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Haroune Houamed

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**Analyse mathématique de quelques modèles tridimensionnels en mécanique
des fluides et en magnétohydrodynamique**

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Pour obtenir le grade de
Docteur en Sciences Mathématiques de l'Université Côte d'Azur

Thèse dirigée par
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Soutenue le 24 septembre 2020

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Analyse mathématique de quelques modèles tridimensionnels en mécanique des fluides et en magnétohydrodynamique

Résumé. Cette thèse est essentiellement consacrée à l'étude de certaines propriétés mathématiques de trois équations : les équations de Navier-Stokes, les équations de Boussinesq et les équations de Hall magnétohydrodynamique, dans le contexte tridimensionnel. Il convient de noter que, pour le moment, la question de la solvabilité globale des équations de Navier-Stokes 3D reste un problème ouvert : nous ne savons pas si des données initiales régulières génèrent, de manière unique, une solution régulière et globale en temps, ou bien si une formation d'une singularité peut se produire à temps fini. Il est donc clair qu'un tel problème est aussi ouvert pour toutes les équations à l'étude dans ce travail.

Le but de cette thèse est d'étudier d'abord certaines conditions liées au phénomène de l'explosion à temps fini des solutions fortes aux équations de Navier-Stokes 3D (**Chapitre II**). Les **Chapitres III, IV et V** sont consacrés à la question du caractère bien posé du système de Boussinesq avec différents cas de dissipations. Nous consacrerons le **Chapitre III** aux équations de Boussinesq complètement visqueuses avec des données initiales axisymétriques appartenant aux espaces critiques de Lebesgue. Pour le même système mais sans dissipation dans l'équation de ρ , travailler dans des espaces critiques est plus délicat. Toutefois, dans le **Chapitre IV** nous parvenons à traiter le cas des données initiales axisymétriques qui sont critiques par rapport à l'échelle de la vitesse. Le résultat principal du **Chapitre V** est l'unicité des solutions (dans une classe d'espaces anisotropes) du système de Boussinesq avec des dissipations partielles (horizontales) dans toutes les équations. Nous utilisons ensuite ce résultat pour améliorer certains théorèmes déjà établis. Enfin, dans le **Chapitre VI** nous établissons quelques résultats du critère bien posé (local en temps ou global pour les données initiales petites) et du comportement à long temps des solutions globales (dans certains espaces fonctionnels) pour un modèle de Hall-MHD.

Mots-clés. Fluide incompressible, magnétohydrodynamique, équations de Navier-Stokes, équations de Boussinesq, Hall-MHD, critères d'explosion, invariance d'échelle, espaces critiques, espaces de Besov, espaces de Sobolev, solutions axisymétriques, problème de Cauchy.

Mathematical analysis of some three-dimensional fluid and magnetohydrodynamic models

Abstract. *This thesis is basically devoted to the study of some mathematical properties of three equations: the Navier-Stokes equations, the Boussinesq equations and the Hall-magnetohydrodynamic equations in the three-dimensional context. It is worth noting that, for the time being, the question of the global solvability of the 3D Navier-Stokes equations remains as an open problem: In particular, we do not know if a regular initial data would whether uniquely generate a global (in time) regular solution, or if a formation of a singularity at finite time can occur. Thus, it is clear that such a problem is still open for all the equations studied in this work.*

*The goal of this thesis is to study first some conditions related to the blow-up phenomenon of the 3D Navier-Stokes equations that can occur at finite time (**Chapter II**). Then, **Chapters III, IV** and **V** are devoted to the question of the wellposedness of the Boussinesq system with different cases of dissipation. More precisely, in **Chapter III** we establish the global wellposedness of the completely viscous Boussinesq equations when the initial data is axisymmetric belonging to the critical Lebesgue spaces. On the other hand, in the case when there is no dissipation in the ρ -equation, it is hard to work in critical spaces. However, in **Chapter IV** we solve the problem globally in time in the situation where the initial data is axisymmetric and critical with respect to the velocity's scaling. The main result in **Chapter V** is the uniqueness of the solution (in some class of anisotropic spaces) to the Boussinesq system with partial dissipation in all the equations. We use then this latest to improve some known results. Finally, in **Chapter VI** we establish several results of the local (or global for small initial data) wellposedness and large time behavior of the global solutions (in several functional spaces) to a model of Hall-MHD.*

keywords. *Incompressible fluid, magnetohydrodynamic, Navier-Stokes equations, Boussinesq equations, Hall-MHD, Blow-up criterion, scaling invariance, critical spaces, Besov spaces, Sobolev spaces, axisymmetric solutions, Cauchy problem.*

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Sometimes, life is unfair, hard and bad things can happen at any time, this is not predictable and this is not like an equation you can solve (at least, not for large times!). A formation of singularities can occur and many people think they can handle it alone, I always thought like that, maybe I still do! But, the point is, it is always great to have others by your side. Words cannot be enough to express my thanks to all the people in my life, but at least maybe now you know that I'm really grateful to all of you. So **Thank you all!**

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*Introduction and Overview of the
main results*

CHAPTER I

Introduction

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I.1 General Introduction and derivation of the Navier-Stokes equations

In this thesis, we intend to present several results in the mathematical context of some fluid mechanic and hydrodynamic models in dimension three. In particular, our study treats some points in question related to the three dimensional Navier-Stokes equations, the Boussinesq equations with different cases of dissipation in the equations and a model of the Hall-magnetohydrodynamic equations. Note that in all these systems, the core equations are the Navier-Stokes ones. In this introduction, we will briefly introduce the derivation of the 3D Navier-Stokes equations and then we will give a short abstract about the main results of this work.

I.1.1 General introduction to the Navier Stokes equations

The Navier-Stokes system is a mathematical model which describes the motion of Newtonian viscous fluid (gaz, liquid), and explains the resistance of a moving body in a fluid. In the 19th century, in particular in 1845, C. Navier and G. Stokes proposed the following model to describe the evolution of a viscous fluid

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla p = f, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \quad (\text{I.1})$$

where $\mu > 0$ is the viscosity of the fluid, $u = (u^1, u^2, u^3)$ represents its velocity, p denotes the pressure forces, and f is the external force. We will see that the Boussinesq equations is an example when the force f is a scalar quantity acting in the e_3 direction in the first equation of (I.1), and satisfying a transport or transport-diffusion equation.

As known in the literature, one may notice the special two terms: $u \cdot \nabla u$ and $\mu \Delta u$ appearing in (I.1), also known as the convective and the viscous term. Roughly speaking, $u \cdot \nabla u$ is non linear, instability generator and held responsible for the turbulent flow of the fluid in certain situations. On the other hand, the term $\mu \Delta u$ is known by its opposite “smoothing” effects. Based on these two terms it came the definition of the Reynold number Re , given by

$$Re \stackrel{\text{def}}{=} \frac{|u \cdot \nabla u|}{|\mu \Delta u|}.$$

As briefly explained in the introduction of [112], the experimental observations propose the existence of a critical number Re^* for which, any flow whose Reynold number satisfies $Re < Re^*$ is said to be a ”laminar flow”. Laminar flow is characterized by fluid particles which are roughly flowing in the same direction. It is also the opposite of a turbulent regime, made of vortices that contradict each other. Such flow is as well the kind of the flows that we like to deal with when we want to circulate a fluid in a pipe or fly an airplane (because it is more stable, and predictable by the equations).

1.1.2 The outlines of the derivation of the Navier-Stokes equations

In this subsection, we examine the physical meaning of the three-dimensional Navier-Stokes equations and perform a simple mathematical derivation based on Newton's second law. While these equations may look intimidating and complicated to a lot of people, all they really are is a statement that the sum of forces is equal to the product of the mass by the acceleration.

The model we derive in this subsection is the following

$$\begin{cases} \rho(\partial_t u + u \cdot \nabla u) = \mu \Delta u - \nabla p + \rho g, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \quad (\text{NS}_\mu)$$

Above, ρ is a scalar function representing the density of the fluid, and $g = (g_x, g_y, g_z)$ some external forces (let us say the gravity force for instance).

The system (NS_μ) is written for a differential element of the fluid which is infinitesimally small, the three forces we are concerned with here are the forces due to the gravity, the forces due to differences in pressure and the forces due to the viscosity of the fluid. Let us also point out that each of these terms is on a per unit volume basis.

If we denote $(\sum F_x, \sum F_y, \sum F_z) = \sum F = F^{ext} + F^{int}$ being respectively the external and the internal forces acting on our model, and similarly $a = (a_x, a_y, a_z)$ denotes the acceleration, then the Newton's second law gives

$$ma = \sum F. \quad (\text{N})$$

In the left hand side we have the mass times the acceleration, keeping in mind that each term above is on a per unit volume basis, that is the external forces might be the weight of something given by

$$F^{ext} = mg.$$

The mass is given by the density times a per unit volume $V = dxdydz$, that is for the external force in right hand side of the equation above we have

$$\frac{F^{ext}}{V} = \rho g.$$

For the left hand side of (N) , after dividing by V , we should examine the quantity ρa .

To make the presentation simple enough, we should deal here only with the x -direction, the same mathematics would apply for the y and z -directions.

By definition of the x -acceleration, we have

$$a_x = \frac{du^1}{dt} = \partial_t u^1 + \partial_x u^1 \partial_t x + \partial_y u^1 \partial_t y + \partial_z u^1 \partial_t z,$$

by using the fact that

$$\partial_t x = u^1, \quad \partial_t y = u^2, \quad \partial_t z = u^3,$$

we end up with

$$a_x = \partial_t u^1 + u \cdot \nabla u^1, \tag{I.2}$$

above, the first term in the right hand-side is known as the local acceleration, and the second one is known as the convective acceleration.

We move now to examine the forces F^{ext} acting on an elementary volume element of the fluid, to do so, let us present formally this element by an infinitesimal cube in \mathbb{R}^3 , see figures below.

Typically, when we draw a free body diagram, gravity will be acting downward in the y -direction, here we are considering an arbitrary case where a component of gravity could act for example in the x -direction. The force due to gravity is then given by

$$F_x^{ext} = mg_x = \rho dx dy dz g_x \tag{I.3}$$

Let us now examine the forces F^{int} acting on the left and the right hand side of our element: we could have a normal stress acting to the right on the right face, and another one acting on the left outward from the left face, we respectively denote these two forces by $\sigma_{xx}(x+dx)$ and $\sigma_{xx}(x)$. On the top and bottom faces, we could have a shear stress acting to the right on the top face, and shear stress acting to left on the bottom face, the notations we use for these forces are, respectively, $\tau_{yx}(y+dy)$ and $\tau_{yx}(y)$, see figure I.1.

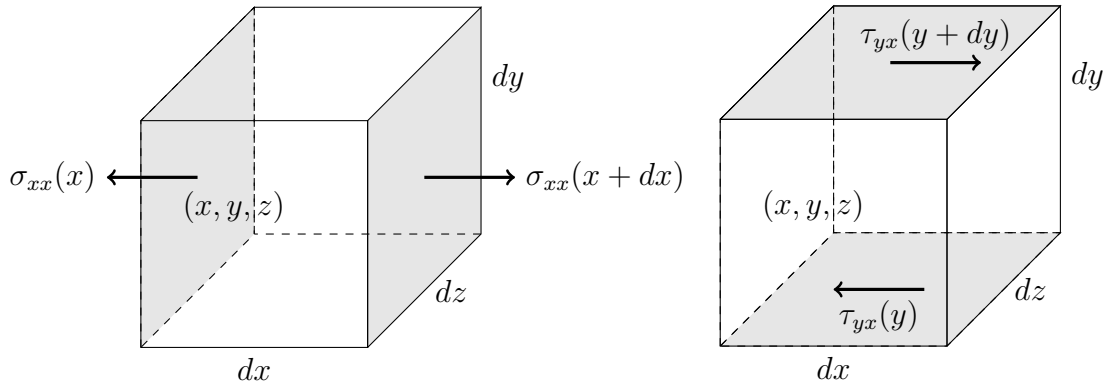


Figure I.1

Similarly, for the front and the back of the cube, we could have as shear forces $\tau_{zx}(z+dz)$ and $\tau_{zx}(z)$ presented in figure I.2.

To resume, the result of all the forces defined above is

$$F_x^{int} = (\sigma_{xx}(x+dx) - \sigma_{xx}(x)) dy dz + (\tau_{yx}(y+dy) - \tau_{yx}(y)) dx dz + (\tau_{zx}(z+dz) - \tau_{zx}(z)) dx dy \tag{I.4}$$

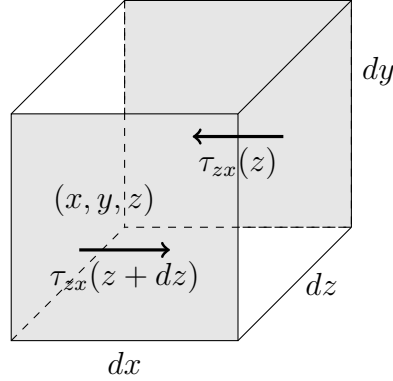


Figure I.2

By substituting (I.2), (I.3) and (I.4) into the Newton's second law formula (N) and dividing the equation by the volume $dx dy dz$, we end up with the following expression

$$\rho g_x + \frac{\sigma_{xx}(x + dx) - \sigma_{xx}(x)}{dx} + \frac{\tau_{yx}(y + dy) - \tau_{yx}(y)}{dy} + \frac{\tau_{zx}(z + dz) - \tau_{zx}(z)}{dz} = \rho a_x.$$

By letting dx, dy, dz approaching zero, the last equation above turns into the differential form

$$\rho g_x + \partial_x \sigma_{xx} + \partial_y \tau_{yx} + \partial_z \tau_{zx} = \rho a_x.$$

By doing the same computation for the y and z -directions, we would come up with the following system

$$\begin{aligned} \rho a_x &= \rho g_x + \partial_x \sigma_{xx} + \partial_y \tau_{yx} + \partial_z \tau_{zx}, \\ \rho a_y &= \rho g_y + \partial_x \tau_{xy} + \partial_y \sigma_{yy} + \partial_z \tau_{zy}, \\ \rho a_z &= \rho g_z + \partial_x \tau_{xz} + \partial_y \tau_{yz} + \partial_z \sigma_{zz}. \end{aligned} \tag{M}$$

The three equations above are known as the equations of motion for a fluid. The next step consists at getting from these equations to the Navier-Stokes equations (NS_μ). To do so we need a way to relate the normal and the shear stress to the viscosity of the fluid and the velocity profiles, and this is done by using the following equations which are the constitutive relations for a Newtonian fluid, which we will not get into here (see [40] for some details)

$$\begin{aligned} \sigma_{xx} &= -p + 2\mu \partial_x u^1, & \tau_{xy} &= \tau_{yx} = \mu(\partial_y u^1 + \partial_x u^2), \\ \sigma_{yy} &= -p + 2\mu \partial_y u^2, & \tau_{yz} &= \tau_{zy} = \mu(\partial_z u^2 + \partial_y u^3), \\ \sigma_{zz} &= -p + 2\mu \partial_z u^3, & \tau_{zx} &= \tau_{xz} = \mu(\partial_x u^3 + \partial_z u^1). \end{aligned} \tag{R}$$

If we accept the relations (R) as being true, then deriving the Navier-Stokes equations form (M) form to the form given by (NS_μ) becomes a series of algebraic manipulations, let us only mention that the incompressibility condition ($\nabla \cdot u = 0$) plays a significant role in the simplification of some terms in this last step.

I.2 Principal results

The main results of this thesis are based on the following papers [80, 65, 82, 41, 81] that we regroup them as follows

Accepted for publication

- *About some possible blow-up conditions for the 3D Navier-Stokes equations.* Accepted for publication in Journal of Differential Equations.
- *On the global wellposedness of axisymmetric Boussinesq system in critical Lebesgue spaces.* Accepted for publication in J. of Discrete & Continuous Dynamical Systems - A. (With A.Hanachi and M.Zerguine).
- *On the global solvability of the axisymmetric Boussinesq system with critical regularity.* Published online in 05 June 2020 in J. of Nonlinear Analysis. (With M.Zerguine).

Submitted (waiting for decision)

- *Wellposedness and long time behavior for the electron inertial Hall-MHD system in Besov and Kato-Herz spaces.*
- *Uniqueness result for the 3-D Navier-Stokes-Boussinesq equations with horizontal dissipation.* (with P.Dreyfuss).

In progress (not included in this thesis)

- On the global wellposedness of the axisymmetric Boussinesq system with measures as initial data (with A.Hanachi and M.Zerguine).
- Profile decomposition to the 3D MHD equations and applications.
- A new proof of the so-called Necas inequality in Lipchitz domain (with P. Dreyfuss).

As it will be explained with more details in the sequel of this work, what we can and what we cannot prove concerning the mathematical analysis of the three dimensional Navier-Stokes equations is the the motivation of all the questions we treat in this thesis. In order to make our introduction simple enough, we choose to postpone the presentation of the known important “technical” results for (NS_μ) and detail them in the Appendix B.

In the rest of this introduction, we briefly outline the statements of the main results in our first five papers listed above.

I.2.1 Some possible blow-up conditions for the 3D Navier-Stokes equations

It is well known that the question of the global resolution (in general) of the three-dimensional Navier-Stokes equations is an open question till today. These equations are given by

$$\begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla p = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} v = 0, \\ (v, \rho)_{t=0} = (v_0, \rho_0), \end{cases} \quad (\text{NS}_\mu)$$

where, v , p and μ represent, respectively, the velocity of the fluid, its pressure, and its cinematic viscosity. It is also well known, due to the work of J. Leray [96], that the equations above have at least one global weak solution u in $L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; \dot{H}^1(\mathbb{R}^3))$ if the initial data is in $L^2(\mathbb{R}^3)$. The uniqueness of such solutions remains till today as an open question. J. Leray proved also in his paper that the construction of a unique regular solution is possible if the initial data is regular enough (saying $u_0 \in H^1(\mathbb{R}^3)$). However, this solution is known only for short time if the size of the initial data is arbitrary large. On the other hand, the maximal life of the solution T^* can be chosen to be $T^* = \infty$ if the initial data is small enough.

The next step in this direction consists of trying to understand what happens to the solution near the maximal time of existence T^* if this latest is finite. The various answers we find in the literature affirm that if $T^* < \infty$ then the solution will blow-up, in some spaces, when we get close to T^* . In this part of our work, we are more interested in *critical* the blow-up criterion in terms of spaces respecting the natural scaling of the 3D Navier-Stokes equations.

As a first example of this kind of results, we can mention the Prodi-Serrin criteria, saying that if $T^* < \infty$, then for all $(p, q) \in [1, \infty) \times [1, \infty]$, with $\frac{2}{p} + \frac{3}{q} = 1$, we have

$$\lim_{t \rightarrow T^*} \int_0^t \|v(\tau, \cdot)\|_{L^q(\mathbb{R}^3)}^p d\tau = \infty.$$

More recently, better blow-up criterion have been proved. These results take in consideration less components of velocity or its matrix gradient. The first result in this direction, taking into account only one component of the velocity in some scaling invariant space, has been proved by J-Y. Chemin and P. Zhang in [30]. their result is the following

$$T^* < \infty \implies \lim_{t \rightarrow T^*} \int_0^t \|v^3(\tau, \cdot)\|_{\dot{H}^{\frac{1}{2} + \frac{2}{p}}(\mathbb{R}^3)}^p d\tau = \infty, \quad \forall p \in (4, 6). \quad (\text{I.5})$$

(I.5) has been generalized, by the same authors together with Z. Zheng, to the case $p \in (4, \infty)$. The special case $p = 2$ has been treated by J-Y. Chemin, I. Gallagher and P. Zhang in [29]. The authors in [29] considered also the case $p = \infty$, which is more harder to deal with compared to the case $p < \infty$, and they showed that if $T^* < \infty$, then it is not possible

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for one component of the velocity to tend to zero too fast (as t tends to T^*) in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. In order to prove a better result, the authors proposed to slightly reinforce the $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ -norm of one component in the horizontal frequencies. More precisely, for an arbitrary constant $E > 0$, they introduced the space $H_{\log_h, E}^{\frac{1}{2}}(\mathbb{R}^3)$ as the sub-space of $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ such that

$$\|a\|_{H_{\log_h, E}^{\frac{1}{2}}}^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} |\xi| \log(E|\xi_h| + e) |\widehat{a}(\xi)|^2 d\xi,$$

and they proved the existence of some constant $c_0 > 0$ such that, if $T^* < \infty$ then

$$\limsup_{t \rightarrow T^*} \|v^3(t, \cdot)\|_{\dot{H}_{\log_h, E}^{\frac{1}{2}}(\mathbb{R}^3)} > c_0. \quad (\text{I.6})$$

Our first result in this thesis consists of proving a similar version of (I.6), but with a vertical logarithmic perturbation. Our result is the following

Theorem I.2.1. *There exists a constant $c_0 > 0$ satisfying: If v is the maximal solution to (NS_μ) in $C([0, T^*]; H^1)$, and if $T^* < \infty$, then for all $E > 0$ we have*

$$\limsup_{t \rightarrow T^*} \|v^3(t, \cdot)\|_{\dot{H}_{\log_v, E}^{\frac{1}{2}}(\mathbb{R}^3)} > c_0, \quad (\text{I.7})$$

where,

$$\|a\|_{\dot{H}_{\log_v, E}^{\frac{1}{2}}}^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} |\xi| \log(E|\xi_v| + e) |\widehat{a}(\xi)|^2 d\xi.$$

As a corollary, we improve a little bit the two results above by replacing the spaces $\dot{H}_{\log_h, E}^{\frac{1}{2}}$ and $\dot{H}_{\log_v, E}^{\frac{1}{2}}$ by the space $\dot{H}_{\log, E}^{\frac{1}{2}}$, defined by its norm

$$\|a\|_{\dot{H}_{\log, E}^{\frac{1}{2}}}^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} |\xi| \log(E \min\{|\xi_h|, |\xi_v|\} + e) |\widehat{a}(\xi)|^2 d\xi.$$

In the second part of [80], we prove also two criterion considering some components of the matrix gradient of the velocity in some critical Besov-type spaces. This results can be summarized in the two theorems below. We will use as notation

$$\omega \stackrel{\text{def}}{=} \nabla \times v, \quad s_p \stackrel{\text{def}}{=} \frac{2}{p} - \frac{1}{2}, \quad \text{and} \quad \mathcal{B}_{q,p} \stackrel{\text{def}}{=} \dot{B}_{q,\infty}^{\frac{3}{q} + \frac{2}{p} - 2}. \quad (\text{I.8})$$

Theorem I.2.2. *Let v be the maximal solution to (NS_μ) in $C([0, T^*]; H^1)$. If $T^* < \infty$, then*

$$\forall p, m \in [2, 4], \quad \forall \alpha \in \left[0, \frac{2}{p} - \frac{1}{2}\right], \quad \forall \beta \in \left[0, \frac{2}{m} - \frac{1}{2}\right], \quad \text{we have}$$

$$\int_0^{T^*} \|\partial_3 v^3(t')\|_{\dot{B}_{2,\infty}^{\alpha, s_p - \alpha}}^p dt' + \int_0^{T^*} \|\omega^3(t')\|_{\dot{B}_{2,\infty}^{\beta, s_m - \beta}}^m dt' = \infty.$$

Theorem I.2.3. *Let v be the maximal solution to (NS_μ) in $C([0, T^*]; H^1)$. If $T^* < \infty$, then for all $q_1, q_2 \in [3, \infty)$, and p_1, p_2 be such that*

$$\frac{3}{q_i} + \frac{2}{p_i} \in (1, 2), \quad i \in \{1, 2\}, \quad (\text{I.9})$$

we have

$$\int_0^{T^*} \|\partial_3 v^3(t')\|_{\mathcal{B}_{q_1, p_1}}^{p_1} dt' + \int_0^{T^*} \|\omega^3(t')\|_{\mathcal{B}_{q_2, p_2}}^{p_2} dt' = \infty.$$

1.2.2 Global wellposedness of the axisymmetric completely-viscous Boussinesq system in critical spaces

In the paper [65], we consider the system of the 3D Boussinesq completely viscous

$$\begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla p = \rho \vec{e}_3, \\ \partial_t \rho + v \cdot \nabla \rho - \kappa \Delta \rho = 0, \\ \operatorname{div} v = 0, \\ (v, \rho)_{t=0} = (v_0, \rho_0), \end{cases} \quad (\text{B}_{\mu, \kappa})$$

where, v designates the velocity of the fluid, p its pressure and ρ the fluctuation of its density around a constant value. $\mu > 0$ and $\kappa > 0$ represent, respectively, the cinematic viscosity and the constant diffusion. We consider in [65] the Cauchy problem of $(\text{B}_{\mu, \kappa})$ for $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^3$, and we are interested in the study of the global strong solutions. It is well known that the question of the global solvability of $(\text{B}_{\mu, \kappa})$ (in the general case) is still open. In our paper [65], we consider the case of the axisymmetric solutions without swirl, where the velocity and the density have the form

$$v(t, x) = v^r(t, r, z)e_r + v^z(t, r, z)e_z, \quad \rho(t, x) = \rho(t, r, z),$$

with, $r^2 \stackrel{\text{def}}{=} x_1^2 + x_2^2$, et $z \stackrel{\text{def}}{=} x_3$. Our reasoning is strongly inspired by the work of Th. Gallay and V. Sverak [56] who dealt with the Navier-Stokes case ($\rho = 0$).

by introducing the vorticity $\omega_\theta \stackrel{\text{def}}{=} \partial_z u^r - \partial_r u^z$ and the auxiliary quantity $\zeta \stackrel{\text{def}}{=} \frac{\omega_\theta}{r}$, we can check that the system $(\text{B}_{\mu, \kappa})$ is equivalent to the following one

$$\begin{cases} \partial_t \zeta + v \cdot \nabla \zeta - \mu \left(\Delta + \frac{2}{r} \partial_r \right) \zeta = -\frac{\partial_r}{r} \rho, \\ \partial_t \rho + v \cdot \nabla \rho - \kappa \Delta \rho = 0. \end{cases} \quad (\tilde{\text{B}}_{\mu, \kappa})$$

If we further assume that $\kappa = \mu$, then we can see that the quantity $\Gamma \stackrel{\text{def}}{=} \kappa \zeta - \frac{\rho}{2}$ satisfies a transport-diffusion like equation (without a force term):

$$\partial_t \Gamma + v \cdot \nabla \Gamma - \mu \left(\Delta + \frac{2}{r} \partial_r \right) \Gamma = 0,$$

which, combined with the second equation in $(\tilde{B}_{\mu,\kappa})$ makes it possible to obtain some estimates a priori global in time on the solutions of the system.

We also consider other interesting quantities to be able to establish more estimates which eventually allow us to construct a unique global solution. More precisely, we find that the quantities $\tilde{\rho} \stackrel{\text{def}}{=} r\rho$ and $\tilde{\Gamma} \stackrel{\text{def}}{=} r\Gamma$ play a very important role in our analysis. The equations which govern these quantities are given by

$$\begin{cases} \partial_t \tilde{\rho} + v \cdot \nabla \tilde{\rho} - \mu \left(\Delta - \frac{1}{r^2} \right) \tilde{\rho} = \frac{v^r}{r} \tilde{\rho} - 2\partial_r \rho, \\ \partial_t \tilde{\Gamma} + v \cdot \nabla \tilde{\Gamma} - \mu \left(\Delta - \frac{1}{r^2} \right) \tilde{\Gamma} = \frac{v^r}{r} \tilde{\Gamma}. \end{cases}$$

Finally, our result can be summarized as follows

Theorem I.2.4. *Let $\mu = \kappa > 0$, and v_0 be a divergence free, axisymmetric without swirl, vector field satisfying $\nabla \times v_0 \in L^1(\mathbb{R}^3)$. Let ρ_0 be a scalar axisymmetric function in $L^1(\mathbb{R}^3)$. Then the system of Boussinesq completely viscous has a unique global solution (u, ρ) be such that, for all $t > 0$*

$$\|\zeta(t)\|_{L^1} + \|\rho(t)\|_{L^1} + t \left(\|r\zeta(t)\|_{L^\infty} + \|r\rho(t)\|_{L^\infty} \right) + t^{\frac{3}{2}} \left(\|\zeta(t)\|_{L^\infty} + \|\rho(t)\|_{L^\infty} \right) \leq C,$$

where, $C > 0$ is a constant depending only on $\|\zeta_0\|_{L^1(\mathbb{R}^3)} + \|\rho_0\|_{L^1(\mathbb{R}^3)}$.

1.2.3 Global wellposedness of the axisymmetric Boussinesq system with no diffusion in the ρ -equation

The paper [82] is a work in collaboration with M. Zerguine, which concerns the study of the existence and the uniqueness of the global solutions of the three-dimensional Boussinesq system, without dissipation in the density equation, in a case of the particular initial data.

