proposition 5.6].

Now we consider the case of the Hilbert scheme of n points on a projective $K3$ surface S , for $n \geq 2$. Bayer and Macrì gave in [5] a more explicit description of movable and ample cones in term of walls of said cones using Mukai's homomorphism $\theta : v^\perp \rightarrow H^2(S^{[n]}, \mathbb{Z})$ with $v = (1, 0, 1 - n) \in \tilde{\Lambda}$, which we define in Definition 2.26. Bayer and Macrì's description holds in general for moduli spaces of semistable sheaves over some projective $K3$ surface S ; in this case Mukai's homomorphism is the one given in Proposition 2.27. In the following we will always consider n an integer, $n \geq 2$.

Theorem 2.99. *The interior of the movable cone $\text{Mov}(S^{[n]})^0$ is one of the open chambers of the decomposition of $\overline{\text{Pos}(S^{[n]})}$ whose walls are the linear subspaces $\theta(v^\perp \cap a^\perp)$ for $a \in H_{alg}^*(S, \mathbb{Z})$ such that*

- $a^2 = -2$ and $(v, a) = 0$ or
- $a^2 = 0$ and $(v, a) = 1$ or
- $a^2 = 0$ and $(v, a) = 2$.

Proof. See [5, Theorem 12.3]. In our setting the Brauer class $\alpha \in \text{Br}(S)$ is 1, so $H_{alg}^*(S, \alpha, \mathbb{Z})$ is $H_{alg}^*(S, \mathbb{Z})$, see [5, Remark 2.6]. Spherical classes are -2 -square classes in $H_{alg}^*(S, \mathbb{Z})$. \square

Theorem 2.100. *The ample cone $\mathcal{A}(S^{[n]})$ is one of the open chambers of the decomposition of $\overline{\text{Pos}(S^{[n]})}$ whose walls are the linear subspaces $\theta(v^\perp \cap a^\perp)$ for $a \in H_{alg}^*(S, \mathbb{Z})$ such that $a^2 \geq -2$ and $0 \leq (v, a) \leq n - 1$.*

Proof. See [5, Theorem 12.1]. \square

In the case of a K3 surface S such that $\text{Pic}(S) = \mathbb{Z}L$ with $L^2 = 2t$, Bayer and Macrì determine a very explicit numerical description of the movable cone of $S^{[n]}$ for $n \geq 2$ through Theorem 2.99. This description depends on the diophantine equation $(n-1)X^2 - tY^2 = 1$; we will always refer to integer solutions when talking about the solvability of it. When $S^{[n]}$ has Picard rank two, the walls inside $\overline{\text{Pos}(S^{[n]})}$ are generated by a single class.

Theorem 2.101. *Let S be a K3 surface such that $\text{Pic}(S) = \mathbb{Z}L$ with $L^2 = 2t$.*

- *If $t(n-1)$ is a square, then*

$$\overline{\text{Mov}(S^{[n]})} = \langle L_n, (n-1)L_n - \sqrt{t(n-1)}\delta \rangle_{\mathbb{R}_{\geq 0}}$$

- *If $t(n-1)$ is not a square and $(n-1)X^2 - tY^2 = 1$ admits a solution, then*

$$\overline{\text{Mov}(S^{[n]})} = \langle L_n, z(n-1)L_n - tw\delta \rangle_{\mathbb{R}_{\geq 0}},$$

where (z, w) is the positive solution of $(n-1)X^2 - tY^2 = 1$ with minimal X .

- *If $t(n-1)$ is not a square and $(n-1)X^2 - tY^2 = 1$ does not admit a solution, then*

$$\overline{\text{Mov}(S^{[n]})} = \langle L_n, zL_n - tw\delta \rangle_{\mathbb{R}_{\geq 0}},$$

where (z, w) is the minimal solution of $X^2 - t(n-1)Y^2 = 1$ with $X \equiv \pm 1 \pmod{n-1}$.

Proof. See [5, Proposition 13.1]. □

Note that, in the third case, a solution satisfying $X \equiv \pm 1 \pmod{n-1}$ always exists by (1.2): if we denote by (u, s) the minimal solution, either $(z, w) = (u, s)$ or $(z, w) = (2t(n-1)s^2 + 1, 2us)$. Al. Cattaneo proved the following.

Proposition 2.102. *Let S be a K3 surface such that $\text{Pic}(S) = \mathbb{Z}L$ with $L^2 = 2t$. The walls defined in Theorem 2.100 which have non-trivial intersection with the interior of the movable cone are spanned by the classes of the form $Xh - 2tY\delta$ which lie in the movable cone, for (X, Y) a positive solution of one of Pell's equations $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ with $X \equiv \pm\alpha \pmod{2(n-1)}$, where the possible values of ρ and α are:*

$$\begin{cases} \rho = -1 \text{ and } 1 \leq \alpha \leq n-1; \\ \rho = 0 \text{ and } 3 \leq \alpha \leq n-1; \\ 1 \leq \rho < \frac{n-1}{4} \text{ and } 4\rho + 1 \leq \alpha \leq n-1. \end{cases} \quad (2.2)$$

Proof. The walls in Theorem 2.100 which does not appear in Theorem 2.99 are in the form $\theta(v^\perp \cap a^\perp)$ for $a \in \tilde{\Lambda}$ satisfying the numerical conditions given in [22, Remark 2.8]. Then [22, Lemma 2.5] explicitly describes those classes in terms of Pell's equations $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ (the bounds on ρ and α come from the bounds in [22, Remark 2.8]). Finally, a direct calculation gives a generator of the wall. □

We specify these constructions for S a projective $K3$ surface; in this case, by Riemann-Roch theorem, prime exceptional divisors are exactly the (-2) -curves on S . This has important consequences.

Proposition 2.103. *Let L be a line bundle on S . Then L is ample if and only if L lies in the positive cone of S and $(L, \Gamma) > 0$ for every (-2) -curve Γ .*

Proof. Recall that for $K3$ surfaces the BBF form is the cup product. We verify that Nakai–Moishezon–Kleiman criterion holds for L : we know by Remark 2.87 that for any $D \in \text{Pos}(S)$ the product of D with L is strictly positive, so we are left to prove that L has positive intersection with any prime exceptional divisor. But we already remarked that those are exactly the (-2) -curves on S . \square

Hence, by Lemma 2.30, in this case the closure of the movable cone $\overline{\text{Mov}(S)}$ and the Nef cone $\text{Nef}(S)$ coincide, and in particular there is only one chamber in the interior of $\text{Mov}(S)$. This has an immediate consequence.

Lemma 2.104. *Every birational map $\alpha : S \dashrightarrow \Sigma$ between projective $K3$ surfaces is in fact biregular.*

Proof. By Remark 2.96, the parallel transport operator α^* sends the fundamental exceptional chamber \mathcal{FE}_Σ inside \mathcal{FE}_S . By Theorem 2.47 then the birational map α is bihomomorphic since movable and Nef cones are the same. \square

Remark 2.105. Two elements $x, y \in \text{Pos}(S)$ inside the positive cone lies in the same chamber if and only if $(x, \Gamma) \cdot (y, \Gamma) > 0$ for every (-2) -class Γ .

Consider Γ a (-2) -class on S : one between Γ and $-\Gamma$ is effective, so no wall intersects the interior of the Nef cone. So we can rewrite Proposition 2.103 as follows.

Proposition 2.106. *Let S be a projective $K3$ surface. The ample cone $\mathcal{A}(S)$ is the chamber of the positive cone which can be written in the form*

$$\{x \in \text{Pos}(S) \mid (x, \Gamma) > 0 \text{ for every effective } (-2)\text{-class } \Gamma\}.$$

Consider now a (-2) -curve Γ on S . The associated reflection R_Γ , see Definition 1.22, acts on the positive cone: to verify this just consider any $x \in \text{Pos}(S)$ and check that $(x, R_\Gamma(z)) > 0$. The reflection fixes Γ^\perp and, since it is an isometry, it preserves the union of the walls, so it acts on the set of chambers in a non-trivial way.

Lemma 2.107. *For any $x \in \text{Pos}(S)$ there exists a finite number of (-2) -curves $\Gamma_1, \dots, \Gamma_n$ such that*

$$(R_{\Gamma_1} \circ \dots \circ R_{\Gamma_n})(x) \in \text{Nef}(S).$$

If moreover there is no (-2) -class Γ such that $(\Gamma, x) = 0$, then

$$(R_{\Gamma_1} \circ \dots \circ R_{\Gamma_n})(x) \in \mathcal{A}(S).$$

Proof. See [64, Chapter 8, Corollary 2.9]. \square

2.4.2 Automorphisms of hyperkähler manifolds and their action in cohomology

Let X be a hyperkähler manifold. The previous section gives us a description of the natural maps

$$\psi_X^A : \text{Aut}(X) \rightarrow O(H^2(X, \mathbb{Z})) \quad \text{and} \quad \psi_X^B : \text{Bir}(X) \rightarrow O(H^2(X, \mathbb{Z}))$$

that send f to the isometry f^* : the image of first one is the group of Hodge monodromy operators sending the ample cone in itself, while the image of the second one is the group of Hodge monodromy operators sending the movable cone in itself. Something more can be said for the deformation type in which we are interested in this thesis, as Huybrechts and Beauville showed.

Theorem 2.108. *For a projective manifold X of $K3^{[n]}$ -type, the maps ψ_X^A and ψ_X^B are injective.*

Proof. For ψ_X^A see [8, Proposition 10] in the case $X \cong S^{[n]}$, then [59, Theorem 2.1] to verify that the kernel of ψ_X^A is invariant by smooth deformation. For ψ_X^B we observe that a birational map fixing the whole second cohomology group fixes also a polarization, so it is in fact biregular. \square

The result above holds also in the non-projective case, see [62, Proposition 9.1]. We are now ready to formulate the following result.

Corollary 2.109. *Let X be a projective hyperkähler manifold of $K3^{[n]}$ -type and let $f \in O(H^2(X, \mathbb{Z}))$ be an isometry.*

- *There exists a birational morphism $\alpha : X \dashrightarrow X$ such that $f = \alpha^*$ if and only if f is a Hodge isometry which is orientation preserving, $\bar{f} = \pm \text{id} \in O(A_{H^2(X, \mathbb{Z})})$ and it sends the movable cone to itself.*
- *There exists a biregular morphism $\alpha : X \rightarrow X$ such that $f = \alpha^*$ if and only if f is an orientation preserving Hodge isometry, $\bar{f} = \pm \text{id} \in O(A_{H^2(X, \mathbb{Z})})$ and it sends an ample class on X to an ample class.*

Moreover, if α exists it is unique.

Proof. By Proposition 2.86 a necessary condition for an isometry f to come from a birational map is that $f \in \text{Mon}_{Hdg}^2(X)$, so it must preserve orientation and act as $\pm \text{id}$ on the discriminant group by Proposition 2.51. The condition on the movable cone (resp. ample classes) comes then from Remark 2.96 (resp. Hodge-theoretic Torelli theorem). On the other hand, an orientation-preserving Hodge isometry f that acts as prescribed on the discriminant group is induced by a birational map by Proposition 2.51 and Remark 2.96, then regularity follows again from Hodge-theoretic Torelli theorem. Finally, the unicity is Theorem 2.108. \square

Consider S a projective $K3$ surface with an automorphism $\alpha \in \text{Aut}(S)$. The associated *natural automorphism* $\alpha^{[n]} \in \text{Aut}(S^{[n]})$, for any $n \geq 2$ is obtained by sending the zero-dimensional subscheme $Z \subset S$ of length n to the zero-dimensional subscheme $\alpha(Z)$.

Clearly $\alpha^{[n]}$ is the identity on $S^{[n]}$ if and only if α itself is the identity on S , so there is an injective

morphism $\text{Aut}(S) \hookrightarrow \text{Aut}(S^{[n]})$.

The diagram

$$\begin{array}{ccc}
\text{Aut}(S) & \xrightarrow{\psi_S^A} & H^2(S, \mathbb{Z}) \\
\downarrow & & \downarrow i \\
\text{Aut}(S^{[n]}) & \xrightarrow{\psi_{S^{[n]}}^A} & H^2(S^{[n]}, \mathbb{Z}) \cong i(H^2(S, \mathbb{Z})) \oplus \mathbb{Z}\delta
\end{array} \tag{2.3}$$

is commutative and $(\alpha^{[n]})^*(\delta) = \delta$ by the geometric description of the exceptional divisor $E = 2\delta$. So the action in cohomology of $\alpha^{[n]}$ is the extension of the action on $i(H^2(S, \mathbb{Z}))$ by the identity. The converse is also true.

Theorem 2.110. *Let S be a projective K3 surface, and take $n \geq 2$ an integer. An automorphism $\alpha \in \text{Aut}(S^{[n]})$ is natural if and only if it leaves globally invariant the exceptional divisor E on $S^{[n]}$.*

Proof. See [17, Theorem 1]. □

Clearly a natural automorphism on $S^{[n]}$ is always biregular by Lemma 2.104.

Now we consider X a hyperkähler manifold and we fix a symplectic 2-form ω on it. A first property of a birational automorphism $\alpha \in \text{Aut}(X)$ comes from its action on the symplectic 2-form: α^* respects the Hodge decomposition, so it sends $\omega \in H^{2,0}(X)$ to a multiple of ω . If α has finite order k , then $\alpha^*\omega = \xi\omega$ with $(\xi)^k = 1$.

Definition 2.111. *An automorphism $\alpha \in \text{Bir}(X)$ is called symplectic if $\alpha^*\omega = \omega$ and non-symplectic otherwise. We say that a group $G \subseteq \text{Bir}(X)$ act symplectically on X if every $\alpha \in G$ is symplectic. We say that a group $G \subseteq \text{Bir}(X)$ act non-symplectically on X if it contains no symplectic automorphism except for the identity.*

Remark 2.112. For every $\alpha \in \text{Aut}(X)$ we call ξ the complex number such that $\alpha^*\omega = \xi\omega$ and $\text{Aut}_s(X)$ the subgroup of symplectic automorphisms of X : there is an exact sequence

$$1 \rightarrow \text{Aut}_s(X) \rightarrow \text{Aut}(X) \xrightarrow{\lambda} \mathbb{C}, \tag{2.4}$$

where λ sends α to ξ . Clearly $\text{Aut}_s(X)$ is normal.

Consider the restriction of the previous sequence to a finite group $G \leq \text{Aut}(X)$; the image of G through λ is finite in \mathbb{C} , hence cyclic. In particular, if G acts non-symplectically then G is cyclic. The same arguments hold for the group of birational automorphisms of X .

The following is clear from (2.3); it can also be deduced geometrically from the description of the 2-form on the Hilbert scheme, given by Beauville in [9, Section 6].

Lemma 2.113. *Let S be a projective K3 surface. An automorphism $\alpha \in \text{Aut}(S)$ is symplectic if and only if the associated natural automorphism $\alpha^{[n]} \in \text{Aut}(S^{[n]})$ is symplectic.*

Definition 2.114. Let X be a hyperkähler manifold. The transcendental lattice of X is the primitive sublattice $\text{Tr}(X) = \text{NS}(X)^\perp \subset H^2(X, \mathbb{Z})$.

Denoted by $\rho(X)$ the Picard rank of X , and $b_2(X)$ its second Betti number, the Néron-Severi lattice has signature $(1, \rho(X) - 1)$, hence T has signature $(2, b_2(X) - \rho(X) - 2)$.

Definition 2.115. Let $\alpha \in \text{Bir}(X)$ be a birational automorphism of finite order on X . The invariant lattice of α is the primitive sublattice $T_\alpha = H^2(X, \mathbb{Z})^{\alpha^*} = \{t \in H^2(X, \mathbb{Z}) \mid \alpha^*(t) = t\}$ and the coinvariant lattice of α is the primitive sublattice $S_\alpha = T_\alpha^\perp$. Given $G \leq \text{Bir}(X)$ finite, the invariant and coinvariant lattices T_G and S_G are defined in the same way.

So, for $\alpha \in \text{Bir}(X)$ finite, there are four (a priori) different sublattices of $H^2(X, \mathbb{Z})$. The relation between them depends on the simplicity of α , as we explain below. With the obvious modifications the same result holds for groups.

Proposition 2.116. If $\alpha \in \text{Bir}(X)$ is symplectic, then $\text{Tr}(X) \subseteq T_\alpha$, $S_\alpha \subseteq \text{NS}(X)$ and S_α is negative definite. If $\alpha \in \text{Bir}(X)$ is non-symplectic, then $T_\alpha \subseteq \text{NS}(X)$, $\text{Tr}(X) \subseteq S_\alpha$ and T_α is hyperbolic.

Proof. We call k the order of α . The sublattice S_α is the kernel of the isometry $\text{id} + \alpha^* + \dots + (\alpha^*)^{k-1}$. If α symplectic, for every $x \in S_\alpha$ we have

$$0 = (x + \alpha^*(x) + \dots + (\alpha^*)^{k-1}(x), \omega) = \sum_{i=1, \dots, k-1} ((\alpha^*)^i(x), (\alpha^*)^i \omega) = k(x, \omega),$$

so x lies in ω^\perp , which is $\text{NS}(X)$. The other inclusion is then obtained from the definition of the invariant lattice T_α .

Consider the real extension of α : any automorphism fixes the Kähler cone, in particular we can always find an α -invariant class, for example $D = \sum_{i=0}^{k-1} (\alpha^*)^i(d)$ for any Kähler class d . Then, by symplecticity, the extension of the invariant lattice contains $P = H^{2,0} \oplus H^{0,2} \oplus \mathbb{R}D$; the coinvariant lattice is then contained in P^\perp , hence it is negative definite since $H^2(X, \mathbb{Z})$ has signature $(3, 19)$.

For α non-symplectic, take $\xi \neq 1$ such that $\alpha^* \omega = \xi \omega$. Then, for every $x \in T_\alpha$,

$$(\omega, x) = (\alpha^*(\omega), \alpha^*(x)) = \xi(\omega, x),$$

so $x \in \text{NS}(X)$. The other inclusion is then obtained from the definition of the coinvariant lattice S_α . For the signature it is sufficient to observe that D , defined as above, is always in the invariant lattice. \square

When X is a manifold of $K3^{[n]}$ -type and $\alpha \in \text{Aut}(X)$ is a symplectic automorphism of prime order on it, few choices are allowed for the coinvariant lattice S_α : Mongardi provided in [82, Corollary 5.1] a classification of these invariant lattices. In the next section we give some additional detail in the case of $K3$ surfaces, which was already proved by Nikulin in [89]. In higher dimension, Mongardi gave the following result for symplectic involutions.

Theorem 2.117. Let X be a manifold of $K3^{[n]}$ -type, $n \geq 1$. The coinvariant lattice of a biregular symplectic involution on X is isomorphic to the rank-eight lattice $E_8(-2)$.

Proof. See [82, Corollary 5.1] for $p = 2$. □

As a first consequence, we have a lower bound for the Picard rank of a manifold of $K3^{[n]}$ -type, $n \geq 1$, if X admits a symplectic involution which is biregular.

Proposition 2.118. *Let X be a manifold of $K3^{[n]}$ -type, $n \geq 1$. If X admits a biregular symplectic involution, then $\text{rk}(\text{Pic}(X)) \geq 8$.*

Proof. Call ι the symplectic involution. Since by Proposition 2.116 the coinvariant lattice S_ι is contained in $\text{NS}(X)$, it is sufficient to verify that S_ι has rank 8. This holds by Theorem 2.117. □

This last result does not always hold when ι is non biregular. As an example, see Proposition 6.6, Item *iii*).

2.4.3 Automorphisms of $K3$ surfaces

Global Torelli theorem for $K3$ surfaces allows the following description of the automorphism group of the very general element in the moduli space of $\langle 2t \rangle$ -polarized $K3$ surfaces \mathcal{K}_{2t} .

Theorem 2.119. *Let Σ be a $K3$ surface such that $\text{Pic}(\Sigma) = \mathbb{Z}L$ with $L^2 = 2t$. If $t \geq 2$ then the automorphism group $\text{Aut}(\Sigma)$ is trivial, while if $t = 1$ we have $\text{Aut}(\Sigma) \cong \mathbb{Z}/2\mathbb{Z}$ and the involution that generates $\text{Aut}(\Sigma)$ is non-symplectic.*

Proof. See [64, Chapter 15, Corollary 2.12], non-symplecticity comes from Proposition 2.118. □

The theorem above relies on lattice theory but, as we explained in Section 2.2.3, in the case $t = 1$ the $K3$ surface Σ is a double covering of the projective plane $\Sigma \rightarrow \mathbb{P}^2$ ramified over a smooth sextic, so the involution that generates $\text{Aut}(\Sigma)$ is the covering involution. We present the equivalent result for very general manifolds of $K3^{[n]}$ -type in Section 2.4.5. Note that the very general element in $\mathcal{M}_{2d,\gamma}^n$ has again rank one, in particular it is not in the form $\Sigma^{[n]}$ for $n \geq 2$.

There is a huge literature about automorphism groups of $K3$ surfaces, from the foundational work by Nikulin, see for example [90], to now. Without any attempt of completeness, here we state the results that will be useful in the thesis; in particular, it is useful to confront the results presented in this section with Section 5.4.1, to see why we chose them and how they are used.

We are mostly interested in finite groups.

Proposition 2.120. *Let (Σ, L) be a polarized $K3$ surface. The subgroup of $\text{Aut}(\Sigma)$ of automorphisms fixing the polarization L is finite.*

Proof. See [64, Chapter 5, Proposition 3.3]. □

The study of automorphisms on a $K3$ surface relies heavily on lattice theory, due to Global Torelli theorem for $K3$ surfaces. In particular, given Σ a $K3$ surface, the existence of automorphisms on Σ can be rephrased in terms of sublattices of $H^2(\Sigma, \mathbb{Z})$.

We treat separately the symplectic and non-symplectic cases. We consider first symplectic automorphisms.

Remark 2.121. As we can expect from Proposition 2.116, the coinvariant lattice of a finite group G acting symplectically on a $K3$ surface Σ contains no (-2) -class, or it would not be possible to choose a polarization on Σ which is invariant with respect to G . To prove it, consider instead $G \leq \text{Aut}(\Sigma)$ acting symplectically with coinvariant lattice $S_G \subset \text{NS}(\Sigma)$ and invariant lattice T_G , and take $\Gamma \in S_G$ with $\Gamma^2 = -2$. By Riemann-Roch theorem either Γ or $-\Gamma$ is effective, cfr. Section 2.2.3; assume that Γ is effective. Then $\sum_{\alpha \in G} \alpha(\Gamma)$ is again effective and it lies in $S_G \cap T_G = 0$, which is impossible.

Theorem 2.122. *Let Σ be a $K3$ surface, and let $G \leq \text{Aut}(\Sigma)$ be a group acting symplectically on Σ . If G is abelian then it is isomorphic to one of the following groups*

$$\begin{aligned} & \mathbb{Z}/n\mathbb{Z} \text{ for } 2 \leq n \leq 8, & (\mathbb{Z}/n\mathbb{Z})^2 \text{ for } n = 2, 3, 4, \\ & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}, & \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}, & (\mathbb{Z}/2\mathbb{Z})^k \text{ for } k = 3, 4. \end{aligned}$$

Proof. See [90, Theorem 4.5]. □

An important result by Nikulin states that, if Σ_1 and Σ_2 are $K3$ surfaces with $G_i \leq \text{Aut}(\Sigma_i)$ abelian subgroups for $i = 1, 2$, and $G_1 \cong G_2$, then the action of G_1 on $H^2(\Sigma_1, \mathbb{Z})$ is isomorphic to the action of G_2 on $H^2(\Sigma_2, \mathbb{Z})$. This is [89, Theorem 4.7].

Remark 2.123. A complete list of the finite groups acting symplectically on some $K3$ surface has been done by Mukai in [85]. Some additional information can be found in [114, Table 2].

Consider a $\langle 2t \rangle$ -polarized $K3$ surface (Σ, L) and p a prime number such that $\mathbb{Z}/p\mathbb{Z} = \langle \alpha \rangle$ acts symplectically on Σ . We know by Theorem 2.122 that $p \in \{2, 3, 5, 7\}$. As usual, we denote by $S_{\mathbb{Z}/p\mathbb{Z}}$ the coinvariant lattice of $\mathbb{Z}/p\mathbb{Z}$ (or equivalently of α).

Remark 2.124. By Proposition 2.116, the Picard rank of Σ , when Σ is projective, has to be at least $1 + \text{rk}(S_{\mathbb{Z}/p\mathbb{Z}})$, in particular it always contains a primitive overlattice of $\mathbb{Z}L \oplus S_{\mathbb{Z}/p\mathbb{Z}}$. The following result, due mainly to Van Geemen and Sarti for $p = 2$ and to Garbagnati and Sarti for $p = 3, 5, 7$, provides a list of some necessary and sufficient conditions, on t and p , for the existence of Σ with minimal Picard rank.

Theorem 2.125. *We consider $p \in \{2, 3, 5, 7\}$ and we call \mathcal{L}_p the lattice $\mathbb{Z}L \oplus S_{\mathbb{Z}/p\mathbb{Z}}$ with $L^2 = 2t$, $t \geq 1$.*

- a) *For $p = 2$ the lattice \mathcal{L}_2 has rank 9, for $p = 3$ the lattice \mathcal{L}_3 has rank 13, for $p = 5$ the lattice \mathcal{L}_5 has rank 17, for $p = 7$ the lattice \mathcal{L}_7 has rank 19.*
- b) *Take $p \in \{2, 3, 5\}$. For every $t \geq 1$ there exists a $\langle 2t \rangle$ -polarized $K3$ surface (Σ, L) with Néron-Severi lattice isomorphic to \mathcal{L}_p . Every $K3$ surface obtained in this way admits a symplectic automorphism of order p that leaves invariant the polarization L .*
- c) *Consider $p = 7$. There exists a $\langle 2t \rangle$ -polarized $K3$ surface (Σ, L) admitting a symplectic automorphism of order 7 that leaves invariant the polarization L if and only if $t \equiv 1, 2, 4 \pmod{7}$.*

Proof. (Sketch) To prove *a)* it is necessary, for any p , to compute the coinvariant lattice in a particular case. Then, by [89, Theorem 4.7], for every $K3$ surface Σ with a symplectic automorphism of order p , the coinvariant lattice is isomorphic to the one of the particular case.

The case $p = 2$ is computed directly on the $K3$ lattice using Global Torelli theorem for $K3$ surfaces: Morrison, in the proof of [83, Theorem 5.7], described an action which interchanges the two copies of $E_8(-1)$ inside Λ_{K3} and acts as the identity on $U^{\oplus 3}$, which has rank 6.

For the other cases, this approach presents some technical problems, since an isomery of order bigger than two is not automatically determined by the choice of invariant and coinvariant lattice. The description is obtained in any case by Garbagnati and Sarti in [44, Theorem 4.1], by means of elliptic fibrations.

For *b)* the case $p = 2$ is [12, Proposition 2.3], the case $p = 3, 5$ is [44, Proposition 5.2]. In any of these results it is stated that there exists a $K3$ surface Σ with a symplectic automorphism α of order p such that the coinvariant lattice of α is the prescribed lattice. We also know, see [12, Proposition 2.3] for $p = 2$ and [44, Proposition 5.1] for $p = 3, 5$, that Σ admits a polarization that can be identified to L in $\text{NS}(\Sigma) = \mathbb{Z}L \oplus S_\alpha$. Since $L \in S_\alpha^\perp$ then L lies in the invariant lattice of α .

For *c)* consider [45, Proposition 6.5]: when $t \equiv 1, 2, 4 \pmod{7}$ the procedure is the same as in *b)*. By the construction made by Garbagnati and Sarti to prove *a)* for $p = 7$, the invariant lattice of a symplectic order-7 automorphism, on a $K3$ surface, is isomorphic to the lattice whose Gram matrix is

$$T = U(7) \oplus \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix},$$

cfr. [44, Theorem 4.1]. When $t \not\equiv 1, 2, 4 \pmod{7}$, Garbagnati and Sarti proved with a direct computation that there is no element inside T whose square is $2t$. This computation can be found in the proof of [45, Proposition 6.5]. \square

Remark 2.126. Van Geemen, Sarti and Garbagnati studied also the case of polarizations coming from an overlattice of \mathcal{L}_p , see for example the second part of [45, Proposition 6.5], where they found $K3$ surfaces whose Néron-Severi lattice is isomorphic to an overlattice of \mathcal{L}_7 . In Section 5.4.1 we make clear why we need invariant polarizations.

In this thesis we will not need an explicit description of the invariant and coinvariant lattices of the automorphisms above; however, these descriptions exist and can be found at [44, Theorem 4.1]. They were computed for $p = 2$ by Morrison in [83], by Garbagnati and Sarti for $p = 3, 5, 7$ in [44]. We presented the case $p = 2$ in Theorem 2.117 for hyperkähler manifolds of $K3^{[n]}$ -type with $n \geq 1$.

Remark 2.127. Consider Σ a $K3$ surface and $G \cong \mathbb{Z}/p\mathbb{Z}$ a group acting symplectically on it such that Σ has Picard rank $\rho(\Sigma) = 1 + \text{rk}(S_G)$; this happens for example in the cases described in Theorem 2.125, Item *b)*, *c)*. We consider the same notation. In this cases it is possible to construct a coarse moduli space of pairs of this type: we will say that (Σ, G) and (Σ', G') are isomorphic if there exists an isomorphism $\Sigma \xrightarrow{\sim} \Sigma'$ under which G and G' are identified. Here we only need the cases $\text{NS}(\Sigma) \cong \mathcal{L}_p$. Take $\mathcal{L}_p \subset \Lambda_{K3}$ and $\Gamma(\mathcal{L}_p) \leq O(\mathcal{L}_p)$ the subgroup of isometries fixing pointwise \mathcal{L}_p : the moduli space of $\langle 2t \rangle$ -polarized $K3$ surfaces which admit a symplectic action of G fixing the

polarization results in a quotient of an open inside $\{[x] \in \mathbb{P}(\mathcal{L}_p^\perp \otimes \mathbb{C}) \mid x^2 = 0, (x, \bar{x}) > 0\}$ by the restriction of $\Gamma(\mathcal{L}_p)$ to \mathcal{L}_p^\perp .

The construction is explained in detail in [34, Section 3] and arranged for the case of symplectic automorphisms in [12, Proposition 2.3] for $p = 2$, in [45, Remark 6.2] otherwise. Similar constructions, but in higher dimension and in the non-symplectic case, will be presented in the next section.

Anyway, it is clear by the construction that these moduli spaces have dimension $20 - \text{rk}(\mathcal{L}_p)$; as we can expect, the very general element in the moduli space have Néron-Severi lattice isomorphic to \mathcal{L}_p .

Now we consider non-symplectic automorphisms. We already know by Remark 2.112 that a group $G \leq \text{Aut}(\Sigma)$ is cyclic if the only symplectic element of G is the identity. As usual, we denote by $\varphi(n)$ the Euler function evaluated at n .

Proposition 2.128. *Let Σ be a K3 surface with Picard rank $\rho(\Sigma)$ and transcendental lattice $\text{Tr}(\Sigma)$. Consider $\alpha \in \text{Aut}(\Sigma)$ of order n , with α acting as a n -th root of unity on a symplectic two-form ω . Then $\varphi(n) \leq \text{rk}(\text{Tr}(\Sigma)) = 22 - \rho(\Sigma)$ and $\varphi(n) \mid \text{rk}(\text{Tr}(\Sigma))$.*

Proof. See [64, Chapter 15, Corollary 1.14]. □

This result, based on the study of the representation of α on the transcendental lattice $\text{Tr}(\Sigma)$, has an immediate consequence.

Corollary 2.129. *The order of α is at most 66. If α has prime order p , then $p \leq 19$.*

Proof. For the first part we observe that $\varphi(n) > 21$ if $n > 66$. The second part is clear. □

As for the existence of non-symplectic automorphisms, we have for example the following, by Zhang.

Proposition 2.130. *For $n \leq 66$, let ξ be a n -th root of unity; there exists a K3 surface Σ , and an automorphism of order n on Σ acting as ξ on a symplectic two-form of Σ , if and only if $n \neq 60$.*

Proof. See [115, Proposition 3.2 and Proposition 3.3]. □

By Theorem 2.122 and Corollary 2.129 we can compute a bound on the order of an automorphism of finite order on a K3 surface, which is $528 = 66 \cdot 8$. Actually, a better bound exists: Keum computed it in [71].

Theorem 2.131 (Keum). *The order of a finite order automorphism on a K3 surface is at most 66.*

Proof. See [71, Theorem 1.2]. □

Remark 2.132. Consider a K3 surface Σ with a non-symplectic automorphism α of finite order acting on it. A complete family of deformation of the pair (Σ, α) has dimension $20 - \text{rk}(T_\alpha) = \text{rk}(S_\alpha) - 2$; in the next section we give additional details on the theory of deformation of automorphisms on hyperkähler manifolds of $K3^{[n]}$ -type.

The fixed locus of automorphisms of prime order has been completely described in terms of invariant and coinvariant lattices of the automorphism. This has been done by Nikulin, see [89], for the symplectic case. Here we only present the case of a symplectic involution acting on some $K3$ surface. Consider a $K3$ surface Σ with a symplectic involution ι on it, and take $x \in \Sigma$ a fixed point for the action of the involution. Then the action of ι on the tangent space of Σ in x has the following diagonal form

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

since the determinant has to be one by symplecticity. So the fixed locus of Σ is discrete, hence finite. Something more can be said.

Proposition 2.133. *Let Σ be a $K3$ surface and ι a symplectic involution acting on it. Then ι has exactly 8 fixed points.*

Proof. See [89, Section 5]. □

We call $\text{Fix}(\iota)$ the fixed locus of ι . We can easily compute the fixed locus of the natural involution $\iota^{[2]} \in \text{Aut}(\Sigma^{[2]})$ (see Section 2.4.2 for the definition): it is composed by all the pairs $p + q$ with $p, q \in \text{Fix}(\iota)$, which give $\binom{8}{2} = 28$ isolated fixed points, and the closure of the set of elements in the form $p + \iota(p)$ for $p \in \Sigma - \text{Fix}(\iota)$, which gives a copy of Σ , so a $K3$ surface. Camere conjectured in [20] that this is the only possibility for the fixed locus of a symplectic involution on a hyperkähler manifold of $K3^{[n]}$ -type; Mongardi proved it to be true.

Theorem 2.134 (Mongardi). *Let X be a hyperkähler manifold of $K3^{[n]}$ -type and ι a symplectic involution acting on it. Then ι fix pointwise exactly 28 points and one $K3$ surface.*

Proof. See [80, Theorem 4.1]. Mongardi proved in [80, Theorem 1.3] that any pair (X, ι) as in the claim can be deformed to a pair $(\Sigma^{[2]}, j^{[2]})$, where Σ is a $K3$ surface and $j \in \text{Aut}(\Sigma)$ is a symplectic involution. As a consequence, the fixed locus of ι deforms to the fixed locus of $j^{[2]}$, which is a $K3$ surface as we said above. □

Remark 2.135. In the symplectic case, the fixed locus of $\alpha \in \text{Aut}(\Sigma)$ of prime order p only depends on p . The left cases are $p = 3, 5, 7$, for which α has respectively six, four and three fixed points.

The non-symplectic case is more complicated: Artebani, Sarti and Taki showed in [3] that, given a $K3$ surface with a non-symplectic automorphism α of order p , for $p > 2$ and prime, the topological structure of the fixed locus of α determines uniquely its action on the second cohomology group of the $K3$ surface. The case $p = 2$ has been treated by Nikulin in [88, Theorem 4.2.2].

A survey of non-symplectic automorphisms of prime order can be found at [3]: Artebani, Sarti and Taki described the fixed locus of any automorphisms of this type in [3, Theorem 0.1]. More informations about fixed loci, invariant and coinvariant lattices can be found at [3] in Images 1 and 2, Table 1 to 7. Beware that in [3] the notation is different from ours, as $S(\alpha)$ is the invariant lattice of α (instead of T_α) and $T(\alpha)$ is its coinvariant lattice (instead of S_α).

2.4.4 Deformation of involutions and moduli spaces

We want to construct a moduli space for hyperkähler manifolds with a biregular involution; as for marked and polarized manifolds, we need to define the corresponding deformation families. Here we will always consider pairs (X, α) where $\alpha \in \text{Aut}(X)$ is an automorphism on a hyperkähler manifold X , or pairs (X, G) where $G \leq \text{Aut}(X)$.

Definition 2.136. *Let X_1, X_2 be hyperkähler manifolds which are deformation equivalent, and let $G_i \leq \text{Aut}(X_i)$. Then (X_1, G_1) is deformation equivalent to (X_2, G_2) if there exists*

- a family of deformation $\mathcal{X} \rightarrow B$,
- two points $b_1, b_2 \in B$ and two isomorphisms $\phi_i : X_i \xrightarrow{\sim} \mathcal{X}_{b_i}$ for $i = 1, 2$,
- a group G isomorphic both to G_1 and G_2 with an action on \mathcal{X} that fixes every fiber, such that the restriction of the action to a fiber is faithful and the action induced through ϕ_i on X_i coincides with G_i for $i = 1, 2$.

The existence of a universal deformation family guarantees that, for every hyperkähler manifold with a finitely generated group acting on it, there exists a family of deformation that carries the action of the group. The argument we give now comes from [14, Section 4], we refer to it for details. We make use of a general result by Horikawa, see [60, Theorem 8.1], that implies that, given $\alpha : X_1 \rightarrow X_2$ a biholomorphic map between manifolds, and let $\mathcal{X} \rightarrow B$ be a family of deformation of X_1 based in $0 \in B$ (i.e. the fiber over 0 is isomorphic to X_1), there exists an open neighborhood $\Delta \ni 0$ inside B , a family of deformation $\mathcal{X}' \rightarrow \Delta$ of X_2 based in 0 and a holomorphic map $\phi : \mathcal{X}_\Delta \rightarrow \mathcal{X}'$ over Δ such that $\phi_0 = \alpha$.

Consider a hyperkähler manifold X with its universal deformation $\mathcal{X} \rightarrow \text{Def}(X)$ and an automorphism $\alpha \in \text{Aut}(X)$; we take the family of deformation $\mathcal{X}' \rightarrow \Delta$ of X and $\phi : \mathcal{X}_\Delta \rightarrow \mathcal{X}'$ as we said above. By universality of $\mathcal{X} \rightarrow \text{Def}(X)$ there exists a unique map $\gamma_\alpha : \Delta \rightarrow \text{Def}(X)$ such that $\mathcal{X}' = \mathcal{X} \times_{\text{Def}(X)} \Delta = \mathcal{X}_\Delta$, so we have a diagram

$$\begin{array}{ccccc} \mathcal{X}_\Delta & \xrightarrow{\phi} & \mathcal{X}_\Delta & \hookrightarrow & \mathcal{X} \\ \downarrow & & \downarrow & & \downarrow \\ \Delta & \xrightarrow{\text{id}} & \Delta & \xrightarrow{\gamma_\alpha} & \text{Def}(X). \end{array}$$

We call $A : \mathcal{X}_\Delta \rightarrow \mathcal{X}$ the composition of the upper maps in the diagram. Now we consider $D(\alpha) = \{t \in \Delta \mid \gamma_\alpha(t) = t\}$. By restricting the families of deformation to $D(\alpha)$ we have a diagram

$$\begin{array}{ccc} \mathcal{X}_{D(\alpha)} & \xrightarrow{A} & \mathcal{X}_{D(\alpha)} \\ \downarrow & & \downarrow \\ D(\alpha) & \xrightarrow{\text{id}} & D(\alpha). \end{array}$$

Note that we can do the same for α^{-1} : up to shrinking Δ and then $D(\alpha)$, the morphism A is biholomorphic by universality of the deformation. Again by universality of $\mathcal{X} \rightarrow \text{Def}(X)$, the deformation family $\mathcal{X}_{D(\alpha)} \rightarrow D(\alpha)$ is complete between the families on which the extension of α has trivial action on the base.

As we remarked below Theorem 2.12, there is a natural isomorphism between the Zariski tangent space $T_0 \text{Def}(X)$ and $H^1(X, TX)$. Under this identification, the morphism γ_α corresponds to the differential $d\alpha$, hence the family $\mathcal{X}_{D(\alpha)} \rightarrow D(\alpha)$ is smooth, and the tangent space $T_0 D(\alpha)$ corresponds to $H^1(X, TX)^{d\alpha}$. We want to link $T_0 D(\alpha) \cong H^1(X, TX)^{d\alpha}$ with the lattices associate to α that we defined in Section 2.4.2: we can do it by using the isomorphism $TX \cong \Omega_X$ induced by a symplectic form ω . We consider the two possible cases separately.

- If α is symplectic, then the isomorphism $TX \cong \Omega_X$ is equivariant, thus $T_0 D(\alpha)$ is isomorphic to $H^1(X, \Omega_X)^{\alpha^*}$.
- If α is non-symplectic, and in particular $\alpha^*\omega = \xi\omega$, then $T_0 D(\alpha)$ corresponds to the eigenspace of $H^1(X, \Omega_X)$ associated to the eigenvalue ξ (the action of α is always diagonalizable, since α has finite order).

Given a group $G \leq \text{Aut}(X)$ generated by $\alpha_1, \dots, \alpha_k \in \text{Aut}(X)$ acting on X , the same construction holds for every generator. It is then sufficient to shrink Δ to extend, using the universality of $\mathcal{X} \rightarrow \text{Def}(X)$, the action of the whole group G on a family based on X . Finally, we take the intersection of the corresponding Δ 's to obtain a deformation family $\mathcal{X}_{D(G)} \rightarrow D(G)$ for the pair (X, G) .

Remark 2.137. Since, by Ehresmann's theorem, the structure of $\langle \alpha \rangle$ -module of $H^2(X, \mathbb{Z})$ is invariant under small deformations, the invariant and coinvariant sublattices T_G and S_G are invariant under small deformations along $D(\alpha)$.

Now we focus to the case of involutions. Consider ι a biregular involution on X .

- if ι is symplectic, then the family of deformation $\mathcal{X}_{D(\iota)} \rightarrow D(\iota)$ of (X, ι) has dimension $\dim D(\iota) = \dim(T_\iota) - 2$;
- if ι is non-symplectic, then the family of deformation $\mathcal{X}_{D(\iota)} \rightarrow D(\iota)$ of (X, ι) has dimension $\dim D(\iota) = \dim(S_\iota) - 2$.

Remark 2.138. Consider some projective $K3$ surface Σ , and an involution $\iota \in \text{Aut}(\Sigma)$. The invariant and coinvariant lattices of the natural involution $\iota^{[n]}$ on $\Sigma^{[n]}$, seen through the injection $i : H^2(\Sigma, \mathbb{Z}) \rightarrow H^2(\Sigma^{[n]}, \mathbb{Z})$, are $T_{\iota^{[n]}} \cong i(T_\iota)$ and $S_{\iota^{[n]}} \cong i(S_\iota) \oplus \mathbb{Z}\delta$, where δ is the generator of the orthogonal complement of the image of i .

Looking at the bases of the families of deformation that we described, this implies that, if ι is symplectic, the very general deformation of $(\Sigma^{[n]}, \iota^{[n]})$ does not come from a local deformation of (Σ, ι) . On the other hand if ι is non-symplectic every small deformation of $(\Sigma^{[n]}, \iota^{[n]})$ is in the form $(\Omega^{[n]}, j^{[n]})$, for (Ω, j) a small deformation of (Σ, ι) .

We present some notations and results from [70] and [21]. In the following we will focus on non-symplectic involutions on hyperkähler manifolds of $K3^{[n]}$ -type; we fix $n \geq 2$ and thus the abstract lattice $\Lambda_{K3^{[n]}}$, which we will denote here by Λ . Recall that we defined the group $\text{Mon}(\Lambda) < O(\Lambda)$ in Definition 2.41.

Definition 2.139. *A sublattice $M \subseteq \Lambda$ is called admissible if*

- *M is hyperbolic and*
- *there exists an involution $\rho \in O(\Lambda)$ such that $\Lambda^\rho = M$.*

We do not consider M as an abstract lattice, but always as a sublattice of Λ , as the same abstract lattice can admit different non-isometric embeddings into a fixed lattice: this is a key point when dealing with connected components of moduli spaces of hyperkähler manifolds.

In the particular case of an involution $\iota \in \text{Aut}(X)$ on a hyperkähler manifold X , the action of ι in cohomology is completely determined by the pair given by invariant and coinvariant lattices (T_ι, S_ι) : since ι acts as the identity on the invariant lattice T_ι and as $-\text{id}$ on the coinvariant lattice S_ι , the whole ι^* is the extension on the overlattice $H^2(X, \mathbb{Z})$ of $\text{id} \oplus (-\text{id})$ on $T_\iota \oplus S_\iota$.

In the same way, the involution ρ is completely determined by the choice of M : we say that ρ is the involution *associated to M* .

We fix a connected component \mathcal{M}_Λ^0 of the moduli space of marked hyperkähler manifolds of $K3^{[n]}$ -type.

Definition 2.140. *Let $M \subset \Lambda$ be an admissible sublattice with associated involution $\rho \in \text{Mon}(\Lambda)$. A marked manifold $(X, \eta) \in \mathcal{M}_\Lambda^0$ admits a (M, ρ) -polarization if there is an involution $\iota \in \text{Aut}(X)$ on X such that $\eta \circ \iota^* = \rho \circ \eta$. Two (M, ρ) -polarized manifolds are isomorphic if they are isomorphic as marked manifolds.*

We denote by $\mathcal{M}_{M, \rho}^n \subseteq \mathcal{M}_\Lambda^0$ the moduli space of (M, ρ) -polarized manifolds.

Note that by Theorem 2.108 the involution ι is uniquely determined by η and ρ ; we will also write (X, η, ι) for a (M, ρ) -polarized manifold, where ι is the involution in the definition, to stress the role of the involution.

By definition of ι the invariant lattice $T_\iota = \eta^{-1}(M)$ is always hyperbolic, hence by Proposition 2.116 the involution ι is always non-symplectic.

A period map can be induced by restriction on $\mathcal{M}_{M, \rho}^n$ of the period map of \mathcal{M}_Λ^0 , so that we obtain again a holomorphic map whose image is contained in $\Omega_{M^\perp} = \{[x] \in \mathbb{P}(M^\perp \otimes \mathbb{C}) \mid x^2 = 0, (x, \bar{x}) > 0\}$, since the symplectic 2-form ω is always orthogonal to the Néron-Severi lattice and hence to the invariant lattice T_ι too, see Proposition 2.116. It turns out, see [70, Proposition 7.8], that the image of the period map

$$F_{M, \rho}^n : \mathcal{M}_{M, \rho}^n \rightarrow \Omega_{M^\perp} \tag{2.5}$$

is $\Omega_{M^\perp}^0 = \Omega_{M^\perp} - \bigcup_{s \in \Delta(M^\perp)} (s^\perp \cap \Omega_{M^\perp})$, where $\Delta(M^\perp)$ is the set of wall divisors of M^\perp which we defined in Proposition 2.93.

Definition 2.141. Consider X a hyperkähler manifold and $\iota \in \text{Aut}(X)$ a non-symplectic involution on it. We say that (X, ι) is of type M if there is a marking $\eta : H^2(X, \mathbb{Z}) \rightarrow \Lambda$ on X such that (X, η, ι) is a (M, ρ) -polarized manifold. Two hyperkähler manifolds (X_i, ι_i) , for $i = 1, 2$, of type M are isomorphic if there exists an isomorphism $f : X_1 \rightarrow X_2$ such that $\iota_2 \circ f = f \circ \iota_1$.

Clearly we can associate a pair of type M to a (M, ρ) -polarized hyperkähler manifold, and for every (X, ι) of type M there exists a (M, ρ) -polarized hyperkähler manifold whose associated pair of type M is (X, ι) . We analyze in detail the relation between Definition 2.141 and Definition 2.140.

If $(X_h, \eta_h, \iota_h) \in \mathcal{M}_{M, \rho}^n$ for $h = 1, 2$ are (M, ρ) -polarized manifolds that are isomorphic, the corresponding pairs (X_h, ι_h) of type M are isomorphic too.

Vice versa, consider (X_1, ι_1) and (X_2, ι_2) isomorphic of type M ; we study the group of monodromy operators induced by isomorphisms $f : X_1 \xrightarrow{\sim} X_2$ between them. Consider two markings such that (X_1, η_1, ι_1) and (X_2, η_2, ι_2) are (M, ρ) -polarized manifolds. We have then a monodromy operator $\eta_1 \circ f \circ \eta_2^{-1} : \Lambda \rightarrow \Lambda$ with $(\eta_1 \circ f \circ \eta_2^{-1})(M) = M$: two choices of marking on (X, ι) of type M differ by an element of the (non-trivial) group

$$\text{Mon}(\Lambda, M) = \{f \in \text{Mon}(\Lambda) \mid f(M) = M\} = \{f \in \text{Mon}(\Lambda) \mid f \circ \rho = \rho \circ f\}. \quad (2.6)$$

In this spirit we consider the following space.

Definition 2.142. We denote by \mathcal{M}_M the quotient $\mathcal{M}_{M, \rho}^n / \text{Mon}(\Lambda, M)$.

The action of $\text{Mon}(\Lambda, M)$ on $\mathcal{M}_{M, \rho}^n$ identifies different (M, ρ) -polarized manifolds which are isomorphic as pairs of type M . Denote by Γ_{M^\perp} the restriction of $\text{Mon}(\Lambda, M)$ to M^\perp : the group Γ_{M^\perp} acts on $\Delta(M^\perp)$ since $\text{Mon}(\Lambda, M)$ preserves M^\perp and a monodromy operator sends wall divisors in wall divisors. The period map is equivariant with respect to the action of $\text{Mon}(\Lambda, M)$. Hence, by definition of \mathcal{M}_M , the period map (2.5) induces a surjective period map

$$P_M : \mathcal{M}_M \rightarrow \Omega_{M^\perp}^0 / \Gamma_{M^\perp}. \quad (2.7)$$

By Remark 2.137, a small deformation of a pair (X, ι) of type M (resp. a (M, ρ) -polarized manifold (X, η, ι)) along the subvariety $D(\iota)$ is again a pair of type M (resp. a (M, ρ) -polarized manifold). The family $\mathcal{X}_{D(\iota)} \rightarrow D(\iota)$ is complete between the families of pairs of type M and (M, ρ) -polarized manifolds.

Remark 2.143. Two (M, ρ) -polarized manifolds are deformation equivalent if and only if the associated pairs of type M are deformation equivalent.

Fix $n \geq 2$, an admissible sublattice $M \subset \Lambda_{K3^{[n]}}$ and a connected component \mathcal{M}_Λ^0 of the moduli space of marked hyperkähler manifolds of $K3^{[n]}$ -type. Joumaah in [70] gives a lattice-theoretic condition for two pairs (X_h, ι_h) with $h = 1, 2$ of type M to be deformation equivalent.

We call *positive cone of M* the cone $\{x \in M_\mathbb{R} \mid x^2 > 0\} \cap \mathcal{C}_X$ and we denote it by \mathcal{C}_M . As we defined in Proposition 2.93, the classes in $\Delta(M)$ are the wall divisors of M ; the *Kähler-type chambers of M* are the connected components of

$$\mathcal{C}_M - \bigcup_{s \in \Delta(M)} s^\perp.$$

We denote by $\text{KT}(M)$ the set of Kähler-type chambers of the lattice M ; the restriction Γ_M of $\text{Mon}(\Lambda_{K3[n]}, M)$ to $M_{\mathbb{R}}$ acts on $\Delta(M)$, hence on $\text{KT}(M)$. Finally, the *invariant Kähler cone* of (X, ι) of type M is the intersection $\mathcal{K}_X^\iota = \mathcal{K}_X \cap (T_\iota \otimes \mathbb{R})$ of the Kähler cone with the real extension of the invariant lattice T_ι .

Theorem 2.144. *Let $M \subset \Lambda_{K3[n]}$ be an admissible sublattice. There is a one-to-one correspondence between equivalence classes of deformation types of pairs of type M and the set $\text{KT}(M)/\Gamma_M$. Equivalently, there is a one-to-one correspondence between equivalence classes of deformation types of (M, ρ) -polarized manifolds and the set $\text{KT}(M)/\Gamma_M$.*

Proof. See [70, Theorem 9.11] and Remark 2.143. The one-to-one correspondence $\mathcal{M}_M \rightarrow \text{KT}(M)/\Gamma_M$ is induced in this way: for (X, ι) of type M , consider any marking η such that (X, η, ι) is a (M, ρ) -polarized manifold. Then (X, ι) is associated to the unique Kähler-type chamber of M that contains the image of the invariant Kähler cone $\eta(\mathcal{K}_X^\iota)$. \square

It is tempting to consider \mathcal{M}_M as moduli space of pairs of type M ; as Joumaaah pointed out, some attention is needed. Fix a class $[\mathcal{K}] \in \text{KT}(M)/\Gamma_M$, and call $\mathcal{M}_{M, \mathcal{K}} \subseteq \mathcal{M}_M$ the locus of pairs of type M whose deformation type is associated to $[\mathcal{K}]$ through Theorem 2.144. We will also say that (X, ι) has deformation type $[\mathcal{K}]$. Moreover, we say that $(X, \iota) \in \mathcal{M}_{M, \mathcal{K}}$ is *simple* if the pullback of the Kähler-type chamber \mathcal{K} is exactly the invariant Kähler cone i.e. $\eta^{-1}(\mathcal{K}) = \mathcal{K}_X^\iota$.

Definition 2.145. *We denote by $\mathcal{M}_{M, \mathcal{K}}^0 \subseteq \mathcal{M}_{M, \mathcal{K}}$ the locus of simple pairs of type M of deformation type $[\mathcal{K}]$.*

Remark 2.146. Joumaaah, in [70, Definition 7.1], called *stable invariant Kähler cone* the pullback $\eta^{-1}(\mathcal{K})$. Elements inside the stable invariant Kähler cone correspond to classes that become ample on a general deformation of (X, ι) , see [70, Proposition 9.4]. If (X, ι) is not simple, there is a birational model X' of X on which ι deform to an involution ι' which is still biregular; the pairs $(X, \iota), (X', \iota')$ belongs to the same $\mathcal{M}_{M, \mathcal{K}}$ and they have same period by the Global Torelli theorem for marked varieties.

Now we consider $\Omega_{M^\perp}^+$ the connected component of Ω_{M^\perp} individuated by \mathcal{M}_Λ^0 and \mathcal{K} . Inside $\text{Mon}(\Lambda, M)$ there is the group of monodromy operators

$$\{f \in \text{Mon}(\Lambda, M) \mid f(\mathcal{K}) = \mathcal{K}\};$$

we denote by $\Gamma_{M^\perp, \mathcal{K}}$ the restriction of this group to M^\perp ; different choices of $[\mathcal{K}]$ may give rise to different groups $\Gamma_{M^\perp, \mathcal{K}}$, see [70, Example 10.9]. By [4, Theorem 10.11] the quotient $\Omega_{M^\perp}^+/\Gamma_{M^\perp, \mathcal{K}}$ is a quasi-projective variety. Joumaaah proved the following.

Theorem 2.147. *The period map*

$$\mathcal{M}_{M, \mathcal{K}}^0 \rightarrow \Omega_{M^\perp}^+/\Gamma_{M^\perp, \mathcal{K}}$$

is injective and surjective. In particular $\mathcal{M}_{M, \mathcal{K}}^0$ is quasi-projective and it is the coarse moduli space of simple pairs of type M of deformation type $[\mathcal{K}]$.

Proof. See [70, Theorem 10.5]. □

Remark 2.148. In this thesis we are mostly interested in the case of rank-one lattices M . In this setting the construction above is much more simple: if M has rank one the set of Kähler-type chambers of M modulo isomorphism, $KT(M)$, has only one element. This means, by Theorem 2.144, that all pairs in \mathcal{M}_M are deformation equivalent as manifolds with an involution.

Corollary 2.149. *When M has rank one, \mathcal{M}_M is the coarse moduli space of pairs of type M .*

Proof. As we remarked above, in this case there is only one possible deformation type $[\mathcal{K}]$, where \mathcal{K} is the unique element of $KT(M)$. Moreover, every pair is simple, so $\mathcal{M}_{M,\mathcal{K}}^0$ is the whole \mathcal{M}_M . The result then follows by Theorem 2.147. □

Admissible sublattices has been completely classified by Boissière, Camere and Sarti in dimension four in [14, Proposition 8.2 and figures 1,2], where they verified that for every admissible sublattice M the moduli spaces $\mathcal{M}_{M,\rho}^2$ is not empty, see [14, Proposition 8.5]. The classification have been generalized for every even dimension by Joumaah, see [69, Proposition 5.1.1], with some inaccuracies which were then fixed by Camere, An. Cattaneo and Al. Cattaneo in [21, Proposition 1.1 and 2.1].

On the other hand, given a lattice M with the associated involution ρ , one could ask whether the corresponding moduli spaces are non-empty. Certainly, the involution ρ should be a monodromy operator, so in particular we should have $\pm\rho \in \tilde{O}(\Lambda)$. Camere, An. Cattaneo and Al. Cattaneo showed that for $\text{rk}(M) \leq 20$ this is in fact a sufficient condition in [21, Theorem 2.3], which is the following.

Theorem 2.150. *For $n \geq 2$, let $M \subset \Lambda_{K3^{[n]}}$ be an admissible sublattice with $\text{rk}(M) \leq 20$ and let ρ be the associated involution. If $\pm\rho \in \tilde{O}(\Lambda_{K3^{[n]}})$ then $\mathcal{M}_{M,\rho}^n$ and \mathcal{M}_M are non-empty.*

Proof. Since M has rank smaller than 20, by Corollary 2.45 we can consider (X, η) a marked manifold of $K3^{[n]}$ -type such that $\text{NS}(X) = \eta^{-1}(M)$: we want to prove that (X, η) is a (M, ρ) -polarized manifold. We can induce an involution $\phi = \eta^{-1} \circ \rho \circ \eta$ on $H^2(X, \mathbb{Z})$. The invariant lattice of ϕ is $\text{NS}(X)$ by construction, thus ϕ fixes an ample class on X since M is hyperbolic. In particular ϕ is orientation preserving, which implies in turn that ϕ is a monodromy operator by Proposition 2.51. Moreover any symplectic form ω lies in the orthogonal complement of the invariant lattice $\text{NS}(X)$ of ϕ , so ϕ respects Hodge decomposition as it acts on ω as $-\text{id}$. So by Theorem 2.47 there exists a birational involution ι on X such that $\iota^* = \phi$; since ϕ fixes an ample class on X the involution is in fact biregular. By construction ι is the involution that realizes the (M, ρ) -polarization for (X, η, ι) . Clearly \mathcal{M}_M is non-empty if and only if $\mathcal{M}_{M,\rho}^n$ is non-empty. □

Remark 2.151. For $n = 2$ or $n - 1$ a power of a prime number, by Remark 2.55 the moduli space $\mathcal{M}_{M,\rho}^n$ is non-empty for every admissible sublattice M with $\text{rk}(M) \leq 20$. The case $n = 2$ has been covered by Boissière, Camere and Sarti in [14, Proposition 8.5] with in addition a description of every admissible sublattice.

For a future use, we give some sufficient condition for an involution $\rho \in O(\Lambda)$ to have non-empty associated moduli spaces.

Proposition 2.152. *Let $u \in \Lambda_{K3^{[n]}}$ be a primitive element whose divisibility is either u^2 or $u^2/2$. Then $\pm R_u \in \tilde{O}(\Lambda_{K3^{[n]}})$ if and only if $u^2 = \pm 2$, or $u^2 = \pm 2(n-1)$. In particular*

- 1) if $u^2 = \pm 2$ then $R_u \in \tilde{O}(\Lambda)$,
- 2) if $u^2 = \pm 2(n-1)$ then $-R_u \in \tilde{O}(\Lambda)$.

Proof. (Sketch) The condition on divisibility is necessary by Definition 1.22. See [51, Corollary 3.4]; the lattice in the statement is $U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1) \rangle$, but the proof can be adapted without change. For the second part see [51, Proposition 3.1]. \square

2.4.5 Involutions whose invariant lattice has rank one

Consider $n \geq 2$ and an admissible sublattice $M \subset \Lambda_{K3^{[n]}}$. Families of (M, ρ) -polarized manifolds are particularly interesting when M has rank one: as a first reason, if $\mathcal{M}_{M, \rho}^n$ is non-empty, then it has dimension $21 - \text{rk}(M)$, as we can check in Remark 2.137. So the smaller the rank of M , the bigger the associated moduli space. Actually, something more can be said.

Lemma 2.153. *Let M, M' be two admissible sublattices in $\Lambda_{K3^{[n]}}$ whose associated involutions are ρ, ρ' . If $\pm \rho, \pm \rho' \in \tilde{O}(\Lambda_{K3^{[n]}})$, then $\mathcal{M}_{M, \rho}^n$ is in the closure of $\mathcal{M}_{M', \rho'}^n$ if and only if $M \supset M'$.*

Proof. This is [21, Lemma 3.6] in the special case of involutions, as was already pointed out in [21, Remark 3.7]. The condition on involutions ensures the non-emptiness of the moduli spaces, cfr. Theorem 2.150; this condition is not requested in [21] since the involutions in the definition of admissible sublattices are asked to be monodromy operators in that setting. \square

Thus, between moduli spaces of type $\mathcal{M}_{M, \rho}^n$, those whose admissible sublattice has rank one are maximal, with respect to the relation “... is in the closure of ...”.

Here we present a necessary and sufficient condition for a divisor on a manifold of $K3^{[n]}$ -type to be the generator of the invariant lattice of a non-symplectic involution on X . Recall that, in the case of involutions, the action in cohomology is uniquely determined by the invariant and coinvariant lattices.

Theorem 2.154. *Let X be a hyperkähler manifold of $K3^{[n]}$ -type. It admits a biregular non-symplectic involution ι whose invariant lattice has rank one if and only if one of the followings hold:*

- i) X admits a polarization D whose square is 2 and divisibility $\gamma = 1$. In this case ι^* acts as $-\text{id}$ on $A_{H^2(X, \mathbb{Z})}$, the invariant lattice is generated by D and the coinvariant lattice is isomorphic to

$$U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2(n-1) \rangle \oplus \langle -2 \rangle;$$

ii) $n \equiv 0 \pmod{4}$ and X admits a polarization D whose square is 2 and divisibility $\gamma = 2$. In this case ι^* acts as $-\text{id}$ on $A_{H^2(X, \mathbb{Z})}$, the invariant lattice is generated by D and the coinvariant lattice is isomorphic to

$$U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle \oplus \begin{bmatrix} -\frac{n}{2} & n-1 \\ n-1 & -2(n-1) \end{bmatrix};$$

iii) -1 is a quadratic residue modulo $n-1$ and X admits a polarization D whose square is $2(n-1)$ and divisibility $\gamma = n-1$. In this case ι^* acts as id on $A_{H^2(X, \mathbb{Z})}$, the invariant lattice is generated by D and the coinvariant lattice is isomorphic to

$$U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle \oplus \langle -2 \rangle.$$

Proof. The existence of an involution, if a divisor exists which satisfies one of the conditions in the statement, is [22, Proposition 5.3]. The rest of the statement is [21, Proposition 3.1]. \square

Remark 2.155. If the involution $\iota \in \text{Aut}(X)$ exists, the induced monodromy operator on the second cohomology group is completely determined by the class of the polarization D . In particular $\iota^* = -R_D$, with

$$\begin{aligned} -R_D : H^2(X, \mathbb{Z}) &\rightarrow H^2(X, \mathbb{Z}) \\ v &\mapsto \frac{(v, D)}{D^2} \cdot D - v, \end{aligned}$$

the opposite of the reflection in the orthogonal of $\mathbb{Z}D$, and whose invariant is $\mathbb{Z}D$. In the three cases the isometry is defined a priori over $H^2(X, \mathbb{C})$ and it restricts to an isometry of $H^2(X, \mathbb{Z})$, cfr. Definition 1.22. In fact, the non-symplecticity of ι comes from the fact that D and any two-form ω are orthogonal.

