Spin representations for Hermitian Lie groups

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Représentations spinorielles pour les groupes Hermitiens

Présentée et soutenue par :

Alexis Gilles

En vue de l'obtention du grade de

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Discipline : Mathématiques

Laboratoire de Mathématiques J.A. Dieudonné (LJAD)

Thèse dirigée par François Labourie

Soutenue le 25 Juin 2019

Devant le jury, composé de :

Julie Déserti	MCF	Université Paris-Diderot	Examinateur
Olivier Guichard	Pr	Université de Strasbourg	Rapporteur
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Résumé

Nous nous intéressons à un cas particulier d'homomorphismes maximaux depuis un groupe de surface dans un groupe de Lie Hermitien de type tube, que nous appelons *maximaux entiers*.

Dans la première partie, nous étudions le cas où le groupe de Lie est localement isomorphe au groupe des isométries directes du plan hyperbolique. Dans ce cas, les homomorphismes maximaux entiers induisent une hyperbolisation de la surface de départ et nous les relions avec les surfaces de Riemann spinorielles, c'est-à-dire les surfaces de Riemann équipées d'un fibré en droite dont une certaine puissance tensorielle est isomorphe au produit tensoriel du fibré canonique et d'un diviseur donné. Si notre surface est fermée, nous associons un entier modulo un autre entier fixé à chaque géodésique, son *nombre de translation*. Nous comptons asymptotiquement le nombre de géodésiques plus petites qu'une longueur donnée et de nombre de translation donné.

Dans la deuxième partie, nous nous intéressons au cas d'un groupe de Lie Hermitien de type tube quelconque. Nous lui associons un certain revêtement fini et dans ce dernier, nommons les représentations : *représentations spinorielles*. Nous montrons alors que l'espace des représentations maximales entières spinorielles s'identifie au produit cartésien de l'espace des représentations maximales entières dans le groupe de Lie initial et d'un sous espace explicite de l'ensemble des homomorphismes du premier groupe d'homologie à coefficients entiers du fibré unitaire tangent de notre surface de départ dans un groupe cyclique fini. L'homéomorphisme construit entre ces deux espaces est de plus équivariant sous l'action du groupe modulaire. Nous sommes donc amenés à expliciter l'action du groupe modulaire sur l'espace des homomorphismes du premier groupe d'homologie du fibré unitaire tangent de notre surface dans un groupe cyclique fini. Pour terminer, ces résultats appliqués à un cas particulier permettront de calculer le nombre de composantes connexes de représentations diagonales dans certains groupes de Lie localement isomorphes au groupe symplectique.

Mots-clés Représentation maximale, groupe de surfaces, structure spinorielle, groupe modulaire, groupe de Lie Hermitien de type tube.

Abstract

We study a particular case of maximal homomorphisms from a surface group into a Hermitian Lie group of tube type, which we call *integral maximal*.

In the first part, we deal with the case when the Lie group is locally isomorphic to the group of isometries of the hyperbolic plane. In this case, integral maximal homomorphisms induce hyperbolizations of the initial surface and we relate them to spin structures on Riemann surfaces, that is to line bundles whose tensor power is isomorphic to the tensor product of the canonical bundle and a given divisor. Fixing such an integral maximal representation, we associate to each geodesic an integer modulo a fixed integer, its *translation number*. We then give, when the surface is closed, the asymptotic growth of the number of geodesics with given translation number.

In the second part, we study the general case of an arbitrary Hermitian Lie group of tube type. Fixing a specific finite cover of such a Lie group, we call the representations into the cover *spin representations* and we show that the space of integral maximal spin representations is homeomorphic to the product of the space of maximal representations into the initial Lie group and an explicit subspace of homomorphisms from the first homology group with integer coefficient of the unit tangent bundle of the surface into a finite cyclic group. The homeomorphism we construct is moreover mapping class group equivariant so that we naturally study the action of the mapping class group on the space of homomorphisms from the first homology group of the unit tangent bundle of the surface into a finite cyclic group. Finally we apply these results to count the number of connected components of diagonal representations into some Lie groups locally isomorphic to the symplectic group.

Keywords Maximal representation, surface group, spin structure, mapping class group, Hermitian Lie group of tube type.

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Contents

Introduction

Structures spinorielles et représentations Soit S une surface fermée de genre $g \geqslant 2$ et X une structure de surface de Riemann sur S. Une *caractéristique théta* ou *structure spinorielle* (theta characteristic ou spin structure en anglais) sur X est une racine carrée du fibré canonique K_X de X. Une telle racine carrée L donne lieu à un revêtement ramifié $L \to L^{\otimes 2} \simeq K_X$ qui ne ramifie que sur l'image de la section nulle $X \to K_X$. Retirant cette image et en notant N° le fibré obtenu à partir d'un fibré N après retrait de l'image de la section nulle, il vient un revêtement à deux feuillets $p : L^\circ \to K_X^\circ$ de K_X° auquel est associé l'homomorphisme suivant

$$\pi_1(K_X^\circ) \xrightarrow{\mathrm{Id}} \pi_1(K_X^\circ)/p_*(\pi_1(L^\circ)) = \mathbb{Z}/2\mathbb{Z}\,.$$

Munissons X d'une structure riemannienne, de sorte que K_X s'identifie au fibré tangent TX de X et oubliant la structure complexe nous obtenons une application

$$\begin{array}{rccc} \Phi : & \{\text{Structures spinorielles sur } X\} & \to & \mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/2\mathbb{Z}) \\ & L & \mapsto & \xi_L\,, \end{array}$$

où TS est le fibré tangent de S et où le théorème de Hurewicz donne l'identification $\mathbb{H}_1(TS^\circ, \mathbb{Z}) \simeq \pi_1(TS^\circ)/[\pi_1(TS^\circ), \pi_1(TS^\circ)]$.

L'application Φ est injective mais n'est pas surjective. Pour décrire son image, rappelons que la projection $TS^\circ \to S$ induit par tiré en arrière une injection de $\mathrm{H}^1(S, \mathbb{Z}/2\mathbb{Z})$ dans $\mathrm{H}^1(TS^\circ, \mathbb{Z}/2\mathbb{Z})$ telle que la suite suivante soit exacte

$$0 \to \mathrm{H}^1(S, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^1(TS^\circ, \mathbb{Z}/2\mathbb{Z}) \xrightarrow{\lambda} \mathbb{Z}/2\mathbb{Z} \to 0$$

où $\lambda(\xi) = \xi(t)$ est l'image par ξ d'une boucle t dans une fibre de TS°. Si L est une structure spinorielle, alors $\xi_L(t) = 1 \mod 2$ et un argument de cardinalité montre que l'image de Φ est exactement l'image réciproque de $1 \in \mathbb{Z}/2\mathbb{Z}$ par λ.

Si f est un difféomorphisme de S préservant l'orientation et L une structure spinorielle sur X, alors le tiré en arrière f^*L est encore une structure spinorielle sur X, qui peut ne pas être L. Dans [Ati71] et [Mum71], les auteurs étudient un invariant des structures spinorielles défini comme la dimension modulo 2 de l'espace $\mathrm{H}^0(X, L)$ des sections holomorphes de la structure spinorielle L. D'un point de vue plus topologique, dans [Joh80] l'auteur associe à L une forme quadratique sur $\mathrm{H}_1(S, \mathbb{Z}/2\mathbb{Z})$ et l'invariant d'Arf (voir [Arf41]) de cette forme quadratique correspond à $\dim \mathrm{H}^0(X, L) \mod 2$.

Le degré du fibré canonique de X étant pair, les structures spinorielles existent toujours. Si $r \in \mathbb{Z}_{>0}$ divise $2g-2$, alors il existe sur X des *structures spinorielles d'ordre r* (r-

spin structures en anglais), c'est-à-dire des fibrés en droite holomorphes L sur X tels que $L^{\otimes r} \simeq K_X$. Dans [Sip82], l'autrice identifie l'ensemble des structures spinorielles d'ordre r sur X avec l'ensemble des homomorphismes $\xi : \mathrm{H}_1(TS^\circ, \mathbb{Z}) \to \mathbb{Z}/r\mathbb{Z}$ tels que $\xi(t) = 1$. Cette identification lui permet d'expliciter l'action d'un ensemble de générateurs de $\Gamma(S)$, le groupe modulaire de S, sur l'ensemble des structures spinorielles d'ordre r sur X.

Si sur la surface de Riemann compacte X on marque s points $x_1, \ldots, x_s$, et si on fixe $r > 0$ et $m_1, \ldots, m_s$ des entiers tels que $r \mid 2g - 2 + s - \sum m_j$, alors une *structure spinorielle d'ordre r et de type* $(m_1, \ldots, m_s)$ sur la surface de Riemann avec points marqués $(X; x_1, \ldots, x_s)$ est la donnée d'un fibré en droite holomorphe $L \to X$ tel que

$$L^{\otimes r} \simeq K_X \otimes (-\sum (m_j - 1)x_j)$$

où $K_X \otimes (-\sum(m_j - 1)x_j)$ est le fibré en droite dont les sections holomorphes sont les sections méromorphes s de K_X telles que le diviseur $\mathrm{div}(s) - \sum(m_j - 1)x_j$ soit positif, c'est-à-dire que s peut avoir un pôle d'ordre au plus $m_j - 1$ en x_j si $m_j - 1 \geqslant 0$ et a un zéro d'ordre au moins $1 - m_j$ en x_j si $m_j - 1 < 0$. Ces structures spinorielles et leurs isomorphismes définissent *l'espace de module des surfaces de Riemann spinorielles de type* $\mathfrak{m} = (m_1, \ldots, m_s)$ $\mathcal{M}_{g,s}^{r,\mathfrak{m}}$ qui est un revêtement régulier d'ordre r^{2g} de l'espace $\mathcal{M}_{g,s}$ de module des surfaces de Riemann avec points marqués.

Dans [Wit92, Wit93], l'auteur utilise l'espace de module $\mathcal{M}_{g,s}^{r,\mathfrak{m}}$ pour énoncer une conjecture généralisant une autre conjecture qu'il avait lui-même formulée dans [Wit91] et qui fut prouvée par M. Kontsevich dans [Kon92]. La conjecture originelle d'E. Witten porte sur les nombres d'intersections de classes tautologiques sur la compactification $\overline{\mathcal{M}}_{g,s}$ qui sont définies comme suit :

Pour chaque point marqué j, il existe un fibré en droite L_j (au sens orbifold) sur l'orbifold $\overline{\mathcal{M}}_{g,s}$ tel que la fibre au dessus d'une surface de Riemann $Y \in \mathcal{M}_{g,s}$ soit le fibré cotangent de Y au point x_j. Soit $\psi_j = c_1(L_j)$ la première classe de Chern du fibré L_j et pour $d_1, \ldots, d_s \in \mathbb{Z}_{\geqslant 0}$ soit

$$\langle \tau_{d_1} \cdots \tau_{d_s} \rangle_g = \int_{\overline{\mathcal{M}}_{g,s}} \psi_1^{d_s} \cdots \psi_s^{d_s} \, .$$

Ce nombre d'intersection est bien défini dès que la somme des d_j égale la dimension complexe de $\mathcal{M}_{g,s}$, soit $3g - 3 + s$. La conjecture originelle d'E. Witten consiste alors en une relation de récurrence entre les différents $\langle \tau_{d_1} \cdots \tau_{d_s} \rangle_g$, qui peut être exprimée en terme d'équations différentielles vérifiées par une fonction génératrice des $\langle \tau_{d_1} \cdots \tau_{d_s} \rangle_g$. Dans [Mir07b], l'autrice relie les nombres d'intersection des classes ψ avec le volume de l'espace de module $\mathcal{M}_{g,s}(L)$ des surfaces hyperboliques de genre g avec s composantes de

bord totalement géodésique de longueur fixée $L = (L_1, \dots, L_s)$. En effet, M. Mirzakhani prouve que le volume de $\mathcal{M}_{g,s}(L)$ est un polynôme

$$\mathrm{Vol}(\mathcal{M}_{g,s}(L)) = \sum_{\substack{\alpha \in \mathbb{Z}_{\geqslant 0}^s \\ |\alpha| \leqslant 3g-3+s}} C_\alpha L^\alpha$$

où pour $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{Z}_{\geqslant 0}$, $L^\alpha = L_1^{\alpha_1} \cdots L_s^{\alpha_s}$ et $|\alpha| = \sum \alpha_j$, et si $|\alpha| = 3g - 3 + s$, c'est-à-dire pour les plus hauts degrés,

$$C_\alpha = \frac{1}{2^{|\alpha|}\alpha_1! \cdots \alpha_s!} \langle \tau_{d_1} \cdots \tau_{d_s} \rangle_g \,.$$

De plus, elle montre dans [Mir07a] que le volume de $\mathcal{M}_{g,s}(L)$ s'exprime en fonction du volume des $\mathcal{M}_{g',s'}(L')$ pour $2g'+s' < 2g+s$, et la relation de récurrence qu'elle obtient pour les C_α implique la conjecture d'E. Witten pour les nombres d'intersections $\langle \tau_{d_1} \cdots \tau_{d_s} \rangle_g$.

Avec l'espoir initial, mais pas encore abouti (et donc non présenté ici), d'étudier les nombres d'intersections d'E. Witten sur $\mathcal{M}_{g,s}^{r,\mathfrak{m}}$, nous avons considéré un espace de « surfaces hyperboliques spinorielles à bord totalement géodésique de longueur fixée » $\mathcal{M}_{g,s}^{r,\mathfrak{m}}(L)$. Pour prouver cela, nous montrons dans la première partie de cette thèse que l'espace $\mathcal{M}_{g,s}^{r,\mathfrak{m}}$ s'identifie au quotient par $\Gamma(S)$ d'un sous-ensemble de la variété de représentation du groupe fondamental de S dans le revêtement connexe à r feuillets de $\mathsf{PSL}_2(\mathbb{R})$.

Soit Spin^r l'unique (à isomorphisme près) revêtement connexe à r feuillets de $\mathsf{PSL}_2(\mathbb{R})$. Si $\overline{g} \in \mathsf{Spin}^r$ est un antécédent de $g \in \mathsf{PSL}_2(\mathbb{R})$, alors $\overline{g}$ est dit *hyperbolique* (resp. *parabolique, elliptique*) si g l'est. Identifiant $\mathbf{P}^1\mathbb{R}$ et le cercle $\mathbb{R}/\mathbb{Z}$, l'action de g sur $\mathbb{R}/\mathbb{Z}$ se relève en une action de $\overline{g}$ sur $\mathbb{R}/r\mathbb{Z}$ telle que le diagramme suivant commute

$$\begin{array}{ccc} \mathbb{R}/r\mathbb{Z} & \xrightarrow{\overline{g}} & \mathbb{R}/r\mathbb{Z} \\ \downarrow & & \downarrow \\ \mathbb{R}/\mathbb{Z} & \xrightarrow{g} & \mathbb{R}/\mathbb{Z}\,. \end{array}$$

Si de plus g est hyperbolique ou parabolique, il fixe un point dans $\mathbb{R}/\mathbb{Z}$ et donc son nombre de rotation est nul. Ainsi, ses différents antécédents $\overline{g}$ dans Spin^r diffèrent par un entier modulo r, leur *nombre de translation modulo* r, noté $\mathrm{Trans}^{(r)}(\overline{g})$, qui est leur nombre de rotation en tant qu'homéomorphismes de $\mathbb{R}/r\mathbb{Z}$.

Soit $\overline{S} = S_g$ une surface orientée connexe compacte de genre g et

$$S = S_{g,s} = \overline{S} - \{p_1, \dots, p_s\}$$

la surface obtenue en retirant s points. Pour tout $j = 1, \ldots, s$, soit c_j une boucle autour du j-ième point retiré. Comme l'action du groupe modulaire sur $\mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{PSL}_2(\mathbb{R}), P)$ est propre, l'action du groupe modulaire sur $\mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}})$ l'est aussi. Dans la première partie de la thèse, nous montrons que le quotient $\mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}})$ s'identifie à $\mathcal{M}_{g,s}^{r,\mathfrak{m}}$.

Théorème
Il existe un difféomorphisme

$$\mathcal{M}_{g,s}^{r,\mathfrak{m}} \simeq \mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}})/\Gamma(S)$$

où $\mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_m)$ *désigne l'espace des classes de conjugaison d'homomorphismes* $\rho : \pi_1(S) \to \mathsf{Spin}^r$ *induisant une hyperbolisation de* S *de volume fini et tels que* $\mathrm{Trans}^{(r)}(\rho(c_j)) = -m_j \mod r$.

Voir le Théorème 1. On peut alors définir $\mathcal{M}_{g,s}^{r,\mathfrak{m}}(L)$ comme le quotient par $\Gamma(S)$ de la variété de représentation relative de $\pi_1(S)$ dans Spin^r contenant les homomorphismes $\rho : \pi_1(S) \to \mathsf{Spin}^r$ induisant une hyperbolisation de S tels que $\rho(c_j)$ soit hyperbolique de distance de translation L_j pour $L_j > 0$, et parabolique pour $L_j = 0$, et ait pour nombre de translation $-m_j$ modulo r ; homomorphismes que nous appellerons *hyperbolisations spinorielles* de S.

Si $\rho : \pi_1(S) \to \mathsf{Spin}^r$ est une hyperbolisation spinorielle de S et Σ est la structure hyperbolique induite sur S, alors à chaque géodésique fermée sur Σ est associée un entier modulo r de la manière suivante : une telle géodésique définit une classe de conjugaison dans $\pi_1(S)$ et le nombre de translation modulo r est invariant par conjugaison.

Pour $L > 0$ et $m \in \mathbb{Z}/r\mathbb{Z}$, soit

$$\begin{aligned} \mathcal{C}(L, m) &= \{\gamma \text{ géodésique fermée}, l_\Sigma(\gamma) < L, \mathrm{Trans}^{(r)}(\gamma) = m\}, \\ C(L, m) &= \mathrm{Card}(\mathcal{C}(L, m)), \end{aligned}$$

où $l_\Sigma(\gamma)$ est la longueur de la géodésique γ. Soit $C(L) = \sum_{m \in \mathbb{Z}/r\mathbb{Z}} C(L, m)$ le nombre total de géodésiques fermées sur Σ de longueur strictement inférieure à L. Si Σ est compacte, un résultat de G. Margulis (voir [Mar69]) donne l'équivalent suivant

$$C(L) \sim \frac{e^L}{L}$$

où $f(L) \sim g(L)$ signifie $\lim_{L\to\infty} f(L)/g(L) = 1$. En reliant la fonction $\gamma \mapsto \mathrm{Trans}^{(r)}(\gamma)$ à un homomorphisme $\mathrm{H}_1(T^1\Sigma, \mathbb{Z}) \to \mathbb{Z}/r\mathbb{Z}$, où $T^1\Sigma$ est le fibré tangent unitaire de Σ, nous utilisons, comme une boite noire, un théorème d'A. Katsuda et T. Sunada [KS90] dans le chapitre 3 pour établir, toujours sous l'hypothèse de compacité de Σ, le théorème suivant

Théorème
Pour tout $m \in \mathbb{Z}/r\mathbb{Z}$, nous avons l'équivalent suivant quand $L \to \infty$

$$C(L,m) \sim \frac{e^L}{rL}\,.$$

De plus, les géodésiques de nombre de rotation fixé sont équiréparties sur Σ au sens suivant : pour toute fonction $f \in C^\infty(\Sigma)$ et $m \in \mathbb{Z}/r\mathbb{Z}$,

$$\lim_{L\to\infty} \frac{1}{C(L,m)} \sum_{\gamma\in\mathcal{C}(L,m)} \frac{1}{l_\Sigma(\gamma)} \int_\gamma f = \int_\Sigma f \,\mathrm{d}\,\mathrm{vol}_\Sigma\ .$$

voir le Théorème 2.

Représentations maximales entières Soit S une surface orientée connexe compacte de genre $g \geqslant 2$ et $h : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$ un homomorphisme. Soit $a_1, b_1, \ldots, a_g, b_g$ une collection génératrice d'éléments de $\pi_1(S)$ telle qu'on ait la présentation suivante

$$\pi_1(S) = \langle a_1, b_1, \ldots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle\,.$$

Soit $\widetilde{\mathsf{PSL}}_2(\mathbb{R})$ le revêtement universel de $\mathsf{PSL}_2(\mathbb{R})$. Puisque $\pi_1(\mathsf{PSL}_2(\mathbb{R}))$ est central dans $\widetilde{\mathsf{PSL}}_2(\mathbb{R})$, le commutateur $[\cdot,\cdot] : \widetilde{\mathsf{PSL}}_2(\mathbb{R}) \times \widetilde{\mathsf{PSL}}_2(\mathbb{R}) \to \widetilde{\mathsf{PSL}}_2(\mathbb{R})$ factorise par

$$\tilde{[\cdot,\cdot]} : \mathsf{PSL}_2(\mathbb{R}) \times \mathsf{PSL}_2(\mathbb{R}) \to \widetilde{\mathsf{PSL}}_2(\mathbb{R})$$

et ainsi $h : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$ définit un unique entier $eu(h)$, appelé *nombre d'Euler de h*, définie par la relation

$$\prod_{i=1}^{g} \tilde{[}h(a_i), h(b_i)\tilde{]} = \tau^{eu(h)}\,,$$

où τ est l'image de $1 \in \pi_1(\mathsf{PSL}_2(\mathbb{R}))$ par $\pi_1(\mathsf{PSL}_2(\mathbb{R})) \to \widetilde{\mathsf{PSL}}_2(\mathbb{R})$.

Dans [Gol80, Gol88], l'auteur démontre que la variété de représentation $\mathrm{Rep}(\pi_1(S), \mathsf{PSL}_2(\mathbb{R}))$ consiste en $4g-3$ composantes connexes caractérisées comme étant les préimages de $\{2-2g, \ldots, 2g-2\}$ par eu et que la composante $eu^{-1}(2g-2)$ s'identifie à l'espace de Teichmüller de S, c'est-à-dire à l'espace des structures de surfaces hyperboliques marquées sur S induisant l'orientation initiale de S.

Si G est un groupe de Lie simple Hermitien de rang réel n, c'est-à-dire si son espace symétrique $\mathcal{X}$ est Hermitien de rang n, alors il existe sur $\mathcal{X}$ une forme symplectique ω invariante sous l'action de G, unique si on impose la courbure sectionnelle holomorphe de

$\mathcal{X}$ d'être égale à -1.

Pour $x \in \mathcal{X}$ et $g_0, g_1, g_2 \in \mathsf{G}$, soit $\Delta(g_0x, g_1x, g_2x)$ le triangle géodésique dans $\mathcal{X}$ ayant pour sommets g_0x, g_1x et g_2x. La fonction $c_\omega : \mathsf{G}^3 \to \mathbb{R}$ définie par

$$c_\omega = \frac{1}{2\pi} \int_{\Delta(g_0x, g_1x, g_2x)} \omega \tag{1}$$

est un cocycle continu et sa classe cohomologique $\kappa_{\mathsf{G},\mathbb{R}}$ dans $\mathrm{H}^2_c(\mathsf{G}, \mathbb{R})$ s'appelle la *classe de Kähler (réelle)* de G. La cohomologie du groupe $\pi_1(S)$ s'identifie à la cohomologie de S, et comme S est compacte, l'intégration sur S donne un isomorphisme $\int_S : \mathrm{H}^2(\pi_1(S), \mathbb{R}) \simeq \mathbb{R}$. Pour tout homomorphisme $h : \pi_1(S) \to \mathsf{G}$, le réel

$$T(h) = \int_S h^* \kappa_{\mathsf{G},\mathbb{R}}$$

ainsi obtenu est *l'invariant de Toledo de* h, et pour $\mathsf{G} = \mathsf{PSL}_2(\mathbb{R})$, on retrouve le nombre d'Euler $eu(h)$. La fonction

$$\begin{array}{rccc} T : & \mathrm{Hom}(\pi_1(S), \mathsf{G}) & \to & \mathbb{R} \\ & h & \mapsto & T(h) \end{array}$$

est continue, invariante par conjugaison et son image est contenue dans l'ensemble d'entiers $\{n(2-2g), \ldots, n(2g-2)\}$.

Les éléments de $\mathrm{Hom}(\pi_1(S), \mathsf{G})$ dont l'invariant de Toledo est maximal, c'est-à-dire égal à $n(2g-2)$, sont les *homomorphismes maximaux*. Si $\mathsf{G} = \mathsf{PSL}_2(\mathbb{R})$, on retrouve exactement les hyperbolisations de S. Si $\mathsf{G} = \mathsf{Spin}^r$, le revêtement connexe de $\mathsf{PSL}_2(\mathbb{R})$ à r feuillets, on retrouve les hyperbolisations spinorielles de S, s'il en existe.

Si $S = S_{g,s}$ est une surface obtenue en retirant $s \geqslant 1$ points d'une surface compacte orientée connexe de genre $g \geqslant 0$, $g > 1 - s/2$, la définition de l'invariant de Toledo est plus compliquée car le groupe $\pi_1(S)$ est libre et donc $\mathrm{H}^2(\pi_1(S), \mathbb{R}) = 0$. Pour palier à ce problème, on utilise la cohomologie bornée : pour G un groupe topologique localement compact et $A = \mathbb{R}, \mathbb{Z}$, la *cohomologie bornée borélienne de* G *à coefficient dans* A, notée $\mathrm{H}^\bullet_{\mathrm{Bb}}(G, A)$, est la cohomologie du complexe de cochaines boréliennes $c : G \times \cdots \times G \to A$ qui sont bornées et invariantes sous l'action $g \cdot c(g_0, \ldots, g_n) = c(gg_0, \ldots, gg_n)$.

Le cocycle c_ω défini par l'équation (1) est borné et donc définit une classe $\kappa^b_{\mathsf{G},\mathbb{R}} \in \mathrm{H}^2_{\mathrm{Bb}}(\mathsf{G}, \mathbb{R})$ dite *classe de Kähler bornée (réelle)* de G. Dans [BIW10] les auteurs construisent une forme linéaire continue $l : \mathrm{H}^2_{\mathrm{b}}(\pi_1(S), \mathbb{R}) \to \mathbb{R}$ et la quantité qu'ils considèrent

$$T_{\mathrm{b}}(h) = l(h^* \kappa^b_{\mathsf{G},\mathbb{R}})$$

généralise l'invariant de Toledo précédent. La fonction

$$\begin{array}{rccc} T_{\mathrm{b}}: & \mathrm{Hom}(\pi_1(S),\mathsf{G}) & \to & \mathbb{R} \\ & h & \mapsto & T_{\mathrm{b}}(h) \end{array}$$

est continue, invariante par conjugaison et son image est exactement l'intervalle réel $[n\chi(S), -n\chi(S)]$, voir [BIW10]. Les homomorphismes h tels que $T_{\mathrm{b}}(h) = -n\chi(S)$ sont dits *maximaux*. Ils sont injectifs et d'image discrète.

Si $\mathsf{G} = \mathsf{PSL}_2(\mathbb{R})$, il existe une unique classe $\kappa^b_{\mathsf{PSL}_2(\mathbb{R}),\mathbb{Z}} \in \mathrm{H}^2_{\mathrm{Bb}}(\mathsf{PSL}_2(\mathbb{R}),\mathbb{Z})$, dite *classe de Kähler bornée entière* de $\mathsf{PSL}_2(\mathbb{R})$, telle que $\kappa^b_{\mathsf{PSL}_2(\mathbb{R}),\mathbb{R}}$ soit l'image de $\kappa^b_{\mathsf{PSL}_2(\mathbb{R}),\mathbb{Z}}$ par le changement de coefficient $\mathrm{H}^2_{\mathrm{Bb}}(\mathsf{PSL}_2(\mathbb{R}),\mathbb{Z}) \to \mathrm{H}^2_{\mathrm{b}}(\mathsf{PSL}_2(\mathbb{R}),\mathbb{R})$. Dans [BIW10], utilisant leurs résultats ainsi que les résultats de [Ghy87], les auteurs observent qu'il existe $\kappa^b_{S,\mathbb{Z}} \in \mathrm{H}^2_{\mathrm{b}}(\pi_1(S),\mathbb{Z})$, dite *classe fondamentale bornée entière de* S, telle que $h : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$ soit maximal si et seulement si $h^*\kappa^b_{\mathsf{PSL}_2(\mathbb{R}),\mathbb{Z}} = \kappa^b_{S,\mathbb{Z}}$. Si $\kappa^b_{S,\mathbb{R}}$ est l'image de $\kappa^b_{S,\mathbb{Z}}$ par changement de coefficient et G un groupe de Lie simple Hermitien de rang n, alors

Théorème ([BIW10])
L'homomorphisme $h : \pi_1(S) \to \mathsf{G}$ est maximal si et seulement si $h^*\kappa^b_{\mathsf{G},\mathbb{R}} = n\kappa^b_{S,\mathbb{R}}$.

Soit G un groupe de Lie simple Hermitien et de type tube, c'est-à-dire que son espace symétrique est biholomorphe à un domaine de la forme $V \oplus i\Omega$ où V est un espace vectoriel réel et $\Omega \subsetneq V$ est un cône ouvert convexe non vide de V. Dans ce cas, comme pour $\mathsf{PSL}_2(\mathbb{R})$, la classe de Kähler bornée réelle de G a une version entière $\kappa^b_{\mathsf{G},\mathbb{Z}} \in \mathrm{H}^2_{\mathrm{b}}(\mathsf{G},\mathbb{Z})$ dite *classe de Kähler bornée entière de* G. Nous introduirons la notion d'homomorphisme *maximal entier* pour les homomorphismes $h : \pi_1(S) \to \mathsf{G}$ dont nous verrons qu'elle est équivalente à ce que $h^*\kappa^b_{\mathsf{G},\mathbb{Z}} = n\kappa^b_{S,\mathbb{Z}}$, voire la Proposition 6.1.12. Si tous les homomorphismes maximaux entiers sont maximaux, la réciproque n'est en général pas vraie. On note $\mathrm{Hom}_{\mathrm{IntMax}}(S,\mathsf{G})$ l'ensemble des homomorphismes entiers maximaux de $\pi_1(S)$ dans G. C'est une union de composantes connexes de l'espace $\mathrm{Hom}_{\mathrm{Max}}(S,\mathsf{G})$ des homomorphismes maximaux de $\pi_1(S)$ dans G.

Oubliant le caractère borné de $\kappa^b_{\mathsf{G},\mathbb{Z}}$, nous obtenons une extension centrale de G par $\mathbb{Z}$, c'est-à-dire une suite exacte

$$0 \to \mathbb{Z} \xrightarrow{i} \hat{\mathsf{G}} \xrightarrow{p} \mathsf{G} \to 1$$

où $i(\mathbb{Z})$ est un sous groupe central de G. Soit $\mathsf{G}_r := \hat{\mathsf{G}}/i(r\mathbb{Z})$, qui est un revêtement fini de G. Si le groupe $\pi_1(S)$ est libre (resp. si $s = 0$ et $r \mid n(2g-2)$), tout homomorphisme $h \in \mathrm{Hom}(\pi_1(S),\mathsf{G})$ admet r^{2g+s-1} (resp. r^{2g}) relevés $\rho \in \mathrm{Hom}(\pi_1(S),\mathsf{G}_r)$, et l'application

$$p_* : \mathrm{Hom}(\pi_1(S),\mathsf{G}_r) \to \mathrm{Hom}(\pi_1(S),\mathsf{G})$$

est un fibré principal de groupe structural $\mathrm{Hom}(\pi_1(S), \mathbb{Z}/r\mathbb{Z})$. Soit

$$A_{g,s}^{r,n} = \{\xi \in \mathrm{Hom}(\mathrm{H}_1(TS^\circ), \mathbb{Z}), \mathbb{Z}/r\mathbb{Z}) : \xi(t) = n \mod r\}$$

où, comme précédemment, t désigne la classe d'homologie d'une boucle dans la fibre de TS°. Restreignant p_* aux homomorphismes entiers maximaux, et en notant $\mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G})$ le quotient de $\mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G})$ par l'action par conjugaison de G, nous obtenons le Théorème 11

Théorème
Soit $S = S_{g,s}$ avec $2g-2+s > 0$, $r \in \mathbb{Z}_{>0}$. On suppose $s > 0$ ou $r|n(2g-2)$. Alors il existe un difféomorphisme équivariant pour l'action de $\Gamma(S)$

$$\Phi : \mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}_r) \xrightarrow{\sim} \mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}) \times A_{g,s}^{r,n} \tag{2}$$

où l'application $\mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}_r) \longrightarrow \mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G})$ est $[\rho] \longmapsto [p_*\rho]$.

À la classe de Kähler bornée entière κ de G est associée une fonction $\mathrm{Trans}_\kappa^{(r)} : \mathsf{G}_r \to \mathbb{R}/r\mathbb{Z}$ et pour $\rho \in \mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}_r)$, $\mathrm{Trans}_\kappa^{(r)} \circ \rho(\gamma) \in \mathbb{Z}/r\mathbb{Z}$ pour tout $\gamma \in \pi_1(S)$. Supposons $s > 0$ et pour tout $1 \leqslant j \leqslant s$, soit c_j une boucle simple lisse autour de la j-ième pointe dans le sens direct, et soit $\overline{c}_j$ son relevé tangentiel à TS°. Pour $\mathfrak{m} = (m_1, \ldots, m_s) \in \mathbb{Z}^s$, soit

$$\mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}_r, \mathfrak{m}) = \{[\rho] \in \mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}_r) : \mathrm{Trans}_\kappa^{(r)}(\rho(c_j)) = -m_j \mod r\}\,.$$

Par un théorème de [BIW10], $\mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}_r, \mathfrak{m})$ est non vide si et seulement si $\sum_{j=1}^n m_j = n(2g-2+s) \mod r$. Soit $I_{g,s}^{r,n} = \{\mathfrak{m} = (m_1, \ldots, m_s) \in \mathbb{Z}/r\mathbb{Z} : \sum_{j=1}^s m_j = n(2g-2+s) \mod r\}$. Pour $\mathfrak{m} \in I_{g,s}^{r,n}$, posons

$$A_{g,s}^{r,n,\mathfrak{m}} = \{\xi \in \mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z}) : \xi(t) = n \mod r,\, \xi(\overline{c}_j) = m_j\, \forall j\}\,.$$

L'action de $\Gamma(S)$ sur $\mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z})$ préserve les $A_{g,s}^{r,n,\mathfrak{m}}$ et nous avons

$$\mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z}) = \coprod_{1 \leqslant r \leqslant n} A_{g,s}^{r,n}$$

et pour $s > 0$

$$A_{g,s}^{r,n} = \coprod_{\mathfrak{m} \in I_{g,s}^{r,n}} A_{g,s}^{r,n,\mathfrak{m}}$$

où $\coprod$ désigne l'union disjointe. Nous obtenons alors un raffinement du théorème précédent, voir encore le Théorème 11.

Théorème
Soit $S = S_{g,s}$ avec $2g-2+s > 0$, $r \in \mathbb{Z}_{>0}$. On suppose $s > 0$. Alors il existe un homéomorphisme équivariant pour l'action de $\Gamma(S)$

$$\Phi : \mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}_r, \mathfrak{m}) \xrightarrow{\sim} \mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}) \times A_{g,s}^{r,n,\mathfrak{m}} . \tag{3}$$

Dans le cas $\mathsf{G} = \mathsf{PSL}_2(\mathbb{R})$, on obtient

$$\mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}}) \simeq \mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{PSL}_2(\mathbb{R}), P) \times A_{g,s}^{r,1,\mathfrak{m}}$$

et pour déterminer le nombre de composantes connexes de $\mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}})/\Gamma(S)$, il suffit de déterminer $A_{g,s}^{r,1,\mathfrak{m}}/\Gamma$. C'est équivalent à ce que fait T. Jarvis dans [Jar00], où il étudie la géométrie de $\mathcal{M}_{g,s}^{r,\mathfrak{m}}$ et en particulier dénombre ses composantes connexes. S'il était déjà connu que $\mathcal{M}_g^2$ a deux composantes connexes (distinguées par l'invariant d'Arf, voir [Ati71, Mum71, Joh80]), T. Jarvis prouve que pour $g \geqslant 2$, $\mathcal{M}_{g,s}^{r,\mathfrak{m}}$ a au plus deux composantes connexes, et en fait deux si et seulement si 2 divise les entiers r et $m_j + 1$ pour tous $j = 1, \ldots, s$. Nous généralisons son calcul et obtenons le nombre d'orbites de l'action de $\Gamma(S)$ sur $A_{g,s}^{r,n,\mathfrak{m}}$, voir le Théorème 10.

Théorème
Soit $s > 0$ et soit

$$l_{g,s}^{r,n,\mathfrak{m}} = \begin{cases} 1 & g = 0 \\ \mathrm{pgcd}(r, m_1 - n, \ldots, m_s - n) & g = 1 \\ \mathrm{pgcd}(r, 2n, m_1 - n, \ldots, m_s - n) & \textit{sinon,} \end{cases}$$

alors

$$\mathrm{Card}(A_{g,s}^{r,n,\mathfrak{m}}/\Gamma(S)) = d(l_{g,s}^{r,n,\mathfrak{m}}) .$$

Pour $s = 0$, nous avons

$$\mathrm{Card}(A_{g,0}^{r,n}/\Gamma(S)) = d(\mathrm{pgcd}(2n, r)) ,$$

où $d(k)$ est le nombre de diviseurs d'un entier k.

Nous utilisons ce résultat pour calculer le nombre de composantes connexes de représentations maximales diagonales dans un revêtements fini de $\mathsf{PSp}_{2n}(\mathbb{R})$ pour n impair.

Si $\mathsf{G} = \mathsf{PSp}_{2n}(\mathbb{R})$, si n est impair et si r est premier avec n, alors le groupe G_r est l'unique revêtement connexe à r feuillets de G, et il existe un unique (à conjugaison près)

couple (Δ, Δ_r) d'homomorphismes tel que le diagramme suivant commute

$$\begin{array}{ccc} \mathsf{Spin}^r & \xrightarrow{\Delta_r} & \mathsf{G}_r \\ \downarrow & & \downarrow \\ \mathsf{PSL}_2(\mathbb{R}) & \xrightarrow{\Delta} & \mathsf{G} \end{array}$$

et que Δ soit la composition $\mathsf{PSL}_2(\mathbb{R}) \to \mathsf{PSL}_2(\mathbb{R})^n \to \mathsf{G}$. Les homomorphismes Δ et Δ_r induisent par tiré en arrière des homomorphismes $\Delta^* : \mathrm{H}^2_{\mathrm{b}}(\mathsf{G}, \mathbb{Z}) \to \mathrm{H}^2_{\mathrm{b}}(\mathsf{PSL}_2(\mathbb{R}), \mathbb{Z})$ et $\Delta_r^* : \mathrm{H}^2_{\mathrm{b}}(\mathsf{G}_r, \mathbb{Z}) \to \mathrm{H}^2_{\mathrm{b}}(\mathsf{Spin}^r, \mathbb{Z})$ tels que le diagramme suivant commute

$$\begin{array}{ccc} \mathrm{H}^2_{\mathrm{b}}(\mathsf{G}, \mathbb{Z}) & \xrightarrow{\Delta_r^*} & \mathrm{H}^2_{\mathrm{b}}(\mathsf{PSL}_2(\mathbb{R}), \mathbb{Z}) \\ \downarrow & & \downarrow \\ \mathrm{H}^2_{\mathrm{b}}(\mathsf{G}_r, \mathbb{Z}) & \xrightarrow{\Delta^*} & \mathrm{H}^2_{\mathrm{b}}(\mathsf{Spin}^r, \mathbb{Z}) \end{array}$$

et de plus $\Delta^* \kappa^b_{\mathsf{G},\mathbb{Z}} = n \kappa^b_{\mathsf{PSL}_2(\mathbb{R}),\mathbb{Z}}$. En particulier, si $h : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$, alors

$$(\Delta \circ h)^* \kappa^b_{\mathsf{G},\mathbb{Z}} = n h^* \kappa^b_{\mathsf{PSL}_2(\mathbb{R}),\mathbb{Z}}$$

et ainsi $\Delta \circ h$ est maximal entier si et seulement si h est une hyperbolisation de S. De même, si $\rho : \pi_1(S) \to \mathsf{Spin}^r$, alors $\Delta_r \circ \rho : \pi_1(S) \to \mathsf{G}_r$ est maximal entier si et seulement si ρ l'est. Les compositions $\Delta \circ h$ et $\Delta_r \circ \rho$ pour h et ρ maximaux sont les *homomorphismes diagonaux Fuchsiens*. Pour $r = 2$, on retrouve le cas des homomorphismes diagonaux Fuchsiens dans $\mathsf{Sp}_{2n}(\mathbb{R})$.