More precisely, we consider the following Boussinesq system:

$$\begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla p = \rho \vec{e}_3, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \partial_t \rho + v \cdot \nabla \rho = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} v = 0, \\ (v, \rho)_{t=0} = (v_0, \rho_0), \end{cases} \quad (\text{B}_\mu)$$

and we study the global Cauchy problem, for axisymmetric initial data without swirl. The main result of our work consists in improving the result of T. Hmidi and F. Rousset cite Hmidi-Rousset in the sense of proving that the system is globally well posed with less initial regularity while working in critical spaces compared to the scale of the velocity. The proof we propose here is based on an idea due to Calderon and which has been used in other papers like cite Abidi. The main result statement is as follows:

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Theorem I.2.1. *Let $v_0 \in H^{\frac{1}{2}}(\mathbb{R}^3) \cap \dot{B}_{3,1}^0(\mathbb{R}^3)$ be a divergence free, axisymmetric without swirl, vector field. Let ρ_0 be a scalar axisymmetric function such that $\rho_0 \in L^2(\mathbb{R}^3) \cap \dot{B}_{3,1}^0(\mathbb{R}^3)$. Then (\mathbf{B}_μ) has a unique global axisymmetric solution satisfying*

$$(v, \rho) \in E_T(H^{\frac{1}{2}}) \cap L_T^\infty(\dot{B}_{3,1}^0) \cap L_T^1(\dot{B}_{3,1}^2) \times L_T^\infty(L^2 \cap \dot{B}_{3,1}^0).$$

The proof of Theorem (I.2.1) is essentially based on another version of the first Theorem proved by H. Abidi in his paper [3] for the 3D Navier-Stokes equations with a forcing term

$$\begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla p = f, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} v = 0, \\ v_{t=0} = v_0. \end{cases} \quad (\text{NS}_\mu)$$

The proof of our version of the Theorem above is the subject of the first part of our paper. The main idea of the proof is also based on a method of splitting in frequencies due to Calderon, and our result consists in considering less spatial regularity for the source term in front of asking a little more integrability in time compared to the work of Abidi [3]. The summary of the result is as follows:

Theorem I.2.2. *Let $v_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$ be a divergence free, axisymmetric without swirl, vector field. Let f be an axisymmetric without swirl vector field such that $f \in L_{loc}^2(\mathbb{R}_+; H^\beta(\mathbb{R}^3))$, for some $\beta > \frac{1}{4}$, Then (NS_μ) has a unique global axisymmetric solution satisfying*

$$v \in C(\mathbb{R}_+; H^{\frac{1}{2}}) \cap L_{loc}^2(\mathbb{R}_+; H^{\frac{3}{2}}).$$

1.2.4 Uniqueness result for an anisotropic model of Boussinesq system

In Chapter V, we consider the 3-D Boussinesq system with horizontal dissipation, where there is no smoothing effect on the vertical derivatives, the system in question will be then

$$\begin{cases} (\partial_t + u \cdot \nabla)u - \Delta_h u + \nabla P = \rho e_3, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ (\partial_t + u \cdot \nabla)\rho - \Delta_h \rho = 0, \\ \operatorname{div} u = 0, \\ (u, \rho)|_{t=0} = (u_0, \rho_0). \end{cases} \quad (\text{B}_h)$$

The partial diffusion in the above system is a natural assumption for several cases of interest in geophysical fluid flows, we refer to [107] for more details. Recently, the Boussinesq system with partial dissipation received a special attention, and a significant progress in its analysis have been made, see for instance [101, 2, 120]. Note that the lack of the vertical gain in this model makes it more difficult to study and establish the same results as in the fully-dissipative case. Our principal goal here is to study the Cauchy problem of this model. More precisely, we prove a stability-type result of solutions $(u, \rho) \in L_T^\infty(H^{0,s} \times H^{0,1-s})$ with $(\nabla_h u, \nabla_h \rho) \in L_T^2(H^{0,s} \times H^{0,1-s})$ and $s \in [1/2, 1]$. The main theorem of Chapter V is then the following

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Theorem I.2.5. *Let $s \in [1/2, 1]$ and $(u, \rho), (v, \eta)$ be two solutions for system (B_h) in*

$$L_{loc}^\infty(\mathbb{R}_+; H^{0,s}) \cap L_{loc}^2(\mathbb{R}_+; H^{1,s}) \times L_{loc}^\infty(\mathbb{R}_+; H^{0,1-s}) \cap L_{loc}^2(\mathbb{R}_+; H^{1,1-s}).$$

Then $(u, \rho) = (v, \eta)$

The technics we use in this Chapter are based on several works dealing with the 3-D Navier Stokes equations with only horizontal dissipation [28, 85, 106]. In particular one should pay attention to the fact that the model is parabolic with respect to the horizontal variables, but it is hyperbolic with respect to the vertical one. Hence, our analysis will be done in spaces that are algebra with respect to the vertical variables, at least for the velocity.

As a consequence, we improve the conditions stated in the paper [101] in order to obtain a global wellposedness result in the case of axisymmetric initial data. Our second result is the following

Theorem I.2.6. *Let $u_0 \in H^1(\mathbb{R}^3)$ be an axisymmetric divergence free vector field without swirl such that $\frac{\omega_0}{r} \in L^2$ and let $\rho_0 \in L^2$ be an axisymmetric function. Then there is a unique global solution (u, ρ) of the system (B_h) . Moreover we have*

$$\begin{aligned} u &\in \mathcal{C}(\mathbb{R}_+; H^1) \cap L_{loc}^2(\mathbb{R}_+; H^{1,1} \cap H^{2,0}), & \frac{\omega}{r} &\in L_{loc}^\infty(\mathbb{R}_+; L^2) \cap L_{loc}^2(\mathbb{R}_+; H^{1,0}), \\ \rho &\in \mathcal{C}(\mathbb{R}_+; L^2) \cap L_{loc}^2(\mathbb{R}_+; H^{1,0}). \end{aligned}$$

We should mention that the main feature in our proof is the choice of the spaces in which we do our estimates. This will be explained with more technical details in the introduction of Chapter V.

At the end of this chapter, we provide another example of a local existence and uniqueness result, where, for the existence part, we do not need to assume a high spacial regularity for the initial density, while the uniqueness part is a direct application of our first theorem above. The statement is the following

Theorem I.2.7. *Let $s \in [1/2, 1]$, $\delta \in [0, s]$ and $(u_0, \rho_0) \in H^{0,s} \times H^{0,\delta}$. We have*

- *There exists $C_0 > 0$ such that if*

$$\|u_0\|_{0,s}^2 + T \|\rho_0\|_{L^2} \left(\|u_0\|_{L^2} + \|\rho_0\|_{L^2} \left(1 + \frac{T}{2}\right) \right) < C_0^2,$$

then (B_h) has at least one solution (u, ρ) in $\mathcal{X}^{s,\delta}(T)$, where

$$\mathcal{X}^{s,\delta}(T) \stackrel{def}{=} L_T^\infty H^{0,s} \cap L_T^2 H^{1,s} \times L_T^\infty H^{0,\delta} \cap L_T^2 H^{1,\delta}.$$

- *The solution is unique if $\delta \geq 1 - s$*

I.2.5 Wellposedness and large time behavior of the electron inertia Hall-MHD system

Finally, Chapter VI is devoted to the study of the wellposedness of the Hall-MHD system augmented by the effect of electron inertia. After simplification of some physical quantities, the system we deal with here is

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla P = (\nabla \times B) \times H, \\ \partial_t H - \Delta H + 2\nabla \times ((\nabla \times B) \times H) = \nabla \times (u \times H) + \nabla \times ((\nabla \times B) \times (\nabla \times u)), \\ H = (Id - \Delta)B, \\ \operatorname{div} u = \operatorname{div} B = 0, \\ (u|_{t=0}, B|_{t=0}) = (u_0, B_0). \end{cases}$$

Our main goal consists in generalizing the wellposedness topic proved in [46] from the Sobolev context to the general Besov spaces.

In particular, our first result deals with small initial data $u_0, B_0, \nabla \times B_0$ in $\dot{B}_{p,r}^{\frac{3}{p}-1}(\mathbb{R}^3)$, for $(p, r) \in [1, \infty) \times [1, \infty]$. The statement of our first result is the following

Theorem I.2.8. *Let $p \in [1, \infty)$, $r \in [1, \infty]$ and $\mathcal{U}_0 \stackrel{def}{=} (u_0, B_0, \nabla \times B_0)$ be in $\dot{B}_{p,r}^{\frac{3}{p}-1}(\mathbb{R}^3)$. There exists $c_0 > 0$ such that, if*

$$\|\mathcal{U}_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} < c_0,$$

then the Hall-MHD model above has a unique global solution \mathcal{U} in $\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})$, with

$$\|\mathcal{U}\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})} < 2c_0.$$

Then we show that we can reduce the required regularity on the initial magnetic field in the first result (in the case $r = 1$) to be only in $\dot{B}_{p,1}^{\frac{3}{p}}(\mathbb{R}^3)$, together with an additional condition on the maximal time of existence. More precisely, we prove

Theorem I.2.9. *Let $T > 0$, $p \in [1, \infty)$ and (u_0, B_0) be two divergence vector fields in $\dot{B}_{p,1}^{\frac{3}{p}-1}(\mathbb{R}^3) \times \dot{B}_{p,1}^{\frac{3}{p}}(\mathbb{R}^3)$, there exists $c_0 > 0$ such that if*

$$\|u_0\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} + (2 + T) \|B_0\|_{\dot{B}_{p,1}^{\frac{3}{p}}} < c_0,$$

then the Hall-MHD system above has a unique solution (u, B) on $(0, T)$ with

$$\|u\|_{\mathcal{L}_T(\dot{B}_{p,1}^{\frac{3}{p}-1})} + (2 + T) \|B\|_{\mathcal{L}_T(\dot{B}_{p,1}^{\frac{3}{p}})} < 2c_0.$$

Our proofs are essentially based on some fixed point arguments applied to a reformulated version of the system above. The estimates we use in this Chapter are in particular a fine

estimates proved for the heat-semi group together with some estimates in the homogeneous-Besov spaces for the non-homogeneous operator $(Id - \Delta)^{-1}$.

Finally, we study the large time behavior of the solution in some particular spaces¹. We show that the \widehat{L}^p (and eventually the L^p) norm of the solution $(u, B, \nabla \times B)$ associated to an initial data in $\widehat{B}_{p,\infty}^{\frac{3}{p}-1}(\mathbb{R}^3)$ is controlled by $t^{-\frac{1}{2}(1-\frac{3}{p})}$, for all $p \in (3, \infty)$, which provides a polynomial decay to zero of the \widehat{L}^p norm of the solution. The last result of this Chapter is then the following

Theorem I.2.10. *Let $p \in (3, \infty)$ and u_0, B_0 be two divergence free vector fields. There exists $c_0 > 0$ such that if*

$$\|e^{t\Delta}u_0\|_{\widehat{K}_p^{1-\frac{3}{p}}} + \|e^{t\Delta}B_0\|_{\widehat{K}_p^{1-\frac{3}{p}}} + \|e^{t\Delta}(\nabla \times B_0)\|_{\widehat{K}_p^{1-\frac{3}{p}}} < c_0,$$

then the Hall-MHD system above has a unique global solution $\mathcal{U} = (u, B, \nabla \times B)$ in $\widehat{K}_p^{1-\frac{3}{p}}$ satisfying

$$\|\mathcal{U}(t, \cdot)\|_{L^p} \lesssim t^{-\frac{1}{2}(1-\frac{3}{p})}.$$

We also point out in Chapter VI some very interesting open questions related to the MHD model in study, we make a link (at least formally) with other models and recent results by comparing this latest to our main results. More details can be also found in our associated paper [81]

1.2.6 Overview and the organization of the thesis

The rest of the thesis is organized as follows: We present our results in the form of three parts, with an additional part as an Appendix. The chapters of each part can be read independently, but some technical lemmata and propositions are proved in the appendix because we believe that these latest can be also useful for more general problems and not only for the models in question in this thesis.

- **Part one About the Navier-Stokes equations:** It contains one chapter in which we prove the results described in the subsection I.2.1.
- **Part two About the Boussinesq system:** This part is composed of three Chapters each one treats, respectively, one of the subsections I.2.2, I.2.3 and I.2.4.
- **Part three About the Hall-MHD system:** The third part of this work is devoted to the proof of the results aforementioned in the subsection I.2.4.
- **Part Four Appendix:** The finale part contains three chapters which are organized as follows:

¹We refer to the Appendix A for the definition of the hat-spaces.

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- **Appendix A** is devoted to recall the definitions of the functional spaces used in all our work, and to provide a general tool box of results which can be of a constant use in the mathematical analysis of PDE's.
- **Appendix B** Is about the Navier-Stokes equations, which is also the technical part associated to the introduction of the Navier-Stokes equations. We begin by recalling some basic notions of the Navier-Stokes equations and we end with briefly presenting some important results on these equations, starting from the work of J.Leray till today.
- Finally, **Appendix C** Is the perspective part of this thesis. In which, we briefly comment on some open questions related to our results and we mention some open questions that interests the author.

Part I

About the Navier-Stokes equations

CHAPTER II

Some possible blow-up conditions for the 3D Navier-Stokes equations

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II.1 Introduction and statement of the main results

In this chapter, we are interested in the study of the possibility of the blow-up for regular solutions to the 3D incompressible Navier Stokes equations

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0, \end{cases} \quad (NS)$$

where the unknowns of the equations $u = (u^1, u^2, u^3)$, p are respectively, the velocity and the pressure of the fluid. We recall that the set of the solutions to (NS) is invariant under the transformation:

$$u_{0,\lambda}(x) \stackrel{\text{def}}{=} \lambda u_0(\lambda x), \quad u_\lambda(t, x) \stackrel{\text{def}}{=} \lambda u_\lambda(\lambda^2 t, \lambda x).$$

That is if u is a solution to (NS) on $[0, T] \times \mathbb{R}^3$ associated to the initial data u_0 , then, for all $\lambda > 0$, u_λ is a solution to (NS) on $[0, \lambda^{-2}T] \times \mathbb{R}^3$ associated to the initial data $u_{0,\lambda}$.

It is well known that (NS) has at least one global weak solution with finite energy

$$\|u(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla u(t')\|_{L^2}^2 dt' \leq \|u_0\|_{L^2}^2 \quad (\text{II.1})$$

This result was proved first by J. Leray in [96]. In dimension three, the uniqueness of such solutions stands to be an open problem. J. Leray proved also in his famous paper [96] that, for more regular initial data, namely for $u_0 \in H^1(\mathbb{R}^3)$, (NS) has a unique local smooth solution. That is to say, there exists $T^* > 0$ and a unique maximal solution u in $L_T^\infty(H^1(\mathbb{R}^3)) \cap L_T^2(H^2(\mathbb{R}^3))$.

The question of the behavior of this solution after T^* remains to be also an open problem.

In order to give a “formally” large picture, let us define the set

$$\chi_T \stackrel{\text{def}}{=} \left((L_T^\infty L^2 \cap L_T^2 H^1) \cdot (L_T^\infty L^2 \cap L_T^2 H^1) \right)', \quad (\text{II.2})$$

where χ_T is the dual of

$$(L_T^\infty L^2 \cap L_T^2 H^1) \cdot (L_T^\infty L^2 \cap L_T^2 H^1) \stackrel{\text{def}}{=} \{uv : (u, v) \in (L_T^\infty L^2 \cap L_T^2 H^1) \times (L_T^\infty L^2 \cap L_T^2 H^1)\}.$$

Multiplying (NS) by $-\Delta u$, and integrating by parts yield to

$$\frac{d}{2dt} \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{H^1}^2 = - \int_{\mathbb{R}^d} (\nabla u \cdot \nabla u | \nabla u).$$

If we suppose that ∇u is already bounded in G_T some sub-space of χ_T , then one may prove that ∇u is bounded in $L_T^\infty L^2 \cap L_T^2 H^1$. This is the case in dimension two where we get, for

free, by the L^2 -energy estimate (II.1) a uniform bound of ∇u in $L_T^2 L^2 \subset ((L_T^4 L^4) \cdot L_T^4 L^4)' \subset \chi_T$.

In the case of dimension three, several works have been done in this direction, establishing a global wellposedness of (NS) under assumptions of the type $\nabla u \in G_T$.

We can set as an example of these results the well known Prodi-Serrin type criterion. Saying that, if $u \in L^p([0, T], L^q(\mathbb{R}^3))$, with $\frac{2}{p} + \frac{3}{q} = 1$ and $q \in]3, \infty]$, then (NS) is globally wellposed.

The limit case where $q = 3$ was proved recently by L. Escauriaza, G. Seregin and V. Sverák in [42], proving that: if $T^* \stackrel{def}{=} T^*(u_0)$ denotes the life span of a regular solution u associated to the initial data u_0 then

$$T^* < \infty \implies \limsup_{t \rightarrow T^*} \|u(t)\|_{L^3(\mathbb{R}^3)} = \infty. \quad (\text{II.3})$$

This was extended to the full limit in time in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ by G. Seregin in [114].

Several works have been done in this direction as well, the recent work of I. Gallagher, G. Koch and F. Planchon [51] for instance extends (II.3) to the critical Besov spaces $\dot{B}_{p,q}^{-1+\frac{3}{p}}$, for $3 < p, q < \infty$.

In another hand, one may notice that the divergence free condition can provide us another type of conditions for the global regularity (let us say anisotropic ones) under conditions on some components of the velocity or its gradient. Several works have been done in this direction, one may see for instance [103, 19, 20, 50, 66, 90, 104, 108, 111, 124] for examples in some scaling invariant spaces or not of Serrin-type regularity criterion, or equivalently proving that, if T^* is finite then

$$\int_0^{T^*} \|u^3(t, \cdot)\|_{L^q}^p = \infty \quad \text{or} \quad \int_0^{T^*} \|\partial_j u^3(t, \cdot)\|_{L^q}^p = \infty.$$

The first result in a scaling invariant space under only one component of the velocity has been proved by J.-Y Chemin and P. Zhang in [30] for $p \in]4, 6[$ and a little bit later by the same authors together with Z. Zhang in [32] for $p \in]4, \infty[$. The case $p = 2$ has been treated very recently by J.-Y Chemin, I. Gallagher and P. Zhang in [29]. As mentioned in [29] such a result in the case of $p = \infty$, assuming it is true, seems to be out of reach for the time being.

However, the authors in [29] proved some results for $p = \infty$. Mainly, they proved that if there is a blow-up at some time $T^* > 0$, then it is not possible for one component of the velocity to tend to 0 too fast. More precisely they proved the following blow-up condition

$$\forall \sigma \in \mathbb{S}^2, \forall t < T^*, \quad \sup_{t' \in [t, T^*[} \|u(t') \cdot \sigma\|_{\dot{H}^{\frac{1}{2}}} \geq c_0 \log^{-\frac{1}{2}} \left(e + \frac{\|u(t)\|_{L^2}^4}{T^* - t} \right)$$

The last result proved in their paper needs reinforcing slightly the $\dot{H}^{\frac{1}{2}}$ norm in some directions. Mainly, without loss of generality, their result can be stated as the following

Theorem II.1.1. *There exists a positive constant c_0 such that if u is a maximal solution of (NS) in $C([0, T^*[, H^1)$, then for all positive real number E we have:*

$$T^* < \infty \implies \limsup_{t \rightarrow T^*} \|u^3(t)\|_{\dot{H}_{\log_h, E}^{\frac{1}{2}}} \geq c_0,$$

where

$$\|a\|_{\dot{H}_{\log_h, E}^{\frac{1}{2}}}^2 \stackrel{def}{=} \int_{\mathbb{R}^3} |\xi| \log(E|\xi_h| + e) |\hat{a}(\xi)|^2 d\xi < \infty, \quad \text{and } \xi_h \stackrel{def}{=} (\xi_1, \xi_2).$$

Motivated by this result, we aim to show that, up to a small modification in the proof of Theorem II.1.1, we can obtain the same blow-up condition in the case $p = \infty$, by slightly reinforcing the $\dot{H}^{\frac{1}{2}}$ norm in the vertical direction instead of the horizontal one. More precisely, we define

Definition II.1.1. *Let E be a positive real number. We define $\dot{H}_{\log_v, E}^{\frac{1}{2}}$ to be the sub space of $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ such that:*

$$a \in \dot{H}_{\log_v, E}^{\frac{1}{2}} \iff \|a\|_{\dot{H}_{\log_v, E}^{\frac{1}{2}}}^2 \stackrel{def}{=} \int_{\mathbb{R}^3} |\xi| \log(E|\xi_v| + e) |\hat{a}(\xi)|^2 d\xi < \infty$$

We will prove

Theorem II.1.2. *There exists a positive constant c_0 such that if u is a maximal solution of (NS) in $C([0, T^*[, H^1)$, then for all positive real number E we have*

$$T^* < \infty \implies \limsup_{t \rightarrow T^*} \|u^3(t)\|_{\dot{H}_{\log_v, E}^{\frac{1}{2}}} \geq c_0.$$

Remark II.1.1. *The blow-up condition stated in Theorem II.1.2 above can be generalized to the following one*

$$\forall \sigma \in \mathbb{S}^2, \quad T^* < \infty \implies \limsup_{t \rightarrow T^*} \|\sigma \cdot u(t)\|_{\dot{H}_{\log_{\tilde{\sigma}}, E}^{\frac{1}{2}}} \geq c_0,$$

where

$$\|a\|_{\dot{H}_{\log_{\tilde{\sigma}}, E}^{\frac{1}{2}}}^2 \stackrel{def}{=} \int_{\mathbb{R}^3} |\xi| \log(E|\xi_{\tilde{\sigma}}| + e) |\hat{a}(\xi)|^2 d\xi < \infty, \quad \text{with } \xi_{\tilde{\sigma}} \stackrel{def}{=} (\xi \cdot \sigma).$$

As a corollary, by combining the arguments leading to prove our Theorem II.1.2 together with the ones leading to Theorem II.1.1 from [29], we can reach the same result under a little bit better condition on one component of the velocity. Let us define the space $\dot{H}_{\log, E}^{\frac{1}{2}}$ by

$$\|a\|_{\dot{H}_{\log, E}^{\frac{1}{2}}}^2 \stackrel{def}{=} \int_{\mathbb{R}^3} |\xi| \log(E \min\{|\xi_h|, |\xi_v|\} + e) |\hat{a}(\xi)|^2 d\xi < \infty.$$

Then we prove

Corollary II.2. *There exists a positive constant c_0 such that, if u is a maximal solution of (NS) in $C([0, T^*[, H^1)$, then for all positive real number E , we have*

$$T^* < \infty \implies \limsup_{t \rightarrow T^*} \|u^3(t)\|_{\dot{H}_{\log, E}^{\frac{1}{2}}} \geq c_0.$$

The other results that we will prove in this Chapter can be seen as some blow-up criterion under scaling invariant conditions on one component of the velocity and one component of the vorticity, whether in some anisotropic Besov spaces of the form $L^p((B_{2, \infty}^\alpha)_h (B_{2, \infty}^{s_p - \alpha})_v)$, for $\alpha \in [0, s_p]$, or $L^p(\mathcal{B}_{q, p})$, where

$$s_p \stackrel{\text{def}}{=} \frac{2}{p} - \frac{1}{2}, \quad \text{and} \quad \mathcal{B}_{q, p} \stackrel{\text{def}}{=} B_{q, \infty}^{\frac{3}{q} + \frac{2}{p} - 2}. \quad (\text{II.4})$$

We will prove

Theorem II.2.1. *Let u be a maximal solution of (NS) in $C([0, T^*]; H^1)$. If $T^* < \infty$, then*

$$\forall p, m \in [2, 4], \quad \forall \alpha \in \left[0, \frac{2}{p} - \frac{1}{2}\right], \quad \forall \beta \in \left[0, \frac{2}{m} - \frac{1}{2}\right], \quad \text{we have:}$$

$$\int_0^{T^*} \|\partial_3 u^3(t')\|_{B_{2, \infty}^{\alpha, s_p - \alpha}}^p dt' + \int_0^{T^*} \|\omega^3(t')\|_{B_{2, \infty}^{\beta, s_m - \beta}}^m dt' = \infty.$$

Theorem II.2.2. *Let u be a maximal solution of (NS) in $C([0, T^*]; H^1)$. If $T^* < \infty$, then for all $q_1, q_2 \in [3, \infty[$, for all p_1, p_2 satisfying*

$$\frac{3}{q_i} + \frac{2}{p_i} \in]1, 2[, \quad i \in \{1, 2\}. \quad (\text{II.5})$$

We have

$$\int_0^{T^*} \|\partial_3 u^3(t')\|_{\mathcal{B}_{q_1, p_1}}^{p_1} dt' + \int_0^{T^*} \|\omega^3(t')\|_{\mathcal{B}_{q_2, p_2}}^{p_2} dt' = \infty.$$

Remark II.2.1. 1. *All the spaces stated in Theorems II.2.1 and Theorem II.2.2 above are scaling invariant spaces under the natural 3-D Navier-Stokes scaling.*

2. *The regularity of the spaces stated in the blow-up conditions in Theorem II.2.2 is negative, more precisely under assumption (II.5), $\frac{3}{q_i} + \frac{2}{p_i} - 2 \in]-1, 0[$. Moreover, the integrability asked for in the associated Besov spaces is always higher than 3, which make these spaces larger than $L_T^p(H^{\frac{2}{p} - \frac{1}{2}})$*

3. *For Theorem II.2.1, taking in mind the embedding $L_T^p(H^{\frac{2}{p} - \frac{1}{2}}) \hookrightarrow L_T^p(B_{2, \infty}^{\alpha, \frac{2}{p} - \frac{1}{2} - \alpha})$, for all $\alpha \in [0, \frac{2}{p} - \frac{1}{2}]$ (see Lemma A.2.6), it is obvious then that the blow-up conditions stated in Theorem II.2.1 implies the ones in $L_T^p(H^{\frac{2}{p} - \frac{1}{2}})$.*

4. In the case $p = 4$ (resp. $m = 4$) in Theorem II.2.1, α (resp. β) is necessary zero, this means that the anisotropic space above is nothing but $L_T^4(\dot{B}_{2,\infty}^0)$, which is still larger than $L_T^4(L^2)$. The proof in this case can be done without any use of anisotropic technics.

The structure of this Chapter is the following: in order to prove each one of Theorems II.1.2, II.2.1 and II.2.2, in a first time we will reduce their proofs to three lemmata. Then we prove each one in its subsection.

We will make use of the following notations: if A and B are two real quantities, the notation $A \lesssim B$ means $A \leq CB$ for some universal constant C which is independent on varying parameters of the problem. $(c_q)_{q \in \mathbb{Z}}$ (resp. $(d_q)_{q \in \mathbb{Z}}$) will be a sequence satisfying $\sum_{q \in \mathbb{Z}} c_q^2 \leq 1$

(resp. $\sum_{q \in \mathbb{Z}} d_q \leq 1$), which is allowed to differ from a line to another one.

Sometimes, we will use the notation

$$\begin{aligned} L_T^r(L_h^p L_v^q) &\stackrel{def}{=} L^r((0, T); L^p(\mathbb{R}_h^2); L^q(\mathbb{R}_v)), & \dot{H}_h^s(\dot{H}_v^t) &\stackrel{def}{=} \dot{H}^{s,t}(\mathbb{R}^3) \\ \|\cdot\|_{\dot{H}_h^s(\dot{H}_v^t)} &\stackrel{def}{=} \|\cdot\|_{\dot{H}^{s,t}(\mathbb{R}^3)}, & \|\cdot\|_{\dot{B}_{p,q}^s} &\stackrel{def}{=} \|\cdot\|_{\dot{B}_{p,q}^s(\mathbb{R}^3)} \end{aligned}$$

In the rest of this Chapter, in order to simplify the notations, we will omit the dot in the notation of the homogeneous dyadic operators. That is to say, we will be using only the homogeneous version of Littlewood-Paley theory defined in the Appendix A, by denoting the associated dyadic blocks Δ_j instead of $\dot{\Delta}_j$.

II.3 Almost critical one component condition

II.3.1 Proof of Theorem II.1.2

In this subsection, we reduce the proof of Theorem II.1.2 to a proof of a crucial inequality. For $i, \ell \in \{1, 2\}$, we denote

$$J_{i\ell}(u, u^3) \stackrel{def}{=} \int_{\mathbb{R}^3} \partial_i u^3 \partial_3 u^\ell \partial_i u^\ell. \quad (\text{II.6})$$

The proof of Theorem II.1.2 is then based on the following lemma

Lemma II.3.1. *There exists $C > 0$ such that, for any $E > 0$, we have:*

$$|J_{i\ell}(u, u^3)| \leq \left(\frac{1}{10} + C \|u^3\|_{H_{\log v, E}^{\frac{1}{2}}} \right) \|\nabla_h u\|_{\dot{H}^1}^2 + C \|u^3\|_{\dot{H}^{\frac{1}{2}}}^2 \frac{\|\partial_3 u^h\|_{L^2}^2}{E^2}.$$

Let us assume for a moment that Lemma II.3.1 holds true, we show first how to derive Theorem II.1.2 from it, then we go back to prove Lemma II.3.1 in the next subsection.

Proof of Theorem II.1.2 Following the idea of [29] we begin by establishing a bound of $\nabla_h u$ in $L_T^\infty(L^2) \cap L_T^2(\dot{H}^1)$, then we use this estimate to establish a bound of $\partial_3 u$ in the same space. To do so we multiply (NS) by $-\Delta_h u$, usual calculation leads then to:

$$\frac{d}{2dt} \|\nabla_h u\|_{L^2}^2 + \|\nabla_h u\|_{\dot{H}^1}^2 = \sum_{i=1}^4 \mathcal{E}_i(u), \quad (\text{II.7})$$

with

$$\begin{aligned} \mathcal{E}_1(u) &\stackrel{\text{def}}{=} - \sum_{i=1}^2 (\partial_i u^h \cdot \nabla_h u^h | \partial_i u^h)_{L^2}, \\ \mathcal{E}_2(u) &\stackrel{\text{def}}{=} - \sum_{i=1}^2 (\partial_i u^h \cdot \nabla_h u^3 | \partial_i u^3)_{L^2}, \\ \mathcal{E}_3(u) &\stackrel{\text{def}}{=} - \sum_{i=1}^2 (\partial_i u^3 \partial_3 u^h | \partial_i u^h)_{L^2}, \\ \mathcal{E}_4(u) &\stackrel{\text{def}}{=} - \sum_{i=1}^2 (\partial_i u^3 \partial_3 u^h | \partial_i u^h)_{L^2}. \end{aligned}$$

A direct computation shows that $\mathcal{E}_1(u)$, $\mathcal{E}_2(u)$ and $\mathcal{E}_4(u)$ can be expressed as a sum of terms of the form

$$I(u) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \partial_i u^3 \partial_j u^k \partial_\ell u^m,$$

where $(j, \ell) \in \{1, 2\}^2$ and $(i, k, m) \in \{1, 2, 3\}^3$.