In general, given a Hodge monodromy operator in the form $-R_u$ for some $u \in \text{Pos}(X)$, it fixes only the exceptional (resp. Kähler-type) chamber in which u lies, so there exists $\iota \in \text{Bir}(X)$ (resp. $\iota \in \text{Aut}(X)$) such that $\iota^* = -R_u$ if and only if $u \in \text{Mov}(X)$ (resp. $\mathcal{A}(X)$). This follows directly from Hodge-theoretic Torelli theorem.

As a second reason to study non-symplectic involutions whose invariant lattice has rank one, consider some $\langle 2t \rangle$ -polarized $K3$ surface S with $t \geq 2$: we show that *if S is very general inside the moduli space \mathcal{K}_{2t} , the automorphism group $\text{Aut}(S^{[n]})$ of $S^{[n]}$ is generated by at most one non-symplectic non-natural involution whose invariant lattice has rank one.* Boissière, An. Cattaneo, Nieper-Wisskirchen and Sarti pointed it out in dimension four in [16] and Al. Cattaneo generalized it to any even dimension in [22]. We will prove it at Corollary 2.158 after some preliminar results.

Consider the Hilbert scheme $S^{[n]}$ for some projective $K3$ surface S and $n \geq 2$. We already defined the injective maps

$$\psi_{S^{[n]}}^A : \text{Aut}(S^{[n]}) \rightarrow O(H^2(S^{[n]}, \mathbb{Z})), \quad \psi_{S^{[n]}}^B : \text{Bir}(S^{[n]}) \rightarrow O(H^2(S^{[n]}, \mathbb{Z})),$$

see Theorem 2.108. We consider an automorphism $\alpha \in \text{Bir}(S^{[n]})$; as the induced monodromy operator α^* acts on the Néron-Severi group of $S^{[n]}$, there is a second pair of maps

$$\overline{\psi_{S^{[n]}}^A} : \text{Aut}(S^{[n]}) \rightarrow O(\text{NS}(S^{[n]})), \quad \overline{\psi_{S^{[n]}}^B} : \text{Bir}(S^{[n]}) \rightarrow O(\text{NS}(S^{[n]}))$$

given by the restriction of α^* to the Néron-Severi lattice of $S^{[n]}$; the kernel of $\overline{\psi_{S^{[n]}}^A}$ is contained in the subgroup of natural automorphisms of $S^{[n]}$ by Theorem 2.110, since a birational morphism acting trivially on $\text{NS}(S^{[n]})$ sends the exceptional divisor E to itself. As in the proof of Theorem 2.108, the kernel of $\overline{\psi_{S^{[n]}}^B}$ is contained in the kernel of $\overline{\psi_{S^{[n]}}^A}$.

Consider now (S, L) such that $\text{Pic}(S) = \mathbb{Z}L$ with $L^2 = 2t$ for $t \geq 1$. This is the form of a very general element in the moduli space of $\langle 2t \rangle$ -polarized K3 surfaces \mathcal{K}_{2t} . In this case $\text{Pic}(S^{[n]}) = i(\text{Pic}(S)) \oplus \mathbb{Z}\delta$ has rank two and it is isomorphic, as a lattice, to $\langle 2t \rangle \oplus \langle -2(n-1) \rangle$.

Lemma 2.156. *Let S be a projective K3 surface such that $\text{Pic}(S) = \mathbb{Z}L$ with $L^2 = 2t$, for $t \geq 1$. For any $n \geq 2$, the kernel of both $\overline{\psi_{S^{[n]}}^A}$ and $\overline{\psi_{S^{[n]}}^B}$ is the subgroup of natural automorphisms on $S^{[n]}$. In particular*

- if $t \geq 2$ both maps are injective,
- if $t = 1$ the kernel of both maps is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

Proof. Let ψ be one of the two maps in the statement. We know that the kernel of ψ sits in the subgroup of natural automorphisms. Consider now $\alpha \in \text{Aut}(S)$ and the associated natural automorphism $\alpha^{[n]} \in \text{Aut}(S^{[n]})$. Since the ample cone of S is generated by L , the latter has to be fixed by α , hence $\alpha^{[n]}$ acts as the identity on both the generators of $\text{NS}(S^{[n]})$, by Theorem 2.110 and the commutativity of (2.3). So $\alpha \in \ker(\psi)$. The rest of the statement follows from Theorem 2.119. \square

If $S^{[n]}$ admits a non-natural biregular automorphism α , it has to be non-symplectic by Proposition 2.118, hence its invariant lattice would be contained in $\text{NS}(S^{[n]})$ and it would have rank one by Lemma 2.156.

Moreover, for $\alpha \in \text{Bir}(S^{[n]})$ the isometry $\overline{\psi_{S^{[n]}}^B}(\alpha) = \alpha_{\text{NS}(S^{[n]})}^*$ should send the movable cone of $S^{[n]}$ to itself by Remark 2.96; if it is biregular, it has moreover to send the ample cone of $S^{[n]}$ to itself. In the following proposition we state some necessary conditions to have a birational involution on $S^{[n]}$. This is [22, Proposition 6.1] for the case of biregular automorphisms, but the proof is the same.

Proposition 2.157. *Let S be a projective K3 surface such that $\text{Pic}(S) = \mathbb{Z}L$ with $L^2 = 2t$, for $t \geq 1$. For $n \geq 2$, let (z, w) be the minimal solution of $X^2 - t(n-1)Y^2 = 1$ with $X \equiv \pm 1 \pmod{n-1}$. If $S^{[n]}$ admits a non-natural birational automorphism, then*

- a) $t(n-1)$ is not a square and $(n-1)X^2 - tY^2 = 1$ has no solution for $n \neq 2$;
- b) $z \equiv \pm 1 \pmod{2(n-1)}$ and $w \equiv 0 \pmod{2}$;
- c) the closure of the movable cone of $S^{[n]}$ is $\overline{\text{Mov}(S^{[n]})} = \langle L_n, zL_n - tw\delta \rangle_{\mathbb{R} \geq 0}$;

d) the birational automorphism is a non-symplectic involution whose action on $\text{NS}(S^{[n]})$ on the base $\{L_n, -\delta\}$ is given by

$$\begin{pmatrix} z & -(n-1)w \\ tw & -z \end{pmatrix} \quad (2.8)$$

which fixes the line spanned by $(n-1)wL_n - (z-1)\delta$.

Proof. Call σ the non-natural birational automorphism. The vector space $\text{NS}(S^{[n]})$ is two dimensional and it is spanned by the two walls that delimit the movable cone, hence an orientation-preserving isometry of $\text{NS}(S^{[n]})$ that fixes them has to be the identity. The birational automorphism σ is non-natural, so $\sigma^*_{|\text{NS}(S^{[n]})}$ has to exchange the two walls; in particular two primitive generators of the walls should have the same square. This can happen only in the third case in Theorem 2.101 except when $n = 2$, for which the second and the third case are the same. This proves a) and c).

To prove b) we use the description of walls given in Theorem 2.101 for the wall of the movable cone generated by $zL_n - tw\delta$ using [22, Lemma 2.5] with $\alpha = 1$.

Finally d) is obtained by imposing $\sigma^*_{|\text{NS}(S^{[n]})}(L_n) = zL_n - tw\delta$ and $\sigma^*_{|\text{NS}(S^{[n]})}(zL_n - tw\delta) = L_n$. \square

Note that by Proposition 2.157, Item d), the involution acts on the set of Kähler-type chambers by reversing their order: a chamber of $\overline{\text{Mov}(S^{[n]})}$ is preserved by the action if and only if it is the central one. So the generator of the invariant lattice $(n-1)wL_n - (z-1)\delta$ lies in the central chamber if the number of chamber is odd, on a wall between two chambers otherwise. Since the ample chamber is the one delimited on one side by L_n , to have a biregular involution we need $\mathcal{A}(S^{[n]}) = \text{Mov}(S^{[n]})$.

Corollary 2.158. *Let S be a projective K3 surface such that $\text{Pic}(S) = \mathbb{Z}L$ with $L^2 = 2t$, for $t \geq 2$. The birational automorphism group $\text{Bir}(S^{[n]})$, hence the biregular automorphism group $\text{Aut}(S^{[n]})$ too, is generated by at most one non-symplectic non-natural involution whose invariant lattice has rank one.*

Proof. The map $\overline{\psi_{S^{[n]}}^B}$ is injective, see Lemma 2.156. It is then sufficient to observe that the involution in Proposition 2.157, Item d) is completely determined by n , t and by z , w , which in turn are completely determined by n , t . \square

A lower bound on t has been obtained by Al. Cattaneo for the existence of biregular involutions on $S^{[n]}$, again for S very general.

Proposition 2.159. *Let S be a projective K3 surface such that $\text{Pic}(S) = \mathbb{Z}L$ with $L^2 = 2t$, for $t \geq 2$. If $t \leq 2n - 3$ every birational involution $\iota \in \text{Bir}(S^{[n]})$ is not biregular.*

Proof. See [22, Proposition 1.1] for the case $t \geq 2$. The result is obtained by observing that the description of the walls of the ample cone, obtained from the matricial description of the isometry ι^* , does not match with the explicit description provided by Bayer and Macrì in [6, Proposition 10.3] for $t \leq 2n - 3$. \square

Remark 2.160. For $t = 1$ the group $\text{Aut}(S^{[n]})$ is always generated by the natural involution coming from S . A priori we could have another non-natural involution as described in Proposition 2.157, Item d), let say ι and call $\tau^{[n]}$ the natural non-symplectic involution. In this case either ι or $\iota \circ \tau^{[n]}$ would be symplectic, which is impossible by Proposition 2.118. However, this does not rule out the existence of a birational involution on $S^{[n]}$: we will show indeed that this situation can occur in Section 6.2. The problem of calculating the group of biregular automorphisms $\text{Aut}(S^{[n]})$ of $S^{[n]}$ when S has Picard rank one has been solved by Boissière, An. Cattaneo, Nieper-Wisskirchen and Sarti in dimension four, see [16]; Al. Cattaneo extended the result to any dimension in [22]. With these results, the problem becomes completely numerical on t and n .

Theorem 2.161. *Let (S, L) be a $\langle 2t \rangle$ -polarized K3 surface such that $\text{Pic}(S) = \mathbb{Z}L$, $t \geq 2$. Let (z, w) be the minimal solution of $X^2 - t(n-1)Y^2 = 1$ with $X \equiv \pm 1 \pmod{n-1}$. Then $\text{Aut}(S^{[n]}) \neq \{\text{id}\}$ if and only if the followings hold:*

- i) $t(n-1)$ is not a square;*
- ii) if $n \neq 2$, $(n-1)X^2 - tY^2 = 1$ has no integer solutions;*

iii) for all integers ρ, α as follows:

- 1) $\rho = 1$ and $1 \leq \alpha \leq n-1$, or*
- 2) $\rho = 0$ and $3 \leq \alpha \leq n-1$, or*
- 3) $1 \leq \rho \leq \frac{n-1}{4}$ and $\max\{4\rho + 1, \lceil 2\sqrt{\rho(n-1)} \rceil\} \leq \alpha \leq n-1$,*

the minimal solution (X, Y) of

$$X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$$

with $X \equiv \pm \alpha \pmod{2(n-1)}$, if it exists, is such that $\frac{Y}{X} \geq \frac{w}{2z}$;

iv) one of the following holds:

- 1) $(n-1)X^2 - tY^2 = -1$ has integer solutions;*
- 2) $X^2 - t(n-1)Y^2 = -1$ has integer solutions.*

In this case $\text{Aut}(S^{[n]})$ is generated by a non-natural non-symplectic involution, whose invariant lattice is generated by $bL_n - a\delta$, where

- (a, b) is the minimal solution of $(n-1)X^2 - tY^2 = -1$ if Item 1) of *iv)* holds,*
- $(a, \frac{b}{n-1})$ is the minimal solution of $X^2 - t(n-1)Y^2 = -1$ if Item 2) of *iv)* holds.*

The two cases are mutually exclusive for $n \geq 3$.

Proof. See [22, Theorem 1.3]. The description of the invariant lattice is given explicitly in the proof of the theorem. To show that at most one between *iv)*, 1) and *iv)*, 2) holds for fixed n, t see Lemma 1.6. \square

Remark 2.162. Theorem 2.161 is a generalization of the case $n = 2$, proved by Boissière, An. Cattaneo, Nieper-Wisskirchen and Sarti. In this setting only $\langle 2 \rangle$ -polarizations provide involutions on the Hilbert scheme $S^{[2]}$, cfr. Theorem 2.154. Condition $c)$ of Theorem 2.161 needs to be verified only for $\rho = -1$, $\alpha = 1$, so that the associated equation is $X^2 - 4tY^2 = 5$; everytime it admits a solution, the minimal solution (X, Y) satisfies $X \equiv \pm\alpha \pmod{2}$ and $\frac{Y}{X} < \frac{w}{2z}$, see [22, Remark 2.4]. The conditions in Theorem 2.161 thus read, for $t \geq 2$,

- i) t is not a square,
- ii) $X^2 - 4tY^2 = 5$ has no solution,
- iii) $X^2 - tY^2 = -1$ admits a solution.

The invariant lattice is then generated by $bL_2 - a\delta$, where (a, b) is the minimal solution of $X^2 - tY^2 = -1$. The result in this form is [16, Theorem 1.1]. Note also that condition $iii)$ implies that $t = \frac{X^2+1}{Y^2}$ is never a square, so condition $i)$ is automatically satisfied once condition $iii)$ holds.

Theorem 2.161 allows, for any $n \geq 2$, to find a family of polarizations t such that the Hilbert scheme of n points on the very general element of \mathcal{K}_{2t} admits a non-natural non-symplectic biregular involution.

Proposition 2.163. *Let $n \geq 2$ and $t = (n-1)k^2 + 1$ for $k \geq \frac{n+3}{2}$. For S a projective K3 surface with $\text{Pic}(S) = \mathbb{Z}L$ with $L^2 = 2t$, there exists a non-natural non-symplectic involution $\iota \in \text{Aut}(S^{[n]})$ with $T_\iota = \langle 2 \rangle$.*

Proof. See [22, Proposition 6.7]. □

We will provide, in Proposition 6.6, families of polarizations for which condition $iv), 2)$ of Theorem 2.161 holds, for every n such that -1 is a quadratic residue modulo $n-1$; the latter is a necessary condition to have such polarizations, as we can immediately check by taking the equation $X^2 - t(n-1)Y^2 = -1$ modulo $n-1$. We will talk, in Chapter 6, about the case of birational involutions on the Hilbert scheme of n points on very general elements of \mathcal{K}_{2t} .

We end this survey with some examples of geometric descriptions of involutions whose invariant lattice has rank one on some Hilbert scheme of n points on K3 surfaces. These are mostly birational automorphisms.

A geometric description that holds for every dimension on a Hilbert scheme of n points on some K3 surface is given by Beauville, see [8, Section 6] for details.

Example 2.164. *Let (Σ, Γ) be a $\langle 2n \rangle$ -polarized K3 surface such that Γ is very ample, and call $\phi_{|\Gamma|} : \Sigma \rightarrow \mathbb{P}^{n+1}$ the associated map. From now on, we consider Σ as a surface of degree $2n$ in \mathbb{P}^{n+1} through $\phi_{|\Gamma|}$.*

We can geometrically describe a birational involution ι on $\Sigma^{[n]}$ in the following way. A general point of $\Sigma^{[n]}$ corresponds to n points p_1, \dots, p_n in general position on Σ : the linear space generated by p_1, \dots, p_n intersects Σ in n other points, say q_1, \dots, q_n . We set $q_1 + \dots + q_n$ to be the image of $p_1 + \dots + p_n$ through ι . If $n \geq 3$, we call ι a Beauville's involution. By [8, Section 6], the birational

involution extends to an involution on the whole space if and only if $n = 2$ and $\Sigma \subset \mathbb{P}^3$ contains no line. If $n = 2$ we call this extension Beauville's involution. In this case ι is non-natural, as we can find two distinct points in $\Sigma \subset \mathbb{P}^3$ such that the line spanned by them intersects Σ only in a single point with multiplicity two.

The Beauville's involution is non-symplectic, its invariant lattice has rank one and it is generated by $\Gamma_n - \delta$, whose square is 2. Vice versa, consider $\Sigma^{[n]}$ some Hilbert scheme of n points on a $K3$ surface Σ and D a divisor on it with $D^2 = 2$. If D can be written as $\Gamma_n - \delta$, with $\Gamma \in \text{Pic}(\Sigma)$ very ample, then the involution associated to D is a Beauville's involution.

Consider the case $n = 2$, $t = 5$; let (Σ, Γ) be a $\langle 10 \rangle$ -polarized $K3$ surface which is Brill-Noether general, see Definition 2.33, and call $\phi_{|\Gamma|} : \Sigma \rightarrow \mathbb{P}^6$ the closed embedding associated to Γ . From now on, by Theorem 2.34 we can consider Σ as the intersection of the Grassmannian $G(2, \mathbb{C}^5) \subset \mathbb{P}^9$, a linear subspace $\mathbb{P}^6 \subset \mathbb{P}^9$ and a quadric hypersurface Q in \mathbb{P}^6 , so that $\Sigma = \mathbb{P}^6 \cap G(2, V_5) \cap Q$.

For $p \in G(2, \mathbb{C}^5)$ we denote by l_p the line in $\mathbb{P}(\mathbb{C}^5)$ that corresponds to the point p . We call *Grassmannian hull* of Σ the Fano three-fold $M_\Sigma = G(2, \mathbb{C}^5) \cap \mathbb{P}^6$ and we say that Σ is *strongly smooth* if M_Σ is smooth. Some theory about Grassmannian hulls can be found at Section 4.4. So Σ is a smooth dimensionally transverse intersection of M_Σ and Q . We will use two facts: for $p, q \in \Sigma$ generic points, the intersection of l_p and l_q is empty; for any general $V_4 \subset V_5$ hyperplane, the restrictions $G(2, V_5) \cap \mathbb{P}(\wedge^2 V_4) \cap \mathbb{P}^6 = G(2, V_4)$ and $Q \cap \mathbb{P}(\wedge^2 V_4)$ are not the same quadric hypersurface. The following construction is presented in [94, Section 4.3], or [66, Section 3.1], for general elements of \mathcal{K}_{10} ; we give moreover a sufficient precise condition for the existence of the involution on (Σ, Γ) .

Example 2.165. *Suppose that $\Sigma \subset \mathbb{P}^6$ is strongly smooth. We can geometrically describe a birational involution ι on $\Sigma^{[2]}$ in the following way. Take two points p, q in general position on Σ : since $l_p \cap l_q = \emptyset$, there is a unique 4-dimensional vector space $V_4 \subset \mathbb{C}^5$ that contains the two lines.*

Consider the intersection $G(2, V_4) \cap \mathbb{P}^6$. By transversality it has degree 2; if it is a surface, its intersection with Σ gives an elliptic pencil of degree 4 in Σ , so that $(\Sigma, \Gamma) \in D_{4,0}$, which is absurd by Proposition 5.12. Hence $G(2, V_4) \cap \mathbb{P}^6$ is a conic, which we call c ; by definition of V_4 , the conic c contains p and q . Now we have $c \cap \Sigma = c \cap Q = \{p, q, p', q'\}$, and we define $p' + q'$ as the image of $p + q$ through ι . We call ι a O'Grady's involution.

Remark 2.166. A priori we have two involutions on $\Sigma^{[2]}$, the one described geometrically above and the one whose action in cohomology is $-R_D$ with $D = \Gamma_2 - 2\delta$. However, as we can see in Section 5.3, the involution comes from the covering involution of a *EPW* sextic, in particular by Corollary 5.20 the invariant lattice is generated exactly by D . By Torelli theorem this means that the two involutions are in fact the same.

So the involution we described is anti-symplectic, its invariant lattice has rank one and it is generated by $\Gamma_2 - 2\delta$. This involution is never biregular, and $\Sigma^{[2]}$ admits no birational model on which the involution is biregular, since the movable cone $\overline{\text{Mov}}(\Sigma^{[2]})$ has exactly two chambers, one spanned by Γ_2 and $\Gamma_2 - 2\delta$, the other by $\Gamma_2 - 2\delta$ and $9L_2 - 20\delta$, as one can verify by Theorem 2.101 and Proposition 2.102 for $n = 2$, $t = 5$.

Remark 2.167. The sufficient conditions we gave on Σ can be rephrased as conditions on its Picard lattice. By Proposition 2.83 and Proposition 5.12 (which we will present later) the construction above holds everytime $(\Sigma, \Gamma) \notin D_{h,0}$ for $h = 1, \dots, 4$. By the previous remark, the action in cohomology is then $-R_D$, where $D = L_2 - 2\delta$.

We will give geometrical descriptions for some other cases in Section 6.6, always for involutions which are not biregular. In Chapter 3 we give a description, although through a deformation, of the involutions whose existence is proved in Proposition 2.163.

Chapter 3

Geometrical descriptions through deformation

3.1 Introduction

The Torelli theorem for hyperkähler manifolds of $K3^{[n]}$ -type allows to treat a lot of problems concerning automorphisms using tools from lattice theory or number theory. Unfortunately, this gives no insight on the geometrical point of view.

In Section 2.4.5 we presented the results of Boissière, An. Cattaneo, Nieper-Wisskirchen and Sarti, and then Al. Cattaneo, which completely described in numerical terms biregular involutions on the Hilbert scheme of n points of a $K3$ surface; in dimension four, aside from Beauville's involution, not much other geometrical realizations are known. Some other descriptions exist that do not use the structure of Hilbert scheme of $S^{[n]}$; the one that we will explain in detail in Chapter 4 arises in the case of Hilbert squares of $K3$ surfaces admitting an ample square-two line bundle on them i.e. classes of polarized manifolds in the form $(S^{[2]}, D) \in \mathcal{M}_{2,1}^2$. In this case, for S very general, the manifold $S^{[2]}$ is isomorphic to the double cover of a sextic hyperplane in \mathbb{P}^5 belonging to the family of *EPW* sextics, and the involution is the corresponding covering involution: we will talk extensively about them in Section 4.2.

Proposition 3.1. *Consider $t \geq 3$. For $(S, L) \in \mathcal{K}_{2t}$ general, the automorphism group $\text{Aut}(S^{[2]})$ is non-trivial if and only if $S^{[2]}$ is isomorphic to a smooth double *EPW* sextic.*

Proof. This is [32, Corollary 7.6]. Consider t such that the conditions in Theorem 2.161 hold, and let (a, b) be the minimal solution of $X^2 - tY^2 = -1$. The elements in the form $(S^{[2]}, bL_2 - a\delta)$, for $(S, L) \in \mathcal{K}_{2t}$ such that $bL_2 - a\delta$ is ample, fill an open subset, which we call \mathcal{N} , inside an irreducible component of the Noether-Lefschetz divisor $\mathcal{C}_{2t} \subset \mathcal{M}_{2,1}^2$, see [32, Proposition 7.1]. Let $\mathcal{U}_{EPW} \subset \mathcal{M}_{2,1}^2$ be the subspace of $\langle 2 \rangle$ -polarized manifolds of $K3^{[2]}$ -type which are isomorphic to a double *EPW* sextic, with the polarization which generates the invariant lattice of the covering involution. By Theorem 4.61 we know that \mathcal{U}_{EPW} is open and its intersection is non-empty with

every connected component of \mathcal{C}_e if $e \geq 12$. The first $t \geq 3$ such that the conditions in Theorem 2.161 hold is 10, so $2t \geq 12$. Hence the general element in the form $(S^{[2]}, bL_2 - a\delta)$ lies in $\mathcal{U}_{EPW} \cap \mathcal{N}$. \square

We fix a sublattice $M \subset \Lambda_{K3^{[2]}}$ isomorphic to the rank-one lattice $\langle 2 \rangle$ whose associate involution is ρ . Consider the integer $t = k^2 + 1$ with $k \geq 3$ and a very general element $(S, L) \in \mathcal{K}_{2t}$. By Proposition 2.163 the Hilbert square $S^{[2]}$ admits a (unique) involution, whose invariant lattice is generated by the square-two line bundle $L_2 - k\delta$: we call ι such involution. When k is odd, Boissière, An. Cattaneo, Markushevich and Sarti in [15] used a natural isomorphism between \mathcal{M}_M and $\mathcal{M}_{2,1}^2$, see [15, Corollary 4.1], to give a description of the involution on $S^{[2]}$ which keeps track of the structure of Hilbert scheme on it; this is [15, Theorem 6.1].

In this chapter we generalise this result in two senses.

- In Section 3.2 we generalize [15, Corollary 4.1] and [32, Proposition 7.1] to any dimension, in Section 3.3 we show how to generalize the approach in [15, Section 6] to any dimension and we use it to describe geometrically, up to a certain deformation, the family of polarizations $t = (n-1)k^2 + 1$ for $k \geq \frac{n+3}{2}$, $n \geq 2$, which we presented in Proposition 2.163. The biggest difference from the four-dimensional case is that the moduli spaces in greater dimension are not necessarily connected. These first two sections are part of a joint work with Al. Cattaneo, which appears in Sections 7 and 8 of

On birational transformations of Hilbert schemes of points on K3 surfaces,
<https://arxiv.org/abs/2005.05187>, submitted.

- In Section 3.4 we consider again the four-dimensional case and we extend it by giving other examples of geometric descriptions through a deformation which keeps track of the structure of Hilbert scheme on $S^{[2]}$.

3.2 Isomorphisms between moduli spaces of polarized manifolds of $K3^{[n]}$ -type

From now on we always consider hyperkähler manifolds of $K3^{[n]}$ -type for some n . Consider n, d, γ positive integers with $n \geq 2$ and $\gamma | \gcd(d, 2(n-1))$; we will make use of Apostolov's characterization of polarized hyperkähler manifolds which lie in the same connected component inside the moduli space $\mathcal{M}_{2d, \gamma}^n$, see Definition 2.69. We presented it in Theorem 2.75.

For any $(X, D) \in \mathcal{M}_{2d, \gamma}^n$ we fix ι_X as in Proposition 2.27 a primitive embedding inside the $O(\tilde{\Lambda})$ -orbit of maps to the Mukai lattice $\tilde{\Lambda}$. Following [2], we defined the polarized rank-two lattice $(T(X, D), \iota_X(D))$ associated to (X, D) ; by Proposition 2.74, two polarized manifolds $(X_1, D_1), (X_2, D_2) \in \mathcal{M}_{2d, \gamma}^n$ are in the same connected component of the moduli space if and only if $(T(X_1, D_1), \iota_{X_1}(D_1)) \cong (T(X_2, D_2), \iota_{X_2}(D_2))$ as lattices with a distinguished vector.

Let X be a manifold of $K3^{[n]}$ -type with a non-symplectic biregular involution $\iota \in \text{Aut}(X)$ such that the invariant lattice T_ι has rank one. As we explained in Theorem 2.154, there exists an ample

primitive class $D \in \mathcal{A}(X)$ generating T_t such that (X, D) belongs to one of the following moduli spaces: $\mathcal{M}_{2,1}^n$; $\mathcal{M}_{2,2}^n$ (non-empty if and only if $n \equiv 0 \pmod{4}$); $\mathcal{M}_{2(n-1),n-1}^n$ (non-empty if and only if -1 is a quadratic residue modulo $n-1$).

Proposition 3.2. *Let $n \geq 2$ and for $r \in \mathbb{N}$ denote by $\rho(r)$ the number of distinct prime divisors of r .*

- *The moduli space $\mathcal{M}_{2,1}^n$ is connected; the polarization type is unique.*
- *If $n \equiv 0 \pmod{4}$, then $\mathcal{M}_{2,2}^n$ is connected; the polarization type is unique.*
- *If $n \equiv 0 \pmod{2}$ and -1 is a quadratic residue modulo $n-1$, then $\mathcal{M}_{2(n-1),n-1}^n$ has $2^{\rho(n-1)-1}$ connected components; the polarization type is unique.*
- *If $n \equiv 1 \pmod{4}$ and -1 is a quadratic residue modulo $n-1$, then $\mathcal{M}_{2(n-1),n-1}^n$ has $2^{\rho(\frac{n-1}{4})}$ connected components.*
- *If $n \equiv 3 \pmod{4}$ and -1 is a quadratic residue modulo $n-1$, then $\mathcal{M}_{2(n-1),n-1}^n$ has $2^{\rho(\frac{n-1}{2})-1}$ connected components.*

Proof. The unicity of the polarization type in the first three cases is granted by Lemma 2.63 and Lemma 2.64. The number of connected component is a direct calculation from Proposition 2.76. \square

Given $n, t \geq 2$ we define the property

(*) : for a projective $K3$ surface S with $\text{Pic}(S) = \mathbb{Z}L$, $L^2 = 2t$, the Hilbert scheme $S^{[n]}$ is equipped with a biregular non-natural automorphism.

Equivalently, this means that n, t satisfy the numerical conditions given in Theorem 2.161.

Take n, t which satisfy (*). Let (a, b) (resp. $(a, \frac{b}{n-1})$) be the integer solution of $(n-1)X^2 - tY^2 = -1$ (resp. $X^2 - t(n-1)Y^2 = -1$) with minimal $X > 0$, depending on which of the two equations is solvable (see Proposition 6.1).

Inside the moduli space \mathcal{K}_{2t} of $\langle 2t \rangle$ -polarized $K3$ surfaces, we denote by $U \subset \mathcal{K}_{2t}$ the subset of elements (S, L) such that $D = bL_n - a\delta \in \text{NS}(S^{[n]})$ is ample.

Remark 3.3. The subset U is non-empty, for example by Theorem 2.161 and Theorem 2.154 it contains all $\langle 2t \rangle$ -polarized $K3$ surfaces of Picard rank one.

We denote by $\mathcal{M}(n, t)$ the *connected component of one of the moduli spaces $\mathcal{M}_{2,1}^n$, $\mathcal{M}_{2,2}^n$, $\mathcal{M}_{2(n-1),n-1}^n$ which contains the polarized manifolds $(S^{[n]}, bL_n - a\delta)$ as above, for $(S, L) \in U$.*

For $(S, L) \in U$, the polarized manifold $(S^{[n]}, D)$ is in one of the moduli spaces $\mathcal{M}_{2,1}^n$, $\mathcal{M}_{2,2}^n$ if $(n-1)X^2 - tY^2 = -1$ is solvable, otherwise $(S^{[n]}, D) \in \mathcal{M}_{2(n-1),n-1}^n$.

Assume that $D^2 = 2$. One can readily check that the divisibility of D is $\gcd(b, 2(n-1))$, hence $(S^{[n]}, D) \in \mathcal{M}_{2,1}^n$ if b is odd while $(S^{[n]}, D) \in \mathcal{M}_{2,2}^n$ if b is even. By Proposition 3.2 these moduli spaces are connected.

If $D^2 = 2(n-1)$, we are interested in determining the connected component $\mathcal{M}(n, t)$ of $\mathcal{M}_{2(n-1), n-1}^n$ which contains $(S^{[n]}, D)$. By Remark 2.28 the canonical embedding $\iota_{S^{[n]}}$ in the Mukai lattice satisfies

$$\begin{aligned} \iota_{S^{[n]}} : H^2(S^{[n]}, \mathbb{Z}) &\hookrightarrow \tilde{\Lambda} = H^*(S, \mathbb{Z}) \\ L_n &\mapsto (0, -L, 0) \\ -\delta &\mapsto (1, 0, n-1). \end{aligned}$$

The image of $\iota_{S^{[n]}}$ is the orthogonal complement of $v = (1, 0, 1-n)$. Let $\beta = \frac{b}{n-1} \in \mathbb{N}$. The primitive lattice $T_n^t = T(S^{[n]}, D)$ is generated by v and $w = \frac{1}{n-1}(av - \iota_{S^{[n]}}(D)) = (0, \beta L, -2a)$. With respect to the basis $\{v, w\}$, the distinguished vector in T_n^t is $\iota_{S^{[n]}}(D) = av - (n-1)w$ and the bilinear form of the lattice is given by the matrix

$$\begin{pmatrix} 2(n-1) & 2a \\ 2a & 2t\beta^2 \end{pmatrix}.$$

In the following we assume that n, t satisfy (*).

Lemma 3.4. *For $n \geq 2$, let $t_1, t_2 \geq 2$ be such that $X^2 - t_i(n-1)Y^2 = -1$ is solvable, and let (a_i, β_i) be the minimal solution. Assuming that both t_1, t_2 satisfy (*), the connected components $\mathcal{M}(n, t_1)$, $\mathcal{M}(n, t_2)$ of $\mathcal{M}_{2(n-1), n-1}^n$ coincide if and only if $a_1 \equiv \pm a_2 \pmod{n-1}$.*

Proof. By Proposition 2.74, the connected components coincide if and only if there exists an isometry $\lambda : T_n^{t_1} \rightarrow T_n^{t_2}$ such that $\lambda(a_1 v_1 - (n-1)w_1) = a_2 v_2 - (n-1)w_2$. In particular, $\lambda(w_1) = \pm w_2$ and $\lambda((n-1)w_1) = (n-1)w_2 - (a_2 \pm a_1)v_2$. Such isometry exists if and only if $a_1 \equiv \pm a_2 \pmod{n-1}$. \square

We now consider the map

$$\begin{aligned} {}^n\phi_{2t} : U &\rightarrow \mathcal{M}(n, t) \\ (S, L) &\mapsto (S^{[n]}, bL_n - a\delta). \end{aligned}$$

Clearly the image of ${}^n\phi_{2t}$ is contained in the Noether–Lefschetz locus of $\mathcal{M}(n, t)$. We defined in Section 2.3.2 the Noether–Lefschetz divisor $\mathcal{C}_{2t} \subset \mathcal{M}(n, t)$, as the locus which parametrizes polarized manifolds (X, D) whose Néron–Severi lattice contains a primitive hyperbolic rank-two sublattice $K \ni D$ and such that $K^\perp \subset H^2(X, \mathbb{Z})$ has discriminant $-2t$. The following result generalize [32, Proposition 7.1].

Proposition 3.5. *Given n, t which satisfy (*), the subset $U \subset \mathcal{K}_{2t}$ is open and the rational map ${}^n\phi_{2t} : \mathcal{K}_{2t} \dashrightarrow \mathcal{M}(n, t)$ is birational onto an irreducible component of \mathcal{C}_{2t} .*

Proof. Consider $(S, L) \in \mathcal{K}_{2t}$ such that $\text{Pic}(S) = \mathbb{Z}L$; we know that $(S, L) \in U$. Take $(\mathcal{X}, \mathcal{D}) \rightarrow \text{Def}(S^{[n]}, bL_n - a\delta)$ the infinitesimal deformation of the pair $(S^{[n]}, bL_n - a\delta)$, and U' a small open neighborhood of (S, L) in \mathcal{K}_{2t} such that

$$\left\{ \coprod_{(\Sigma, H) \in U'} (\Sigma^{[n]}, bH_n - a\delta_\Sigma) \right\} \hookrightarrow (\mathcal{X}, \mathcal{D})$$

is a subfamily of deformations of $(S^{[n]}, bL_n - a\delta)$. Since ampleness is an open property (see [52, corollary 9.6.4]), up to shrinking U' we have $U' \subset U$.

To verify that ${}^n\phi_{2t}$ is generically injective consider again (S, L) whose Picard rank is 1, and consider some (Σ, H) such that there exists $\alpha : \Sigma^{[n]} \xrightarrow{\sim} S^{[n]}$ which does not come from an isomorphism on the $K3$ surfaces. The movable cone of $S^{[n]}$, resp. $\Sigma^{[n]}$, is the convex hull of L_n , resp. H_n , and some other $r_S \in \text{NS}(S^{[n]})$, resp. $r_\Sigma \in \text{NS}(\Sigma^{[n]})$; for details see Theorem 2.101. If α^* would send δ_S to δ_Σ , then its restriction $\alpha^*|_{i(H^2(S^{[2]}, \mathbb{Z}))} : i(H^2(S^{[2]}, \mathbb{Z})) \rightarrow i(H^2(\Sigma^{[2]}, \mathbb{Z}))$ would induce a Hodge isometry between the second cohomology groups of S and Σ ; by the Global Torelli theorem for $K3$ surfaces the $K3$ surfaces would be isomorphic, also as polarized manifolds since S has Picard rank one.

So we have instead $\alpha(L_n) = r_\Sigma$ and $\alpha(r_S) = H_n$; if we take ι the involution on $S^{[n]}$, we have that $\alpha \circ \iota$ sends L_n on H_n , so δ_S is sent on δ_Σ . Again, this means that (S, L) and (Σ, H) are isomorphic.

To verify that the image of ${}^n\phi_{2t}$ lies in \mathcal{C}_{2t} consider again $(S, L) \in \mathcal{K}_{2t}$ whose Picard rank is one: it is then sufficient to observe that $\text{NS}(S^{[n]})^\perp \cong \text{Tr}(S)$. \square

Consider any $t \geq 2$. For $(S, L) \in \mathcal{K}_{2t}$ such that $\text{Pic}(S) = \mathbb{Z}L$, for every $n \geq 2$ there is at most one choice of a line bundle on $S^{[n]}$ that satisfies conditions in Theorem 2.154, see Corollary 2.158. So, though there are many possible polarizations on $S^{[n]}$, there is only one such that $S^{[n]}$, with this polarization, lies in $\mathcal{M}_{2d, \gamma}^n$ for $(d, \gamma) \in \{(1, 1), (1, 2), (n-1, n-1)\}$, and the connected component of $\mathcal{M}_{2d, \gamma}^n$, in which it lies, is uniquely determined by n and t . We already denoted it by $\mathcal{M}(n, t)$.

Remark 3.6. Consider n such that -1 is a quadratic residue modulo $n-1$. In Proposition 6.6 we will show that there always exists some $t > 2$ such that n, t satisfy $(*)$, and in particular (iv) , 2); moreover for every connected component \mathcal{M} of $\mathcal{M}_{2(n-1), n-1}^n$, we can choose t such that $\mathcal{M}(n, t) = \mathcal{M}$.

As an example, for every $a > 2$ such that $a^2 \equiv -1 \pmod{n-1}$ there exists t such that $(a, 1)$ is the minimal solution of $X^2 - t(n-1)Y^2 = -1$. By Lemma 6.16, we can choose a big enough such that the line bundle $bL_n - a\delta$ is ample on the very general Hilbert scheme of n points on a $K3$ surface $(S, L) \in \mathcal{K}_{2t}$. The number of classes in $\mathbb{Z}/(2(n-1)\mathbb{Z})$ whose square is 1 is twice the number of connected component of $\mathcal{M}_{\gamma, 2d}^n$ (cfr. [50, proof of Proposition 3.6], and [2, Proposition 3.1]). By Lemma 3.4, a_1 and a_2 give rise to t_1 and t_2 such that $\mathcal{M}(n, t_1) = \mathcal{M}(n, t_2)$ if and only if $a_1 \equiv \pm a_2 \pmod{n-1}$. This means that the number of connected components reached in this way is exactly the whole number of connected components of $\mathcal{M}_{2(n-1), n-1}^n$.

In the following proposition we always consider $\Lambda_{K3^{[n]}}$ as the sublattice of $\tilde{\Lambda}$ which is orthogonal to the vector $v = (1, 0, 1-n)$. We will call $[h]_{\tilde{\mathcal{O}}}$ the $\tilde{\mathcal{O}}(\Lambda_{K3^{[n]}})$ -orbit of an element of $\Lambda_{K3^{[n]}} \subset \tilde{\Lambda}$. The following is an extension of Lemma 3.4 to the case of manifolds of $K3^{[n]}$ -type.

Lemma 3.7. *Take $(X_1, D_1), (X_2, D_2) \in \mathcal{M}_{2d, \gamma}^n$, then they lie in the same connected component if and only if $[\iota_{X_1}(D_1)]_{\tilde{\mathcal{O}}} = \pm[\iota_{X_2}(D_2)]_{\tilde{\mathcal{O}}}$.*

Proof. The only non-trivial case is $\gamma = d = n-1$. As we verified above, for any connected component of $\mathcal{M}_{2(n-1), n-1}^n$ there is always a Hilbert scheme of n points on a $K3$ surface that lies, as a polarized variety, in it. So, by Theorem 2.73, for $k = 1, 2$ we can choose $\iota_{X_k} : H^2(X_k, \mathbb{Z}) \rightarrow \tilde{\Lambda}$

such that $\iota_{X_k}(H^2(X_k, \mathbb{Z})) = v^\perp$.

If the two polarized varieties belong to the same component, there is some parallel transport operator $g : H^2(X_1, \mathbb{Z}) \rightarrow H^2(X_2, \mathbb{Z})$ sending D_1 to D_2 . By Theorem 2.73, we can take $\iota_{X_1} = \iota_{X_2} \circ g$, and so $\iota_{X_1}(D_1) = \iota_{X_2}(D_2)$. In particular the two vectors lie in the same $\tilde{O}(\Lambda_{K3^{[n]}})$ -orbit.

Conversely, take $\lambda \in O(\Lambda_{K3^{[n]}})$ such that $\pm \lambda \in \tilde{O}(\Lambda_{K3^{[n]}})$ and $\lambda(h_1) = h_2$, where $h_k = \iota_{X_k}(D_k)$, $k = 1, 2$. We call $M_{\Lambda_{K3^{[n]}}}$ and $M_{\Lambda_n^\perp}$ the subgroups associated to the embeddings of the two lattices in $\tilde{\Lambda}$ (see Section 1.3.3). Since $\Lambda_n^\perp = \mathbb{Z}v \cong \langle 2(n-1) \rangle$, by Theorem 1.34 an isometry of $\Lambda_{K3^{[n]}}$ extends to an isometry of $\tilde{\Lambda}$ if and only if it acts as $\pm \text{id}$ on $M_{\Lambda_{K3^{[n]}}}$. This is the case for λ , so there exists some $\tilde{\lambda} \in O(\tilde{\Lambda})$ such that $\tilde{\lambda}(h_1) = h_2$, $\tilde{\lambda}(v) = \pm v$. Since $T(X_k, h_k)$, for $k = 1, 2$, is the saturated in $\tilde{\Lambda}$ of $\mathbb{Z}h_k \oplus \mathbb{Z}v$, we also have $(T(X_1, D_1), h_1) \cong (T(X_2, D_2), h_2)$. \square

Note that, since $\iota_X : H^2(X, \mathbb{Z}) \rightarrow \Lambda_{K3^{[n]}} \subset \tilde{\Lambda}$ can be chosen as a marking on $(X, D) \in \mathcal{M}_{2d, \gamma}^n$, the polarized manifold (X, D) has polarization type $[h]$ if and only if $[\iota_X(D)] = [h]$ as $O(\Lambda_{K3^{[n]}})$ -orbits of elements of $\Lambda_{K3^{[n]}}$.

For $n \geq 2$, we denote by Λ the lattice $\Lambda_{K3^{[n]}}$ and we fix a connected component \mathcal{M}_Λ^0 of the moduli space of marked hyperkähler manifolds of $K3^{[n]}$ -type \mathcal{M}_Λ . We use results and notations from Section 2.4.4.

Let $\rho \in O(\Lambda)$ be an involution such that $\Lambda^\rho = \mathbb{Z}h$, with h primitive of square $2d > 0$ and divisibility γ for some $(d, \gamma) \in \{(1, 1), (1, 2), (n-1, n-1)\}$. We call M the sublattice $\mathbb{Z}h$. Notice that $\rho = -R_h$, hence the space \mathcal{M}_M , see Definition 2.142, is non-empty by Proposition 2.152 and Theorem 2.150.

Remark 3.8. Since M has rank one, all pairs in \mathcal{M}_M are deformation equivalent as manifolds with an involution and \mathcal{M}_M is the coarse moduli space of pairs of type M , see Remark 2.148 and Corollary 2.149.

The following statement generalizes [15, Corollary 4.1] for $n = 2$. As usual, we denote by $\mathcal{M}_{[h]}$ the moduli space of polarized manifolds of $K3^{[n]}$ -type with polarization type $[h]$, which we defined in Definition 2.62.

Theorem 3.9. *For $(d, \gamma) \in \{(1, 1), (1, 2), (n-1, n-1)\}$, let $\rho \in O(\Lambda)$ be an involution such that Λ^ρ is generated by a primitive element h of square $2d$ and divisibility γ . Then \mathcal{M}_M is isomorphic to a connected component of $\mathcal{M}_{[h]} \subset \mathcal{M}_{2d, \gamma}^n$.*

Proof. By Theorem 2.150 and Proposition 2.152 the moduli space \mathcal{M}_M is non-empty. Let (X, ι) be a pair of type M . By definition, there exists a marking $\eta : H^2(X, \mathbb{Z}) \rightarrow \Lambda$ such that (X, η, ι) is a (M, ρ) -polarized manifold, in particular $\rho = \eta \circ \iota^* \circ \eta^{-1}$. If D is the primitive ample line bundle which generates the invariant lattice $T_\iota = \eta^{-1}(\Lambda^\rho)$, then $(X, D) \in \mathcal{M}_{2d, \gamma}^n$. Note that different choices of η differ by an element of $\text{Mon}(\Lambda, M)$, cfr. (2.6), so the line bundle D is well defined. The image of the map

$$\begin{aligned} \alpha : \mathcal{M}_M &\rightarrow \mathcal{M}_{2d, \gamma}^n \\ (X, \iota) &\mapsto (X, D) \end{aligned}$$

is contained in a connected component $\mathcal{M}^0 \subset \mathcal{M}_{2d,\gamma}^n$. Indeed, if $(X_1, \iota_1), (X_2, \iota_2)$ are pairs of type M , call η_1 and η_2 two markings such that $(X_1, \eta_1, \iota_1), (X_2, \eta_2, \iota_2)$ are (M, ρ) -polarized manifolds. Then $g = \eta_2^{-1} \circ \eta_1$ is a parallel transport operator, because (X_1, η_1) and (X_2, η_2) lie in the same connected component \mathcal{M}_Λ^0 of \mathcal{M}_Λ by definition of \mathcal{M}_M . Moreover, the sublattice T_{ι_1} is sent through g to T_{ι_2} , hence $g(D_1) = D_2$, so g is a polarized parallel transport operator.

The connected component \mathcal{M}^0 is determined by the explicit choice of the isometry ρ . By construction, $\mathcal{M}^0 \subset \mathcal{M}_{[h]}$ with $\Lambda^\rho = \mathbb{Z}h$ as in the statement.

Let now (X, D) be a polarized manifold in \mathcal{M}^0 . By Theorem 2.154 there is a (unique) non-symplectic involution $\iota \in \text{Aut}(X)$ such that $\iota^* = -R_D$. Since (X, D) is in the same connected component of $\alpha((X', \iota'))$ for any $(X', \iota') \in \mathcal{M}_M$, and $\rho = -R_h$, there exists a marking $\eta : H^2(X, \mathbb{Z}) \rightarrow \Lambda$ such that $(X, \eta) \in \mathcal{M}_\Lambda^0$ and $\rho = \eta \circ \iota^* \circ \eta^{-1}$, hence we have a well-defined map

$$\begin{aligned} \mathcal{M}^0 &\rightarrow \mathcal{M}_M \\ (X, D) &\mapsto (X, \iota) \end{aligned}$$

which is the inverse of α . Again, different markings η give rise to non-isomorphic $(X, \eta, \iota) \in \mathcal{M}_{M,\rho}^n$, but all of them are sent to the same class by the quotient $\mathcal{M}_{M,\rho}^n \rightarrow \mathcal{M}_M$ given by Definition 2.142. \square

Remark 3.10. Let $(d, \gamma) \in \{(1, 1), (1, 2), (n-1, n-1)\}$ and fix a connected component $\mathcal{M}^0 \subset \mathcal{M}_{2d,\gamma}^n$. For $(X, D) \in \mathcal{M}^0$ let $\eta : H^2(X, \mathbb{Z}) \rightarrow \Lambda$ be a marking such that $(X, \eta) \in \mathcal{M}_\Lambda^0$ and define $\rho = -R_{\eta(D)} \in O(\Lambda)$ and $M = \Lambda^\rho$. Then it is clear from the proof of the previous proposition that \mathcal{M}^0 is the connected component of $\mathcal{M}_{2d,\gamma}^n$ which is isomorphic to \mathcal{M}_M via the map α .

Consider two involutions $\rho_1 = -R_{h_1}$ and $\rho_2 = -R_{h_2}$ on Λ , with associated invariant lattices $\mathbb{Z}h_1$ and $\mathbb{Z}h_2$. The moduli spaces $\mathcal{M}_{\mathbb{Z}h_k}$ for $k = 1, 2$ are isomorphic if and only if h_1 and h_2 , or h_1 and $-h_2$, are in the same $O(\Lambda)$ -orbit. This implies that, in fact, \mathcal{M}_M as in Theorem 3.9 is isomorphic to any connected component of $\mathcal{M}_{[h]}$.

3.3 Some geometric descriptions for the involution on the Hilbert scheme of a very general surface

In this section we refer to Beauville's involution, which we described in Example 2.164.

Take n, t which satisfy (*). If $(n-1)X^2 - tY^2 = -1$ admits solutions let (a, b) be the minimal one, otherwise we take (a, b) such that $(a, \frac{b}{n-1})$ is the minimal solution of $X^2 - t(n-1)Y^2 = -1$. Consider the rational map ${}^n\phi_{2t} : \mathcal{K}_{2t} \dashrightarrow \mathcal{M}(n, t)$ and $U \subset \mathcal{K}_{2t}$ its domain of definition as in Proposition 3.5.

Proposition 3.11. *Consider $n \geq 2, t \geq 2$ which satisfy (*). Assume that there exists $(\Omega, \Theta) \in \mathcal{K}_{2t}$ with a birational involution $j \in \text{Bir}(\Omega^{[n]})$ which acts on $H^2(\Omega^{[n]}, \mathbb{Z})$ as $-R_D$, for $D = b\Theta_n - a\delta$. Then for every $(S, L) \in U$ there exists a deformation family of hyperkähler manifolds $\pi : \mathcal{X} \rightarrow B$ over an analytic connected base B , a line bundle \mathcal{L} on \mathcal{X} and points $0, p \in B$ such that*

- $(\pi^{-1}(0), \mathcal{L}|_{\pi^{-1}(0)}) \cong (\Omega^{[n]}, b\Theta_n - a\delta)$ and $(\pi^{-1}(p), \mathcal{L}|_{\pi^{-1}(p)}) \cong (S^{[n]}, bL_n - a\delta)$;

- for every $s \in B - \{0\}$ the pair $(\pi^{-1}(s), \mathcal{L}|_{\pi^{-1}(s)})$ is of the form $(\Sigma^{[n]}, bH_n - a\delta)$ for some $(\Sigma, H) \in U$.

Proof. Since U is path-connected, for any $(S_1, L_1), (S_2, L_2) \in U$ there exists a polarized deformation of $K3$ surfaces $\pi' : (\mathcal{X}', \mathcal{H}) \rightarrow B'$ between (S_1, L_1) and (S_2, L_2) with $B' \subset U$. We define $B = {}^n\phi_{2t}(B')$: passing to the Hilbert schemes, this gives a polarized deformation $\pi : (\mathcal{X}, \mathcal{F}) \rightarrow B$ between $(S_1^{[n]}, b(L_1)_n - a\delta)$ and $(S_2^{[n]}, b(L_2)_n - a\delta)$, whose fibers are all of the form $(\Sigma^{[n]}, bH_n - a\delta)$ for $(\Sigma, H) \in U$.

Consider now (Ω, Θ) as in the statement. If D is ample then $(\Omega, \Theta) \in U$, hence the previous argument allows us to conclude. On the other hand, if j is not biregular then D is movable; a priori we do not know if the general deformation of D is still movable. However, since U is open, we can still find a family of hyperkähler manifolds with a line bundle $\pi : (\mathcal{X}, \mathcal{L}) \rightarrow B$ such that the fiber over $0 \in B$ is $(\Omega^{[n]}, D)$, while for all $s \in B, s \neq 0$ the fiber $\pi^{-1}(s)$ is of the form $(\Sigma^{[n]}, bH_n - a\delta)$ for $(\Sigma, H) \in U$. \square

Combined with the results of Section 3.2, the proposition gives a description of a deformation path of hyperkähler manifolds with an involution, from $(\Omega^{[n]}, j)$ to $(S^{[n]}, \sigma)$, where σ is the non-symplectic involution of the Hilbert scheme of n points on a very general $\langle 2t \rangle$ -polarized $K3$ surface S . This deformation path, say ζ , has the advantage to be explicit, in the sense that it is induced by a deformation of $K3$ surfaces from Ω to S . Hence

- the path ζ lies in a 19-dimensional subspace of the 20-dimensional moduli space \mathcal{M}_M , with $M \cong \langle 2d \rangle$;
- through the isomorphism in Theorem 3.9, the path ζ , exceptly maybe for $(\Omega^{[2]}, b\Theta_2 - a\delta)$, lies in the same Noether–Lefschetz divisor \mathcal{C}_{2t} as $(S^{[n]}, D)$, where D is the ample line bundle generating the invariant lattice T_σ ;
- every fiber of the deformation is a Hilbert scheme on some $K3$ surface, so we can keep track of the action of the involution, on length n subschemes of $K3$ surfaces, along the deformation path. Given $p_1 + \dots + p_n$ a general point on $S^{[n]}$, we denote by p'_1, \dots, p'_n the points on Ω that correspond by deformation, along ζ , to $p_1, \dots, p_n \in S$. The image of $p_1 + \dots + p_n$ with respect to the involution on $S^{[n]}$ is then the point corresponding by deformation to $p''_1 + \dots + p''_n = j(p'_1 + \dots + p'_n)$.

As an application of Proposition 3.11, we give a geometrical description (up to deformation) of the biregular involutions of Hilbert schemes of very general $K3$ surfaces of degree $2t = 2((n - 1)k^2 + 1)$, which we presented in Proposition 2.163. This extends [15, Theorem 6.1] for $n = 2$ and k odd.

We fix $n \geq 2$. For $k \geq 2$ we consider the hyperbolic rank-two lattice M_k whose Gram matrix, with respect to a suitable basis, is

$$\begin{pmatrix} 2n & 2n + k - 1 \\ 2n + k - 1 & 2n \end{pmatrix}.$$

By Proposition 2.57 there exists a K3 surface Ω whose Picard lattice is isomorphic to M_k . Let $d_1, d_2 \in \text{Pic}(\Omega)$ be generators such that $d_1^2 = d_2^2 = 2n$, $(d_1, d_2) = 2n + k - 1$.

Lemma 3.12. *For $n \geq 2$ and $k \geq 3$, there exists $\Theta \in \text{Pic}(\Omega)$ ample with $\Theta^2 = 2((n-1)k^2 + 1)$. The Hilbert scheme $\Omega^{[n]}$ has a birational involution whose action on $H^2(\Omega^{[n]}, \mathbb{Z})$ is $-R_{(\Theta_n - k\delta)}$.*

Proof. Let $L = (k+1)d_1 - d_2 \in \text{Pic}(\Omega)$, which satisfies $L^2 = 2((n-1)k^2 + 1)$. Consider the chamber decomposition of $\text{Pos}(\Omega)$ where the walls are orthogonal to (-2) -classes: we make use of Remark 2.105 to show that d_1, d_2, L are in the interior of the same chamber. Any (-2) -class in $\text{NS}(\Omega)$ is of the form $\Gamma = (\frac{-(2n+k-1)y \pm \sqrt{\Delta}}{2n})d_1 + yd_2$, with $y \in \mathbb{Z}$ and $\Delta = y^2(k-1)(4n+k-1) - 4n$. What we want to prove is that the sign of (Γ, d_2) and (Γ, L) is always determined by the sign before $\sqrt{\Delta}$, and that neither d_2 nor L lie in the orthogonal of Γ . By convexity of chambers, this will hold for $d_1 = \frac{d_2 + L}{k+1}$ too.

We have

$$\begin{aligned} 2n \cdot (N, d_2) &= y(4n^2 - (2n+k-1)^2) \pm \sqrt{\Delta}(2n+k-1) \\ 2n \cdot (N, L) &= y(4n^2 - (2n+k-1)^2) \pm \sqrt{\Delta}(2kn-k+1). \end{aligned}$$

Since $2n+k-1 < 2kn-k+1$ for $k \geq 2$, it is sufficient to consider (N, d_2) . The condition on d_2 is equivalent, after some calculations, to

$$(2n+k-1)^2 > \frac{4n^3 y^2}{ny^2 - 1},$$

which holds true for $k \geq 3$.

By Lemma 2.107 there exists an isometry $\varrho \in H^2(\Omega, \mathbb{Z})$ which maps d_1, d_2, L to the ample cone. We denote $D_i = \varrho(d_i)$ and $\Theta = \varrho(L)$.

For $i = 1, 2$, we observe that D_i is very ample by Lemma 2.82: as Ω has Picard rank two, the fact that $(\Omega, \Theta) \in D_{h,0}$, for $h = 1$ or $h = 2$, would imply that the discriminant of $\text{Pic}(\Omega)$ is either 1 or 4, which never holds.

As a consequence, we have two distinct embeddings $\phi_{|D_i|} : \Omega \hookrightarrow \mathbb{P}^{n+1}$ whose image is a surface of degree $D_i^2 = 2n$. Each of these embeddings gives rise to a Beauville's involution $\iota_i \in \text{Bir}(\Omega^{[n]})$, which acts on $H^2(\Omega^{[n]}, \mathbb{Z})$ as the reflection fixing the line spanned by $(D_i)_n - \delta$. Now we take $j = \iota_1 \iota_2 \iota_1 \in \text{Bir}(\Omega^{[n]})$: its invariant lattice is generated by $\iota_1^*((D_2)_n - \delta) = \Theta_n - k\delta$. \square

Recall that for $t = (n-1)k^2 + 1$ the minimal solution of $(n-1)X^2 - tY^2 = -1$ is $(k, 1)$. Thus, from the lemma and Proposition 3.11 we obtain the following geometric description.

Proposition 3.13. *For $n \geq 2$ and $k \geq \frac{n+3}{2}$, the non-natural, non-symplectic involution of the Hilbert scheme of n points on a K3 surface of Picard rank one in $\mathcal{K}_{2((n-1)k^2+1)}$ can be described, up to a deformation as in Proposition 3.11, as the conjugation of a birational Beauville's involution with respect to another birational Beauville's involution.*

Remark 3.14. • The proposition holds for all $k \geq 3$ such that the birational involution which generates $\text{Bir}(S^{[n]})$, for S a $\langle 2((n-1)k^2 + 1) \rangle$ -polarized K3 surface of Picard rank one, is biregular (see Conjecture 6.15).

- For $k = 2$ (i.e. $t = 4n - 3$) the wall that is orthogonal to the (-2) -class $d_1 - d_2$ separates d_1 and d_2 , therefore the construction of Lemma 3.12 cannot be performed. We will show in Section 6.3 that this value of t is n -irregular (see Definition 6.7 and Lemma 6.14), hence Proposition 3.11 is not applicable.

For $n = 2$ both Beauville's involutions in Proposition 3.13 are biregular, as one can show that there are no lines on the quartic surfaces $\psi|_{D_i}(\Omega) \subset \mathbb{P}^3$. For k odd, this construction is [15, Theorem 6.1].

3.4 The four-dimensional case

In the case $n = 2$ we push further our construction. The moduli space $\mathcal{M}_{2,1}^2 \cong \mathcal{M}_M$ is connected by Proposition 2.76.

Given $t \geq 2$, the property (*) becomes

(**) : for a projective $K3$ surface S with $\text{Pic}(S) = \mathbb{Z}L$, $L^2 = 2t$, the Hilbert square $S^{[2]}$ is equipped with a biregular non-natural automorphism.

This is equivalent to ask that t satisfies the numerical conditions given in Remark 2.162. The integers $t \leq 100$ that satisfy such conditions are $t = 2, 10, 13, 17, 26, 37, 50, 53, 58, 65, 73, 74, 82, 85, 97$. For $t = 2$ the involution on the very general element S with $\text{Pic}(S) = \mathbb{Z}L$, $L^2 = 4$ is a Beauville's involution.

The integers $t \leq 100$ for which Boissière, An. Cattaneo, Nieper-Wisskirchen and Sarti described the involution on the very general element in \mathcal{K}_{2t} , up to a deformation as in Proposition 3.11, are $t = 10, 26, 50, 82$. In Proposition 3.13 we add $t = 17, 37, 65$.

Now take any t such that $X^2 - tY^2 = -1$ admits a solution; it lies in at least one sequence $t_k^{b,m} = \frac{(a_k^m)^2 + 1}{b^2}$ indexed by k , where

- b is chosen such that -1 is a quadratic residue modulo b^2 (or $b = 1$),
- $m \in \{1, \dots, b - 1\}$ is fixed such that $m^2 \equiv -1 \pmod{b^2}$,
- $a_k^m = b^2 k + m$ is another sequence indexed by k .

In particular, for every p prime dividing b , we have $p \equiv 1 \pmod{4}$ (b is always odd for $n = 2$).

Proposition 3.15. *Consider $b, \{t_k^{b,m}\}_{k \in \mathbb{N}}$ as above: for $k \geq 3$, the integer $t_k^{b,m}$ satisfies (**).*

Proof. A solution, not necessarily minimal, of $X^2 - t_k^{b,m} Y^2 = -1$ is (a_k^m, b) . By Remark 2.162 we only need to prove that $X^2 - 4t_k^{b,m} Y^2 = 5$ does not admit any solution for $k \geq 3$; any solution is in the form $(\frac{U}{b}, V)$ with $U^2 - 4((a_k^m)^2 + 1)V^2 = 5b^2$. We study solutions of Pell's equation $X^2 - 4(a^2 + 1)Y^2 = m$ for a general m in the case $4(a^2 + 1) > m^2$, so that we are in the hypothesis of Corollary 1.8. The continued fraction expansion of $\sqrt{4(a^2 + 1)}$ is

$$[2a; \overline{a, 4a}],$$

and the first two convergents are $2a/1$ and $(2a^2 + 1)/a$. We take numerator and denominator of these convergents as, respectively, n and d ; the expression $n^2 - 4(a^2 + 1)d^2$ gives us 1 and -4 . This means that $X^2 - 4(a^2 + 1)Y^2 = m$ has solutions if and only if m is in the forms h^2 or $-4h^2$, for some $h \in \mathbb{N}$; as $5b^2$ is always positive but never a square, Pell's equation $P_{4((a_k^m)^2 + 1)}(5b^2)$ admits no solution for $4((a_k^m)^2 + 1) > 25b^4$. Since $a_k^m = b^2k + m$, the latter holds true for $k \geq 3$. \square

So for every sequence $\{t_k^{b,m}\}_{k \in \mathbb{N}}$ there are at most three k 's such that $t_k^{b,m}$ does not satisfy Theorem 2.161; for example for $t_k^{1,1} = k^2 + 1$ we know that $(**)$ holds for $k \neq 2$, $t_2^{1,1} = 5$, see Proposition 2.163 for $n = 2$.

Previously we worked with polarizations that appear on the sequence $t_k^{1,1}$. Here we study some polarizations on a sequence $t_k^{b,m}$ where $b > 1$ i.e. some polarization which is not in the form $a^2 + 1$.

Consider M_a as in the previous section, cfr. (3.3); we index with a instead of k because we will consider the subsequence of a 's in the form $a_k^{18} = 25k + 18$, this time indexed by k . For simplicity we consider again d_1, d_2 as generators whose associated Gram matrix is (3.3). We look for an overlattice of M_a on which the element $L \in M_a$ defined in Lemma 3.12 is no more primitive.

Remark 3.16. A direct computation on the divisibility of $L = (a + 1)d_1 - d_2$ shows that there exists an overlattice of M_a , such that L is not primitive in it, if and only if $a \equiv 18 \pmod{25}$. In this case $\frac{1}{5}L$ is primitive in this overlattice.