Soit $\mathrm{Hom}_\Delta(S, \mathsf{G})$ et $\mathrm{Hom}_\Delta(S, \mathsf{G}_r)$ les plus petites unions de composantes connexes de, respectivement, $\mathrm{Hom}_{\mathrm{Max}}(S, \mathsf{G})$ et $\mathrm{Hom}_{\mathrm{Max}}(S, \mathsf{G}_r)$ contenant les homomorphismes diagonaux Fuchsiens. Les éléments de Hom_Δ sont dit *maximaux diagonaux*. Sachant que parmi les homomorphismes maximaux, les homomorphismes maximaux entiers restent maximaux entiers lors de déformations continues, tout homomorphisme maximal diagonal est maximal entier.

Soit $\mathrm{Rep}_\Delta(S, \mathsf{G})$ et $\mathrm{Rep}_\Delta(S, \mathsf{G}_r)$ les quotients de $\mathrm{Hom}_\Delta(S, \mathsf{G})$ et $\mathrm{Hom}_\Delta(S, \mathsf{G}_r)$ par conjugaison. L'homéomorphisme $\Gamma(S)$-équivariant (3) se restreint en un difféomorphisme

$$\mathrm{Rep}_\Delta(S, \mathsf{G}_r) \xrightarrow{\sim} \mathrm{Rep}_\Delta(S, \mathsf{G}) \times A^{r,n}_{g,s}$$

et nous en déduisons

Corollaire

Le nombre de composantes connexes de $\mathrm{Rep}_\Delta(S, \mathsf{G}_r)/\Gamma(S)$ *est égal à* $\mathrm{Card}(A^{r,n}_{g,s}/\Gamma(S))$*, où*

$$\mathrm{Card}(A^{r,n}_{g,s}/\Gamma(S)) = \begin{cases} r^{s-1} & g = 0,\ s \geqslant 3 \\ \sum_{d|r} d^{s-1} & g = 1,\ s \geqslant 1 \\ r^{s-1} + (r/2)^{s-1} & g \geqslant 2,\ s \geqslant 1,\ 2 \mid r \\ r^{s-1} & g \geqslant 2,\ s \geqslant 1,\ 2 \nmid r \\ 2 & g \geqslant 2,\ s = 0,\ 2 \mid r \mid 2g-2 \\ 1 & g \geqslant 2,\ s = 0,\ 2 \nmid r \mid 2g-2 \\ 0 & g \geqslant 2,\ s = 0,\ r \nmid 2g-2 \end{cases}$$

voir le Corollaire 12.

Résumé des sections Dans le premier chapitre, nous fixons les notations et définitions des structures spinorielles et des représentations associées pour, dans le deuxième chapitre, prouver le Théorème 1. Dans le troisième chapitre nous prouvons le Corollaire 2 sur le comptage des géodésiques de nombre de rotation fixé, ce qui conclut la première partie.

La deuxième partie commence, après l'énoncé des résultats dans le quatrième chapitre, par l'étude de l'action du groupe modulaire sur $\mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z})$ et la preuve du Théorème 10 constitue le cinquième chapitre. Le sixième chapitre traite des représentations maximales entières et nous y prouvons le Théorème 11. Dans le septième et dernier chapitre, nous appliquons nos résultats pour prouver le Corollaire 12.

Part I

r-spin structures and representations

Statement of results

Let X be a compact Riemann surface of genus g and $Y = (X; x_1, \dots, x_s)$ a pointed Riemann surface with $2g-2+s > 0$. If $r \in \mathbb{Z}_{>0}$ and $\mathfrak{m} = (m_1, \dots, m_s) \in \mathbb{Z}^s$, a *$r$-spin structure of type $\mathfrak{m}$ on Y* is a pair $(\mathcal{L}, f)$ where $\mathcal{L} \to X$ is an invertible sheaf and f is a sheaf isomorphism

$$f : \mathcal{L}^{\otimes r} \longrightarrow \omega_X(\Delta)$$

where ω_X is the canonical sheaf of X, $\Delta = -\sum_1^s (m_j - 1) \cdot x_j$ a divisor on X and $\omega_X(\Delta)$ the sheaf of meromorphic sections of ω_X with $\mathrm{div}(s) + \Delta \geqslant 0$. For degree reason, such a r-spin structure exists if and only if $r \mid 2g - 2 + s - \sum m_j$, see for instance [Jar00].

A *r-spin Riemann surface* is a pointed Riemann surface equipped with a r-spin structure. Isomorphism classes of r-spin Riemann surfaces of type $\mathfrak{m}$, genus g and with s points form a moduli space $\mathcal{M}_{g,s}^{r,\mathfrak{m}}$, see [Jar00, Chi08]. Forgetting the r-spin structure yields a regular r^{2g}-cover

$$\mathcal{M}_{g,s}^{r,\mathfrak{m}} \longrightarrow \mathcal{M}_{g,s}$$

of the moduli space $\mathcal{M}_{g,s}$ of genus g Riemann surfaces with s points.

The uniformization theorem yields a bijection between $\mathcal{M}_{g,s}$ and isomorphism classes of hyperbolic structures on S with cusps at each puncture. The holonomy of such a hyperbolic structure is a *maximal homomorphism* $\pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$ sending loops about each puncture to positive parabolic elements, where maximal homomorphisms are a special case of homomorphisms from a surface group into a Hermitian Lie group and were investigated by M. Burger, A. Iozzi and A. Wienhard in [BIW10]. More precisely, let

$$\mathrm{Rep}_{\mathrm{Max}}(S, \mathsf{PSL}_2(\mathbb{R}), P) = \mathrm{Hom}_{\mathrm{Max}}(S, \mathsf{PSL}_2(\mathbb{R}), P)/\mathsf{PSL}_2(\mathbb{R})$$

be the space of conjugacy classes of maximal homomorphisms $\rho : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$ sending loops about punctures to parabolic elements. Let $\Gamma(S)$ be the (pure) *mapping class group* of S, that is the group of orientation-preserving diffeomorphisms of S fixing each puncture, up to isotopy. The mapping class group $\Gamma(S)$ acts on $\mathrm{Rep}_{\mathrm{Max}}(S, \mathsf{PSL}_2(\mathbb{R}), P)$ and the uniformisation theorem tells us that the quotient is analytically diffeomorphic to $\mathcal{M}_{g,s}$.

In the first part we show that the uniformisation diffeomorphism lifts and we identify $\mathcal{M}_{g,s}^{r,\mathfrak{m}}$ with a subspace of the quotient of the representation variety of $\pi_1(S)$ into the unique connected r-cover of $\mathsf{PSL}_2(\mathbb{R})$. Before stating the correspondance, we need a few definitions.

The fundamental group of $\mathsf{PSL}_2(\mathbb{R})$ is isomorphic to $\mathbb{Z}$ and we call Spin^r the quotient of $\widetilde{\mathsf{PSL}}_2(\mathbb{R})$, the universal cover of $\mathsf{PSL}_2(\mathbb{R})$, by $r\mathbb{Z} < \pi_1(\mathsf{PSL}_2(\mathbb{R}))$. The Lie group Spin^r is

a connected simple Lie group which fits into the following exact sequence

$$0 \to \mathbb{Z}/r\mathbb{Z} \to \mathsf{Spin}^r \to \mathsf{PSL}_2(\mathbb{R}) \to 1\,.$$

For $g \in \mathsf{PSL}_2(\mathbb{R})$, let $\mathrm{Rot}(g) \in \mathbb{R}/\mathbb{Z}$ be the *rotation number* of g as the rotation number of a homeomorphism of the circle. Since $\mathrm{Rot} : \mathsf{PSL}_2(\mathbb{R}) \to \mathbb{R}/\mathbb{Z}$ is continuous, we may lift it in a unique way in $\mathrm{Trans} : \widetilde{\mathsf{PSL}}_2(\mathbb{R}) \to \mathbb{R}$ with $\mathrm{Trans}(e) = 0$. If $g \in \mathsf{Spin}^r$, the translation numbers of the lifts of g to $\widetilde{\mathsf{PSL}}_2(\mathbb{R})$ differ by some multiple of r, so that we have a well defined continuous function

$$\mathrm{Trans}^{(r)} : \mathsf{Spin}^r \to \mathbb{R}/r\mathbb{Z}$$

which is the translation number function of Spin^r acting on $\mathbb{R}/r\mathbb{Z}$. Moreover if $g \in \mathsf{PSL}_2(\mathbb{R})$ is parabolic or hyperbolic, then g fixes a point on the circle and $\mathrm{Rot}(g) = 0$, so that the translation numbers of its lifts are integers.

Let $\mathrm{Rep}_{\mathrm{Max}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}})$ be the set of conjugacy classes of maximal homomorphisms $\rho : \pi_1(S) \to \mathsf{Spin}^r$ such that, if c_j is a loop about p_j in the positive direction, $\rho(c_j)$ is parabolic and has translation number $-m_j \mod r$. As a consequence of Theorem 13 of [BIW10], we have that for $\mathrm{Rep}_{\mathrm{Max}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}})$ to be non-empty, one must have $\sum m_j = 2g - 2 + s \mod r$. In Chapter 2 we prove the following statement.

Theorem 1
Let S be a genus g compact oriented connected surface with s points removed and $2g-2+s > 0$, $r > 0$ some integer and $\mathfrak{m} = (m_1, \ldots, m_s) \in \mathbb{Z}^r$ such that $\sum_{j=1}^s m_j = 2g - 2 + s \mod r$.

There exists an analytic diffeomorphism

$$\mathrm{Hol}_r : \mathcal{M}_{g,s}^{r,\mathfrak{m}} \longrightarrow \mathrm{Rep}_{\mathrm{Max}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}})/\Gamma(S)$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathcal{M}_{g,s}^{r,\mathfrak{m}} & \xrightarrow{\mathrm{Hol}_r} & \mathrm{Rep}_{\mathrm{Max}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}})/\Gamma(S) \\ \downarrow & & \downarrow \\ \mathcal{M}_{g,s} & \xrightarrow{\mathrm{Hol}} & \mathrm{Rep}_{\mathrm{Max}}(S, \mathsf{PSL}_2(\mathbb{R})_2, P)/\Gamma(S)\,. \end{array}$$

This point of view on r-spin structures allows us to apply a result of A. Kastuda and T. Sunada [KS90] to get the following theorem for the case $s = 0$. Let S be a closed genus g surface and $\rho : \pi_1(S) \to \mathsf{Spin}^r$ a faithful and discrete homomorphism. The composition $\pi_1(S) \xrightarrow{\rho} \mathsf{Spin}^r \to \mathsf{PSL}_2(\mathbb{R})$ yields a hyperbolic structure Σ on S. To a closed geodesic $\gamma \subset \Sigma$, we associate its length $\ell_\Sigma(\gamma)$ and its *translation number* given by $\mathrm{Trans}^{(r)}(\rho([\gamma]))$,

where $[\gamma] \subset \pi_1(S)$ is the conjugacy class associated to the geodesic γ. To count the number of closed geodesics of length less than L and with given translation number, we introduce the following notations for $L > 0$ and $m \in \mathbb{Z}$

$$\begin{aligned}\mathcal{C}(L,m) &= \{\gamma \text{ closed geodesic on } \Sigma,\ \ell_\Sigma(\gamma) < L,\ \mathrm{Trans}(\rho([\gamma])) = m \mod r\}\,,\\ C(L,m) &= \mathrm{Card}(\mathcal{C}(L,m))\,.\end{aligned}$$

From work of Margulis [Mar69, Mar04], it follows easily that

$$\sum_{m=0}^{r-1} C(L,m) \sim \frac{\exp(L)}{L}\,.$$

As an application of Theorems 1 and 3 by A. Kastuda and T. Sunada in [KS90], we prove in Chapter 3 a more precise result

Theorem 2
Let S be a genus g compact oriented connected surface, $g \geqslant 2$. Let $r \in \mathbb{N}$ be some integer with $r \mid 2g-2$ and $\rho : \pi_1(S) \to \mathsf{Spin}^r$ a faithful and discrete homomorphism, Σ the associated hyperbolic structure on S.

Then for all $m \in \{0, \dots, r-1\}$,

$$C(L,m) \sim \frac{\exp(L)}{rL}\,,$$

and the geodesics with translation number m modulo r are equidistributed *in the following sense: for all $f \in C^\infty(S)$,*

$$\lim_{L\to\infty} \frac{1}{C(L,m)} \sum_{\gamma\in\mathcal{C}(L,m)} \frac{1}{\ell_\Sigma(\gamma)} \int_\gamma f = \int_\Sigma f \,\mathrm{d}\,\mathrm{vol}\,,$$

where

$$\int_\gamma f = \int_{t\in I} f(c(t))|\dot{c}(t)|\mathrm{d}t$$

for $c : I \to \Sigma$ a parametrization of γ.

Chapter 1

Definitions and notations

1.1 Uniformization theorem for pointed Riemann surfaces

1.1.1 Riemann surfaces

definition and first examples

Definition 1.1.1.

- A *Riemann surface* X is the data of a second countable connected Hausdorff topological space X_0 and a collection of charts (U_i, ϕ_i) where for all i, U_i is an open subset of X_0, $\phi_i : U_i \to \mathbb{C}$ is a homeomorphism on its image and for all i, j such that $U_i \cap U_j \neq \emptyset$,

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$$

  is a biholomorphism.

- For two Riemann surfaces X and Y, a continuous map $f : X \to Y$ is *holomorphic* if it is holomorphic in every chart.

- Two Riemann surfaces are *equivalent* if there exists a biholomorphism between them.

Example 1.1.1. The complex plane $\mathbb{C}$ is a Riemann surface. Any open set of $\mathbb{C}$ is also a Riemann surface, in particular the half plane $\mathbb{H} = \{\tau \in \mathbb{C},\, \Im(\tau) > 0\}$ and the unit disc $\mathbb{D} = \{z \in \mathbb{C},\, |z| < 1\}$ are Riemann surfaces. The Riemann sphere $\mathbb{P}^1\mathbb{C} = \mathbb{C} \cup \{\infty\}$ is also a Riemann surface, where the charts are given by $(\mathbb{C}, \mathrm{Id})$ and $(\mathbb{C} \cup \{\infty\} \setminus \{0\}, z \mapsto 1/z)$.

Compact case A *compact Riemann surface* is a Riemann surface whose underlying topological space is compact.

Since biholomorphic maps are in particular smooth orientation preserving diffeomorphisms, every Riemann surface is a connected smooth oriented manifold of real dimension 2.

If S is a compact oriented connected smooth surface, the *genus* of S is the maximal number of non intersecting loops one may draw one S without disconnecting S. The underlying space of a compact Riemann surface is uniquely characterized by its genus, as shown by

Theorem 3 (Classification of compact surfaces)
Every compact orientable connected surface S is homeomorphic to S_g for $g \in \mathbb{Z}_{\geqslant 0}$ the genus of S, where S_g is the sphere with g handles attached. Two surfaces with different genus are not homeomorphic.

Definition 1.1.2. A *pointed Riemann surface* $(X; x_1, \ldots, x_s)$ is the data of a compact Riemann surface X together with s distinct numbered points $x_1, \ldots, x_s \in X$. A *biholomorphism of pointed Riemann surfaces* $f : (X; x_1, \ldots, x_s) \to (X'; x'_1, \ldots, x'_s)$ is a biholomorphism from X to X' such that $f(x_j) = x'_j$ for all $j = 1, \ldots, s$.

Uniformization theorem We saw in Example 1.1.1 three Riemann surfaces: the half plane $\mathbb{H}$, the complex plane $\mathbb{C}$ and the Riemann sphere $\mathbb{P}^1\mathbb{C}$. They are, up to biholomorphism, the only simply connected Riemann surfaces, as is stated by the

Theorem 4 (Uniformization theorem)
Let X be a simply connected Riemann surface. Then X is biholomorphic either to $\mathbb{H}$, $\mathbb{C}$ or $\mathbb{P}^1\mathbb{C}$.

As a consequence, the universal cover of any Riemann surface is either the half plane, the complex plane of the Riemann sphere.

Biholomorphisms of a Riemann surface Every Riemann surface is either the Riemann sphere, a quotient of $\mathbb{C}$ by a group of biholomorphisms or a quotient of $\mathbb{H}$ by a group of biholomorphisms. We have the following classical fact.

Proposition 1.1.1. *The group of biholomorphisms of the Riemann sphere is $\mathsf{PSL}_2(\mathbb{C})$, where for $z \in \mathbb{C} \cup \{\infty\}$ and $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathsf{PSL}_2(\mathbb{C})$,*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot z = \frac{az + b}{cz + d}.$$

Biholomorphisms of $\mathbb{H}$ (resp. $\mathbb{C}$) are biholomorphisms of $\mathbb{P}^1\mathbb{C}$ mapping $\mathbb{H}$ (resp. $\mathbb{C}$) to itself, that is

- *The group of biholomorphisms of* $\mathbb{H}$ *is* $\mathsf{PSL}_2(\mathbb{R})$.
- *The group of biholomorphisms of* $\mathbb{C}$ *is the affine group of maps of the form* $z \mapsto az+b$ *where* $a, b \in \mathbb{C}$ *and* $a \neq 0$.

We will be mostly interested in compact pointed Riemann surfaces whose biholomorphisms group is finite.

Proposition 1.1.2. *Let* X *be a compact pointed Riemann surface of genus* g *and with* s *points. The group of biholomorphisms of* X *is finite if and only if* $2g-2+s>0$.

Moduli space, Teichmüller space and mapping class group

Let $\overline{S} = S_g$ be a compact connected oriented genus g surface and let $P = \{p_1, \dots, p_s\}$ be s numbered distinct points on $\overline{S}$, with $2g-2+s>0$. Let $S = S_{g,s} = \overline{S} - P$. We will say that $S_{g,s}$ is a *compact connected oriented surface with* s *points removed* or *with* s *punctures*.

Teichmüller space A *marked pointed Riemann surface over* S is the data of (Y, ϕ) where $Y = (X; x_1, \dots, x_s)$ is a pointed Riemann surface and $\phi : \overline{S} \to X$ is a diffeomorphism with $\phi(p_j) = x_j$.

Two marked pointed Riemann surfaces (Y, ϕ) and (Y', ϕ') are *equivalent* if there exists a biholomorphism $h : Y \to Y'$ and an orientation preserving diffeomorphism $f : \overline{S} \to \overline{S}$ fixing each p_j and isotopic to the identity such that the following diagram commutes

$$\begin{array}{ccc} \overline{S} & \xrightarrow{\phi} & Y \\ \downarrow f & & \downarrow h \\ \overline{S} & \xrightarrow{\phi'} & Y' . \end{array}$$

Let $\mathscr{T}(S)$ be the *Teichmüller space* of S, that is the space of all marked pointed Riemann surfaces up to equivalence. Teichmüller space comes with a natural complex structure and

Theorem 5
Teichmüller space is biholomorphic to a complex ball of dimension $3g-3+s$.

Mapping class group The *mapping class group* $\Gamma(\overline{S}, P)$ *of* $(\overline{S}, P)$ is the group of orientation preserving diffeomorphisms of $\overline{S}$ that fix P pointwise up to isotopy fixing P

$$\Gamma(\overline{S}, P) = \mathrm{Diff}^+(\overline{S}, P) / \mathrm{Diff}_0(\overline{S}, P) .$$

We will most often write $\Gamma(S) = \Gamma(\overline{S}, P)$, remembering the order of the cusps. The mapping class group acts on Teichmüller space: if $f \in \mathrm{Diff}^+(\overline{S}, P)$, the equivalence class in Teichmüller space of

$$f \cdot (Y, \phi) = (Y, \phi \circ f^{-1})$$

depends only of the isotopy class of f.

Theorem 6
The mapping class group acts properly discontinuously on Teichmüller space.

Moduli space The quotient $\mathcal{M}_{g,s} = \mathscr{T}(S)/\Gamma(S)$ is the *moduli space of pointed Riemann surfaces of genus g and s points*. It is the space of biholomorphism classes of compact pointed genus g Riemann surfaces with s points.

1.1.2 Hyperbolic surfaces and holonomy

The hyperbolic plane

Hyperbolic surfaces are 2 dimensional manifold modelled on the hyperbolic plane.

The half plane model of the hyperbolic plane is the Riemannian manifold $\mathbb{H} = \{x + iy,\ (x, y) \in \mathbb{R} \times \mathbb{R}_{>0}\}$ equipped with the metric

$$ds^2 = \frac{dx^2 + dy^2}{y^2}\,.$$

The surface $\mathbb{H}$ is the only (up to isometries) simply connected complete Riemannian surface of constant sectionnal curvature equal to -1. Its group of orientation-preserving isometries is $\mathsf{PSL}_2(\mathbb{R})$.

Boundary at infinity The *boundary at infinity* $\partial_\infty \mathbb{H}$ of $\mathbb{H}$ is the set of equivalence classes of geodesic rays where two rays are equivalent if they are at bounded Hausdorff distance. The boundary at infinity of $\mathbb{H}$ naturally identifies with $\mathbf{P}^1\mathbb{R} = \mathbb{R} \cup \{\infty\}$ and isometries of $\mathbb{H}$ extend continuously on $\mathbf{P}^1\mathbb{R}$, acting by homography.

Isometries of the hyperbolic plane Let $g = [A] \in \mathsf{PSL}_2(\mathbb{R})$ be a non trivial isometry of $\mathbb{H}$. We say that g is *hyperbolic* if $|\mathrm{Tr}(A)| > 2$, *parabolic* if $|\mathrm{Tr}(A)| = 2$ and *elliptic* if $|\mathrm{Tr}(A)| < 2$.

If g is hyperbolic, g is conjugated to

$$H_\lambda = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{bmatrix}$$

for a unique $\lambda > 1$, has no fixed point in $\mathbb{H}$ and fixes two points in $\mathbb{P}^1\mathbb{R}$, one attracting and one repulsive. If g is parabolic, g is conjugated to

$$P_\varepsilon = \begin{bmatrix} 1 & \varepsilon \\ 0 & 1 \end{bmatrix}$$

for a unique $\varepsilon \in \{+1, -1\}$, has no fixed point in $\mathbb{H}$ and fixes one point in $\mathbf{P}^1\mathbb{R}$. We will say that g is *positive* parabolic (resp. *negative* parabolic) if g is conjugated to P_1 (resp. to P_{-1}). If g is elliptic, g is conjugated to

$$R_\theta = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

for a unique $\theta \in (0, \pi)$, has a fixed point in $\mathbb{H}$ and no fixed point in $\mathbf{P}^1\mathbb{R}$.

Rotation number For $g \in \mathsf{PSL}_2(\mathbb{R})$, the *rotation number of* g is $\mathrm{Rot}(g) \in \mathbb{R}/\mathbb{Z}$ defined as the rotation number of the homeomorphism of the circle $\mathbf{P}^1\mathbb{R}$ induced by g. The rotation number is the only function $f : \mathsf{PSL}_2(\mathbb{R}) \to \mathbb{R}/\mathbb{Z}$ such that

- f is central, that is $f(ghg^{-1}) = f(h)$ for all $g, h \in \mathsf{PSL}_2(\mathbb{R})$;
- $f(H_\lambda) = f(P_\varepsilon) = 0$;
- $f(R_\theta) = \theta/\pi$.

Existence and uniqueness of such a function follows from the classification of isometries of $\mathbb{H}$. As a consequence, $\mathrm{Rot} : \mathsf{PSL}_2(\mathbb{R}) \to \mathbb{R}/\mathbb{Z}$ is continuous since

$$\mathrm{Rot}([A]) = \arccos\left(\frac{1}{2}\min\left(|\mathrm{Tr}(A)|, 2\right)\right) \mod \mathbb{Z}$$

where $[A] \mapsto \min(|\mathrm{Tr}(A)|, 2)$ is continuous.

Hyperbolic surfaces

Definition 1.1.3.

A *hyperbolic surface* Σ is a smooth connected oriented surface equipped with a complete Riemannian metric of constant curvature -1.

The universal cover of a hyperbolic surface is isometric to the hyperbolic plane, thus every hyperbolic surface is a quotient $\mathbb{H}/\Lambda$ for Λ some subgroup of $\mathsf{PSL}_2(\mathbb{R})$. If the volume of Σ is finite, we have

Theorem (Gauss-Bonnet formula)
Let Σ be a finite volume hyperbolic surface. Then

$$\mathrm{Area}(\Sigma) = -2\pi\chi(\Sigma)\,.$$

In particular, the Euler characteristic of a hyperbolic surface is negative. The underlying topological space of a finite volume hyperbolic surface is some $S_{g,s}$ for $2g-2+s>0$.

Fricke space

Let $S = S_{g,s}$ be a compact oriented connected surface with s punctures and suppose $2g-2+s>0$.

A *marked hyperbolic surface of finite volume* (Σ, ϕ) is the data of a hyperbolic surface Σ of finite volume and a diffeomorphism $\phi : S \to \Sigma$.

Two marked hyperbolic surfaces of finite volume (Σ_1, ϕ_1) and (Σ_2, ϕ_2) are *equivalent* if there exists an isometry $I : \Sigma_1 \to \Sigma_2$ and an orientation preserving diffeomorphism $f : S \to S$ fixing each puncture and isotopic to the identity such that the following diagram commutes

$$\begin{array}{ccc} S & \xrightarrow{\phi_1} & \Sigma_1 \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle I} \\ S & \xrightarrow{\phi_2} & \Sigma_2\,. \end{array}$$

The *Fricke space* $\mathfrak{F}(S)$ is the space of all marked finite volume hyperbolic surfaces up to equivalence.

Representation variety

Let $x \in S$ and $\pi_1(S) = \pi_1(S, x)$ be the fundamental group of S based at x. The group $\pi_1(S)$ admits the following presentation

$$\pi_1(S) = \left\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_s \middle| \prod [a_i, b_i] \prod c_j = 1 \right\rangle \tag{1.1}$$

where $[a,b] = aba^{-1}b^{-1}$. It is free if and only if $s>0$.

For any Lie group G, let $\mathrm{Hom}(\pi_1(S), \mathsf{G})$ be the space of homomorphism $\rho : \pi_1(S) \to \mathsf{G}$. Such a homomorphism is determined by the image of the generators of $\pi_1(S)$, so that if $s=0$, $\mathrm{Hom}(\pi_1(S), \mathsf{G})$ identifies with a subset of $2g$ copies of G

$$\mathrm{Hom}(\pi_1(S), \mathsf{G}) = \left\{ A_1, B_1, \ldots, A_g, B_g \in \mathsf{G},\, \prod [A_i, B_i] = 1 \right\}$$

and if $s > 0$, $\pi_1(S)$ is a free group on $2g + s - 1$ generators and we have

$$\mathrm{Hom}(\pi_1(S), \mathsf{G}) \simeq \mathsf{G}^{2g+s-1}\,.$$

For $g \in \mathsf{G}$ and $\rho \in \mathrm{Hom}(\pi_1(S), \mathsf{G})$ we may define the conjugate of ρ by g to be

$$\begin{array}{rccc} g \cdot \rho : & \pi_1(S) & \to & \mathsf{G} \\ & \gamma & \mapsto & g\rho(\gamma)g^{-1}\,. \end{array}$$

The *representation variety* $\mathrm{Rep}(\pi_1(S), \mathsf{G})$ of $\pi_1(S)$ into G is the quotient of $\mathrm{Hom}(\pi_1(S), \mathsf{G})$ by G

$$\mathrm{Rep}(\pi_1(S), \mathsf{G}) = \mathrm{Hom}(\pi_1(S), \mathsf{G})/\mathsf{G}\,.$$

Note that with this definition, some points of $\mathrm{Rep}(\pi_1(S), \mathsf{G})$ might not be closed for the quotient topology, however we will be interested in the subset of maximal representations $\mathrm{Rep}_{\mathrm{Max}}(S, \mathsf{G}) \subset \mathrm{Rep}(\pi_1(S), \mathsf{G})$ which is Hausdorff for the quotient topology.

Relative representation variety Suppose $s > 0$ and let c_j be a loop in the positive direction around the jth puncture for all j. Inside $\pi_1(S)$ the conjugacy class of c_j is well defined and if $C_j \subset \mathsf{G}$ is a conjugacy class, we define the *relative character variety* of S to be

$$\mathrm{Rep}(\pi_1(S), \mathsf{G}, C) = \{[\rho] \in \mathrm{Rep}(\pi_1(S), \mathsf{G}),\ \rho(c_j) \in C_j\ \forall j\}\ .$$

Mapping class group and outer automorphims The group $\mathrm{Aut}(\pi_1(S))$ of automorphisms of $\pi_1(S)$ acts on $\mathrm{Rep}(\pi_1(S), \mathsf{G})$ by precomposition. For $\eta \in \pi_1(S)$, let $I_\eta : \pi_1(S) \to \pi_1(S)$, $\gamma \mapsto \eta\gamma\eta^{-1}$ be an inner automorphism. For all $\gamma \in \pi_1(S)$ and $\rho : \pi_1(S) \to \mathsf{G}$,

$$\begin{aligned} (I_\eta \cdot \rho)(\gamma) &= \rho(\eta^{-1}\gamma\eta) \\ &= \rho(\eta)^{-1}\rho(\gamma)\rho(\eta) \\ &= (\rho(\eta) \cdot \rho)(\gamma) \end{aligned}$$

so that ρ and $I_\eta \cdot \rho$ define the same point in $\mathrm{Rep}(\pi_1(S), \mathsf{G})$, and we get an action of $\mathrm{Out}(\pi_1(S))$ on $\mathrm{Rep}(\pi_1(S), \mathsf{G})$ where

$$\mathrm{Out}(\pi_1(S)) = \mathrm{Aut}(\pi_1(S))/\,\mathrm{Inn}(\pi_1(S))$$

is the quotient of $\mathrm{Aut}(\pi_1(S))$ by the normal subgroup $\mathrm{Inn}(\pi_1(S))$ of inner automorphisms.

Let f be an orientation-preserving diffeomorphism of S sending the jth puncture to

itself for all j. Post-composition by f yields

$$f_* : \pi_1(S, x) \to \pi_1(S, f(x))\,.$$

Identifying $\pi_1(S, x)$ with $\pi_1(S, f(x))$ up to some inner automorphism, we have a well defined map

$$\mathrm{Diff}^+(S) \longrightarrow \mathrm{Out}(\pi_1(S))$$

and if f and g are isotopic they induce, up to some inner automorphism, the same automorphism of $\pi_1(S)$, thus we get a homomorphism

$$\sigma : \Gamma(S) \longrightarrow \mathrm{Out}(\pi_1(S))$$

which is always injective. More precisely, we have

Theorem 7 (Dehn-Nielsen-Baer)
If $S = S_g$ is a compact surface, then the following sequence is exact

$$1 \to \Gamma(S) \xrightarrow{\sigma} \mathrm{Out}(\pi_1(S)) \to \mathbb{Z}/2\mathbb{Z} \to 0\,.$$

For $S = S_{g,s}$ with $s > 0$, the following sequence is exact

$$1 \to \Gamma(S) \xrightarrow{\sigma} \mathrm{Out}^\star(\pi_1(S)) \to \mathbb{Z}/2\mathbb{Z} \to 0\,,$$

where $\mathrm{Out}^\star(\pi_1(S))$ is the subgroup of $\mathrm{Out}(\pi_1(S))$ preserving the conjugacy class of the simple closed curves around each punctures.

In particular, $\Gamma(S)$ acts on the relative character varieties.

Holonomy of a marked hyperbolic surface Let (Σ, ϕ) be a marked hyperbolic surface with finite volume. The universal cover of Σ is isometric to $\mathbb{H}$, so that $\Sigma = \mathbb{H}/\Lambda$ for $i : \Lambda < \mathsf{PSL}_2(\mathbb{R})$ a torsion free lattice. The marking ϕ yields an isomorphism $\phi_* : \pi_1(S) \to \Lambda$ and we get a homomorphism $h = i \circ \phi_* : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$. Moreover, because Σ has finite volume, each peripheral element is sent to a positive parabolic one. With P denoting the conjugacy class of positive parabolic elements in $\mathsf{PSL}_2(\mathbb{R})$, the resulting map

$$\mathrm{Hol} : \mathfrak{F}(S) \to \mathrm{Rep}_{\mathrm{Max}}(\pi_1(S), \mathsf{PSL}_2(\mathbb{R}), P)$$

is well defined. The holonomy map gives a natural topology on $\mathfrak{F}(S)$. A stronger statement is the following, see [FM12], Section 10.6.3.

Theorem 8
The Fricke space is a smooth real manifold diffeomorphic to an open ball of dimension $6g-6+2s$.

1.1.3 Pointed Riemann surfaces and finite volume hyperbolic surfaces

Since the orientation-preserving isometries of the hyperbolic plane are exactly its biholomorphisms and and since every local isometry from $\mathbb{H}$ to $\mathbb{H}$ is a biholomorphism, every hyperbolic surface is a Riemann surface, and two isometric hyperbolic surfaces are biholomorphic when seen as Riemann surfaces.

We have the following converse

Theorem 9 (Uniformization for pointed Riemann surfaces)
Let $(X; x_1, \dots, x_s)$ *be a pointed Riemann surface where* X *has genus* g *and* $2g-2+s>0$. *Up to isometries, there exists a unique hyperbolic surface structure with finite volume* Σ *on* $S_{g,s}$ *such that* $X \setminus \{x_1, \dots, x_s\}$ *is biholomorphic to* Σ *seen as a Riemann surface.*

Through this identification, pointed points are sent to cusps. In particular, we have the following identification between Teichmüller space and the Fricke space.

$$\mathscr{T}(S) \simeq \mathfrak{F}(S)$$

and taking the quotient by $\Gamma(S)$,

$$\begin{aligned}\mathcal{M}_{g,s} &\simeq \{\text{Isometry classes of finite volume hyperbolic surfaces of genus } g \text{ with } s \text{ cusps}\}\\ &\simeq \mathrm{Rep}(\pi_1(S), \mathsf{PSL}_2(\mathbb{R}), P)/\Gamma(S)\,.\end{aligned}$$

1.2 r-spin structures and homomorphisms into Spin^r

1.2.1 First definitions

Sheaves, line bundles, divisors

Sheaves Let Z be any Riemann surface.

Definition 1.2.1. A *sheaf* $\mathcal{F}$ over Z is the data of a non empty set $\mathcal{F}(U)$ for every open set $U \subset Z$ and of *restriction maps* $s_{U,V} : \mathcal{F}(V) \to \mathcal{F}(U)$ for every open sets $U \subset V \subset Z$, such that

- For all open set U, $s_{U,U} = \mathrm{Id}_{\mathcal{F}(U)}$;

- For all open sets $U \subset V \subset W$, $s_{U,W} = s_{U,V} \circ s_{V,W}$.

And for $U = \bigcup_\alpha U_\alpha$,

- For all $(F_\alpha)_\alpha \in (\mathcal{F}(U_\alpha))_\alpha$ such that for all α, β, $s_{U_\alpha \cap U_\beta, U_\alpha}(F_\alpha) = s_{U_\alpha \cap U_\beta, U_\beta}(F_\beta)$, there exists $F \in \mathcal{F}(U)$ such that $s_{U_\alpha, U}(F) = F_\alpha$;
- For all $F, G \in \mathcal{F}(U)$, if $s_{U_\alpha, U}(F) = s_{U_\alpha, U}(G)$ for all α, then $F = G$.

Sheaves most often come with structure. A sheaf $\mathcal{F}$ is said to be of *abelian groups* (resp. *rings, modules, algebras*) if the sets $\mathcal{F}(U)$ are abelian groups (resp. rings, modules, algebras) and the maps $s_{U,V}$ are morphisms of these structures. In these cases we require $\mathcal{F}(\emptyset) = \{0\}$.

If $\iota : Z \to X$ is a holomorphic map between two Riemann surfaces and if $\mathcal{F}$ is a sheaf on Z, the *push forward* sheaf $\iota_*\mathcal{F}$ of $\mathcal{F}$ by ι is a sheaf on X given by, for every $U \subset X$ open set,

$$\iota_*\mathcal{F}(U) = \mathcal{F}(\iota^{-1}(U)) .$$

Example 1.2.1. For any open set $U \subset Z$, let $\mathcal{O}_Z(U)$ be the ring of holomorphic functions $U \to \mathbb{C}$. Equipped with the restriction maps $s_{U,V} : f \mapsto f|_U$, $\mathcal{O}_Z$ is a sheaf of rings.

If $\mathbb{D}$ is the unit disk in $\mathbb{C}$, $\mathbb{D}^\times$ the punctured unit disk and $\iota : \mathbb{D}^\times \to \mathbb{D}$ the inclusion, then the push forward sheaf

$$\iota_*\mathcal{O}_{\mathbb{D}^\times}$$

is a sheaf on $\mathbb{D}$ which is not isomorphic to $\mathcal{O}_\mathbb{D}$. Indeed, there are more holomorphic functions $\mathbb{D}^\times \to \mathbb{C}$ than holomorphic functions $\mathbb{D} \to \mathbb{C}$.

A sheaf $\mathcal{F}$ over Z is a *$\mathcal{O}_Z$-module* if for every open set U, $\mathcal{F}(U)$ is a $\mathcal{O}_Z(U)$-module and the module multiplication is compatible with the restriction maps.

If $\iota : Z \to X$ is a holomorphic map between two Riemann surfaces and if $\mathcal{F}$ is a sheaf on Z, the *push forward* sheaf $\iota_*\mathcal{F}$ of $\mathcal{F}$ by ι is a sheaf on X given by, for every $U \subset X$ open set,

$$\iota_*\mathcal{F}(U) = \mathcal{F}(\iota^{-1}(U)) .$$

Vector bundle

Definition 1.2.2. Let A be a smooth manifold, $K = \mathbb{R}$ or $K = \mathbb{C}$ and $\ell \in \mathbb{Z}_{>0}$. A *vector bundle of rank l over A* is the data of a smooth manifold E and a smooth surjective map $p : E \to A$ together with a dimension ℓ vector space structure on each $E_a := p^{-1}(a)$ for all $a \in A$, such that there exists a covering $(U_\alpha)_\alpha$ of A and smooth diffeomorphisms, called *trivializations*, $\phi_\alpha : E|_{U_\alpha} := p^{-1}(U_\alpha) \to U_\alpha \times K^\ell$ such that

- The following diagram commutes

$$\begin{array}{ccc} E|_{U_\alpha} & \xrightarrow{\phi_\alpha} & U_\alpha \times K^\ell \\ & \searrow^{p} & \downarrow \mathrm{pr}_1 \\ & & U_\alpha \,; \end{array}$$

- For all $a \in U_\alpha$, $E_a \xrightarrow{\phi_\alpha} \{a\} \times K^\ell$ is a linear isomorphism.

We call A the *base*, E_a the *fiber* and E the *total space*.

If $p : E \to A$ is a K-vector bundle of rank l and U_α a covering of A such that $E|_{U_\alpha}$ is trivial for all α, we get *transition functions* $g_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathsf{GL}_l(K)$ for all U_α, U_β with $U_\alpha \cap U_\beta \neq \emptyset$ such that

$$\begin{array}{rccc} \phi_\alpha \circ \phi_\beta^{-1} \; : & U_\alpha \cap U_\beta \times K^\ell & \longrightarrow & U_\alpha \cap U_\beta \times K^\ell \\ & (x, v) & \longmapsto & (x, g_{\alpha,\beta}(x)v)\,. \end{array}$$

The data of the transition functions determines the bundle.

A *section* of a vector bundle $p : E \to A$ is a function $s : A \to E$ such that for all $a \in A$, $s(a) \in E_a$. The *zero section of* E is the section s such that for all $a \in A$, $s(a) = 0_{E_a}$. Let $E^\circ = E - s(A)$. If E has real rank 2 and comes with an orientation, say E is the tangent bundle of an oriented surface or is a holomorphic bundle, then a *loop around the origin* in E° is a loop going once in a fiber of E° in the positive direction. We will write t_E for the homotopy or homology class (context will tell) of such a loop, and call such a class *the* loop around the origin.