Next, by duality, product rules and then interpolation, for any $p \in (1, +\infty]$, one may easily show that¹

$$\begin{aligned} I(u) &\lesssim \|\nabla_h u^3\|_{\dot{H}^{\frac{2}{p}-\frac{1}{2}}} \|\partial_j u^k \partial_\ell u^m\|_{\dot{H}^{\frac{1}{2}-\frac{2}{p}}} \\ &\lesssim \|u^3\|_{\dot{H}^{\frac{2}{p}+\frac{1}{2}}} \|\nabla_h u\|_{\dot{H}^{1-\frac{1}{p}}}^2 \\ &\lesssim \|u^3\|_{\dot{H}^{\frac{2}{p}+\frac{1}{2}}} \|\nabla_h u\|_{L^2}^{\frac{2}{p}} \|\nabla_h u\|_{\dot{H}^1}^{2-\frac{2}{p}}. \end{aligned}$$

In particular, for $p = \infty$ we have:

$$I(u) \lesssim \|u^3\|_{H^{\frac{1}{2}}} \|\nabla_h u\|_{\dot{H}^1}^2. \quad (\text{II.8})$$

¹Notice that $I(u)$ provides a global bound if $u^3 \in L^p(\dot{H}^{\frac{2}{p}+\frac{1}{2}})$ for some $p \in (1, \infty]$. It is in fact the term $\mathcal{E}_3(u)$ which poses a problem, and this is why this method doesn't give a complete answer to the regularity criteria under one component only in the case $p = 2$ as mentioned in [29].

The term $\mathcal{E}_3(u)$, can be estimated by using Lemma II.3.1, to obtain

$$\mathcal{E}_3(u) \leq \left(\frac{1}{10} + C \|u^3\|_{H_{\log v, E}^{\frac{1}{2}}} \right) \|\nabla_h u\|_{\dot{H}^1}^2 + C \|u^3\|_{\dot{H}^{\frac{1}{2}}}^2 \frac{\|\partial_3 u^h\|_{L^2}^2}{E^2}.$$

We define then

$$T_* \stackrel{\text{def}}{=} \sup \left\{ T \in [0, T^* [/ \sup_{t \in [0, T]} \|u^3(t)\|_{H_{\log v, E}^{\frac{1}{2}}} \leq \frac{1}{4C} \right\}.$$

Therefore, for all $t \leq T_*$, relation (II.7) together with estimate (II.8), Lemma II.3.1 and the classical L^2 –energy estimate lead to

$$\|\nabla_h u(t)\|_{L^2}^2 + \int_0^t \|\nabla_h u(s)\|_{\dot{H}^1}^2 ds \leq \|\nabla_h u_0\|_{L^2}^2 + \frac{\|u_0\|_{L^2}^2}{E^2}. \quad (\text{II.9})$$

On the other hand, as explained in [29], multiplying (NS) by $-\partial_3^2 u$, integrating over \mathbb{R}^3 , integration by parts together with the divergence free condition lead to

$$\begin{aligned} \frac{d}{2dt} \|\partial_3 u\|_{L^2}^2 + \|\partial_3 u\|_{\dot{H}^1}^2 &\lesssim \|\partial_3 u\|_{L^6} \|\nabla_h u\|_{L^3} \|\partial_3 u\|_{L^2} \\ &\lesssim \frac{1}{2} \|\partial_3 u\|_{\dot{H}^1}^2 + C \|\nabla_h u\|_{L^2} \|\nabla_h u\|_{\dot{H}^1} \|\partial_3 u\|_{L^2}^2. \end{aligned}$$

(II.9) above leads then to a bound for u in $L_{T_*}^\infty(\dot{H}^1)$.

Thus, by contraposition, if the quantity $\|u(t)\|_{\dot{H}^1}$ blows-up at a finite time $T^* > 0$, then

$$\forall t \in [0, T^*[: \sup_{s \in [0, t]} \|u^3(s)\|_{\dot{H}^{\frac{1}{2}}} > c_0, \stackrel{\text{def}}{=} \frac{1}{4C}$$

which gives the desired result by passing to the limit $t \rightarrow T^*$.

Theorem II.1.2 is proved. □

II.3.2 Proof of the main Lemma II.3.1

Before presenting the proof of Lemma II.3.1, we shall prove two simple lemmata based on which frequencies dominate the horizontal or the vertical ones.

The first lemma can be used when the horizontal frequencies control the vertical ones

Lemma II.3.2. *Let s, t be two real numbers, let f be a regular function, we define f^G as*

$$f^G \stackrel{\text{def}}{=} \sum_{q \leq k} \Delta_k^h \Delta_q^v f.$$

Then we have:

$$\|\partial_3 f^G\|_{\dot{H}^{s,t}} \lesssim \|\nabla_h f\|_{\dot{H}^{s,t}}$$

Proof Let us use Plancherel-Parseval identity to write:

$$\|\partial_3 f^{\mathbf{G}}\|_{H^{s,t}}^2 \approx \int_{\mathbb{R}^3} |\xi_h|^{2s} |\xi_v|^{2t} \left| \sum_{q \leq k} |\xi_v| \varphi_k^h(\xi) \varphi_q^v(\xi) \right|^2 |\hat{f}(\xi)|^2 d\xi, \quad (\text{II.10})$$

where: $\varphi_k^h \stackrel{\text{def}}{=} \varphi(2^{-k}|\xi_h|)$, $\varphi_q^v \stackrel{\text{def}}{=} \varphi(2^{-q}|\xi_v|)$, and φ is the function given in the Appendix to define the dyadic blocks. Thus, using the support properties of φ_k^h , φ_q^v , and the condition $q < k$, we infer that, for all $\xi = (\xi_h, \xi_v) \in \text{Supp}(\varphi_k^h) \times \text{supp}(\varphi_q^v)$

$$|\xi_v| \lesssim 2^q \leq 2^k \lesssim |\xi_h|. \quad (\text{II.11})$$

Plugging (II.11) into (II.10) concludes the proof of the lemma. \square

The second lemma that we will need is useful to estimate some parts of the anisotropic Bony's decomposition for functions having a dominated vertical frequencies compared to the horizontal ones, and which are supported away from zero horizontally in Fourier side.

Lemma II.3.3. *Let f be a regular function, and $E > 0$. We define $f_{\sharp, E^{-1}}^{\mathbf{B}}$ as*

$$f_{\sharp, E^{-1}}^{\mathbf{B}} \stackrel{\text{def}}{=} \sum_{k < q} \Delta_k^h \Delta_q^v f_{\sharp, E^{-1}},$$

where

$$f_{\sharp, E^{-1}} \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{B_h^c(0, E^{-1})} \hat{f}).$$

Above, $B_h(0, R)$ refers to the two dimensional ball of radius equals to R . Then we have the following estimates

$$\|\Delta_q^v S_{j-1}^h(f_{\sharp, E^{-1}}^{\mathbf{B}})\|_{L_v^2(L_h^\infty)} \lesssim (\log(E2^q + e))^{\frac{1}{2}} c_q \|\nabla_h f\|_{L^2(\mathbb{R}^3)}. \quad (\text{II.12})$$

$$\|S_{q-1}^v S_{j-1}^h(f_{\sharp, E^{-1}}^{\mathbf{B}})\|_{L_v^\infty(L_h^\infty)} \lesssim (\log(E2^q + e))^{\frac{1}{2}} c_q 2^{\frac{q}{2}} \|\nabla_h f\|_{L^2(\mathbb{R}^3)}. \quad (\text{II.13})$$

Proof According to the support properties we have

$$\Delta_q^v S_{j-1}^h(f_{\sharp, E^{-1}}^{\mathbf{B}}) = \left(\Delta_q^v S_{j-1}^h \sum_{i \in \{-1, 0, 1\}} S_{q-1+i}^h \Delta_{q+i}^v \right) f_{\sharp, E^{-1}},$$

therefore, due to Bernstein's inequality, we can write

$$\begin{aligned} \|\Delta_q^v S_{j-1}^h(f_{\sharp, E^{-1}}^{\mathbf{B}})\|_{L_v^2(L_h^\infty)} &\lesssim \sum_{E^{-1} \lesssim 2^k \lesssim 2^q} 2^k \|\Delta_k^h \Delta_q^v f\|_{L^2(\mathbb{R}^3)} \\ &\lesssim \left(\sum_{E^{-1} \lesssim 2^k \lesssim 2^q} c_k \right) c_q \|\nabla_h f\|_{L^2(\mathbb{R}^3)} \\ &\lesssim (\log(E2^q + e))^{\frac{1}{2}} c_q \|\nabla_h f\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

Thus the first inequality is proved. For the second, one we first write

$$\|S_{q-1}^v S_{j-1}^h(f_{\sharp, E^{-1}}^{\mathbf{B}})\|_{L_v^\infty(L_h^\infty)} \lesssim \sum_{m \leq q} 2^{\frac{m}{2}} \|\Delta_m^v S_{j-1}^h(f_{\sharp, E^{-1}}^{\mathbf{B}})\|_{L_v^2(L_h^\infty)},$$

inequality (II.12) gives then

$$\begin{aligned} 2^{-\frac{q}{2}} \|S_{q-1}^v S_{j-1}^h(f_{\sharp, E^{-1}}^{\mathbf{B}})\|_{L_v^\infty(L_h^\infty)} &\lesssim \left(\sum_{m \leq q} 2^{\frac{1}{2}(m-q)} (\log(E2^m + e))^{\frac{1}{2}} c_m \right) \|\nabla_h f\|_{L^2(\mathbb{R}^3)} \\ &\lesssim (\log(E2^q + e))^{\frac{1}{2}} c_q \|\nabla_h f\|_{L^2(\mathbb{R}^3)} \end{aligned}$$

Inequality (II.13) follows. □

Now we are in position to prove lemma II.3.1.

Proof of lemma II.3.1 Let us write, for $E \in \mathbb{R}^+$ and $a \in \mathcal{S}'(\mathbb{R}^3)$:

$$a_{\flat, E^{-1}} \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{B_h(0, E^{-1})} \hat{a}), \quad a_{\sharp, E^{-1}} \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{B_h^c(0, E^{-1})} \hat{a}).$$

Based on this decomposition, we write

$$J_{il}(u, u^3) = J_E^\flat + J_E^\sharp,$$

where

$$J_E^\flat \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \partial_i u^3 \partial_3 u_{\flat, E^{-1}}^\ell \partial_i u^\ell, \quad \text{and} \quad J_E^\sharp \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \partial_i u^3 \partial_3 u_{\sharp, E^{-1}}^\ell \partial_i u^\ell.$$

The main point consists in estimating J_E^\sharp . Using Bony's decomposition with respect to the horizontal variables, to write

$$\begin{aligned} J_E^\sharp &= J_E^{\sharp, 1} + J_E^{\sharp, 2} \quad \text{with} \\ J_E^{\sharp, 1} &\stackrel{\text{def}}{=} \int_{\mathbb{R}_v} \left(\int_{\mathbb{R}_h^2} \partial_i u^\ell(x_h, x_3) \tilde{T}_{\partial_i u^3(x_h, x_3)}^h \partial_3 u_{\sharp, E^{-1}}^\ell(x_h, x_3) dx_h \right) dx_3, \\ J_E^{\sharp, 2} &\stackrel{\text{def}}{=} \int_{\mathbb{R}_v} \left(\sum_{k \in \mathbb{Z}} \int_{\mathbb{R}_h^2} \Delta_k^h \partial_i u^\ell(x_h, x_3) \tilde{\Delta}_k^h T_{\partial_3 u_{\sharp, E^{-1}}^\ell(x_h, x_3)}^h \partial_i u^3(x_h, x_3) dx_h \right) dx_3, \end{aligned}$$

and

$$\tilde{T}_a^h b \stackrel{\text{def}}{=} \sum_{k \in \mathbb{Z}} S_{k+2} a \Delta_k b.$$

$J_E^{\sharp, 1}$ can be estimated by duality then by using some product laws, we obtain

$$\begin{aligned} J_E^{\sharp, 1} &\lesssim \|\nabla_h u\|_{L_v^\infty(\dot{H}_h^{\frac{1}{2}})} \left\| \tilde{T}_{\partial_i u^3}^h \partial_3 u_{\sharp, E^{-1}}^\ell \right\|_{L_v^1(\dot{H}_h^{-\frac{1}{2}})} \\ &\lesssim \|\nabla_h u\|_{L_v^\infty(\dot{H}_h^{\frac{1}{2}})} \|\nabla_h u^3\|_{L_v^2(\dot{H}_h^{-\frac{1}{2}})} \|\partial_3 u\|_{L_v^2(\dot{H}_h^1)}. \end{aligned}$$

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Using then the inequality: $\|\nabla_h u\|_{L^\infty(\dot{H}_h^{\frac{1}{2}})} \lesssim \|\nabla_h u\|_{\dot{H}^1(\mathbb{R}^3)}$ (see Lemma A.2.7), we infer that

$$J_E^{\sharp,1} \lesssim \|u^3\|_{\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)} \|\nabla_h u\|_{\dot{H}^1(\mathbb{R}^3)}^2. \quad (\text{II.14})$$

In order to estimate $J_E^{\sharp,2}$, we split it into a sum of a good term $J_E^{\sharp,2,\mathbf{G}}$ and a bad one $J_E^{\sharp,2,\mathbf{B}}$ based on the dominated frequencies of $\partial_3 u^\ell$

$$\begin{aligned} \partial_3 u_{\sharp,E-1}^\ell &= \partial_3 u_{\sharp,E-1}^{\ell,\mathbf{G}} + \partial_3 u_{\sharp,E-1}^{\ell,\mathbf{B}} \quad \text{with} \\ \partial_3 u_{\sharp,E-1}^{\ell,\mathbf{G}} &\stackrel{\text{def}}{=} \sum_{q \leq k} \Delta_k^h \Delta_q^v \partial_3 u_{\sharp,E-1}^\ell \quad \text{and} \quad \partial_3 u_{\sharp,E-1}^{\ell,\mathbf{B}} \stackrel{\text{def}}{=} \sum_{k < q} \Delta_k^h \Delta_q^v \partial_3 u_{\sharp,E-1}^\ell \end{aligned}$$

The good term can be easily estimated without using the fact that $u_{\sharp,E-1}^\ell$ contains only the high horizontal frequencies, but just by making use of the fact that the horizontal frequencies control the vertical ones. We proceed as follows, by using the product Lemma A.2.3, we find:

$$J_E^{\sharp,2,\mathbf{G}} \lesssim \|\nabla_h u^3\|_{H_h^{-\frac{1}{2}}(L_v^2)} \left\| \partial_3 u_{\sharp,E-1}^{\ell,\mathbf{G}} \right\|_{H_h^{\frac{3}{4}}(H_v^{\frac{1}{4}})} \|\nabla_h u\|_{H_h^{\frac{3}{4}}(H_v^{\frac{1}{4}})}.$$

Lemma II.3.2 gives then

$$J_E^{\sharp,2,\mathbf{G}} \lesssim \|\nabla_h u^3\|_{H_h^{-\frac{1}{2}}(L_v^2)} \|\nabla_h u\|_{H_h^{\frac{3}{4}}(H_v^{\frac{1}{4}})}^2,$$

which yields finally, by using Lemma A.2.6

$$J_E^{\sharp,2,\mathbf{G}} \lesssim \|u^3\|_{H^{\frac{1}{2}}} \|\nabla_h u\|_{H^1}^2. \quad (\text{II.15})$$

In order to estimate the bad term $J_E^{\sharp,2,\mathbf{B}}$, we use the Bony's decomposition with respect to vertical variables to infer that

$$J_E^{\sharp,2,\mathbf{B}} \lesssim \sum_{q,k \in \mathbb{Z}} \|\Delta_k^h \Delta_q^v \nabla_h u\|_{L^2(\mathbb{R}^3)} (\mathcal{I}_{k,q}^{(1)} + \mathcal{I}_{k,q}^{(2)} + \mathcal{I}_{k,q}^{(3)}), \quad (\text{II.16})$$

where

$$\begin{aligned} \mathcal{I}_{k,q}^{(1)} &\stackrel{\text{def}}{=} \left\| S_{k-1}^h S_{q-1}^v (\partial_3 u_{\sharp,E-1}^{\ell,\mathbf{B}}) \right\|_{L_v^\infty(L_h^\infty)} \|\Delta_k^h \Delta_q^v \nabla_h u^3\|_{L^2(\mathbb{R}^3)}, \\ \mathcal{I}_{k,q}^{(2)} &\stackrel{\text{def}}{=} \left\| S_{k-1}^h \Delta_q^v (\partial_3 u_{\sharp,E-1}^{\ell,\mathbf{B}}) \right\|_{L_v^2(L_h^\infty)} \|\Delta_k^h S_{q-1}^v \nabla_h u^3\|_{L_v^\infty(L_h^2)}, \\ \mathcal{I}_{k,q}^{(3)} &\stackrel{\text{def}}{=} 2^{\frac{q}{2}} \sum_{j \geq q - N_0} \left\| S_{k-1}^h \Delta_j^v (\partial_3 u_{\sharp,E-1}^{\ell,\mathbf{B}}) \right\|_{L_v^2(L_h^\infty)} \left\| \Delta_k^h \tilde{\Delta}_j^v \nabla_h u^3 \right\|_{L^2(\mathbb{R}^3)}. \end{aligned}$$

The estimates of these terms are based on Lemma II.3.3, by taking $f_{\sharp,E-1}^{\mathbf{B}} = \partial_3 u_{\sharp,E-1}^{\ell,\mathbf{B}}$

We use inequality (II.13) from Lemma II.3.3 to estimate $\mathcal{I}_{k,q}^{(1)}$, which gives

$$\begin{aligned} \mathcal{I}_{k,q}^{(1)} &\lesssim (\log(E2^q + e))^{\frac{1}{2}} c_q 2^{\frac{q}{2}} \|\nabla_h \partial_3 u\|_{L^2(\mathbb{R}^3)} 2^k \|\Delta_k^h \Delta_q^v u^3\|_{L^2(\mathbb{R}^3)} \\ &\lesssim c_q^2 c_k 2^{\frac{q}{2}} 2^{\frac{k}{2}} \|\nabla_h \partial_3 u\|_{L^2(\mathbb{R}^3)} \|u^3\|_{\dot{H}_{\log v,E}^{\frac{1}{2}}}. \end{aligned}$$

Thus, we obtain

$$\mathcal{I}_{k,q}^{(1)} \lesssim c_k 2^{\frac{q}{2}} 2^{\frac{k}{2}} \|\nabla_h \partial_3 u\|_{L^2(\mathbb{R}^3)} \|u^3\|_{\dot{H}_{\log v, E}^{\frac{1}{2}}}. \quad (\text{II.17})$$

In order to estimate $\mathcal{I}_{k,q}^{(2)}$, we use inequality (II.12) to infer that

$$\mathcal{I}_{k,q}^{(2)} \lesssim (\log(E2^q + e))^{\frac{1}{2}} c_q \|\nabla_h \partial_3 u\|_{L^2(\mathbb{R}^3)} 2^k \|\Delta_k^h S_{q-1}^v u^3\|_{L_v^\infty(L_h^2)} \quad (\text{II.18})$$

$$\lesssim c_q 2^{\frac{k}{2}} \|\nabla_h \partial_3 u\|_{L^2(\mathbb{R}^3)} (\log(E2^q + e))^{\frac{1}{2}} 2^{\frac{k}{2}} \|\Delta_k^h S_{q-1}^v u^3\|_{L_v^\infty(L_h^2)}. \quad (\text{II.19})$$

Next, we use the following estimate

$$2^{-\frac{q}{2}} (\log(E2^q + e))^{\frac{1}{2}} 2^{\frac{k}{2}} \|\Delta_k^h S_{q-1}^v u^3\|_{L_v^\infty(L_h^2)} \lesssim \sum_{m \leq q} (2^{(m-q)} \log(E2^q + e))^{\frac{1}{2}} 2^{\frac{k}{2}} \|\Delta_k^h \Delta_m^v u^3\|_{L^2(\mathbb{R}^3)},$$

together with the fact that

$$\begin{aligned} \log(E2^q + e) &\leq \log(2^{q-m}(E2^m + e)), \quad \forall m \leq q \\ &\leq \log(E2^m + e) + (q - m) \\ &\lesssim \log(E2^m + e)(1 + (q - m)) \end{aligned}$$

to achieve

$$2^{-\frac{q}{2}} (\log(E2^q + e))^{\frac{1}{2}} 2^{\frac{k}{2}} \|\Delta_k^h S_{q-1}^v u^3\|_{L_v^\infty(L_h^2)} \lesssim \sum_{m \leq q} (\sigma_{q-m} c_m) c_k \|u^3\|_{\dot{H}_{\log v, E}^{\frac{1}{2}}},$$

where $\sigma_j \stackrel{\text{def}}{=} \frac{\sqrt{1+j}}{2^{\frac{j}{2}}} \in \ell_j^1(\mathbb{N})$. By using convolution inequality, we deduce that

$$\mathcal{I}_{k,q}^{(2)} \lesssim c_k 2^{\frac{q}{2}} 2^{\frac{k}{2}} \|\nabla_h \partial_3 u\|_{L^2(\mathbb{R}^3)} \|u^3\|_{\dot{H}_{\log v, E}^{\frac{1}{2}}}. \quad (\text{II.20})$$

Finally, in order to estimate $\mathcal{I}_{k,q}^{(3)}$, we use again inequality (II.13) from Lemma II.3.3, we obtain

$$\begin{aligned} \mathcal{I}_{k,q}^{(3)} &\lesssim 2^{\frac{q}{2}} 2^{\frac{k}{2}} \left(\sum_{j \geq q-N_0} (\log(E2^j + e))^{\frac{1}{2}} c_j 2^{\frac{k}{2}} \|\Delta_k^h \Delta_j^v u^3\|_{L^2(\mathbb{R}^3)} \right) \|\nabla_h \partial_3 u\|_{L^2(\mathbb{R}^3)} \\ &\lesssim c_k 2^{\frac{q}{2}} 2^{\frac{k}{2}} \|\nabla_h \partial_3 u\|_{L^2(\mathbb{R}^3)} \|u^3\|_{\dot{H}_{\log v, E}^{\frac{1}{2}}}. \end{aligned}$$

Together with (II.17) and (II.20) yield

$$\sum_{i \in \{1,2,3\}} \mathcal{I}_{k,q}^{(i)} \lesssim c_k 2^{\frac{q}{2}} 2^{\frac{k}{2}} \|\nabla_h u\|_{H^1(\mathbb{R}^3)} \|u^3\|_{\dot{H}_{\log v, E}^{\frac{1}{2}}}.$$

Plugging this last one into (II.16) gives

$$\begin{aligned} J_E^{\sharp,2,\mathbf{B}} &\lesssim \left(\sum_{k,q \in \mathbb{Z}} c_k 2^{\frac{q}{2}} 2^{\frac{k}{2}} \|\Delta_k^h \Delta_q^v \nabla_h u\|_{L^2(\mathbb{R}^3)} \right) \|\nabla_h u\|_{\dot{H}^1(\mathbb{R}^3)} \|u^3\|_{\dot{H}_{\log v, E}^{\frac{1}{2}}} \\ &\lesssim \|\nabla_h u\|_{\dot{H}_h^{\frac{1}{2}}(\dot{B}_{2,1}^{\frac{1}{2}})_v} \|\nabla_h u\|_{\dot{H}^1(\mathbb{R}^3)} \|u^3\|_{\dot{H}_{\log v, E}^{\frac{1}{2}}}. \end{aligned}$$

Lemma A.2.6 then gives

$$J_E^{\sharp,2,\mathbf{B}} \lesssim \|\nabla_h u\|_{\dot{H}^1(\mathbb{R}^3)}^2 \|u^3\|_{\dot{H}_{\log v, E}^{\frac{1}{2}}}. \quad (\text{II.21})$$

From (II.14), (II.15) and (II.21) we deduce

$$J_E^{\sharp} \lesssim \|\nabla_h u\|_{\dot{H}^1(\mathbb{R}^3)}^2 \|u^3\|_{\dot{H}_{\log v, E}^{\frac{1}{2}}}.$$

J_E^{\flat} can be estimated along the same lines as in [29], by using the product law $(\dot{B}_{2,1}^1)_h \cdot \dot{H}_h^{\frac{1}{2}} \subset \dot{H}_h^{\frac{1}{2}}$, together with the embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L_v^\infty(\dot{H}_h^{\frac{1}{2}})$ (see Lemma A.2.7), we infer that

$$\begin{aligned} J_E^{\flat} &\lesssim \|\nabla_h u^3\|_{L_v^2(\dot{H}^{-\frac{1}{2}})_h} \|\partial_3 u_{v, E^{-1}}^\ell \partial_i u^\ell\|_{L_v^2(\dot{H}_h^{\frac{1}{2}})} \\ &\lesssim \|u^3\|_{\dot{H}_h^{\frac{1}{2}}} \|\partial_3 u_{v, E^{-1}}^\ell\|_{L_v^2(\dot{B}_{2,1}^1)_h} \|\nabla_h u\|_{L_v^\infty(\dot{H}_h^{\frac{1}{2}})} \\ &\lesssim \|u^3\|_{\dot{H}_h^{\frac{1}{2}}} \|\nabla_h u\|_{\dot{H}^1} \frac{\|\partial_3 u\|_{L^2}}{E} \\ &\lesssim \frac{1}{100} \|\nabla_h u\|_{\dot{H}^1} + C \|u^3\|_{\dot{H}_h^{\frac{1}{2}}}^2 \frac{\|\partial_3 u\|_{L^2}^2}{E^2} \end{aligned}$$

Lemma II.3.1 is then proved. □

II.3.3 Proof of Corollary II.2

We end this section by giving the idea to prove the Corollary II.2. As aforementioned, the proof of Corollary II.2 is simple if we apply the arguments leading to prove Theorems II.1.1 and II.1.1.

Indeed, all the machinery of the proof relies on estimating the trilinear term $J_{i\ell}(u, u^3)$ given by (II.6). As the proofs of Theorems II.1.2 and II.1.1 are respectively based on Lemma II.3.1, and Lemma 2.3 from [29], the proof of Corollary II.2 can be reduced to the proof of the following Lemma

Lemma II.3.4. *There exists $C > 0$ such that, for any $E > 0$, we have:*

$$|J_{i\ell}(u, u^3)| \leq \left(\frac{1}{5} + C \|u^3\|_{\dot{H}_{\log v, E}^{\frac{1}{2}}} \right) \|\nabla_h u\|_{\dot{H}^1}^2 + C \|u^3\|_{\dot{H}_h^{\frac{1}{2}}}^2 \frac{\|\partial_3 u\|_{L^2}^2}{E^2}.$$

Proof

We recall the expression of $J_{i,\ell}(u, u^3)$

$$J_{i\ell}(u, u^3) \stackrel{\text{def}}{=} \int_{\mathbb{R}^3} \partial_i u^3 \partial_3 u^\ell \partial_i u^\ell.$$

Let us write

$$u^3 = u_h^3 + u_v^3,$$

with

$$u_h^3 \stackrel{\text{def}}{=} \mathcal{F}^{-1}(1_{|\xi_v| \leq |\xi_h|} \widehat{u^3}(\xi)), \quad u_v^3 \stackrel{\text{def}}{=} \mathcal{F}^{-1}(1_{|\xi_h| < |\xi_v|} \widehat{u^3}(\xi)).$$

This decomposition gives in particular

$$J_{i\ell}(u, u^3) = J_{i\ell}(u, u_h^3) + J_{i\ell}(u, u_v^3),$$

and

$$\|u^3\|_{\dot{H}_{\log, E}^{\frac{1}{2}}} = \|u_h^3\|_{\dot{H}_{\log v, E}^{\frac{1}{2}}} + \|u_v^3\|_{\dot{H}_{\log h, E}^{\frac{1}{2}}}.$$

The proof of Lemma II.3.4 is achieved by applying Lemma II.3.1 to $J_{i\ell}(u, u_h^3)$ and Lemma 2.3 from [29] to $J_{i\ell}(u, u_v^3)$. \square

II.4 Critical two components criterion

II.4.1 Proof of the main Theorems II.2.1 and II.2.2

In order to prove Theorem II.2.1 and Theorem II.2.2, we will be using, respectively, the following two lemmata, the proofs of which are the purpose of the next subsection

Lemma II.4.1. *For all $p \in]2, 4[$, for all $\alpha \in]0, s_p[$, where $s_p = \frac{2}{p} - \frac{1}{2}$, we have:*

$$|(fg|g)_{L^2}| \leq \frac{1}{10} \|g\|_{\dot{H}^1(\mathbb{R}^3)}^2 + C \|f\|_{B_{2,\infty}^{\alpha, s_p - \alpha}}^p \|g\|_{L^2(\mathbb{R}^3)}^2.$$

The second lemma reads as follows

Lemma II.4.2. *For any $p, q \in [1, \infty]$ satisfying $\frac{3}{q} + \frac{2}{p} \in]1, 2[$, we have*

$$|(fg|g)_{L^2}| \leq \frac{1}{10} \|g\|_{\dot{H}^1(\mathbb{R}^3)}^2 + C \|f\|_{B_{q,p}^p} \|g\|_{L^2(\mathbb{R}^3)}^2.$$

In the rest of this subsection, we outline the proofs of Theorem II.2.1 and Theorem II.2.2, assuming the two lemmata above hold true

Proof of Theorem II.2.1 Following for example the idea of [125], we multiply (NS) by $-\Delta u$ and we integrate in space to obtain

$$\begin{aligned} \frac{d}{2dt} \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 &= \int_{\mathbb{R}^3} (u \cdot \nabla u) \cdot \Delta u \\ &= - \int_{\mathbb{R}^3} \sum_{i,j,k=1}^3 \partial_k u^j \partial_j u^i \partial_k u^i. \end{aligned}$$

For the time being, we don't know how to deal with the tri-linear term on the right hand-side above in order to obtain a global-estimate of u in $L_T^\infty \dot{H}_x^1 \cap L_T^2 \dot{H}_x^2$, so to close the estimates the idea is similar to the one in Theorems II.1.1 and II.1.2, and it consists in looking at this term as a bi-linear operator acting on $(L_T^\infty \dot{H}_x^1 \cap L_T^2 \dot{H}_x^2)^2$ after assuming a condition which allows to control some components of the matrix $\partial_i u^j$.