Lemma 3.17. *Take $a_k^{18} = 25k + 18$, and consider the sequence $t_k^{5,18} = \frac{(a_k^{18})^2 + 1}{25}$. For every $k \geq 0$, $t_k^{5,18}$ satisfies $(**)$.*

Proof. By Proposition 3.15 the statement holds for $k \geq 3$. We are left to verify $(**)$ for $k = 0, 1, 2$, which can be checked directly with computer algebra calculations. \square

We call M'_k the even overlattice of $M_{a_k^{18}}$ obtained by adding $\frac{1}{5}L = (5k + \frac{19}{5})d_1 - \frac{1}{5}d_2$: M'_k is generated by $d_3 = \frac{1}{5}d_1 + \frac{1}{5}d_2$ and d_1 . We have $d_1^2 = 4$, $d_3^2 = \frac{1}{25}(d_1 + d_2)^2 = 2(k + 1)$ and $(d_1, d_3) = 5(k + 1)$. A Gram matrix of M'_k for the base $\{d_1, d_3\}$ is thus

$$\begin{pmatrix} 4 & 5(k + 1) \\ 5(k + 1) & 2(k + 1) \end{pmatrix}. \quad (3.1)$$

Now we fix $k \geq 0$. By Proposition 2.57 there exists a $K3$ surface Ω' whose Picard lattice is isomorphic to M'_k . Note that Ω' is not isomorphic to Ω , since their Picard lattices does not have the same discriminant.

Some computer algebra calculations show that the first cases that admit (-2) -classes are $k + 1 = 1, 3, 11, 17, 19, 33$.

Lemma 3.18. *For $k \geq 0$ there exists $\Theta' \in \text{Pic}(\Omega')$ ample with $(\Theta')^2 = 2t_k^{5,18}$. The Hilbert scheme $(\Omega')^{[2]}$ has a biregular involution whose action on $H^2((\Omega')^{[2]}, \mathbb{Z})$ is $-R_{(5\Theta'_2 - a_k^{18}\delta)}$.*

Proof. Now $L = (a_k^{18} + 1)d_1 - d_2$ is no more a primitive vector, so we take $L' = \frac{1}{5}L = (5k+4)d_1 - d_3$. Note that $L'^2 = 2t_k^{5,18}$.

Some calculations, very similar to those of Lemma 3.12, show that d_1, d_2 and L are still in the same chamber, so by Lemma 2.107 there exists an isometry $\varrho \in H^2(\Omega, \mathbb{Z})$ which maps d_1, d_2, L to the ample cone. We denote $D_i = \varrho(d_i)$ for $i = 1, \dots, 3$ and $\Theta' = \varrho(L')$. We verify through Theorem 2.31 that they are very ample, since there is no elliptic pencil on Ω' .

So we can consider $(\Omega', \Theta') \in \mathcal{K}_{2t_k^{5,18}}$; the involutions on $(\Omega')^{[2]}$ associated to $(D_1)_2 - \delta$ and $(D_2)_2 - \delta$, say ι_1 and ι_2 , are Beauville's involutions; the invariant lattice of the conjugate of ι_2 with respect to ι_1 , that we call j , is then spanned by $5\Theta'_2 - a_k^{18}\delta$.

To prove that j is biregular, it is sufficient to prove that Ω' does not contain any line through one of the embeddings given by D_1 or D_2 . To prove it for D_1 a straight calculation is sufficient. For $D_2 = -D_1 + 5D_3$ we take a (-2) -class N and we impose $(D_2, N) = 1$. After some computation we get

$$4(1 \pm \sqrt{\Delta}) = 5(k+1)[8y + 5(\pm\sqrt{\Delta} - 5(k+1)y)] \quad (3.2)$$

with $\Delta = [5(k+1)y]^2 - 8(2(k+1)y^2 + 1)$.

For $k+1$ even there is no (-2) -curve on Ω' . Consider instead the case $k+1$ odd: by (3.2) either $(k+1) = 1$ or $\sqrt{\Delta} \equiv \pm 1$ modulo $(k+1)$. In the last case we obtain that 9 has to be zero modulo $k+1$. So we are left to check by direct computation the cases $k+1 = 1, 3, 9$, which can be easily done. \square

Something got swept under the carpet in the proof of Lemma 3.18: a priori it may happens that, for some k , the pair $(a_k^{18}, 5)$ is not the minimal solution of $P_{t_k^{5,18}}(-1)$, so that $5L_2 - a_k^{18}\delta$ is not ample on $S^{[2]}$ for (S, L) a general deformation of $(\Omega', \Theta') \in \mathcal{K}_{2t_k^{5,18}}$, see Proposition 3.5. However, in this case the minimal solution is in the form $(a^2 + 1, 1)$, and we can take (Ω, Θ) as in Lemma 3.12.

Proposition 3.19. *For $t_k^{5,18} = \frac{(a_k^{18})^2 + 1}{25} = 25k^2 + 36k + 13$, $k \geq 0$, the non-natural, non-symplectic involution of the Hilbert square on a K3 surface of Picard rank one in $\mathcal{K}_{2t_k^{5,18}}$ can be described, up to a deformation as in Proposition 3.11, as the conjugation of a biregular Beauville's involution with respect to another biregular Beauville's involution.*

We cannot use further the same idea, as everytime we try to conjugate a Beauville's involution with another Beauville's involution we fall in the previous cases. Given two such involutions ι_1 and ι_2 on a K3 surface S , there exists an isometry α between the sublattice generated by the invariant lattices of them and a sublattice of M'_k for some k . The ample generators of the two invariant lattices would be sent by α to d_1 and d_2 , so the invariant of $\iota_1 \circ \iota_2 \circ \iota_1$ would be either Θ' or $5\Theta'$: we already studied all such configurations.

Lemma 3.20. *Take $a_k^{99} = 169k + 99$, and consider the sequence $t_k^{13,99} = \frac{(169k+99)^2 + 1}{169}$. For every $k \geq 0$, the integer $t_k^{13,99}$ satisfies (**).*

Proof. By Proposition 3.15 the statement holds for $k \geq 3$. We are left to verify (**) for $k = 0, 1, 2$, which can be checked directly with computer algebra calculations. \square

We will find some other construction for polarizations on the sequence $t_k^{13,99}$. We refer to O'Grady's involution, which we described in Example 2.165.

Consider the lattice M_k whose Gram matrix, with respect to a suitable basis, is

$$\begin{pmatrix} 4 & a_k^{99} + 6 \\ a_k^{99} + 6 & 10 \end{pmatrix}. \quad (3.3)$$

Let $D_1, D_2 \in \text{Pic}(\Omega)$ be generators such that $D_1^2 = 4, D_2^2 = 10, (D_1, D_2) = a_k^{99} + 6$. We call L the vector $(a_k^{99} + 2)D_1 - D_2$, primitive in M_k . We denote by M'_k the overlattice obtained from M_k by adding $h \in M_k^*$ such that $D_2 = 10D_1 + 13h$. The Gram matrix of M'_k , with respect to the base $\{D_1, h\}$, is

$$\begin{pmatrix} 4 & 13k + 5 \\ 13k + 5 & -10(k + 1) \end{pmatrix} \quad (3.4)$$

The elements D_1, D_2 are still primitive in M'_k , while there exists $\Theta = (13k + 7)D_1 - h$ primitive such that $13\Theta = L$. Note that $L^2 = 2((a_k^{99})^2 + 1)$ while $\Theta^2 = 2t_k^{13,99}$. We denote by Ω a K3 surface whose Picard lattice is isomorphic to M'_k , whose existence is granted by Proposition 2.57. From now on, we only consider the case $k \equiv 0, 1, 3, 5, 7, 9 \pmod{10}$, to keep calculations simpler. In this case there is no (-2) -curve on Ω ; this can be checked by taking, for $\Gamma \in \text{Pic}(\Omega)$, $\frac{1}{2}\Gamma^2$ modulo 2 and 5.

Lemma 3.21. *For $k \equiv 0, 1, 3, 5, 7, 9 \pmod{10}$, the line bundle $\Theta \in \text{Pic}(\Omega)$ is ample. The Hilbert square $\Omega^{[2]}$ has a birational involution whose action on $H^2(\Omega^{[2]}, \mathbb{Z})$ is $-R_{(13\Theta_2 - a_k^{99}\delta)}$.*

Proof. For these numerical conditions there is no (-2) -curve on Ω , so $\mathcal{A}(\Omega) = \text{Pos}(\Omega)$. Hence Θ, D_1 and D_2 are ample line bundles. A straight calculation shows that $\text{Pic}(\Omega)$ is not isomorphic to a lattice whose discriminant is 1, 4, 9, 16. Hence

- the line bundle D_1 is very ample, see Lemma 2.82. For our numerical conditions Ω does not contain any line with respect to the embedding associated to D_1 . So we have a biregular Beauville's involution ι_1 on $\Omega^{[2]}$ that acts on cohomology as $\iota_1^* = -R_{(D_1)_2 - \delta}$;
- $(\Omega, D_2) \in \mathcal{K}_{10}$ satisfies the sufficient conditions given in Remark 2.167, so $\Omega^{[2]}$ carries an O'Grady's involution ι_2 as in Example 2.165, that acts on cohomology as $\iota_2^* = -R_{(D_2)_2 - 2\delta}$.

The composition $j = \iota_1 \iota_2 \iota_1$ is then a non-symplectic birational involution whose invariant lattice contains $\iota_1^*((D_2)_2 - 2\delta) = L_2 - a_k^{99}\delta$ and is spanned by $13\Theta_2 - a_k^{99}\delta$. \square

Here we should really pay attention to the k 's for which $(a_k^{99}, 13)$ is *not* the minimal solution of $X^2 - t_k^{13,99}Y^2 = -1$. These are the cases where $t_k^{13,99}$ is either in the form $m^2 + 1, 25m^2 + 36m + 13$ or $25m^2 + 14m + 2$. In the first two cases we can refer to Proposition 3.13 for $n = 2$ and Proposition 3.19.

Proposition 3.22. *For $t_k^{13,99} = \frac{(169k+99)^2+1}{169} = 169k^2 + 198k + 58$ for $k \equiv 0, 1, 3, 5, 7, 9 \pmod{10}$, the non-natural, non-symplectic involution on the Hilbert square on a K3 surface of Picard rank one in $\mathcal{K}_{2t_k^{13,99}}$ can be described, up to a deformation as in Proposition 3.11, as the conjugation of an O'Grady's involution with respect to a biregular Beauville's involution, except if $t_k^{13,99}$ is in the form $m^2 + 1, 25m^2 + 36m + 13$ or $25m^2 + 14m + 2$.*

Remark 3.23. If $t_k^{13,99}$ is in the form $m^2 + 1$ for $m \geq 3$, or $25m^2 + 36m + 13$ for $m \geq 0$, then the involution on the Hilbert square of a very general element of $\mathcal{K}_{t_k^{13,99}}$ can be described, up to a deformation as in Proposition 3.11, as the conjugation of a biregular Beauville's involution with respect to another biregular Beauville's involution.

We does not have a description for $t_k^{13,99}$ in the form $25m^2 + 14m + 2$ that does not lie in the sequences $t_m^{1,1}$ or $t_m^{5,18}$.

The first column of the following Table lists the t 's with $t \leq 100$ such that **(**)** hold. The second column gives the minimal solution of $P_t(-1)$, as we call the Pell's equation $X^2 - tY^2 = -1$. The last column sums up Proposition 3.13 for $n = 2$, Proposition 3.19 and Proposition 3.22 when $t \leq 100$. We write $-$ when we are not able to produce a geometric description.

t	minimal solution of $P_t(-1)$	Description up to a deformation as in Proposition 3.11
10	(3,1)	$\iota_1 \iota_2 \iota_2$, with ι_1 and ι_2 biregular Beauville's involutions
13	(18,5)	$\iota_1 \iota_2 \iota_2$, with ι_1 and ι_2 biregular Beauville's involutions
17	(4,1)	$\iota_1 \iota_2 \iota_2$, with ι_1 and ι_2 biregular Beauville's involutions
26	(5,1)	$\iota_1 \iota_2 \iota_2$, with ι_1 and ι_2 biregular Beauville's involutions
37	(6,1)	$\iota_1 \iota_2 \iota_2$, with ι_1 and ι_2 biregular Beauville's involutions
50	(7,1)	$\iota_1 \iota_2 \iota_2$, with ι_1 and ι_2 biregular Beauville's involutions
53	(182,25)	-
58	(99,13)	$\iota_1 \iota_2 \iota_2$, with ι_1 a biregular Beauville's involution and ι_2 an O'Grady's involution
65	(8,1)	$\iota_1 \iota_2 \iota_2$, with ι_1 and ι_2 biregular Beauville's involutions
73	(1068,125)	-
74	(43,5)	$\iota_1 \iota_2 \iota_2$, with ι_1 and ι_2 biregular Beauville's involutions
82	(9,1)	$\iota_1 \iota_2 \iota_2$, with ι_1 and ι_2 biregular Beauville's involutions
85	(378,41)	-
97	(5604,569)	-

Chapter 4

Eisenbud-Popescu-Walter sextics and Gushel-Mukai varieties: an introduction

4.1 Introduction

We present a family of varieties called Eisenbud-Popescu-Walter sextics, whose properties were studied firstly by Eisenbud, Popescu and Walter in [37, Example 9.3]. Once fixed a volume form on $\bigwedge^6 V_6$ for $V_6 \cong \mathbb{C}^6$, we have a symplectic form on $\bigwedge^3 V_6$. Given some Lagrangian subspace A of $\bigwedge^3 V_6$, there is a canonical way to associate to it a subvariety of $\mathbb{P}(V_6)$, which is an irreducible sextic hypersurface for a general choice of A . The sextics obtained in this way are called Eisenbud-Popescu-Walter sextics or, more frequently, *EPW* sextics.

Any *EPW* sextic admits a canonical double cover which coincides with the universal cover outside the ramification locus. This double cover is called double *EPW* sextic.

The importance of *EPW* sextics in the study of hyperkähler manifolds lies in the fact that, again for a general choice of A , the natural double cover of the sextic is a hyperkähler fourfold of $K3^{[2]}$ -type. This result is proved by O'Grady in [96, Theorem 1.1]. Moreover, O'Grady proved that the general element in the moduli space of $\langle 2 \rangle$ -polarized varieties of $K3^{[2]}$ -type admits a geometric description as a double *EPW* sextic, see for example [98, Theorem 4.25]. Here lies an important reason to study double covers of *EPW* sextics: we know, by the deformation argument given in Corollary 2.14, that elements of the moduli space $\mathcal{M}_{2,1}^2$ are still hyperkähler manifolds, but this does not give an insight of its geometrical structure, other than cohomologically-wise. On the contrary, double *EPW* sextics are quite concrete objects.

Moreover, *EPW* sextics provide a generalization of the description of $\langle 2 \rangle$ -polarized $K3$ surfaces as double covers of the plane, ramified on a sextic curve, through the map induced by the polarization, which we presented in Section 2.2.3. As we said, a general element (X, D) in $\mathcal{M}_{2,1}^2$ is the double

cover of a general *EPW* double sextic, with the $\langle 2 \rangle$ -polarization given by the generator of the invariant lattice of the covering involution. Then the map $\phi_{|D|}$ is the map inducing the ramified double cover of the *EPW* sextic.

On the other hand, ordinary Gushel-Mukai varieties are subvarieties of $\mathbb{P}(\wedge^2 V_5)$, for $V_5 \cong \mathbb{C}^5$, obtained by intersecting a general and big enough linear subspace $\mathbb{P}(W)$ of $\mathbb{P}(\wedge^2 V_5)$ with the Grassmannian $G(2, V_5)$ and a general quadric hypersurface. The dimension of an ordinary Gushel-Mukai variety is a number between 1 and 5, and we will be mostly interested in Gushel-Mukai varieties whose dimension is at least two; Gushel-Mukai varieties of dimension two correspond exactly to Brill-Noether general *K3* surfaces (see Theorem 2.34).

The relation between Gushel-Mukai varieties and *EPW* sextics was clear since [96, Section 6], where a particular double *EPW* sextic was proved to be isomorphic to the Hilbert square of a Brill-Noether general *K3* surface. In this way O'Grady proved by a deformation argument that the general double *EPW* sextic is a hyperkähler manifold. Moreover, in [67, Lemma 2.2 and Lemma 4.15], Iliev and Manivel showed a way to obtain a *EPW* sextic from a general ordinary Gushel-Mukai variety of dimension 3, 4 or 5. For the fourfold case, they showed a way to provide the associated double *EPW* sextic of a holomorphic symplectic 2-form without appealing on deformation theory, but inducing the two-form on the cover by a two-form on the Hilbert scheme of conics of the Gushel-Mukai variety.

Finally, in the series of papers [27], [29], [30], [31], Debarre and Kuznetsov studied systematically the interplay between Gushel-Mukai varieties and *EPW* sextics. In particular, given a Gushel-Mukai variety, they provided a way to explicitly construct a vector space V_6 and a Lagrangian $A \subset \mathbb{L}\mathbb{G}(\wedge^3 V_6)$, and vice versa, given a Lagrangian $A \subset \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ and an hyperplane $V_5 \subset V_6$, they give a way to construct an ordinary Gushel-Mukai variety in $\mathbb{P}(\wedge^2 V_5)$, whose dimension depends on the choice of V_5 . An important point of all these constructions is that there is not a 1 : 1 correspondence between isomorphism classes of ordinary Gushel-Mukai varieties and *EPW* sextics: while we can associate a sextic to a Gushel-Mukai variety, given a *EPW* sextic we have to choose an hyperplane $V_5 \subset V_6$ to fix an isomorphism class of Gushel-Mukai varieties. In [27, Theorem 3.23 and Theorem 3.25], Debarre and Kuznetsov provide a description of the set of Gushel-Mukai varieties of dimension bigger than two with associated the same Lagrangian A .

In this chapter we give the description of those varieties, which is pretty technical. At the end of the chapter, after some auxiliary construction, we present Theorem 3.6 of [27], the fundamental link between the two types of varieties. While we do not explain the whole proof of the theorem, we give an insight of the construction, which will be useful later. Sometimes in the chapter we make references about definitions and results that will be presented later: the goal is to show the natural interplay between the families we will describe.

All the results in this chapter are already known, except some considerations in Section 4.3.4. Moreover we propose an alternative proof of the finiteness of the automorphism group of an *EPW* sextic with no non-zero decomposable vector, see Proposition 4.28.

Known results come mainly from the series of papers on *EPW* sextics written by O’Grady ([96], [95], [98], [99], [100], [102]), from the already cited series of papers on Gushel-Mukai varieties by Debarre and Kuznetsov, from Iliev and Manivel’s work [67] and Piontkowski and Van de Ven’s work [104].

4.2 *EPW* sextics

4.2.1 Definition of *EPW* sextics

Given V a complex vector space we call V^\vee the dual of V , that is $\text{Hom}(V, \mathbb{C})$. For $i : W \hookrightarrow V$ a vector subspace, we write $i^\vee : V^\vee \rightarrow W^\vee$ for the canonical projection given by the pullback of linear functions.

Given a complex vector space V , the Grassmannian of k -dimensional vector subspaces in V will be denoted by $G(k, V)$. In particular, consider $k = 2$ and $V = V_5$ a five dimensional vector space, on which we fix a volume form $\text{vol} : \bigwedge^5 V_5 \rightarrow \mathbb{C}$. We know that a vector $\eta \in \bigwedge^2 V_5$ lies in $G(2, V_5)$ if and only if $\eta \wedge \eta = 0$. By the isomorphism $V_5 \cong (\bigwedge^4 V_5)^\vee$ induced by the volume form, this is equivalent to ask that $v \wedge \eta \wedge \eta = 0$ for every $v \in V_5$.

Definition 4.1. *The Pfaffian quadric P_v , for some $v \in V_5$, on $\bigwedge^2 V_5$ is the symmetric bilinear form $x \mapsto \text{vol}(v \wedge x \wedge x)$. For every $v \in V_5$, we write $Q(v)$ for the projective quadric hypersurface associated to P_v .*

We write then $G(2, V_5) = \bigcap_{v \in V_5} Q(v)$; the vector space

$$H^0(\mathbb{P}(\bigwedge^2 V_5), \mathcal{I}_{G(2, V_5)}(2))$$

is isomorphic to V_5 through the isomorphism that sends $v \in V_5$ to the symmetric bilinear form $P_v = \text{vol}(v \wedge _ \wedge _)$. A Pfaffian quadric P_v has rank 6 for every $v \in V_5$.

O’Grady provided slightly different points of view on the construction of Eisenbud-Popescu-Walter sextics, see for example [96], [98] and [100]; here we follow mainly [100], with a view to [28], by Debarre and Kuznetsov. Moreover, we refer to [39] and [27] for some results. Let V_6 be a 6-dimensional vector space over \mathbb{C} . We fix a volume form on $\bigwedge^6 V_6$: this induces a symplectic form on $\bigwedge^3 V_6$, that is

$$\begin{aligned} \omega : (\bigwedge^3 V_6) \times (\bigwedge^3 V_6) &\rightarrow \mathbb{C} \\ (u, v) &\mapsto \text{vol}(u \wedge v). \end{aligned}$$

Since ω is symplectic, the induced map $\bigwedge^3 V_6 \xrightarrow{\omega} (\bigwedge^3 V_6)^\vee$ is an isomorphism.

Given two vector subspaces $U_1, U_2 \subseteq \bigwedge^3 V_6$, we will make use of the linear map

$$\omega_{U_1, U_2} : U_1 \rightarrow U_2^\vee \tag{4.1}$$

$$u \mapsto \omega(u, -)|_{U_2} \tag{4.2}$$

given by the composition of the inclusion $U_1 \hookrightarrow \bigwedge^3 V_6$ with the dual of the inclusion $U_2 \hookrightarrow \bigwedge^3 V_6$.

Definition 4.2. An isotropic subspace $i : I \hookrightarrow \bigwedge^3 V_6$ is a vector subspace of $\bigwedge^3 V_6$ on which the restriction of the symplectic form is zero, or equivalently a subspace such that $\omega_{I,I}$ is zero. A Lagrangian subspace $i : A \hookrightarrow \bigwedge^3 V_6$ is an isotropic subspace whose dimension is maximal, or equivalently $\omega_{A,A}$ induces an exact sequence

$$0 \rightarrow A \xrightarrow{i} \bigwedge^3 V_6 \xrightarrow{i^\vee \circ \omega} A^\vee \rightarrow 0.$$

We can always find a base of V_6 such that the matrix associated to ω is in the form

$$\left[\begin{array}{c|c} 0 & I_{10} \\ \hline -I_{10} & 0 \end{array} \right],$$

where I_{10} is the 10-dimensional identity matrix. It is then clear that a Lagrangian subspace has always dimension 10.

Let $G(10, \bigwedge^3 V_6)$ be the Grassmannian of 10-dimensional vector subspaces of $\bigwedge^3 V_6$. We call $\mathbb{L}G(\bigwedge^3 V_6) \subset G(10, \bigwedge^3 V_6)$ the *symplectic Grassmannian*, parametrizing Lagrangian subspaces with respect to the volume form in $\bigwedge^3 V_6$; since two volume forms differ by a non-zero constant, $\mathbb{L}G(\bigwedge^3 V_6)$ does not depend on the choice of the volume form. The symplectic Grassmannian is a well-understood object; for simplicity we refer to [100, Corollary 1.2] for the following result.

Theorem 4.3. *The symplectic Grassmannian $\mathbb{L}G(\bigwedge^3 V_6)$ is a subvariety of $G(10, \bigwedge^3 V_6)$. Moreover, it is irreducible and smooth of dimension 55.*

Definition 4.4. For every $v \in V_6 - \{0\}$, consider

$$F_v = v \wedge \bigwedge^2 V_6.$$

Clearly $F_v \hookrightarrow \bigwedge^3 V_6$ is a Lagrangian subspace for every v . Notice that, for any $\lambda \in \mathbb{C}^*$, F_v and $F_{\lambda v}$ are equal, so we can define a vector bundle $F \subset \mathcal{O}_{\mathbb{P}(V_6)} \otimes \bigwedge^3 V_6$ on $\mathbb{P}(V_6)$ whose fiber on $[v]$ is F_v .

Given $v \in V_6 - \{0\}$, we fix a decomposition $V_6 \cong \mathbb{C}v \oplus V_5$ for some 5-dimensional subvector space $V_5 \subset V_6$; this induces a decomposition $\bigwedge^3 V_6 \cong \bigwedge^3 V_5 \oplus F_v$ and every element of F_v can be written in the form $v \wedge \eta$ for some $\eta \in \bigwedge^2 V_5$. We have then an induced isomorphism of vector spaces

$$\rho : F_v \rightarrow \bigwedge^2 V_5 \tag{4.3}$$

$$v \wedge \eta \mapsto \eta. \tag{4.4}$$

Hence F is a 10-dimensional vector bundle. We will sometimes make use of this decomposition.

Definition 4.5. A Lagrangian subbundle of $\mathcal{O}_{\mathbb{P}(V_6)} \otimes \bigwedge^3 V_6$ is a vector bundle such that every fiber is a Lagrangian subspace of $\bigwedge^3 V_6$.

The vector bundle F is a Lagrangian subbundle of $\mathcal{O}_{\mathbb{P}(V_6)} \otimes \bigwedge^3 V_6$, and for $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ the trivial vector bundle $\mathcal{O}_{\mathbb{P}(V_6)} \otimes A$ is Lagrangian too.

Let $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ be a Lagrangian subspace in $\bigwedge^3 V_6$. Let us consider the map of vector bundles

$$\lambda_A : F \rightarrow \mathcal{O}_{\mathbb{P}(V_6)} \otimes A^\vee$$

such that $(\lambda_A)_{[v]} = \omega_{F_v, A}$ for every $[v] \in \mathbb{P}(V_6)$. Since A is a Lagrangian subspace of $\bigwedge^3 V_6$, the map λ_A is injective on $[v] \in \mathbb{P}(V_6)$ if and only if $F_v \cap A = 0$.

Remark 4.6. The map λ_A is not injective as a map between vector bundles, as we will check in Lemma 4.7. However, the induced map on sheaves is indeed injective.

Note that the rank of F and $\mathcal{O}_{\mathbb{P}(V_6)} \otimes A^\vee$ is the same and it is equal to 10. We can thus consider the map

$$\det(\lambda_A) : \det(F) \rightarrow \det(\mathcal{O}_{\mathbb{P}(V_6)} \otimes A^\vee).$$

We define the determinantal variety

$$Y_A = Z(\det(\lambda_A)).$$

In general, Y_A can be the whole space $\mathbb{P}(V_6)$: consider, as an example, $A = F_v$ for some $[v] \in \mathbb{P}(V_6)$. However O'Grady gave some precise sufficient condition on A to have $Y_A \neq \mathbb{P}(V_6)$, which we present in Proposition 4.11. O'Grady verified that this condition holds for A general in $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$.

Lemma 4.7. *We have an isomorphism $\det(F) \cong \mathcal{O}_{\mathbb{P}(V_6)}(-6)$, so that, when Y_A is not the whole space, the hypersurface Y_A is a sextic. In particular, Y_A is never empty.*

Proof. This computation on Chern classes is made explicit by Ferretti in [39, Section 1.1]. We call \mathcal{S} the tautological subbundle on $\mathbb{P}(V_6)$ and \mathcal{Q} the tautological quotient bundle. The fiber-wise isomorphisms (4.3) glue together to an isomorphism of vector bundles $F \cong \mathcal{S} \otimes \bigwedge^2 \mathcal{Q}$. Call h the class of a hyperplane section on $\mathbb{P}(V_6)$; via the splitting principle for Chern classes, see [58, Appendix A.3], and the natural exact sequence of line bundles

$$0 \rightarrow \mathcal{S} \rightarrow \mathcal{O}_{\mathbb{P}(V_6)} \otimes V_6 \rightarrow \mathcal{Q} \rightarrow 0,$$

we obtain $c_1(\mathcal{S}) = -h$, $c_1(\bigwedge^2 \mathcal{Q}) = 4h$. Thus

$$c_1(\det(F)) = 10c_1(\mathcal{S}) + c_1(\bigwedge^2 \mathcal{Q}) = -6h$$

As for the second part of the statement, λ_A is a section of the line bundle

$$\mathrm{Hom}(F, \mathcal{O}_{\mathbb{P}(V_6)} \otimes A^\vee) \cong F^\vee \otimes (\mathcal{O}_{\mathbb{P}(V_6)} \otimes A^\vee) \cong F^\vee \otimes A^\vee,$$

so $Y_A = Z(\det(\lambda_A)) = Z(\det(F^\vee))$ is non-empty and it is a sextic. \square

Definition 4.8. *When $Y_A \neq \mathbb{P}(V_6)$, the sextic hypersurface Y_A is called Eisenbud Popescu Walter (EPW) sextic.*

We can associate to $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ a stratification of $\mathbb{P}(V_6)$: for every $k \geq 0$,

$$Y_A^{\geq k} = \{[v] \in \mathbb{P}(V_6) \mid \dim(F_v \cap A) \geq k\}. \quad (4.5)$$

Consider $Y_A^{\geq k}$ as a subset of $\mathbb{P}(V_6)$; obviously the support of $Y_A^{\geq 0}$ is equal to $\mathbb{P}(V_6)$, and the support of $Y_A^{\geq 1}$ is the *EPW sextic* Y_A . The subspace A is Lagrangian, so $Y_A^{\geq k}$ can be seen as the locus of points at which the map $\lambda_A : F \rightarrow \mathcal{O}_{\mathbb{P}(V_6)} \otimes A^\vee$ has corank at least k i.e. the *Lagrangian degeneracy locus* of the Lagrangian vector bundles F and $\mathcal{O}_{\mathbb{P}(V_6)} \otimes A^\vee$. For details on this point of view, see [28, Section 4]. This description allows to provide to $Y_A^{\geq k}$ a structure of scheme: on an open trivializing F , it is the support of the vanishing of all the $(k+1) \times (k+1)$ -minors of a matrix representing λ_A ; we will always consider $Y_A^{\geq k}$ with this structure, which corresponds to that of $\mathbb{P}(V_6)$ and Y_A for $k = 0, 1$. For every k , the variety $Y_A^{\geq k+1}$ is a closed subvariety of $Y_A^{\geq k}$. We define

$$Y_A^k := \{[v] \in \mathbb{P}(V_6) \mid \dim(F_v \cap A) = k\}.$$

Consider $B \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ such that $A \cap B = 0$. The set

$$U = \{[v] \in \mathbb{P}(V_6) \mid F_v \cap B = 0\}$$

is open in $\mathbb{P}(V_6)$, as it is the inverse image, through the map $[v] \mapsto F_v$, of the open subset of $\mathbb{L}\mathbb{G}(\wedge^3 V_6)$ of Lagrangian subspaces that have trivial intersection with A . On U we can see $Y_A^{\geq k}$ as a *quadratic degeneracy locus*, which is the locus of points where a family of quadratic forms $\mathbf{q} : (\mathcal{O}_{\mathbb{P}(V_6)} \otimes B) \times (\mathcal{O}_{\mathbb{P}(V_6)} \otimes B) \rightarrow \mathbb{C}$ has corank at least k . For any $[v] \in U$ the maps $\omega_{F_v, B}$ and $\omega_{B, A}$ are isomorphisms, respectively by the definition of U and the choice of B . So we can consider an isomorphism $\mathcal{O}_{\mathbb{P}(V_6)} \otimes B^\vee \rightarrow \mathcal{O}_{\mathbb{P}(V_6)} \otimes B$ whose fiber on $[v]$ is the composition

$$B^\vee \xrightarrow{\omega_{F_v, B}^{-1}} F_v \xrightarrow{\omega_{F_v, A}} A^\vee \xrightarrow{\omega_{B, A}^{-1}} B; \quad (4.6)$$

it induces a bilinear form \mathbf{q} on $\mathcal{O}_{\mathbb{P}(V_6)} \otimes B$. A direct computation shows that \mathbf{q} is symmetric, moreover $Y_A^{\geq k}$ is exactly the quadratic degeneracy locus associated to \mathbf{q} . To verify this, observe that the corank of (4.6) and λ_A coincide on every $[v] \in U$, since $(\lambda_A)_{[v]} = \omega_{F_v, A}$ and $\omega_{F_v, B}$ and $\omega_{B, A}$ are isomorphisms.

This is a local description, but it is straightforward to see that, for B, B' such that $A \cap B = A \cap B' = 0$, the descriptions agree on the intersection of the associated open subsets of $\mathbb{P}(V_6)$.

Remark 4.9. In this last setting the codimension of $Y_A^{\geq k}$ inside $\mathbb{P}(V_6)$ is at most $\frac{k(k+1)}{2}$ when $Y_A^{\geq k}$ is non-empty, see [100, Section 1.2] or [56, Section 1, a)]. When the codimension is exactly $\frac{k(k+1)}{2}$ we say that $Y_A^{\geq k}$ has *expected dimension*; in this case $Y_A^{\geq k}$ is Cohen-Macaulay, cfr. [35, Theorem 18.18, Remark 2].

The following easy fact has been proved by O'Grady, see for example [98, Claim 3.7].

Lemma 4.10. *Consider $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ such that it exists $[v] \in Y_A^{\geq 4}$. Then $\mathbb{P}(A) \cap G(3, V_6) \neq \emptyset$.*

Proof. Fix $V_5 \subset V_6$ such that $V_6 \cong \mathbb{C}v \oplus V_5$ and consider the map $\rho : F_v \rightarrow \Lambda^2 V_5$ as in Equation (4.3). The intersection $\mathbb{P}(F_v) \cap G(3, V_6)$ is sent, through the isomorphism, to $G(2, V_5)$, which is a codimension-six subvariety inside $\mathbb{P}(\Lambda^2 V_5) \cong \mathbb{P}^9$. Since $\mathbb{P}(\rho(F_v \cap A))$ has dimension at least three by hypothesis, there exists at least one $[v \wedge \eta] \in \mathbb{P}(A) \cap G(3, V_6)$ such that $[\eta] \in \mathbb{P}(\rho(F_v \cap A)) \cap G(2, V_5)$. \square

Proposition 4.11. *Take $A \in \mathbb{L}\mathbb{G}(\Lambda^3 V_6)$. If $\mathbb{P}(A) \cap G(3, V_6) = \emptyset$, then Y_A is a EPW sextic.*

Proof. See [100, Corollary 1.5]. Actually O’Grady proved much more i.e. that Y_A is a EPW sextic if there exists at least one $v \in V_6 - \{0\}$ such that $\mathbb{P}(F_v) \cap G(3, V_6)$ is empty. However, for our purpose the stronger hypothesis we request is sufficient. \square

A EPW sextic is never smooth, and O’Grady described its singularities in terms of the fibration $\{Y_A^{\geq k}\}_{k \geq 1}$.

Definition 4.12. *We call Θ_A the locus of the classes of decomposable vectors in $\mathbb{P}(A)$, that is*

$$\Theta_A = \mathbb{P}(A) \cap G(3, V_6).$$

Any element of Θ_A can be seen as a 3-dimensional vector space inside V_6 , or as a vector of $\Lambda^3 V_6$ through the Plücker embedding. We refer to $[v_1 \wedge v_2 \wedge v_3]$ in the second case and to $W = \langle v_1, v_2, v_3 \rangle \subset V_6$ in the first case. Consider $W \in \Theta_A$: for every $w \in W$ we have $F_w \cap A \neq 0$, so $\mathbb{P}(W) \subset Y_A$.

Proposition 4.13. *If $Y_A \neq \mathbb{P}(V_6)$, we have*

$$\text{Sing}(Y_A) = Y_A^{\geq 2} \cup \left(\bigcup_{W \in \Theta_A} \mathbb{P}(W) \right).$$

Proof. See [100, Corollary 1.5]. \square

We will be not so much interested in EPW sextics associated to a Lagrangian subspace A which contains a non-zero decomposable vector. However, Lagrangian subspaces containing decomposable vectors are studied, for example in relation to Morin’s problem, see [100] or [101].

4.2.2 The singular locus of an EPW sextic

It will be useful, for our study of EPW sextics, to know the structure of their singular locus in the case $\Theta_A = \emptyset$: in this case we have $\text{Sing}(Y_A) = Y_A^2$, by Proposition 4.13, and $Y_A^3 = Y_A^{\geq 3}$ by Lemma 4.10. Now we study the singular locus $Y_A^{\geq 2}$.

Proposition 4.14. *Let $A \in \mathbb{L}\mathbb{G}(\Lambda^3 V_6)$. Suppose that $[v] \in Y_A^2$ and that $A \cap F_v$ does not contain a non-zero decomposable element. Then Y_A^2 is smooth and 2-dimensional in a neighborhood of $[v]$.*

Proof. See [100, Proposition 1.9]. \square

From now on we consider a Lagrangian subspace $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ that contains no non-zero decomposable vector.

Theorem 4.15. *Let $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ be a Lagrangian subspace that contains no non-zero decomposable vector. Then Y_A^3 is finite and smooth (as a scheme).*

Proof. See [98, Claim 3.7]. □

When Y_A^3 is non-empty, it does not have the expected dimension, cfr. Remark 4.9.

Remark 4.16. Proposition 4.11, Proposition 4.13, Proposition 4.14 and Theorem 4.15 rely on a local description, in a neighborhood of $[v] \in \mathbb{P}(V_6)$, of the *EPW* sextic in terms of the Pfaffian quadrics (cfr. Definition 4.1) of $F_v \cong \wedge^2 V_5$ for some V_5 such that (4.3) holds. The link with the description of a *EPW* sextic as a quadratic degeneracy locus is [100, Proposition 1.3]. We do not introduce them now to keep the dissertation as simple as possible, but they will play a key role in Section 4.5.

Proposition 4.17 (O’Grady). *Let $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ be a Lagrangian subspace which contains no non-zero decomposable vector. Then $Y_A^{\geq 2}$ is a normal Cohen-Macaulay degree-40 surface in $\mathbb{P}(V_6)$; its singular locus is the finite smooth scheme Y_A^3 .*

Proof. Call h the class of a hyperplane section on $\mathbb{P}(V_6)$. In Lemma 4.7 we checked that the first Chern class of F is $-6h$; here we will use a more precise description given by O’Grady in [96, Equation 1.8]: the total Chern class of the vector bundle F is $c(F) = 1 - 6h + 18h^2 - 34h^3 + \dots$. If $Y_A^{\geq 2}$ is empty, or if it has the expected dimension, that is 2 (see Remark 4.9), by [42, Formule 6.7] its cohomology class as a cycle is $2c_3(F) - c_1(F)c_2(F) = 40h^3$. In particular it is non-trivial, so $Y_A^{\geq 2}$ is non-empty.

So by Remark 4.9 the dimension of $Y_A^{\geq 2}$ is at least 2; by Theorem 4.15 there exists at least one point inside $Y_A^2 = Y_A^3 - Y_A^{\geq 2}$, so by Proposition 4.14 the variety Y_A^2 is a surface. Again by Theorem 4.15 the variety $Y_A^{\geq 2}$ is a surface too, as it is obtained by adding a finite number of points to Y_A^2 . By the calculation of the cohomology class we also have that it has degree 40.

By Remark 4.9 we know that $Y_A^{\geq 2}$ is Cohen-Macaulay and it is smooth in codimension two, since by Proposition 4.14 it is smooth on the open Y_A^2 . This proves that $Y_A^{\geq 2}$ is normal, as a Cohen-Macaulay variety which is smooth in codimension one.

We already proved $Y_A^3 \supseteq \text{Sing}(Y_A^{\geq 2})$; a degeneration locus is singular where the rank of the associated map drops, so we have the last inclusion we need, $Y_A^3 \subseteq \text{Sing}(Y_A^{\geq 2})$. □

Since Y_A is an hypersurface, it is Cohen-Macaulay. We have, by Proposition 4.17, that its singular locus has codimension 2: this means that Y_A is normal.

Corollary 4.18 (O’Grady). *Let $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ be a Lagrangian subspace which contains no non-zero decomposable vector. Then Y_A is an integral normal hypersurface in $\mathbb{P}(V_6)$.*

Recall that, while $Y_A^{\geq 2}$ is always normal, see Proposition 4.17, it is smooth if and only if $Y_A^3 = \emptyset$.

Proposition 4.19. *Let $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ be a Lagrangian subspace which contains no non-zero decomposable vector. Then the surface $Y_A^{\geq 2}$ is integral i.e. irreducible and reduced. If $Y_A^{\geq 2}$ is smooth then*

$$2K_{Y_A^{\geq 2}} = \mathcal{O}_{Y_A^{\geq 2}}(6),$$

where $K_{Y_A^{\geq 2}}$ is its canonical bundle, in particular it is a surface of generale type.

Proof. See [27, Theorem B.2] for the irreducibility. When $Y_A^{\geq 2}$ is smooth see [39, Proposition 1.13] for the canonical of $Y_A^{\geq 2}$. \square

4.2.3 The dual of an EPW sextic

One important property of EPW sextics is that they admit a dual: the volume form on $\wedge^6 V_6$ induces a volume form on $\wedge^6 V_6^\vee \cong (\wedge^6 V_6)^\vee$ too, and thus a symplectic form on $\wedge^3 V_6^\vee \cong (\wedge^3 V_6)^\vee$, so that we can consider Lagrangian subspaces in $\mathbb{L}\mathbb{G}(\wedge^3 V_6^\vee)$ too. This has been pointed out by O’Grady, for an exhaustive treatment see for example [100, Section 1.6].

Given $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ a Lagrangian subspace, we define

$$A^\perp = \{\phi \in (\wedge^3 V_6)^\vee \mid \phi(a) = 0 \text{ for every } a \in A\}.$$

By the definition of the volume form on the dual, the subspace A^\perp is Lagrangian too, so we can define an isomorphism between varieties

$$\begin{aligned} \delta : \mathbb{L}\mathbb{G}(\wedge^3 V_6) &\rightarrow \mathbb{L}\mathbb{G}((\wedge^3 V_6)^\vee) \\ A &\mapsto A^\perp. \end{aligned}$$

Since A^\perp is a Lagrangian subspace, we can define a EPW sextic on the dual too. For any $v \in V_6$, the fiber F_v of the vector bundle F is a Lagrangian subspace, so we can check the image of F_v under δ .

Consider $v \in V_6$, and $V_5 \subset V_6$ as in (4.3); we have an isomorphism of vector spaces between V_5 and the orthogonal complement $(\mathbb{C}v)^\perp = \{\phi \in V_6^\vee \mid \phi(v) = 0\}$. Fix a basis \mathcal{B} of V_5 , seen as the orthogonal complement of $\mathbb{C}v$. For every $v \wedge \eta \in F_v$ and $\varepsilon_1, \dots, \varepsilon_3 \in \mathcal{B}$ we have

$$(\varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_3)(v \wedge \eta) = \pm \varepsilon_1(v) \cdot (\varepsilon_2 \wedge \varepsilon_3)(\eta) = 0.$$

This means that $\wedge^3 V_5 \in \mathbb{L}\mathbb{G}((\wedge^3 V_6)^\vee)$ is contained in $\delta(F_v)$. Both vector spaces have dimension 10, so they coincide, hence $\delta(F_v) = \wedge^3 V_5$.

Remark 4.20. We have the following description of the EPW strata associated to A^\perp . Consider $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$, then

$$\begin{aligned} Y_{A^\perp}^{\geq k} &= \{[V_5] \in \mathbb{P}(V_6^\vee) \mid \dim(\wedge^3 V_5 \cap A) \geq k\} \\ Y_{A^\perp}^k &= \{[V_5] \in \mathbb{P}(V_6^\vee) \mid \dim(\wedge^3 V_5 \cap A) = k\}. \end{aligned}$$

Because of the characterization of $\delta(F_v)$ that we gave above, it is sufficient to prove that the map δ leaves invariant the dimension of the intersection between two Lagrangian subspaces, which we verify now.

Take $A_1, A_2 \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$, and call $K \subset (\wedge^3 V_6)^\vee$ the orthogonal complement of $A_1 + A_2$. Clearly $\dim K = \dim(A_1 \cap A_2)$ and $\wedge^3 V_6 \cong (A_1 + A_2) \oplus K$. Now for $h = 1, 2$ consider \widetilde{A}_h the subspaces of $\wedge^3 V_6$ such that $A_h = (A_1 \cap A_2) \oplus \widetilde{A}_h$: since $A_1^\perp \cong \widetilde{A}_2 \oplus K$ and vice versa, we have $A_1^\perp \cap A_2^\perp = K$.

The following result can be found, for example, in [100, Section 1.6].

Lemma 4.21. *Consider $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ a Lagrangian subspace. Then A contains no non-zero decomposable vectors if and only if A^\perp contains no non-zero decomposable vectors.*

Proof. Consider a Lagrangian subspace $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ that contains a decomposable vector, let say that there exists a three-dimensional $W \subset V_6$ such that $\wedge^3 W \subset A$. Then we have $\wedge^3 W^\perp \subset A^\perp$. \square

Corollary 4.22. *If A contains no non-zero decomposable vector, both Y_A and Y_{A^\perp} are EPW sextics.*

Proof. This comes directly from Proposition 4.11 and the lemma above. \square

Definition 4.23. *Consider $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ such that A does not contain any non-zero decomposable vector. We call Y_{A^\perp} the dual EPW sextic of the EPW sextic Y_A .*

It is natural to ask for the relation between Y_{A^\perp} and Y_A . An interesting property is that, when $\Theta_A = \emptyset$, the associated EPW sextic Y_A is projectively dual to the dual EPW sextic Y_{A^\perp} , in the classical sense: it is the closure of the union of the tangent spaces of the smooth points of Y_A .

Consider the flag variety $\text{Fl}(1, 5; V_6)$ of flags (v, V_5) where $v \in V_6$, $V_5 \subset V_6$ is a hyperplane such that $v \in V_5$. Inside it we have the scheme

$$\widehat{Y}_A = \{(v, V_5) \in \text{Fl}(1, 5; V_6) \mid A \cap (v \wedge \bigwedge^2 V_5) \neq 0\}.$$

Note that if $(v, V_5) \in \widehat{Y}_A$ then $A \cap F_v \neq 0$ and $A \cap \wedge^3 V_5 \neq 0$, so we have the natural projections induced by the flag variety,

$$\begin{array}{ccc} & \widehat{Y}_A & \\ p \swarrow & & \searrow q \\ Y_A & & Y_{A^\perp}. \end{array} \quad (4.7)$$

Theorem 4.24. *Consider $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$. If A contains no non-zero decomposable vectors, then the hypersurfaces $Y_A \subset \mathbb{P}(V_6)$ and $Y_{A^\perp} \subset \mathbb{P}(V_6^\vee)$ are projectively dual, $Y_A^\vee = Y_{A^\perp}$.*

Moreover \widehat{Y}_A is irreducible and $q \circ p^{-1}$ is a birational isomorphism that sends $[v] \in Y_A$ to its tangent space, so that the map $q \circ p^{-1}$ realizes the projective duality between these two hypersurfaces.

Proof. See [27, Proposition B.3]. \square

In fact, to have the projective duality it is sufficient to ask that A contains finitely many decomposable vectors, as Debarre and Kuznetsov verified, always in [27, Proposition B.3].

The *conormal variety* of Y_A is the closure of the locus of pairs (p, T) inside $\mathbb{P}(V_6) \times \mathbb{P}(V_6^\vee)$ such that $p \in Y_A^1$ (so that p lies in the smooth locus of Y_A) and T is the tangent space of Y_A at p , see [36, Section 10.6]. The proof of Theorem 4.24 implies that, when A contains no non-zero decomposable vectors, the conormal variety of Y_A is the closure of

$$\{([v], [V_5]) \mid [v] \in Y_A^1, (v, V_5) \in \widehat{Y_A}\}.$$

So the birational map $q \circ p^{-1} : Y_A \dashrightarrow Y_{A^\perp}$ is the *Gauss map* i.e. in coordinates, the map $Y_A \dashrightarrow \mathbb{P}(V_6^\vee)$ sending $p \in Y_A^1$ to the class of the partial derivatives of the equation of Y_A .

Later we will describe some properties of the action of the map $\delta : \mathbb{L}\mathbb{G}(\wedge^3 V_6) \rightarrow \mathbb{L}\mathbb{G}((\wedge^3 V_6)^\vee)$ on the moduli space of *EPW sextics*.

4.2.4 Automorphisms of *EPW sextics*

Here we present some results about automorphisms on *EPW sextics*. We will refer to some results about double covers of *EPW sextics*, which we present in Section 4.3. Most of the results are from O'Grady, although we refer mostly to [27, Section B.3], where Debarre and Kuznetsov summarized what is known about this topic. The first part of Theorem 4.26 comes from [81, Proposition 1.4.7]: we will expand on this in Proposition 4.67.

Take $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ such that Y_A is a *EPW sextic* i.e. it is not the whole projective space. Then the Picard group $\text{Pic}(Y_A)$ is generated by $\mathcal{O}_{Y_A}(1)$, by the Lefschetz hyperplane theorem, see [53, Exposé XII, Corollary 3.7]. Recall that Y_A is a *EPW sextic* if A contains no non-zero decomposable vector, see Proposition 4.11.

We make use of the following result, which is obtained by Debarre and Kuznetsov through the study of the linear systems $|\mathcal{O}_{Y_A^{\geq 2}}(1) - \mathcal{O}_{Y_A}(E)|$ and $|\mathcal{O}_{Y_{A^\perp}}(E')|$, where E (resp. E') is the exceptional divisor of the birational morphism p (resp. q) in (4.7).

Proposition 4.25. *Let $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ be a Lagrangian subspace which contains no non-zero decomposable vector. Then the surface $Y_A^{\geq 2}$ is not contained in any hyperplane of $\mathbb{P}(V_6)$.*

Proof. See [27, Lemma B.6]. □

Theorem 4.26. *Consider $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ and $A' \in \mathbb{L}\mathbb{G}(\wedge^3 V_6')$ such that Y_A and $Y_{A'}$ are *EPW sextics*.*

- 1) *any linear isomorphism $g \in \text{PGL}(V_6)$ such that $(\wedge^3 g)(A) = A'$ induces an isomorphism $Y_A \xrightarrow{\sim} Y_{A'}$.*

If, moreover, A contains no non-zero decomposable vectors then

- 2) *A' contains no non-zero decomposable vectors,*
- 3) *any isomorphism $Y_A \xrightarrow{\sim} Y_{A'}$ comes from a linear morphism $g \in \text{PGL}(V_6)$ such that $(\wedge^3 g)(A) = A'$.*

Proof. See [27, Proposition B.8]. For 1) it is sufficient to observe that $g : \mathbb{P}(V_6) \rightarrow \mathbb{P}(V'_6)$ sends Y_A^k to $Y_{A'}^k$ since

$$\dim(F_v \cap A) = \dim\left(\left(\bigwedge^3 g\right)(F_v) \cap \left(\bigwedge^3 g\right)(A)\right) = \dim(F_{g(v)} \cap A').$$

To prove 2) observe that, since Y_A and $Y_{A'}$ has Picard rank one, any isomorphism $Y_A \xrightarrow{\sim} Y_{A'}$ comes from a linear morphism $g \in PGL(V_6)$. Clearly $\text{Sing}(Y_A)$ and $\text{Sing}(Y_{A'})$ are isomorphic; we know by Proposition 4.19 that $\text{Sing}(Y_A) = Y_A^{\geq 2}$, hence $\text{Sing}(Y_{A'})$, is an irreducible surface and is not a plane, which cannot happen if A contains a non-zero decomposable vector, see Proposition 4.13.

For 3) we need the double covers that we will define in Section 4.3; recall that Y_A and $Y_{A'}$ are normal by Corollary 4.18. Let X_A and $X_{A'}$ be the associated double covers; the restrictions $f_A^{-1}(Y_A^1) \rightarrow Y_A^1$ and $f_{A'}^{-1}(Y_{A'}^1) \rightarrow Y_{A'}^1$ are universal covers, see Corollary 4.39. Consider now an isomorphism $\alpha : Y_A \xrightarrow{\sim} Y_{A'}$. It sends the singular locus of Y_A to the singular locus of $Y_{A'}$, hence Y_A^1 to $Y_{A'}^1$; by the property of the universal covers, the automorphism lifts to an automorphism $f_A^{-1}(Y_A^1) \xrightarrow{\sim} f_{A'}^{-1}(Y_{A'}^1)$, which can be extended in a unique way to the branched cover. So there exists an automorphism $\tilde{\alpha}$ such that the following diagram

$$\begin{array}{ccc} X_A & \xrightarrow{\tilde{\alpha}} & X_{A'} \\ \downarrow f_A & & \downarrow f_{A'} \\ Y_A & \xrightarrow{\alpha} & Y_{A'} \end{array}$$

commutes, in other words $X_A \rightarrow Y_A$ and $X_{A'} \rightarrow Y_{A'}$ are isomorphic as covers, see Section 1.4. By Proposition 4.44 this implies the claim. \square

In this thesis we only deal with Lagrangian subspaces $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ that contains no non-zero decomposable vectors. In this setting, the following characterization of the automorphisms of Y_A holds; the proof comes directly from Theorem 4.26.

Corollary 4.27. *Consider $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ a Lagrangian subspace that contains no non-zero decomposable vectors. Then $\text{Aut}(Y_A) = PGL_A(V_6)$, where*

$$PGL_A(V_6) = \left\{ g \in PGL(V_6) \mid \left(\bigwedge^3 g\right)(A) = A \right\}.$$

The finiteness of the group $\text{Aut}(Y_A)$, when $\mathbb{P}(A) \cap G(3, V_6) = \emptyset$, is obtained by O'Grady in the context of the study of stability of points, inside $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$, for the action of $PGL(V_6)$ (by Corollary 4.27, for Lagrangian subspaces without decomposable vectors we have $\text{Aut}(Y_A) \subset PGL(V_6)$). The statement given by O'Grady is [99, Theorem 2.1.1]: O'Grady described through flag conditions the locus of non-stable elements of $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ and listed them in [99, Table (1)]: a case-by-case analysis of the flag conditions, by means of Lemma 4.10 or easily-rearranged versions of it, shows that all Lagrangian subspaces in the list contain at least one non-zero decomposable vector.

Here we present an alternative proof of this result, which does not appeal to the classification made by O'Grady. Instead, our proof is mostly based on the study of the surface $Y_A^{\geq 2}$.

Proposition 4.28. *If A contains no non-zero decomposable vectors then $\text{Aut}(Y_A)$ is a finite group.*

Proof. We begin with the case of $Y_A^3 = \emptyset$; in this case $Y_A^2 = Y_A^{\geq 2}$ is an integral smooth surface of general type, see Proposition 4.19. Any morphism of Y_A sends Y_A^2 to Y_A^2 since it is the singular locus of the *EPW* sextic. So we have a restriction morphism $\rho : \text{Aut}(Y_A) \cong \text{PGL}_A(V_6) \rightarrow \text{Aut}(Y_A^2)$; by Proposition 4.25 the surface spans the whole $\mathbb{P}(V_6)$, hence ρ is injective. Then it is sufficient to observe that the automorphism group of a surface of general type is finite, see [113] (a more general statement can be found in [54, Theorem 1.1]). In the case of $Y_A^{\geq 2}$ singular, then we can use the link between Gushel-Mukai varieties and *EPW* sextics, which we will present in Section 4.5.2: by Remark 5.42, the automorphism group of Y_A sits in a short exact sequence between two finite groups, hence it is finite too. \square

As we will show in Section 4.3.3, Lagrangian subspaces that do not contain any non-zero decomposable vector fill the complement of an irreducible divisor inside $\mathbb{L}\mathbb{G}(\wedge^3 V_6)$.

Proposition 4.29. *If A contains no non-zero decomposable vectors then $\text{Aut}(Y_A) = \{\text{id}\}$ for $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ very general.*

Proof. See [27, Theorem B.9, Item b)]. \square

4.3 Double *EPW* sextics

4.3.1 Definition of Double *EPW* sextics

The importance of *EPW* sextics comes mostly from the fact that, in the general case, they admits a ramified double cover which is a hyperkähler manifold. The existence of the double cover was proved by O’Grady in [96]. The rank-one, reflexive and 2-torsion Weil divisor associated to the double cover of a *EPW* sextic (see Lemma 1.43) is well presented in [96, Section 4] or [95, Sections 1.1 and 1.2]. The construction of this sheaf is also recalled in section 2.2 of Ferretti’s PhD thesis, see [38]. Here we present Debarre and Kuznetsov’s approach to the construction, which is very similar but more general. We will refer to the results we presented in Section 1.4 to give an insight of the theory of double covers of Lagrangian intersection loci. For a complete treatment of the topic see [28].

As we already remarked, the scheme $Y_A^{\geq k}$ is the locus of points at which the map $\lambda_A : F \rightarrow \mathcal{O}_{\mathbb{P}(V_6)} \otimes A^\vee$ has corank at least k i.e. the *Lagrangian degeneracy locus* of F and $\mathcal{O}_{\mathbb{P}(V_6)} \otimes A^\vee$. The *first Lagrangian cointersection sheaf* is the cokernel of the restriction of λ_A to Y_A ; we denote

$$\mathcal{R} = \left(\text{Coker}(\omega_{\mathcal{A}_1, \mathcal{A}_2}|_{Y_A}) \right)^{\vee\vee}.$$

The goal of this section is to present the following, of which we sketch a proof after some preliminary results. As usual, the set of non-zero decomposable vectors inside A is denoted by Θ_A .

Theorem 4.30. (*O'Grady*) Consider $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ such that $\Theta_A = \emptyset$. Then there is a unique double cover $f_A : X_A \rightarrow Y_A$ with branch locus $Y_A^{\geq 2}$ such that

$$(f_A)_* \mathcal{O}_{X_A} \cong \mathcal{O}_{Y_A} \oplus \mathcal{R}(-3), \quad (4.8)$$

where \mathcal{R} is the first Lagrangian cointersection sheaf. The variety X_A is integral and normal, and it is smooth away from $f_A^{-1}(Y_A^3)$. In particular, if $Y_A^3 = \emptyset$ then X_A is smooth.

Definition 4.31. We call double EPW sextic the double cover X_A , and ι_A the associated covering involution (see Section 1.4 for the action of the involution on the structure sheaf of X_A).

To make the construction clearer, we present a more general construction. Let \mathcal{V} be a vector bundle of rank $2n$ on a normal variety S and let $\omega : \mathcal{V} \times \mathcal{V} \rightarrow \mathcal{O}_S$ be a family of symplectic forms on \mathcal{V} . For any two subbundles $\mathcal{U}_1 \hookrightarrow \mathcal{V}$ and $\mathcal{U}_2 \hookrightarrow \mathcal{V}$ we denote by $\omega_{\mathcal{U}_1, \mathcal{U}_2}$ the map that, on the fiber over $s \in S$, is $\omega_{(\mathcal{U}_1)_s, (\mathcal{U}_2)_s}$ (see (4.1) for the definition). A *Lagrangian subbundle* in \mathcal{V} is a rank- n vector subbundle $\mathcal{A} \hookrightarrow \mathcal{V}$ such that $\omega_{\mathcal{A}, \mathcal{A}}$ vanishes on the whole S .

Definition 4.32. Let \mathcal{A}_1 and \mathcal{A}_2 be Lagrangian subbundles of \mathcal{V} . The Lagrangian degeneracy loci of \mathcal{A}_1 and \mathcal{A}_2 are the schemes

$$S_k = \{s \in S \mid (\omega_{\mathcal{A}_1, \mathcal{A}_2})_s \text{ has corank at least } k\}.$$

The *Lagrangian intersection locus* S_k is the locus of points at which the intersection of the fibers of \mathcal{A}_1 and \mathcal{A}_2 has dimension at least k . Clearly $S_0 = S$.

Definition 4.33. For $k \geq 0$, let S_k be the corresponding Lagrangian degeneracy loci of \mathcal{A}_1 and \mathcal{A}_2 . The Lagrangian cointersection sheaf $\mathcal{C}(\mathcal{A}_1, \mathcal{A}_2)$ is the cokernel of the restriction of $\omega_{\mathcal{A}_1, \mathcal{A}_2}$ to S_k ,

$$\mathcal{C}_k = \text{Coker}(\omega_{\mathcal{A}_1, \mathcal{A}_2})|_{S_k}$$

Consider the map defined at $s \in S$ as $(\mathcal{A}_1)_s \oplus (\mathcal{A}_2)_s \xrightarrow{\text{id} \oplus -\text{id}} \mathcal{V}_s$. The order of \mathcal{A}_1 and \mathcal{A}_2 is not clear from the notations S_k and \mathcal{C}_k ; the reason is that the Lagrangian degeneration locus S_k is the locus of $s \in S$ at which the map above has corank at least k . In the same way, the k -th Lagrangian cointersection sheaf \mathcal{C}_k is the cokernel of the same map, restricted to S_k .

Now we assume that S_k is normal for every $k \geq 0$ and that $S_k^0 = S_k - S_{k+1}$ is dense in S_k . Fixed $k \geq 0$, we consider the sheaf $\mathcal{R}_k = (\bigwedge^k \mathcal{C}_k)^{\vee\vee}$; since cokernel and alternating product of coherent sheaves are coherent, \mathcal{R}_k is a rank-one reflexive sheaf.

Theorem 4.34. [28, Theorem 4.2] Assume that S_k is normal, and that $\text{codim}_{S_k}(S_{k+1}) \geq 2$. For each line bundle \mathcal{M} on S_k such that

$$(\mathcal{L}^{\otimes (-n-k)} \otimes \det(\mathcal{A}_1) \otimes \det(\mathcal{A}_2))|_{S_k} \cong \mathcal{M}^{\otimes 2}, \quad (4.9)$$

there is a double cover $f_{\mathcal{M}} : \tilde{S}_k \rightarrow S_k$, with S_k normal, that satisfies the following two properties:

a) there is an isomorphism

$$f_{\mathcal{M}*}\mathcal{O}_{\tilde{S}_k} \cong \mathcal{O}_{S_k} \oplus (\mathcal{M} \otimes \mathcal{R}_k),$$

b) the morphism $f_{\mathcal{M}}$ is étale over the dense open subset S_k^0 .

If all invertible functions on S_k are squares, such a double cover is unique up to isomorphism.

Proof. (Sketch) The main point is that, under these hypothesis, the reflexive sheaf $\mathcal{M} \otimes \mathcal{R}_k$ is self dual. The statement come then directly from the statement of Theorem 1.39. \square

We present a result that will be useful for the description of the double cover of a *EPW* sextic. It comes from a local description of the Lagrangian degeneracy loci as quadratic degeneracy loci, something we already presented in the particular case of *EPW* strata, see Section 4.2.1 and Remark 4.16. As in Remark 4.9, we say that $S_i - S_{i+1}$ has *expected dimension* if its codimension in S is exactly $i(i+1)/2$.

Proposition 4.35. *Assume that S is smooth, that $S_i - S_{i+1}$ is nonsingular of expected dimension for each $i \in \{k, k+1\}$, and that $S_{k+2} = \emptyset$. For each choice of a line bundle \mathcal{M} on S_k satisfying (4.9), giving rise to a double cover $f_{\mathcal{M}} : \tilde{S}_k \rightarrow S_k$, we have*

- the scheme \tilde{S}_k is smooth,
- the branch locus of $f_{\mathcal{M}}$ is equal to S_{k+1} ,
- the preimage of the branch locus is the first order infinitesimal neighborhood of the ramification locus.

Proof. See [28, Corollary 4.8]. \square

We can now prove the main result of this section.

Proof of Theorem 4.30. Take $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ such that $\Theta_A = \emptyset$. We rewrite Theorem 4.34 in the setting of *EPW* sextics. Take

- $\mathbb{P}(V_6)$ as the normal variety S ;
- the trivial vector bundle $\mathcal{O}_{\mathbb{P}(V_6)} \otimes \wedge^3 V_6$ as \mathcal{V} ;
- the constant family of symplectic forms ω on $\mathcal{O}_{\mathbb{P}(V_6)} \otimes \wedge^3 V_6$;
- the trivial subvector bundle $A \otimes \mathcal{O}_{\mathbb{P}(V_6)}$ as the Lagrangian subbundle \mathcal{A}_1 ;
- the vector bundle F as the Lagrangian subbundle \mathcal{A}_1 (see Definition 4.4).

