Polynomials in one variable and signs of their coefficients and real roots

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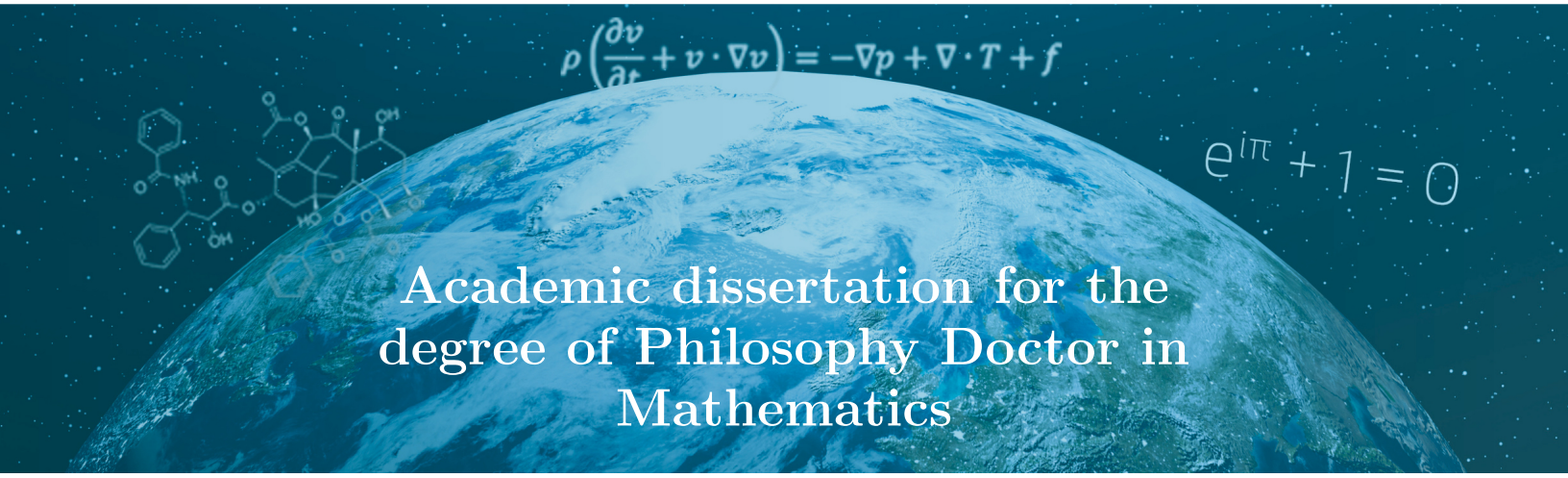

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Abstract (French version)

Les polynômes sont très utilisés en mathématiques appliquées. Les modèles utilisant des polynômes fournissent un cadre analytique pour la modélisation mécanique, physique, économique ou financière.

Pour un polynôme réel à une variable, nous considérons d'un côté les signes de ses coefficients et de l'autre les signes de ses racines réelles. La règle de Descartes impose des contraintes sur le nombre de racines strictement positives exprimées en termes du nombre de changements de signe dans la suite des coefficients du polynôme. Cela permet de déduire des restrictions sur le nombre de racines négatives.

Une *suite de signe (SS)* de longueur $d+1$ est une séquence finie de signes "+" et/ou "−". Pour le polynôme $P := x^d + a_{d-1}x^{d-1} + \cdots + a_0$, $(a_j \in \mathbb{R}^*)$ nous disons que la séquence $(1, a_{d-1}, \ldots, a_0)$ définit la SS σ si $\sigma = (+, \text{sgn}\,(a_{d-1}), \ldots, \text{sgn}\,(a_0))$. Comme nous ne considérons que les polynômes moniques, le premier signe de la SS est un "+". Soit c (respectivement p) le nombre de changements (respectivement préservations) de signe dans la séquence $(1, a_{d-1}, \ldots, a_0)$ et soit *pos* (respectivement *neg*) le nombre de racines positives (respectivement négatives) de P compté avec multiplicité. D'aprés la règle de Descartes on a $pos \leq c$ et, en appliquant cette règle à $P(-x)$, $neg \leq p$. Une observation faite par Fourier dit que: $c - pos \in 2\mathbb{Z}$ et $p - neg \in 2\mathbb{Z}$. On dit que (pos, neg) est une *paire admissible (PA)* pour σ si la règle de Descartes est satisfaite (*pos* est le nombre de racines positives et *neg* le nombre de racines négatives). Un couple (SS, PA) est dit *réalisable* s'il existe un polynôme monique dont la séquence de coefficients définit la SS σ et qui a exactement *pos* racines positives et *neg* racines négatives, toutes distinctes.

Pour un degré donné d, nous recherchons les couples (SS, PA) non réalisables. Le premier résultat de notre travail donne des conditions suffisantes pour l'existence ou la non-existence de polynômes réels de degré d avec deux changements de signe dans la SS. Pour $d \leq 10$ et pour les SSs avec deux changements de signe, nous donnons la réponse exhaustive à la question.

Jusqu'en degré 8, pour tous les cas non réalisables, l'un des nombres *pos* ou *neg* est 0. Le deuxième résultat présenté dans cette thèse dit que pour $d = 9$, il existe un couple non réalisable (SS, PA) dans lequel les deux composantes de la PA sont non nulles (et c'est le seul couple en degré 9).

Le troisième résultat porte sur un polynôme et ses dérivées de tous ordres. Dans ce cas, nous avons une contrainte supplémentaire qui est celle du théorème de Rolle. Une liste de PAs du polynôme P et de ses dérivées compatibles avec la règle de Descartes et le théorème de Rolle, est appelée *séquence de paires admissibles (SPA)*. Dans cette partie, nous avons expliqué la (non) réalisabilité de toutes les SPAs possibles en degré ≤ 5.

Le quatrième résultat de cette thèse concerne l'explication de la non-réalisabilité à l'aide d'images montrant le discriminant. Nous expliquons la non-existence du seul cas non réalisable pour $d = 5$ et l'existence de tous les autres cas au moyen d'images montrant le discriminant de la famille des polynômes $x^5 + x^4 + ax^3 + bx^2 + cx + d$.

Abstract (English version)

Polynomials are very used in applied mathematics. Polynomial models provide an analytic framework for mechanics, physics, economy or financial modeling. They play an important role in a growing range of applications.

For a real polynomial in one variable, one can consider on the one side the signs of its coefficients and on the other the signs of its possible real roots. The Descartes' rule of signs imposes restrictions on the number of its positive roots expressed via the number of sign changes in its sequence of coefficients. This allows to deduce restrictions on the number of negative roots as well.

A *sign pattern (SP)* of length $d+1$ is a finite sequence of "+" and/or "−" signs. For the polynomial $P := x^d + a_{d-1}x^{d-1} + \cdots + a_0$, $(a_j \in \mathbb{R}^*)$ we say that the sequence $(1, a_{d-1}, \ldots, a_0)$ defines the SP σ if $\sigma = (+, \mathrm{sgn}(a_{d-1}), \ldots, \mathrm{sgn}(a_0))$. As we consider only monic polynomials, the first sign of the SP is a "+". Denote by c and p the numbers of sign changes and sign preservations in the sequence $(1, a_{d-1}, \ldots, a_0)$ and by *pos* and *neg* the numbers of positive and negative roots of P counted with multiplicity. According to Descartes' rule of signs, one has $pos \leq c$ and, applying this rule to the polynomial $P(-x)$ one has $neg \leq p$. Fourier's observation is tantamount to saying that: $c - pos \in 2\mathbb{Z}$ and $p - neg \in 2\mathbb{Z}$. We say that (pos, neg) is an *admissible pair (AP)* for σ if Descartes' rule is satisfied (*pos* is the number of positive roots and *neg* the number of negative roots). A given couple (SP, AP) is *realizable* if there exists a monic polynomial whose sequence of coefficients defines the SP σ and which has exactly *pos* positive and exactly *neg* negative roots, all of them distinct.

Question 1. *For a given degree d, what are the non-realizable couples (SP,AP) ?*

We explore this question and find some sufficient conditions for the (non)existence of degree d real polynomials with two sign changes in the SP. For $d \leq 10$ and for SPs with two sign changes, we give the exhaustive answer to Question 1.

Up to degree 8, for all the non-realizable cases, one of the numbers *pos* or *neg* is 0. The next result presented in this thesis is that for $d = 9$, there is a non-realizable couple (SP, AP) in which both components of the AP are nonzero (and this is the only couple in degree 9).

We formulate a third problem when a polynomial and its derivatives of all orders are considered. In this case we have one more constraint due to Rolle's theorem. A list of APs of the polynomial P and its derivatives compatible with Descartes' rule and Rolle's theorem is called *sequence of admissible pairs (SAP)*. In this part, one has explained the realizability of all possible SAPs in degree up to 5.

The fourth result of this thesis concerns the explanation of the non-realizability of certain couples (SP, AP) using pictures showing the discriminant set. We explain the non-existence of the only non-realizable case for $d = 5$ and the existence of all other cases by means of pictures showing the discriminant set of the family of polynomials $x^5 + x^4 + ax^3 + bx^2 + cx + d$ together with the coordinate axes.

Introduction

1. Descartes' rule of signs – formulation and history

In his book La Géométrie from 1637, René Descartes states that a real univariate polynomial has not more positive roots than the number of sign changes in the sequence of its coefficients. In 1890, Fourier completes this rule in [10] (see p. 294 therein) by saying that the difference between the number of positive roots and the number of sign changes of the coefficients is a multiple of 2. Fourier's remark has often been erroneously attributed to C. F. Gauss and his book [2] in which the remark is absent. In the present thesis we consider problems stemming from Descartes' rule of signs as completed by Fourier.

We consider monic real univariate polynomials with no vanishing coefficients:

$$P := x^d + a_{d-1}x^{d-1} + \cdots + a_0,\ a_j \in \mathbb{R}^* \ .$$

Descartes' rule of signs imposes some conditions on the number of positive roots of the polynomial. Indeed, denote by c and p the numbers of sign changes and sign preservations in the sequence $(1, a_{d-1}, \ldots, a_0)$ and by pos and neg the numbers of positive and negative roots of P counted with multiplicity which means that:

$$c + p = d \ .$$

According to Descartes' rule of signs,

$$pos \leq c. \tag{1.1}$$

Applying this rule to the polynomial $P(-x)$ gives

$$neg \leq p. \tag{1.2}$$

One can observe that the wording of this rule is very similar to the Intermediate Value Theorem, which says that a continuous function must have at least one root in a given interval if the sign of the function changes in that interval. Fourier's observation is tantamount to saying that:

$$c - pos \in 2\mathbb{Z} \tag{1.3}$$

and

$$p - neg \in 2\mathbb{Z} \ . \tag{1.4}$$

Notice that without the assumption the coefficients a_j to be nonzero conditions (1.2) do not hold true – for the polynomial $x^2 - 1$, one has $c = 1$, $p = 0$ and $neg = 1$. It is also clear that

$$\text{sgn } a_0 = (-1)^{pos} \ . \tag{1.5}$$

2. A problem of realizability and $\mathbb{Z}_2 \times \mathbb{Z}_2$-action

Descartes' rule gives only necessary conditions in the sense that if one knows the positions of the positive and negative coefficients of a degree d polynomial, one is not sure that for all cases of values of *pos* and *neg* compatible with conditions (1.1), (1.2), (1.3), (1.4) and (1.5), there exist monic polynomials with such values of *pos* and *neg*.

Definition 1. (i) A *sign pattern (SP)* of length $d+1$ is a finite sequence of plus and/or minus signs. For the polynomial P, we say that the sequence (1, a_{d-1}, ..., a_0) defines the SP σ if $\sigma = (+, \mathrm{sgn}(a_{d-1}), \ldots, \mathrm{sgn}(a_0))$. As we consider only monic polynomials, the first sign of the SP is a "+". For a given SP σ with c sign changes and p sign preservations, we call the pair (c, p) the *Descartes' pair* of σ.

(ii) We say that (pos, neg) is an *admissible pair (AP)* for σ if the conditions (1.1), (1.2), (1.3), (1.4) and (1.5) are satisfied.

Definition 2. We say that a given couple (SP, AP) is *realizable* if there exists a monic polynomial whose sequence of coefficients defines the SP σ and which has exactly *pos* positive and exactly *neg* negative roots, all of them distinct.

The notions SP, AP and Descartes' pair are introduced by B. Z. Shapiro in [8].

Example 1. For $d = 4$, the SP $\sigma = (+, -, +, +, +)$ has four APs: $(2, 2)$, $(0, 2)$, $(2, 0)$ and $(0, 0)$, the first of which is its Descartes' pair.

At this point, a natural question comes to mind:

Question 1. *For a given SP σ, are all admissible pairs (pos, neg) realizable by polynomials with the SP σ?*

The answer is "No" and the first example of non-realizabilty was found by D. Grabiner (see [3]). Namely, he has shown that the admissible pair $(0, 2)$ is not realizable with the sign pattern $\sigma_1 := (+, -, -, -, +)$.

It means that one cannot find a polynomial

$$Q(x) = a_4x^4 - a_3x^3 - a_2x^2 - a_1x + a_0 \ ,$$

where $a_j > 0$, with two negative roots and with no positive roots.

Grabiner's argument is quite simple: he observes that a fourth-degree polynomial following the SP σ_1 with only two negative roots could be factored as $a(x^2 + bx + c)(x^2 - sx + t)$ with $a, b, c, s, t > 0$, $s^2 < 4t$, and $b^2 \geq 4c$. The product of these factors equals:

$$a(x^4 + (b - s)x^3 + (t + c - bs)x^2 + (bt - cs)x + ct) \ .$$

To get the SP σ_1, one needs to have $b < s$ and $bt < cs$, which gives $b^2t < s^2c$ and thus $b^2/c < s^2/t$. But one has $b^2/c \geq 4 > s^2/t$.

Remark 1. One can easily prove that Grabiner's case is the first case of non-realizabilty which means that for degree 1, 2 and 3 all cases are realizable. Indeed for degree $d = 1$ one has two couples (SP, AP) $(((+, +), (0, 1))$ and $((+, -), (1, 0))$, for degree $d = 2$ one has 6 and for degree $d = 3$ one has 16 couples. And one can easily find a polynomial for each such couple (for example, the polynomial $P(x) = x + 1$ for the couple $((+, +), (0, 1))$.

An automatic corollary of Grabiner's result is that the couple $((+,+,-,+,+),(2,0))$ is also non-realizable by a degree 4 polynomial. Indeed, if it were realizable by a polynomial $R(x)$, then Grabiner's case would be realizable by the polynomial $R(-x)$. In order not to consider separately such cases we introduce the following definition:

Definition 3. One defines the natural $\mathbb{Z}_2 \times \mathbb{Z}_2$-action on the space of monic polynomials (and as a consequence on the space of couples (SP, AP) as well) as follows:

(1) The first generator g_1 acts by changing the signs of all monomials in second, fourth etc. position which for polynomials means $P(x) \to (-1)^d P(-x)$; the AP (pos, neg) becomes (neg, pos);

(2) The second generator g_2 acts by reading the SP backwards which for polynomials means $P(x) \to x^d P(1/x)/P(0)$. One divides by $P(0)$ in order to obtain again a monic polynomial; the AP (pos, neg) remains (pos, neg).

Remark 2. The generators g_1 and g_2 are commuting involutions. In the space of couples (SP, AP), the orbits of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action are of length 2 or 4. Grabiner's result concerns an orbit of length 2:

$$((+,-,-,-,+),(0,2)) \quad \text{and} \quad ((+,+,-,+,+),(2,0)) \ ;$$

Both SPs are center symmetric, so applying the second generator of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action does not introduce a new couple. One can show that for $d = 4$, Grabiner's result is the only example (modulo the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action) of non-realizability of a couple (SP, AP).
For $d = 3$, an example of an orbit of length 4 is the following one:

$$\begin{array}{ll} ((+,-,-,-),(1,2)) \ , & ((+,+,+,-),(1,2)) \ , \\ ((+,-,+,+),(2,1)) \quad \text{and} & ((+,+,-,+),(2,1)) \ . \end{array}$$

There exist no orbits of length 1 because if the a SP begins with $(+,+)$, then applying the first generator of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action gives a SP beginning with $(+,-)$ and vice versa.

3. Resolution of the realizability problem for $d \leq 8$

As one now knows that the answer to Question 1 is negative, one can ask a new question:

Question 2. *For a given degree d, what are the non-realizable couples (SP,AP) ?*

For $d \leq 4$, Grabiner's example is the only example of a non-realizable couple (SP, AP) modulo the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action (see Remarks 1 and 2). For $d = 5$ or 6, A. Albouy and Y. Fu have classified all such non-realizable cases in [1]:

Degree	Sign Pattern	Admissible Pair
5	$(+\ -\ -\ -\ -+)$	(0,3)
6	$(+\ -\ -\ -\ -\ +\ +)$	(0,4)
	$(+\ -\ -\ -\ -\ -\ +)$	(0,2) (0,4)
	$(+\ +\ -\ +\ +\ +\ +)$	(2,0)

Then, for $d = 7$, J. Forsgård, V. P. Kostov and B. Z. Shapiro gave the complete list of non-realizable cases (see [8]):

Degree	Sign Pattern	Admissible Pair
7	$(+\,-\,-\,-\,-\,+\,+\,+)$	(0,5)
	$(+\,-\,-\,-\,-\,-\,+\,+)$	(0,5)
	$(+\,-\,-\,-\,-\,-\,-\,+)$	(0,3) (0,5)
	$(+\,+\,-\,-\,-\,-\,+\,+)$	(0,5)
	$(+\,-\,-\,-\,-\,+\,-\,+)$	(0,3)

They also showed in [8] that, for $d = 8$, the following cases are not realizable:

Degree	Sign Pattern	Admissible Pair
8	$(+\,-\,-\,-\,-\,+\,+\,+\,+)$	(0,6)
	$(+\,-\,-\,-\,-\,-\,+\,+\,+)$	(0,6)
	$(+\,-\,-\,-\,-\,-\,-\,+\,+)$	(0,6)
	$(+\,-\,-\,-\,-\,-\,-\,-\,+)$	(0,2) (0,4) (0,6)
	$(+\,+\,-\,+\,+\,+\,+\,+\,+)$	(2,0)
	$(+\,+\,-\,-\,-\,-\,-\,+\,+)$	(0,6)
	$(+\,+\,+\,+\,-\,+\,+\,+\,+)$	(2,0)
	$(+\,+\,+\,+\,-\,+\,-\,+\,+)$	(2,0) (4,0)
	$(+\,-\,-\,-\,+\,-\,-\,-\,+)$	(0,2) (0,4)

In [14], V. P. Kostov, showed that for $d = 8$ these cases and the cases of the table bellow are all cases of non-realizability modulo the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action:

Degree	Sign Pattern	Admissible Pair
8	$(+\,+\,-\,-\,-\,-\,+\,+\,+)$	(0,6)
	$(+\,+\,-\,+\,-\,-\,-\,+\,+)$	(4,0)
	$(+\,-\,-\,-\,-\,-\,-\,+\,+)$	(0,4)
	$(+\,+\,+\,-\,-\,+\,-\,+\,+)$	(4,0)
	$(+\,+\,+\,+\,-\,+\,-\,-\,+)$	(4,0)
	$(+\,+\,-\,+\,-\,-\,-\,-\,+)$	(4,0)

Thus for $d \leq 8$, one knows the exhaustive answer to Question 2.

4. Examples of realizability and non-realizability for arbitrary degrees

The following *concatenation lemma* (see [8, Lemma 14]) is used to justify the realizability of certain couples (SP, AP).

Lemma 1. *Suppose that the monic polynomials P_1 and P_2 of degrees d_1 and d_2 with SPs $\bar{\sigma_1} = (+, \hat{\sigma_1})$ and $\bar{\sigma_2} = (+, \hat{\sigma_2})$, respectively, realize the pairs (pos_1, neg_1) and (pos_2, neg_2). (Here the SPs $\hat{\sigma_1}$ and $\hat{\sigma_2}$ are obtained from $\bar{\sigma_1}$ and $\bar{\sigma_2}$ by deleting the first sign $+$.) Then*

- *If the last position of $\hat{\sigma_1}$ is $+$, then for any $\epsilon > 0$ small enough, the polynomial $\epsilon^{d_2} P_1(x)P_2(x/\epsilon)$ realizes the SP $(+, \hat{\sigma_1}, \hat{\sigma_2})$ and the pair $(pos_1 + pos_2, neg_1 + neg_2)$.*

- *If the last position of $\hat{\sigma_1}$ is $-$, then for any $\epsilon > 0$ small enough, the polynomial $\epsilon^{d_2} P_1(x)P_2(x/\epsilon)$ realizes the SP $(+, \hat{\sigma_1}, -\hat{\sigma_2})$ and the pair $(pos_1 + pos_2, neg_1 + neg_2)$. (Here $-\sigma$ is the SP obtained from σ by changing each $+$ by $-$ and vice versa.)*

Remark 3. Each SP is realizable with its Descartes' pair. The proof of this can be found in [13] (see Remark 5 therein).

It is not clear whether it is possible, for a given degree d, to formulate an exhaustive answer to Question 2. Attempts have been made to give sufficient conditions for realizability or non-realizability. Thus, for even degrees d, the following result holds true, see [8, Proposition 4]:

Theorem 1. *For d even, consider SPs satisfying the following three conditions:*

- *the sign of the constant term (i.e., the last entry) is $+$;*
- *the signs of all odd monomials are $+$;*
- *among the remaining signs of even monomials there are $l \geq 1$ minuses (at arbitrary positions).*

Then, for any such SP, the APs $(2, 0)$, $(4, 0)$, ..., $(2l, 0)$, and only they, are non-realizable. (Using the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action one obtains more such examples.)

Other sufficient conditions concern odd degrees d. Namely, for a fixed odd degree $d \geq 5$ and $1 \leq k \leq (d-3)/2$, denote by σ_k the SP beginning with two pluses followed by k pairs "$-, +$" and then by $d - 2k - 1$ minuses. Its Descartes' pair equals $(2k + 1, d - 2k - 1)$. The following theorem is proved in [15] (see Theorem 6 therein).

Theorem 2. *(1) The SP σ_k is not realizable with any of the pairs $(3, 0)$, $(5, 0)$, ..., $(2k + 1, 0)$;*

(2) the SP σ_k is realizable with the pair $(1, 0)$;

(3) the SP σ_k is realizable with any of the pairs $(2l + 1, 2r)$, $l = 0, 1, \ldots, k, r = 1, 2, \ldots, (d - 2k - 1)/2$.

Another case in which important results on (non)-realizability are obtained is the one when the SP contains exactly two sign changes.

Notation 1. We denote by $\Sigma_{m,n,q}$ the SP consisting of $m \geq 1$ pluses followed by $n \geq 1$ minuses followed by $q \geq 1$ pluses, where $m + n + q = d + 1$. For a given polynomial P, we denote by P^R the corresponding *reverted polynomial*, i.e. $P^R := x^d P(1/x)$ (see about the second generator of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action in Definition 3). If the polynomial P defines the SP $\Sigma_{m,n,q}$ then, P^R defines the SP $\Sigma_{m,n,q}$. The roots of P^R are the reciprocals of the roots of P.

The following proposition is Proposition 6 of [8], taking into account the correction introduced in [9].

Proposition 1. *(1) The SP $\Sigma_{m,n,q}$ is not realizable with the AP $(0, d - 2)$ if*

$$\kappa := \frac{d-m-1}{m} \times \frac{d-q-1}{q} \geq 4;$$

(2) The SP $\Sigma_{m,n,q}$ is realizable with any AP of the form $(2, v)$ except in the case when d and m are even and $n = 1$ (hence q is even).

5. New results in the case of 2 sign changes

The first new result of this thesis concern the SP $\Sigma_{m,n,q}$ for small values of m and n. The proofs are given in [6].

Theorem 3. *(See Theorem 3 of* [6]*.)*

(1) For $n = 1$, $d \geq 2$, and $n = 2$, $d \geq 3$, any SP $\Sigma_{m,n,q}$ is realizable with the AP $(0, d-2)$.

(2) For $n = 3$ and $d \geq 5$, any SP $\Sigma_{m,3,q}$ is realizable with the AP $(0, d-2)$. For $d = 4$, the SP $(+,-,-,-,+)$ is not realizable with the AP $(0, 2)$.

(3) For $n = 4$, the SP $\Sigma_{m,4,q}$ is realizable with the AP $(0, d-2)$ if $q \geq m \geq 3$ and $d \geq 10$ or $q > m = 2$ and $d \geq 11$.

(4) For $m = 1$ and $n \geq 4$, the SP $\Sigma_{1,n,q}$ is not realizable with the AP $(0, d-2)$.

Proposition 2. *(See Proposition 1 of* [6]*.)*

(1) For $d = 9$, the SPs $\Sigma_{3,4,3}$ and $\Sigma_{2,4,4}$ are not realizable with the AP $(0, 7)$.

(2) For $d = 10$, the SP $\Sigma_{2,4,5}$ is not realizable with the AP $(0, 8)$.

Remarks 1. (1) For $d = 9$ and 10, Theorem 3 and Proposition 2 cover all cases of couples $(\Sigma_{m,n,q}, (0, d-2))$.

(2) If a SP $\Sigma_{m,n,q}$ is realizable with the AP $(0, d-2)$, then it is also realizable with the APs $(0, d-4)$, $(0, d-6)$, ... Indeed, if a polynomial P realizes the couple $(\Sigma_{m,n,q}, (0, d-2))$, then one can assume that all critical levels of P are distinct. Hence in the family of polynomials $P + t$, $t \geq 0$, one encounters for suitable values of t polynomials with exactly $d-4$, $d-6$, ... negative distinct roots and with no positive roots.

Another result of [6], valid for any values of m, n and q, is the following theorem:

Theorem 4. *(See Theorem 4 of* [6]*.)*

The SP $\Sigma_{m,n,q}$ is realizable with the AP $(0, d-2)$ if

$$A(d, m, n) := -dn^2 + 4dm + 4dn - 4m^2 - 4mn - 4d + 4m > 0 \ .$$

6. Examples of non-existence with both components of the AP non-zero

As we can see, for both $d = 4$ and 5 there is, respectively, a single non-realizable case (up to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action). For $d = 6$, 7 and 8 there are respectively 4, 6 and 19 non-realizable cases. In all of them, one of the numbers *pos* or *neg* is 0. It was conjectured in [8] (see Conjecture 12 therein) that non-realizable cases of couples

(SP, AP) exist only when one of the components of the AP is 0. However, it was shown in [12] that for $d = 11$, the following couple (SP, AP) is not realizable:

$$(\ (+,-,-,-,-,-,+,+,+,+,+,-) \ , \ (1,8) \ ) \ .$$

The next result presented in this thesis is an improvement of the result of [12]. Namely, for $d = 9$,we set $\sigma^0 = (+,-,-,-,-,+,+,+,+,-)$. The following theorem is Theorem 1 of [5]:

Theorem 5. *(1) The SP σ^0 is not realizable with the AP $(1,6)$.*

(2) Modulo the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, for $d \leq 9$, this is the only non-realizable couple (SP, AP) in which both components of the AP are nonzero.

7. Sequences of admissible pairs

We formulate now the third problem which is studied in this thesis. For a polynomial P and its SP σ_0, let us define the sequences of SPs σ_0, σ_1, ..., σ_{d-1} corresponding to the polynomial P and to its derivatives of order $\leq d-1$ (the SP σ_j is obtained from σ_{j-1} by deleting the last component).

We denote by (c_k, p_k) and (pos_k, neg_k) the Descartes' and admissible pairs for the SPs σ_k, $k = 0, \ldots, d-1$. In this case, Rolle's theorem implies that

$$\begin{gathered} pos_{k+1} \geq pos_k - 1 \ , \ neg_{k+1} \geq neg_k - 1 \\ \text{and} \qquad pos_{k+1} + neg_{k+1} \geq pos_k + neg_k - 1 \ . \end{gathered} \tag{7.6}$$

Definition 4. For a given SP σ_0 of length $d+1$, and for $k = 0, \ldots, d-1$, suppose that the pair (pos_k, neg_k) satisfies the conditions (1.1) – (7.6). Then we say that

$$((pos_0, neg_0), (pos_1, neg_1), \ldots, (pos_{d-1}, neg_{d-1})) \tag{7.7}$$

is a *sequence of admissible pairs (SAP)* (i.e. a sequence of APs for the SP σ_0 in the sense of these conditions). We say that a SAP is *realizable* if there exists a polynomial P the signs of whose coefficients define the SP σ_0 and such that for $k = 0, \ldots, d-1$, the polynomial $P^{(k)}$ has exactly pos_k positive and neg_k negative roots, all of them being simple.

Remark 4. A given SAP defines a unique SP, this follows from condition (1.5). Indeed, the SAP (7.7) defines the SP (beginning with a +) :

$$(\ + \ , \ (-1)^{pos_{d-1}} \ , \ (-1)^{pos_{d-2}} \ , \ \ldots \ , \ (-1)^{pos_0} \ ) \ .$$

However, for a given SP there are, in general, several possible SAPs.

Example 2. For $d = 2$ and for the SP $(+,+,+)$, there are two possible SAPs, namely, $((0,2),(0,1))$ and $((0,0),(0,1))$. For $d = 3$ and for the SP $(+,+,+,+)$, there are three possible SAPs:

$$((0,3),(0,2),(0,1)) \ , \ ((0,1),(0,2),(0,1)) \text{ and } ((0,1),(0,0),(0,1)) \ .$$

For $d = 4$ and for the SP $(+,+,+,+,+)$, this number is 7:

$$\begin{array}{lll} ((0,4),(0,3),(0,2),(0,1)) & , & ((0,2),(0,3),(0,2),(0,1)) \quad , \\ ((0,2),(0,1),(0,2),(0,1)) & , & ((0,2),(0,1),(0,0),(0,1)) \quad , \\ ((0,0),(0,3),(0,2),(0,1)) & , & ((0,0),(0,1),(0,2),(0,1)) \\ \text{and} & & ((0,0),(0,1),(0,0),(0,1)) \quad . \end{array}$$

The next six numbers (denoted by $A(d)$), obtained as numbers of SAPs compatible with the all-pluses SP of length $d+1$, are:

$$12 \ , \ 30 \ , \ 55 \ , \ 143 \ , \ 273 \ , \ 728 \ .$$

Example 3. Consider the couple (SP, AP), $C := ((+,+,-,+,+),\ (0,2))$. It can be extended in two ways into a couple (SP, SAP):

$$(\quad (+,+,-,+,+) \quad , \quad (0,2) \quad , \quad (2,1) \quad , \quad (1,1) \quad , \quad (0,1) \quad) \quad \text{and}$$

$$(\quad (+,+,-,+,+) \quad , \quad (0,2) \quad , \quad (0,1) \quad , \quad (1,1) \quad , \quad (0,1) \quad) \quad .$$

Indeed, by Rolle's theorem, the derivative of a polynomial realizing the couple C has at least one negative root. Condition (1.5) implies that this derivative (which is of degree 3) has an even number of positive roots. This gives the two possibilities $(2,1)$ and $(0,1)$ for (pos_1, neg_1). The second derivative has a positive and a negative root. Indeed, it is a degree 2 polynomial with positive leading and negative last coefficient.

The next question which is considered in the present thesis is the following one:

Question 3. *For a given degree d, which SAPs are realizable?*

Examples 2 and 3 explain why Question 3 is a refinement of Question 2. In what follows, we may often write down the couples (SP, SAP), not just the SAPs. There are examples of couples (SP, SAP) which are not realizable and whose non-realizability follows from the one of the corresponding couples (SP, AP), where AP$= (pos_0, neg_0)$:

Example 4. For $d = 4$, the couple (SP, SAP)

$$(7.8) \qquad (\ (+,+,-,+,+)\ ,\ (2,0)\ ,\ (2,1)\ ,\ (1,1)\ ,\ (0,1)\ )$$

is not realizable, because the couple $((+,+,-,+,+)\ ,\ (2,0))$ is not realizable (this follows from Grabiner's example see the lines that follow Question 1 and see Remark 2). Hence for $d = 5$, the following couples (SP, SAP) are not realizable:

(7.9)

$$\begin{array}{l} (\quad (+,+,-,+,+,+) \quad , \quad (2,1) \quad , \quad (2,0) \quad , \quad (2,1) \quad , \quad (1,1) \quad , \quad (0,1) \quad)\ , \\ (\quad (+,+,-,+,+,+) \quad , \quad (0,1) \quad , \quad (2,0) \quad , \quad (2,1) \quad , \quad (1,1) \quad , \quad (0,1) \quad)\ , \\ (\quad (+,+,-,+,+,-) \quad , \quad (3,0) \quad , \quad (2,0) \quad , \quad (2,1) \quad , \quad (1,1) \quad , \quad (0,1) \quad)\ , \\ (\quad (+,+,-,+,+,-) \quad , \quad (1,0) \quad , \quad (2,0) \quad , \quad (2,1) \quad , \quad (1,1) \quad , \quad (0,1) \quad)\ . \end{array}$$

Indeed, if one of them were realizable by a degree 5 polynomial P, then the derivative P' would realize the case (7.8). For $d = 5$, the following couple (SP, SAP) is

also not realizable, because the couple (SP, AP), $((+,+,-,+,-,-)\ ,\ (3,0)\ )$ is not realizable, see the first table in Subsection 3:

$$(\ (+,+,-,+,-,-)\ ,\ (3,0)\ ,\ (3,1)\ ,\ (2,1)\ ,\ (1,1)\ ,\ (0,1)\ )\ . \tag{7.10}$$

With regard to Question 3, our principal result is the following theorem:

Theorem 6. *(1) For $d = 1$, 2 and 3, all couples (SP, SAP) are realizable.*
(2) For $d = 4$, the only couple (SP, SAP) which is not realizable is (7.8).
(3) For $d = 5$, the only couples (SP, SAP) which are not realizable are (7.9) and (7.10).

Remark 5. As we see, for degrees up to 5, the questions of realizability of couples (SP, AP) and (SP, SAP) (or just SAP, see Remark 4), i.e. Questions 2 and 3, have the same answers in the following sense. For $d \leq 5$, denote by (σ, B), a couple (SP, SAP), where B is defined by formula (7.7). Then the couple (σ, B) is not realizable if and only if there exists k, $0 \leq k \leq d-1$, such that the couple (SP, AP) $(\sigma_k^\dagger, (pos_k, neg_k))$ is not realizable, where $\sigma^\dagger$ is the SP obtained from σ by deleting its last $d-k-1$ components. It would be interesting to know whether this is the case for any degree d. However, the number of cases to be considered with regard to Question 3 increases too rapidly with d, see Example 2, so already for $d = 6$ obtaining the answer to this question seems quite difficult.

8. Geometric illustration

The fourth result of this thesis concern the explanation of the non-realizability using pictures showing the discriminant set. For degree 5 polynomials, it has been proved by A. Albouy and Y. Fu in [1] that there is exactly one example of non-realizability of a couple (SP, AP) modulo the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, namely

$$(\ (+,+,-,+,-,-)\ ,\ (3,0)\ )\ . \tag{8.11}$$

In [15] the *discriminant set* of the family of polynomials $x^4 + x^3 + ax^2 + bx + c$ is represented (i.e. the set of values of the triple (a, b, c) for which the polynomial has a multiple real root) and thus the non-realizability of the only non-realizable case for degree 4 polymials is explained geometrically. In [7] we give such an explanation for the case (8.11). We explain this non-existence and the existence in all other cases with $d = 5$ by means of pictures showing the discriminant set of the family of polynomials $x^5 + x^4 + ax^3 + bx^2 + cx + d$ together with the coordinate axes.

References

[1] A. Albouy, Y. Fu: Some remarks about Descartes' rule of signs. Elem. Math., 69 (2014), 186–194. Zbl 1342.12002, MR3272179

[2] C. F. Gauss: Beweis eines algebraischen Lehrsatzes. J. Reine Angew. Math. 3 (1828) 1–4; Werke 3, 67–70, Göttingen, 1866. ERAM 003.0089cj, MR1577673

[3] D. J. Grabiner: Descartes Rule of Signs: Another Construction. Am. Math. Mon. 106 (1999), 854–856. Zbl 0980.12001, MR1732666

[4] H. Cheriha, Y. Gati and V. P. Kostov: Descartes' rule of signs, Rolle's theorem and sequences of admissible pairs (submitted).

[5] H. Cheriha, Y. Gati and V. P. Kostov: A nonrealization theorem in the context of Descartes' rule of signs. St. Kliment Ohridski - Faculty of Mathematics and Informatics (accepted).

[6] H. Cheriha, Y. Gati and V. P. Kostov: On Descartes' rule for polynomials with two variations of signs (submitted).

[7] H. Cheriha, Y. Gati and V. P. Kostov: Degree 5 polynomials and Descartes' rule of signs (submitted).

[8] J. Forsgård, B. Shapiro and V. P. Kostov: Could René Descartes have known this? Exp. Math. 24 (4) (2015), 438-448. Zbl 1326.26027, MR3383475

[9] J. Forsgård, B. Shapiro and V. P. Kostov, Corrigendum: Could René Descartes have known this? Exp. Math. 28 (2) (2019), 255-256. 10.1080/10586458.2017.1417775.

[10] J. Fourier: Sur l'usage du théorème de Descartes dans la recherche des limites des racines. Bulletin des sciences par la Société philomatique de Paris (1820) 156–165, 181–187; œuvres 2, 291–309, Gauthier - Villars, 1890.

[11] V. Jullien: Descartes La "Geometrie" de 1637.

[12] V. P. Kostov: Polynomials, sign patterns and Descartes' rule of signs, Mathematica Bohemica, vol. 144 (2019), issue 1, pp. 39-67.

[13] V. P. Kostov: Descartes' rule of signs and moduli of roots, Publicationes Mathematicae Debrecen. Panor. Synth. 33 (2011), vi + 141 p. SMF. Zbl 1259.12001, MR2952044

[14] V. P. Kostov: On realizability of sign patterns by real polynomials, Czechoslovak Math. J. 68 (2018) 853-874.

[15] V. P. Kostov and B. Shapiro: Polynomials, sign patterns and Descartes' rule, Acta Universitatis Matthiae Belii, series Mathematics Issue 2019, 1-11, ISSN 1338-7111.

CHAPTER 1

Polynomials with two variations of signs

ON DESCARTES' RULE FOR POLYNOMIALS WITH TWO VARIATIONS OF SIGNS

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Abstract. For sequences of $d+1$ signs $+$ and $-$ beginning with a $+$ and having exactly two variations of sign, we give some sufficient conditions for the (non)existence of degree d real univariate polynomials with such signs of the coefficients and having given numbers of positive and negative roots compatible with Descartes' rule of signs.

Key words: real polynomial in one variable; Descartes' rule of signs; sign pattern

AMS classification: 26C10, 30C15

1. Introduction

In the present paper we consider a problem which is a natural continuation of Descartes' rule of signs. The latter states that the number of positive roots of a real univariate polynomial (counted with multiplicity) is majorized by the number of sign changes in the sequence of its coefficients. We focus on polynomials without zero coefficients. Such a polynomial (say, of degree d) is representable in the form $P := x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$, $a_j \in \mathbb{R}^*$. Denoting by c and p the numbers of sign changes and sign preservations in the sequence 1, a_{d-1}, ..., a_1, a_0 and by pos and neg the number of positive and negative roots of P (hence $c+p = pos+neg = d$) one obtains the conditions

$$\begin{array}{lll} pos \le c & , \quad neg \le p & , \quad c+p=d \\ c-pos \in 2\mathbb{N} & , \quad p-neg \in 2\mathbb{N} & , \quad (-1)^{pos} = \operatorname{sgn}(a_0) \end{array} \tag{1.1}$$

(the condition $neg \le p$ results from Descartes' rule applied to the polynomial $P(-x)$).

We call *sign pattern (SP)* a sequence of $+$ or $-$ signs of length $d+1$ beginning with a $+$. We say that the polynomial P defines the SP $(+, \operatorname{sgn}(a_{d-1}), \ldots, \operatorname{sgn}(a_1), \operatorname{sgn}(a_0))$. A pair (pos, neg) satisfying conditions (1.1) is called *admissible*. An admissible pair (AP) is called *realizable* if there exists a polynomial P with exactly pos positive distinct and exactly neg negative distinct roots.

Example 1. For $c = 0$, the all-pluses SP is realizable with any AP (which is of the form $(0, d-2k)$, $k = 0, 1, \ldots, [d/2]$, where $[\alpha]$ denotes the integer part of $\alpha \in \mathbb{R}$). Indeed, one can construct a polynomial P with d distinct negative roots and $d-1$ distinct critical levels. Then in the family of polynomials $P+t$, $t > 0$, one encounters polynomials with exactly $d-2$, $d-4$, ..., $d-2[d/2]$ negative distinct

roots and with no positive roots (as t increases, the polynomial $P+t$ loses two-by-two its real roots; each time two coalescing real roots give birth to a complex conjugate pair).

Theorem 1. *If there is just one variation in the sequence of signs, then Descartes' conclusion cannot be improved.*

In technical terms the theorem means that any SP with $c=1$ is realizable with any AP of the form $(1, d-1-2k)$, $k \leq [(d-1)/2]$. The following theorem is proved in a slightly more general setting in [2]:

Theorem 2. [Grabiner] *Whatever be the sequence of signs, there are polynomials with this sequence of signs and with the maximal numbers of positive and of negative roots (all of them distinct) allowed by Descartes' rule.*

This means that any SP is realizable with the AP (c, p). Theorems 1 and 2 are proved in Section 3. Theorem 1 shows that in terms of the value of c, the first truly nontrivial case is $c=2$. Its study is the object of the present paper. We should point out that due to the possibility to consider instead of the polynomial $P(x)$ the polynomial $P(-x)$ (this change exchanges the quantities c and p and the quantities *pos* and *neg*), it suffices to consider (for a given degree d) the cases with $c \leq [d/2]$.

2. Polynomials with two variations of sign

The main results of this paper concern polynomials with two variations of sign, i.e. defining SPs with $c=2$.

Notation 1. We denote by $\Sigma_{m,n,q}$ the SP consisting of $m \geq 1$ pluses followed by $n \geq 1$ minuses followed by $q \geq 1$ pluses, where $m+n+q = d+1$. For a given polynomial P, we denote by P^R the corresponding *reverted* polynomial, i.e. $P^R := x^d P(1/x)$. If the polynomial P defines the SP $\Sigma_{m,n,q}$, then P^R defines the SP $\Sigma_{q,n,m}$. The roots of P^R are the reciprocals of the roots of P.