Let us recall the Biot-Savart law identity which allows to write the so-called div-curl decomposition of u^h as

$$u^h = \nabla_h^\perp \Delta_h^{-1}(\omega^3) - \nabla_h \Delta_h^{-1}(\partial_3 u^3). \quad (\text{II.22})$$

Identity (II.22) insures that, for $(i, j) \in \{1, 2\}^2$, $\partial_i u^j$ can be writing in terms of ω^3 and $\partial_3 u^3$, modulo some anisotropic Fourier-multipliers of order zero, more precisely we have, for $(i, j) \in \{1, 2\}^2$

$$\partial_i u^j = \mathcal{R}_{i,j} \omega_3 + \tilde{\mathcal{R}}_{i,j} \partial_3 u^3,$$

where $\tilde{\mathcal{R}}_{i,j}$ and $\mathcal{R}_{i,j}$ are zero-order Fourier multipliers bounded from L^q into L^q for all q in $]1, \infty[$. On the other hand, the quantity $\partial_k u^j \partial_j u^i \partial_k u^i$ contains always, at least, one term of the form $\partial_i u^j$ with $(i, j) \in \{1, 2\}^2$ or $i = j = 3$, we infer that

$$\frac{d}{2dt} \|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 \lesssim \sum_{\substack{(l,k,m,n) \in \{1,2,3\} \\ (i,j) \in \{1,2\}}} \left| \int_{\mathbb{R}^3} (\mathcal{R}_{i,j} \omega_3 + \tilde{\mathcal{R}}_{i,j} \partial_3 u^3) \partial_k u^l \partial_m u^n \right|. \quad (\text{II.23})$$

Lemma II.4.2 gives then

$$\frac{d}{2dt} \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{\dot{H}^1}^2 \leq \frac{1}{10} \|\nabla u\|_{\dot{H}^1}^2 + C \left(\|\partial_3 u_3\|_{\dot{B}_{2,\infty}^{\alpha, s_p - \alpha}}^p + \|\omega^3\|_{\dot{B}_{2,\infty}^{\beta, s_m - \beta}}^m \right) \|\nabla u\|_{L^2}^2.$$

Gronwall's lemma leads then to

$$\|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(t')\|_{\dot{H}^1}^2 dt' \lesssim \|\nabla u_0\|_{L^2} \exp \left[C \int_0^t \left(\|\partial_3 u_3(t')\|_{\dot{B}_{2,\infty}^{\alpha, s_p - \alpha}}^p + \|\omega^3(t')\|_{\dot{B}_{2,\infty}^{\beta, s_m - \beta}}^m \right) dt' \right]. \quad (\text{II.24})$$

That is if, for some α, β, p, m satisfying the hypothesis of Theorem II.2.1, the quantity in the right hand side of (II.24) is finite, then u is bounded in $L_T^\infty(\dot{H}^1)$. By contraposition, if there is a blow-up of the \dot{H}^1 norm at some finite T^* then, for all α, β, p, m as given by Theorem II.2.1

$$\int_0^{T^*} \left(\|\partial_3 u_3(t')\|_{\dot{B}_{2,\infty}^{\alpha, s_p - \alpha}}^p + \|\omega^3(t')\|_{\dot{B}_{2,\infty}^{\beta, s_m - \beta}}^m \right) dt' = \infty.$$

Theorem II.2.1 is then proved. □

Proof of Theorem II.2.2 The proof of Theorem II.2.2 does not differ a lot from the previous one. We resume from (II.23), applying Lemma II.4.1 gives²

$$\frac{d}{2dt} \|\nabla u\|_{L^2}^2 + \|\nabla u\|_{\dot{H}^1}^2 \leq \frac{1}{10} \|\nabla u\|_{\dot{H}^1}^2 + C \left(\|\partial_3 u_3\|_{\mathcal{B}_{q_1, p_1}^{p_1}}^{p_1} + \|\omega^3\|_{\mathcal{B}_{q_2, p_2}^{p_2}}^{p_2} \right) \|\nabla u\|_{L^2}^2.$$

Next, integrating in time interval $[0, t]$, and applying Gronwall's lemma gives

$$\|\nabla u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(t')\|_{\dot{H}^1}^2 dt' \lesssim \|\nabla u_0\|_{L^2} \exp \left[C \int_0^t \left(\|\partial_3 u_3\|_{\mathcal{B}_{q_1, p_1}^{p_1}}^{p_1} + \|\omega^3\|_{\mathcal{B}_{q_2, p_2}^{p_2}}^{p_2} \right) dt' \right].$$

Same arguments as in the conclusion of the previous Theorem lead to the desired result. Theorem II.2.2 is Proved. □

II.4.2 Proof of the main lemmata II.4.1 and II.4.2

The purpose of this subsection is to outline the idea of the proof of Lemma II.4.1 and Lemma II.4.2

Proof of lemma II.4.1 Let $p \in [2, 4]$ and $\alpha \in [0, \frac{2}{p} - \frac{1}{2}]$. We define q and θ such that

$$\frac{2}{q} \stackrel{def}{=} 1 - \frac{1}{p}, \tag{II.25}$$

$$\theta \stackrel{def}{=} \frac{1}{2} + \frac{\alpha}{2} - \frac{1}{p}. \tag{II.26}$$

One may check that

$$q \in \left] \frac{8}{3}, 4 \right[\subset]2, \infty[\quad \text{and} \quad \theta \in \left[\frac{1}{2} - \frac{1}{p}, \frac{1}{4} \right] \subset \left[0, \frac{1}{2} \right] \subset \left[0, \frac{2}{q} \right],$$

which allow us to use the following embedding, due to Lemmata A.2.6 and A.2.9

$$L_T^\infty(L^2(\mathbb{R}^3)) \cap L_T^2(\dot{H}^1(\mathbb{R}^3)) \hookrightarrow L_T^q(\dot{B}_{2,1}^{\frac{2}{q}}(\mathbb{R}^3)) \hookrightarrow L_T^q((\dot{B}_{2,1}^{\frac{2}{q}-\theta})_h(\dot{B}_{2,1}^\theta)_v). \tag{II.27}$$

Thus, by using Lemma A.2.3, if $g \in (\dot{B}_{2,1}^{\frac{2}{q}-\theta})_h(\dot{B}_{2,1}^\theta)_v$ then $g \cdot g \in (\dot{B}_{2,1}^{\frac{4}{q}-2\theta-1})_h(\dot{B}_{2,1}^{2\theta-\frac{1}{2}})_v$. By virtue of (II.25), (II.26) and embedding (II.27), we infer that

$$\|g \cdot g\|_{(\dot{B}_{2,1}^{-\alpha})_h(\dot{B}_{2,1}^{-\frac{2}{p}+\frac{1}{2}+\alpha})_v} \lesssim \|g\|_{(\dot{B}_{2,1}^{\frac{2}{q}-\theta})_h(\dot{B}_{2,1}^\theta)_v}^2.$$

²Note that the case $q_i = \infty$ is considered in the estimates proved in Lemma II.4.2. However, we did not say anything about this case in Theorem II.2.2 due to the lack of continuity of Riesz operators $\mathcal{R}_{i,j}$ and $\tilde{\mathcal{R}}_{i,j}$ from L^∞ into L^∞ .

which gives by duality, embedding (II.27) and Lemma A.2.9

$$\begin{aligned} |(fg|g)_{L^2}| &\lesssim \|f\|_{(\dot{B}_{2,\infty}^\alpha)_h(\dot{B}_{2,\infty}^{\frac{2}{p}-\frac{1}{2}-\alpha})_v} \|g \cdot g\|_{(\dot{B}_{2,1}^{-\alpha})_h(\dot{B}_{2,1}^{-\frac{2}{p}+\frac{1}{2}+\alpha})_v} \\ &\lesssim \|f\|_{(\dot{B}_{2,\infty}^\alpha)_h(\dot{B}_{2,\infty}^{sp-\alpha})_v} \|g\|_{\dot{B}_{2,1}^{\frac{2}{q}}}^2 \\ &\lesssim \|f\|_{(\dot{B}_{2,\infty}^\alpha)_h(\dot{B}_{2,\infty}^{sp-\alpha})_v} \|g\|_{L^2}^{\frac{2}{p}} \|g\|_{\dot{H}^1}^{2(1-\frac{2}{p})}. \end{aligned}$$

Finally, we obtain

$$|(fg|g)_{L^2}| \leq \frac{1}{10} \|g\|_{\dot{H}^1}^2 + C \|f\|_{(\dot{B}_{2,\infty}^\alpha)_h(\dot{B}_{2,\infty}^{sp-\alpha})_v}^p \|g\|_{L^2}^2.$$

Lemma II.4.1 is proved. □

Proof of lemma II.4.2 According to Lemma A.2.9, in particular inequality (A.15) gives

$$\|g(t, \cdot)\|_{\dot{B}_{2,1}^{\frac{2}{m}}(\mathbb{R}^3)} \lesssim \|g(t, \cdot)\|_{\dot{H}^1(\mathbb{R}^3)}^{\frac{2}{m}} \|g(t, \cdot)\|_{L^2(\mathbb{R}^3)}^{1-\frac{2}{m}}, \quad \forall m \in]2, \infty[. \quad (\text{II.28})$$

We use then the Bony's decomposition to study the product $g \cdot g$.

Let $(q, p) \in [1, \infty]^2$ satisfying

$$q \in [3, \infty] \quad \text{and} \quad \frac{3}{q} + \frac{2}{p} \in]1, 2[.$$

Let (m_1, m_2) be in $[2, \infty] \times]2, \infty[$, given by

$$\frac{2}{3m_1} \stackrel{\text{def}}{=} \frac{1}{q} \quad \text{and} \quad 2\left(1 - \frac{1}{m_2}\right) \stackrel{\text{def}}{=} \frac{3}{q} + \frac{2}{p} \in]1, 2[\iff m_2 \in]2, \infty[. \quad (\text{II.29})$$

Let us define the real number N_{m_1} associated to the embedding $\dot{H}^{\frac{2}{m_1}}(\mathbb{R}^3)$ in $L^{N_{m_1}}(\mathbb{R}^3)$

$$\frac{1}{N_{m_1}} \stackrel{\text{def}}{=} \frac{1}{2} - \frac{2}{3m_1} \in \left[\frac{1}{6}, \frac{1}{2}\right].$$

Let us also define r to be the conjugate of q , that is

$$\frac{1}{r} \stackrel{\text{def}}{=} 1 - \frac{1}{q} \in \left[\frac{2}{3}, 1\right].$$

We write

$$\Delta_j(g \cdot g) = 2\Delta_j T_g(g) + \Delta_j R(g, g)$$

where T and R are the operators associated to the Bony's decomposition, defined in the Appendix A.

We turn now to estimate the two parts of $\Delta_j(g \cdot g)$. We have

$$\begin{aligned} \|\Delta_j T_g(g)\|_{L^r} &\lesssim \|S_{j-1}g\|_{L^{N_{m_1}}} \|\Delta_j g\|_{L^2} \\ &\lesssim \|g\|_{L^{N_{m_1}}} d_j 2^{-j \frac{2}{m_2}} \|g\|_{\dot{B}_{2,1}^{\frac{2}{m_2}}} \end{aligned}$$

using then the embedding

$$\|g\|_{L^{N_{m_1}}} \lesssim \|g\|_{\dot{H}^{\frac{2}{m_1}}} \quad (\text{II.30})$$

together with the interpolation inequality (II.28) gives

$$\|\Delta_j T_g(g)\|_{\dot{B}_{r,1}^{\frac{2}{m_2}}} \lesssim 2^{-j \frac{2}{m_2}} d_j \|g\|_{\dot{H}^1}^{\frac{2}{m_1} + \frac{2}{m_2}} \|g\|_{L^2}^{2 - \frac{2}{m_1} + \frac{2}{m_2}} \quad (\text{II.31})$$

For the remainder term, we proceed almost similarly

$$\begin{aligned} \|\Delta_j R(g, g)\|_{L^r} &\lesssim \sum_{j' \geq j-5} \left\| \tilde{\Delta}_{j'} g \right\|_{L^{N_{m_1}}} \|\Delta_{j'} g\|_{L^2} \\ &\lesssim 2^{-j \frac{2}{m_2}} \sum_{j' \geq j-5} (d_{j'} 2^{-(j'-j) \frac{2}{m_2}}) \|g\|_{L^{N_{m_1}}} \|g\|_{\dot{B}_{2,1}^{\frac{2}{m_2}}}, \end{aligned}$$

where $\tilde{\Delta}_{j'} \stackrel{\text{def}}{=} \sum_{i \in \{-1, 0, 1\}} \Delta_{j'+i}$. By convolution inequality, interpolation inequality (II.28) and the embedding one (II.30), we get

$$\|\Delta_j R(g, g)\|_{L^r} \lesssim 2^{-j \frac{2}{m_2}} d_j \|g\|_{\dot{H}^1}^{\frac{2}{m_1} + \frac{2}{m_2}} \|g\|_{L^2}^{2 - \frac{2}{m_1} + \frac{2}{m_2}}$$

which gives, together with (II.31)

$$\|\Delta_j(g \cdot g)\|_{L^r} \lesssim 2^{-j \frac{2}{m_2}} d_j \|g\|_{\dot{H}^1}^{\frac{2}{m_1} + \frac{2}{m_2}} \|g\|_{L^2}^{2 - \frac{2}{m_1} + \frac{2}{m_2}}$$

On the other hand, by duality, we get

$$\begin{aligned} |(fg|g)_{L^2}| &\lesssim \sum_{j \in \mathbb{Z}} \|\Delta_j f\|_{L^q} \|\Delta_j(g \cdot g)\|_{L^r} \\ &\lesssim \sum_{j \in \mathbb{Z}} (2^{-j \frac{2}{m_2}} \|\Delta_j f\|_{L^q} d_j) \|g\|_{\dot{H}^1}^{\frac{2}{m_1} + \frac{2}{m_2}} \|g\|_{L^2}^{2 - \frac{2}{m_1} + \frac{2}{m_2}} \\ &\lesssim \|f\|_{\dot{B}_{q,\infty}^{-\frac{2}{m_2}}} \|g\|_{\dot{H}^1}^{\frac{2}{m_1} + \frac{2}{m_2}} \|g\|_{L^2}^{2 - \frac{2}{m_1} + \frac{2}{m_2}}. \end{aligned}$$

By virtue of (II.29) we have

$$1 - \left(\frac{1}{m_1} + \frac{1}{m_2} \right) = \frac{1}{p} \quad \text{and} \quad -\frac{2}{m_2} = \frac{3}{q} + \frac{2}{p} - 2.$$

This gives

$$|(fg|g)_{L^2}| \leq \frac{1}{10} \|g\|_{\dot{H}^1}^2 + C \|f\|_{\mathcal{B}_{q,p}^p} \|g\|_{L^2}^2.$$

Lemma II.4.2 is proved. \square

Part II

About the Boussinesq system

CHAPTER III

G.W-P of the completely-viscous axisymmetric Boussinesq system in critical spaces

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III.1 Introduction

The description of the state of a moving stratified fluid in dimension three under the Boussinesq approach, taking into account the friction forces is determined by the distribution of the fluid velocity $v(t, x)$ with free-divergence located in position x at a time t , the scalar function $\rho(t, x)$ designates either the temperature in the context of thermal convection, or the mass density in the modeling of geophysical fluids and $p(t, x)$ is the pressure which relates v and ρ through an elliptic equation. This provides us to the Cauchy problem for the Boussinesq system,

$$\begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla p = \rho \vec{e}_z, & \text{if } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \partial_t \rho + v \cdot \nabla \rho - \kappa \Delta \rho = 0, & \text{if } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} v = 0, \\ (v, \rho)|_{t=0} = (v_0, \rho_0). \end{cases} \quad (\mathbf{B}_{\mu, \kappa})$$

Above, μ and κ are two non-negative parameters which can be seen as the inverse of Reynolds numbers and $\rho \vec{e}_z$ models the influence of the buoyancy force in the fluid motion in the vertical direction $\vec{e}_z = (0, 0, 1)$.

Boussinesq flows are ubiquitous in various nature phenomenon, such as oceanic circulations, atmospheric fronts or katabatic winds, industry such as fume cupboard ventilation or dense gas dispersion, see, e.g. [107].

Let us notice that the Navier-Stokes equations are obtained as particular case from $(\mathbf{B}_{\mu, \kappa})$ when the initial density is constant. Such equations are given as follows

$$\begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla p = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} v = 0, \\ v|_{t=0} = v_0. \end{cases} \quad (\mathbf{NS}_\mu)$$

An important breakthrough was J. Leray's paper in the thirties of last century where he showed the global existence of weak solutions in energy space for any dimension. Nevertheless, the uniqueness of such solutions has been till now an open question, unless for the two-dimensional case. Lately, the local wellposedness issue in the setting of mild solutions for (\mathbf{NS}_μ) was done by H. Fujita and T. Kato [44] for initial data belonging to the critical Sobolev space $\dot{H}^{\frac{1}{2}}$ in the sense of *scale invariance*. More similar results are established in several functional spaces like L^3 , $\dot{B}_{p, \infty}^{-1+\frac{3}{p}}$ and BMO^{-1} . It should be noted that these types of solutions are globally well posed in a time for initial data sufficiently small with respect to the viscosity, except in two-dimensional, see [94, 113]. In a similar way, especially in dimension two of spaces, the system $(\mathbf{B}_{\mu, \kappa})$ was tackled by many authors in various functional spaces and different values for the parameters κ and μ . For the connected subject, we refer to some selected references [70, 72, 73, 76, 92, 100, 119].

Before discussing some theoretical underpinnings results on the wellposedness topic for the viscous Boussinesq system $(\mathbf{B}_{\mu, \kappa})$ in three-dimensional, first let us point out again that the

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topic of global existence and uniqueness for (NS_μ) in the general case is till now an open problem in PDEs. It is therefore incumbent upon us to seek out a subclass of vector fields which in turn leads to some conservation quantities, and so the global wellposedness result. Such subclass involves to rewrite (NS_μ) under vorticity formulation by applying the "curl" to the momentum equation, which is defined by $\omega = \nabla \times v$. Thus, we have:

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega - \mu \Delta \omega = \omega \cdot \nabla v & \text{if } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \omega|_{t=0} = \omega_0. \end{cases} \quad (\text{III.1})$$

For three-dimensional flow the situation is more complicated due to the presence of stretching term $\omega \cdot \nabla v$, which contributes additional drawbacks for the fluid motion.

The subclass we look for requires to assume that the velocity is an axisymmetric vector field without *swirl* in the sense that v can be decomposed in the cylindrical coordinates (r, θ, z) as follows:

$$v(t, x) = v^r(t, r, z) \vec{e}_r + v^z(t, r, z) \vec{e}_z,$$

where for every $x = (x_1, x_2, z) \in \mathbb{R}^3$ we have

$$x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad r \geq 0, \quad 0 \leq \theta < 2\pi.$$

Above, the triplet $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$ represents the usual frame of unit vectors in the radial, azimuthal and vertical directions with the notation

$$\vec{e}_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0 \right), \quad \vec{e}_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0 \right), \quad \vec{e}_z = (0, 0, 1).$$

For these flows, the vorticity ω takes the form $\omega \stackrel{\text{def}}{=} \omega_\theta \vec{e}_\theta$, with

$$\omega_\theta = \partial_z v^r - \partial_r v^z. \quad (\text{III.2})$$

Taking advantage of $\text{div } v = 0$, the velocity field can be determined clearly in the half-space $\Omega = \{(r, z) \in \mathbb{R}^2 : r > 0\}$ by solving the following elliptic system

$$\begin{cases} \partial_r v^r + \frac{1}{r} v^r + \partial_z v^z = 0, \\ \partial_z v^r - \partial_r v^z = \omega_\theta, \end{cases} \quad (\text{III.3})$$

under homogeneous boundary conditions $v^r = \partial_r v^z = 0$. The differential system (III.3) is wellknown as the *axisymmetric Biot-Savart* law associated to (NS_μ) , see Section III.2.1.

A few computations claim that the term $\omega \cdot \nabla v$ becomes $\frac{v^r}{r} \omega_\theta$ and that ω_θ evolves,

$$\partial_t \omega_\theta + (v \cdot \nabla) \omega_\theta - \mu \Delta \omega_\theta + \mu \frac{\omega_\theta}{r^2} = \frac{v^r}{r} \omega_\theta, \quad (\text{III.4})$$

with the notations $v \cdot \nabla = v^r \partial_r + v^z \partial_z$ and $\Delta = \partial_r^2 + \frac{1}{r} \partial_r + \partial_z^2$. By setting $\Pi = \frac{\omega_\theta}{r}$, we discover that Π satisfies

$$\partial_t \Pi + v \cdot \nabla \Pi - \mu \left(\Delta + \frac{2}{r} \partial_r \right) \Pi = 0, \quad \Pi|_{t=0} = \Pi_0. \quad (\text{III.5})$$

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Since the dissipative operator $(\Delta + \frac{2}{r}\partial_r)$ has a good sign, thus the L^p -norms are time bounded, that is for $t \geq 0$

$$\|\Pi(t)\|_{L^p} \leq \|\Pi_0\|_{L^p}, \quad p \in [1, \infty]. \quad (\text{III.6})$$

Under this pattern, M. Ukhoviskii and V. Yudovich [116], independently O. Ladyzhenskaya [91] succeed to recover (NS_μ) globally in time, whenever $v_0 \in H^1$ and $\omega_0, \frac{\omega_0}{r} \in L^2 \cap L^\infty$. This result was relaxed later by S. Leonardi, J. Målek, J. Necās and M. Pokorný for $v_0 \in H^2$ and weakened recently in [3] by H. Abidi for $v_0 \in H^{\frac{1}{2}}$.

The majority of aforementioned results are accomplished within the framework of finite energy solutions. For the solutions with infinite energy, in particular in two dimensions many results dealing with the global wellposedness problem have been obtained by numerous authors. Particularly, worth mentioning that Giga, Miyakawa and Osada have been established in [60] that (NS_μ) admits a unique global solutions for initial vorticity is measure. Lately, M. Ben-Artzi [10] has shown that (NS_μ) is globally well posed whenever the initial vorticity ω_0 belongs to critical Lebesgue space $L^1(\mathbb{R}^2)$ who proposed a new formalism based on elementary comparison principles for linear parabolic equations. While, the uniqueness was relaxed later by H. Brezis [13]. Thereafter, this result was improved by I. Gallagher and Th. Gallay [48], where they constructed solutions globally in time, under the assumption that ω_0 is a finite measure. For more details about this subject we refer the reader to the references [34, 55, 58].

More recently, the global wellposedness problem for (III.4) was revisited in three-dimensional by Th. Gallay and V. Šverák who established in [56] that (NS_μ) possesses a unique global solutions if the initial velocity is an axisymmetric vector field and its vorticity lying the critical space $L^1(\Omega)$. In addition, they were extending their results in more general case, i.e., ω_0 is a finite measure, where Ω is half-plane $\Omega = \{(r, z) \in \mathbb{R}^2 : r > 0, z \in \mathbb{R}\}$ endowed with the product measure $drdz$. Actually, their paradigm uses specifically the standard fixed point method to show in particular the local wellposedness for the system (III.4) under the form

$$\partial_t \omega_\theta + \text{div}_*(v\omega_\theta) = \mu \left(\Delta - \frac{1}{r^2} \right) \omega_\theta$$

combined with the special structure of the vorticity, in particular, the axisymmetric Biot-Svart law, where $\text{div}_* f = \partial_r f^r + \partial_z f^z$. They showed that the local solutions constructed can be extended to the global one by exploiting some a priori estimates for ω_θ in various norms. We point out that the uniqueness topic for initial vorticity is measure was done by the same authors under some smallness condition on the punctual part. Furthermore, they provide also an asymptotic behavior study for positive vorticity with a finite impulse.

For $(\text{B}_{\mu,\kappa})$, the global regularity in dimension three of spaces has received a considerable attention. As shown in [36], for $\kappa = 0$ R. Danchin and M. Paicu investigated that $(\text{B}_{\mu,\kappa})$ is locally well posed in time in any dimension in the framework of Fujita-Kato's solutions. Next, in the axisymmetric case, H. Abidi, T. Hmidi and S. Keraani proved in [5] that

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$(B_{\mu,\kappa})$ is globally well posed by rewriting it under vorticity-density formulation:

$$\begin{cases} \partial_t \omega_\theta + v \cdot \nabla \omega_\theta - \frac{v^r}{r} \omega_\theta = (\Delta - \frac{1}{r^2}) \omega_\theta - \partial_r \rho, \\ \partial_t \rho + v \cdot \nabla \rho = 0, \\ (\omega_\theta, \rho)|_{t=0} = (\omega_0, \rho_0). \end{cases} \quad (\text{III.7})$$

Consequently, the quantity $\Pi = \frac{\omega_\theta}{r}$ solves the equation

$$\partial_t \Pi + v \cdot \nabla \Pi - (\Delta + \frac{2}{r} \partial_r) \Pi = -\frac{\partial_r \rho}{r}. \quad (\text{III.8})$$

They assumed that $v_0 \in H^1(\mathbb{R}^3)$, $\Pi_0 \in L^2(\mathbb{R}^3)$, $\rho_0 \in L^2 \cap L^\infty$ with $\text{supp } \rho_0 \cap (Oz) = \emptyset$ and $P_z(\text{supp } \rho_0)$ is a compact set in \mathbb{R}^3 , especially to dismiss the violent singularity of $\frac{\partial_r \rho}{r}$, with P_z being the orthogonal projector over (Oz) . Those results are improved later by T. Hmidi and F. Rousset in [74] for $\kappa > 0$ by removing the assumption on the support of the density. Their strategy is deeply based on the coupling between the two equations of the system (III.9) by introducing a new unknown which is called *coupled function*, this latest will play a significant role in our analysis as well, we shall come back to this later. In the same way, by using the advantage of the coupled function, C. Miao and X. Zheng studied in [101] the Cauchy problem of the system in question where they replaced the full dissipation by an horizontal one in all the equations. More recently in [41] P. Dreyfuss and H. Houamed treated the system in question, where they replaced the full dissipation by an horizontal one in all the equations, and proved the global wellposedness if the axisymmetric initial data (v_0, Π_0, ρ_0) lies in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$. In the same direction, in [82] H. Houamed and M. Zerouine succeed to solve $(B_{\mu,\kappa})$ globally in time for $\kappa = 0$ and axisymmetric initial data $(v_0, \rho_0) \in (H^{\frac{1}{2}} \cap \dot{B}_{3,1}^0)(\mathbb{R}^3) \times (L^2 \cap \dot{B}_{3,1}^0)(\mathbb{R}^3)$, by essentially combining the works of [3, 36, 74].

In the present Chapter, we want to conduct the same results recently obtained in [56] for the viscous Boussinesq system $(B_{\mu,\kappa})$ expressed by the following vorticity-density formulation.

$$\begin{cases} \partial_t \omega_\theta + v \cdot \nabla \omega_\theta - \frac{v^r}{r} \omega_\theta = (\Delta - \frac{1}{r^2}) \omega_\theta - \partial_r \rho, \\ \partial_t \rho + v \cdot \nabla \rho - \kappa \Delta \rho = 0, \\ (\omega_\theta, \rho)|_{t=0} = (\omega_0, \rho_0), \end{cases} \quad (\text{III.9})$$

for initial data (ω_0, ρ_0) in the critical Lebesgue space $L^1(\Omega) \times L^1(\mathbb{R}^3)$, with the following notations

$$\|\omega_\theta\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |\omega_\theta(r, z)|^p dr dz \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty), \\ \text{essup}_{(r,z) \in \Omega} |\omega_\theta(r, z)| & \text{if } p = \infty \end{cases}$$

and

$$\|\Pi\|_{L^p(\mathbb{R}^3)} = \begin{cases} \left(\int_\Omega |\Pi(r, z)|^p r dr dz \right)^{\frac{1}{p}} & \text{if } p \in [1, \infty), \\ \text{essup}_{(r,z) \in \Omega} |\Pi(r, z)| & \text{if } p = \infty. \end{cases}$$

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Let us notice that the spaces $L^1(\Omega)$ and $L^1(\mathbb{R}^3)$ are scale invariant, in the sense

$$\|\lambda^2 \omega_0(\lambda \cdot)\|_{L^1(\Omega)} = \|\omega_0\|_{L^1(\Omega)}, \quad \|\lambda^3 \rho_0(\lambda \cdot)\|_{L^1(\mathbb{R}^3)} = \|\rho_0\|_{L^1(\mathbb{R}^3)}$$

for any $(\omega_0, \rho_0) \in L^1(\Omega) \times L^1(\mathbb{R}^3)$ and any $\lambda > 0$. This is derived from the fact

$$\omega_\theta(t, r, z) \mapsto \lambda^2 \omega_\theta(\lambda^2 t, \lambda r, \lambda z), \quad \rho(t, r, z) \mapsto \lambda^3 \rho(\lambda^2 t, \lambda r, \lambda z).$$

is a symmetry of (III.9).