Looking at the corresponding Lagrangian degeneracy locus, with these data we have $\lambda_A = \omega_{\mathcal{A}_1, \mathcal{A}_2}$, $Y_A^{\geq k} = S_k$ and $Y_A^k = S_k^0 = S_k - S_{k+1}$. Moreover, we know by Proposition 4.17 and Corollary 4.18 that $Y_A = S_1$ is a normal fourfold and $Y_A^{\geq 2} = S_2$ has codimension 2, so the hypotheses of Theorem 4.34 hold. Here the discriminant group of \mathcal{A}_1 is trivial, so (4.9) becomes, by Lemma 4.7,

$$\mathcal{O}_{Y_A^{\geq k}}(-6) \cong \mathcal{M}^{\otimes 2}$$

for some line bundle \mathcal{M} on $Y_A^{\geq k}$. For this reason, we take $\mathcal{M} = \mathcal{O}_{Y_A^{\geq k}}(-3)$. Finally, unicity of X_A comes from the fact that, by connectedness of X_A , every functions on Y_A are constant functions, which in turn are squares.

By Theorem 4.34 for every $k \in \{0, \dots, 3\}$ there exists a double cover of $Y_A^{\geq k}$, ramified over $Y_A^{\geq k+1}$: here we consider the double cover $f_A : X_A \rightarrow Y_A$ associated to $Y_A^{\geq 1}$. By construction (4.8) holds and X_A is normal. As in the statement, we call \mathcal{R} the first Lagrangian cointersection sheaf: the branch locus of f_A is non-empty so $\mathcal{R}(-3)$ is not trivial, hence X_A is integral by Lemma 1.41.

The fact that the branch locus of f_A is the whole Y_A^2 comes from Proposition 4.35 with $S = \mathbb{P}(V_6) - Y_A^3$, and smoothness away from $f_A^{-1}(Y_A^3)$ as well. \square

O'Grady built the double cover X_A in [96]. In fact, even when A contains some decomposable vectors, the double cover $f_A : X_A \rightarrow Y_A$ can be constructed, provided that $Y_A \neq \mathbb{P}(V_6)$. See [98, Proposition 1.9] for the general definition of the multiplication $\mathcal{R}(-3) \times \mathcal{R}(-3) \rightarrow \mathcal{O}_{Y_A}$ that leads to the definition of the double cover, see again Section 1.4. In any case, the double cover of a EPW sextic Y_A is singular at any point of $\mathbb{P}(W) \subset Y_A$ for $W \in \Theta_A$, see for example the proof of [98, Proposition 3.4]: since f_A is étale outside $Y_A^{\geq 2}$, the only non-trivial case is when a point of $\mathbb{P}(W)$ lies in $Y_A^{\geq 2}$.

We consider again A that does not contain any decomposable vector; by Theorem 4.30 the singularity locus is contained in X_A^3 . A local study of a point in X_A^3 allows to say more.

Theorem 4.36. *Let $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ be a Lagrangian subspace such that $\Theta_A = \emptyset$. Then the singular locus of the double EPW sextic X_A is*

$$\text{Sing}(X_A) = f_A^{-1}(Y_A^3)$$

so that X_A is smooth if and only if $Y_A^3 = \emptyset$.

Proof. See [98, Claim 1.13]. \square

4.3.2 Double EPW sextics as hyperkähler manifolds

General double EPW sextics are hyperkähler manifolds of $K3^{[2]}$ -type. O'Grady proved it in [95]; here we propose another proof, always by O'Grady, that appears in [98].

Theorem 4.37. [98, Theorem 4.25] *Let $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ be a Lagrangian subspace such that $\Theta_A = \emptyset$, and $Y_A^3 = \emptyset$. Then X_A is a hyperkähler manifold of $K3^{[2]}$ -type.*

Proof. The proof is divided in some steps.

- a) A double *EPW* sextic X_A is smooth if and only if Θ_A and $Y_A^{[3]}$ are empty: this is Theorem 4.36 when A contains no non-zero decomposable vectors. We explain at the end of the last section the case $\Theta_A \neq \emptyset$.
- b) Every two smooth double *EPW* sextics are deformation equivalent: O'Grady proved in [98, Proposition 3.1] that, for every $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6) - \Sigma$ such that $Y_A \neq \mathbb{P}(V_6)$, there exists a small open $\mathcal{U} \subset \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ which parametrizes double covers of *EPW* sextics associated to Lagrangian subspaces inside \mathcal{U} . Note that we should take \mathcal{U} such that, for every $B \in \mathcal{U}$, the associated Y_B is not the whole $\mathbb{P}(V_6)$. Since $\mathbb{L}\mathbb{G}(\wedge^3 V_6)$, hence $\mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$, is irreducible, every two smooth double covers are deformation equivalent. For details about the family of deformation $\mathcal{X}_A \rightarrow \mathcal{U}$ of X_A see Section 5.3.2.
- c) Any smooth Kähler deformation of a hyperkähler manifold is still a hyperkähler manifold (see Proposition 2.13), so to prove the claim it is sufficient to show a particular X_A which is deformation equivalent to the Hilbert square of some *K3* surface. By Remark 1.40, every double cover is projective, hence Kähler.
- c) An explicit example of a Hilbert square $S^{[2]}$ which deforms to a double *EPW* sextic is provided at Corollary 5.30. The *K3* surface has to be chosen in this way: let V_5 be a 5-dimensional vector space, $G(2, V_5) \subset \mathbb{P}(\wedge^2 V_5)$ the Grassmannian of planes in V_5 through the Plücker embedding. Choose $W \subset \wedge^2 V_5$ a general 7-dimensional vector space, and $Q \subset \mathbb{P}(W)$ a general quadric hypersurface in $\mathbb{P}(W)$. Then the dimensionally transverse intersection

$$S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$$

with the polarization induced by the line bundle $\mathcal{O}_{\mathbb{P}(W)}(1)$ is a Brill-Noether general *K3* surface.

□

Remark 4.38. The embedded *K3* surface appearing in Theorem 4.37 is an example of *ordinary Gushel-Mukai variety*. We will see in Section 4.5.2 that there is a systematic relation between the family of Gushel-Mukai varieties and *EPW* sextics.

A simple corollary of Theorem 4.37 is the following.

Corollary 4.39. *Let $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ be a Lagrangian subspace such that $\Theta_A = \emptyset$, and $Y_A^{[3]} = \emptyset$. The étale cover $f_A^{-1}(Y_A^1) \rightarrow Y_A^1$ is the universal cover of the smooth fourfold Y_A^1 . Moreover, $\pi_1(Y_A^1) = \pi_1(Y_A) \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. Since X_A is a hyperkähler manifold, it is simply connected. The fundamental groups of X_A^1 is again trivial, since $f_A^{-1}(Y_A^{\geq 2})$ has codimension 2, so $f_A^{-1}(Y_A^1) \rightarrow Y_A^1$ is a universal cover. The universal cover of Y_A^1 is a double cover, so its fundamental group has cardinality two. The same hold for Y_A , since $Y_A^{\geq 2}$ has codimension 2 inside Y_A . □

Since X_A is of type $K3^{[2]}$ if $\Theta_A = \emptyset$ and $Y_A^3 = \emptyset$, its second integral cohomology group is isomorphic, as a lattice equipped with the BBF form, to $\Lambda_{K3^{[2]}} = U^{\oplus 3} \oplus E_8(-1) \oplus \langle -2 \rangle$. By Remark 1.40 we have a natural polarization on X_A , that is $f_A^* \mathcal{O}_{Y_A}(1)$. Note that, by Lefschetz hyperplane theorem, see [53, Exposé XII, Corollary 3.7], this is the only primitive polarization on X_A that comes from Y_A .

Definition 4.40. For $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ with $\Theta_A = \emptyset$, we denote by $D_A \in \text{Pic}(X_A)$ the pullback, on the double cover X_A , of the line bundle $\mathcal{O}_{Y_A}(1)$ through $f_A : X_A \rightarrow Y_A$.

When X_A is smooth, we can compute the square of the natural polarization on X_A with respect to the BBF form.

Lemma 4.41. For $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ with $\Theta_A = \emptyset$ and $Y_A^3 = \emptyset$, the polarization D_A on X_A has square 2 with respect to the BBF form.

Proof. By definition of BBF form, see Theorem 2.3, the equality $c_{X_A}(\alpha, \alpha)^2 = \int_X \alpha^4$ holds, where c_{X_A} is the Fujiki constant; in the case of manifolds of $K3^{[n]}$ -type this last is equal to 3. Then the claim follows, since Y_A is a sextic and $f_A : X_A \rightarrow Y_A$ is a double cover, so $\int_{X_A} D_A^4 = 12$. \square

Any double EPW sextic X_A admits a covering involution $\iota_A \in \text{Aut}(X_A)$. Moreover, when X_A is smooth we have $D_A^2 = 2$ by Lemma 4.41, so there exists a non-symplectic involution on X_A whose action in cohomology is $-R_{D_A}$, cfr. Theorem 2.154. By definition the polarization D_A is ι_A -invariant. A priori it is difficult to deduce the action in cohomology of an automorphism which is described geometrically. However, in this case, it is always possible to prove that ι_A is exactly the involution acting as $-R_{D_A}$.

Lemma 4.42. For $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ with $\Theta_A = \emptyset$ and $Y_A^3 = \emptyset$, the covering involution acts in cohomology as the inverse of the reflection with respect to $\mathbb{Z}D_A$. In particular ι_A is non-symplectic.

Proof. By Remark 1.35, the invariant lattice of ι_A is spanned by D_A , since, as we said above, the Picard group of Y_A is generated by the restriction to Y_A of the line bundle $\mathcal{O}_{\mathbb{P}(V_6)}(1)$. So $\iota_A^* \in O(H^2(X_A, \mathbb{Z}))$ is the extension to the whole $H^2(X_A, \mathbb{Z})$ of the isometry $\text{id} \oplus (-\text{id})$ on $\mathbb{Z}D_A \oplus (\mathbb{Z}D_A)^\perp$, so it is exactly $-R_{D_A}$. The non-symplecticity follows since D_A and any two-form ω are orthogonal. \square

Remark 4.43. By definition of the line bundle D_A , the map $\phi_{|D_A|} : X_A \dashrightarrow \mathbb{P}(H^0(X_A, D_A)^\vee) \cong \mathbb{P}^5$ is biregular and factorizes as

$$\phi_{|D_A|} : X_A \xrightarrow{2:1} X_A / \langle \iota_A \rangle \cong Y_A \hookrightarrow \mathbb{P}^5.$$

The first aim of [96] was indeed to study the map $\phi_{|D|}$ for (X, D) a general $\langle 2 \rangle$ -polarized manifold of $K3^{[2]}$ -type: O'Grady proved that (X, D) is in the form (X_A, D_A) for some Lagrangian subspace $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$, see Theorem 4.56.

More in general, in the next section we study the moduli space of smooth double EPW sextics with the polarization D_A . A key point for this is the following result, proved by O'Grady in [99].

Proposition 4.44. *Consider two Lagrangian subspaces $A, A' \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ such that $\Theta_A = \Theta_{A'} = \emptyset$. Then the double covers $f_A : X_A \rightarrow Y_A$ and $f_{A'} : X_{A'} \rightarrow Y_{A'}$ are isomorphic if and only if there exists some $\alpha \in PGL(V_6)$ such that $(\bigwedge^3 \alpha)(A) = A'$. Moreover, any isomorphism between covers comes from a linear morphism $g \in PGL(V_6)$ such that $(\bigwedge^3 g)(A) = A'$.*

Proof. See [99, Proposition 1.2.1]. The second part is not explicitly stated by O'Grady in the claim of the proposition, but it comes from the proof; this has been pointed out by Debarre and Kuznetsov in [27, proof of Proposition B.8, b)]. The idea is that, once the existence of an isomorphism between $X_A \rightarrow Y_A$ and $X_{A'} \rightarrow Y_{A'}$ is imposed, one can reduce to the case $Y_A = Y_{A'}$, since the isomorphism of covers descends to the *EPW* sextics and the Picard rank of these last is one by Lefschetz hyperplane theorem, see [53, Exposé XII, Corollary 3.7]. We can see $Y_A = Y_{A'}$ inside a projective space $\mathbb{P}(\mathbb{C}^6)$. Then O'Grady proved that the isomorphism induced on $\bigwedge^3 \mathbb{C}^6$ by the automorphism of covers sends A to A' and is scalar, so $A = A'$. \square

Remark 4.45. When X_A and $X_{A'}$ are manifolds of $K3^{[2]}$ -type, Proposition 4.44 can be rephrased in the following way: *the $\langle 2 \rangle$ -polarized manifolds (X_A, D_A) and $(X_{A'}, D_{A'})$ are isomorphic if and only if there exists some $\alpha \in PGL(V_6)$ such that $(\bigwedge^3 \alpha)(A) = A'$. Indeed, the two coverings are isomorphic if and only if there is an isomorphism $\phi : X_A \rightarrow X_{A'}$ which commutes with the covering involutions. By Torelli theorem this can be checked in cohomology; since the covering involutions acts respectively as $-R_{D_A}$ and $-R_{D_{A'}}$ in cohomology, $\phi : X_A \rightarrow X_{A'}$ commutes with them if and only if ϕ^* sends $D_{A'}$ to D_A .*

Actually, Proposition 4.44 holds everytime a double cover over Y_A can be constructed: it is sufficient to ask $Y_A \neq \mathbb{P}(V_6)$. In general, most of these results can be generalized to *EPW* sextics with good properties. For example, In Theorem 5.22, which we will present later, O'Grady proved that, in the case $Y_A^3 \neq \emptyset$ (but $\Theta_A = \emptyset$), there exist a $K3$ surface S and a birational morphism $S^{[2]} \rightarrow X_A$ whose definition locus has codimension 2. This proves that also in this case the second cohomology group $H^2(X_A, \mathbb{Z})$ can be endowed with a lattice structure and is isomorphic to $\Lambda_{K3^{[2]}}$ with this structure.

4.3.3 Moduli space of double *EPW* sextics

We start by studying the parameter space of *EPW* sextics, that is the Lagrangian Grassmannian $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$. By Theorem 4.36 and Theorem 4.37, we should describe the subspace of Lagrangian subspaces $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ such that A either contains some non-zero decomposable vector, or $Y_A^{\geq 3} \neq \emptyset$.

Definition 4.46. *We denote*

$$\Sigma = \{A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6) \mid \mathbb{P}(A) \cap G(3, V_6) \neq \emptyset\},$$

the set of Lagrangian subspaces containing a non-zero decomposable vector, and

$$\Delta = \{A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6) \mid Y_A^{\geq 3} \neq \emptyset\}.$$

the set of Lagrangian subspaces whose associated third stratum is not empty.

Remark 4.47. Clearly we can rewrite Definition 4.46 for $\mathbb{L}\mathbb{G}(\wedge^3 V_6^\vee)$. For simplicity we will call Σ and Δ the corresponding sets on $\mathbb{L}\mathbb{G}(\wedge^3 V_6^\vee)$ too. For every property and definition we give for $\Sigma, \Delta \subset \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ we assume the same for $\Sigma, \Delta \subset \mathbb{L}\mathbb{G}(\wedge^3 V_6^\vee)$.

Both Σ and Δ admit a structure of divisor of $\mathbb{L}\mathbb{G}(\wedge^3 V_6)$.

Theorem 4.48. *The subsets Σ, Δ are distinct irreducible divisors in $\mathbb{L}\mathbb{G}(\wedge^3 V_6)$. For $A \in \Delta$ general, the set $Y_A^{\geq 3}$ is a singleton.*

Proof. See [100, Proposition 2.1 Item 1)] for Σ , and [98, Proposition 2.2] for Δ . In the proof of [98, Proposition 2.2], O’Grady checks by a dimension count that $\Sigma \cap \Delta$ is not a divisor, so the two divisors does not coincide. \square

Remark 4.49. We can rephrase Lemma 4.10 as follows: for $A \in \Delta - \Sigma$ we have $Y_A^{\geq 3} = Y_A^3$. In particular this holds for $A \in \Delta$ general.

Definition 4.50. *We denote $\mathbb{L}\mathbb{G}(\wedge^3 V_6)^0 = \mathbb{L}\mathbb{G}(\wedge^3 V_6) - (\Delta \cup \Sigma)$.*

We can now riformulate Theorem 4.37 and Theorem 4.36 as follows.

Theorem 4.51 (O’Grady). *Consider $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$. Then X_A is a hyperkähler manifold of $K3^{[n]}$ -type if and only if $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$.*

Since Σ and Δ are divisors in $\mathbb{L}\mathbb{G}(\wedge^3 V_6)$, for A general the double EPW sextic X_A is a hyperkähler manifold.

We defined in Section 4.2.3 the map $\delta : \mathbb{L}\mathbb{G}(\wedge^3 V_6) \rightarrow \mathbb{L}\mathbb{G}((\wedge^3 V_6)^\vee)$, which sends a Lagrangian subspace A to its orthogonal complement A^\perp .

Proposition 4.52 (O’Grady). *The map $\delta : \mathbb{L}\mathbb{G}(\wedge^3 V_6) \rightarrow \mathbb{L}\mathbb{G}((\wedge^3 V_6)^\vee)$ sends Σ to Σ and does not send Δ to Δ .*

Proof. The claim about Σ is Lemma 4.21. The proof of the second part is given in [96, Section 6] by O’Grady: we rewrite it using the language of [67] and [27], for definitions and results we refer to Section 4.5.2 and Section 5.3.2. We prove that there exists $A \notin \Delta$ such that $\delta(A) \in \Delta$. Consider a Brill-Noether general (10)-polarized $K3$ surface S , seen as an ordinary Gushel-Mukai variety through the embedding $\phi_{|L|}$ i.e. $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$, where $V_5 \cong \mathbb{C}^5$ and $G(2, V_5)$ is the Grassmannian of planes inside V_5 , $W \subset \wedge^2 V_5$ is a 7-dimensional complex vector space and Q is a projective quadric hypersurface inside $\mathbb{P}(W)$. For (S, L) very general we can suppose that S has Picard rank one, so that, by Proposition 5.31, the moduli space $\{F \text{ is a stable sheaf on } S \mid \text{rk}(F) = 2, c_1(F) = L, c_2(F) = 5\} / \sim$ is isomorphic to the double EPW sextic X_A for some Lagrangian space $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$. To be more precise, in this setting $V_6 = H^0(\mathbb{P}(W), \mathcal{I}_S(2))$ is the space of quadrics through S . Since X_A is smooth, we have $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$. On the other hand, by Theorem 5.22 the dual double EPW sextic $X_{A^\perp} = X_{\delta(A)}$ has always at least one singular point, that corresponds through the map $c : S^{[2]} \rightarrow X_{\delta(A)}$ to the image of the plane $P_S \subset S^{[2]}$. Thus $\delta(A) \notin \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$ and in particular $\delta(A) \in \Delta$, since $Y_{\delta(A)}^3 \neq \emptyset$. \square

In Chapter 5 we will often refer to a third divisor inside $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$, different from Σ and Δ by Proposition 4.52.

Definition 4.53. We denote by $\Pi \subset \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ the irreducible divisor defined as the pullback through δ of $\Delta \subset \mathbb{L}\mathbb{G}(\bigwedge^3 V_6^\vee)$.

Remark 4.54. O'Grady calls $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)^{00}$ the open subspace $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)^0 - \Pi$.

We present the construction of the moduli space of covers $X_A \rightarrow Y_A$ i.e. the moduli space of $\langle 2 \rangle$ -polarized manifolds (X_A, D_A) , see Remark 4.45; this has been studied in detail by O'Grady in [99]. By Proposition 4.44 we are interested in the action of $PGL(V_6)$ on $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$. There is a unique linearization of this action, see [99, Section 1.3], so we can define unambiguously the GIT quotient

$$\mathbb{L}\mathbb{G}(\bigwedge^3 V_6) // PGL(V_6). \quad (4.10)$$

Clearly Δ and Σ are invariant under the action of $PGL(V_6)$, hence Π too. Moreover, by Proposition 4.28 the open subset $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6) - \Sigma$ is contained in the stable locus of $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ so the same holds for $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)^0 \subset \mathbb{L}\mathbb{G}(\bigwedge^3 V_6) - \Sigma$, the locus of Lagrangian subspaces whose associated double EPW sextic is smooth. Hence the GIT quotient $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)^0 // PGL(V_6)$ is the moduli space of double covers $X_A \rightarrow Y_A$ with X_A smooth, and $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6) // PGL(V_6)$ is a compactification of it. For an exhaustive description of the non-stable locus of $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ see [99, Theorem 2.1.1 and Table 1].

Now we consider $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)^0 \subset \mathbb{L}\mathbb{G}(\bigwedge^3 V_6) - \Sigma$ as a space of $\langle 2 \rangle$ -polarized manifolds of $K3^{[2]}$ -type i.e. elements of $\mathcal{M}_{2,1}^2$ (see Definition 2.69).

Definition 4.55. We denote by $\mathcal{U}_{EPW} \subset \mathcal{M}_{2,1}^2$ the locus of pairs (X, D) isomorphic to (X_A, D_A) for some $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)^0$.

Recall that, by the calculations made by Apostolov, the moduli space $\mathcal{M}_{2,1}^2$ is connected, or equivalently irreducible, see Proposition 3.2. Its dimension is 20. As we can see below, a general element inside $\mathcal{M}_{2,1}^2$ is a double EPW sextic with the natural polarization.

Theorem 4.56 (O'Grady). *The set \mathcal{U}_{EPW} is open and dense inside $\mathcal{M}_{2,1}^2$.*

Proof. For what we said above, we have a birational map

$$\mathbb{L}\mathbb{G}(\bigwedge^3 V_6) // PGL(V_6) \dashrightarrow \mathcal{M}_{2,1}^2 \quad (4.11)$$

$$[A] \mapsto (X_A, D_A) \quad (4.12)$$

which restricts to a regular map on $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)^0 // PGL(V_6) \rightarrow \mathcal{M}_{2,1}^2$; for regularity see [102, Theorem 0.2]. We consider this restriction: it is injective by Remark 4.45, hence its dimension equals the dimension of $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)^0 // PGL(V_6)$. We know, cfr. Theorem 4.3, that $\dim(\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)) = 55$. Since $\dim(PGL(V_6)) = 35$ we have $\dim(\mathbb{L}\mathbb{G}(\bigwedge^3 V_6) // PGL(V_6)) = 55 - 35 = 20$; as we said above,

$\mathbb{L}\mathbb{G}(\wedge^3 V_6)^0 \subset \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ is open and contained in the stable locus, hence it has dimension 20 too. So \mathcal{U}_{EPW} is open inside $\mathcal{M}_{2,1}^2$, and dense since $\mathcal{M}_{2,1}^2$ is irreducible. \square

The morphism $\delta : \mathbb{L}\mathbb{G}(\wedge^3 V_6) \rightarrow \mathbb{L}\mathbb{G}(\wedge^3 V_6^\vee)$ induces a birational involution

$$\mathcal{M}_{2,1}^2 \dashrightarrow \mathcal{M}_{2,1}^2 \tag{4.13}$$

$$[(X_A, D_A)] \mapsto [(X_{A^\perp}, D_{A^\perp})], \tag{4.14}$$

which is biregular on the image of $(\mathbb{L}\mathbb{G}(\wedge^3 V_6)^0 - \Pi) // \mathrm{PGL}(V_6)$ inside $\mathcal{M}_{2,1}^2$: the divisor $\Pi // \mathrm{PGL}(V_6)$ is sent to $\Delta // \mathrm{PGL}(V_6)$ which is outside the locus of classes of smooth double covers. As O'Grady showed, elements of $(\Delta - \Sigma) // \mathrm{PGL}(V_6)$ correspond to a family of hyperkähler manifolds with a big and nef line bundle. The construction is made in [95] and [98], and uses Theorem 5.22 which we will present in Chapter 5.

Remark 4.57. The involution defined in (4.13) is the birational involution defined in Section 2.3.4, see [95, Theorem 1.1]. To simplify notation, from now on we call δ the birational involution, on $\mathcal{M}_{2,1}^2$, induced by δ through (4.11).

We know that \mathcal{U}_{EPW} is open by a dimension count, but to understand the complement of it inside $\mathcal{M}_{2,1}^2$ is much more difficult. Consider the period map

$$P : \mathcal{M}_{2,1}^2 \rightarrow \mathcal{P}_{2,1}^2,$$

which, by the Global Torelli theorem for polarized hyperkähler manifolds of $K3^{[n]}$ -type, is an open embedding since $\mathcal{M}_{2,1}^2$ is irreducible. Recall that we define $\mathcal{C}_e = P^{-1}(\mathcal{D}_e)$, see Section 2.3.2. O'Grady proved the following.

Theorem 4.58. *The intersection $\mathcal{U}_{EPW} \cap \mathcal{C}_4$ is empty.*

Proof. See [102, Section 4.3]. \square

Remark 4.59. By Proposition 3.5, the very general element inside \mathcal{D}_4 is a polarized Hilbert square $(S^{[2]}, L_2 - \delta)$, with $(S, L) \in \mathcal{K}_4$. The involution on $(S^{[2]}, L_2 - \delta)$ is a Beauville involution, cfr. Example 2.164, so Theorem 4.58 means that the map $\phi_{|L_2 - \delta|} : S^{[2]} \rightarrow \mathbb{P}^5$ never induces a double cover of an *EPW* sextic inside \mathbb{P}^5 . Anyway, if we call $\iota \in \mathrm{Aut}(S^{[2]})$ the Beauville involution, the map associated to the linear system $|L_2 - \delta|$ factors as

$$\phi_{|L_2 - \delta|} : S^{[2]} \rightarrow S^{[2]} / \langle \iota \rangle \xrightarrow{3:1} G(2, \mathbb{C}^4) \hookrightarrow \mathbb{P}^5,$$

and $3G(2, \mathbb{C}^4)$ can be seen as a degenerate *EPW* sextic, for details see [39, Section 3.2].

As Debarre and Macrì proved, see Theorem 2.79, the image of the period map is the complement of \mathcal{D}_2 , \mathcal{D}_8 and one (out of two) irreducible component of \mathcal{D}_{10} . Moreover, by Theorem 4.58 we also know that the image of \mathcal{U}_{EPW} through the period map has no intersection with \mathcal{D}_4 . The notation below comes from [26, Section 7.1 and Section 7.3] and our definition is motivated by [29, Remark 5.29].

Definition 4.60. We call \mathcal{D}'_{10} the irreducible component of \mathcal{D}_{10} which is the closure of the image, through the period map, of $(\Pi - \Delta - \Sigma) // \mathrm{PGL}(V_6) \subset \mathcal{U}_{EPW}$. We denote by \mathcal{D}''_{10} the other irreducible component of \mathcal{D}_{10} , or equivalently $\mathcal{D}''_{10} = \delta(\mathcal{D}'_{10})$, see Remark 2.85.

By definition, the divisor \mathcal{D}''_{10} does not lie in the image of the period map. We can consider the moduli space $(\mathcal{M}_{2,1}^2)^\circ$ of pairs (X, D) with X hyperkähler manifold and D a big and nef square-2 divisor on X , and the associated extended period map $P^\circ : (\mathcal{M}_{2,1}^2)^\circ \rightarrow \mathcal{P}_{2,1}^2$. The general element of $\Delta // \mathrm{PGL}(V_6)$ sits in this moduli space, and by the work of O'Grady, see [96] and [98], the period map extends to $(\Delta - \Sigma) // \mathrm{PGL}(V_6) \subset (\mathcal{M}_{2,1}^2)^\circ$ and $P^\circ \circ \delta = \delta \circ P^\circ$ (the period point of X_A is the one of $S^{[2]}$ given in Theorem 5.22). In this setting, we can define \mathcal{D}'_{10} as the closure of $P^\circ((\Pi - \Sigma) // \mathrm{PGL}(V_6))$ and \mathcal{D}''_{10} as the closure of $P^\circ((\Delta - \Sigma) // \mathrm{PGL}(V_6))$.

The following result is stated by Debarre and Macrì in [32, Example 6.3].

Theorem 4.61 (Debarre, Iliev, Manivel, Macrì). *Consider $e \geq 12$ such that \mathcal{D}_e is non-empty. Then the intersection of \mathcal{U}_{EPW} with any irreducible component of \mathcal{D}_e is non-empty.*

Proof. (Sketch) This proof make use of the relation between EPW sextics and Gushel-Mukai varieties (see Definition 4.85), which we will present in Section 4.5.2: given a smooth Gushel-Mukai variety Z , we will call $A(Z)$ the Lagrangian subspace associated to Z through Theorem 4.106. Moreover, the definition of the period point of a smooth Gushel-Mukai variety of dimension four is needed; this definition is one of the main points of [29] and is treated in [29, Section 5.1]; for simplicity we also refer to [31, Proposition 6.1]. For us, it is sufficient to know that the period point of a smooth 4-dimensional Gushel-Mukai lies in the period domain $\mathcal{P}_{2,1}^2$ associated to $\mathcal{M}_{2,1}^2$, and that the period point of Z coincides with the period point of $(X_{A(Z)}, D_{A(Z)})$. In particular, to prove the claim it is sufficient to find a smooth Gushel-Mukai Z_e whose period point lies in \mathcal{D}_e . Finally [26, Theorem 8.1] shows a method to construct, from a Brill-Noether general $K3$ surface, a Gushel-Mukai variety Z_e whose period point lies in \mathcal{D}_e . \square

Nothing more is known about the complement of \mathcal{U}_{EPW} . O'Grady conjectured that $\mathcal{M}_{2,1}^2 = \mathcal{U}_{EPW} \cup \mathcal{D}_4$; in this case we would also have that the image of \mathcal{U}_{EPW} through the period map would be the image of $\mathcal{M}_{2,1}^2$ minus \mathcal{D}_4 .

We end with a result by Debarre and Macrì, which shows that double EPW sextics appear in a wide range of moduli spaces of polarized manifolds of $K3^{[n]}$ -type; obviously in these cases the polarization will not be the one coming from the cover. By Lemma 2.64 we know that $\mathcal{M}_{2,d,2}^2$ is non-empty if and only if $d \equiv -1 \pmod{4}$.

Proposition 4.62. *Let d be a positive square-free integer such that $d \equiv -1 \pmod{4}$. There are infinitely many even $e > \frac{d}{2}$ such that $\mathcal{C}_e \subset \mathcal{M}_{2,d,2}^2$ is non-empty and the general point of it corresponds to a double EPW sextics. The union of these divisors \mathcal{C}_e is dense in $\mathcal{M}_{2,d,2}^2$.*

Proof. See [32, Corollary 7.16] and [32, Corollary 7.18]. \square

4.3.4 Automorphisms of double EPW sextics

Smooth double EPW sextics are hyperkähler manifolds of $K3^{[2]}$ -type, so everything we said in Section 2.4 holds for them. Anyway, for $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$ we have a structure of double cover $f_A : X_A \rightarrow Y_A$, or equivalently a natural $\langle 2 \rangle$ -polarization $D_A \in \text{Pic}(X_A)$, see Remark 4.45, so it is natural to ask whether it is possible to find automorphisms on X_A which commutes with the covering involution ι_A , so that they descend to automorphisms on Y_A . The answer of this question is clear and can be turned to a lattice-theoretic statement: the covering involution acts on $H^2(X_A, \mathbb{Z})$ as $-R_{D_A}$, see Lemma 4.42, so by Theorem 2.108 for any $G \subset \text{Aut}(X_A)$ the covering involution commutes with G if and only if $g^*(D_A) = D_A$ for any $g \in G$ i.e. D_A lies in the invariant lattice T_G .

The question can be refined. Consider again a smooth double EPW sextic X_A , with an automorphism α on it such that α does not commute with the covering involution ι_A . Does there exists $A' \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$ such that $X_A \rightarrow Y_A$ and $X_{A'} \rightarrow Y_{A'}$ are isomorphic as double covers and α commutes with the covering involution $\iota_{A'}$? Again, there is a lattice-theoretic answer; as usual, we denote by $\mathcal{A}(X_A)$ the ample cone of X_A and by T_α the invariant lattice of α .

Lemma 4.63. *Consider $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$ and $\alpha \in \text{Aut}(X_A)$. There exists $A' \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$ as above if and only if there is an automorphism $g \in \text{Aut}(X_A)$ such that $g^*(D_A)$ lies in the invariant lattice T_α . Equivalently, $A' \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ as above exists if and only if there is an orientation-preserving Hodge isometry of $H^2(X_A, \mathbb{Z})$ sending D_A into $T_\alpha \cap \mathcal{A}(X_A)$.*

Proof. If $g \in \text{Aut}(X_A)$ such that $g^*(D_A) \in T_\alpha$ exists, then $(X_A, g^*(D_A)) \in \mathcal{M}_{2,1}^2$. Moreover, we call $\phi_{|D_A|}$ and $\phi_{|g^*(D_A)|}$ the maps associated to D_A and $g^*(D_A)$, and $H^0(g)$ the map induced on the global section of the line bundles; we have a commutative diagram

$$\begin{array}{ccc} X_A & \xrightarrow{g} & X_A \\ \phi_{|D_A|} \downarrow & & \downarrow \phi_{|g^*(D_A)|} \\ \mathbb{P}(H^0(X_A, D_A)^\vee) & \xrightarrow{H^0(g)^\vee} & \mathbb{P}(H^0(X_A, g^*(D_A))^\vee) \end{array}$$

so $(X_A, g^*(D_A)) \in \mathcal{U}_{EPW}$ and α commutes with the involution associated to $g^*(D_A)$, that is $g^{-1} \circ \iota_A \circ g$.

Conversely, if A' exists we can identify X_A and $X_{A'}$. By Remark 4.45, under this identification $D_{A'}$ is a square-2 line bundle such that there exists an automorphism $g \in \text{Aut}(X_A)$ sending D_A to $D_{A'}$; since $\iota_{A'}$ commutes with α we have the claim.

Finally, by Corollary 2.109 a Hodge isometry of $H^2(X_A, \mathbb{Z})$ is induced by an automorphism of X_A if and only if it preserves the orientation and the ample cone, the condition on the action of the discriminant being automatically fulfilled since the discriminant group of $H^2(X_A, \mathbb{Z})$ has cardinality two. \square

Note that, if we does not impose the isomorphism of X_A and $X_{A'}$ to be an isomorphism of double cover, but only an isomorphism between varieties, answering the question above is much

more difficult. In fact, if Lemma 4.63 fails, this is equivalent to ask if, given any manifold of $K3^{[2]}$ -type X with an automorphism $\alpha \in \text{Aut}(X)$, then X admits a $\langle 2 \rangle$ -polarization $D \in \text{Pic}(X) \cap T_\alpha$ such that (X, D) is isomorphic to (X_A, D_A) for some $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)^0$.

Unfortunately, for the moment we cannot hope to have a lattice-theoretic answer to the question if we remove the hypothesis of the isomorphism between covers: once we consider a α -invariant $\langle 2 \rangle$ -polarization D on X_A , we are not able to verify if (X_A, D) lies in \mathcal{U}_{EPW} , since the complement of it inside $\mathcal{M}_{2,1}^2$ is not known, see Section 4.3.3. On the other hand, we have some sufficient condition for an automorphism α on X_A to *not* descend to any *EPW* sextic, which are:

- a) T_α does not contain any square-two element, or
- b) for every $\langle 2 \rangle$ -polarization $D \in \text{Pic}(X) \cap T_\alpha$, the period point of the polarized manifold $(X, D) \in \mathcal{M}_{2,1}^2$ lies in \mathcal{D}_4 , or
- c) α has infinite order.

The first sufficient condition is clear, for the second one see Theorem 4.58. For the third one we observe that any automorphism of an *EPW* sextic without non-zero decomposable vectors has finite order, see Proposition 4.28.

We give an example of c). Consider $A, A' \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)^0$ such that $X_A = X_{A'}$ with two distinct covering involutions ι_A and $\iota_{A'}$ whose associated ample line bundles spanning the invariant lattice are, respectively, D_A and $D_{A'}$. Clearly the two involutions cannot commute: we consider $\alpha = \iota_A \circ \iota_{A'}$, in particular we study its action on the sublattice $R = \mathbb{Z}D_A + \mathbb{Z}D_{A'}$ to show that $\alpha^n \neq \text{id}$ for every $n \geq 1$. We denote $k = (D_A, D_{A'})$, we compute the matrix associated to $\alpha^* = (-R_{A'}) \circ (-R_A)$ with respect to the basis $\{D_A, D_{A'}\}$ and we obtain

$$\begin{pmatrix} k^2 - 1 & k \\ -k & -1 \end{pmatrix},$$

where $k \geq 3$ since the lattice R has to be hyperbolic. The minimal polynomial of the matrix is never a product of cyclotomic polynomials, hence α^* has not finite order on R . So α has infinite order by Corollary 2.109 and it cannot descend to any *EPW* sextic.

To find an example of a double *EPW* sextic with two distinct covering involutions we can consider, for any fixed $d \equiv -1 \pmod{4}$, a very general element (X_A, L) inside a divisor $\mathcal{C}_e \subset \mathcal{M}_{2d,2}^2$ as in Proposition 4.62. As Debarre and Macrì pointed out in [32, Corollary 7.16] and below, in this situation $\text{Aut}(X_A)$ is isomorphic to the infinite dihedral group D_∞ and contains infinitely many covering involutions $\{\iota_{A_m}\}_{m \in \mathbb{Z}}$. The group D_∞ is the semidirect product $\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$ with $(n, a) \cdot (k, b) = (n + (-1)^a k, a + b)$, and $(m, 1) \in D_\infty$ is associated to the involution ι_{A_m} (see [32, Proposition A.3] for the description of the action in cohomology of ι_{A_m}). As we chose (X_A, L) very general, for every $m \in \mathbb{Z}$ the quotient $X_{A_m}/\langle \iota_{A_m} \rangle \cong Y_{A_m}$ is an *EPW* sextic, whose automorphism group is trivial since no element of D_∞ commutes with $(m, 1)$ except itself. A way to find concrete examples for d, e is presented in the proof of [32, Corollary 7.18].

Remark 4.64. In the example above, two elements $(n, 1), (k, 1) \in D_\infty \cong \text{Aut}(X_A)$ are conjugate if and only $n \equiv k \pmod{2}$, so there is, up to isomorphism, exactly one Lagrangian subspace $A' \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)^0$ such that $X_A \cong X_{A'}$ but $Y_A \not\cong Y_{A'}$.

Remark 4.65. Consider a Lagrangian subspace $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$ and a group G acting on X_A . If G contains the covering involution ι_A , then the whole G descends to Y_A if and only if it is finite. Clearly if the group commutes with ι it is finite. On the other hand, if there is $g \in G$ that does not commute with ι_A , then X_A admits another covering involution $\iota_{A'} = g^{-1} \circ \iota_A \circ g$, associated to $(X_A, g^*(D_A))$: since $\alpha = \iota_A \circ \iota_{A'}$ lies in G , then G is infinite as in the example above. In particular, $\text{Aut}(X_A)$ is finite if and only if there is only one covering involution on X_A .

It would be interesting to find a double *EPW* sextic X_A with an automorphism which does not descend to any *EPW* sextic, and that does not lie in the group of automorphisms generated by automorphisms that commute with some covering involution $\iota_{A'}$, for $A' \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$ such that $X_A \cong X_{A'}$.

Another point of view is possible. Given an automorphism on Y_A , it is always possible to lift it to the double cover X_A .

Definition 4.66. Given $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ such that $A \notin \Sigma$, we denote by $\text{Aut}_{D_A}(X_A) \subseteq \text{Aut}(X_A)$ the group of automorphisms fixing the polarization D_A , or equivalently

$$\text{Aut}_{D_A}(X_A) = \{\alpha \in \text{Aut}(X_A) \mid \alpha \circ \iota_A = \iota_A \circ \alpha\}.$$

If A has no non-zero decomposable vectors, there is a short exact sequence

$$1 \rightarrow \{\text{id}, \iota_A\} \rightarrow \text{Aut}_{D_A}(X_A) \xrightarrow{\lambda} \text{Aut}(Y_A) \rightarrow 1. \quad (4.15)$$

The idea is the same of the proof of Theorem 4.26. Any automorphism $\theta \in \text{Aut}(Y_A)$ fixes the singular locus $Y_A^{\geq 2}$, hence it sends Y_A^1 to itself; by Corollary 4.39 the étale cover $f_A^{-1}(Y_A^1) \rightarrow Y_A^1$ is a universal cover, so θ lifts to an automorphism on $f_A^{-1}(Y_A^1)$, which can be extended in a unique way to the branched cover. Hence the morphism λ in (4.15) is surjective. To compute the kernel it is sufficient to observe that the kernel of the restriction of λ to Y_A^1 is isomorphic to the fundamental group of Y_A^1 , which has cardinality two. Since Y_A^1 is open inside the *EPW* sextic Y_A , the kernel of λ has again cardinality two, and it is generated by the covering involution ι_A .

The following result is a slight generalization of [81, Proposition 1.4.7], that is Item *a*) below. For simplicity, we will consider 1 as a first root of unity, to include the case of a symplectic action in Item *b*) below.

Proposition 4.67. Consider $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$ and $G \subseteq \text{Aut}(Y_A)$.

- a) If G acts trivially on a section of the canonical bundle of Y_A , then there is an injective morphism $\mu : \text{Aut}(Y_A) \rightarrow \text{Aut}_{D_A}(X_A)$ such that $\{\text{id}, \iota_A\} \times \mu(G) \subseteq \text{Aut}(X_A)$ with $\mu(G)$ acting symplectically on X_A .
- b) If G has odd order, then there is an injective morphism $\mu : \text{Aut}(Y_A) \rightarrow \text{Aut}_{D_A}(X_A)$ such that $\{\text{id}, \iota_A\} \times \mu(G) \subseteq \text{Aut}(X_A)$. Moreover, consider n such that $\alpha \in G$ acts as a n -th primitive root of unity on a section of the canonical bundle of Y_A . Then $\mu(\alpha)$ acts as a n -th primitive root of unity on a symplectic 2-form ω on X_A .

Proof. We consider σ_Y a section of the canonical bundle of Y_A , so that $f_A^*\sigma_Y$ is a section of the canonical bundle of X_A . We fix a symplectic 2-form ω on X_A such that $\omega^{\otimes 2} = f_A^*\sigma_Y$.

- a) For every $\alpha \in G$, by (4.15) there exists $\tilde{\alpha} \in \text{Aut}_{D_A}(X_A)$ such that $f_A \circ \tilde{\alpha} = \alpha \circ f_A$; this last equality assures that $\tilde{\alpha}$ acts trivially on $f_A^*\sigma_Y$. So $(\tilde{\alpha}^*\omega)^{\otimes 2} = \omega^{\otimes 2}$, hence $\tilde{\alpha}^*$ act as ± 1 on ω . We define $\mu(\alpha)$ as the one between $\tilde{\alpha}$ and $\iota_A \circ \tilde{\alpha}$ that acts trivially on ω .
- b) Consider $\alpha \in G$ of order k ; by (4.15) there are two elements $\tilde{\alpha}_h \in \text{Aut}_{D_A}(X_A)$, for $h = 1, 2$, such that $f_A \circ \tilde{\alpha}_h = \alpha \circ f_A$, which differ by ι_A . Since α has odd order one between them, say $\tilde{\alpha}_1$, has order k , while $\tilde{\alpha}_2$ has order $2k$. We define then $\mu(\alpha) = \tilde{\alpha}_1$.

Now let ζ be a n -th primitive root of unity such that $\alpha^*\sigma_Y = \zeta\sigma_Y$. We have

$$(\mu(\alpha))^*(f_A^*\sigma_Y) = f_A^*(\alpha^*\sigma_Y) = \zeta(f_A^*\sigma_Y).$$

So $(\mu(\alpha)^*\omega)^{\otimes 2} = \zeta\omega^{\otimes 2}$, thus $\mu(\alpha)$ acts on ω as $\pm\sqrt{\zeta}$. If $n \neq 1$, since k (hence n) is odd, $(\sqrt{\zeta})^k = -1$, while $-\sqrt{\zeta}$ is a n -th primitive root of unity. So $\mu(\alpha)$ should act on ω as $-\sqrt{\zeta}$ (and $\tilde{\alpha}_2$ as $\sqrt{\zeta}$). If $n = 1$ we have $\mu(\alpha)^*\omega = \omega$, again because k is odd (or by the same calculation as above, with $\sqrt{\zeta} = -1$).

In both cases, μ is a section of the restriction of (4.15) to G , so inside $\text{Aut}(X_A)$ we have a semidirect product between $\{\text{id}, \iota_A\}$ and G . Since it is contained in $\text{Aut}_{D_A}(X_A)$ the product is actually direct. \square

Remark 4.68. In the case *b*) of Proposition 4.67 an automorphism α acts on a section of the canonical as a n -th root of unity, and so does $\mu(\alpha)$ on the 2-form on X_A , but the two roots of unity are actually different, they are respectively ζ and $-\sqrt{\zeta}$.

The case of groups with even order is more complicated. At least for involutions we have the following, proved by Camere in [20, Lemma 9] and [20, Remark 10].

Lemma 4.69. *Consider $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)^0$. For any involution $\alpha \in \text{Aut}(Y_A)$ there are exactly two involutions $\tilde{\alpha}_h$, for $h = 1, 2$, on $\text{Aut}(X_A)$ such that $f_A \circ \tilde{\alpha}_h = \alpha \circ f_A$. One of them acts symplectically on X_A , the other non-symplectically.*

Proof. The existence of the two involutions is guaranteed by Equation (4.15). They differ by ι_A , so if $\tilde{\alpha}_1$ acts as the identity on X_A the other acts as -1 , and vice versa. \square

In this thesis we concentrate mostly on automorphisms of double *EPW* sextics lifted from an *EPW* sextic. We will talk again about automorphisms of (double) *EPW* sextics in Section 5.4.3.

4.4 Grassmannian hulls

4.4.1 A survey on smooth Grassmannian hulls

We fix a 5-dimensional complex vector space V_5 and we choose a volume form $\text{vol} : \bigwedge^5 V_5 \rightarrow \mathbb{C}$.

Definition 4.70. Let $W \subseteq \Lambda^2 V_5$ be a linear subspace of dimension at least 7. We call Grassmannian hull the intersection

$$M = \mathbb{P}(W) \cap G(2, V_5).$$

Since both $\mathbb{P}(W)$ and $G(2, V_5)$ are smooth, hence Cohen-Macaulay, the Grassmannian hull M is a dimensionally transverse intersection, see Definition 1.1, if and only if $\text{codim}_{\mathbb{P}(\Lambda^2 V_5)}(M) = \text{codim}_{\mathbb{P}(\Lambda^2 V_5)}(\mathbb{P}(W)) + \text{codim}_{\mathbb{P}(\Lambda^2 V_5)}(G(2, V_5)) = \text{codim}_{\mathbb{P}(\Lambda^2 V_5)}(\mathbb{P}(W)) + 3$, which is equivalent to ask that $\dim M = \dim W - 4$. Note that, with this definition, the dimension of a Grassmannian hull which is a dimensionally transverse intersection is always at least 3.

Remark 4.71. When M is a dimensionally transverse intersection it has degree 5 inside $\mathbb{P}(W)$, as a linear section of $G(2, V_5)$, which has degree 5 inside $\mathbb{P}(\Lambda^2 V_5)$.

Lemma 4.72. Take $W \subseteq \Lambda^2 V_5$ such that the Grassmannian hull M is a dimensionally transverse intersection, then M can be seen as an intersection of quadrics. We have

$$H^0(\mathbb{P}(W), \mathcal{I}_M(2)) \cong V_5.$$

Proof. Consider $v \in V_5 - \{0\}$ with the associated Pfaffian quadric P_v and projective quadric hypersurface $Q(v)$. Setting $\tilde{P}_v = (P_v)|_{\mathbb{P}(W)}$ and $\tilde{Q}(v) = Q(v) \cap \mathbb{P}(W)$ we have then $M = \bigcap_{v \in V_5} \tilde{Q}(v)$. Note that for every $v \in V_5 - \{0\}$ the hypersurface $\tilde{Q}(v)$ is a quadric: if P_v vanishes on W for some v , the Grassmannian hull cannot be a dimensionally transverse intersection. The vector space $H^0(\mathbb{P}(W), \mathcal{I}_M(2))$ is isomorphic to V_5 through the isomorphism

$$\begin{aligned} V_5 &\rightarrow H^0(\mathbb{P}(W), \mathcal{I}_M(2)) \\ v &\mapsto \tilde{P}_v. \end{aligned}$$

Since M can be written as an intersection of projective quadrics in the form $\tilde{Q}(v)$, every element of $H^0(\mathbb{P}(W), \mathcal{I}_M(2))$ can be described as a linear composition of elements in the form \tilde{P}_v . We are left to prove that the restriction $\rho : H^0(\mathbb{P}(W), \mathcal{I}_{G(2, V_5)}(2)) \rightarrow H^0(\mathbb{P}(W), \mathcal{I}_M(2))$, which sends P_v to \tilde{P}_v , is injective, but this is equivalent to ask that no P_v vanish on W . \square

We recall that, as we defined in Section 4.2.3, the orthogonal complement of $W \subseteq \Lambda^2 V_5$ is the complex vector space $W^\perp = \{\phi \in (\Lambda^2 V_5)^\vee \mid \phi(w) = 0 \text{ for every } w \in W\}$. The volume form on $\Lambda^5 V_5$ gives an isomorphism $(\Lambda^2 V_5)^\vee \cong \Lambda^3 V_5$; through this identification, we have

$$W^\perp = \{\eta \in \bigwedge^3 V_5 \mid \eta \wedge w = 0 \text{ for every } w \in W\}. \quad (4.16)$$

Inside $\mathbb{P}(\Lambda^2 V_5^\vee)$ there is another Grassmannian hull associated to M , which is $\mathbb{P}(W^\perp) \cap G(2, V_5^\vee)$. Note that it is likely to be empty, as the codimension of W in $\Lambda^2 V_5^\vee$ is the dimension of W : if $\mathbb{P}(W^\perp) \cap G(2, V_5^\vee)$ is a dimensionally transverse intersection, it has codimension $8 + \dim(M)$, which is always strictly bigger than 10.

In particular this holds for W general, hence for W general the Grassmannian hull M is a dimensionally transverse intersection, as we note in the following result, which is classical.

Proposition 4.73. *The Grassmannian hull M is dimensionally transverse and smooth if and only if $\mathbb{P}(W^\perp) \cap G(2, V_5^\vee) = \emptyset$.*

Proof. This is [27, Proposition 2.22], rearranged to the case $\dim W \geq 7$. □

From now on we are mostly interested in Grassmannian hulls that are smooth dimensionally transverse intersection; the reason will be clear by Definition 4.92 and Theorem 4.110.

Remark 4.74. Consider $v \in V_5 - \{0\}$ and choose a decomposition $V_5 = \mathbb{C}v \oplus U$. The kernel of the Pfaffian quadric P_v is the four dimensional space $v \wedge U$.

Lemma 4.75. *Let $M = \mathbb{P}(W) \cap G(2, V_5)$ be a smooth Grassmannian hull of dimension n which is a dimensionally transverse intersection. Take $v \in V_5$, and let \tilde{P}_v be the restriction, on W , of the associated Pfaffian quadric P_v . Then*

$$\ker(\tilde{P}_v) = \ker(P_v) \cap W$$

so that P_v has rank at most 6 and corank at least $\dim W - 6$, with equality for v general.

Proof. The last statement is just a dimension count once the equality is proved, so we only need to prove the former. Consider the description of the kernel of P_v that we gave in Remark 4.74. A priori we could write the kernel of the quadric \tilde{P}_v as

$$\left((v \wedge U) \oplus \left\{ \eta \in \bigwedge^2 U \mid \eta \wedge w = 0 \text{ for every } w \in W \right\} \right) \cap W.$$

Consider $u \in U$, $\xi \in \bigwedge^2 U$ such that $\eta = v \wedge u + \xi \in W$ belongs to the kernel of P_v ; in particular $v \wedge \eta \wedge \eta = v \wedge (\xi \wedge \xi)$ is zero, so $[\xi] \in G(2, U)$ if $\xi \neq 0$. In this case we choose a basis $\{e_1, \dots, e_5\}$ on V_5 such that $v = e_1$ and $\xi = e_2 \wedge e_3$. If u does not belong to the space spanned by e_2 and e_3 we can rearrange the base so that $\eta = e_1 \wedge e_4 + e_2 \wedge e_3$: in this case the condition $w \wedge \eta = 0$ imposes five independent constraints on w , so no vector in this form can lie in the kernel of P_v . On the other hand, if $u = \alpha e_1 + \beta e_2$ the condition $w \wedge \eta = 0$ for $w = \sum_{i < j} \lambda_{ij} e_i \wedge e_j$ gives a system where the constraint $\lambda_{45} = 0$ appears. Thus the linear map $e_1 \wedge e_4 \wedge e_5 \wedge - : W \rightarrow \mathbb{C}$ (or more precisely $\text{vol}(e_1 \wedge e_4 \wedge e_5 \wedge -)$), which is $(e_2 \wedge e_3)^\vee$, lies in W^\perp and by Proposition 4.73 the Grassmannian hull M is not smooth and dimensionally transverse, against the hypothesis. So ξ is forced to be zero. □

A *smooth Fano variety* is a smooth projective variety whose anticanonical bundle is ample. Given X a smooth Fano variety and K_X its canonical bundle, the greatest integer r such that K_X is divisible by r in $\text{Pic}(X)$ is the *index* of X . If $\text{Pic}(X)$ has rank one then X is said to be a *prime* Fano variety. Grassmannian hulls are a classical example of Fano varieties.

Proposition 4.76. *Let M be a smooth Grassmannian hull which is a dimensionally transverse intersection of dimension n . Then M is a smooth prime Fano manifold of index $n - 1$ and degree 5.*

Proof. The degree of M is 5 anytime it is a dimensionally transverse intersection, cfr. Remark 4.71. When M is smooth it is prime by Lefschetz hyperplane theorem, see [53, Exposé XIII, Corollary 3.7]. We compute the index of M by adjunction. Let H be the line bundle induced by restriction of the hyperplane class of $\mathbb{P}(W)$: the canonical bundle of $G(2, V_5)$ is $-5H$ and the codimension of W inside $\bigwedge^2 V_5$ is $6 - n$, hence we have $K_M = -5H + (6 - n)H = -(n - 1)H$ modulo numerical equivalence. \square

For three-dimensional Grassmannian hulls we can say more. The following result can be found in [68]; for convenience we refer also to [98].

Proposition 4.77. *Let M be a smooth three-dimensional Grassmannian hulls that is a dimensionally transverse intersection.*

- 1) M does not contain any plane.
- 2) the Fano variety $F(M)$ parametrizing lines on M , with the reduced scheme structure, is isomorphic to \mathbb{P}^2 .
- 3) Any other three-dimensional Grassmannian hull that is a dimensionally transverse intersection is smooth if and only if isomorphic to M .

Proof. If M would contain a plane, it would be a divisor on M and it would not be numerically equivalent to the hyperplane section, since M has degree 5 inside $\mathbb{P}(W)$; this is not possible since M is prime by Proposition 4.76. For the rest of the proof, see [98, Proposition 5.2, Item 2) and 3)]. \square

Given a basis $\{e_1, \dots, e_5\}$ on V_5 , we have an induced basis on the space of matrices $M_{5,5}(\mathbb{C})$, which is $\{E_{i,j}\}_{1 \leq i,j \leq 5}$, where $E_{i,j}$ has 1 in the position (i, j) and zero otherwise. Since $(\bigwedge^2 V_5)^\vee \cong \bigwedge^2 V_5^\vee$, any element of $(\bigwedge^2 V_5)^\vee$ can be seen as an antisymplectic form on V_5 . In coordinates the linear form $(e_i \wedge e_j)^\vee$ corresponds to the matrix $\frac{1}{2}(E_{i,j} - E_{j,i})$.

Now we take a hyperplane $H \in (\bigwedge^2 V_5)^\vee$, corresponding to an antisymplectic matrix in the form $A = \sum_{i,j} \lambda_{i,j}(E_{i,j} - E_{j,i})$. Then $p \wedge q \in G(2, V_5)$ lies in $\mathbb{P}(H)$ if and only if $p^t A q = 0$: we can verify directly by choosing a base on V_5 such that $p \wedge q = e_1 \wedge e_2$. This can be found in [104, Section 1].

Remark 4.78. Consider ϕ an automorphism of V_5 , and fix a basis on V_5 . Let $T \in \text{GL}(V_5)$ be the matrix associated to ϕ in these coordinates. For a hyperplane H on $\bigwedge^2 V_5$ the image $(\bigwedge^2 \phi)(H)$ is associated to the antisymmetric matrix ${}^t T^{-1} A T^{-1}$; given $M = \mathbb{P}(W) \cap G(2, V_5)$ a Grassmannian hull, the automorphism $\bigwedge^2 \phi$ fix M if and only if for every hyperplane H containing W , and associated antisymmetric matrix A , the antisymmetric matrix ${}^t T^{-1} A T^{-1}$ is still associated to a hyperplane containing W .

Since an antisymmetric matrix has even rank, every element in $(\bigwedge^2 V_5)^\vee$, seen as an antisymplectic matrix, has corank at least one. A simple computation, see for example [104, Corollary 1.6], shows that the dual Grassmannian $G(2, V_5^\vee)$ is exactly the locus of matrices whose rank drops. By Proposition 4.73 a Grassmannian hull $\mathbb{P}(W) \cap G(2, V_5)$ is a smooth dimensionally transverse

intersection if and only if $\mathbb{P}(W^\perp)$ contains only matrices whose corank is one. So in this case we can define the *center map*:

$$\begin{aligned} c : \mathbb{P}(W^\perp) &\rightarrow \mathbb{P}(V_5) \\ [A] &\mapsto [\ker(A)]; \end{aligned}$$

For $A \in W^\perp$ we call *center* of A the associated point of $\mathbb{P}(V_5)$.

4.4.2 Automorphisms of smooth Grassmannian hulls

In this section we present some results by Piontkowski and Van de Ven, see [104]. Let $M = \mathbb{P}(W) \cap G(2, V_5)$ be a smooth Grassmannian hull which is a dimensionally transverse intersection. If M has dimension 6 then it is simply the Grassmannian, the dual W^\perp is empty and $\text{Aut}(M) = \text{PGL}(V_5)$. Now we focus on the cases where M has dimension at most 5.

Theorem 4.79. *Let $W \subseteq \bigwedge^2 V_5$ be a vector subspace such that $M = \mathbb{P}(W) \cap G(2, V_5)$ is a smooth dimensionally transverse intersection. Then every automorphism of M is in the form $\bigwedge^3 \phi$ for some $\phi \in \text{PGL}(V_5)$, so that we have a canonical inclusion $\text{Aut}(M) \subseteq \text{Aut}(G(2, V_5)) = \text{PGL}(V_5)$. Moreover $\text{Aut}(M) = \text{Aut}(G(2, V_5)) \cap \text{Aut}(\mathbb{P}(W))$.*

Proof. See [104, Theorem 1.2]. Just note that the theorem in [104] holds for M general, and the condition of genericity is exactly equivalent to M being smooth and dimensionally transverse as an intersection, by Proposition 4.73. The numerical condition on codimension is always fulfilled in our setting. \square

The result above, together with Remark 4.78, allows to study the automorphism group of a smooth and dimensionally transverse Grassmannian hull, as we explain below.

See [104, Section 5] for a detailed treatment of this case. Consider M a 5-dimensionally smooth Grassmannian hull which is a dimensionally transverse intersection. In this case the dual $\mathbb{P}(W^\perp)$ is a point. Take any $v \in W^\perp$: we can choose a basis $\{e_1, \dots, e_5\}$ on V_5 such that the antisymmetric matrix associated to v is

$$A = \left[\begin{array}{cc|cc|c} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right].$$

Note that the matrix

$$\Omega = \left[\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array} \right].$$

induces a skew-symmetric bilinear form on the space spanned by $\{e_1, \dots, e_4\}$; its rank is maximal. In these coordinates the center of A is $c = [e_5]$. Any $T \in \text{GL}(V_5)$ induces an action on $G(2, V_5)$; this

action sends M to M if and only if T fixes the center of W , so that there exists $\lambda \in \mathbb{C}^*$ such that ${}^tT^{-1}AT^{-1} = \lambda A$, see Remark 4.78. An explicit computation shows that T must be in the form

$$T = \left[\begin{array}{c|c} \bar{T} & 0 \\ \hline {}^t a & b \end{array} \right],$$

where $b \in \mathbb{C}^*$, $a \in \mathbb{C}^4$ and $\lambda {}^t\bar{T}A\bar{T} = A$. We have $[\det(\bar{T})] = \pm \frac{1}{\lambda^2}$, we can normalize \bar{T} so that ${}^t\bar{T}A\bar{T} = A$. Recall that the *complex symplectic group* is

$$\mathrm{Sp}(4, \mathbb{C}) = \{M \in \mathrm{GL}(4) \mid {}^tM\Omega M = \Omega\}.$$

Once \bar{T} is normalized, a representant in the class of $[T] \in \mathrm{PGL}(V_5)$ is fixed once we quotient by $\langle -\mathrm{id} \rangle$.

Proposition 4.80. [104, Proposition 5.2] *Let M be a smooth Grassmannian hull which is a dimensionally transverse intersection of dimension 5. There exists a basis on V_5 such that the automorphism group $\mathrm{Aut}(M) \subset \mathrm{Aut}(G(2, V_5)) \cong \mathrm{PGL}(V_5)$ is in the form*

$$\left\{ \left[\begin{array}{c|c} \bar{T} & 0 \\ \hline {}^t a & b \end{array} \right], \bar{T} \in \mathrm{Sp}(4, \mathbb{C}), a \in \mathbb{C}^4, b \in \mathbb{C}^* \right\} / \langle -\mathrm{id} \rangle.$$

The subgroup of matrices in the form

$$\left[\begin{array}{c|c} I_4 & 0 \\ \hline {}^t a & 1 \end{array} \right]$$

where I_4 is the identity matrix, is normal in $\mathrm{Aut}(M)$, so there is a short exact sequence

$$1 \rightarrow \mathbb{C}^4 \rightarrow \mathrm{Aut}(M) \rightarrow (\mathrm{Sp}(4, \mathbb{C}) \times \mathbb{C}^*) / \{\pm 1\} \rightarrow 1. \quad (4.17)$$

The normality of the subgroup is proved with a straightforward calculation.

See [104, Section 6] for a detailed treatment of this case. Consider M a 4-dimensionally smooth Grassmannian hull which is a dimensionally transverse intersection. In this case the dual $\mathbb{P}(W^\perp)$ is a line inside $\mathbb{P}(\bigwedge^2 V_5)$.

Proposition 4.81. *The center map is an embedding, whose image is a conic in $\mathbb{P}(V_5)$.*

Proof. See [104, Proposition 6.3]. □

To describe the automorphism group of M , the trickiest part is to find a good basis for V_5 . Fix A, B skew-symmetric matrices such that, up to the canonical automorphism $(\bigwedge^2 V_5)^\vee \cong \bigwedge^2 V_5^\vee$, we can write $W^\perp = \lambda A - \mu B$ for $\lambda, \mu \in \mathbb{C}$. By Proposition 4.81 we can consider a basis on V_5 such that the center map is

$$(\lambda : \mu) \mapsto (0 : 0 : \lambda^2 : \lambda\mu : \mu^2)$$

Leaving e_3, e_4, e_5 invariant, we can furthermore choose the first two element of the basis such that A and B are in the form

$$A = \left[\begin{array}{cccc|c} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad B = \left[\begin{array}{cc|cc|c} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 \end{array} \right].$$

Call V_3 the vector space spanned by e_3, e_4 and e_5 , so that $\pi = \mathbb{P}(V_3)$ is the plane spanned by the conic of centers. We can also write $V_5 \cong V_3^\perp \oplus V_3$. By Remark 4.78 and Theorem 4.79, any automorphism of M has to fix the conic of centers in $\mathbb{P}(V_5)$ and send π to π ; the action on V_3^\perp is the induced action on the hyperplanes containing π . By [55, 10.12], the action on V_3 and V_3^\perp is then induced by an automorphism $\phi \in PGL(W^\perp)$ in the following way: given a matrix

$$t = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

that represents ϕ , the induced action on $\mathbb{P}(V_5)$ needs to be in the form

$$T = \begin{bmatrix} \alpha t_2^{-1} & 0 \\ S & t_3 \end{bmatrix}.$$

with $\alpha \in \mathbb{C}^*$, $S \in M_{3,2}(\mathbb{C})$,

$$t_2 = \begin{bmatrix} d & c \\ b & a \end{bmatrix} \quad t_3 = \begin{bmatrix} d^2 & 2cd & d^2 \\ bd & ad+bc & ac \\ b^2 & 2ab & a^2 \end{bmatrix}.$$

A simple computation gives necessary and sufficient conditions on S .