For small values of m or n, we have the following result:

Theorem 3. *(1) For $n=1$, $d \geq 2$, and for $n=2$, $d \geq 3$, any SP $\Sigma_{m,n,q}$ is realizable with the AP $(0, d-2)$.*

(2) For $n=3$ and $d \geq 5$, any SP $\Sigma_{m,3,q}$ is realizable with the AP $(0, d-2)$. For $d=4$, the SP $\Sigma_{1,3,1}$ is not realizable with the AP $(0,2)$.

(3) For $n=4$, the SP $\Sigma_{m,4,q}$ is realizable with the AP $(0, d-2)$ if $q \geq 3$, $m \geq 3$ and $d \geq 10$, or if $m=2$ and $q \geq 6$ (hence $d \geq 11$).

(4) For $m=1$ and $n \geq 4$, the SP $\Sigma_{1,n,q}$ is not realizable with the AP $(0, d-2)$.

Theorem 3 is proved in Section 4.

Remarks 1. (1) If a SP $\Sigma_{m,n,q}$ is realizable with the AP $(0, d-2)$, then it is realizable with any AP of the form $(0, d-2k)$, $k=1, \ldots, [(d-2)/2]$. Indeed, if a polynomial P with distinct nonzero roots realizes the SP $\Sigma_{m,n,q}$, then one can perturb P to make all its critical levels distinct. In the family $P+t$ one encounters (for suitable positive values of t) polynomials with exactly $d-2k$ distinct negative roots and no positive ones, for $k=1, \ldots, [(d-2)/2]$. As $t \geq 0$, the constant term of the polynomial P is positive hence $P+t$ defines the SP $\Sigma_{m,n,q}$.

(2) The exhaustive answer to the question which couples (SP, AP) are realizable for $d \leq 8$ is given in [2], [1], [4] and [5]. From the results in these papers one deduces

that for $5 \leq d \leq 8$, the SP $\Sigma_{m,4,q}$ is not realizable with the AP $(0, d-2)$. For $d \geq 9$, $n \leq 4$ and $c = 2$, the only cases when the AP is $(0, d-2)$ and which are not covered by Theorem 3 are the ones of $\Sigma_{3,4,3}$ and $\Sigma_{2,4,4}$ for $d = 9$ and of $\Sigma_{2,4,5}$ for $d = 10$. These cases are settled by Proposition 1.

(3) The following result is proved in [4] (see Proposition 6 therein): *If* $\kappa := ((d-m-1)/m)((d-q-1)/q) \geq 4$, *then the SP* $\Sigma_{m,n,q}$ *is not realizable with the AP* $(0, d-2)$. This seems to be the only result concerning nonrealizability of the couple $(\Sigma_{m,n,q}, (0, d-2))$ known up to now. Part (4) of Theorem 3 implies nonrealizability of cases which are not covered by the cited result. These are $\Sigma_{1,4,d-4}$ for $d \geq 11$ (with $\kappa = 3(d-2)/(d-4)$ which is $\leq 27/7 < 4$ for $d \geq 11$).

(4) In [6] it is shown that for $d = 11$, the SP with three variations of sign $\Sigma_{1,5,5,1}$ is not realizable with the AP $(1, 8)$. This is the first known example of nonrealizability in which both components of the AP are nonzero. In [3] it is shown that for $d = 9$, the SP $\Sigma_{1,4,4,1}$ is not realizable with the AP $(1, 6)$.

(5) In [7] and [8] *hyperbolic polynomials* (i.e. real polynomials with all roots real) with one or two sign variations are considered. The following problem is studied there: When the moduli of all the roots of a hyperbolic polynomial are arranged in the increasing order on the real half-line, at which position(s) can be the modulus/moduli of its positive root(s) depending on the position(s) of the sign change(s) in the sequence of coefficients? Other results about hyperbolic polynomials in one variable can be found in [9].

Proposition 1. *(1) For* $d = 9$, *the SPs* $\Sigma_{3,4,3}$ *and* $\Sigma_{2,4,4}$ *are not realizable with the AP* $(0, 7)$.

(2) For $d = 10$, *the SP* $\Sigma_{2,4,5}$ *is not realizable with the AP* $(0, 8)$.

Proposition 1 is proved in Section 5. Our next result contains sufficient conditions for realizability of a SP $\Sigma_{m,n,q}$ with the AP $(0, d-2)$:

Theorem 4. *The SP* $\Sigma_{m,n,q}$ *is realizable with the AP* $(0, d-2)$ *if*

$$\mathcal{L}(d, m, n) := -dn^2 + 4dm + 4dn - 4m^2 - 4mn - 4d + 4m > 0 \ . \tag{2.2}$$

Theorem 4 is proved in Section 6.

Remarks 2. (1) Condition (2.2) is sharp in the following sense: in the two nonrealizable cases $(\Sigma_{3,4,3}, (0, 7))$ and $(\Sigma_{2,4,5}, (0, 8))$ (see Proposition 1) one has $\mathcal{L}(d, m, n) = 0$.

(2) The condition of realizability (2.2) can be compared with the condition of nonrealizability $\kappa \geq 4$ (see part (3) of Remarks 1). To this end the latter can be given the following equivalent form:

$$3dm - dn - 3m^2 - 3mn + 2d + 3m + n - 2 \leq 0 \ .$$

(3) As the SPs $\Sigma_{m,n,q}$ and $\Sigma_{q,n,m}$ are simultaneously (non)realizable with the AP $(0, d-2)$ (see the definition of P^R in Notation 1), one can assume that $m \leq q$ hence $m \leq [d/2]$. Condition (2.2) can be presented in the form

$$\mathcal{M}d - \mathcal{N}m > 0 \ \text{ with } \ \mathcal{M} := 4m + 4n - n^2 - 4 \ \text{ and } \ \mathcal{N} := 4m + 4n - 4$$

which allows, for given $n = n_0$, to find m_0 such that for $m \geq m_0$, one has $\mathcal{N}/\mathcal{M} < 2$. Then for $m \geq m_0$ and $d \geq 2m + n_0$, condition (2.2) is fulfilled and the corresponding SP is realizable with the AP $(0, d-2)$.

3. Proof of Theorems 1 and 2

Proof of Theorem 1. We remind the formulation of a *concatenation lemma* (see [4]):

Lemma 1. *Suppose that the monic polynomials P_1 and P_2 of degrees d_1 and d_2 with SPs $(+,\sigma_1)$ and $(+,\sigma_2)$ respectively realize the pairs (pos_1, neg_1) and (pos_2, neg_2). Here σ_j denote what remains of the SPs when the initial sign $+$ is deleted. Then*

(1) if the last position of σ_1 is $+$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2} P_1(x)P_2(x/\varepsilon)$ realizes the SP $(1,\sigma_1,\sigma_2)$ and the pair $(pos_1 + pos_2, neg_1 + neg_2)$;

(2) if the last position of σ_1 is $-$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2} P_1(x)P_2(x/\varepsilon)$ realizes the SP $(1,\sigma_1,-\sigma_2)$ and the pair $(pos_1+pos_2, neg_1+neg_2)$. Here $-\sigma_2$ is obtained from σ_2 by changing each $+$ by $-$ and vice versa.

For $d = 1$, the SP $(+,+)$ (resp. $(+,-)$) is realizable with the AP $(0,1)$ (resp. $(1,0)$) by the polynomial $x+1$ (resp. $x-1$). Applying Lemma 1 with P_1 and P_2 of the form $x \pm 1$ one realizes for $d = 2$ all the three SPs with $c = 0$ or $c = 1$ with the APs of the form $(0,2)$ or $(1,1)$.

Suppose that for $d = d_0 \geq 2$ all SPs with $c = 0$ or $c = 1$ are realizable by monic polynomials (denoted by P). Then to realize for $d = d_0 + 1$ a SP with $c = 0$ or $c = 1$ with the pair $(0, d_0 + 1)$ or $(1, d_0)$ it suffices to apply Lemma 1 with $P_1 = P$ and with $P_2 = x - 1$ (resp. $P_2 = x + 1$) if $c = 0$ and the last two signs of the SP defined by P are $(+,-)$ (resp. if $c = 1$ and these last two signs are $(-,-)$).

To realize for $c = 1$ a SP with any AP $(1, d-1-2k)$, $k \leq [(d-1)/2]$, it suffices to perturb a polynomial P realizing this SP with the pair $(1, d-1)$ so that all critical levels become distinct and then choose suitable values of $t > 0$ in the family of polynomials $P - t$. □

Proof of Theorem 2. One has to apply $d-1$ times Lemma 1. When it is applied for the first time one sets $P_1 := x - 1$ (resp. $P_1 := x + 1$) if the second entry of the SP is $-$ (resp. $+$). Each time the polynomial P_2 equals $x - 1$ or $x + 1$. □

4. Proof of Theorem 3

Part (1). For $d = 2$ and $d = 3$, the polynomials

$$(x-1)^2 + 1 = x^2 - 2x + 2 \ , \ \ (x+2)((x-2)^2 + 2) = x^3 - 2x^2 - 2x + 12$$

realize the APs $(0,0)$ and $(0,1)$ with the SPs $\Sigma_{1,1,1}$ and $\Sigma_{1,2,1}$ respectively. If a polynomial P realizes a SP $\Sigma_{m,n,q}$ (with $n = 1$ or 2) with the AP $(0, d-2)$, then the concatenation Q of P with $x+1$ realizes the SP $\Sigma_{m,n,q+1}$ with the AP $(0, d-1)$, and the polynomial $Q^R := x^d Q(1/x)$ (the reverted of Q) realizes the SP $\Sigma_{q+1,n,m}$ with the AP $(0, d-1)$. Thus by means of concatenation and reversion one can realize all SPs $\Sigma_{m,1,q}$ and $\Sigma_{m,2,q}$ with the AP $(0, d-2)$.

Part (2). For $d = 4$, the nonrealizability of the SP $\Sigma_{1,3,1}$ with the AP $(0,2)$ is proved in [2]. For $d = 5$, the SP $\Sigma_{1,3,2}$ is realizable with the AP $(0,3)$, see [1]. To prove the first claim of part (2) one has to combine concatenation and reversion as in the proof of part (1) (applied to $\Sigma_{1,3,2}$).

Part (3). For $d = 10$, the polynomial

$$(x+1)^8(x^2-2.49x+1.56) \;=$$

$$x^{10}+5.51x^9+9.64x^8-1.24x^7-25.76x^6-30.94x^5$$

$$-2.24x^4+25.64x^3+24.76x^2+9.99x+1.56$$

defines the SP $\Sigma_{3,4,4}$. (The quadratic factor is without real roots.) One can perturb its 8-fold root at -1 so that the latter splits into 8 negative simple roots. Thus the perturbation realizes this SP with the AP $(0,8)$. Similarly, for $d=11$, a suitable perturbation of the polynomial

$$(x+1)^9(x^2-4.69x+5.5) \;=$$

$$x^{11}+4.31x^{10}-0.71x^9-35.34x^8-69.96x^7-2.94x^6$$

$$+186.06x^5+335.04x^4+302.16x^3+156.79x^2+44.81x+5.5$$

realizes the SP $\Sigma_{2,4,6}$ with the AP $(0,9)$. As in the proof of parts (1) and (2), one deduces the realizability of all SPs as claimed by part (3) by applying concatenation and reversion.

Part (4). Suppose that the SP $\Sigma_{1,n,q}$ with $n\geq 4$ is realizable by the polynomial

$$P(x)=(x^{d-2}+e_1x^{d-3}+\cdots+e_{d-2})(x^2-zx+y)$$

where $d\geq 5$, $z^2<4y$ and $e_j>0$ is the jth elementary symmetric function of the moduli a_j of the negative roots of P. (As $y>0$, the coefficient of x of the quadratic factor must be negative, otherwise all coefficients of P will be positive, so $z>0$). Thus one obtains the conditions

$$\begin{array}{lcl|lcl|lcl|lcl|l}
z^2 & < & 4y & & & & & & & & & & , \\
e_1-z & < & 0 & , & \text{i. e.} & z & > & e_1 & > & 0 & & & , \\
e_2-e_1z+y & < & 0 & , & \text{i. e.} & e_1z & > & y+e_2 & & & & & , \\
e_3-e_2z+e_1y & < & 0 & , & \text{i. e.} & e_2z & > & e_3+e_1y & & & & & \text{and} \\
e_4-e_3z+e_2y & < & 0 & , & \text{i. e.} & e_3z & > & e_4+e_2y & > & 0 & & & .
\end{array}$$

Keeping in mind that $e_j>0$, $y>0$ and $z>0$, one gets

$$z \;>\; (e_4+e_2y)/e_3 \quad \text{and} \quad z \;<\; 2\sqrt{y} \quad , \quad \text{i. e.}$$

$$2e_3\sqrt{y} \;>\; e_4+e_2y \quad \text{and} \quad T(\sqrt{y}) \;:=\; e_2y-2e_3\sqrt{y}+e_4 \;<\; 0 \quad .$$

The quadratic polynomial T has a positive discriminant $e_3^2-e_2e_4$ (this follows from Newton's inequalities). Hence it has two real roots, so the last inequality implies

$$\sqrt{y}<\frac{e_3+\sqrt{e_3^2-e_2e_4}}{e_2}$$

and as $\sqrt{y}>\dfrac{z}{2}>\dfrac{e_1}{2}$, one deduces the condition

$$e_1e_2-2e_3<\sqrt{4e_3^2-4e_2e_4}\;. \tag{4.3}$$

One has $e_1e_2 - 2e_3 > 0$. Indeed, every product $a_ia_ja_k$ is encountered exactly two times in $2e_3$ and three times in e_1e_2 (and there are also the products $a_i^2a_j$ in e_1e_2). Hence one can take squares of both hand-sides of inequality (4.3) and then divide by e_2 to obtain the condition

$$e_1^2e_2 + 4e_4 < 4e_1e_3 \ . \tag{4.4}$$

We are going to show that for $d \geq 3$,

$$e_1^2e_2 + 4e_4 > 4e_1e_3 \tag{4.5}$$

which contradiction proves part (4). For $d = 3$, one has $e_1e_2 \geq 9e_3$ (see Proposition 2 on page 2 of [10]). Suppose that (4.5) holds true up to degree $d \geq 3$. We proceed by induction on d. Recall that we denote by $(-a_j)$ the negative roots of P. For degree $d+1$, we have to show that

$$\begin{array}{l} (a_{d-1} + e_1)^2(a_{d-1}e_1 + e_2) + 4(a_{d-1}e_3 + e_4) \quad > \\ \\ 4(a_{d-1} + e_1)(a_{d-1}e_2 + e_3) \qquad\qquad\qquad\qquad\qquad\quad , \end{array}$$

where e_j are the elementary symmetric functions of the quantities $a_1, \ldots, a_{d-2}$, which is simplified to

$$a_{d-1}^3e_1 + 2a_{d-1}^2e_1^2 + a_{d-1}e_1^3 > 3a_{d-1}^2e_2 + 2a_{d-1}e_1e_2 \ . \tag{4.6}$$

Newton's inequality $e_1^2 \geq \dfrac{2d}{d-1}e_2$ implies the following ones:

$$\begin{array}{rcl} 2a_{d-1}^2e_1^2 & \geq & 4a_{d-1}^2\frac{d}{d-1}e_2 \quad \text{and} \\ a_{d-1}e_1^3 & \geq & \frac{2d}{d-1}a_{d-1}e_1e_2 \end{array} \tag{4.7}$$

From inequalities (4.7) we conclude that

$$\begin{array}{rcl} a_{d-1}^3e_1 + 2a_{d-1}^2e_1^2 + a_{d-1}e_1^3 & > & 4a_{d-1}^2\frac{d}{d-1}e_2 + \frac{2d}{d-1}a_{d-1}e_1e_2 \\ \\ & > & 3a_{d-1}^2e_2 + 2a_{d-1}e_1e_2 \end{array}$$

which proves (4.6) and hence (4.5) as well.

5. Proof of Proposition 1

We give in detail the proof of part (1). For part (2), we point out only the differences w.r.t. the proof of part (1). These differences are only technical in character. In order to give easily references to the different parts of the proof, the latter are marked by 1^0, 2^0, ..., 6^0.

Proof of part (1) of Proposition 1. 1^0. Suppose that there exists a polynomial $P := RQ$, where

$$R := (x + u_1) \cdots (x + u_7) \ , \ u_j > 0 \ , \ \text{and} \ Q := x^2 + rx + s \ ,$$

which realizes one of the two SPs $\Sigma_{3,4,3}$ or $\Sigma_{2,4,4}$ with the AP $(0, 7)$. We set $P := \sum_{j=0}^{9} p_jx^j$ and $Q := (x-a)^2 + b$, $a \in \mathbb{R}$, $b \geq 0$. We show that for $b = 0$, there exists no polynomial satisfying the conditions

$$p_3 < 0 \ , \ p_6 < 0 \ , \ \text{resp.} \ p_4 < 0 \ , \ p_7 < 0 \ . \tag{5.8}$$

Hence this holds true also for $b > 0$ because $P = R \cdot Q|_{b=0} + bR$, and the polynomial R has all coefficients positive. This in turn implies that for $b \geq 0$, there exists no polynomial P realizing the SP $\Sigma_{3,4,3}$ or $\Sigma_{2,4,4}$. So from now on we concentrate on the case $b = 0$.

2^0. Suppose that a polynomial P with $b = 0$ and $u_1 \geq u_2 \geq \cdots \geq u_7 \geq 0$ satisfying the left or right couple of inequalities (5.8) exists. We make the change of variables $x \mapsto u_1 x$ and after this we multiply P by $(1/u_1)^9$ (these changes preserve the signs of the coefficients), so now we are in the case $u_1 = 1$. Denote by $\Delta \subset \mathbb{R}^7_+ = \{(u_2, u_3, \ldots, u_7, a)\}$ the set on which one has conditions (5.8). The closure $\overline{\Delta}$ of this set is compact. Indeed, one has $p_1 \geq 0$ hence

$$1 + u_2 + \cdots + u_7 - 2a \geq 0 \text{ and } u_j \leq 1 \text{ hence } a \in [0, 7/2] \ .$$

The set $\overline{\Delta}$ can be stratified according to the multiplicity vector of the variables $(u_2, \ldots, u_7)$ and the possible equalities $u_j = 0$, $u_i = 1$ and/or $a = 0$. Suppose that the set $\overline{\Delta}$ contains a polynomial satisfying the inequalities (5.8).

Remarks 3. (1) For this polynomial one has $a > 0$, otherwise all its coefficients are nonnegative. One has also $u_j > 0$, $j = 2, \ldots, 7$. Indeed, in the case of $\Sigma_{3,4,3}$ (resp. $\Sigma_{2,4,4}$), if three or more (resp. if four or more) of the variables u_j are 0, then the polynomial P has less than two sign changes in the sequence of its coefficients and by the Descartes rule of signs P cannot have two positive roots counted with multiplicity. For $\Sigma_{3,4,3}$, if exactly one or two of the variables u_j equal 0, then the polynomial P is the product of x with a polynomial defining the SP $\Sigma_{3,4,2}$ or of x^2 with a polynomial defining the SP $\Sigma_{3,4,1}$. However these SPs are not realizable with the APs $(0, 6)$ or $(0, 5)$ respectively, see [5] and [4]. For $\Sigma_{2,4,4}$, if exactly one, two or three of the variables u_j equal 0, then P is the product of x, x^2 or x^3 with a polynomial defining respectively the SP $\Sigma_{2,4,3}$, $\Sigma_{2,4,2}$ or $\Sigma_{2,4,1}$ which is not realizable with the AP $(0, 6)$, $(0, 5)$ or $(0, 4)$, see [5], [4] and [1].

(2) The set $\overline{\Delta}$ being compact the quantity $p_3 + p_6$, resp. $p_4 + p_7$, attains its minimum $-\delta$ on it ($\delta > 0$). Consider the set $\Delta^\bullet \subset \overline{\Delta}$ on which one has $p_3 + p_6 \leq -\delta/2$, resp. $p_4 + p_7 \leq -\delta/2$. On this set one has $a \geq 2^{-9}\delta$. Indeed, $P = x^2 R - 2axR + a^2 R$, so any coefficient of P is not less than $-2a\sigma$, where σ is the sum of all coefficients of R (they are all nonnegative); clearly $\sigma \leq 2^7$ (follows from $u_j \in [0, 1]$).

(3) There exists $\delta_* > 0$ such that on the set $\Delta^\bullet$, one has also $u_j \geq \delta_*$. This follows from part (1) of the present remarks.

3^0. We need some technical lemmas:

Lemma 2. *The minimum of the quantity $p_3 + p_6$, resp. $p_4 + p_7$, is not attained at a point of the set $\Delta^\bullet$ with three or more distinct and distinct from 1 among the quantities u_j, $2 \leq j \leq 7$.*

The lemmas used in the proof of part (1) of Proposition 1 are proved after the proof of part (1).

Lemma 3. *Conditions (5.8) fail for $u_1 = u_2 = \cdots = u_7 = 1$ and any $a > 0$.*

Thus to prove Proposition 1 we have to consider only the case when exactly one or two of the quantities u_j are distinct from 1. We use the following result:

Lemma 4. *For $d \geq 4$, set $P := RQ$, where $R := \prod_{i=1}^{d-2}(x+u_i)$, $u_i > 0$, $Q := (x-a)^2$. Then the coefficients p_j of P, $j = 2, \ldots, d-2$, are quadratic polynomials in a with positive leading coefficients and with two distinct positive roots.*

4^0. Further we consider several different cases according to the multiplicity of u_{j_0}, the smallest of the variables u_j. In the proofs we use linear changes $x \mapsto \chi x$, $\chi > 0$, followed by $P \mapsto \chi^{-9}P$. These changes preserve the signs of the coefficients; the condition $u_1 = 1$ is lost and the condition $u_{j_0} = 1$, $j_0 \neq 1$, is obtained. The aim of this is to have more explicit computations. In all the cases the polynomial R is of the form $R = (x+1)^{s_1}(x+v)^{s_2}(x+w)^{s_3}$, $s_1 + s_2 + s_3 = 7$, and one has $v > 1$, $w > 1$, but v and w are not necessarily distinct and we do not suppose that $v > w$ or $v < w$ (which permits us to assume that $s_2 \geq s_3$). Allowing the equality $v = w$ means treating together cases of exactly two or exactly three distinct quantities u_j (counting also $u_1 = 1$). We list the triples (s_1, s_2, s_3) defining the cases:

$$(5,1,1) \quad , \quad (4,2,1) \quad , \quad (3,3,1) \quad , \quad (3,2,2) \quad , \quad (2,4,1) \quad ,$$

$$(2,3,2) \quad , \quad (1,5,1) \quad , \quad (1,4,2) \qquad \text{and} \qquad (1,3,3) \quad .$$

The cases when there are exactly two different quantities u_j one of which is $u_1 = 1$ can be coded in a similar way. E.g. $(5,2)$ means that $R = (x+1)^5(x+u)^2$, $u > 1$. The nonrealizability of these cases follows automatically from the one of the above 9 ones (when v and w coalesce), with the only exception of $R = (x+1)^6(x+w)$ (the case $(6,1)$).

Lemma 5. *Conditions (5.8) fail in case* $(6,1)$.

5^0. We consider the SP $\Sigma_{2,4,4}$ first. We compute using MAPLE the resultant $\mathrm{Res}\,(p_4, p_7, a)$ as a function of v and w. Then we set $v := 1 + V$, $w := 1 + W$, $V > 0$, $W > 0$. In all 9 cases this resultant is a polynomial in V and W with all coefficients positive. Hence for no value of $V > 0$ and $W > 0$ do the coefficients p_4 and p_7 vanish together.

In all 9 cases, the leading coefficients of p_4 and p_7 considered as quadratic polynomials in a are positive. In fact, they are polynomials in v and w with all coefficients positive. For $v = w = 2$, we compute the two roots $y_1 < y_2$ of p_4 and the two roots $y_3 < y_4$ of p_7. In all 9 cases, one has $y_1 < y_2 < y_3 < y_4$. By continuity, these inequalities hold true for all values of $v > 1$ and $w > 1$. Hence the intervals (y_1, y_2) and (y_3, y_4) on which p_4 and p_7 are negative do not intersect for any $v > 1$, $w > 1$. This proves the proposition in the case of $\Sigma_{2,4,4}$.

6^0. Consider now the SP $\Sigma_{3,4,3}$. Recall that the polynomials $P(x)$ and $x^9P(1/x)$ have one and the same numbers of positive and negative roots. Their roots are mutually reciprocal and they define the same SP. Hence the non-realizability of the case $(5,1,1)$ (resp. $(4,2,1)$, or $(3,3,1)$, or $(3,2,2)$) implies the one of $(1,5,1)$ (resp. $(2,4,1)$ and $(1,4,2)$, or $(1,3,3)$, or $(2,3,2)$).

As in the case of $\Sigma_{2,4,4}$, we express $\mathrm{Res}\,(p_3, p_6, a)$ as a polynomial of v and w, and then of V and W. In cases $(5,1,1)$, $(4,2,1)$ and $(3,2,2)$, this resultant has a single monomial with negative coefficient, this is UV. We give the monomials VW, V^2 and W^2 for these three cases:

$$(5,1,1) \quad -9408VW + 28224V^2 + 28224W^2 \quad ,$$

$$(4,2,1) \quad -18816VW + 47040V^2 + 28224W^2 \quad ,$$

$$(3,2,2) \quad -37632VW + 47040V^2 + 47040W^2 \quad .$$

The discriminants of these quadratic homogeneous polynomials are negative hence they are nonnegative (and positive for $V > 0$, $W > 0$). In the case of $(3,3,1)$, there are exactly two monomials with negative coefficients, namely VW and V^2W. The resultant equals

$$(-28224VW+56448V^2+28224W^2)+V(-42336VW+127008W^2+282240V^2)+\cdots$$

(we skip all other monomials; their coefficients are positive). The two quadratic homogeneous polynomials have negative discriminants, so they are positive for $V > 0$, $W > 0$.

The rest of the reasoning goes by exact analogy with the case of $\Sigma_{2,4,4}$.

□

Proof of Lemma 2. Denote by v_1, v_2 and v_3 three distinct and distinct from 1 of the variables u_j. We prove that one can choose v_1^*, v_2^*, v_3^*, $a^* \in \mathbb{R}$ such that the infinitesimal change $v_j \mapsto v_j + \varepsilon v_j^*$, $j = 1, 2, 3$, $a \mapsto a + \varepsilon a^*$, $\varepsilon > 0$, results in $p_\mu \mapsto p_\mu + \varepsilon p_\mu^* + o(\varepsilon)$, $p_\nu \mapsto p_\nu + \varepsilon p_\nu^* + o(\varepsilon)$, where $(\mu, \nu) = (3,6)$ or $(4,7)$ and $p_\mu^* < 0$, $p_\nu^* < 0$. Hence locally the quantity $p_\mu + p_\nu$ is not minimal.

Set $P := (x+v_1)^{\alpha_1}(x+v_2)^{\alpha_2}(x+v_3)^{\alpha_3}(x-a)^2P^\dagger$, where a, $-v_1$, $-v_2$ and $-v_3$ are not roots of $P^\dagger$ and α_j are the multiplicities of throots $-v_j$ of P. Set $P_{v_j} := P/(x+v_j)$, $P_a := P/(x-a)$, $P_{v_i,v_j} := P/((x+v_i)(x+v_j))$, $P_{a,v_j} := P/((x+a)(x+v_j))$ etc. Then the above infinitesimal change transforms P into

$$P + \varepsilon\tilde{P} + o(\varepsilon) \ , \ \text{ where } \ \tilde{P} := \sum_{j=1}^{3} \alpha_j v_j^* P_{v_j} - 2a^* P_a \ .$$

We show that one can choose v_j^* and a^* such that the coefficients of x^μ and x^ν of the polynomial $\tilde{P}$ (where $(\mu,\nu) = (3,6)$ or $(4,7)$) are both negative from which the lemma follows. To this end we observe that each of the polynomials P_{v_j} and P_a is a linear combination of $P^\diamond := P_{v_1,v_2,v_3,a} := x^5 + Ax^4 + Bx^3 + Cx^2 + Dx + E$, $xP^\diamond$, $x^2P^\diamond$ and $x^3P_{v,w,a}$.

We consider first the case of $\Sigma_{3,4,3}$, i.e $(\mu,\nu) = (3,6)$. The 2-vectors of coefficients of x^3 and x^6 of the polynomials $P^\diamond$, $xP^\diamond$, $x^2P^\diamond$ and $x^3P^\diamond$ equal $(B,0)$, $(C,1)$, (D,A) and (E,B) respectively. For $B \neq 0$, the first two of them are not collinear. As $E \neq 0$ (see parts (2) and (3) of Remarks 3), for $B = 0$, the second and fourth of these vectors are not collinear and the choice of v_j^* and a^* is possible.

If $(\mu,\nu) = (4,7)$, then the 2-vectors of coefficients of x^4 and x^7 equal $(A,0)$, $(B,0)$, $(C,1)$ and (D,A). One has either $A \neq 0$ or $B \neq 0$. Indeed, the polynomial $P^\diamond$ has all roots real and by Rolle's theorem this is the case of $(P^\diamond)'$ and $(P^\diamond)''$ as well. If $A = B = C = D = 0 \neq E$ (resp. $A = B = C = 0 \neq D$ or $A = B = 0 \neq C$), then $P^\diamond$ (resp. $(P^\diamond)'$ or $(P^\diamond)''$) has not all roots real. Thus either $(A,0)$, $(C,1)$ or $(B,0)$, $(C,1)$ are not collinear and the choice of v_j^* and a^* is possible.

□

Proof of Lemma 3. For the polynomial $(x+1)^7(x-a)^2$, we list its coefficients p_3, p_4, p_6 and p_7 and their roots:

$$\begin{array}{rclcrcl} p_3 & = & 7-42a+35a^2 & , & p_4 & = & 21-70a+35a^2 \quad , \\ 0.2 & , & 1 & & 0.36\ldots & , & 1.63\ldots \\ p_6 & = & 35-42a+7a^2 & , & p_7 & = & 21-14a+a^2 \quad . \\ 1 & , & 5 & & 1.70\ldots & , & 12.2\ldots \end{array}$$

Hence for no value of $a \geq 0$ does one have the left or the right two of conditions (5.8) together.

□

Proof of Lemma 4. Set $R := r_{d-2}x^{d-2} + r_{d-3}x^{d-3} + \cdots + r_0$, $r_j > 0$, $r_{d-2} = 1$. The polynomial R has $d-2$ negative roots. Hence Newton's inequalities hold true:

$$(5.9)\quad \left(r_k/\binom{d-2}{k}\right)^2 \geq \left(r_{k-1}/\binom{d-2}{k-1}\right)\left(r_{k+1}/\binom{d-2}{k+1}\right)\ ,\ k=1,\ldots,d-3\ .$$

The coefficient p_{k+1} equals $a^2r_{k+1} - 2ar_k + r_{k-1}$, $k = 1, \ldots, d-3$, $r_{k+1} > 0$. This quadratic polynomial has two distinct positive roots if and only if $r_k^2 > r_{k-1}r_{k+1}$. These inequalities result from (5.9) because $\binom{d-2}{k}^2 > \binom{d-2}{k-1}\binom{d-2}{k+1}$ (the latter inequality is equivalent to $((k+1)/k)((d-1-k)/(d-2-k)) > 1$ which is true).

□

Proof of Lemma 5. In case (6,1), with $P = (x+1)^6(x+w)(x-a)^2$, one has

$$\begin{array}{rcll} p_3 & = & 1+6w-12a-30wa+15a^2+20wa^2 & , \\ p_4 & = & 6+15w-30a-40wa+20a^2+15wa^2 & , \\ p_6 & = & 20+15w-30a-12wa+6a^2+wa^2 & \text{and} \\ p_7 & = & 15+6w-12a-2wa+a^2 & . \end{array}$$

For $w = 1$, the roots of p_4 (resp. of p_7) equal $0.36\ldots$ and $1.63\ldots$ (resp. $1.70\ldots$ and $12.29\ldots$). As $\mathrm{Res}\,(p_4, p_7, a) = 7056 + 2520w + 540w^2 + 3960w^3 + 1800w^4$ has no positive roots, for any $w > 0$ fixed, the two intervals of values of a, for which $p_4 < 0$ or $p_7 < 0$, do not intersect. Hence the couple of conditions $p_4 < 0$, $p_7 < 0$ fails.

One has $\mathrm{Res}\,(p_3, p_6, a) = 7056(w-1)^2(w+1)^2$, so only for $w = 1$ do the polynomials p_3 and p_6 have a root in common. For $w = 1/2$, $w = 1$ and $w = 2$, the roots of p_3 and p_6 equal respectively

$$\begin{array}{lllcllcll} w=1/2 & 0.17\ldots & , & 0.90\ldots & \text{and} & 0.91\ldots & , & 4.62\ldots & ; \\ w=1 & 0.2 & , & 1 & \text{and} & 1 & , & 5 & ; \\ w=2 & 0.21\ldots & , & 1.09\ldots & \text{and} & 1.10\ldots & , & 5.64\ldots & . \end{array}$$

Hence again the intervals of values of a for which $p_3 < 0$ or $p_6 < 0$ do not intersect and the couple of conditions $p_3 < 0$, $p_6 < 0$ fails.

□

Proof of part (2) of Proposition 1. 1^0. In the analog of part 1^0 of the proof of part (1), we set $R := (x+u_1)\cdots(x+u_8)$, $u_j > 0$, and the analog of inequalities (5.8) reads $p_5 < 0$, $p_8 < 0$.

2^0. In the analog of part 2^0 we make the change of variables $x \mapsto u_1 x$ and then we multiply P by $(1/u_1)^{10}$. We denote by $\Delta \subset \mathbb{R}^8_+ = \{(u_2, u_3, \dots, u_8, a)\}$ the set on which one has the conditions $p_5 < 0$, $p_8 < 0$. On the closure $\overline{\Delta}$ of this set one has $p_1 \geq 0$ hence

$$1 + u_2 + \cdots + u_8 - 2a \geq 0 \text{ and } u_j \leq 1 \text{ hence } a \in [0,4] \ .$$

The analog of Remarks 3 reads:

Remarks 4. (1) One has $u_j > 0$, $j = 2$, ..., 8. Indeed, if exactly one of the quantities u_j is 0, then $P = xY$, where the polynomial Y defines the SP $\Sigma_{2,4,4}$ which by part (1) of Proposition 1 is impossible. If more than one of the quantities u_j is 0, then see part (1) of Remarks 3 about $\Sigma_{2,4,4}$.

(2) In the proof of part (2) of Proposition 1 we define the set $\Delta^\bullet \subset \overline{\Delta}$ as the one on which one has $p_5 + p_8 \leq -\delta/2$. On this set one has $a \geq 2^{-10}\delta$. Indeed, as $P = x^2R - 2axR + a^2R$, any coefficient of P is not less than $-2a\sigma$, where σ is the sum of all coefficients of R (they are all nonnegative); clearly $\sigma \leq 2^8$ (follows from $u_j \in [0,1]$).

3^0. The analog of Lemma 2 reads: *The minimum of the quantity $p_5 + p_8$ is not attained at a point of the set $\Delta^\bullet$ with three or more distinct and distinct from* 1 *among the quantities u_j, $2 \leq j \leq 8$.*

The proof is much the same as the one of Lemma 2. One sets $(\mu, \nu) = (5, 8)$. Each of the polynomials P_{v_j} and P_a is a linear combination of $P^\diamond := P_{v_1,v_2,v_3,a} := x^6 + Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex + F$, $xP^\diamond$, $x^2P^\diamond$ and $x^3P_{v,w,a}$. The 2-vectors of coefficients of x^5 and x^8 of the polynomials $P^\diamond$, $xP^\diamond$, $x^2P^\diamond$ and $x^3P^\diamond$ equal $(A,0)$, $(B,0)$, $(C,1)$ and (D,A) respectively. If $A \neq 0$ or $B \neq 0$, there are two noncollinear among the first three of these vectors and the choice of v_j^* and a^* is possible. If $A = B = 0$, then, as $F \neq 0$, either the polynomial $P^\diamond$ or one of its derivatives is not with all roots real which is a contradiction.

The analog of Lemma 3 reads: *Conditions $p_5 < 0$, $p_8 < 0$ fail for $u_1 = \cdots = u_8 = 1$ and any $a > 0$.*

Here's the proof of this. For the polynomial $(x+1)^8(x-a)^2$, we list its coefficients p_5, p_8 and their roots:

$$p_5 \quad = \quad 28(2 - 5a + 2a^2) \quad , \quad p_8 \quad = \quad 28 - 16a + a^2 \quad ,$$

$$0.5 \quad , \quad 2 \qquad\qquad\qquad\qquad 2 \quad , \quad 14$$

Hence for no value of $a \geq 0$ does one have $p_5 < 0$, $p_8 < 0$.

We remind that Lemma 4 is formulated for any $d \geq 4$.

4^0. In the analog of part 4^0 of the proof, one has $R = (x+1)^{s_1}(x+v)^{s_2}(x+w)^{s_3}$, $s_1+s_2+s_3 = 8$, and one has to consider the following cases of exactly three different quantities u_j:

$$(6,1,1) \quad , \quad (5,2,1) \quad , \quad (4,3,1) \quad , \quad (4,2,2) \quad , \quad (3,4,1) \quad , \quad (3,3,2) \quad ,$$

$$(2,5,1) \quad , \quad (2,4,2) \quad , \quad (2,3,3) \quad , \quad (1,6,1) \quad , \quad (1,5,2) \text{ and } (1,4,3) \quad .$$

The cases with exactly two different quantities u_j are treated in the same way. The exceptional case is the one with $R = (x+1)^7(x+w)$ (the case $(7,1)$).

Lemma 6. *The conditions $p_5 < 0$, $p_8 < 0$ fail in case $(7,1)$.*

Proof. Set $P := (x+1)^7(x+w)(x-a)^2$. Then

$$p_5 = 21 + 35w - 70a - 70wa + 35a^2 + 21wa^2 \text{ and } p_8 = 21 + 7w - 14a - 2wa + a^2 .$$

One has $\mathrm{Res}\,(p_5, p_8, a) = 3969(w+2)^2(w-1)^2$. We list the roots of p_5 and p_8 for $w = 1/2$, $w = 1$ and $w = 2$:

$$\begin{array}{lllllllll} w = 1/2 & 0.45\ldots & , & 1.850\ldots & \text{and} & 1.865\ldots & , & 13.13\ldots & ; \\ w = 1 & 0.5 & , & 2 & \text{and} & 2 & , & 14 & ; \\ w = 2 & 0.54\ldots & , & 2.18\ldots & \text{and} & 2.21\ldots & , & 15.78\ldots & . \end{array}$$

As in the proof of Lemma 5, we conclude that the conditions $p_5 < 0$, $p_8 < 0$ fail for $w > 0$. □

5^0. We compute Res (p_5, p_8, a) as a function of v and w and then set $v := 1+V$, $w := 1+W$. Our aim is to show that in all 12 cases, the leading coefficients of p_5 and p_8 considered as quadratic polynomials in a are positive. The rest of the reasoning is done by analogy with part 5^0 of the proof of part (1) of Proposition 1.

6^0. It is in the analog of 6^0 that there is much more technical work to be done. Of the twelve cases listed in 4^0, in three there is a single monomial with a negative coefficient, and this is UV. We list the coefficients of the monomials UV, U^2 and V^2 of the cases $(6,1,1)$, $(5,2,1)$ and $(4,2,2)$ respectively:

$$(-10206, 35721, 35721)\ ,\ (-20412, 61236, 35721)\ ,\ (-40824, 61236, 61236)\ .$$

Everywhere in 6^0 quadratic and biquadratic polynomials have negative discriminants. There are four cases in which exactly two monomials have negative signs, namely $(4,3,1)$, $(3,4,1)$, $(3,3,2)$ and $(2,4,2)$ in which we give only the monomials forming quadratic homogeneous polynomials with negative discriminants (multiplied by 1 or U); we skip all other monomials (their coefficients are positive):

$$(-30618UV + 76545U^2 + 35721V^2) + U(-10206UV + 221130U^2 + 91854V^2)$$

$$(-40824UV + 81648U^2 + 35721V^2) + U(-81648UV + 326592U^2 + 122472V^2)$$

$$(-61236UV + 76545U^2 + 61236V^2) + U(-20412UV + 221130U^2 + 81648V^2)$$

$$(-81648UV + 81648U^2 + 61236V^2) + U(-163296UV + 326592U^2 + 108864V^2)$$

In the cases $(2,5,1)$ and $(1,6,1)$ there are four and five negative monomials respectively. These cases are treated in a similar way:

$$(-51030UV + 76545U^2 + 35721V^2) + U(-187110UV + 391230U^2 + 153090V^2)$$

$$+U^2(-245430UV + 868725U^2 + 297270V^2)$$

$$+U^3(-86670UV + 1094472U^2 + 352350V^2)$$

and

$$(-6804UV + 6804U^2 + 3969V^2) + U(-22680UV + 28728U^2 + 12474V^2)$$

$$+U^2(-29052UV + 50436U^2 + 15849V^2) + U^4(-3252UV + 24628U^2 + 3672V^2)$$

$$+U^3(-16848UV + 47088U^2 + 10368V^2)\ .$$

In the case $(2, 3, 3)$, there are four negative monomials which we include in polynomials as follows:

$$(-91854UV + 76545U^2 + 76545V^2) + (-59778U^2V^2 + 273375U^4 + 273375V^4)$$

$$(-30618U^2V + 221130U^3 + 221130V^3 - 30618UV^2)\ .$$

For the third polynomial in brackets its corresponding inhomogeneous polynomial

$$-30618x^2 + 221130x^3 + 221130 - 30618x$$

has one negative and two complex conjugate roots. For a univariate real polynomial with positive leading coefficient and having only negative and complex conjugate roots we say that it is of *type P*. It is clear that the homogeneous polynomial corresponding to a type P univariate polynomial (we say that it is also of type P) is nonnegative.