At this stage, we are ready to state the main result of this Chapter. To be precise, we will prove the following theorem.

Theorem III.1.1. *Let $(\omega_0, \rho_0) \in L^1(\Omega) \times L^1(\mathbb{R}^3)$ be an axisymmetric initial data, then the system (III.9) for $\kappa = 1$ admits a unique global mild solution. More precisely, we have:*

$$(\omega_\theta, r\rho) \in \left(C^0([0, \infty); L^1(\Omega)) \cap C^0((0, \infty); L^\infty(\Omega)) \right)^2, \quad (\text{III.10})$$

$$\rho \in C^0([0, \infty); L^1(\mathbb{R}^3)) \cap C^0((0, \infty); L^\infty(\mathbb{R}^3)). \quad (\text{III.11})$$

Furthermore, for every $p \in [1, \infty]$, there exists some constant $\tilde{K}_p(D_0) > 0$, for which, and for all $t > 0$ the following statements hold.

$$\|(\omega_\theta(t), r\rho(t))\|_{L^p(\Omega) \times L^p(\Omega)} \leq t^{-(1-\frac{1}{p})} \tilde{K}_p(D_0), \quad (\text{III.12})$$

$$\|\rho(t)\|_{L^p(\mathbb{R}^3)} \leq t^{-\frac{3}{2}(1-\frac{1}{p})} \tilde{K}_p(D_0), \quad (\text{III.13})$$

where

$$D_0 \stackrel{\text{def}}{=} \|(\omega_0, \rho_0)\|_{L^1(\Omega) \times L^1(\mathbb{R}^3)}.$$

A few comments about Theorem III.1.1 are given by the following remarks.

Remark III.1.2. By axisymmetric scalar function we mean again a function that depends only on the variable (r, z) but not on the angle variable θ in cylindrical coordinates. We can check that the axisymmetric structure is preserved through the time in the way that if (v_0, ρ_0) is axisymmetric without swirl, then the obtained solution is also.

Remark III.1.3. The hypothesis $\omega_\theta \in L^1(\Omega)$ doesn't imply generally that the associated velocity v is in $L^2(\Omega)$ space. Consequently, the classical energy estimate is not available to provide a uniform bound for the velocity.

The proof is organized in two parts. The first one cares with the local well posedness topic for (III.9) in the spirit of [56] due to Th. Gallay and V. Svérak . We make use of fixed point-method for an equivalent system (III.9) on product space equipped with an adequate norm with the help of the axisymmetric Biot-Savart law and some norm estimates in terms of the velocity and the vorticity. But in our context, we should deal carefully with the additional term $\frac{\partial_r \rho}{r}$ which contributes a singularity over the axis (Oz) . The remedy is to

hide this term by exploiting the coupling structure of the system (III.9) for $\kappa = 1$ and introducing new unknown functions Γ and $\tilde{\Gamma}$ in the spirit of [74] by setting $\Gamma = \frac{\omega_\theta}{r} - \frac{\rho}{2}$, and $\tilde{\Gamma} = r\Gamma$. A straightforward computation shows that Γ and $\tilde{\Gamma}$ solve, respectively

$$\begin{cases} \partial_t \Gamma + v \cdot \nabla \Gamma - (\Delta + \frac{2}{r} \partial_r) \Gamma = 0, \\ \Gamma_{t=0} = \Gamma_0. \end{cases} \quad (\text{III.14})$$

$$\begin{cases} \partial_t \tilde{\Gamma} + v \cdot \nabla \tilde{\Gamma} - \frac{v^r}{r} \tilde{\Gamma} - (\Delta - \frac{1}{r^2}) \tilde{\Gamma} = 0, \\ \tilde{\Gamma}_{t=0} = r\Gamma_0. \end{cases} \quad (\text{III.15})$$

In fact, in the second part, we shall investigate some a priori estimates for all the unknowns in order to establish the global regularity for the system in question. Significant properties of the new unknowns, such as the maximum principle, are gained in this transition and Γ evolves a similar equation and keeps the same boundary conditions as Π in the case of the axisymmetric Navier-Stokes without swirl, see (III.5). As a consequence, the function Γ (and eventually $\tilde{\Gamma}$ in Ω after some technical computations) satisfies an estimate as (III.6) which will be crucial in the process of deriving the global regularity of our solutions.

For the reader's convenience, we provide a brief headline of this Chapter. In section 2, we briefly depict the framework that exists regarding the axisymmetric Biot-Savart law. Many results could be spent in explaining this framework in detail, in particular, the relation between the velocity vector field and its vorticity by means of the stream function. Along the way, we recall some weighted estimates which will be helpful in the sequel. Afterwards, we focus in the linear equation of (III.9) and some characterization of their associated semi-group, in particular the $L^p \rightarrow L^q$ estimate as in two-dimension space. Section 3, mainly treats the local well posedness topic for the system (III.9). The main tool is the fixed point argument on the product space combined with a few technics about the semi-group estimates. In section 4, we investigate some global a priori estimates by coupling the system (III.9) and introducing the new unknowns Γ and $\tilde{\Gamma}$. Considering these latest quantities will be helpful to derive the global estimates for the equivalent system (III.9) and consequently the system $(B_{\mu,\kappa})$.

III.2 Introduction to the axisymmetric without swirl vector fields and to the heat-semi group operator

In this section, we recall some basic tools which will be employed in the subsequent sections. In particular, we develop the Biot-Savart law in the framework of axisymmetric vector fields, and we study the linear equation associated to the system (III.9), usually specialized to the local existence.

III.2.1 The tool box of Biot-Savart law

Recalling that in the cylindrical coordinates and in the class of axisymmetric vector fields without swirl the velocity is given by $v = (v^r, 0, v^z)$ with v^r and v^z are independent of θ -variable, ω_θ its vorticity defined from Ω into \mathbb{R} by $\omega_\theta = \partial_z v^r - \partial_r v^z$ and the divergence-free condition $\operatorname{div} v = 0$ turns out to be

$$\partial_r(rv^r) + \partial_z(rv^z) = 0.$$

In this case, it is not difficult to build a scalar function $\Omega \ni (r, z) \mapsto \psi(r, z) \in \mathbb{R}$ which is called *axisymmetric stream function* and satisfies

$$v^r = -\frac{1}{r}\partial_z\psi, \quad v^z = \frac{1}{r}\partial_r\psi. \quad (\text{III.16})$$

Consequently, one obtains that ψ evolves the following linear elliptic inhomogeneous equation

$$-\frac{1}{r}\partial_r^2\psi + \frac{1}{r^2}\partial_r\Psi - \frac{1}{r}\partial_z^2\psi = \omega_\theta,$$

with the boundary conditions $\psi(0, z) = \partial_r\psi(0, z) = 0$. By setting $\mathcal{L} = -\frac{1}{r}\partial_r^2 + \frac{1}{r^2}\partial_r - \frac{1}{r}\partial_z^2$, one finds the following boundary value problem

$$\begin{cases} \mathcal{L}\psi(r, z) = \omega_\theta(r, z) & \text{if } (r, z) \in \Omega \\ \psi(r, z) = \partial_r\psi(r, z) = 0 & \text{if } (r, z) \in \partial\Omega, \end{cases} \quad (\text{III.17})$$

where $\partial\Omega = \{(r, z) \in \mathbb{R}^2 : r = 0\}$. It is evident that \mathcal{L} is an elliptic operator of second order, then according to [115], \mathcal{L} is invertible with an inverse \mathcal{L}^{-1} . Consequently, the above boundary value problem admits a unique solution given by

$$\Psi(r, z) \stackrel{\text{def}}{=} \mathcal{L}^{-1}\omega_\theta(r, z) = \int_{-\infty}^{\infty} \int_0^{\infty} \frac{\sqrt{\tilde{r}r}}{2\pi} F\left(\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{\tilde{r}r}\right) \omega_\theta(\tilde{r}, \tilde{z}) d\tilde{r}d\tilde{z}, \quad (\text{III.18})$$

where the function $F : (0, \infty) \rightarrow \mathbb{R}$ is expressed as follows.

$$F(s) = \int_0^\pi \frac{\cos \alpha d\alpha}{(2(1 - \cos \alpha) + s)^{1/2}}. \quad (\text{III.19})$$

Since, F cannot be expressed as an elementary function, but it contributes some asymptotic properties near $s = 0$ and $s = \infty$ listed in the following proposition. For more details about the proof, see [43, 115].

Proposition III.2.1. *Let F be the function defined by (III.19), then the following assertions hold.*

- (i) $F(s) = \frac{1}{2} \log \frac{1}{s} + \log 8 - 2 + O\left(s \log \frac{1}{s}\right)$ and $F'(s) = -\frac{1}{2s} + O\left(\log \frac{1}{s}\right)$ as $s \rightarrow 0^+$.

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(ii) $F(s) = \frac{\pi}{2s^{3/2}} + O\left(\frac{1}{s^{5/2}}\right)$ and $F'(s) = -\frac{3\pi}{4s^{5/2}} + O\left(\frac{1}{s^{7/2}}\right)$ as $s \rightarrow \infty$.

(iii) For every $k \in \mathbb{N}^*$, we have

$$|F(s)| \lesssim \min\left(\left(\frac{1}{s}\right)^\epsilon, \left(\frac{1}{s}\right)^{\frac{1}{2}}, \left(\frac{1}{s}\right)^{\frac{3}{2}}\right), \quad \epsilon \in]0, \frac{1}{2}[,$$

and

$$|F^{(k)}(s)| \lesssim \min\left(\left(\frac{1}{s}\right)^k, \left(\frac{1}{s}\right)^{k+\frac{1}{2}}, \left(\frac{1}{s}\right)^{k+\frac{3}{2}}\right), \quad s \in]0, \infty[.$$

(iv) The maps $s \mapsto s^\alpha F(s)$ and $s \mapsto s^\beta F'(s)$ are bounded for $0 < \alpha \leq \frac{3}{2}$ and $1 \leq \beta \leq \frac{5}{2}$ respectively.

Now, let

$$\mathcal{K}(r, z, \tilde{r}, \tilde{z}) = \frac{\sqrt{\tilde{r}r}}{2\pi} F\left(\frac{(r - \tilde{r})^2 + (z - \tilde{z})^2}{\tilde{r}r}\right). \quad (\text{III.20})$$

Thus in view of (III.18), Ψ takes the form

$$\Psi(r, z) = \int_{-\infty}^{\infty} \int_0^{\infty} \mathcal{K}(r, z, \tilde{r}, \tilde{z}) \omega_\theta(\tilde{r}, \tilde{z}) d\tilde{r} d\tilde{z},$$

with \mathcal{K} can be seen as the kernel of the integral representation above. The last formula together with (III.16) claim that there exists a genuine connection between the velocity and its vorticity, namely the *axisymmetric Biot-Savart* law which reads as follows

$$v^r(r, z) = \int_{-\infty}^{\infty} \int_0^{\infty} \mathcal{K}_r(r, z, \tilde{r}, \tilde{z}) \omega_\theta(\tilde{r}, \tilde{z}) d\tilde{r} d\tilde{z}, \quad v^z(r, z) = \int_{-\infty}^{\infty} \int_0^{\infty} \mathcal{K}_z(r, z, \tilde{r}, \tilde{z}) \omega_\theta(\tilde{r}, \tilde{z}) d\tilde{r} d\tilde{z}. \quad (\text{III.21})$$

Here, with the notation $\xi^2 = \frac{(r - \tilde{r})^2 + (z - \tilde{z})^2}{\tilde{r}r}$ we have

$$\mathcal{K}_r(r, z, \tilde{r}, \tilde{z}) = -\frac{1}{\pi} \frac{z - \tilde{z}}{r^{3/2} \tilde{r}^{1/2}} F'(\xi^2) \quad (\text{III.22})$$

and

$$\mathcal{K}_z(r, z, \tilde{r}, \tilde{z}) = \frac{1}{\pi} \frac{r - \tilde{r}}{r^{3/2} \tilde{r}^{1/2}} F'(\xi^2) + \frac{1}{4\pi} \frac{\tilde{r}^{1/2}}{r^{3/2}} (F(\xi^2) - 2\xi^2 F'(\xi^2)). \quad (\text{III.23})$$

A worthwhile properties of the kernels \mathcal{K}_r and \mathcal{K}_z are given in the following result. For more details about the proof see [56].

Proposition III.2.2. *Let $(r, z, \tilde{r}, \tilde{z}) \in \Omega \times \Omega$, then we have*

$$|\mathcal{K}_r(r, z, \tilde{r}, \tilde{z})| + |\mathcal{K}_z(r, z, \tilde{r}, \tilde{z})| \leq \frac{C}{((r - \tilde{r})^2 + (z - \tilde{z})^2)^{1/2}}. \quad (\text{III.24})$$

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Now, we state the first consequence of the above result, in particular the $L^p \rightarrow L^q$ between the velocity and its vorticity, specifically we establish.

Proposition III.2.3. *Let v be an axisymmetric velocity vector associated to the vorticity ω_θ via the axisymmetric Biot-Savart law (III.21). Then the following assertions hold.*

(i) *Let $(p, q) \in (1, 2) \times (2, \infty)$, and $\frac{1}{p} - \frac{1}{q} = \frac{1}{2}$. If $\omega_\theta \in L^p(\Omega)$, then $v \in (L^q(\Omega))^2$ and*

$$\|v\|_{L^q(\Omega)} \leq C \|\omega_\theta\|_{L^p(\Omega)}. \quad (\text{III.25})$$

(ii) *Let $(p, q) \in [1, 2) \times (2, \infty]$, we define $\sigma \in]0, 1[$ by $\frac{1}{2} = \frac{\sigma}{p} + \frac{1-\sigma}{q}$. Then for $\omega_\theta \in L^p(\Omega) \cap L^q(\Omega)$, we have $v \in (L^\infty(\Omega))^2$ and*

$$\|v\|_{L^\infty(\Omega)} \leq C \|\omega_\theta\|_{L^p(\Omega)}^\sigma \|\omega_\theta\|_{L^q(\Omega)}^{1-\sigma}. \quad (\text{III.26})$$

Proof

(i) Combining (III.21) and (III.24), we get

$$|v^r(r, z)| \leq C \int_{-\infty}^{\infty} \int_0^{\infty} \frac{|\omega_\theta(\tilde{r}, \tilde{z})|}{((r - \tilde{r})^2 + (z - \tilde{z})^2)^{1/2}} d\tilde{r} d\tilde{z},$$

and

$$|v^z(r, z)| \leq C \int_{-\infty}^{\infty} \int_0^{\infty} \frac{|\omega_\theta(\tilde{r}, \tilde{z})|}{((r - \tilde{r})^2 + (z - \tilde{z})^2)^{1/2}} d\tilde{r} d\tilde{z}.$$

The last two integrals on the right-hand side are singular integrals, so by hypothesis $\frac{1}{p} - \frac{1}{q} = \frac{1}{2}$, Hardy-Littlewood-Sobolev inequality, see e.g. [23, Theorem 6.1.3] yields the desired estimate.

(ii) Let $R > 0$, then in view of (III.24) we have.

$$|v(r, z)| \lesssim \int_{\Omega_R} \frac{|\omega_\theta(r - \tilde{r}, z - \tilde{z})|}{(\tilde{r}^2 + \tilde{z}^2)^{\frac{1}{2}}} d\tilde{r} d\tilde{z} + \int_{\Omega \setminus \Omega_{1R}} \frac{|\omega_\theta(r - \tilde{r}, z - \tilde{z})|}{(\tilde{r}^2 + \tilde{z}^2)^{\frac{1}{2}}} d\tilde{r} d\tilde{z},$$

where $\Omega_R = \{(r, z) \in \Omega : 0 < r \leq R, -R \leq z \leq R\}$. Thus, Hölder's inequality implies

$$|v(r, z)| \lesssim \|\omega_\theta\|_{L^q(\Omega)} R^{1-\frac{2}{q}} + \|\omega_\theta\|_{L^p(\Omega)} \frac{1}{R^{\frac{2}{p}-1}}.$$

It is enough to take $R = (\|\omega_\theta\|_{L^p(\Omega)} / \|\omega_\theta\|_{L^q(\Omega)})^\ell$, with $\ell = \frac{\sigma}{1-2/q} = \frac{1-\sigma}{2/p-1}$, then easy computations achieve the estimate. \square

In the axisymmetric case the weighted estimates practice a decisive role to bound some quantities like $r^\alpha v$ in Lebesgue spaces for some α . Now, we state some of them, the proof of which can be found in [43, 56].

Proposition III.2.4. *Let $\alpha, \beta \in [0, 2]$ be such that $\beta - \alpha \in [0, 1)$, and assume that $p, q \in (1, \infty)$ satisfy*

$$\frac{1}{p} - \frac{1}{q} = \frac{1 + \alpha - \beta}{2}.$$

If $r^\beta \omega_\theta \in L^p(\Omega)$, then $r^\alpha v \in (L^q(\Omega))^2$ and the following bound holds

$$\|r^\alpha v\|_{L^q(\Omega)} \leq C \|r^\beta \omega_\theta\|_{L^p(\Omega)}. \quad (\text{III.27})$$

III.2.2 Characterizations of the semi-groups associated with the linearized equation

We focus on studying the linearized boundary initial value problem associated to the system (III.9) and we state some properties of their semi-groups. Specifically, we consider

$$\begin{cases} \partial_t \omega_\theta - \left(\Delta - \frac{1}{r^2} \right) \omega_\theta = 0, & \text{in } \Omega, \\ \partial_t \rho - \Delta \rho = 0, & \text{in } \mathbb{R}^3, \\ (\omega_\theta, \rho)|_{t=0} = (\omega_0, \rho_0), \end{cases} \quad (\text{III.28})$$

with $\Omega = \{(r, z) \in \mathbb{R}^2 : r > 0\}$ is the half-space by prescribing the homogeneous Dirichlet conditions at the boundary $r = 0$ for ω_θ variable. For $(\omega_0, \rho_0) \in L^1(\Omega) \times L^1(\mathbb{R}^3)$, the solution of (III.28) is given explicitly by

$$\begin{cases} \omega_\theta(t) = \mathbb{S}_1(t) \omega_0, \\ \rho(t) = \mathbb{S}_2(t) \rho_0, \end{cases}$$

where $(\mathbb{S}_1(t))_{t \geq 0}$ and $(\mathbb{S}_2(t))_{t \geq 0}$ being respectively the semi-groups or evolution operators associated to the dissipative operators $(\Delta - \frac{1}{r^2})$ and Δ .

Such are characterized by the following explicit formula, namely we have.

Proposition III.2.5. *The family $(\mathbb{S}_1(t), \mathbb{S}_2(t))_{t \geq 0}$ associated to (III.28) is expressed by the following*

$$\begin{cases} (\mathbb{S}_1(t) \omega_0)(r, z) = \frac{1}{4\pi t} \int_\Omega \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_1\left(\frac{t}{r\tilde{r}}\right) e^{-\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{4t}} \omega_0(\tilde{r}, \tilde{z}) d\tilde{r} d\tilde{z}, \\ (\mathbb{S}_2(t) \rho_0)(r, z) = \frac{1}{4\pi t} \int_\Omega \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_2\left(\frac{t}{r\tilde{r}}\right) e^{-\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{4t}} \rho_0(\tilde{r}, \tilde{z}) d\tilde{r} d\tilde{z}, \end{cases} \quad (\text{III.29})$$

where the functions $(0, +\infty) \ni t \mapsto \mathcal{N}_1(t), \mathcal{N}_2(t) \in \mathbb{R}$ are defined by

$$\begin{cases} \mathcal{N}_1(t) = \frac{1}{\sqrt{\pi t}} \int_{-\pi/2}^{\pi/2} e^{-\frac{\sin^2 \alpha}{t}} \cos(2\alpha) d\alpha, \\ \mathcal{N}_2(t) = \frac{1}{\sqrt{\pi t}} \int_{-\pi/2}^{\pi/2} e^{-\frac{\sin^2 \alpha}{t}} d\alpha. \end{cases} \quad (\text{III.30})$$

Proof

We assume that (ω_θ, ρ) solves (III.28), then a straightforward computations claim that (ω, ρ) , with $\omega = \omega_\theta \vec{e}_\theta$, satisfies the usual heat equation $\partial_t \omega - \Delta \omega = 0$ and $\partial_t \rho - \Delta \rho = 0$ in \mathbb{R}^3 with initial data $(\omega(0, \cdot), \rho(0, \cdot))$. Therefore, for every $t > 0$ we have

$$\begin{cases} \omega(t, x) = \frac{1}{(4\pi t)^{3/2}} \int_{\mathbb{R}^3} e^{-\frac{|x-\tilde{x}|^2}{4t}} \omega(0, \tilde{x}) d\tilde{x}, \\ \rho(t, x) = \frac{1}{(4\pi t)^{3/2}} \int_{\mathbb{R}^3} e^{-\frac{|x-\tilde{x}|^2}{4t}} \rho(0, \tilde{x}) d\tilde{x}. \end{cases} \quad (\text{III.31})$$

We will develop each term in the cylindrical basis $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$ by writing $x = (r \cos \theta, r \sin \theta, z)$ and $\tilde{x} = (\tilde{r} \cos \tilde{\theta}, \tilde{r} \sin \tilde{\theta}, \tilde{z})$, hence the first equation of (III.31) takes the form

$$\begin{aligned} \omega_\theta(t, r, z) \begin{pmatrix} -\sin \theta \\ \cos \theta \\ 0 \end{pmatrix} &= \frac{1}{(4\pi t)^{3/2}} \int_0^\infty \int_{\mathbb{R}} \int_{-\pi}^\pi e^{-\frac{|x-\tilde{x}|^2}{4t}} \omega_0(\tilde{r}, \tilde{z}) \begin{pmatrix} -\sin \tilde{\theta} \\ \cos \tilde{\theta} \\ 0 \end{pmatrix} \tilde{r} d\tilde{\theta} d\tilde{z} d\tilde{r} \\ &= I_1. \end{aligned} \quad (\text{III.32})$$

Since, $|x - \tilde{x}|^2 = (r - \tilde{r})^2 + (z - \tilde{z})^2 + 4r\tilde{r} \sin^2(\frac{\theta - \tilde{\theta}}{2})$, thus we have

$$I_1 = \frac{1}{(4\pi t)} \int_0^\infty \int_{\mathbb{R}} \left(\frac{1}{(4\pi t)^{1/2}} \int_{-\pi}^\pi e^{-\frac{-r\tilde{r} \sin^2 \frac{\theta - \tilde{\theta}}{2}}{t}} \begin{pmatrix} -\sin \tilde{\theta} \\ \cos \tilde{\theta} \\ 0 \end{pmatrix} \tilde{r} d\tilde{\theta} \right) e^{-\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{4t}} \omega_0(\tilde{r}, \tilde{z}) d\tilde{z} d\tilde{r}. \quad (\text{III.33})$$

To treat I_1 , we set $\alpha = \frac{\theta - \tilde{\theta}}{2}$ then we have

$$\begin{aligned} \frac{1}{\sqrt{4\pi t}} \int_{-\pi}^\pi e^{-\frac{-r\tilde{r} \sin^2 \frac{\theta - \tilde{\theta}}{2}}{t}} (-\sin \tilde{\theta}) \tilde{r} d\tilde{\theta} &= -\frac{1}{\sqrt{\pi t}} \int_{\theta/2 - \pi/2}^{\theta/2 + \pi/2} e^{-\frac{-r\tilde{r} \sin^2 \alpha}{t}} (\sin \theta \cos 2\alpha - \cos \theta \sin 2\alpha) \tilde{r} d\alpha \\ &= -\frac{1}{\sqrt{\pi t}} \int_{-\pi/2}^{+\pi/2} e^{-\frac{-r\tilde{r} \sin^2 \alpha}{t}} (\sin \theta \cos 2\alpha) \tilde{r} d\alpha \\ &\quad - \frac{1}{\sqrt{\pi t}} \int_{-\pi/2}^{+\pi/2} e^{-\frac{-r\tilde{r} \sin^2 \alpha}{t}} (\cos \theta \sin 2\alpha) \tilde{r} d\alpha. \end{aligned}$$

For $t \in (0, \infty)$, define

$$\mathcal{N}_1(t) = \frac{1}{\sqrt{\pi t}} \int_{-\pi/2}^{\pi/2} e^{-\frac{\sin^2 \alpha}{t}} \cos(2\alpha) d\alpha.$$

Then the last estimate becomes

$$\frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{-\pi}^\pi e^{-\frac{-r\tilde{r} \sin^2 \frac{\theta - \tilde{\theta}}{2}}{t}} (-\sin \tilde{\theta}) \tilde{r} d\tilde{\theta} = \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_1\left(\frac{t}{r\tilde{r}}\right) (-\sin \theta).$$

Similarly,

$$\frac{1}{(4\pi t)^{\frac{1}{2}}} \int_{-\pi}^\pi e^{-\frac{-r\tilde{r} \sin^2 \frac{\theta - \tilde{\theta}}{2}}{t}} \cos \tilde{\theta} \tilde{r} d\tilde{\theta} = \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_1\left(\frac{t}{r\tilde{r}}\right) \cos \theta.$$

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Combining the last two estimates and plugging them in I_1 we reach the desired estimate.

For the second equation in (III.31) we express the formula of ρ by

$$\rho(t, r, z) = \frac{1}{4\pi t} \int_{\Omega} \left(\frac{1}{2\sqrt{\pi t}} \int_{-\pi}^{\pi} e^{-\frac{r\tilde{r}\sin^2\left(\frac{\theta-\tilde{\theta}}{2}\right)}{t}} \tilde{r} d\tilde{\theta} \right) e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{4t}} \rho(0, \tilde{r}, \tilde{z}) d\tilde{r} d\tilde{z}. \quad (\text{III.34})$$

Setting

$$I_2 = \frac{1}{2\sqrt{\pi t}} \int_{-\pi}^{\pi} e^{-\frac{r\tilde{r}\sin^2\left(\frac{\theta-\tilde{\theta}}{2}\right)}{t}} \tilde{r} d\tilde{\theta}.$$

The same variable $\alpha = \frac{\theta-\tilde{\theta}}{2}$ allows us to write

$$I_2 = \frac{1}{\sqrt{\pi t}} \int_{-\pi/2}^{\pi/2} e^{-\frac{r\tilde{r}\sin^2\alpha}{t}} \tilde{r} d\alpha = \sqrt{\frac{\tilde{r}}{r}} \mathcal{N}_2\left(\frac{t}{r\tilde{r}}\right),$$

with \mathcal{N}_2 is defined for $t > 0$ by

$$\mathcal{N}_2(t) = \frac{1}{\sqrt{\pi t}} \int_{-\pi/2}^{\pi/2} e^{-\frac{\sin^2\alpha}{t}} d\alpha.$$

Plugging I_2 in (III.34), we get the result. This ends the proof of the Proposition. \square

The following Proposition provides some asymptotic behavior of the functions \mathcal{N}_1 and \mathcal{N}_2 near 0 and ∞ , which will be fundamental in the sequel.

Proposition III.2.6. *Let $\mathcal{N}_1, \mathcal{N}_2 : (0, \infty) \rightarrow \mathbb{R}$ be the functions defined in (III.30). Then the following statements hold.*

- (i) $\mathcal{N}_1(t) = 1 - \frac{3t}{4} + O(t^2)$ and $\mathcal{N}_1'(t) = -\frac{3}{4} + O(t)$ when $t \uparrow 0$;
- (ii) $\mathcal{N}_1(t) = \frac{\pi^{1/2}}{4t^{3/2}} + O\left(\frac{1}{t^{5/2}}\right)$ and $\mathcal{N}_1'(t) = -\frac{3\pi^{1/2}}{8t^{5/2}} + O\left(\frac{1}{t^{7/2}}\right)$ when $t \uparrow \infty$;
- (iii) $\mathcal{N}_2(t) = 1 + \frac{t}{4} + O(t^2)$ and $\mathcal{N}_2'(t) = \frac{1}{4} + O(t)$ when $t \uparrow 0$;
- (iv) $\mathcal{N}_2(t) = \frac{\pi^{1/2}}{t^{1/2}} - \frac{\pi^{1/2}}{2t^{3/2}} + O\left(\frac{1}{t^{5/2}}\right)$ and $\mathcal{N}_2'(t) = -\frac{\pi^{1/2}}{2t^{3/2}} + \frac{3\pi}{4t^{5/2}} + O\left(\frac{1}{t^{7/2}}\right)$ when $t \uparrow \infty$.

Proof

(i) Substituting $\zeta = \frac{\sin\alpha}{\sqrt{t}}$ in \mathcal{N}_1 , we shall have

$$\begin{aligned} \mathcal{N}_1(\zeta) &= \frac{1}{\sqrt{\pi}} \int_{-\frac{1}{\sqrt{t}}}^{\frac{1}{\sqrt{t}}} e^{-\zeta^2} \frac{1-2t\zeta^2}{\sqrt{1-t\zeta^2}} d\zeta \\ &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{\sqrt{t}}} e^{-\zeta^2} \frac{1-2t\zeta^2}{\sqrt{1-t\zeta^2}} d\zeta \\ &= \frac{2}{\sqrt{\pi}} \left(\int_0^{\frac{1}{2\sqrt{t}}} e^{-\zeta^2} \frac{1-2t\zeta^2}{\sqrt{1-t\zeta^2}} d\zeta + \int_{\frac{1}{2\sqrt{t}}}^{\frac{1}{\sqrt{t}}} e^{-\zeta^2} \frac{1-2t\zeta^2}{\sqrt{1-t\zeta^2}} d\zeta \right) \\ &= \text{II}_1 + \text{II}_2. \end{aligned}$$

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Note that $\lim_{t \uparrow 0} \text{II}_2 = 0$, so the behavior of \mathcal{N}_1 near 0 comes from II_1 . Hence, let us deal with II_1 , we insert the Taylor expansion of the function $\zeta \rightarrow \frac{1}{\sqrt{1-t\zeta^2}}$ in the integral of II_1 to obtain

$$\text{II}_1 = \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{2\sqrt{t}}} e^{-\zeta^2} \left(1 - \frac{3}{2}t\zeta^2 - t^2\zeta^4\right) d\zeta + O(t^3).$$

It is straightforward to show that

$$\int_0^\infty e^{-\zeta^2} d\zeta = \frac{\sqrt{\pi}}{2}, \quad \int_0^\infty \zeta^2 e^{-\zeta^2} d\zeta = \frac{\sqrt{\pi}}{4}, \quad \int_0^\infty \zeta^4 e^{-\zeta^2} d\zeta = 3\frac{\sqrt{\pi}}{8}.$$

Consequently, $\lim_{t \uparrow 0} \text{II}_1 = 1$. Combining all the previous quantities, we find the asymptotic behavior of II_1 near 0, that is,

$$\text{II}_1 = 1 - \frac{3}{4}t + O(t^2).$$

By derivation of II_1 , we find the behavior of \mathcal{N}_1' .