A vector bundle is said *complex* if its fibers are complex vector spaces, *real* if its fibers are real vector spaces.

If its base is a complex manifold, a complex vector bundle is *holomorphic* if the trivializations are holomorphic. A *(Euclidean, Hermitian) metric* on a vector bundle $p : E \to A$ is a smooth function $|\cdot| : E \to \mathbb{R}_{\geqslant 0}$ that restricts to a (Euclidean, Hermitian) metric on each fiber.

Over a complex manifold, by a *line bundle* we will mean a complex holomorphic vector bundle of rank 1.

Invertible sheaf If $L \to X$ is a line bundle over a complex manifold X, we construct $\mathcal{F}_L$ the *sheaf of holomorphic sections of* L by setting

$$\mathcal{F}_L(U) = \{\text{Holomorphic sections of } L|_U\}\,.$$

The sheaf $\mathcal{F}_L$ is a $\mathcal{O}_X$-module and if $L|_U \simeq U \times \mathbb{C}$ is trivial, then $(\mathcal{F}_L)|_U$ is isomorphic to $(\mathcal{O}_X)|_U$ as $(\mathcal{O}_X)|_U$-modules.

Definition 1.2.3. A sheaf $\mathcal{F}$ over X is said to be invertible if it is locally isomorphic to $\mathcal{O}_X$ as a $\mathcal{O}_X$-module, that is if for all $x \in X$, there exists an open set $U \subset X$ containing x and a sheaf isomorphism

$$\mathcal{F}|_U \simeq (\mathcal{O}_X)|_U \,.$$

If $\mathcal{F}$ is an invertible sheaf and U_α is a covering of X such that $\mathcal{F}|_{U_\alpha}$ is trivial, we get transition functions

$$g_{\alpha,\beta} : \mathcal{O}_X(U_\alpha \cap U_\beta) \to \mathcal{O}_X(U_\alpha \cap U_\beta)$$

and for all $x \in U_\alpha \cap U_\beta$, $g_{\alpha,\beta}$ is the multiplication by a non zero complex number. The collection $g_{\alpha,\beta}$ allows one to construct a line bundle $L \to X$ such that $\mathcal{F}_L = \mathcal{F}$ and we get the usual correspondence

Proposition 1.2.1. *Invertible sheaves and line bundles over X are in a bijective and natural correspondence.*

If $p : E \to A$ is a vector bundle and $f : B \to A$ a continuous map, the *pull back bundle* $f^*p : f^*E \to B$ is the bundle given by the total space

$$f^*E = \{(b, e) \in B \times E,\ f(b) = p(e)\} \,.$$

If $p : L \to X$ is a line bundle with associated sheaf of sections $\mathcal{L}$ and $\iota : Z \to X$ is holomorphic, the pull back ι^*L is again a line bundle and we define the $\iota^*\mathcal{L}$ to be the invertible sheaf associated with the line bundle ι^*L.

From two line bundles L_1, L_2 over Z we can construct the *tensor product* $L_1 \otimes L_2 \to Z$ defined by multiplying the transition function of L_1 and L_2. If $\mathcal{L}_1$ and $\mathcal{L}_2$ are the invertible sheaves of sections of L_1 and L_2 respectively, we get an invertible sheaf $\mathcal{L}_1 \otimes \mathcal{L}_2$ of sections of $L_1 \otimes L_2$.

Divisors A *divisor* on a Riemann surface X is an element of the free abelian group generated by the points of X. A divisor Δ is *non negative*, and we write $\Delta \geqslant 0$, if it can be written $\Delta = \sum n_j x_j$ with $x_j \in X$ and $n_j \in \mathbb{Z}_{\geqslant 0}$.

If s is a global meromorphic section of an invertible sheaf over a compact Riemann surface X, s has a finite number of poles and zeros and we define the divisors associated to s to be

$$\mathrm{div}(s) = \sum_{s(p)=0} n_p p - \sum_{p \text{ pole of } s} n_p p$$

where n_p is the multiplicity of p as a zero or pole of s.

If Δ is a divisor on X, we associate to Δ the sheaf $\mathcal{O}(\Delta)$ of meromorphic functions f on X such that $\mathrm{div}(f)+\Delta \geqslant 0$. If V_α is a covering of X and u_α a meromorphic function on V_α with $\mathrm{div}(u_\alpha) = \Delta$ on V_α, then $\mathcal{O}|_{V_\alpha} = u_\alpha^{-1}\mathcal{O}_X$ and thus $\mathcal{O}(\Delta)$ is an invertible sheaf. The line bundle L_Δ associated to $\mathcal{O}(\Delta)$ is given by the covering V_α and the transition function $g_{\alpha\beta} = u_\alpha/u_\beta$. We will often write $(\Delta) := L_\Delta$.

We have the sheaf isomorphism

$$\begin{aligned} \mathcal{O}(\Delta) &\longrightarrow \mathcal{O}(L_\Delta) \\ f &\longmapsto s \end{aligned}$$

locally given by $s = fu_\alpha$. We define s_Δ the *canonical section of* L_Δ to be the image of the constant meromorphic function $f = 1$ by this isomorphism. In particular, $\mathrm{div}(s_\Delta) = \Delta$. If $\Delta = \sum a_j x_j$ is a divisor on X and $Z = X \setminus \{x_1, \ldots, x_s\}$, pulling back the bundle (Δ) to Z we get the trivial bundle. Indeed, on the open set $Z \subset X$, the canonical section is constant equal to 1 and thus trivializes the bundle.

For any sheaf $\mathcal{F}$, we will write $\mathcal{F}(\Delta)$ for the sheaf of meromorphic sections s of $\mathcal{F}$ with $\mathrm{div}(s) + \Delta \geqslant 0$. It is naturally isomorphic to $\mathcal{F} \otimes \mathcal{O}(\Delta)$.

1.2.2 Definition of a r-spin structure

For X a Riemann surface, the *canonical sheaf* ω_X of X is the sheaf of holomorphic 1-forms. It is the invertible sheaf of sections of the canonical bundle K_X.

Definition 1.2.4.

- Let $r \in \mathbb{Z}_{>0}$ and $m_1, \ldots, m_s \in \mathbb{Z}$. A r-spin structure of type $(m_1, \ldots, m_s)$ on a pointed Riemann surface $(X; x_1, \ldots, x_s)$ is the data $(\mathcal{L}, f)$ of an invertible sheaf $\mathcal{L}$ and a sheaf isomorphism

$$f : \mathcal{L}^{\otimes r} \xrightarrow{\simeq} \omega_X(-\sum (m_j - 1)x_j)\,.$$

- A *r-spin Riemann surface* is a pointed Riemann surface equipped with a r-spin structure.

- Two r-spin Riemann surfaces $(X_1; x_1, \ldots, x_l, \mathcal{L}_1, f_1)$ and $(X_2; y_1, \ldots, y_l, \mathcal{L}_2, f_2)$ of a given type are *isomorphic* if there exists a pair (φ, ε) where $\varphi : (X_1; x_1, \ldots, x_l) \xrightarrow{\sim} (X_2; y_1, \ldots, y_l)$ is a biholomorphism and $\varepsilon : (\varphi^*\mathcal{L}_2, \varphi^* f_2) \xrightarrow{\sim} (\mathcal{L}_1, f_1)$ is an isomorphism of r-spin structures, that is $\varepsilon : \varphi^*\mathcal{L}_2 \to \mathcal{L}_1$ is an isomorphism of sheaves and

the following diagram commutes

$$\begin{array}{ccc} \varphi^*\mathcal{L}_2^{\otimes r} & \xrightarrow{\quad\varepsilon^{\otimes r}\quad} & \mathcal{L}_1^{\otimes r} \\ \downarrow{\scriptstyle \varphi^* f_2} & & \downarrow{\scriptstyle f_1} \\ \varphi^*\omega_{X_2}(-\sum(m_j-1)y_j) & \longrightarrow & \omega_{X_1}(-\sum(m_j-1)x_j))\,. \end{array}$$

Remark 1.2.1. For a given type $(m_1,\dots,m_s)$, r-spin structures exist if and only if $r \mid 2g-2+s-\sum m_j$ for degree reason. If non empty, the set of r-spin structures of type $(m_1,\dots,m_s)$ over X is an affine space over the space of r-roots of $\mathcal{O}_X$ which contains r^{2g} elements.

Let $\mathcal{M}_{g,s}^{r,\mathfrak{m}}$ be the *moduli space of r-spin Riemann surfaces*, that is the set of equivalence classes of r-spin compact pointed Riemann surfaces of genus g with s points and type $\mathfrak{m}=(m_1,\dots,m_s)$.

If for all j, $m'_j = m_j \mod r$ and if $\mathcal{L}'$ is a r-spin structure of type $(m'_1,\dots,m'_s)$, then $\mathcal{L}=\mathcal{L}'(\sum \frac{m'_j-m_j}{r}x_j)$ is a r-spin structure of type $(m_1,\dots,m_s)$ and the moduli spaces $\mathcal{M}_{g,s}^{r,\mathfrak{m}}$ and $\mathcal{M}_{g,s}^{r,\mathfrak{m}'}$ are canonically isomorphic. For this reason, we most often restrict ourself to the case where $0 \leqslant m_j \leqslant r-1$.

The forgetful map $\mathcal{M}_{g,s}^{r,\mathfrak{m}} \to \mathcal{M}_{g,s}$ forgetting about the r-spin structure is a regular r^{2g}-cover.

1.2.3 r-spin hyperbolizations

Let $S=S_{g,s}$ with $2g-2+s>0$ and let a_i,b_i,c_j be the generators of $\pi_1(S)$ given in Equation (1.1). We assume moreover that each c_j is represented by a loop winding around the jth puncture once in the positive direction.

Covers of $\mathsf{PSL}_2(\mathbb{R})$

The fundamental group of $\mathsf{PSL}_2(\mathbb{R})$ is isomorphic to $\mathbb{Z}$. Let τ be the relative homotopy class of the loop

$$t\in[0,1] \mapsto R_{\pi t} \in \mathsf{PSL}_2(\mathbb{R})\,.$$

The fundamental group $\pi_1(\mathsf{PSL}_2(\mathbb{R}))$ is generated by τ and we have the exact sequence

$$0 \to \mathbb{Z} \xrightarrow{1\mapsto\tau} \widetilde{\mathsf{PSL}}_2(\mathbb{R}) \to \mathsf{PSL}_2(\mathbb{R}) \to 1$$

where $\widetilde{\mathsf{PSL}}_2(\mathbb{R})$, the universal cover of $\mathsf{PSL}_2(\mathbb{R})$, acts on $\mathbb{R}$ lifting the action of $\mathsf{PSL}_2(\mathbb{R})$ on $\mathbf{P}^1\mathbb{R}$. The center of $\widetilde{\mathsf{PSL}}_2(\mathbb{R})$ is exactly $\{\tau^k,\, k \in \mathbb{Z}\}$. Non central elements are called *elliptic, parabolic* or *hyperbolic* depending on their image in $\mathsf{PSL}_2(\mathbb{R})$.

The continuous function $\mathrm{Rot} : \mathsf{PSL}_2(\mathbb{R}) \to \mathbb{R}/\mathbb{Z}$ lifts in a unique continuous function

$$\mathrm{Trans} : \widetilde{\mathsf{PSL}}_2(\mathbb{R}) \to \mathbb{R}$$

such that $\mathrm{Trans}(\mathrm{Id}) = 0$.

For $r \in \mathbb{Z}_{>0}$, let

$$\mathsf{Spin}^r = \widetilde{\mathsf{PSL}}_2(\mathbb{R})/\langle\tau^r\rangle$$

be the unique connected r-cover of $\mathsf{PSL}_2(\mathbb{R})$. With this notation, $\mathsf{Spin}^2 = \mathsf{SL}_2(\mathbb{R})$. For $g \in \mathsf{Spin}^r$, the *translation number modulo* r of g is a well defined element $\mathrm{Trans}^{(r)}(g) \in \mathbb{R}/r\mathbb{Z}$ equal to $\mathrm{Trans}(\overline{g}) \mod r$ for any $\overline{g} \in \widetilde{\mathsf{PSL}}_2(\mathbb{R})$ lifting g.

Lifting hyperbolizations to Spin^r

Let

$$p_* : \mathrm{Hom}(\pi_1(S), \mathsf{Spin}^r) \to \mathrm{Hom}(\pi_1(S), \mathsf{PSL}_2(\mathbb{R}))$$

be the map induced by the projection $p : \mathsf{Spin}^r \to \mathsf{PSL}_2(\mathbb{R})$.

Closed case and Euler number For a moment we assume that $S = S_g$ is a closed surface of genus $g > 1$. Because τ is central in $\widetilde{\mathsf{PSL}}_2(\mathbb{R})$, if $x, y \in \widetilde{\mathsf{PSL}}_2(\mathbb{R})$ the commutator $[x, y] \in \widetilde{\mathsf{PSL}}_2(\mathbb{R})$ depends only on the image of x and y in $\mathsf{PSL}_2(\mathbb{R})$. Hence a well defined map

$$\tilde{[\cdot,\cdot]} : \mathsf{PSL}_2(\mathbb{R}) \times \mathsf{PSL}_2(\mathbb{R}) \to \widetilde{\mathsf{PSL}}_2(\mathbb{R})\,.$$

Since here S is closed, $\prod_{i=1}^g [a_i, b_i] = 1$ and for $h \in \mathrm{Hom}(\pi_1(S_g), \mathsf{PSL}_2(\mathbb{R}))$, we have the *Euler number* of h, defined as the only integer $eu(\rho)$ such that

$$\tau^{eu(\rho)} = \prod \tilde{[\rho(a_i), \rho(b_i)]}\,.$$

The Euler number is a conjugacy invariant and W. Goldman showed in his thesis [Gol80]

Theorem (W. Goldman)
The homomorphism h is the holonomy of a hyperbolic structure on S_g if and only if we have the equality $eu(h) = 2g - 2$.

The Euler number tells us whether or not a homomorphism $h \in \mathrm{Hom}(\pi_1(S_g), \mathsf{PSL}_2(\mathbb{R}))$ has antecedents in $\mathrm{Hom}(\pi_1(S), \mathsf{Spin}^r)$. Indeed, h lifts to Spin^r if and only if $r \mid eu(h)$.

General case Back to the general case, that is $S = S_{g,s}$. If $s > 0$, the group $\pi_1(S)$ is free on $2g + s - 1$ generators and any homomorphism $h : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$ has r^{2g+s-1} lifts to Spin^r.

As an immediate consequence of Theorem 12 of [BIW10], we have

Corollary
Let $\rho \in \mathrm{Hom}(\pi_1(S), \mathsf{Spin}^r)$ be any lift of a hyperbolization of S. Then

$$-\sum_j \mathrm{Trans}^{(r)}(\rho(c_j)) = 2g - 2 + s \mod r\,. \tag{1.2}$$

If $h : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$ is a hyperbolization of S, the elements $h(c_j) \in \mathsf{PSL}_2(\mathbb{R})$ are either parabolic or hyperbolic. In particular, they have rotation number 0. If $\rho : \pi_1(S) \to \mathsf{Spin}^r$ yields an hyperbolization $p_*\rho : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$, we call ρ a *r-spin hyperbolization* and write $\mathrm{Hom}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r)$ for the space of all r-spin hyperbolizations of S in Spin^r. For any collection $\mathfrak{m} = (m_1, \dots, m_s) \in \{0, \dots, r-1\}^s$, let

$$\mathrm{Hom}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, \mathfrak{m}) = \{\rho \in \mathrm{Hom}_{\mathrm{Hyp}}(\pi_1(S), \mathsf{Spin}^r),\ \forall j\ \mathrm{Trans}^{(r)}(\rho(c_j)) = -m_j \mod r\}$$

and

$$\mathrm{Hom}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}}) = \{\rho \in \mathrm{Hom}_{\mathrm{Hyp}}(\pi_1(S), \mathsf{Spin}^r, \mathfrak{m}),\ \forall j\, \rho(c_j) \text{ is positive parabolic}\}\,.$$

As a consequence of Formula (1.2), one has that the spaces $\mathrm{Hom}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, \mathfrak{m})$ and $\mathrm{Hom}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}})$ are non empty if and only if $\sum m_j = 2g - 2 + s \mod r$, where $\sum m_j = 0$ whenever $s = 0$.

Affine structure on r-spin hyperbolizations Let

$$\mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, \mathfrak{m}) \quad \text{and} \quad \mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}})$$

be the quotient of $\mathrm{Hom}_{\mathrm{Hyp}}(\pi_1(S), \mathsf{Spin}^r, \mathfrak{m})$ and $\mathrm{Hom}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}})$ by Spin^r. Let

$$A^r = \{\xi \in \mathrm{Hom}(\pi_1(S), \mathbb{Z}/r\mathbb{Z}),\ \forall j\, \xi(c_j) = 0\}\,.$$

Using the fact that

$$0 \to \mathbb{Z}/r\mathbb{Z} \to \mathsf{Spin}^r \to \mathsf{PSL}_2(\mathbb{R}) \to 1$$

is a central extension, we deduce the following

Proposition 1.2.2. *Let $\mathfrak{m} = (m_1, \dots, m_s) \in \mathbb{Z}^s$ such that $\sum_{j=1}^s = 2g - 2 + s \mod r$.*

Then $\mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}})$ *is a* A^r*-principal bundle over* $\mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{PSL}_2(\mathbb{R}), P)$*, that is the* $\mathbb{Z}/r\mathbb{Z}$*-module* A^r *acts freely and transitively on the fibers of*

$$p_* : \mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}}) \to \mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{PSL}_2(\mathbb{R}), P).$$

Moreover, since $\mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{PSL}_2(\mathbb{R}), P)$ is the Teichmüller space, it is simply connected and the cover p_* must be trivial, so that we have the identification

$$\mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}}) \simeq \mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{PSL}_2(\mathbb{R}), P) \times A^r \,.$$

However this identification is not $\Gamma(S)$-equivariant.

Chapter 2

Proof of Theorem 1

Let S be a compact connected oriented genus g surface with s punctures, where $2g-2+s > 0$. We introduce marked r-spin Riemann surfaces in order to, later, define their holonomy.

Definition 2.0.1. A *marked* r-spin Riemann surface of type $\mathfrak{m} = (m_1, \ldots, m_s)$ is the data of a r-spin Riemann surface $(X; x_1, \ldots, x_s, \mathcal{L}, f)$ together with a diffeomorphism $\phi : (\overline{S}; p_1, \ldots, p_s) \to (X; x_1, \ldots, x_s)$.

An isomorphism of marked r-spin Riemann surfaces of type $\mathfrak{m}$ between $(\phi_1, Y_1, \mathcal{L}_1, f_1)$ and $(\phi_2, Y_2, \mathcal{L}_2, f_2)$ is an *isomorphism* of r-spin Riemann surfaces (φ, ε) between $(Y_1, \mathcal{L}_1, f_1)$ and $(Y_2, \mathcal{L}_2, f_2)$ where the diffeomorphism

$$\phi_2^{-1} \circ \varphi \circ \phi_1 : S \to S$$

is isotopic to the identity.

Let $\mathscr{T}^{r,\mathfrak{m}}(S)$ be the space of all marked r-spin Riemann surfaces of type $\mathfrak{m}$ up to isomorphism. The aim of this section is to prove the following proposition, which will imply Theorem 1.

Proposition 2.0.1. *There exists a $\Gamma(S)$-equivariant analytic diffeomorphism*

$$\mathrm{Hol}_r : \mathscr{T}^{r,\mathfrak{m}}(S) \longrightarrow \mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, \mathfrak{m})$$

such that the following diagram commutes

$$\begin{array}{ccc} \mathscr{T}^{r,\mathfrak{m}}(S) & \xrightarrow{\mathrm{Hol}_r} & \mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}}) \\ \downarrow & & \downarrow \\ \mathscr{T}(S) & \xrightarrow{\mathrm{Hol}} & \mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{PSL}_2(\mathbb{R}), P)\,. \end{array}$$

The main difficulty of the proposition is to deal with what happens at the pointed points. Consequently, we start by a local study above the disc.

2.1 Local case

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$, x its origin, $0 \leqslant m < r$ and $u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \in \mathsf{PSL}_2(\mathbb{R})$. We identify $\mathbb{H}/\langle u \rangle$ and $\mathbb{D}^\times = \mathbb{D} \setminus \{x\}$ through $\Psi : \tau \mapsto \exp(2i\pi\tau)$. Let $\iota : \mathbb{D}^\times \to \mathbb{D}$ be the inclusion.

Being the isometry group of $\mathbb{H}$, $\mathsf{PSL}_2(\mathbb{R})$ acts on the tangent bundle $T\mathbb{H}$ of $\mathbb{H}$ and by duality on the cotangent bundle $K_{\mathbb{H}} = T^*\mathbb{H}$ of $\mathbb{H}$. For $x \in \mathbb{H}$, $\alpha \in T_x^*H$ one has $g_*\alpha \in T_{gx}^*\mathbb{H}$ and for $X \in T_{gx}\mathbb{H}$

$$g_*\alpha(X) = \alpha((Tg)^{-1}X)\,.$$

Let $\omega_{\mathbb{H}}$ be the sheaf of holomorphic sections of $K_{\mathbb{H}}$. Since elements of $\mathsf{PSL}_2(\mathbb{R})$ are biholomorphic maps, $\mathsf{PSL}_2(\mathbb{R})$ acts on $\omega_{\mathbb{H}}$ and for $\sigma \in \omega_{\mathbb{H}}(\mathbb{H})$, $g \in \mathsf{PSL}_2(\mathbb{R})$ and $x \in \mathbb{H}$ one has

$$(g \cdot \sigma)(x) = g_*(\sigma(g^{-1}x))\,.$$

We construct an invertible sheaf $\mathcal{L}_{\mathbb{H}}$ on $\mathbb{H}$ such that $\mathcal{L}_{\mathbb{H}}^{\otimes r} = \omega_{\mathbb{H}}$. To do so, we construct a line bundle $L_{\mathbb{H}} \to \mathbb{H}$ whose sheaf of holomorphic sections will be $\mathcal{L}_{\mathbb{H}}$.

Let $i \in \mathbb{H}$ be the square root of unity. The projection

$$\begin{array}{ccc} \mathsf{PSL}_2(\mathbb{R}) & \to & \mathbb{H} \\ g & \mapsto & gi \end{array}$$

defines a principal $\mathbb{R}/\mathbb{Z}$ bundle where $v \in \mathbb{R}/\mathbb{Z}$ acts on $\mathsf{PSL}_2(\mathbb{R})$ via $v \cdot g = gR_{-v\pi}$ for all $g \in \mathsf{PSL}_2(\mathbb{R})$. From the circle bundle $\mathsf{PSL}_2(\mathbb{R}) \to \mathbb{H}$ we construct a line bundle $\mathsf{PSL}_2(\mathbb{R}) \times_{\mathbb{R}/\mathbb{Z}} \mathbb{C}$ in the following way

$$\mathsf{PSL}_2(\mathbb{R}) \times_{\mathbb{R}/\mathbb{Z}} \mathbb{C} = \{(g, \lambda) \in \mathsf{PSL}_2(\mathbb{R}) \times \mathbb{C}\} \Big/ \mathbb{R}/\mathbb{Z}$$

where $v \in \mathbb{R}/\mathbb{Z}$ acts on (g, λ) by $v \cdot (g, \lambda) = (gR_{-v\pi}, e^{-2i\pi v})$. Choosing any 1-form $\alpha \in T_i^*\mathbb{H} \setminus \{0\}$, we get a bundle isomorphism

$$\begin{array}{ccc} \mathsf{PSL}_2(\mathbb{R}) \times_{\mathbb{R}/\mathbb{Z}} \mathbb{C} & \to & T^*\mathbb{H} \\ [g, \lambda] & \mapsto & \lambda g_*\alpha\,, \end{array}$$

and note that $v \in \mathbb{R}/\mathbb{Z}$ acts on $\mathsf{PSL}_2(\mathbb{R}) \times_{\mathbb{R}/\mathbb{Z}} \mathbb{C}$ via

$$u \cdot [g, \lambda] = [gR_{v\pi}, \lambda] = [g, \lambda e^{-2i\pi v}] .$$

Let $(R^{(r)}_{v\pi})_{v\in\mathbb{R}}$ be the 1-parameter subgroup of Spin^r such that the image of $R^{(r)}_{v\pi}$ in $\mathsf{PSL}_2(\mathbb{R})$ is $R_{v\pi}$. We now define $L_{\mathbb{H}}$ as the line bundle associated to the circle bundle $\mathsf{Spin}^r \to \mathbb{H}$

$$L_{\mathbb{H}} = \mathsf{Spin}^r \times_{\mathbb{R}/r\mathbb{Z}} \mathbb{C}$$

where $v \in \mathbb{R}/r\mathbb{Z}$ acts on (g, λ) via $v \cdot (f, \lambda) = (fR^{(r)}_{-v\pi}, e^{-\frac{2i\pi v}{r}}\lambda)$. Then $L_{\mathbb{H}}$ is a line bundle over $\mathbb{H}$ and for all $f \in \mathsf{Spin}^r$ with image g in $\mathsf{PSL}_2(\mathbb{R})$, the following diagram commutes

$$\begin{array}{ccccc} L_{\mathbb{H}} & \longrightarrow & T^*\mathbb{H} & \longrightarrow & \mathbb{H} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g} \\ L_{\mathbb{H}} & \longrightarrow & T^*\mathbb{H} & \longrightarrow & \mathbb{H}, \end{array}$$

where the map $L_{\mathbb{H}} \to T^*\mathbb{H} \simeq \mathsf{PSL}_2(\mathbb{R}) \times_{\mathbb{R}/\mathbb{Z}} \mathbb{C}$ is $[f, \lambda] \mapsto [g, \lambda^r]$. The projection $\mathsf{Spin}^r \to \mathsf{PSL}_2(\mathbb{R})$ induces an isomorphism $L_{\mathbb{H}}^{\otimes r} \xrightarrow{\sim} K_{\mathbb{H}}$.

The sheaf $\mathcal{L}_{\mathbb{H}}$ of holomorphic sections of $L_{\mathbb{H}}$ comes with an action of Spin^r lifting the action of $\mathsf{PSL}_2(\mathbb{R})$ on $\omega_{\mathbb{H}}$.

Recall that on $K_{\mathbb{H}}$ we have the metric dual to the hyperbolic metric on $T\mathbb{H}$

$$K_{\mathbb{H}} \xrightarrow{|\cdot|_{K_{\mathbb{H}}}} \mathbb{R}_{\geqslant 0}$$

and the composition

$$|\cdot|_{\mathcal{L}_{\mathbb{H}}} : L_{\mathbb{H}} \longrightarrow K_{\mathbb{H}} \xrightarrow{|\cdot|_{K_{\mathbb{H}}}} \mathbb{R}_{\geqslant 0}$$

is a metric on $L_{\mathbb{H}}$. For $\xi \in \mathbf{P}^1\mathbb{R}$ a point at infinity of $\mathbb{H}$ and $U \subset \mathbb{H}$ some open set, a section of $\mathcal{L}_{\mathbb{H}}(U)$ is said to be *bounded near* ξ is there exists a horoball B centered at ξ and a constant $C > 0$ such that for all $\tau \in B \cap U$, $|s(\tau)|_{L_{\mathbb{H}}} < C$.

Recall that $\infty \in \mathbf{P}^1\mathbb{R}$ is the point at infinity of $\mathbb{H}$ that is fixed by u. For $g \in \mathsf{Spin}^r$ a preimage of u and $U \subset \mathbb{D}$, let

$$\mathcal{L}_g(U) = \{s \in \mathcal{L}_{\mathbb{H}}((\iota \circ \Psi)^{-1}(U)),\ g_*s = s \text{ and } s \text{ is bounded near } \infty\}$$

be the sheaf on g-invariant sections of $\mathcal{L}_{\mathbb{H}}$ that are bounded near ∞. It is a subsheaf of the pushforward sheaf $(\iota \circ \Psi)_*\mathcal{L}_{\mathbb{H}}$ on $\mathbb{D}$.

Lemma 2.1.1. *The sheaves $\mathcal{L}_g$ are all invertible. Moreover, if $\mathcal{L}$ is an invertible sheaf on*

$\mathbb{D}$ *and* $0 < m \leqslant r$*, then* $\mathcal{L}^{\otimes r}$ *is isomorphic to* $\omega_{\mathbb{D}}((1-m)x)$ *if and only if* $\mathcal{L}$ *is isomorphic to* $\mathcal{L}_g$ *where* $\mathrm{Trans}^{(r)}(g) = -m \mod r$ *and* $x \in \mathbb{D}$ *is the origin.*

Proof. Let $dz \in \omega_{\mathbb{D}}(\mathbb{D})$ be the differential of the function $z \in \mathbb{D} \mapsto z \in \mathbb{C}$. We have the sheaf isomorphisms

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{D}} & \to & \omega_{\mathbb{D}} \\ f & \mapsto & f dz \end{array}$$

and

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{D}} & \to & \omega_{\mathbb{D}}((1-m)x) \\ f & \mapsto & f(z) z^{m-1} dz \,. \end{array}$$

Let $d\tau \in \omega_{\mathbb{H}}(\mathbb{H})$ be the differential of the function $\tau \in \mathbb{H} \mapsto \tau \in \mathbb{C}$ and $\sigma_0 \in \mathcal{L}_{\mathbb{H}}(\mathbb{H})$ such that $\sigma_0^{\otimes r} = d\tau$. There are r possible choices for σ_0, indexed by the r roots of unity. Let $g_0 \in \mathsf{Spin}^r$ be a lift of u with $\mathrm{Trans}^{(r)}(g_0) = 0$. We have

$$(g_0 \cdot \sigma_0)(\tau) = (g_0)_* \sigma_0(u^{-1}\tau) = \sigma_0(\tau)$$

and now if g is a lift of u with $\mathrm{Trans}^{(r)}(g) = -m \mod r$ for some $0 < m \leqslant r$, then $g = g_0 R^{(r)}_{-m}$ and

$$(g \cdot \sigma_0)(\tau) = e^{2i\pi m/r} \sigma_0(\tau) \,.$$

For $k \in \mathbb{Z}$, we define a global section σ_k of $\mathcal{L}_{\mathbb{H}}$ by

$$\sigma_k(\tau) = \exp\left(\frac{2i\pi k \tau}{r}\right) \sigma_0(\tau) \,.$$

$$\begin{aligned} (g \cdot \sigma_k)(\tau) &= g_* \sigma_k(g^{-1}\tau) \\ &= \exp\left(\frac{2i\pi m}{r}\right) \exp\left(\frac{2i\pi k(\tau-1)}{r}\right) \sigma_0(\tau) \\ &= \exp\left(\frac{2i\pi (m-k)}{r}\right) \sigma_k(\tau) \,, \end{aligned}$$

and σ_k is g-invariant if and only if $k = m \mod r$.

Pick $k \in \{1, \ldots, r\}$ so that $r \mid k-m$. Being g-invariant, σ_k yields a section of $(\iota \circ \Psi)_* \mathcal{L}_{\mathbb{H}}$, and every global section of $(\iota \circ \Psi)_* \mathcal{L}_{\mathbb{H}}$ is of the form $f(\tau)\sigma_k(\tau)$ where $f(\tau+1) = f(\tau)$ for $f : \mathbb{H} \to \mathbb{C}$ some holomorphic function. Such a function f may be written $f(\tau) = F(z)$

where $z = e^{2i\pi\tau}$ is the coordinate on $\mathbb{D}^\times$. We compute the norm of $f\sigma_k$ as a section of $\mathcal{L}_{\mathbb{H}}$:

$$\begin{aligned} |f(\tau)\sigma_k(\tau)|_{\mathcal{L}_{\mathbb{H}}} &= \left|f(\tau)e^{2i\pi k\tau/r}\right||\sigma_0|_{\mathcal{L}_{\mathbb{H}}} \\ &= |F(z)||z|^{k/r}\frac{-2\pi}{\log|z|}\,, \end{aligned}$$

because, with $\Im\tau$ the imaginary part of τ,

$$\begin{aligned} |\sigma_0|_{L_{\mathbb{H}}} &= |d\tau|_{K_{\mathbb{H}}} \\ &= \frac{1}{\Im\tau} \\ &= \frac{-2\pi}{\log|z|}\,. \end{aligned}$$

Sections of $\mathcal{L}_g$ are exactly sections of $(\iota\circ\Psi)_*\mathcal{L}_{\mathbb{H}}$ whose norm is bounded, so that $f\sigma_k$ is a section of $\mathcal{L}_g$ if and only if the singularity of F at the origin of $\mathbb{D}$ is a removable one, that is if and only if F extends to a holomorphic function $\mathbb{D}\to\mathbb{C}$. Finally, we get a sheaf isomorphism

$$\mathcal{O}_{\mathbb{D}} \longrightarrow \mathcal{L}_g,\, F \longmapsto F\sigma_k\,.$$

Which shows $\mathcal{L}_g$ to be an invertible sheaf.

We now compute $\mathcal{L}_g^{\otimes r}$ by first computing $\sigma_k^{\otimes r}$ using the fact that $2i\pi d\tau = dz/z$ since $z = \exp(2i\pi\tau)$.

$$\begin{aligned} \sigma_k^{\otimes r} &= e^{2i\pi k\tau}\sigma_0^{\otimes r} \\ &= \frac{-i}{2\pi}z^k\frac{dz}{z} \\ &= \frac{-i}{2\pi}z^{k-1}dz\,. \end{aligned}$$

So that we have a sheaf isomorphism

$$\mathcal{L}_g^{\otimes r} \longrightarrow \omega_{\mathbb{D}}((1-k)x)$$

where x is the origin of $\mathbb{D}$.

Inversely, if $\mathcal{L}$ is an invertible sheaf above $\mathbb{D}$ such that $\mathcal{L}^{\otimes r} = \omega_{\mathbb{D}}((1-k)x)$. We construct a section $\sigma\in\mathcal{L}(\mathbb{D})$

then σ_k is a global section of $\mathcal{L}$ and $\mathcal{L} = \mathcal{L}_g$ where $\mathrm{Trans}^{(r)}(g) + k = 0 \mod r$.

Indeed, let $L_{\mathbb{H}}$ and $K_{\mathbb{H}}$ be the line bundles respectively associated to the sheaves $\mathcal{L}_{\mathbb{H}}$ and $\omega_{\mathbb{H}}$ and define similarly L and $K_{\mathbb{D}}$ to be associated with $\mathcal{L}$ and $\omega_{\mathbb{D}}$. The line bundle

$\Psi^*L|_{\mathbb{D}^\times} \to \mathbb{H}$ is a r-root of $K_{\mathbb{H}}$ and (up to multiplication by a r-root of unity), we have $\Psi^*L|_{\mathbb{D}^\times} = L_{\mathbb{H}}$. Let $\sigma : \mathbb{H} \to \Psi^*L|_{\mathbb{D}^\times}$ be a section such that

$$((\iota \circ \Psi)_*\sigma)^{\otimes r} = \frac{-i}{2\pi} z^{k-1} dz$$

as a section of $L^{\otimes r} = K_{\mathbb{D}} \otimes ((1-k)x)$. To construct σ, one first observes that the section $\frac{-i}{2\pi} z^{k-1} dz$, when restricted to $\mathbb{D}^\times$, never vanishes, so that is it a section of $(K_{\mathbb{D}} \otimes (1-k)x))|^\circ_{\mathbb{D}^\times}$, where for a line bundle N, N° is the bundle N with the zero section removed. Since $L^\circ \to (K_{\mathbb{D}} \otimes ((1-k)x))|^\circ_{\mathbb{D}^\times}$ is a regular cover, the map

$$\begin{array}{ccc} \mathbb{H} & \to & (K_{\mathbb{D}} \otimes ((1-k)x))^\circ|_{\mathbb{D}^\times} \\ \tau & \mapsto & \frac{-i}{2\pi} z^{k-1} dz \end{array}$$

where $z = e^{2i\pi\tau}$ lifts in a map $\sigma : \mathbb{H} \to L$ and

$$\begin{array}{ccc} \mathcal{L}_g & \longrightarrow & \mathcal{L} \\ f\sigma_k & \longmapsto & f\sigma \end{array}$$

is a sheaf isomorphism. □

2.2 The holonomy map

2.2.1 Lifting the hyperbolization

Let $(Y, \phi, \mathcal{L}, f)$ be a marked r-spin Riemann surface of type $\mathfrak{m}$ where $Y = (X; x_1, \ldots, x_s)$ is a compact pointed Riemann surface. Forgetting the r-spin structure, we get an hyperbolization $h \in \mathrm{Hom}_{\mathrm{Hyp}}(S, \mathsf{PSL}_2(\mathbb{R}), P)$ of (Y, ϕ). Let $Z = \mathbb{H}/h(\pi_1(S))$ be the hyperbolic surface with cusps equipped with its Riemann surface structure such that $\iota : Z \to X$ is a biholomorphism on its image $X \setminus \{x_1, \ldots, x_s\}$.

We view the r-spin structure as a line bundle L over X. Removing the zero sections of the line bundles L and K_Z we get, respectively, L° and K_Z° and restricting L° to $Z \subset X$, we get a r-cover

$$\theta : L^\circ|_Z \to K_Z^\circ$$

where we identified $(K_X \otimes (-\sum(m_j - 1)x_j))|_Z$ with K_Z through the pullback by the inclusion $\iota : Z \to X$.

The cover

$$L^\circ_{\mathbb{H}} \to K^\circ_{\mathbb{H}} \to K^\circ_Z$$

sends the loop around the origin $t_{L_{\mathbb{H}}}$ of $L_{\mathbb{H}}$ to r times the loop around the origin in K_Z, that is to $\theta_*(t_{L|_Z}) = t_{K_Z}^r$ in $\pi_1(K_Z^\circ)$. Consequently, by homotopy lifting property, we get a r-cover $L_{\mathbb{H}}^\circ \to L^\circ|_Z$ such that the following diagram commutes

$$\begin{array}{ccc} L_{\mathbb{H}}^\circ & \longrightarrow & K_{\mathbb{H}}^\circ \\ \downarrow & & \downarrow \\ L^\circ|_Z & \longrightarrow & K_Z^\circ \end{array}$$

The deck transformation group of both $L_{\mathbb{H}}^\circ \to L^\circ|_Z$ and $K_{\mathbb{H}}^\circ \to K_Z^\circ$ is $\pi_1(S)$. Thus we get $\rho : \pi_1(S) \to \mathsf{Spin}^r$ lifting the hyperbolization $h : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$ such that

$$L^\circ|_Z = L_{\mathbb{H}}^\circ/\rho(\pi_1(S))\,.$$

The homomorphism so obtained is the *holonomy* $\mathrm{Hol}_r(Y, \phi, \mathcal{L}, f)$ of the marked r-spin Riemann surface.

2.2.2 Constructing a r-spin structure from a homomorphism

In this section, starting from $\rho \in \mathrm{Hom}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}})$, we construct a r-spin structure whose holonomy is ρ. Let $h : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$ be the hyperbolization of S induced by ρ and (Y, ϕ) the associated point of Teichmüller space where $Y = (X; x_1, \ldots, x_s)$. Let $Z = \mathbb{H}/h(\pi_1(S))$ the associated Riemann surface and $q : \mathbb{H} \to Z$ its universal cover, $\iota : Z \to X$ the inclusion.

Taking the sheaf of ρ invariant sections of $\mathcal{L}_{\mathbb{H}}$, we get a sheaf on X that is invertible if and only if $s = 0$. Otherwise, we need to discard some sections. To do so, let $Q \subset \mathbf{P}^1\mathbb{R}$ be the set of parabolic fixed points of h, that is the set of $\xi \in \mathbf{P}^1\mathbb{R}$ such that there exists $\gamma \in \pi_1(S)$, $h(\gamma)$ is parabolic and $h(\gamma)\xi = \xi$.