In the case $(1, 5, 2)$, there are seven negative monomials:

$$(-102060UV + 76545U^2 + 61236V^2) + U(-374220UV + 391230U^2 + 136080V^2)$$

$$+(-490860U^3V + 868725U^4 + 10530U^2V^2 + 369360UV^3 + 79704V^4)$$

$$+U^2(513540V^3 - 210600UV^2 - 173340U^2V + 1094472U^3)$$

$$+U^2(-215190U^2V^2 + 855450U^4 + 372915V^4)$$

$$+U^4V(-64116UV + 300060U^2 + 176760V^2)\ .$$

The third and fourth of the polynomials in brackets are of type P hence nonnegative.

Finally, in the case $(1, 4, 3)$ we have also seven negative monomials:

$$(-122472UV + 81648U^2 + 76545V^2) + (-383940U^2V^2 + 565056U^4 + 273375V^4)$$

$$+(326592U^3 - 244944U^2V - 40824UV^2 + 221130V^3)+$$

$$+U(-359649U^2V^2 + 552096U^4 + 557928V^4)$$

$$+(-75816U^3V^3 + 332928U^6 + 79065V^6)$$

$$+U(-15066U^3V^3 + 126720U^6 + 138096V^6)\ .$$

The third and the last two polynomials in brackets are of type P. □

6. Proof of Theorem 4

Consider the polynomial $P = (x+1)^{d-2}(x^2 - zx + y)$, where the quadratic factor has no real roots, i.e. $z^2 < 4y$. Hence $y > 0$ and $z > 0$ (otherwise all coefficients of P must be positive). If the polynomial P defines the SP $\Sigma_{m,n,q}$, then any perturbation of P with $d-2$ distinct negative roots close to -1 defines also the SP $\Sigma_{m,n,q}$. We expand P in powers of x:

$$P := x^d + \Sigma_{j=1}^{d} p_j x^{d-j} \ ,$$

where $p_j = C_{d-2}^{j} - C_{d-2}^{j-1} z + C_{d-2}^{j-2} y$ with $C_\mu^\nu = 0$ if $\mu < \nu$. The coefficients of P define the SP $\Sigma_{m,n,q}$, so

$$p_j > 0 \ \text{ for } \ j = 1, \dots, m-1 \ \text{ and for } \ j = m+n, \dots, d \ , \ \text{ and}$$

$$p_j < 0 \ \text{ for } \ j = m, m+1, \dots, m+n-1 \ .$$

The latter inequalities (combined with $z < 2\sqrt{y}$) yield:

$$C_{d-2}^{j} + C_{d-2}^{j-2} y < C_{d-2}^{j-1} z < 2 C_{d-2}^{j-1} \sqrt{y}, \quad j = m, m+1, \dots, m+n-1 \ .$$

This means that :

$$\frac{C_{d-2}^{j-1} - \sqrt{\delta_{j-1}}}{C_{d-2}^{j}} < \sqrt{y} < \frac{C_{d-2}^{j-1} + \sqrt{\delta_{j-1}}}{C_{d-2}^{j}}, \tag{6.10}$$

where $\delta_{j-1} := (C_{d-2}^{j-1})^2 - C_{d-2}^{j-2} C_{d-2}^{j} > 0$. Indeed, the polynomial $C_{d-2}^{j} - 2C_{d-2}^{j-1}\sqrt{y} + C_{d-2}^{j-2} y$ (quadratic in $\sqrt{y}$) has a positive discriminant δ_{j-1} hence its value is negative precisely when $\sqrt{y}$ is between its roots. Set

$$Q^{\pm}(k) := (C_{d-2}^{k} \pm \sqrt{\delta_k}) / C_{d-2}^{k-1} \ . \tag{6.11}$$

Lemma 7. *One has* $Q^{\pm}(k) = \dfrac{d-k-1}{k}(1 \pm A(k))$, *where*

$$A(k) = \sqrt{1 - \frac{k(d-k-2)}{(k+1)(d-k-1)}} = \sqrt{\frac{(d-1)}{(k+1)(d-k-1)}} \ .$$

Lemma 8. *The quantities* $Q^{\pm}(k)$ *are decreasing functions in* k *(for* $k = 1, 2, \dots, [\frac{d}{2}]$*).*

Lemmas 8 and 7 are proved after the proof of Theorem 4. It follows from Lemma 8 that one can find a value of y satisfying conditions (6.10) if

$$Q^{-}(m-1) < Q^{+}(m+n-2) \tag{6.12}$$

or equivalently

$$a - f < aB + fG \ , \tag{6.13}$$

where

$$a = \frac{d-m}{m-1} > 0 \qquad B = \sqrt{1 - \frac{(m-1)(d-m-1)}{m(d-m)}}$$

$$f = \frac{d-m-n+1}{m+n-2} > 0 \qquad G = \sqrt{1 - \frac{(m+n-2)(d-m-n)}{(m+n-1)(d-m-n+1)}}$$

One has

$$a - f = \frac{(n-1)(d-1)}{(m+n-2)(m-1)} > 0$$

which permits to take squares in (6.13) to obtain the condition :

$$H := \frac{(a-f)^2 - (aB)^2 - (fG)^2}{2af} < GB \tag{6.14}$$

which is equivalent to (6.12). If $H < 0$, then (6.14) is trivially true. If $H \geq 0$, then (6.14) is equivalent to $H^2 < (GB)^2$, i.e. to (2.2) (the latter equivalence can be proved using MAPLE). Theorem 4 is proved.

Proof of Lemma 7.

$$\begin{aligned}
\delta_k &= (C^k_{d-2})^2 - C^{k-1}_{d-2} C^{k+1}_{d-2} \\
&= \left(\frac{(d-2)\dots(d-k-1)}{k!}\right)^2 - \frac{(d-2)\dots(d-k)}{(k-1)!} \cdot \frac{(d-2)\dots(d-k-2)}{(k+1)!} \\
&= \left(\frac{(d-2)\dots(d-k)(d-k-1)(k+1)}{(k+1)!}\right)^2 \\
&\qquad - \frac{(d-2)\dots(d-k)k(k+1)(d-2)\dots(d-k-2)}{((k+1)!)^2} \\
&= \left(\frac{(d-2)\dots(d-k)}{(k+1)!}\right)^2 \{(d-k-1)^2(k+1)^2 \\
&\qquad - k(k+1)(d-k-1)(d-k-2)\} \\
&= \left(\frac{(d-2)!}{(d-k-1)!(k+1)!}\right)^2 (k+1)(d-1)(d-k-1) \ .
\end{aligned}$$

We substitute this expression of δ_k in (6.11) to obtain

$$\begin{aligned}
Q^{\pm}(k) &= \frac{(d-k-1)!(k-1)!}{(d-2)!} \cdot \left(\frac{(d-2)!}{(d-k-2)!k!} \pm \sqrt{\delta_k}\right) \\
&= \frac{(d-k-1)}{k} \pm \frac{1}{k(k+1)} \sqrt{(k+1)(d-1)(d-k-1)} \\
&= \frac{(d-k-1)}{k} \pm \sqrt{\frac{(d-k-1)^2}{k^2} - \frac{(d-k-1)(d-k-2)}{k(k+1)}} \\
&= \frac{(d-k-1)}{k} \left(1 \pm \sqrt{1 - \frac{k(d-k-2)}{(k+1)(d-k-1)}}\right) \ .
\end{aligned}$$

□

Proof of Lemma 8. Both factors $\dfrac{d-k-1}{k}$ and $1+A(k)$ of $Q^+(k)$ are decreasing in k (for $k=1,2,\ldots,[\frac{d}{2}]$) hence $Q^+(k)$ is also decreasing. We represent the quantity $Q^-(k)$ in the form

$$Q^-(k)=\left(\frac{d-k-2}{k+1}\right)/\left(1+\sqrt{\frac{d-1}{(k+1)(d-k-1)}}\right)$$

The inequality $Q^-(k)>Q^-(k+1)$ is equivalent to

$$\tfrac{d-k-2}{k+1}+\tfrac{d-k-2}{k+1}\sqrt{\tfrac{d-1}{(k+2)(d-k-2)}} \quad > \quad \tfrac{d-k-3}{k+2}+\tfrac{d-k-3}{k+2}\sqrt{\tfrac{d-1}{(k+1)(d-k-1)}}$$

This follows from $\dfrac{d-k-2}{k+1}>\dfrac{d-k-3}{k+2}$, $\dfrac{1}{(k+1)\sqrt{k+2}}>\dfrac{1}{(k+2)\sqrt{k+1}}$ and $(d-k-2)(d-k-1)>(d-k-3)^2$. □

References

[1] A. Albouy, Y. Fu: Some remarks about Descartes' rule of signs. Elem. Math., 69 (2014), 186–194. Zbl 1342.12002, MR3272179

[2] D. J. Grabiner: Descartes Rule of Signs: Another Construction. Am. Math. Mon. 106 (1999), 854–856. Zbl 0980.12001, MR1732666

[3] H. Cheriha, Y. Gati and V. P. Kostov: A nonrealization theorem in the context of Descartes' rule of signs, St. Kliment Ohridski - Faculty of Mathematics and Informatics (accepted).

[4] J. Forsgård, B. Shapiro and V. P. Kostov: Could René Descartes have known this? Exp. Math. 24 (4) (2015), 438-448. Zbl 1326.26027, MR3383475

[5] V. P. Kostov: On realizability of sign patterns by real polynomials, Czechoslovak Math. J. 68 (143) (2018), no. 3, 853874.

[6] V. P. Kostov: Polynomials, sign patterns and Descartes' rule of signs, Mathematica Bohemica 144 (2019), No. 1, 39-67.

[7] V. P. Kostov: Descartes' rule of signs and moduli of roots, Publicationes Mathematicae Debrecen 96, 1-2 (2020), 12.

[8] V. P. Kostov: Moduli of roots of hyperbolic polynomials and Descartes' rule of signs, arXiv:1909.05532.

[9] V. P. Kostov: Topics on hyperbolic polynomials in one variable. Panor. Synth. 33 (2011), vi + 141 p. SMF.

[10] T. P. Mitev: New inequalities between elementary symmetric polynomials, J. of Inequalities in Pure and Appl. Math. 4 (2) (2003), 11 p.

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CHAPTER 2

Non-realizability with both components of the admissible pairs non-zero

A NONREALIZATION THEOREM IN THE CONTEXT OF DESCARTES' RULE OF SIGNS

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Abstract. For a real degree d polynomial P with all nonvanishing coefficients, with c sign changes and p sign preservations in the sequence of its coefficients ($c + p = d$), Descartes' rule of signs says that P has $pos \leq c$ positive and $neg \leq p$ negative roots, where $pos \equiv c(\mod 2)$ and $neg \equiv p(\mod 2)$. For $1 \leq d \leq 3$, for every possible choice of the sequence of signs of coefficients of P (called sign pattern) and for every pair (pos, neg) satisfying these conditions there exists a polynomial P with exactly pos positive and neg negative roots (all of them simple); that is, all these cases are realizable. This is not true for $d \geq 4$, yet for $4 \leq d \leq 8$ (for these degrees the exhaustive answer to the question of realizability is known) in all nonrealizable cases either $pos = 0$ or $neg = 0$. It was conjectured that this is the case for any $d \geq 4$. For $d = 9$, we show a counterexample to this conjecture: for the sign pattern $(+,-,-,-,-,+,+,+,+,-)$ and the pair $(1,6)$ there exists no polynomial with 1 positive, 6 negative simple roots and a complex conjugate pair and, up to equivalence, this is the only case for $d = 9$.

1. Introduction

In his work La Géométrie published in 1637, René Descartes (1596-1650) announces his classical rule of signs which says that for the real polynomial $P(x, a) := x^d + a_{d-1}x^{d-1} + \cdots + a_0$, the number c of sign changes in the sequence of its coefficients serves as an upper bound for the number of its positive roots. When roots are counted with multiplicity, then the number of positive roots has the same parity as c. One can apply these results to the polynomial $P(-x)$ to obtain an upper bound on the number of negative roots of P. For a given c, one can find polynomials P with c sign changes with exactly c, $c-2$, $c-4$, ... positive roots. One should observe that by doing so one does not impose any restrictions on the number of negative roots.

Remark 1. It is mentioned in [1] that 18th century authors used to count roots with multiplicity while omitting the parity conclusion; later this conclusion was attributed (see [3]) to a paper of Gauss of 1828 (see [7]), although it is absent there, but was published by Fourier in 1820 (see p. 294 in [6]).

In the present paper we consider polynomials P without zero coefficients. We denote by p the number of sign preservations in the sequence of coefficients of P, and by pos_P (resp. neg_P) the number of positive and negative roots of P. Thus the following condition must be fulfilled:

$$pos_P \leq c\ ,\ pos_P \equiv c\,(\mathrm{mod}\ 2)\ ,\ neg_P \leq p\ ,\ neg_P \equiv p\,(\mathrm{mod}\ 2)\ . \tag{1.1}$$

Definition 1. A *sign pattern* is a finite sequence σ of $(\pm)$-signs; we assume that the leading sign of σ is $+$. For a given sign pattern of length $d+1$ with c sign changes and p sign preservations, we call (c,p) its *Descartes pair*, $c+p=d$. For a given sign pattern σ with Descartes pair (c,p), we call (pos, neg) an *admissible pair* for σ if conditions (1.1), with $pos_P = pos$ and $neg_P = neg$, are satisfied.

It is natural to ask the following question: *Given a sign pattern σ of length $d+1$ and an admissible pair (pos, neg) can one find a degree d real monic polynomial the signs of whose coefficients define the sign pattern σ and which has exactly pos simple positive and exactly neg simple negative roots ?* When the answer to the question is positive we say that the couple $(\sigma, (pos, neg))$ is *realizable*.

For $d = 1$, 2 and 3, the answer to this question is positive, but for $d = 4$ D. J. Grabiner showed that this is not the case, see [8]. Namely, for the sign pattern $\sigma^* := (+,+,-,+,+)$ (with Descartes pair $(2,2)$), the pair $(2,0)$ is admissible, see (1.1), but the couple $(\sigma^*, (2,0))$ is not realizable. Indeed, for a monic polynomial $P_4 := x^4 + a_3x^3 + \cdots + a_0$ with signs of the coefficients defined by σ^* and having exactly two positive roots $u < v$ one has $a_j > 0$ for $j \neq 2$, $a_2 < 0$ and $P_4((u+v)/2) < 0$. Hence $P_4(-(u+v)/2) < 0$ because $a_j((u+v)/2)^j = a_j(-(u+v)/2)^j$, $j = 0$, 2, 4 and $0 < a_j((u+v)/2)^j = -a_j(-(u+v)/2)^j$, $j = 1$, 3. As $P_4(0) = a_0 > 0$, there are two negative roots $\xi < -(u+v)/2 < \eta$ as well.

Definition 2. We define *the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action* on couples of the form (sign pattern, admissible pair) by its two generators. Denote by $\sigma(j)$ the jth component of the sign pattern σ. The first of the generators replaces the sign pattern σ by σ^r, where σ^r stands for the *reverted* (i.e. read from the back) sign pattern multiplied by $\sigma(1)$, and keeps the same pair (pos, neg). This generator corresponds to the fact that the polynomials $P(x)$ and $x^d P(1/x)/P(0)$ are both monic and have the same numbers of positive and negative roots. The second generator exchanges *pos* with *neg* and changes the signs of σ corresponding to the monomials of odd (resp. even) powers if d is even (resp. odd); the rest of the signs are preserved. We denote the new sign pattern by σ_m. This generator corresponds to the fact that the roots of the polynomials (both monic) $P(x)$ and $(-1)^d P(-x)$ are mutually opposite, and if σ is the sign pattern of P, then σ_m is the one of $(-1)^d P(-x)$.

Remark 2. For a given sign pattern σ and an admissible pair (pos, neg), the couples $(\sigma, (pos, neg))$, $(\sigma^r, (pos, neg))$, $(\sigma_m, (neg, pos))$ and $((\sigma_m)^r, (neg, pos))$ are simultaneously realizable or not. One has $(\sigma_m)^r = (\sigma^r)_m$.

Modulo the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action Grabiner's example is the only nonrealisable couple (sign pattern, admissible pair) for $d = 4$. All cases of couples (sign pattern, admissible pair) for $d = 5$ and 6 which are not realizable are described in [1]. For $d = 7$, this is done in [5] and for $d = 8$ in [5] and [11]. For $d = 5$, there is a single nonrealizable case (up to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action). The sign pattern is $(+,+,-,+,-,-,)$ and the admissible pair is $(3,0)$. For $n = 6$, 7 and 8 there are respectively 4, 6, and 19 nonrealizable cases. In all of them one of the numbers *pos* or *neg* is 0. In the present paper we show that for $d = 9$ this is not so.

Notation 1. For $d = 9$, we denote by σ^0 the following sign pattern (we give on the first and third lines below respectively the sign patterns σ^0 and σ^0_m while the line in the middle indicates the positions of the monomials of odd powers):

$$\begin{array}{rcccccccccccc} \sigma^0 & = & (& + & - & - & - & - & + & + & + & + & - &) \\ & & & & 9 & & 7 & & 5 & & 3 & & 1 & & \\ \sigma^0_m & = & (& + & + & - & + & - & - & + & - & + & + &) \end{array}$$

In a sense σ^0 is centre-antisymmetric – it consists of one plus, five minuses, five pluses and one minus.

Theorem 1. *(1) The sign pattern σ^0 is not realizable with the admissible pair $(1, 6)$.*

(2) Modulo the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, for $d \leq 9$, this is the only nonrealizable couple (sign pattern, admissible pair) in which both components of the admissible pair are nonzero.

Remark 3. It is shown in [10] that for $d = 11$, the admissible pair $(1, 8)$ is not realizable with the sign pattern (+ - - - - - + + + + + -). Hence Theorem 1 shows an example of a nonrealisable couple, with both components of the admissible pair different from zero, in the least possible degree (namely, 9).

Section 2 contains comments concerning the above result and realizability of sign patterns and admissible pairs in general. Section 3 contains some technical lemmas which allow to simplify the proof of Theorem 1. The plan of the proof of part (1) of Theorem 1 is explained in Section 4. The proof results from several lemmas whose proofs can be found in Section 5. The proof of part (2) of Theorem 1 is given in Section 8.

2. Comments

It seems that the problem to classify, for any degree d, all couples (sign pattern, admissible pair) which are not realizable, is quite difficult. This is confirmed by Theorem 1. For the moment, only certain sufficient conditions for realizability or nonrealizability have been formulated:

- in [5] and [13] series of nonrealizable cases were found, for $d \geq 4$, even and for $d \geq 5$, odd respectively;
- in [5] sufficient conditions are given for the nonrealizability of sign patterns with exactly two sign changes.
- in [4] sufficient conditions are given for realizability the nonrealizability of sign patterns with exactly two sign changes.

Remark 4. For $d \leq 8$, all couples (sign pattern, admissible pairs) with $pos \geq 1$, $neg \geq 1$, are realizable. That is, in the examples of nonrealizability given in [5] and [13] one has either $pos = 0$ or $neg = 0$, so the question to construct an example of nonrealizability with $pos \neq 0 \neq neg$ was a challenging one

The result in [5] about sign patterns with exactly two sign changes, consisting of m pluses followed by n minuses followed by q pluses, with $m + n + q = d + 1$, is formulated in terms of the following quantity:

$$\kappa := \frac{d-m-1}{m} \cdot \frac{d-q-1}{q} \ .$$

Lemma 1. *For $\kappa \geq 4$, such a sign pattern is not realizable with the admissible pair $(0, d-2)$. The sign pattern is realizable with any admissible pair of the form $(2, v)$.*

Lemma 1 coincides with Proposition 6 of [5]. One can construct new realizable cases with the help of the following concatenation lemma (see its proof in [5]):

Lemma 2. *Suppose that the monic polynomials P_j of degrees d_j and with sign patterns of the form $(+, \sigma_j)$, $j = 1, 2$ (where σ_j contains the last d_j components of the corresponding sign pattern) realize the pairs (pos_j, neg_j). Then*

(1) if the last position of σ_1 is $+$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2} P_1(x)P_2(x/\varepsilon)$ realizes the sign pattern $(+, \sigma_1, \sigma_2)$ and the pair $(pos_1+pos_2, neg_1+neg_2)$;

(2) if the last position of σ_1 is $-$, then for any $\varepsilon > 0$ small enough, the polynomial $\varepsilon^{d_2} P_1(x)P_2(x/\varepsilon)$ realizes the sign pattern $(+, \sigma_1, -\sigma_2)$ and the pair $(pos_1 + pos_2, neg_1 + neg_2)$ (here $-\sigma_2$ is obtained from σ_2 by changing each $+$ by $-$ and vice versa).

Remark 5. If Lemma 2 were applicable to the case treated in Theorem 1, then this case would be realizable and Theorem 1 would be false. We show here that Lemma 2 is indeed inapplicable. It suffices to check the cases $\deg P_1 \geq 5$, $\deg P_2 \leq 4$ due to the centre-antisymmetry of σ^0 and the possibility to use the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action. In all these cases the sign pattern of the polynomial P_1 has exactly two sign changes (including the first sign $+$, the four minuses that follow and the next between one and four pluses). With the notation from Lemma 1, these cases are $m = 1$, $n = 4$, $q = 1, \ldots, 4$. The respective values of κ are 9, 6, 5 and 9/2. All of them are > 4. By Descartes' rule the polynomial P_1 can have either 0 or 2 positive roots. In the case of 2 positive roots, Lemma 2 implies that its concatenation with P_2 has at least 2 positive roots which is a contradiction. Hence P_1 has no positive roots. The polynomials P_1 and P_2 define sign patterns with $3+q-1$ and $4-q$ sign preservations respectively. The polynomial P_1 has $\leq 1+(q-1)$ negative roots (see Lemma 1) and P_2 has $\leq 4-q$ ones. Therefore he concatenation of P_1 and P_2 has ≤ 6 negative roots and a polynomial realizing the couple $(\sigma^0, (1, 6))$ (if any) could not be represented as a concatenation of P_1 and P_2. This, of course, does not a priori mean that such a polynomial does not exist.

3. Preliminaries

Notation 2. By S we denote the set of tuples $a \in \mathbb{R}^9$ for which the polynomial $P(x, a) = x^9 + a_8x^8 + \cdots + a_0$ realizes the pair $(1, 6)$ and the signs of its coefficients define the sign pattern σ_0.

We denote by T the subset of S for which $a_8 = -1$. For a polynomial $P \in S$, the conditions $a_9 = 1$, $a_8 = -1$ can be obtained by rescaling the variable x and by multiplying P by a nonzero constant (a_9 is the leading coefficient of P).

Lemma 3. *For $a \in \bar{S}$, one has $a_j \neq 0$ for $j = 7, 6, 3, 2$, and one does not have $a_4 = 0$ and $a_5 = 0$ simultaneously.*

Proof of Lemma 3. For $a_j = 0$ (where j is one of the indices $7, 6, 3, 2$) there are less than 6 sign changes in the sign pattern σ^0_m. Descartes' rule of signs implies that the polynomial $P(., a)$ has less than 6 negative roots counted with multiplicity. The same is true for $a_5 = a_4 = 0$. □

Lemma 4. *For $a \in \bar{S}$, one has $a_0 \neq 0$.*

Remark 6. A priori the set $\bar{S}$ can contain polynomials with all roots real and nonzero. The positive ones can be either a triple root or a double and a simple roots (but not three simple roots). If $a \in S$, then $P(x, a)$ has the maximal possible number of negative roots (equal to the number of sign preservations in the sign

pattern). If $a^{'} \in \bar{S}$, then the polynomial $Q(x, a^{'})$ is the limit of polynomials $Q(x, a)$ with $a \in S$. In the limit as a $\to a^{'}$, the complex conjugate pair can become a double positive, but not a double negative root, because there are no 8 sign preservations in the sign pattern.

Proof of Lemma 4. Suppose that for $P \in \bar{S}$, one has $a_0 = 0$ and for $j \neq 0$, $a_j \neq 0$. Hence the polynomial $P_1 := P/x$ has 6 negative roots and either 0 or 2 positive roots. We show that 0 positive roots is impossible. Indeed, the polynomial P_1 defines a sign pattern with exactly 2 sign changes. Suppose that all negative roots are distinct. If P_1 has no positive roots, then one can apply Lemma 1, according to which, as one has $\kappa = 9/2 > 4$, such a polynomial does not exist. If P_1 has a negative root $-b$ of multiplicity $m > 1$, then its perturbation

$$P_{1,\epsilon} := (x + b + \epsilon)P_1/(x + b) \ , \ 0 < \epsilon \ll 1 \ ,$$

defines the same sign pattern and instead of the root $-b$ of multiplicity m has a root $-b$ of multiplicity $m - 1$ and a simple root $b - \epsilon$. After finitely many such perturbations, one is in the case when all negative roots are distinct.

If P_1 has 2 positive roots, then this is a double positive root g, see Remark 6. In this case, we add to P_1 a linear term $\pm\epsilon x$ (with ϵ small enough in order not to change the sign pattern) to make the double root bifurcate into a complex conjugate pair. The sign is chosen depending on whether P_1 has a minimum or a maximum at g. After this, if there are multiple negative roots, we apply perturbations of the form $P_{1,\epsilon}$.

Suppose that $a_1 = a_0 = 0$, and that for $j \geq 2$, $a_j \neq 0$. Then one considers the polynomial $P_2 := P/x^2$. It defines a sign pattern with two sign changes and one has $\kappa = 5 > 4$. Hence it has 2 positive roots, otherwise one obtains a contradiction with Lemma 1.

Suppose now that exactly one of the coefficients a_4 or a_5 is 0. We assume this to be a_4, for a_5 the reasoning is similar. Suppose also that either $a_1 \neq 0$, $a_0 = 0$ or $a_1 = a_0 = 0$, and that for $j \geq 2$, $j \neq 4$, one has $a_j \neq 0$. We treat in detail the case $a_1 \neq 0$, $a_0 = 0$, the case $a_1 = a_0 = 0$ is treated by analogy. We first make the double positive root if any bifurcate into a complex conjugate pair as above. This does not change the coefficient a_4. After this instead of perturbations $P_{1,\epsilon}$ we use perturbations preserving the condition $a_4 = 0$. Suppose that $P_1 = (x - b)^m Q_1 Q_2$, where Q_1 and Q_2 are monic polynomials, deg $Q_2 = 2$, Q_2 having a complex conjugate pair of roots, Q_1 having $6 - m$ negative roots counted with multiplicity. Then we set:

$$P_1 \mapsto P_1 + \epsilon(x - b)^{m-1}(x + h_1)(x + h_2)Q_1 \ ,$$

where the real numbers h_i are distinct, different from any of the roots of P and chosen in such a way that the coefficient δ of x^3 of P_1 is 0. Such a choice is possible, because all coefficients of the polynomial $(x + b)^{m-1}Q_1$ are positive, hence δ is of the form $A+(h_1+h_2)B+Ch_1h_2$, where $A > 0$, $B > 0$ and $C > 0$. The result of the perturbation is a polynomial P_1 having six negative distinct roots and a complex conjugate pair; its coefficient of x^3 is 0. By adding a small positive number to this coefficient, one obtains a polynomial P_1 with roots as before and defining the sign pattern $(+ - - - - - + + + +)$. For this polynomial one has $\kappa = 9/2 > 4$ which contradicts Lemma 1.

In the case $a_1 = a_0 = 0$, the polynomial P_1 thus obtained has five negative distinct roots, a complex conjugate pair and a root at 0. One adds small positive numbers to its constant term and to its coefficient of x^3 and one proves in the same way that such a polynomial does not exist. □

Remark 7. One deduces from Lemmas 3 and 4 that for a polynomial in $\bar{T}$ exactly one of the following conditions holds true:

(1) all its coefficients are nonvanishing;

(2) exactly one of them is vanishing and this coefficient is either a_1 or a_4 or a_5;

(3) exactly two of them are vanishing, and these are either a_1 and a_4 or a_1 and a_5.

Lemma 5. *There exists no real degree 9 polynomial satisfying the following conditions:*

- *the signs of its coefficients define the sign pattern σ^0,*
- *it has a complex conjugate pair with nonpositive real part,*
- *it has a single positive root,*
- *it has negative roots of total multiplicity 6.*

Proof. Suppose that such a monic polynomial P exists. We can write P in the form $P = P_1P_2P_3$, where $\deg P_1 = 6$.

All roots of P_1 are negative hence $P_1 = \sum_{j=0}^{6} \alpha_j x^j$, $\alpha_j > 0$, $\alpha_6 = 1$; $P_2 = x - w$, $w > 0$; $P_3 = x^2 + \beta_1 x + \beta_0$, $\beta_j \geq 0$, $\beta_1^2 - 4\beta_0 < 0$.

By Descartes' rule of signs, the polynomial $P_1P_2 = \sum_{j=0}^{7} \gamma_j x^j$, $\gamma_7 = 1$, has exactly one sign change in the sequence of its coefficients. It is clear that as $0 > a_8 = \gamma_6 + \beta_1$, and as $\beta_1 \geq 0$, one must have $\gamma_6 < 0$. But then $\gamma_j < 0$ for $j = 0, \ldots, 6$. For $j = 2$, 3 and 4, one has $a_j = \gamma_{j-2} + \beta_1\gamma_{j-1} + \beta_0\gamma_j < 0$ which means that the signs of a_j do not define the sign pattern σ^0. □

Remark 8. It follows from Lemma 5 that polynomials of $\bar{T}$ can only have negative roots of total multiplicity 6 and positive roots of total multiplicity 1 or 3 (i.e., either one simple, or one simple and one double or one triple positive root); these polynomials have no root at 0 (Lemma 4). Indeed, when approaching the boundary of T, the complex conjugate pair can coalesce to form a double positive (but never nonpositive) root; the latter might eventually coincide with the simple positive root.

4. Plan of the proof of part (1) of Theorem 1

Suppose that there exists a monic polynomial $P(x, a^*)|_{a_8^*=-1}$ with signs of its coefficients defined by the sign pattern σ^0, with 6 distinct negative, a simple positive and two complex conjugate roots.

Then for a close to $a^* \in \mathbb{R}^8$, all polynomials $P(x, a)$ share with $P(x, a^*)$ these properties. Therefore the interior of the set T is nonempty. In what follows we denote by Γ the connected component of T to which a^* belongs. Denote by $-\delta$ the value of a_7 for $a = a^*$ (recall that this value is negative).

Lemma 6. *There exists a compact set $K \subset \bar{\Gamma}$ containing all points of $\bar{\Gamma}$ with $a_7 \in [-\delta, 0)$. Hence there exists $\delta_0 > 0$ such that for every point of $\bar{\Gamma}$, one has $a_7 \leq -\delta_0$, and for at least one point of K and for no point of $\bar{\Gamma}\backslash K$, the equality $a_7 = -\delta_0$ holds.*

Proof. Suppose that there exists an unbounded sequence $\{a^n\}$ of values $a \in \bar{\Gamma}$ with $a_7^n \in [-\delta, 0)$. Hence one can perform rescalings $x \mapsto \beta_n x$, $\beta_n > 0$, such that the largest of the moduli of the coefficients of the monic polynomials $Q_n := (\beta_n)^{-9} P(\beta_n x, a^n)$ equals 1. These polynomials belong to $\bar{S}$, not necessarily to $\bar{T}$ because a_8 after the rescalings, in general, is not equal to -1. The coefficient of x^7 in Q_n equals $a_7^n/(\beta_n)^2$. The sequence $\{a^n\}$ si unbounded, so there exists a subsequence β_{n_k} tending to ∞. This means that the sequence of monic polynomials $Q_{n_k} \in \bar{S}$ with bounded coefficients has a polynomial in $\bar{S}$ with $a_7 = 0$ as one of its limit points which contradicts Lemma 3.

Hence the moduli of the roots and the tuple of coefficients a_j of $P(x, a) \in \bar{\Gamma}$ with $a_7 \in [-\delta, 0)$ remain bounded from which the existence of K and δ_0 follows. □

The above lemma implies the existence of a polynomial $P_0 \in \bar{\Gamma}$ with $a_7 = -\delta_0$. We say that P_0 is *a_7-maximal*. Our aim is to show that no polynomial of $\bar{\Gamma}$ is a_7-maximal which contradiction will be the proof of Theorem 1.

Definition 3. A real univariate polynomial is *hyperbolic* if it has only real (not necessarily simple) roots. We denote by $H \subset \bar{\Gamma}$ the set of hyperbolic polynomials in $\bar{\Gamma}$. Hence these are monic degree 9 polynomials having positive and negative roots of respective total multiplicities 3 and 6 (vanishing roots are impossible by Lemma 3). By $U \subset \bar{\Gamma}$ we denote the set of polynomials in $\bar{\Gamma}$ having a complex conjugate pair, a simple positive root and negative roots of total multiplicity 6. Thus $\bar{\Gamma} = H \cup U$ and $H \cap U = \emptyset$. We denote by U_0, U_2, $U_{2,2}$, U_3 and U_4 the subsets of U for which the polynomial $P \in U$ has respectively 6 simple negative roots, one double and 4 simple negative roots, at least two negative roots of multiplicity ≥ 2, one triple and 3 simple negative roots and a negative root of multiplicity ≥ 4.

The following lemma on hyperbolic polynomials is proved in [10]. It is used in the proofs of the other lemmas.

Lemma 7. *Suppose that V is a hyperbolic polynomial of degree $d \geq 2$ with no root at 0. Then:*

(1) V does not have two or more consecutive vanishing coefficients.

(2) If V has a vanishing coefficient, then the signs of its surrounding two coefficients are opposite.

(3) The number of positive (of negative) roots of V is equal to the number of sign changes in the sequence of its coefficients (in the one of $V(-x)$).

By a sequence of lemmas we consecutively decrease the set of possible *a_7-maximal* polynomials until in the end it turns out that this set must be empty. The proofs of the lemmas of this section except Lemma 6 are given in Sections 5 (Lemmas 8 – 12), 6 (Lemma 13) and 7 (Lemmas 14 – 16).

Lemma 8. *(1) No polynomial of $U_{2,2} \cup U_4$ is a_7-maximal.*

(2) For each polynomial of U_3, there exists a polynomial of U_0 with the same values of a_7, a_5, a_4 and a_1.

(3) For each polynomial of $U_0 \cup U_2$, there exists a polynomial of $H \cup U_{2,2}$ with the same values of a_7, a_5, a_4 and a_1.

Lemma 8 implies that if there exists an a_7-maximal polynomial in $\bar{\Gamma}$, then there exists such a polynomial in H. So from now on, we aim at proving that H contains no such polynomial hence H and $\bar{\Gamma}$ are empty.

Lemma 9. *There exists no polynomial in H having exactly two distinct real roots.*

Lemma 10. *The set H contains no polynomial having one triple positive root and negative roots of total multiplicity* 6.

Lemma 10 and Remark 6 imply that a polynomial in H (if any) satisfies the following condition:

Condition A. Any polynomial $P \in H$ has a double and a simple positive roots and negative roots of total multiplicity 6.

Lemma 11. *There exists no polynomial $P \in H$ having exactly three distinct real roots and satisfying the conditions $\{a_1 = 0, a_4 = 0\}$ or $\{a_1 = 0, a_5 = 0\}$.*

It follows from the lemma and from Lemma 3 that a polynomial $P \in H$ having exactly three distinct real roots (hence a double and a simple positive and an 6-fold negative one) can satisfy at most one of the conditions $a_1 = 0$, $a_4 = 0$ and $a_5 = 0$.

Lemma 12. *No polynomial in H having exactly three distinct real roots is a_7-maximal.*

Thus an a_7-maximal polynomial in H (if any) must satisfy Condition A and have at least four distinct real roots.

Lemma 13. *The set H contains no polynomial having a double and a simple positive roots and exactly two distinct negative roots of total multiplicity* 6*, and which satisfies either the conditions $\{a_1 = a_4 = 0\}$ or $\{a_1 = a_5 = 0\}$.*

At this point we know that an a_7-maximal polynomial of H satisfies Condition A and one of the two following conditions:

Condition B. It has exactly four distinct real roots and satisfies exactly one or none of the equalities $a_1 = 0$, $a_4 = 0$ or $a_5 = 0$.

Condition C. It has at least five distinct real roots.

Lemma 14. *The set H contains no a_7-maximal polynomial satisfying Conditions A and B.*

Therefore an a_7-maximal polynomial in H (if any) must satisfy Conditions A and C.

Lemma 15. *The set H contains no a_7-maximal polynomial having exactly five distinct real roots.*

Lemma 16. *The set H contains no a_7-maximal polynomial having at least six distinct real roots.*

Hence the set H contains no a_7-maximal polynomial at all. It follows from Lemma 8 that there is no such polynomial in $\bar{\Gamma}$. Hence $\bar{\Gamma} = \emptyset$.

5. Proofs of Lemmas 7, 8, 9, 10, 11 and 12

Proof of Lemma 7: Part (1). Suppose that a hyperbolic polynomial V with two or more vanishing coefficients exists. If V is degree d hyperbolic, then $V^{(k)}$ is also hyperbolic for $1 \le k < d$. Therefore we can assume that V is of the form $x^\ell L + c$, where $\deg L = d - \ell$, $\ell \ge 3$, $L(0) \ne 0$ and $c = V(0) \ne 0$. If V is hyperbolic and $V(0) \ne 0$, then such is also $W := x^d V(1/x) = cx^d + x^{d-\ell} L(1/x)$ and also $W^{(d-\ell)}$

which is of the form $ax^\ell + b$, $a \neq 0 \neq b$. However given that $\ell \geq 3$, this polynomial is not hyperbolic.

For the proof of part (2) we use exactly the same reasoning, but with $\ell = 2$. The polynomial $ax^2 + b$, $a \neq 0 \neq b$, is hyperbolic if and only if $ab < 0$.

To prove part (3) we consider the sequence of coefficients of $V := \sum_{j=0}^{d} v_j x^j$, $v_0 \neq 0 \neq v_d$. Set $\Phi := \sharp\{k | v_k \neq 0 \neq v_{k-1}, v_k v_{k-1} < 0\}$, $\Psi := \sharp\{k | v_k \neq 0 \neq v_{k-1}, v_k v_{k-1} > 0\}$ and $\Lambda := \sharp\{k | v_k = 0\}$. Then $\Phi + \Psi + 2\Lambda = d$. By Descartes' rule of signs the number of positive (of negative) roots of V is $pos_V \leq \Phi + \Lambda$ (resp. $neg_V \leq \Psi + \Lambda$). As $pos_V + neg_V = d$, one must have $pos_V = \Phi + \Lambda$ and $neg_V = \Psi + \Lambda$. It remains to notice that $\Phi + \Lambda$ is the number of sign changes in the sequence of coefficients of V (and $\Psi + \Lambda$ of $V(-x)$), see part (2) of the lemma. □

Proof of Lemma 8: Part (1). A polynomial of $U_{2,2}$ or U_4 respectively is representable in the form:

$$P^\dagger := (x+u)^2(x+v)^2 S\Delta \quad \text{and} \quad P^* := (x+u)^4 S\Delta \ ,$$

where $\Delta := (x^2 - \xi x + \eta)(x - w)$ and $S := x^2 + Ax + B$. All coefficients u, v, w, ξ, η, A, B are positive and $\xi^2 - 4\eta < 0$ (see Lemma 5); for A and B this follows from the fact that all roots of $P^\dagger/\Delta$ and P^*/Δ are negative. (The roots of $x^2 + Ax + B$ are not necessarily different from $-u$ and $-v$.) We consider the two Jacobian matrices

$$J_1 := (\partial(a_8, a_7, a_1, a_4)/\partial(\xi, \eta, w, u)) \text{ and } J_2 := (\partial(a_8, a_7, a_1, a_5)/\partial(\xi, \eta, w, u)) \ .$$

In the case of $P^\dagger$ their determinants equal

$$\begin{aligned} \det J_1 &= (A^2u^2v + 2A^2uv^2 + 2Au^2v^2 + Auv^3 + 2ABu^2 + 5ABuv \\ &\quad + 2ABv^2 + 3Bu^2v + 2Buv^2 + Bv^3 + 2B^2u + B^2v)\Pi \ , \\ \det J_2 &= (A^2uv + Au^2v + 2Auv^2 + 2ABu \\ &\quad + ABv + 2Bu^2 + 4Buv + 2Bv^2)\Pi \ , \end{aligned}$$

where $\Pi := -2v(w+u)(-\eta - w^2 + w\xi)(\xi u + \eta + u^2)$.

These determinants are nonzero. Indeed, each of the factors is either a sum of positive terms or equals $-\eta - w^2 + w\xi < -\xi^2/4 - w^2 + w\xi = -(\xi/2 - w)^2 \leq 0$. Thus one can choose values of (ξ, η, w, v) close to the initial one (u, A and B remain fixed) to obtain any values of (a_8, a_7, a_1, a_4) or (a_8, a_7, a_1, a_5) close to the initial one. In particular, with $a_8 = -1$, $a_1 = a_4 = 0$ or $a_8 = -1$, $a_1 = a_5 = 0$ while a_7 can have values larger than the initial one. Hence this is not an a_7-maximal polynomial. (If the change of the value of (ξ, η, w, v) is small enough, the values of the coefficients a_j, $j = 0$, 2, 3, 5 or 4 and 6 can change, but their signs remain the same.) The same reasoning is valid for P^* as well in which case one has

$$\begin{aligned} \det J_1 &= (3A^2u^2 + 3Au^3 + 9ABu + 6Bu^2 + 3B^2)M \ , \\ \det J_2 &= (A^2u + 3Au^2 + 3AB + 8Bu)M \ , \end{aligned}$$

with $M := -4u^2(w+u)(-\eta - w^2 + w\xi)(\xi u + \eta + u^2)$.

To prove part (2), we observe that if the triple root of $P \in U_3$ is at $-u < 0$, then in case when P is increasing (resp. decreasing) in a neighbourhood of $-u$ the

polynomial $P - \varepsilon x^2(x+u)$ (resp. $P + \varepsilon x^2(x+u)$), where $\varepsilon > 0$ is small enough, has three simple roots close to $-u$; it belongs to $\bar{\Gamma}$, its coefficients a_j, $2 \neq j \neq 3$, are the same as the ones of P, the signs of a_2 and a_3 are also the same.