Theorem 4.82. [104, Theorem 6.6] *Let M be a smooth Grassmannian hull which is a dimensionally transverse intersection of dimension 4. There exists a basis on V_5 such that the automorphism group $\text{Aut}(M) \subset \text{Aut}(G(2, V_5)) \cong \text{PGL}(V_5)$ is in the form*

$$\begin{bmatrix} \alpha I_3 & 0 \\ S & I_2 \end{bmatrix} \cdot \begin{bmatrix} t_2^{-1} & 0 \\ 0 & t_3 \end{bmatrix}, \quad (4.18)$$

where $\alpha \in \mathbb{C}^*$, I_n is the n -dimensional identity matrix, t_2 and t_3 described above and $S = (s_{i,j})_{\substack{1 \leq i \leq 3 \\ 1 \leq j \leq 2}}$ such that $s_{i,j} = s_{h,k}$ if $i+j = h+k$. The subgroup of matrices in the form

$$\begin{bmatrix} \alpha I_3 & 0 \\ S & I_2 \end{bmatrix},$$

fixing pointwise the center conic, is normal in $\text{Aut}(M)$, so that there is a short exact sequence

$$1 \rightarrow \mathbb{C}^4 \rtimes \mathbb{C}^* \rightarrow \text{Aut}(M) \rightarrow \text{PGL}(2, \mathbb{C}) \rightarrow 1$$

The normality of the subgroup of automorphisms fixing pointwise the center conic, and its identification with $\mathbb{C}^4 \rtimes \mathbb{C}^*$, can be proved with a straightforward calculation. By a typo, in [104, Theorem 6.6] the normal group is $\mathbb{C}^* \rtimes \mathbb{C}^4$ instead of $\mathbb{C}^4 \rtimes \mathbb{C}^*$.

See [104, Section 7] for a detailed treatment of this case. Consider $M = \mathbb{P}(W) \cap G(2, V_5)$ a 3-dimensionally smooth Grassmannian hull that is a dimensionally transverse intersection. In this case the dual $\mathbb{P}(W^\perp)$ is a plane.

Proposition 4.83. *The center map is an embedding of degree 4, whose image is a smooth projected Veronese surface V .*

Proof. See [104, Proposition 7.2]. □

We recall the definition of projected Veronese surface. The Veronese surface is the image of the map

$$\begin{aligned} \mathbb{P}(W^\perp) &\rightarrow \mathbb{P}(\mathrm{Sym}^2(W^\perp)) \\ v &\mapsto \mathbb{P}(v \cdot v) \end{aligned}$$

and its secant variety in $\mathrm{Sym}^2(W^\perp)$ is the locus of elements in the form $\mathbb{P}(v \cdot w)$. Choose a hyperplane $\mathbb{P}^4 \subset \mathbb{P}(\mathrm{Sym}^2(W^\perp))$ and a point p outside the Veronese surface, the projected Veronese surface $V \subset \mathbb{P}^4$ is the image of the Veronese surface through a projection from p to \mathbb{P}^4 , and it is smooth if and only if p does not lie in the secant variety of the Veronese surface, or equivalently the conic corresponding to p is irreducible and reduced, i.e. smooth. So we should see V_5 as a hyperplane inside $\mathrm{Sym}^2(W^\perp)$. In this setting the projection point p should correspond to a smooth conic on $\mathbb{P}(W^\perp)$, say C_p . Clearly an automorphism of C_p induces a transformation ϕ on $\mathbb{P}(\mathrm{Sym}^2(W^\perp))$ and on $\mathbb{P}(V_5)$ since the projection point is fixed. Moreover the automorphism is equivariant with respect to the center map c , hence it induces an automorphism on the projected Veronese surface V ; this last is the locus of centers of W , so ϕ induces an automorphism of the Grassmannian hull M . Vice versa, any automorphism on M acts on $\mathbb{P}(W^\perp)$ and $\mathbb{P}(V_5)$ in a way that respects the center map, in particular it respects the projection from p and so it sends C_p to itself. Putting all this together we have the following.

Theorem 4.84. [104, Theorem 7.5] *Let $M = \mathbb{P}(W) \cap G(2, V_5)$ be a smooth Grassmannian hull which is a dimensionally transverse intersection of dimension 3. There exists a conic $C_p \subset \mathbb{P}(W^\perp)$ such that $\mathrm{Aut}(M) \cong \mathrm{Aut}(C_p) \cong \mathrm{PGL}(2, \mathbb{C})$.*

4.5 Gushel-Mukai varieties

4.5.1 Definition of Gushel-Mukai varieties

We summarize here some results by Debarre and Kuznetsov, from [27]. We fix a 5-dimensional complex vector space V_5 and $G(2, V_5)$ the Grassmannian of 2-vector spaces inside V_5 , embedded in $\mathbb{P}(\wedge^2 V_5) \cong \mathbb{P}^9$ through the Plücker embedding.

Definition 4.85. Consider a vector subspace $W \subseteq \bigwedge^2 V_5$ of dimension $\dim W \geq 6$, and a quadric hypersurface $Q \subset \mathbb{P}(W)$. We call ordinary Gushel-Mukai (GM) intersection the scheme

$$Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q.$$

If Z is integral (i.e. reduced and irreducible) and

$$\dim(Z) = \dim(W) - 5$$

then we call Z an ordinary GM variety.

When Z is a GM variety, then both Q and $\mathbb{P}(W) \cap G(2, V_5)$ are Cohen-Macaulay schemes, so Z is a dimensionally transverse intersection inside $G(2, V_5)$ (cfr. Definition 1.1 for the definition): it is a complete intersection inside a Grassmannian instead of a projective space.

Remark 4.86. When Z is an ordinary GM variety it has degree 10 inside $\mathbb{P}(W)$, as a dimensionally transverse intersection of $\mathbb{P}(W) \cap G(2, V_5)$, whose degree is 5, and a quadric hyperplane of $\mathbb{P}(W)$.

Consider Z normal of dimension n . The canonical bundle of $G(2, V_5)$ is $-5H$, the codimension of W inside $\bigwedge^2 V_5$ is $5 - n$ and Q is a quadric hypersurface. Call K_Z the canonical bundle of Z : summing up we obtain $K_Z = -5H + (5 - n)H + 2H = -(n - 2)H$ modulo numerical equivalence. Hence, when $n \geq 3$, the ordinary GM variety Z is a Fano variety of index $n - 2$ and degree 10 (see Section 4.4.1 for the definition of index of a Fano variety).

Remark 4.87. Another class of GM intersections, hence varieties, exist. Let K be a vector space of dimension k for $k \geq 0$, and $C_K G(2, V_5)$ be the cone over the Grassmannian in $\mathbb{P}(K \oplus \bigwedge^2 V_5)$. Let W a n -dimensional vector subspace of $\bigwedge^2 V_5 \oplus K$, $n \geq 6$, and Q a quadric hypersurface of $\mathbb{P}(W)$. We say then that $Z = \mathbb{P}(W) \cap C_K G(2, V_5) \cap Q$ is a *special GM intersection*, and a *special GM variety* when $\dim(Z) = n - 5 \geq 1$.

When dealing with normal special GM variety, some extra attention must be taken, as they are not necessarily locally complete intersections. In [27, Definition 2.25 and Proposition 2.26], Debarre and Kuznetsov gave necessary and sufficient conditions for a normal special GM variety to be a locally complete intersection; normal ordinary GM varieties are always locally complete intersections. However, in this thesis we will never work with special GM varieties.

Definition 4.88. Let $Z = \mathbb{P}(W') \cap G(2, V'_5) \cap Q'$ and $Z = \mathbb{P}(W') \cap G(2, V'_5) \cap Q'$ be ordinary GM varieties. An isomorphism between them is a linear map $\phi : \mathbb{P}(W) \rightarrow \mathbb{P}(W')$ such that $\phi(Z) = Z'$. Given an ordinary GM variety Z , we denote by $\text{Aut}(Z, \mathbb{P}(W))$ the group of automorphisms of Z .

Here we follow a different approach than Debarre and Kuznetsov, considering always GM varieties as embedded varieties. On the other hand, Debarre and Kuznetsov defined polarized GM varieties as pairs (Z, H) where Z is a GM variety with a polarization i.e. an ample line bundle on it; with this definition, the group $\text{Aut}(Z, \mathbb{P}(W))$ corresponds to the group $\text{Aut}_H(Z)$ of automorphisms fixing the polarization induced on Z by the restriction of $\mathcal{O}_{\mathbb{P}(W)}(1)$.

By definition, an ordinary GM variety has dimension at most 5. If $\dim(Z) \geq 3$ and Z is smooth, by Lefschetz hyperplane theorem, see [53, Exposé XII, Corollary 3.7], the Picard group of Z has rank one, hence there exists only one choice of a (primitive) polarization on Z . In all these cases, we

have $\text{Aut}(Z) = \text{Aut}_H(Z) = \text{Aut}(Z, \mathbb{P}(W))$, where we denoted by $\text{Aut}(Z)$ the automorphism group of Z as an abstract variety.

Remark 4.89. If Z is a 2-dimensional ordinary GM variety, call H the polarization given by the restriction of $\mathcal{O}_{\mathbb{P}(W)}(1)$: the polarized variety (Z, H) is a Brill-Noether general $K3$ surface, see Theorem 2.34. Again by Theorem 2.34, every Brill-Noether general $K3$ surface can be seen as a smooth ordinary GM variety of dimension 2. We will say more about this in Section 5.2.

In this thesis we will not say much about 1-dimensional GM varieties. For the sake of completeness, we just refer to [27, Proposition 2.11], presented below.

Proposition 4.90. *The canonical model of a smooth projective curve is a GM variety if and only if it is a Clifford general curve of genus 6.*

Every ordinary GM variety has an associated Grassmannian hull; for a short survey about these objects see Section 4.4.

Definition 4.91. *Let $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ be an ordinary GM variety. The Grassmannian hull of Z is the intersection $M_Z = \mathbb{P}(W) \cap G(2, V_5)$.*

Definition 4.92. *A GM variety Z is strongly smooth if both Z and its associated Grassmannian hull $M_Z = \mathbb{P}(W) \cap G(2, V_5)$ are dimensionally transverse and smooth.*

We define the Grassmannian hull of Z when the latter is a dimensionally transverse intersection, in particular it has dimension $n = \dim(W) - 5$. This means that the Grassmannian hull M_Z has always dimension $n + 1$, as Q is a hypersurface, hence it is always a dimensionally transverse intersection of degree 5 (cfr. Remark 4.71). In particular the only condition to check, for a smooth ordinary GM variety Z , to be strongly smooth, is that M_Z is smooth too. Moreover, since Z is irreducible, the same holds for M_Z .

Proposition 4.93. *Let Z be a smooth ordinary GM variety of dimension $n \geq 2$. Then Z is strongly smooth if $n \geq 3$. If $n \leq 2$ then the Grassmannian hull M_Z has finitely many rational double points.*

Proof. See [27, Proposition 2.20]. □

Every ordinary GM variety Z is an intersection of quadric hypersurfaces by Lemma 4.72, since $Z = M_Z \cap Q$. If we denote by γ the bilinear symmetric form whose zero locus is the quadric hypersurface Q , we also have

$$H^0(\mathbb{P}(W), \mathcal{I}_Z(2)) \xrightarrow{\sim} V_5 \oplus \mathbb{C}\gamma \tag{4.19}$$

$$\tilde{P}_v = \text{vol}(v \wedge _ \wedge _) \mapsto v \tag{4.20}$$

$$\gamma \mapsto \gamma. \tag{4.21}$$

We can then consider a 6-dimensional V_6 containing V_5 , endowed with a linear map $q : V_6 \rightarrow \text{Sym}^2 W^\vee$ such that $q(v) = \tilde{P}_v$ for any $v \in V_5 - \{0\}$. Denote by $Q(x)$ the projective hyperplane quadric associated to $q(x)$; for any $x \in V_6 - V_5$ we have $Z = M_Z \cap Q(x)$; note that the definition of *GM* variety does not depend on the choice of x .

In general, when an embedded variety is cut out by quadrics, we can define the associated *excess conormal sheaf*. This sheaf plays an important role to check whether a certain intersection of quadrics is a *GM* variety. Here we give an idea of its structure and basic properties, for an exhaustive treatment see [27, Appendix A].

Let $Z \subset \mathbb{P}(W)$ be a subvariety of a projective space which is an intersection of quadrics, and

$$V_Z = H^0(\mathbb{P}(W), \mathcal{I}_Z(2))$$

the vector space of quadrics through Z . There is a map $V_Z \otimes \mathcal{O}_{\mathbb{P}(W)}(-2) \rightarrow \mathcal{I}_Z$, defined as

$$\begin{aligned} (V_Z \otimes \mathcal{O}_{\mathbb{P}(W)}(-2))(U) &\rightarrow \mathcal{I}_Z(U) \\ Q \otimes s &\mapsto Q|_U \cdot s \end{aligned}$$

for $U \subset \mathbb{P}(W)$ open. This map is surjective since $\mathcal{I}_Z(2)$ is globally generated, and the composition with the map $\mathcal{I}_Z \rightarrow \frac{\mathcal{I}_Z}{\mathcal{I}_Z^2}$ is again surjective. The support of the latter is contained in Z , and if we restrict to Z we have a surjective morphism of sheaves

$$V_Z \otimes \mathcal{O}_Z(-2) \rightarrow \frac{\mathcal{I}_Z}{\mathcal{I}_Z^2} = \mathcal{N}_{Z/\mathbb{P}(W)}^\vee,$$

where $\mathcal{N}_{Z/\mathbb{P}(W)}^\vee$ is, as usual, the conormal sheaf of Z with respect to $\mathbb{P}(W)$.

Definition 4.94. *Let $Z \subset \mathbb{P}(W)$ be an intersection of quadrics. We call excess conormal sheaf of Z , denoted by $\mathcal{E}\mathcal{N}_{Z/\mathbb{P}(W)}^\vee$, the kernel of the natural surjective morphism $V_Z \otimes \mathcal{O}_Z(-2) \rightarrow \mathcal{N}_{Z/\mathbb{P}(W)}^\vee$.*

We thus have a short exact sequence

$$0 \rightarrow \mathcal{E}\mathcal{N}_{Z/\mathbb{P}(W)}^\vee \rightarrow V_Z \otimes \mathcal{O}_Z(-2) \rightarrow \mathcal{N}_{Z/\mathbb{P}(W)}^\vee \rightarrow 0.$$

The excess conormal sheaf is locally free on the points on which Z is a locally complete intersection, see [27, Lemma A.2], hence on the whole Z for a normal ordinary *GM* variety. Let H be the polarization induced on Z by the restriction of $\mathcal{O}_{\mathbb{P}(W)}(1)$.

The *twisted excess conormal sheaf* is $\mathcal{U}_Z = \mathcal{E}\mathcal{N}_{Z/\mathbb{P}(W)}^\vee(2H)$; for a normal ordinary *GM* variety it induces the closed embedding

$$Z \hookrightarrow G(2, V_5), \tag{4.22}$$

which is called *Gushel map*, see [27, Section 2.5].

It is natural to ask if an abstract variety admits an embedding as a *GM* variety inside $G(2, V_5)$: here we present the intrinsic characterization of normal *GM* varieties given by Debarre and Kuznetsov in [27]. While in this thesis we always consider embedded *GM* varieties, this result is a key point

to define *GM data*, which are a set of vector spaces and linear maps which can be associated to any normal *GM* variety. These collections of objects are very useful to handle and will play an important role in Section 4.5.2. As usual, a polarized variety (Z, H) is a variety with an ample line bundle on it.

Theorem 4.95. *A normal integral polarized projective variety (Z, H) of dimension $n \geq 1$ is a polarized GM variety if and only if the following conditions hold:*

- 1) $H^n = 10$ and $K_Z = -(n-2)H$ (modulo numerical equivalence);
- 2) H is very ample and the vector space $W = H^0(Z, H)^\vee$ has dimension $n+5$;
- 3) Z is an intersection of quadrics in $\mathbb{P}(W)$ and the space $V_6 = H^0(\mathbb{P}(W), \mathcal{I}_Z(2)) \subset \text{Sym}^2 W^\vee$ of quadrics throught Z has dimension 6;
- 4) the twisted excess conormal sheaf \mathcal{U}_Z of Z in $\mathbb{P}(W)$ is simple i.e. $\text{Hom}(\mathcal{U}_Z, \mathcal{U}_Z) \cong \mathbb{C}$.

Proof. (Sketch) See [27, Theorem 2.3]. We only show that 1) – 4) hold for normal ordinary *GM* varieties. Consider a normal ordinary *GM* variety $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$, with the polarization H given by the restriction of $\mathcal{O}_{\mathbb{P}(W)}(1)$.

- 1) This is Remark 4.86.
- 2) Clearly H is very ample. By taking a resolution for $\mathcal{O}_{G(2, V_5)}$ it is possible to obtain $H^0(Z, H) = H^0(\mathbb{P}(W), \mathcal{O}_{\mathbb{P}(W)}(1))$ which in turn is isomorphic to W^\vee .
- 3) This is Equation (4.19).
- 4) See [27, Theorem 2.3].

We refer again to Debarre and Kuznetsov for the proof that a normal polarized variety satisfying 1) – 4) is a *GM* intersection. Finally, the conditions for Z to be a *GM* variety are provided by the fact that $\dim(H^0(Z, \mathcal{O}_Z(H)))$ is $(n+5)$ -dimensional, and that Z is by hypotheses a variety, hence integral. \square

Remark 4.96. Some vector spaces and linear maps, other than W, V_6, V_5 and $q : V_6 \rightarrow \text{Sym}^2 W^\vee$, come into play in the other direction of the proof of Theorem 4.95. They are

- a 1-dimensional vector space L ;
- a linear map $\mu : W \otimes L \rightarrow \bigwedge^2 V_5$;
- a linear isomorphism $\epsilon : \bigwedge^5 V_5 \rightarrow L^{\otimes 2}$.

On a normal *GM* variety, the space L can be thought of as $(V_6/V_5)^\vee$. The restriction of the projectivization of the linear map μ induces the Gushel map $Z \rightarrow G(2, V_5)$. It is injective if and only if the *GM* variety is ordinary: its kernel K is the vector space on which we take the cone $C_K G(2, V_5)$ in Remark 4.87. (As we will see also in Theorem 4.106, W can be naturally seen as a subspace of $\bigwedge^2 V_5 \otimes L^\vee$, hence the tensorization with L in the definition of μ).

The map ϵ comes from a short exact sequence

$$0 \rightarrow V_5 \rightarrow V_6 \rightarrow (L^\vee)^{\otimes 2} \otimes \bigwedge^6 V_6 \rightarrow 0, \quad (4.23)$$

which in turn appears by [27, Proposition A.3] and by simplicity of the twisted excess conormal sheaf \mathcal{U}_Z . From (4.23) we obtain an isomorphism $\bigwedge^5 V_5 \otimes \bigwedge^6 V_6 \otimes (L^\vee)^{\otimes 2} \rightarrow \bigwedge^6 V_6$, thus an isomorphism $\epsilon : \bigwedge^5 V_5 \rightarrow L^{\otimes 2}$. In some sense it can thought of as a volume form on $\bigwedge^5 V_5$.

In the statement of Theorem 4.95 and Remark 4.96 some vector spaces and maps are presented. We collect them in the next definition. Following Debarre and Kuznetsov, for the sake of readability we does not change notation when we take, for example, $\mu \otimes \text{id}_L$ instead of μ or $\epsilon \otimes \text{id}_{\bigwedge^4 V_5^\vee}$ instead of ϵ .

Definition 4.97. *An ordinary GM data $(W, V_6, V_5, L, \mu, q, \epsilon)$ of dimension n consists of*

- a $(n + 5)$ -dimensional vector space W ;
- a 6-dimensional vector space V_6 ;
- a hyperplane V_5 of V_6 ;
- a 1-dimensional vector space L ;
- an injective linear map $\mu : W \otimes L \hookrightarrow \bigwedge^2 V_5$;
- a linear map $q : V_6 \rightarrow S^2 W^\vee$;
- a linear isomorphism $\epsilon : \bigwedge^5 V_5 \rightarrow L^{\otimes 2}$,

such that the following diagram commutes

$$\begin{array}{ccc} V_5 & \hookrightarrow & V_6 \\ \downarrow \epsilon & & \downarrow q \\ L^{\otimes 2} \otimes \bigwedge^4 V_5^\vee & \xrightarrow{\text{Sym}^2 \mu^\vee} & \text{Sym}^2 W^\vee \end{array}$$

that is

$$q(v)(w_1, w_2) = \epsilon(v \wedge \mu(w_1) \wedge \mu(w_2)) \quad (4.24)$$

for all $v \in V_5$, $w_1, w_2 \in W$.

In the definition above, (4.24) corresponds to asking that any quadric in the form $q(v)$ for $v \in V_5 - \{0\}$ is \tilde{P}_v i.e. it is the restriction to W of the Pfaffian quadric P_v . As we can expected from Remark 4.96, special GM data are defined in the same way by Debarre and Kuznetsov in [27], except that the map μ is no more injective.

Definition 4.98. Consider two ordinary GM data sets, $(W, V_6, V_5, L, \mu, q, \epsilon)$ and $(W', V'_6, V'_5, L', \mu', q', \epsilon')$. An isomorphism between them is a triple of linear isomorphisms $\phi_W : W \rightarrow W'$, $\phi_V : V_6 \rightarrow V'_6$, $\phi_L : L \rightarrow L'$ such that $\phi_V(V_5) = V'_5$, $\epsilon' \circ \bigwedge^5 \phi_V = \phi_L^{\otimes 2} \circ \epsilon$, and the following diagrams commute

$$\begin{array}{ccc}
V_6 & \xrightarrow{q} & \mathrm{Sym}^2 W^\vee \\
\phi_V \downarrow & & \uparrow \mathrm{Sym}^2 \phi_V^\vee \\
V'_6 & \xrightarrow{q'} & \mathrm{Sym}^2 (W')^\vee
\end{array}
\qquad
\begin{array}{ccc}
W \otimes L & \xrightarrow{\mu} & \bigwedge^2 V_5 \\
\phi_W \otimes \phi_L \downarrow & & \downarrow \bigwedge^2 (\phi_V|_{V_5}) \\
W' \otimes L' & \xrightarrow{\mu'} & \bigwedge^2 V'_5.
\end{array}
\tag{4.25}$$

We denote by $\mathrm{Aut}(W, V_6, V_5, L, \mu, q, \epsilon)$ the group of automorphisms of the GM data $(W, V_6, V_5, L, \mu, q, \epsilon)$ i.e. the subgroup of $(g_W, g_V, g_L) \in GL(W) \times GL(V_6) \times \mathbb{C}^*$ such that

$$g_V(V_5) = V'_5 \quad \det(g_V|_{V_5}) = g_L^2 \quad (\mathrm{Sym}^2 g_W^\vee) \circ q \circ g_V = q \quad \left(\bigwedge^2 g_V \right) \circ \mu = \mu \circ (g_W \otimes g_V).$$

Now we consider a normal ordinary GM variety $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$: the study by Debarre and Kuznetsov which we presented in Theorem 4.95, see also Remark 4.96, allows to associate a data set $(W, V_6, V_5, L, \mu, q, \epsilon)$ to it.

Lemma 4.99. The collection $(W, V_6, V_5, L, \mu, q, \epsilon)$ associated to a normal GM variety is a set of GM data. The data associated to isomorphic ordinary GM variety are isomorphic ordinary GM data.

Proof. The first part is clear. For the second part one attention is needed: we consider two ordinary GM varieties $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ and $Z' = \mathbb{P}(W') \cap G(2, V'_5) \cap Q'$ as polarized varieties, with the polarization H (resp. H') induced by $\mathcal{O}_{\mathbb{P}(W)}(1)$ (resp. $\mathcal{O}_{\mathbb{P}(W')}(1)$) and an automorphism $(\phi, \psi) : (Z, H) \rightarrow (Z', H')$, where $\phi : Z \xrightarrow{\sim} Z'$ is an automorphism of varieties and $\psi : H \xrightarrow{\sim} \phi^* H'$ an isomorphism of line bundles. This automorphism induces a linear morphism between W and W' , hence two isomorphisms, one between V_6 and V'_6 the other between L and L' . These automorphisms are compatible with the (twisted) excess conormal sheaves of Z and Z' , in particular the isomorphisms are compatible with the Gushel maps, see Equation (4.22). So quadrics from V_5 are sent to quadrics from V'_5 . The isomorphisms are compatible with the maps q, μ, ϵ . \square

By Theorem 4.95, a GM data $(W, V_6, V_5, L, \mu, q, \epsilon)$ provide all needed to produce a GM intersection, which a priori is not a variety nor is normal. In particular, (4.24) assures that quadrics in V_5 cut out the linear section of $G(2, V_5)$.

In the case of special GM data, we call K the kernel of μ ; there is a natural isomorphism $L \otimes W \cong \mu(W \otimes L) \oplus K$. Consider then the injective map $\bar{\mu} : \mu(L \otimes W) \oplus K \rightarrow \bigwedge^2 V_5 \oplus K$: the associated special GM intersection then lies in $C_K G(2, V_5)$ through the map $\bar{\mu}$. Again, quadrics in V_5 cut out the linear section of the cone $C_K G(2, V_5)$ over the Grassmannian. The "linear part" of the cone is given by elements in the kernel of μ : for every $v \in V_5$, $w \in \ker(\mu)$, we have $q(v)(w, w) = 0$ by (4.24). As usual, in the following statement we only refer to ordinary GM variety; we write $Q(v)$ for the projective quadric associated to $q(v)$.

Lemma 4.100. *Let $(W, V_6, V_5, L, \mu, q, \epsilon)$ be some ordinary GM data of dimension n . We consider $\mathbb{P}(W) = \mathbb{P}(W \otimes L)$ as a subspace of $\mathbb{P}(\wedge^2 V_5)$ through μ . Then*

$$Z = \bigcap_{v \in V_6} Q(v) \subset \mathbb{P}(W)$$

is an ordinary GM intersection, which is a GM variety if and only if Z is integral of dimension n . Isomorphic ordinary GM data of dimension n gives rise to isomorphic ordinary GM intersections.

Proof. By (4.24), inside $\mathbb{P}(W)$ we have a Grassmannian hull $M = \bigcap_{v \in V_5} Q(v) \subset \mathbb{P}(W)$. Now we choose any $x \in V_6 - V_5$ and we call Q the projective quadric associated to $q(x)$. We have then $Z = M \cap Q$, hence Z is an ordinary GM intersection whose Grassmannian hull is $M_Z = M$. By definition Z is an ordinary GM variety if and only if it is integral of dimension n , cfr. Definition 4.85. The last part of the statement is clear. \square

Remark 4.101. The two constructions in Lemma 4.99 and Lemma 4.100 are mutually inverse.

We explicit the link between the automorphism groups of an ordinary GM variety and the associated GM data. As in the proof of Lemma 4.99, we consider a GM variety as a polarized variety (Z, H) ; the associated automorphism group, which we denote $\text{Aut}(Z, H)$, consists of pairs (ϕ, ψ) with $\phi : Z \xrightarrow{\sim} Z$ an automorphism of varieties and $\psi : H \xrightarrow{\sim} \phi^* H'$ an isomorphism of line bundles. Debarre and Kuznetsov proved the following result, see [27, Corollary 2.10].

Lemma 4.102. *Let Z be an ordinary GM variety with associated GM data $(W, V_6, V_5, L, \mu, q, \epsilon)$. Then there is an exact sequence*

$$1 \rightarrow \mathbb{G}_m \rightarrow \text{Aut}(W, V_6, V_5, L, \mu, q, \epsilon) \rightarrow \text{Aut}(Z, \mathbb{P}(W)) \rightarrow 1,$$

where $\mathbb{G}_m \cong \mathbb{C}^$ is the subgroup of $\text{Aut}(Z, H)$ containing automorphisms of polarized varieties acting trivially on Z and by dilatation on H .*

Proof. In the proof of Lemma 4.99 we show a way to associate an automorphism of GM data to an element of $\text{Aut}(Z, H)$; this construction admits a converse, so that we have an isomorphism of groups $\text{Aut}(Z, H) \cong \text{Aut}(W, V_6, V_5, L, \mu, q, \epsilon)$. Then the statement follow, since there is a natural short exact sequence

$$1 \rightarrow \mathbb{G}_m \rightarrow \text{Aut}(Z, H) \rightarrow \text{Aut}(Z, \mathbb{P}(W)) \rightarrow 1.$$

\square

Debarre and Kuznetsov summarize Lemma 4.100, Lemma 4.99, Remark 4.101 and Lemma 4.102 in the following result, written in categorical language; as above, in the statement polarized GM varieties are considered, with automorphism group $\text{Aut}(Z, H)$. A *groupoid* is a category in which every arrow is an isomorphism.

Theorem 4.103. [27, Theorem 2.9] *Let $(W, V_6, V_5, L, \mu, q, \epsilon)$ be a set of ordinary GM data of dimension n . Then*

$$Z = \bigcap_{v \in V_6} Q(v) \subset \mathbb{P}(W)$$

is an ordinary GM intersection. Conversely, given a normal ordinary GM variety $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$, the collection $(W, V_6, V_5, L, \mu, q, \epsilon)$ associated in Lemma 4.99 is an ordinary GM data of dimension n . Those constructions are mutually inverse and define an equivalence between

- the groupoids of normal ordinary polarized GM varieties (Z, H) of dimension n , and their isomorphisms;
- the groupoid of ordinary GM data $(W, V_6, V_5, L, \mu, q, \epsilon)$ of dimension n such that Z is a normal GM variety of dimension n , and their isomorphisms.

Let Z be a normal ordinary GM variety and let $(W, V_6, V_5, L, \mu, q, \epsilon)$ be the associated GM data. Recall that $\text{Aut}(W, V_6, V_5, L, \mu, q, \epsilon)$ is a subgroup of $GL(W) \times GL(V_6) \times \mathbb{C}^*$; since $\text{Aut}(Z, \mathbb{P}(W))$ is the quotient of $\text{Aut}(W, V_6, V_5, L, \mu, q, \epsilon)$ by \mathbb{G}_m , see Lemma 4.102 we have a natural morphism

$$\text{Aut}(Z, \mathbb{P}(W)) \rightarrow PGL(V_6) \quad (4.26)$$

$$[(g_W, g_V, g_L)] \mapsto [g_V]. \quad (4.27)$$

Corollary 4.104. *Let Z be a normal ordinary GM variety. The natural morphism $\text{Aut}(Z, \mathbb{P}(W)) \rightarrow PGL(V_6)$ is injective.*

Proof. Consider $(g_W, g_V, g_L) \in \text{Aut}(W, V_6, V_5, L, \mu, q, \epsilon)$ such that $[g_V]$ is the identity. Then g_V is scalar, and its restriction to V_5 , hence $\bigwedge^2 V_5$, is scalar too. By (4.25) then the linear maps g_W is scalar too. But the action of (g_W, g_V, g_L) on Z is exactly the restriction of $[g_W] \in PGL(W)$ to Z , cfr. Definition 4.88. \square

4.5.2 A correspondence between data sets

We define now another set of data. As in Section 4.2.1, everytime we will consider a 6-dimensional complex vector space V_6 , we endow it with a volume form on $\bigwedge^6 V_6$, which induces a symplectic form on $\bigwedge^3 V_6$. We will have then a symplectic Grassmannian $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ parametrizing Lagrangian subspaces of $\bigwedge^3 V_6$. As before, the definition of $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ does not depend on a particular volume form, so we will never stress the choice of it.

Definition 4.105. *An ordinary Lagrangian data is a collection (V_6, V_5, A) , where*

- V_6 is a 6-dimensional complex vector space,
- $V_5 \subset V_6$ is a hyperplane,
- $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ is a Lagrangian subspace.

The ordinary Lagrangian data (V_6, V_5, A) and (V'_6, V'_5, A') are isomorphic if there is a linear isomorphism $\phi: V_6 \rightarrow V'_6$ such that $\phi(V_5) = V'_5$ and $(\bigwedge^3 \phi)(A) = A'$.

Consider some Lagrangian data (V_6, V_5, A) , and set $L^\vee = V_6/V_5$: this induces a decomposition $\bigwedge^3 V_6 = \bigwedge^3 V_5 \oplus \bigwedge^2 V_5 \otimes L^\vee$. If we call λ_3 the projection to the second summand, we have a chain of maps

$$A \hookrightarrow \bigwedge^3 V_6 \xrightarrow{\lambda_3} \bigwedge^2 V_5 \otimes L^\vee.$$

Now define $W = \lambda_3(A) \subseteq \bigwedge^2 V_5 \otimes L^\vee$, a linear subspace whose codimension is $\dim(A \cap \bigwedge^3 V_5)$. We easily provided four linear subspaces W, V_6, V_5, L , as in Definition 4.97. Defining the maps, especially q , to have some complete ordinary GM data, needs some additional work. Note that, once a vector $x \in V_6 - V_5$ is fixed, there is a natural isomorphism

$$F_x \xrightarrow{\sim} \bigwedge^2 V_5 \otimes L^\vee \quad (4.28)$$

$$x \wedge \eta \mapsto \eta \otimes [x] \quad (4.29)$$

where F_x is the Lagrangian subspace defined in Definition 4.4. We defined W as a subspace of $\bigwedge^2 V_5 \otimes L^\vee$; under the identification (4.28), the image of $W \otimes L$ through the inclusion $\mu : W \otimes L \rightarrow \bigwedge^2 V_5$ will be defined as $\rho(W)$, with $\rho : F_x \rightarrow \bigwedge^2 V_5$ as in Equation (4.3).

The next result, by Debarre and Kuznetsov, sums up the results by O'Grady, [96, Section 5 and Section 6], [98, Section 4], in dimension 2 and Iliev and Manivel, [67, Section 2] in dimension 5 and [67, Section 4] in dimension 3, 4. Some other results from [67] will be presented later.

Theorem 4.106. *There is a bijection between the set of isomorphism classes of ordinary GM data and the set of isomorphism classes of Lagrangian data sets.*

Proof. (Sketch) Rather than giving a precise account of the proof, we show the way to extract one type of data from the other, referring to [27, Theorem 3.6] for an exhaustive proof.

Let (V_6, V_5, A) be Lagrangian data. We note $L^\vee = V_6/V_5$, and V_6, V_5 in the definition of GM data will be the same V_6, V_5 . As we said above, we call $\lambda_3 : \bigwedge^3 V_6 \rightarrow \bigwedge^2 V_5 \otimes L^\vee$ the projection induced by the decomposition $\bigwedge^3 V_6 = \bigwedge^3 V_5 \oplus \bigwedge^2 V_5 \otimes L^\vee$. Then we can define $W = \lambda_3(A) \subseteq \bigwedge^2 V_5 \otimes L^\vee$. By definition of λ_3 we have $\ker((\lambda_3)|_A) = A \cap \bigwedge^3 V_5$. So there is a commutative diagram

$$\begin{array}{ccc} A & \hookrightarrow & \bigwedge^3 V_6 \xrightarrow{\lambda_3} \bigwedge^2 V_5 \otimes L^\vee \\ & \searrow \lambda_3 & \nearrow \tilde{\mu} \\ & & W \end{array}$$

We called $\tilde{\mu}$ the second map in the factorization of $A \rightarrow V_5 \otimes L^\vee$.

We are left to define an injective linear map $\mu : W \otimes L \rightarrow \bigwedge^2 V_5$, an isomorphism $\epsilon : \bigwedge^5 V_5 \rightarrow L^{\otimes 2}$ and an injective linear map $q : V_6 \rightarrow \text{Sym}^2 W^\vee$.

We choose an arbitrary isomorphism

$$\epsilon : \bigwedge^5 V_5 \rightarrow L^{\otimes 2}$$

and we define $\mu = \tilde{\mu} \otimes \text{id}_L : W \otimes L \rightarrow \bigwedge^2 V_5 \otimes L^\vee \otimes L = \bigwedge^2 V_5$. As for q , we will need for $p = 3, 4$ the short exact sequences

$$0 \rightarrow \bigwedge^p V_5 \rightarrow \bigwedge^p V_6 \xrightarrow{\lambda_p} \bigwedge^{p-1} V_5 \otimes L^\vee \rightarrow 0 \quad (4.30)$$

induced by the decomposition $\bigwedge^p V_6 = \bigwedge^p V_5 \oplus \bigwedge^{p-1} V_5 \otimes L^\vee$, which we already needed for $p = 3$ to define W and μ . For every $v \in V_6$, we define a bilinear form $\tilde{q}(v)$

$$\begin{aligned} \tilde{q}(v) : A \times A &\rightarrow \mathbb{C} \\ (\xi_1, \xi_2) &\mapsto -\epsilon(\lambda_4(v \wedge \xi_1) \wedge \lambda_3(\xi_2)). \end{aligned}$$

Debarre and Kuznetsov show that the fact that A is Lagrangian implies that $\tilde{q}(v)$ is symmetric for every $v \in V_6$; this proof relies on [27, Proposition C.1], and more in general on the correspondence between Lagrangian subspaces and quadric hypersurfaces inside $\mathbb{P}(\bigwedge^3 V_6)$, well recalled in [27, Appendix C]. Moreover, the kernel of $\tilde{q}(v)$ contains $\ker((\lambda_3)|_A)$, so it induces the symmetric bilinear form which we were searching,

$$\begin{aligned} q(v) : W \times W &\rightarrow \mathbb{C} \\ (\lambda_3(\xi_1), \lambda_3(\xi_2)) &\mapsto -\epsilon(\lambda_4(v \wedge \xi_1) \wedge \lambda_3(\xi_2)). \end{aligned}$$

Again for the Lagrangian property of A one can verify that, for $v \in V_5$, the symmetric bilinear form becomes $q(v) = \epsilon(v \wedge \lambda_3(\xi_1) \wedge \lambda_3(\xi_2))$, which in turn is equal to $\epsilon(v \wedge \tilde{\mu}(\xi_1) \wedge \tilde{\mu}(\xi_2))$ by definition of $\tilde{\mu}$; since $\mu = \tilde{\mu} \otimes \text{id}_L$ the equality holds by replacing $\tilde{\mu}$ with μ as well. We have thus obtained $(W, V_6, V_5, L, \mu, q, \epsilon)$, that by construction are *GM* data. Finally, while the choice of ϵ is arbitrary, different choices of ϵ give rise to isomorphic *GM* data.

We present now the inverse construction. Consider some ordinary *GM* data $(W, V_6, V_5, L, \mu, q, \epsilon)$. As in the other construction, we already have V_6 and V_5 and we are left to find some Lagrangian $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$. We fix again $L^\vee = V_6/V_5$. We consider two linear maps. The first one is

$$f : \bigwedge^3 V_5 \oplus (V_6 \otimes W \otimes L) \rightarrow \text{Hom}(W, L) \quad (4.31)$$

such that the image of $(\eta, v \otimes w \otimes \ell)$ is the map that sends w' to $\epsilon(\eta \wedge \mu(w')) + q(v)(w, w') \otimes \ell$. We keep notation simple, but note that, in the definition of f , the map μ is considered as a linear map from W to $\bigwedge^2 V_5 \otimes L^\vee$ and ϵ as a map from $\bigwedge^2 V_5 \otimes L^\vee$ to L . The second one is

$$g : \bigwedge^3 V_5 \oplus (V_6 \otimes W \otimes L) \rightarrow \bigwedge^3 V_6 \quad (4.32)$$

$$(\eta, v \otimes w \otimes \ell) \mapsto \eta + v \wedge \mu(w \otimes \ell). \quad (4.33)$$

We can then define $A = g(\ker(f))$, which can be proved to be a Lagrangian subspace inside $\bigwedge^3 V_6$, again by the correspondence between Lagrangian subspaces and quadric hypersurfaces inside $\mathbb{P}(\bigwedge^3 V_6)$, see [27, Lemma C.2]. A direct computation shows that the two constructions are one the inverse of the other. \square

The construction in Theorem 4.106 can be combined with Lemma 4.99 and Lemma 4.100 to associate, to any normal ordinary *GM* variety, some Lagrangian data, and vice versa to associate a *GM* intersection to some Lagrangian data. Note that, as in Lemma 4.100, the *GM* intersection obtained from (V_6, V_5, A) is not necessarily normal, or even a *GM* variety. An important sufficient

condition, for Lagrangian data, to provide a GM intersection that is, in fact, a smooth GM variety is provided by Debarre and Kuznetsov in [27, Theorem 3.14]. We will present it in our setting as Theorem 4.110. However, the GM intersection associated to the GM data extracted in Theorem 4.106 can be obtained by direct calculation from the proof of the theorem, which is pretty explicit.

Recall that in this thesis we always consider GM intersection as embedded in $G(2, V_5)$.

Definition 4.107. *Given a normal ordinary GM variety $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$, we denote by $(V_6(Z), V_5(Z), A(Z))$ the Lagrangian data associated by Theorem 4.106 to the GM data obtained by Lemma 4.99 from Z . We call $A(Z)$ the Lagrangian subspace associated to Z .*

Given some Lagrangian data (V_6, V_5, A) , we denote by $Z(V_5, A)$ the ordinary GM intersection obtained by Lemma 4.100 from the GM data associated by Theorem 4.106 to (V_6, V_5, A) . When not otherwise specified, we write $Z(V_5, A) = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ where Q is the projective quadric associated to $q(x)$ for any $x \in V_6 - V_5$.

The reason for which we call $Z(V_5, A)$ the associated GM intersection, instead of $Z(V_6, V_5, A)$, is that in practice we will identify all the six-dimensional vector spaces in Lagrangian data sets, to simplify notation.

This construction by Debarre and Kuznetsov extends with some changes to special normal GM varieties, see remark 4.87, extensively presented throughout [27, Section 3.2]. To do it, it is necessary to introduce *extended Lagrangian data*, which are in the form (V_6, V_5, A, A_1) where $A_1 \subset \mathbb{C} \oplus (V_6/V_5)^\vee$ is a one-dimensional vector space. In this situation, if $A_1 = (V_6/V_5)^\vee$ then the associated GM variety is ordinary and the construction is the same explained above, while for $A_1 \neq (V_6/V_5)^\vee, \mathbb{C}$ the construction is adapted to provide a special GM variety. Finally, if $A_1 = \mathbb{C}$ the adapted construction can be utilized to obtain a GM intersection, but it results in something that it is not even a locally complete intersection.

The situation is completely symmetric for GM varieties, ordinary GM varieties giving rise to Lagrangian data in the form $(V_6, V_5, (V_6/V_5)^\vee)$ and special GM varieties to Lagrangian data in the form (V_6, V_5, A_1) for $A_1 \neq L, \mathbb{C}$. The only exception is that Lagrangian data in the form (V_6, V_5, \mathbb{C}) cannot be reached.

In this thesis we will only work with strongly smooth ordinary GM varieties, so it is sufficient for us to present the construction for ordinary data.

We collect some results about the bijection given in Theorem 4.106, proved by Debarre and Kuznetsov. Here we start to see the interplay between the collections of data.

Proposition 4.108. *Let (V_6, V_5, A) be some Lagrangian data, and $Z(V_5, A)$ the corresponding ordinary GM intersection. We denote by $(W, V_6, V_5, L, \mu, q, \epsilon)$ the GM data associated to $Z(V_5, A)$ and, for $x \in V_6 - \{0\}$, by F_x the Lagrangian subspace defined in Definition 4.4.*

- 1) *the vector space W has dimension $\dim(W) = 10 - \dim(A \cap \bigwedge^3 V_5)$*
- 2) *For $x \in V_6 - V_5$, we have $\ker(q(x)) = A \cap F_x$.*

3) If $Z(V_5, A)$ is a GM variety, then $\dim(Z(V_5, A)) = 5 - \dim(A \cap \bigwedge^3 V_5)$.

Proof. See [27, Proposition 3.9]. For 1) we only need to observe that W is defined as the image, through λ_3 , of A , and the kernel of $(\lambda_3)|_A$ is $A \cap \bigwedge^3 V_5$.

For 2), once we fix $x \in V_6 - V_5$, we have an isomorphism $F_x \rightarrow \bigwedge^2 V_5 \otimes L^\vee$ as in Equation (4.28) which induces a decomposition $\bigwedge^3 V_6 \cong \bigwedge^3 V_5 \oplus F_x$. Note that both subspaces are Lagrangian. We call again λ_3 the projection map $\bigwedge^3 V_6 \rightarrow F_x$; we have then $W = \lambda_3(A) \subseteq F_x$. We need the correspondence between Lagrangian subspaces and quadric hypersurfaces inside $\bigwedge^3 V_6$: by [27, Proposition C.1] the kernel of the quadratic form $\tilde{q} : A \times A \rightarrow \mathbb{C}$, defined in the proof of Theorem 4.106, is $\ker(\tilde{q}(x)) = (A \cap \bigwedge^3 V_5) \oplus (A \cap F_x)$. Since $q(x)$ is the factorization of $\tilde{q}(x)$ on F_x , the kernel of $q(x)$ is $\lambda_3(\ker(\tilde{q}(x))) = A \cap F_x$.

The proof of 2) is a dimension count: by definition of GM variety we have $\dim Z(V_5, A) = \dim(W) - 5$ and we know, from Item 1), that $\dim(W) = 10 - \dim(A \cap \bigwedge^3 V_5)$; summing up we obtain the statement. \square

Remark 4.109. Proposition 4.108, Item 2), does not hold, a priori, for $x \in V_5$; this is clear from the proof, as it relies heavily on the decomposition $V_6 = \mathbb{C}x \oplus V_5$. In Lemma 4.75 we computed the rank of the quadric when $x \in V_5$.

Consider Z an ordinary GM variety, and $(V_6(Z), V_5(Z), A(Z))$ the associated Lagrangian data. We fix a volume form on $\bigwedge^6 V_6(Z)$; we defined, inside $\mathbb{L}\mathbb{G}(\bigwedge^3 V_6(Z))$, the divisor Σ of Lagrangian subspaces that contains no non-zero decomposable vectors, see Definition 4.46. This is the setting for the result below (we defined strongly smooth GM varieties in Definition 4.92).

Theorem 4.110. *If $\dim(A(Z) \cap \bigwedge^3 V_5) \leq 3$, then Z is a strongly smooth ordinary GM variety if and only if $A(Z) \notin \Sigma$. Moreover, if $A(Z) \notin \Sigma$ then $[V_5] \in \mathbb{P}(V_6^\vee(Z))$ lies in $Y_{A(Z)^\perp}^{5-n}$, with $n = \dim(Z)$.*

Proof. For the first part see [27, Theorem 3.14]. For the second part observe that if $A \notin \Sigma$ then the ordinary GM intersection Z is in fact an ordinary GM variety, so $\dim(Z) = 5 - \dim(A \cap \bigwedge^3 V_5)$ by Proposition 4.108, Item 3). The statement follows once we consider the description of the stratification $\{Y_{A^\perp}^{\geq k}\}_{k \geq 1}$ given in Remark 4.20. \square

Corollary 4.111. *Consider a 6-dimensional complex vector space V_6 , and fix a volume form on $\bigwedge^3 V_6$. Suppose that $A \notin \Sigma$. Then, for every hyperplane $V_5 \subset V_6$, the GM variety associated to the Lagrangian data (V_6, V_5, A) is strongly smooth of dimension n , where $[V_5] \in Y_{A^\perp}^{5-n}$.*

Corollary 4.112. *Consider a 6-dimensional complex vector space V_6 , and fix a volume form on $\bigwedge^3 V_6$. Suppose that $A \notin \Sigma$. Then, for every $[V_5] \in Y_{A^\perp}^3$, the GM variety $Z(A, V_5)$, endowed with the polarization given by the restriction of $\mathcal{O}_{\mathbb{P}(W)}(1)$, is a Brill-Noether general K3 surface (see Definition 2.33).*

Remark 4.113. Although GM curves exist, in this thesis we never deal with them. The reason is clear from Corollary 4.111: we are interested in EPW sextics Y_A whose double covering is a hyperkähler manifold, in particular by Theorem 4.51 we have that $A \notin \Sigma$. Then, by Lemma 4.10,

any ordinary GM intersection (hence variety) associated to A has dimension at least 2. In other words, ordinary GM curves are never strongly smooth.

There is another way to obtain the EPW sextic associated to some strongly smooth ordinary GM variety. Consider $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ a normal ordinary GM variety of dimension n , with $V_6 = H^0(\mathbb{P}(W), \mathcal{I}_Z(2)) \subset \text{Sym}^2 W^\vee$ the space of quadrics cutting Z . We denote by $\widetilde{\text{Disc}}(Z) \subseteq |V_6|$ the scheme of projective quadrics, inside the linear system $|V_6|$, that are singular. Note that, by Lemma 4.75, the general element inside the hyperplane $|V_5|$ has corank $\dim W - 6 = n - 1$ and every element has corank at least $n - 1$. Thus, for $n \geq 2$, $|V_5|$ is always contained in $\widetilde{\text{Disc}}(Z)$ with multiplicity $\dim W - 6 = n - 1$. Here we denote by $|V_5|$ and $|V_6|$ the linear systems, to stress the fact that the elements inside them are quadric hypersurfaces.

Definition 4.114. *Let $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ be a normal ordinary GM variety. If $\widetilde{\text{Disc}}(Z) \neq |V_6|$, we call discriminant of Z the scheme*

$$\text{Disc}(Z) = \widetilde{\text{Disc}}(Z) - (n - 1)|V_5|.$$

If $\widetilde{\text{Disc}}(Z) = |V_6|$ we define $\text{Disc}(Z) = |V_6|$.

The scheme $\widetilde{\text{Disc}}(Z)$ has degree $10 - (5 - n) = 6 + (n - 1)$ inside $|V_6|$, hence $\text{Disc}(Z)$ is a sextic hypersurface when it is not the whole linear system (cfr. [67, Sections 2.1 and 2.2]). Call $A(Z) \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ the Lagrangian subspace associated to Z in Theorem 4.106: Iliev and Manivel were the first to prove that, in the general case, $\text{Disc}(Z)$ is isomorphic to the EPW sextic $Y_{A(Z)}$ for $n \in \{3, 4, 5\}$. For $n = 2$ the proof is part of the first version of O’Grady’s proof that X_A is a hyperkähler manifold for Z general. The case $n = 2$ can be found at [96, Section 5] and the higher-dimensional case will be summarized below.

Anyway, in the setting given by Debarre and Kuznetsov, a manageable proof can be given that works in every dimension. As usual, we give the result for ordinary GM varieties; the proof is identical in the case of special GM varieties.

Proposition 4.115. [27, Corollary 3.17] *Let Z be an ordinary GM variety which is strongly smooth of dimension $n \geq 2$, and let $A(Z)$ be the associated Lagrangian subspace. Then $\text{Disc}(Z) = Y_{A(Z)}$.*

Proof. Consider $x \in V_6 - V_5$: Proposition 4.108, Item 2) assures that $[x]$ lies in $Y_{A(Z)}$ if and only if the projective quadric associated to $q(x)$ is not smooth. This is sufficient to prove that $Y_{A(Z)}$ and $\text{Disc}(Z)$ coincide as sets outside $|V_5|$.

Proposition 4.108, Item 3), implies that $\dim(A \cap \bigwedge^3 V_5) = 5 - n \leq 3$, so asking the strongly-smoothness of Z is equivalent to ask that $A(Z) \notin \Sigma$, see Theorem 4.110. Hence we know by Proposition 4.11 that $Y_{A(Z)}$ is a EPW sextic i.e. a sextic hypersurface inside $|V_6|$. Both $\text{Disc}(Z)$ and $Y_{A(Z)}$ are sextic hypersurfaces, and they coincide as sets outside $|V_5|$, so they are equal. \square

Remark 4.116. When $\text{Disc}(Z) \neq |V_6|$, we can always consider $Q \in |V_6| - |V_5| - \text{Disc}(Z)$, so that Z is the dimensionally transverse intersection of the Grassmannian hull M_Z and the smooth quadric Q . This happens, for example, everytimes Z is smooth of dimension at least 3, or strongly smooth of dimension 2.

Consider any 4-dimensional vector subspace $V_4 \subset V_5$. For any $v \in V_5 - \{0\}$, the restriction of the Pfaffian quadric P_v to $\Lambda^2 V_4$ always gives rise to the Plücker quadric $G(2, V_4) \subset \mathbb{P}(\Lambda^2 V_4)$: this happens since $G(2, V_5)$ restricts to $G(2, V_4)$ which is a quadric hypersurface in $\mathbb{P}(\Lambda^2 V_4)$. So the whole linear system $|V_6|$ cuts a pencil of quadrics in $\mathbb{P}(W \cap \Lambda^2 V_4)$, and a hyperplane of $|V_6|$ a single quadric in $\mathbb{P}(W \cap \Lambda^2 V_4)$. Iliev and Manivel provided the following description of the dual *EPW* sextic $Y_{A(Z)^\perp} \cong \text{Disc}(Z)^\vee$ inside $|V_6|^\vee$.

Proposition 4.117. *The dual hypersurface $\text{Disc}(Z)^\vee \subset |V_6|^\vee$ is the locus parametrizing the hyperplanes $H \in |V_6|^\vee$ for which there exists V_4 such that, for every $\sigma \in H$, the restriction of σ to $\mathbb{P}(W \cap \Lambda^2 V_4)$ is singular.*

Proof. See [67, Proposition 2.4]. □

Definition 4.118. *We call Plücker point the point $|V_5| \in |V_6|^\vee$.*

Now we sketch the version of Proposition 4.115 proved in [67]. Iliev and Manivel showed that, for Z a general ordinary smooth *GM* variety of dimension n , the scheme $\text{Disc}(Z)$ is a general *EPW* sextic if $n \in \{3, 4, 5\}$. More precisely, for $n = 5$ they provided an explicit Lagrangian subspace $A \in \mathbb{L}\mathbb{G}(\Lambda^3 V_6)$ and they proved that $Y_A = \text{Disc}(Z)$, see [67, Proposition 2.1]. A computation shows directly that, in this case, A is the same Lagrangian subspaces $A(Z)$ given in Theorem 4.106: we made this computation in Remark 5.8.

In dimension $n = 3, 4$ the result can be obtained by deformation and degeneration. Consider a $(n+1)$ -dimensional smooth *special GM* variety $Z' = \mathbb{P}(W) \cap C_K G(2, V_5) \cap Q \subset \mathbb{P}(K \oplus \Lambda^2 V_5)$ with $K \cong \mathbb{C}$; the linear section $Z = \mathbb{P}(W \cap \Lambda^2 V_5) \cap G(2, V_5) \cap Q \subset \mathbb{P}(\Lambda^2 V_5)$ is again a *GM* variety, this time n -dimensional and ordinary. Iliev and Manivel proved that the discriminants of Z and Z' are isomorphic under the restriction morphism $H^0(\mathbb{P}(W), \mathcal{I}_{Z'}(2)) \rightarrow H^0(\mathbb{P}(W \cap \Lambda^2 V_5), \mathcal{I}_Z(2))$, see [67, Lemma 4.14] (another proof can be found at [27, Lemma 3.3]). Then they showed that Z' can be seen as a degeneration of a family of $(n+1)$ -dimensional ordinary *GM* varieties; the discriminant $\text{Disc}(Z')$ is then the degeneration of the discriminants of the corresponding ordinary *GM* varieties in the family. This is [67, Lemma 4.15]. So $\text{Disc}(Z) \cong \text{Disc}(Z')$ is the degeneration of a family of *EPW* sextics, associated to ordinary *GM* varieties of higher dimension. This results are summarized in [67, Corollary 4.17].

4.5.3 Automorphisms of Gushel-Mukai varieties

In Theorem 4.106 a relation between *EPW* sextics and ordinary *GM* varieties is established, but we said nothing about the relation between the respective automorphism groups. The goal of this section is to show that, at least in the cases in which we are interested, automorphisms of $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ correspond to automorphisms of $Y_{A(Z)}$ leaving invariant the point $\mathbb{P}(V_5) \in \mathbb{P}(V_6^\vee)$, where as usual V_6 is the space of projective quadric hypersurfaces of $\mathbb{P}(W)$ containing Z . Again, we follow Debarre and Kuznetsov in [27].

Let Z be a normal ordinary *GM* variety. In Corollary 4.104 we checked that the natural map $\text{Aut}(Z, \mathbb{P}(W)) \rightarrow PGL(V_6)$ is injective, now we compute its image. In the following we consider

as usual an isomorphism between two *GM* varieties as the restriction of a linear map between the corresponding linear subspaces cutting them (see Definition 4.88).

Proposition 4.119. *Let $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ and $Z' = \mathbb{P}(W') \cap G(2, V'_5) \cap Q'$ be two normal ordinary *GM* varieties, with associated Lagrangian data respectively (V_6, V_5, A) and (V'_6, V'_5, A') . Then*

- 1) *every isomorphism $\phi : Z \rightarrow Z'$ induces an isomorphism $g : V_6 \rightarrow V'_6$ such that $(\bigwedge^3 g)(A) = A'$ and $g(V_5) = V'_5$;*
- 2) *every isomorphism $g : V_6 \rightarrow V'_6$ such that $(\bigwedge^3 g)(A) = A'$ and $g(V_5) = V'_5$ is induced by an isomorphism between Z and Z' .*

Proof. See [27, Proposition 3.12, Item a) and b)]. For 1), as in the proof of Lemma 4.99, the isomorphism $\phi : Z \rightarrow Z'$ induces an isomorphism between the corresponding *GM* data. This isomorphism induces in turn, by Theorem 4.106, an isomorphism of *EPW* data, in particular it sends V_5 to V'_5 and maps A to A' . The proof of 2) is just the converse of the one of 1). \square

Corollary 4.120. *Let Z be a normal ordinary *GM* variety with associated Lagrangian data (V_6, V_5, A) . Then $\text{Aut}(Z, \mathbb{P}(W)) = PGL_{A, V_5}(V_6)$, where*

$$PGL_{A, V_5}(V_6) = \left\{ g \in PGL(V_6) \mid \left(\bigwedge^3 g \right)(A) = A, g(V_5) = V_5 \right\}.$$

Proof. [27, Proposition 3.12, Item c)]. The automorphism is induced by the map $\text{Aut}(Z, \mathbb{P}(W)) \rightarrow PGL(V_6)$, which is injective by Corollary 4.104 and whose image is computed in Proposition 4.119. \square

The last description allows to relate the automorphisms of Z and Y_A . Recall that also $\text{Aut}(Y_A)$ can be seen as a subgroup of $PGL(V_6)$, see Corollary 4.27.

Proposition 4.121. *Let $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ and $Z' = \mathbb{P}(W') \cap G(2, V'_5) \cap Q'$ be two normal ordinary *GM* varieties, with associated Lagrangian data respectively (V_6, V_5, A) and (V'_6, V'_5, A') .*

- a) *Every isomorphism $\phi : Z \rightarrow Z'$ induces an isomorphism $g : V_6 \rightarrow V'_6$ such that $g(V_5) = V'_5$, that restricts to an isomorphism $g : Y_A \rightarrow Y_{A'}$.*
- b) *If Z is smooth and $\dim(Z) \geq 3$, then every isomorphism $g : V_6 \rightarrow V'_6$ such that $g(Y_A) = Y_{A'}$ and $g(V_5) = V'_5$ is induced by an isomorphism between Z and Z' .*
- c) *If Z is strongly smooth and $\dim(Z) = 2$, then every isomorphism $g : V_6 \rightarrow V'_6$ such that $g(Y_A) = Y_{A'}$ and $g(V_5) = V'_5$ is induced by an isomorphism between Z and Z' .*
- d) *If Z is smooth and $\dim(Z) \geq 3$, or strongly smooth and $\dim(Z) = 2$, the automorphism group of $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ is isomorphic to the group of automorphisms of Y_A fixing the Plücker point $|V_5| \in |V_6|^\vee$.*

Proof. See [27, Proposition 3.19, Item $a) - c)$]. For $a)$ take Proposition 4.119, Item 1), and Theorem 4.26, Item 1). For $c)$ we know that, by Proposition 4.108, Item 3), we have $\dim(A \cap \bigwedge^3 V_5) = 5 - n \leq 3$, so asking the strongly-smoothness of Z is equivalent to ask $A(Z) \notin \Sigma$, see Theorem 4.110. Then the statement follows by Theorem 4.26, Item 3) and Proposition 4.119, Item 2). The proof of $b)$ is identical to that of $c)$, once we point out that smoothness and strongly-smoothness are equivalent when $\dim(Z) \geq 3$, see Proposition 4.93. Finally, $d)$ is clear from 4.27 and Corollary 4.120. \square

Corollary 4.122. *Let Z be a smooth ordinary GM variety of dimension $n \geq 3$. Then the automorphism group $\text{Aut}(Z, \mathbb{P}(W))$ is finite, and trivial if Z is very general. Also the automorphism group of Z as an abstract variety $\text{Aut}(Z)$ is finite, and it is equal to $\text{Aut}(Z, \mathbb{P}(W))$.*

Proof. See [27, Proposition 3.19, Item $c) - d)$]. The group $PGL_{A, V_5}(V_6) \cong \text{Aut}(Z, \mathbb{P}(W))$ is a subgroup of $PGL_A(V_6) \cong \text{Aut}(Y_A)$, which is finite by Proposition 4.28. To prove the statement for Z very general it is sufficient to observe that ordinary GM varieties correspond, through Theorem 4.106, to very general Lagrangian subspaces A . Then we conclude by Proposition 4.29. The rest of the statement has already been stated in Section 4.5.1 and relies on the Lefschetz hyperplane theorem to show that Z is a prime Fano manifold. \square

Remark 4.123. Finiteness of the automorphism group $\text{Aut}(Z, \mathbb{P}(W))$, and triviality for Z very general, hold as well in the case of 2-dimensional smooth GM varieties and is classical, cfr. Proposition 2.120 and Theorem 2.119.

Chapter 5

Sextics whose dual has a point of multiplicity three

5.1 Introduction

A general $\langle 10 \rangle$ -polarized $K3$ surface (S, L) can be seen as a 2-dimensional smooth ordinary GM variety through the map induced by the line bundle L ; as we saw in Section 4.5.2, this means that we can associate a Lagrangian subspace $A(S)$ to S . For simplicity we fix a 6-dimensional complex vector space V_6 such that $A(S) \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ for every general $(S, L) \in \mathcal{K}_{10}$. We called Π the divisor of $\mathbb{L}\mathbb{G}(\wedge^3 V_6)$ parametrizing Lagrangian subspaces that can be obtained from a $K3$ surface as in Theorem 4.106, and $\mathbb{L}\mathbb{G}(\wedge^3 V_6)^0 \subset \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ the open subset parametrizing Lagrangian subspaces whose associated double EPW sextic is a hyperkähler manifold. Since $\Pi \cap \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$ is open in Π , the general $\langle 10 \rangle$ -polarized $K3$ surface (S, L) is associated to a Lagrangian subspace $A(S)$ such that $X_{A(S)}$ is a hyperkähler manifold.

Provided that a $K3$ surface $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ is strongly smooth, by Proposition 4.121 the automorphism group of the EPW sextic $Y_{A(S)}$ contains the group of automorphisms of S induced by $\mathbb{P}(W)$. Our first aim is to use the theory of automorphisms over $K3$ surfaces, which we presented in Section 2.4.3, to provide automorphisms on the EPW sextics $Y_{A(S)}$ given by the $K3$ surface S ; then these automorphisms lift to automorphisms of the double EPW sextic $X_{A(S)}$. The problem is that the general element inside \mathcal{K}_{10} has trivial automorphism group, so we need to provide explicit conditions on S to give rise to a double EPW sextic which is a hyperkähler manifold.