Proposition 2.2.1. *For $\rho \in \mathrm{Hom}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}})$, let $\mathcal{L}_\rho$ be the sheaf over X defined by*

$$\begin{aligned} \mathcal{L}_\rho(U) = \{ & s \in (\iota \circ q)_* \mathcal{L}_{\mathbb{H}}(U),\ s \text{ is bounded near all } \xi \in Q \\ & \text{and for all } \gamma \in \pi_1(S),\ \rho(\gamma)_* s = s \} \end{aligned}$$

for $U \subset X$ any open set. Then $(\mathcal{L}_\rho, f)$ defines a r-spin structure of type $\mathfrak{m}$ over X where f is the map induced by $\mathcal{L}_{\mathbb{H}}^{\otimes r} \simeq \omega_{\mathbb{H}}$, and the holonomy of $(Y, \phi, \mathcal{L}_\rho, f)$ is conjugated to ρ.

Proof. Intuitively, $\mathcal{L}_\rho$ is the sheaf of ρ-invariant sections thar are bounded near pointed points. In particular, the local study done in Lemma 2.1.1 applies and for every open set

D small enough containing a pointed point, $\mathcal{L}_\rho|_D$ is indeed trivial and

$$\mathcal{L}_\rho^{\otimes r}|_D = \omega_X(-\sum(m_j-1)x_j)|_D\,.$$

If U is a small open set not containing any x_j, the boundedness condition is empty and the result is immediate. □

In the process of proving Theorem 1 we have shown that the following diagram commutes

$$\begin{array}{ccc} \mathscr{T}^{r,\mathfrak{m}}(S) & \xrightarrow{\mathrm{Hol}_r} & \mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r, P_{\mathfrak{m}}) \\ \downarrow & & \downarrow \\ \mathscr{T}(S) & \xrightarrow{\mathrm{Hol}} & \mathrm{Rep}_{\mathrm{Hyp}}(S, \mathsf{PSL}_2(\mathbb{R}), P)\,. \end{array}$$

The regularity of Hol_r can be check locally since the vertical lines are regular covers.

Chapter 3

Application: counting closed geodesics with given rotation number

Let Σ be a compact connected oriented hyperbolic surface. A free homotopy class of closed curves on Σ defines on the one hand a unique geodesic on Σ and one the other hand a unique conjugacy class in $\pi_1(S)$. Since $\mathrm{Trans}^{(r)} : \mathsf{Spin}^r \to \mathbb{R}$ is conjugacy invariant, if $\rho : \pi_1(S) \to \mathsf{Spin}^r$ is a r-spin hyperbolization inducing the hyperbolic structure Σ and if γ is a closed geodesic on Σ we may define the translation number of γ as $\mathrm{Trans}^{(r)}(\rho([\gamma])) \in \mathbb{Z}/r\mathbb{Z}$ where $[\gamma]$ is the conjugacy class associated to γ.

3.1 A theorem of Katsuda and Sunada

We first recall a weak version of Theorems 1 and 3 by A. Katsuda and T. Sunada appearing in [KS90]. Let ϕ_t be the geodesic flow on $M = T^1\Sigma$ the unit tangent bundle of Σ. Given any surjective homomorphism $\psi : \mathrm{H}_1(M, \mathbb{Z}) \to \mathbb{Z}/r\mathbb{Z}$, we define, for $m \in \mathbb{Z}/r\mathbb{Z}$ and $L > 0$,

$$\begin{aligned} \mathcal{C}(L, m) &= \{x \text{ closed orbit of length } l(x) < L,\ \psi([x]) = m\}\ , \\ C(L, m) &= \mathrm{Card}(\mathcal{C}(L, m))\,, \end{aligned}$$

where $[x]$ denotes the homology class of the orbit x. Here since ϕ is the geodesic flow on the hyperbolic surface Σ, we have

$$\mathcal{C}(L, m) = \{\gamma \text{ closed geodesic on } \Sigma,\ \ell_\Sigma(\gamma) < L,\ \psi([\gamma]) = m\}\ .$$

For $L \to \infty$, we write $f(L) \sim g(L)$ whenever $\lim \frac{f(L)}{g(L)} = 1$.

Theorem (A. Katsuda, T. Sunada)
For all $m \in \mathbb{Z}/r\mathbb{Z}$, as $L \to \infty$, we have

$$C(L,m) \sim \frac{\exp(L)}{rL}\,.$$

Moreover, for all $f \in C^\infty(M)$, for all $m \in \mathbb{Z}/r\mathbb{Z}$, the following equality holds

$$\lim_{L\to\infty} \frac{1}{C(L,m)} \sum_{\gamma\in\mathcal{C}(L,m)} \frac{1}{\ell_\Sigma(\gamma)} \int_\gamma f = \int_\Sigma f \,\mathrm{d}\,\mathrm{vol}_\Sigma\,.$$

3.2 Translation number and homomorphism

To prove Theorem 2, we relate $\mathrm{Trans}^{(r)} \circ \rho : \pi_1(S) \to \mathsf{Spin}^r$ to a homomorphism from $\mathrm{H}_1(TS^\circ, \mathbb{Z})$ to $\mathbb{Z}/r\mathbb{Z}$, where ρ is a r-spin hyperbolization of S. The fact that a r-spin structure yield a homomorphism $\mathrm{H}_1(TS^\circ, \mathbb{Z}) \to \mathbb{Z}/r\mathbb{Z}$ was already observed by P. Sipe in [Sip82] in the closed case. We relate such a homomorphism to the translation number function modulo r in order to prove Theorem 2.

In TS°, recall a *loop in the fiber* is a simple curve in any fiber that is positively oriented, where the orientation comes from the orientation of S. The image of such a loop in $\pi_1(TS^\circ)$ defines a unique element, noted t_{TS} or t when there is no ambiguity and called *the loop in the fiber*. We also write t_{TS} or t again for the homology class of such a loop.

Let x be a smooth curve on S. The *tangential lift of x* is the curve $\overline{x}$ in TS° given by $\overline{x} = (x, \dot{x})$.

Proposition 3.2.1. *Let $\rho \in \mathrm{Hom}_{\mathrm{Hyp}}(S, \mathsf{Spin}^r)$ and equip S with the associated hyperbolic metric. There exists a homomorphism $\xi : \mathrm{H}_1(TS^\circ, \mathbb{Z}) \to \mathbb{Z}/r\mathbb{Z}$ such that*

1. *$\xi(t_{TS}) = 1$;*
2. *For all closed geodesic γ,*

$$\xi(\overline{\gamma}) = -\,\mathrm{Trans}^{(r)}(\rho([\gamma]))\,.$$

In order to define the needed homomorphism, we need the following lemmas, which will also be useful in the next part.

For the following two lemmas we don't assume S to be closed, that is $S = S_{g,s}$ with $2g - 2 + s > 0$.

Lemma 3.2.1. *Let $x_0 \in S$ and $X_0 \in T_{x_0}S^\circ$ be two base points. Fix any hyperbolic metric on S. Let $g \in \pi_1(S, x_0)$. Assume g is non peripheral (resp. peripheral) and let γ be the only geodesic (resp. a horocycle) freely homotopic to any curve representing g. Then there exists a unique $s(g) \in \pi_1(TS^\circ, X_0)$ such that*

- *$s(g)$ is an antecedent of g,*
- *any curve representing $s(g)$ is freely homotopic to the tangential lift of γ.*

Moreover, the function $s : g \in \pi_1(S, x_0) \mapsto s(g) \in \pi_1(TS^\circ, X_0)$ does not depend on the chosen hyperbolic metric.

Proof. First we prove the uniqueness.

Let g_1 and g_2 be two such lifts.

Because $p(g_1) = p(g_2) = g$, there exists $k \in \mathbb{Z}$, $g_1 = g_2 t^k$. It suffices to show $k = 0$. Since any curve representing g_1 and g_2 are freely homotopic, g_1 and g_2 are conjugated: there exists $x \in \pi_1(TS^\circ)$, $xg_1x^{-1} = g_2$. In particular in $\pi_1(S)$ we have

$$p(x)gp(x)^{-1} = g\,.$$

Thus, we have two integers $\alpha, \beta \in \mathbb{Z}\backslash\{0\}$ such that $p(x)^\alpha = g^\beta$. In particular, $p(g_1^{-\beta}x^\alpha) = 1$ and let $\delta \in \mathbb{Z}$ such that $g_1^{-\beta}x^\alpha = t^\delta$. Since $xg_1x^{-1} = g_2 = g_1t^{-k}$ we have

$$\begin{aligned} g_1t^{-k\alpha} &= x^\alpha g_1 x^{-\alpha} \\ &= g_1^\beta t^\delta g_1 t^{-\delta} g_1^{-\beta} \\ &= g_1\,. \end{aligned}$$

And thus $k = 0$ and $g_1 = g_2$.

Let us show the existence. let $\mathrm{Conj}(\gamma) \subset \pi_1(S)$ be the conjugacy class associated to γ, that is the conjugacy class of g. Let $\mathrm{Conj}(\overline{\gamma}) \subset \pi_1(TS^\circ)$ be the conjugacy class of the tangential lift $\overline{\gamma}$ of γ. Since

$$0 \to \mathbb{Z} \to \pi_1(TS^\circ) \xrightarrow{p} \pi_1(S) \to 0$$

is a central extension, p restricts to a bijection $p|_{\mathrm{Conj}(\overline{\gamma})} : \mathrm{Conj}(\overline{\gamma}) \to \mathrm{Conj}(\gamma)$. Indeed, the proof of the uniqueness of $s(g)$ shows $p|_{\mathrm{Conj}(\overline{\gamma})}$ to be injective. To show the surjectivity, let $h \in \mathrm{Conj}(\gamma)$ and suppose $h \notin p(\mathrm{Conj}(\overline{\gamma}))$. Since $h \in \mathrm{Conj}(\gamma)$, we have

$$p^{-1}(h) \subset p^{-1}(\mathrm{Conj}(\gamma)) = \bigcup_{k\in\mathbb{Z}} t^k\,\mathrm{Conj}(\overline{\gamma})$$

so that $h \in \bigcup p(t^k \operatorname{Conj}(\overline{\gamma})) = p(\operatorname{Conj}(\overline{\gamma}))$, contradiction. Now we can define $s(g)$ to be $(p|_{\operatorname{Conj}(\overline{\gamma})})^{-1}(g)$, since $g \in \operatorname{Conj}(\gamma)$, thus showing the existence.

The function that associate a geodesic in a free homotopy class to a hyperbolic metric is continuous and thus the constructed lift depends continuously on the metric. By discreetness and by connectedness of the space of hyperbolic metric on S, the lift does not depends on the metric chosen to construct it. □

Recall that $\tau \in \widetilde{\mathsf{PSL}}_2(\mathbb{R})$ is the image of the loop $t \in [0,1] \mapsto R_{t\pi} \in \mathsf{PSL}_2(\mathbb{R})$ by the injection $\pi_1(\mathsf{PSL}_2(\mathbb{R})) \to \widetilde{\mathsf{PSL}}_2(\mathbb{R})$.

Lemma 3.2.2. *There exists a continuous function*

$$\begin{array}{ccc} \operatorname{Hom}_{\mathrm{Hyp}}(S, \mathsf{PSL}_2(\mathbb{R})) & \to & \operatorname{Hom}(\pi_1(TS^\circ), \widetilde{\mathsf{PSL}}_2(\mathbb{R})) \\ h & \mapsto & \tilde{h} \end{array}$$

where for $h \in \operatorname{Hom}_{\mathrm{Hyp}}(S, \mathsf{PSL}_2(\mathbb{R}))$, $\tilde{h}$ is the only homomorphism such that

- *For any closed geodesic γ on S, $\operatorname{Trans}(\tilde{h}(\overline{\gamma})) = 0$;*
- *$\tilde{h}(t_{TS}) = \tau$;*
- *the following diagram commutes*

$$\begin{array}{ccc} \pi_1(TS^\circ) & \xrightarrow{\tilde{h}} & \widetilde{\mathsf{PSL}}_2(\mathbb{R}) \\ \downarrow & & \downarrow \\ \pi_1(S) & \xrightarrow{h} & \mathsf{PSL}_2(\mathbb{R}) \,. \end{array}$$

Proof. Uniqueness of $\tilde{h}$ is immediate. For the existence, let $h \in \operatorname{Hom}_{\mathrm{Hyp}}(S, \mathsf{PSL}_2(\mathbb{R}))$ and $\Sigma = \mathbb{H}/h(\pi_1(S))$. Let $T^1\Sigma$ be the unit tangent bundle of Σ. Let $X_0 \in T^1\Sigma$ be a based point, $\tilde{X}_0 \in T^1\mathbb{H}$ a lift of X_0. We identify $\mathsf{PSL}_2(\mathbb{R})$ with $T^1\mathbb{H}$ through $g \mapsto g\tilde{X}_0$ and the universal cover of $T^1\Sigma$ with $\widetilde{\mathsf{PSL}}_2(\mathbb{R})$, where the universal cover is defined through paths based at X_0.

Suppose γ is a unit length geodesic on Σ with $\dot{\gamma}(0) = X_0$. Let ν be the lift of γ to $\widetilde{\mathsf{PSL}}_2(\mathbb{R})$ starting at the identity. Then the end point of ν corresponds to $\tilde{h}(\gamma)$, and since $\nu(t)$ projects in $\mathsf{PSL}_2(\mathbb{R})$ (identified with $T^1\mathbb{H}$) to an hyperbolic element for all t, we have $\operatorname{Rot}(\gamma(t)) = 0$ for all t, thus $\operatorname{Trans}(\tilde{h}(\overline{\gamma})) = 0$. The proposition follows because Trans is invariant by conjugacy. □

We can now proceed with the proof of Proposition 3.2.1, which will enable us to use Theorem 3.1 and thus finish the proof of Theorem 2.

Proof (of Proposition 3.2.1). Let $h : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$ be the composition $\pi_1(S) \xrightarrow{\rho} \mathsf{Spin}^r \to \mathsf{PSL}_2(\mathbb{R})$ and $\tilde{h} : \pi_1(TS^\circ) \to \mathsf{Spin}^r$ be the lift of h obtained by Lemma 3.2.2 above and composition with $\widetilde{\mathsf{PSL}}_2(\mathbb{R}) \to \mathsf{Spin}^r$. Let $p : \pi_1(TS^\circ) \to \pi_1(S)$ be the projection. For all $x \in \pi_1(S)$ we define $\xi(x) \in \mathbb{Z}/r\mathbb{Z}$ to be such that

$$\tau^{\xi(v)} = \tilde{h}(x)(\rho(p(x)))^{-1}$$

where τ is the generator of the center of Spin^r. We check that $\xi : \pi_1(TS^\circ) \to \mathbb{Z}/r\mathbb{Z}$ is indeed a homomorphism: for $a, b \in \pi_1(TS^\circ)$, we have

$$\begin{aligned}
\tau^{\xi(ab)} &= \tilde{h}(ab)(\rho(p(ab)))^{-1} \\
&= \tilde{h}(a)\tilde{h}(b)\rho(p(b))^{-1}\rho(p(a))^{-1} \\
&= \tilde{h}(a)\tau^{\xi(b)}\rho(p(a))^{-1} \\
&= \tau^{\xi(a)+\xi(b)}\,.
\end{aligned}$$

We call again $\xi : \mathrm{H}_1(TS^\circ, \mathbb{Z}) \to \mathbb{Z}/r\mathbb{Z}$ the homomorphism obtained via the Hurewicz Theorem, and if γ is a closed geodesic on S (equipped with the hyperbolic structure from ρ), then with $\overline{x} \in \pi_1(TS^\circ)$ freely homotopic to $\overline{\gamma}$, $\tau^{\xi(\overline{\gamma})}\rho(p(\overline{x})) = \tilde{h}(\overline{x})$ thus

$$0 = \mathrm{Trans}^{(r)}(\tilde{h}(\overline{x})) = \xi(\overline{\gamma}) + \mathrm{Trans}^{(r)}(\rho(p(\overline{x})))\,. \qquad \square$$

Part II

Integral Maximal r-spin representations

Chapter 4

Introduction and statement of results

4.1 The action of the mapping class group on the space $\mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z})$

Let $g, s \in \mathbb{Z}_{\geqslant 0}$ such that $2g - 2 + s > 0$. Let S be a genus g connected compact oriented surface with s points $p_1, \ldots, p_s$ removed, TS the tangent bundle of S and TS° the tangent bundle of S where we removed the zero section. Let $x_0 \in S$ be a base point and a_i, b_i, c_j be a collection of smooth simple closed curves on S such that, writing (a, b) for the algebraic intersection number of the curves a and b, we have

- $(a_i, b_j) = \delta_{i,j}$,
- $(a_i, a_j) = (b_i, b_j) = 0$,
- c_j is freely homotopic to a loop around p_j winding in the positive direction,
- The curves a_i, b_i, c_j intersect at x_0 and, writing again a_i, b_i, c_j for the relative homotopy classes, we have the following presentation of $\pi_1(S)$

$$\pi_1(S) = \left\langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_s | \prod_{i=1}^{n} [a_i, b_i] \prod_{j=1}^{s} c_j = 1 \right\rangle .$$

Let t be the loop around the origin in TS°. Lifting the curves a_i, b_i, c_j to TS° tangentially, we get a generating set $\{\overline{a}_i, \overline{b}_i, \overline{c}_j, t\}$ of $\mathrm{H}_1(TS^\circ, \mathbb{Z})$ satisfying the only relation $\sum \overline{c}_j = (2g - 2 + s)t$.

The mapping class group $\Gamma(S)$ acts on the space $\mathrm{Hom}(\mathrm{H}_1(TS^\circ,\mathbb{Z}),\mathbb{Z}/r\mathbb{Z})$ and for $n\in\mathbb{Z}_{\geqslant 0}$, $\mathfrak{m}=(m_1,\dots,m_s)\in\mathbb{Z}^s$, the space

$$A_{g,s}^{r,n,\mathfrak{m}}=\{\xi\in\mathrm{Hom}(\mathrm{H}_1(TS^\circ,\mathbb{Z}),\mathbb{Z}/r\mathbb{Z}),\xi(t)=n \mod r,\ \forall j,\ \xi(\bar{c}_j)=m_j \mod r\}$$

is a union of orbits which is empty whenever $\sum m_j\neq(2g-2+s)n \mod r$.

Let $\ell_{g,s}^{r,n,\mathfrak{m}}\in\mathbb{Z}_{\geqslant 0}$ be defined as

$$\ell_{g,s}^{r,n,\mathfrak{m}}=\begin{cases}1 & g=0\\ \gcd(r,m_1-n,\dots,m_s-n) & g=1\\ \gcd(r,2n,m_1-n,\dots,m_s-n) & \text{else.}\end{cases}$$

If $s=0$ then $g\geqslant 2$ and $l_{g,s}^{r,n,\mathfrak{m}}=\gcd(2n,r)$.

One of our main results is

Theorem 10 (Mapping class group action)
Let $n,g,s,r,m_1,\dots,m_s\in\mathbb{Z}$ with $g,s\geqslant 0$, $r,n>0$, $2g-2+s>0$ and $n(2g-2+s)=\sum m_j \mod r$. Let $\ell=\ell_{g,s}^{r,n,\mathfrak{m}}$. The action of the mapping class group $\Gamma(S)$ on $A_{g,s}^{r,n,\mathfrak{m}}$ decomposes into $d(\ell)$ orbits, where $d(k)$ is the number of divisors of the integer k, including 1 and k.

4.2 Integral maximal representations and a mapping class group equivariant map

Let G be a simple Hermitian Lie group of tube type. To G is associated $\kappa=\kappa_{\mathsf{G},\mathbb{Z}}\in\mathrm{H}^2_{\mathrm{B}}(\mathsf{G},\mathbb{Z})$ its integral Kähler class. The class κ yields in the usual way a central extension of G by $\mathbb{Z}$

$$0\to\mathbb{Z}\xrightarrow{i}\mathsf{G}_\kappa\to\mathsf{G}\to 1\,.$$

Associated to κ and defined in [BIW10] Section 7 are the *rotation number* Rot_κ and *translation number* Trans_κ. They are continuous and homogeneous quasi-morphisms such that the following diagram commutes

$$\begin{array}{ccc}\mathsf{G}_\kappa & \xrightarrow{\mathrm{Trans}_\kappa} & \mathbb{R}\\ \downarrow & & \downarrow\\ \mathsf{G} & \xrightarrow{\mathrm{Rot}_\kappa} & \mathbb{R}/\mathbb{Z}\,,\end{array}$$

and they generalize the Rot and Trans associated to $\mathsf{PSL}_2(\mathbb{R})$ that we used in the first part. Let $\mathsf{G}_r=\mathsf{G}_\kappa/i(r\mathbb{Z})$ be the r-cover of G associated to κ and let $\mathrm{Trans}^{(r)}:\mathsf{G}_r\to\mathbb{R}/r\mathbb{Z}$. Let

$h : \pi_1(S) \to \mathsf{G}$ be a *maximal homomorphism* as defined in [BIW10] Definition 3.5, or equivalently see Definition 6.1.10, and assume moreover that for all $x \in \pi_1(S)$, $\mathrm{Rot}_\kappa(\rho(x)) = 0$. We will call such homomorphisms *integral maximal*, although the definition we will adopt will be slightly different. If $\rho : \pi_1(S) \to \mathsf{G}_r$ is a lift of h, then $\mathrm{Trans}^{(r)}_\kappa \circ \rho$ takes values into $\mathbb{Z}/r\mathbb{Z}$ and for $\mathfrak{m} = (m_1, \dots, m_s) \in \mathbb{Z}^s$, let

$$\mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G}_r, \mathfrak{m}) = \{\rho \in \mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G}_r),\ \forall j\ \mathrm{Trans}^{(r)}_\kappa(\rho(c_j)) = -m_j \mod r\}$$

be the subspace of integral maximal homomorphisms with assigned translation number around each puncture. As a consequence of the proof of Lemma 8.10 and of Theorem 12 of [BIW10], $\mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G}_r, \mathfrak{m})$ is a union of connected components of $\mathrm{Hom}_{\mathrm{Max}}(S, \mathsf{G}_r)$ and it is non empty if and only if $\sum m_j = n(2g - 2 + s) \mod r$.

Let $\mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G})$ be the quotient of $\mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G})$ by G and $\mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}_r, \mathfrak{m})$ the quotient of $\mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G}_r, \mathfrak{m})$ by G_r.

For $n, r \geqslant 1$, let

$$A^{r,n}_{g,s} = \{\xi \in \mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z}),\ \xi(t_{TS}) = n \mod r\}\,.$$

Our result is

Theorem 11 (Equivariant map)
Let G be a Hermitian Lie group of tube type and real rank n. Let S be a connected compact oriented surface of genus g minus s numbered and distinct points $(p_1, \dots, p_s)$ with $2g - 2 + s > 0$. Let κ be the integral Kähler class of G and

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \mathsf{G}_\kappa \xrightarrow{p} \mathsf{G} \longrightarrow 1$$

the associated central extension.

Let $r > 0$ be an integer and G_r be the quotient of G_κ by $\iota(r\mathbb{Z})$.

There exists a mapping class group equivariant homeomorphism

$$\mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}_r) \xrightarrow{\Phi} \mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}) \times A^{r,n}_{g,s}\,,$$

where $\mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}_r) \to \mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G})$ is $\rho \mapsto p \circ \rho$. For $\mathfrak{m} = (m_1, \dots, m_s) \in \mathbb{Z}^s$, the restriction of Φ to $\mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}_r, \mathfrak{m})$ yields a mapping class group equivariant homeomorphism

$$\mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}_r, \mathfrak{m}) \xrightarrow{\Phi} \mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}) \times A^{r,n,\mathfrak{m}}_{g,s}\,.$$

Remark 4.2.1. The integral maximal condition is a necessary one to construct Φ the way we do it and this construction cannot be extended to the whole representation variety.

4.3 Application

Let $\Delta : \mathsf{PSL}_2(\mathbb{R}) \to \mathsf{PSp}_{2n}(\mathbb{R})$ be the diagonal embedding. A homomorphism of the form

$$\pi_1(S) \xrightarrow{h} \mathsf{PSL}_2(\mathbb{R}) \xrightarrow{\Delta} \mathsf{PSp}_{2n}(\mathbb{R})$$

is called *diagonal Fuchsian* whenever h is a hyperbolization of S and where $\Delta : \mathsf{PSL}_2(\mathbb{R}) \to \mathsf{PSp}_{2n}(\mathbb{R})$ is the composition of

$$\begin{array}{ccc} \mathsf{PSL}_2(\mathbb{R}) & \to & (\mathsf{PSL}_2(\mathbb{R}))^n \\ A & \mapsto & (A, \dots, A) \end{array}$$

and the inclusion $(\mathsf{PSL}_2(\mathbb{R}))^n \subset \mathsf{PSp}_{2n}(\mathbb{R})$ given by blocks. Deformations of diagonal Fuchsian homomorphisms inside the space of maximal homomorphisms are called *diagonal homomorphisms*. Let

$$\mathrm{Hom}_\Delta(S, \mathsf{PSp}_{2n}(\mathbb{R})) \quad \text{and} \quad \mathrm{Rep}_\Delta(S, \mathsf{PSp}_{2n}(\mathbb{R}))$$

be respectively the space of diagonal homomorphisms into $\mathsf{PSp}_{2n}(\mathbb{R})$ and their quotient by $\mathsf{PSp}_{2n}(\mathbb{R})$. These spaces are connected components of, respectively, $\mathrm{Hom}_{\mathrm{Max}}(S, \mathsf{PSp}_{2n}(\mathbb{R}))$ and $\mathrm{Rep}_{\mathrm{Max}}(S, \mathsf{PSp}_{2n}(\mathbb{R}))$.

If n is odd, $\pi_1(\mathsf{PSp}_{2n}(\mathbb{R})) = \mathbb{Z}$ and we can show that the r-cover $\mathsf{PSp}_{2n}(\mathbb{R})_r$ associated to $\kappa = \kappa_{\mathsf{PSp}_{2n}(\mathbb{R}),\mathbb{Z}}$ is connected whenever $\gcd(r, n) = 1$ (see Lemma 6.1.10), so that it must be the only one such cover of $\mathsf{PSp}_{2n}(\mathbb{R})$. In [GW10], the authors show that for $n \geqslant 3$ odd and S closed, $\mathrm{Hom}_{\mathrm{Max}}(S, \mathsf{PSp}_{2n}(\mathbb{R}))$ consists of 3 connected components, one of them being $\mathrm{Hom}_\Delta(S, \mathsf{PSp}_{2n}(\mathbb{R}))$, see their Theorem 8. They also show, under the same hypothesis, that $\mathrm{Rep}_{\mathrm{Max}}(S, \mathsf{Sp}_{2n}(\mathbb{R}))/\Gamma(S)$ has 6 connected components, two of them consisting of maximal diagonal representations, see their Theorem 10.

If S is closed, the mapping class group $\Gamma(S)$ acts properly discontinuously on the space $\mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}_r)$, see [Wie06] Theorem 1.1. If S has punctures, the notion of connected components on the quotient $\mathrm{Rep}_\Delta(S, \mathsf{G}_r)/\Gamma(S)$ still holds. The previous theorems allow us to determine the number of connected components of diagonal representations into $\mathsf{PSp}_{2n}(\mathbb{R})_r$ up to mapping class group action. There are r^{2g+s-1} connected components of diagonal representations in $\mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{PSp}_{2n}(\mathbb{R})_r)$ if $s > 0$, r^{2g} if $s = 0$ and $r \mid 2n(g-1)$ and 0 otherwise. After mapping class group action, we get

Corollary 12
Suppose r and n are coprime, $n > 2$ odd and let G be the finite r-cover of $\mathsf{PSp}_{2n}(\mathbb{R})$. Then

the space

$$\mathrm{Rep}_{\Delta}(S, \mathsf{G})/\Gamma(S)$$

has $d(g, s, r)$ *connected components, where*

$$d(g,s,r) = \begin{cases} r^{s-1} & g = 0,\ s \geqslant 3 \\ \sum_{d|r} d^{s-1} & g = 1,\ s \geqslant 1 \\ r^{s-1} + (r/2)^{s-1} & g \geqslant 2,\ s \geqslant 1,\ 2 \mid r \mid 2g-2 \\ r^{s-1} & g \geqslant 2,\ s \geqslant 1,\ 2 \nmid r \ \textit{or}\ r \nmid 2g-2 \\ 2 & g \geqslant 2,\ s = 0,\ 2 \mid r \mid 2g-2 \\ 1 & g \geqslant 2,\ s = 0, 2 \nmid r \mid 2g-2 \\ 0 & g \geqslant 2,\ s = 0, r \nmid 2g-2\,. \end{cases}$$

Chapter 5

Action of the mapping class group on $A_{g,s}^{r,n,\mathfrak{m}}$

In this chapter we prove Theorem 10.

5.1 Notations and definitions

5.1.1 Mapping class group acting on homology

Let $\overline{S}$ be a compact connected oriented surface of genus g and $P = \{p_1, \dots, p_s\}$ be s distinct points on $\overline{S}$. Let $S = \overline{S} - P$ be the punctured surface. We assume $2g - 2 + s > 0$. Recall from Part I that the *mapping class group* of S is the group $\Gamma(S)$ of orientation-preserving diffeomorphisms of $\overline{S}$ fixing P pointwise, up to isotopy fixing P.

For any vector bundle $p : E \to B$, let *punctured vector bundle* $E^\circ \to B$ be the restriction of p to the total space E minus the image of the zero section. If $TS \to S$ is the tangent bundle of S, the punctured vector bundle $TS^\circ \to S$ is homotopy equivalent to the unit tangent bundle of S equipped with any metric.

If f is a diffeomorphism of $\overline{S}$ fixing P pointwise, the tangent application $Tf : TS \to TS$ yields a map $Tf : TS^\circ \to TS^\circ$ and

$$\begin{array}{rccc} Tf_* : & \mathrm{H}_1(TS^\circ, \mathbb{Z}) & \to & \mathrm{H}_1(TS^\circ, \mathbb{Z}) \\ & [\gamma] & \mapsto & [Tf(\gamma)] \end{array}$$

is a well-defined homomorphism which depends only on the isotopy class of f, so that we get a natural action of $\Gamma(S)$ on $\mathrm{H}_1(TS^\circ, \mathbb{Z})$ and on $\mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z})$ by precomposition.

In this part we explicit the action of $\Gamma(S)$ on $\mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z})$ and in particular

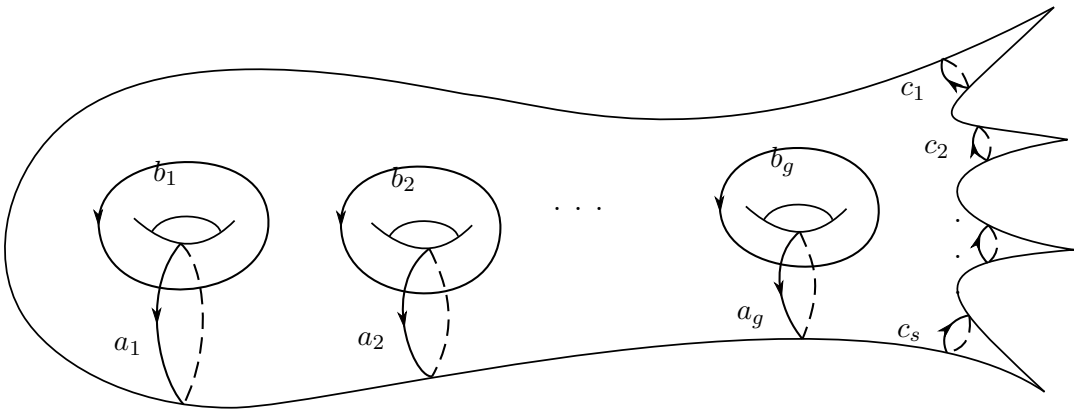


Figure 5.1 – A standard collection of curves on S

enumerates the orbits.

We need to single out some elements of $\pi_1(TS^\circ)$ that will serve as a generating set and give coordinates for the action of $\Gamma(S)$. If the set of punctures P is non-empty, $\pi_1(S)$ is isomorphic to $\mathbb{F}_{2g+s-1}$ the free group on $2g+s-1$ generators and $\pi_1(TS^\circ)$ is isomorphic to $\mathbb{F}_{2g+s-1} \times \mathbb{Z}$, but the isomorphisms are not natural and the action of the mapping class group difficult to compute in such coordinates.

5.1.2 Standard collection of curves on S

For $p \in P$ a puncture of S, a *loop around* p is a simple loop c going around p in the positive direction so that the connected component of the complement of c in S containing the puncture p is a punctured disc.

Let a_i, b_i, c_j for $i = 1, \ldots, g$ and $j = 1, \ldots, s$ be smooth simple closed curves on S such that, writing (a, b) for the algebraic intersection number of the curves a and b, we have

- $(a_i, b_j) = \delta_{i,j}$;
- $(a_i, a_j) = (b_i, b_j) = 0$;
- c_j is a loop about p_j .

We call any such collection of curves a *standard collection of curves on* S, see Figure 5.1. Fixing some base point x_0 on S, each curve of a standard collection of curves defines a conjugacy class in $\pi_1(S, x_0)$. We choose an element in the conjugacy class defined by each

curve of a fixed collection such that we have the usual presentation of $\pi_1(S)$

$$\pi_1(S) = \left\langle a_1, b_1, \dots, a_g, b_g, c_1, \dots, c_s | \prod_{i=1}^{n} [a_i, b_i] \prod_{j=1}^{s} c_j = 1 \right\rangle,$$

where a generator $x = a_i, b_i, c_j$ in the presentation is the relative homotopy class of some curve freely homotopic to x.

5.1.3 A generating set for $\mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z})$

The projection $TS^\circ \to S$ yields a homomorphism $p : \pi_1(TS^\circ) \to \pi_1(S)$ that is a $\mathbb{Z}$-central extension, in other words the following sequence is exact

$$0 \to \mathbb{Z} \xrightarrow{\iota} \pi_1(TS^\circ) \xrightarrow{p} \pi_1(S) \to 0$$

and $\iota(\mathbb{Z}) \subset \pi_1(TS^\circ)$ is a central subgroup.

Recall that for x a smooth closed curve on S, the *tangential lift* of x is the curve $\overline{x} = (x, \dot{x})$ in TS°.

Lemma 5.1.1. *Let $S = S_{g,s}$ with $\chi(S) < 0$ and $s > 0$. For $j = 1, \dots, s$ let c_j be a smooth simple loop around the puncture p_j and $\overline{c}_j$ its tangential lift to TS° and let t denote the loop around the origin in TS°.*

We have the following relation in $\mathrm{H}_1(TS^\circ, \mathbb{Z})$

$$\sum \overline{c}_j = (2g - 2 + s)t\,.$$

Proof. Let x_0 be some point on S and X be a nowhere zero vector field on $S \setminus \{x_0\}$ such that the $\overline{c}_j$ are orbits of X. We may see X as a singular vector field on the compact genus g surface $\overline{S}$ compactifying S. On $\overline{S}$, X has a singularity at each p_j with index 1 and a singularity at x_0, and the sum of all indexes must be equal to $\chi(\overline{S}) = 2 - 2g$, so the singularity at x_0 is of order $2 - 2g - s$ and the boundary of the image of X in TS° is $\overline{c}_1 + \cdots + \overline{c}_s + \chi(S)t$. □

Recall from Part I that for $g \in \pi_1(S)$ we denote $s(g) \in \pi_1(TS^\circ)$ the lift of g given by Lemma 3.2.1.

Lemma 5.1.2. *The fundamental group $\pi_1(TS^\circ)$ admits the following presentation*

$$\pi_1(TS^\circ) = \Big\langle \overline{a}_1, \overline{b}_1, \dots, \overline{a}_g, \overline{b}_g, \overline{c}_1, \dots, \overline{c}_s, t \,|\, t \text{ is central and } \prod_{i=1}^{g} [\overline{a}_i, \overline{b}_i] \prod_{j=1}^{s} \overline{c}_j = t^{-\chi(S)} \Big\rangle ,$$

where each generator $\overline{a}_i, \overline{b}_i, \overline{c}_j$ is the relative homotopy class of a loop freely homotopic to the curve $\overline{a}_i, \overline{b}_i, \overline{c}_j$.

Remark 5.1.1. If the surface S has non-empty boundary, by changing $\overline{c}_1$ in $\overline{c}_1 \cdot t^{\chi(S)+k}$ for some $k \in \mathbb{Z}$, we would change the relation $\prod_{i=1}^{g} [\overline{a}_i, \overline{b}_i] \prod_{j=1}^{s} \overline{c}_j = t^{-\chi(S)}$ in $\prod_{i=1}^{g} [\overline{a}_i, \overline{b}_i] \prod_{j=1}^{s} \overline{c}_j = t^k$. In the lemma, we get $t^{-\chi(S)}$ because of the specific choice of generators, which contrasts with the closed case.

Proof. By the Hurewicz theorem, the abelianisation of $\pi_1(TS^\circ)$ is isomorphic to the first homology group $\mathrm{H}_1(TS^\circ, \mathbb{Z})$. So to prove the lemma it is enough to show that in $\mathrm{H}_1(TS^\circ, \mathbb{Z})$, one has

$$\overline{c}_1 + \cdots + \overline{c}_s = -\chi(S) t \,,$$

which was done in Lemma 5.1.1 before. □

5.2 Mapping class group action

We now prove Theorem 10, which will be a consequence of Proposition 5.2.1 and of Lemma 5.2.2.

Let $\psi = [f] \in \Gamma(S)$ be a mapping class represented by a diffeomorphism f. We consider the action of ψ on $\mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z}) = \mathrm{Hom}(\pi_1(TS^\circ), \mathbb{Z}/r\mathbb{Z})$ defined by, for $\xi \in \mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z})$ and $\alpha = [a] \in \mathrm{H}_1(TS^\circ, \mathbb{Z})$,

$$\psi \cdot \xi(\alpha) = \xi([Tf^{-1}(a)]) \,.$$

As a direct consequence of the definition of action of Tf on TS°, we see that the action of ψ on $\mathrm{H}_1(TS^\circ, \mathbb{Z})$ satisfies $\psi(t) = t$ and $[f](\overline{a}) = [\overline{f(a)}]$ where $t = t_{TS}$ is the loop around the origin and $a \subset S$ is a smooth closed curve.

Let $r, n \in \mathbb{Z}_{>0}$ be two integers and $\mathfrak{m} = (m_1, \dots, m_s) \in \mathbb{Z}^s$. We defined

$$A_{g,s}^{r,n,\mathfrak{m}} = \{\xi \in \mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z}),\, \xi(t) = n \mod r \text{ and } \forall j,\, \xi(\overline{c}_j) = m_j \mod r\} \,.$$

Since $(2g - 2 + s)t = \sum \overline{c}_j$ ($= 0$ if $s = 0$) in $\mathrm{H}_1(TS^\circ, \mathbb{Z})$, $A_{g,s}^{r,n,\mathfrak{m}}$ is non-empty if and only if

$n(2g-2+s) = \sum m_j$ and when non-empty, $A_{g,s}^{r,n,\mathfrak{m}}$ consists of r^{2g} elements characterized by the images of $\overline{a}_i$ and $\overline{b}_i$ in $\mathbb{Z}/r\mathbb{Z}$.

The space $\mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z})$ is the disjoint union of the $A_{g,s}^{r,n,\mathfrak{m}}$ for $n = 1, \ldots, r$ and $m_j \in \{0, \ldots, r-1\}$ with $\sum m_j = n(2g-2+s) \mod r$.

5.2.1 Affine structure on $A_{g,s}^{r,n,\mathfrak{m}}$

The composition of the projection $TS^\circ \to S$ and the inclusion $S \to \overline{S}$ yields an injective $\mathbb{Z}/r\mathbb{Z}$-module homomorphism

$$\mathrm{Hom}(\mathrm{H}_1(\overline{S}, Z), \mathbb{Z}/r\mathbb{Z}) \to \mathrm{Hom}(\mathrm{H}_1(TS^\circ, Z), \mathbb{Z}/r\mathbb{Z})$$

whose image is exactly $A_{g,s}^{r,0,\mathfrak{o}}$. In particular, $A_{g,s}^{r,0,\mathfrak{o}}$ is a $\mathbb{Z}/r\mathbb{Z}$-module.