For the proof of part (3), we observe first that 1) for $x < 0$ the polynomial P has three maxima and three minima and 2) for $x > 0$ one of the following three things holds true: either $P' > 0$, or there is a double positive root γ of P', or P' has two positive roots $\gamma_1 < \gamma_2$ (they are both either smaller than or greater than the positive root of P). Suppose first that $P \in U_0$. Consider the family of polynomials $P - t$, $t \geq 0$. Denote by t_0 the smallest value of t for which one of the three things happens: either $P - t$ has a double negative root v (hence a local maximum), or $P - t$ has a triple positive root γ or $P - t$ has a double and a simple positive roots (the double one is at γ_1 or γ_2). In the second and third cases one has $P - t_0 \in H$. In the first case, if $P - t_0$ has another double negative root, then $P - t_0 \in U_{2,2}$ and we are done. If not, then consider the family of polynomials

$$P_s := P - t_0 - s(x^2 - v^2)^2(x^2 + v^2) = P - t_0 - s(x^6 - v^2x^4 - x^2v^4 + v^6) \ , \ s \geq 0 \ .$$

The polynomial $-(x^6 - v^2x^4 - x^2v^4 + v^6)$ has double real roots at $\pm v$ and a complex conjugate pair. It has the same signs of the coefficients of x^6, x^4 and 1 as $P - t_0$ and P. The rest of the coefficients of $P - t_0$ and P_s are the same. As s increases, the value of P_s for every $x \neq \pm v$ decreases. So for some $s = s_0 > 0$ for the first time one has either $P_s \in U_{2,2}$ (another local maximum of P_s becomes a double negative root) or $P_s \in H$ (P_s has positive roots of total multiplicity 3, but not three simple ones). This proves part (3) for $P \in U_0$.

If $P \in U_2$ and the double negative root is a local minimum, then the proof of part (3) is just the same. If this is a local maximum, then one skips the construction of the family $P - t$ and starts constructing the family P_s directly. □

Proof of Lemma 9: Suppose that such a polynomial exists. Then it must be of the form $P := (x+u)^6(x-w)^3$, $u > 0$, $w > 0$. The conditions $a_8 = -1$ and $a_1 > 0$ read:

$$6u - 3w = -1 \ \text{ and } \ 3u^5w^2(u - 2w) > 0 \ .$$

In the plane of the variables (u, w) the domain $\{u > 0, w > 0, u - 2w > 0\}$ does not intersect the line $6u - 3w = -1$ which proves the lemma. □

Proof of Lemma 10: Represent the polynomial in the form $P = (x+u_1)\cdots(x+u_6)(x-\xi)^3$, where $u_j > 0$ and $\xi > 0$. The numbers u_j are not necessarily distinct. The coefficient a_8 then equals $u_1 + \cdots + u_6 - 3\xi$. The condition $a_8 = -1$ implies $\xi = \xi_* := (u_1 + \cdots + u_6 + 1)/3$. Denote by $\tilde{a}_1$ the coefficient a_1 expressed as a function of $(u_1, \ldots, u_6, \xi)$. Using computer algebra (say, MAPLE) one finds $27\tilde{a}_1|_{\xi=\xi_*}$:

$$27\tilde{a}_1|_{\xi=\xi_*} = -(-3u_1\cdots u_6 + X + Y)(u_1 + \cdots + u_6 + 1)^2 \ ,$$

where $Y := u_1\cdots u_6(1/u_1 + \cdots + 1/u_6)$ and $X := u_1\cdots u_6 \sum_{1\leq i,j\leq 6, i\neq j} u_i/u_j$ (the sum X contains 30 terms). We show that $a_1 < 0$ which by contradiction proves the lemma. The factor $(u_1 + \cdots + u_6 + 1)^2$ is positive. The factor $\Xi := -3u_1\cdots u_6 + X + Y$ contains a single monomial with a negative coefficient, namely, $-3u_1\cdots u_6$. Consider the sum

$$
\begin{aligned}
& -3u_1\cdots u_6 + u_1^2u_3u_4u_5u_6 + u_2^2u_3u_4u_5u_6 + u_1u_3^2u_4u_5u_6 + u_2u_3u_4^2u_3u_5u_6 \\
+ \quad & u_1u_3u_4u_5^2u_6 + u_2u_3u_4u_5u_6^2 \\
\\
= \quad & u_3u_4u_5u_6((u_1-u_2)^2+u_1u_2) + u_3u_4u_5u_6((u_1-u_2)^2+u_1u_2) \\
+ \quad & u_3u_4u_5u_6((u_1-u_2)^2+u_1u_2) \qquad\qquad > \quad 0
\end{aligned}
$$

(the second and third monomials are in X). Hence Ξ is representable as a sum of positive quantities, so $\Xi > 0$ and $a_1 < 0$. □

Proof of Lemma 11: Suppose that such a polynomial exists. Then it must be of the form $(x+u)^6(x-w)^2(x-\xi)$, where $u>0$, $w>0$, $\xi>0$, $w\neq\xi$. One checks numerically (say, using MAPLE), for each of the two systems of algebraic equations $a_8=-1$, $a_1=0$, $a_4=0$ and $a_8=-1$, $a_1=0$, $a_5=0$, that each real solution (u,w,ξ) or (u,v,w) contains a nonpositive component. □

Proof of Lemma 12: Making use of Condition A formulated after Lemma 10, we consider only polynomials of the form $(x+u)^6(x-w)^2(x-\xi)$. Consider the Jacobian matrix

$$J_1^* := (\partial(a_8,a_7,a_1)/\partial(u,w,\xi)) \ .$$

Its determinant equals $-12u^4(u+w)(u-5w)(\xi-w)(k+u)$. All factors except $u-5w$ are nonzero. Thus for $u\neq 5w$, one has $\det J_1\neq 0$, so one can fix the values of a_8 and a_1 and vary the one of a_7 arbitrarily close to the initial one by choosing suitable values of u, w and ξ. Hence the polynomial is not a_7-maximal. For $u=5w$, one has $a_3=-2500w^5(\xi+5w)<0$ which is impossible. Hence there exist no a_7-maximal polynomials which satisfy only the condition $a_1=0$ or none of the conditions $a_1=0$, $a_4=0$ or $a_5=0$. To see that there exist no such polynomials satisfying only the condition $a_4=0$ or $a_5=0$ one can consider the matrices $J_4^* := (\partial(a_8,a_7,a_4)/\partial(u,w,\xi))$ and $J_5^* := (\partial(a_8,a_7,a_5)/\partial(u,w,\xi))$. Their determinants equal respectively

$$-60u(u+w)(2u-w)(\xi-w)(\xi+u) \ \text{ and } \ -12u(u+w)(5u-w)(\xi-w)(\xi+u) \ .$$

They are nonzero respectively for $2u\neq w$ and $5u\neq w$, in which cases in the same way we conclude that the polynomial is not w_7-maximal. If $u=w/2$, then $a_1=-(1/64)w^7(10\xi-w)$ and $a_8=w-\xi$. As $a_1>0$ and $a_8<0$, one has $w>10\xi$ and $\xi>w>10\xi$ which is a contradiction. If $w=5u$, then $a_6=20u^2(u+\xi)>0$ which is again a contradiction.

□

6. Proof of Lemma 13

The multiplicities of the negative roots of P define the following a priori possible cases:

$$\text{A) } (5,1) \ , \ \text{B) } (4,2) \text{ and C) } (3,3) \ .$$

In all of them the proof is carried out simultaneously for the two possibilities $\{a_1=a_4=0\}$ and $\{a_1=a_5=0\}$. In order to simplify the proof we fix one of the roots to be equal to -1 (this can be achieved by a change $x\mapsto\beta x$, $\beta>0$, followed by

$P \mapsto \beta^{-9}P$). This allows to deal with one less parameter. By doing so we can no longer require that $a_8 = -1$, but only that $a_8 < 0$.

Case A). We use the following parametrization:

$$P = (x+1)^5(sx+1)(tx-1)^2(wx-1)\ ,\ s>0\ ,\ t>0\ ,\ w>0\ ,\ t\neq w\ ,$$

i.e. the negative roots of P are at -1 and $-1/s$ and the positive ones at $1/t$ and $1/w$.

The condition $a_1 = w+2t-s-5 = 0$ yields $s = w+2t-5$. For $s = w+2t-5$, one has

$$a_3 = a_{32}w^2 + a_{31}w + a_{30}\ , \qquad a_4 = a_{42}w^2 + a_{41}w + a_{40}\ ,$$

where $a_{32} = -2t+5$, $a_{31} = -(2t-5)^2$, $a_{30} = -2t^3+20t^2-50t+40$

and $a_{42} = t^2-10t+10$, $a_{41} = 2t^3-25t^2+70t-50$, $a_{40} = -10t^3+55t^2-100t+45$.

The coefficient a_{30} has a single real root $6.7245\ldots$ hence $a_{30} < 0$ for $t > 6.7245\ldots$. On the other hand,

$$a_{32}w^2 + a_{31}w = w(-2t+5)(w+2t-5) = w(-2t+5)s$$

which is negative for $t > 6.7245\ldots$. Thus the inequality $a_3 > 0$ fails for $t > 6.7245\ldots$. Observing that $a_{41} = (2t-5)a_{42}$ one can write

$$a_4 = (w+2t-5)wa_{42} + a_{40} = swa_{42} + a_{40}\ .$$

The real roots of a_{42} (resp. a_{40}) equal $1.127\ldots$ and $8.872\ldots$ (resp. $0.662\ldots$). Hence for $t \in [1.127\ldots, 8.872\ldots]$, the inequality $a_4 > 0$ fails. There remains to consider the possibility $t \in (0, 1.127\ldots)$.

It is to be checked directly that for $s = w+2t-5$, one has

$$a_8/t = 10t^2w + 5tw^2 - 2t^2 - 29tw - 2w^2 + 5t + 10w = (5t-2)ws + t(5-2t)$$

which is nonnegative (hence $a_8 < 0$ fails) for $t \in [2/5, 5/2]$. Similarly

$$a_6 = a_6^* w(w+2t-5) + a_6^\dagger = a_6^* ws + a_6^\dagger\ , \quad \text{where}$$
$$a_6^* = 10t^2 - 20t + 5\ , \qquad a_6^\dagger = -5(t-1)(4t^2-9t+1)\ .$$

The real roots of a_6^* (resp. $a_6^\dagger$) equal $1.707\ldots > 2/5 = 0.4$ and $0.293\ldots$ (resp. $1 > 2/5$, $0.117\ldots$ and $2.133\ldots$) hence for $t \in (0, 2/5)$ one has $a_6^* > 0$ and $a_6^\dagger > 0$, i.e. $a_6 > 0$ and the equality $a_6 = 0$ or the inequality $a_6 < 0$ is impossible. □

Case B). We parametrize P as follows:

$$P = (x+1)^4(Tx^2+Sx-1)^2(wx-1)\ ,\ T>0\ ,\ w>0\ .$$

In this case we presume S to be real, not necessarily positive. The factor $(Tx^2 + Sx - 1)^2$ contains the double positive and negative roots of P.

From $a_1 = w+2S-4 = 0$ one finds $S = (4-w)/2$. For $S = (4-w)/2$, one has

$$\begin{array}{ll} a_8/T = (4w-1)T + 4w - w^2 \ , & \\ a_5 = a_{52}T^2 + a_{51}T + a_{50} \ , & \text{where} \\ a_{52} = w - 4 \ , & \\ a_{51} = -4w^2 + 10w - 16 & \text{and} \\ a_{50} = (3/2)w^3 - 9w^2 + 16w - 12 \ . & \end{array}$$

Suppose first that $w > 1/4$. The inequality $a_8 < 0$ is equivalent to

$$T < T_0 := (w^2 - 4w)/(4w - 1) \ .$$

As $T > 0$, this implies $w > 4$.

For $T = T_0$, one obtains $a_5 = 3C/2(4w-1)^2$, where the numerator $C := 6w^5 - 40w^4 + 85w^3 - 54w^2 + 32w - 8$ has a single real root $0.368\ldots$. Hence for $w > 4$, one has $C > 0$ and $a_5|_{T=T_0} > 0$. On the other hand, $a_{50} = a_5|_{T=0}$ has a single real root $3.703\ldots$, so for $w > 4$ one has $a_5|_{T=0} > 0$. For $w > 4$ fixed, and for $T \in [0, T_0]$, the value of the derivative

$$\partial a_5/\partial T = (2w - 8)T - 4w^2 + 10w - 16$$

is maximal for $T = T_0$; this value equals

$$-2(7w^3 - 14w^2 + 21w - 8)/(4w - 1)$$

which is negative because the only real root of the numerator is $0.510\ldots$. Thus $\partial a_5/\partial T < 0$ and a_5 is minimal for $T = T_0$. Hence the inequality $a_5 < 0$ fails for $w > 1/4$. For $w = 1/4$ one has $a_8 = 15/16 > 0$.

So suppose that $w \in (0, 1/4)$. In this case the condition $a_8 < 0$ implies $T > T_0$. For $T = T_0$ one gets

$$a_4 = 3D/2(4w-1)^2 \ , \ \text{where } D := 8w^5 - 32w^4 + 54w^3 - 85w^2 + 40w - 6$$

has a single real root $2.719\ldots$. Hence for $w \in (0, 1/4)$ one has $D < 0$ and $a_4|_{T=T_0} < 0$. The derivative $\partial a_4/\partial T = -w^2 - 2T - 4$ being negative one has $a_4 < 0$ for $w \in (0, 1/4)$, i.e. the inequality $a_4 > 0$ fails. □

Case C). We set

$$P := (x+1)^3(sx+1)^3(tx-1)^2(wx-1) \ , \ s > 0 \ , \ t > 0 \ , \ w > 0 \ , \ t \neq w \ .$$

The condition $a_1 = w + 2t - 3s - 3 = 0$ implies $s = s_0 := (w + 2t - 3)/3$. For $s = s_0$, one has $27a_8 = t(w + 2t - 3)^2 H^*$, where

$$H^* := 6wt^2 - 2t^2 + 3w^2t - 5wt + 3t + 6w - 2w^2 \ . \tag{6.2}$$

We show first that for $s = s_0$, the case $a_1 = a_5 = 0$ is impossible. To fix the ideas, we represent on Fig. 1 the sets $\{H^* = 0\}$ (solid curve) and $\{a_5^* = 0\}$ (dashed curve), where $a_5^* := a_5|_{s=s_0}$. Although we need only the nonnegative values of t and w, we show these curves also for the negative values of the variables to make things more clear. (The lines $t = 2/3$ and $w = 1/3$ are asymptotic lines for the set $\{H^* = 0\}$). For $t \geq 0$ and $w \geq 0$, the only point, where $H^* = a_5^* = 0$, is the point $(0; 3)$. However, at this point one has $a_8 = 0$, i.e. this does not correspond to the required sign pattern.

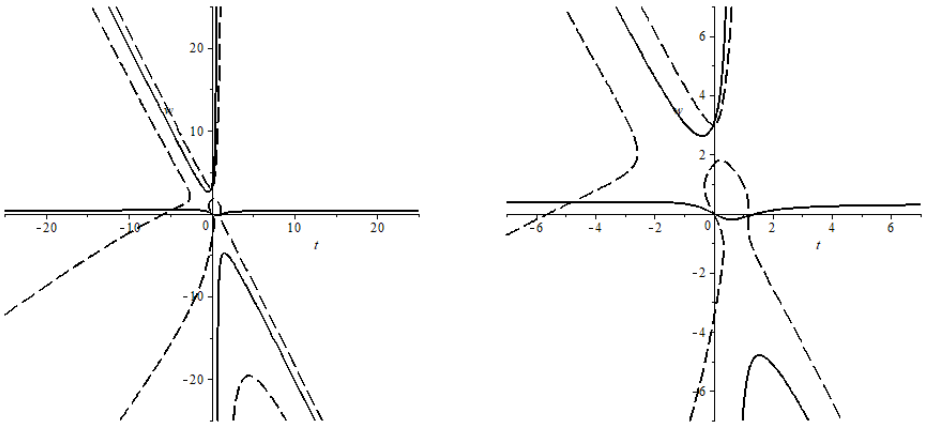

FIGURE 1. The sets $\{H^* = 0\}$ (solid curve) and $\{a_5^* = 0\}$ (dashed curve), with 3 and 4 connected components respectively.

Lemma 17. *(1) For $(t, w) \in \Omega_1 \cup \Omega_2$, where $\Omega_1 = [3/2, \infty) \times [1/3, \infty)$ and $\Omega_2 = [0, 3/2] \times [0, 3]$, one has $H^* \geq 0$.*

(2) For $(t, w) \in \Omega_3 := [3/2, \infty) \times [0, 1/3]$, one has $a_5^* < 0$.

(3) For $(t, w) \in \Omega_4 := [0, 3/2] \times [3, \infty)$, the two conditions $H^* < 0$ and $a_5^* = 0$ do not hold simultaneously.

Lemma 17 (which is proved after the proof of Lemma 13) implies that in each of the sets Ω_j, $1 \leq j \leq 4$, at least one of the two conditions $H^* < 0$ (i. e. $a_8 < 0$) and $a_5^* = 0$ fails. There remains to notice that $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 = \{t \geq 0, w \geq 0\}$.

Now, we show that for $s = s_0$, the case $a_1 = a_4 = 0$ is impossible. On Fig. 2 we show the sets $\{H^* = 0\}$ (solid curve) and $\{a_4^* = 0\}$ (dashed curve), where $a_4^* := a_4|_{s=s_0}$. We use the notation introduced in Lemma 17. By part (1) of Lemma 17 the case $a_1 = a_4 = 0$ is impossible for $(t, w) \in \Omega_1 \cup \Omega_2$.

Lemma 18. *(1) For $(t, w) \in \Omega_3$, one has $a_4^* > 0$.*

(2) For $(t, w) \in \Omega_4$, the two conditions $H^* < 0$ and $a_4^* = 0$ do not hold simultaneously.

Thus the couple of conditions $H^* < 0$, $a_4^* = 0$ fails for $t \geq 0$, $w \geq 0$. This proves Lemma 13. Lemma 18 is proved after Lemma 17 .

□

Proof of Lemma 17. Part (1). Consider the quantity H^* as a polynomial in the variable w:

$$H^* = b_2 w^2 + b_1 w + b_0 \ , \quad \text{where} \quad b_2 = 3t - 2 \ ,$$

$$b_1 = 6t^2 - 5t + 6 \qquad \text{and} \qquad b_0 = -2t(t - 3/2) \ .$$

Its discriminant $\Delta_w := b_1^2 - 4b_0 b_2 = 9(2t^2 - 3t + 2)(2t^2 + t + 2)$ is positive for any real t. This is why for $t \neq 2/3$, the polynomial H^* has 2 real roots; for $t = 2/3$, it is a linear polynomial in w and has a single real root $-5/24$. When H^* is considered

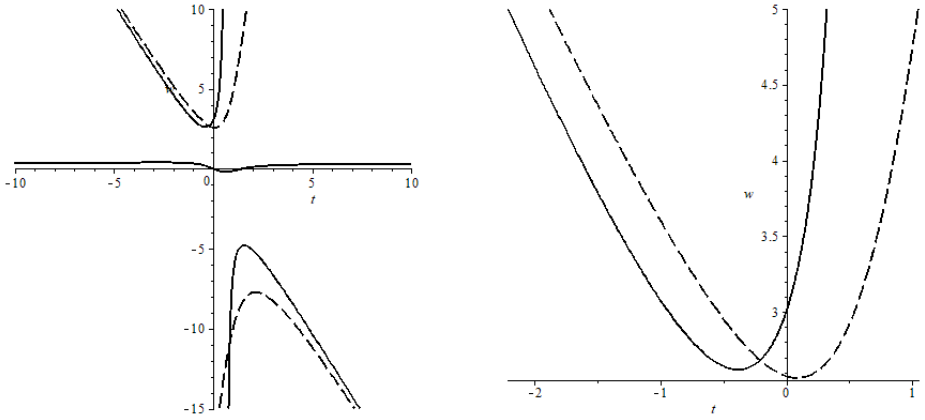

FIGURE 2. The sets $\{H^* = 0\}$ (solid curve) and $\{a_4^* = 0\}$ (dashed curve), with 3 and 2 connected components respectively.

as a polynomial in the variable t, one sets

$$H^* := c_2 t^2 + c_1 t + c_0 \ , \quad \text{where} \quad c_2 = 6w - 2 \ , \tag{6.3}$$
$$c_1 = 3w^2 - 5w + 3 \qquad \text{and} \qquad c_0 = -2w(w-3) \ .$$

Its discriminant

$$\Delta_t := c_1^2 - 4c_0c_2 = 9(w^2 + 5w + 1)(w^2 - 3w + 1)$$

is negative if and only if $w \in (-4.79\ldots, 0.20\ldots) \cup (-0.38\ldots, 2, 61\ldots)$. One checks directly that $H^*|_{w=1/3} = (5/3)t + 16/9$ which is positive for $t \geq 0$. Next, one has $H^*|_{w=0} = b_0$ which is negative for $t > 3/2$. Finally, for $t > 3/2$, the ratio b_0/b_2 is negative which means that for $t > 3/2$ fixed, the polynomial H^* has one positive and one negative root, so the positive root belongs to the interval $(0, 1/3)$ (because $H^*|_{w=1/3} > 0$). Hence $H^* \geq 0$ for $(t, w) \in \Omega_1$ and $H^* > 0$ for (t, w) in the interior of Ω_1.

Suppose now that $(t, w) \in [0, 3/2] \times [0, 3]$. For $t \in (2/3, 3/2]$ fixed, one has $b_2 > 0$, $b_1/b_2 > 0$ and $b_0/b_2 > 0$ which implies that H^* has two negative roots, and for $(t, w) \in (2/3, 3/2] \times [0, 3]$, one has $H^* > 0$. For $t \in [0, 2/3)$ fixed, one has $b_2 < 0$, $b_1/b_2 < 0$, $b_0/b_2 < 0$ and H^* has a positive and a negative root; given that $b_2 < 0$, H^* is positive between them. For $w = 3$ and $t \geq 0$, one has $H^* = t(16t + 15) \geq 0$, with equality only for $t = 0$. Therefore $H^* > 0$ for $(t, w) \in [0, 2/3) \times [0, 3]$. And for $t = 2/3$, one obtains $H^* = (16/3)w + 10/9$ which is positive for $w \geq 0$.

Part (2). One has

$$\begin{aligned} a_5^* \quad = \quad & -8t^5 + 8t^4 w + 6t^3 w^2 - 4t^2 w^3 - 2tw^4 - 24t^4 \\ & -66t^3 w - 63t^2 w^2 - 12tw^3 + 3w^4 + 84t^3 + 153t^2 w \\ & +90tw^2 - 3w^3 - 144t^2 - 144tw - 36w^2 + 108t + 54w \ . \end{aligned}$$

Consider a_5^* as a polynomial in w. Set $R_w :=\mathrm{Res}(a_5^*, \partial a_5^*/\partial w, w)/2125764$. Then $R_w = (2t-3)R_w^1 R_w^2$, where

$$\begin{aligned} R_w^1 &= 32t^5 + 16t^4 - 80t^3 + 184t^2 - 142t - 63 \ , \\ R_w^2 &= 10t^{10} - 80t^9 + 365t^8 - 928t^7 + 1564t^6 - 1788t^5 \\ &\quad +1345t^4 - 668t^3 + 208t^2 - 40t + 4 \ . \end{aligned}$$

The real roots of R_w^1 (resp. R_w^2) equal $-2.56\ldots$, $-0.30\ldots$ and $1.18\ldots$ (resp. $0.34\ldots$ and $1.16\ldots$). That is, the largest real root of R_w is $3/2$. One has

$$a_5^*|_{w=0} = -4t(2t^4 + 6t^3 - 21t^2 + 36t - 27) \ ,$$

with real roots equal to $-5.55\ldots$, 0 and $1.18\ldots$. This means that for $t > 3/2$, the signs of the real roots of a_5^* do not change and their number (counted with multiplicity) remains the same. For $t = 3/2$ and $t = 2$, one has

$$a_5^* = -30w^3 - (45/2)w^2 - (243/4) \quad \text{and} \quad a_5^* = -w^4 - 43w^3 - 60w^2 - 22w - 328$$

respectively, which quantities are negative. Hence $a_5^* < 0$ for $t \geq 3/2$ from which part (2) follows.

Part (3). Consider the resultant

$$\begin{aligned} R^\flat &:= \mathrm{Res}(H^*, a_5^*, t) = -52488w(w-3)R^\sharp(w^2 - w + 1)^2 \ , \quad \text{where} \\ R^\sharp &:= 5w^6 - 16w^5 + 40w^4 - 23w^3 + 61w^2 - 16w - 2 \ . \end{aligned}$$

The real roots of $R^\sharp$ equal $-0.09\ldots$ and $0.37\ldots$; the factor $w^2 - w + 1$ has no real roots. Thus the largest real root of $R^\flat$ equals 3. For $w = 3$, one has

$$a_5^* = -4t^2(2t^3 + 15t + 90) \leq 0 \ ,$$

with equality if and only if $t = 0$. For $w > 3$ and $t \geq 0$, the sets $\{H^* = 0\}$ and $\{a_5^* = 0\}$ do not intersect (because $R^\flat < 0$). We showed in the proof of part (1) of the lemma that the discriminant Δ_t is positive for $w \geq 3$. Hence each horizontal line $w = w_0 > 3$ intersects the set $\{H^* = 0\}$ for two values of t; one of them is positive and one of them is negative (because $c_0/c_2 < 0$); we denote them by t_+ and t_-.

The discriminant $R_t :=\mathrm{Res}(a_5^*, \partial a_5^*/\partial t, t)$ equals $2176782336(w-3)R_t^1 R_t^2$, where

$$\begin{aligned} R_t^1 &:= 5w^{12} + 50w^{11} + 100w^{10} - 2513w^9 + 10781w^8 - 25932w^7 + 46604w^6 \\ &\quad -70411w^5 + 86678w^4 - 82706w^3 + 65264w^2 - 43104w + 16896 \ , \\ R_t^2 &:= 8w^4 + 154w^3 - 68w^2 - 239w - 352 \ . \end{aligned}$$

The factor R_t^1 is without real roots. The real roots of R_t^2 (both simple) equal $-19.61\ldots$ and $1.81\ldots$. Hence for each $w = w_0 > 3$, the polynomial a_5^* has one and the same number of real roots. Their signs do not change with t. Indeed, a_5^* is a degree 5 polynomial in t, with leading coefficient and constant term equal to -8 and $3w(w-3)(w^2 + 2w - 6)$ respectively; the real roots of the quadratic factor equal $-3.64\ldots$ and $1.64\ldots$.

For $w_0 > 3$, the polynomial a_5^* has exactly 3 real roots $t_1 < t_2 < t_3$. For any $w_0 > 3$, the signs of these roots and of the roots $t_\pm$ of H^* and the order of these 5 numbers on the real line are the same. For $w = 4$, one has

$$t_1 = -3.3\ldots < t_- = -1.6\ldots < t_2 = -0.8\ldots < t_+ = 0.2\ldots < t_3 = 0.3\ldots$$

Hence the only positive root t_3 of a_5^* belongs to the domain where $H^* > 0$. Hence one cannot have $a_5^* = 0$ and $H^* < 0$ at the same time. The lemma is proved.

□

Proof of Lemma 18. Part (1). One has

$$\begin{aligned} a_4^* \;:=\; & -20t^4 - 22t^3w - 30t^2w^2 - 10tw^3 + w^4 + 66t^3 + 45t^2w + 36tw^2 \\ & +15w^3 - 135t^2 - 54tw - 54w^2 + 108t + 54w - 81\ . \end{aligned}$$

Consider a_4^* as a polynomial in t. Its discriminant $\Delta_t^\bullet :=\mathrm{Res}(a_4^*, \partial a_4^*/\partial t, t)$ is of the form $170061120\Delta^\flat\Delta^\sharp(w^2 - w + 1)^2$, where

$$\begin{aligned} \Delta^\flat \;:=\; & 9w^4 + 48w^3 + 82w^2 + 56w + 205 \quad \text{and} \\ \Delta^\sharp \;:=\; & 3w^4 + 14w^3 - 63w^2 + 51w - 82\ . \end{aligned}$$

Only the factor $\Delta^\sharp$ has real roots, and these equal $w_- := -7.72\ldots$ and $w_+ := 2.56\ldots$; they are simple. For $w \in (w_-, w_+)$, the quantity a_4^* is negative. Indeed, $a_4^*|_{w=0} = -20t^4 + 66t^3 - 135t^2 + 108t - 81$ which polynomial has no real roots; hence this is the case of $a_4^*|_{w=w_0}$ for any $w_0 \in (w_-, w_+)$. This proves part (1), because the set Ω_3 belongs to the strip $\{w_- < w < w_+\}$.

Part (2). The discriminant $\mathrm{Res}(a_4^*, H^*, t)$ equals $-26244R^\triangle(w^2 - w + 1)^2$ whose factor

$$R^\triangle := 2w^6 + 16w^5 - 61w^4 + 23w^3 - 40w^2 + 16w - 5$$

has exactly two real (and simple) roots which equal $-10.90\ldots$ and $2.68\ldots$. Hence for $w \geq 3 > w_+$,

(1) the sets $\{H^* = 0\}$ and $\{a_4^* = 0\}$ do not intersect;

(2) the numbers of positive and negative roots of H^* and a_4^* do not change; for H^* this follows from formula (6.3); for a_4^* whose leading coefficient as a polynomial in t equals -20, this results from $a_4^*|_{t=0} = w^4 + 15w^3 - 54w^2 + 54w - 81$ whose real roots $-18.1\ldots$ and $2.5\ldots$ (both simple) are < 3.

Hence for $w = w_0 \geq 3$, one has $h_- < A_- < 0 \leq h_+ < A_+$, where h_- and h_+ (resp. A_- and A_+) are the two roots of $H^*|_{w=w_0}$ (resp. of $a_4^*|_{w=w_0}$), with equality only for $w_0 = 3$. It is sufficient to check this string of inequalities for one value of w_0, say, for $w_0 = 4$, in which case one obtains

$$h_- = -1.63\ldots < A_- = -1.26\ldots < h_+ = 0.22\ldots < A_+ = 0.85\ldots\ .$$

Hence for $w = w_0 \geq 3$, the only positive root of the polynomial $a_4^*|_{w=w_0}$ belongs to the domain $\{H^* > 0\}$. This proves part (2) of the Lemma.

□

7. Proofs of Lemmas 14, 15 and 16

Proof of Lemma 14:

Notation 3. If ζ_1, ζ_2, ..., ζ_k are distinct roots of the polynomial P (not necessarily simple), then by P_{ζ_1}, P_{ζ_1,ζ_2}, ..., $P_{\zeta_1,\zeta_2,...,\zeta_k}$ we denote the polynomials

$$P/(x-\zeta_1)\ ,\ P/(x-\zeta_1)(x-\zeta_2)\ ,\ \dots\ ,\ P/(x-\zeta_1)(x-\zeta_2)\dots(x-\zeta_k)\ .$$

Denote by u, v, w and t the four distinct roots of P (all nonzero). Hence

$$P=(x-u)^m(x-v)^n(x-w)^p(x-t)^q\ ,\ m+n+p+q=9\ .$$

For $j=1$, 4 or 5, we show that the Jacobian 3×4-matrix $J:=(\partial(a_8,a_7,a_j)/\partial(u,v,w,t))^t$ (where a_8, a_7, a_j are the corresponding coefficients of P expressed as functions of (u,v,w,t)) is of rank 3. (The entry in position $(2,3)$ of J is $\partial a_7/\partial w$.) Hence one can vary the values of (u,v,w,t) in such a way that a_8 and a_j remain fixed (the value of a_8 being -1) and a_7 takes all possible nearby values. Hence the polynomial is not a_7-maximal.

The entries of the four columns of J are the coefficients of x^8, x^7 and x^j of the polynomials $-mP_u=\partial P/\partial u$, $-nP_v$, $-pP_w$ and $-qP_t$. By abuse of language we say that the linear space $\mathcal{F}$ spanned by the columns of J is generated by the polynomials P_u, P_v, P_w and P_t. As

$$P_{u,v}=\frac{P_u-P_v}{v-u}\ ,\ P_{u,w}=\frac{P_u-P_w}{w-u}\ \text{ and }\ P_{u,t}=\frac{P_u-P_t}{t-}\ ,$$

one can choose as generators of $\mathcal{F}$ the quadruple $(P_u,\,P_{u,v},\,P_{u,w},\,P_{u,t})$; in the same way one can choose $(P_u,\,P_{u,v},\,P_{u,v,w},\,P_{u,v,t})$ or $(P_u,\,P_{u,v},\,P_{u,v,w},\,P_{u,v,w,t})$ (the latter polynomials are of respective degrees 8, 7, 6 and 5). As $(x-t)P_{u,v,w,t}=P_{u,v,w}$, $(x-w)P_{u,v}=P_{u,v,w}$ etc. one can choose as generators the quadruple

$$\psi:=(x^3P_{u,v,w,t}\ ,\ x^2P_{u,v,w,t}\ ,\ xP_{u,v,w,t}\ ,\ P_{u,v,w,t})\ .$$

Set $P_{u,v,w,t}:=x^5+Ax^4+\cdots+G$. The coefficients of x^8, x^7 and x^5 of the quadruple ψ define the matrix $J^*:=\begin{pmatrix}1&0&0&0\\A&1&0&0\\D&C&B&A\end{pmatrix}$. Its columns span the space $\mathcal{F}$ hence $\operatorname{rank}J^*=\operatorname{rank}J$. As at least one of the coefficients B and A is nonzero (Lemma 7) one has $\operatorname{rank}J^*=3$ and the lemma follows (for the case $j=6$). In the cases $j=5$ and $j=1$ the last row of J^* equals respectively $(\ E\ D\ C\ B\ )$ and $(\ 0\ 0\ G\ F\ )$ and in the same way $\operatorname{rank}J^*=3$.

□

Proof of Lemma 15: We are using Notation 3 and the method of proof of Lemma 14. Denote by u, v, w, t, h the five distinct real roots of P (not necessarily simple). Thus using Lemma 10 one can assume that

$$(7.4)\ \ P=(x+u)^\ell(x+v)^m(x+w)^n(x-t)^2(x-h)\ ,\ u,v,w,t,h>0\ ,\ \ell+m+n=6\ .$$

Set $J:=(\partial(a_8,a_7,a_j,a_1)/\partial(u,v,w,t,h))^t$, $j=4$ or 5. The columns of J span a linear space $\mathcal{L}$ defined by analogy with the space $\mathcal{F}$ of the proof of Lemma 14, but spanned by 4-vector-columns.

Set $P_{u,v,w,t,h} := x^4 + ax^3 + bx^2 + cx + d$. Consider the vector-column

$$(0, 0, 0, 0, 1, a, b, c, d)^t \ .$$

The similar vector-columns defined when using the polynomials $x^s P_{u,v,w,t,h}$, $1 \leq s \leq 4$, instead of $P_{u,v,w,t,h}$ are obtained from this one by successive shifts by one position upward. To obtain generators of $\mathcal{L}$ one has to restrict these vector-columns to the rows corresponding to x^8 (first), x^7 (second), x^j ($(9-j)$th) and x (eighth row).

Further we assume that $a_1 = 0$. If this is not the case, then at most one of the conditions $a_4 = 0$ and $a_5 = 0$ is fulfilled and the proof of the lemma can be finished by analogy with the proof of Lemma 14.

Consider the case $j = 5$. Hence the rank of J is the same as the rank of the matrix

$$M := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ c & b & a & 1 & 0 \\ 0 & 0 & 0 & d & c \end{pmatrix} \begin{matrix} x^8 \\ x^7 \\ x^5 \\ x \end{matrix} \ .$$

One has $\operatorname{rank} M = 2 + \operatorname{rank} N$, where $N = \begin{pmatrix} a & 1 & 0 \\ 0 & d & c \end{pmatrix}$. Given that $d \neq 0$, see Lemma 4, one can have $\operatorname{rank} N < 2$ only if $a = c = 0$. We show that the condition $a = c = 0$ leads to the contradiction that one must have $a_8 > 0$. We set $u = 1$ to reduce the number of parameters, so we require only the inequality $a_8 < 0$, but not the equality $a_8 = -1$, to hold true. We have to consider the following cases for the values of the triple (ℓ, m, n) (see (7.4)): 1) $(4, 1, 1)$, 2) $(3, 2, 1)$ and 3) $(2, 2, 2)$. Notice that

$$P_{u,v,w,t,h}|_{u=1} = (x+1)^{\ell-1}(x+v)^{m-1}(x+w)^{n-1}(x-t) \ .$$

In case 1) one has

$$a = 3 - t \ , \ b = 3 - 3t \ , \ c = 1 - 3t \text{ and } d = -t \ , \tag{7.5}$$

so the condition $a = c = 0$ leads to the contradiction $3 = t = 1/3$.

In case 2) one obtains

$$a = 2 + v - t \ , \ b = 1 + 2v - (2 + v)t \ , \ c = v - (1 + 2v)t \text{ and } d = -vt \ . \tag{7.6}$$

Thus, the condition $a = c = 0$ yields $v = -1$, $t = 1$. This is also a contradiction because v must be positive.

In case 3) one gets

$$\begin{aligned} &a = 1 + v + w - t \ , && b = v + (1 + v)w - (1 + v + w)t \ , \\ &c = vw - (v + (1 + v)w)t \quad \text{and} \quad && d = -vwt \ . \end{aligned} \tag{7.7}$$

When one expresses v and w as functions of t from the system of equations $a = c = 0$, one obtains two possible solutions: $v = t$, $w = -1$ and $v = -1$, $w = t$. In both cases one of the variables $(v,\ w)$ is negative which is a contradiction.

Now consider the case $j = 4$. The matrices M and N equal respectively

$$M := \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 \\ d & c & b & a & 1 \\ 0 & 0 & 0 & d & c \end{pmatrix} \ , \ N = \begin{pmatrix} b & a & 1 \\ 0 & d & c \end{pmatrix} \ .$$

One has $\operatorname{rank} N < 2$ only for $b = 0$, $d = ac$ (because $d \neq 0$).

In case 1) these conditions lead to the contradiction $1 = t = (3/2) \pm \sqrt{5}/2$ see (7.5).

In case 2) one expresses the variable t from the condition $b = 0$: $t = t^\bullet := (1 + 2v)/(2 + v)$. Set $a^\bullet := a|_{t=t^\bullet}$, $c^\bullet := c|_{t=t^\bullet}$ and $d^\bullet := d|_{t=t^\bullet}$. The quantity $d^\bullet - a^\bullet c^\bullet$ equals $3(v^2 + v + 1)^2/(2 + v)^2$ which vanishes for no $v \geq 0$. So case 2) is also impossible.

In case 3) the condition $b = 0$ implies $t = t^\triangle := (vw + v + w)/(1 + v + w)$. Set $a^\triangle := a|_{t=t^\triangle}$, $c^\triangle := c|_{t=t^\triangle}$ and $d^\triangle := d|_{t=t^\triangle}$. The quantity $d^\triangle - a^\triangle c^\triangle$ equals $(w^2 + w + 1)(v^2 + v + 1)(v^2 + vw + w^2)/(1 + v + w)^2$ which is positive for any $v \geq 0$, $w \geq 0$. Hence case 3) is impossible. The lemma is proved.

□

Proof of Lemma 16: We use the same ideas and notation as in the proof of Lemma 15. Six of the six or more real roots of P are denoted by (u, v, w, t, h, q). The space $\mathcal{L}$ is defined by analogy with the one of the proof of Lemma 15. The Jacobian matrix J is of the form

$$J := (\partial(a_8, a_7, a_j, a_1)/\partial(u, v, w, t, h, q))^t \ .$$

Set $P_{u,v,w,t,h,q} := x^3 + ax^2 + bx + c$ and consider the vector-column

$$(0, 0, 0, 0, 0, 1, a, b, c)^t \ .$$

Its successive shifts by one position upward correspond to the polynomials $x^s P_{u,v,w,t,h,q}$, $s \leq 5$. In the case $j = 5$ the matrices M and N look like this:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 \\ c & b & a & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & b \end{pmatrix} \ , \ N = \begin{pmatrix} a & 1 & 0 & 0 \\ 0 & 0 & c & b \end{pmatrix} \ .$$

One has $\operatorname{rank} M = 2 + \operatorname{rank} N$ and $\operatorname{rank} N = 2$, because at least one of the two coefficients b and c is nonzero (Lemma 7). Hence $\operatorname{rank} M = 4$ and the lemma is proved by analogy with Lemmas 14 and 15. In the case $j = 4$ the matrices M and N look like this:

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ a & 1 & 0 & 0 & 0 & 0 \\ 0 & c & b & a & 1 & 0 \\ 0 & 0 & 0 & 0 & c & b \end{pmatrix} \ , \ N = \begin{pmatrix} b & a & 1 & 0 \\ 0 & 0 & c & b \end{pmatrix} \ .$$

The matrix N is of rank 4, because either $b \neq 0$ or $b = 0$ and both a and c are nonzero (Lemma 7). Hence $\operatorname{rank} M = 4$.

□

8. Proof of part (2) of Theorem 1

We remind that we consider polynomials with positive leading coefficients. For $d = 9$, we denote by σ a sign pattern and by σ^* the shortened sign pattern (obtained from σ by deleting its last component)

Lemma 19. *For $d = 9$, if $pos \geq 2$ and $neg \geq 2$, then such a couple (sign pattern, admissible pair) is realizable.*

Proof. Suppose that the last two components of σ are equal (resp. different). Then the pair $(pos, neg - 1)$ (resp. $(pos - 1, neg)$) is admissible for the sign pattern σ^* and the couple $(\sigma^*, (pos\ ,\ neg - 1))$ (resp. $(\sigma^*, (pos - 1\ ,\ neg))$) is realizable by some degree 8 polynomial P, see Remark 4. Hence the couple $(\sigma, (pos\ ,\ neg))$ is realizable by the concatenation of the polynomials P and $x + 1$ (resp. P and $x - 1$). □

Lemma 19 implies that in any nonrealizable couple with $pos > 0$ and $neg > 0$, one of the numbers pos, neg equals 1. Using the the standard $\mathbb{Z}_2 \times \mathbb{Z}_2$-action (i.e changing if necessary $P(x)$ to $-P(-x)$) one can assume that $pos = 1$. This implies that the last component of the sign pattern is $-$.