(ii) The Mac Laurin's expansion of the function $y \mapsto e^{-y}$ at 0 gives

$$e^{-\frac{\sin^2 \alpha}{t}} = 1 - \frac{\sin^2 \alpha}{t} + O\left(\frac{1}{t^2}\right).$$

Thus we get

$$\mathcal{N}_1(t) = \frac{1}{\sqrt{\pi t}} \int_{-\pi/2}^{\pi/2} \left(1 - \frac{\sin^2 \alpha}{t}\right) \cos 2\alpha d\alpha + O\left(\frac{1}{t^{5/2}}\right).$$

After an easy computations we achieve the estimate.

(iii) To prove this assertion, setting $y = \frac{\sin \alpha}{\sqrt{t}}$ in \mathcal{N}_2 and we split the integral into two parts, one has

$$\mathcal{N}_2(t) = \frac{2}{\sqrt{\pi}} \left(\int_0^{\frac{1}{2\sqrt{t}}} \frac{e^{-y^2}}{\sqrt{1-ty^2}} dy + \int_{\frac{1}{2\sqrt{t}}}^{\frac{1}{\sqrt{t}}} \frac{e^{-y^2}}{\sqrt{1-ty^2}} dy \right).$$

We follow the same steps as \mathcal{N}_1 . For the second integral in right-hand side, we have

$$\begin{aligned} \frac{2}{\sqrt{\pi}} \int_{\frac{1}{2\sqrt{t}}}^{\frac{1}{\sqrt{t}}} \frac{e^{-y^2}}{\sqrt{1-ty^2}} dy &= \frac{2}{\sqrt{\pi}} \int_{\frac{1}{2\sqrt{t}}}^{\frac{1}{\sqrt{t}}} \frac{e^{-y^2}}{(\sqrt{1-\sqrt{t}y})(\sqrt{1+\sqrt{t}y})} dy \\ &\leq C e^{-\frac{1}{4t}} \int_{\frac{1}{2\sqrt{t}}}^{\frac{1}{\sqrt{t}}} \frac{1}{\sqrt{1-\sqrt{t}y}} dy. \end{aligned}$$

Let us observe that the last estimate goes to 0 as $t \uparrow 0$, so the asymptotic behavior of \mathcal{N}_2 near 0 comes only from the first integral. To be precise, it is clear that $t \mapsto \frac{1}{\sqrt{1-ty^2}}$ is bounded function whenever $0 < y < \frac{1}{2\sqrt{t}}$ and

$$\lim_{t \uparrow 0} \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{2\sqrt{t}}} e^{-y^2} dy = 1.$$

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Thus, the expansion of the function $x \mapsto (1-x)^{-\frac{1}{2}}$ for $x = ty^2$ enables us to write

$$\begin{aligned}\mathcal{N}_2(t) &= \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{2\sqrt{t}}} e^{-y^2} \left(1 + \frac{ty^2}{2}\right) dy + O(t^2) \\ &= 1 - \frac{t}{4} + O(t^2).\end{aligned}$$

(iv) Using Taylor series of the function e^{-y} near 0, for $y = \frac{\sin^2 \alpha}{t}$, we get

$$\mathcal{N}_2(t) = \frac{1}{\sqrt{\pi t}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(1 - \frac{\sin^2 \alpha}{t}\right) d\alpha + O\left(\frac{1}{t^{5/2}}\right),$$

thus the expansions of \mathcal{N}_2 and \mathcal{N}_2' follow by an easy computation. □

Some remarks from [56] are recalled below

Remark III.2.1. (i) We believe that the function $t \mapsto \mathcal{N}_1(t)$ is decreasing over $]0, \infty[$, but the proof seems very hard.

(ii) The functions $t \mapsto t^\alpha \mathcal{N}_1(t)$, $t \mapsto t^\alpha \mathcal{N}_2(t)$ and $t \mapsto t^\beta \mathcal{N}_1'(t)$, $t \mapsto t^\beta \mathcal{N}_2'(t)$ are bounded for $0 \leq \alpha \leq \frac{1}{2}$ and $0 \leq \beta \leq \frac{3}{2}$.

Other nice properties of $(\mathbb{S}_i(t))_{t \geq 0}$, with $i = 1, 2$, in particular the estimate $L^p \rightarrow L^q$ are given in the following result.

Proposition III.2.7. *The family $((\mathbb{S}_1(t), \mathbb{S}_2(t))_{t \geq 0}$ associated to (III.28) is a strongly continuous semi-group of bounded linear operators in $L^p(\Omega) \times L^p(\Omega)$ for any $p \in [1, \infty]$. Furthermore, for $1 \leq p \leq q \leq \infty$ the following assertions hold.*

(i) For $(\omega_0, \rho_0) \in L^p(\Omega) \times L^p(\Omega)$, we have for every $t > 0$

$$\|(\mathbb{S}_1(t)\omega_0, \mathbb{S}_2(t)\rho_0)\|_{L^q(\Omega) \times L^q(\Omega)} \leq \frac{C}{t^{\frac{1}{p} - \frac{1}{q}}} \|(\omega_0, \rho_0)\|_{L^p(\Omega) \times L^p(\Omega)}. \quad (\text{III.35})$$

(ii) For $f = (f^r, f^z) \in L^p(\Omega) \times L^p(\Omega)$, we have for every $t > 0$

$$\|\mathbb{S}_1(t) \operatorname{div}_* f\|_{L^q(\Omega)} \leq \frac{C}{t^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}}} \|f\|_{L^p(\Omega)}. \quad (\text{III.36})$$

(iii) For $f = (f^r, f^z) \in L^p(\Omega) \times L^p(\Omega)$, we have every $t > 0$

$$\|\mathbb{S}_2(t) \operatorname{div} f\|_{L^q(\Omega)} \leq \frac{C}{t^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q}}} \|f\|_{L^p(\Omega)}. \quad (\text{III.37})$$

Here, $\operatorname{div}_* f = \partial_r f^r + \partial_z f^z$ (resp. $\operatorname{div} f = \partial_r f^r + \partial_z f^z + \frac{f^r}{r}$) stands the divergence operator over \mathbb{R}^2 (resp. the divergence operator over \mathbb{R}^3 in the axisymmetric case).

Proof

(i) We follow the proof of [56] with minor modifications, for this aim let $(r, z), (\tilde{r}, \tilde{z}) \in \Omega$, we will prove the following worth while estimates

$$\begin{cases} \frac{1}{4\pi t} \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_1\left(\frac{t}{r\tilde{r}}\right) e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{4t}} \leq \frac{C}{t} e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{5t}}, \\ \frac{1}{4\pi t} \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_2\left(\frac{t}{r\tilde{r}}\right) e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{4t}} \leq \frac{C}{t} e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{5t}}. \end{cases} \quad (\text{III.38})$$

We distinguish two cases $\tilde{r} \leq 2r$ and $\tilde{r} > 2r$.

• $\tilde{r} \leq 2r$. Employing the fact $t \mapsto (t^\alpha \mathcal{N}_1(t), t^\alpha \mathcal{N}_2(t))$ is bounded for $\alpha \in [0, \frac{1}{2}]$, see, (ii)-Remark III.2.1 and $t \mapsto e^{-t}$ is decreasing, we get the result.

• $\tilde{r} > 2r$. The remark $\tilde{r} \leq 2((r-\tilde{r})^2+(z-\tilde{z})^2)^{\frac{1}{2}}$, a new use of $t \mapsto (t^\alpha \mathcal{N}_1(t), t^\alpha \mathcal{N}_2(t))$ is bounded for $\alpha \in [0, \frac{1}{2}]$ and $te^{-\frac{t^2}{4}} \leq Ce^{-\frac{t^2}{5}}$ for $t \geq 0$ lead to

$$\begin{aligned} \frac{1}{4\pi t} \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_i\left(\frac{t}{r\tilde{r}}\right) e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{4t}} &\leq \frac{C}{t} \left(\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{4t}\right)^{\frac{1}{2}} e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{4t}} \\ &\leq \frac{C}{t} e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{5t}}, \quad i \in \{1, 2\}. \end{aligned}$$

Next, from (III.38) and the last estimate we write

$$\begin{aligned} |\mathbb{S}_1(t)\omega_0| + |\mathbb{S}_2(t)\rho_0| &\leq \frac{1}{4\pi t} \int_{\Omega} \left| \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_1\left(\frac{t}{r\tilde{r}}\right) e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{4t}} \omega_0(\tilde{r}, \tilde{z}) \right| d\tilde{r}d\tilde{z} \\ &\quad + \frac{1}{4\pi t} \int_{\Omega} \left| \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_2\left(\frac{t}{r\tilde{r}}\right) e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{4t}} \rho_0(\tilde{r}, \tilde{z}) \right| d\tilde{r}d\tilde{z} \\ &\leq \frac{C}{t} \int_{\Omega} e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{5t}} (|\omega_0(\tilde{r}, \tilde{z})| + |\rho_0(\tilde{r}, \tilde{z})|) d\tilde{r}d\tilde{z}. \end{aligned}$$

The last line can be seen as a convolution product, then Young's inequality gives the desired estimate.

(ii) By definition for every $(r, z) \in \Omega$, we have

$$\begin{aligned} (\mathbb{S}_1(t)\text{div}_*f)(r, z) &= \frac{1}{4\pi t} \int_{\Omega} \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_1\left(\frac{t}{r\tilde{r}}\right) e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{4t}} (\partial_{\tilde{r}}f^r(\tilde{r}, \tilde{z}) + \partial_{\tilde{z}}f^z(\tilde{r}, \tilde{z})) d\tilde{r}d\tilde{z} \\ &= \frac{1}{4\pi t} \int_{\Omega} \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_1\left(\frac{t}{r\tilde{r}}\right) e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{4t}} \partial_{\tilde{r}}f^r(\tilde{r}, \tilde{z}) d\tilde{r}d\tilde{z} \\ &\quad + \frac{1}{4\pi t} \int_{\Omega} \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_1\left(\frac{t}{r\tilde{r}}\right) e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{4t}} \partial_{\tilde{z}}f^z(\tilde{r}, \tilde{z}) d\tilde{r}d\tilde{z} \\ &= \text{II}_1 + \text{II}_2. \end{aligned}$$

After an integration by parts, it happens

$$\text{II}_1 = \frac{1}{4\pi t} \int_{\Omega} \frac{\tilde{r}^{1/2}}{r^{1/2}} \left(\frac{t}{r\tilde{r}^2} \mathcal{N}'_1\left(\frac{t}{r\tilde{r}}\right) - \left(\frac{1}{2\tilde{r}} + \left(\frac{r-\tilde{r}}{2t} \right) \right) \mathcal{N}_1\left(\frac{t}{r\tilde{r}}\right) \right) e^{-\frac{(r-\tilde{r})^2+(z-\tilde{z})^2}{4t}} f^r(\tilde{r}, \tilde{z}) d\tilde{r}d\tilde{z},$$

and

$$\mathbb{II}_2 = -\frac{1}{4\pi t} \int_{\Omega} \frac{\tilde{r}^{1/2}}{r^{1/2}} \left(\frac{z - \tilde{z}}{2t} \right) \mathcal{N}_1 \left(\frac{t}{r\tilde{r}} \right) e^{-\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{4t}} f^z(\tilde{r}, \tilde{z}) d\tilde{r}d\tilde{z}.$$

We proceed by the same manner as above. The fact that the functions $\mathcal{N}_1, \mathcal{N}_1'$ and $t \mapsto t^\alpha \mathcal{N}_1(t), t \mapsto t^\alpha \mathcal{N}_1'(t)$ are bounded, see Remark III.2.1, one finds

$$|\mathbb{II}_1| \leq \frac{C}{t^{3/2}} \int_{\Omega} e^{-\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{5t}} |f^r(\tilde{r}, \tilde{z})| d\tilde{r}d\tilde{z},$$

and

$$|\mathbb{II}_2| \leq \frac{C}{t^{3/2}} \int_{\Omega} e^{-\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{5t}} |f^z(\tilde{r}, \tilde{z})| d\tilde{r}d\tilde{z}.$$

Together with Young's inequality, we obtain (III.36).

(iii) Let $(r, z) \in \Omega$, then we have

$$\begin{aligned} \mathbb{S}_2(t) \operatorname{div} f(r, z) &= \frac{1}{4\pi t} \int_{\Omega} \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_2 \left(\frac{t}{r\tilde{r}} \right) e^{-\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{4t}} \left(\partial_{\tilde{r}} f^r(\tilde{r}, \tilde{z}) + \partial_{\tilde{r}} f^z(\tilde{r}, \tilde{z}) + \frac{1}{\tilde{r}} f^r(\tilde{r}, \tilde{z}) \right) d\tilde{r}d\tilde{z} \\ &= \frac{1}{4\pi t} \int_{\Omega} \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_2 \left(\frac{t}{r\tilde{r}} \right) e^{-\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{4t}} \partial_{\tilde{r}} f^r(\tilde{r}, \tilde{z}) \tilde{r} d\tilde{z} \\ &\quad + \frac{1}{4\pi t} \int_{\Omega} \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_2 \left(\frac{t}{r\tilde{r}} \right) e^{-\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{4t}} \partial_{\tilde{z}} f^z(\tilde{r}, \tilde{z}) \tilde{r} d\tilde{z} \\ &\quad + \frac{1}{4\pi t} \int_{\Omega} \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_2 \left(\frac{t}{r\tilde{r}} \right) e^{-\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{4t}} \frac{1}{\tilde{r}} f^r(\tilde{r}, \tilde{z}) d\tilde{r}d\tilde{z} \\ &= \mathbb{III}_3 + \mathbb{III}_4 + \mathbb{III}_5. \end{aligned} \tag{III.39}$$

The two terms \mathbb{III}_3 and \mathbb{III}_4 ensue by the same argument as in (ii). It remains to treat the term \mathbb{III}_5 in the following way

$$\begin{aligned} \mathbb{III}_5 &= \frac{1}{4\pi t} \int_{\Omega} \frac{\tilde{r}^{1/2}}{r^{1/2}} \mathcal{N}_2 \left(\frac{t}{r\tilde{r}} \right) e^{-\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{4t}} \frac{1}{\tilde{r}} f^r(\tilde{r}, \tilde{z}) d\tilde{r}d\tilde{z} \\ &= \frac{1}{4\pi t} \int_{\Omega} \frac{1}{(r\tilde{r})^{1/2}} \mathcal{N}_2 \left(\frac{t}{r\tilde{r}} \right) e^{-\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{4t}} f^r(\tilde{r}, \tilde{z}) d\tilde{r}d\tilde{z}. \end{aligned} \tag{III.40}$$

The fact that $(\cdot/r\tilde{r})^{1/2} \mathcal{N}_2(\cdot/r\tilde{r})$ is bounded guided to

$$|\mathbb{III}_5| \leq \frac{C}{t^{3/2}} \int_{\Omega} e^{-\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{4t}} |f^r(\tilde{r}, \tilde{z})| d\tilde{r}d\tilde{z}.$$

By plugging the last estimate in (III.40) and combine it with (III.39), it follows

$$|\mathbb{S}_2(t) \operatorname{div} f| \leq \frac{C}{t^{3/2}} \int_{\Omega} e^{-\frac{(r-\tilde{r})^2 + (z-\tilde{z})^2}{5t}} (|f^r(\tilde{r}, \tilde{z})| + |f^z(\tilde{r}, \tilde{z})|) d\tilde{r}d\tilde{z}.$$

Then a new use of Young's inequality leads to the result.

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To reach our claim, it remains to establish that $\mathbb{R}_+ \ni t \mapsto \mathbb{S}_1(t)$ (resp. $\mathbb{R}_+ \ni t \mapsto \mathbb{S}_2(t)$) is continuous on $L^p(\Omega)$ (resp. on $L^p(\Omega)$). We restrict ourselves only to treat $(\mathbb{S}_1(t))_{t \geq 0}$. Let $\omega_0 \in L^p(\Omega)$ and define its extension on \mathbb{R}^2 by $\tilde{\omega}_0$ which equal to 0 outside of Ω . Thus, in view the change of variables $\tilde{r} = r + \sqrt{t}\vartheta$ and $\tilde{z} = z + \sqrt{t}\gamma$, the statement (III.29) takes the form

$$(\mathbb{S}_1(t)\omega_0)(r, z) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \left(1 + \frac{\sqrt{t}\vartheta}{r}\right)^{1/2} \mathcal{N}_1\left(\frac{t}{r(r + \sqrt{t}\vartheta)}\right) e^{-\frac{\vartheta^2 + \gamma^2}{4}} \tilde{\omega}_0(r + \sqrt{t}\vartheta, z + \sqrt{t}\gamma) d\vartheta d\gamma.$$

We write

$$\mathbb{S}_1(t)\omega_0(r, z) - \omega_0(r, z) = \frac{1}{4\pi} \int_{\mathbb{R}^2} e^{-\frac{\vartheta^2 + \gamma^2}{4}} \Upsilon(t, r, z, \vartheta, \gamma) d\vartheta d\gamma, \quad (\text{III.41})$$

where

$$\Upsilon(t, r, z, \vartheta, \gamma) = \left(1 + \frac{\sqrt{t}\vartheta}{r}\right)^{1/2} \mathcal{N}_1\left(\frac{t}{r(r + \sqrt{t}\vartheta)}\right) \tilde{\omega}_0(r + \sqrt{t}\vartheta, z + \sqrt{t}\gamma) - \tilde{\omega}_0(r, z).$$

Taking the L^p -estimate of (III.41), then with the aid of the following Minkowski's integral formula in general case

$$\left(\int_{X_1} \left(\int_{X_2} F(x_1, x_2) d\lambda_2(x_2)\right)^p d\lambda_1(x_1)\right)^{1/p} \leq \int_{X_2} \left(\int_{X_1} F(x_1, x_2)^p d\lambda_1(x_1)\right)^{1/p} d\lambda_2(x_2),$$

one obtains for $p \in [1, \infty)$ that

$$\|\mathbb{S}_1(t)\omega_0(r, z) - \omega_0(r, z)\|_{L^p(\Omega)} \leq \frac{1}{4\pi} \int_{\mathbb{R}^2} e^{-\frac{\vartheta^2 + \gamma^2}{4}} \|\Upsilon(t, r, z, \vartheta, \gamma)\|_{L^p(\Omega)} d\vartheta d\gamma.$$

Since

$$\frac{1}{4\pi} \int_{\mathbb{R}^2} e^{-\frac{\vartheta^2 + \gamma^2}{4}} d\vartheta d\gamma = 1,$$

All we need to check is $\|\Upsilon(t, r, z, \vartheta, \gamma)\|_{L^p(\Omega)} \rightarrow 0$ as $t \uparrow 0$. To do this, let $r > 0$ and $r + \sqrt{t}\vartheta > 0$, as we have seen above we have

$$\left(1 + \sqrt{t}\frac{\vartheta}{r}\right)^{1/2} \mathcal{N}_1\left(\frac{t}{r(r + \sqrt{t}\vartheta)}\right) \leq C(1 + |\vartheta|).$$

Therefore

$$\begin{aligned} \|\Upsilon(t, \cdot, \cdot, \vartheta, \gamma)\|_{L^p(\Omega)} &\leq C(1 + |\vartheta|) (\|\omega_0(\cdot + \sqrt{t}\vartheta, \cdot + \sqrt{t}\gamma)\|_{L^p(\Omega)} + \|\omega_0\|_{L^p(\Omega)}) \\ &\leq C(1 + |\vartheta|) \|\omega_0\|_{L^p(\Omega)}. \end{aligned}$$

On the other hand, we have $\left(1 + \frac{\sqrt{t}\vartheta}{r}\right)^{1/2} \mathcal{N}_1\left(\frac{t}{r(r + \sqrt{t}\vartheta)}\right)$ goes to 1 as $t \uparrow 0$. Thus, Lebesgue's dominated convergence asserts for $(\vartheta, \gamma) \in \mathbb{R}^2$ that $\|\Upsilon(t, r, z, \vartheta, \gamma)\|_{L^p(\Omega)} \rightarrow 0$ when $t \uparrow 0$. A new use of Lebesgue's dominated convergence, we finally deduce

$$\lim_{t \uparrow 0} \|\mathbb{S}_1(t)\omega_0(r, z) - \omega_0(r, z)\|_{L^p(\Omega)} \rightarrow 0, \quad (\text{III.42})$$

which accomplishes the proof. □

In the spirit of Proposition 3.5 in [56], another weighted estimates for the linear semi-group (III.29) are shown in the following proposition, the proof of which can be done by the same reasoning as in the previous proposition,

Proposition III.2.8. *Let $1 \leq p \leq q \leq \infty$, $i \in \{1, 2\}$ and $(\alpha, \beta) \in [-1, 2]$, with $\alpha \leq \beta$. Assume that $r^\beta f \in L^p(\Omega)$, then*

$$\|r^\alpha \mathbb{S}_i(t)f\|_{L^q(\Omega)} \leq \frac{C}{t^{\frac{1}{p} - \frac{1}{q} + \frac{(\beta - \alpha)}{2}}} \|r^\beta f\|_{L^p(\Omega)}. \quad (\text{III.43})$$

In addition, if $(\alpha, \beta) \in [-1, 1]$, $\alpha \leq \beta$ and $r^\beta f \in L^p(\Omega)$, then

$$\|r^\alpha \mathbb{S}_i(t)\text{div}_* f\|_{L^q(\Omega)} \leq \frac{C}{t^{\frac{1}{2} + \frac{1}{p} - \frac{1}{q} + \frac{(\beta - \alpha)}{2}}} \|r^\beta f\|_{L^p(\Omega)}. \quad (\text{III.44})$$

We end this section by recalling the following classical estimate on the heat kernel in dimension three, the proof of which is left to the reader.

Proposition III.2.9. *Let $1 \leq p \leq q \leq \infty$. Assume that $f \in L^p(\mathbb{R}^3)$, then*

$$\|\mathbb{S}_2(t)f\|_{L^q(\mathbb{R}^3)} \leq \frac{C}{t^{\frac{3}{2}(\frac{1}{p} - \frac{1}{q})}} \|f\|_{L^p(\mathbb{R}^3)}. \quad (\text{III.45})$$

III.3 Local wellposedness in critical Lebesgue spaces

We will explore the aforementioned results and some preparatory topics in the previous sections. We begin by the the construction of the unique solution (III.9).

III.3.1 The fixed point argument (Construction of the local solution)

In order to scrutinize the local wellposedness issue for the system (III.9), we rewrite it in view of the divergence-free condition in the following form

$$\begin{cases} \partial_t \omega_\theta + \text{div}_*(v\omega_\theta) = \left(\partial_r^2 + \partial_z^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} \right) \omega_\theta - \partial_r \rho & \text{if } (t, r, z) \in \mathbb{R}_+ \times \Omega, \\ \partial_t \rho + \text{div}(v\rho) - \Delta \rho = 0 & \text{if } (t, r, z) \in \mathbb{R}_+ \times \Omega, \\ (\omega_\theta, \rho)|_{t=0} = (\omega_0, \rho_0). \end{cases} \quad (\text{III.46})$$

The direct treatment of the local wellposedness topic for (III.46) in the spirit of [56] for initial data (ω_0, ρ_0) in the critical space $L^1(\Omega) \times L^1(\mathbb{R}^3)$ contributes many technical difficulties.

This motivates to add the following new unknown $\tilde{\rho} \stackrel{\text{def}}{=} r\rho$ which solves

$$\partial_t \tilde{\rho} + \text{div}_*(v\tilde{\rho}) = \left(\partial_r^2 + \partial_z^2 + \frac{1}{r} \partial_r - \frac{1}{r^2} \right) \tilde{\rho} - 2\partial_r \rho. \quad (\text{III.47})$$

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We remark that $\tilde{\rho}$ satisfies the same equation as ω_θ with additional source term and their variations are in Ω .

To achieve our topic we will handle with the following equivalent integral formulation.

$$\begin{cases} \omega_\theta(t) = \mathbb{S}_1(t)\omega_0 - \int_0^t \mathbb{S}_1(t-\tau) \operatorname{div}_*(v(\tau)\omega_\theta(\tau)) d\tau - \int_0^t \mathbb{S}_1(t-\tau) \partial_r \rho(\tau) d\tau \\ \tilde{\rho}(t) = \mathbb{S}_1(t)\tilde{\rho}_0 - \int_0^t \mathbb{S}_1(t-\tau) \operatorname{div}_*(v(\tau)\tilde{\rho}(\tau)) d\tau - 2 \int_0^t \mathbb{S}_1(t-\tau) \partial_r \rho(\tau) d\tau \\ \rho(t) = \mathbb{S}_2(t)\rho_0 - \int_0^t \mathbb{S}_2(t-\tau) \operatorname{div}(v(\tau)\rho(\tau)) d\tau. \end{cases} \quad (\text{III.48})$$

In order to analyze the above system, we will be working in the following Banach spaces.

$$\begin{aligned} X_T &= \left\{ f \in C^0((0, T], L^{4/3}(\Omega)) : \|f\|_{X_T} < \infty \right\}, \\ Y_T &= \left\{ g \in C^0((0, T], L^{4/3}(\Omega)) : \|g\|_{Y_T} < \infty \right\}, \\ Z_T &= \left\{ h \in C^0((0, T], L^{4/3}(\mathbb{R}^3)) : \|h\|_{Z_T} < \infty \right\}, \end{aligned}$$

equipped with the following norms

$$\|f\|_{X_T} = \sup_{0 < t \leq T} t^{1/4} \|f(t)\|_{L^{4/3}(\Omega)}, \quad \|g\|_{Y_T} = \sup_{0 < t \leq T} t^{1/4} \|rg(t)\|_{L^{4/3}(\Omega)}, \quad \|h\|_{Z_T} = \sup_{0 < t \leq T} t^{3/8} \|h(t)\|_{L^{4/3}(\mathbb{R}^3)}.$$

Now, our task is to prove the following result.

Proposition III.3.1. *Let $(\omega_0, \rho_0) \in L^1(\Omega) \times L^1(\mathbb{R}^3)$, then there exists $T = T(\omega_0, \rho_0)$ such that (III.48) admits a unique local solution satisfying*

$$(\omega_\theta, \rho) \in C((0, T]; X_T) \times C((0, T]; Y_T \cap Z_T). \quad (\text{III.49})$$

Proof

We will proceed by the fixed point theorem in the product space $\mathcal{X}_T = X_T \times X_T \times Z_T$ equipped by the norm

$$\|(\omega_\theta, \tilde{\rho}, \rho)\|_{\mathcal{X}_T} \stackrel{def}{=} \|\omega_\theta\|_{X_T} + \|\tilde{\rho}\|_{X_T} + \|\rho\|_{Z_T}.$$

Notice that by definition, we have ¹

$$\|\tilde{\rho}\|_{X_T} = \|\rho\|_{Y_T}. \quad (\text{III.50})$$

For $t \geq 0$, define the free part $(\omega_{\text{lin}}(t), \tilde{\rho}_{\text{lin}}(t), \rho_{\text{lin}}(t)) = (\mathbb{S}_1(t)\omega_0, \mathbb{S}_1(t)(r\rho_0), \mathbb{S}_2(t)\rho_0)$, where $(\mathbb{S}_1(t), \mathbb{S}_2(t))$ is given in Proposition III.2.5. In accordance with the (i)-Proposition III.2.7, it is not difficult to check that for $(\omega_0, \rho_0) \in L^1(\Omega) \times L^1(\mathbb{R}^3)$, we have for $T > 0$

$$\sup_{0 < t \leq T} t^{1/4} \|\omega_{\text{lin}}(t)\|_{L^{\frac{4}{3}}(\Omega)} \leq C \|\omega_0\|_{L^1(\Omega)}, \quad (\text{III.51})$$

¹By assuming (III.50) we skip few parts of the proof for simplicity, see remark III.3.1 for more details.

and

$$\sup_{0 < t \leq T} t^{1/4} \|\tilde{\rho}_{\text{lin}}(t)\|_{L^{\frac{4}{3}}(\Omega)} \leq C \|r\rho_0\|_{L^1(\Omega)} = C \|\rho_0\|_{L^1(\mathbb{R}^3)}. \quad (\text{III.52})$$

On the other hand, the fact that

$$\|\rho_{\text{lin}}(t)\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} = \|r^{\frac{3}{4}}\rho_{\text{lin}}(t)\|_{L^{\frac{4}{3}}(\Omega)}$$

together with (III.43) stated in Proposition III.2.8, we further get

$$\sup_{0 < t \leq T} t^{3/8} \|\rho_{\text{lin}}(t)\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} \leq C \|r\rho_0\|_{L^1(\Omega)} = C \|\rho_0\|_{L^1(\mathbb{R}^3)}. \quad (\text{III.53})$$

Combining (III.51), (III.52) and (III.53) to obtain that $(\omega_{\text{lin}}, \tilde{\rho}_{\text{lin}}, \rho_{\text{lin}}) \in \mathcal{X}_T$.