For L a $\mathbb{Z}/r\mathbb{Z}$-module, by an *affine $\mathbb{Z}/r\mathbb{Z}$-module over L* we mean a transitive and free L-space, that is a set K equipped with an addition $+ : L \times K \to K$ satisfying:

1. $\lambda_1 + (\lambda_2 + k) = (\lambda_1 + \lambda_2) + k$ for $\lambda_i \in L$, $k \in K$;
2. $\forall k_1, k_2 \in K, \exists! \lambda \in L, k_2 = \lambda + k_1$.

Lemma 5.2.1. *The space $A_{g,s}^{r,n,\mathfrak{m}}$ is an affine $\mathbb{Z}/r\mathbb{Z}$-module over $A_{g,s}^{r,0,\mathfrak{o}}$.*

Proof. Recall $A_{g,s}^{r,0,\mathfrak{o}}$ consists of those homomorphisms $\zeta : \mathrm{H}_1(TS^\circ, \mathbb{Z}) \to \mathbb{Z}/r\mathbb{Z}$ with $\zeta(\overline{c}_j) = \zeta(t) = 0$. In particular, if $\xi \in A_{g,s}^{r,n,\mathfrak{m}}$ then $\xi + \zeta \in A_{g,s}^{r,n,\mathfrak{m}}$ and the difference of two elements in $A_{g,s}^{r,n,\mathfrak{m}}$ is in $A_{g,s}^{r,0,\mathfrak{o}}$. □

Let $\mathcal{A}$ be the group of affine transformations of $A_{g,s}^{r,n,\mathfrak{m}}$. Every element of the mapping class group acts on $A_{g,s}^{r,n,\mathfrak{m}}$ by affine transformation and $A_{g,s}^{r,0,\mathfrak{o}}$ acts on $A_{g,s}^{r,n,\mathfrak{m}}$ by translation, thus we get the diagram

$$\begin{array}{ccccccccc} & & & & & & \Gamma(S) & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & A_{g,s}^{r,0,\mathfrak{o}} & \longrightarrow & \mathcal{A} & \longrightarrow & \mathsf{GL}_{2g}(\mathbb{Z}/r\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

where the line is exact since as an affine $\mathbb{Z}/r\mathbb{Z}$ module, $A_{g,s}^{r,n,\mathfrak{m}}$ is isomorphic to $(\mathbb{Z}/r\mathbb{Z})^{2g}$.

On $\mathrm{H}_1(\overline{S}, \mathbb{Z})$ we have a symplectic pairing: the algebraic intersection pairing i which yields a symplectic pairing i^* on $\mathrm{Hom}(\mathrm{H}_1(\overline{S}, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z})$ and we equip $A_{g,s}^{r,0,\mathfrak{o}}$ with this pairing. The mapping class group preserves the symplectic pairing i^* and the image of $\Gamma(S) \to \mathsf{GL}_{2g}(\mathbb{Z}/r\mathbb{Z})$ is $\mathsf{Sp}_{2g}(\mathbb{Z}/r\mathbb{Z})$ (see [FM12] Chapter Six).

Let $\mathrm{Tor}(S)$ be the *Torelli group* of S, that is the subgroup of the mapping class group acting trivially on $\mathrm{H}_1(S,\mathbb{Z})$. Since $A_{g,s}^{r,0,\mathfrak{o}}$ lies in $\mathrm{Hom}(\mathrm{H}_1(S,\mathbb{Z}),\mathbb{Z}/r\mathbb{Z})$, elements of $\mathrm{Tor}(S)$ act trivially on $A_{g,s}^{r,0,\mathfrak{o}}$ and thus the image of $\mathrm{Tor}(S)$ in $\mathcal{A}$ is contained in the subgroup of translation by elements of $A_{g,s}^{r,0,\mathfrak{o}}$. More precisely, we have

Proposition 5.2.1. *The image of the Torelli group of S inside $\mathcal{A}$ is the subgroup of translation $\gcd(2n, m_1-n,\dots,m_s-n)A_{g,s}^{r,0,\mathfrak{o}}$ if $g>1$ and $\gcd(m_1-n,\dots,m_s-n)A_{1,s}^{r,0,\mathfrak{o}}$ if $g=1$.*

More precisely, if E is the image of $\Gamma(S)$ in $\mathcal{A}$, then the following sequences are exact:
For $g>1$,

$$0\to \gcd(2n, m_1-n,\dots,m_s-n)A_{g,s}^{r,0,\mathfrak{o}}\to E\to \mathsf{Sp}_{2g}(\mathbb{Z}/r\mathbb{Z})\to 1\,.$$

For $g=1$,

$$0\to \gcd(m_1-n,\dots,m_s-n)A_{1,s}^{r,0,\mathfrak{o}}\to E\to \mathsf{Sp}_{2}(\mathbb{Z}/r\mathbb{Z})\to 1\,.$$

We prove Proposition 5.2.1 in Section 5.3.2.

Let $\ell=\ell_{g,s}^{r,n,\mathfrak{m}}$ where we recall

$$\ell_{g,s}^{r,n,\mathfrak{m}}=\begin{cases}1 & g=0\\ \gcd(r,m_1-n,\dots,m_s-n) & g=1\\ \gcd(r,2n,m_1-n,\dots,m_s-n) & \text{else}\end{cases}$$

If $\ell=1$, Theorem 10 follows since under the Torelli group there is already 1 orbit. Otherwise, since $l\mid r$ we have a natural map $A_{g,s}^{r,n,\mathfrak{m}}\xrightarrow{[r/l]}A_{g,s}^{\ell,n,\mathfrak{m}}$ induced by the reduction $\mathbb{Z}/r\mathbb{Z}\xrightarrow{\mathrm{mod}\,\ell}\mathbb{Z}/l\mathbb{Z}$ and, by Proposition 5.2.1, the action of the Torelli group is transitive on each fiber so that to conclude we need to know the number of orbits of the action on the base $A_{g,s}^{\ell,n,\mathfrak{m}}$, which is the object of the following lemma.

Lemma 5.2.2. *If $l=l_{g,s}^{r,n,\mathfrak{m}}$, the action of $\Gamma(S)$ on $A_{g,s}^{\ell,n,\mathfrak{m}}$ has $d(\ell)$ orbits.*

Thus to finish the proof of Theorem 10, we just need to prove Lemma 5.2.2 and Proposition 5.2.1, which is the object of the next section.

5.3 Explicit action of a generating set of the mapping class group

5.3.1 Dehn twists and bounding pairs

Dehn twists

Let x be a simple closed curve on S. Let j be an orientation-preserving embedding of the cylinder $[-1,1]\times\mathbb{S}^1$ into S with $j(\{0\}\times\mathbb{S}^1)=x$. The (right) *Dehn twist T_x about the loop x* is the homeomorphism of S defined by

$$T_x(z)=z \text{ if } x\notin \mathrm{Im}(j)\,,$$
$$T_x(j(t,\theta))=j(t,\theta+(t+1)\pi)\,.$$

This homeomorphism is homotopic to a orientation preserving diffeomorphism of S which we will also denote by T_x. The mapping class $[T_x]$ is independent of the orientation and the homotopy class of x and of the embedding j. Dehn twists act on $\mathrm{H}_1(S,\mathbb{Z})$ via the formula

$$T_x([a])=[T_x(a)]=[a+i(a,x)x]\,,$$

for any loop a on S and where $[\cdot]$ denotes the homology class in $\mathrm{H}_1(S,\mathbb{Z})$ and i the algebraic intersection number.

Lemma 5.3.1. *Let x and a be two smooth simple closed curves on S, $\overline{a}$ and $\overline{x}$ their respective lifts to TS°, $[\overline{a}]$ and $[\overline{x}]$ their homology classes in $\mathrm{H}_1(TS^\circ,\mathbb{Z})$. The Dehn twist T_x acts on $\mathrm{H}_1(TS^\circ,\mathbb{Z})$ in the following manner*

$$T_x([\overline{a}])=[\overline{T_x(a)}]=[\overline{a}+i(a,x)\overline{x}]\,. \tag{5.1}$$

Proof. The first equality comes from the definition of the action of the diffeomorphism T_x on TS°. For the second one, we deform x and a so that each crossing is transverse. Then, working locally at each crossing, one can construct a homology between $\overline{T_x(a)}$ and $\overline{a}+i(a,x)\overline{x}$. □

Bounding pairs

A *bounding pair* on S is the data of two simple closed curves γ_1,γ_2 on S such that $\gamma_1\cup\gamma_2$ separates the surface into two subsurfaces, γ_1 and γ_2 are non trivial and non homotopic in S and $[\gamma_1+\gamma_2]=0$ in $\mathrm{H}_1(\overline{S},\mathbb{Z})$. A *twist about a bounding pair* (γ_1,γ_2) is the composition $T_{\gamma_1}\circ T_{\gamma_2}^{-1}$ of a Dehn twist about γ_1 and the inverse of a Dehn twist about γ_2.

The following proposition is a special case of a theorem by A. Putman, see [Put07] Theorem 1.3.

Proposition 5.3.1 (A. Putman)**.** *The Torelli group is generated by twists about bounding pairs and Dehn twists about separating simple closed curves that are trivial in* $\mathrm{H}_1(\overline{S}, \mathbb{Z})$*.*

5.3.2 Proofs of Proposition 5.2.1 and Lemma 5.2.2

To study the action of the Torelli group on $A_{g,s}^{r,n,\mathfrak{m}}$, we first observe that Dehn twists about separating simple closed curves act trivially on $A_{g,s}^{r,n,\mathfrak{m}}$. Indeed, for any separating simple closed curve γ, one can find a standard collection of curves such that all have trivial intersection with γ, thus T_γ acts trivially on $A_{g,s}^{r,n,\mathfrak{m}}$.

We fix a standard collection of curves (a_i, b_i, c_j) on S. Let $k \in \{1, \dots, g\}$ and suppose (γ_1, γ_2) is a bounding pair cutting S in two subsurfaces S_1 and S_2 with orientation of γ_1 and γ_2 inherited from the orientation of S_1 and such that $i(b_k, \gamma_1) = 1 = -i(b_k, \gamma_2)$ and for all $i \neq k$, all $j = 1, \dots, g$ and $l = 1, 2$, $i(a_j, \gamma_l) = i(b_i, \gamma_l) = 0$. Applying twice the formula 5.1, we get that the action of the twist $W = T_{\gamma_1} \circ T_{\gamma_2}^{-1}$ about (γ_1, γ_2) on $\mathrm{H}_1(TS^\circ, \mathbb{Z})$ is

$$\begin{aligned} W(\overline{b}_k) &= \overline{b}_k + \overline{\gamma}_1 + \overline{\gamma}_2 \\ W(\overline{b}_i) &= \overline{b}_i, \ \forall i \neq k \\ W(\overline{a}_i) &= \overline{a}_i, \ \forall i \,. \end{aligned}$$

Applying Lemma 5.1.1 to the surface S_1, we obtain

$$(2g_1 + s_1)t = \overline{\gamma}_1 + \overline{\gamma}_2 + \sum_{i=1}^{s_1} \overline{c}_{j_i}$$

where g_1 is the genus of S_1 and $\{p_{j_i}\}_{i=1,\dots,s_1}$ is the set of punctures of S that lie on S_1.

In particular, for $\xi \in A_{g,s}^{r,n,\mathfrak{m}}$, we have

$$\begin{aligned} \xi(\overline{\gamma}_1 + \overline{\gamma}_2) &= n(2g_1 + s_1) + \sum_{i=1}^{s_1} -m_{j_i} \\ &= 2ng_1 + \sum_{i=1}^{s_1} (-m_{j_i} + n) \,. \end{aligned}$$

So for all $\xi \in A_{g,s}^{r,n,\mathfrak{m}}$ one has

$$W \cdot \xi = \xi + (2ng_1 - \sum_{i=1}^{s_1}(m_{j_i} - n))b_k^*(\xi)$$

where $(a_1^*, b_1^*, \ldots, a_g^*, b_g^*)$ is the basis of $A_{g,s}^{r,0,\mathfrak{o}}$ dual to the basis $(a_1, b_1, \ldots, a_g, b_g)$ of $\mathrm{H}_1(\overline{S}, \mathbb{Z})$, that is W acts on $A_{g,s}^{r,n,\mathfrak{m}}$ by translation by $(2ng_1 - \sum_{i=1}^{s_1}(m_{j_i} - n))b_k^*$.

For any $x \in \{a_1, b_1, \ldots, a_g, b_g\}$, any $I \subset \{1, \ldots, n\}$ and any $g_1 \in \{0, \ldots, g-1\}$ such that $2g_1 + \mathrm{Card}(I) > 0$, there exists a bounding pair (γ_1, γ_2) bounding a surface S_1 of genus g_1, containing the punctures of S indexed by I and such that among $\{a_1, b_1, \ldots, a_g, b_g\}$, γ_1 and γ_2 intersect x only, and once, see Figure 5.2. The action of the twist about this bounding pair is then by translation by $(2ng_1 - \sum_{i \in I}(m_i - n))x^*$. Thus if $g > 1$, the image of the Torelli group in the group of translation $A_{g,s}^{r,0,\mathfrak{o}}$ is the subgroup

$$\left(2n\mathbb{Z} \oplus \bigoplus_j (m_j - n)\,\mathbb{Z}\right) A_{g,s}^{r,0,\mathfrak{o}} = \gcd(2n, r, m_1 - n, \ldots, m_s - n) A_{g,s}^{r,0,\mathfrak{o}}$$

since $A_{g,s}^{r,0,\mathfrak{o}}$ is a $\mathbb{Z}/r\mathbb{Z}$ module. For $g = 1$ or $g = 0$, for every bounding pair, at least one of the two obtained subsurfaces has genus 0 so that modulo r, the translation obtained is of the form $(-\sum_{i=1}^{s_1} m_{j_i} - n)x^*$ and thus the image of the Torelli group in the group of translation $A_{g,s}^{r,0,\mathfrak{o}}$ is the subgroup

$$\gcd(r, m_1 - n, \ldots, m_s - n) A_{g,s}^{r,0,\mathfrak{o}}\,,$$

which conclude the proof of Proposition 5.2.1.

Proof of Lemma 5.2.2

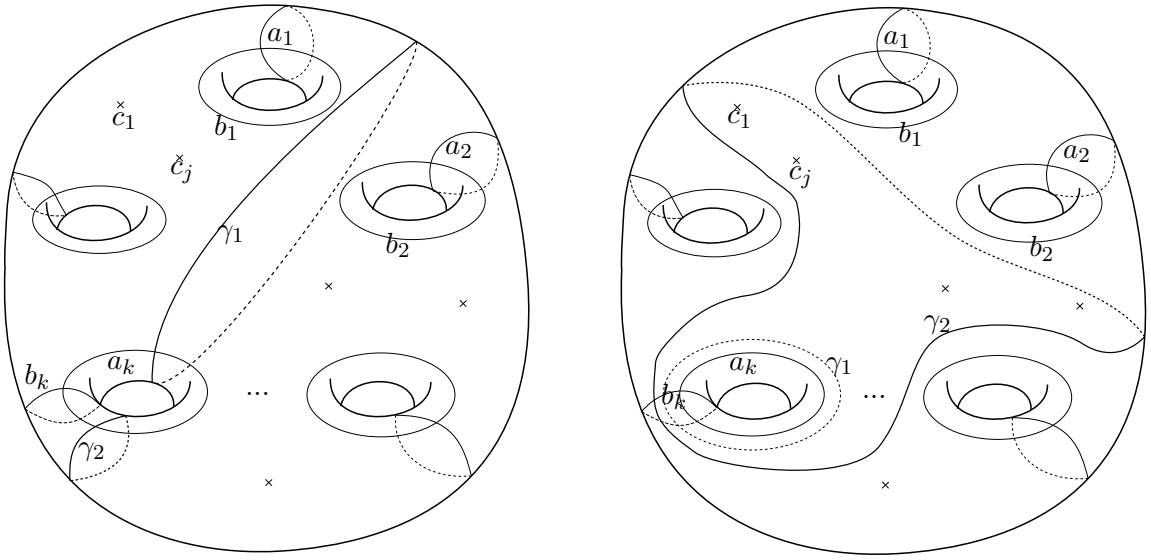
Let $\ell = \ell_{g,s}^{r,n,\mathfrak{m}}$ and recall $d(\ell)$ is the number of divisors of ℓ. Since $\ell \mid m_j - n$ for all j, we have

$$A_{g,s}^{\ell,n,\mathfrak{m}} = A_{g,s}^{\ell,n,\mathfrak{n}}$$

where $\mathfrak{n} = (n, \ldots, n)$. We have four possibilities for $A_{g,s}^{\ell,n,\mathfrak{n}}$, each needing a specific proof.

1. $g = 0$ and in this case, $A_{g,s}^{\ell,n,\mathfrak{n}}$ consists only of one element. Indeed, $\{\overline{c}_1, \ldots, \overline{c}_s, t\}$ is then a generating set of $\mathrm{H}_1(TS^\circ, \mathbb{Z})$ and thus $\xi \in A_{g,s}^{\ell,n,\mathfrak{n}}$ is uniquely determined;

2. $g > 1$ and $\ell \mid n$, in this case $m_i = m_i - n = 0 \mod \ell$ and $A_{g,s}^{\ell,n,\mathfrak{n}} = A_{g,s}^{\ell,0,\mathfrak{o}}$, where $\mathfrak{o} = (0, \ldots, 0)$. The action of the mapping class group on $A_{g,s}^{r,n,\mathfrak{n}}$ is exactly the action



Example for $x = a_k$ Example for $x = b_k$

Figure 5.2 – Examples of bounding pairs

of $\mathsf{Sp}_{2g}(\mathbb{Z}/l\mathbb{Z})$ on $\mathbb{Z}/l\mathbb{Z}^{2g}$, which has indeed $d(\ell)$ orbits and the greatest commun divisors of the coordinates yields a complete invariant.

3. $g = 1$ and proving the statement in the case $s = 1$ will be enough.

4. $g > 1$ and $\ell \nmid n$, and in this case we first deal with the case where n is a power of 2 and then deduce the general case.

Proof for $g = 1$ Assume first $s = 1$, so that $\Gamma(S) = \mathsf{SL}_2(\mathbb{Z})$ and every element of $\Gamma(S)$ can be written as a word in the two Dehn twists T_a and T_b and using the coordinates $\xi \in A_{1,1}^{\ell,n,\mathfrak{n}} \mapsto (\xi(\overline{a}), \xi(\overline{b})) \in \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$, one sees that $\gcd(\ell, \xi(\overline{a}), \xi(\overline{b}))$ is a complete invariant.

If $s > 1$, keeping only one puncture of S, we get another surface S' with genus 1 and 1 puncture along with a map $\psi \in \Gamma(S) \mapsto \psi' \in \Gamma(S')$ such that the following diagram commutes

$$\begin{array}{ccc} A_{1,s}^{\ell,n,\mathfrak{n}} & \longrightarrow & A_{1,1}^{\ell,n,\mathfrak{n}} \\ \downarrow \psi & & \downarrow \psi' \\ A_{1,s}^{\ell,n,\mathfrak{n}} & \longrightarrow & A_{1,1}^{\ell,n,\mathfrak{n}} \end{array}$$

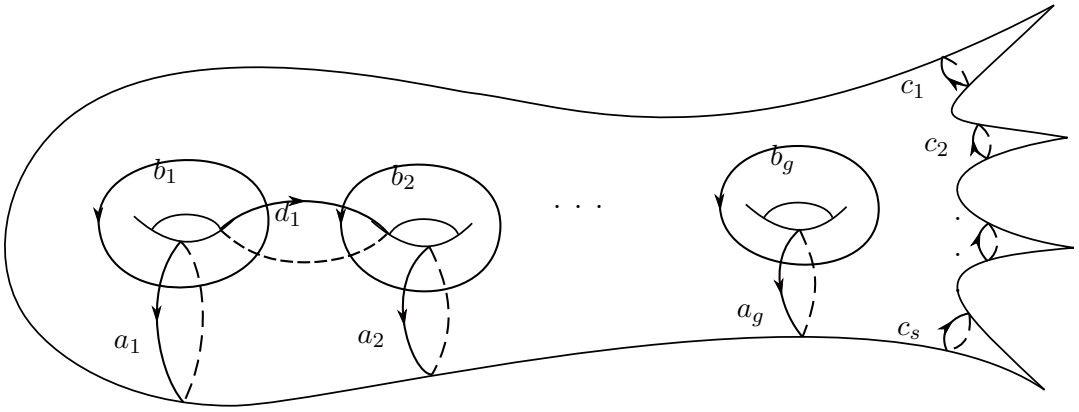


Figure 5.3 – A standard collection of curves on S and the curves d_i .

where the lines are affine isomorphisms. Thus there are as well $d(\ell)$ orbits of $\Gamma(S)$ acting on $A_{g,s}^{\ell,n,\mathfrak{n}}$.

Proof for $\ell \nmid n$ and $g > 1$ We first need to describe the actions of some particular Dehn twists in the coordinates given by

$$\xi \in A_{g,s}^{2n,n,\mathfrak{n}} \longmapsto (\xi(\overline{a}_i), \xi(\overline{b}_i))_{1 \leqslant i \leqslant g} \in ((\mathbb{Z}/2n\mathbb{Z})^2)^g = (\mathbb{Z}/2n\mathbb{Z})^{2g} .$$

Let d_i be a simple closed curve homologous to $a_{i+1} - a_i$, see figure 5.3. Lifting d_i to TS° in $\overline{d}_i$ we get $\overline{d}_i = \overline{a}_{i+1} - \overline{a}_i + t$ (applying Lemma 5.1.1 to the pair of pants bounded by a_i, a_{i+1}, d_i). For $i, j \in \mathbb{Z}$, let $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise.

Lemma 5.3.2. *From formula 5.1 for the action of a Dehn twist, the action of $T_{a_j}, T_{b_j}, T_{d_j}$ on $A_{g,s}^{2n,n,\mathfrak{n}}$ expressed in coordinates is*

- $T_{a_j}\left((\alpha_i, \beta_i)_i\right) = (\alpha_i, \beta_i + \delta_{i,j}\alpha_j)_i$;
- $T_{b_j}\left((\alpha_i, \beta_i)_i\right) = (\alpha_i - \delta_{i,j}\beta_j, \beta_i)_i$;
- $T_{d_j}\left((\alpha_i, \beta_i)_i\right) = (\alpha_i, \beta_i + (\delta_{i,j} - \delta_{i+1,j})(\alpha_{j+1} - \alpha_j + n))_i$.

Note that this set of elements was use in a similar fashion by O. Randal-Williams in [RW14], his situation being close to ours in the case $n = 1$. The fact that together with the Torelli group, the Dehn twists T_{a_i}, T_{b_1} and T_{d_i} generate the mapping class group will make two following proofs simpler, see [FM12].

We can now proceed with the proof of Lemma 5.2.2. By definition of ℓ we have $\ell \mid 2n$. We want to show that $\Gamma(S)$ acting on $A_{g,s}^{\ell,n,\mathfrak{n}}$ has $d(\ell)$ orbits. To do so, we decompose $A_{g,s}^{\ell,n,\mathfrak{n}}$ into a product depending on the prime decomposition of ℓ as follows.

For k any integer, let $k = \prod_{p \text{ prime}} p^{\nu_p(k)}$ be the prime decomposition of k. The isomorphism

$$\mathbb{Z}/\ell\mathbb{Z} \simeq \prod_{p \text{ prime}} \mathbb{Z}/p^{\nu_p(\ell)}\mathbb{Z}$$

yields an isomorphism

$$\begin{array}{rccc} \mathcal{C}: & A_{g,s}^{\ell,n,\mathfrak{n}} & \xrightarrow{\sim} & \prod_{p \text{ prime}} A_{g,s}^{p^{\nu_p(\ell)},n,\mathfrak{n}} \\ & \xi & \mapsto & (\xi_p)_{p \text{ prime}} \end{array}$$

with $\xi_p = \xi \mod p^{\nu_p(\ell)}$ for p prime.

For $f \in \mathrm{Diff}^+(S)$, $\xi \in A_{g,s}^{\ell,n,\mathfrak{n}}$ and $a \in \mathrm{H}_1(TS^\circ, \mathbb{Z})$,

$$\begin{aligned} \mathcal{C}(f \cdot \xi)(a) &= \mathcal{C}(\xi \circ f^{-1})(a) \\ &= \left(\xi(f^{-1}(a)) \mod p^{\nu_p(\ell)}\right)_p \\ &= \left(f \cdot \xi_p(a)\right)_p \\ &= f \cdot \mathcal{C}(\xi)(a). \end{aligned}$$

As a consequence, the following diagram commutes

$$\begin{array}{ccc} A_{g,s}^{\ell,n,\mathfrak{n}} & \xrightarrow{\mathcal{C}} & \prod_{p \text{ prime}} A_{g,s}^{p^{\nu_p(\ell)},n,\mathfrak{n}} \\ \downarrow f & & \downarrow f \\ A_{g,s}^{\ell,n,\mathfrak{n}} & \xrightarrow{\mathcal{C}} & \prod_{p \text{ prime}} A_{g,s}^{p^{\nu_p(\ell)},n,\mathfrak{n}} \end{array}$$

where f acts on the element $(\xi_p)_p \in \prod_{p \text{ prime}} A_{g,s}^{p^{\nu_p(\ell)},n,\mathfrak{n}}$ by $f \cdot (\xi_p)_p = (f \cdot \xi)_p$.

As a consequence, we get a well-defined map

$$A_{g,s}^{\ell,n,\mathfrak{n}}/\Gamma(S) \to \prod_{p \text{ prime}} \left(A_{g,s}^{p^{\nu_p(\ell)},n,\mathfrak{n}}/\Gamma(S)\right),$$

which we show is a bijection.

Lemma 5.3.3. *The isomorphism $\mathcal{C}$ induces a bijection*

$$A_{g,s}^{\ell,n,\mathfrak{n}}/\Gamma(S) \simeq \prod_{p \text{ prime}} \left(A_{g,s}^{p^{\nu_p(\ell)},n,\mathfrak{n}}/\Gamma(S)\right) .$$

Proof. During the proof, p will always be a prime integer. Indexing over the prime integers will really be indexing over the prime integers that divide ℓ.

Surjectivity comes from the surjectivity of $\mathcal{C}$. For the injectivity, let $\xi, \zeta \in A_{g,s}^{\ell,n,\mathfrak{n}}$ and let $(\xi_p)_p = \mathcal{C}(\xi)$, $(\zeta_p)_p = \mathcal{C}(\zeta)$ be their images by $\mathcal{C}$. Let us assume that there is $f_p \in \Gamma(S)$ such that for all p, $\xi_p = f_p\zeta_p$. We want to show that there exists $f \in \Gamma(S)$ such that $\xi = f\zeta$.

It is enough to deal with the case where the action of f_p on $A_{g,s}^{p^{\nu_(\ell)},n,\mathfrak{n}}$ is trivial for all p but one, say q. We may further assume that, when acting on $A_{g,s}^{q^{\nu_q(\ell)},n,\mathfrak{n}}$, the action of f_q is the action of ϕ where ϕ is either T_{a_i}, T_{b_i} or T_{d_i}, since together with the Torelli group they generate the mapping class group.

Let $\overline{q} \in \mathbb{Z}$ be such that $\overline{q} = 1 \mod q^{\nu_q(\ell)}$ and $\overline{q} = 0 \mod p^{\nu_p(\ell)}$ for $p \neq q$. Using the formula for the action of elementary Dehn twists, we deduce that $\phi^{\overline{q}}$ induces the same action as f_q on $A_{g,s}^{q^{\nu_q(\ell)},n,\mathfrak{n}}$ and acts trivially on $A_{g,s}^{p^{\nu_p(\ell)},n,\mathfrak{n}}$ for $p \neq q$, thus $\xi_p = \phi^{\overline{q}}\zeta_p$ for all p, that is $\mathcal{C}(\xi) = \mathcal{C}(\phi^{\overline{q}}\zeta)$, and using the injectivity of $\mathcal{C}$ we deduce $\xi = \phi^{\overline{q}}\zeta$ in $A_{g,s}^{\ell,n,\mathfrak{n}}$. □

For $p > 2$ prime, $p^{\nu_p(\ell)} \mid n$ since $p^{\nu_p(\ell)} \mid 2n$ and 2 is coprime with p, so $A_{g,s}^{p^{\nu_p(\ell)},n,\mathfrak{n}} = A_{g,s}^{p^{\nu_p(\ell)},0,\mathfrak{o}}$ and $A_{g,s}^{p^{\nu_p(\ell)},n,\mathfrak{n}}/\Gamma(S)$ has exactly $d(p^{\nu_p(\ell)}) = \nu_p(\ell) + 1$ elements.

Thus to prove that $A_{g,s}^{\ell,n,\mathfrak{n}}/\Gamma(S)$ has $d(\ell)$ elements, all that remains it to show $A_{g,s}^{2^{\nu_2(\ell)},n,\mathfrak{n}}/\Gamma(S)$ has $\nu_2(\ell) + 1$ elements: indeed, we would then have

$$\begin{aligned} \mathrm{Card}(A_{g,s}^{\ell,n,\mathfrak{n}}/\Gamma(S)) &= \prod_{p \text{ prime}} \mathrm{Card}(A_{g,s}^{p^{\nu_p(\ell)},n,\mathfrak{n}}/\Gamma(S)) \\ &= \prod_{p \text{ prime}} (\nu_p(\ell) + 1) = d(\ell) . \end{aligned}$$

If n is a power of 2, since $\ell \mid 2n$ and $\ell \nmid n$, we have $2n = \ell$, and the following lemma will conclude.

Lemma 5.3.4. *Suppose n is a power of 2. Then the reduction modulo n $A_{g,s}^{2n,n,\mathfrak{n}} \to A_{g,s}^{n,n,\mathfrak{n}} = A_{g,s}^{n,0,\mathfrak{o}}$ yields a surjective map*

$$A_{g,s}^{2n,n,\mathfrak{n}}/\Gamma(S) \xrightarrow{j} A_{g,s}^{n,n,\mathfrak{n}}/\Gamma(S)$$

such that the orbit of $0 \in A_{g,s}^{n,0,\mathfrak{o}}$ has 2 antecedents by j and all the other orbits have only

one antecedent.

In particular, $A_{g,s}^{2n,n,\mathfrak{n}}/\Gamma(S)$ *has* $d(n)+1=d(2n)$ *elements.*

Proof. We identify $(\mathbb{Z}/2n\mathbb{Z})^{2g}$ with $A_{g,s}^{2n,n,\mathfrak{n}}$ and $(\mathbb{Z}/n\mathbb{Z})^{2g}$ with $A_{g,s}^{n,0,\mathfrak{o}}$ during the proof. The map $j : A_{g,s}^{2n,n,\mathfrak{n}} \to A_{g,s}^{n,n,\mathfrak{n}}$ is, in these coordinates, the reduction modulo n $(\mathbb{Z}/2n\mathbb{Z})^{2g} \to (\mathbb{Z}/n\mathbb{Z})^{2g}$. Let $\xi \in A_{g,s}^{2n,n,\mathfrak{n}}$ such that $j(\xi)=(0,\dots,0)$, that is

$$\xi=(\epsilon_1,\dots,\epsilon_{2g}),\ \epsilon_i\in\{0,n\}\,.$$

Using T_{a_i} and T_{b_i}, we transform the block $(\epsilon_{2i-1},\epsilon_{2i})$ into $(0,\varepsilon_i)$ where $\varepsilon_i=0$ if $\epsilon_{2i-1}=\epsilon_{2i}=0$ and $\varepsilon_i=n$ otherwise. Then repeated use of T_{d_i} in decreasing order of i's shows that ξ is in the same $\Gamma(S)$-orbit as

$$\xi_\varepsilon=(0,\varepsilon,0,0,\dots,0)$$

where $\varepsilon\in\{0,n\}$ depends on ξ. In particular, $j^{-1}(0)$ has at most 2 elements. It remains to show that ξ_0 and ξ_n are not in the same $\Gamma(S)$ orbits. We consider the following function

$$\begin{array}{rccc} I: & j^{-1}(0) & \to & \mathbb{Z}/2\mathbb{Z} \\ & (\epsilon_1,\dots,\epsilon_{2g}) & \mapsto & \mathrm{Card}\{i,(\epsilon_{2i-1},\epsilon_{2i})=(0,0)\} \mod 2 \end{array}.$$

We claim that I is invariant by $\Gamma(S)$. To show this, it is enough to consider the actions of T_{a_i},T_{b_i},T_{d_i} because together with elements of the Torelli group (which acts trivially here) they generate $\Gamma(S)$.

Indeed, let $(\alpha_1,\beta_1,\dots,\alpha_g,\beta_g)\in j^{-1}(0)$. For $i,j=1,\dots,g$, either $(\alpha_i,\beta_i)=(0,0)$ in which case

$$T_{a_j}((\alpha_i,\beta_i))=T_{b_j}((\alpha_i,\beta_i))=(0,0)$$

or $(\alpha_i,\beta_i)\neq(0,0)$ and

$$\begin{aligned} T_{a_j}((\alpha_i,\beta_i))&\neq(0,0)\\ T_{b_j}((\alpha_i,\beta_i))&\neq(0,0). \end{aligned}$$

Overall, the total number of blocks $(0,0)$ doesn't change.

Now applying T_{d_j}, either $\alpha_{j+1}-\alpha_j+n=0 \mod 2n$ and

$$T_{d_j}((\alpha_1,\beta_1,\dots,\alpha_g,\beta_g))=(\alpha_1,\beta_1,\dots,\alpha_g,\beta_g)$$

or $\alpha_{j+1} - \alpha_j + n = n \mod 2n$ and

$$T_{g_j}(\alpha_j, \beta_j, \alpha_{j+1}, \beta_{j+1}) = (\alpha_j, \beta_j + n, \alpha_{j+1}, \beta_{j+1} + n)\,,$$

so that if $\alpha_{j+1} = \alpha_j = n$, the total number of $(0,0)$ blocks doesn't change and if $\alpha_{j+1} = \alpha_j = 0$, the total number of $(0,0)$ blocks changes by ± 2, since here $\alpha_i = \alpha_{i+1} \mod 2n$.

In every cases, $I(T_x(\xi)) = I(\xi)$ for any $x = a_i, b_i, d_i$, proving the function I to be mapping class group invariant.

Since $I(\xi_0) = 0 \neq 1 = I(\xi_n)$, ξ_0 and ξ_n are not $\Gamma(S)$-equivalent, thus proving $\mathrm{Card}(j^{-1}(0)) = 2$.

Every mapping class group orbits in $A_{g,s}^{n,0,\mathfrak{o}}$ may be represented by a $\zeta_k = (0, k, 0, \ldots, 0) \in A_{g,s}^{n,0,\mathfrak{o}}$ with $k \mid n$, $0 \leqslant k < n$. Suppose $k > 0$ and let $\xi \in j^{-1}(\zeta_k)$, that is

$$\xi = (\epsilon_1, \epsilon_2 + k, \epsilon_3 \ldots, \epsilon_{2g}),\ \epsilon_i \in \{0, n\}\,.$$

By the same earlier process, ξ is in the same orbit as

$$\xi_{\varepsilon_1, \varepsilon_2 + k} = (\varepsilon_1, \varepsilon_2 + k, 0, \ldots, 0)$$

where $\varepsilon_1, \varepsilon_2 \in \{0, n\}$. Well-chosen repeated applications of T_{a_1} and T_{b_1} will send ξ_{α_1, β_1} to $\xi_{0,K}$ where $K = \gcd(\alpha_1, \beta_1, 2n)$. Using the fact that n is a power of 2 and k is a proper divisor of n, a little computation shows $\gcd(\varepsilon_1, \varepsilon_2 + k, 2n) = k$ for all $\varepsilon_1, \varepsilon_2 \in \{0, n\}$. In particular, $j^{-1}(\zeta_k)$ consists of only one element, the orbit of $\xi_{0,k}$. □

Lastly we prove that $A_{g,s}^{2^{\nu_2(\ell)}, n, \mathfrak{n}}$ has $d(2^{\nu_2(\ell)})$ orbits for any n. We write $n = 2^{\nu_2(n)}k$ where $\gcd(k, 2) = 1$. Let $\overline{k} \in \mathbb{Z}/2^{\nu_2(\ell)}\mathbb{Z}$ be the inverse of k modulo $2^{\nu_2(\ell)}$. Linear multiplication by $\overline{k}$ in the $\mathbb{Z}/2^{\nu_2(\ell)}\mathbb{Z}$-module $\mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/2^{\nu_2(\ell)}\mathbb{Z})$ restricts to a $\Gamma(S)$-equivariant affine isomorphism

$$A_{g,s}^{2^{\nu_2(\ell)}, n, \mathfrak{n}} \xrightarrow{\sim} A_{g,s}^{2u, u, \mathfrak{u}}$$

where $u = 2^{\nu_2(\ell)-1} = 2^{\nu_2(n)}$ and $\mathfrak{u} = (u, \ldots, u)$, and we conclude by applying Lemma 5.3.4.

Chapter 6

Integral maximal representations

6.1 Preliminaries

6.1.1 Background material on group cohomology

In this section we review bounded and unbounded, Borel or continuous group cohomology with value in an abelian group, giving more attention to the cohomology of a Lie group and of a surface group.

6.1.2 Group cohomology

Let $R = \mathbb{R}, \mathbb{Z}$ or $\mathbb{R}/\mathbb{Z}$. For any locally compact topological group G we have the complex

$$(C^\bullet(G,R), d) = (\{f : G^{\bullet+1} \to R\}, d)$$

where the differential of $f \in C^n(G,R)$ is given by

$$(df)(g_0, \dots, g_n) = \sum_{i=0}^{n} (-1)^i f(g_0, \dots, \widehat{g_i}, \dots g_n)\,.$$

The cohomology of $(C^\bullet(G,R), d)$ is trivial in positive degree and we consider the subcomplex $C^\bullet(G,R)^G$ of G-invariant cochains, where for $g, g_0, \dots, g_n \in G$, $g \cdot f(g_0, \dots, g_n) = f(g^{-1}g_0, \dots, g^{-1}g_n)$. The complex $(C^\bullet(G,R)^G, d)$ is called the *homogeneous complex*.

One may as well consider the following complex, called *inhomogeneous*

$$(C(G^\bullet, R), \delta)$$

where, for $g_0, \dots, g_n \in G$, the differential of a cochain f is defined by

$$(\delta f)(g_0, \dots, g_n) = f(g_2, \dots, g_n) + \sum_{i=1}^{n-1} (-1)^i f(g_1, \dots, g_i g_{i+1}, \dots, g_n) + (-1)^n f(g_1, \dots, g_{n-1}) \,.$$

An inhomogeneous cocycle is said *normalized* if $c(\mathrm{Id}_G, g_2, \dots, g_n) = 0$ for all $g_2, \dots, g_n$, where Id_G is the identity of G. Every inhomogeneous cocycle differs from a normalized one by a coboundary and we will always assume our inhomogeneous cocycles to be normalized.

The inhomogeneous and homogeneous complexes are isomorphic. Indeed, to any homogeneous $f \in C(G^{n+1}, R)^G$, we associate $V^n(f) \in C(G^n, R)$ defined by

$$V^n f(g_1, \dots, g_n) = f(1, g_1, g_1 g_2, \dots, g_1 \cdots g_n)$$

and $V^n : (C^{n+1}(G, R)^G, d) \to (C^n(G, R), \delta)$ is a complex isomorphism whose inverse U^n is given by

$$U^n f(g_0, \dots, g_n) = f(g_0^{-1} g_1, \dots, g_{n-1}^{-1} g_n)$$

Both the inhomogeneous and homogeneous representations will be used. The cohomology of various sub-complexes of $(C^\bullet(G, R)^G, d)$ yields the cohomology theories we will be interested in.

Definition 6.1.1. The *regular* (resp. *continous, bounded, Borel, bounded continuous, Borel bounded) cohomology of G with coefficient in R*, noted $\mathrm{H}^\bullet(G, R)$ (resp. $\mathrm{H}^\bullet_{\mathrm{c}}(G, R)$, $\mathrm{H}^\bullet_{\mathrm{b}}(G, R)$, $\mathrm{H}^\bullet_{\mathrm{B}}(G, R)$, $\mathrm{H}^\bullet_{\mathrm{bc}}(G, R)$, $\mathrm{H}^\bullet_{\mathrm{Bb}}(G, R)$), is the cohomology of the complex of $(C^\bullet(G, R)^G, d)$ (resp. of the sub-complex of continuous, bounded, Borel, bounded continuous, Borel bounded) cochains.