Lemma 20. *For $d = 9$, if $pos = 1$, $neg \geq 2$ and the last two components of σ are $(-\ ,\ -)$, then such a couple $(\sigma, (pos\ ,\ neg))$ is realizable.*

Proof. The couple $(\sigma^*, (pos\ ,\ neg-1))$ is realizable by some polynomial P, see Remark 4. Hence the concatenation of P and $x+1$ realizes the couple $(\sigma, (pos\ ,\ neg))$. □

Hence for any nonrealizable couple $(\sigma, (pos\ ,\ neg))$, one has $pos = 1$, $neg \geq 2$ and the last two components of σ are $(+\ ,\ -)$. Thus, the couple $(\sigma^*, (0\ ,\ neg))$ is nonrealizable; The first and the last components of σ^* are $+$. There are 19 such couples modulo the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, see [11]:

Case	Sign pattern	Admissible pair(s)
A	$(++------+)$	$(0,6)$
B	$(+-------+)$	$(0,6)$
C	$(++++----+)$	$(0,6)$
D	$(+++-----+)$	$(0,6)$
E	$(+-+---+-+)$	$(0,2)$
F	$(+-+-+---+)$	$(0,2)$
$G1-G2$	$(+-+-----+)$	$(0,2)\ ,\ (0,4)$
$H1-H2$	$(+---+---+)$	$(0,2)\ ,\ (0,4)$
$I1-I3$	$(+--------+)$	$(0,2)\ ,\ (0,4)\ ,\ (0,6)$

$$\begin{array}{lll} J & (++\,+---+\,+) & (0,6) \\ K & (+----+--+) & (0,4) \\ L & (+-------+) & (0,4) \\ M & (+-+\,+----+) & (0,4) \\ N & (+-+------+) & (0,4) \\ Q & (+----+-+\,+) & (0,4) \end{array}$$

To obtain all couples $(\sigma^*,\ (0\ ,\ neg))$ giving rise to nonrealizable couples $(\sigma,\ (1\ ,\ neg))$ by concatenation with $x-1$, one has to add to the above list of cases $(A-Q)$ the cases obtained from them by acting with the first generator of the $\mathbb{Z}_2\times\mathbb{Z}_2$-action, i.e. the one replacing σ by σ^r, see Definition 2. The second generator (the one replacing σ by σ^m) has to be ignored, because it exchanges the two components of the admissible pair and the condition $pos=1$ could not be maintained. The cases that are to be added are denoted by (A^r-Q^r). E.g.

$$N^r\quad (++\,---+-+)\quad (0,4)\ .$$

One can observe that, due to the center-symmetry of certain sign patterns, one has $A=A^r$, $E=E^r$, $Hj=Hj^r$, $j=1,\ 2$ and $Ij=Ij^r$, $j=1,\ 2,\ 3$.

With the only exception of case C^r, we show that all cases $(A-Q)$ and (A^r-Q^r), are realizable which proves part (2) of the theorem. We do this by means of Lemma 2. We explain this first for the following cases:

$$B^r\ ,\quad C\ ,\quad D\ ,\quad E\ ,\quad F\ ,\quad F^r\ ,\quad G1\ ,\quad G1^r\ ,\quad G2\ ,\quad G2^r\ ,\quad H1\ ,\quad H2\ ,$$

$$I1\ ,\quad I2\ ,\quad I3\ ,\quad K\ ,\quad K^r\ ,\quad L^r\ ,\quad M\ ,\quad M^r\ ,\quad N^r\quad \text{and}\quad Q^r\ .$$

In all of them the last three components of σ are $(-+-)$, and we set $P_2^\dagger:=x^2-x+1$ (see part (2) of Lemma 2). The polynomial $P_2^\dagger$ has no real roots and defines the sign pattern $\sigma^\dagger:=(+-+)$. Denote by $\tilde\sigma$ the sign pattern obtained from σ by deleting its two last components. Hence $(1,neg)$ is an admissible pair for the sign pattern $\tilde\sigma$, and the couple $(\tilde\sigma,(1,neg))$ is realizable by some degree 7 monic polynomial $\tilde P_1$, see Remark 4. By Lemma 2 the concatenation of $\tilde P_1$ and $P_2^\dagger$ realizes the couple $(\sigma,(1,neg))$.

In cases A, B, J, L, N and Q, the last four components of the sign pattern σ are $(-++-)$. We set $P_2^\triangle:=(x+2)((x^2-2)+1)=x^3-2x^2-3x+10$. Hence $P_2^\triangle$ realizes the couple $((+--+),(0,1))$. Denote by $\sigma^\triangle$ the sign pattern obtained from σ by deleting its three last components. Hence $(1,neg-1)$ is an admissible pair for the sign pattern $\sigma^\triangle$, and the couple $(\sigma^\triangle,(1,neg-1))$ is realizable by some degree 6 monic polynomial $P_1^\triangle$, see Remark 4. By Lemma 2 the concatenation of $P_1^\triangle$ and $P_2^\triangle$ realizes the couple $(\sigma,(1,neg))$.

In the two remaining cases D^r and J^r, the last six components of σ are $(--++\,+-)$. The sign pattern $\sigma^\ddagger:=(++\,---+)$ is realizable by some degree 5 polynomial $P_2^\ddagger$, see [1]. Denote by $\sigma^\diamond$ the sign pattern obtained from σ by deleting

its five last components. Hence in cases D^r and J^r one has $\sigma^\diamond = (+----)$ and $\sigma^\diamond = (++---)$ respectively. Thus the couple $(\sigma^\diamond, (1,3))$ is realizable by some monic degree 4 polynomial $P_1^\diamond$ (see Remark 4), and the concatenation of $P_1^\diamond$ and $P_2^\ddagger$ realizes the couple $(\sigma, (1, neg))$. Part (2) of Theorem 1 is proved.

References

[1] *A. Albouy, Y. Fu*: Some remarks about Descartes' rule of signs. Elem. Math. **69** (2014), 186–194. Zbl 1342.12002, MR3272179.

[2] *B. Anderson, J. Jackson, M. Sitharam*: Descartes rule of signs revisited. Am. Math. Mon. **105** (1998), 447–451. Zbl 0913.12001, MR1622513.

[3] *F. Cajori*: A history of the arithmetical methods of approximation to the roots of numerical equations of one unknown quantity. Colorado College Publication, Science Series **12–7** (1910), 171–215.

[4] *H. Cheriha, Y. Gati, V. P. Kostov*: Descartes' rule of signs, Rolle's theorem and sequences of admissible pairs. (submitted). arXiv:1805.04261.

[5] *J. Forsgård, B. Shapiro, V. P. Kostov*: Could René Descartes have known this? Exp. Math. **24**, No. 4 (2015), 438-448. Zbl 1326.26027, MR3383475.

[6] *J. Fourier*: Sur lusage du théorème de Descartes dans la recherche des limites des racines. Bulletin des sciences par la Société philomatique de Paris (1820) 156–165, 181–187; œuvres 2, 291–309, Gauthier- Villars, 1890.

[7] *C. F. Gauss*: Beweis eines algebraischen Lehrsatzes. J. Reine Angew. Math. **3** (1828) 1–4; Werke 3, 67–70, Göttingen, 1866. ERAM 003.0089cj, MR1577673.

[8] *D. J. Grabiner*: Descartes' Rule of Signs: Another Construction. Am. Math. Mon. **106** (1999) 854–856. Zbl 0980.12001, MR1732666.

[9] *V. P. Kostov*: Topics on hyperbolic polynomials in one variable. Panoramas et Synthèses **33** (2011), vi + 141 p., SMF. Zbl 1259.12001, MR2952044.

[10] *V. P. Kostov*, Polynomials, sign patterns and Descartes' rule of signs. Mathematica Bohemica 144 (2019), No. 1, 39-67.

[11] *V. P. Kostov*: On realizability of sign patterns by real polynomials. Czechoslovak Math. J. (to appear) arXiv:1703.03313.

[12] *V. P. Kostov*: On a stratification defined by real roots of polynomials. Serdica Math. J. **29**, No. 2 (2003), 177–186. Electronic preprint math.AG/0208219. Zbl 1049.12002, MR1992553

[13] *V. P. Kostov, B. Shapiro*: Polynomials, sign patterns and Descartes' rule. Acta Universitatis Matthiae Belii, series Mathematics (2019), 1-11, ISSN 1338-7111.

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CHAPTER 3

Sequences of admissible pairs

DESCARTES' RULE OF SIGNS, ROLLE'S THEOREM AND SEQUENCES OF COMPATIBLE PAIRS

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Abstract. Consider the sequence s of the signs of the coefficients of a real univariate polynomial P of degree d. Descartes' rule of signs gives compatibility conditions between s and the pair (r^+, r^-), where r^+ is the number of positive roots and r^- the number of negative roots of P. It was recently asked if there are other compatibility conditions, and the answer was given in the form of a list of incompatible triples $(s; r^+, r^-)$ which begins at degree $d = 4$ and is known up to degree 8. In this paper we raise the question of the compatibility conditions for $(s; r_0^+, r_0^-; r_1^+, r_1^-; \dots; r_{d-1}^+, r_{d-1}^-)$, where r_i^+ (resp. r_i^-) is the number of positive (resp. negative) roots of the i-th derivative of P. We prove that up to degree 5, there are no other compatibility conditions than the Descartes conditions, the above recent incompatibilities for each i, and the trivial conditions given by Rolle's theorem.

Key words: real polynomial in one variable; sign pattern; Descartes' rule of signs; Rolle's theorem

AMS classification: 26C10; 30C15

1. Introduction

We consider real univariate polynomials and the possible numbers of real positive and negative roots for them and for their derivatives. Without loss of generality we consider only monic polynomials and we limit ourselves to the generic case when neither of the coefficients of the polynomial is 0, i.e. we consider the family of polynomials $P := x^d + a_{d-1}x^{d-1} + \cdots + a_0$, x, $a_j \in \mathbb{R}^*$.

Denote by c and p the numbers of sign changes and sign preservations in the sequence $(1, a_{d-1}, \dots, a_0)$ and by r^+ and r^- the numbers of positive and negative roots of P counted with multiplicity. Descartes' rule of signs, completed by an observation made by Fourier (see [2], [3], [4] and [5]), states that

$$r^+ \leq c \text{ and } c - r^+ \in 2\mathbb{Z} \ . \tag{1.1}$$

Applying this rule to the polynomial $P(-x)$ one gets

$$r^- \leq p \text{ and } p - r^- \in 2\mathbb{Z} \ . \tag{1.2}$$

Notice that without the assumption the coefficients a_j to be nonzero conditions (1.2) do not hold true – for the polynomial $x^2 - 1$ one has $c = 1$, $p = 0$ and $r^- = 1$. It is clear that

$$\text{sgn } a_0 = (-1)^{r^+} \ . \tag{1.3}$$

Definition 1. A *sign pattern* of length $d+1$ is a finite sequence of plus and/or minus signs. (As we consider only monic polynomials, the first sign is a $+$.) We say that the sequence $(1,\ a_{d-1},\ \ldots,\ a_0)$ defines the sign pattern σ if $\sigma = (+,\ \mathrm{sgn}(a_{d-1}),\ \ldots,\ \mathrm{sgn}(a_0))$. For a given sign pattern σ with c sign changes and p sign preservations, we call the pair (c,p) the *Descartes' pair* of σ and we say that a pair (r^+, r^-) is *compatible* for σ if the conditions (1.1) and (1.2) are satisfied. We say that a given couple (sign pattern, compatible pair) is *realizable* if there exists a monic polynomial whose sequence of coefficients defines the sign pattern σ and which has exactly r^+ positive and exactly r^- negative roots, all of them simple.

For $d = 1$, 2 and 3, all couples (sign pattern, compatible pair) are realizable (this is easy to check). For $d = 4$, there are only two cases of couples (sign pattern, compatible pair) which are not realizable (see [9]):

$$((+,+,-,+,+),(2,0)) \quad \text{and} \quad ((+,-,-,-,+),(0,2))\ . \tag{1.4}$$

For $d = 5$, there are also only two nonrealizable couples (sign pattern, compatible pair), see [1]:

$$((+,+,-,+,-,-),(3,0)) \quad \text{and} \quad ((+,-,-,-,-,+),(0,3))\ . \tag{1.5}$$

The question which such couples are realizable is completely solved for $d = 6$ in [1], for $d = 7$ in [7] and for $d = 8$ partially in [7] and completely in [10]. In [6] and [11] an example of nonrealizability is given for $d = 9$ and $d = 11$ respectively, and when both components of the compatible pair are nonzero.

The signs of the coefficients a_j define the sign patterns $\sigma_0,\ \sigma_1,\ \ldots,\ \sigma_{d-1}$ corresponding to the polynomial P and to its derivatives of order $\le d-1$ (the sign pattern σ_j is obtained from σ_{j-1} by deleting the last component). We denote by (c_k, p_k) and (r_k^+, r_k^-) the Descartes' and compatible pairs for the sign patterns σ_k, $k = 0,\ \ldots,\ d-1$. Rolle's theorem implies that

$$\begin{aligned} &r_{k+1}^+ \ge r_k^+ - 1 \qquad , \qquad r_{k+1}^- \ge r_k^- - 1 \\ &\text{and} \qquad r_{k+1}^+ + r_{k+1}^- \ge r_k^+ + r_k^- - 1\ . \end{aligned} \tag{1.6}$$

It can happen that $P^{(k+1)}$ has more real roots than $P^{(k)}$. E. g. this is the case of $P = x^3 + 3x^2 - 8x + 10 = (x+5)((x-1)^2+1)$, because $P' = 3x^2 + 6x - 8$ has one positive and one negative root. It is always true that

$$r_{k+1}^+ + r_{k+1}^- + 3 - r_k^+ - r_k^- \in 2\mathbb{N}\ . \tag{1.7}$$

Definition 2. For a given sign pattern σ_0 of length $d+1$, and for $k = 0,\ \ldots,\ d-1$, suppose that the pair (r_k^+, r_k^-) satisfies the conditions (1.1) – (1.3) and (1.6) – (1.7). Then we say that $((r_0^+, r_0^-),\ \ldots,\ (r_{d-1}^+, r_{d-1}^-))$ $(*)$ is a *sequence of compatible pairs* (i.e. a sequence of pairs compatible for the sign pattern σ_0 in the sense of these conditions). We say that a sequence of compatible pairs is *realizable* if there exists a polynomial P the signs of whose coefficients define the sign pattern σ_0 and such that for $k = 0,\ \ldots,\ d-1$, the polynomial $P^{(k)}$ has exactly r_k^+ positive and r_k^- negative roots, all of them being simple.

Remark 1. The sequence of compatible pairs $(*)$ defines the sign pattern σ_0. This follows from condition (1.3). Given a sequence of compatible pairs $((r_0^+, r_0^-), \ldots, (r_{d-1}^+, r_{d-1}^-))$, the corresponding sign pattern (beginning with a $+$) equals

$$(+ , (-1)^{r_{d-1}^+} , (-1)^{r_{d-2}^+} , \ldots , (-1)^{r_0^+}) .$$

However, for a given sign pattern there are, in general, several possible sequences of compatible pairs. The following example gives an idea how fast the number of sequences of pairs compatible with a given sign pattern might grow with d: for $d = 2$ and for the sign pattern $(+,+,+)$, there are two possible sequences of compatible pairs, namely, $((0,2),(0,1))$ and $((0,0),(0,1))$. For $d = 3$ and for the sign pattern $(+,+,+,+)$, there are three possible sequences of compatible pairs:

$$((0,3),(0,2),(0,1)) , ((0,1),(0,2),(0,1)) \text{ and } ((0,1),(0,0),(0,1)) .$$

For $d = 4$ and for the sign pattern $(+,+,+,+,+)$, this number is 7:

$$\begin{array}{ccc} ((0,4),(0,3),(0,2),(0,1)) & , & ((0,2),(0,3),(0,2),(0,1)) \quad , \\ ((0,2),(0,1),(0,2),(0,1)) & , & ((0,2),(0,1),(0,0),(0,1)) \quad , \\ ((0,0),(0,3),(0,2),(0,1)) & , & ((0,0),(0,1),(0,2),(0,1)) \\ \text{and} & & ((0,0),(0,1),(0,0),(0,1)) \quad . \end{array}$$

The next six numbers (denoted by $A(d)$), obtained as numbers of sequences of pairs compatible with the all-pluses sign pattern of length $d+1$, are:

$$12 , 30 , 55 , 143 , 273 , 728 .$$

They coincide with the terms of sequence A047749 of The On-line Encyclopedia of Integer Sequences founded by N. J. A. Sloane in 1964. To be more precise, sequence A047749 begins like this: 1, 1, 1, 2, 3, 7, 12, 30, 55, 143, ... Its terms are defined as $\binom{3m}{m}/(2m+1)$ if $n = 2m$ and as $\binom{3m+1}{m+1}/(2m+1)$ if $n = 2m+1$. It would be interesting to (dis)prove that this formula applies to all numbers $A(d)$ for $d \in \mathbb{N}$. We prove a weaker statement (see Proposition 1) which implies that the numbers $A(d)$ grow faster than the numbers $[d/2]+1$ of compatible pairs (r_0^+, r_0^-) compatible with the all-pluses sign pattern of length $d+1$. These compatible pairs are $(0, d-2r)$, $r = 0, \ldots, [d/2]$ (the integer part of $d/2$).

Proposition 1. *For $d \geq 2$ even, one has $A(d) \geq 2A(d-1)$. For $d \geq 3$ odd, one has $A(d) \geq 3A(d-1)/2$.*

Proof. For $d = 2$ and 3 the proposition is to be checked straightforwardly. Suppose that $d \geq 4$. Denote by $h_{d,m}$ the number of sequences of compatible pairs with $(r_0^+, r_0^-) = (0, m)$. Set $h_{d,m} := 0$ for $m > d$. Hence $h_{d,d} = 1$ and

$$h_{d,m} = \begin{cases} h_{d,m+2} & \text{if } d \text{ is even and } m = 0 \\ h_{d,m+2} + h_{d-1,m-1} & \text{in all other cases} \end{cases} .$$

This can be deduced from conditions (1.6) and (1.7). Thus if d is even, then one deduces from the above formulas that

$$h_{d,2} = h_{d,0} = h_{d-1,d-1} + h_{d-1,d-3} + \cdots + h_{d-1,1} = A(d-1) ,$$

and as $h_{d,d} = 1 > 0$, one obtains $A(d) > 2A(d-1)$. If d is odd, then

$$h_{d,3} = h_{d-1,d-1} + h_{d-1,d-3} + \cdots + h_{d-1,2} \quad \text{and}$$
$$h_{d,1} = h_{d-1,d-1} + h_{d-1,d-3} + \cdots + h_{d-1,2} + h_{d-1,0} = A(d-1) \quad .$$

As $d-1$ is even, one has $h_{d-1,2} = h_{d-1,0}$, so $h_{d,3} > A(d-1)/2$ and $A(d) > h_{d,3} + h_{d,1} > 3A(d-1)/2$. □

In what follows, for the sake of making things more explicit, we write down often the couples (sign pattern, sequence of compatible pairs), not just the sequences of compatible pairs.

Example 1. Consider the couple (sign pattern, compatible pair) $C := ((+,+,-,+,+), (0,2))$. It can be extended in two ways into a couple (sign pattern, sequence of compatible pairs):

$$(\quad (+,+,-,+,+) \quad , \quad (0,2) \quad , \quad (2,1) \quad , \quad (1,1) \quad , \quad (0,1) \quad) \quad \text{and}$$

$$(\quad (+,+,-,+,+) \quad , \quad (0,2) \quad , \quad (0,1) \quad , \quad (1,1) \quad , \quad (0,1) \quad) \quad .$$

Indeed, by Rolle's theorem, the derivative of a polynomial realizing the couple C has at least one negative root. Condition (1.3) implies that this derivative (which is of degree 3) has an even number of positive roots. This gives the two possibilities $(2,1)$ and $(0,1)$ for (r_1^+, r_1^-). The second derivative has a positive and a negative root. Indeed, it is a degree 2 polynomial with positive leading and negative last coefficient. The realizability of the above two couples (sign pattern, sequence of compatible pairs) is justified in the proof of Theorem 1.

Our first result is the following proposition:

Proposition 2. *For any given sign pattern of length $d+1$, $d \geq 1$, there exists a unique sequence of compatible pairs such that $r_0^+ + r_0^- = d$. This sequence of compatible pairs is realizable. For the given sign pattern, this pair (r_0^+, r_0^-) is its Descartes' pair.*

Proof. The condition $r_0^+ + r_0^- = d$ implies that if a polynomial P realizes a sequence of compatible pairs with the given sign pattern, then $r_0^+ = c$ and $r_0^- = p$, i.e. the compatible pair (r_0^+, r_0^-) is the Descartes' pair for the given sign pattern. Next, one has $r_1^+ \geq r_0^+ - 1$ and $r_1^- \geq r_0^- - 1$, see (1.6). As $\deg P' = d-1$, this means that $r_1^+ + r_1^- \geq d-2$, i.e. at least $d-2$ of the roots of the polynomial P' are real. So the remaining one root is also real (hence $r_1^+ + r_1^- = d-1$) and its sign is defined by condition (1.3). Continuing like this one proves uniqueness of the sequence of compatible pairs satisfying the condition $r_0^+ + r_0^- = d$.

Now we show by induction on d that any given sign pattern is realizable with its Descartes' pair. For $d = 1$ this is evident. Suppose that a sign pattern σ of length $d+1$ is realizable with its Descartes' pair by a polynomial P. Denote by κ the last component of σ (hence $\kappa = +$ or $\kappa = -$). Consider the sign patterns σ^* and $\sigma^\dagger$ defined in Proposition 3. For $\varepsilon > 0$ small enough, the polynomial $P(x)(x+\varepsilon)$ defines the sign pattern σ^* for $\kappa = +$ and $\sigma^\dagger$ for $\kappa = -$, and vice versa for $P(x)(x-\varepsilon)$. Indeed, for ε small enough, the coefficients of x^{d+1}, x^d, …, x of $P(x)(x \pm \varepsilon)$ have the same signs as the coefficients of x^d, x^{d-1}, …, 1 of P (because the former equal 1, $a_{d-1} \pm \varepsilon$, $a_{d-2} \pm \varepsilon a_{d-1}$, …, $a_0 \pm \varepsilon a_1$). The sign of the last coefficient equals $\pm\kappa$

in the case of $P(x)(x \pm \varepsilon)$. Thus one realizes the sign patterns σ^* and $\sigma^\dagger$ of length $d+2$.

□

Remarks 1. (1) Consider a sign pattern of length $d+1$, $d \geq 1$, and a sequence of compatible pairs with $(r_0^+, r_0^-) = (d-1, 1)$ (resp. $(r_0^+, r_0^-) = (1, d-1)$). By Proposition 2, this couple (sign pattern, sequence of compatible pairs) is realizable by some polynomial P. But then all other sequences of compatible pairs with the same pairs (r_k^+, r_k^-), $k = 1, \ldots, d-1$, and with $(r_0^+, r_0^-) = (d-1-2\nu, 1)$ (resp. $(r_0^+, r_0^-) = (1, d-1-2\nu)$), $\nu = 1, \ldots, [(d-1)/2]$, are also realizable with this sign pattern. Indeed, by adding a small linear term εx to the polynomial P (without changing the sign pattern of its coefficients) one can obtain the condition the critical values of P to be distinct. In the case $(r_0^+, r_0^-) = (1, d-1)$, the constant term of P is negative, see (1.3). Hence in the family $P - v$, $v > 0$ (defining the same sign pattern for all values of v) one encounters polynomials with exactly one positive and exactly $d-1$, $d-3$, …, $d-2[(d-1)/2]$ negative roots for suitable values of v. In the case $(r_0^+, r_0^-) = (d-1, 1)$, the sign of the constant term equals $(-1)^{d-1}$ and in the family $P + (-1)^{d-1}v$ one encounters polynomials with exactly one negative and exactly $d-1$, $d-3$, …, $d-2[(d-1)/2]$ positive roots.

(2) In the same way, if $(r_0^+, r_0^-) = (d, 0)$ (resp. $(r_0^+, r_0^-) = (0, d)$), then this couple (sign pattern, sequence of compatible pairs) is realizable by some polynomial P, and all couples (sign pattern, sequence of compatible pairs) with the same sign pattern, the same pairs (r_k^+, r_k^-), $k = 1, \ldots, d-1$, and with $(r_0^+, r_0^-) = (d-2\nu, 0)$ (resp. $(r_0^+, r_0^-) = (0, d-2\nu)$), $\nu = 1, \ldots, [d/2]$, are also realizable.

There are examples of couples (sign pattern, sequence of compatible pairs) which are not realizable:

Example 2. For $d = 4$, the couple (sign pattern, sequence of compatible pairs)

$$(1.8) \qquad (\ (+,+,-,+,+)\ ,\ (2,0)\ ,\ (2,1)\ ,\ (1,1)\ ,\ (0,1)\ )$$

is not realizable because the first of the two couples (sign pattern, compatible pair) (1.4) is not realizable. Hence for $d = 5$, the following couples (sign pattern, sequence of compatible pairs) are not realizable:

$$(1.9)$$
$$(\ (+,+,-,+,+,+)\ ,\ (2,1)\ ,\ (2,0)\ ,\ (2,1)\ ,\ (1,1)\ ,\ (0,1)\ ),$$
$$(\ (+,+,-,+,+,+)\ ,\ (0,1)\ ,\ (2,0)\ ,\ (2,1)\ ,\ (1,1)\ ,\ (0,1)\ ),$$
$$(\ (+,+,-,+,+,-)\ ,\ (3,0)\ ,\ (2,0)\ ,\ (2,1)\ ,\ (1,1)\ ,\ (0,1)\ ),$$
$$(\ (+,+,-,+,+,-)\ ,\ (1,0)\ ,\ (2,0)\ ,\ (2,1)\ ,\ (1,1)\ ,\ (0,1)\ ).$$

For $d = 5$, the following couple (sign pattern, sequence of compatible pairs) is also not realizable, see the first of the nonrealizable couples (sign pattern, compatible pair) in (1.5):

$$(1.10) \qquad (\ (+,+,-,+,-,-)\ ,\ (3,0)\ ,\ (3,1)\ ,\ (2,1)\ ,\ (1,1)\ ,\ (0,1)\ ).$$

Remark 2. When couples (sign pattern, compatible pair) are studied, one can use a second symmetry to reduce the number of cases to be considered. This symmetry stems from the fact that the polynomials $P(x)$ and its reverted one $(\mathrm{sgn}(a_0))x^dP(1/x)$ have one and the same numbers of positive and negative roots. Up to a sign, the sign pattern defined by the latter polynomial is the one defined by P, but read backward. In the present paper we cannot use reversion, because the two ends of a sign pattern do not play the same role – we differentiate w.r.t. of x which makes disapear one by one the coefficients of the lowest degree monomials.

The main result of the present paper is the following theorem:

Theorem 1. *For $d \in \mathbb{N}^*$, we consider $(2d+1)$-tuples of the form $(s; r_0^+, r_0^-; r_1^+, r_1^-; \ldots; r_{d-1}^+, r_{d-1}^-)$, where s is a sequence of $d+1$ signs $+$ or $-$ beginning with a $+$ and r_i^+ (resp. r_i^-) is the number of positive (resp. negative) roots of the ith derivative of a degree d monic univariate real polynomial P with signs of the coefficients defined by the sequence s. Then:*

(1) For $d = 1$, 2 and 3, for any such $(2d+1)$-tuple compatible with Descartes' rule of signs and with Rolle's theorem, there exists such a polynomial P, with all real roots of $P^{(k)}$ simple, $0 \le k \le d-1$;

(2) For $d = 4$, for any such 9-tuple compatible with Descartes' rule of signs and with Rolle's theorem and different from the 9-tuple (1.8) and the one obtained from it via the change $x \mapsto -x$, there exists such a polynomial P, with all real roots of $P^{(k)}$ simple, $0 \le k \le d-1$;

(3) For $d = 5$, for any such 11-tuple compatible with Descartes' rule of signs and with Rolle's theorem and different from the 11-tuples (1.9) and (1.10) and the ones obtained from them via the change $x \mapsto -x$, there exists such a polynomial P, with all real roots of $P^{(k)}$ simple, $0 \le k \le d-1$.

Remark 3. As we see, for degrees up to 5, the questions of realizability of couples (sign pattern, compatible pair) and (sign pattern, sequence of compatible pairs) (or just sequence of compatible pairs, see Remark 1) have the same answers. The much more numerous cases of sequences of compatible pairs compared to couples (sign pattern, compatible pair) as d grows (see Remark 1 and Proposition 1) indicate that it is not unlikely these answers to be different for some $d \ge 6$.

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2. Some auxiliary results

In the proof of Theorem 1 we use the following proposition:

Proposition 3. *Suppose that the couple (σ, U) is realizable by a polynomial P, where σ is a sign pattern of length $d+1$ and U is a sequence of compatible pairs. Denote by σ^* (resp. by $\sigma^\dagger$) the sign pattern of length $d+2$ obtained from σ by adding a sign $+$ (resp. $-$) to its right. Then*

(1) for d even, the couple $(\sigma^*, ((0,1), U))$ (resp. $(\sigma^\dagger, ((1,0), U))$) is realizable.

(2) for d odd, the couple $(\sigma^*,\ ((0,0),\ U))$ (resp. $(\sigma^\dagger,\ ((1,1),\ U))$) is realizable.

Proof. Denote by Q some polynomial such that $Q' = P$. Suppose that d is even. Then for $A > 0$ sufficiently large, the polynomial $Q + A$ (resp. $Q - A$) has a single real root which is simple and negative (resp. simple and positive), so $Q + A$ realizes the sequence of compatible pairs $((0,1),\ U)$ with the sign pattern σ^* (resp. $Q - A$ realizes the sequence of compatible pairs $((1,0),\ U)$ with the sign pattern $\sigma^\dagger$).

Suppose that d is odd. Then for $A > 0$ sufficiently large, the polynomial $Q + A$ has no real roots and realizes the sequence of compatible pairs $((0,0),\ U)$ with the sign pattern σ^* (resp. the polynomial $Q - A$ has a single positive and a single negative root, both simple, so it realizes the sequence of compatible pairs $((1,1),\ U)$ with the sign pattern $\sigma^\dagger$. □

Another proposition which implies part of the proof of Theorem 1 reads:

Proposition 4. *For $d = 5$, consider the sequences of compatible pairs in which $(r_2^+, r_2^-) = (0,1)$ or $(1,0)$. All these sequences of compatible pairs are realizable (with the sign patterns which they define, see Remark 1).*

Proof. First of all we explicit the sequences of compatible pairs with $(r_2^+, r_2^-) = (1,0)$ or $(1,0)$. It is clear that when the sign pattern σ_0 begins with two signs $+$, then for $((r_3^+, r_3^-),\ (r_4^+, r_4^-))$ one has the three possibilities

$$((0,2),(0,1))\ ,\ ((1,1),(0,1))\ \text{and}\ ((0,0),(0,1))\ . \tag{2.11}$$

Proposition 3 allows not to consider the case $(r_0^+, r_0^-) = (0,1)$ or $(1,0)$ because then the couple (sign pattern, sequence of compatible pairs) is realizable. In particular, one needs not to consider the situation when $(r_1^+, r_1^-) = (0,0)$, because then $(r_0^+, r_0^-) = (0,1)$ or $(1,0)$, see (1.6) and (1.7). Therefore if $(r_2^+, r_2^-) = (1,0)$, then there exist the following four possible choices for $((r_0^+, r_0^-),\ (r_1^+, r_1^-))$:

$$((3,0),(2,0))\ ,\ ((2,1),(2,0))\ ,\ ((2,1),(1,1))\ \text{and}\ ((1,2),(1,1))\ . \tag{2.12}$$

For $(r_2^+, r_2^-) = (0,1)$, the possibilities are also four:

$$((0,3),(0,2))\ ,\ ((1,2),(0,2))\ ,\ ((1,2),(1,1))\ \text{and}\ ((2,1),(1,1))\ . \tag{2.13}$$

Combining the possibilities (2.11) with each of the choices (2.12) (resp. (2.13)) one obtains 12 sequences of compatible pairs with $(r_2^+, r_2^-) = (1,0)$ and 12 with $(r_2^+, r_2^-) = (0,1)$.

To realize a sequence of compatible pairs with $(r_2^+, r_2^-) = (1,0)$ we consider the polynomial $T := x^3 - 1$ having a single real root 1. If we choose P'' to equal T, and P' to equal $x^4/4 - x + 0.1$, then P' has two positive roots $\lambda_1 := 0.10\ldots$ and $\lambda_2 := 1.55\ldots$ and a complex conjugate pair. One can represent P in the form $\int_{\lambda_1}^x P'(t)dt + \varepsilon$. For $\varepsilon = 0$, it has a double root at λ_1, a simple one $> \lambda_1$ and a complex conjugate pair. Hence for $\varepsilon > 0$ small enough, it has three positive simple roots and a conjugate pair.

Finally we set $P := \int_{\lambda_1}^x P'(t)dt + \varepsilon + \theta_1 x^4 + \theta_2 x^3$, where $\theta_j \in \mathbb{R}^*$ are small enough (much smaller than ε) and such that the polynomial P''' realizes the necessary couple (2.11). The sign pattern begins with two signs $+$, so one should have $\theta_1 >$

0. It is clear that P realizes the sequence of compatible pairs whose first three compatible pairs are $(3,0)$, $(2,0)$ and $(1,0)$.

If one sets $P := \int_{\lambda_2}^{x} P'(t)dt - \varepsilon + \theta_1 x^4 + \theta_2 x^3$, then the real roots of $P|_{\varepsilon=\theta_1=\theta_2=0}$ are $-0.96\ldots$ (simple) and λ_2 (double), so P realizes the sequence of compatible pairs whose first three compatible pairs are $(2,1)$, $(2,0)$ and $(1,0)$.

If one sets $P'' := T$ and $P' := x^4/4 - x - 0.1$, then the real roots of P' are $\mu_1 := -0.099\ldots$ and $\mu_2 := 1.6\ldots$. If we set $P := \int_{\mu_2}^{x} P'(t)dt - \varepsilon + \theta_1 x^4 + \theta_2 x^3$, then P realizes the sequence of compatible pairs whose first three compatible pairs are $(2,1)$, $(1,1)$ and $(1,0)$. If we set $P := \int_{\mu_1}^{x} P'(t)dt + \varepsilon + \theta_1 x^4 + \theta_2 x^3$, then P realizes the sequence of compatible pairs whose first three compatible pairs are $(1,2)$, $(1,1)$ and $(1,0)$.

To realize a sequence of compatible pairs with $(r_2^+, r_2^-) = (0,1)$ we consider the polynomial $U := x^3 + 1$ having a single real root (-1). By analogy we set $P'' := U$ and obtain the polynomial $P' = x^4/4 + x - 0.1$ having roots $\nu_1 := -1.6\ldots = -\mu_2$ and $\nu_2 := 0.09\ldots = -\mu_1$. Then $P := \int_{\nu_1}^{x} P'(t)dt + \varepsilon + \theta_1 x^4 + \theta_2 x^3$ realizes the sequence of compatible pairs whose first three compatible pairs are $(1,2)$, $(1,1)$ and $(0,1)$, and $P := \int_{\nu_2}^{x} P'(t)dt - \varepsilon + \theta_1 x^4 + \theta_2 x^3$ realizes the sequence of compatible pairs whose first three compatible pairs are $(2,1)$, $(1,1)$ and $(0,1)$.

If we set $P'' := U$ and $P' = x^4/4 + x + 0.1$, then the roots of P' equal $\rho_1 := -1.5\ldots = -\lambda_2$ and $\rho_2 := -0.1\ldots = -\lambda_1$. Thus $P := \int_{\rho_1}^{x} P'(t)dt + \varepsilon + \theta_1 x^4 + \theta_2 x^3$ realizes the sequence of compatible pairs whose first three compatible pairs are $(2,1)$, $(1,1)$ and $(0,1)$, and $P := \int_{\rho_2}^{x} P'(t)dt - \varepsilon + \theta_1 x^4 + \theta_2 x^3$ realizes the sequence of compatible pairs whose first three compatible pairs are $(0,3)$, $(0,2)$ and $(0,1)$. □

The following lemma allows to construct examples of realizability of couples (sign pattern, sequence of compatible pairs) by deforming polynomials with multiple roots.

Lemma 1. *Consider the polynomials $S := (x+1)^3(x-a)^2$ and $T := (x+a)^2(x-1)^3$, $a > 0$. Their coefficients of x^4 are positive if and only if respectively $a < 3/2$ and $a > 3/2$. The coefficients of the polynomial S define the sign pattern*

$$(+,+,+,+,-,+) \quad \text{for} \quad a \in (\ 0\ ,\ (3-\sqrt{6})/3\ ) \quad ,$$

$$(+,+,+,-,-,+) \quad \text{for} \quad a \in (\ (3-\sqrt{6})/3\ ,\ 3-\sqrt{6}\ ) \quad ,$$

$$(+,+,-,-,-,+) \quad \text{for} \quad a \in (\ 3-\sqrt{6}\ ,\ 2/3\ ) \quad \text{and}$$

$$(+,+,-,-,+,+) \quad \text{for} \quad a \in (\ 2/3\ ,\ 3/2\ ) \quad .$$

The coefficients of T define the sign pattern

$$(+,+,-,+,+,-) \quad \text{for} \quad a \in (\ 3/2,\ (3+\sqrt{6})/3\ ) \quad ,$$

$$(+,+,-,-,+,-) \quad \text{for} \quad a \in (\ (3+\sqrt{6})/3\ ,\ 3+\sqrt{6}\ ) \quad \text{and}$$

$$(+,+,+,-,+,-) \quad \text{for} \quad a > 3+\sqrt{6} \quad .$$

Proof. The proof of the lemma is straightforward – we list the coefficients of the polynomials S and T (without the leading one) and below them their roots. For the polynomial S, the list looks like this:

$$3-2a \quad , \quad 3-6a+a^2 \quad , \quad 1-6a+3a^2 \quad , \quad -2a+3a^2 \quad , \quad a^2$$

$$3/2 \qquad 3\pm\sqrt{6} \qquad (3\pm\sqrt{6})/3 \qquad 0\ ,\ 2/3 \qquad 0$$

and one has the following order of these roots on the real line (we list the roots and their approximative values):

$$0 \ < \ \tfrac{3-\sqrt{6}}{3} \ < \ 3-\sqrt{6} \ < \ \tfrac{2}{3} \ < \ \tfrac{3}{2} \ < \ \tfrac{3+\sqrt{6}}{3} \ < \ 3+\sqrt{6}\ .$$

$$0.18\ldots \qquad 0.55\ldots \qquad 0.66\ldots \qquad 1.5 \qquad 1.81\ldots \qquad 5.44\ldots$$

For the polynomial T, we obtain the following list:

$$2a-3 \quad , \quad 3-6a+a^2 \quad , \quad -1+6a-3a^2 \quad , \quad -2a+3a^2 \quad , \quad -a^2$$

$$3/2 \qquad 3\pm\sqrt{6} \qquad (3\pm\sqrt{6})/3 \qquad 0\ ,\ 2/3 \qquad 0\ .$$

□

Finally, we make use of two more propositions to prove Theorem 1:

Proposition 5. *For $d=5$, all sequences of compatible pairs with $r_1^+ + r_1^- = 4$ and with the exception of the one defined by (1.10) are realizable.*

Remark 4. In the proofs of Propositions 5 and 6, when a given case is realizable by a given polynomial, we list in a line the approximations of the real roots of the polynomial and its first three derivatives. The roots of one and the same derivative are separated by commas, between the roots of the different derivatives we put semicolons. We do not give the roots of the fourth derivatives which are always negative because all sign patterns begin with $(+,+)$.

Proof of Proposition 5. We observe first that one cannot have $(r_1^+, r_1^-) = (4,0)$, because then the coefficient of x^3 in P' (and hence the coefficient of x^4 in P) must be negative. Therefore we have to consider four cases.

Case 1. $(r_1^+, r_1^-) = (0,4)$. Hence $(r_2^+, r_2^-) = (0,3)$, $(r_3^+, r_3^-) = (0,2)$ and $(r_4^+, r_4^-) = (0,1)$. There are six possibilities for (r_0^+, r_0^-), and their relizability results as follows: for $(0,5)$ and $(1,4)$ (resp. for $(0,3)$ and $(1,2)$ or for $(0,1)$ and $(1,0)$) from Proposition 2 (resp. from Remarks 1 or Proposition 3).

Case 2. $(r_1^+, r_1^-) = (1,3)$. Hence $r_0^+ = 0$ or 1, see (1.6). By condition (1.6), there are two possibilities:

Case 2a. $(r_2^+, r_2^-) = (0,3)$, $(r_3^+, r_3^-) = (0,2)$ and $(r_4^+, r_4^-) = (0,1)$. There are seven possible values of (r_0^+, r_0^-). For five of them we find out that:

i) $(2,3)$ and $(1,4)$ are realizable by Proposition 2;

ii) $(1,2)$ is realizable by Remarks 1;

iii) $(0,1)$ and $(1,0)$ are realizable by Proposition 3.

To deal with the sixth possibility $(r_0^+, r_0^-) = (0,3)$ we use Lemma 1. Consider the polynomial S with $a \in (0, (3-\sqrt{6})/3)$, and its deformation $S_1 := S + \varepsilon(x^2+x)$, where $\varepsilon > 0$ is sufficiently small. The polynomial S_1 has a root at -1 at which the first derivative is negative. Hence to the left and right of this root there are two more negative roots (because $S_1(0) = a^2 > 0$). On the other hand S_1 has no positive roots (because for $x > 0$, one has $S(x) \geq 0$ and $x^2 + x > 0$). The roots of

S_1 are close to the roots of S, so S_1 has a complex conjugate pair close to a and realizes the sixth possibility.

The last of the seven possibilities for (r_0^+, r_0^-) is $(2,1)$. We consider again the polynomial S with $a \in (0, (3-\sqrt{6})/3)$. Hence $S_2 := S - \varepsilon$ has two real positive roots close to a and a simple negative root close to -1. For $0 < \eta \ll \varepsilon$, the polynomial $S_3 := S_2 - \eta x$ has two real positive roots close to a and a simple negative root close to -1; its derivative has two simple roots close to -1 and a simple root close to a. The fourth root of S_2' must also be real, and as the constant term of S_2' is negative, this root must be negative. Thus the seventh possibility is realizable by the polynomial S_3.