Next, define the following quantity which will be useful in the contraction.

$$\Lambda(\omega_0, \rho_0, T) = C \|(\omega_{\text{lin}}, \tilde{\rho}_{\text{lin}}, \rho_{\text{lin}})\|_{\mathcal{X}_T}. \quad (\text{III.54})$$

We claim that $\Lambda(\omega_0, \rho_0, T) \rightarrow 0$ when $T \uparrow 0$. To do this, we employ the fact $(L^{4/3}(\Omega) \cap L^1(\Omega)) \times (L^{4/3}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3))$ is a dense space in $L^1(\Omega) \times L^1(\mathbb{R}^3)$. Then for every $\varepsilon > 0$ and every $(\omega_0, \rho_0) \in L^1(\Omega) \times L^1(\mathbb{R}^3)$ there exists $(\phi, \psi) \in (L^{4/3}(\Omega) \cap L^1(\Omega)) \times (L^{4/3}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3))$ such that

$$\|(\omega_0, \rho_0) - (\phi, \psi)\|_{L^1(\Omega) \times L^1(\mathbb{R}^3)} < \varepsilon.$$

On account of (i)-Proposition III.2.7 we write

$$\begin{aligned} \|\omega_{\text{lin}}(t)\|_{L^{4/3}(\Omega)} &= \|\mathbb{S}_1(t)(\omega_0 - \phi + \phi)\|_{L^{4/3}(\Omega)} \\ &\leq \|\mathbb{S}_1(t)(\omega_0 - \phi)\|_{L^{4/3}(\Omega)} + \|\mathbb{S}_1(t)\phi\|_{L^{4/3}(\Omega)} \\ &\leq \frac{C}{t^{1/4}} \|\omega_0 - \phi\|_{L^1(\Omega)} + C \|\phi\|_{L^{4/3}(\Omega) \cap L^1(\Omega)}. \end{aligned}$$

Multiply the both sides by $t^{1/4}$ and taking the supremum over $(0, T]$ to get

$$\begin{aligned} \sup_{0 < t \leq T} t^{1/4} \|\omega_{\text{lin}}(t)\|_{L^{4/3}(\Omega)} &\leq C \|\omega_0 - \phi\|_{L^1(\Omega)} + CT^{1/4} \|\phi\|_{L^{4/3}(\Omega) \cap L^1(\Omega)} \\ &\leq C\varepsilon + CT^{1/4} \|\phi\|_{L^{4/3}(\Omega) \cap L^1(\Omega)}. \end{aligned}$$

Thus, by setting

$$C_0(\omega_0, T) = \sup_{0 < t \leq T} t^{1/4} \|\omega_{\text{lin}}(t)\|_{L^{4/3}(\Omega)}, \quad (\text{III.55})$$

and let T (resp. ε) goes to 0, one deduces

$$\lim_{T \uparrow 0} C_0(\omega_0, T) = 0. \quad (\text{III.56})$$

By the same reasoning as above, it holds

$$\sup_{0 < t \leq T} t^{1/4} \|\tilde{\rho}_{\text{lin}}(t)\|_{L^{4/3}(\Omega)} \leq C\varepsilon + CT^{1/4} \|\phi\|_{L^{4/3}(\Omega) \cap L^1(\Omega)},$$

with

$$C_1(\tilde{\rho}_0, T) = \sup_{0 < t \leq T} t^{1/4} \|\tilde{\rho}_{\text{lin}}(t)\|_{L^{4/3}(\Omega)}. \quad (\text{III.57})$$

Likewise

$$\lim_{T \uparrow 0} C_1(\tilde{\rho}_0, T) = 0. \quad (\text{III.58})$$

For ρ_{lin} , a new use of Propositions III.2.8 and III.2.9 yield

$$\begin{aligned} \|\rho_{\text{lin}}(t)\|_{L^{4/3}(\mathbb{R}^3)} &= \|\mathbb{S}_2(t)(\rho_0 - \psi + \psi)\|_{L^{4/3}(\mathbb{R}^3)} \\ &\leq \|\mathbb{S}_2(t)(\rho_0 - \psi)\|_{L^{4/3}(\mathbb{R}^3)} + \|\mathbb{S}_2(t)\psi\|_{L^{4/3}(\mathbb{R}^3)} \\ &\leq \frac{C}{t^{3/8}} \|\rho_0 - \psi\|_{L^1(\mathbb{R}^3)} + \|\psi\|_{L^{4/3}(\mathbb{R}^3)} \end{aligned}$$

Now, we multiply the both sides by $t^{3/8}$ and we take the supremum over $(0, T]$ to deduce

$$\begin{aligned} \sup_{0 < t \leq T} t^{3/8} \|\rho_{\text{lin}}(t)\|_{L^{4/3}(\mathbb{R}^3)} &\leq C \|\rho_0 - \psi\|_{L^1(\mathbb{R}^3)} + CT^{3/8} \|\psi\|_{L^{4/3} \cap L^1(\mathbb{R}^3)} \\ &\leq C\varepsilon + CT^{3/8} \|\psi\|_{L^{4/3} \cap L^1(\mathbb{R}^3)}. \end{aligned} \quad (\text{III.59})$$

Similarly, by putting

$$C_2(\rho_0, T) = \sup_{0 < t \leq T} t^{3/8} \|\rho_{\text{lin}}(t)\|_{L^{4/3}(\mathbb{R}^3)}, \quad (\text{III.60})$$

we shall obtain that

$$\lim_{T \uparrow 0} C_2(\rho_0, T) = 0. \quad (\text{III.61})$$

Collecting (III.56), (III.58) and (III.61), so that by (III.54), we end up with

$$\lim_{T \uparrow 0} \Lambda(\omega_0, \rho_0, T) = 0.$$

Now, we are ready to contract the integral formulation (III.48) in \mathcal{X}_T . Doing so, define for $(\omega_\theta, \tilde{\rho}, \rho) \in \mathcal{X}_T$ the map

$$(0, T] \ni t \mapsto \mathcal{T}(t)(\omega_\theta, \tilde{\rho}, \rho) \in L^{4/3}(\Omega) \times L^{4/3}(\Omega) \times L^{4/3}(\mathbb{R}^3)$$

by

$$\mathcal{T}(t)(\omega_\theta, \tilde{\rho}, \rho) = \begin{pmatrix} \int_0^t \mathbb{S}_1(t-\tau) \operatorname{div}_*(v(\tau)\omega_\theta(\tau)) d\tau + \int_0^t \mathbb{S}_1(t-\tau) \partial_r \rho(\tau) d\tau \\ \int_0^t \mathbb{S}_1(t-\tau) \operatorname{div}_*(v(\tau)\tilde{\rho}(\tau)) d\tau + 2 \int_0^t \mathbb{S}_1(t-\tau) \partial_r \rho(\tau) d\tau \\ \int_0^t \mathbb{S}_2(t-\tau) \operatorname{div}(v(\tau)\rho(\tau)) d\tau \end{pmatrix}. \quad (\text{III.62})$$

We aim at estimating $\mathcal{T}(t)(\omega_\theta, \tilde{\rho}, \rho)$ in $L^{4/3}(\Omega) \times L^{4/3}(\Omega) \times L^{4/3}(\mathbb{R}^3)$. Due to the similarity of the first two lines of (III.62), we will restrict ourselves to analyze the first and the third

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ones. For $\int_0^t \mathbb{S}_1(t-\tau) \operatorname{div}_*(v(\tau)\omega_\theta(\tau)) d\tau$, we employ (III.35) in Proposition III.2.7 and Hölder's inequality with respect to time to obtain

$$\begin{aligned} \left\| \int_0^t \mathbb{S}_1(t-\tau) \operatorname{div}_*(v(\tau)\omega_\theta(\tau)) d\tau \right\|_{L^{\frac{4}{3}}(\Omega)} &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}+1-\frac{3}{4}}} \|v(\tau)\omega_\theta(\tau)\|_{L^1(\Omega)} d\tau \\ &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}} \|v(\tau)\|_{L^4(\Omega)} \|\omega_\theta(\tau)\|_{L^{\frac{4}{3}}(\Omega)} d\tau. \end{aligned}$$

Thanks to (III.25), it follows that

$$\begin{aligned} \left\| \int_0^t \mathbb{S}_1(t-\tau) \operatorname{div}_*(v(\tau)\omega_\theta(\tau)) d\tau \right\|_{L^{\frac{4}{3}}(\Omega)} &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{3}{4}}} \|\omega_\theta(\tau)\|_{L^{\frac{4}{3}}(\Omega)}^2 d\tau \\ &\lesssim \int_0^t \frac{d\tau}{(t-s)^{\frac{3}{4}} \tau^{\frac{1}{2}}} \|\omega_\theta\|_{X_T}^2 \\ &\lesssim t^{-\frac{1}{4}} \|\omega_\theta\|_{X_T}^2. \end{aligned}$$

We show next how to estimate $\int_0^t \mathbb{S}_1(t-\tau) \partial_r \rho(\tau) d\tau$ in $L^{\frac{4}{3}}(\Omega)$. In view of Proposition III.2.8 for $\alpha = 0$ and $\beta = \frac{3}{4}$, we get

$$\begin{aligned} \left\| \int_0^t \mathbb{S}_1(t-\tau) \partial_r \rho(\tau) d\tau \right\|_{L^{\frac{4}{3}}(\Omega)} &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}+\frac{3-0}{4-2}}} \|r^{\frac{3}{4}} \rho\|_{L^{\frac{4}{3}}(\Omega)} d\tau \\ &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{7}{8}}} \|\rho\|_{L^{\frac{4}{3}}(\mathbb{R}^3)} d\tau \\ &\lesssim \int_0^t \frac{d\tau}{(t-\tau)^{\frac{7}{8}} \tau^{\frac{3}{8}}} \|\rho\|_{Z_T} \\ &\lesssim t^{-\frac{1}{4}} \|\rho\|_{Z_T}. \end{aligned}$$

The above estimates combined with (III.54) provide the following inequality

$$\|\omega_\theta\|_{X_T} \leq \Lambda(\omega_0, \rho_0, T) + C\|\omega_\theta\|_{X_T}^2 + C\|\rho\|_{Z_T}. \quad (\text{III.63})$$

As explained above, the estimate of $\int_0^t \mathbb{S}_1(t-\tau) \operatorname{div}_*(v(\tau)\tilde{\rho}(\tau)) d\tau$ can be done along the same lines, so we have

$$\left\| \int_0^t \mathbb{S}_1(t-\tau) \operatorname{div}_*(v(\tau)\tilde{\rho}(\tau)) d\tau \right\|_{L^{\frac{4}{3}}(\Omega)} \lesssim t^{-\frac{1}{4}} \|\omega_\theta\|_{X_T} \|\tilde{\rho}\|_{X_T}, \quad (\text{III.64})$$

we deduce that

$$\|\tilde{\rho}\|_{X_T} \leq \Lambda(\omega_0, \rho_0, T) + C\|\omega_\theta\|_{X_T} \|\tilde{\rho}\|_{X_T} + C\|\rho\|_{Z_T}. \quad (\text{III.65})$$

Let us move to estimate the last line in (III.62). Under the remark $\operatorname{div}(v\rho) = \frac{v^r}{r}\rho + \operatorname{div}_*(v\rho)$, we write

$$\begin{aligned} \int_0^t \|\mathbb{S}_2(t-\tau) \operatorname{div}(v(\tau)\rho(\tau))\|_{L^{4/3}(\mathbb{R}^3)} d\tau &= \int_0^t \left\| \mathbb{S}_2(t-\tau) \left(\frac{v^r(\tau)}{r} \rho(\tau) \right) \right\|_{L^{4/3}(\mathbb{R}^3)} d\tau \\ &\quad + \int_0^t \|\mathbb{S}_2(t-\tau) \operatorname{div}_*(v(\tau)\rho(\tau))\|_{L^{4/3}(\mathbb{R}^3)} d\tau \end{aligned} \quad (\text{III.66})$$

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So, for the first term, we shall apply (III.43) stated in Proposition III.2.8 for $\alpha = \frac{3}{4}$ and $\beta = 2$ to get

$$\begin{aligned}
 \int_0^t \left\| \mathbb{S}_2(t-\tau) \left(\frac{v^r(\tau)}{r} \rho(\tau) \right) \right\|_{L^{4/3}(\mathbb{R}^3)} d\tau &= \int_0^t \left\| r^{3/4} \mathbb{S}_2(t-\tau) \left(\frac{v^r(\tau)}{r} \rho(\tau) \right) \right\|_{L^{4/3}(\Omega)} d\tau \\
 &\lesssim \int_0^t \frac{1}{(t-\tau)^{1-3/4+(2-3/4)/2}} \|v^r(\tau) r \rho(\tau)\|_{L^1(\Omega)} d\tau \\
 &\lesssim \int_0^t \frac{1}{(t-\tau)^{7/8}} \|v^r(\tau)\|_{L^4(\Omega)} \|\tilde{\rho}(\tau)\|_{L^{4/3}(\Omega)} d\tau \\
 &\lesssim \int_0^t \frac{1}{(t-\tau)^{7/8} \tau^{1/2}} \|\omega_\theta\|_{X_T} \|\tilde{\rho}\|_{X_T} d\tau \\
 &\lesssim t^{-3/8} \|\omega_\theta\|_{X_T} \|\tilde{\rho}\|_{X_T}.
 \end{aligned}$$

Therefore

$$t^{3/8} \int_0^t \left\| \mathbb{S}_2(t-\tau) \left(\frac{v^r(\tau)}{r} \rho(\tau) \right) \right\|_{L^{4/3}(\mathbb{R}^3)} d\tau \lesssim \|\omega_\theta\|_{X_T} \|\tilde{\rho}\|_{X_T}.$$

The second term of the r.h.s. in (III.66), will be done by a similar way as above, but we employ (III.44) in Proposition III.2.8 for $\alpha = \frac{3}{4}$ and $\beta = 1$, one may write

$$t^{3/8} \int_0^t \left\| \mathbb{S}_2(t-\tau) \operatorname{div}_*(v(\tau) \rho(\tau)) \right\|_{L^{4/3}(\mathbb{R}^3)} d\tau \lesssim \|\omega_\theta\|_{X_T} \|\tilde{\rho}\|_{X_T}.$$

Gathering the last two estimates and inserting them in (III.66), one has

$$t^{3/8} \int_0^t \left\| \mathbb{S}_2(t-\tau) \operatorname{div}(v(\tau) \rho(\tau)) \right\|_{L^{4/3}(\mathbb{R}^3)} d\tau \lesssim \|\omega_\theta\|_{X_T} \|\tilde{\rho}\|_{X_T},$$

combined with (III.60), it follows

$$\|\rho\|_{Z_T} \leq \Lambda(\omega_0, \rho_0, T) + \|\omega_\theta\|_{X_T} \|\tilde{\rho}\|_{X_T}. \quad (\text{III.67})$$

Collecting (III.63), (III.65) and (III.67) we finally find the nonlinear system

$$\|\omega_\theta\|_{X_T} \leq \Lambda(\omega_0, \rho_0, T) + C \|\omega_\theta\|_{X_T}^2 + \|\rho\|_{Z_T}. \quad (\text{III.68})$$

$$\|\tilde{\rho}\|_{X_T} \leq \Lambda(\omega_0, \rho_0, T) + C \|\omega_\theta\|_{X_T} \|\tilde{\rho}\|_{X_T} + \|\rho\|_{Z_T}. \quad (\text{III.69})$$

$$\|\rho\|_{Z_T} \leq \Lambda(\omega_0, \rho_0, T) + C \|\omega_\theta\|_{X_T} \|\tilde{\rho}\|_{X_T}. \quad (\text{III.70})$$

In order to better justify the contraction argument, let us denote

$$\mathcal{B}_T(R) \stackrel{\text{def}}{=} \{(a, b) \in X_T \times X_T : \|(a, b)\|_{X_T \times X_T} < R\}.$$

and we claim, for R, T sufficiently small, $(\omega_\theta, \tilde{\rho}) \in \mathcal{B}_T(R)$.

By substituting (III.70) into (III.68) and (III.69), the contraction argument is satisfied if

$$3\Lambda(\omega_0, \rho_0, T) + \tilde{C}R^2 < R.$$

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Since $\Lambda(\omega_0, \rho_0, T) \rightarrow 0$ when $T \uparrow 0$, then an usual argument leads to the existence of $T^* > 0$ for which $\|\omega_\theta\|_{X_T} + \|\tilde{\rho}\|_{X_T}$ remains bounded by R for all $T < T^*$. Finally by substituting this latest in (III.70) we deduce that $\|\rho\|_{Y_T}$ remains bounded as well for all $T < T^*$. The local existence and uniqueness follow then from classical fixed-point arguments. For the continuity of the solution, we will postpone the proof after another asymptotic properties, this completes the proof. \square

Remark III.3.1. In fact in the proof of the local existence above we skip two steps by assuming that $\tilde{\rho} = r\rho$, the rigorous proof should be as the following: In a first step instead of dealing with (III.48), we need first to solve the system

$$\begin{cases} \omega_\theta(t) = \mathbb{S}_1(t)\omega_0 - \int_0^t \mathbb{S}_1(t-\tau) \operatorname{div}_*(v(\tau)\omega_\theta(\tau)) d\tau - \int_0^t \mathbb{S}_1(t-\tau) \partial_r \rho(\tau) d\tau \\ \tilde{\rho}(t) = \mathbb{S}_1(t)\tilde{\rho}_0 - \int_0^t \mathbb{S}_1(t-\tau) \operatorname{div}_*(v(\tau)\tilde{\rho}(\tau)) d\tau - 2 \int_0^t \mathbb{S}_1(t-\tau) \partial_r \rho(\tau) d\tau \\ \rho(t) = \mathbb{S}_2(t)\rho_0 - \int_0^t \mathbb{S}_2(t-\tau) \operatorname{div}(v(\tau)\tilde{\rho}(\tau)) d\tau. \end{cases} \quad (\text{III.71})$$

by following the idea explained in the proof above, and finally we check that $\tilde{\rho} = r\rho$ by solving a heat-type equation evolving the quantity $\tilde{\rho} - r\rho$ with 0 initial data.

Remark III.3.2. In the light of remark 4.2 from [56], the local time of existence T given by Proposition III.3.1 above can not be bounded from below by using only the norm $\|(\omega_0, \rho_0)\|_{L^1(\Omega) \times L^1(\mathbb{R}^3)}$. However, in the case where $(\omega_0, \rho_0) \in (L^1(\Omega)) \times L^1(\mathbb{R}^3) \cap (L^p(\Omega) \times L^p(\mathbb{R}^3))$, for some $p > 1$, it is easy to explicitly provide a lower bound on T from an upper bound of $\|(\omega_0, \rho_0)\|_{L^p(\Omega) \times L^p(\mathbb{R}^3)}$ by making use of Propositions III.2.7, III.2.8, and III.2.9.

III.3.2 The bootstrap argument (More properties for the local solution)

We supply the above local wellposedness result by the following properties of the solution constructed in the previous part. Especially, we will prove.

Proposition III.3.2. *For any $p \in (1, \infty]$, we have*

$$\begin{aligned} \lim_{t \uparrow 0} t^{(1-\frac{1}{p})} \|\omega_\theta(t)\|_{L^p(\Omega)} &= 0, \\ \lim_{t \uparrow 0} t^{(1-\frac{1}{p})} \|r\rho(t)\|_{L^p(\Omega)} &= 0, \\ \lim_{t \uparrow 0} t^{\frac{3}{2}(1-\frac{1}{p})} \|\rho(t)\|_{L^p(\mathbb{R}^3)} &= 0. \end{aligned}$$

For, $p = 1$, the above quantities are just bounded as t tends to 0.

Proof

The proof is based principally on a bootstrap argument similar to that of [56]. For this aim, we will use the notations

$$N_p(f, T) \stackrel{\text{def}}{=} \sup_{0 < t \leq T} t^{(1-\frac{1}{p})} \|f\|_{L^p(\Omega)}, \quad J_p(f, T) \stackrel{\text{def}}{=} \sup_{0 < t \leq T} t^{\frac{3}{2}(1-\frac{1}{p})} \|f\|_{L^p(\mathbb{R}^3)}.$$

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$$M_p(f_0, T) \stackrel{\text{def}}{=} \sup_{0 < t \leq T} t^{(1-\frac{1}{p})} \|\mathbb{S}_1(t)f_0\|_{L^p(\Omega)}, \quad F_p(f_0, T) \stackrel{\text{def}}{=} \sup_{0 < t \leq T} t^{\frac{3}{2}(1-\frac{1}{p})} \|\mathbb{S}_2(t)f_0\|_{L^p(\mathbb{R}^3)}.$$

From the properties of the semi-groups \mathbb{S}_1 and \mathbb{S}_2 , we have for all $p \in (1, \infty]$

$$\lim_{T \uparrow 0} M_p(\omega_0, T) = \lim_{T \uparrow 0} M_p(r\rho_0, T) = \lim_{T \uparrow 0} F_p(\rho_0, T) = 0. \quad (\text{III.72})$$

In addition, from the local existence the desired inequalities hold also for $p = \frac{4}{3}$, assuming for a moment that the $L^1(\Omega) \times L^1(\mathbb{R}^3)$ -norm of $(\omega_\theta(t), \rho(t))$ is bounded for all t small enough (for $t < T$, with T denotes the local time of existence given by the local existence theory). Thus, by interpolation the proposition in question holds for all $p \in (1, \frac{4}{3}]$. In order to extend it to the other values of p we consider the Duhamel formula (III.48), and we will argue as in the local existence part, thus we omit some steps to make the presentation simpler. In view of Proposition III.2.8, we write

$$\begin{aligned} \|\omega_\theta(t)\|_{L^p(\Omega)} &\leq \|\mathbb{S}_1(t)\omega_0\|_{L^p(\Omega)} + C \int_0^{\frac{t}{2}} \frac{\|\omega_\theta\|_{L^q(\Omega)}^2}{(t-\tau)^{\frac{2}{q}-\frac{1}{p}}} d\tau + C \int_{\frac{t}{2}}^t \frac{\|\omega_\theta(\tau)\|_{L^{q_1}(\Omega)} \|\omega_\theta(\tau)\|_{L^{q_2}(\Omega)}}{(t-\tau)^{\frac{1}{q_1}+\frac{1}{q_2}-\frac{1}{p}}} d\tau \\ &\quad + C \int_0^{\frac{t}{2}} \frac{\|\rho(\tau)\|_{L^{\frac{4}{3}}(\mathbb{R}^3)}}{(t-\tau)^{\frac{1}{2}+\frac{3}{4}-\frac{1}{p}+\frac{3}{8}}} d\tau + C \int_{\frac{t}{2}}^t \frac{\|\rho(\tau)\|_{L^p(\mathbb{R}^3)}}{(t-\tau)^{\frac{1}{2}+\frac{1}{2p}}} d\tau. \end{aligned}$$

Under the conditions

$$\frac{1}{2} \leq \frac{2}{q} - \frac{1}{p}, \quad \frac{1}{2} \leq \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{p} < 1, \quad (\text{III.73})$$

we shall obtain

$$N_p(\omega_\theta, T) \leq M_p(\omega_0, T) + C_{p,q} N_q(\omega_\theta, T)^2 + C_{q_1, q_2} N_{q_1}(\omega_\theta, T) N_{q_2}(\omega_\theta, T) + C_p J_{\frac{4}{3}}(\rho, T) + C_p J_p(\rho, T). \quad (\text{III.74})$$

We recall that $\tilde{\rho}$ evolves almost the same equation as ω_θ , so we have

$$N_p(\tilde{\rho}, T) \leq M_p(\tilde{\rho}_0, T) + C_{p,q} N_q(\omega_\theta, T) N_q(\tilde{\rho}, T) + C_{q_1, q_2} N_{q_1}(\omega_\theta, T) N_{q_2}(\tilde{\rho}, T) + C_p J_{\frac{4}{3}}(\rho, T) + C_p J_p(\rho, T). \quad (\text{III.75})$$

Finally, to claim similar estimate for $J_p(\rho, T)$, first we write

$$\|\rho(t)\|_{L^p(\mathbb{R}^3)} \leq \|\mathbb{S}_2(t)\omega_0\|_{L^p(\mathbb{R}^3)} + C \int_0^{\frac{t}{2}} \frac{\|\omega_\theta\|_{L^{\frac{4}{3}}(\Omega)} \|\tilde{\rho}\|_{L^{\frac{4}{3}}(\Omega)}}{(t-\tau)^{\frac{1}{2}+1-\frac{1}{p}+\frac{1-\frac{1}{p}}{2}}} d\tau + C \int_{\frac{t}{2}}^t \frac{\|\omega_\theta(\tau)\|_{L^{q_1}(\Omega)} \|\tilde{\rho}(\tau)\|_{L^{q_2}(\Omega)}}{(t-\tau)^{\frac{1}{2}+\frac{1}{q_1}-\frac{1}{p}+\frac{1-\frac{1}{p}}{2}}} d\tau,$$

with

$$\frac{1}{\alpha} = \frac{1}{q_1} + \frac{1}{q_2} - \frac{1}{2}.$$

Under the additional condition on p, q_1, q_2

$$\frac{1}{q_1} + \frac{1}{q_2} - \frac{3}{2p} < \frac{1}{2}$$

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and for $q = \frac{4}{3}$, we obtain

$$J_p(\rho, T) \leq F_p(\rho_0, T) + C_p N_{\frac{4}{3}}(\omega_\theta, T) N_{\frac{4}{3}}(\tilde{\rho}, T) + C_{q_1, q_2} N_{q_1}(\omega_\theta, T) N_{q_2}(\tilde{\rho}, T). \quad (\text{III.76})$$

Plugging (III.76) in (III.74) and (III.75) for $q = \frac{4}{3}$, and by denoting

$$U_p(T) \stackrel{\text{def}}{=} N_p(\omega_\theta, T) + N_p(\tilde{\rho}, T), \quad V_p(T) \stackrel{\text{def}}{=} M_p(\omega_0, T) + M_p(\tilde{\rho}_0, T) + F_p(\rho_0, T),$$

we deduce, that

$$U_p(T) \leq C_{p, q_1, q_2} (V_p(T) + U_{\frac{4}{3}}(T)^2 + J_{\frac{4}{3}}(\rho, T) + U_{q_1}(T) U_{q_2}(T)).$$

Now, to cover all the range $p \in (\frac{4}{3}, \infty)$, we proceed by the following bootstrap algorithm:

- For $q_1 = q_2 = \frac{4}{3}$ we can check that $U_p(T) \rightarrow 0$ as $T \rightarrow 0$ for all $1 < p < \frac{3}{2}$.
- Next, by taking $q_1 = q_2$ sufficiently close to $\frac{3}{2}$, we obtain the same result, for all $p < \frac{9}{5}$.
- For $q_1 = q_2 = \frac{8}{5}$, the estimate in question will hold then for all $p < 2$.
- Taking q_1 sufficiently close to 2, the result follows for all $p < \frac{3}{2}q_2$ and for all $q_2 < 2$.
- Finally, we define the sequence p_n by $p_0 = \frac{4}{3}$ and p_n sufficiently close to $\frac{3}{2}p_{n-1}$, by induction, we find that p_n is sufficiently close to $(\frac{3}{2})^n p_0$. Hence, letting n goes to ∞ , we can cover all the range $p < \infty$, and thus we obtain

$$U_p(T) \rightarrow 0, \quad T \uparrow 0, \quad \text{for all } p \in (1, \infty).$$

Finally, substituting this latest into (III.76), leads to

$$J_p(\rho, T) \rightarrow 0, \quad T \uparrow 0, \quad \text{for all } p \in (1, \infty). \quad (\text{III.77})$$

In order to treat the case $p = \infty$, we need to pay some more attention to the equation of the density. We propose the following procedure: For the equations of ω_θ and $\tilde{\rho}$, we just need to chose $q_1 = \frac{3}{2}$, $q_2 = 4$ and $p = \infty$ in (III.74) and (III.75). Indeed, remark that this choice of (q_1, q_2, p) is admissible by the relation (III.73). This leads to the following estimates

$$N_\infty(\omega_\theta, T) \leq f_1(T) + C J_\infty(\rho, T). \quad (\text{III.78})$$

$$N_\infty(\tilde{\rho}, T) \leq f_2(T) + C J_\infty(\rho, T), \quad (\text{III.79})$$

with

$$f_1(T) = M_\infty(\omega_0, T) + C N_{\frac{4}{3}}(\omega_\theta, T)^2 + C N_{\frac{3}{2}}(\omega_\theta, T) N_4(\omega_\theta, T) + C J_{\frac{4}{3}}(\rho, T) \xrightarrow{T \rightarrow 0} 0,$$

$$f_2(T) = M_\infty(\tilde{\rho}_0, T) + C N_{\frac{4}{3}}(\omega_\theta, T)^2 + C N_{\frac{3}{2}}(\omega_\theta, T) N_4(\omega_\theta, T) + C J_{\frac{4}{3}}(\rho, T) \xrightarrow{T \rightarrow 0} 0.$$

Thus, all we need in order to close our estimates is to deal with $J_\infty(\rho, T)$ as follows: By using again the properties of the heat-semi group as in the case $p < \infty$, we infer that

$$\|\rho(t)\|_{L^\infty(\mathbb{R}^3)} \leq \|\mathbb{S}_2(t)\rho_0\|_{L^\infty(\mathbb{R}^3)} + C \int_0^t \frac{\|\omega_\theta\|_{L^{\frac{4}{3}}(\Omega)} \|\tilde{\rho}\|_{L^{\frac{4}{3}}(\Omega)}}{(t-\tau)^{\frac{1}{2} + \frac{3}{2}}} d\tau + C \int_{\frac{t}{2}}^t \frac{\|u(\tau)\|_{L^\infty(\mathbb{R}^3)} \|\rho(\tau)\|_{L^q(\mathbb{R}^3)}}{(t-\tau)^{\frac{1}{2} + \frac{3}{2q}}} d\tau. \quad (\text{III.80})$$

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In order to insure that the last term on the r.h.s above is finite, we need to chose p such that $\frac{1}{2} + \frac{3}{2q} < 1$. Let us for, instance, take $q = 6$.