The Borel versions are most useful when $A = \mathbb{Z}$ since the continuous versions might not be relevant. The inclusions of complexes "continuous $\to$ Borel" and "bounded $\to$ unbounded" yields *comparison maps* in the corresponding cohomology theories and the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}^\bullet_{\mathrm{bc}}(G, R) & \longrightarrow & \mathrm{H}^\bullet_{\mathrm{c}}(G, R) \\ \downarrow & & \downarrow \\ \mathrm{H}^\bullet_{\mathrm{Bb}}(G, R) & \longrightarrow & \mathrm{H}^\bullet_{\mathrm{B}}(G, R) \,. \end{array}$$

If Γ is any discrete group, then every cochains on Γ is Borel and continuous, so that $\mathrm{H}^\bullet_{\mathrm{B}}(\Gamma, R) = \mathrm{H}^\bullet_{\mathrm{c}}(\Gamma, R) = \mathrm{H}^\bullet(\Gamma, R)$ and $\mathrm{H}^\bullet_{\mathrm{Bb}}(\Gamma, R) = \mathrm{H}^\bullet_{\mathrm{bc}}(\Gamma, R) = \mathrm{H}^\bullet_{\mathrm{b}}(\Gamma, R)$. For a countable CW-complex X, let $\mathrm{H}_\bullet(X, R)$ and $\mathrm{H}^\bullet(X, R)$ be respectively the singular homology and

cohomology of X with R-coefficient. One may consider the subcomplex of bounded singular cochains on X and get the singular bounded cohomology of X, denoted $\mathrm{H}^\bullet_\mathrm{b}(X,R)$. If $\pi_1(X)$ is the fundamental group of X and X is aspherical, one has isomorphisms g_X between the (bounded) cohomology of $\pi_1(X)$ and of X, such that the following diagram with isomorphisms for horizontal lines and comparison maps as vertical lines commutes

$$\begin{array}{ccc} \mathrm{H}^\bullet_\mathrm{b}(\pi_1(X),R) & \xrightarrow{g_X} & \mathrm{H}^\bullet_\mathrm{b}(X,R) \\ \downarrow & & \downarrow \\ \mathrm{H}^\bullet(\pi_1(X),R) & \xrightarrow{g_X} & \mathrm{H}^\bullet(X,R)\,. \end{array}$$

The following facts will be useful:

1. Every cochain with $\mathbb{R}/\mathbb{Z}$-coefficient is bounded and thus $\mathrm{H}^\bullet_\mathrm{c}(G,\mathbb{R}/\mathbb{Z}) = \mathrm{H}^\bullet_\mathrm{bc}(G,\mathbb{R}/\mathbb{Z})$ and $\mathrm{H}^\bullet_\mathrm{B}(G,\mathbb{R}/\mathbb{Z}) = \mathrm{H}^\bullet_\mathrm{Bb}(G,\mathbb{R}/\mathbb{Z})$ for G locally compact topological group, with similar equalities for singular cohomology of CW-complexes;

2. The short exact sequence
$$0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$$
yields long exacts sequences in Borel and Borel bounded group cohomology and in singular and bounded singular cohomology, all compatible with respect to the comparison maps and Gromov isomorphisms;

3. By using regularization operators defined in [Bla79] one can show $\mathrm{H}^\bullet_\mathrm{bc}(G,\mathbb{R}) = \mathrm{H}^\bullet_\mathrm{Bb}(G,\mathbb{R})$;

4. For $f : G \to R$, f is a inhomogeneous 2-cocycle if and only if f is a homomorphism, as $\delta f(x,y) = f(x) - f(xy) + f(y)$. From this one deduce that $\mathrm{H}^1_\mathrm{c}(G,R) = \mathrm{Hom}_\mathrm{c}(G,R)$ is the space of continuous homomorphisms;

5. if $G = G_1 \times G_2$ then we have linear isomorphisms between the cohomology of G with coefficient in R and the cartesian product of the cohomology of G_1 and G_2 with coefficient in R, for every cohomology theory and in a natural way with respect to the comparison maps and change of coefficient.

6. If G is compact or abelian, $\mathrm{H}^n_\mathrm{Bb}(G,\mathbb{R}) = \mathrm{H}^n_\mathrm{bc}(G,\mathbb{R}) = 0$ for $n > 0$.

Continuous central extensions and Borel cohomology

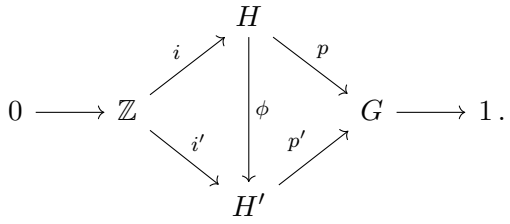
Let G be a locally compact topological group. A *(continuous) central extension of G by $\mathbb{Z}$* is a locally compact topological group H together with continuous homomorphisms

$i : \mathbb{Z} \to H$ and $p : H \to G$ such that the following sequence is exact

$$0 \to \mathbb{Z} \xrightarrow{i} H \xrightarrow{p} G \to 1$$

and $i(\mathbb{Z})$ is a central subgroup of H.

Two central extensions (H, i, p) and (H', i', p') are *equivalent* if there exists an isomorphism $\phi : H \to H'$ such that the following diagram commutes



If $\mathbb{Z} \xrightarrow{i} H \xrightarrow{p} G$ is a central extension and $\sigma : G \to H$ is a Borel section of p, for $x, y \in G$ we define $d\sigma(x, y) \in \mathbb{Z}$ to be the only integer such that

$$\sigma(g)\sigma(g') = \sigma(gg')i(d\sigma(g, g'))\,.$$

For $g_0, g_1, g_2 \in G$, we have

$$\begin{aligned}\sigma(g_0)\sigma(g_1)\sigma(g_2) &= \sigma(g_0g_1)\sigma(g_2)i(d\sigma(g_0, g_1)) \\ &= \sigma(g_0g_1g_2)i(d\sigma(g_0g_1, g_2) + d\sigma(g_0, g_1))\end{aligned}$$

as well as

$$\begin{aligned}\sigma(g_0)\sigma(g_1)\sigma(g_2) &= \sigma(g_0)\sigma(g_1g_2)i(d\sigma(g_1, g_2)) \\ &= \sigma(g_0g_1g_2)i(d\sigma(g_0, g_1g_2) + d\sigma(g_1, g_2))\end{aligned}$$

thus

$$d\sigma(g_1, g_2) - d\sigma(g_0g_1, g_2) + d\sigma(g_0, g_1g_2) - d\sigma(g_0, g_1) = 0$$

that is $d\sigma$ is an inhomogeneous Borel 2-cocycle. It is not hard to see that if σ' is a different section of $p : H \to G$, then there exists a function $\xi : G \to \mathbb{Z}$ such that

$$d\sigma = d\sigma' + \delta\xi\,.$$

Indeed, $\xi(g)$ is the only integer such that $\sigma(g) = \sigma'(g)i(\xi(g))$. In conclusion, the central

extension $H \to G$ yields a well-defined class in $\mathrm{H}^2_{\mathrm{B}}(G, \mathbb{Z})$ and one can see that two equivalent central extensions yield the same class.

Conversely, let G be any locally compact topological group and $\kappa \in \mathrm{H}^2_{\mathrm{B}}(G, \mathbb{Z})$. Suppose c is an inhomogeneous cocycle representing κ. We define the group $G \times_c \mathbb{Z}$ such that:

- the elements of $G \times_c \mathbb{Z}$ are pairs $(g, k) \in G \times \mathbb{Z}$;
- $(g, k) \cdot (g', k') = (gg', k + k' + c(g, g'))$ for all $g, g' \in G$ and $k, k' \in \mathbb{Z}$.

Before defining the topology on $G \times_c \mathbb{Z}$, we recall the following fact.

Lemma 6.1.1. *Let G be a locally compact topological group equipped with a Haar measure. For $X \subset G$, let $X^{-1}X = \{x^{-1}y : x, y \in X\}$. Then the sets $X^{-1}X$ for X Borel set of positive measure form a basis of neighborhoods of the identity of G.*

We equip $G \times_c \mathbb{Z}$ with the σ-algebra such that

$$\begin{array}{ccc} G & \to & G \times_c \mathbb{Z} \\ g & \mapsto & (g, 0) \end{array}$$

and

$$\begin{array}{ccc} (G \times_c \mathbb{Z})^2 & \to & G \times_c \mathbb{Z} \\ (x, y) & \mapsto & x^{-1}y \end{array}$$

are measurable. Let μ be a left invariant Haar measure on G and ν be the only measure on $G \times_c \mathbb{Z}$ such that

- ν is left $G \times_c \mathbb{Z}$-invariant;
- $\nu(\{(g, k) : g \in B\}) = \mu(B)$ for every $k \in \mathbb{Z}$ and $B \subset G$ borelien.

To define a topology compatible with the group structure on $G \times_c \mathbb{Z}$ we need only to specify a basis of neighborhoods of the identity. The basis of neighborhoods we take is the family

$$\left\{X^{-1}X\right\}_X$$

for $X \subset G \times_c \mathbb{Z}$ measurable with $\nu(X) > 0$. Using Lemma 6.1.1, one deduces that the projection $p : G \times_c \mathbb{Z} \to G$ is continuous.

Moreover, $G \times_c \mathbb{Z}$ is Hausdorff. Indeed, for $Y \subset G$ a measurable set with $\mu(X) > 0$ let

$$X_Y = \{(g, 0) : g \in Y\} \ .$$

Since G is Hausdorff, one has

$$\bigcap_{\mu(Y)>0} Y^{-1}Y = \{\mathrm{Id}_{\mathrm{G}}\}$$

and we deduce, using the equality $\cap((X_Y)^{-1}X_Y) = (X_{\cap Y})^{-1}X_{\cap Y}$, that

$$\bigcap_{\mu(Y)>0} (X_Y)^{-1}X_Y = \{\mathrm{Id}_{\mathrm{G}\times_c\mathbb{Z}}\} \ .$$

Equipped with this topology, $G\times_c\mathbb{Z}$ is a locally compact topological group and $p : G\times_c\mathbb{Z} \to G$ is a covering map.

Since $0 \to \mathbb{Z} \to G \times_c \mathbb{Z} \to G \to 1$ is exact, we get a continuous central extension

$$0 \to \mathbb{Z} \xrightarrow{i} G \times_c \mathbb{Z} \xrightarrow{p} G \to 1$$

whose associated Borel class is κ.

In general, for a locally compact topological group G and a class $\kappa \in \mathrm{H}^2_{\mathrm{Bb}}(G,\mathbb{Z})$ we denotes by G_κ the central extension associated to κ.

Rotation numbers

These rotation numbers where introduced in [BIW10] Section 7. A lot of what follows here is an adaptation to our situation of their results.

We define the *rotation number* and *translation number* associated to a bounded Borel integral class. Let G be a locally compact topological group. For any $B < G$ closed subgroup we have the long exact sequence in Borel bounded cohomology associated to the short exact sequence of coefficient $0 \to \mathbb{Z} \to \mathbb{R} \to \mathbb{R}/\mathbb{Z} \to 0$

$$0 \to \mathrm{H}^1_{\mathrm{Bb}}(B,\mathbb{R}/\mathbb{Z}) \to \mathrm{H}^2_{\mathrm{Bb}}(B,\mathbb{Z}) \to \mathrm{H}^2_{\mathrm{Bb}}(B,\mathbb{R}) \to \cdots .$$

Using the facts from Section 6.1.2, we have

$$\mathrm{H}^1_{\mathrm{Bb}}(B,\mathbb{R}/\mathbb{Z}) = \mathrm{Hom}_{\mathrm{c}}(B,\mathbb{R}/\mathbb{Z}) \quad \text{and} \quad \mathrm{H}^2_{\mathrm{Bb}}(B,\mathbb{R}) = \mathrm{H}^2_{\mathrm{bc}}(B,\mathbb{R})$$

and in the end the long exact sequence of coefficient becomes

$$0 \to \mathrm{Hom}_{\mathrm{c}}(B,\mathbb{R}/\mathbb{Z}) \xrightarrow{\mathfrak{d}} \mathrm{H}^2_{\mathrm{Bb}}(B,\mathbb{Z}) \to \mathrm{H}^2_{\mathrm{bc}}(B,\mathbb{R}) \to \cdots . \tag{6.1}$$

If $\kappa \in \mathrm{H}^2_{\mathrm{Bb}}(G,\mathbb{Z})$, let $\kappa_{\mathbb{R}}$ be its image in $\mathrm{H}^2_{\mathrm{bc}}(G,\mathbb{R})$. If $\kappa_{\mathbb{R}}|_B = 0$ let $f_B : B \to \mathbb{R}/\mathbb{Z}$ be the unique continuous homomorphism such that $\mathfrak{d}(f_B) = \kappa|_B$. Applying this to $B = \overline{\langle g\rangle}$ for any $g \in G$, and using the fact that $\mathrm{H}^2_{\mathrm{bc}}(B,\mathbb{R}) = 0$ since $B = \overline{\langle g\rangle}$ is abelian, we define, as in [BIW10] Section 7, the rotation number of g as follows.

Definition 6.1.2. The *rotation number* of $g \in G$ with respect to $\kappa \in \mathrm{H}^2_{\mathrm{Bb}}(G, \mathbb{Z})$ is

$$\mathrm{Rot}_\kappa(g) = f_{\overline{\langle g \rangle}}(g)\,.$$

The following properties follow from the fact that the exact sequence (6.1) is natural with respect to group homomorphisms.

Lemma 6.1.2.

- $\mathrm{Rot}_\kappa : G \to \mathbb{R}/\mathbb{Z}$ *is conjugacy invariant;*
- *if* $\kappa_{\mathbb{R}}|_B = 0$ *then* $\mathrm{Rot}_{\kappa|_B}$ *is a continuous homomorphism such that* $\mathfrak{d}(\mathrm{Rot}_{\kappa|_B}) = \kappa|_B$ *;*
- *if* $f : G_1 \to G_2$ *is a continuous homomorphism and* $\kappa_2 \in \mathrm{H}_{\mathrm{Bb}}(G_2, \mathbb{Z})$ *then*

$$\mathrm{Rot}_{f^*\kappa_2}(g_1) = \mathrm{Rot}_{\kappa_2}(f(g_1))\,;$$

- $\mathrm{Rot}_{\kappa_1+\kappa_2} = \mathrm{Rot}_{\kappa_1} + \mathrm{Rot}_{\kappa_2}$ *for all* $\kappa_1, \kappa_2 \in \mathrm{H}^2_{\mathrm{Bb}}(G, \mathbb{Z})$.

Translation numbers

Let $f : G \to \mathbb{R}$ be any function. We say that f is a *quasimorphism* if the function

$$\delta f(x, y) = f(xy) - f(x) - f(y)$$

is bounded on $G \times G$. We say that f is *homogeneous* if $f(x^n) = nf(x)$ for all $n \in \mathbb{Z}$ and $x \in G$.

Lemma 6.1.3. *Let* $f : G \to \mathbb{Z}$ *be a quasimorphism. For* $x \in G$ *let*

$$Hf(x) = \lim_n \frac{f(x^n)}{n}\,.$$

Then $Hf - f$ *is bounded and* Hf *is a homogeneous quasimorphism.*

Proof. It is enough to show that $Hf - f$ is bounded. Let $n \in \mathbb{Z}_{>0}$. Using the quasimorphism property we have

$$f(x^n) = nf(x) + \sum_{i=1}^{n-1} \delta f(x, x^i)$$

and since δf is bounded, $Hf - f$ is bounded. □

The following lemma is Lemma 7.4 of [BIW10].

Lemma 6.1.4. *Every Borel homogeneous quasimorphism is continuous.*

Let $\kappa \in \mathrm{H}^2_{\mathrm{Bb}}(G, \mathbb{Z})$ and let c be a bounded inhomogeneous cocycle representing κ. Let $\mathbb{Z} \xrightarrow{i} G_\kappa \xrightarrow{p} G$ be the central extension associated to c as in 6.1.2, forgetting that c is bounded. Let f_0 be

$$\begin{array}{rccc} f_0 : & G_\kappa & \to & \mathbb{Z} \\ & (g,k) & \mapsto & k\,. \end{array}$$

By definition, $\delta f_0 = c$ so f_0 is a quasimorphism. We define the *translation number function with respect to* κ to be

$$\mathrm{Trans}_\kappa = H f_0\,.$$

Proposition 6.1.1. *The function* $\mathrm{Trans}_\kappa : G_\kappa \to \mathbb{R}$ *does not depend on the choice of the bounded cocycle* c *used to define it and is a continuous and homogeneous quasimorphism. Moreover for* $h \in G_\kappa$ *and* $n \in \mathbb{Z}$*,*

$$\mathrm{Trans}_\kappa(i(n)h) = n + \mathrm{Trans}_\kappa(h)$$

and

$$\mathrm{Trans}_\kappa(h) = \mathrm{Rot}_\kappa(p(h)) \mod \mathbb{Z}\,.$$

In particular, Rot_κ *is continuous.*

Proof. The fact that Trans_κ is a continuous homogeneous quasimorphism follows from the fact that f_0 is a Borel quasimorphism and Lemma 6.1.4.

If c and c' are bounded cocycle both representing κ as a bounded class, then $c - c' = \delta\xi$ for $\xi : G \to \mathbb{Z}$ some bounded function. If $G_\kappa = G \times_c \mathbb{Z}$, let $f_0(g,k) = k$ and $f_1 : G \times_c \mathbb{Z}$ be defined by

$$f_1 : G \times_c \mathbb{Z} \xrightarrow{\phi} G \times_{c'} \mathbb{Z} \xrightarrow{f_0} \mathbb{Z}$$

where ϕ is the isomorphism

$$\begin{array}{rccc} \phi : & G \times_c \mathbb{Z} & \to & G \times_{c'} \mathbb{Z} \\ & (g,k) & \mapsto & (g, k + \xi(g)) \end{array}$$

so that $f_1(g,k) = k + \xi(g)$. Since ξ is bounded,

$$H f_1 = H f_0\,,$$

so Trans_κ depends only on κ.

The equality

$$\mathrm{Trans}_\kappa(i(n)h) = n + \mathrm{Trans}_\kappa(h)$$

follows from the fact that $i(\mathbb{Z})$ is central and f_0 already satisfies it.

Lastly we show the relation between Trans and Rot. By the last part of Lemma 6.1.2 we need to prove it for $G = \mathbb{Z}$ only. Let $q : \mathbb{R} \to \mathbb{R}/\mathbb{Z}$ be the projection so that we want to show $q \circ Hf_0 = \mathrm{Rot}_\kappa$ and it is enough to show Because Hf_0 is homogeneous and continuous, $q \circ Hf_0$ is a group homomorphism and we claim that $\mathfrak{d}(q \circ Hf_0) = \kappa$. Indeed, let $\sigma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be the section with value in $[0, 1)$ so that $\sigma \circ q \circ Hf_0$ is a Borel function from our group $\mathbb{Z}$ to the coefficient module $\mathbb{R}$. By definition of $\mathfrak{d}$ we have

$$\mathfrak{d}(q \circ Hf_0) = -\delta(\sigma \circ q \circ Hf_0) = \delta f_0 - \delta(f_0 - Hf_0 + \sigma \circ q \circ Hf_0)$$

because $\delta Hf_0 = 0$ since Hf_0 is a homomorphism. The function $f_0 - Hf_0 + \sigma \circ q \circ Hf_0$ takes integral value because f and $Hf_0 - \sigma \circ q \circ Hf_0$ do and is bounded because $f_0 - Hf_0$ and $\sigma \circ q \circ Hf_0$ are. So in $\mathrm{H}^2_{\mathrm{Bb}}(\mathbb{Z}, \mathbb{Z})$ we get

$$\mathfrak{d}(q \circ Hf_0) = [df] = [c] = \kappa\,.$$

So by definition of the rotation number, $\mathrm{Rot}_\kappa(g) = q \circ Hf_0(g)$. □

Example 6.1.1. Let $G = \mathrm{Homeo}^+(\mathbb{R}/\mathbb{Z})$ be the group of orientation preserving homeomorphisms of the circle. Let $\tilde{G} = \mathrm{Homeo}^+_{\mathbb{Z}}(\mathbb{R})$ be the group of orientation-preserving homeomorphism of $\mathbb{R}$ that commutes with the translation $T : x \mapsto x + 1$. For $g \in G$ let $\tilde{g} \in \tilde{G}$ be the only lift of g with $0 \leqslant \tilde{g}(0) < 1$. The *bounded Euler class* $e^b \in \mathrm{H}^2_{\mathrm{Bb}}(G, \mathbb{Z})$ is the bounded cohomology class of the cocycle c defined by, for all $g, h \in G$,

$$\tilde{g}\tilde{h} = \widetilde{gh}\tau^{c(g,h)}\,.$$

The usual rotation number of a homeomorphism of the circle $g \in G$ is exactly $\mathrm{Rot}_{e^b}(g)$. If $\iota : \mathsf{PSL}_2(\mathbb{R}) \to G$ is the action of $\mathsf{PSL}_2(\mathbb{R})$ on the circle and if $\kappa = \iota^* e^b$, then $(\mathsf{PSL}_2(\mathbb{R}))_\kappa$ is isomorphic to $\widetilde{\mathsf{PSL}}_2(\mathbb{R})$ and the translation number function of the first Part is Trans_κ, or more precisely the following diagram commutes

$$\begin{array}{ccc} \widetilde{\mathsf{PSL}}_2(\mathbb{R}) & & \\ \downarrow \simeq & \searrow^{\mathrm{Trans}} & \\ (\mathsf{PSL}_2(\mathbb{R}))_\kappa & \xrightarrow[\mathrm{Trans}_\kappa]{} & \mathbb{R}\,. \end{array}$$

6.1.3 Lie groups of Hermitian type

For this section the reference is [Hel01].

Definition 6.1.3. A *(Riemannian globally) symmetric space* is a connected riemannian

manifold $\mathcal{X}$ such that for all $x \in \mathcal{X}$, there exists an isometry s_x of $\mathcal{X}$ fixing x and with $T_x s_x = -\operatorname{Id}_{T_x\mathcal{X}}$. The isometry s_x is called a *geodesic symmetry*,

For $\mathcal{X}$ a symmetric space, let $\operatorname{Isom}(\mathcal{X})$ be its group of isometries equipped with the compact open topology and $\mathsf{G} = \operatorname{Isom}_\circ(\mathcal{X})$ the connected component containing the identity.

The group G admits a unique Lie group structure compatible with its topology.

A simply connected symmetric space $\mathcal{X}$ *has a Euclidean factor* if there exists $k \in \mathbb{Z}_{>0}$ and $\mathcal{X}'$ a symmetric space such that $\mathcal{X}$ is isometric to the product $\mathbb{R}^k \times \mathcal{X}'$.

Proposition 6.1.2. *Let $\mathcal{X}$ be a simply connected symmetric space and $\mathsf{G} = \operatorname{Isom}_\circ(\mathcal{X})$. Then G is semisimple if and only if $\mathcal{X}$ has no Euclidean factor.*

If $\mathcal{X}$ has no Euclidean factor then we say that

1. $\mathcal{X}$ is *of compact type* if the sectional curvature of $\mathcal{X}$ is non-negative.
2. $\mathcal{X}$ is *of non-compact type* if the sectional curvature of $\mathcal{X}$ is non-positive.

Proposition 6.1.3. *Every simply connected symmetric space is isometric to a unique product of a Euclidean space, a symmetric space of non-compact type and a symmetric space of compact type.*

A simply connected symmetric space $\mathcal{X}$ is said *irreducible* if $\mathcal{X}$ is not an Euclidean space and whenever $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ then $\mathcal{X} = \mathcal{X}_1$ or $\mathcal{X} = \mathcal{X}_2$. Irreducible symmetric spaces are either of compact type or of non-compact type.

Every simply connected symmetric space decomposes into a product of irreducible symmetric spaces:

Proposition 6.1.4 (De Rham decomposition)**.** *Let $\mathcal{X}$ be a simply connected symmetric space. Then there exists a unique $(k, l) \in (\mathbb{Z}_{\geqslant 0})^2$ and non isometric irreducible symmetric spaces $\mathcal{X}_1, \ldots, \mathcal{X}_l$ unique up to isometries and permutations such that*

$$\mathcal{X} \simeq \mathbb{R}^k \times \mathcal{X}_1 \times \cdots \times \mathcal{X}_l \,.$$

Moreover, we have a Lie group isomorphism

$$\operatorname{Isom}_\circ(\mathcal{X}) \simeq \operatorname{Isom}_\circ(\mathbb{R}^k) \times \operatorname{Isom}_\circ(\mathcal{X}_1) \times \cdots \times \operatorname{Isom}_\circ(\mathcal{X}_l) \,.$$

Proposition 6.1.5. *Let $\mathcal{X}$ be a non-compact symmetric space and $\mathsf{G} = \operatorname{Isom}_\circ(\mathcal{X})$. Then the group G is semisimple with trivial center. Moreover, G is simple if and only if $\mathcal{X}$ is irreducible.*

A *flat of rank k* in a non-compact type symmetric space $\mathcal{X}$ is an isometric embedding $\mathbb{R}^k \to \mathcal{X}$. A flat is *maximal* if maximal for the inclusion. Every flat is contained in a maximal flat and all maximal flats are conjugated under the action of G. The *rank* of $\mathcal{X}$ is the dimension of a maximal flat. Since geodesics are flats of rank 1, the rank of $\mathcal{X}$ is at least 1.

A *maximal compact subgroup* of a Lie group is a compact subgroup which is maximal for the inclusion. In a Lie group, they are all conjugated. Let $\mathcal{X}$ be a symmetric space of the non-compact type, $\mathsf{G} = \mathrm{Isom}_\circ(G)$, $x \in \mathcal{X}$ and $\mathsf{K} = \mathsf{G}_x = \{g \in \mathsf{G},\, g(x) = x\}$.

Proposition 6.1.6. *The subgroup $\mathsf{K} < \mathsf{G}$ is a maximal compact subgroup and as G-spaces we have*

$$\mathsf{G}/\mathsf{K} \simeq \mathcal{X}$$

induced by $g \in \mathsf{G} \mapsto g \cdot x \in \mathcal{X}$.

If G is a connected semisimple Lie group with no compact factor and finite center and K a maximal compact subgroup one can equip the homogeneous space G/K with a symmetric space structure. The resulting symmetric space is called the *symmetric space of* G and the action of G on G/K is isometric and yields a surjective homomorphism $\mathsf{G} \to \mathrm{Isom}_\circ(\mathsf{G}/\mathsf{K})$ which has finite kernel.

Lie group and cohomology

The various cohomology theories we introduced in Section 6.1.2 behave nicely for connected semisimple Lie groups with finite center and no compact factor. We summarize the main facts we will need. Let G be a connected semisimple Lie group with finite center and no compact factor, $\mathcal{X}$ its symmetric space and K a maximal compact subgroup of G.

The following proposition follows closely Proposition 7.7 of [BIW10], we summarize the proof.

Proposition 6.1.7. *The following diagram is commutative, the horizontal lines are exact sequences and the vertical lines are comparison maps and are isomorphisms*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathrm{H}^2_{\mathrm{Bb}}(\mathsf{G},\mathbb{Z}) & \xrightarrow{i_{\mathrm{b}}} & \mathrm{H}^2_{\mathrm{bc}}(\mathsf{G},\mathbb{R}) & \longrightarrow & \mathrm{H}^2_{\mathrm{bc}}(\mathsf{G},\mathbb{R}/\mathbb{Z}) \\
 & & \downarrow & & \downarrow & & \downarrow{=} \\
0 & \longrightarrow & \mathrm{H}^2_{\mathrm{B}}(\mathsf{G},\mathbb{Z}) & \xrightarrow{i} & \mathrm{H}^2_{\mathrm{c}}(\mathsf{G},\mathbb{R}) & \longrightarrow & \mathrm{H}^2_{\mathrm{c}}(\mathsf{G},\mathbb{R}/\mathbb{Z})\,.
\end{array}$$

Moreover, the images of i and i_{b} are lattices.

Proof. Because G is semisimple, every homomorphism $\mathsf{G} \to \mathbb{R}/\mathbb{Z}$ is trivial and thus $\mathrm{Hom}(\mathsf{G}, \mathbb{R}/\mathbb{Z}) = 0$. The horizontal lines are then the exact sequence of coefficient (6.1). The fact that the comparison map $\mathrm{H}^2_{\mathrm{bc}}(\mathsf{G}, \mathbb{R}) \to \mathrm{H}^2_{\mathrm{c}}(\mathsf{G}, \mathbb{R})$ is an isomorphism is proven in [BM02]. The fact that the comparison map $\mathrm{H}^2_{\mathrm{Bb}}(\mathsf{G}, \mathbb{Z}) \to \mathrm{H}^2_{\mathrm{B}}(\mathsf{G}, \mathbb{Z})$ is also an isomorphism follows from the commutativity of the diagram and the fact that $\mathrm{H}^2_{\mathrm{c}}(\mathsf{G}, \mathbb{R}/\mathbb{Z}) = \mathrm{H}^2_{\mathrm{bc}}(\mathsf{G}, \mathbb{R}/\mathbb{Z})$. Finally, the image of i_{b} is a lattice because the image of i is.

□

Lemma 6.1.5. *Any G-invariant k-form on $\mathcal{X}$ is closed.*

Proof. This proof comes from [BILW05] Lemma 2.1. Because the action of G on $\mathcal{X}$ is through $\mathrm{Isom}_\circ(\mathcal{X})$ we might as well assume $\mathsf{G} = \mathrm{Isom}_\circ(\mathcal{X})$. Let $\omega \in \Omega^k(\mathcal{X})^{\mathsf{G}}$ be a G-invariant k-form on $\mathcal{X}$ and $s \in \mathrm{Isom}(\mathcal{X})$ be the geodesic symmetry at some point $x \in \mathcal{X}$. For $g \in \mathsf{G}$, because G is normal in $\mathrm{Isom}(\mathcal{X})$, we have $sgs \in \mathsf{G}$ and thus by G invariance of ω we have $sgs\omega = \omega$, that is

$$g(s\omega) = s\omega$$

thus $s\omega$ is also G invariant. Moreover $T_x s = -\mathrm{Id}$ and thus $(s\omega)_x = (-1)^k \omega_x$. By G invariance the equality holds for all x, that is $(-1)^k \omega = s\omega$. Applying it to both ω and $d\omega$ we get

$$d\omega = (-1)^k d(s\omega) = (-1)^k s d\omega = -(-1)^{k+1} s d\omega = -d\omega\,. \qquad \square$$

If $\omega \in \Omega^2(\mathcal{X})^{\mathsf{G}}$ is a G-invariant 2-form on $\mathcal{X}$ and $x \in \mathcal{X}$ then the function

$$c_\omega(g_0, g_1, g_2) = \frac{1}{2\pi} \int_{\Delta(g_0 x, g_1 x, g_2 x)} \omega\,, \tag{6.2}$$

where $\Delta(g_0 x, g_1 x, g_2 x)$ is the geodesic triangle with vertex $g_0 x$, $g_1 x$ and $g_2 x$, is a homogeneous G-invariant 2-cocycle which is moreover bounded and whose bounded cohomology class does not depend on x and the map

$$\begin{array}{ccc} \Omega^2(\mathcal{X})^{\mathsf{G}} & \longrightarrow & \mathrm{H}^2_{\mathrm{c}}(\mathsf{G}, \mathbb{R}) \\ \omega & \longmapsto & [c_\omega] \end{array}$$

is an isomorphism called *Van Est isomorphism*, see [vE53].

We recall the following fact

Lemma 6.1.6. *If G is a simple connected Lie group with finite center, then $\mathrm{H}^2_{\mathrm{c}}(\mathsf{G}, \mathbb{R})$ has dimension 0 or 1.*

Whether the dimension of $\mathrm{H}^2_{\mathrm{c}}(\mathsf{G}, \mathbb{R})$ is 1 or 0 depends on whether $\mathcal{X}$ is Hermitian or not. Hermitian symmetric spaces are particular cases of Kähler manifolds.

Complex and Kähler structures

Definition 6.1.4.
1. A *complex manifold* is a pair $(M, \mathcal{A})$ consisting of a smooth manifold M together with an atlas of charts $\mathcal{A}$ such that each chart (U, ϕ) consists of an open set $U \subset M$ and a diffeomorphism $\phi : U \to \phi(U) \subset \mathbb{C}^n$ and where every transition map is a biholomorphism.

2. An *almost complex structure* J on a manifold M is a tensor $J : TM \to TM$ such that $J^2 = -\operatorname{Id}_{\mathrm{TM}}$.

On $\mathbb{C}^n$ there is a natural almost complex structure given by the multiplication by i in the tangent space. It is preserved by biholomorphisms and thus every complex manifold has a natural almost complex structure. An almost complex structure induced by a complex manifold structure is said *integrable*. Conversely, we have the following theorem, for which we refer to [NN57].

Theorem 13 (Newlander, Nirenberg)
An almost complex structure J on M is integrable if and only if for all X, Y vector fields on M we have

$$[JX, JY] = J[JX, Y] + J[X, JY] + [X, Y]\,.$$

Definition 6.1.5. Let (M, J) be an almost complex manifold. A riemannian structure g on M is said *Hermitian* if for all X, Y vector fields on M we have

$$g(JX, JY) = g(X, Y)\,.$$

If (M, J, g) is a complex Hermitian manifold, that is if J is an integrable almost complex structure, then we define the following tensors:

- a hermitian metric
$$h(X, Y) = g(X, Y) + ig(X, JY)\,;$$
- a 2-form
$$\omega(X, Y) = g(X, JY)\,.$$

Definition 6.1.6. A complex Hermitian manifold is *Kähler* if the associated 2-form is closed.

Hermitian symmetric spaces

Definition 6.1.7. Let $\mathcal{X}$ be a connected complex manifold with a Hermitian structure. We say that $\mathcal{X}$ is a *Hermitian symmetric space* if $\mathcal{X}$ is simply connected and if for every

$x \in \mathcal{X}$ there exists s_x a biholomorphic isometry fixing x and with $T_x s_x = -\operatorname{Id}_{T_x\mathcal{X}}$.

Forgetting the complex structure, we see that every Hermitian symmetric space is a symmetric space of even dimension.

The complex structure together with the Riemannian metric of a Hermitian symmetric space yields an invariant 2-form with is closed by invariance, thus

Proposition 6.1.8. *Every Hermitian symmetric space is Kähler.*

Definition 6.1.8. A Lie group G is said to be of *Hermitian type* if G is a semisimple Lie group with finite center and whose symmetric space is Hermitian of non-compact type.

Proposition 6.1.9. *Let G be a simple non-compact Lie group and $\mathcal{X}$ its (irreducible) symmetric space. Are equivalent*

1. *G is of Hermitian type;*
2. *$\mathrm{H}^2_c(\mathsf{G},\mathbb{R})$ has dimension 1;*
3. *$\mathrm{H}^2_{\mathrm{B}}(\mathsf{G},\mathbb{Z})$ has rank 1;*
4. *$\pi_1(G)$ has a $\mathbb{Z}$ factor;*
5. *the center of a maximal compact subgroup of G is not discrete.*

Proof. That 1 and 5 are equivalent is proved in [Hel01], Chap.VIII Theorem 6.1.

Assertions 2 and 3 are equivalent since $\mathrm{H}^2_{\mathrm{B}}(\mathsf{G},\mathbb{Z})$ is a lattice in $\mathrm{H}^2_{\mathrm{c}}(\mathsf{G},\mathbb{R})$ by Proposition 6.1.7.

We show 3 and 4 to be equivalent. Suppose $\pi_1(\mathsf{G}) = A \times \mathbb{Z}$ and let $\mathsf{G}_1 = \tilde{\mathsf{G}}/A$ where $\tilde{G}$ is the universal cover of G. The following sequence is exact

$$0 \to \mathbb{Z} \to \mathsf{G}_1 \to \mathsf{G} \to 1$$

and G_1 is connected, thus $\mathrm{H}^2_{\mathrm{B}}(\mathsf{G},\mathbb{Z})$ is non trivial. Conversely, if $\kappa \in \mathrm{H}^2_{\mathrm{B}}(\mathsf{G},\mathbb{Z})$ is non trivial, then seeing κ as a bounded integral class, let $f = \mathrm{Rot}_\kappa|_{\mathsf{K}}$ which is a homomorphism $\mathsf{K} \to \mathbb{R}/\mathbb{Z}$ because K is compact and thus $\mathrm{H}^2_{\mathrm{bc}}(\mathsf{K},\mathbb{R}) = 0$. The homomorphism f is non trivial because $\kappa \neq 0$ and

$$\begin{array}{ccc} \mathrm{H}^2_{\mathrm{Bb}}(\mathsf{G},\mathbb{Z}) & \to & \mathrm{Hom}_{\mathrm{c}}(\mathsf{K},\mathbb{R}/\mathbb{Z}) \\ \kappa & \mapsto & \mathrm{Rot}_\kappa|_{\mathsf{K}} \end{array}$$

is an isomorphism, see [BIW10] Proposition 7.7(2). Thus we get a surjective homomorphism $\pi_1(\mathsf{G}) = \pi_1(\mathsf{K}) \to \mathbb{Z}$.

If G is Hermitian, its symmetric space has a symplectic form ω which is moreover G invariant and thus $\Omega^2(\mathcal{X})^{\mathsf{G}}$ is non trivial, so by Van Est isomorphism $\mathrm{H}^2_{\mathrm{c}}(\mathsf{G},\mathbb{R})$ has positive dimension, hence dimension 1, thus 1 implies 2. Conversely, let ω be a non trivial G-invariant 2-form on a symmetric space $\mathcal{X}$. Notice that ω has to be nondegenerate. Indeed, ω is K-invariant and the action of K on $T_x\mathcal{X}$ is irreducible because $\mathcal{X}$ is irreducible. The kernel of ω is a linear subspace of $T_x\mathcal{X}$ which is stable under K, so it must be trivial since ω_x is non trivial. Thus 2 implies 1, since ω is then a G-invariant symplectic form and together with the Riemannian metric we get an integrable almost complex structure and thus a Kähler structure. □

Maximal polydisk

Let G be a Hermitian type Lie group and $\mathcal{X}$ its symmetric space. Let n be the rank of $\mathcal{X}$.

- A *polydisk* in $\mathcal{X}$ is a totally geodesic and holomorphic enbedding $t : \mathbb{D}^k \to \mathcal{X}$.
- A *maximal polydisk* in $\mathcal{X}$ is a polydisk whose image is maximal for the inclusion. All maximal polydisks are conjugated and of (complex) dimension n.
- A *diagonal disk* in $\mathcal{X}$ is the image of the diagonal $\mathbb{D} \subset \mathbb{D}^n$ under a maximal polydisk $\mathbb{D}^n \to \mathcal{X}$.

If $d : \mathbb{D} \to \mathcal{X}$ is a diagonal disk associated to a maximal polydisk $t : \mathbb{D}^n \to \mathcal{X}$, then for some $r > 0$ there exists homomorphisms

$$\begin{array}{ccc} \mathsf{Spin}^r & \longrightarrow & \mathsf{Spin}^r \times \cdots \times \mathsf{Spin}^r \\ & \searrow^{\delta} & \downarrow{\scriptstyle \tau} \\ & & \mathsf{G}\,. \end{array}$$

such that t is τ-equivariant and d is δ-equivariant and

$$\tau^*(\kappa^b_{\mathsf{G},\mathbb{R}}) = \kappa^b_{(\mathsf{Spin}^r)^n,\mathbb{R}} \qquad \text{and} \qquad \delta^*\kappa^b_{\mathsf{G},\mathbb{R}} = n\kappa^b_{\mathsf{Spin}^r,\mathbb{R}}\,.$$

Hermitian symmetric space of tube type

Definition 6.1.9. A Hermitian symmetric space is of *tube type* if it is biholomorphic to a domain $V \oplus i\Omega$ where V is a real vector space and $\Omega \subsetneq V$ is a convex open non-empty cone.