Case 2b. $(r_2^+, r_2^-) = (1,2)$, $(r_3^+, r_3^-) = (0,2)$ and $(r_4^+, r_4^-) = (0,1)$ or $(r_2^+, r_2^-) = (1,2)$, $(r_3^+, r_3^-) = (1,1)$ and $(r_4^+, r_4^-) = (0,1)$ (we consider the two possibilities together). The pair (r_0^+, r_0^-) can take the following values:

i) $(1,4)$ or $(2,3)$ – the cases are realizable by Proposition 2;

ii) $(1,2)$ – the case is realizable by Remarks 1;

iii) $(0,1)$ or $(1,0)$ – the cases are realizable by Proposition 3;

iv) $(0,3)$ – for $(r_3^+, r_3^-) = (0,2)$, the case is realizable by the polynomial

$$\begin{aligned} G &:= (x+1.01)(x+1)(x+0.99)((x-0.3)^2+0.01) \\ &= x^5 + 2.40x^4 + 1.2999x^3 - 0.50004x^2 - 0.299950x + 0.099990 \end{aligned}$$

$$\begin{aligned} \text{roots}: \; & -1.01\,,\ -1\,,\ -0.99\,; \qquad -1.0\ldots\,,\ -0.9\ldots\,,\ -0.2\ldots\,,\ 0.2\ldots\,; \\ & -1.0\ldots\,,\ -0.5\ldots\,,\ 0.09\ldots\,; \quad -0.7\ldots\,,\ -0.1\ldots\,; \end{aligned}$$

for $(r_3^+, r_3^-) = (1,1)$, the case is realizable by the polynomial

$$\begin{aligned} H &:= (x+1.01)(x+1)(x+0.99)((x-0.6)^2+0.01) \\ &= x^5 + 1.80x^4 - 0.2301x^3 - 1.48998x^2 - 0.089917x + 0.369963 \end{aligned}$$

$$\begin{aligned} \text{roots}: \; & -1.01\,,\ -1\,,\ -0.99\,; \qquad -1.0\ldots\,,\ -0.9\ldots\,,\ -0.03\ldots\,,\ 0.5\ldots\,; \\ & -1.0\ldots\,,\ -0.4\ldots\,,\ 0.3\ldots\,; \quad -0.7\ldots\,,\ 0.03\ldots\,. \end{aligned}$$

v) $(2,1)$ – for $(r_3^+, r_3^-) = (0,2)$, the case is realizable by the polynomial $K := x^5 + 20x^4 + 0.6x^3 - 5x^2 - x + 0.5$

$$\begin{aligned} \text{roots}: \; & -19.9\ldots\,,\ 0.2\ldots\,,\ 0.4\ldots\,; \quad -15.9\ldots\,,\ -0.30\ldots\,,\ -0.10\ldots\,,\ 0.38\ldots\,; \\ & -11.9\ldots\,,\ -0.2\ldots\,,\ 0.1\ldots\,; \quad -7.9\ldots\,,\ -0.007\ldots\,; \end{aligned}$$

for $(r_3^+, r_3^-) = (1,1)$, the case is realizable by the polynomial

$$\begin{aligned} L &:= ((x+1.01)(x+1)(x+0.99)+0.1)((x-0.6)^2-0.01) \\ &= x^5 + 1.80x^4 - 0.2501x^3 - 1.44998x^2 - 0.269915x + 0.384965 \end{aligned}$$

$$\begin{aligned} \text{roots}: \; & -1.4\ldots\,,\ 0.5\ldots\,,\ 0.7\ldots\,; \quad -1.1\ldots\,,\ -0.76\ldots\,,\ -0.09\ldots\,,\ 0.6\ldots\,; \\ & -1.0\ldots\,,\ -0.4\ldots\,,\ 0.3\ldots\,; \quad -0.7\ldots\,,\ 0.03\ldots\,. \end{aligned}$$

Case 3. $(r_1^+, r_1^-) = (2,2)$. There are two cases to consider:

Case 3a. $(r_2^+, r_2^-) = (2,1)$, $(r_3^+, r_3^-) = (1,1)$ and $(r_4^+, r_4^-) = (0,1)$. (One cannot have $(r_3^+, r_3^-) = (2,0)$, because in this case the coefficient of x in P''' hence the one of x^4 in P must be negative.) There are eight possible values of (r_0^+, r_0^-):

i) $(3,2)$ or $(2,3)$ – realizability follows from Proposition 2;

ii) $(0,1)$ or $(1,0)$ – realizability results from Proposition 3;

iii) $(3,0)$ or $(1,2)$ – realizability is deduced from Lemma 1 as follows. Consider for some fixed $a \in (3/2, (3+\sqrt{6})/3)$ the polynomial T and its deformation

$$T_\varepsilon := (x-1)(x-1-\varepsilon)(x-1+\varepsilon)(x+a)^2 \ , \ 0 < \varepsilon \ll 1 \ .$$

It has two critical values attained for some $x \in (1-\varepsilon, 1)$ and for some $x \in (1, 1+\varepsilon)$. These values are $O(\varepsilon)$. Hence one can choose ε and $\eta > 0$ small enough so that the polynomial $T_\varepsilon + \eta$ (resp. $T_\varepsilon - \eta$) realizes the sequence of compatible pairs with $(r_0^+, r_0^-) = (3,0)$ (resp. with $(r_0^+, r_0^-) = (1,2)$).

iv) $(2,1)$ – we realize the sequence of compatible pairs by the polynomial

$$N := x^5 + 2x^4 - 60x^3 + 0.05x^2 + x + 5 \ .$$

roots : $-8.8\ldots$, $0.4\ldots$, $6.8\ldots$; $-6.8\ldots$, $-0.07\ldots$, $0.07\ldots$, $5.2\ldots$;
$-4.8\ldots$, $0.0002\ldots$, $3.6\ldots$; $-2.8\ldots$, $2.0\ldots$.

v) $(0,3)$ – we realize the sequence of compatible pairs by the polynomial

$$D := x^5 + 0.01x^4 - 1.9990x^3 + 0.059990x^2 + 0.99940005x + 0.0000019999$$

roots : $-1.1\ldots$, $-0.8\ldots$, $-0.000002\ldots$; $-1.0\ldots$, $-0.4\ldots$, $0.4\ldots$, $0.9\ldots$;
$-0.7\ldots$, $0.01\ldots$, $0.7\ldots$; $-0.4\ldots$, $0.4\ldots$.

Case 3b. $(r_2^+, r_2^-) = (1,2)$, $(r_3^+, r_3^-) = (0,2)$ and $(r_4^+, r_4^-) = (0,1)$ or $(r_2^+, r_2^-) = (1,2)$, $(r_3^+, r_3^-) = (1,1)$ and $(r_4^+, r_4^-) = (0,1)$ (we consider the two possibilities in parallel). There are seven possible values for (r_0^+, r_0^-), the same as in Case 3a.

i) For $(3,2)$, $(2,3)$, $(0,1)$ and $(1,0)$, the answers why these cases are realizable are the same as in Case 3a.

ii) For $(3,0)$ and $(1,2)$, we use Lemma 1. Consider the polynomial T with $a > 3+\sqrt{6}$ (for $(r_3^+, r_3^-) = (0,2)$) or $a \in ((3+\sqrt{6})/3, 3+\sqrt{6})$ (for $(r_3^+, r_3^-) = (1,1)$). The cases are realizable by the polynomials $T_\varepsilon \pm \eta$ as in Case 3a.

iii) For $(2,1)$, and when $(r_3^+, r_3^-) = (1,1)$, the case is realizable by the polynomial

$$\Lambda := x^5 + 0.2x^4 - 6x^3 - 0.05x^2 + 0.01x + 0.5 \ .$$

roots : $-2.5\ldots$, $0.4\ldots$, $2.3\ldots$; $-1.9\ldots$, $-0.02\ldots$, $0.02\ldots$, $1.8\ldots$;
$-1.4\ldots$, $-0.002\ldots$, $1.2\ldots$; $-0.81\ldots$, $0.73\ldots$.

For $(2,1)$, and when $(r_3^+, r_3^-) = (0,2)$, we realize the case by the polynomial

$$\Xi := x^5 + 2.25x^4 + 1.0166666666x^3 - 0.45x^2 + 0.025x + 0.0015 \ .$$

roots : $-0.03\ldots$, $0.13\ldots$, $0.18\ldots$; $-1.0\ldots$, $-0.9\ldots$, $0.03\ldots$, $0.1\ldots$;
$-0.9\ldots$, $-0.4\ldots$, $0.09\ldots$; $-0.7\ldots$, $-0.1\ldots$.

iv) For $(0,3)$, and when $(r_3^+, r_3^-) = (1,1)$, we realize the case by a deformation of the polynomial S from Lemma 1 with $a \in (2/3, 3/2)$, namely

$$S_\varepsilon := (x+1-\varepsilon)(x+1)(x+1+\varepsilon)((x-a)^2+\varepsilon)\ ,\ 0<\varepsilon \ll 1\ .$$

For $(0,3)$, and when $(r_3^+, r_3^-) = (0,2)$, we realize the case by the polynomial

$$\Phi := x^5 + 2.4x^4 + 0.481x^3 - 0.8510x^2 + 0.08529x + 0.01729\ .$$

roots : $-1.9\ ,\ -1\ ,\ -0.1\ ;$ $\quad -1.6\ldots\ ,\ -0.6\ldots\ ,\ 0.05\ldots\ ,\ 0.2\ldots\ ;$
$\quad -1.2\ldots\ ,\ -0.3\ldots\ ,\ 0.1\ldots\ ;$ $\quad -0.9\ldots\ ,\ -0.05\ldots\ .$

Case 4. $(r_1^+, r_1^-) = (3,1)$. Hence the sign pattern is of the form $(+,+,-,+,-,\pm)$, because the sign pattern defined by P' must have three sign changes. Thus $(r_2^+, r_2^-) = (2,1)$, $(r_3^+, r_3^-) = (1,1)$ and $(r_4^+, r_4^-) = (0,1)$. There are seven possibilities for (r_0^+, r_0^-) out of which $(4,1)$ and $(3,2)$ (resp. $(2,1)$) are realizable by Proposition 2 (resp. by Remarks 1) while the realizability of $(0,1)$ and $(1,0)$ results from Proposition 3. We realize the case $(r_0^+, r_0^-) = (1,2)$ by the polynomial

$$U := x^5 + x^4 - 9.01x^3 + 10.97x^2 - 4.05x - 0.01\ .$$

roots : $-4.0\ldots\ ,\ -0.002\ldots\ ,\ 1.2\ldots\ ;$ $\quad -3.0\ldots\ ,\ 0.2\ldots\ ,\ 0.8\ldots\ ,\ 1.0\ldots\ ;$
$\quad -2.1\ldots\ ,\ 0.5\ldots\ ,\ 1.0\ldots\ ;$ $\quad -1.1\ldots\ ,\ 0.7\ldots\ .$

The case $(r_0^+, r_0^-) = (3,0)$ is not realizable, see (1.10) in Example 2.

□

Proposition 6. *For $d = 5$, all sequences of compatible pairs with $r_1^+ + r_1^- = 2$ and with the exception of the four sequences of compatible pairs defined by (1.9) are realizable.*

Proof. We are considering neither the cases with $(r_0^+, r_0^-) = (0,1)$ or $(1,0)$ (which have been treated by Proposition 3) nor the ones with $r_0^+ + r_0^- = 5$ (see Proposition 2) nor the ones with $(r_2^+, r_2^-) = (0,1)$ or $(1,0)$ (which have been settled by Proposition 4). Therefore we are going to limit ourselves to the situations in which $r_0^+ + r_0^- = 3$ and $r_2^+ + r_2^- = 3$. It is impossible to have $(r_2^+, r_2^-) = (3,0)$, because this would mean that the coefficient of x^2 in P'' (hence the one of x^4 in P) must be negative. So three cases have to be examined (defined by (r_2^+, r_2^-)):

Case A. $(r_2^+, r_2^-) = (0,3)$. Hence $(r_3^+, r_3^-) = (0,2)$ and $(r_4^+, r_4^-) = (0,1)$. Observe first that one cannot have $(r_1^+, r_1^-) = (2,0)$, because then P'' should have at least one positive root. Therefore $(r_1^+, r_1^-) = (0,2)$ or $(1,1)$. For $(r_1^+, r_1^-) = (0,2)$, we realize the cases $(r_0^+, r_0^-) = (0,3)$ and $(r_0^+, r_0^-) = (1,2)$ by the polynomials $\tilde{P}$ and P_* respectively:

$$\tilde{P} := x^5 + 20x^4 + 40x^3 + 5x^2 + x + 0.5$$

roots : $-17.7\ldots\ ,\ -2.1\ldots\ ,\ -0.2\ldots\ ;$ $\quad -14.3\ldots\ ,\ -1.5\ldots\ ;$
$\quad -10.9\ldots\ ,\ -1.0\ldots\ ,\ -0.04\ldots\ ;$ $\quad -7.4\ldots\ ,\ -0.5\ldots\ ;$

$$P_* := x^5 + 20x^4 + 40x^3 + 5x^2 + x - 0.5\ .$$

For $k \geq 1$, the roots of $P_*^{(k)}$ and $\tilde{P}^{(k)}$ are the same due to $\tilde{P} - P_* \equiv 1$. The roots of P_* equal $-17.7\ldots$, $-2.1\ldots$ and $0.1\ldots$.

For $(r_1^+, r_1^-) = (1,1)$, we realize the cases $(r_0^+, r_0^-) = (2,1)$ and $(r_0^+, r_0^-) = (1,2)$ by the polynomials $\tilde{Q}$ and Q_*:

$$\tilde{Q} := x^5 + 100x^4 + 20x^3 + 0.5x^2 - x + 0.005$$

$$\begin{aligned} \text{roots}: \quad & -99.7\ldots\ ,\ 0.005\ldots\ ,\ 0.1\ldots\ ; && -79.8\ldots\ ,\ 0.09\ldots\ ; \\ & -59.8\ldots\ ,\ -0.09\ldots\ ,\ -0.009\ldots\ ; && -39.9\ldots\ ,\ -0.05\ldots\ ; \end{aligned}$$

$$Q_* := x^5 + 30x^4 + 20x^3 + 5x^2 - x - 0.5$$

$$\begin{aligned} \text{roots}: \quad & -29.3\ldots\ ,\ -0.3\ldots\ ,\ 0.2\ldots\ ; && -23.4\ldots\ ,\ 0.06\ldots\ ; \\ & -17.6\ldots\ ,\ -0.18\ldots\ ,\ -0.15\ldots\ ; && -11.8\ldots\ ,\ -0.1\ldots\ . \end{aligned}$$

Case B. $(r_2^+, r_2^-) = (1,2)$, $(r_3^+, r_3^-) = (1,1)$ and $(r_4^+, r_4^-) = (0,1)$.

Case B1. $(r_1^+, r_1^-) = (0,2)$. We realize the case $(r_0^+, r_0^-) = (0,3)$ by the polynomial

$$J_\sharp := x^5 + 9x^4 - 0.8x^3 - 0.0073x^2 + 96x + 36$$

$$\begin{aligned} \text{roots}: \quad & -8.9\ldots\ ,\ -2.2\ldots\ ,\ -0.3\ldots\ ; && -7.2\ldots\ ,\ -1.4\ldots\ ; \\ & -5.4\ldots\ ,\ -0.002\ldots\ ,\ 0.04\ldots\ ; && -3.6\ldots\ ,\ 0.02\ldots\ . \end{aligned}$$

We realize the case $(r_0^+, r_0^-) = (1,2)$ by the polynomial

$$V_\flat := x^5 + 9x^4 - 0.8x^3 - 0.0073x^2 + 96x - 36$$

$$\begin{aligned} \text{roots}: \quad & -8.9\ldots\ ,\ -2.5\ldots\ ,\ 0.3\ldots\ ; && -7.2\ldots\ ,\ -1.4\ldots\ ; \\ & -5.4\ldots\ ,\ -0.002\ldots\ ,\ 0.04\ldots\ ; && -3.6\ldots\ ,\ 0.02\ldots\ . \end{aligned}$$

Case B2. $(r_1^+, r_1^-) = (1,1)$. We realize the case $(r_0^+, r_0^-) = (2,1)$ by the polynomial

$$P_\sharp := x^5 + 0.2x^4 - 6x^3 - 0.05x^2 - 0.1x + 0.05$$

$$\begin{aligned} \text{roots}: \quad & -2.5\ldots\ ,\ 0.1\ldots\ ,\ 2.3\ldots\ ; && -1.9\ldots\ ,\ 1.8\ldots\ ; \\ & -1.4\ldots\ ,\ -0.002\ldots\ ,\ 1.2\ldots\ ; && -0.8\ldots\ ,\ 0.7\ldots\ . \end{aligned}$$

We realize the case $(r_0^+, r_0^-) = (1,2)$ by the polynomial

$$\begin{aligned} P_\circ := \ & x^5 + 0.2x^4 - 6x^3 - 0.05x^2 - 0.1x - 0.05 \\ \text{roots}: \quad & -2.5\ldots\ ,\ -0.1\ldots\ ,\ 2.3\ldots\ ; && -1.9\ldots\ ,\ 1.8\ldots\ ; \\ & -1.4\ldots\ ,\ -0.002\ldots\ ,\ 1.2\ldots\ ; && -0.8\ldots\ ,\ 0.7\ldots\ . \end{aligned}$$

Case B3. $(r_1^+, r_1^-) = (2,0)$. To realize the case $(r_0^+, r_0^-) = (3,0)$ we consider the polynomial

$$\begin{aligned} W_\flat := \ & x^5 + 4.4x^4 - 19.295x^2 + 13.22x - 1.1295 \\ \text{roots}: \quad & 0.1\ ,\ 0.6\ldots\ ,\ 1.3\ldots\ ; && 0.3\ldots\ ,\ 1.0\ldots\ ; \\ & -2.2\ldots\ ,\ -1.1\ldots\ ,\ 0.7\ldots\ ; && -1.7\ldots\ ,\ 0\ . \end{aligned}$$

As we see, all real roots of $W_\flat^{(k)}$, $k \leq 4$, are simple. Hence for $\varepsilon > 0$ sufficiently close to 0, the polynomial $W_\flat - \varepsilon x^3$ realizes this case.

To realize the case $(r_0^+, r_0^-) = (2,1)$ we construct first the polynomial

$$W_\sharp := x^5 + 4.6x^4 - 17.495x^2 + 8.74x + 1.0485$$

$$\begin{array}{rll} \text{roots}: & -0.1\,,\ 0.6\ldots\,,\ 1.3\ldots\,; & 0.2\ldots\,,\ 1.0\ldots\,; \\ & -2.4\ldots\,,\ -0.9\ldots\,,\ 0.7\ldots\,; & -1.84\,,\ 0\,. \end{array}$$

We realize the case by the polynomial $W_\sharp - \varepsilon x^3$.

Case C. $(r_2^+, r_2^-) = (1,2)$, $(r_3^+, r_3^-) = (0,2)$ and $(r_4^+, r_4^-) = (0,1)$.

Case C1. $(r_1^+, r_1^-) = (0,2)$. We realize the case $(r_0^+, r_0^-) = (0,3)$ by the polynomial

$$P_\flat := x^5 + 9x^4 + 3x^3 - 0.73x^2 + 96x + 36$$

$$\begin{array}{rll} \text{roots}: & -8.4\ldots\,,\ -2.5\ldots\,,\ -0.3\ldots\,; & -6.8\ldots\,,\ -1.6\ldots\,; \\ & -5.2\ldots\,,\ -0.2\ldots\,,\ 0.05\ldots\,; & -3.5\ldots\,,\ -0.08\ldots\,. \end{array}$$

We realize the case $(r_0^+, r_0^-) = (1,2)$ by the polynomial

$$T_\flat := x^5 + 20x^4 + 80x^3 - 0.02x^2 + x - 0.5$$

$$\begin{array}{rll} \text{roots}: & -14.4\ldots\,,\ -5.5\ldots\,,\ 0.1\ldots\,; & -11.9\ldots\,,\ -4.0\ldots\,; \\ & -9.4\ldots\,,\ -2.5\ldots\,,\ 0.00008\ldots\,; & -6.8\ldots\,,\ -1.1\ldots\,. \end{array}$$

Case C2. $(r_1^+, r_1^-) = (1,1)$. We realize the case $(r_0^+, r_0^-) = (2,1)$ by the polynomial

$$S_\flat := x^5 + 9x^4 + 3x^3 - 0.73x^2 - 96x + 36$$

$$\begin{array}{rll} \text{roots}: & -8.7\ldots\,,\ 0.3\ldots\,,\ 1.8\ldots\,; & -6.9\ldots\,,\ 1.2\ldots\,; \\ & -5.2\ldots\,,\ -0.2\ldots\,,\ 0.05\ldots\,; & -3.5\ldots\,,\ -0.08\ldots\,. \end{array}$$

We realize the case $(r_0^+, r_0^-) = (1,2)$ by the polynomial

$$U_\flat := x^5 + 20x^4 + 0.06x^3 - 0.05x^2 - x - 0.5$$

$$\begin{array}{rll} \text{roots}: & -19.9\ldots\,,\ -0.3\ldots\,,\ 0.4\ldots\,; & -15.9\ldots\,,\ 0.2\ldots\,; \\ & -11.9\ldots\,,\ -0.02\ldots\,,\ 0.01\ldots\,; & -7.9\ldots\,,\ -0.0007\ldots\,. \end{array}$$

Case C3. $(r_1^+, r_1^-) = (2,0)$. We realize the case $(r_0^+, r_0^-) = (3,0)$ by the polynomial $W_\flat + \varepsilon x^3$, and the case $(r_0^+, r_0^-) = (2,1)$ by the polynomial $W_\sharp + \varepsilon x^3$, with $W_\flat$ and $W_\sharp$ as defined in Case B3. One cannot have $(r_0^+, r_0^-) = (1,2)$ or $(0,3)$, see (1.6).

Case D. $(r_2^+, r_2^-) = (2,1)$. One cannot have $(r_3^+, r_3^-) = (2,0)$, because then the coefficient of x in P''' (hence the one of x^4 in P) should be negative. Therefore $(r_3^+, r_3^-) = (1,1)$ and $(r_4^+, r_4^-) = (0,1)$. The possibility $(r_1^+, r_1^-) = (2,0)$ has not to be considered – it gives rise to the four sequences of compatible pairs (1.9). So we have to treat two possibilities;

Case D1. $(r_1^+, r_1^-) = (1,1)$. Hence $(r_0^+, r_0^-) = (1,2)$ or $(2,1)$, see (1.6). We realize the case $(r_0^+, r_0^-) = (1,2)$ by the polynomial

$$P_\dagger := x^5 + 0.2x^4 - 6x^3 + 0.05x^2 - 0.01x - 0.5$$

$$\begin{array}{rll} \text{roots}: & -2.5\ldots\,,\ -0.4\ldots\,,\ 2.3\ldots\,; & -1.9\ldots\,,\ 1.8\ldots\,; \\ & -1.4\ldots\,,\ 0.002\ldots\,,\ 1.2\ldots\,; & -0.8\ldots\,,\ 0.7\ldots\,. \end{array}$$

We realize the case $(r_0^+, r_0^-) = (2, 1)$ by the polynomial

$$K_\flat := x^5 + 9x^4 - 0.8x^3 + 0.0073x^2 - 96x + 36$$

$$\begin{array}{rll} \text{roots}: & -9.2\ldots\ , & 0.3\ldots\ , \ 1.9\ldots\ ; \qquad -7.3\ldots\ , \ 1.3\ldots\ ; \\ & -5.4\ldots\ , & 0.003\ldots\ , \ 0.04\ldots\ ; \quad -3.6\ldots\ , \ 0.02\ldots\ . \end{array}$$

Case D2. $(r_1^+, r_1^-) = (0, 2)$. Hence $(r_0^+, r_0^-) = (0, 3)$ or $(1, 2)$, see (1.6). We realize the case $(r_0^+, r_0^-) = (0, 3)$ by the polynomial

$$J_\flat := x^5 + 9x^4 - 0.8x^3 + 0.0073x^2 + 96x + 36$$

$$\begin{array}{rll} \text{roots}: & -8.9\ldots\ , & -2.2\ldots\ , \ -0.3\ldots\ ; \quad -7.2\ldots\ , \ -1.4\ldots\ ; \\ & -5.4\ldots\ , & 0.003\ldots\ , \ 0.04\ldots\ ; \quad -3.6\ldots\ , \ 0.02\ldots\ . \end{array}$$

We realize the case $(r_0^+, r_0^-) = (1, 2)$ by the polynomial

$$K_\sharp := x^5 + 9x^4 - 0.8x^3 + 0.0073x^2 + 96x - 36$$

$$\begin{array}{rll} \text{roots}: & -8.9\ldots\ , & -2.5\ldots\ , \ 0.3\ldots\ ; \qquad -7.2\ldots\ , \ -1.4\ldots\ ; \\ & -5.4\ldots\ , & 0.003\ldots\ , \ 0.04\ldots\ ; \quad -3.6\ldots\ , \ 0.02\ldots\ . \end{array}$$

□

3. Proof of Theorem 1

Proof of part (1) of Theorem 1. For $d = 1$, the only possible couple (sign pattern, sequence of compatible pairs) modulo the $\mathbb{Z}_2$-action and an example of a polynomial which realizes it is:

$$((+,+),(0,1)) \ \text{ realizable by } \ x + 1\ .$$

For $d = 2$, there are three couples (sign pattern, sequence of compatible pairs) (we list also the derivatives):

Couple	P	P'	
$((+,+,+),(0,2),(0,1))$	$(x+1)(x+2) = x^2 + 3x + 2$	$2x + 3$	,
$((+,+,+),(0,0),(0,1))$	$(x+1)^2 + 1 = x^2 + 2x + 2$	$2x + 2$	and
$((+,+,-),(1,1),(0,1)$	$(x+2)(x-1) = x^2 + x - 2$	$2x + 1$	.

For $d = 3$, there are 10 such couples (we list them together with P, P' and P''):

$$((+,+,+,+) \quad , \quad (0,3),(0,2),(0,1))$$

$x^3 + 6x^2 + 11x + 6 =$	$3x^2 + 12x + 11 =$	$6x + 12 =$
$(x+1)(x+2)(x+3)$	$3(x + 2 + 1/\sqrt{3})(x + 2 - 1/\sqrt{3})$	$6(x+2)$

$$((+,+,+,+) \quad , \quad (0,1),(0,2),(0,1))$$

$$\begin{array}{lll} x^3+5x^2+8x+6= & 3x^2+10x+8= & 6x+10= \\ (x+3)((x+1)^2+1) & 3(x+2)(x+4/3) & 6(x+5/3) \end{array}$$

$$((+,+,+,+) \quad , \quad (0,1),(0,0),(0,1))$$

$$\begin{array}{lll} x^3+3x^2+13x+11= & 3x^2+6x+13= & 6x+6= \\ (x+1)((x+1)^2+10) & 3(x+1)^2+10 & 6(x+1) \end{array}$$

$$((+,+,+,-) \quad , \quad (1,2),(0,2),(0,1))$$

$$\begin{array}{lll} x^3+4x^2+x-6= & 3x^2+8x+1= & 6x+8= \\ (x+3)(x+2)(x-1) & 3(x+\frac{4+\sqrt{13}}{3})(x+\frac{4-\sqrt{13}}{3}) & 6(x+4/3) \end{array}$$

$$((+,+,+,-) \quad , \quad (1,0),(0,2),(0,1))$$

$$\begin{array}{lll} x^3+3x^2+x-5= & 3x^2+6x+1= & 6x+6= \\ (x-1)((x+2)^2+1) & 3(x+1+\sqrt{2/3})(x+1-\sqrt{2/3}) & 6(x+1) \end{array}$$

$$((+,+,+,-) \quad , \quad (1,0),(0,0),(0,1))$$

$$\begin{array}{lll} x^3+3x^2+4x-8= & 3x^2+6x+4= & 6x+6= \\ (x-1)((x+2)^2+4) & 3(x+1)^2+1 & 6(x+1) \end{array}$$

$$((+,+,-,+) \quad , \quad (2,1),(1,1),(0,1))$$

$$\begin{array}{lll} x^3+x^2-10x+8= & 3x^2+2x-10= & 6x+2= \\ (x-1)(x-2)(x+4) & 3(x+\frac{1-\sqrt{31}}{3})(x+\frac{1+\sqrt{31}}{3}) & 6(x+1/3) \end{array}$$

$$((+,+,-,+) \quad , \quad (0,1),(1,1),(0,1))$$

$$\begin{array}{lll} x^3+2x^2-6x+8= & 3x^2+4x-6= & 6x+4= \\ ((x-1)^2+1)(x+4) & 3(x+\frac{2-\sqrt{22}}{3})(x+\frac{2+\sqrt{22}}{3}) & 6(x+2/3) \end{array}$$

$$((+,+,-,-) \quad , \quad (1,2),(1,1),(0,1))$$

$$\begin{array}{lll} x^3+x^2-4x-4= & 3x^2+2x-4= & 6x+2= \\ (x-2)(x+1)(x+2) & 3(x+\frac{1-\sqrt{13}}{3})(x+\frac{1+\sqrt{13}}{3}) & 6(x+1/3) \end{array}$$

$$((+,+,-,-) \quad , \quad (1,0),(1,1),(0,1))$$

$$\begin{array}{lll} x^3+x^2-0.5x-1.5= & 3x^2+2x-0.5= & 6x+2= \\ (x-1)((x+1)^2+0.5) & 3(x+\frac{1+\sqrt{2.5}}{3})(x+\frac{1-\sqrt{2.5}}{3}) & 6(x+1/3) \end{array}$$

□

Proof of part (2) of Theorem 1. We make use of Propositions 3 and 2 and of Remarks 1. Hence when the compatible pair for P is of the form $(1,1)$ or $(0,0)$, then realizability of the sequence of compatible pairs follows from Proposition 3. When $r_0^+ + r_0^- = 4$, realizability follows from Proposition 2. When the Descartes pair of the sign pattern equals $(0,4)$ and $(r_0^+, r_0^-) = (0,2)$, realizability follows from Remarks 1. We present the proof of realizability of the remaining cases by listing

the sign patterns in the lexicographic order. In the proof ε and η denote positive and sufficiently small numbers.

1. $((+,+,+,+,+),(0,2),(0,1),(0,2),(0,1))$. We set $P'' := (x+1)^2 - \varepsilon$. Hence P'' has two negative roots and P''' has a simple negative root. Set $P' := \int_{-2}^{x} P''(t)dt$. Hence $P'(0) > 0$ and P' has a single root which equals -2. Then we set $P := \int_{-2-\eta}^{x} P'(t)dt$.

2. $((+,+,+,+,+),(0,2),(0,1),(0,0),(0,1))$. For $x \in [-3,-0.5]$, the graphs of the polynomial $P^\ddagger := (x+1)(x+2)(1+\varepsilon x^2)$ and of its first and second derivatives are close to the graphs respectively of $(x+1)(x+2) = x^2+3x+2$, $2x+3$ and 2. It is clear that $P^\ddagger$ has a complex conjugate pair of roots. As

$$(P^\ddagger)' = (2x+3)(1+\varepsilon x^2) + 2\varepsilon x(x+1)(x+2) = 2x+3+2\varepsilon x(2x+1)(x+1)\ ,$$

for $\varepsilon > 0$ small enough, the polynomial $(P^\ddagger)'$ has a single real root which is close to $-3/2$, and $(P^\ddagger)'' = 2(1+\varepsilon(6x^2+6x+1))$ has no real root. Obviously, $(P^\ddagger)''' = \varepsilon(12x+6)$ has one negative root.

3. $((+,+,+,-,+),(2,0),(1,2),(0,2),(0,1))$. One sets

$$P' := (x-0.25)((x+1)^2-\varepsilon) = x^3+1.75x^2+0.5x-0.25+O(\varepsilon)\ ,$$

and then $P = \int_{0.25}^{x} P'(t)dt - \eta$.

4. $((+,+,+,-,+),(2,0),(1,0),(0,2),(0,1))$. We set $P'' := (x+1)^2-\varepsilon$, $P' := \int_1^x P''(t)dt$ and $P := \int_1^x P'(t)dt - \eta$.

5. $((+,+,+,-,+),(2,0),(1,0),(0,0),(0,1))$. We set $P := x^4 - x + \varepsilon + \eta x^2 + \eta^2 x^3$. Hence $P'' = 12x^2 + 6\eta^2 x + 2\eta$ has no real root and $P''' = 24x + 6\eta^2$ has a negative root. The polynomial $T := x^4 - x + \varepsilon$ has two positive roots and a complex conjugate pair, so for $0 < \eta \ll \varepsilon$ this is also the case of P. As for T', it has a single real root $1/4^{1/3}$, so P' has a single real root close to $1/4^{1/3}$.

6. $((+,+,+,-,+),(0,2),(1,2),(0,2),(0,1))$. Set

$$P' := (x-0.5)(x+1)(x+3) = x^3+3.5x^2+x-1.5\ .$$

One has $|\int_{-3}^{-1} P'(t)dt| > |\int_{-1}^{0.5} P'(t)dt|$, because the graph of P' is symmetric w.r.t. the point $(-7/6, P'(-7/6))$ with $P'(-7/6) > 0$. Hence P has minima at -3 and 0.5 and $P(-3) < P(0.5)$. Thus one can choose $a \in \mathbb{R}$ such that $P := \int_0^x P'(t)dt + a$ two negative simple roots and no nonnegative root.

7. $((+,+,-,+,+),(0,2),(2,1),(1,1),(0,1))$. One sets

$$P' := (x+3)((x-1)^2-\varepsilon) = x^3+x^2-5x+3+O(\varepsilon) \text{ and } P := \int_{-3-\eta}^{x} P'(t)dt\ .$$

8. $((+,+,-,+,+),(0,2),(0,1),(1,1),(0,1))$. One sets

$$P' := (x+1)((x-0.25)^2+\varepsilon) = x^3+0.5x^2-0.25x+0.0625+O(\varepsilon) \text{ and } P := \int_{-1}^{x} P'(t)dt - \eta\,.$$

9. $((+,+,-,-,+),(2,0),(1,2),(1,1),(0,1))$. One sets

$$P' := (x-1.5)((x+1)^2-\varepsilon) = x^3+0.5x^2-2x-1.5+O(\varepsilon) \text{ and } P := \int_{1.5}^{x} P'(t)dt-\eta\ .$$

10. $((+,+,-,-,+),(2,0),(1,0),(1,1),(0,1))$. One sets

$$P' := (x-1)((x+1)^2+\varepsilon) = x^3+x^2-x-1+O(\varepsilon) \text{ and } P := \int_{1}^{x} P'(t)dt-\eta\ .$$

11. $((+,+,-,-,+),(0,2),(1,2),(1,1),(0,1))$. One sets

$$P' := (x-1)(x+2)(x+\varepsilon) = x^3+(1+\varepsilon)x^2+(-2+\varepsilon)x-2\varepsilon\ .$$

Thus $|\int_{-2}^{-\varepsilon} P'(t)dt| > |\int_{-\varepsilon}^{1} P'(t)dt|$. One can choose η such that for $P := \int_{-2-\eta}^{x} P'(t)dt$ one has $P(0) > 0$ and P has two negative and no nonnegative root.

□

Proof of part (3) of Theorem 1. There are three possible values for the sum $r_1^+ + r_1^-$, namely, 0, 2 and 4. If $r_1^+ + r_1^- = 0$, then $r_0^+ + r_0^- = 1$ (see (1.6)) and the realizability of such a case results from Proposition 4. If $r_1^+ + r_1^- = 2$ or 4, then realizability follows from Proposition 5 or 6. □

References

[1] A. Albouy, Y. Fu: Some remarks about Descartes' rule of signs. Elem. Math., 69 (2014), 186–194. Zbl 1342.12002, MR3272179

[2] B. Anderson, J. Jackson, M. Sitharam: Descartes rule of signs revisited. Am. Math. Mon. 105 (1998), 447–451. Zbl 0913.12001, MR1622513

[3] F. Cajori: A history of the arithmetical methods of approximation to the roots of numerical equations of one unknown quantity. Colorado College Publication, Science Series 12–7 (1910), 171–215.

[4] J. Fourier: Sur l'usage du théorème de Descartes dans la recherche des limites des racines. Bulletin des sciences par la Société philomatique de Paris (1820) 156–165, 181–187; œuvres 2, 291–309, Gauthier-Villars, 1890.

[5] C. F. Gauss: Beweis eines algebraischen Lehrsatzes. J. Reine Angew. Math. 3 (1828) 1–4; Werke 3, 67–70, Göttingen, 1866. ERAM 003.0089cj, MR1577673

[6] H. Cheriha, Y. Gati and V. P. Kostov: A nonrealization theorem in the context of Descartes' rule of signs, St. Kliment Ohridski - Faculty of Mathematics and Informatics (accepted).

[7] J. Forsgård, B. Shapiro and V. P. Kostov: Could René Descartes have known this? Exp. Math. 24 (4) (2015), 438-448. Zbl 1326.26027, MR3383475

[8] J. Forsgård, B. Shapiro and V. P. Kostov: Corrigendum: Could René Descartes have known this? Exp. Math. 28 (2) (2019), 255-256. 10.1080/10586458.2017.1417775

[9] D. J. Grabiner: Descartes Rule of Signs: Another Construction. Am. Math. Mon. 106 (1999), 854–856. Zbl 0980.12001, MR1732666

[10] V. P. Kostov: On realizability of sign patterns by real polynomials, Czechoslovak Math. J. 68 (2018) 853-874.

[11] V. P. Kostov: Polynomials, sign patterns and Descartes' rule of signs, Mathematica Bohemica, 144 (1) (2019) 39-67.

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CHAPTER 4

Geometric illustration

DEGREE 5 POLYNOMIALS AND DESCARTES' RULE OF SIGNS

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Abstract. For a univariate real polynomial without zero coefficients, Descartes' rule of signs (completed by an observation of Fourier) says that its numbers *pos* of positive and *neg* of negative roots (counted with multiplicity) are majorized respectively by the numbers c and p of sign changes and sign preservartions in the sequence of its coefficients, and that the differences $c - pos$ and $p - neg$ are even numbers. For degree 5 polynomials, it has been proved by A. Albouy and Y. Fu that there exist no such polynomials having three distinct positive and no negative roots and whose signs of the coefficients are $(+,+,-,+,-,-)$ (or having three distinct negative and no positive roots and whose signs of the coefficients are $(+,-,-,-,-,+)$). For degree 5 and when the leading coefficient is positive, these are all cases of numbers of positive and negative roots (all distinct) and signs of the coefficients which are compatible with Descartes' rule of signs, but for which there exist no such polynomials. We explain this non-existence and the existence in all other cases with $d = 5$ by means of pictures showing the discriminant set of the family of polynomials $x^5 + x^4 + ax^3 + bx^2 + cx + d$ together with the coordinate axes.

Key words: real polynomial in one variable; sign pattern; Descartes' rule of signs

1. Introduction

Consider a univariate real polynomial $P(x) := \sum_{j=0}^{d} a_j x^j$, $a_d \neq 0$, with c sign changes in the sequence of its coefficients. The classical Descartes' rule of signs says that the number *pos* of its positive roots is not larger than c, see [10]. Fourier (see [8]) has observed also that if roots are counted with multiplicity, then the number $c - pos$ is even. In the present paper we consider polynomials with all coefficients nonzero. In this case if one considers the polynomial $P(-x)$ and applies Descartes' rule to it, one finds that for the number *neg* of negative roots of P, (counted with multiplicity) one obtains $neg \leq p$, where p is the number of sign preservations in the sequence of coefficients (hence $c+p = d$); moreover, the number $p - neg$ is even. Descartes' rule of signs gives only necessary conditions about the possible values of the numbers *pos* and *neg* when the numbers c and p are known. To explain what sufficient conditions means we need the following definition:

Definition 1. For a given degree d, a *sign pattern (SP)* is a sequence of $d+1$ signs ($+$ or $-$). We assume the first of them to be a $+$, because without loss of generality we consider only monic polynomials. Given the degree d and a SP, we denote by c and p the numbers of sign changes and sign preservations in the SP and we call the pair (c, p) *Descartes' pair*. Any pair (pos, neg) satisfying the conditions

$$pos \leq c\ ,\ c - pos \in 2\mathbb{Z}\ ,\ neg \leq p\ ,\ p - neg \in 2\mathbb{Z} \tag{1.1}$$

is called *admissible pair (AP)* for the given SP. In particular, the Descartes' pair is an AP. A given couple (SP, AP) is *realizable* if there exists a monic degree d polynomial the signs of whose coefficients define the given SP and which has exactly *pos* positive and exactly *neg* negative roots, all of them simple.

To give sufficient conditions in the context of Descartes' rule of signs means to give the answer to the following realization problem:

Problem 1. *For a given degree d, which couples (SP, AP) are realizable and which are not?*

The answer to this problem is known for $d \leq 8$. For $d \leq 3$, all couples (SP, AP) are realizable. For $d = 4$, the answer to it is due to D. Grabiner, see [9], for $d = 5$ and 6 it is due to A. Albouy and Y. Fu, see [1], and for $d = 7$ and 8, it was given by J. Forsgård, V. P. Kostov and B. Z. Shapiro, see [6] and [12].

Remark 1. In order to reduce the number of couples (SP, AP) to be considered one can use the following $\mathbb{Z}_2 \times \mathbb{Z}_2$-*action*. Its first generator g_1 changes a given polynomial $P(x)$ to $(-1)^d P(-x)$ thereby changing every second sign of the SP and replacing the AP (pos, neg) by the AP (neg, pos). The second generator g_2 changes $P(x)$ to $P^R(x) := x^d P(1/x)/P(0)$ which means reading the SP backward and preserving the AP (the roots of the *reverted polynomial* P^R are the reciprocals of the roots of P). The generators g_1 and g_2 are commuting involutions. Given a couple (SP, AP) (denoted by λ), the couples λ and $g_1(\lambda)$ are always different, because the second signs of their SPs are different, but one might have $g_2(\lambda) = \lambda$ or $g_1 g_2(\lambda) = \lambda$.

Thus orbits of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action consist of 4 or 2 couples (SP, AP). E.g. for $d = 2$, one has the orbit $((+,-,-),(1,1))$, $((+,+,-),(1,1))$ of length 2; for $d = 3$, the orbit

$$((+,+,+,-)\ ,\ (1,2))\ , \qquad ((+,-,+,+)\ ,\ (2,1))\ ,$$

$$((+,-,-,-)\ ,\ (1,2))\ , \qquad ((+,+,-,+)\ ,\ (2,1))$$

is of length 4. It is clear that all 4 or 2 couples (SP, AP) of a given orbit are simultaneously (non)realizable.