- Strongly smoothness is related to a divisorial condition on \mathcal{K}_{10} by Greer, Li and Tian in [47, Lemma 1.5].
- Once S is strongly smooth, in Section 5.2 we give necessary and sufficient conditions on $\text{NS}(S)$ for $X_{A(S)}$ to be a hyperkähler manifold, both in geometrical and in lattice-theoretic terms, see Corollary 5.6 and Corollary 5.13. These are the first main results of this chapter.

The results in the first section rely on the study of the quadrics containing a the $K3$ surface and on a careful analysis of the exterior algebra involved in the relation between GM varieties and Lagrangian subspaces.

In the second section we present the main results of [98]. As we can see, the condition that we provide in Theorem 5.2, for the smoothness of double EPW sextics, fits naturally in the hypotheses for the results given by O'Grady.

In the third section we study the automorphism groups of EPW sextics, mostly by using the results from Section 4.4 and the fact that the automorphism groups of GM varieties and EPW sextics are finite, at least in our setting. This allows to give bounds on the automorphism group of GM varieties, see Theorem 5.40, Proposition 5.43 and Proposition 5.44. Bounds on the automorphism group are also found for some families of GM varieties described by Debarre, Iliev and Manivel in [26], see Corollary 5.57. On the other hand, by means of the results in the first section, we can induce automorphisms on double EPW sextics from $K3$ surfaces and on other GM varieties.

In the fourth section we study the fiber of the map $\mathcal{K}_{10} \dashrightarrow \mathcal{M}_{2,1}^2$ which maps a general $\langle 10 \rangle$ -polarized $K3$ surface to the associated double EPW sextic with the natural polarization.

All the results in this chapter are original, except for the ones presented in Section 5.3.

5.2 $K3$ surfaces whose associated double EPW sextic is hyperkähler

In this section we want to find precise divisorial conditions on \mathcal{K}_{10} to have a Brill-Noether general $K3$ surface whose associated EPW sextic Y_A admits a double cover which is a hyperkähler manifold; these are presented in Corollary 5.6 and Corollary 5.13. We recall that $(S, L) \in \mathcal{K}_{10}$ is Brill-Noether general if and only if S can be seen, through the morphism $\phi_{|L|} : S \rightarrow \mathbb{P}^6$, as a smooth ordinary Gushel-Mukai variety of dimension two, see Theorem 2.34. More precisely, consider $(S, L) \in \mathcal{K}_{10}$ which is Brill-Noether general. Then (cfr. Theorem 4.95):

- $\phi_{|L|}$ is a closed embedding;
- $W = H^0(S, L)^\vee$ is 7-dimensional and $V_6 = H^0(\mathbb{P}(W), \mathcal{I}_{\phi_{|L|}(S)}(2))$ has dimension 6;
- there exist a hyperplane $V_5 \subset V_6$ and an injective linear map $\mu : W \otimes (V_6/V_5)^\vee \hookrightarrow \bigwedge^2 V_5$ such that the Grassmannian hull of (S, L) is

$$M_S = \mathbb{P}(\mu(W \otimes (V_6/V_5)^\vee)) \cap G(2, V_5);$$

- for any $x \in V_6 - V_5$, let $Q(x)$ be the associated projective quadric hypersurface. Then

$$(\mu \circ \phi_{|L|})(S) = M_S \cap \mu(Q(x))$$

and S can be seen, through the closed embedding $\mu \circ \phi_{|L|}$, as the dimensionally transverse intersection of M_S and $\mu(Q(x))$, see Definition 4.85 (and Definition 1.1 for dimensionally transverse intersection).

Remark 5.1. For the sake of readability, we will drop some notation: we see a Brill-Noether general $(S, L) \in \mathcal{K}_{10}$ as a smooth ordinary Gushel-Mukai 2-dimensional variety

$$S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$$

with Q a non-Pfaffian projective quadric hypersurface, $M_S = \mathbb{P}(W) \cap G(2, V_5)$ its Grassmannian hull and L the line bundle individuated by the restriction of $\mathcal{O}_{\mathbb{P}(W)}(1)$. We also know that every smooth surface obtained in this way is a Brill-Noether general $K3$ surface.

We state here our results, the proof of the first one will be presented at the end of the section. Recall that $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ is said to be strongly smooth if the associated Grassmannian hull $M_S = \mathbb{P}(W) \cap G(2, V_5)$ is a smooth threefold. In this case, given $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ a Lagrangian subspace and an element $[v] \in \mathbb{P}(V_6)$, then $[v] \in Y_A^k$ where $k = \dim(A \cap F_v)$.

Theorem 5.2. *Let $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ be a strongly smooth $K3$ surface. Let $A(S)$ be the associated Lagrangian subspace. Then*

$$|Y_{A(S)}^3 \cap \mathbb{P}(V_5)| = |\{\ell \subset S \mid \ell \text{ line}\}|,$$

in particular if S contains no line then $Y_{A(S)}^3 \cap \mathbb{P}(V_5) = \emptyset$.

We state and proof the second main result of this section. For details about linear spaces inside a projective quadric hypersurface, which are a key component of the proof below, see for example [48, Section 6.1, Linear spaces on quadrics].

Consider a linear space $P \subset \mathbb{P}^n$ and a variety $V \subset \mathbb{P}^n$ with $P \cap V = \emptyset$. We call *cone of L over V* the variety that, as a set, is the union of the lines spanned by a point on L and a point on V . For the structure of cone of a singular quadric in general see [48, Section 6.1, Rank of a quadric]; in the following we have a quadric hypersurface that is the cone of $L \cong \mathbb{P}^2$ over a smooth quadric surface.

Proposition 5.3. *Let $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ be a $K3$ surface, and let $A(S)$ be the associated Lagrangian subspace. If $Y_{A(S)}^3 - \mathbb{P}(V_5) \neq \emptyset$, then S contains a quintic elliptic pencil.*

Proof. Consider $x \in V_6 - V_5$ such that $[x] \in Y_A^3$: by Proposition 4.108, Item 2), the corank of $q(x)$ is 3. Let $Q(x)$ be the projective quadric hypersurface associated to $q(x)$: then S can be seen as the dimensionally transverse intersection of $M_S = \mathbb{P}(W) \cap G(2, V_5)$ and $Q(x)$.

Now we study $Q(x)$: it is a cone of $\mathbb{P}(\ker(q(x))) = \mathbb{P}(A \cap F_x)$ over a smooth quadric surface in \mathbb{P}^3 . We call C this surface. It has two families of lines on it, two lines inside C intersect if and only if they do not lie in the same family. We fix $\{\ell_t\}_{t \in \mathbb{P}^1}$ one of the two families. The cone of $\mathbb{P}(\ker(q(x)))$ over ℓ_t is a \mathbb{P}^4 which we denote by π_t . A codimension count, by transversality of M_S and $Q(x)$, shows that $M_S \cap \pi_t = S \cap \pi_t$ is a curve on S ; again by transversality it has degree 5.

Finally, consider $(\pi_t \cap S)$ and $(\pi_{t'} \cap S)$ for $t \neq t'$. We have $\pi_t \cap \pi_{t'} = \mathbb{P}(\ker(q(x)))$, as the two lines

ℓ_t and $\ell_{t'}$ are in the same family, so $(\pi_t \cap S) \cap (\pi_{t'} \cap S) = \mathbb{P}(\ker(q(x))) \cap S$. But the latter has to be empty, since $\mathbb{P}(\ker(q(x)))$ is the singularity locus of $Q(x)$ and S is smooth and dimensionally transverse intersection. So the pencil $\{S \cap \pi_t\}_{t \in \mathbb{P}^1}$ is elliptic. \square

Remark 5.4. Proposition 5.3 implies that, if $Y_{A(S)}^3 - \mathbb{P}(V_5) \neq \emptyset$, then $\text{Pic}(S)$ contains the lattice spanned by the linear section L and the class of $\pi_t \cap S$, whose Gram matrix with respect to this basis is

$$\begin{bmatrix} 10 & 5 \\ 5 & 0 \end{bmatrix}. \quad (5.1)$$

There are two families of lines on the quadric surface C , the other one provides a second elliptic pencil $\{\tilde{\pi}_t \cap S\}_{t \in \mathbb{P}^1}$. The class of $(\pi_t \cap S) + (\tilde{\pi}_t \cap S)$ is the hyperplane class L , and $(\pi_t \cap S, \tilde{\pi}_t \cap S) = 5$.

Remark 5.5. Theorem 5.2 can be seen as an analogue in dimension two of [29, Theorem 4.7], which describes the lines inside a smooth GM variety of dimension at least 3. The request of strongly smoothness instead of smoothness, in our statement, is coherent with Proposition 4.93: in the case of [29, Theorem 4.7] the two properties are equivalent.

Corollary 5.6. *Let $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ be a strongly smooth K3 surface. Let $A(S)$ be the associated Lagrangian subspace. If S contains neither lines nor quintic elliptic pencils, then $A(S) \in \mathbb{L}\mathbb{G}(\Lambda^3 V_6)^0$. In particular, the double EPW sextic $X_{A(S)}$ is a hyperkähler manifold.*

Proof. Recall that $\mathbb{L}\mathbb{G}(\Lambda^3 V_6)^0 = \mathbb{L}\mathbb{G}(\Lambda^3 V_6) - \Delta - \Sigma$, see Definition 4.46. We know by Theorem 4.110 that $A(S) \notin \Sigma$ when S is strongly smooth, so we are left to verify that $A(S) \notin \Delta$ i.e. $Y_{A(S)}^3 = \emptyset$. But the latter holds by Theorem 5.2 and Proposition 5.3. The rest of the statement follows then by Theorem 4.51. \square

After the proof of Theorem 5.2 we present another formulation of Corollary 5.6 in terms of divisors on the moduli space \mathcal{K}_{10} . In Section 5.5 we study the map from an open subspace of \mathcal{K}_{10} which associates the Lagrangian subspace $A(S)$, hence the hyperkähler manifold $X_{A(S)}$, to S ; its domain is given by Corollary 5.13.

In Theorem 4.106, Debarre and Kuznetsov provided a very explicit way to associate a Lagrangian subspace $A \in \mathbb{L}\mathbb{G}(\Lambda^3 V_6)$ to a K3 surface $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ which is an ordinary Gushel-Mukai variety, where V_6 is the space of quadric hypersurfaces of $\mathbb{P}(W)$ containing S . Already O'Grady showed the link between these varieties in [96] and [98], but the description made in [27] permits a more precise study. In the following remark we show an advantage of this description.

Remark 5.7. Given some Lagrangian data (V_6, V_5, A) , Theorem 4.106 allows to calculate the equation of $Z(V_5, A) \subseteq \mathbb{P}(\Lambda^2 V_5)$. We choose an element $x \in V_6 - V_5$ so that we have an isomorphism $\Lambda^2 V_5 \otimes L^\vee \cong F_x$ as in (4.28), hence $\Lambda^3 V_6 \cong \Lambda^3 V_5 \oplus F_x$. We compose the projection to the second component of $\Lambda^3 V_6$ with the isomorphism $\rho : F_x \rightarrow \Lambda^2 V_5$ of (4.3): the sequence (4.30) for $p = 3$ becomes

$$0 \rightarrow \bigwedge^3 V_5 \rightarrow \bigwedge^3 V_6 \cong \bigwedge^3 V_5 \oplus F_x \xrightarrow{\lambda_3} \bigwedge^2 V_5 \rightarrow 0. \quad (5.2)$$

For $p = 4$ the sequence (4.30) can be rewritten as

$$0 \rightarrow \bigwedge^4 V_5 \rightarrow \bigwedge^4 V_6 \xrightarrow{\lambda_4} \bigwedge^3 V_5 \rightarrow 0$$

as well. We will thus consider W as a subspace of $\bigwedge^2 V_5$, by dropping the injective map μ , and we identify $\bigwedge^5 V_5$ with \mathbb{C} through the volume form on $\bigwedge^5 V_5$ induced by the one on V_6 . The linear space W is then obtained simply by projection, $W = \lambda_3(A)$, and it has dimension $10 - \dim(A \cap \bigwedge^3 V_5)$, see Proposition 4.108, Item 1). The Grassmannian $G(2, V_5)$ is obtained by intersecting the projective quadric hypersurfaces associated to the Pfaffian quadrics $P_v = v \wedge \cdot \wedge \cdot$, see Definition 4.1. An explicit equation of P_v can be easily found with a straight calculation. As for the quadric hypersurface Q , we consider the same $x \in V_6 - V_5$ as before and we call $Q(x)$ the projective quadric hypersurface associated to $q(x) \in \text{Sym}^2 W^\vee$. We have

$$Z(V_5, A) = \mathbb{P}(W) \cap G(2, V_5) \cap Q(x).$$

Consider $w \in W$: since W is the image of A through λ_3 , there exists $\eta \in \bigwedge^3 V_5$ such that $\eta + x \wedge w \in A$. By definition of $q : V_6 \rightarrow \text{Sym}^2 W^\vee$ given in Theorem 4.106, the element $w = \lambda_3(\eta + x \wedge w)$ lies in $Q(x)$ if and only if

$$0 = \lambda_4(x \wedge \eta) \wedge w = \eta \wedge w.$$

Remark 5.8. If we consider instead some ordinary GM intersection $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$, with associated GM data $(W, V_6, V_5, L, \mu, q, \epsilon)$, it is possible to reconstruct the associated Lagrangian subspace $A(Z)$. We consider again $W \subseteq \bigwedge^2 V_5 \otimes L^\vee$, and we fix $x \in V_6 - V_5$ such that $Q = Q(x)$. As above, we have a decomposition $\bigwedge^3 V_6 \cong \bigwedge^3 V_5 \oplus F_x$ and (5.2) holds. We see $\bigwedge^3 V_5$ as the dual of $\bigwedge^2 V_5$ through the linear map $\epsilon : \bigwedge^5 V_5 \rightarrow L^{\otimes 2}$ inside the GM data of Z : more precisely, ϵ induces $(\bigwedge^3 V_5 \otimes L^\vee) \cong (\bigwedge^2 V_5 \otimes L^\vee)^\vee$. Recall that in (4.31) the linear map μ has W as domain and $\bigwedge^2 V_5 \otimes L^\vee$ as codomain. We have then the following description of $A(Z)$:

$$\left\{ (\eta, w) \in \bigwedge^3 V_5 \oplus W \mid w \in W, \epsilon(\eta \wedge \mu(\cdot))|_W = -q(x)(w, \cdot) \otimes [x] \text{ as elements of } W^\vee \right\}. \quad (5.3)$$

Now we drop some notation, to make calculations simpler, in particular we forget ϵ and, as in Remark 5.7, we consider again $W \subseteq \bigwedge^2 V_5$ and we see $\bigwedge^3 V_5$ as the dual of $\bigwedge^2 V_5$. Moreover, the inclusion $W \hookrightarrow \bigwedge^2 V_5$ induces a short exact sequence of vector spaces

$$0 \rightarrow W^\perp \rightarrow \left(\bigwedge^2 V_5 \right)^\vee \rightarrow W^\vee \rightarrow 0,$$

so that we can consider $(\bigwedge^2 V_5)^\vee \cong W^\perp \oplus W^\vee$. Putting all together we can rewrite $A(Z) \subset W^\perp \oplus W^\vee \oplus F_x$ as

$$\left\{ (\xi, -q(x)(w, \cdot), x \wedge w) \in W^\perp \oplus W^\vee \oplus F_x \mid \xi \in W^\perp, w \in W \right\}. \quad (5.4)$$

From this description it is clear that, under the identification $\bigwedge^3 V_5 \cong (\bigwedge^2 V_5)^\vee$, we have an inclusion $W^\perp \subset A(Z)$. Since $\dim(A(Z) \cap \bigwedge^3 V_5) = 10 - \dim(W)$ by Proposition 4.108, the two spaces are

the same i.e. $A(Z) \cap \bigwedge^3 V_5 = W^\perp$.

Consider the linear map $x \wedge W \rightarrow W^\vee$ that sends $x \wedge w$ to $-q(x)(w, -)$: its graphic $\Gamma_x \subset W^\vee \oplus W$ allows to give a decomposition $A(Z) = W^\perp \oplus \Gamma_x \subset W^\perp \oplus W^\vee \oplus F_x$.

In particular, if Z has dimension 5, we can see by (5.3) the Lagrangian subspace $A(Z)$ as the graph Γ_x of a linear map from $\bigwedge^2 V_5$ to $(\bigwedge^2 V_5)^\vee$, which is Iliev and Manivel's description in [67, Proposition 2.1]. However, in this section we are interested in the 2-dimensional case.

Let $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ be a strongly smooth $K3$ surface and M_S its Grassmannian hull. Recall that Grassmannian hulls of strongly smooth Brill-Noether general $K3$ surfaces are isomorphic, see Proposition 4.77, Item 3).

We need a characterization of the lines inside M_S . For $v \in V_5 - \{0\}$, $V_3 \ni v$ a 3-dimensional vector space inside V_5 , we define the 2-dimensional vector space $L_{v, V_3} = \{v \wedge t \mid t \in V_3\}$. Every line inside $G(2, V_5)$ is in the form $\mathbb{P}(L_{v, V_3})$, see for example [29, Section 4.1] or [24, Section 2, The Variety of Lines on $\mathbb{G}(1, N) \cap H^l$]. Hence the same description holds for lines inside M_S . For every $v \in V_5 - \{0\}$, we denote by P_v the Pfaffian quadric associated to v .

Lemma 5.9. *Consider $v \in V_5 - \{0\}$. There exists a three dimensional vector space $V_3 \ni v$ such that the line $\mathbb{P}(L_{v, V_3}) \subset G(2, V_5)$ is contained in M_S if and only if $\ker(P_v) \cap W$ has dimension exactly 2. In this case $L_{v, V_3} = \ker(P_v) \cap W$.*

Proof. Given v , we fix a decomposition $V_5 = \mathbb{C}v \oplus U$; we know that $\ker(P_v) = v \wedge U$, see Remark 4.74, hence any subspace in the form $v \wedge V_3$ is contained in $\ker(P_v)$. If $\ker(P_v) \cap W$ has dimension two then its projectivization is a line on M_S in the form $\mathbb{P}(L_{v, V_3})$. On the other side, if $\mathbb{P}(L_{v, V_3}) \subset M_S$ then $L_{v, V_3} \subset W \cap \ker(P_v)$, so this last subspace has dimension at least 2. Finally the dimension cannot be bigger than 2, or M_S would contain a plane, which is absurd by Proposition 4.77, Item 1). \square

We will make use of the following classical characterization of linear subspaces inside a projective quadric.

Remark 5.10. Given a symmetric bilinear form $q \in \text{Sym}^2 W^\vee$ with associated projective quadric Q , a linear subspace $\mathbb{P}(V) \subset \mathbb{P}(W)$ is contained in Q if and only if $q|_{V \times V} = 0$.

We are ready to prove the first main result of this section.

Proof of Theorem 5.2. To simplify notation we call A the Lagrangian subspace $A(S)$ associated to S . Fix $x \in V_6 - V_5$, so that $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q(x)$, where $Q(x)$ is the projective quadric hypersurface associated to the bilinear form $q(x)$; as in Remark 5.7 we have $\bigwedge^3 V_6 \cong (\bigwedge^2 V_5)^\vee \oplus F_x$ and a linear map $\lambda_3 : (\bigwedge^2 V_5)^\vee \oplus F_x \rightarrow \bigwedge^2 V_5$ which sends $(\phi, x \wedge w)$ to w .

For any $v \in V_5 - \{0\}$ we consider a decomposition $V_5 = \mathbb{C}v \oplus U$. Since $F_v = (v \wedge \bigwedge^2 U) \oplus v \wedge U$, we can write

$$\lambda_3(A) = W \quad \lambda_3(F_v) = v \wedge U \quad \ker((P_v)|_W) = (v \wedge U) \cap W.$$

Moreover, by Remark 5.8 we have $A \cap (\bigwedge^2 V_5)^\vee = W^\perp$; looking at the decomposition of F_v , we can also write $F_v \cap (\bigwedge^2 V_5)^\vee = v \wedge \bigwedge^2 U = \ker(P_v)^\perp$. We are interested in the dimension of $A \cap F_v$: clearly (5.2) induces an equality

$$\dim(A \cap F_v) = \dim(\lambda_3(A \cap F_v)) + \dim(A \cap F_v \cap (\bigwedge^2 V_5)^\vee). \quad (5.5)$$

We have $\lambda_3(A \cap F_v) \subseteq \ker(P_v) \cap W$; as in the proof of Lemma 5.9, the dimension of $\ker(P_v) \cap W$ is at most 2. In particular:

- 1) If $\dim(\ker(P_v) \cap W) = 2$, then $\ker(P_v) + W$ has dimension 9, hence there exists only one hyperplane containing both of them, so $\dim(W^\perp \cap \ker(P_v)^\perp) = 1$;
- 2) If $\dim(\ker(P_v) \cap W) = 1$, then $\ker(P_v) + W$ has dimension 10, hence there is no hyperplane containing both of them, so $W^\perp \cap \ker(P_v)^\perp = 0$.

Now we look again at Equation (5.5): since $A \cap F_v \cap (\bigwedge^2 V_5)^\vee = W^\perp \cap \ker(P_v)^\perp$, to have $\dim(A \cap F_v) = 3$ we must consider the vectors $v \in V_5$ for which Item 1) above happens.

We consider v such that $\dim(A \cap F_v) = 3$ i.e. $[v] \in Y_A^3$; in particular this means that Item 1) above holds, and $\lambda_3(A \cap F_v) = \lambda_3(A) \cap \lambda_3(F_v)$, so that the RHS of Equation (5.5) can be three. The projective space $\ell = \mathbb{P}(\lambda_3(A \cap F_v))$ is a line inside M_S ; to prove that it is contained in S we prove $\ell \subset Q(x)$. We denoted by $q(x)$ the bilinear symmetric map associated to $Q(x)$; by Remark 5.7, we know that $q(x)(w, w) = 0$ if and only if there there exists $\eta \in \bigwedge^3 V_5$ such that $\eta + x \wedge w \in A$ and $\eta \wedge w = 0$.

Consider any $v \wedge u$ inside $\lambda_3(A \cap F_v)$: by hypothesis, there exists $\eta \in \bigwedge^3 V_5$ such that $\eta + x \wedge v \wedge u \in A \cap F_v$. This means that $\eta \in F_v$, so $\eta \wedge v \wedge u = 0$. In particular $v \wedge u \in Q(x)$, hence the whole line $\mathbb{P}(\lambda_3(A \cap F_v))$ is contained in S .

On the other hand, if $\mathbb{P}(L_{v, V_3}) \subset S$ for some $v \in V_5$, then by Lemma 5.9 we have $L_{v, V_3} = \ker(P_v) \cap W \cong \mathbb{C}^2$, since the line is contained in M_S too. So we are in the first case above i.e. there exists some non-zero $\eta \in W^\perp \cap \ker(P_v)^\perp$. Then to prove $[v] \in Y_A^3 \cap \mathbb{P}(V_5)$ we need to prove that $\lambda_3(A \cap F_v) = \lambda_3(A) \cap \lambda_3(F_v) = W \cap \ker(P_v)$.

Consider $v \wedge u \in W \cap \ker(P_v)$. We make use of (5.4): under the decomposition $(\bigwedge^2 V_5)^\vee \cong W^\perp \oplus W^\vee$, the element

$$\alpha = (0, -q(x)(v \wedge u, -), x \wedge v \wedge u) \in W^\perp \oplus W^\vee \oplus F_x$$

lies in A and $\lambda_3(\alpha) = v \wedge u$. To prove that $\alpha \in F_v$, so that we have $v \wedge u \in \lambda_3(A \cap F_v)$, we show that $\alpha \wedge \eta = 0$ for every $\eta \in F_v$, then we can conclude, since F_v is Lagrangian. As α is zero on the first component, we have $\alpha \wedge \eta = 0$ for every η outside W , so we are left to prove $0 = \alpha \wedge \eta = -q(x)(v \wedge u, \eta)$ for $\eta \in F_v \cap W = L_{v, V_3}$. But, since $\mathbb{P}(L_{v, V_3}) \subset S$ by hypothesis, the line $\mathbb{P}(L_{v, V_3})$ is contained in $Q(x)$: as we recalled in Remark 5.10, this happens if and only if $q(x)|_{L_{v, V_3} \times L_{v, V_3}} = 0$. In particular $q(x)(v \wedge u, -) = 0$ on L_{v, V_3} .

Finally, to prove the one-to-one correspondence between points of $Y_A^3 \cap \mathbb{P}(V_5)$ and lines inside S , it is sufficient to observe that two lines $\mathbb{P}(L_{v, V_3})$ and $\mathbb{P}(L_{v', V_3'})$ intersect in at most one point if $[v] \neq [v']$. \square

Remark 5.11. Call \tilde{P}_v the restriction of the Pfaffian quadric P_v to $\mathbb{P}(W)$. By Lemma 4.75 we also have $\ker(P_v) \cap W = \ker(\tilde{P}_v)$. As in the proof above, by a dimension count for any $v \in V_5$ we have $\dim(\ker(\tilde{P}_v)) \geq 1$, even if $\dim(F_v \cap A) = 0$ for $[v]$ general. We already noticed it in the description of Y_A as the *discriminant locus* of S , see Definition 4.114.

We return to $\langle 10 \rangle$ -polarized $K3$ surfaces inside \mathcal{K}_{10} . We need the following result, proved by Greer, Li and Tian in [47], which we already used in Section 3.4.

Proposition 5.12. *Let $(S, L) \in \mathcal{K}_{10}$ be a Brill-Noether general $K3$ surface. Then (S, L) is strongly smooth if and only if $(S, L) \notin D_{4,0}$.*

Proof. See [47, Lemma 1.5]. □

For convenience, in the following we sum up various results: the first is due to Mukai, the second to Greer, Li and Tian. The last one summarizes Remark 5.4 and Corollary 5.6.

Corollary 5.13. *Let $(S, L) \in \mathcal{K}_{10}$ be a $\langle 10 \rangle$ -polarized $K3$ surface. If (S, L) is Brill-Noether general, let $A(S) \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ be the Lagrangian subspace associated to (S, L) . Then*

- 1) $(S, L) \notin D_{h,0}$ for $h \in \{1, 2, 3\}$ if and only if (S, L) is Brill-Noether general;
- 2) if (S, L) is Brill-Noether general, then $(S, L) \notin D_{4,0}$ if and only if (S, L) is strongly smooth, if and only if $A(S) \notin \Sigma$.

Moreover, if (S, L) is Brill-Noether general and strongly smooth, then $A(S) \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$ (i.e. $Y_A^3 = \emptyset$) if $(S, L) \notin D_{x,y}$, with $(x, y) \in \{(1, -2), (5, 0)\}$. In particular

- 3) if (S, L) is Brill-Noether general and strongly smooth, then $Y_{A(S)}^3 \cap \mathbb{P}(V_5) = \emptyset$ if and only if $(S, L) \notin D_{1,-2}$;
- 4) if (S, L) is Brill-Noether general and strongly smooth, then $Y_{A(S)}^3 - \mathbb{P}(V_5) = \emptyset$ if $(S, L) \notin D_{5,0}$.

Proof. For 1) see Proposition 2.83 and Theorem 2.34. For 2) see Proposition 5.12. The rest of the statement comes directly from Theorem 5.2, Remark 5.4 and Corollary 5.6. Lines on $S = \mathbb{P}(W) \cap G2, V_5) \cap Q$ correspond to sublattices of $\text{NS}(S)$ whose Gram matrix is

$$\begin{bmatrix} 10 & 1 \\ 1 & -2 \end{bmatrix}. \tag{5.6}$$

To prove 3) it is then sufficient to show that sublattices of this type admit no non-trivial overlattice. We call Λ the lattice associated to (5.6): some computer algebra calculation, see Remark 1.31, shows that the quadratic form on A_Λ is $q_\Lambda = \frac{\mathbb{Z}}{3\mathbb{Z}}(2/3) \oplus \frac{\mathbb{Z}}{7\mathbb{Z}}(6/7)$, and by a direct calculation the only isotropic group in A_Λ is $\{0\}$, hence the claim, cfr. Proposition 1.30. For 4) see Remark 5.4: as above, we compute the overlattices of (5.1). The only non-trivial overlattice is U , but this case is ruled out in this setting, since (S, L) is Brill-Noether general, cfr. Proposition 2.83. □

At this point, if $(S, L) \notin \mathcal{D}_{5,0}$ then finding whether the Lagrangian subspace associated to a $\langle 10 \rangle$ -polarized $K3$ surface (S, L) gives rise to a hyperkähler manifold becomes a lattice-theoretic problem on $\text{NS}(S)$.

Remark 5.14. According to Proposition 5.3, when a strongly smooth $K3$ surface $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ admits a quintic elliptic pencil, there is still room for the associated Lagrangian subspace $A(S)$ to lie in $\mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$. As the condition to admit such a pencil is divisorial on \mathcal{K}_{10} ($(S, L) \in D_{5,0}$), it is independent from the other conditions we impose. So the question about polarized $K3$ surfaces inside $D_{5,0}$ is: is the quintic elliptic pencil cut, on S , by a family of linear spaces on a projective quadric $Q(x)$? If the answer is *yes*, then $[x] \in Y_A^3$ hence $A(S) \notin \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$. On the other side, since $\mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$ is open inside $\mathbb{L}\mathbb{G}(\wedge^3 V_6)$, it is sufficient to find one example where the answer is *no* to deduce that $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$ for (S, L) general inside $D_{5,0}$.

5.3 Desingularization

5.3.1 Conics on Brill-Noether general $K3$ surfaces

In this and the next section we summarize some results coming from O’Grady’s work in [98], integrating it with Debarre and Kuznetsov’s approach from [27] and with Theorem 5.2. We have three goals: firstly, to provide some tools that will be useful later. Secondly, to show how our results fit in a natural way with [98], where a necessary condition often requested appears in Theorem 5.2. The condition is that the projective model of the $K3$ surface does not contain lines. Thirdly, the result from [98] which we present in Corollary 5.30 is the key point in the version of [98] of the proof that smooth double EPW sextics are hyperkähler manifolds of $K3^{[n]}$ -type. This is the proof which we presented in Section 4.3.2.

Here we focus on Lagrangian subspaces A inside $\mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$ whose dual has $Y_{A^\perp}^3 \neq \emptyset$. Those Lagrangian subspaces are easier to study because of their explicit link with Hilbert squares of $K3$ surfaces, and they appear both in [96] and [95]. Also in [66] and [75] they were used as a starting point to study EPW sextics and their coverings.

Remark 5.15. Never in this section we mention the fact that the double covering X_A , for $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$, is a hyperkähler manifold. Actually, it is the other way round: these results are used in Theorem 4.37 to prove that X_A is a hyperkähler manifold.

See Definition 4.53 for the definition of the divisors Δ, Σ, Π inside $\mathbb{L}\mathbb{G}(\wedge^3 V_6)$. Consider a Lagrangian subspace $A \in \Pi - \Sigma - \Delta$; the latter is open inside the divisor Π . We fix moreover $[V_5] \in Y_{A^\perp}^3$, so that we have a Lagrangian data set (V_6, V_5, A) . Let $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ be the associated GM intersection: by Corollary 4.111 then S is a strongly smooth $K3$ surface. We call M_S its Grassmannian hull. We fixed a projective quadric hypersurface Q such that S is the dimensionally transverse intersection of M_S and Q . As usual, we have $H^0(\mathbb{P}(W), \mathcal{I}_S(2)) = V_6$,

which in turn can be seen as $V_6 = V_5 \oplus \mathbb{C}\gamma$ where γ is a bilinear form associated to Q , see (4.19). The Plücker point $|V_5| \in |V_6|^\vee$ will play an important role in the following constructions.

Remark 5.16. For compatibility, A is actually the dual of the Lagrangian subspace that O’Grady called A in [98].

Consider the Hilbert square $S^{[2]}$. For any zero-dimensional subscheme $Z \subset S$ of length 2, we denote by $\langle Z \rangle$ the line spanned by it. By strong smoothness, the Fano variety $F(M_S)$ of lines inside M_S is isomorphic to \mathbb{P}^2 , see Proposition 4.77, Item 2). It can be seen as a subvariety of $S^{[2]}$ through the closed embedding, defined by O’Grady,

$$F(M_S) \rightarrow S^{[2]} \tag{5.7}$$

$$\ell \mapsto \ell \cap Q. \tag{5.8}$$

Clearly the map would not be defined on lines lying in S , but Theorem 5.2 guarantees that there is no line on S , as $A \notin \Delta$. Following O’Grady, we denote by $P_S \subset S^{[2]}$ the image of the map above.

O’Grady defined another map, cfr. [98, (4.2.5)], to the locus of hyperplanes of projective quadrics cutting S i.e. $|V_6|^\vee$, that is

$$g : S^{[2]} \rightarrow |V_6|^\vee \tag{5.9}$$

$$Z \mapsto \{\sigma \in |V_6| \text{ such that } \langle Z \rangle \subset \sigma\}. \tag{5.10}$$

A priori this map is defined outside the subvarieties $\ell^{(2)} \subset S^{[2]}$ for $\ell \subset S$ line (the Hilbert square of a curve coincide with its 2-symmetric power). However, by Theorem 5.2 the map is everywhere defined. Consider the open subset

$$S_\star^{[2]} = S^{[2]} - \left(P_S \cup \bigcup_{C \subset S \text{ conic}} C^{(2)} \right). \tag{5.11}$$

In the definition of $S_\star^{[2]}$, when C is not smooth, we delete all $Z \in S^{[2]}$ such that Z is contained in C .

Proposition 5.17. *The fiber of g on the Plücker point $|V_5| \in |V_6|^\vee$ is P_S . The fiber on a point inside $S_\star^{[2]}$ has cardinality at most 2, and the generic fiber has cardinality exactly 2.*

Proof. The fiber on the Plücker point can be computed directly: every projective quadric cutting the Grassmannian hull contains the lines inside the Grassmannian hull. For the second part see [98, Proposition 4.20, Item 1)]. \square

Call L the line bundle on S defined by the restriction of $\mathcal{O}_{\mathbb{P}(W)}(1)$; O’Grady proved in [94, Claim 5.16] that the pullback of the line bundle $\mathcal{O}_{|V_6|}(1)$ is the square-2 line bundle

$$g^* \mathcal{O}_{|V_6|}(1) = L_2 - 2\delta. \tag{5.12}$$

In particular, the map g is the map induced by the complete linear system $|L_2 - 2\delta|$. The following result was proved by O’Grady in [98, Proposition 4.21]; here we propose a slightly different version of the proof. We make use of the characterization of the dual of the discriminant given by Iliev and Manivel, that we reported in Proposition 4.117.

Proposition 5.18. *The image of $g : S^{[2]} \rightarrow |V_6|^\vee$ is the sextic hypersurface $\text{Disc}(S)^\vee \cong Y_{A^\perp}$.*

Proof. Consider a zero-dimensional subscheme $Z \subset S$ of length 2 in the form $p + q$ with p, q in general position, $p + q \in S_\star^{[2]}$ defined in (5.11). We have then $l_p \cap l_q = \emptyset$, so that $\langle Z \rangle = V_4$ is a 4-dimensional vector space (see Example 2.165 for the notation) and $W \cap \bigwedge^2 V_4 \cong \mathbb{C}^3$. Take σ inside the hyperplane $g(Z)$: the intersection $\sigma \cap \mathbb{P}(W \cap \bigwedge^2 V_4)$ is a conic and it contains the line $\langle Z \rangle$ by definition of g , hence it is a singular conic. So $g(Z) \in \text{Disc}(S)^\vee$ by Proposition 4.117.

The image of $S^{[2]}$ has dimension four since g has finite generic fibers by Proposition 5.17, so the result follows as $\text{Disc}(S)^\vee$ is irreducible by Corollary 4.18. \square

In Example 2.165 we described *O'Grady's involution*, a birational non-natural involution on $S^{[2]}$. Call ι such involution and $\mathcal{A} \subset S_\star^{[2]}$ its definition locus.

Proposition 5.19. *On the open subset \mathcal{A} we have*

$$g|_{\mathcal{A}} \circ \iota = g|_{\mathcal{A}}.$$

The induced map $\mathcal{A}/\langle \iota \rangle \rightarrow g(\mathcal{A})$ is an isomorphism.

Proof. See [98, Proposition 4.20, Item 1) and 2)] for the first part, and for the fact that $\mathcal{A}/\langle \iota \rangle \rightarrow g(\mathcal{A})$ is bijective. Finally, $g(\mathcal{A})$ is normal by Corollary 4.18, since it is an open subset of $\text{Disc}(Z)^\vee$, so the birational map is in fact an isomorphism by Zariski's main theorem. \square

As a corollary, the invariant lattice of ι can be computed. We used this result in Section 3.4.

Corollary 5.20 (O'Grady). *The invariant lattice of ι is spanned by $g^* \mathcal{O}_{\text{Disc}(S)^\vee}(1) = L_2 - 2\delta$.*

Proof. By Proposition 2.10, the open subset \mathcal{A} has codimension 2, hence $\text{Pic}(S^{[2]}) \cong \text{Pic}(\mathcal{A})$; the isomorphism is ι -equivariant. We know that $g(\mathcal{A})$ is normal, also by Proposition 5.17 the morphism $\mathcal{A} \rightarrow g(\mathcal{A})$ is a double cover, thus by Remark 1.35 the invariant of ι on \mathcal{A} is $g^* \mathcal{O}_{g(\mathcal{A})}(1) = (L_2 - 2\delta)|_{\mathcal{A}}$. \square

We need the following result.

Proposition 5.21. *Given a zero-dimensional subscheme $Z \subset M_S$ of length 2, there exists a unique conic on M_S containing Z .*

Proof. See [98, Corollary 5.5]. \square

The first main result that we pick from [98] is the following, which brings together [98, Claim 4.19] and some steps of the proof of [98, Theorem 4.15]; the proof is considerably simplified by the machinery provided in Section 4.5.2. Recall that, since $A \in \Pi$, the double *EPW* sextic X_{A^\perp} is singular, although normal, and $\text{Sing}(X_{A^\perp}) = f_{A^\perp}^{-1}(Y_{A^\perp}^3)$; the Plücker point $|V_5|$ corresponds to $[V_5] \in Y_{A^\perp}^3$ through the isomorphism $\text{Disc}(Z)^\vee \cong Y_{A^\perp}$.

To simplify notation, from now on we consider g as a map whose image is Y_{A^\perp} instead of $\text{Disc}(Z)^\vee$. We denote, as usual, by $f_{A^\perp} : X_{A^\perp} \rightarrow Y_{A^\perp}$ the double cover.

Theorem 5.22. (*O'Grady*) Let C_1, \dots, C_k be the smooth conics contained in S . Then $P_S, C_1^{(2)}, \dots, C_N^{(2)}$ are pairwise disjoint. The morphism $g : S^{[2]} \rightarrow Y_{A^\perp}$ factorizes as

$$S^{[2]} \xrightarrow{c} X_{A^\perp} \xrightarrow{f_{A^\perp}} Y_{A^\perp}, \quad (5.13)$$

where c is an isomorphism on the complement of $P_S \cup C_1^{(2)} \cup \dots \cup C_N^{(2)}$ onto the smooth locus of X_{A^\perp} . Moreover, the morphism c contracts each of $P_S, C_1^{(2)}, \dots, C_k^{(2)}$ to a point of $\text{Sing}(X_{A^\perp})$, and $\text{Sing}(X_{A^\perp}) = \{c(P_S), c(C_1^{(2)}), \dots, c(C_N^{(2)})\}$.

Proof. The proof consists of two parts. We start by proving that there exists a contraction $\tilde{c} : S^{[2]} \rightarrow N$ to a normal variety N that contracts each of $P_S, C_1^{(2)}, \dots, C_k^{(2)}$ to a point; this proves also that $\text{Sing}(N) = \{c(P_S), c(C_1^{(2)}), \dots, c(C_N^{(2)})\}$ and that \tilde{c} is an isomorphism on the complement of $P_S \cup C_1^{(2)} \cup \dots \cup C_N^{(2)}$ onto the smooth locus of N . This first part of the proof comes from [98, Claim 4.19].

We already know by Proposition 5.21 that $C_i^{(2)} \cap C_j^{(2)} = \emptyset$ if $i \neq j$. By definition of P_S , see (5.7), if there exists $Z \in P_S \cap C_i^{(2)}$ for some i , then $\langle Z \rangle \subset M_S$. Moreover $C_i \subset S$ hence $C_i \subset M_S$, and, since C_i is smooth by hypothesis, we have $\langle Z \rangle \not\subset C_i$. Consider any quadric σ cutting M_S : since $\sigma \cap \langle C_i \rangle$ contains both C_i and $\langle Z \rangle$, the whole plane $\langle C_i \rangle$ lies inside σ : since M_S is cut out by quadrics, cfr. Lemma 4.72, this is impossible since M_S contains no plane by Proposition 4.77, Item 1). So each of $P_S, C_1^{(2)}, \dots, C_k^{(2)}$ can be contracted to a point and N exists.

We prove that g factorizes through $\tilde{c} : S^{[2]} \rightarrow N$. For this, it is sufficient to prove that g is constant on P_S and on $C_i^{(2)}$ for every i . For P_S see Proposition 5.17; for any $Z \in C_i^{(2)}$ we have $g(Z) = \{\sigma \in |V_6| \text{ such that } \langle C_i \rangle \subset \sigma\}$. This is again [98, Claim 4.19]. We call $\bar{g} : N \rightarrow Y_{A^\perp}$ the induced map. Moreover, the O'Grady's involution on $S^{[2]}$, which is birational, descends to a everywhere defined, biregular involution of N , which we denote by $\bar{\iota}$: this is [98, Proposition 4.20, Item 3)], since by Theorem 5.2 there is no line on S , as $A \notin \Delta$. Clearly $\bar{g} \circ \bar{\iota} = \bar{g}$ by Proposition 5.19; exactly as in Proposition 5.19, the induced map $N/\langle \bar{\iota} \rangle \rightarrow Y_{A^\perp}$ is an isomorphism.

Now we prove $N \cong X_{A^\perp}$: this is part of the proof of [98, Theorem 4.15]. Call N^0 the complement of the fixed locus of $\bar{\iota}$: as we said before, Y_{A^\perp} can be seen as the quotient of N by a non-trivial involution, so the morphism $\bar{g}|_{N^0}$ is an étale double cover over the smooth locus $Y_{A^\perp} = Y_A^1$. We know by Corollary 4.39 that $\pi_1(Y_A^1) = \mathbb{Z}/2\mathbb{Z}$, so $\bar{g}|_{N^0} : N^0 \rightarrow Y_A^1$ is the universal cover of Y_A^1 . But the same holds for the restriction to $f_{A^\perp}^{-1}(Y_{A^\perp}^1)$ of f_{A^\perp} , see again Corollary 4.39. So $f_{A^\perp}^{-1}(Y_{A^\perp}^1)$ and N^0 are isomorphic as covers of $Y_{A^\perp}^1$; furthermore, both X_A and N are normal completions of the universal cover such that the extended map is finite, hence $\bar{g} : N \rightarrow Y_{A^\perp}$ and $X_{A^\perp} \rightarrow Y_{A^\perp}$ are isomorphic as covers of Y_{A^\perp} . In particular the isomorphism sends the singular locus of X_A to the singular locus of N , so, through the isomorphism, we have $\text{Sing}(X_A) = \{c(P_S), c(C_1^{(2)}), \dots, c(C_N^{(2)})\}$. \square

Remark 5.23. The image of P_S through the map g is the Plücker point $[V_5] \in Y_{A^\perp}^3$. For every choice of $[V'_5] \in Y_{A^\perp}^3$ the construction is identical: for $[V'_5] \in Y_{A^\perp}^3$ we call $S' = \mathbb{P}(W') \cap G(2, V'_5) \cap Q'$ the K3 surface associated to (V_6, V'_5, A) . We have again a map $g' : (S')^{[2]} \rightarrow Y_{A^\perp}$ which factorizes

as $(S')^{[2]} \xrightarrow{c'} X_{A^\perp} \xrightarrow{f_{A^\perp}} Y_{A^\perp}$; the difference is that g' sends $P_{S'}$ to $[V_5']$, which is then the Plücker point of S' inside $|H^0(\mathbb{P}(W'), \mathcal{I}_{S'}(2))|^\vee \cong |V_6|^\vee$.

Remark 5.24. Although the results in this section come mainly from [98], we heavily used Theorem 5.2 to ensure that, for $A \notin \Delta$, the hypothesis requested by O'Grady were satisfied. See the definition of P_S at Equation (5.7), the definition locus of g at Equation (5.9), the proof of Theorem 5.22.

5.3.2 Families of desingularizations

This section follows pretty closely [98]: the goal is to present [98, Theorem 4.15], which states that, for $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ without lines, the Hilbert square $S^{[2]}$ is a desingularization of X_{A^\perp} . This is Theorem 5.28.

Consider $A^\perp \in \mathbb{L}\mathbb{G}(\wedge^3 V_6) - \Sigma$ a Lagrangian subspace which does not contain any non-zero decomposable vector (we take A^\perp instead of A for consistency with the last section). The subset $\mathbb{L}\mathbb{G}(\wedge^3 V_6) - \Sigma$ is open in $\mathbb{L}\mathbb{G}(\wedge^3 V_6)$; for a small open neighborhood \mathcal{U} of A^\perp we have the natural family of deformation $\mathcal{Y}_{\mathcal{U}} \rightarrow \mathcal{U}$ whose fiber over a Lagrangian subspace $B \in \mathcal{U}$ is the *EPW* sextic Y_B . Note that $Y_B \neq \mathbb{P}(V_6)$ by Proposition 4.11. O'Grady proved in [98, Proposition 3.1] that, for any $A^\perp \in \mathbb{L}\mathbb{G}(\wedge^3 V_6) - \Sigma$, there exist

- a neighborhood \mathcal{U} of A^\perp inside $\mathbb{L}\mathbb{G}(\wedge^3 V_6) - \Sigma$ with the natural family $\mathcal{Y}_{\mathcal{U}} \rightarrow \mathcal{U}$;
- a family $\rho_{\mathcal{U}} : \mathcal{X}_{\mathcal{U}} \rightarrow \mathcal{U}$ whose fiber over a Lagrangian subspace $B \in \mathcal{U}$ is the double *EPW* sextic X_B ;
- a morphism of families with base \mathcal{U}

$$\begin{array}{ccc} \mathcal{X}_{\mathcal{U}} & \xrightarrow{f_{\mathcal{U}}} & \mathcal{Y}_{\mathcal{U}} \\ & \searrow \rho_{\mathcal{U}} & \swarrow \\ & \mathcal{U} & \end{array}$$

such that, for every Lagrangian subspace $B \in \mathcal{U}$, the morphism between the fibers in the covering map $f_B : X_B \rightarrow Y_B$.

Such a family is called *tautological family parametrized by \mathcal{U}* . We fix a tautological family parametrized by \mathcal{U} ; it induces two types of schemes composed by subvarieties of $\mathcal{Y}_{\mathcal{U}}$, again parametrized by \mathcal{U} : $\mathcal{Y}_{\mathcal{U}}^{\geq d} \rightarrow \mathcal{U}$ (resp. $\mathcal{Y}_{\mathcal{U}}^d \rightarrow \mathcal{U}$) whose fiber over $B \in \mathcal{U}$ is the subvariety $Y_B^{\geq d} \subseteq Y_B$ (resp. $Y_B^d \subseteq Y_B$). Following O'Grady, we also define

$$\mathcal{W}_{\mathcal{U}} = f_{\mathcal{U}}^{-1}(\mathcal{Y}_{\mathcal{U}}^3).$$

By looking at the ramification locus of f_B , for $B \in \mathcal{U}$, we have that the morphism $f_{\mathcal{U}} : \mathcal{X}_{\mathcal{U}} \rightarrow \mathcal{Y}_{\mathcal{U}}$ induces an isomorphism $\mathcal{W}_{\mathcal{U}} \cong \mathcal{Y}_{\mathcal{U}}^3$. Note that, since $\mathcal{U} \cap \Sigma = \emptyset$, by Lemma 4.10 we have $\mathcal{Y}_{\mathcal{U}}^3 = \mathcal{Y}_{\mathcal{U}}^{\geq 3}$. As we can expect from Theorem 4.36, the singularity locus of $\mathcal{X}_{\mathcal{U}}$ is $\mathcal{W}_{\mathcal{U}}$, see [98, Proposition 3.2].

Now we take the blow-up of $\mathcal{X}_{\mathcal{U}}$ along $\mathcal{W}_{\mathcal{U}}$: we call it $\pi_{\mathcal{U}} : \bar{\mathcal{X}}_{\mathcal{U}} \rightarrow \mathcal{X}_{\mathcal{U}}$, and we denote by $E_{\mathcal{U}}$ the exceptional locus of it. The blow-up $\bar{\mathcal{X}}_{\mathcal{U}}$ turns out to be smooth, see [98, Claim 3.8].

Proposition 5.25. *Up to restricting \mathcal{U} to a small neighborhood in the classical topology, $E_{\mathcal{U}} \rightarrow \mathcal{Y}_{\mathcal{U}}^3$ is a locally trivial fibration. For $B \in \mathcal{U} \cap \Delta$, the fiber over a point $[v] \in (\mathcal{Y}_{\mathcal{U}}^3)_B \cong Y_B^3$ is isomorphic to $\mathbb{P}(B \cap F_v)^\vee \times \mathbb{P}(B \cap F_v)^\vee$.*

Proof. See [98, Claim 3.8]. □

By definition $\mathbb{P}(B \cap F_v) \cong \mathbb{P}^2$ for $[v] \in Y_B^3$, while if $B \notin \Sigma$ then $(\mathcal{Y}_{\mathcal{U}}^3)_B$ is empty.

Definition 5.26. *Consider $A^\perp \in \mathbb{L}\mathbb{G}(\wedge^3 V_6) - \Sigma$ with $Y_{A^\perp}^3 = \{[v_0], \dots, [v_s]\}$. Consider $A^\perp \in \mathcal{U}$ a small neighborhood in the classical topology that verify Proposition 5.25. We call choice of a \mathbb{P}^2 -fibration ϵ for X_{A^\perp} the choice, for every $i \in \{0, \dots, s\}$, of a projection $E_{\mathcal{U}}([v_i]) \rightarrow \mathbb{P}(A^\perp \cap F_{v_i})$.*

We choose a \mathbb{P}^2 -fibration ϵ for X_{A^\perp} : it induce a factorization as in the commutative diagram

$$\begin{array}{ccc} E_{\mathcal{U}} & \xrightarrow{\quad} & F_{\mathcal{U}} \\ & \searrow & \swarrow \\ & \mathcal{Y}_{\mathcal{U}}^3 & \end{array} \tag{5.14}$$

where $E_{\mathcal{U}} \rightarrow \mathcal{Y}_{\mathcal{U}}^3$ is the \mathbb{P}^2 -fibration in Proposition 5.25, $E_{\mathcal{U}} \rightarrow F_{\mathcal{U}}$ is the \mathbb{P}^2 -fibration induced by the choice of ϵ and $F_{\mathcal{U}} \rightarrow \mathcal{Y}_{\mathcal{U}}^3$ is again a \mathbb{P}^2 -fibration. Recall that $f_{\mathcal{U}}$ induces an isomorphism $\mathcal{W}_{\mathcal{U}} \cong \mathcal{Y}_{\mathcal{U}}^3$, so (5.14) induces a morphism between families, parametrized by \mathcal{U} ,

$$\begin{array}{ccc} \bar{\mathcal{X}}_{\mathcal{U}} & \xrightarrow{\quad} & \mathcal{X}_{\mathcal{U}}^\epsilon \\ & \searrow \pi_{\mathcal{U}} & \swarrow c_{\mathcal{U}}^\epsilon \\ & \mathcal{X}_{\mathcal{U}} & \end{array} \tag{5.15}$$

The existence of the diagram is ensured by [98, Claim 3.8]. We consider the family of deformation $\rho_{\mathcal{U}}^\epsilon : \mathcal{X}_{\mathcal{U}}^\epsilon \rightarrow \mathcal{U}$, with the morphism $\rho_{\mathcal{U}}^\epsilon$ given by the composition of $c_{\mathcal{U}}^\epsilon$ and $\rho_{\mathcal{U}} : \mathcal{X}_{\mathcal{U}} \rightarrow \mathcal{U}$, defined above as the composition of $f_{\mathcal{U}}$ and the natural morphism $\mathcal{Y}_{\mathcal{U}} \rightarrow \mathcal{U}$. Then $g_{\mathcal{U}}^\epsilon = f_{\mathcal{U}} \circ c_{\mathcal{U}}^\epsilon : \mathcal{X}_{\mathcal{U}}^\epsilon \rightarrow \mathcal{Y}_{\mathcal{U}}$ is a morphism of families with base \mathcal{U} .

Finally, let $B \in \mathcal{U}$. We define $X_B^\epsilon = (\rho_{\mathcal{U}}^\epsilon)^{-1}(B)$ and

$$c_B^\epsilon = (c_{\mathcal{U}}^\epsilon)|_{X_B^\epsilon} : X_B^\epsilon \rightarrow X_B \quad g_B^\epsilon = (g_{\mathcal{U}}^\epsilon)|_{X_B^\epsilon} : X_B^\epsilon \rightarrow Y_B.$$

On X_B^ϵ we define the line bundle $\mathcal{O}_{X_B^\epsilon}(1) = (g_B^\epsilon)^* \mathcal{O}_{Y_B}(1)$. We do not take the neighborhood \mathcal{U} into account, in the definitions, since they do not depend on it. The variety X_B^ϵ is smooth, see [98, Proposition 3.10, Item 1)]. When B does not lie in $\Delta \cup \Sigma$ the scheme \mathcal{Y}_B^3 is empty, hence we trivially have an isomorphism of double covers between $X_B^\epsilon \rightarrow Y_B$ and $X_B \rightarrow Y_B$, or equivalently an isomorphism of polarized varieties

$$(X_B, \mathcal{O}_{X_B}(1)) \cong (X_B^\epsilon, \mathcal{O}_{X_B^\epsilon}(1)),$$

where $\mathcal{O}_{X_B}(1)$ is the pullback of the line bundle $\mathcal{O}_{Y_B}(1)$. Consider now $B \in \Delta - \Sigma$. In this case $c_B^\epsilon : X_B^\epsilon \rightarrow X_B$ is an isomorphism outside Y_B^3 and the fiber over $[v_i] \in Y_B^3$ is a plane \mathbb{P}^2 for every

$i \in \{0, \dots, s\}$. More precisely $(c_B^\epsilon)^{-1}([v_i]) \cong \mathbb{P}(A \cap F_{v_i})$, see [98, Proposition 3.10, Item 1]); the isomorphism depends on the choice of the \mathbb{P}^2 -fibration ϵ . by definition the line bundle $\mathcal{O}_{X_B}(1)$ is trivial on $(c_B^\epsilon)^{-1}([v_i])$ for every $i \in \{0, \dots, s\}$, in particular it is not ample. To describe X_B^ϵ when $B \in \Sigma$ we need the following result.

Proposition 5.27 (O’Grady). *If ϵ' is another choice of a \mathbb{P}^2 -fibration for X_{A^\perp} , there exists a commutative diagram*

$$\begin{array}{ccc} X_{A^\perp}^\epsilon & \overset{\text{-----}}{\longrightarrow} & X_{A^\perp}^{\epsilon'} \\ & \searrow g_{A^\perp}^\epsilon & \swarrow g_{A^\perp}^{\epsilon'} \\ & & Y_{A^\perp}. \end{array} \quad (5.16)$$

in which the birational map is the flop of a collection of $(g_{A^\perp}^\epsilon)^{-1}([v_i])$. Conversely, every flop of a collection of $(g_{A^\perp}^\epsilon)^{-1}([v_i])$'s is isomorphic to one $X_{A^\perp}^{\epsilon'}$ for some choice of a \mathbb{P}^2 -fibration ϵ' for X_{A^\perp} .

Proof. See [98, Proposition 3.10, Item 3)]. □

Consider $A \in \Pi - \Delta$, hence $A^\perp \in \Delta - \Pi$ (cfr. Remark 4.47 and Definition 4.53 for the notation). In Theorem 5.22 we introduce the map $c : S^{[2]} \rightarrow X_{A^\perp}$, which is an isomorphism on the complement of $P_S \cup C_1^{(2)} \cup \dots \cup C_N^{(2)}$ onto the smooth locus of X_A and contracts a plane over every point inside $f_{A^\perp}^{-1}(Y_{A^\perp}^3)$.

Theorem 5.28. [98, Theorem 4.15, Item 1)] *There exists a choice of a \mathbb{P}^2 -fibration ϵ for X_{A^\perp} such that*

$$(S^{[2]}, L_2 - 2\delta) \cong (X_{A^\perp}^\epsilon, \mathcal{O}_{X_{A^\perp}^\epsilon}(1)).$$

Proof. For every choice of a \mathbb{P}^2 -fibration ϵ' for X_{A^\perp} , we have a commutative diagram

$$\begin{array}{ccc} S^{[2]} & \xrightarrow{c} & X_{A^\perp} & \xleftarrow{c_{A^\perp}^{\epsilon'}} & X_{A^\perp}^{\epsilon'} \\ & \searrow g & \downarrow f_{A^\perp} & \swarrow g_{A^\perp}^{\epsilon'} & \\ & & Y_{A^\perp} & & \end{array}$$

which induces a birational morphism $\psi_{\epsilon'} : S^{[2]} \dashrightarrow X_{A^\perp}$ that sends $L_2 - 2\delta$ to $\mathcal{O}_{X_B^\epsilon}(1)$. Clearly $\psi_{\epsilon'}$ is an isomorphism away from $P_S \cup C_1^{(2)} \cup \dots \cup C_N^{(2)}$ and is a flop of a collection of irreducible components of it. Hence by Proposition 5.27 there exists a choice of a \mathbb{P}^2 -fibration ϵ for X_{A^\perp} such that ψ_ϵ is biregular. □

Remark 5.29. A generalization of Theorem 5.28 holds for $A \in \Pi \cap \Delta$: in this case a birational isomorphism $S^{[2]} \dashrightarrow X_B^\epsilon$, sending $L_2 - 2\delta$ to $\mathcal{O}_{X_B^\epsilon}(1)$, still exists and it is never biregular, see [98, Theorem 4.15, Item 2)] and [98, Remark 4.16].

So consider $A^\perp \in \Delta - \Pi$ and a neighborhood \mathcal{U} such that a tautological family parametrized by \mathcal{U} exists. Fix any point $[V_5] \in Y_{A^\perp}^3$ and call (S, L) the associated Brill-Noether general $K3$ surface. Let ϵ be a choice of a \mathbb{P}^2 -fibration for X_{A^\perp} such that Theorem 5.28 holds.

Then, for every Lagrangian subspace B inside the open subset $\mathbb{L}\mathbb{G}(\wedge^3 V_6)^0 \cap \mathcal{U}$, so that $(X_B^\epsilon, \mathcal{O}_{X_B^\epsilon}(1)) \cong (X_B, \mathcal{O}_{X_B}(1))$, we know by the choice of \mathcal{U} that $(X_B, \mathcal{O}_{X_B}(1))$ and $(S^{[2]}, L_2 - 2\delta)$ are deformation equivalent by Theorem 5.28. So the following result holds.

Corollary 5.30. *There exists a Lagrangian subspace $B \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$ for which the double EPW sextic X_B is a hyperkähler manifold of $K3^{[2]}$ -type.*

In fact every two polarized double EPW sextics $(X_A, \mathcal{O}_{X_A}(1))$ and $(X_B, \mathcal{O}_{X_B}(1))$ are deformation equivalent, hence for every $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0$ the double EPW sextic X_A is a hyperkähler manifold of $K3^{[2]}$ -type, cfr. Section 4.3. We stated Corollary 5.30 in this way because we need it for Theorem 4.37.

We end with an independent result, from which O'Grady inferred Corollary 5.30 in [96], by means of a combined result by Huybrechts, O'Grady and Yoshioka which can be found, for example, at [5, Theorem 3.6].

Proposition 5.31. *Let $(S, L) \in \mathcal{K}_{10}$ be a Brill-Noether general $K3$ surface, and let $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ be the associated Lagrangian subspace. If there exists an integer h such that L has divisibility $10h$ inside $\text{NS}(S)$, then the moduli space*

$$X = \{F \text{ is a stable sheaf on } S \mid \text{rk}(F) = 2, c_1(F) = L, c_2(F) = 5\} / \sim$$

is isomorphic to X_A . In particular X_A is smooth.

Proof. This is [96, Section 5]. For details see also O'Grady's construction in [94, Subsection 5.4]; the assumption on the divisibility of L is [94, Hypothesis (4.8)] for ℓ the first Chern class of L , hence $(\ell, D) = 10$. \square

5.4 Automorphisms

5.4.1 Automorphisms of Brill-Noether general $K3$ surfaces

As in Section 5.2, we identify a Brill-Noether general $K3$ surface $(S, L) \in \mathcal{K}_{10}$ with the ordinary GM variety of dimension 2 in the form $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ obtained by the closed embedding $\phi_{|L|}$. In particular we consider a strongly smooth S i.e a Brill-Noether general $K3$ surface whose associated Grassmannian hull M_S is a smooth threefold.

The automorphism group of the GM variety $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$, which we denoted by $\text{Aut}(S, \mathbb{P}(W))$ in Definition 4.88, corresponds to the group of automorphisms of the abstract surface S fixing the polarization L ; by Proposition 2.120 this group is finite.

Remark 5.32. Consider $\alpha \in \text{Aut}(S, \mathbb{P}(W))$: by definition it comes from a linear isomorphism of $\mathbb{P}(W)$. Moreover it is induced by an automorphism (g_W, g_V, g_L) of ordinary GM data, see Lemma 4.102. By (4.25) in the definition of automorphism of GM data, the linear isomorphism

$g_W \in \mathrm{PGL}(W \otimes L) = \mathrm{PGL}(W)$ is the restriction of $\bigwedge^2 (g_{V|_{V_5}})$ to $\mathbb{P}(W)$; in particular the restriction morphism $\rho : \{\alpha \in \mathrm{Aut}(\mathbb{P}(W)) \mid \alpha(S) = S\} \rightarrow \mathrm{Aut}(S, \mathbb{P}(W))$ factorizes through $\mathrm{Aut}(M_S)$,

$$\rho : \{\alpha \in \mathrm{Aut}(\mathbb{P}(W)) \mid \alpha(S) = S\} \xrightarrow{f} \mathrm{Aut}(M_S) \xrightarrow{g} \mathrm{Aut}(S, \mathbb{P}(W)).$$

We know by Theorem 4.95, Item 2), that S is not contained in any hyperplane inside $\mathbb{P}(W)$, so $g \circ f$ is injective. Then f is injective too (we already knew it, as a part of the proof of Theorem 4.79 which we did not present). By definition of $\mathrm{Aut}(S, \mathbb{P}(W))$ the morphism $g \circ f$ is also surjective, so it is an isomorphism of group, hence $\mathrm{Aut}(S, \mathbb{P}(W))$ can be seen as a subgroup of $\mathrm{Aut}(M_S)$ through $f \circ (g \circ f)^{-1}$; note that g is never an isomorphism, as $\mathrm{Aut}(M_S)$ is always infinite, cfr. Theorem 4.84.

We never use the fact that S has dimension 2: the inclusion $\mathrm{Aut}(Z, \mathbb{P}(W)) \subset \mathrm{Aut}(M_Z)$ holds for strongly smooth ordinary GM varieties in any dimension.

We denote by D_n the dihedral group of order $2n$, by \mathfrak{S}_n the symmetric group on n objects and by A_n the alternating group on n objects.