A Hermitian Lie group is of *tube type* if its symmetric space is. Hermitian Lie groups of tube type include $\mathsf{Sp}_{2n}(\mathbb{R})$, $\mathsf{SU}(n,n)$ and $\mathsf{SO}(2,n)$.

The following fact can be found in [BIW10] remark 7.11, as a consequence of [CK07].

Proposition 6.1.10. *If* G *is a Hermitian Lie group of tube type, then the real Kähler class and the real bounded Kähler class are respectively the image of an integral class* $\kappa_{\mathsf{G},\mathbb{Z}} \in \mathrm{H}^2_{\mathrm{B}}(\mathsf{G},\mathbb{Z})$ *called the* integral Kähler class *of* G *and of an integral bounded class* $\kappa^b_{\mathsf{G},\mathbb{Z}} \in \mathrm{H}^2_{\mathrm{Bb}}(\mathsf{G},\mathbb{Z})$ *called the* bounded integral Kähler class *of* G.

Example 6.1.2. The Lie group $\mathsf{PSL}_2(\mathbb{R})$ is Hermitian of tube type, indeed its symmetric space is biholomorphic to the upper-half plane which is $\mathbb{R} \oplus i\mathbb{R}_{>0}$. Let $\kappa = \kappa^b_{\mathsf{PSL}_2(\mathbb{R}),\mathbb{Z}}$ be the bounded integral class of $\mathsf{PSL}_2(\mathbb{R})$, $e^b \in \mathrm{H}^2_{\mathrm{Bb}}(\mathrm{Homeo}^+(\mathbb{R}/\mathbb{Z}),\mathbb{Z})$ be the bounded Euler class and $\iota : \mathsf{PSL}_2(\mathbb{R}) \to \mathrm{Homeo}^+(\mathbb{R}/\mathbb{Z})$ the section such that $0 \leqslant \mathrm{Trans}_{e^b}(\iota(g)) < 1$. Then $\mathrm{Rot}_\kappa = \mathrm{Rot}_{\iota^* e^b}$ and thus $\kappa|_{\mathsf{K}} = \iota^* e^b|_{\mathsf{K}}$ where K is a maximal compact subgroup of $\mathsf{PSL}_2(\mathbb{R})$. In unbounded Borel cohomology, the restriction

$$\mathrm{H}_{\mathrm{B}}(\mathsf{PSL}_2(\mathbb{R}),\mathbb{Z}) \to \mathrm{H}_{\mathrm{B}}(\mathsf{K},\mathbb{Z})$$

is an isomorphism. Thus, using Proposition 6.1.7 we conclude that $\iota^* e^b = \kappa$.

6.1.4 The bounded fundamental class of a surface

For surface groups and surfaces, bounded cohomology behave very differently than the unbounded version. Let $S = S_{g,s}$ with $\chi(S) < 0$.

Bounded fundamental class

If $s > 0$, the fundamental group of S is free on $2g + s - 1$ generators and the cohomology groups $\mathrm{H}^n(\pi_1(S), R)$ vanish for $n > 1$ and $R = \mathbb{R}, \mathbb{Z}, \mathbb{R}/\mathbb{Z}$, and if $s = 0$, we have $\mathrm{H}^2(\pi_1(S), R) \simeq R$. However,

Proposition 6.1.11. *Let* $S = S_{g,s}$ *with* $\chi(S) < 0$. *Then* $\mathrm{H}^2_{\mathrm{b}}(\pi_1(S),\mathbb{Z})$ *and* $\mathrm{H}^2_{\mathrm{b}}(\pi_1(S),\mathbb{R})$ *are infinite dimensional.*

If $\kappa = \kappa^b_{\mathsf{PSL}_2(\mathbb{R}),\mathbb{Z}}$ is the integral bounded Kähler class of $\mathsf{PSL}_2(\mathbb{R})$ and if $h_1, h_2 : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$ are hypebolizations of S, it is shown in [BIW10] that the actions of h_1 and h_2 on the circle are semiconjugated in the sense of E. Ghys [Ghy87] and as a consequence in $\mathrm{H}^2_{\mathrm{b}}(\pi_1(S),\mathbb{Z})$ we have

$$h_1^*(\kappa) = h_2^*(\kappa)\,.$$

The class

$$\kappa^b_{S,\mathbb{Z}} \in \mathrm{H}^2_{\mathrm{b}}(\pi_1(S), \mathbb{Z})$$

so obtained is called the *integral bounded fundamental class* of S, and its image $\kappa^b_{S,\mathbb{R}} \in \mathrm{H}^2_{\mathrm{b}}(\pi_1(S), \mathbb{R})$ by the change of coefficient map is called the *real bounded fundamental class* of S.

Note that the function $\mathrm{Rot}_{\kappa^b_{S,\mathbb{Z}}} : \pi_1(S) \to \mathbb{R}/\mathbb{Z}$ is trivial. Indeed, if $h : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$ is a hyperbolization of S, then for all $\gamma \in \pi_1(S)$,

$$\mathrm{Rot}_{\kappa^b_{S,\mathbb{Z}}}(\gamma) = \mathrm{Rot}_{\kappa^b_{\mathsf{PSL}_2(\mathbb{R}),\mathbb{Z}}}(h(\gamma)) = 0 \tag{6.3}$$

because $h(\gamma)$ fixes a point in $\mathbb{R}/\mathbb{Z}$.

Definition of T^nS° and s_n

Recall that TS° is the tangent bundle of S with the zero section removed. Let n be a positive integer. We construct a real punctured plane bundle T^nS° together with a n-cover $p^n : TS^\circ \to T^nS^\circ$.

To this end, let Z be a Riemann surface structure on S. The tangent bundle TZ of Z is a complex line bundle and forgetting the complex structure we recover the tangent bundle TS of S. We consider the line bundle $TZ^{\otimes n}$ where the tensor product is over $\mathbb{C}$. The natural bundle map

$$\begin{array}{rccc} p^n : & TZ & \longrightarrow & TZ^{\otimes n} \\ & v & \longmapsto & v^{\otimes n} \end{array}$$

is a n-cover with singularity at the zero section and for a small open set $U \in Z$ we have the following commutative diagram

$$\begin{array}{ccc} TZ|_U & \xrightarrow{\sim} & U \times \mathbb{C} \\ \downarrow{\scriptstyle p^n} & & \downarrow{\scriptstyle (\mathrm{Id}, z \mapsto z^n)} \\ T^nZ|_U & \xrightarrow{\sim} & U \times \mathbb{C}\,. \end{array}$$

Let T^nS be the underlying real structure of $TZ^{\otimes n}$. The singular n-cover $p^n : TS \to T^nS$ yields a regular n-cover

$$p^n : TS^\circ \to T^nS^\circ$$

after restriction to TS°.

In Lemma 3.2.1 we defined $s : \pi_1(S) \to \pi_1(TS^\circ)$ section of $\pi_1(TS^\circ) \to \pi_1(S)$. If S is equipped with any hyperbolic structure, recall that $s(g) \in \pi_1(TS^\circ)$ is the only antecedent

of g that can be represented by a loop freely homotopic to the tangential lift $\overline{\gamma}$ where γ is the only geodesic freely homotopic to a curve representing g.

Let s_n be the composition

$$s_n : \pi_1(S) \xrightarrow{s} \pi_1(TS^\circ) \xrightarrow{p_*^n} \pi_1(T^nS^\circ)\,.$$

We use the section s_n to get a particular presentation for $\pi_1(T^nS^\circ)$ from a standard collection of curves on S.

Presentation of $\pi_1(T^nS^\circ)$

Let $(\alpha_i, \beta_i, \gamma_j)$ be a standard collection of curves on S and $a_i, b_i, c_j \in \pi_1(S)$ the relative homotopy classes of loops freely homotopic to $\alpha_i, \beta_i, \gamma_j$ respectively, such that we have the usual presentation of $\pi_1(S)$.

Lemma 6.1.7. *The group $\pi_1(T^nS^\circ)$ has the following presentation*

$$\begin{aligned}\pi_1(T^nS^\circ) =&\big\langle s_n(a_1), s_n(b_1), \dots, s_n(a_g), s_n(b_g), s_n(c_1), \dots, s_j(c_s), t \,\big|\\ &t \text{ is central and } \prod_{i=1}^{g}[s_n(a_i), s_n(b_i)] \prod_{j=1}^{s} s_n(c_j) = t^{-n\chi(S)}\big\rangle\,,\end{aligned}$$

where $t = t_{T^nS}$ is the loop around the origin.

Proof. We proved the case $n = 1$ in Lemma 5.1.2. The general case is a consequence of it as the relation

$$\prod_{i=1}^{g}[s_n(a_i), s_n(b_i)] \prod_{j=1}^{s} s_n(c_j) = t^{-n\chi(S)}$$

is the image by p_*^n of the corresponding relation in $\pi_1(TS^\circ)$. □

Some properties of s_n

Recall from Section 6.1.2 that if

$$0 \to \mathbb{Z} \xrightarrow{i} G \xrightarrow{p} H \to 1$$

is a central extension, if $\sigma : H \to G$ is any section of p, then $d\sigma : H \times H \to \mathbb{Z}$ defined by

$$\sigma(x)\sigma(y) = \sigma(xy)i(dp(x, y))$$

is an inhomogeneous 2-cocycle.

If $s > 0$, the cocycle ds_n is trivial in $\mathrm{H}^2_{\mathrm{B}}(\pi_1(S), \mathbb{Z})$ since this cohomology group is trivial. However, ds_n is actually bounded and not trivial in $\mathrm{H}^2_{\mathrm{Bb}}(\pi_1(S), \mathbb{Z})$ and

Lemma 6.1.8. *The 2-cocycle ds_n is bounded and represents $n\kappa^b_{S,\mathbb{Z}}$ in $\mathrm{H}^2_{\mathrm{Bb}}(\pi_1(S), \mathbb{Z})$.*

Proof. It is enough to prove it for $n = 1$ since $ds_n = nds$ where $s = s_1$. Let $\sigma : \mathsf{PSL}_2(\mathbb{R}) \to \widetilde{\mathsf{PSL}}_2(\mathbb{R})$ be the only section with $0 \leqslant \mathrm{Trans}_\kappa(\sigma(x)) < 1$ for all $x \in \mathsf{PSL}_2(\mathbb{R})$ and where $\kappa = \kappa^b_{\mathsf{PSL}_2(\mathbb{R}),\mathbb{Z}}$ is the bounded integral Kähler class of $\mathsf{PSL}_2(\mathbb{R})$. Because σ is a section of $\widetilde{\mathsf{PSL}}_2(\mathbb{R}) \to \mathsf{PSL}_2(\mathbb{R})$, the cocycle $d\sigma$ represents the integral Kähler class of $\mathsf{PSL}_2(\mathbb{R})$. Moreover, a little computation yields

$$d\sigma(x, y) = \mathrm{Trans}_\kappa(\sigma(x)\sigma(y)) - \mathrm{Trans}_\kappa(\sigma(xy)) .$$

Since Trans_κ is a quasimorphism and $\mathrm{Trans}_\kappa(\sigma(z))$ is bounded uniformly in z, $d\sigma$ is bounded and thus represents κ the integral bounded fundamental class of $\mathsf{PSL}_2(\mathbb{R})$.

We now show $h^*d\sigma = ds$ for $h : \pi_1(S) \to \mathsf{PSL}_2(\mathbb{R})$ a hyperbolization of S, which will conclude, since $\kappa^b_{S,\mathbb{Z}} = h^*\kappa = h^*[d\sigma]$. Let $\tilde{h} : \pi_1(TS^\circ) \to \widetilde{\mathsf{PSL}}_2(\mathbb{R})$ be the lift of h given by Lemma 3.2.2. In particular, if γ is a closed geodesic whose relative homotopy class is $x \in \pi_1(S)$, then the relative homotopy class of the tangential lift $\overline{\gamma}$ of γ is $s(x)$ (see Lemma 3.2.1). In particular we get

$$\tilde{h}(s(x)) = \sigma(h(x)) .$$

Applying $\tilde{h}$ to the equality $s(x)s(y) = s(xy)t_{TS}^{ds(x,y)}$ one gets

$$ds(x, y) = d\sigma(h(x), h(y))$$

for all $x, y \in \pi_1(S)$, thus proving $[ds] = [h^*d\sigma] = h^*\kappa = \kappa^b_{S,\mathbb{Z}}$. □

The section s_n enjoys some more properties. Let $X_0 \in T^nS^\circ$ be the base point of $\pi_1(T^nS^\circ)$ and x_0 its image in S, so that $\pi_1(S) = \pi_1(S, x_0)$ and $\pi_1(T^nS^\circ) = \pi_1(T^nS^\circ, X_0)$. If f is a diffeomorphism of S, Tf acts on TS and on T^nS for all n. We write $T^nf : T^nS \to T^nS$ and the class of T^nf up to inner automorphism depends only on the isotopy class of f.

Lemma 6.1.9. *1. Let $f \in \mathrm{Diff}^+(S)$. Suppose moreover that $T^nf(X_0) = X_0$ so that we have automorphisms $f_* : \pi_1(S, x_0) \to \pi_1(S, x_0)$ and $T^nf_* : \pi_1(T^nS^\circ, X_0) \to \pi_1(T^nS^\circ, X_0)$. Then for all $x \in \pi_1(S)$,*

$$s_n(f_*(x)) = (T^nf)_*(s_n(x)) .$$

2. Let $k \in \mathbb{Z}$ and $x \in \pi_1(S)$. Then

$$s_n(x^k) = s_n(x)^k \,.$$

Proof. Since s_n is the composition of the section

$$s = s_1 : \pi_1(S) \to \pi_1(TS^\circ)$$

and the homomorphism

$$p_*^n : \pi_1(TS^\circ) \to \pi_1(T^nS^\circ),\ c \mapsto c^n$$

and since for all $v \in TS^\circ$,

$$(Tf(v))^n = T^n f(v)\,,$$

it is enough to prove the lemma for $n = 1$.

Let first show $s(x^k) = (s(x))^k$. Suppose S is equipped with any finite volume hyperbolic metric and suppose first that x is a non peripheral element. Let γ be the only geodesic associated to x. By definition, $s(x)$ is the only element of $\pi_1(TS^\circ)$ both antecedent of x and freely homotopic to the tangential lift of γ, see Lemma 3.2.1. Since first taking the tangential lift $\overline{\gamma}$ of γ and then travelling k times along $\overline{\gamma}$ is the same as taking the tangential lift of the geodesic γ travelled k times, we get $s(x^k) = s(x)^k$. If x is peripheral, we do the same thing with a horocycle instead of the geodesic. Since $\tilde{h}$ is a homomorphism, we get $\tilde{h}(s_n(x^k)) = (\tilde{h}(s_n(x)))^k$.

Then we show the equivariance under f. Let $x \in \pi_1(S)$ and γ be the only geodesic on S (or a horocycle if x is peripheral) freely homotopic to x. By definition of s, on the one hand $s(f_*(x)) \in \pi_1(TS^\circ)$ is the only antecedent of $f_*(x) \in \pi_1(S)$ that is freely homotopic in TS° to $\overline{f(\gamma)}$, the tangential lift of $f(\gamma)$. One the other hand, $Tf_*(s(x)) \in \pi_1(TS^\circ)$ is the only antecedent of $f_*(x) \in \pi_1(S)$ that is freely homotopic to $Tf_*(\overline{\gamma})$. But the chain rule implies

$$\overline{f_*(\gamma)} = Tf_*(\overline{\gamma})$$

thus proving the lemma. □

6.1.5 Integral maximal homomorphisms

Let G be a simple Hermitian Lie group of tube type, $\mathcal{X}$ its irreducible symmetric space of rank $n = n_{\mathcal{X}}$, $\kappa^b_{\mathsf{G},\mathbb{Z}}$ its bounded integral Kähler class and $\kappa^b_{\mathsf{G},\mathbb{R}}$ its bounded real Kähler class.

Let $S = S_{g,s}$ be a genus g compact oriented connected surface with s punctures, $\kappa^b_{S,\mathbb{Z}} \in$

$\mathrm{H}^2_{\mathrm{b}}(\pi_1(S),\mathbb{Z})$ its bounded integral fundamental class and $\kappa^b_{S,\mathbb{R}} \in \mathrm{H}^2_{\mathrm{bc}}(\pi_1(S),\mathbb{R})$ its bounded real fundamental class.

The definition of a maximal homomorphism we adopt differs but is equivalent to the one used in [BIW10], see their Corollary 8.6.

Definition 6.1.10. 1. A homomorphism $\rho : \pi_1(S) \to \mathsf{G}$ is *maximal* if

$$\rho^* \kappa^b_{\mathsf{G},\mathbb{R}} = n\kappa^b_{S,\mathbb{R}}\,.$$

2. A homomorphism $\rho : \pi_1(S) \to \mathsf{G}$ is *integral maximal* if

$$\rho^* \kappa^b_{\mathsf{G},\mathbb{Z}} = n\kappa^b_{S,\mathbb{Z}}\,.$$

Proposition 6.1.12. *Let G be a simple Hermitian Lie group of Hermitian type and $\kappa = \kappa^b_{\mathsf{G},\mathbb{Z}}$ its bounded integral Kähler class. Let $\rho : \pi_1(S) \to \mathsf{G}$ be a homomorphism. Then the following statements are equivalent*

1. *ρ is integral maximal;*
2. *ρ is maximal and $\mathrm{Rot}_\kappa \circ \rho \equiv 0$.*

Proof. First assume ρ is integral maximal. Then ρ is maximal and for $\gamma \in \pi_1(S)$,

$$\begin{aligned}\mathrm{Rot}_\kappa(\rho(\gamma)) &= \mathrm{Rot}_{\rho^*\kappa}(\gamma)\\ &= \mathrm{Rot}_{n\kappa^b_{S,\mathbb{Z}}}(\gamma)\\ &= n\,\mathrm{Rot}_{\kappa^b_{S,\mathbb{Z}}}(\gamma) = 0\,,\end{aligned}$$

using equation 6.3 for the last equality. Now assume ρ to be maximal and $\mathrm{Rot}_\kappa \circ \rho \equiv 0$. Let $\beta = \rho^*\kappa^b_{\mathsf{G},\mathbb{Z}} - n\kappa^b_{S,\mathbb{Z}}$. Because ρ is maximal, taking real coefficient we get $\beta_{\mathbb{R}} = 0$ and using the exact sequence of coefficient (6.1) we find $f \in \mathrm{Hom}_{\mathrm{c}}(\pi_1(S),\mathbb{R}/\mathbb{Z})$ such that

$$\mathfrak{d}(f) = \beta = \rho^*\kappa^b_{\mathsf{G},\mathbb{Z}} - n\kappa^b_{S,\mathbb{Z}}\,.$$

Now if $\gamma \in \pi_1(S)$, we get

$$f(\gamma) = \mathrm{Rot}_{\rho^*\kappa^b_{\mathsf{G},\mathbb{Z}}}(\gamma) - \mathrm{Rot}_{n\kappa^b_{S,\mathbb{Z}}}(\gamma) = \mathrm{Rot}_{\kappa^b_{\mathsf{G},\mathbb{Z}}}(\rho(\gamma)) = 0\,.$$

So $f \equiv 0$ and $\beta = 0$, that is $\rho^*\kappa^b_{\mathsf{G},\mathbb{Z}} = n\kappa^b_{S,\mathbb{Z}}$. □

Let $\mathrm{Hom}_{\mathrm{Max}}(S,\mathsf{G})$ be the space of maximal homomorphisms and $\mathrm{Hom}_{\mathrm{IntMax}}(S,\mathsf{G})$ the space of integral maximal homomorphisms. We write S instead of $\pi_1(S)$ to remind us

of the dependence in $\kappa^b_{S,\mathbb{Z}}$ which cannot be defined with $\pi_1(S)$ alone as soon as S is not compact. Every maximal homomorphism is reductive (see Theorem 5 of [BIW10]) thus the action of G by conjugacy on $\mathrm{Hom}_{\mathrm{Max}}(S,\mathsf{G})$ is proper and we let

$$\mathrm{Rep}_{\mathrm{Max}}(S,\mathsf{G}) = \mathrm{Hom}_{\mathrm{Max}}(S,\mathsf{G})/\mathsf{G} \qquad \text{and} \qquad \mathrm{Rep}_{\mathrm{IntMax}}(S,\mathsf{G}) = \mathrm{Hom}_{\mathrm{IntMax}}(S,\mathsf{G})/\mathsf{G}$$

be the representation varieties.

If ρ is a homomorphism into $\mathsf{PSL}_2(\mathbb{R})$ or its cover Spin^r then ρ is maximal if and only if ρ is integral maximal. Indeed, if ρ is maximal and $\gamma \in \pi_1(S)$, then $\rho(\gamma)$ is either parabolic or hyperbolic and in both cases has rotation number 0. We don't know in general whether or not every maximal homomorphism in a tube type Hermitian Lie group is integral.

If $\mathsf{G} = \mathrm{Isom}_\circ(\mathcal{X})$ is a simple Hermitian Lie group of tube type then integral maximal homomorphisms exist: indeed if $\Delta : \mathsf{PSL}_2(\mathbb{R}) \to \mathsf{G}$ is a diagonal disk, then $\Delta^*\kappa^b_{\mathsf{G},\mathbb{Z}} = n\kappa^b_{\mathsf{PSL}_2(\mathbb{R}),\mathbb{Z}}$ and thus if h is a hyperbolization of S, then $\Delta \circ h$ is integral maximal.

Let $\rho : \pi_1(S) \to \mathsf{G}$ be a maximal homomorphism. In Lemma 10.8 of [BIW10], the authors show that there exists an integer e_{G} depending on G only such that

$$\forall \gamma \in \pi_1(S),\ \mathrm{Rot}_{\kappa^b_{\mathsf{G},\mathbb{Z}}}(\rho(\gamma)) \in e_{\mathsf{G}}^{-1}\mathbb{Z}/\mathbb{Z}\,. \tag{6.4}$$

As a consequence, we get

Proposition 6.1.13. *The spaces* $\mathrm{Hom}_{\mathrm{IntMax}}(S,\mathsf{G})$ *and* $\mathrm{Rep}_{\mathrm{IntMax}}(S,\mathsf{G})$ *are a union of connected components of, respectively,* $\mathrm{Hom}_{\mathrm{Max}}(S,\mathsf{G})$ *and* $\mathrm{Rep}_{\mathrm{Max}}(S,\mathsf{G})$.

Proof. Let κ be the integral bounded Kähler class of G, $h \in \mathrm{Hom}_{\mathrm{IntMax}}(S,\mathsf{G})$ any maximal integral homomorphism and $\rho \in \mathrm{Hom}_{\mathrm{Max}}(S,\mathsf{G})$ be any maximal homomorphism. The real class in $\mathrm{H}^2_{\mathrm{bc}}(\mathsf{G},\mathbb{R})$ corresponding to the integral class $\rho^*\kappa - h^*\kappa \in \mathrm{H}^2_{\mathrm{Bb}}(\mathsf{G},\mathbb{Z})$ vanishes and thus $\rho^*\kappa - h^*\kappa = \mathfrak{d}(R_\rho)$ for some $R_\rho \in \mathrm{Hom}_c(\pi_1(S),\mathbb{R}/\mathbb{Z})$, see the long exact sequence 6.1. For $\gamma \in \pi_1(S)$, from the definition of the rotation numbers one gets

$$R_\rho(\gamma) = \mathrm{Rot}_\kappa(\rho(\gamma)) - \mathrm{Rot}_\kappa(h(\gamma))\,.$$

The continuous homomorphism R_ρ takes value in $e_{\mathsf{G}}^{-1}\mathbb{Z}/\mathbb{Z}$ because of equation 6.4, and we obtain a continuous application

$$\mathcal{R} : \rho \in \mathrm{Hom}_{\mathrm{Max}}(S,\mathsf{G}) \mapsto R_\rho \in \mathrm{Hom}_c(\pi_1(S), e_{\mathsf{G}}^{-1}\mathbb{Z}/\mathbb{Z})$$

where $\mathrm{Hom}_c(\pi_1(S), e_{\mathsf{G}}^{-1}\mathbb{Z}/\mathbb{Z})$ is discrete and

$$\mathrm{Hom}_{\mathrm{IntMax}}(S,\mathsf{G}) = \mathcal{R}^{-1}(0)$$

is open and close as preimage of an open and close set. □

Integral maximal homomorphisms and lifts

Let

$$0 \to \mathbb{Z} \xrightarrow{i} \mathsf{G}_\kappa \xrightarrow{p} \mathsf{G} \to 1$$

be the $\mathbb{Z}$-central extension of G associated to $\kappa = \kappa^b_{\mathsf{G},\mathbb{Z}}$ (forgetting that it is bounded). Let $\mathrm{Trans}_\kappa : \mathsf{G}_\kappa \to \mathbb{R}$ be the translation number function.

For $r \in \mathbb{Z}_{>0}$, let

$$\mathsf{G}_r = \mathsf{G}_\kappa / i(r\mathbb{Z})$$

and $\mathrm{Trans}^{(r)}_\kappa : \mathsf{G}_r \to \mathbb{R}/r\mathbb{Z}$ be the projection of $\mathrm{Trans}_\kappa : \mathsf{G}_\kappa \to \mathbb{R}$. The following sequence is exact

$$0 \to \mathbb{Z}/r\mathbb{Z} \xrightarrow{i_r} \mathsf{G}_r \xrightarrow{p_r} \mathsf{G} \to 1\,. \tag{6.5}$$

The group G_r is a semisimple Lie group with finite center and Hermitian of tube type, however it needs not be connected.

Lemma 6.1.10. *Let n be the rank of the symmetric space of G. If n and r are coprime then G_r is connected.*

Moreover, the pull back homomorphisms

$$p_r^* : \mathrm{H}^2_{\mathrm{Bb}}(\mathsf{G}_r, \mathbb{Z}) \to \mathrm{H}^2_{\mathrm{Bb}}(\mathsf{G}, \mathbb{Z}) \quad \textit{and} \quad p_r^* : \mathrm{H}^2_{\mathrm{bc}}(\mathsf{G}_r, \mathbb{R}) \to \mathrm{H}^2_{\mathrm{bc}}(\mathsf{G}, \mathbb{R})$$

are isomorphisms compatible with the change of coefficient and comparison maps.

Proof. Let $\mathcal{B}_{>0}(\mathsf{G})$ be the set of Borel sets of G with positive Haar measure. For $B \in \mathcal{B}_{>0}(\mathsf{G})$ and $k \in \mathbb{Z}$ let

$$B_k = \{(g, k) : g \in B\}$$

so that $B_k \subset \mathsf{G}_\kappa$ is a measurable set with positive measure, see 6.1.2. Recall that the topology on G_κ is the for which the family

$$\left\{(B_k)^{-1} B_k\right\}_{B \in \mathcal{B}_{>0}(\mathsf{G}),\, k \in \mathbb{Z}}$$

is a basis of neighborhood of the identity.

First suppose $n = 1$, that is G is locally isomorphic to $\mathsf{PSL}_2(\mathbb{R})$. In this case, and using notation from section 6.1.2, G_κ is isomorphic to $\widetilde{\mathsf{PSL}}_2(\mathbb{R})$ through

$$\begin{array}{rccc} \varphi : & \widetilde{\mathsf{PSL}}_2(\mathbb{R}) & \to & \mathsf{G}_\kappa \\ & g & \mapsto & (q(g), \mathrm{E}\left(\mathrm{Trans}_\kappa\left(g\right)\right)) \end{array}$$

where $q : \widetilde{\mathsf{PSL}}_2(\mathbb{R}) \to \mathsf{G}$ is the universal cover and $\mathrm{E}(x) = \max\{k \in \mathbb{Z} : k \leqslant x\}$. Indeed, the cocycle

$$c : (g, g') \mapsto \mathrm{E}(\mathrm{Trans}_\kappa(gg')) - \mathrm{E}(\mathrm{Trans}_\kappa(g)) - \mathrm{E}(\mathrm{Trans}_\kappa(g'))$$

is equal to $d\sigma$, where $\sigma : H \to \widetilde{\mathsf{PSL}}_2(\mathbb{R})$ is the only section with $0 \leqslant \mathrm{Trans}_\kappa(\sigma(g)) < 1$ for all $g \in H$, so that c is a bounded Borel cocycle representing κ and thus φ is a homomorphism. Moreover φ is measurable and maps measurable sets with positive measure to measurable sets with positive measure, thus φ is continuous at the identity and thus continuous. In a similar fashion, one checks that

$$\begin{array}{ccc} G_\kappa & \to & \widetilde{\mathsf{PSL}}_2(\mathbb{R}) \\ (g,k) & \mapsto & \sigma(g)\tau^k \end{array}$$

is a continuous homomorphism inverse of φ, hence the claimed isomorphism.

Let now G be any Hermitian Lie group of tube type and n be the rank of its symmetric space. Let $\Delta : \mathsf{H} \to \mathsf{G}$ be a diagonal disc where $\mathsf{H} = \mathsf{Spin}^\ell$ is a finite cover of $\mathsf{PSL}_2(\mathbb{R})$. In particular, $\Delta^* \kappa_{\mathsf{G},\mathbb{Z}} = n\kappa_{\mathsf{H},\mathbb{Z}}$.

First we show that $\mathsf{H}_{n\kappa_{\mathsf{H},\mathbb{Z}}}$ has n connected components. The following homomorphism

$$\begin{array}{rcl} \widetilde{\mathsf{PSL}}_2(\mathbb{R}) \simeq \mathsf{H}_{\kappa_{\mathsf{G},\mathbb{Z}}} & \longrightarrow & \mathsf{H}_{n\kappa_{\mathsf{G},\mathbb{Z}}} \\ (h,k) & \longmapsto & (h,nk) \end{array}$$

is measurable and thus continuous and its image image $\mathsf{H}_{(0)} = \{(h, nk) : h \in \mathsf{H},\, k \in \mathbb{Z}\}$ is connected. Moreover, as a subgroup of $\mathsf{H}_{n\kappa_{\mathsf{H},\mathbb{Z}}}$, $\mathsf{H}_{(0)}$ is locally closed, hence closed. It is also open since it is the complement of the union

$$(\mathrm{Id}_\mathsf{H}, 1) \cdot \mathsf{H}_{(0)} \cup \ldots \cup (\mathrm{Id}_\mathsf{H}, n-1) \cdot \mathsf{H}_{(0)}\,,$$

where Id_H is the identity element of H. If r is an integer coprime with n, let

$$\mathsf{H}_r = \mathsf{H}_{n\kappa_{\mathsf{G},\mathbb{Z}}}/r\mathbb{Z}\,.$$

Let us show that the map $\Xi : \mathsf{H}_{(0)} \to \mathsf{H}_r$ is surjective. Given $k \in \mathbb{Z}$, since r and n are coprime there exists $l \in \mathbb{Z}$ so that $ln = k \mod r$. Thus $(h,k) \in \mathsf{H}_r$ is the image of $(h, ln) \in \mathsf{H}_{(0)}$. It follows that Ξ is surjective hence H_r is connected.

Now we consider the homomorphism $\Delta_r : \mathsf{H}_r \to \mathsf{G}_r$ induced by $\Delta : \mathsf{H} \to \mathsf{G}$. Its image is connected because H_r is, so that $(\mathrm{Id}_\mathsf{G}, k)$ and $(\mathrm{Id}_\mathsf{G}, k')$ lie in the same connected component for any $k, k' \in \mathbb{Z}/r\mathbb{Z}$. If $g \in \mathsf{G}$, one has $(g,0)(\mathrm{Id}_\mathsf{G}, k) = (g,k)$ because the inhomogeneous

cocycle c representing $\kappa_{\mathsf{G},\mathbb{Z}}$ is normalized, that is $c(\mathrm{Id}_{\mathsf{G}}, g) = 0$ for all $g \in \mathsf{G}$, and since the left multiplication by $(g, 0)$ in G_r is continuous, (g, k) and (g, k') are in the same connected component.

To show that G_r is connected, we now only need to show that for $g, g' \in \mathsf{G}$, $(g, 0)$ and $(g', 0)$ are in the same connected component. Pick any continuous path $g_t \in \mathsf{G}$ with $g_0 = g$ and $g_1 = g'$, and lift it to $\tilde{g}_t \in \mathsf{G}_r$ so that $\tilde{g}_0 = (g, 0)$. Then $\tilde{g}_1 = (g', k)$ for some k, where (g', k) is in the same connected component as $(g', 0)$. Thus $(g, 0)$ and $(g', 0)$ are path connected and G_r is connected.

Lastly, $p_r^* : \mathrm{H}^2_{\mathrm{bc}}(\mathsf{G}_r, \mathbb{R}) \to \mathrm{H}^2_{\mathrm{bc}}(\mathsf{G}, \mathbb{R})$ is a isomorphism because both are equal, through Van Est isomorphism, to $\Omega^2(\mathcal{X})^{\mathsf{G}}$. □

Lifting an integral maximal homomorphism yields an integral maximal homomorphism. More precisely we have the following lemma.

Lemma 6.1.11. *Assume G_r is connected. Let $\rho : \pi_1(S) \to \mathsf{G}_r$. Then ρ is integral maximal if and only if $p_r \circ \rho : \pi_1(S) \to \mathsf{G}$ is.*

Proof. Since G_r is connected and with $p_r : \mathsf{G}_r \to \mathsf{G}$ the projection, we have

$$p_r^* \kappa^b_{\mathsf{G},\mathbb{Z}} = \kappa^b_{\mathsf{G}_r,\mathbb{Z}}$$

since the symmetric space of G_r is the symmetric space of G.

Thus for $\rho : \pi_1(S) \to \mathsf{G}_r$, one has

$$\rho^* \kappa^b_{\mathsf{G}_r,\mathbb{Z}} = n\kappa^b_{S,\mathbb{Z}} \quad \text{if and only if} \quad \rho^* \circ p_r^* \kappa^b_{\mathsf{G},\mathbb{Z}} = (p_r \circ \rho)^* \kappa^b_{\mathsf{G},\mathbb{Z}} = n\kappa^b_{S,\mathbb{Z}}\,. \qquad \square$$

If G_r is not connected, we define integral maximal homomorphism to satisfies the previous lemma:

Definition 6.1.11. A homomorphism $\rho : \pi_1(S) \to \mathsf{G}_r$ is *(integral) maximal* if $p_r \circ \rho : \pi_1(S) \to \mathsf{G}$ is.

Let $\rho \in \mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G})$. By exactness of the sequence $0 \to \mathbb{Z}/r\mathbb{Z} \to \mathsf{G}_r \xrightarrow{p_r} \mathsf{G} \to 0$, the set

$$V^r_\rho := \{\eta \in \mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G}_r),\, p_r \circ \rho = \eta\}$$

of lifts of ρ to G_r is, if not empty, an affine space over $\mathrm{Hom}(\pi_1(S), \mathbb{Z}/r\mathbb{Z})$. Indeed if $\eta_1, \eta_2 \in V^r_\rho$, there exists a unique homomorphism $\beta : \pi_1(S) \to \mathbb{Z}/r\mathbb{Z}$ such that

$$\eta_1(\gamma) = \eta_2(\gamma) \cdot i_r(\beta(\gamma))\,,$$

where $i_r : \mathbb{Z}/r\mathbb{Z} \to \mathsf{G}_r$.

We refers to [BIW10] for a proof of the following fact, as it is a consequence of their Theorem 12.

Proposition 6.1.14. *Suppose S is not compact and for all j, let c_j be a simple loop in the positive direction around the jth puncture. Let $\rho \in \mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G})$ be an integral maximal homomorphism and $\tilde{\rho} : \pi_1(S) \to \mathsf{G}_\kappa$ any lift of ρ. Then*

$$-\sum_j \mathrm{Trans}_\kappa(\tilde{\rho}(c_j)) = n(2g-2+s)\,.$$

And if $\eta : \pi_1(S) \to \mathsf{G}_r$ lifts ρ, then

$$-\sum_j \mathrm{Trans}_\kappa^{(r)}(\tilde{\rho}(c_j)) = n(2g-2+s) \mod r\,.$$

If the surface S is closed, $\pi_1(S)$ is not free. Let $\{a_i, b_i\}_i$ be generators of $\pi_1(S)$ with the usual only relation $\prod_{i=1}^g [a_i, b_i] = 1$.

Lemma 6.1.12. *Suppose S is a closed surface. Let $\rho \in \mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G})$ and $A_\ell, B_\ell \in \mathsf{G}_\kappa$ be lifts of $\rho(a_\ell), \rho(b_\ell)$ respectively. Then*

$$\prod_{\ell=1}^g [A_\ell, B_\ell] = i(n(2g-2))\,.$$

As a consequence, if S is closed, an integral maximal homomorphism lifts to G_r if and only if $r \mid n(2g-2)$.

Proof. Let $\nu = \rho^* \kappa^b_{\mathsf{G},\mathbb{Z}}$ be the pull back of the regular Kähler class of G. By definition and naturalness of the comparison maps with respect to pull back, we have $\nu = n\kappa_{S,\mathbb{Z}}$. If $[S] \in \mathrm{H}_1(S, \mathbb{Z})$ is the fundamental class of S we have

$$\begin{aligned} \langle \nu | [S] \rangle &= n \langle \kappa_{S,\mathsf{G}} | [S] \rangle \\ &= n(2g-2)\,, \end{aligned}$$

thus the obstruction to lifting ρ is $n(2g-2)$. □

6.2 Lifting an integral maximal homomorphism

For this whole section we fix G a simple Hermitian Lie group of tube type, $\kappa = \kappa^b_{\mathsf{G},\mathbb{Z}}$ its integral bounded Kähler class,

$$0 \to \mathbb{Z} \xrightarrow{i} \mathsf{G}_\kappa \xrightarrow{p} \mathsf{G} \to 1$$

the central extension of G associated to κ (forgetting that κ is bounded) and $\tau = i(1) \in \mathsf{G}_\kappa$.

Recall the n cover $TS^\circ \to T^nS^\circ$ and the section $s_n : \pi_1(S) \to \pi_1(T^nS^\circ)$ such that ds_n is a bounded inhomogeneous cocycle representing $n\kappa^b_{S,\mathbb{Z}} \in \mathrm{H}^2_{\mathrm{Bb}}(\pi_1(S), \mathbb{Z})$.

The aim of this section is the following proposition.

Proposition 6.2.1. *Let G be a simple Hermitian Lie group of tube type and real rank n, $\kappa = \kappa^b_{\mathsf{G},\mathbb{Z}}$ its integral bounded Kähler class. Let T^nS° be the punctured plane bundle n-covered by the punctured tangent bundle of S.*

For any integral maximal homomorphism $h : \pi_1(S) \to \mathsf{G}$, there exists a unique $\tilde{h} : \pi_1(T^nS^\circ) \to \mathsf{G}_\kappa$ such that

- *$\tilde{h}$ lifts h, that is $\tilde{h}(t_{T^nS}) = \tau$ and the following diagram commutes*

$$\begin{array}{ccc} \pi_1(T^nS^\circ) & \xrightarrow{\tilde{h}} & \mathsf{G}_\kappa \\ \downarrow & & \downarrow \\ \pi_1(S) & \xrightarrow{h} & \mathsf{G}\,; \end{array}$$

- *for all $x \in \pi_1(S)$, $\mathrm{Trans}_k(\tilde{h}(s_n(x))) = 0$.*

The map $\mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G}) \to \mathrm{Hom}(\pi_1(T^nS^\circ), \mathsf{G}_\kappa)$, $h \mapsto \tilde{h}$ is continuous and yields another continuous map $\mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}) \to \mathrm{Rep}(\pi_1(T^nS^\circ), \mathsf{G}_\kappa)$ such that for every mapping class $\phi \in \Gamma(S)$ and every $h \in \mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G})$

$$\phi \cdot [\tilde{h}] = \widetilde{\phi \cdot [h]}$$

where here $[\cdot]$ denotes the conjugacy class.