In each of the cases $d = 4$ and $d = 5$ there is exactly one example of non-realizability of a couple (SP, AP) modulo the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, namely

$$(1.2) \qquad (\ (+,+,-,+,+)\ ,\ (2,0)\ ) \ \text{ and } \ (\ (+,+,-,+,-,-)\ ,\ (3,0)\ )\ ,$$

see [9] and [1] respectively. For each of these two couples (SP, AP) one has $g_2(\lambda) = \lambda$, see Remark 1, so they define orbits of length 2. For $d = 6$, 7 and 8, there are respectively 4, 6 and 19 non-realizable cases modulo the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, see [1], [6] and [12].

Proposition 1. *For $d = 5$, there are 22 realizable and no non-realizable orbits of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action of length 4 and 13 realizable and one non-realizable orbits of length 2.*

The proposition is proved in Section 3 after Remarks 3.

In [13] the *discriminant set* of the family of polynomials $x^4 + x^3 + ax^2 + bx + c$ is represented (i.e. the set of values of the triple (a, b, c) for which the polynomial

has a multiple real root) and thus the non-realizability of the first of the two cases (1.2) is explained geometrically. In the present paper we give such an explanation for the non-realizability of the second of these cases and of the realizability of all other cases with $d = 5$. One can assume that the first two signs of the SP are $(+,+)$. Recall that $a_d = 1$. The change $P(x) \mapsto P(a_{d-1}x)/a_{d-1}^d$ transforms $P(x)$ into $x^d + x^{d-1} + \cdots$, i.e. one can normalize the first two coefficients. So we consider the 4-parameter family of polynomials

$$P(x, a, b, c, d) := x^5 + x^4 + ax^3 + bx^2 + cx + d \ , \tag{1.3}$$

with a, b, c, $d \in \mathbb{R}$. We denote by Δ the *discriminant set*

$$\Delta \ := \ \{ \ (a, b, c, d) \in \mathbb{R}^4 \mid \mathrm{Res}(P, \partial P/\partial x, x) = 0 \ \} \ . \tag{1.4}$$

A polynomial of the family P has a multiple real root exactly if $(a, b, c, d) \in \Delta$. Our aim is to explain by means of pictures of the set Δ why the second of the cases (1.2) is not realizable. These pictures are given in Section 3. In Section 2 we remind some properties of the set Δ and we explain the notation used on the pictures.

2. Properties of the discriminant set

The set Δ partitions $\mathbb{R}^4 \setminus \Delta$ into three open domains, in which a polynomial of the family P has 5, 3 or 1 simple real roots and hence 0, 1 or 2 conjugate pairs respectively (for properties of discriminants see [2]). On the figures these domains are indicated by the letters h, t and s respectively. We remind that polynomials of the domain h (i.e. with all roots real) are called *hyperbolic*; the set of values of the parameters (a, b, c, d) for which the polynomial P is hyperbolic is called the *hyperbolicity domain* of the family (1.3). The domain h, contrary to the domains t and s, is not present on all figures, and when it is present, it is bounded; it is a curvilinear quadrigon or triangle, see part (2) of Remarks 2. The set Δ and the coordinate hyperplanes together partition the set

$$\mathbb{R}^4 \ \setminus \ \{ \ \Delta \ \cup \ \{ \ a = 0 \ \} \ \cup \ \{ \ b = 0 \ \} \ \cup \ \{ \ c = 0 \ \} \ \cup \ \{ \ d = 0 \ \} \ \} \tag{2.5}$$

into open domains in each of which both the number of real roots and the signs of the coefficients of the polynomial P remain the same; in fact, the real roots are distinct and nonzero hence the number of positive and negative roots is the same in each of the domains. The non-realizability of the second of the cases (1.2) is explained by the absence of the corresponding domain.

Remark 2. It would be interesting to (dis)prove that each of the open domains of the set (2.5) is contractible and that to each realizable case (SP, AP) there corresponds exactly one of these domains.

Remark 3. For $d = 6$, 7 and 8, the following *neighbouring property* holds true (the property can be checked directly using the results of [1], [6] and [12]): For each two non-realizable orbits C_0, C_* of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action one can find a finite string of such orbits C_1, C_2, …, $C_s = C_*$ such that for each two of the orbits C_i and C_{i+1} of this string there exist couples (SP, AP) $C_i^0 \in C_i$, $C_{i+1}^0 \in C_{i+1}$, such that either the SPs of C_i^0 and C_{i+1}^0 differ only by one sign and their APs are the same, or their SPs and one of the components of their APs are the same while the other components of the APs differ by ± 2.

Example: for $d = 6$, the non-realizable cases are the ones of the orbits of the following couples (SP, AP), see [1]:

$$\begin{array}{ccccc} (\ (+,+,-,+,-,-,+)\ ,\ (4,0)\ ) & , & (\ (+,+,-,+,-,+,+)\ ,\ (2,0)\ ) \\ (\ (+,+,-,+,-,+,+)\ ,\ (4,0)\ ) & \text{and} & (\ (+,+,-,+,+,+,+)\ ,\ (2,0)\ )\ . \end{array}$$

It is clear that they are neighbouring.

If the couples C_i^0 and C_{i+1}^0 were realizable, then they would correspond to two domains of the set (2.5) separated by a hypersurface, either by Δ or by one of the coordinate hyperplanes.

For $d = 9$, the neighbouring property does not hold true. Indeed, for $d = 9$, there exists a single non-realizable case (modulo the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action) with both components of the AP nonzero, this is the couple $C_\sharp := ((+,-,-,-,-,+,+,+,+,-),(1,6))$, see [3]. There exist non-realizable cases in which one of the components of the AP equals 0, see [4]. However there are no non-realizable couples in which the AP equals $(1,8)$, $(1,4)$, $(3,6)$, $(8,1)$, $(4,1)$ or $(6,3)$. For $d = 9$, there exist non-realizable cases in which one of the components of the AP equals 0, see [5]. Hence if C_0 is an orbit of a couple (SP, AP) with one of the components of the AP equal to 0 and C_* is the orbit of the couple $C_\sharp$, then one cannot construct the string of orbits C_i.

The set Δ is *stratified*. Its strata are defined by the *multiplicity vectors* of the real roots of the polynomials of the family P (in the case of two conjugate pairs, we do not specify whether these pairs are distinct or not). The notation which we use for the strata should be clear from the following example:

Example 1. There is a single stratum $\mathcal{T}_5$ corresponding to a polynomial with a five-fold real root. This polynomial is

$$(x + 1/5)^5 = x^5 + x^4 + 2x^3/5 + 2x^2/25 + x/125 + 1/3125 \tag{2.6}$$

and the stratum $\mathcal{T}_5$ is of dimension 0 in $\mathbb{R}^4$. The strata $\mathcal{T}_{4,1}$, $\mathcal{T}_{3,2}$, $\mathcal{T}_{2,3}$ and $\mathcal{T}_{1,4}$ are of dimension 1 in $\mathbb{R}^4$; they correspond to polynomials of the form $(x-x_1)^m(x-x_2)^{5-m}$, where $x_1 < x_2$ and $m = 4$, 3, 2 and 1 respectively; hence $mx_1+(5-m)x_2 = -1$. The stratum $\mathcal{T}_3$ is of dimension 2 and corresponds to polynomials $(x-x_1)^3(x^2+ux+v)$, where $u^2 < 4v$.

Remarks 1. (1) The dimension of a stratum is equal to the number of distinct roots (real or complex) minus 1; we subtract 1, because the sum of all roots equals (-1). Thus $\mathcal{T}_{4,1}$, $\mathcal{T}_{3,2}$, $\mathcal{T}_{2,3}$ and $\mathcal{T}_{1,4}$ are the only strata of dimension 1. As $d = 5$, i.e. as d is odd, there is always at least one real root, so a stratum corresponding to polynomials having at least one conjugate pair (hence to polynomials having ≥ 3 distinct roots) is of dimension ≥ 2. This is the case of the stratum $\mathcal{T}_3$.

(2) The tangent space at any point of any stratum of dimension 1, 2 or 3 is transversal to the space *Obcd*, *Ocd* or *Od* respectively. This follows from [11, Theorem 2].

On Fig. 1 and Fig. 2 we show the projections in the (a, b)-plane of the strata $\mathcal{T}_5$, $\mathcal{T}_{4,1}$, $\mathcal{T}_{3,2}$, $\mathcal{T}_{2,3}$ and $\mathcal{T}_{1,4}$. The union of the projections of the three strata $\mathcal{T}_{4,1}$, $\mathcal{T}_5$ and $\mathcal{T}_{1,4}$ (resp. $\mathcal{T}_{3,2}$, $\mathcal{T}_5$ and $\mathcal{T}_{2,3}$) is an algebraic curve drawn by a solid (resp. dashed) line and having a cusp at the projection of $\mathcal{T}_5$; the coordinates of the projection of $\mathcal{T}_5$ are (2/5, 2/25), see (2.6). When following a vertical line (i.e. parallel to

the b-axis) from below to above, the projections of the strata are intersected in the following order: $\mathcal{T}_{4,1}$, $\mathcal{T}_{3,2}$, $\mathcal{T}_{2,3}$, $\mathcal{T}_{1,4}$. These projections and the a- and b-axes define 15 open *zones* in $\mathbb{R}^2$ (the space Oab), denoted by A, B, ..., M, N and P.

The SPs which we use begin with $(+,+)$. In the right upper corner of Fig. 1 the notation $\sigma = (+,+,+,+,\tilde{\sigma})$ means that when one chooses the values of the variables (a,b) from the first quadrant, then this defines the SP σ, in which $\tilde{\sigma}$ stands for the couple of signs of the variables (c,d) (and similarly for the other three corners of Fig. 1). Recall that in the plane, the four open quadrants correspond to the following couples of signs of the two coordinates:

$$\text{I}: \ (+,+), \ \text{II}: \ (-,+), \ \text{III}: \ (-,-) \text{ and } \text{IV}: \ (+,-) \ .$$

Notation 1. Further in the text, we use the following notation: $\sigma_{i,j}$ means that the signs of the variables (a,b) correspond to the ith and the ones of the variables (c,d) to the jth quadrant. Thus the SPs $(+,+,-,+,+,-)$ and $(+,+,+,+,-,-)$ are denoted by $\sigma_{2,4}$ and $\sigma_{1,3}$ respectively.

We explain now the meaning of the pictures. On Fig. 1, 2, 3 and 4 we represent the plane (a,b); the a-axis is horizontal and the b-axis is vertical. On the rest of the figures we represent the plane (c,d); the c-axis is horizontal and the d-axis is vertical. We fix a value (a_0,b_0) of the couple (a,b) from one of the domains A, ..., N, P, and we draw the set $\Delta^\sharp := \Delta|_{(a,b)=(a_0,b_0)}$. The figures thus obtained resemble the ones given in [15] in relationship with the butterfly catastrophe. Indeed, in the definition of the latter one uses a degree 5 monic polynomial family S with vanishing coefficient of x^4. The family $P(x,a,b,c,d)$, see (1.3), is obtained from S via the shift $x \mapsto x+1/5$ which means making an upper-triangular affine transformation in the space of coefficients. The convexity of the curves shown on the figures results from the following theorem of I. Méguerditchian, see [14, Proposition 1.3.3], which is a generalization of a result of B. Chevallier, see [5].

Theorem 1. *Locally the discriminant set Δ at a point, where it is smooth, belongs entirely to one of the two half-spaces defined by its tangent hyperplane, namely, the one, where the polynomial P has two more real roots.*

We use also another result of [14]:

Lemma 1. [Lemma about the product] *Suppose that P_1, ..., P_s are monic polynomials, where for $i \neq j$, the polynomials P_i and P_j have no root in common. Set $P := P_1 \cdots P_s$. Then there exist open neighbourhoods U_k of P_k and U of P such that for $Q_k \in U_k$, the mapping*

$$U_1 \times \cdots \times U_s \to U \quad , \quad (Q_1, \ldots, Q_s) \mapsto Q_1 \cdots Q_s$$

is a diffeomorphism.

Remarks 2. (1) From Lemma 1 one can deduce what local singularities of the sets $\Delta^\sharp$ can be encountered. At a point (a,b,c,d) for which the polynomial $\tilde{P} := P(x,a,b,c,d)$ has one double and one or three simple roots the set $\Delta^\sharp$ is smooth. Indeed, according to Lemma 1 in this case the set $\Delta^\sharp$ is locally diffeomorphic to the cartesian product of the discriminant set of the family $x^2 + ux + v$, u, $v \in \mathbb{R}$ (which is the curve $u^2 = 4v$), and $\mathbb{R}^2$.

At a point where $\tilde{P}$ has a triple real root and 2 or 0 simple real roots, the set $\Delta^\sharp$ is diffeomorphic to the cartesian product of a semi-cubic parabola (i.e. a cusp) and

$\mathbb{R}^2$. Indeed, the discriminant set of the family of polynomials $S_1 := x^3 + ux + v$, u, $v \in \mathbb{R}$, is the curve $27v^2+4u^3 = 0$. The family $S_2 := x^3+wx^2+u^*x+v^*$ is obtained from S_1 via the shift $x \mapsto x+w/3$; here $u^* = u+w^2/3$ and $v^* = v+uw/3+w^3/27$. On the figures cusp points are denoted by κ, λ and μ.

At a point where $\tilde{P}$ has one simple and two double roots, the set $\Delta^\sharp$ is locally diffeomorphic to the cartesian product of two transversally intersecting smooth curves and $\mathbb{R}^2$, see Lemma 1. On the figures, such points are denoted by ϕ, ψ or θ.

At a point where $\tilde{P}$ has a triple and a double real roots, the set $\Delta^\sharp$ is locally diffeomorphic to a cartesian product of $\mathbb{R}^2$ and the union of a semi-cubic parabola and a smooth arc passing through the cusp point and transversal to the geometric tangent at the cusp point. Such points belong to the strata $\mathcal{T}_{2,3}$ and $\mathcal{T}_{3,2}$. We do not show such sets $\Delta^\sharp$ on the pictures.

Finally, if $\tilde{P}$ has a quadruple and a simple real roots (such points belong to the strata $\mathcal{T}_{1,4}$ and $\mathcal{T}_{4,1}$), then locally the set $\Delta^\sharp$ is diffeomorphic to the cartesian product of a swallowtail and $\mathbb{R}$. For a picture of a swallowtail see [15].

On the figures the letters α and ω denote the "infinite branches" of the sets $\Delta^\sharp$.

There are no vertical tangent lines at any point of any of the sets $\Delta^\sharp$, see part (2) of Remarks 1.

(2) On the figures the hyperbolicity domain (denoted by h) is represented by the following curvilinear triangles or quadrigons:

$\lambda\mu\phi$	Figures	6 (right),	8 (left),	16	and 17 (right) ;
$\lambda\theta\psi\phi$	Figures	7 (left),	9 (left),	15	and 18 (left) ;
$\lambda\theta\kappa$	Figures	11,	14,	19 (left)	.

How the set Δ looks like near the origin is justified by the following lemma:

Lemma 2. *In the space of the parameters (a, b, c, d), a point with $c = d = 0$ belongs to the discriminant set Δ. If $b \neq 0 = c = d$, then the set Δ is tangent to the hyperplane $d = 0$. For $b > 0$ and $b < 0$ the set belongs locally to the half-plane $d \geq 0$ and $d \leq 0$ respectively.*

Proof. For $c = d = 0$, the number 0 is a double root of P which proves the first claim of the lemma. For $b \neq 0 = c = d$, the polynomial P is locally representable in the form $(x+\varepsilon)^2(g+hx+ux^2+x^3)$,where $g \neq 0$. Hence $c = 2\varepsilon g+\varepsilon^2 h$ and $d = g\varepsilon^2$, i.e. the set Δ is locally defined by an equation of the form $d = c^2/4g + o(c^2)$ which proves the second claim. For $b > 0$ and $b < 0$ one has $g > 0$ and $g < 0$ respectively which proves the last statement of the lemma. □

3. Pictures representing the discriminant set

On Figures 5-20 we show the set $\Delta^\sharp$ for values of (a, b) from the different zones shown on Figures 1 and 2. After each figure we indicate which cases are realizable in the domains h, t and s, see Notation 1. Whenever a figure consists of two pictures, the one on the right is a detailed picture close to the origin. Under each of Fig. 5-20 we indicate the cases which are realizable in the different parts of the plane (c, d) delimited by the coordinate axes and the corresponding set $\Delta^\sharp$. E.g. when after Fig. 4 under "domain t" we write "5 , 9 $\sigma_{2,1}$ (2,1) , (0,3)", this means that in the two parts of the domain t in the first quadrant the cases ($\sigma_{2,1}$,(2,1)) and ($\sigma_{2,1}$, (0,3)) are realizable. The numbers 5 and 9 are numbers of different cases. These numbers are attributed in the order of appearance of the cases. When one and the

same case appears in different zones, then it bears the same number. There are two figures corresponding to zone E, see the lines following Fig. 9.

Remarks 3. (1) The following four rules hold true. They allow to define by continuity the case (SP, AP) which is realizable in any domain of the (c, d)-plane for (a, b) fixed.

i) When the c-axis is crossed at a point not belonging to the set $\Delta^\sharp$ and different from the origin, then exactly one real root changes sign. When a hyperplane $a = 0$, $b = 0$ or $c = 0$ is crossed, then only the corresponding sign in the SP changes.

ii) On all pictures, in the part of the domain s which is above the c-axis, the only real root of the polynomial P is negative. Indeed, for a, b and c fixed and $d > 0$ large enough, the polynomial P has only one real root which is simple and negative.

iii) At a cusp point belonging to the closure of the domain t (but not h) the triple root of P has the same sign as the single real root in the adjacent s-domain.

iv) Suppose that a point follows the arc of the set $\Delta^\sharp$ which passes through the point $(0, 0)$ in the plane (c, d). Then when the point passes from the first into the second quadrant or vice versa, a double real root of P changes sign.

v) The AP corresponding to a point of the set (2.5) and belonging to the domain h is the Descartes' pair which is defined by the SP. This allows, for each of the Figures 5-20 containing the domain h, to find the cases realizable in each of the parts of the domain h.

vi) At a self-intersection point ϕ or ψ one of the open sectors defined by the two intersecting arcs of $\Delta^\sharp$ belongs to the domain s and its opposite sector belongs to the domain h. The other two sectors (denoted here by S_1 and S_2) belong to the domain t. When a point moves from ϕ or ψ into S_1, then one of the two double roots of P gives birth to a complex conjugate pair of roots. When a point moves from ϕ or ψ into S_2, then the other one of the two double roots of P gives birth to such a pair. This follows from Lemma 1.

(2) To the possible Descartes' pairs of the SPs beginning with $(+, +)$, i.e. to $(0, 5)$, $(1, 4)$, $(2, 3)$, $(3, 2)$ and $(4, 1)$, there correspond 3, 3, 4, 4 and 3 possible APs respectively. E. g. to the Descartes' pair $(2, 3)$ there correspond the possible APs $(2, 3)$, $(0, 3)$, $(2, 1)$ and $(0, 1)$. This means that for the four quadrants in the plane (a, b) one obtains the following numbers of a priori possible couples (SP, AP) (we list also the zones of each quadrant on Fig. 1-2 and the possible couples (SP, Descartes' pair) for the given quadrant):

$$\begin{array}{lllll}
\text{I}: & 13 & L, M, N, P & (\sigma_{1,1}, (0, 5)), (\sigma_{1,2}, (2, 3)), (\sigma_{1,3}, (1, 4)), (\sigma_{1,4}, (1, 4)) \ , \\
\text{II}: & 15 & A, B, C & (\sigma_{2,1}, (2, 3)), (\sigma_{2,2}, (4, 1)), (\sigma_{2,3}, (3, 2)), (\sigma_{2,4}, (3, 2)) \ , \\
\text{III}: & 15 & D, E, F, G & (\sigma_{3,1}, (2, 3)), (\sigma_{3,2}, (2, 3)), (\sigma_{3,3}, (1, 4)), (\sigma_{3,4}, (3, 2)) \ , \\
\text{IV}: & 15 & H, I, J, K & (\sigma_{4,1}, (2, 3)), (\sigma_{4,2}, (2, 3)), (\sigma_{4,3}, (1, 4)), (\sigma_{4,4}, (3, 2)) \ .
\end{array}$$

The non-realizable couple (the second of couples (1.2)) corresponds to quadrant II.

(3) On Fig. 3-4 we show the projections in the plane (a, b) of the strata $\mathcal{T}_5$ and $\mathcal{T}_{i,j}$, $1 \leq i, j \leq 4$, $i + j = 5$, (see Fig. 1-2), and of the set $\mathcal{M}$ of values of (a, b) for

which the set $\Delta^\sharp$ has a self-intersection at $c = d = 0$. This is the set for which the polynomial $x^3 + x^2 + ax + b$ has a multiple root. It is drawn by a dotted line. It

(a) has a cusp point $(1/3, 1/27)$ situated on the projection of the stratum $\mathcal{T}_{3,2}$;

(b) is tangent to the projection of $\mathcal{T}_{2,3}$ at $(a, b) = (1/4, 0)$;

(c) is tangent to the projection of $\mathcal{T}_{1,4}$ at $(a, b) = (0, 0)$;

(d) intersects the projection of $\mathcal{T}_{3,2}$ and $\mathcal{T}_{2,3}$ at $(a, b) =$

$$(-8 - 4\sqrt{10})/15, (-152(2 + \sqrt{10}) + 52)/675) \quad = \quad (-1.37\ldots, -1.39\ldots) \text{ and}$$

$$(-8 + 4\sqrt{10})/15, (-152(2 - \sqrt{10}) + 52)/675) \quad = \quad (0.30\ldots, 0.03\ldots)$$

respectively;

(e) has no point in common with the projection of the stratum $\mathcal{T}_{4,1}$.

At these points the polynomial P and the set $\Delta^\sharp$ have respectively

(a) P: a triple negative root and a double root at 0, $\Delta^\sharp$: a cusp point at $c = d = 0$ with a non-horizontal tangent line at it and a smooth arc with a horizontal tangent line at $c = d = 0$;

(b) P: a double negative root and a triple root at 0, $\Delta^\sharp$: a cusp at $c = d = 0$ with a horizontal tangent line at it and a smooth arc with a non-horizontal tangent line at $c = d = 0$;

(c) P: a negative a simple root and a quadruple root at 0, $\Delta^\sharp$: a 4/3-singularity with a horizontal tangent line at $c = d = 0$;

(d) at $\mathcal{T}_{3,2}$: P: a simple positive, a double negative and a double 0 root, $\Delta^\sharp$: two transversally intersecting arcs at $(0; 0)$ one of which with a horizontal tangent line; at $\mathcal{T}_{2,3}$: P: a simple and a double negative and a double 0 root, $\Delta^\sharp$: two transversally intersecting arcs at $(0; 0)$ one of which with a horizontal tangent line.

We do not include the set $\mathcal{M}$ in the partition of the plane (a, b) into zones in order to keep the number of figures to be drawn reasonably low. Some changes of the relative position of the cusps of the set $\Delta^\sharp$ and the coordinate axes c and d as the values of a and b change are commented between the figures.

(4) Two SPs, one corresponding to the third and one to the fourth quadrant in the (a, b)-plane, begin by $(+, +, -, -)$ and $(+, +, +, -)$ respectively. Hence if their last two signs are the same, then they contain one and the same number of sign changes and sign preservations. This means that one and the same APs correspond to them. Therefore the two couples (SP, AP), $(\sigma_{3,j}, (k_1, k_2))$ and $(\sigma_{4,j}, (k_1, k_2))$, are simultaneously realizable.

Proof of Proposition 1. We use Notation 1 and the definition of the generators g_1 and g_2 of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action, see Remark 1. We consider only SPs beginning with $(+, +)$ which means that we deal with halves of orbits (the other halves are with SPs beginning with $(+, -)$, see Remark 1). We consider the action of g_1 and g_2 not only on couples (SP, AP), but also just on SPs. Thus

$$g_2(\sigma_{2,1}) = \sigma_{4,1} \text{ and } g_2(\sigma_{1,3}) = \sigma_{3,3} \ . \tag{3.7}$$

The Descartes' pair corresponding to $\sigma_{2,1}$ (resp. $\sigma_{1,3}$) equals $(2, 3)$ (resp. $(1, 4)$), see part (2) of Remarks 3. We denote by ρ any of the APs, so $\rho = (2, 3)$, $(0, 3)$, $(2, 1)$ or $(0, 1)$ (resp. $\rho = (1, 4)$, $(1, 2)$ or $(1, 0)$). Thus $((\sigma_{2,1}, \rho), (\sigma_{4,1}, \rho))$ (resp. $((\sigma_{1,3}, \rho), (\sigma_{3,3}, \rho))$) are half-orbits; the other halves are of the form $(g_1((\sigma_{2,1}, \rho)), g_1((\sigma_{4,1}, \rho)))$ (resp. $(g_1((\sigma_{1,3}, \rho)), g_1((\sigma_{3,3}, \rho)))$), see Remark 1. Thus we have described $4 + 3 = 7$ orbits of length 4. Next, one has

(3.8)
$$g_1g_2(\sigma_{1,2}) = \sigma_{4,4}\ ,\ g_1g_2(\sigma_{2,2}) = \sigma_{1,4}\ ,\ g_1g_2(\sigma_{3,2}) = \sigma_{2,4} \text{ and } g_1g_2(\sigma_{4,2}) = \sigma_{3,4}\ .$$

The half-orbits in the case of $\sigma_{1,2}$ are of the form $((\sigma_{1,2}, \rho), (\sigma_{4,4}, \rho^R))$, where ρ^R is obtained from ρ by exchanging the two components, similarly for the other cases in (3.8). Thus one obtains $4 + 3 + 4 + 4 = 15$ more orbits of length 4, see part (2) of Remarks 3.

Finally, one obtains

(3.9) $$g_2(\sigma_{1,1}) = \sigma_{1,1}\ ,\ g_2(\sigma_{2,3}) = \sigma_{2,3}\ ,\ g_2(\sigma_{3,1}) = \sigma_{3,1} \text{ and } g_2(\sigma_{4,3}) = \sigma_{4,3}\ ,$$

so by analogy with the SPs involved in (3.7) one obtains half-orbits of length 1 hence orbits of length 2. Their quantity is $3 + 4 + 4 + 3 = 14$. One of them is the only non-realizable orbit, see the second couple in (1.2). There remains to notice that each possible SP $\sigma_{i,j}$ participates in exactly one of the equalities (3.7), (3.8) and (3.9), and to remind that the generators g_1 and g_2 are commuting involutions. Hence we have described all orbits of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-action. □

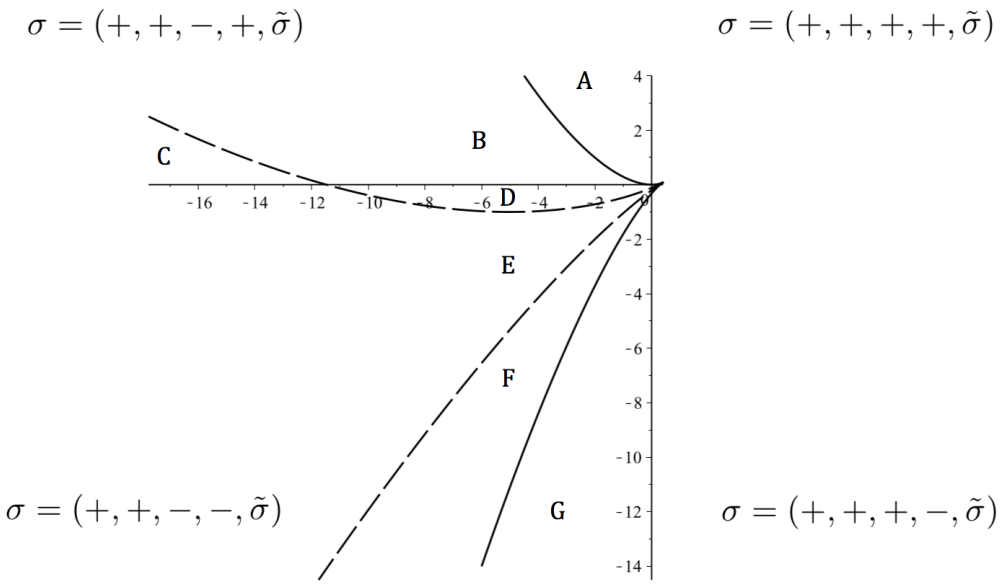

FIGURE 1. The projection of the discriminant locus of P to the plane of the parameters (a, b).

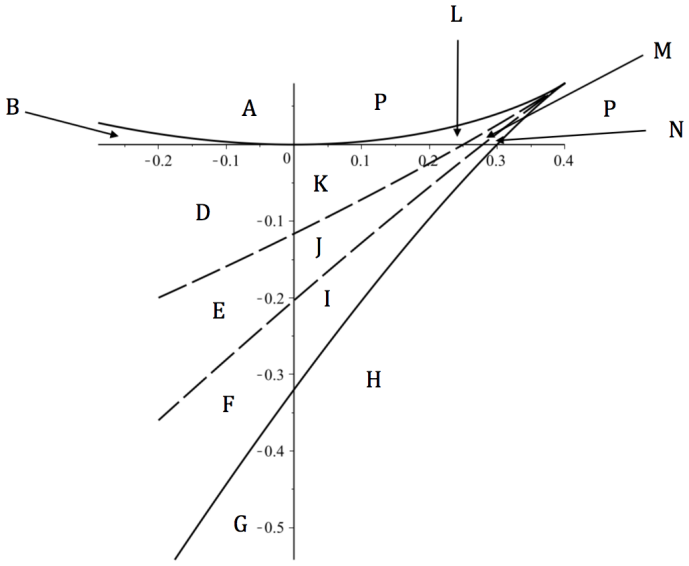

FIGURE 2. Picture of the projection in the plane (a, b) of the discriminant locus of P with an enlarged portion near the cusp point.

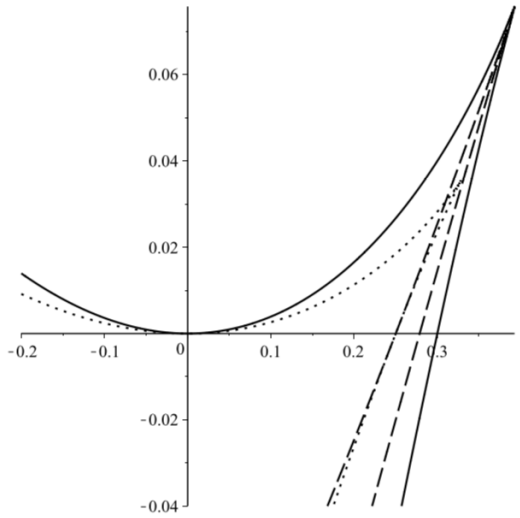

FIGURE 3. The projections in the plane (a, b) of the strata $\mathcal{T}_5$ and $\mathcal{T}_{i,j}$, $1 \leq i, j \leq 4$, $i + j = 5$, and of the set $\mathcal{M}$.

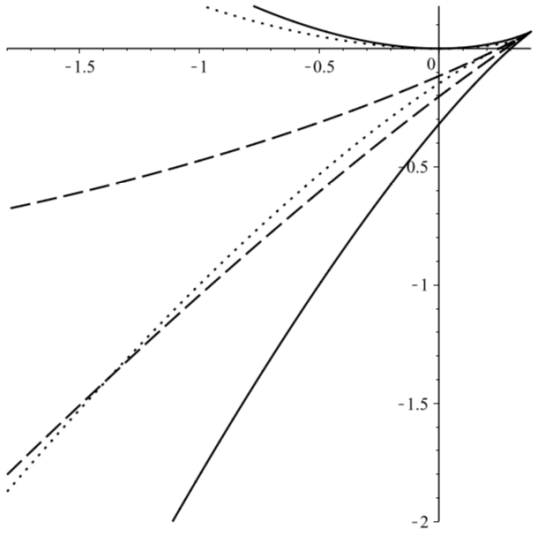

FIGURE 4. The projections in the plane (a, b) of the strata $\mathcal{T}_5$ and $\mathcal{T}_{i,j}$, $1 \leq i, j \leq 4$, $i + j = 5$, and of the set $\mathcal{M}$ (with enlarged portion near the cusp points).

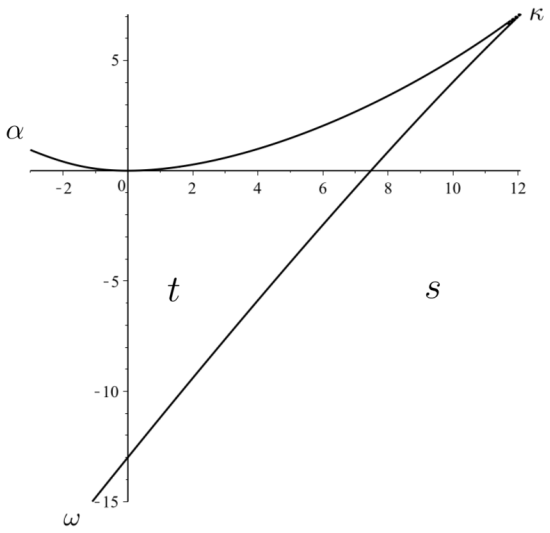

FIGURE 5. Zone A: The set $\Delta^\sharp$ for $a = -2$ and $b = 3$.

	domain s			domain t	
1	$\sigma_{2,1}$	(0,1)	5	$\sigma_{2,1}$	(0,3)
2	$\sigma_{2,2}$	(0,1)	6	$\sigma_{2,2}$	(2,1)
3	$\sigma_{2,3}$	(1,0)	7	$\sigma_{2,3}$	(1,2)
4	$\sigma_{2,4}$	(1,0)	8	$\sigma_{2,4}$	(1,2)

Cases 2 and 6 are realizable in the open second quadrant, above and below the set $\Delta^\sharp$ respectively; cases 3 and 7 in the open third quadrant, below and above $\Delta^\sharp$ respectively; cases 4 and 8 in the open fourth quadrant, below and above $\Delta^\sharp$ respectively. Cases 1 and 5 occupy the open first quadrant minus the set $\Delta^\sharp$, case 5 the curvilinear triangle and case 1 the rest of the quadrant.

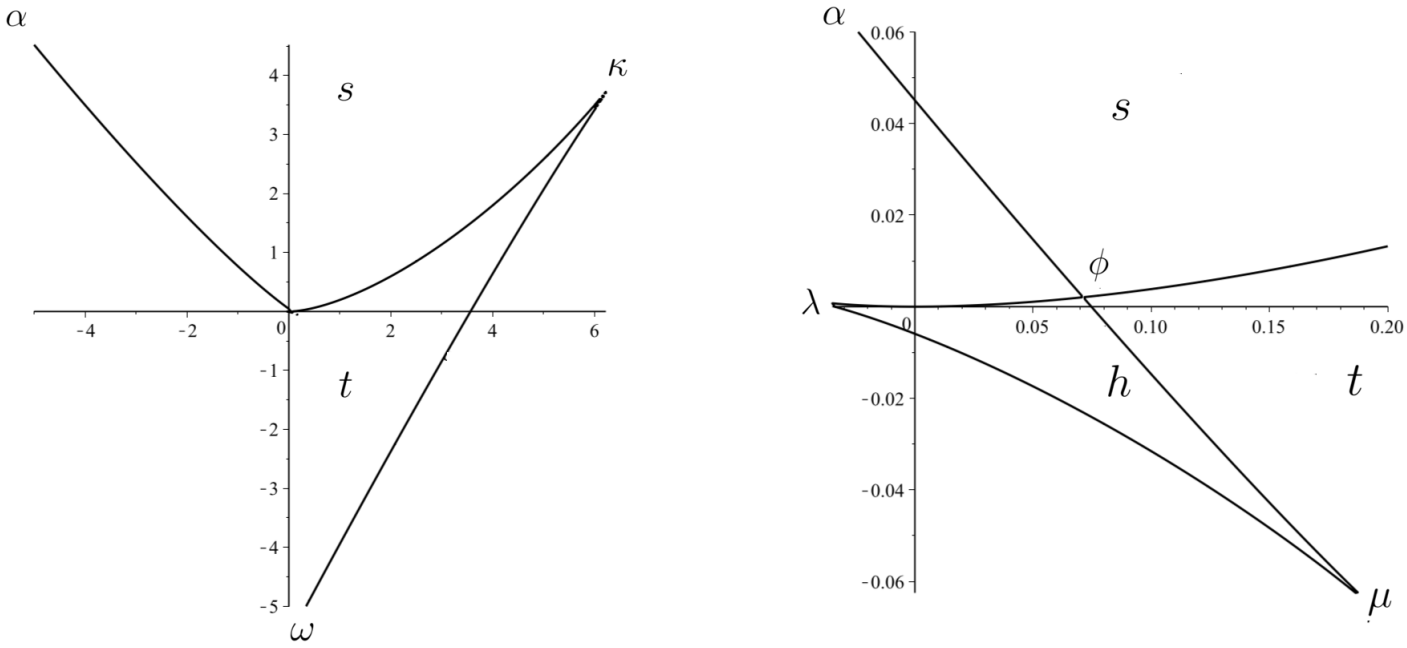

FIGURE 6. Zone B: The set $\Delta^\sharp$ for $a = -2$ and $b = 0.5$.

	domain s			domain t			domain h	
1	$\sigma_{2,1}$	(0,1)	5 , 9	$\sigma_{2,1}$	(0,3) , (2,1)	10	$\sigma_{2,1}$	(2,3)
2	$\sigma_{2,2}$	(0,1)	6	$\sigma_{2,2}$	(2,1)	11	$\sigma_{2,2}$	(4,1)
3	$\sigma_{2,3}$	(1,0)	7	$\sigma_{2,3}$	(1,2)	12	$\sigma_{2,3}$	(3,2)
4	$\sigma_{2,4}$	(1,0)	8	$\sigma_{2,4}$	(1,2)	13	$\sigma_{2,4}$	(3,2)

The infinite branch ω intersects the negative d-half-axis.

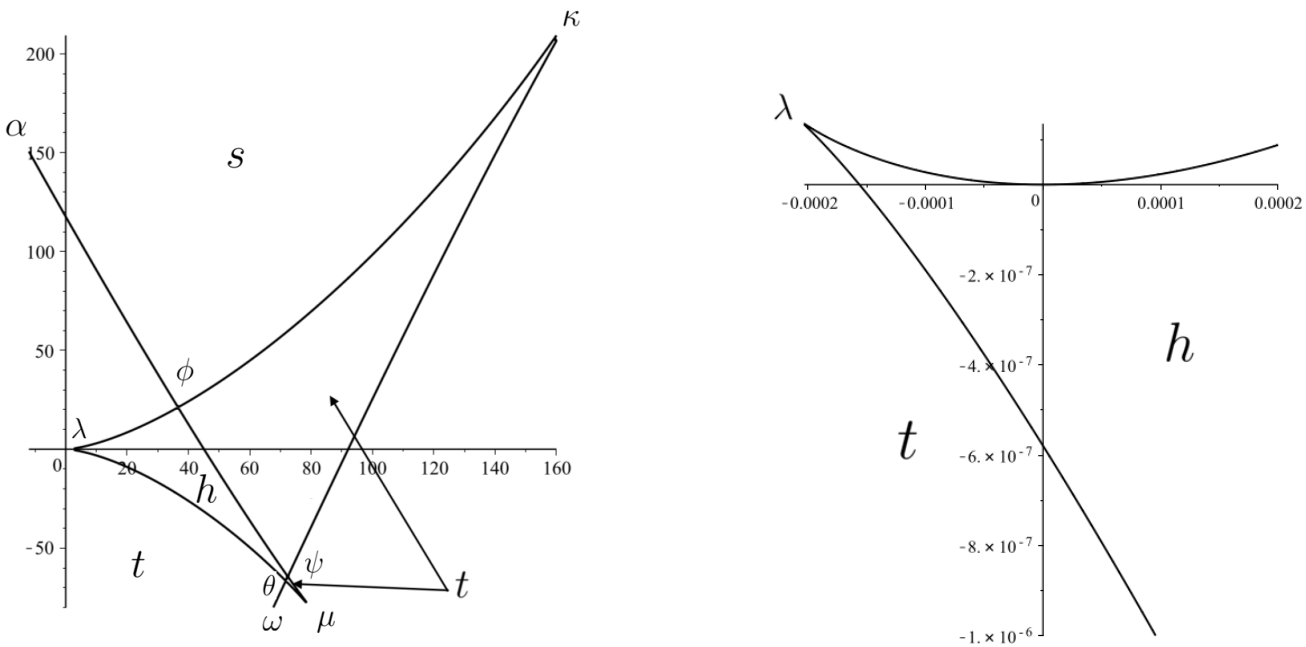

FIGURE 7. Zone C: The set $\Delta^\sharp$ for $a = -16$ and $b = 0.1$.

domain s			domain t			domain h		
1	$\sigma_{2,1}$	(0,1)	5 , 9	$\sigma_{2,1}$	(0,3) , (2,1)	10	$\sigma_{2,1}$	(2,3)
2	$\sigma_{2,2}$	(0,1)	6	$\sigma_{2,2}$	(2,1)	11	$\sigma_{2,2}$	(4,1)
3	$\sigma_{2,3}$	(1,0)	7	$\sigma_{2,3}$	(1,2)	12	$\sigma_{2,3}$	(3,2)
4	$\sigma_{2,4}$	(1,0)	8 , 14	$\sigma_{2,4}$	(1,2) , (3,0)	13	$\sigma_{2,4}$	(3,2)

On Fig. 7, the infinite branch ω intersects the negative d-half-axis. The intersection of the domain t with the fourth quadrant consists of three curvilinear triangles. In the one which borders the third quadrant the AP is $(1, 2)$ (as in the intersection of the domain t with the third quadrant), in the one which belongs entirely to the interior of the fourth quadrant it is $(3, 0)$ and in the one which borders the first quadrant it is again $(1, 2)$.