Theorem 5.33. *Let $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ be a Brill-Noether general $K3$ surface which is strongly smooth. Then $\mathrm{Aut}(S, \mathbb{P}(W))$ is isomorphic to one of the following groups:*

$$\mathbb{Z}/n\mathbb{Z} \text{ for } 1 \leq n \leq 66, \quad D_n \text{ for } 2 \leq n \leq 66, \quad A_4, \quad \mathfrak{S}_4, \quad A_5. \quad (5.17)$$

In particular, if an abelian subgroup $G \subseteq \mathrm{Aut}(S, \mathbb{P}(W))$ acts symplectically on S , then it is either cyclic of order at most 8 or isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

Proof. By Remark 5.32 we have $G \subset M_S$. Since S is strongly smooth, by Theorem 4.84 the automorphism group of M_S is isomorphic to $\mathrm{PGL}(2, \mathbb{C})$. The groups in (5.17) are exactly the finite subgroups inside $\mathrm{PGL}(2, \mathbb{C})$, up to isomorphism, with the bound on the order given by Theorem 2.131: as we said above $\mathrm{Aut}(S, \mathbb{P}(W))$ is finite, hence the first part of the statement is proved.

For the second part it is sufficient to check the groups that are both in (5.17) and in (2.122). \square

Corollary 5.13, Item 1) – 2), gives necessary and sufficient conditions for a $\langle 10 \rangle$ -polarized $K3$ surface to be a strongly smooth Brill-Noether general $K3$ surface, in particular families of ordinary GM varieties of dimension 2 with an automorphism can be studied.

Proposition 5.34. *For $p = 2, 3, 5$ there exist strongly smooth Brill-Noether general $K3$ surfaces $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$, of Picard rank respectively 9, 13 and 17, admitting a symplectic automorphism $\alpha \in \mathrm{Aut}(S, \mathbb{P}(W))$ of order p .*

There is no strongly smooth Brill-Noether general $K3$ surfaces $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ admitting a symplectic action of the cyclic group $\mathbb{Z}/7\mathbb{Z}$.

Proof. Consider Theorem 2.125. To prove the statement for $p = 2, 3, 5$ it is sufficient to look at the Néron-Severi group of the corresponding $K3$ surfaces given in Theorem 2.125, which is $\mathcal{L}_p = \mathbb{Z}L \oplus S_{\mathbb{Z}/p\mathbb{Z}}$ with L ample. For every $x \in \mathbb{Z}$, $s \in S_{\mathbb{Z}/p\mathbb{Z}}$, the product $(L, xL + s)$ is a multiple of 10, in particular $(S, L) \notin D_{h,0}$ for $h = 1, 2, 3, 4$. Then, by Corollary 5.13, S can be seen as a

2-dimensional ordinary GM variety which is strongly smooth. Consider α a generator of $\mathbb{Z}/p\mathbb{Z}$; since $L \in S_{\mathbb{Z}/p\mathbb{Z}}^\perp$, the polarization is invariant for the action of α , hence $\alpha \in \text{Aut}(S, \mathbb{P}(W))$. The case $p = 7$ is clear from Theorem 2.125, Item c). \square

Remark 5.35. The moduli spaces of $\langle 10 \rangle$ -polarized $K3$ surfaces admitting a symplectic action of $\mathbb{Z}/p\mathbb{Z}$ for $p = 2$ (resp. 3 or 5) have dimension 11 (resp. 7 or 3), cfr. Remark 2.127. By Corollary 5.13, strongly smooth Brill-Noether general elements fill an open subset inside any of these moduli spaces, which is the complement of the locus of $\langle 10 \rangle$ -polarized $K3$ surfaces (S, L) inside the divisors $D_{h,0} \in \mathcal{K}_{10}$ for $h = 1, 2, 3, 4$.

Remark 5.36. An interesting property of $K3$ surfaces is that sometimes the action of a group H induces the action of a bigger group G with $H \leq G$. In [43, Corollary 2.15], Garbagnati showed that a $K3$ surface admits $\mathbb{Z}/5\mathbb{Z}$ as group of symplectic automorphisms if and only if it admits the dihedral group of ten elements D_5 as a group of symplectic automorphisms. In particular she proved that $S_{\mathbb{Z}/5\mathbb{Z}} \cong S_{D_5}$, so the polarization L is again invariant. This means that $D_5 \subseteq \text{Aut}(S, \mathbb{P}(W))$ if and only if $\mathbb{Z}/5\mathbb{Z} \subseteq \text{Aut}(S, \mathbb{P}(W))$.

Some other similar situations have been described for $\mathbb{Z}/p\mathbb{Z}$ acting non-symplectically: by Dillies in [33] for $p = 3$, by Garbagnati and Sarti in [46] for every prime. Anyway, in these cases L is no more an invariant polarization for D_{2p} , so the action of the bigger group does not give rise to automorphisms on the GM variety.

5.4.2 Bounds for automorphisms of EPW sextics

Automorphisms of EPW sextics are an interesting topic *per se*; moreover they are strictly connected to automorphisms of their double covers, see Section 4.3.4, and they have some interesting relations with the automorphism group of the strongly smooth GM varieties that can be associated to them, see Section 4.5.3. Here we give some new results about them.

Fix a complex vector space V_6 with a volume form, so that we can define the symplectic Grassmannian $\mathbb{L}\mathbb{G}(\wedge^3 V_6)$. Recall that there is a natural isomorphism between the groups $\text{PGL}(V_6)$ and $\text{PGL}(V_6^\vee)$: a linear map $\phi \in \text{PGL}(V_6)$ is sent to $(\phi^{-1})^\vee$, which maps a hyperplane $[V_5]$ to $[\phi(V_5)]$. To simplify notation, from now on we always identify $\text{PGL}(V_6)$ and $\text{PGL}(V_6^\vee)$ through this isomorphism.

Lemma 5.37. *Take $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)$ such that $A \notin \Sigma$. Then $\text{Aut}(Y_A) = \text{Aut}(Y_{A^\perp})$.*

Proof. By Corollary 4.27, an automorphism $\alpha \in \text{PGL}(V_6)$ restricts to an automorphism of Y_A if and only if $\wedge^3 \alpha$ fixes A . So consider $[V_5] \in Y_{A^\perp}$ i.e. $[V_5]$ such that $\wedge^3 V_5 \cap A \neq 0$, see Remark 4.20 for this characterization of a dual EPW sextic. Since α is an automorphism we have

$$0 \neq \left(\bigwedge^3 \alpha\right) \left(\bigwedge^3 V_5 \cap A\right) = \left(\bigwedge^3 \alpha\right) (V_5) \cap A,$$

hence $[\alpha(V_5)] \in Y_{A^\perp}$, so α restricts to an automorphism of Y_{A^\perp} . We have then $\text{Aut}(Y_A) \subseteq \text{Aut}(Y_{A^\perp})$, by symmetry we also have the equality. \square

In fact, the same calculations as above show that every automorphism of Y_A fixes both Y_A^k and $Y_{A^\perp}^k$ for $k \geq 0$.

We consider $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ a Brill-Noether general $K3$ surface, seen as a smooth ordinary GM variety. We call $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)$ the associated Lagrangian subspace, as in Theorem 4.106. By Proposition 4.108, Item 3), the intersection $A \cap \bigwedge^3 V_5$ has dimension 3, so if we impose S to be strongly smooth we have $A \notin \Sigma$ by Theorem 4.110.

Definition 5.38. Let $\mathcal{C} = \{C_1, \dots, C_N\}$ be the set of smooth conics inside $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$. We denote by $\text{Aut}_{\mathcal{C}}(S, \mathbb{P}(W)) \leq \text{Aut}(S, \mathbb{P}(W))$ the subgroup of automorphisms sending C_h to C_h for every $h \in \{1, \dots, N\}$.

As before, we denote by \mathfrak{S}_n the symmetric group on n objects. Note that clearly the list of groups at (5.17) contains any subgroup of a group in the list, so $\text{Aut}_{\mathcal{C}}(S, \mathbb{P}(W))$ should appear in (5.17), too.

Remark 5.39. If S contains no lines, we always have $|Y_{A^\perp}^3| = |\mathcal{C}| + 1$ by Theorem 5.22.

Theorem 5.40. Let $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ be a strongly smooth Brill-Noether general $K3$ surface with N smooth conics on it, with associated Lagrangian data (V_6, V_5, A) . Suppose that S contains no line: the automorphism group of Y_A sits in an exact sequence

$$1 \rightarrow \text{Aut}_{\mathcal{C}}(S, \mathbb{P}(W)) \rightarrow \text{Aut}(Y_A) \xrightarrow{\phi} \mathfrak{S}_{N+1} \quad (5.18)$$

where ϕ sends $\alpha \in \text{Aut}(Y_A)$ to its action on $Y_{A^\perp}^3$.

Proof. The morphism ϕ in (5.18) is well defined by Lemma 5.37 and its kernel is the subgroup of automorphisms of Y_A fixing *pointwise* $Y_{A^\perp}^3$.

Now we fix an automorphism $\alpha \in \text{Aut}(S, \mathbb{P}(W))$, we call $\tilde{\alpha}$ the natural action induced on $V_6 = H^0(\mathbb{P}(W), \mathcal{I}_S(2))$. Since, by hypothesis, there is no line on S , we make use of Theorem 5.22, so we have a commutative diagram

$$\begin{array}{ccc} S^{[2]} & \xrightarrow{c} & X_{A^\perp} \\ & \searrow g & \downarrow f_{A^\perp} \\ & & Y_{A^\perp} \longleftrightarrow \mathbb{P}(V_6^\vee). \end{array}$$

Consider $\mathcal{C} = \{C_1, \dots, C_N\}$ and let $P_S \cong \mathbb{P}^2$ be the scheme parametrizing lines on the Grassmannian hull of S , see (5.7); again by Theorem 5.22 we have $Y_{A^\perp}^3 = \{g(P_S), g(C_1^{(2)}), \dots, g(C_N^{(2)})\}$. A straightforward computation, using the definition of g , gives $g \circ \alpha^{[2]} = \tilde{\alpha} \circ g$.

On the other hand, we can consider $\text{Aut}(S, \mathbb{P}(W))$ as the subgroup of $\text{Aut}(Y_A) = \text{Aut}(Y_{A^\perp})$ fixing $[V_5] \in Y_{A^\perp}^3$, see Proposition 4.121 since S is strongly smooth: by (4.25) the action of α on $\mathbb{P}(V_6)$ is exactly $\tilde{\alpha}$. Hence the permutation of \mathcal{C} given by α induces a permutation on $\{P_S, C_1^{(2)}, \dots, C_N^{(2)}\}$ and descends to a permutation of $Y_{A^\perp}^3$ which is the same as the one induced by α as an element of $\text{Aut}(Y_A)$ (note that $\alpha^{[2]}$ always fixes P_S by Remark 5.32). In particular $\alpha \in \text{Aut}(S, \mathbb{P}(W)) \subseteq \text{Aut}(Y_A)$ fixes pointwise $Y_{A^\perp}^3$ if and only if $\alpha \in \text{Aut}_{\mathcal{C}}(S, \mathbb{P}(W))$. \square

Note that if we drop the condition on lines on S we do not have a link between the cardinalities of \mathcal{C} and $Y_{A^\perp}^3$.

Remark 5.41. When S contains no conics, or equivalently when $Y_{A^\perp}^3$ is a singleton, we have $\text{Aut}(S, \mathbb{P}(W)) \cong \text{Aut}(Y_A)$.

By Theorem 5.2 we know that, under the hypothesis of Theorem 5.40, the set $Y_A^3 \cap \mathbb{P}(V_5)$ is empty. If moreover S contains no quintic elliptic pencils, see Proposition 5.3, then $Y_A^3 = \emptyset$, hence X_A is a hyperkähler manifold. In this case we give in Lemma 5.48 a relation between the symplecticity of an automorphism of S and the symplecticity of a corresponding automorphism on X_A , which is the expected one.

Remark 5.42. By Lemma 5.37 we can see Theorem 5.40 as follows: if $A \in \Delta - \Sigma$ (or $A \in \Pi - \Sigma$), take N such that $|Y_A^3| = N + 1$ (resp. $|Y_{A^\perp}^3| = N + 1$) Then there is an exact sequence

$$1 \rightarrow G \rightarrow \text{Aut}(Y_A) \xrightarrow{\phi} \mathfrak{S}_{N+1} \quad (5.19)$$

where ϕ sends $\alpha \in \text{Aut}(Y_A)$ to its action on Y_A^3 (resp. $Y_{A^\perp}^3$) and G is one of the groups in the list at (5.17).

A similar idea can be used to provide a bound in the general case i.e. $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6) - \Sigma$, if we add a condition on the action of a group.

Proposition 5.43. Consider $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6) - \Sigma$ and $G \subseteq \text{Aut}(Y_A)$. Suppose that G fixes a point $[v] \in Y_A^2$ or a point $[V_5] \in Y_{A^\perp}^2$. Then there exists a group H among

$$\mathbb{Z}/k\mathbb{Z} \text{ for } k \geq 1, \quad D_k \text{ for } k \geq 2, \quad A_4, \quad \mathfrak{S}_4, \quad A_5 \quad (5.20)$$

and an integer $r \geq 1$ such that G sits in an exact sequence

$$1 \rightarrow \mathbb{Z}/r\mathbb{Z} \rightarrow G \rightarrow H \rightarrow 1. \quad (5.21)$$

Proof. We consider the case of G fixing $[V_5] \in Y_{A^\perp}^2$, the dual case is identical with A^\perp instead of A and $v \in Y_A^2 \subset \mathbb{P}((V_6^\vee)^\vee)$ by Lemma 5.37. We consider the Lagrangian data (V_6, V_5, A) and we call $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ the associated ordinary GM intersection: by hypothesis $A \notin \Sigma$ and $\dim(A \cap \bigwedge^3 V_5) = 2$, so Z is a strongly smooth ordinary GM variety of dimension 3 by Corollary 4.111.

By Proposition 4.121, Item *d*), we can see G as a subgroup of $\text{Aut}(Z, \mathbb{P}(W))$. Now we prove that (5.21) holds for any group of automorphisms of Z . Call M_Z the 4-dimensional, smooth, Grassmannian hull of Z : by Theorem 4.82 there is a short exact sequence

$$1 \rightarrow \mathbb{C}^4 \rtimes \mathbb{C}^* \rightarrow \text{Aut}(M_Z) \xrightarrow{\pi} \text{PGL}(2, \mathbb{C}) \rightarrow 1. \quad (5.22)$$

The group G is finite, see Corollary 4.122, and is a subgroup of $\text{Aut}(M_Z)$ by Remark 5.32, hence (5.22) restricts to a short exact sequence

$$1 \rightarrow N \rightarrow G \xrightarrow{\pi} H \rightarrow 1$$

with $N = G \cap (\mathbb{C}^4 \rtimes \mathbb{C}^*)$ and $H = \pi(G)$. Since G is finite, the group H is a finite subgroup of $\mathrm{PGL}(2, \mathbb{C})$ hence it has to appear in (5.20). As for N , it is finite inside $\mathbb{C}^4 \rtimes \mathbb{C}^*$, so it is isomorphic to a finite subgroup of \mathbb{C}^* . \square

The action of $G \subseteq \mathrm{Aut}(Y_A)$, restricted to $\mathbb{P}(V_5)$, is described in a suitable basis at (4.18).

Recall that the *spin group* $\mathrm{Spin}(n)$, for $n \geq 2$, is the universal cover of the special orthogonal group $\mathrm{SO}(n)$. As it is clear from Proposition 4.80, the quotient by $\{\pm 1\}$ in the result below is induced by the natural map $\mathrm{GL}(V_5) \twoheadrightarrow \mathrm{PGL}(V_5)$.

Proposition 5.44. *Consider $A \in \mathrm{LG}(\wedge^3 V_6) - \Sigma$ and $G \subseteq \mathrm{Aut}(Y_A)$. Suppose that G fixes a point $[v] \in Y_A^1$ or a point $[V_5] \in Y_{A^\perp}^1$. Then G is the quotient by $\{\pm 1\}$ of a group \tilde{G} such that there is an exact sequence*

$$1 \rightarrow \mathbb{Z}/r\mathbb{Z} \rightarrow \tilde{G} \rightarrow H \rightarrow 1$$

for some $r \geq 1$ and H a finite subgroup inside $\mathrm{Spin}(5)$.

Proof. The idea of the proof is identical to that of Proposition 5.43: in the same way, we only need to prove that the claim holds for a group G acting on some strongly smooth ordinary GM variety of dimension 4. Any such group is a finite subgroup of $\mathrm{Aut}(M)$, where M is a 5-dimensional smooth Grassmannian hull which is a dimensionally transverse intersection. Consider (4.17) restricted to G , we have

$$1 \rightarrow N \rightarrow G \xrightarrow{\pi} H \rightarrow 1$$

with $N = G \cap \mathbb{C}^4$ and $H = \pi(G) \subset (\mathrm{Sp}(4, \mathbb{C}) \times \mathbb{C}^*) / \{\pm 1\}$. Clearly $N = \{1\}$ as G is finite, hence G is isomorphic to the quotient of a finite subgroup of $\mathrm{Sp}(4, \mathbb{C}) \times \mathbb{C}^*$. The symplectic group $\mathrm{Sp}(4, \mathbb{C})$ is a non-compact, connected Lie group: any finite subgroup of $\mathrm{Sp}(4, \mathbb{C})$ lies inside its maximal compact subgroup, which is the compact symplectic group $\mathrm{Sp}(2)$, given by the intersection of $\mathrm{Sp}(4, \mathbb{C})$ with the unitary group $U(4)$. In turn $\mathrm{Sp}(2)$ is isomorphic to $\mathrm{Spin}(5)$. \square

Remark 5.45. By definition of $\mathrm{Spin}(5)$ there is a short exact sequence

$$1 \rightarrow \mathbb{Z}/2\mathbb{Z} \rightarrow \mathrm{Spin}(5) \rightarrow \mathrm{SO}(5) \rightarrow 1.$$

As we did in Proposition 5.43 and Proposition 5.44, it is possible to study finite subgroups of $\mathrm{Spin}(5)$ through their projection on $\mathrm{SO}(5)$: finite subgroups of $\mathrm{SO}(5)$ have been classified by Mecchia and Zimmermann in [79], see [79, Corollary 2].

Remark 5.46. Under the hypotheses of Proposition 5.44 for some $G \subseteq \mathrm{Aut}(Y_A)$, there is always a lifting $\mu : G \rightarrow \mathrm{Aut}(X_A)$: the restriction of G to Y_A^1 fixes a point $p \in Y_A^1$, so the action of G lifts to the universal cover $X_A^1 \rightarrow Y_A^1$ by the unique lifting property, and then it extends to the whole X_A , hence to the branch and ramification loci, in a unique way. When $\alpha \in G$ has odd order, the lift coincides with the one given in Proposition 4.67, Item *b*).

5.4.3 Lifting automorphisms to double EPW sextics

We take $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0 - \Pi$, so that both X_A and X_{A^\perp} are hyperkähler manifolds. For $G \subseteq \text{Aut}(Y_A) = \text{Aut}(Y_{A^\perp})$, we have an extension

$$1 \rightarrow \{\text{id}, \iota_A\} \rightarrow \tilde{G} \rightarrow G \rightarrow 1$$

with $\tilde{G} \subseteq \text{Aut}_{D_A}(X_A)$ (see Definition 4.66), and a similar extension of G acting on X_{A^\perp} . A priori we cannot say that the two extensions are isomorphic. However, the following result assures that this happens, at least under the conditions of Proposition 4.67. As in Proposition 4.67, we will consider 1 as a first root of unity, to include the case of a symplectic action.

Lemma 5.47. *Consider $A \in \mathbb{L}\mathbb{G}(\wedge^3 V_6)^0 - \Pi$ and let ω_A and ω_{A^\perp} be two symplectic 2-forms, respectively on X_A and X_{A^\perp} . We take $G \subseteq \text{Aut}(Y_A) = \text{Aut}(Y_{A^\perp})$ such that one of the two conditions of Proposition 4.67 holds for G acting on Y_A , so that G lifts to an action on X_A . Then G lifts to an action on X_{A^\perp} too.*

Moreover, let $\mu_A : G \rightarrow \text{Aut}(X_A)$ and $\mu_{A^\perp} : G \rightarrow \text{Aut}(X_{A^\perp})$ be the liftings of G , respectively to $\text{Aut}(X_A)$ and $\text{Aut}(X_{A^\perp})$. Then for every $\alpha \in G$ the automorphism $\mu_A(\alpha)$ acts as a n -th primitive root of unity on ω_A if and only if $\mu_{A^\perp}(\alpha)$ acts as a n -th primitive root of unity on ω_{A^\perp} .

Proof. If Proposition 4.67, Item *b*), holds for $G \subseteq \text{Aut}(Y_A)$, then G , seen as a group acting on Y_{A^\perp} , lifts automatically to $\text{Aut}(X_{A^\perp})$, since Item *b*) depends only on the cardinality of G .

We prove the second part of the statement in this case. We refer to Section 4.2.3: consider the birational map $\mathcal{G} = q \circ p^{-1}$ defined in Theorem 4.24, which is the Gauss map on Y_A , sending $p \in Y_A^1$ to its tangent space. Consider a section σ_Y of the canonical bundle of Y_A ; we can choose ω_A such that $f_A^* \sigma_Y = \omega_A^{\otimes 2}$. By the same kind of computations made in Proposition 4.67, we know that $\mu_A(\alpha)$ acts as a n -th primitive root of unity on ω_A if and only if α acts as a primitive n -th root of unity, say ζ , on σ_Y . Now we call U the intersection between the smooth locus of Y_A and the pullback through \mathcal{G} of the smooth locus of $Y_{A^\perp}^1$. The pullback of $(\sigma_Y)|_U$ through \mathcal{G}^{-1} extends to a section of the canonical bundle of Y_{A^\perp} by Hartogs' theorem, since $\mathcal{G}(U)$ has codimension 2 and the pullback agrees with a section of the canonical bundle of Y_A on $\mathcal{G}(U)$. Since \mathcal{G} and α commute, we have

$$\alpha^*(\mathcal{G}^* \sigma_Y) = \mathcal{G}^*(\alpha^* \sigma_Y) = \zeta \mathcal{G}^* \sigma_Y,$$

hence α acts as a n -th primitive root of unity on a section of the canonical of Y_{A^\perp} . We can then apply Proposition 4.67 on $G \subseteq \text{Aut}(Y_{A^\perp})$.

For the case of G satisfying Proposition 4.67, Item *a*), the proof is very similar: we prove as above that G acts trivially on a section of the canonical bundle of Y_{A^\perp} , then we apply Proposition 4.67. \square

We consider instead the case in which $A \in \Pi - (\Delta \cup \Sigma)$. In this case, by Theorem 4.106 we can associate a $K3$ surface, which we call S , to the Lagrangian data (V_6, V_5, A) with $[V_5] \in Y_{A^\perp}^3$. By Corollary 4.111 we know that $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ is strongly smooth, so by Proposition 4.121 the group $\text{Aut}(S, \mathbb{P}(W))$ (see Definition 4.88) injects in $\text{Aut}(Y_A)$ as the group of automorphisms fixing the hyperplane $\mathbb{P}(V_5)$. Consider $\alpha \in \text{Aut}(S, \mathbb{P}(W))$: in the following we call $\tilde{\alpha} \in \text{Aut}(Y_A)$ the

automorphism induced on Y_A by α , see Proposition 4.119, Item 1). We consider again 1 as a first root of unity.

Lemma 5.48. *Consider $\alpha \in \text{Aut}(S, \mathbb{P}(W))$ acting as a n -th primitive root of unity on a symplectic 2-form on S . If n is odd, the morphism $\tilde{\alpha} \in \text{Aut}(Y_A)$ lifts to an automorphism of X_A which acts as a n -th primitive root of unity on a symplectic 2-form on X_A .*

Proof. We know that $\alpha \in \text{Aut}(S, \mathbb{P}(W))$ acts on a 2-form on S in the same way as the natural automorphism $\alpha^{[2]}$ acts on a 2-form on $S^{[2]}$. Then, by the same calculation of the proof of Theorem 5.40, we have that $\tilde{\alpha} \in \text{Aut}(Y_A)$ acts as a n -th primitive root of unity on a section of the canonical bundle of Y_A . So by Proposition 4.67, Item b), we have the claim. \square

Of course involutions on S are a particular case, as there are always a symplectic and a non-symplectic involution on X_A descending to an involution on Y_A , see Lemma 4.69.

Now we focus on $K3$ surfaces, for any $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ we call A the associated Lagrangian subspace. We know that, for S general, the double EPW sextic X_A is a hyperkähler manifold. However, also for S general, the automorphism group $\text{Aut}(S, \mathbb{P}(W))$ is trivial. So to find automorphisms on some S that induce automorphisms on a hyperkähler manifold, the tricky part is to control the generality. By Corollary 5.6 or Corollary 5.13 we are able to give at least sufficient conditions for S to be associated to a X_A which is a hyperkähler manifold. By Lemma 5.48 we also relate the symplecticity of automorphisms on S with the symplecticity of the induced automorphisms on X_A . We are able to give some results, appealing to lattice theory and the classification of automorphisms on $K3$ surfaces given in Section 2.4.3.

Proposition 5.49. *For every $p = 2, 3, 5$ there exist Lagrangian subspaces $A_p \in \Pi-(\Sigma \cup \Delta)$ such that the EPW sextic Y_{A_p} (hence $Y_{A_p^\perp}$) admits an automorphism of order p , which lifts to a symplectic automorphism $\tilde{\alpha}$ of order p on the hyperkähler manifold X_{A_p} .*

Proof. By Lemma 5.48, we only need to prove that the strongly smooth $K3$ surfaces in Proposition 5.34, seen as elements $(S, L) \in \mathcal{K}_{10}$, do not lie in $D_{5,0}$ nor $D_{1,-2}$. Then by Corollary 5.13 the double EPW sextic X_A is a hyperkähler manifold. By a straightforward calculation, the condition $(S, L) \notin D_{5,0} \cup D_{1,-2}$ holds if we consider strongly smooth $K3$ surfaces whose Néron-Severi group is isomorphic to $\mathcal{L}_p = \mathbb{Z}L \oplus S_{\mathbb{Z}/p\mathbb{Z}}$, with $S_{\mathbb{Z}/p\mathbb{Z}}$ the coinvariant lattice of an order- p symplectic automorphism on the $K3$ surface. The claim about $Y_{A_p^\perp}$ comes from Lemma 5.37. \square

Remark 5.50. For $p = 5$, everytime X_A admits a symplectic automorphism induced by S , it admits a symplectic action of the dihedral group D_5 , see Remark 5.36.

We proved the existence of automorphisms on double EPW sextics in a non-geometrical way. On the other hand, everytime we have a geometric description for the automorphism of the strongly smooth $K3$ surface $S = \mathbb{P}(W) \cap G(2, V_5) \cap Q$, it is pretty easy to deduce the description of its action on Y_A , which comes from the linear action induced in the standard way on $V_6 = H^0(\mathbb{P}(W), \mathcal{I}_S(2))$.

However, it is not easy to provide explicit examples of actions on strongly smooth $K3$ surfaces whose associated X_A is smooth: a standard way is to fix a linear isomorphism $g \in \text{GL}(V_5)$ and choose

a 7-dimensional vector space $W \subset \bigwedge^2 V_5$ which is invariant under the action of $\bigwedge^2 g$, obtaining thus a Grassmannian hull $\mathbb{P}(W) \cap G(2, V_5)$ with a morphism α , which is the restriction of $\bigwedge^2 g$. Then we should find an α -invariant quadric hypersurface Q of $\mathbb{P}(W)$ to produce an ordinary GM variety. The difficult part is that the kernel of Q needs to have dimension at most two by Proposition 4.108, Item 2) to keep X_A smooth.

Remark 5.51. Explicit examples of automorphisms on EPW sextics, lifting to the double cover to symplectic automorphisms, are given by Mongardi in [81, Chapter 4]. The Lagrangian subspaces considered in these cases do not lie in Π .

Remark 5.52. In the case of a $K3$ surface S considered in the proof of Proposition 5.49, the polarization L has divisibility zero modulo 10 inside $NS(S)$, so, by Proposition 5.31, we knew that the associated X_A was a hyperkähler manifold without making use of Corollary 5.13. Anyway, Corollary 5.13 allows to give a more precise description of the family of deformation of $K3$ surfaces inducing automorphisms on smooth double EPW sextics: it is the family described in Remark 5.35 intersected $\mathcal{K}_{10} - (D_{5,0} \cup D_{1,-2})$.

Moreover, a further study of actions of groups induced on smooth double EPW sextics by Brill-Noether general $K3$ surfaces cannot be done by means of Proposition 5.31.

5.4.4 Automorphisms of Gushel-Mukai varieties of dimension at least 3

In Section 5.4.1 we described automorphisms of an ordinary strongly smooth GM variety of dimension 2 i.e. automorphisms of a $\langle 10 \rangle$ -polarized, strongly smooth, Brill-Noether general $K3$ surface. Now we study the case of smooth, hence strongly smooth by Proposition 4.93, ordinary GM varieties of dimension 3 and 4. In these cases the Picard group has rank one, so the whole automorphism group of these GM varieties, seen as abstract varieties, is equal to the group of automorphisms coming from the linear space cutting the Grassmannian. By Corollary 4.122 their automorphism group is always finite.

Lemma 5.53. *Let $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ be a smooth ordinary GM variety of dimension n . Consider $G \subseteq \text{Aut}(Z, \mathbb{P}(W))$.*

- If $n = 3$ then there exists a group H among

$$\mathbb{Z}/k\mathbb{Z} \text{ for } k \geq 1, \quad D_k \text{ for } k \geq 2, \quad A_4, \quad \mathfrak{S}_4, \quad A_5 \quad (5.23)$$

and an integer r such that G sits in an exact sequence

$$1 \rightarrow \mathbb{Z}/r\mathbb{Z} \rightarrow G \rightarrow H \rightarrow 1. \quad (5.24)$$

- If $n = 4$ then G is the quotient by $\{\pm 1\}$ of a group \tilde{G} such that there is an exact sequence

$$1 \rightarrow \mathbb{Z}/r\mathbb{Z} \rightarrow \tilde{G} \rightarrow H \rightarrow 1$$

for some $r \geq 1$ and H a finite subgroup inside $\text{Spin}(5)$.

Proof. The proof is contained in the proofs of Proposition 5.44 and Proposition 5.43: it amounts to say that the automorphism group of Z is contained in the automorphism group of its Grassmannian hull M_Z , and to study the finite subgroups of M_Z . \square

We give further conditions for groups acting on ordinary GM varieties belonging to some particular families. We will need the following definitions.

For $v \in V_5 - \{0\}$, $V_4 \ni v$ a 4-dimensional vector space inside V_5 , we define the 3-dimensional vector space $L_{v, V_4} = \{v \wedge t \mid t \in V_4\}$.

Definition 5.54. A plane $\pi \subset G(2, V_5)$ is a σ -plane if it is in the form $\mathbb{P}(L_{v, V_4})$ for some 4-dimensional vector space $V_4 \subset V_5$ and $v \in V_4$, a τ -plane if it is contained in $\mathbb{P}(\bigwedge^2 U_3)$ for some 3-dimensional $U_3 \subset V_5$.

A τ -quadric surface in $G(2, V_5)$ is a linear section of $G(2, V_4)$ for some 4-dimensional $V_4 \subset V_5$.

Planes inside $G(2, V_5)$ are either σ -planes either τ -planes, but they cannot be of both types. In this thesis we are mostly interested in σ -planes: called $F_2^\sigma(G(2, V_5)) \subset G(3, \bigwedge^2 V_5)$ the Hilbert scheme which parameterizes σ -planes inside $G(2, V_5)$, there is a natural map $f : F_2^\sigma(G(2, V_5)) \rightarrow \mathbb{P}(V_5)$ sending $\mathbb{P}(L_{v, V_4})$ to $[v]$ (if $\mathbb{P}(L_{v, V_4}) = \mathbb{P}(L_{v', V_4'})$ we have $[v] = [v']$). For any ordinary GM variety $Z \subset G(2, V_5)$, we call $F_2^\sigma(Z)$ the Hilbert scheme which parameterizes σ -planes inside Z , and we denote by $f_Z : F_2^\sigma(Z) \rightarrow \mathbb{P}(V_5)$ the restriction of f to $F_2^\sigma(Z)$.

Debarre and Kuznetsov give in [29] a characterization of smooth ordinary GM varieties of dimension 4 containing a σ -plane. As usual, we denote by $\mathbb{P}(V_5) \subset \mathbb{P}(V_6)$ the projective linear space associated to V_5 , by $[V_5] \in \mathbb{P}(V_6^\vee)$ the corresponding point of the dual space.

Theorem 5.55 (Debarre, Kuznetsov). *Let $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ be a smooth ordinary GM variety of dimension 4 and let (V_6, V_5, A) be its associated Lagrangian data. Then $f_Z : F_2^\sigma(Z) \rightarrow \mathbb{P}(V_5)$ factors as*

$$f_Z : F_2^\sigma(Z) \xrightarrow{\tilde{f}_Z} Y_A^3 \cap \mathbb{P}(V_5) \hookrightarrow \mathbb{P}(V_5),$$

where \tilde{f}_Z is bijective over $Y_A^3 \cap \mathbb{P}(V_5)$, except at most over one point, whose fiber has cardinality 2.

Proof. See [29, Theorem 4.3, Item c)]. Consider the birational maps p, q defined in (4.7): following the notation used by Debarre and Kuznetsov, we define $\Sigma_1(Z) = p(q^{-1}([V_5])) \subset Y_A \cap \mathbb{P}(V_5)$: as $[V_5] \in Y_{A^\perp}^1$ by Proposition 4.108, Item 3), in our case $\Sigma_1(Z)$ is a point. The one point on which the fiber has cardinality bigger than one is $\Sigma_1(Z)$, if it lies in Y_A^3 . \square

For other results about linear spaces contained in a smooth GM variety see [29, Section 4]; the analogue for τ -planes of Theorem 5.55 is [29, Theorem 4.5, Item c)].

Theorem 5.56 (Debarre, Iliev, Manivel, Kuznetsov). *Let $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ be a smooth ordinary GM variety of dimension 4 and let (V_6, V_5, A) be its associated Lagrangian data. If Z contains a τ -quadric surface, then $A \in \Pi$.*

Proof. See [29, Remark 5.29]. Consider the double *EPW* sextic X_A : the period point of X_A , say \mathbf{p} , lies in \mathcal{D}'_{10} by [26, Section 7.3] and [31, Proposition 6.1]. If X_A is smooth then \mathbf{p} lies in the image of \mathcal{U}_{EPW} through the period map $P : \mathcal{M}_{2,1}^2 \rightarrow \mathcal{P}_{2,1}^2$, so $A \in \Pi$ by Definition 4.60, since P is an open embedding by Theorem 2.68. If X_A is singular we still have $A \notin \Sigma$ by Proposition 4.108, Item 3), and Theorem 4.110. The extended period map P° is defined over $(\mathbb{L}\mathbb{G}(\bigwedge^3 V_6) - \Sigma) // \mathrm{PGL}(V_6) \subset (\mathcal{M}_{2,1}^2)^\circ$ and it is still an open embedding by [102, Theorem 0.3], so in this case we have $A \in \Pi \cap \Delta$. \square

We can give a further bound for the automorphism group of ordinary *GM* varieties of dimension 4 described above.

Corollary 5.57. *Let $Z = \mathbb{P}(W) \cap G(2, V_5) \cap Q$ be a smooth ordinary *GM* variety of dimension 4 and associated Lagrangian subspace A . If one of the following holds:*

- *Z contains a σ -plane,*
- *Z contains a τ -quadric surface,*

then, for every $G \subseteq \mathrm{Aut}(Z, \mathbb{P}(W))$, there is an exact sequence

$$1 \rightarrow N \rightarrow G \rightarrow \mathfrak{G}_n$$

where $n = |Y_A^3|$ (resp. $n = |Y_{A^\perp}^3|$) if Z contains a σ -plane (resp. a τ -quadric surface), and N is one of the groups in the list at (5.17).

Proof. This is an immediate consequence of Remark 5.42 and Theorem 5.55 for σ -planes, Theorem 5.56 for τ -quadric surfaces. The sequence (5.19) restricts to the sequence in the claim for $G \subseteq \mathrm{Aut}(Z, \mathbb{P}(W)) \subseteq \mathrm{Aut}(Y_A) = \mathrm{Aut}(Y_{A^\perp})$. \square

The coarse moduli spaces of smooth *GM* varieties of dimension 4 is constructed in [31, Theorem 5.15]. Inside it, the families of ordinary *GM* varieties containing a σ -plane (resp. a τ -quadric surface) has codimension 2 (resp. 1), see [27, Remark 5.29]; the general members of both families are rational, see [26, Proposition 7.1] and [26, Proposition 7.4].

Remark 5.58. Consider a smooth *GM* variety of dimension at least 3, with associated Lagrangian subspace A . More in general, the bound given in Corollary 5.57 holds everytime either Y_A^3 or $Y_{A^\perp}^3$ is non-empty. For example, it can be applied to smooth *special GM* sixfolds containing a \mathbb{P}^3 by [29, Theorem 4.2]; in this case the modified version of Corollary 4.120 for smooth special *GM* varieties should be considered, see [27, Proposition 3.12, Item c''].

The relation with the automorphism group of the associated *EPW* sextic can also be used the other way round.

Let $S = \mathbb{P}(W^S) \cap G(2, V_5^S) \cap Q^S$ be a strongly smooth *K3* surface admitting an involution j^S and let (V_6, V_5^S, A) be the associated Lagrangian data; we choose S such that X_A is a hyperkähler manifold, see Proposition 5.49. By Proposition 4.108, Item 3) we have $[V_5^S] \in Y_{A^\perp}^3$. We call $j \in \mathrm{Aut}(Y_A)$ the automorphism induced on Y_A by j^S . By Lemma 4.69, there exists a symplectic involution $\tilde{j} \in \mathrm{Aut}(X_A)$ such that $f_A \circ \tilde{j} = j \circ f_A$.

Remark 5.59. The fixed locus of \tilde{j} is the union of a $K3$ surface and 28 points by Theorem 2.134; we call it F_1 . The fixed locus of $\tilde{j} \circ \iota_A$ is a smooth (not necessarily connected) surface by [10, Section 2], since $\tilde{j} \circ \iota_A$ is non-symplectic; we call it F_2 . If we call R the ramification locus of ι_A , we have $F_1 \cap F_2 = F_1 \cap R = F_2 \cap R$.

The fixed locus of j is the union of $f_A(F_1)$ and the locus of points $f_A(x) \in Y_A$ such that $\tilde{j}(x) = \iota_A(x)$ i.e. $f_A(F_2)$. We call $F \subset Y_A$ this union.

Now we consider every point of Y_A as $[V_5]$ for $V_5 \in (V_6^\vee)^\vee$. For every $[V_5] \in F$, the ordinary strongly smooth GM variety associated to the Lagrangian data (V_6^\vee, V_5, A^\perp) admits an involution induced by j , see Proposition 4.119, Item 2).

We call $Z(V_5, A^\perp)$ the GM variety associated to (V_6^\vee, V_5, A^\perp) : if $[V_5] \in Y_A^2$ then $Z(V_5, A^\perp)$ is a threefold, otherwise it is a fourfold (see Corollary 4.111). Let $\pi \subset \mathbb{P}(V_6)$ be the hyperplane parametrizing hyperplanes of $\mathbb{P}(V_6^\vee)$ containing $[V_5^S] \in Y_{A^\perp}^3$: if $[V_5] \in (F \cap \pi) - Y_A^2$ then $Z(V_5, A^\perp)$ is a smooth fourfold containing a σ -plane by Theorem 5.55; as we said above $Z(V_5, A^\perp)$ admits an involution induced by j .

In general, by Proposition 4.121, Item *d*), the study of the fixed locus of automorphisms of EPW sextics allows to find examples of groups acting on GM varieties.

5.5 $K3$ surfaces with same associated Lagrangian subspaces

The following definition is given by Debarre and Kuznetsov in [27].

Definition 5.60. *We call period partners two GM varieties of the same dimension whose associated Lagrangian subspaces are isomorphic.*

Let Z, Z' be ordinary, strongly smooth GM varieties, with associated Lagrangian data respectively (V_6, V_5, A) and (V_6, V_5', A') (we already fixed an automorphism between the 6-dimensional spaces). We suppose that they are period partners: as a direct consequence of Proposition 4.119, this holds if and only if there exists an isomorphism $\phi \in \text{PGL}(V_6)$ such that $(\bigwedge^3 \phi)(A) = A'$ (in this case the dimension of Z and Z' is the same by Proposition 4.108, Item 3)). Then, by Corollary 4.27, the set of isomorphism classes of ordinary period partners of a strongly smooth GM variety Z of dimension n , with associated Lagrangian data (V_6, V_5, A) , is in bijection with $Y_{A^\perp}^{5-n} / \text{PGL}(V_6)_A$. If we want to consider also special GM varieties, we have a bijection between classes of period partners of Z and $(Y_{A^\perp}^{5-n} \cup Y_{A^\perp}^{6-n}) / \text{PGL}(V_6)_A$, because of the modified version of Proposition 4.108, Item 3) for the special case, cfr. [27, Proposition 3.9, Item *c*)]. This is [27, Theorem 3.23], with the hypothesis of smoothness of Z strengthened to include the case $n = 2$.

In dimension $n = 4, 6$, Debarre and Kuznetsov constructed, in [27, Section 5.1], a period map from the coarse moduli spaces of smooth GM varieties of dimension n to the period domain $\mathcal{P}_{2,1}^2$. Then families of period partners in dimension n coincide with families of smooth GM varieties of same dimension with the same period point, by [31, Proposition 6.1] and Theorem 2.68.

In dimension 2 the situation is much simpler. The moduli space of smooth ordinary GM varieties of dimension 2 is open inside \mathcal{K}_{10} (see Corollary 5.13). We have to require explicitly

strongly smoothness, because Proposition 4.93 fails, in exchange Greer, Li and Tian provided a divisorial condition on \mathcal{K}_{10} for it, which is Proposition 5.12, so the situation is still manageable. By means of O’Grady’s results described in Section 5.3, we have the following characterization of period partners in dimension 2.

Lemma 5.61. *Let $(S, L), (S', L') \in \mathcal{K}_{10}$ be strongly smooth Brill-Noether general K3 surfaces such that $(S, L), (S', L') \notin D_{1,-2}$. Then (S, L) and (S', L') are period partners if and only if there exists a birational isomorphism $(S')^{[2]} \dashrightarrow S^{[2]}$ sending $L_2 - 2\delta$ to $L'_2 - 2\delta'$.*

Proof. Consider (S, L) and (S', L') associated to the same Lagrangian subspace A . Then $S^{[2]}$ and $(S')^{[2]}$ are desingularizations of X_{A^\perp} , and $L_2 - 2\delta$ and $L'_2 - 2\delta'$ are the pullback of the line bundle given by the hyperplane section of Y_{A^\perp} , see Theorem 5.28: the desingularization maps are induced by two (not necessarily isomorphic) choices of a \mathbb{P}^2 -fibration on X_{A^\perp} . By Proposition 5.27 these desingularizations are birationally isomorphic, and the commutativity of (5.16) gives the condition on the line bundles.

On the other hand, we consider $(S^{[2]}, L_2 - \delta) \dashrightarrow ((S')^{[2]}, L'_2 - \delta')$ and we call A and A' the Lagrangian subspaces associated to (S, L) and (S', L') . The period points of X_{A^\perp} and $X_{(A')^\perp}$ are the same as their desingularizations $S^{[2]} \rightarrow X_{A^\perp}$ and $(S')^{[2]} \rightarrow X_{(A')^\perp}$, in particular they share the same period point. But, by [102, Theorem 0.3], the extension to $(\mathbb{L}\mathbb{G}(\bigwedge^3 V_6) - \Sigma) // \mathrm{PGL}(V_6)$ of the period map for naturally polarized EPW sextics is an open embedding, hence $A^\perp = (A')^\perp$, so $A = A'$. \square

Remark 5.62. Strongly smooth special GM varieties of dimension 2 do not exist by [27, Proposition 3.9, Item c)], and Lemma 4.10.

Now we focus on moduli spaces for the case $n = 2$. We fix a 6-dimensional complex space V_6 ; for simplicity, Lagrangian data will always be in the form (V_6, V_5, A) with the V_6 that we chose above. For any strongly smooth special GM variety S of dimension 2, we call $A(S)$ the associated Lagrangian subspace. We define a rational map

$$\tau : \mathcal{K}_{10} \dashrightarrow \mathcal{M}_{2,1}^{2,1} \tag{5.25}$$

$$(S, L) \mapsto (X_{A(S)}, D_{A(S)}). \tag{5.26}$$

We denote by \mathcal{V}_{10} the definition locus of τ i.e. the open set containing K3 surfaces whose associated double EPW sextic is smooth, and we consider the open set

$$\mathcal{V}'_{10} = \mathcal{K}_{10} - (D_{1,0} \cup D_{2,0} \cup D_{3,0} \cup D_{4,0} \cup D_{1,-2}).$$

By Corollary 5.13 we have $\mathcal{V}_{10} \subseteq \mathcal{V}'_{10}$ and $\mathcal{V}'_{10} - D_{5,0} \subseteq \mathcal{V}_{10}$, see also Remark 5.14. The image of τ is exactly $(\Pi - \Delta - \Sigma) // \mathrm{PGL}(V_6)$: clearly $Y_{A(S)^\perp}^3 \neq \emptyset$ for any S by Proposition 4.108, Item 3), so $A(S) \in \Pi$, and $A \in \mathbb{L}\mathbb{G}(\bigwedge^3 V_6)^0$ by definition of τ . On the other side, consider $A \in \Pi - \Delta - \Sigma$: by definition of Π there exists $[V_5] \in Y_{A^\perp}^3$, and the associated GM intersection S is a strongly smooth K3 surface by Corollary 4.111, since $A \notin \Sigma$. Clearly $A(S) = A$.

The very general element inside the image of τ is described by O’Grady, see Proposition 5.31.

Definition 5.63. For $N \geq 0$, we denote

$$\mathcal{C}[N] = \{(S, L) \in \mathcal{V}_{10} \text{ such that } \phi_{|L|}(S) \text{ contains exactly } N \text{ conics}\}$$

and

$$\tilde{\Pi}[N] = \{(X_A, D_A) \in (\Pi - \Delta - \Sigma) // \mathrm{PGL}(V_6) \text{ such that } |Y_{A^\perp}^3| = N + 1\}.$$

Since $\mathcal{V}_{10} \cap D_{1,-2} = \emptyset$, there is no line on $\phi_{|L|}(S)$, so, by Remark 5.39, there is a family of maps

$$\tau_N = \tau_{|\mathcal{C}[N]} : \mathcal{C}[N] \rightarrow \tilde{\Pi}[N] \quad \text{for } N \geq 0.$$

Remark 5.64. The fiber of τ_N over $(X_A, D_A) \in \tilde{\Pi}[N]$ has cardinality at most $N + 1$ and it is isomorphic to the quotient $Y_{A^\perp}^3 / \mathrm{PGL}(V_6)_A$, or alternatively to the quotient of $Y_{A^\perp}^3$ by the image of $\phi : \mathrm{Aut}(Y_A) \rightarrow \mathfrak{G}_{N+1}$ defined in Theorem 5.40.

Clearly the general element of \mathcal{V}_{10} lies in $\mathcal{C}[0]$, hence τ induces a birational morphism onto $\Pi - (\Delta \cup \Sigma) // \mathrm{PGL}(V)$. The Néron Severi lattice of any $(S, L) \in \mathcal{C}[1]$ admits a sublattice R , generated by L and the class of the unique conic of $\phi_{|L|}(S)$, whose Gram matrix is

$$\begin{bmatrix} 10 & 2 \\ 2 & -2 \end{bmatrix}.$$

This sublattice does not admit primitive overlattices: this can be done with some computer algebra, see Remark 1.31 and Proposition 1.30. Hence we have $\mathcal{C}[1] = D_{2,-2} \cap \mathcal{V}_{10}$. Moreover, the very general element of $D_{2,-2}$ has Picard lattice isomorphic to R , hence it lies in \mathcal{V}_{10} . So $\mathcal{C}[1]$ has codimension 1 in \mathcal{K}_{10} , and $\tilde{\Pi}[1]$ has codimension 1 in $(\Pi - \Delta - \Sigma) // \mathrm{PGL}(V_6)$, 2 in $\mathcal{M}_{2,1}^2$.

Remark 5.65. The fiber over a very general element of $\tilde{\Pi}[1]$ has cardinality 2. If the fiber over $(X_A, D_A) \in \tilde{\Pi}[1]$ has cardinality 1, then there exists an involution of Y_{A^\perp} exchanging the two elements inside $Y_{A^\perp}^3$. We call $j \in \mathrm{Aut}(X_A)$ the symplectic involution induced on X_A , see Lemma 4.69 and Lemma 5.37: it commutes with the covering involution ι_A , so by Theorem 2.117 the Néron-Severi group of X_A contains $\langle 2 \rangle \oplus E_8(-2)$. We call G the group generated by ι_A and j . A deformation family of maximal dimension of (X_A, G) has then dimension 12, cfr. Section 2.4.4, so a very general small deformation of (X_A, D_A) inside $\Pi // \mathrm{PGL}(V_6)$ does not have any involution commuting with the covering involution.

Remark 5.66. The divisor $\Pi // \mathrm{PGL}(V_6)$ is birational to \mathcal{K}_{10} , hence unirational, see [49, Theorem 4.1]. It is not normal at the very general point of $\tilde{\Pi}[2]$, since the fiber at this point has cardinality 2 i.e. it is not connected.

Chapter 6

Birational involutions

6.1 Introduction

In Section 2.4.5 we presented a classification of the group of biregular automorphisms of the Hilbert scheme $S^{[n]}$, which has been given by Boissière, An. Cattaneo, Nieper-Wisskirchen and Sarti in the case $n = 2$, see Remark 2.162, and by Al. Cattaneo for all $n \geq 3$, see Theorem 2.161. In particular, $\text{Aut}(S^{[n]})$ is either trivial or generated by an involution which is non-symplectic. These results provide explicit numerical conditions on the parameters n and t for the existence of a non-trivial biregular automorphism on $S^{[n]}$. The first aim of this chapter is to give a similar classification for the group $\text{Bir}(S^{[n]})$ of birational automorphisms of the Hilbert scheme, thus generalizing the results of Debarre and Macrì [32] for the case $n = 2$ and of Debarre [25].

In the first Section we combine the description of the movable cone of $S^{[n]}$ due to Bayer and Macrì in [5], which we presented in Theorem 2.101, with Markman's results [77] on monodromy operators on $H^2(S^{[n]}, \mathbb{Z})$ (see Section 2.4.1) to obtain numerical necessary and sufficient conditions on n and t for the existence of birational automorphisms on $S^{[n]}$. These conditions are presented in Theorem 6.3.

It is interesting to notice that, differently from the biregular case, birational involutions of $S^{[n]}$ can be symplectic, for (infinitely many) suitable choices of n, t . In Proposition 6.6 we give examples of sequences of degrees of polarizations $t = t_k(n)$, indexed by $k \in \mathbb{N}$, which realize all cases in Theorem 6.3.

The closed movable cone of a hyperkähler manifold X can be decomposed in open chambers, each of which is the ample cone of a hyperkähler birational model of X . These chambers are permuted by the action of birational automorphisms, and, by the Hodge-theoretic Torelli theorem, biregular automorphisms are exactly those which preserve the ample cone of X . If X admits a birational involution, it is natural to ask whether it becomes biregular on any of these birational models. We confront this problem for Hilbert schemes $S^{[n]}$ in the second section, by using the explicit description of walls between chambers coming from Theorem 2.101 and Proposition 2.102. For a fixed $n \geq 2$, we say that a value $t \geq 1$ is *n-irregular* if, for a very general $\langle 2t \rangle$ -polarized K3 surface

S , the group $\text{Bir}(S^{[n]})$ contains an involution which is not biregular on any hyperkähler birational model of $S^{[n]}$. We provide a numerical characterization of n -irregular values in Propositions 6.9 and 6.11. In particular, we verify that symplectic birational involutions remain strictly birational on all birational models. On the other hand, in the case of non-symplectic birational automorphisms there are only finitely many n -irregular t 's for a fixed $n \geq 2$ (see Corollary 6.13).

The third section provides an in-depth analysis of the case $n = 3$. By studying solutions of two particular Pell's equations we show that, when $\text{Bir}(S^{[3]})$ is not trivial, the number of chambers in the movable cone of $S^{[3]}$ (i.e. the number of distinct hyperkähler birational models) is either one, two, three or five. For $n = 2$ it is known that the number of chambers in $\text{Mov}(S^{[2]})$ is at most three. As n increases, computing an upper bound for the number of chambers becomes more difficult, since the walls arise from the solutions of an increasing number of Pell's equations. The new results for $n = 3$ are summed up in Proposition 6.22.

In the fourth section we show how Theorem 6.3 can be generalized to study birational maps between Hilbert schemes of points on two distinct $K3$ surfaces of Picard rank one (which need to be Fourier-Mukai partners). In the literature, isomorphic Hilbert schemes of points on two non-isomorphic $K3$ surfaces are called (strongly) ambiguous. The main result of the section is Theorem 6.25, an improved version of the criterion [78, Theorem 2.2] for the determination of ambiguous pairs (up to isomorphism or birational equivalence).

Finally, in the fifth section we give new geometrical examples of birational involutions as in Theorem 6.3, for Hilbert schemes of points on quartic surfaces. Finding similar constructions is an interesting problem, since the Torelli-like results used in the proof of Theorem 6.3 give no insight on the geometry level.

This chapter is part of a joint work with Al. Cattaneo, which appears in Sections 3 – 5 of

On birational transformations of Hilbert schemes of points on $K3$ surfaces,
<https://arxiv.org/abs/2005.05187>, submitted.

6.2 Birational automorphisms of $S^{[n]}$

We use notations and results from Section 2.4.5. Let S be a projective $K3$ surface with $\text{Pic}(S) = \mathbb{Z}L$, $L^2 = 2t$, $t \geq 1$. For $n \geq 2$, let $S^{[n]}$ be the Hilbert scheme of n points on S and $\{L_n, -\delta\}$ a basis for $\text{NS}(S^{[n]}) \subset H^2(S^{[n]}, \mathbb{Z})$; for the definition of L_n see as usual Definition 2.22. As usual, given a lattice L , we denote by A_L its discriminant group, endowed with the finite form induced by L (see Section 1.3). For an isometry $f \in O(L)$ we denote by \bar{f} the induced isomorphism on A_L .

Recall that in this situation the maps

$$\overline{\psi_{S^{[n]}}^A} : \text{Aut}(S^{[n]}) \rightarrow O(\text{NS}(S^{[n]})) \quad \overline{\psi_{S^{[n]}}^B} : \text{Bir}(S^{[n]}) \rightarrow O(\text{NS}(S^{[n]})),$$

whose kernel is the subgroup of natural automorphisms on $S^{[n]}$, are injective for $t \geq 2$; for $t = 1$ both kernels are generated by the covering involution on S (see Lemma 2.156). By Corollary 2.158, this means that for $t \geq 2$ the group $\text{Bir}(S^{[n]})$ is isomorphic either to $\{\text{id}\}$ or to $\mathbb{Z}/2\mathbb{Z}$. For $t = 1$ the

group is isomorphic either to $\mathbb{Z}/2\mathbb{Z}$ or to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

If there exists $\sigma \in \text{Bir}(S^{[n]})$ a non-natural involution, as we checked in Proposition 2.157, $t(n-1)$ is not a square, $(n-1)X^2 - tY^2 = 1$ has no integer solutions (if $n \neq 2$) and $\overline{\text{Mov}(S^{[n]})} = \langle L_n, zL_n - tw\delta \rangle_{\mathbb{R}_{>0}}$, where (z, w) is the minimal solution of $z^2 - t(n-1)w^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$; in particular, we have $z \equiv \pm 1 \pmod{2(n-1)}$ and $w \equiv 0 \pmod{2}$. In this situation the isometry σ^* is the reflection of $\text{NS}(S^{[n]})$ which fixes the line spanned by $(n-1)wL_n - (z-1)\delta$. We denote by $\nu = bL_n - a\delta \in \text{NS}(S^{[n]})$ the primitive generator of this line with $a, b > 0$, $(a, b) = 1$.

Proposition 6.1. *Let S be a projective K3 surface with $\text{Pic}(S) = \mathbb{Z}L$, $L^2 = 2t$, $t \geq 1$. For $n \geq 2$, let $\sigma \in \text{Bir}(S^{[n]})$ be a non-natural automorphism and $\nu = bL_n - a\delta \in \text{NS}(S^{[n]})$ be the primitive generator of the line fixed by σ^* with $a, b > 0$. Then one of the following holds:*

- $\sigma^*|_{\text{Tr}(S^{[n]})} = -\text{id}$ and
 - either $\overline{\sigma^*} = -\text{id}$ and $\nu^2 = 2$, i.e. (a, b) is an integer solution of $(n-1)X^2 - tY^2 = -1$;
 - or $\overline{\sigma^*} = \text{id}$, $\nu^2 = 2(n-1)$ and $\text{div}(\nu) = n-1$, i.e. $(a, \frac{b}{n-1})$ is an integer solution of $X^2 - t(n-1)Y^2 = -1$.
- $n \geq 9$, $\sigma^*|_{\text{Tr}(S^{[n]})} = \text{id}$, $\overline{\sigma^*} = -\text{id}$ and $\nu^2 = 2t$, i.e. the minimal solution of $X^2 - t(n-1)Y^2 = 1$ is $(b, \frac{a}{t})$ and $b \not\equiv \pm 1 \pmod{n-1}$.

Proof. Let (z, w) be the minimal solution of $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$; then, as we recalled before, $z \equiv \pm 1 \pmod{2(n-1)}$ and $w \equiv 0 \pmod{2}$. By Proposition 2.86, the isometry σ^* is a Hodge monodromy operator, hence it acts on $A_{H^2(S^{[n]}, \mathbb{Z})}$ as $\pm \text{id}$ by Proposition 2.51. Moreover, since the transcendental sublattice $\text{Tr}(S^{[n]}) \cong \text{Tr}(S)$ has odd rank, $\sigma^*|_{\text{Tr}(S^{[n]})} = \pm \text{id}$, see [64, Chapter 3, Corollary 3.5]. If $t \neq 1$, by imposing that $\sigma^*|_{\text{Tr}(S^{[n]})}$ glues with the isometry $\sigma^*|_{\text{NS}(S^{[n]})}$ of the form (2.8), we conclude (by Theorem 1.34) that $\sigma^*|_{\text{Tr}(S^{[n]})} = \text{id}$ if and only if $z \equiv 1 \pmod{2t}$ and $\sigma^*|_{\text{Tr}(S^{[n]})} = -\text{id}$ if and only if $z \equiv -1 \pmod{2t}$. On the other hand $\overline{\sigma^*} = \text{id}$ if and only if $z \equiv -1 \pmod{2(n-1)}$ and $\overline{\sigma^*} = -\text{id}$ if and only if $z \equiv 1 \pmod{2(n-1)}$, for all $t \geq 1$ (see [22, Remark 5.2]).

If σ^* acts as $-\text{id}$ on $\text{Tr}(S^{[n]})$ and $t \geq 2$, then the statement follows as in the proof of [22, Proposition 5.1] and by [22, Lemma 6.3].

Assume that $z \equiv 1 \pmod{2t}$ (hence, $\sigma^*|_{\text{Tr}(S^{[n]})} = \text{id}$ or $t = 1$). By Lemma 1.6 and by the hypothesis that $(n-1)X^2 - tY^2 = 1$ has no integer solutions we have $z \equiv 1 \pmod{2(n-1)}$, i.e. σ^* acts as $-\text{id}$ on the discriminant group of $H^2(S^{[n]}, \mathbb{Z})$. Moreover, $(z, w) = (2t(n-1)s^2 + 1, 2us)$, where (u, s) is the minimal solution of $X^2 - t(n-1)Y^2 = 1$. The axis of the reflection $\sigma^*|_{\text{NS}(S^{[n]})}$ is spanned by

$$(n-1)wL_n - (z-1)\delta = 2s(n-1)(uL_n - ts\delta).$$

Since $\text{gcd}(u, ts) = 1$, we conclude that the primitive generator of the axis of the reflection is $\nu = uL_n - ts\delta$, whose square is $2t$. By looking at quadratic residues modulo $n-1$, we see that the minimal solution (u, s) of $X^2 - t(n-1)Y^2 = 1$ can have $u \not\equiv \pm 1 \pmod{n-1}$ only if $n \geq 9$. \square

Remark 6.2. Let ω be a generator of $H^{2,0}(S^{[n]})$. Then $\text{NS}(S^{[n]}) = \omega^\perp \cap H^2(S^{[n]}, \mathbb{Z})$, therefore an automorphism $\sigma \in \text{Bir}(S^{[n]})$ is symplectic if and only if $\sigma^*|_{\text{Tr}(S^{[n]})} = \text{id}$. As a consequence of

Proposition 2.118 any symplectic non-trivial birational automorphism of $S^{[n]}$ is not biregular. We will return to this point in Section 6.3.

Theorem 6.3. *Let S be a projective K3 surface with $\text{Pic}(S) = \mathbb{Z}L$, $L^2 = 2t$ and $n \geq 2$.*

If $t \geq 2$, there exists a non-trivial birational automorphism $\sigma : S^{[n]} \dashrightarrow S^{[n]}$ if and only if $t(n-1)$ is not a square and the minimal solution $(X, Y) = (z, w)$ of Pell's equation $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$ satisfies $w \equiv 0 \pmod{2}$ and $(z, z) \equiv (j, k) \in \frac{\mathbb{Z}}{2(n-1)\mathbb{Z}} \times \frac{\mathbb{Z}}{2t\mathbb{Z}}$ with $(j, k) \in \{(1, 1), (1, -1), (-1, -1)\}$. If so $\text{Bir}(S^{[n]}) = \langle \sigma \rangle \cong \mathbb{Z}/2\mathbb{Z}$ and

- *if $(j, k) = (1, -1)$, then σ is non-symplectic with invariant lattice $T_\sigma \cong \langle 2 \rangle$;*
- *if $(j, k) = (-1, -1)$, then σ is non-symplectic with invariant lattice $T_\sigma \cong \langle 2(n-1) \rangle$;*
- *if $(j, k) = (1, 1)$, then σ is symplectic with coinvariant lattice isomorphic to $\langle -2(n-1) \rangle$, and the invariant lattice contains the transcendental lattice.*

If $t = 1$, let $(X, Y) = (a, b)$ be the integer solution of $(n-1)X^2 - Y^2 = -1$ with smallest $a, b > 0$. If $n-1$ is a square or $b \equiv \pm 1 \pmod{n-1}$, then $\text{Bir}(S^{[n]}) = \text{Aut}(S^{[n]}) \cong \mathbb{Z}/2\mathbb{Z}$. Otherwise $n \geq 9$, $\text{Bir}(S^{[n]}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and $\text{Aut}(S^{[n]}) \cong \mathbb{Z}/2\mathbb{Z}$.

Proof. We start with the case $t \geq 2$. By Proposition 6.1, Lemma 1.6 and the previous discussion, the numerical conditions in the statement of the theorem are necessary for the existence of a birational automorphism of $S^{[n]}$ with non-trivial action on $\text{NS}(S^{[n]})$ (i.e. a non-trivial automorphism, since $t \neq 1$). Assume now that these conditions holds. If $n \neq 2$ the equation $(n-1)X^2 - tY^2 = 1$ has no integer solutions, since $(j, k) \neq (-1, 1)$ (Lemma 1.6). We therefore have $\overline{\text{Mov}}(S^{[n]}) = \langle L_n, zL_n - tw\delta \rangle_{\mathbb{R}_{\geq 0}}$ by Theorem 2.101. Using again Lemma 1.6, we give the following definitions depending on (j, k) .