6.2.1 Construction of $\tilde{h}$

From now on and until the end of the section, σ will denote the section of $p : \mathsf{G}_\kappa \to \mathsf{G}$ defined by

$$\text{For all } g \in \mathsf{G},\ 0 \leqslant \mathrm{Trans}_\kappa(\sigma(g)) < 1\,.$$

Lemma 6.2.1. *Let $h : \pi_1(S) \to \mathsf{G}$ be integral maximal. Let $A_i = \sigma(h(a_i))$, $B_i = \sigma(h(b_i))$ and $C_j = \sigma(h(c_j))$. Then*

$$\prod_{i=1}^{g}[A_i, B_i]\prod_{j=1}^{s} C_j = \tau^{n(2g-2+s)}$$

Proof. We first suppose $s > 0$. In this case, let $h_0 : \pi_1(S) \to \mathsf{G}_\kappa$ be any lift of h. One at least must exist since $\pi_1(S)$ is free. Proposition 6.1.14 yields

$$-\sum_j \mathrm{Trans}_\kappa(h_0(c_j)) = n(2g-2+s)\,.$$

We may assume $h_0(a_i) = A_i$ and $h_0(b_i) = B_i$ for all i. Indeed, for all i, one has $p(h_0(a_i)) = h(a_i) = p(A_i)$ and $p(h_0(b_i)) = h(a_i) = p(B_i)$, so that there exists $\alpha_i, \beta_i \in \mathbb{Z}$ with

$$h_0(a_i) = A_i\tau^{\alpha_i} \qquad \text{and} \qquad h_0(b_i) = B_i\tau^{\beta_i}$$

and we define $h_0' : \pi_1(S) \to \mathsf{G}_\kappa$ in the following way

$$\begin{aligned} h_0'(a_i) &= h_0(a_i)\tau^{-\alpha_i} \\ h_0'(b_i) &= h_0(b_i)\tau^{-\beta_i} \\ h_0'(c_j) &= h_0(c_j)\,. \end{aligned}$$

Since τ is central in G_κ, h_0' is a homomorphism lifting h and moreover $h_0'(a_i) = A_i$ and $h_0'(b_i) = B_i$.

Since h is integral maximal, we have $\mathrm{Rot}_\kappa(h(x)) = 0$ for all $x \in \pi_1(S)$ and thus $\mathrm{Trans}_\kappa(h_0(c_j)) = \mathrm{Trans}_\kappa(C_j) = 0 \mod \mathbb{Z}$. Let $m_j = -\mathrm{Trans}_\kappa(h_0(c_j)) \in \mathbb{Z}$. We have

$$\tau^{m_j} h_0(c_j) = C_j$$

consequently

$$\begin{aligned} \prod_{i=1}^{g}[A_i, B_i]\prod_{j=1}^{s} C_j &= \prod_{i=1}^{g}[h_0(a_i), h_0(b_i)]\prod_{j=1}^{s} h_0(c_j)\tau^{m_j} \\ &= \prod_{j=1}^{s}\tau^{m_j} \\ &= \tau^{n(2g-2+s)}\,. \end{aligned}$$

If $s = 0$, that is the surface is closed, we use Lemma 6.1.12. □

6.2.2 Proof of Proposition 6.2.1

Let $h \in \mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G})$. We first define $\tilde{h}$ on a generating set. Let $\tilde{h}$ be the only homomorphism $\pi_1(T^n S^\circ) \to \mathsf{G}_\kappa$ such that for all $x \in \{a_i, b_i, c_j\}$, $\tilde{h}(s_n(x)) = \sigma(h(x))$, that is the image of $s(x)$ is the only antecedent of $h(x)$ with translation number equal to 0, and $\tilde{h}(t_{T^n S}) = \tau$. By Lemma 6.2.1

$$\prod_{i=1}^{g} [\tilde{h}(a_i), \tilde{h}(b_i)] \prod_{j=1}^{s} \tilde{h}(c_j) = \tilde{h}(t_{T^n S})^{n(2g-2+s)}$$

so $\tilde{h}$ is indeed a homomorphism.

Proposition 6.2.2. *For all $x \in \pi_1(S)$, $\tilde{h}(s_n(x)) = \sigma(h(x))$.*

Proof. The cocycle $d\sigma$ represents $\kappa^b_{\mathsf{G},\mathbb{Z}}$ and since h is integral maximal, the pullback $h^* d\sigma$ represents $n\kappa^b_{S,\mathbb{Z}}$.

Moreover, if $x, y \in \pi_1(S)$ by definition of ds_n we have

$$s_n(x)s_n(y) = s_n(xy)t^{ds_n(x,y)}$$

and thus

$$\tilde{h}(s_n(x))\tilde{h}(s_n(y)) = \tilde{h}(s_n(xy))\tau^{ds_n(x,y)}$$

because $\tilde{h}$ is a homomorphism and $\tilde{h}(t_{T^n S}) = \tau$. As a consequence we get the equality $d(\tilde{h} \circ s) = ds_n$.

Using the commutativity of

$$\begin{array}{ccc} \pi_1(T^n S^\circ) & \xrightarrow{\tilde{h}} & \mathsf{G}_\kappa \\ \downarrow & & \downarrow \\ \pi_1(S) & \xrightarrow{h} & \mathsf{G}\,; \end{array}$$

we construct $\beta : \pi_1(S) \to \mathbb{Z}$ such that for $x \in \pi_1(S)$

$$\tilde{h}(s_n(x)) = \sigma(h(x))\tau^{\beta(x)}$$

In $\mathrm{H}^2_{\mathrm{Bb}}(\pi_1(S), \mathbb{Z})$ we have

$$0 = [d(\tilde{h} \circ s_n)] - [d(\sigma \circ h)] = [d\beta]$$

so β is at bounded distance from a homomorphism $\pi_1(S) \to \mathbb{Z}$. We show that actually

$\beta = 0$.

First we show that β is homogeneous. It is enough to show that for all $x \in \pi_1(S)$ and $k \in \mathbb{Z}$,

$$\tilde{h}(s_n(x^k)) = (\tilde{h}(s_n(x)))^k \text{ and } \sigma(h(x^k)) = (\sigma(h(x)))^k.$$

The equality $s_n(x^k) = (s_n(x))^k$ was proven in Lemma 6.1.9.

Then we show $\sigma(h(x^k)) = (\sigma(h(x)))^k$. We write $\kappa = \kappa^b_{\mathsf{G},\mathbb{Z}}$. Since h is integral maximal, we have $\mathrm{Rot}_\kappa(h(x)) = 0$. Thus $\sigma(h(x))$ is the only lift of $h(x)$ with translation number 0. Since Trans_κ is homogeneous, $\sigma(h(x))^k$ has also translation number 0, which shows $(\sigma(h(x)))^k = \sigma(h(x^k))$.

Since $\beta(x)$ is the only integer such that

$$\tilde{h}(s_n(x)) = \sigma(h(x))\tau^{\beta(x)}$$

we get $\beta(x^k) = k\beta(x)$.

Suppose $\beta = \lambda + \epsilon$ where $\lambda : \pi_1(S) \to \mathbb{Z}$ is a homomorphism and $\epsilon : \pi_1(S) \to \mathbb{Z}$ is bounded. Since β is homogeneous, so is ϵ, thus ϵ being bounded it must be 0 and $\beta = \lambda$ is a homomorphism.

By definition of $\tilde{h}$, both $\tilde{h} \circ s_n$ and $\sigma \circ h$ agree on the generating set of $\pi_1(S)$ so for all $1 \leqslant i \leqslant g$ and $1 \leqslant j \leqslant s$,

$$\beta(a_i) = \beta(b_i) = \beta(c_j) = 0\,.$$

So β is a homomorphism trivial on a generating set, hence trivial. □

Proposition 6.2.3 (Mapping class group invariance)**.** *For all $h \in \mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G})$, for all $\phi \in \Gamma(S)$,*

$$\phi \cdot [\tilde{h}] = \widetilde{\phi \cdot [h]}\,.$$

Proof. Let f be an orientation-preserving diffeomorphism of S whose mapping class is ϕ and such that $f(x_0) = x_0$ and $Tf(X_0) = X_0$, where $\pi_1(S) = \pi_1(S, x_0)$ and $\pi_1(T^nS^\circ) = \pi_1(T^nS^\circ, X_0)$.

We use the uniqueness of $\tilde{h}$ as the only lift of h with $\mathrm{Trans}_\kappa(\tilde{h}(s_n(x))) = 0$ for all $x \in \pi_1(S)$. Clearly both $f \cdot \tilde{h}$ and $\widetilde{f \cdot h}$ lift $f \cdot h$ and for all $x \in \pi_1(S)$,

$$\mathrm{Trans}_\kappa(\widetilde{f \cdot h}(s_n(x))) = 0\,.$$

It remains to show $\mathrm{Trans}_\kappa(f \cdot \tilde{h}(s_n(x))) = 0$, and applying Lemma 6.1.9 to f^{-1}

$$\begin{aligned}\mathrm{Trans}_\kappa(f \cdot \tilde{h}(s_n(x))) &= \mathrm{Trans}_\kappa(\tilde{h}(T^n f_*^{-1}(s_n(x))))\\ &= \mathrm{Trans}_\kappa(\tilde{h}(s_n(f_*^{-1}(x)))\\ &= 0\end{aligned}$$

because by construction of $\tilde{h}$, $\mathrm{Trans}_\kappa \circ \tilde{h} \circ s_n = 0$. □

This last proof concludes the proof of Proposition 6.2.1. Indeed,

$$\mathrm{Trans}_\kappa \circ \tilde{h} \circ s_n = \mathrm{Trans}_\kappa \circ \sigma \circ h = 0$$

because h is integral maximal so for $x \in \pi_1(S)$, $\mathrm{Trans}_\kappa(\sigma(h(x))) \equiv 0 \mod \mathbb{Z}$ and by definition of σ, we have $0 \leqslant \mathrm{Trans}_\kappa(\sigma(h(x))) < 1$.

6.3 Construction of Φ

In this section we construct the map

$$\begin{array}{ccc}\mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G}_r) & \longrightarrow & \mathrm{Hom}(\mathrm{H}_1(TS^\circ, \mathbb{Z}), \mathbb{Z}/r\mathbb{Z})\\ \rho & \longmapsto & \xi_\rho\end{array}$$

for Theorem 11.

6.3.1 Sections and homomorphisms

Let $\rho \in \mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G}_r)$ and let $h \in \mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G})$ be the composition of ρ with the projection $\mathsf{G}_r \to \mathsf{G}$. Lifting h in $\tilde{h} : \pi_1(T^n S^\circ) \to \mathsf{G}_\kappa$ as in Proposition 6.2.1, we get a homomorphism

$$\begin{array}{rccccc}\eta : & \pi_1(T^n S^\circ) & \to & & \mathsf{G}_r & \\ & x & \mapsto & \tilde{h}(x) & \mod \langle \tau^r \rangle\,.\end{array}$$

Let $\Lambda = \mathrm{Im}(\eta)$ be the image of η in G_r and τ_r be the image of $\tau \in \mathsf{G}_\kappa$ in G_r.

Lemma 6.3.1. *The image of $\rho : \pi_1(S) \to \mathsf{G}_r$ is a normal subgroup of Λ and the map*

$$\zeta : \Lambda/\rho(\pi_1(S)) \to \mathbb{Z}/r\mathbb{Z}$$

that associates 1 to the class of τ_r is an isomorphism.

Proof. For $v \in \pi_1(T^n S^\circ)$ and $x \in \pi_1(S)$ the projection of v, let $k \in \mathbb{Z}$ such that $\eta(v) = \rho(x)\tau_r^k$.

Then for any $y \in \pi_1(S)$, we have

$$\begin{aligned}\eta(v)\rho(y)\eta(v)^{-1} &= \rho(x)\tau_r^k \rho(y)(\rho(x)\tau_r^k)^{-1} \\ &= \rho(xyx^{-1})\,,\end{aligned}$$

so that $\rho(\pi_1(S))$ is a normal subgroup in Λ. Observe that $\eta(t^r_{T^n S}) = \tau_r^r = \mathrm{Id}_{\mathsf{G}_r}$ and $\eta : \pi_1(T^n S^\circ) \to \mathsf{G}_r$ induces an isomorphism

$$\pi_1(T^n S^\circ)/\langle t^r_{T^n S^\circ}\rangle \longrightarrow \Lambda$$

and the exact sequence

$$0 \to \mathbb{Z}/r\mathbb{Z} \to \pi_1(T^n S^\circ)/\langle t^r_{T^n S^\circ}\rangle \to \pi_1(S) \to 1$$

yields the exact sequence

$$0 \to \mathbb{Z}/r\mathbb{Z} \to \Lambda \to \rho(\pi_1(S)) \to 1$$

so that $\rho(\pi_1(S))/\Lambda$ contains r elements. Since $\zeta : \Lambda/\rho(\pi_1(S)) \to \mathbb{Z}/r\mathbb{Z}$ is a homomorphism with $\zeta(\tau_r) = 1$, it is an isomorphism. □

Let ξ_ρ be the homomorphism obtained by the following composition

$$\xi_\rho : \pi_1(TS^\circ) \to \pi_1(T^n S^\circ) \xrightarrow{\eta} \Lambda/\rho(\pi_1(S)) \xrightarrow{\zeta} \mathbb{Z}/r\mathbb{Z}\,.$$

Proposition 6.3.1. *Let $\rho \in \mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G}_r)$, $h = p \circ \rho \in \mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G})$ its projection and $\xi_\rho \in \mathrm{Hom}(\pi_1(TS^\circ), \mathbb{Z}/r\mathbb{Z})$ as above. Let $x \in \pi_1(S)$ and $s(x) \in \pi_1(TS^\circ)$ its lift. Then*

$$\xi_\rho(s(x)) = -\operatorname{Trans}^{(r)}_\kappa(\rho(x)) \quad \textit{and} \quad \xi_\rho(t_{TS}) = n\,.$$

Moreover, $\rho \mapsto \xi_\rho$ is continuous.

Proof. Let $x \in \pi_1(S)$. Because $\eta(s_n(x))$ and $\rho(x)$ are both send to $h(x)$ by $p : \mathsf{G}_r \to \mathsf{G}$, there exists $k \in \mathbb{Z}/r\mathbb{Z} \simeq \ker(p)$ such that

$$\eta(s_n(x))\rho(x)^{-1} = \tau_r^k\,.$$

From the equation above, $k = \operatorname{Trans}^{(r)}_\kappa(\eta(s_n(x))) - \operatorname{Trans}^{(r)}_\kappa(\rho(x)) = -\operatorname{Trans}^{(r)}_\kappa(\rho(x))$. By

definition, we also have $\xi_\rho(s(x)) = k$. Thus the first assertion is proved.

Let $q : \pi_1(TS^\circ) \to \pi_1(S)$ be the projection. For $x \in \pi_1(TS^\circ)$ one has

$$\rho(q(x))\tau^{\xi_\rho(x)} = \tilde{h}(s_n(x)) \mod \langle \tau^r \rangle .$$

Since $\tilde{h}$ depends continuously on ρ, $\rho \mapsto \xi_\rho$ is continuous. □

Recall that $A^{r,n}_{g,s} = \{\xi \in \mathrm{Hom}(\pi_1(TS^\circ), \mathbb{Z}/r\mathbb{Z}) : \xi(t) = n \mod r\}$.

Proposition 6.3.2. *Let $p_r : \mathsf{G}_r \to \mathsf{G}$ be the projection. The function*

$$\begin{array}{rccc} \Phi : & \mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}_r) & \longrightarrow & \mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}) \times A^{r,n}_{g,s} \\ & [\rho] & \longmapsto & ([p_r \circ \rho], \xi_\rho) \end{array}$$

is a $\Gamma(S)$-equivariant homeomorphism.

Proof. We show Φ is a homomorphism at the level of $\mathrm{Hom}_{\mathrm{IntMax}}$. To show that Φ is a bijection, we explicit the inverse. Let $h \in \mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G})$ and $\xi \in A^{r,n}_{g,s}$. Let $\tilde{h}_r : \pi_1(T^nS^\circ) \to \mathsf{G}_r$ be the lift of h given by Proposition 6.2.1 composed with the projection $\mathsf{G}_\kappa \to \mathsf{G}_r$. For $x \in \pi_1(TS^\circ)$ let $s_n(x) \in \pi_1(T^nS^\circ)$ be its image. Let $\eta : \pi_1(TS^\circ) \to \mathsf{G}_r$ be the homomorphism given by, for $x \in \pi_1(TS^\circ)$,

$$\eta(x) = \tilde{h}_r(s_n(x))\tau_r^{-\xi(x)} .$$

because τ_r is central, η is indeed a homomorphism. Moreover,

$$\begin{aligned} \eta(t_{TS}) &= \tilde{h}_r(t^n_{T^nS^\circ})\tau_r^{-\xi(t_{TS^\circ})} \\ &= \tau_r^{n-n} = \mathrm{Id}_{\mathsf{G}_r} . \end{aligned}$$

Let $q : TS^\circ \to S$ be the projection. Then

$$\ker(q_*) = \langle t_{TS^\circ} \rangle ,$$

and it follows that $\eta = q_* \circ \rho$ where $\rho : \pi_1(S) \to \mathsf{G}_r$ such that $\Phi(\rho) = (h, \xi)$. Thus the map $(h, \xi) \mapsto \rho$ is the inverse of Φ which must be bijective.

The continuity of Φ is a consequence of the continuity of $\rho \mapsto \xi_\rho$ which has been proved in Proposition 6.3.1.

The fact that $\rho \mapsto p_r \circ \rho$ is equivariant is obvious. For $\rho \mapsto \xi_\rho$, let $q : \pi_1(TS^\circ) \to \pi_1(S)$ be the projection and recall that if $\Phi(\rho) = (h, \xi)$ and if $\tilde{h}_r : \pi_1(T^nS^\circ) \to \mathsf{G}_r$, then for

$\phi = [f] \in \Gamma(S)$,

$$\begin{aligned}
\tau^{\phi^{-1}\cdot\xi(x)} &= \rho(q(f_* x)) \cdot \tilde{h}_r(f_* s_n(x))) \\
&= \rho \circ f_*(q(x)) \cdot \tilde{h}_r \circ f_*(s_n(x)) \\
&= \phi \cdot \rho(q(x)) \cdot \widetilde{h \circ f_*}_r(s_n(x)) \qquad \text{(Proposition 6.2.1)} \\
&= \tau^{\zeta(x)}
\end{aligned}$$

where $\zeta = \xi_{\phi\cdot\rho}$. Thus $\phi^{-1} \cdot \xi = \xi_{\phi\cdot\rho}$ and

$$\Phi(\phi \cdot \rho) = \phi \cdot \Phi(\rho)\,,$$

which concludes the last part of the proof of Theorem 11. □

Chapter 7

Applications

In this chapter we prove Corollary 12. It will be a consequence of Proposition 7.1.1 and Proposition 7.2.1.

7.1 Diagonal representations

Let $\delta : \mathsf{SL}_2(\mathbb{R}) \to \mathsf{Sp}_{2n}(\mathbb{R})$ be the composition

$$\mathsf{SL}_2(\mathbb{R}) \to \mathsf{SL}_2(\mathbb{R})^n \to \mathsf{Sp}_{2n}(\mathbb{R})$$

where the first map is the diagonal and the second is the injective homomorphism given by blocks. The composition

$$\mathsf{SL}_2(\mathbb{R}) \xrightarrow{\delta} \mathsf{Sp}_{2n}(\mathbb{R}) \to \mathsf{PSp}_{2n}(\mathbb{R})$$

has kernel $\{\pm \mathrm{Id}\}$ and we call $\Delta : \mathsf{PSL}_2(\mathbb{R}) \to \mathsf{PSp}_{2n}(\mathbb{R})$ the induced homomorphism. Note that Δ stabilizes a diagonal disk in the symmetric space of $\mathsf{PSp}_{2n}(\mathbb{R})$.

From now on, let $\mathsf{G} = \mathsf{PSp}_{2n}(\mathbb{R})$ and $\mathcal{X}$ be its symmetric space.

Lemma 7.1.1. *Let $r \in \mathbb{Z}_{>0}$. Assume r and n are coprime, where n is the rank of $\mathcal{X}$. There exists a unique $\Delta_r : \mathsf{Spin}^r \to \mathsf{G}_r$ such that the following diagram commutes*

$$\begin{array}{ccc} \mathsf{Spin}^r & \xrightarrow{\Delta_r} & \mathsf{G}_r \\ \downarrow & & \downarrow \\ \mathsf{PSL}_2(\mathbb{R}) & \xrightarrow{\Delta} & \mathsf{G}\,. \end{array}$$

Proof. Let c_{G} be a bounded Borel cocycle whose class is the bounded integral Kähler class of G, so that

$$\mathsf{G}_r = \mathsf{G} \times_{c_{\mathsf{G}}} \mathbb{Z}/r\mathbb{Z}$$

using notation from Section 6.1.2. With these notations, we have $\mathsf{Spin}^r = \mathsf{PSL}_2(\mathbb{R}) \times_c \mathbb{Z}/r\mathbb{Z}$ with c representing the Kähler class of $\mathsf{PSL}_2(\mathbb{R})$. Because $\Delta^* \kappa_{\mathsf{G},\mathbb{Z}} = n\kappa_{\mathsf{PSL}_2(\mathbb{R}),\mathbb{Z}}$, we may assume $nc = \Delta^* c_{\mathsf{G}}$. Then Δ_r is given by, for $(g,k) \in \mathsf{PSL}_2(\mathbb{R}) \times_c \mathbb{Z}/r\mathbb{Z} = \mathsf{Spin}^r$

$$\Delta_r((g,k)) = (\Delta g, nk)\,.$$

Using $\gcd(r,n) = 1$ one checks that it is indeed injective. The commutativity of the diagram follows from the definition. □

Definition 7.1.1.

- A *diagonal Fuchsian* homomorphism into G_r is the composition $\Delta \circ \rho$ where $\rho : \pi_1(S) \to \mathsf{Spin}^r$ is a r-spin hyperbolization of S;
- A *diagonal homomorphism* is any maximal homomorphism $\pi_1(S) \to \mathsf{G}_r$ that can be deformed to a diagonal Fuchsian one.

Let $\mathrm{Hom}_\Delta(S, \mathsf{G}_r)$ and $\mathrm{Rep}_\Delta(S, \mathsf{G}_r)$ be respectively the spaces of diagonal homomorphisms and its quotient by G_r. We have

$$\mathrm{Hom}_\Delta(S, \mathsf{G}) \subset \mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G})\,,$$

and $\mathrm{Hom}_\Delta(S, \mathsf{G}_r)$ is a union of connected components of $\mathrm{Hom}_{\mathrm{IntMax}}(S, \mathsf{G}_r)$.

Since $\mathrm{Hom}_{\mathrm{Max}}(S, \mathsf{PSL}_2(\mathbb{R}))$ is connected, $\mathrm{Hom}_\Delta(S, \mathsf{G})$ is connected.

Before proving Corollary 12, we need the following proposition

Proposition 7.1.1. *Assume* $\gcd(r,n) = 1$. *The map*

$$\Phi : \mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}_r) \to \mathrm{Rep}_{\mathrm{IntMax}}(S, \mathsf{G}) \times A_{g,s}^{r,n}$$

from Theorem 11 restricts to

$$\Phi : \mathrm{Rep}_\Delta(S, \mathsf{G}_r) \xrightarrow{\sim} \mathrm{Rep}_\Delta(S, \mathsf{G}) \times A_{g,s}^{r,n}$$

and with $\Gamma(S)$ *the mapping class group of* S,

$$\mathrm{Rep}_\Delta(S, \mathsf{G}_r)/\Gamma(S)$$

has $\mathrm{Card}(A_{g,s}^{r,n}/\Gamma(S))$ *connected components.*

We compute $\mathrm{Card}(A_{g,s}^{r,n}/\Gamma(S))$ in the next session.

Proof. The second part of the proposition is a consequence of the first and the fact that $\mathrm{Rep}_\Delta(S,\mathsf{G})$ consists of only one connected component. The first part is a direct consequence of Lemma 7.1.1 above because it shows that the projection of a diagonal homomorphism $\pi_1(S)\to\mathsf{G}_r$ is again a diagonal homomorphism $\pi_1(S)\to\mathsf{G}$. □

Note that, thanks to Lemma 6.1.10, G_r is connected whenever $\gcd(r,n)=1$. If moreover n is odd, then $\pi_1(\mathsf{PSp}_{2n}(\mathbb{R}))=\mathbb{Z}$ and G_r is isomorphic to H the unique connected r-cover of G. and using Proposition 7.1.1 we get a mapping class group equivariant homeomorphism

$$\mathrm{Rep}_\Delta(S,\mathsf{H})\to\mathrm{Rep}_\Delta(S,\mathsf{G})\times A_{g,s}^{r,n}$$

so that $\mathrm{Rep}_\Delta(S,\mathsf{H})/\Gamma(S)$ has $\mathrm{Card}(A_{g,s}^{r,n}/\Gamma(S))$ connected components and Corollary 12 follows from Proposition 7.2.1.

7.2 Computation of $\mathrm{Card}(A_{g,s}^{r,n}/\Gamma(S))$

In this section we compute the number of orbits for the mapping class group action on $A_{g,s}^{r,n}$.

Let $r,n\geqslant 1$, $g,s\geqslant 0$ be integers such that $2g-2+s>0$ and let $S=S_{g,s}$. Let c_j be loops in the positive direction about each puncture and $\overline{c}_j$ their tangential lift to TS°. For positive s, let

$$I_{g,s}^{r,n}:=\left\{(m_1,\dots,m_s)\in\{0,\dots,r-1\}^s\,,\ m_1+\cdots+m_s=n(2g-2+s)\right\}.$$

For $\mathfrak{m}\in I_{g,s}^{r,n}$ we defined

$$\ell_{g,s}^{r,n,\mathfrak{m}}=\begin{cases}1 & g=0\\ \gcd(r,m_1-n,\dots,m_s-n) & g=1\\ \gcd(r,2n,m_1-n,\dots,m_s-n) & \text{else.}\end{cases}$$

As in Chapter 5, we set

$$\begin{aligned}A_{g,s}^{r}&=\mathrm{Hom}(\mathrm{H}_1(TS^\circ,\mathbb{Z}),\mathbb{Z}/r\mathbb{Z});\\ A_{g,s}^{r,n}&=\left\{\xi\in A_{g,s}^{r}:\ \xi(t_{TS})=n\mod r\right\};\\ A_{g,s}^{r,n,\mathfrak{m}}&=\left\{\xi\in A_{g,s}^{r,n}:\ \xi(\overline{c}_j)=m_j,\ 1\leqslant j\leqslant s\right\}.\end{aligned}$$

So that

$$A_{g,s}^{r} = \coprod_{n\in\mathbb{Z}/r\mathbb{Z}} A_{g,s}^{r,n}$$
$$A_{g,s}^{r,n} = \coprod_{\mathfrak{m}\in I_{g,s}^{r,n}} A_{g,s}^{r,n,\mathfrak{m}}$$

where $\coprod$ is the disjoint union, and

$$A_{g,s}^{r}/\Gamma(S) = \coprod_{n\in\mathbb{Z}/r\mathbb{Z}} (A_{g,s}^{r,n}/\Gamma(S))$$
$$A_{g,s}^{r,n}/\Gamma(S) = \coprod_{\mathfrak{m}\in I_{g,s}^{r,n}} (A_{g,s}^{r,n,\mathfrak{m}}/\Gamma(S))$$

where, by Theorem 10,

$$\mathrm{Card}(A_{g,s}^{r,n,\mathfrak{m}}/\Gamma(S)) = \ell_{g,s}^{r,n,\mathfrak{m}}$$

and for $s = 0$,

$$\mathrm{Card}(A_{g,0}^{r,n}/\Gamma(S)) = \gcd(r, 2n)\,.$$

We compute the total number of orbits of Γ acting on $A_{g,s}^{r,n}$ where $\gcd(r, n) = 1$.

Proposition 7.2.1 (Total number of orbits)**.** *Let*

$$d(g, s, r, n) = \mathrm{Card}(A_{g,s}^{r,n}/\Gamma(S))$$

be the total number of orbits of the mapping class group action on $A_{g,s}^{r,n}$. If r and n are coprime, then

$$d(g,s,r,n) = \begin{cases} r^{s-1} & g = 0,\ s \geqslant 3 \\ \sum_{d|r} d^{s-1} & g = 1,\ s \geqslant 1 \\ r^{s-1} + (r/2)^{s-1} & g \geqslant 2,\ s \geqslant 1,\ 2 \mid r \\ r^{s-1} & g \geqslant 2,\ s \geqslant 1,\ 2 \nmid r \\ 2 & g \geqslant 2,\ s = 0,\ 2 \mid r \mid 2g-2 \\ 1 & g \geqslant 2,\ s = 0,\ 2 \nmid r \mid 2g-2 \\ 0 & g \geqslant 2,\ s = 0,\ r \nmid 2g-2\,. \end{cases}$$

The cases where $s = 0$ are straightforward, indeed if $r \nmid 2g-2$, then $r \nmid n(2g-2)$ and

in this case $A_{g,0}^{r,n}$ is empty. If $r \mid 2g-2$, we have

$$\mathrm{Card}(A_{g,0}^{r,n}/\Gamma(S)) = \gcd(r, 2n) = \gcd(r, 2)$$

where the last equality is a consequence of $\gcd(r, n) = 1$.

For positive s, we have

$$d(g, s, r, n) = \sum_{\mathfrak{m} \in I_{g,s}^{r,n}} \ell_{g,s}^{r,n,\mathfrak{m}}$$

and

Lemma 7.2.1. *Suppose $s > 0$ and $\gcd(r, n) = 1$. Then*

$$\sum_{\mathfrak{m} \in I_{g,s}^{r,n}} \ell_{g,s}^{r,n,\mathfrak{m}} = \begin{cases} r^{s-1} & g = 0,\ s \geqslant 3 \\ \sum_{d \mid r} d^{s-1} & g = 1 \\ r^{s-1} + (r/2)^{s-1} & g \geqslant 2,\ 2 \mid r \\ r^{s-1} & g \geqslant 2,\ 2 \nmid r \end{cases}$$

Proof. We do each case separately.

Case $g = 0, s \geqslant 3$. In this case, $\ell_{g,s}^{r,n,\mathfrak{m}} = 1$ for all $\mathfrak{m} \in I_{g,s}^{r,n}$ and $\sum_{\mathfrak{m} \in I_{g,s}^{r,n}} 1 = \mathrm{Card}(I_{g,s}^{r,n}) = r^{s-1}$.

Case $g = 1$. This is a consequence of Lemma 7.2.2 below.

Case $g \geqslant 2$, $2 \mid r$. In this case, since $\gcd(2n, r) = 2$, we have that $\ell_{g,s}^{r,n,\mathfrak{m}} = 1$ if one of the $m_j - n$ is odd and $\ell_{g,s}^{r,n,\mathfrak{m}} = 2$ otherwise. If $m_1 - n, \ldots, m_{s-1} - n$ are all even then

$$m_s - n = -(m_1 - n) - \ldots - (m_{s-1} - n) + n(2g-2) \mod r$$

is even as well because $2 \mid r$. Thus we get

$$\begin{aligned} \sum_{\mathfrak{m} \in I_{g,s}^{r,n}} \ell_{g,s}^{r,n,\mathfrak{m}} &= \sum_{\substack{\mathfrak{m} \in I_{g,s}^{r,n} \\ \forall j,\, 2 \mid m_j - n}} 2 + \sum_{\substack{\mathfrak{m} \in I_{g,s}^{r,n} \\ \exists j,\, 2 \nmid m_j - n}} 1 \\ &= 2(r/2)^{s-1} + (r^{s-1} - (r/2)^{s-1}) \\ &= r^{s-1} + (r/2)^{s-1}. \end{aligned}$$

Case $g \geqslant 2$, $2 \nmid r$. Since $2 \nmid r$, and because r and n are coprime, then r and $2n$ are coprime and thus $\ell_{g,s}^{r,n,\mathfrak{m}} = 1$ for all $\mathfrak{m} \in I_{g,s}^{r,n}$, hence the result.

□

Lemma 7.2.2. *Let $r, s, n > 0$. Then*

$$\sum_{\substack{0 \leqslant m_1, \ldots, m_s < r \\ m_1 + \cdots + m_s = sn \mod r}} d(\gcd(r, m_1 - n, \ldots, m_s - n)) = \sum_{d|r} d^{s-1}$$

Proof. For such m_j we have

$$\gcd(r, m_1 - n, \ldots, m_s - n) = \gcd(r, m_1 - n, \ldots, m_{s-1} - n)$$

since $\sum_{j=1}^{s}(m_j - n) = 0 \mod r$. Changing indices we have

$$\sum_{0 \leqslant m_1, \ldots, m_{s-1} < 0} d(\gcd(r, m_1 - n, \ldots, m_{s-1} - n)) = \sum_{0 \leqslant m_1, \ldots, m_{s-1} < 0} d(\gcd(r, m_1, \ldots, m_{s-1}))\,.$$

Now let

$$f(r,k) = \mathrm{Card}\,\{(H, x_1, \ldots, x_k) : H \text{ subgroup of } \mathbb{Z}/r\mathbb{Z},\ x_1, \ldots, x_k \in H\}\ .$$

Indexing subgroups of $\mathbb{Z}/r\mathbb{Z}$ by divisors of r and then changing indices, we have

$$\begin{aligned} f(r,k) &= \sum_{d|r} \left(\frac{r}{d}\right)^k \\ &= \sum_{d|r} d^k\,. \end{aligned}$$

But from the definition of $f(r,k)$ we have

$$f(r,k) = \mathrm{Card}\,\{(H, x_1, \ldots, x_k) : x_1, \ldots, x_k \in \mathbb{Z}/r\mathbb{Z},\ \langle x_1, \ldots, x_k\rangle < H\}\ .$$

For $x_1, \ldots, x_k \in \mathbb{Z}/r\mathbb{Z}$, the integer $\gcd(r, x_1, \ldots, x_k)$ is well-defined and $d(\gcd(r, x_1, \ldots, x_k))$ is exactly the number of subgroups of $\mathbb{Z}/r\mathbb{Z}$ containing $x_1, \ldots, x_k$. Thus

$$\begin{aligned} \sum_{x_1, \ldots, x_k \in \mathbb{Z}/r\mathbb{Z}} d(\gcd(r, x_1, \ldots, x_k)) &= \sum_{x_1, \ldots, x_k \in \mathbb{Z}/r\mathbb{Z}} \mathrm{Card}\,\{H < \mathbb{Z}/r\mathbb{Z} : x_1, \ldots, x_k \in H\} \\ &= \mathrm{Card}\,\{(H, x_1, \ldots, x_r) : x_1, \ldots, x_k \in H < \mathbb{Z}/r\mathbb{Z}\} \\ &= f(r,k)\,. \end{aligned}$$

Taking $k = s - 1$ concludes the proof. □

Bibliography

[Arf41] Cahit Arf. Untersuchungen über quadratische Formen in Körpern der Charakteristik 2. I. *Journal für die Reine und Angewandte Mathematik*, 183:148–167, 1941.

[Ati71] Michael F. Atiyah. Riemann surfaces and spin structures. *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série*, 4:47–62, 1971.

[BILW05] Marc Burger, Alessandra Iozzi, François Labourie, and Anna Wienhard. Maximal representations of surface groups: symplectic Anosov structures. *Pure and Applied Mathematics Quarterly*, 1(3, Special Issue: In memory of Armand Borel. Part 2):543–590, 2005.

[BIW10] Marc Burger, Alessandra Iozzi, and Anna Wienhard. Surface group representations with maximal Toledo invariant. *Annals of Mathematics. Second Series*, 172(1):517–566, 2010.

[Bla79] Philippe Blanc. Sur la cohomologie continue des groupes localement compacts. *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série*, 12(2):137–168, 1979.

[BM02] M. Burger and N. Monod. Continuous bounded cohomology and applications to rigidity theory. *Geometric and Functional Analysis*, 12(2):219–280, 2002.

[Chi08] Alessandro Chiodo. Stable twisted curves and their r-spin structures. *Université de Grenoble. Annales de l'Institut Fourier*, 58(5):1635–1689, 2008.

[CK07] Jean-Louis Clerc and Khalid Koufany. Primitive du cocycle de Maslov généralisé. *Mathematische Annalen*, 337(1):91–138, 2007.

[FM12] Benson Farb and Dan Margalit. *A primer on mapping class groups*, volume 49 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 2012.

[Ghy87] Étienne Ghys. Groupes d'homéomorphismes du cercle et cohomologie bornée. In *The Lefschetz centennial conference, Part III (Mexico City, 1984)*, volume 58 of *Contemp. Math.*, pages 81–106. Amer. Math. Soc., Providence, RI, 1987.

[Gol80] William Mark Goldman. *Discontinuous groups and the Euler class*. ProQuest LLC, Ann Arbor, MI, 1980.

[Gol88] William M. Goldman. Topological components of spaces of representations. *Inventiones Mathematicae*, 93(3):557–607, October 1988.

[GW10] Olivier Guichard and Anna Wienhard. Topological invariants of Anosov representations. *Journal of Topology*, 3(3):578–642, 2010.

[Hel01] Sigurdur Helgason. *Differential geometry, Lie groups, and symmetric spaces*, volume 34 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2001.

[Jar00] Tyler J. Jarvis. Geometry of the moduli of higher spin curves. *International Journal of Mathematics*, 11(5):637–663, 2000.

[Joh80] Dennis Johnson. Spin structures and quadratic forms on surfaces. *The Journal of the London Mathematical Society. Second Series*, 22(2):365–373, 1980.

[Kon92] Maxim Kontsevich. Intersection theory on the moduli space of curves and the matrix Airy function. *Communications in Mathematical Physics*, 147(1):1–23, 1992.

[KS90] Atsushi Katsuda and Toshikazu Sunada. Closed orbits in homology classes. *Institut des Hautes Études Scientifiques. Publications Mathématiques*, (71):5–32, 1990.

[Mar69] G. A. Margulis. Certain applications of ergodic theory to the investigation of manifolds of negative curvature. *Akademija Nauk SSSR. Funkcional\cprime nyi Analiz i ego Priloženija*, 3(4):89–90, 1969.

[Mar04] Grigoriy A. Margulis. *On some aspects of the theory of Anosov systems*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004.

[Mir07a] Maryam Mirzakhani. Simple geodesics and Weil-Petersson volumes of moduli spaces of bordered Riemann surfaces. *Inventiones Mathematicae*, 167(1):179–222, 2007.

[Mir07b] Maryam Mirzakhani. Weil-Petersson volumes and intersection theory on the moduli space of curves. *Journal of the American Mathematical Society*, 20(1):1–23, 2007.

[Mum71] David Mumford. Theta characteristics of an algebraic curve. *Annales Scientifiques de l'École Normale Supérieure. Quatrième Série*, 4:181–192, 1971.

[NN57] A. Newlander and L. Nirenberg. Complex analytic coordinates in almost complex manifolds. *Annals of Mathematics. Second Series*, 65:391–404, 1957.

[Put07] Andrew Putman. Cutting and pasting in the Torelli group. *Geometry & Topology*, 11:829–865, 2007.

[RW14] Oscar Randal-Williams. Homology of the moduli spaces and mapping class groups of framed, r-Spin and Pin surfaces. *Journal of Topology*, 7(1):155–186, 2014.

[Sip82] Patricia L. Sipe. Roots of the canonical bundle of the universal Teichmüller curve and certain subgroups of the mapping class group. *Mathematische Annalen*, 260(1):67–92, 1982.

[vE53] W. T. van Est. Group cohomology and Lie algebra cohomology in Lie groups. I, II. *Nederl. Akad. Wetensch. Proc. Ser. A. \bf 56 = Indagationes Math.*, 15:484–492, 493–504, 1953.

[Wie06] Anna Wienhard. The action of the mapping class group on maximal representations. 120:179–191, 2006.

[Wit91] Edward Witten. Two-dimensional gravity and intersection theory on moduli space. In *Surveys in differential geometry (Cambridge, MA, 1990)*, pages 243–310. Lehigh Univ., Bethlehem, PA, 1991.

[Wit92] Edward Witten. The N matrix model and gauged WZW models. *Nuclear Physics. B. Theoretical, Phenomenological, and Experimental High Energy Physics. Quantum Field Theory and Statistical Systems*, 371(1-2):191–245, 1992.

[Wit93] Edward Witten. Algebraic geometry associated with matrix models of two-dimensional gravity. In *Topological methods in modern mathematics (Stony Brook, NY, 1991)*, pages 235–269. Publish or Perish, Houston, TX, 1993.