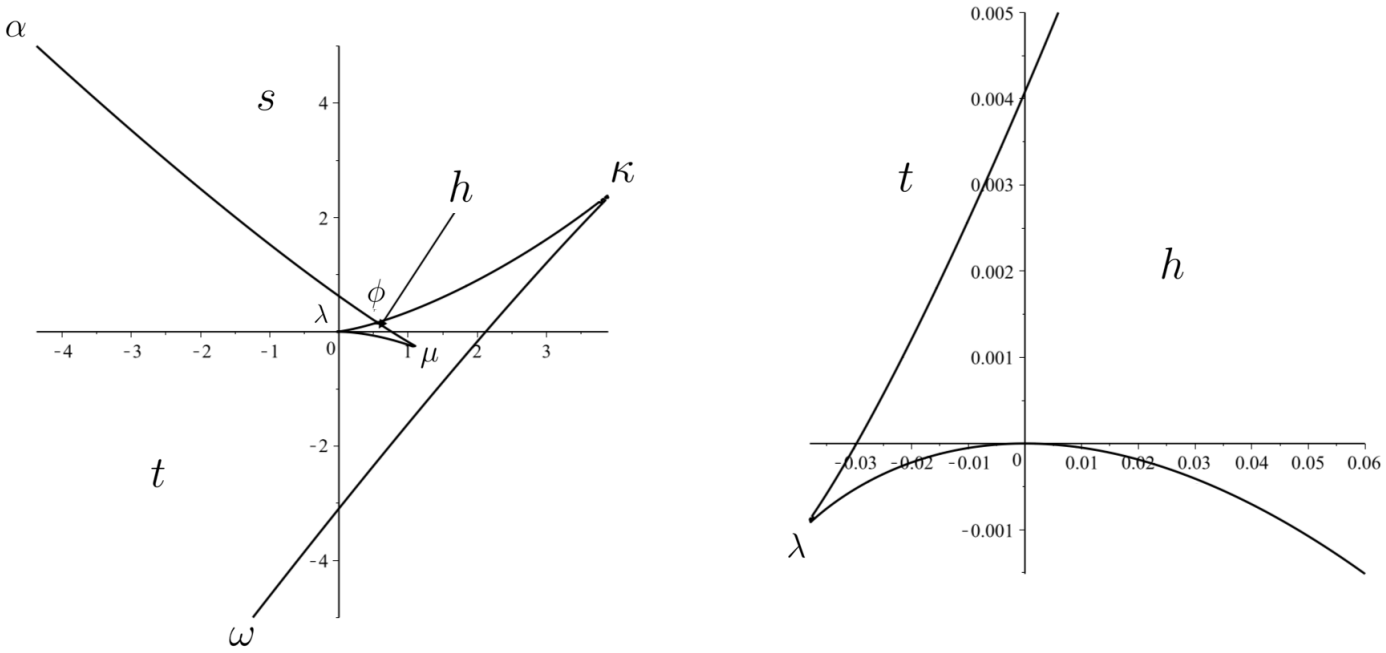

FIGURE 8. Zone D: The set $\Delta^\sharp$ for $a = -2$ and $b = -0.5$.

domain s			domain t			domain h		
15	$\sigma_{3,1}$	(0,1)	19 , 20	$\sigma_{3,1}$	(0,3) , (2,1)	24	$\sigma_{3,1}$	(2,3)
16	$\sigma_{3,2}$	(0,1)	21	$\sigma_{3,2}$	(2,1)	25	$\sigma_{3,2}$	(2,3)
17	$\sigma_{3,3}$	(1,0)	22	$\sigma_{3,3}$	(1,2)	26	$\sigma_{3,3}$	(1,4)
18	$\sigma_{3,4}$	(1,0)	23	$\sigma_{3,4}$	(1,2)	27	$\sigma_{3,4}$	(3,2)

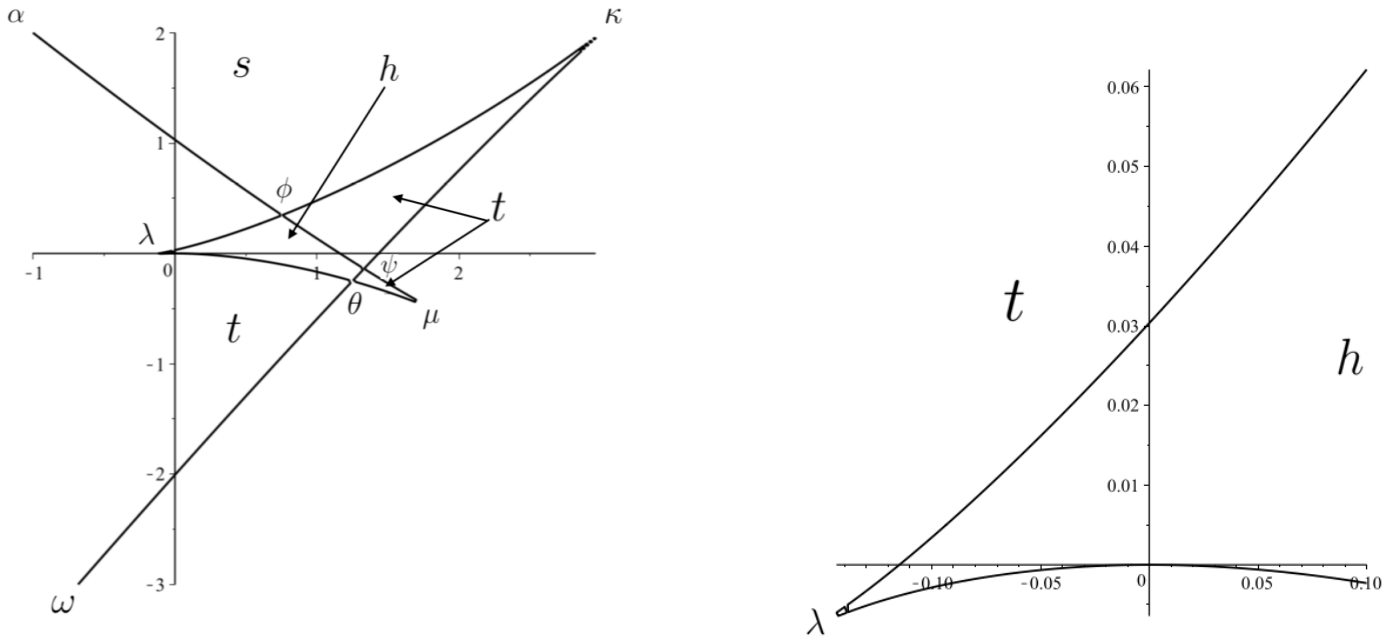

FIGURE 9. Zone E: The set $\Delta^\sharp$ for $a = -2$ and $b = -1$.

domain s			domain t			domain h		
15	$\sigma_{3,1}$	(0,1)	19 , 20	$\sigma_{3,1}$	(0,3) , (2,1)	24	$\sigma_{3,1}$	(2,3)
16	$\sigma_{3,2}$	(0,1)	21	$\sigma_{3,2}$	(2,1)	25	$\sigma_{3,2}$	(2,3)
17	$\sigma_{3,3}$	(1,0)	22	$\sigma_{3,3}$	(1,2)	26	$\sigma_{3,3}$	(1,4)
18	$\sigma_{3,4}$	(1,0)	23 , 28	$\sigma_{3,4}$	(1,2) , (3,0)	27	$\sigma_{3,4}$	(3,2)

On Fig. 9 the self-intersection point ϕ is to the right while on Fig. 10 it is to the left of the d-axis. This is why case 29 is present only on the second of these figures. On Fig. 9 there are two domains corresponding to case 23. If one compares Fig. 9 with Fig. 8 one sees that for $a = -2$, as b increases from -1 to -0.5, the two domains of case 23 fuse in one single domain.

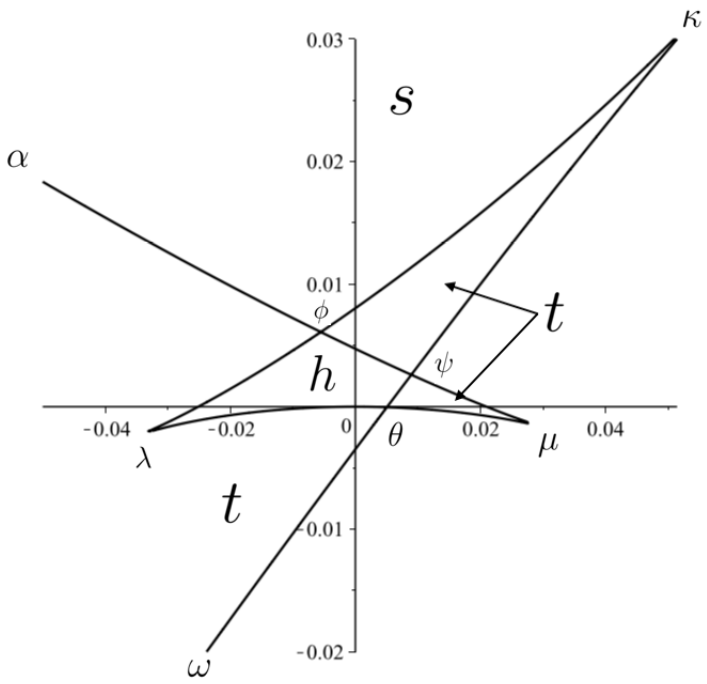

FIGURE 10. Zone E: The set $\Delta^\sharp$ for $a = -0.014$ and $b = -0.15$.

domain s			domain t			domain h		
15	$\sigma_{3,1}$	(0,1)	19 , 20	$\sigma_{3,1}$	(0,3) , (2,1)	24	$\sigma_{3,1}$	(2,3)
16	$\sigma_{3,2}$	(0,1)	21 , 29	$\sigma_{3,2}$	(2,1) , (0,3)	25	$\sigma_{3,2}$	(2,3)
17	$\sigma_{3,3}$	(1,0)	22	$\sigma_{3,3}$	(1,2)	26	$\sigma_{3,3}$	(1,4)
18	$\sigma_{3,4}$	(1,0)	23 , 28	$\sigma_{3,4}$	(1,2) , (3,0)	27	$\sigma_{3,4}$	(3,2)

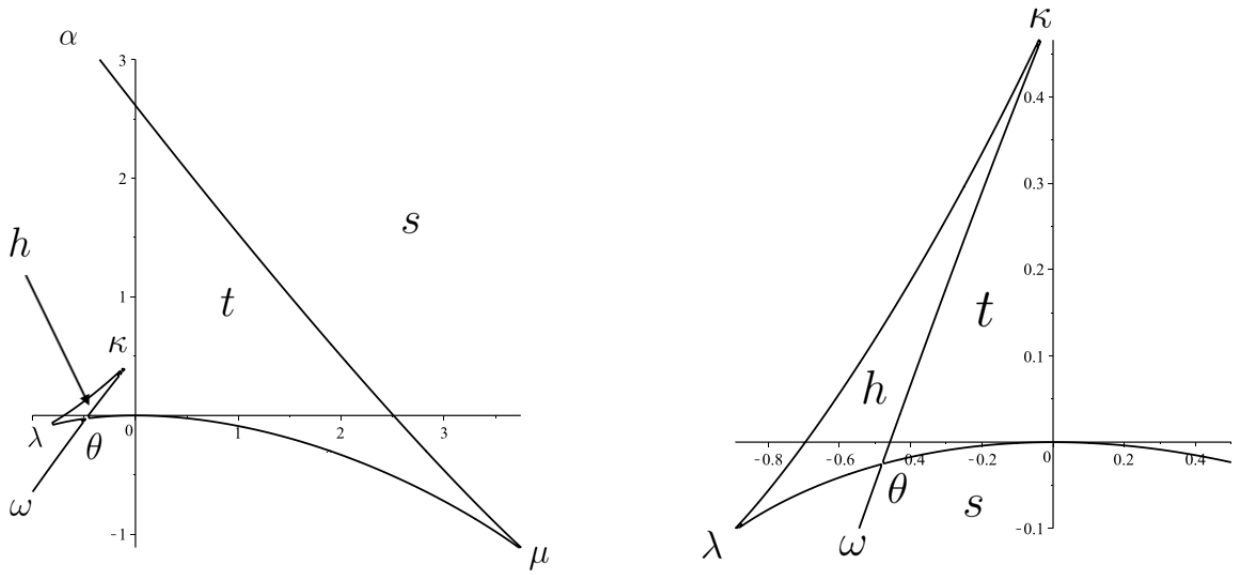

FIGURE 11. Zone F: The set $\Delta^\sharp$ for $a = -2$ and $b = -2.5$.

domain s			domain t			domain h		
15	$\sigma_{3,1}$	(0,1)	20	$\sigma_{3,1}$	(2,1)			
16	$\sigma_{3,2}$	(0,1)	21	$\sigma_{3,2}$	(2,1)	25	$\sigma_{3,2}$	(2,3)
17	$\sigma_{3,3}$	(1,0)	22	$\sigma_{3,3}$	(1,2)	26	$\sigma_{3,3}$	(1,4)
18	$\sigma_{3,4}$	(1,0)	28	$\sigma_{3,4}$	(3,0)			

The intersection of the domain t with the second quadrant consists of two parts. In both of them one and the same case is realizable.

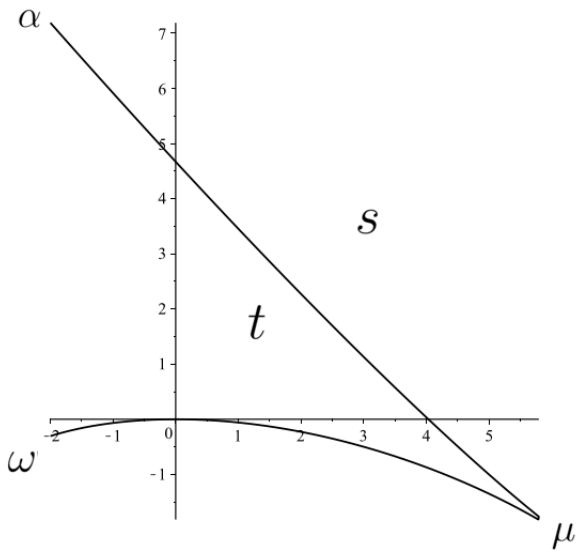

FIGURE 12. Zone G: The set $\Delta^\sharp$ for $a = -2$ and $b = -4$.

	domain s			domain t	
15	$\sigma_{3,1}$	(0,1)	20	$\sigma_{3,1}$	(2,1)
16	$\sigma_{3,2}$	(0,1)	21	$\sigma_{3,2}$	(2,1)
17	$\sigma_{3,3}$	(1,0)	22	$\sigma_{3,3}$	(1,2)
18	$\sigma_{3,4}$	(1,0)	28	$\sigma_{3,4}$	(3,0)

When comparing Fig. 11 and Fig. 12 it becomes clear that for $a = -2$ and for some $b = b_* \in (-4, -2.5)$, in the corresponding picture of the set $\Delta^\sharp$, the domain h intersects only the third, but not the second quadrant.

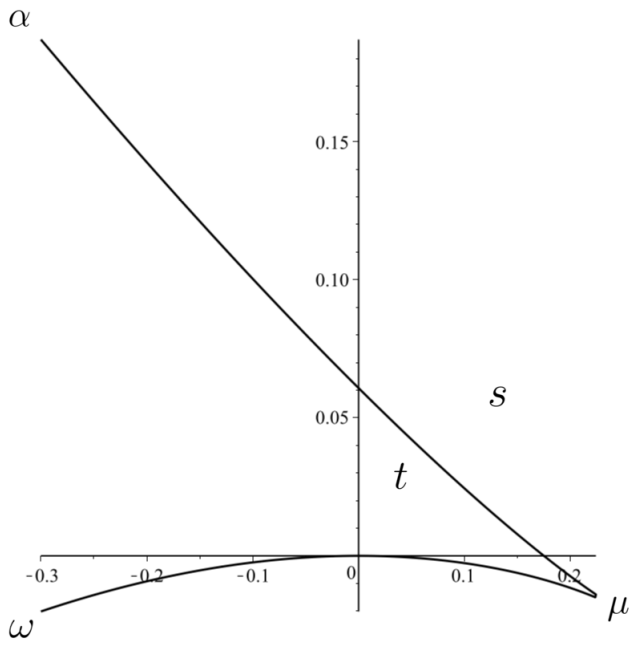

FIGURE 13. Zone H: The set $\Delta^\sharp$ for $a = 1$ and $b = -1$.

	domain s			domain t	
30	$\sigma_{4,1}$	(0,1)	34	$\sigma_{4,1}$	(2,1)
31	$\sigma_{4,2}$	(0,1)	35	$\sigma_{4,2}$	(2,1)
32	$\sigma_{4,3}$	(1,0)	36	$\sigma_{4,3}$	(1,2)
33	$\sigma_{4,4}$	(1,0)	37	$\sigma_{4,4}$	(3,0)

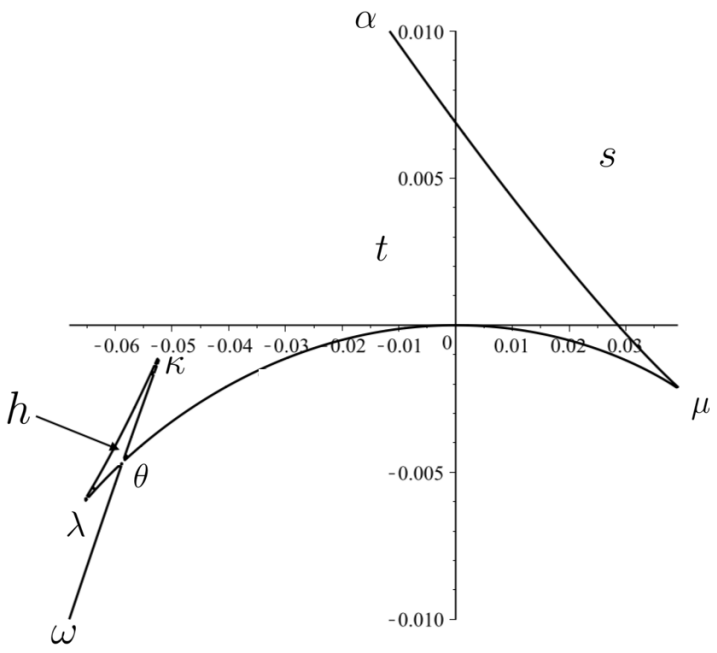

FIGURE 14. Zone I: The set $\Delta^\sharp$ for $a = 0.05$ and $b = -0.2$.

domain s			domain t			domain h		
30	$\sigma_{4,1}$	(0,1)	34	$\sigma_{4,1}$	(2,1)			
31	$\sigma_{4,2}$	(0,1)	35	$\sigma_{4,2}$	(2,1)			
32	$\sigma_{4,3}$	(1,0)	36	$\sigma_{4,3}$	(1,2)	38	$\sigma_{4,3}$	(1,4)
33	$\sigma_{4,4}$	(1,0)	37	$\sigma_{4,4}$	(3,0)			

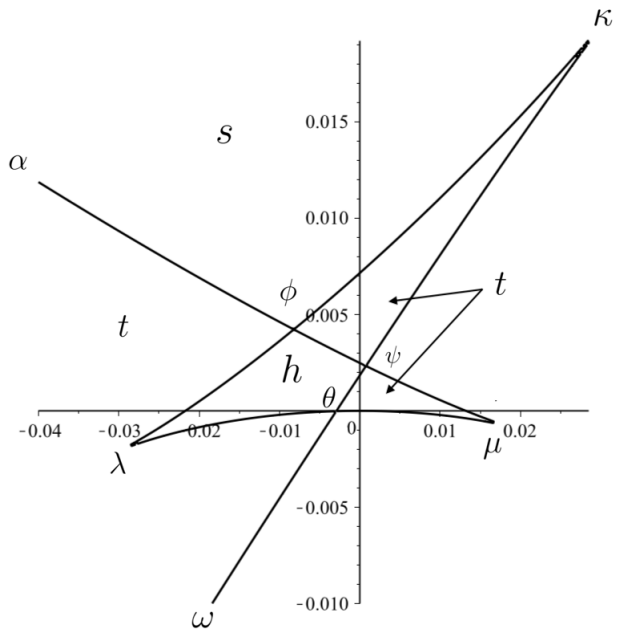

FIGURE 15. Zone J: The set $\Delta^\sharp$ for $a = 0.05$ and $b = -0.12$.

domain s			domain t			domain h		
30	$\sigma_{4,1}$	(0,1)	34 , 39	$\sigma_{4,1}$	(2,1) , (0,3)	41	$\sigma_{4,1}$	(2,3)
31	$\sigma_{4,2}$	(0,1)	35 , 40	$\sigma_{4,2}$	(2,1) , (0,3)	42	$\sigma_{4,2}$	(2,3)
32	$\sigma_{4,3}$	(1,0)	36	$\sigma_{4,3}$	(1,2)	38	$\sigma_{4,3}$	(1,4)
33	$\sigma_{4,4}$	(1,0)	37	$\sigma_{4,4}$	(3,0)			

By comparing Figures 14 and 15 it becomes clear that for $a = 0.05$ and for some $b = b^\flat \in (-0.2, -0.12)$, the domain h intersects the second and the third, but not the first quadrant.

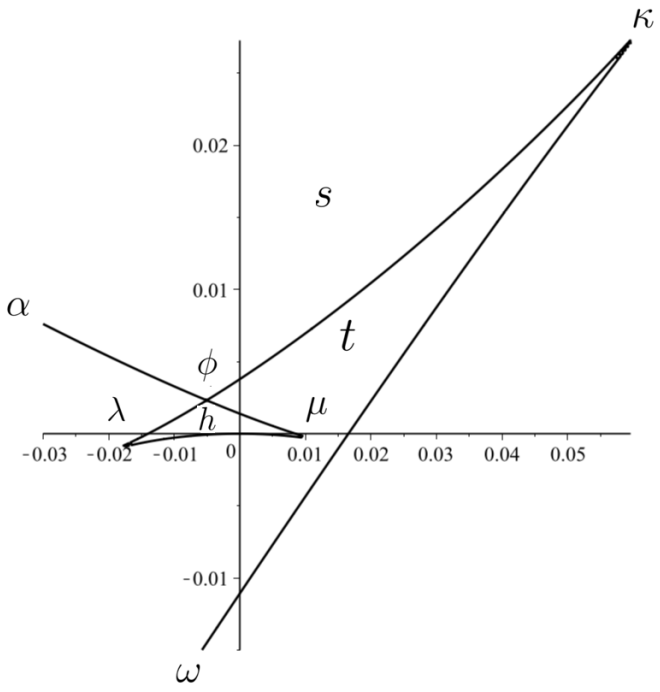

FIGURE 16. Zone K: The set $\Delta^\sharp$ for $a = 0.05$ and $b = -0.09$.

domain s			domain t			domain h		
30	$\sigma_{4,1}$	(0,1)	39	$\sigma_{4,1}$	(0,3)	41	$\sigma_{4,1}$	(2,3)
31	$\sigma_{4,2}$	(0,1)	35 , 40	$\sigma_{4,2}$	(2,1) , (0,3)	42	$\sigma_{4,2}$	(2,3)
32	$\sigma_{4,3}$	(1,0)	36	$\sigma_{4,3}$	(1,2)	38	$\sigma_{4,3}$	(1,4)
33	$\sigma_{4,4}$	(1,0)	43	$\sigma_{4,4}$	(1,2)	44	$\sigma_{4,4}$	(3,2)

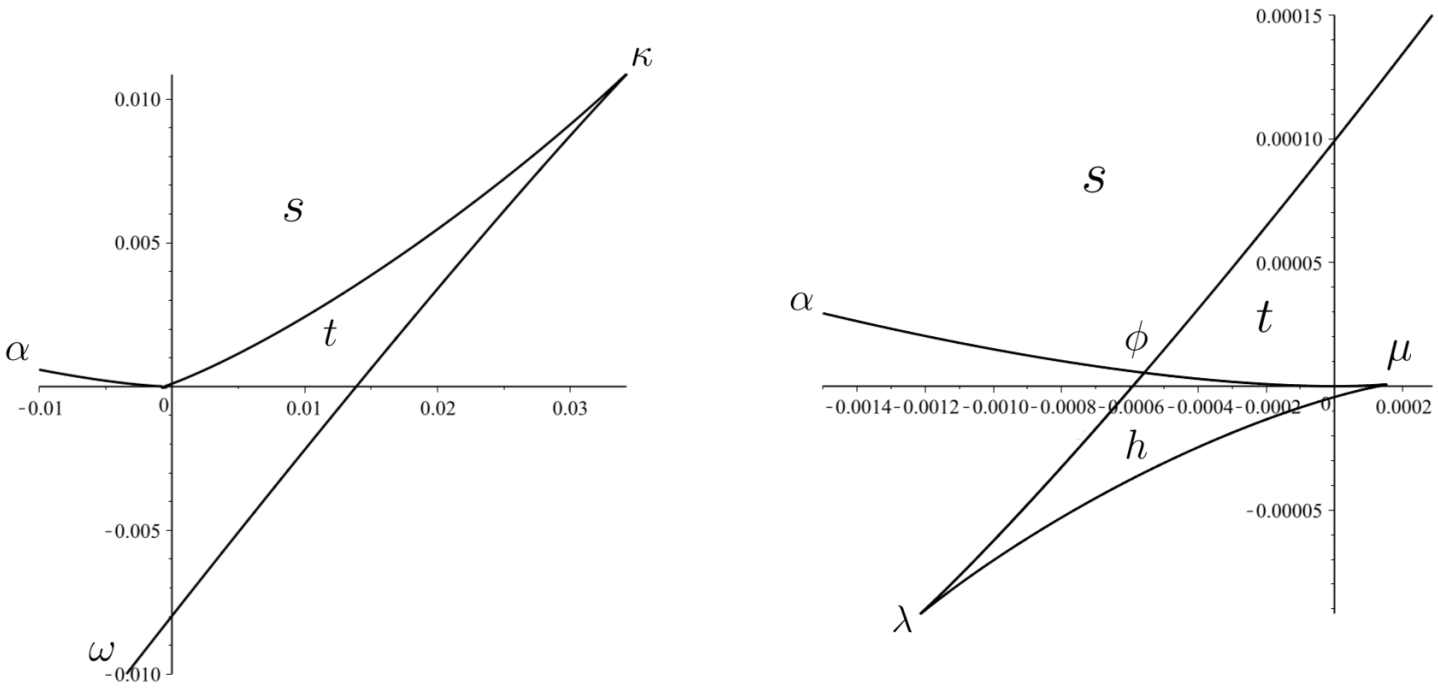

FIGURE 17. Zone L: The set $\Delta^\sharp$ for $a = 0.22$ and $b = 0.01$.

domain s			domain t			domain h		
45	$\sigma_{1,1}$	(0,1)	49	$\sigma_{1,1}$	(0,3)	54	$\sigma_{1,1}$	(0,5)
46	$\sigma_{1,2}$	(0,1)	50 , 51	$\sigma_{1,2}$	(0,3) , (2,1)	55	$\sigma_{1,2}$	(2,3)
47	$\sigma_{1,3}$	(1,0)	52	$\sigma_{1,3}$	(1,2)	56	$\sigma_{1,3}$	(1,4)
48	$\sigma_{1,4}$	(1,0)	53	$\sigma_{1,4}$	(1,2)	57	$\sigma_{1,4}$	(1,4)

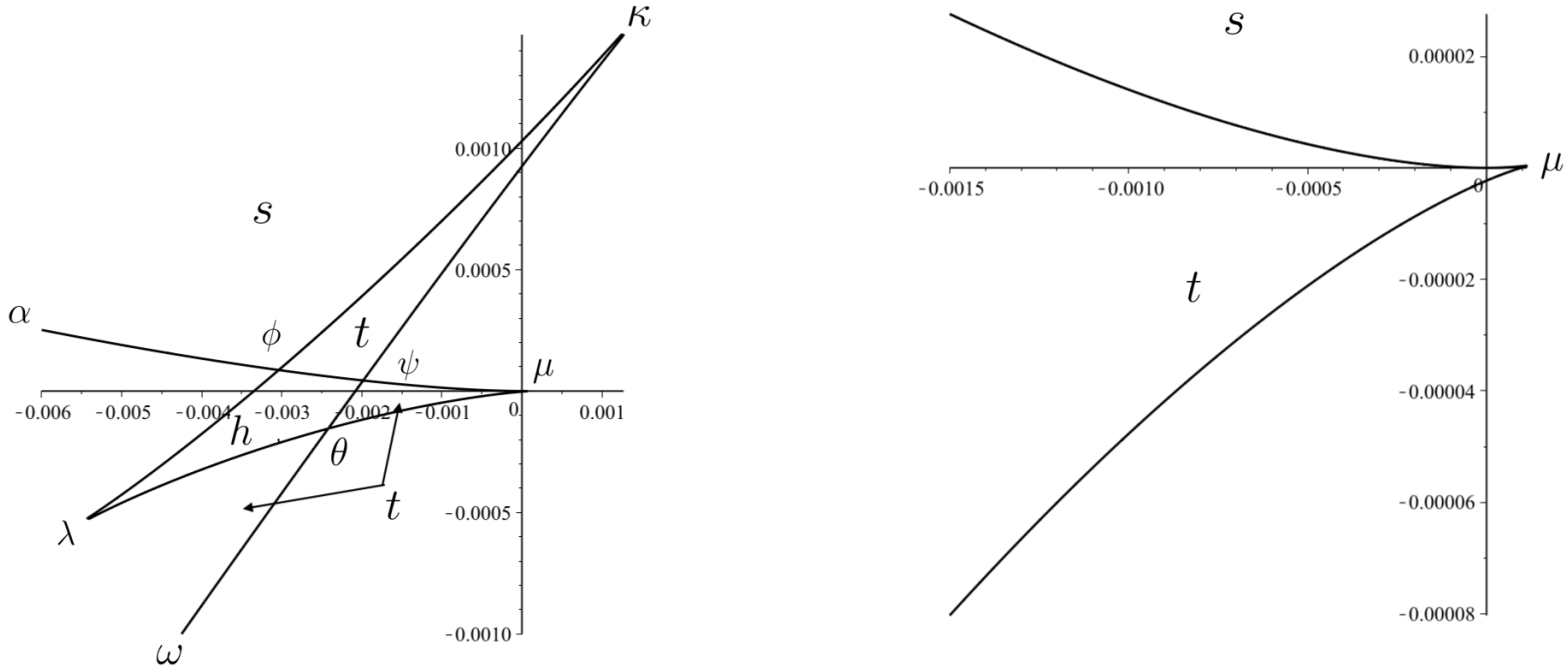

FIGURE 18. Zone M: The set $\Delta^\sharp$ for $a = 0.28$ and $b = 0.01$.

domain s			domain t			domain h		
45	$\sigma_{1,1}$	(0,1)	49	$\sigma_{1,1}$	(0,3)			
46	$\sigma_{1,2}$	(0,1)	50 , 51	$\sigma_{1,2}$	(0,3) , (2,1)	55	$\sigma_{1,2}$	(2,3)
47	$\sigma_{1,3}$	(1,0)	52	$\sigma_{1,3}$	(1,2)	56	$\sigma_{1,3}$	(1,4)
48	$\sigma_{1,4}$	(1,0)	53	$\sigma_{1,4}$	(1,2)			

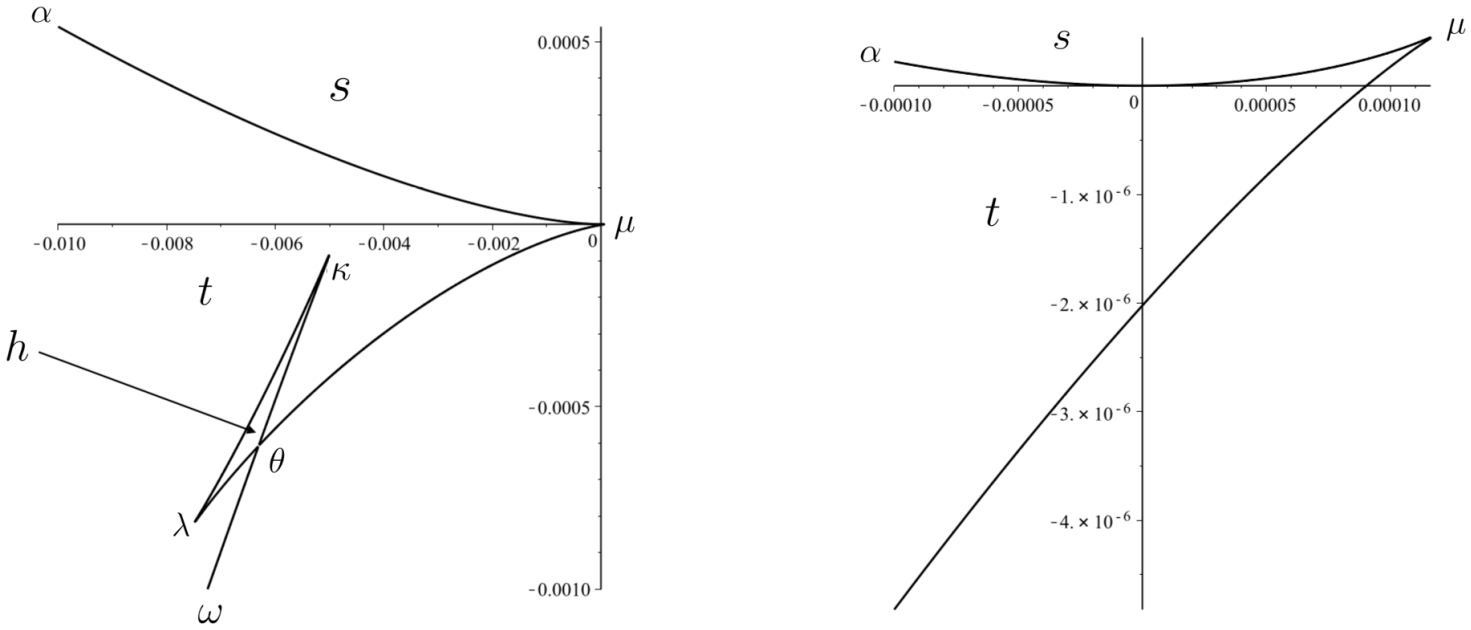

FIGURE 19. Zone N: The set $\Delta^\sharp$ for $a = 0.295$ and $b = 0.01$.

domain s			domain t			domain h		
45	$\sigma_{1,1}$	(0,1)	49	$\sigma_{1,1}$	(0,3)			
46	$\sigma_{1,2}$	(0,1)	51	$\sigma_{1,2}$	(2,1)			
47	$\sigma_{1,3}$	(1,0)	52	$\sigma_{1,3}$	(1,2)	56	$\sigma_{1,3}$	(1,4)
48	$\sigma_{1,4}$	(1,0)	53	$\sigma_{1,4}$	(1,2)			

When comparing Figures 18 and 19 it becomes clear that for $b = 0.01$, there exist two values $0.28 < a_\dagger < a_\natural < 0.295$ of a such that for $a = a_\dagger$, the cusp κ is in the second quadrant, but still above the arc $\mu\alpha$ (so the self-intersection points ϕ and ψ exist), and for $a = a_\natural$, the cusp point κ is in the second quadrant and under the arc $\mu\alpha$.

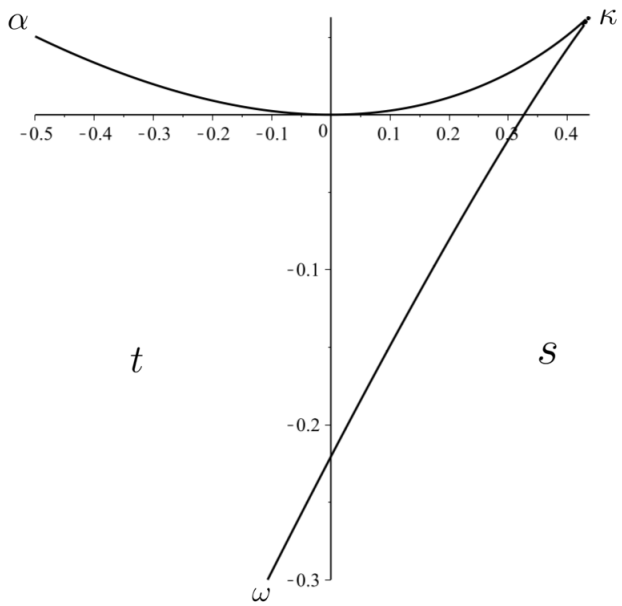

FIGURE 20. Zone P: The set $\Delta^\sharp$ for $a = 1$ and $b = 1$.

	domain s			domain t	
45	$\sigma_{1,1}$	(0,1)	49	$\sigma_{1,1}$	(0,3)
46	$\sigma_{1,2}$	(0,1)	51	$\sigma_{1,2}$	(2,1)
47	$\sigma_{1,3}$	(1,0)	52	$\sigma_{1,3}$	(1,2)
48	$\sigma_{1,4}$	(1,0)	53	$\sigma_{1,4}$	(1,2)

REFERENCES

[1] A. Albouy and Y. Fu, Some remarks about Descartes' rule of signs. Elem. Math., 69 (2014), 186–194. Zbl 1342.12002, MR3272179.
[2] V. I. Arnold, S. M. Gusein-Zade and A. N. Varchenko, Singularities of Differentiable Maps, Birkhäuser; 2012 edition (May 24, 2012).
[3] H. Cheriha, Y. Gati and V. P. Kostov, A nonrealization theorem in the context of Descartes' rule of signs, St. Kliment Ohridski - Faculty of Mathematics and Informatics (accepted).
[4] H. Cheriha, Y. Gati and V. P. Kostov, On Descartes' rule for polynomials with two variations of signs (submitted).
[5] B. Chevallier, Courbes maximales de Harnack et discriminant. (French) [Maximal Harnack curves and the discriminant] Séminaire sur la géométrie algébrique réelle, Tome I, II, 41-65, Publ. Math. Univ. Paris VII, 24, Univ. Paris VII, Paris, 1986.
[6] J. Forsgård, B. Shapiro and V. P. Kostov, Could René Descartes have known this? Exp. Math. 24 (4) (2015), 438-448. Zbl 1326.26027, MR3383475.
[7] J. Forsgård, B. Shapiro and V. P. Kostov, Corrigendum: Could René Descartes have known this? Exp. Math. 28 (2) (2019), 255-256. 10.1080/10586458.2017.1417775.
[8] J. Fourier, Sur l'usage du théorème de Descartes dans la recherche des limites des racines. Bulletin des sciences par la Société philomatique de Paris (1820) 156–165, 181–187; œuvres 2, 291–309, Gauthier - Villars, 1890.

[9] D. J. Grabiner, Descartes Rule of Signs: Another Construction. Am. Math. Mon. 106 (1999), 854–856. Zbl 0980.12001, MR1732666.
[10] V. Jullien, Descartes La "Geometrie" de 1637.
[11] V. P. Kostov, On a stratification defined by real roots of polynomials, Serdica Mathematical Journal 29:2 (2003) 177-186.
[12] V. P. Kostov, On realizability of sign patterns by real polynomials, Czechoslovak Math. J. 68 (2018) 853-874.
[13] V. P. Kostov and B. Shapiro, Polynomials, sign patterns and Descartes' rule, Acta Universitatis Matthiae Belii, series Mathematics Issue 2019, 1-11, ISSN 1338-7111.
[14] I. Méguerditchian, Thesis - Géométrie du discriminant réel et des polynômes hyperboliques, thesis defended in 1991 at the University Rennes 1.
[15] T. Poston and I. Stewart, Catastrophe theory and its applications. With an appendix by D. R. Olsen, S. R. Carter and A. Rockwood. Surveys and Reference Works in Mathematics, No. 2. Pitman, London-San Francisco, Calif.-Melbourne: distributed by Fearon-Pitman Publishers, Inc., Belmont, Calif., 1978. xviii+491 pp. ISBN: 0-273-01029-8.

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Conclusion

In the present thesis we deal with problems stemming from Descartes' rule of signs as completed by Fourier. We consider a monic real degree d polynomial P. We denote by c and p the numbers of sign changes and sign preservations in the sequence of its coefficients (the sequence of corresponding signs is called *sign pattern*) and by *pos* and *neg* the numbers of positive and negative roots of P counted with multiplicity. One has $pos \leq c$, $neg \leq p$ and the numbers $c - pos$, $p - neg$ are even integers.

Descartes' rule gives only necessary conditions. But if one knows the positions of the positive and negative coefficients of a degree d polynomial, one is not sure that for all cases of values of *pos* and *neg* compatible with the above constraints (we say also *admissible*) there exist monic polynomials with such values of *pos* and *neg*. We ask the question: for a given sign pattern σ and for any compatible pair (pos, neg), can one find a polynomial that has exactly *pos* positive roots and *neg* negative roots ? If this is the case, then we say that the couple $(\sigma,\ (pos,\ neg))$ is *realizable*.

In the first part of the thesis, we consider the non-realizable cases for $d = 9$, 10 and 11. The highest number of non-realizable cases corresponds to sign patterns with two sign changes. In this situation we find new sufficient conditions for realizability and non-realizability.

In a second part, we show that there exist no real degree 9 polynomials with exactly one positive and exactly six negative roots and with signs of the coefficients $(+,-,-,-,-,+,+,+,+,-)$. This is an example of non-realizability of lowest possible degree, and for degree 9 it is the only one modulo equivalence, in which both numbers *pos* and *neg* are positive.

The third problem which we study concerns polynomials and their derivatives of all orders. For a polynomial P and its sign pattern σ_0, we define the sequences of sign patterns σ_0, σ_1, ..., σ_{d-1} corresponding to the polynomial P and to its derivatives of order $\leq d-1$. For the kth derivative one can choose a pair (pos_k, neg_k) compatible for the sign pattern σ_k. Here in addition to Descartes' rule and Fourier's observation we have a constraint due to Rolle's theorem. We say that a sequence $((pos_0, neg_0), \ldots, (pos_{d-1}, neg_{d-1}))$ is *realizable* if there exists a polynomial P the signs of whose coefficients define the sign pattern σ_0 and such that for $k = 0, \ldots, d-1$, the polynomial $P^{(k)}$ has exactly pos_k positive and neg_k negative roots, all of them simple. We prove that up to degree 5 all the non-realizable sequences arise from the non-realizable cases of the (sign pattern, compatible pair) realization problem.

Finally, for $d = 5$, we explain the realizability and non-realizability of the couples (sign pattern, compatible pair) by means of pictures of the discriminant set of the family of polynomials of degree 5.