- (i) If $(j, k) = (1, -1)$, then $(n-1)X^2 - tY^2 = -1$ has integer solutions. Let (a, b) be the solution with smallest $a > 0$ and set $\nu = bL_n - a\delta$. By Lemma 1.6 we have $(z, w) = (2(n-1)a^2 + 1, 2ab)$, since we are assuming $t \geq 2$. As $\nu^2 = 2$, we can define $\phi = -R_\nu \in O(H^2(S^{[n]}, \mathbb{Z}))$. Then $\bar{\phi} = -\text{id} \in O(A_{H^2(S^{[n]}, \mathbb{Z})})$ by Proposition 2.152, Item 1).
- (ii) If $(j, k) = (-1, -1)$, then $X^2 - t(n-1)Y^2 = -1$ has integer solutions. Let (a, b) be the minimal solution and set $\nu = (n-1)bL_n - a\delta$. Then $(z, w) = (2t(n-1)b^2 - 1, 2ab)$. The class ν has square $2(n-1)$ and divisibility $n-1$ (an explicit computation can be found in the proof of [22, Lemma 6.3]), so we can define $\phi = -R_\nu \in O(H^2(S^{[n]}, \mathbb{Z}))$ and $\bar{\phi} = \text{id}$ by Proposition 2.152, Item 2).
- (iii) If $(j, k) = (1, 1)$, let (b, a) be the minimal solution of $X^2 - t(n-1)Y^2 = 1$ and set $\nu = bL_n - ta\delta$. Then $(z, w) = (2t(n-1)a^2 + 1, 2ab)$. We define $\gamma = (n-1)aL_n - b\delta$ and we observe that $\gamma^2 = -2(n-1)$, while the divisibility of γ in $H^2(S^{[n]}, \mathbb{Z})$ is either $n-1$ or $2(n-1)$. This implies that $\phi = R_\gamma \in O(H^2(S^{[n]}, \mathbb{Z}))$ and moreover $\bar{\phi} = -\text{id}$ by Proposition 2.152, Item 2).

Notice that there is no ambiguity in the definitions of ν and ϕ when $n = 2$. By construction, $\phi|_{\text{Tr}(S^{[n]})} = -\text{id}$ in cases (i), (ii) and $\phi|_{\text{Tr}(S^{[n]})} = \text{id}$ in case (iii). Thus ϕ extends to a Hodge isometry

of $H^2(S^{[n]}, \mathbb{C})$. Moreover, ϕ is orientation-preserving, as we will prove using Remark 2.54. Indeed, let $\text{sn}_{\mathbb{R}} = \text{sn}_{\mathbb{R}}^{H^2(S^{[n]}, \mathbb{Z})}$ be the real spinor norm of $H^2(S^{[n]}, \mathbb{Z})$, which we defined in Section 2.3.1. If $\phi = -R_{\nu}$ we have

$$\text{sn}_{\mathbb{R}}(\phi) = \text{sn}_{\mathbb{R}}(-\text{id}) \cdot \text{sn}_{\mathbb{R}}(R_{\nu}) = -\text{sn}_{\mathbb{R}}(R_{\nu}) = \text{sgn}(\nu^2) = +1$$

where we used the fact that $H^2(S^{[n]}, \mathbb{Z})$ has signature $(3, 20)$. If $\phi = R_{\gamma}$, then $\text{sn}_{\mathbb{R}}(\phi) = -\text{sgn}(\gamma^2) = +1$. We conclude, by Proposition 2.51, that $\phi \in \text{Mon}_{\text{Hdg}}^2(S^{[n]})$ (in the case $\phi = R_{\gamma}$, also cfr. [77, Proposition 9.12]).

From the relations between (a, b) and (z, w) that we remarked for each of the three pairs (j, k) , it is immediate to check that $\nu \in \text{Mov}(S^{[n]})$ and that $\phi|_{\text{NS}(S^{[n]})}$ is the isometry (2.8) fixing the line spanned by ν , hence $\phi(\text{Mov}(S^{[n]})) = \text{Mov}(S^{[n]})$. Then, by Corollary 2.109, there exists a (unique) birational automorphism $\sigma \in \text{Bir}(S^{[n]})$ such that $\sigma^* = \phi$.

We are left to discuss the case $t = 1$. For all $n \geq 2$ the group $\text{Aut}(S^{[n]})$ contains the natural involution induced by the generator of $\text{Aut}(S)$. Here, the equation $(n-1)X^2 - tY^2 = -1$ has always integer solutions, while $X^2 - t(n-1)Y^2 = -1$ has no integer solutions if we assume that $(n-1)X^2 - tY^2 = 1$ has none.

Let (z, w) be the minimal solution of $z^2 - (n-1)w^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$. If there exists a non-natural automorphism then $w \equiv 0 \pmod{2}$, which implies $z \equiv 1 \pmod{2t}$. Thus, by the proof of Proposition 6.1, (z, w) is not the minimal solution of the equation, i.e. $b \not\equiv \pm 1 \pmod{n-1}$. On the other hand, if we assume $b \equiv \pm 1 \pmod{n-1}$ then $(n-1)X^2 - Y^2 = 1$ has no integer solutions by Lemma 1.5. The existence of a symplectic automorphism follows then as in the proof for $t \geq 2$, case (iii). By Proposition 2.118, this automorphism is not biregular. \square

If S is a $\langle 2 \rangle$ -polarized $K3$ surface of Picard rank one, the smallest n such that $\text{Bir}(S^{[n]}) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ is $n = 9$.

Remark 6.4. Notice that $t = 1$ is the only value of t such that the isometry (2.8) of $\text{NS}(S^{[n]})$ can be glued to both $+\text{id}, -\text{id} \in O(\text{Tr}(S^{[n]}))$. This follows from Theorem 1.34, since the discriminant group of $\text{Tr}(S^{[n]})$ is $\frac{\mathbb{Z}}{2i\mathbb{Z}}$. If $t = 1$, let $\nu = bL_n - a\delta \in \text{NS}(S^{[n]})$ be as in the proof of Theorem 6.3. Then $(-R_{\nu}) \oplus (-\text{id}) \in O(\text{NS}(S^{[n]}) \oplus \text{Tr}(S^{[n]}))$ extends to $-R_{\nu} \in O(H^2(S^{[n]}, \mathbb{Z}))$, while $(-R_{\nu}) \oplus \text{id}$ extends to R_{γ} , with $\gamma = (n-1)aL_n - b\delta$. If $n-1$ is not a square and $b \not\equiv \pm 1 \pmod{n-1}$, both isometries R_{γ} and $-R_{\nu}$ of the cohomology lattice are Hodge monodromy operators which lift to (distinct) non-natural birational automorphisms of $S^{[n]}$. The lift of R_{γ} is symplectic while the lift of $-R_{\nu}$ is non-symplectic. Each automorphism is obtained by composing the other with $\iota^{[n]}$, where ι is the covering involution which generates $\text{Aut}(S)$. Indeed, $(\iota^{[n]})^*|_{\text{NS}(S^{[n]})} = \text{id}$ and $(\iota^{[n]})^*|_{\text{Tr}(S^{[n]})} = -\text{id}$ (since $\iota^{[n]}$ is natural and $\iota^{[n]} \neq \text{id}$).

Remark 6.5. If for $n \geq 2, t \geq 2$ there exists a non-trivial birational automorphism on $S^{[n]}$, it is biregular if and only if n, t satisfy condition c) of Theorem 2.161, i.e. for all integers ρ, α as follows:

- 1) $\rho = -1$ and $1 \leq \alpha \leq n-1$, or
- 2) $\rho = 0$ and $3 \leq \alpha \leq n-1$, or

3) $1 \leq \rho \leq \frac{n-1}{4}$ and $\max\{4\rho + 1, \lceil 2\sqrt{\rho(n-1)} \rceil\} \leq \alpha \leq n-1$,

the minimal solution (X, Y) of

$$X^2 - (4t(n-1))Y^2 = \alpha^2 - 4\rho(n-1)$$

with $X \equiv \pm\alpha \pmod{2(n-1)}$, if it exists, is such that $\frac{Y}{X} \geq \frac{w}{2z}$.

This is equivalent to ask $\mathcal{A}(S^{[n]}) = \text{Mov}(S^{[n]})$. In particular, if $t \leq 2n-3$ the automorphism is not biregular by Proposition 2.159.

The following proposition provides examples of polarization degrees for the $K3$ surface S so that $S^{[n]}$ has a non-natural birational automorphism, for each of the three different actions on cohomology (see Proposition 6.1).

Proposition 6.6. *Fix $n \geq 2$. For $t \geq 1$, we denote by S a projective $K3$ surface with $\text{Pic}(S^{[n]}) = \mathbb{Z}L$, $L^2 = 2t$.*

- (i) *There exist infinitely many t 's such that $\text{Bir}(S^{[n]})$ contains a non-natural, non-symplectic involution σ with invariant lattice $T_\sigma \cong \langle 2 \rangle$, e.g. $t = (n-1)k^2 + 1$ for $k \geq 1$.*
- (ii) *There exists t such that $\text{Bir}(S^{[n]})$ contains a non-natural, non-symplectic involution σ with invariant lattice $T_\sigma \cong \langle 2(n-1) \rangle$ if and only if -1 is a quadratic residue modulo $n-1$. If so, this happens for infinitely many t 's, e.g. $t = \frac{[(n-1)k+q]^2+1}{n-1} = (n-1)k^2 + 2qk + \frac{q^2+1}{n-1}$ for $k, q \geq 1$ and $q^2 \equiv -1 \pmod{n-1}$.*
- (iii) *There exists t such that $\text{Bir}(S^{[n]})$ contains a non-natural, symplectic involution σ if and only if $n-1 = \frac{q^2-1}{h}$ for some $q \geq 3, h \not\equiv 0 \pmod{q \pm 1}$. If so, this happens for infinitely many t 's, e.g. $t = \frac{[(n-1)k+q]^2-1}{n-1} = (n-1)k^2 + 2qk + h$ for $k \geq 1$. For every t the coinvariant lattice is isomorphic to $\langle -2(n-1) \rangle$ and the invariant lattice contains the transcendental lattice.*

Proof. As before, we denote by (z, w) the minimal solution of $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$.

- (i) If $t = (n-1)k^2 + 1$ and $k \geq 1$, then $t(n-1)$ is not a square and $(z, w) = (2t-1, 2k) = (2(n-1)k^2+1, 2k)$ by Lemma 1.5, since $(k, 1)$ is the solution of $(n-1)X^2 - tY^2 = -1$ with smallest $X > 0$. We conclude with Theorem 6.3.
- (ii) If there exists t such that $\text{Bir}(S^{[n]})$ contains a non-natural, non-symplectic involution σ with $H^2(S^{[n]}, \mathbb{Z})^{\sigma^*} \cong \langle 2(n-1) \rangle$, then by Theorem 6.3 and Lemma 1.6 the equation $X^2 - t(n-1)Y^2 = -1$ is solvable, hence -1 is a quadratic residue modulo $n-1$. On the other hand, let $t = (n-1)k^2 + 2qk + \frac{q^2+1}{n-1}$ for $k, q \geq 1$ and $q^2 \equiv -1 \pmod{n-1}$. The minimal solution of $X^2 - t(n-1)Y^2 = -1$ is $(q + (n-1)k, 1)$, therefore $t(n-1)$ is not a square and $(z, w) = (-1 + 2t(n-1), 2q + 2(n-1)k)$, again by Lemma 1.5. The existence of the involution and its action on cohomology follow then from Theorem 6.3.

(iii) If $\text{Bir}(S^{[n]})$ contains a symplectic involution σ , then by Theorem 6.3 and Lemma 1.6 the minimal solution (u, s) of $X^2 - t(n-1)Y^2 = 1$ has $u \not\equiv \pm 1 \pmod{n-1}$. Hence $n-1 = \frac{u^2-1}{ts^2} \geq 8$ and $u \pm 1 \nmid ts^2$. Let now $t = (n-1)k^2 + 2qk + h$ for k, q, h as in the statement. The minimal solution of $X^2 - t(n-1)Y^2 = 1$ is $(u, s) = (q + (n-1)k, 1)$. Since $h \not\equiv 0 \pmod{q \pm 1}$, we have $q \not\equiv \pm 1 \pmod{n-1}$, therefore $(z, w) = (1 + 2t(n-1), 2q + 2(n-1)k)$, which allows us to conclude by using Theorem 6.3. \square

6.3 Decomposition of the movable cone and automorphisms of birational models

We presented in Definition 2.97 the wall-and-chamber decomposition of the movable cone $\text{Mov}(S^{[n]})$; we refer to Proposition 2.102 for an explicit description of the walls that separates the chambers when $S^{[n]}$ has Picard rank 2. The open chambers are the ample cones of hyperkähler birational models of $S^{[n]}$.

Definition 6.7. For $n \geq 2$, let S be a projective K3 surface with $\text{Pic}(S) = \mathbb{Z}H$, $H^2 = 2t$, $t \geq 1$. The value t is said to be n -irregular if there exists a birational automorphism $\sigma \in \text{Bir}(S^{[n]})$ such that for all hyperkähler birational models $g : S^{[n]} \dashrightarrow X$ the involution $g \circ \sigma \circ g^{-1} \in \text{Bir}(X)$ is not biregular.

Remark 6.8. If the decomposition of $\overline{\text{Mov}(S^{[n]})}$ has an odd number of chambers, and there exists a non-natural automorphism $\sigma \in \text{Bir}(S^{[n]})$, then the generator ν of the axis of the reflection $\sigma^*|_{\text{NS}(S^{[n]})}$ is in the interior of one of the chambers. This follows from the fact that σ^* acts on the set of chambers as in Proposition 2.157, Item d), hence if their number is odd, one of them is preserved by σ^* , and therefore it contains ν . This implies that there exists an hyperkähler birational model (the one corresponding to the preserved chamber) on which the involution becomes biregular, by the Hodge-theoretic Torelli theorem. We already noticed in Section 2.4.5 that $\overline{\text{Mov}(S^{[n]})}$ has only one chamber if and only if $\sigma \in \text{Aut}(S^{[n]})$. On the other hand, if there is an even number of chambers in the decomposition of $\overline{\text{Mov}(S^{[n]})}$, then ν lies on one of the walls in the interior of the cone, and therefore σ is not biregular on any of the hyperkähler birational models of $S^{[n]}$.

This implies that t is n -irregular if and only if $\text{Bir}(S^{[n]})$ contains a non-natural birational automorphism and the wall-and-chamber decomposition of $\text{Mov}(S^{[n]})$ has an even number of chambers.

Proposition 6.9. For $n \geq 2$ and $t \geq 1$, assume that there exists a non-natural birational automorphism $\sigma \in \text{Bir}(S^{[n]})$ and either $t = 1$ or $\sigma^*|_{\text{Tr}(S^{[n]})} = \text{id}$. Then t is n -irregular.

Proof. By the proof of Proposition 6.1, if $t = 1$ or $\sigma^*|_{\text{Tr}(S^{[n]})} = \text{id}$ the axis of the reflection $\sigma^*|_{\text{NS}(S^{[n]})}$ is spanned by the class $\nu = bL_n - ta\delta$, where (b, a) is the minimal solution of $X^2 - t(n-1)Y^2 = 1$. Moreover, $n \geq 9$ and $\overline{\text{Mov}(S^{[n]})} = \langle L_n, zL_n - tw\delta \rangle_{\mathbb{R}_{\geq 0}}$ with $(z, w) = (2t(n-1)a^2 + 1, 2ab)$. Since $(b+1)(b-1) = t(n-1)a^2$, we have $c = \max\{\text{gcd}(n-1, b-1), \text{gcd}(n-1, b+1)\} \geq 2$. We define $\alpha = \max\{4, \frac{2(n-1)}{c}\}$, which is an even integer such that $4 \leq \alpha \leq n-1$. We consider Pell's equation $X^2 - 4t(n-1)Y^2 = \alpha^2$: the pair $(X, Y) = (b\alpha, a\frac{\alpha}{2})$ is a solution with the property $\frac{2Y}{X} = \frac{a}{b}$. We also have $X = b\alpha \equiv \pm\alpha \pmod{2(n-1)}$ by construction, hence ν lies on the wall (in the interior of the movable cone) spanned by the class $XL_n - 2tY\delta$. \square

By the previous proposition, a non-natural birational automorphism of $S^{[n]}$ which acts on $\text{Tr}(S^{[n]})$ as id (or as $\pm \text{id}$ if $t = 1$) is not biregular on any hyperkähler birational model of $S^{[n]}$, which is stronger than what we stated in Remark 6.2. By Proposition 6.6, for n fixed the number of n -irregular values t for which this happens is either zero or infinite.

Example 6.10. For any odd $k \geq 5$ with $k \not\equiv \pm 1 \pmod{8}$, define $t = \frac{k^2-1}{8} \in \mathbb{N}$. We can readily check that $S^{[9]}$ admits a symplectic non-natural involution by Theorem 6.3, hence t is 9-irregular by Proposition 6.9.

For non-symplectic automorphisms when $t \neq 1$, the behaviour is different. We now show that, for a fixed $n \geq 2$, there is only a finite number of n -irregular values t for which the non-natural birational automorphism of $S^{[n]}$ acts as $-\text{id}$ on $\text{Tr}(S^{[n]})$, and we provide an algorithm to compute them.

Proposition 6.11. Let $t \geq 2$ and $n \geq 2$, such that $t(n-1)$ is not a square and $(n-1)X^2 - tY^2 = 1$ has no integer solutions (if $n \neq 2$). Assume that one between $(n-1)X^2 - tY^2 = -1$ and $X^2 - t(n-1)Y^2 = -1$ admits integer solutions; define $\ell = 1$ if the first equation is solvable or $\ell = n-1$ if the second one is, and let (a, b) be the solution with smallest $a > 0$. Then t is n -irregular if and only if there exists a pair (α, ρ) as in (2.2) and a positive solution (X, Y) of Pell's equation $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ with $X \equiv \pm \alpha \pmod{2(n-1)}$ such that

$$4t\ell Y^2 = (\alpha^2 - 4\rho(n-1))a^2.$$

Proof. Let $\nu = \ell bL_n - a\delta \in \text{Mov}(S^{[n]})$ be the class of square 2ℓ as in the proof of Theorem 6.3. Then ν lies on one of the walls in the interior of $\text{Mov}(S^{[n]})$ if and only if $\frac{a}{\ell b} = 2t\frac{Y}{X}$, where (X, Y) is a positive solution of one of the equations $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ with $X \equiv \pm \alpha \pmod{2(n-1)}$ and (α, ρ) as in (2.2). We write the condition on the slopes as:

$$\frac{Y}{X} \cdot \frac{\ell b}{2(n-1)a} = \frac{1}{4t(n-1)}$$

and we observe that $(2(n-1)a)^2 - 4t(n-1)(\ell b)^2 = -4\ell(n-1)$. Hence this becomes:

$$\sqrt{\frac{1}{4t(n-1)} - \frac{\alpha^2 - 4\rho(n-1)}{4t(n-1)X^2}} \sqrt{\frac{1}{4t(n-1)} + \frac{4\ell(n-1)}{4t(n-1)(2(n-1)a)^2}} = \frac{1}{4t(n-1)}.$$

If we rearrange the equation, we obtain:

$$\ell X^2 - (n-1)(\alpha^2 - 4\rho(n-1))a^2 = \ell(\alpha^2 - 4\rho(n-1)) \quad (6.1)$$

i.e. $4t\ell Y^2 = (\alpha^2 - 4\rho(n-1))a^2$. □

Remark 6.12. • Notice that, by (6.1) and $(n-1)a^2 - t(\ell b)^2 = -\ell$, we have $X^2 = (\alpha^2 - 4\rho(n-1))t\ell b^2$. However, $t\ell$ has to divide $\alpha^2 - 4\rho(n-1)$ (because $t\ell$ is coprime with a), hence $\alpha^2 - 4\rho(n-1) = t\ell r^2$ for some $r \in \mathbb{N}$. This gives an easy way to compute a (finite) list of candidates for the n -irregular values t , among the divisors of $\alpha^2 - 4\rho(n-1)$ for (α, ρ) as in (2.2). In particular, $t \leq (n-1)(n+3)$.

- In order to check if a value $t \geq 2$ is n -irregular in Proposition 6.11, it is enough to consider the pairs (α, ρ) such that $\alpha^2 - 4\rho(n-1) > 0$ and the positive solutions (X, Y) with smallest X in each equivalence class of solutions of $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$, otherwise the wall spanned by $XL_n - 2tY\delta$ is not in the interior of the movable cone.

Corollary 6.13. *For a fixed $n \geq 2$, the number of n -irregular values t for which $\text{Bir}(S^{[n]})$ contains a non-symplectic birational automorphism is finite. In particular, for $n \leq 8$ there is only a finite number of n -irregular values t .*

Proof. By Proposition 6.1, assuming $t \geq 2$ a non-natural automorphism $\sigma \in \text{Bir}(S^{[n]})$ satisfies $\sigma^*|_{\text{Tr}(S^{[n]})} = -\text{id}$ if and only if the minimal solution of $X^2 - t(n-1)Y^2 = 1$ has $X \equiv \pm 1 \pmod{n-1}$. In particular, this always holds for $n \leq 8$. Then, by Theorem 6.3, either $(n-1)X^2 - tY^2 = 1$ or $X^2 - t(n-1)Y^2 = 1$ is solvable. The statement follows from Proposition 6.11 and Remark 6.12. \square

In Table 6.1, for $n \leq 14$ we list all n -irregular values t as in Corollary 6.13. We separate the values of t for which the middle wall of the movable cone is spanned by a primitive class ν of square 2 (i.e. $\ell = 1$), from those where the generator has square $2(n-1)$ and divisibility $n-1$ (i.e. $\ell = n-1$). For $n \neq 9, 12$, the values t in the table are all the n -irregular values, 9 and 12 being the only $n \leq 14$ which satisfy Proposition 6.6 (iii). We observe that, for $n \leq 5$, all n -irregular values are of the form $t = n$ or $t = 4n - 3$.

Table 6.1: n -irregular values t for $n \leq 14$ as in Corollary 6.13.

n	n -irregular t 's s.t. $\nu^2 = 2$	n -irregular t 's s.t. $\nu^2 = 2(n-1)$
2		5
3	3, 9	/
4	4, 13	/
5	5, 17	/
6	6, 9, 21	/
7	7, 25, 49	/
8	2, 4, 8, 11, 16, 29, 37	/
9	1, 9, 33, 57	/
10	10, 13, 37, 61, 85	/
11	11, 19, 41, 49, 121	/
12	3, 4, 5, 12, 15, 25, 27, 45, 125	/
13	1, 13, 49	/
14	14, 17, 22, 38, 49, 53, 77, 121, 133	5

Lemma 6.14. *For all $n \geq 3$, the value $t = n$ is n -irregular. For all $n \geq 2$, the value $t = 4n - 3$ is n -irregular.*

Proof. Notice that $t = n$ and $t = 4n - 3$ are values of the form $t = (n-1)k^2 + 1$, for $k = 1, 2$ respectively. By Proposition 6.6 the Hilbert scheme $S^{[n]}$ has a non-natural birational automorphism which acts as $-\text{id}$ on $\text{Tr}(S^{[n]})$. The minimal solution of $(n-1)X^2 - tY^2 = -1$ is $(a, b) = (k, 1)$.

- If $t = n$ and $n \geq 3$, consider the equation $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ for $\rho = -1$ and $\alpha = 2$. Its minimal solution is $(X, Y) = (2n, 1)$, which satisfies $X \equiv \alpha \pmod{2(n-1)}$.
- If $t = 4n - 3$ and $n \geq 2$, consider the equation $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ for $\rho = -1$ and $\alpha = 1$. Its minimal solution is $(X, Y) = (4n - 3, 1)$, which satisfies $X \equiv \alpha \pmod{2(n-1)}$.

In both cases the relation $4tY^2 = (\alpha^2 - 4\rho(n-1))a^2$ holds, hence the statement follows from Proposition 6.11. \square

In the case $t = n$, it is known that the automorphism of $S^{[n]}$ which acts as the reflection in the (only) wall contained in the interior of $\text{Mov}(S^{[n]})$ is Beauville's involution, see Example 2.164, which is biregular if and only if $n = 2$ since $\phi_{|L|}(S)$ never contains lines.

In Proposition 6.6 we showed that for $t = (n-1)k^2 + 1$, $k \geq 1$ the Hilbert scheme $S^{[n]}$ has a non-symplectic birational involution. We observed in Lemma 6.14 that the involution is not biregular for $k = 1, 2$, while by Proposition 2.163 it is biregular (i.e. $\mathcal{A}(S^{[n]}) = \text{Mov}(S^{[n]})$) whenever $k \geq \frac{n+3}{2}$. It seems however that this lower bound on k is far from optimal.

Conjecture 6.15. *Let $n \geq 2$ and $t = (n-1)k^2 + 1$, for $k \in \mathbb{N}$. Let S be an algebraic K3 surface with $\text{Pic}(S) = \mathbb{Z}L$, $L^2 = 2t$. If $k \geq 3$, then the non-natural non-symplectic involution which generates $\text{Bir}(S^{[n]})$ is biregular.*

Al. Cattaneo checked computationally that the conjecture holds for all $n \leq 14$.

Proposition 2.163 can be adapted also to the other family of polarizations in Proposition 6.6 giving rise to non-symplectic birational involutions.

Lemma 6.16. *Let $n \geq 3$ and $q \in \mathbb{N}$ be such that $q^2 \equiv -1 \pmod{n-1}$. Let $t = (n-1)k^2 + 2qk + \frac{q^2+1}{n-1}$ for $k \geq 1$ and S an algebraic K3 surface with $\text{Pic}(S) = \mathbb{Z}L$, $L^2 = 2t$. If $k \geq \frac{n+3}{2}$, then the non-natural non-symplectic involution which generates $\text{Bir}(S^{[n]})$ is biregular.*

Proof. By Proposition 6.6 we just need to prove that $\mathcal{A}(S^{[n]}) = \text{Mov}(S^{[n]})$. Define $Q = (n-1)k + q$. For $k \geq \frac{n+3}{2}$ we have $4t(n-1) > (n+3)^2(n-1)^2 \geq (\alpha^2 - 4\rho(n-1))^2$ for all (α, ρ) as in (2.2), hence the solutions of $X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$ are encoded in the convergents of the continued fraction of $\sqrt{4t(n-1)} = 2\sqrt{Q^2 + 1} = [2Q; \overline{Q, 4Q}]$, see Corollary 1.8. FARE: CHIUDI QUI MAGARI The proof then follows as for [22, Proposition 6.7]. \square

6.4 Kähler-type chambers and automorphisms for $n = 3$

It is known that for a K3 surface S of Picard rank one the number of Kähler-type chambers in the decomposition of $\overline{\text{Mov}}(S^{[2]})$ is $d \in \{1, 2, 3\}$ (see for instance [25, Example 3.18]). We now detail the computation of the number of chambers for $n = 3$ in the cases where $\overline{\text{Mov}}(S^{[3]}) = \langle L_3, zL_3 - tw\delta \rangle_{\mathbb{R}_{\geq 0}}$, with (z, w) the minimal solution of $X^2 - 2tY^2 = 1$; note that $z \equiv \pm 1 \pmod{2}$ always holds. By Proposition 2.157, these are necessary conditions to have $\text{Bir}(S^{[3]}) \neq \{\text{id}\}$.

For an integer $t \geq 2$ such that $2t$ is not a square, consider the following two generalized Pell's equations:

$$\begin{aligned} P_{8t}(9) : X^2 - 8tY^2 &= 9 \\ P_{8t}(12) : X^2 - 8tY^2 &= 12. \end{aligned}$$

Remark 6.17. • If we write the two equations as $X^2 - 8tY^2 = 8 + \alpha^2$, with $\alpha \in \{1, 2\}$, then it is clear that all solutions (X, Y) satisfy $X \equiv \pm\alpha \pmod{4}$.

- If (X, Y) is a solution of $P_{8t}(9)$ and (X', Y') is a solution of $P_{8t}(12)$, then $\frac{Y}{X} \neq \frac{Y'}{X'}$ since $\sqrt{\frac{12}{9}}$ is not rational.
- Assume that $P_{8t}(12)$ is solvable and let (X, Y) be a solution. Then $X = 2Z$ for some $Z \in \mathbb{Z}$ such that (Z, Y) is a solution of $P_{2t}(3) : Z^2 - 2tY^2 = 3$. Notice that $Z \equiv 1 \pmod{2}$, $Y \equiv 1 \pmod{2}$ and $t \equiv 1 \pmod{2}$, since 3 is not a quadratic residue modulo 4. By [86, Theorem 110], if $P_{2t}(3)$ is solvable it has one equivalence class of solutions if $t \equiv 0 \pmod{3}$, otherwise it has two classes of solutions, which are conjugate. Then one can easily see that the same holds for $P_{8t}(12)$.

The next two lemmas give bounds for the number of equivalence classes of solutions of $P_{8t}(9)$ and $P_{8t}(12)$, depending on t .

Lemma 6.18. *If $t \equiv 3 \pmod{18}$, then $P_{8t}(12)$ is either not solvable or it has one class of solutions. If $t \equiv 5, 11, 17 \pmod{18}$, then $P_{8t}(12)$ is either not solvable or it has two classes of solutions. In all other cases, $P_{8t}(12)$ is not solvable.*

Proof. By Remark 6.17 the equation $P_{8t}(12)$, if it is solvable, has one class of solutions when $t \equiv 0 \pmod{3}$, two classes otherwise. We also remarked that solutions (X, Y) of $P_{8t}(12)$ are of the form $(2Z, Y)$ for (Z, Y) solution of $P_{2t}(3)$, and that $P_{8t}(12)$ is not solvable if $t \equiv 0 \pmod{2}$.

If $t \equiv 1 \pmod{3}$, write $t = 3q + 1$ for $q \in \mathbb{N}_0$ and assume that (Z, Y) is a solution of $P_{2t}(3)$. Then $Z^2 - (6q + 2)Y^2 = 3$, i.e. $Z^2 \equiv 2Y^2 \pmod{3}$. This implies that $Z \equiv Y \equiv 0 \pmod{3}$, which gives a contradiction. Hence, $P_{8t}(12)$ has no solutions.

If $t \equiv 0 \pmod{9}$, write $t = 9q$ for $q \in \mathbb{N}$. If (X, Y) is a solution of $P_{8t}(12)$, then $X = 6X'$ for some $X' \in \mathbb{N}$ and $3(X')^2 - 6qY^2 = 1$, which is impossible. Hence, $P_{8t}(12)$ has no solutions.

If $t \equiv 6 \pmod{9}$, write $t = 9q + 6$ for $q \in \mathbb{N}_0$. If (X, Y) is a solution of $P_{8t}(12)$, then $X = 3X'$ for some $X' \in \mathbb{N}$ and $3(X')^2 - 8(3q + 2)Y^2 = 4$. This implies $Y^2 \equiv -1 \pmod{3}$, which cannot be. Hence, $P_{8t}(12)$ has no solutions. \square

Lemma 6.19. *If $t \equiv 1 \pmod{3}$ or $t \equiv 3, 6 \pmod{9}$, then all solutions of $P_{8t}(9)$ are equivalent to $(X, Y) = (3, 0)$. If $t \equiv 0 \pmod{9}$, then $P_{8t}(9)$ has either one, two or three classes of solutions. If $t \equiv 2 \pmod{3}$, then $P_{8t}(9)$ has either one or three classes of solutions.*

Proof. If $t = 3q + 1$, for $q \in \mathbb{N}_0$, then $X^2 \equiv 2Y^2 \pmod{3}$, hence all solutions (X, Y) are of the form $(3X', 3Y')$, with $(X')^2 - 8t(Y')^2 = 1$. This is now a standard Pell's equation, which has only one class of solutions, thus the same holds for $P_{8t}(9)$.

If $t = 3q$ for $q \in \mathbb{N}$ and $(q, 3) = 1$, then $X = 3X'$ for some $X' \in \mathbb{Z}$ such that $3(X')^2 - 8qY^2 = 3$. Since $3 \nmid q$ we need $(X, Y) = (3X', 3Y')$ with $(X')^2 - 24q(Y')^2 = 1$. There exists only one class of solutions (X', Y') for this standard Pell's equation, thus also the solutions of $P_{8t}(9)$ form a single class.

Assume now that $t = 9q$ for $q \in \mathbb{N}$. Then a solution (X, Y) of $P_{8t}(9)$ is of the form $(X, Y) = (3X', Y)$ with $(X')^2 - 8qY^2 = 1$. Let (z, w) be the minimal solution of $P_{8t}(1) : z^2 - 8tw^2 = 1$. Notice that the solutions of $P_{8t}(1)$ are the pairs $(X', \frac{Y}{3})$ for all solutions (X', Y) of $(X')^2 - 8qY^2 = 1$ such that $Y \equiv 0 \pmod{3}$. Let (a, b) be the minimal solution of $(X')^2 - 8qY^2 = 1$. Since this is a standard Pell's equation, by (1.2) its next two solutions (for increasing values of X') are $(a^2 + 8qb^2, 2ab)$ and $(a^3 + 24qab^2, 8qb^3 + 3a^2b)$. We observe that one among these first three positive solutions (X', Y) has $Y \equiv 0 \pmod{3}$, since either a^2 (hence, a) or $8qb^2 = a^2 - 1$ is divisible by 3. The positive solution (X', Y) with this property and smallest X' is therefore equal to $(z, 3w)$, thus the corresponding solution $(X, Y) = (3X', Y)$ of $P_{8t}(9)$ satisfies $\frac{Y}{X} = \frac{w}{z}$, i.e. it is the first positive solution in the same equivalence class of $(3, 0)$. We conclude that $P_{8t}(9)$ has either one, two or three classes of solutions.

Finally, assume that $t \equiv 2 \pmod{3}$. Let (z, w) be the minimal solution of $P_{8t}(1)$ and let $(u_1, v_1), (u_2, v_2)$ be two positive solutions of $P_{8t}(9)$ such that, for $i = 1, 2$:

$$0 < u_i \leq 3\sqrt{\frac{z+1}{2}}, \quad 0 < v_i \leq \frac{3w}{\sqrt{2(z+1)}}. \quad (6.2)$$

By (1.3), this is equivalent to ask that either (u_i, v_i) or $(-u_i, v_i)$ is a fundamental solution of $P_{8t}(9)$, different from $(3, 0)$. Thus u_1, v_1, u_2, v_2 are not divisible by three. From $u_1^2 - 8tv_1^2 = 9$ and $u_2^2 - 8tv_2^2 = 9$ we get

$$u_1v_2 \equiv \pm u_2v_1 \pmod{9}, \quad u_1u_2 \equiv \pm 8tv_1v_2 \pmod{9} \quad (6.3)$$

where the signs in the two congruences coincide. If we now multiply $u_1^2 - 8tv_1^2 = 9$ and $u_2^2 - 8tv_2^2 = 9$ member by member, we obtain

$$\left(\frac{u_1u_2 \mp 8tv_1v_2}{9}\right)^2 - 8t\left(\frac{u_1v_2 \mp u_2v_1}{9}\right)^2 = 1 \quad (6.4)$$

where by (6.3) the two squares in the LHS term are integers. If we assume that (u_1, v_1) and (u_2, v_2) are distinct, then $u_1v_2 \mp u_2v_1 \neq 0$. Since (z, w) is the minimal solution of $P_{8t}(1)$, from (6.4) we have $|u_1v_2 \mp u_2v_1| \geq 9w$. However, from (6.2) we compute $|u_1v_2 \mp u_2v_1| < 9w$, which is a contradiction. We conclude $u_1 = u_2, v_1 = v_2$. Thus, there are at most three classes of solutions for $P_{8t}(9)$: the class of $(3, 0)$ and possibly the classes of (u_1, v_1) and $(-u_1, v_1)$. Notice that the latter two classes are always distinct: in order for them to coincide we would need

$$\frac{u_1^2 + 8tv_1^2}{9} \in \mathbb{Z}, \quad \frac{2u_1v_1}{9} \in \mathbb{Z}$$

which does not happen, since $(u_1, 3) = (v_1, 3) = 1$. Hence, the equation has either one or three classes of solutions. \square

We remark that all cases in the statements of Lemma 6.18 and Lemma 6.19 occur, for suitable values of t .

Let S be a projective $K3$ surface such that $\text{Pic}(S) = \mathbb{Z}L$, $L^2 = 2t$, $t \geq 2$. In the case $n = 3$, Proposition 2.157, Items $a) - c)$ reads: if $\text{Bir}(S^{[3]}) \neq \{\text{id}\}$ then

- $2t$ is not a square, $2X^2 - tY^2 = 1$ has no integer solutions;
- the minimal solution of $X^2 - 2tY^2 = 1$ has $Y \equiv 0 \pmod{2}$. We call (z, w) the corresponding minimal solution of $P_{8t}(1) : z^2 - 8tw^2 = 1$;
- $\overline{\text{Mov}(S^{[3]})} = \langle L_3, zL_3 - 2tw\delta \rangle_{\mathbb{R}_{\geq 0}}$, with (z, w) as above.

By Proposition 2.102 for $n = 3$, the walls in the interior of the movable cone are the rays through $XL_3 - 2tY\delta$, for (X, Y) positive solution of $X^2 - 8tY^2 = 8 + \alpha^2$ such that $\alpha \in \{1, 2\}$, $X \equiv \pm\alpha \pmod{4}$ and $0 < \frac{Y}{X} < \frac{w}{z}$. By Remark 6.17, this implies that the number of chambers in the interior of the cone coincides with the number of combined equivalence classes of solutions for $P_{8t}(9)$ and $P_{8t}(12)$ (the class of the solution $(3, 0)$ of $P_{8t}(9)$ determines the two extremal rays of the movable cone). Hence, Lemma 6.18 and Lemma 6.19 give the following result.

Proposition 6.20. *Let $t \geq 2$ such that $2t$ is not a square, $2X^2 - tY^2 = 1$ has no integer solutions and the minimal solution of $X^2 - 2tY^2 = 1$ has $Y \equiv 0 \pmod{2}$. The following table provides the numbers of classes of solutions for the equations $P_{8t}(9)$ and $P_{8t}(12)$ and the number of chambers in the movable cone of $S^{[3]}$, for S a projective $K3$ surface such that $\text{Pic}(S) = \mathbb{Z}L$, $L^2 = 2t$.*

$t \pmod{18}$	0	1	2	3	4	5	6	7	8
# eq. classes $P_{8t}(9)$	1, 2, 3	1	1, 3	1	1	1, 3	1	1	1, 3
# eq. classes $P_{8t}(12)$	0	0	0	0, 1	0	0, 2	0	0	0
# chambers	1, 2, 3	1	1, 3	1, 2	1	1, 3, 5	1	1	1, 3

$t \pmod{18}$	9	10	11	12	13	14	15	16	17
# eq. classes $P_{8t}(9)$	1, 2, 3	1	1, 3	1	1	1, 3	1	1	1, 3
# eq. classes $P_{8t}(12)$	0	0	0, 2	0	0	0	0	0	0, 2
# chambers	1, 2, 3	1	1, 3, 5	1	1	1, 3	1	1	1, 3, 5

In all cases where $\text{Mov}(S^{[3]})$ has no walls in its interior (i.e. there is only one chamber), then any birational automorphism is biregular by the Hodge-theoretic Torelli theorem .

Corollary 6.21. *Let S be a projective $K3$ surface such that $\text{Pic}(S) = \mathbb{Z}L$, $L^2 = 2t$, $t \geq 1$. If $t \equiv 1, 4, 6, 7, 10, 12, 13, 15, 16 \pmod{18}$, then $\text{Bir}(S^{[3]}) = \text{Aut}(S^{[3]})$.*

From Theorem 6.3, Corollary 6.13 and the results of this section we conclude the following.

Proposition 6.22. *Let S be a projective $K3$ surface such that $\text{Pic}(S) = \mathbb{Z}L$, $L^2 = 2t$. If $t = 1$, then $\text{Bir}(S^{[3]}) = \text{Aut}(S^{[3]}) \cong \mathbb{Z}/2\mathbb{Z}$. If $t \geq 2$, then $\text{Bir}(S^{[3]}) \neq \{\text{id}\}$ if and only if:*

- $2t$ is not a square;

- $2X^2 - tY^2 = 1$ has no integer solutions;
- either $2X^2 - tY^2 = -1$ or $X^2 - 2tY^2 = -1$ has integer solutions.

If $\text{Bir}(S^{[3]}) \neq \{\text{id}\}$, let d be the number of chambers in the decomposition of $\overline{\text{Mov}(S^{[3]})}$. Then $d \in \{1, 2, 3, 5\}$ and one of the following holds:

- $d = 1$ and $\text{Bir}(S^{[3]}) = \text{Aut}(S^{[3]}) \cong \mathbb{Z}/2\mathbb{Z}$;
- $d = 2$, $t = 3$ or $t = 9$, $\text{Aut}(S^{[3]}) = \{\text{id}\}$ and $\text{Bir}(S^{[3]}) \cong \mathbb{Z}/2\mathbb{Z}$;
- $d = 3, 5$, $\text{Aut}(S^{[3]}) = \{\text{id}\}$ and $\text{Bir}(S^{[3]}) \cong \mathbb{Z}/2\mathbb{Z}$

If $t \neq 3, 9$ and $\sigma \in \text{Bir}(S^{[3]})$, there exists an hyperkähler sixfold X and a birational map $g : S^{[3]} \dashrightarrow X$ such that $g \circ \sigma \circ g^{-1} \in \text{Aut}(X)$.

Table 6.2 lists the number d of chambers in the decomposition of $\overline{\text{Mov}(S^{[3]})}$ and the structure of the groups $\text{Aut}(S^{[3]})$, $\text{Bir}(S^{[3]})$, for a projective $K3$ surface S with $\text{Pic}(S) = \mathbb{Z}L$, $L^2 = 2t$, $1 \leq t \leq 30$ when $\text{Bir}(S^{[3]}) \neq \{\text{id}\}$.

Table 6.2: Chambers and automorphisms for $n = 3$.

t	d	$\text{Aut}(S^{[3]})$	$\text{Bir}(S^{[3]})$
1	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
3	2	$\{\text{id}\}$	$\mathbb{Z}/2\mathbb{Z}$
5	3	$\{\text{id}\}$	$\mathbb{Z}/2\mathbb{Z}$
9	2	$\{\text{id}\}$	$\mathbb{Z}/2\mathbb{Z}$
11	5	$\{\text{id}\}$	$\mathbb{Z}/2\mathbb{Z}$
13	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
19	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
25	1	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$
27	3	$\{\text{id}\}$	$\mathbb{Z}/2\mathbb{Z}$
29	3	$\{\text{id}\}$	$\mathbb{Z}/2\mathbb{Z}$

6.5 Ambiguous Hilbert schemes and birational models

Remark 6.23. This section, as the whole chapter, is part of a joint work with Al. Cattaneo, presented in [11]. Although Fourier-Mukai (FM) partners are a fundamental tool for the results in this section, we decided to not present them in this thesis, as we need them only here. We refer for example to [61] for a treatment of this topic.

Having classified the group of birational automorphisms of $S^{[n]}$, for a $K3$ surface S with Picard rank one, we explain in this section how the same approach can be used to study the more general problem of whether there exists a $K3$ surface Σ , again of Picard rank one, and a birational map $\phi : S^{[n]} \dashrightarrow \Sigma^{[n]}$ which do not come from an isomorphism $S \rightarrow \Sigma$. This is related to the notion of

(strong) ambiguity for Hilbert schemes of points on $K3$ surfaces, which for $n = 2$ (and ϕ biregular) has been investigated in [32] and [116]. Some of the results of this section overlap (even though they are proved differently) with those of [78], where birationality of derived equivalent Hilbert schemes of $K3$ surfaces is studied.

It is known that, if $S^{[n]}$ and $\Sigma^{[n]}$ are birational, then S and Σ are Fourier–Mukai partners and $S^{[n]}$ and $\Sigma^{[n]}$ are derived equivalent (see [106, Proposition 10]). If $\text{Pic}(S) = \mathbb{Z}H$ with $H^2 = 2t$, $t \geq 1$, then by [103, Proposition 1.10] the number of non-isomorphic FM partners of S is $2^{\rho(t)-1}$, where $\rho(t)$ denotes the number of prime divisors of t (and $\rho(1) = 1$).

We can classify the Fourier–Mukai partners Σ of S as follows (for details see [103, Section 4]). The overlattice $L = H^2(\Sigma, \mathbb{Z})$ of $\text{Tr}(S) \oplus \text{NS}(S)$ (with integral Hodge structure defined by setting $L^{2,0} = \text{Tr}(S)^{2,0}$) corresponds to an isotropic subgroup $I_L \subset A_{\text{Tr}(S)} \oplus A_{\text{NS}(S)} = \frac{\mathbb{Z}}{2t\mathbb{Z}}(-\frac{1}{2t}) \oplus \frac{\mathbb{Z}}{2t\mathbb{Z}}(\frac{1}{2t})$ (as in Proposition 1.30, for more details see [90, Section 1.4]). Since $\text{NS}(S)$ and $\text{Tr}(S)$ are primitive in L , the group I_L is of the form $I_L = I_a = \langle \epsilon + a\eta \rangle$ for some $a \in (\mathbb{Z}/2t\mathbb{Z})^\times$, $a^2 \equiv 1 \pmod{4t}$, where ϵ (respectively, η) is a generator of $A_{\text{Tr}(S)}$ (respectively, $A_{\text{NS}(S)}$) on which the finite quadratic form takes value $-\frac{1}{2t}$ (respectively, $+\frac{1}{2t}$). For each $a \in (\mathbb{Z}/2t\mathbb{Z})^\times$, $a^2 \equiv 1 \pmod{4t}$, there exists a $K3$ surface Σ_a (unique up to isomorphism) and an Hodge isometry $H^2(\Sigma_a, \mathbb{Z}) \xrightarrow{\sim} L_a$, where the latter is of $\text{Tr}(S) \oplus \text{NS}(S)$ defined by I_a . Moreover, $\Sigma_a \cong \Sigma_b$ if and only if $b \equiv \pm a \pmod{2t}$. Indeed, $2^{\rho(t)-1}$ (which is the number of non-isomorphic FM partners of S) is the cardinality of $\left\{ a \in (\mathbb{Z}/2t\mathbb{Z})^\times, a^2 \equiv 1 \pmod{4t} \right\} / \pm \text{id}$.

Remark 6.24. If there exists a birational map $\phi : S^{[n]} \dashrightarrow \Sigma^{[n]}$ as above, then $\phi^*(\text{Mov}(\Sigma^{[n]})) = \text{Mov}(S^{[n]})$, see Remark 2.96. Let $\{h_S, \delta_S\}$ and $\{h_\Sigma, \delta_\Sigma\}$ be the canonical bases for $\text{NS}(S^{[n]})$ and $\text{NS}(\Sigma^{[n]})$ respectively (see Section 2.4.5). Then by Theorem 2.110 the map ϕ is induced by an isomorphism of the underlying $K3$ surfaces if and only if $\phi^*(\delta_\Sigma) = \delta_S$. As a consequence, if ϕ does not come from an isomorphism $S \rightarrow \Sigma$ then $\phi^*(h_\Sigma)$ is the primitive generator of the extremal ray of $\text{Mov}(S^{[n]})$ not spanned by h_S . This implies that Σ is uniquely defined, up to isomorphism: in fact, if we also have $\phi' : S^{[n]} \dashrightarrow (\Sigma')^{[n]}$ which does not come from an isomorphism $S \rightarrow \Sigma'$, then $\phi' \circ \phi^{-1} : \Sigma^{[n]} \dashrightarrow (\Sigma')^{[n]}$ is induced by an isomorphism $\Sigma \rightarrow \Sigma'$, since its pullback maps $h_{\Sigma'}$ to h_Σ (hence, also $\delta_{\Sigma'}$ to δ_Σ).

Theorem 6.25. *Let S be a projective $K3$ surface such that $\text{Pic}(S) = \mathbb{Z}L$, with $L^2 = 2t$, $t \geq 1$. For $n \geq 2$, let (z, w) be the minimal solution of $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$. There exists a $K3$ surface Σ and a birational map $\phi : S^{[n]} \dashrightarrow \Sigma^{[n]}$ which is not induced by an isomorphism $S \rightarrow \Sigma$ if and only if:*

- $t(n-1)$ is not a square;
- if $n \neq 2$, $(n-1)X^2 - tY^2 = 1$ has no integer solutions;
- $z \equiv \pm 1 \pmod{2(n-1)}$ and $w \equiv 0 \pmod{2}$.

If so, the $K3$ surfaces S and Σ are isomorphic if and only if $z \equiv \pm 1 \pmod{2t}$. Moreover, ϕ is biregular if and only if, for all integers ρ, α as follows:

- $\rho = -1$ and $1 \leq \alpha \leq n-1$, or

- $\rho = 0$ and $3 \leq \alpha \leq n - 1$, or
- $1 \leq \rho < \frac{n-1}{4}$ and $\max \left\{ 4\rho + 1, \left\lceil 2\sqrt{\rho(n-1)} \right\rceil \right\} \leq \alpha \leq n - 1$

if Pell's equation

$$X^2 - 4t(n-1)Y^2 = \alpha^2 - 4\rho(n-1)$$

is solvable, the minimal solution (X, Y) with $X \equiv \pm\alpha \pmod{2(n-1)}$ satisfies $\frac{Y}{X} \geq \frac{w}{2z}$.

Proof. If there exists a birational map $\phi : S^{[n]} \dashrightarrow \Sigma^{[n]}$ which does not come from an isomorphism $S \rightarrow \Sigma$ then both extremal rays of $\text{Mov}(S^{[n]})$ correspond to Hilbert–Chow contractions (Remark 6.24). By [5, Theorem 5.7], $t(n-1)$ is not a square, $(n-1)X^2 - tY^2 = 1$ has no integer solutions (if $n \neq 2$) and $z \equiv \pm 1 \pmod{2(n-1)}$, $w \equiv 0 \pmod{2}$. On the other hand, assume that these conditions are satisfied. Let $a \in (\mathbb{Z}/2t\mathbb{Z})^\times$, $a^2 \equiv 1 \pmod{4t}$, such that $S \cong \Sigma_a$. Since $z^2 - t(n-1)w^2 = 1$ and w is even, we can consider the FM partner Σ_{za} of S . As $\text{Tr}(S^{[n]}) \cong \text{Tr}(S)$ and $\text{NS}(S^{[n]}) \cong \text{NS}(S) \oplus \langle -2(n-1) \rangle$, the groups I_a and I_{za} (defined at the beginning of the section) can also be seen as isotropic subgroups of $A_{\text{Tr}(S^{[n]})} \oplus A_{\text{NS}(S^{[n]})}$ (in particular, we assume $2t\eta = h \in \text{NS}(S^{[n]})$). It is then immediate to check that the overlattices of $\text{Tr}(S^{[n]}) \oplus \text{NS}(S^{[n]})$ defined by these two subgroups are $H^2(\Sigma_a^{[n]}, \mathbb{Z})$ and $H^2(\Sigma_{za}^{[n]}, \mathbb{Z})$ respectively. Let $\mu \in O(\text{NS}(S^{[n]}))$ be the isometry (2.8). We have that $\text{id} \oplus \mu \in O(\text{Tr}(S^{[n]}) \oplus \text{NS}(S^{[n]}))$ extends to a Hodge isometry $\psi : H^2(\Sigma_a^{[n]}, \mathbb{Z}) \rightarrow H^2(\Sigma_{za}^{[n]}, \mathbb{Z})$, because $\bar{\mu}(a\eta) = za\eta \in A_{\text{NS}(S^{[n]})}$ (here we use the fact that w is even). Notice that the discriminant group of $H^2(S^{[n]}, \mathbb{Z})$ is generated by the class of $\frac{\delta}{2(n-1)}$ and $\bar{\mu} \left(\frac{\delta}{2(n-1)} \right) = -\frac{(n-1)wh}{2(n-1)} - \frac{z\delta}{2(n-1)} = \pm \frac{\delta}{2(n-1)}$. By Theorem 1.34, ψ extends to a Hodge isometry $H^*(\Sigma_a, \mathbb{Z}) \rightarrow H^*(\Sigma_{za}, \mathbb{Z})$ between the Mukai lattices of the two K3 surfaces Σ_a, Σ_{za} . We conclude that $\Sigma_a^{[n]}$ and $\Sigma_{za}^{[n]}$ are birationally equivalent, by Corollary 2.109.

As stated before, Σ_a and Σ_{za} are isomorphic if and only if $a \equiv \pm za \pmod{2t}$, i.e. $z \equiv \pm 1 \pmod{2t}$. The isomorphism $\phi : S^{[n]} \dashrightarrow \Sigma^{[n]}$ is biregular if and only if the Hodge isometry $\phi^* : H^2(\Sigma^{[n]}, \mathbb{Z}) \rightarrow H^2(S^{[n]}, \mathbb{Z})$ maps ample classes to ample classes, by Corollary 2.109. Since ϕ^* permutes the two extremal rays of the movable cones, this is equivalent to asking that there is only one chamber in the decomposition of $\overline{\text{Mov}}(S^{[n]})$ i.e. that the movable cone of $S^{[n]}$ coincides with the ample cone. The last part of the statement follows then as in the proof of [22, Theorem 6.4]. \square

Remark 6.26. If $S \cong \Sigma$ the existence of a birational map $S^{[n]} \dashrightarrow \Sigma^{[n]}$ which does not come from an isomorphism of the K3 surfaces is equivalent to the existence of a non-natural birational automorphism in $\text{Bir}(S^{[n]})$. Indeed, in this case Theorem 6.25 gives the same conditions of Theorem 6.3.

Remark 6.27. It can be readily checked that the conditions in the first part of Theorem 6.25 are equivalent to those of [78, Theorem 2.2]. Write $z = 2(n-1)k + \epsilon$ and $w = 2h$ for $k, h \in \mathbb{N}$ and $\epsilon \in \{\pm 1\}$. Then, $z^2 - t(n-1)w^2 = 1$ implies $k(k(n-1) + \epsilon) = th^2$. Since the two factors in the LHS term are coprime, there exist $p, q, r, s \in \mathbb{N}$ such that $k = sp^2$, $k(n-1) + \epsilon = rq^2$, $h = pq$, $t = rs$. In particular, $(n-1)sp^2 - rq^2 = \pm 1$, which is what is requested in [78, Theorem 2.2]. The FM partner Σ of S such that there exists a birational map $S^{[n]} \dashrightarrow \Sigma^{[n]}$ not coming from an

isomorphism $S \rightarrow \Sigma$ is the moduli space of sheaves $M_S(p^2s, pqH, q^2r) \cong M_S(s, H, r)$. Notice that $z \equiv \pm 1 \pmod{2t}$ (i.e. $\Sigma \cong S$) if and only if $\{r, s\} = \{1, t\}$.

Corollary 6.28. *Let S be a projective K3 surface with $\text{Pic}(S) = \mathbb{Z}L$, $L^2 = 2t$ and $n \geq 2$ an integer such that $S^{[n]}$ admits an involution which is not biregular. Assume that t is not n -irregular, hence there exists an hyperkähler birational model X of $S^{[n]}$ with a biregular involution. Then X is not isomorphic to the Hilbert scheme of n points on a K3 surface.*

Proof. Let (z, w) be the minimal solution of $X^2 - t(n-1)Y^2 = 1$ with $z \equiv \pm 1 \pmod{n-1}$. By contradiction, assume that $X \cong \Sigma^{[n]}$ for some K3 surface Σ . The groups $\text{Aut}(\Sigma^{[n]})$ and $\text{Aut}(S^{[n]})$ are different so S is not isomorphic to Σ , while $\Sigma^{[n]}$ and $S^{[n]}$ are birational. So $z \not\equiv \pm 1 \pmod{2t}$ by Theorem 6.25, which is in contrast with Theorem 6.3. \square

By Proposition 2.159, Corollary 6.28 is applicable for the values $t \leq 2n-3$ which satisfy Theorem 6.3 and which are not n -irregular (see Table 6.1), e.g. $(n, t) = (6, 2)$.

6.6 Geometrical constructions

Let S be a $\langle 2t \rangle$ -polarized K3 surface of Picard rank one and $n \geq 2$. Theorem 6.3 allows us to determine the values n, t for which $\text{Bir}(S^{[n]}) \neq \{\text{id}\}$, however it does not provide any indication on how to construct these automorphisms geometrically. If $t = n$, then $\text{Bir}(S^{[n]})$ is generated by Beauville's (non-symplectic) involution, see Example 2.164. For $2 \leq n \leq 10$, $2 \leq t \leq 7$, $t \neq n$, the pairs (n, t) such that $\text{Bir}(S^{[n]}) \neq \{\text{id}\}$ are the following:

$$(n, t) = (2, 5), (3, 5), (4, 7), (6, 2), (8, 2), (8, 4), (9, 3), (9, 5).$$

The involution which generates $\text{Bir}(S^{[n]})$ is symplectic for $(n, t) = (9, 3), (9, 5)$, non-symplectic in the other cases.

The general K3 surface S of degree $2t = 10$ is a Brill-Noether general K3 surface, which we described in Theorem 2.34; the birational involution of $S^{[2]}$ is O'Grady's involution, see Example 2.165, while a geometric description for the involution of $S^{[3]}$ has been provided in [25, Example 4.12].

We give new constructions of the non-symplectic involutions for $(n, t) = (6, 2), (8, 2)$. Let \mathcal{K}_4 be the moduli space of $\langle 4 \rangle$ -polarized K3 surfaces. Consider $(S, L) \in \mathcal{K}_4$ such that $(S, L) \notin D_{h,0}$ with $h = 1, 2$: by Lemma 2.82 the line bundle L is very ample and S can be embedded as a smooth quartic surface in \mathbb{P}^3 . In the following we will consider S as a smooth quartic surface in \mathbb{P}^3 , with the polarization $L \in \text{Pic}(S)$ induced by the class of $\mathcal{O}_{\mathbb{P}^3}(1)$.

Example 6.29 ($n = 6, t = 2$). *Let $S \subset \mathbb{P}^3$ be a smooth quartic surface which does not contain any twisted cubic curve, which holds everytime $(S, L) \notin D_{3,-2}$. If $p_1, \dots, p_6 \in S$ are in general linear position there exists a single rational normal curve (i.e. a twisted cubic) passing through them, which we denote by C_3 . Since C_3 does not lie on S , the intersection $S \cap C_3$ consists of twelve points. By associating the p_i 's with the six residual points of intersection, we obtain a birational involution of $S^{[6]}$.*

For $n = 6, t = 2$ the equation $X^2 - t(n - 1)Y^2 = -1$ is solvable, with minimal solution $(a, b) = (3, 1)$. Thus for $(S, H) \in \mathcal{K}_4$ with $\text{Pic}(S) = \mathbb{Z}L$, the birational non-symplectic involution σ which generates $\text{Bir}(S^{[6]})$ satisfies $H^2(S^{[6]}, \mathbb{Z})^{\sigma^*} = \mathbb{Z}\nu \cong \langle 10 \rangle$ with $\nu = 5h - 3\delta$. By Proposition 2.159 the involution is not biregular.

Example 6.30 ($n = 8, t = 2$). Let $S \subset \mathbb{P}^3$ be a smooth quartic surface such that, if a curve $C \subset \mathbb{P}^3$ of degree four is contained in two different quadric surfaces, then $C \not\subset S$. This holds for instance outside the divisors $D_{4,0}, D_{4,2}$ in \mathcal{K}_4 (notice that we do not have to exclude $D_{4,-2}$, since a rational quartic curve in \mathbb{P}^3 is contained in a unique quadric surface, cfr. [58, Exercice 6.1]). Consider $p_1, \dots, p_8 \in S$ in general linear position. The linear system of quadric surfaces in \mathbb{P}^3 passing through p_1, \dots, p_8 is a pencil. Its base locus is a quartic curve C_4 , given by the intersection of two different divisors in the pencil. By the initial assumption C_4 is not contained in S , hence C_4 intersects S along sixteen points, eight of which are the p_i 's. This gives rise to a birational involution of $S^{[8]}$.

For $n = 8, t = 2$ the equation $(n - 1)X^2 - tY^2 = -1$ is solvable, with minimal solution $(a, b) = (1, 2)$. Thus for $(S, H) \in \mathcal{K}_4$ with $\text{Pic}(S) = \mathbb{Z}H$, the birational non-symplectic involution σ which generates $\text{Bir}(S^{[8]})$ satisfies $H^2(S^{[8]}, \mathbb{Z})^{\sigma^*} = \mathbb{Z}\nu \cong \langle 2 \rangle$ with $\nu = 2h - \delta$. By Proposition 2.159 the involution is not biregular.

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Résumé

L'objectif de cette thèse est d'étudier le groupe d'automorphismes de certaines classes de variétés hyperkähleriennes.

La première partie de ce travail de thèse porte sur les doubles *EPW* sextiques, une famille de variétés hyperkähleriennes qui, dans le cas général, sont équivalentes par déformation au schéma de Hilbert de deux points sur une surface *K3*. Notamment nous avons utilisé le lien que ces variétés ont avec les variétés de Gushel-Mukai, qui sont des variétés de Fano dans une Grassmannienne si leur dimension est plus grande que deux, des surface *K3* si la dimension est deux. Nous avons donné des bornes sur la structure d'un certain sous-groupe d'automorphismes d'une double *EPW* sextique sous condition d'existence d'un point fixe pour l'action du groupe. Si une surface *K3* peut être associée à une double *EPW* sextique X , nous avons donné des conditions explicites sur le groupe de Picard de la surface pour que X soit une variété hyperkählienne. Cela permet d'utiliser le théorème de Torelli pour une surface *K3* pour démontrer l'existence de quelques automorphismes sur la double *EPW* sextique. Toujours dans le cas d'existence d'une surface *K3* associée à X , nous avons amélioré la borne obtenue précédemment sur les automorphismes de X , en donnant un lien explicite avec le nombre de coniques sur la surface *K3*. Nous avons montré que la symplecticité d'un automorphisme sur X dépend de la symplecticité d'un automorphisme correspondant sur la surface *K3*.

Dans la deuxième partie de ce travail de thèse, en collaboration avec Alberto Cattaneo, nous avons étudié le groupe d'automorphismes birationnels sur le schéma de Hilbert des points sur une surface *K3* projective, dans le cas générique. Nous avons ensuite étudié les cas où il existe un modèle birationnel où ces automorphismes sont réguliers. Nous avons décrit de façon géométrique quelques involutions dont nous avons prouvé l'existence auparavant. Nous avons établi un isomorphisme entre une composante connexe de l'espace de modules des variétés de type $K3^{[n]}$ avec une polarisation, et l'espace de modules des variétés de même type avec une involution dont le rang de l'invariant est un.

Mots-clés : géométrie algébrique, variétés hyperkähleriennes, automorphismes, transformations birationnelles, surfaces *K3*, doubles *EPW* sextiques, variétés Gushel-Mukai, représentation géométrique.

Abstract

The aim of this thesis is to study the group of automorphisms of certain classes of hyperkähler manifolds.

The first part of this thesis deals with double *EPW* sextics, a family of hyperkähler manifolds which in the general case are equivalent by deformation to the Hilbert scheme of two points on a *K3* surface. In particular we use the relation that these manifolds have with Gushel-Mukai varieties, which are Fano varieties in a Grassmannian if their dimension is greater than two, *K3* surfaces if the dimension is two. We give bounds on the structure of a certain subgroup of automorphisms of a double *EPW* sextic under conditions of the existence of a fixed point for the action of the group. If some *K3* surface can be associated with a double *EPW* sextic X , we give explicit conditions on the Picard group of the surface so that X is a hyperkähler manifold. This allows us to use Torelli's theorem for *K3* surfaces to demonstrate the existence of some automorphisms on the double *EPW* sextic. Again in the case of the existence of a *K3* surface associated with X , we improve the bound obtained previously on the automorphisms of X , giving an explicit link with the number of conics on the *K3* surface. We show that the symplecticity of an automorphism on X depends on the symplecticity of a corresponding automorphism on the *K3* surface.

In the second part of this thesis, in collaboration with Alberto Cattaneo, we study the group of birational automorphisms on Hilbert scheme of points on a *K3* projective surface, in the generic case. Then we study the cases in which there is a birational model where these automorphisms are regular. We describe in a geometrical way some involutions whose existence we proved previously. We establish an isomorphism between a connected component of the moduli space of varieties of type $K3^{[n]}$ with a polarization, and the moduli space of varieties of the same type with an involution whose rank of the invariant is one.

Keywords : algebraic geometry, hyperkähler manifolds, automorphisms, birational transformations, *K3* surfaces, double *EPW* sextics, Gushel-Mukai varieties, geometric representation.