On the other hand, let us remark that, due to the Biot-Savart law, we have , for some $1 < m < 2 < \ell < \infty$

$$\|u(\tau)\|_{L^\infty(\Omega)} \lesssim \|\omega_\theta(\tau)\|_{L^m(\Omega)}^\alpha \|\omega_\theta(\tau)\|_{L^\ell(\Omega)}^{1-\alpha}, \quad \text{for } \alpha = \frac{m \ell - 2}{2 \ell - m} \in (0, 1).$$

Which gives after simplification, thanks to the previous estimates in the case $p < \infty$

$$\|u(\tau)\|_{L^\infty(\Omega)} \lesssim \tau^{-\frac{1}{2}}. \quad (\text{III.81})$$

Finally, by using (III.81), and by taking $q = 6$ in (III.80), we obtain

$$t^{\frac{3}{2}} \|\rho(t)\|_{L^\infty(\mathbb{R}^3)} \leq t^{\frac{3}{2}} \|\mathbb{S}_2(t)\rho_0\|_{L^\infty(\mathbb{R}^3)} + CN_{\frac{4}{3}}(\omega_\theta, T)N_{\frac{4}{3}}(\tilde{\rho}, T) + \sup_{\tau \in (0, T)} \left\{ t^{\frac{1}{2}} \|u(\tau)\|_{L^\infty(\Omega)} \right\} J_6(\rho, T). \quad (\text{III.82})$$

It is easy then to conclude that

$$J_\infty(\rho, T) \rightarrow 0, \quad T \uparrow 0, \quad (\text{III.83})$$

and eventually we get as well

$$N_\infty(\omega_\theta, T) + N_\infty(\tilde{\rho}, T) \rightarrow 0, \quad T \uparrow 0. \quad (\text{III.84})$$

This ends the proof of Proposition III.3.2 provided that we prove that, there exists some $C_0 > 0$ for which

$$\|(\omega_\theta(t), \tilde{\rho}(t), \rho(t))\|_{L^1(\Omega) \times L^1(\Omega) \times L^1(\mathbb{R}^3)} \leq C_0 \|(\omega_0, \rho_0)\|_{L^1(\Omega) \times L^1(\mathbb{R}^3)}.$$

From the definition of $\tilde{\Gamma}$, we have

$$\|\omega_\theta(t)\|_{L^1(\Omega)} \leq \left\| \tilde{\Gamma}(t) \right\|_{L^1(\Omega)} + \|\tilde{\rho}(t)\|_{L^1(\Omega)}$$

and since $\tilde{\rho} = r\rho$, our claim is equivalent to

$$\left\| (\tilde{\Gamma}(t), \rho(t)) \right\|_{L^1(\Omega) \times L^1(\mathbb{R}^3)} \leq C_0 \|(\omega_0, \rho_0)\|_{L^1(\Omega) \times L^1(\mathbb{R}^3)}. \quad (\text{III.85})$$

Let us then prove (III.85), we will restrict our selves to the estimates of the non linear terms since the linear part can be dealt with by applying the properties of the semi-groups proved in the previous section, thus according to the equations of $\tilde{\Gamma}$ and ρ , we need to show that

$$\int_0^t \left\| \mathbb{S}_1(t - \tau) \operatorname{div}_*(v\tilde{\Gamma})(\tau) \right\|_{L^1(\Omega)} d\tau \lesssim \|(\omega_0, \rho_0)\|_{L^1(\Omega) \times L^1(\mathbb{R}^3)}, \quad (\text{III.86})$$

and

$$\int_0^t \|\mathbb{S}_2(t-\tau) \operatorname{div}(v\rho)(\tau)\|_{L^1(\mathbb{R}^3)} d\tau \lesssim \|(\omega_0, \rho_0)\|_{L^1(\Omega) \times L^1(\mathbb{R}^3)}. \quad (\text{III.87})$$

For (III.86), Holder's inequality, Biot Savart law and the definition of the X_T norm, we infer that

$$\begin{aligned} \int_0^t \left\| \mathbb{S}_1(t-\tau) \operatorname{div}_*(v\tilde{\Gamma})(\tau) \right\|_{L^1(\Omega)} d\tau &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} \|v(\tau)\|_{L^4(\Omega)} \left\| \tilde{\Gamma}(\tau) \right\|_{L^{\frac{4}{3}}(\Omega)} d\tau \\ &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} \|\omega_\theta(\tau)\|_{L^{\frac{4}{3}}(\Omega)} \left\| \tilde{\Gamma}(\tau) \right\|_{L^{\frac{4}{3}}(\Omega)} d\tau \\ &\lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}} \tau^{\frac{1}{2}}} d\tau \|\omega_\theta\|_{X_T} \left\| \tilde{\Gamma} \right\|_{X_T} \\ &\lesssim \|\omega_\theta\|_{X_T} \left\| \tilde{\Gamma} \right\|_{X_T}. \end{aligned}$$

For (III.87), we have

$$\|\mathbb{S}_2(t-\tau) \operatorname{div}(v\rho)(\tau)\|_{L^1(\mathbb{R}^3)} = \|r\mathbb{S}_2(t-\tau) \operatorname{div}(v\rho)(\tau)\|_{L^1(\Omega)},$$

then we use first Proposition III.2.8 to infer that

$$\int_0^t \|r\mathbb{S}_2(t-\tau) \operatorname{div}(v\rho)(\tau)\|_{L^1(\Omega)} d\tau \lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} \|vr\rho(\tau)\|_{L^1(\Omega)} d\tau,$$

then, using the definition of $\tilde{\rho} = r\rho$, Holder inequality yields

$$\|\mathbb{S}_2(t-\tau) \operatorname{div}(v\rho)(\tau)\|_{L^1(\mathbb{R}^3)} \lesssim \int_0^t \frac{1}{(t-\tau)^{\frac{1}{2}}} \|v(\tau)\|_{L^4(\Omega)} \|\tilde{\rho}(\tau)\|_{L^{\frac{4}{3}}(\Omega)} d\tau,$$

the rest of the estimate is then similar to the proof of (III.87) above by replacing $\tilde{\Gamma}$ by $\tilde{\rho}$. Finally, (III.86) and (III.87) follow from the local existence theory given by Proposition III.3.1. □

Our last task of this section is to reach the continuity of the solution stated in (III.10) and (III.11) of the main Theorem III.1.1. For this aim, we briefly outline the continuity of ω_θ , the rest of quantities can be treated along the same lines. So, we will show that

$$\omega_\theta \in C^0([0, T^*]; L^p(\Omega)), \quad \forall p \in [1, \infty).$$

To do so, let $0 < t_0 \leq t < T^*$, so we have

$$\omega_\theta(t) - \omega_\theta(t_0) = (\mathbb{S}_1(t-t_0) - \mathbb{I})\omega_\theta(t_0) - \int_{t_0}^t \mathbb{S}_1(t-\tau) \operatorname{div}_*(v(\tau)\omega_\theta(\tau)) d\tau - \int_{t_0}^t \mathbb{S}_1(t-\tau) \partial_\tau \rho(\tau) d\tau. \quad (\text{III.88})$$

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The first term (free part) is derived by the same manner as in (III.42), that is to say,

$$\lim_{t \uparrow t_0} \|(\mathbb{S}_1(t - t_0) - \mathbb{I})\omega_\theta(t_0, \cdot)\|_{L^p(\Omega)} \rightarrow 0. \quad (\text{III.89})$$

Concerning the second term in the r.h.s of (III.88), (III.36) in Proposition III.2.5 provides

$$\left\| \int_{t_0}^t \mathbb{S}_1(t - \tau) \operatorname{div}_*(v(\tau)\omega_\theta(\tau)) d\tau \right\|_{L^p(\Omega)} \lesssim \int_{t_0}^t \frac{1}{(t - \tau)^{\frac{1}{2}}} \|v(\tau)\|_{L^\infty(\Omega)} \|\omega_\theta(\tau)\|_{L^p(\Omega)} d\tau.$$

By virtue of the following interpolation estimate, see, Proposition 2.3 in [56], we have for some $1 < q_1 < 2 < q_2 < \infty$

$$\|v(\tau)\|_{L^\infty(\Omega)} \lesssim \|\omega_\theta(\tau)\|_{L^{q_1}}^\sigma \|\omega_\theta(\tau)\|_{L^{q_2}}^{1-\sigma}, \quad \text{with } \sigma = \frac{q_1}{2} \frac{q_2 - 2}{q_2 - q_1} \in (0, 1),$$

one may conclude that

$$\begin{aligned} & \left\| \int_{t_0}^t \mathbb{S}_1(t - \tau) \operatorname{div}_*(v(\tau)\omega_\theta(\tau)) d\tau \right\|_{L^p(\Omega)} \\ & \lesssim \operatorname{esssup}_{\tau \in (t_0, T^*)} \left(\|\omega_\theta(\tau)\|_{L^p(\Omega)} \|\omega_\theta(\tau)\|_{L^{q_1}}^\sigma \|\omega_\theta(\tau)\|_{L^{q_2}}^{1-\sigma} \right) \int_{t_0}^t \frac{d\tau}{(t - \tau)^{\frac{1}{2}}} \\ & \lesssim \operatorname{esssup}_{\tau \in (t_0, T^*)} \left(\|\omega_\theta(\tau)\|_{L^p(\Omega)} \|\omega_\theta(\tau)\|_{L^{q_1}}^\sigma \|\omega_\theta(\tau)\|_{L^{q_2}}^{1-\sigma} \right) (t - t_0)^{\frac{1}{2}}, \end{aligned}$$

which is sufficient to obtain

$$\lim_{t \uparrow t_0} \left\| \int_{t_0}^t \mathbb{S}_1(t - \tau) \operatorname{div}_*(v(\tau)\omega_\theta(\tau)) d\tau \right\|_{L^p(\Omega)} = 0. \quad (\text{III.90})$$

Let us move to the last term of (III.88) for which we distinguish two cases for p . For $p \in (1, \infty)$, (III.44) stated in Proposition III.2.8 for $\alpha = 0$ and $\beta = \frac{1}{p}$ yield

$$\left\| \int_{t_0}^t \mathbb{S}_1(t - \tau) \partial_r \rho(\tau) d\tau \right\|_{L^p(\Omega)} \lesssim \int_{t_0}^t \frac{1}{(t - \tau)^{\frac{1}{2} + \frac{1}{2p}}} \|r^{\frac{1}{p}} \rho(\tau)\|_{L^p(\Omega)} d\tau, \quad (\text{III.91})$$

and the fact that $\rho \in L^\infty((0, T^*); L^p(\mathbb{R}^3))$ ensures that

$$\left\| \int_{t_0}^t \mathbb{S}_1(t - \tau) \partial_r \rho(\tau) d\tau \right\|_{L^p(\Omega)} \lesssim \operatorname{esssup}_{\tau \in (t_0, T^*)} \|\rho(\tau)\|_{L^p(\mathbb{R}^3)} (t - t_0)^{\frac{1}{2}(1 - \frac{1}{p})}$$

combined with (III.91), one has

$$\lim_{t \uparrow t_0} \left\| \int_{t_0}^t \mathbb{S}_1(t - \tau) \partial_r \rho(\tau) d\tau \right\|_{L^p(\Omega)} = 0. \quad (\text{III.92})$$

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For the case $p = 1$, we will work with $\tilde{\Gamma}$ instead of ω_θ to avoid the source term $\partial_r \rho$. The fact $\|r\rho\|_{L^1(\Omega)} = \|\rho\|_{L^1(\mathbb{R}^3)}$ leads to

$$\|\omega_\theta(t) - \omega_\theta(t_0)\|_{L^1(\Omega)} \leq \|\tilde{\Gamma}(t) - \tilde{\Gamma}(t_0)\|_{L^1(\Omega)} + \|\rho(t) - \rho(t_0)\|_{L^1(\mathbb{R}^3)},$$

so, the continuity of $\|\omega_\theta(\cdot)\|_{L^p(\Omega)}$ relies then on the continuity of $\|\tilde{\Gamma}(\cdot)\|_{L^p(\Omega)}$ and $\|\rho(\cdot)\|_{L^1(\mathbb{R}^3)}$. On the one hand, seen that the equation of $\tilde{\Gamma}$ governs the same equation to that of ω_θ , but without the source term $\partial_r \rho$, hence we follow then the same approach as above to prove that

$$\lim_{t \uparrow t_0} \|\tilde{\Gamma}(t) - \tilde{\Gamma}(t_0)\|_{L^1(\Omega)} = 0. \quad (\text{III.93})$$

On the other hand, ρ solve a transport-diffusion equation, for which the continuity property is well known to hold, thus we skip the details. Therefore

$$\lim_{t \uparrow t_0} \|\omega_\theta(t) - \omega_\theta(t_0)\|_{L^1(\Omega)} = 0.$$

Combining the last estimate with (III.89), (III.90) and (III.92), we achieve the result. In the case $p = 1$ and $t_0 = 0$, we should just be careful about the term $\partial_r \rho$ which can be completely avoided in the estimates by using the coupled functions Γ and $\tilde{\Gamma}$, the details are left to the reader. \square

III.4 Proof of the global estimates

To reach the global existence for the local solution constructed in sections III.3, we will establish some a priori estimates in Lebesgue spaces. For this target, let

$$(\omega_\theta, r\rho, \rho) \in C^0([0, T]; L^p(\Omega) \times L^p(\Omega) \times L^p(\mathbb{R}^3)), \quad p \in [1, \infty), \quad T \in (0, T^*).$$

be a solution of the integral formulation (III.48) and so does (ω_θ, ρ) to the differential equation (III.46) associated to the initial data $(\omega_0, \rho_0) \in L^1(\Omega) \times L^1(\mathbb{R}^3)$, where T^* denotes the maximal time of existence. The key idea here is to use the coupling functions $\Gamma = \Pi - \frac{\rho}{2}$, $\tilde{\Gamma} = r\Gamma$ following [74] with $\Pi = \frac{\omega_\theta}{r}$, combined with the approach of [56]. Some familiar computations show that Γ evolves

$$\begin{cases} \partial_t \Gamma + v \cdot \nabla \Gamma - (\Delta + \frac{2}{r} \partial_r) \Gamma = 0 & \text{if } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \Gamma|_{t=0} = \Gamma_0, \end{cases} \quad (\text{III.94})$$

while $\tilde{\Gamma}$ solves

$$\begin{cases} \partial_t \tilde{\Gamma} + \text{div}_*(v \tilde{\Gamma}) - (\Delta - \frac{1}{r^2}) \tilde{\Gamma} = 0 & \text{if } (t, r, z) \in \mathbb{R}_+ \times \Omega, \\ \tilde{\Gamma}|_{t=0} = \tilde{\Gamma}_0. \end{cases} \quad (\text{III.95})$$

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The role of the new function Γ (resp. $\tilde{\Gamma}$) for the viscous Boussinesq system ($B_{\mu,\kappa}$) is the same that Π (resp. ω_θ) for the Navier-Stokes equations (NS_μ). For this aim, it is quite natural to treat carefully the properties of Γ and $\tilde{\Gamma}$.

The starting point of our analysis says that Γ enjoys the strong maximum principle. We will prove the following.

Proposition III.4.1. *We assume that $\Gamma_0(x_1, x_2, z) > 0$ (or, < 0), then $\Gamma(t, x_1, x_2, z) > 0$ (or, < 0) for any $(x_1, x_2, z) \in \mathbb{R}^3$ and $t > 0$.*

Proof

We follow the formalism recently accomplished in [43]. Up to a regularization of Γ by standard method we can achieve the result as follows: we suppose that $\Gamma_0(x_1, x_2, z) > 0$ (likewise the case $\Gamma_0(x_1, x_2, z) < 0$). Due to the singularity of the term $\frac{2}{r}\partial_r\Gamma$, we can not apply directly the maximum principle. To surmount this hitch, we can be appropriately interpreted the term $\Delta + \frac{2}{r}\partial_r$ as the Laplacian in \mathbb{R}^5 . Thus we recast (III.94) in $]0, \infty[\times \mathbb{R}^5$ by setting

$$\bar{\Gamma}(t, x_1, x_2, x_3, x_4, z) = \Gamma\left(t, \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}, z\right)$$

and

$$\bar{v}(t, x_1, x_2, x_3, x_4, z) = v^r\left(t, \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}, z\right)\bar{e}_r + v^z\left(t, \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}, z\right)\bar{e}_z.$$

Above,

$$r = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}, \quad \bar{e}_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r}, \frac{x_4}{r}, 0\right), \quad \bar{e}_z = (0, 0, 0, 0, 1)$$

Thus the equation (III.94) becomes

$$\begin{cases} \partial_t \bar{\Gamma} + \bar{v} \cdot \nabla_5 \bar{\Gamma} - \Delta_5 \bar{\Gamma} = 0 & \text{if } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^5, \\ \bar{\Gamma}|_{t=0} = \bar{\Gamma}_0, \end{cases} \quad (\text{III.96})$$

where ∇_5 and Δ_5 designate the gradient and Laplacian operators over \mathbb{R}^5 respectively. Consequently, by the strong maximum principle for (III.96), we deduce that

$$\bar{\Gamma} > 0 \quad \text{in }]0, \infty[\times \mathbb{R}^5,$$

which leads to

$$\Gamma > 0 \quad \text{in }]0, \infty[\times \mathbb{R}^3.$$

Thus, the proof is completed. □

The second result cares with the classical L^p -estimate for Γ and showing that $t \mapsto \|\Gamma(t)\|_{L^p(\mathbb{R}^3)}$ is strictly decreasing function for $p \in [1, \infty]$. We will establish the following.

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Proposition III.4.2. *Let v be a smooth divergence-free vector field on \mathbb{R}^3 and Γ be smooth solution of (III.94). Then the following assertion holds.*

$$\|\Gamma(t)\|_{L^p(\mathbb{R}^3)} \leq \|\Gamma_0\|_{L^p(\mathbb{R}^3)}, \quad p \in [1, \infty]. \quad (\text{III.97})$$

In particular, for $p \in [1, \infty]$ the map $t \mapsto \|\Gamma(t)\|_{L^p(\mathbb{R}^3)}$ is strictly decreasing.

Proof

Thanks to the Proposition III.4.1, we can assume that $\Gamma_0 > 0$, thus we have $\Gamma(t) > 0$ for $t \in [0, T]$. We developp an integration by parts and taking into account the Γ -equation, the fact that $\text{div } v = 0$ and the boundary condition over $\partial\Omega$, one has

$$\begin{aligned} \frac{d}{dt} \|\Gamma(t)\|_{L^p(\mathbb{R}^3)}^p &= p \int_{\Omega} \partial_t \Gamma(t) \Gamma^{p-1}(t) r dr dz & (\text{III.98}) \\ &= -p \int_{\Omega} v \cdot (\nabla \Gamma) \Gamma^{p-1} r dr dz + p \int_{\Omega} (\Delta \Gamma) \Gamma^{p-1} r dr dz + 2p \int_{\Omega} (\partial_r \Gamma) \Gamma^{p-1} dr dz \\ &= -p(p-1) \int_{\Omega} |\nabla \Gamma|^2 \Gamma^{p-2} r dr dz + \int_{\Omega} \partial_r \Gamma^p dr dz \\ &= -p(p-1) \int_{\Omega} |\nabla \Gamma|^2 \Gamma^{p-2} r dr dz + \int_{\mathbb{R}} \Gamma^p(t, 0, z) \eta_r dz \\ &= -p(p-1) \int_{\Omega} |\nabla \Gamma|^2 \Gamma^{p-2} r dr dz - \int_{\mathbb{R}} \Gamma^p(t, 0, z) dz < 0. \end{aligned}$$

where $\eta = (\eta_r, \eta_z) = (-1, 0)$ is a outward normal vector over Ω . Thus, integrating in time to obtain the aimed estimate for positive solutions.

Generally if Γ_0 changes its sign, we proceed as follows: we split $\Gamma(t) = \Gamma^+(t) - \Gamma^-(t)$, where Γ^{\pm} solves the following linear equation with the same velocity

$$\begin{cases} \partial_t \Gamma^{\pm} + v \cdot \nabla \Gamma^{\pm} - (\Delta + \frac{2}{r} \partial_r) \Gamma^{\pm} = 0 & \text{if } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \Gamma^{\pm}|_{t=0} = \max(\pm \Gamma_0, 0) \geq 0. \end{cases} \quad (\text{III.99})$$

Arguing as above to obtain that Γ^{\pm} satisfies (III.97). Thus we have:

$$\begin{aligned} \|\Gamma(t)\|_{L^p(\mathbb{R}^3)} &\leq \|\Gamma^+(t)\|_{L^p(\mathbb{R}^3)} + \|\Gamma^-(t)\|_{L^p(\mathbb{R}^3)} & (\text{III.100}) \\ &\leq \|\Gamma_0^+\|_{L^p(\mathbb{R}^3)} + \|\Gamma_0^-\|_{L^p(\mathbb{R}^3)} = \|\Gamma_0\|_{L^p(\mathbb{R}^3)}. \end{aligned}$$

If $\Gamma_0 \neq 0$, we distinguish two cases: $\Gamma_0 > 0$ or $\Gamma_0 < 0$. For this two cases the last inequality is strict and consequently (III.97) is also strict. Therefore, $t \mapsto \|\Gamma(t)\|_{L^1(\mathbb{R}^3)}$ is strictly decreasing for $t = 0$, and analogously we deduce that is strictly decreasing over $[0, T]$. \square

Now, we state a result which deals with the asymptotic behavior of the coupled function Γ in Lebegue spaces $L^p(\mathbb{R}^3)$. Specifically, we have.

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Proposition III.4.3. *Let $\rho_0, \frac{\omega_0}{r} \in L^1(\mathbb{R}^3)$, then for any smooth solution of (III.94) and $1 \leq p \leq \infty$, we have*

$$\|\Gamma(t)\|_{L^p(\mathbb{R}^3)} \leq \frac{C}{t^{\frac{3}{2}(1-1/p)}} \|\Gamma_0\|_{L^1(\mathbb{R}^3)}, \quad (\text{III.101})$$

where $\Gamma_0 = \Pi_0 - \frac{\rho_0}{2}$.

Proof

Due to (III.97), the estimate (III.101) is valid for $p = 1$.

From the estimate (III.98) we have for $p = 2^n$

$$\begin{aligned} \frac{d}{dt} \|\Gamma(t)\|_{L^p(\mathbb{R}^3)}^p &= p \int_{\Omega} \partial_t \Gamma(t) \Gamma^{p-1}(t) r dr dz & (\text{III.102}) \\ &= -p(p-1) \int_{\Omega} |\nabla \Gamma|^2 \Gamma^{p-2} r dr dz - \int_{-\infty}^{\infty} \Gamma^p(t, 0, z) dz \\ &\leq -p(p-1) \int_{\mathbb{R}^3} |\nabla \Gamma|^2 \Gamma^{p-2} dx \\ &= -p(p-1) \int_{\Omega} \left| \frac{2}{p} \nabla \Gamma^{\frac{p}{2}} \right|^2 r dr dz = -\frac{4(p-1)}{p} \int_{\Omega} |\nabla \Gamma^{\frac{p}{2}}|^2 r dr dz. \end{aligned}$$

Thanks to the well known Nash's inequality in general case

$$\int_{\mathbb{R}^N} |f|^2 dx \leq C \left(\int_{\mathbb{R}^N} |\nabla f|^2 dx \right)^{1-\gamma} \left(\int_{\mathbb{R}^N} |f| dx \right)^{2\gamma}, \quad \gamma = \frac{2}{N+2}. \quad (\text{III.103})$$

one obtains for $N = 3$

$$-\frac{d}{dt} \int_{\Omega} \Gamma^p(t) r dr dz \geq \frac{4(p-1)}{p} C \left(\int_{\Omega} |\Gamma^{\frac{p}{2}}| r dr dz \right)^{-4/3} \left(\int_{\Omega} \Gamma^p r dr dz \right)^{5/3}.$$

To simplify the presentation, setting $E_p(t) = \|\Gamma(t)\|_{\mathbb{R}^3}^p = \int_{\Omega} |\Gamma(t)|^p r dr dz$, then the last inequality becomes

$$-\frac{d}{dt} E_p(t) \geq \frac{4(p-1)}{p} C E_{p/2}^{-4/3}(t) E_p^{5/3}(t) \quad (\text{III.104})$$

We prove (III.101) for $p = 2^n$ with non-negative integers n by induction. Assume that (III.101) is true for $q = 2^k$ with $k \geq 0$, and let $p = 2^{k+1}$. Combined with (III.104)

$$-\frac{d}{dt} E_p(t) \geq \frac{4(p-1)C}{p} (C_q^q t^{-\frac{3}{2}(q-1)} \|\Gamma_0\|_{L^1(\mathbb{R}^3)}^q)^{-4/3} E_p^{5/3}(t).$$

Thus we have

$$\begin{aligned} \frac{3}{2} \frac{d}{dt} \left(E_p(t) \right)^{-2/3} &= \frac{-\frac{d}{dt} E_p(t)}{E_p^{5/3}(t)} \geq \frac{4(p-1)C}{p} C_q^{-4q/3} \|\Gamma_0\|_{L^1(\mathbb{R}^3)}^{-4q/3} t^{2(q-1)} \\ &= \frac{4(p-1)C}{p} C_q^{-2p/3} \|\Gamma_0\|_{L^1(\mathbb{R}^3)}^{-2p/3} t^{(p-2)}. \end{aligned}$$

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Hence, integrating in time the last inequality yields

$$E_p^{-2/3}(t) \geq E_p^{-2/3}(0) + \frac{8C}{3p} C_q^{-2p/3} \|\Gamma_0\|_{L^1(\mathbb{R}^3)}^{-2p/3} t^{p-1}.$$

After a few easy computations, we derive the following

$$\|\Gamma(t)\|_{L^p(\mathbb{R}^3)} = E_p^{\frac{1}{p}}(t) \leq \left(\frac{3p}{8C}\right)^{\frac{3}{2p}} C_q \|\Gamma_0\|_{L^1(\mathbb{R}^3)} t^{-3/2(1-1/p)}.$$

By setting $C_p = \left(\frac{3p}{8C}\right)^{\frac{3}{2p}} C_q$, then (III.101) remains true for $p = 2^{k+1}$. Let us observe that

$$\begin{aligned} C_p = \left(\frac{3p}{8C}\right)^{\frac{3}{2p}} C_q &= \left(\frac{3}{8C}\right)^{\frac{3}{2^{k+2}}} 2^{\frac{3(k+1)}{2^{k+2}}} C_{2^k} \\ &\leq \left(\frac{3}{8C}\right)^{\frac{3}{4} \sum_{k \geq 0} \frac{1}{2^k}} 2^{\frac{3}{4} \sum_{k \geq 0} \frac{k+1}{2^k}} C_1 \stackrel{def}{=} C_\infty \end{aligned}$$

which means that C_∞ is independent of p . Letting $p \rightarrow \infty$, we deduce that

$$\|\Gamma(t)\|_{L^\infty(\mathbb{R}^3)} \leq C_\infty t^{-3/2} \|\Gamma_0\|_{L^1(\mathbb{R}^3)}. \quad (\text{III.105})$$

For the other values of p , we proceed by complex interpolation to get

$$\|\Gamma(t)\|_{L^p(\mathbb{R}^3)} \leq C \|\Gamma(t)\|_{L^1(\mathbb{R}^3)}^{1/p} \|\Gamma(t)\|_{L^\infty(\mathbb{R}^3)}^{1-1/p},$$

combined with (III.105), so the proof is completed. \square

Next, we recall some a priori estimates for ρ -equation in Lebesgue spaces. To be precise, we have.

Proposition III.4.4. *Let $\rho_0 \in L^1(\mathbb{R}^3)$ and $p \in [1, \infty]$, then there exists some nonnegative universal constant $C_p > 0$ depending only on p such that for any smooth solution of ρ -equation in (III.9), we have*

- (i) $\|\rho(t)\|_{L^p(\mathbb{R}^3)} \leq \|\rho_0\|_{L^p(\mathbb{R}^3)},$
- (ii) $\|\rho(t)\|_{L^p(\mathbb{R}^3)} \leq \frac{C_p}{t^{\frac{3}{2}(1-\frac{1}{p})}} \|\rho_0\|_{L^1(\mathbb{R}^3)}.$

Proof

(i) Can be done by a routine computations as shown in Proposition III.4.2, while (ii) can be obtained along the same way as Proposition III.4.3.

We should mention also that the constant C_p is bounded with respect to p (see the proof of Proposition III.4.3), and according to the proof of Proposition III.4.3 C_∞ is given by

$$C_\infty \stackrel{def}{=} \left(\frac{3}{8C}\right) 2^{\frac{3}{4} \sum_{k \geq 0} \frac{k+1}{2^k}} C_1 < \infty \quad (\text{III.106})$$

\square

Now we will prove another type of estimates for the quantities $\tilde{\Gamma}, \tilde{\rho}$ and ω_θ which is the real purpose of this part. Namely, we establish.

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Proposition III.4.5. *Let $\rho_0, \frac{\omega_0}{r} \in L^1(\mathbb{R}^3)$ and $p \in [1, \infty]$, then there exist a non-negative constants \tilde{C}_p, K_p , depending only on p and the initial data, such that for any smooth solution of (III.95), (III.47) and (III.46), we have*

- (i) $\|\tilde{\Gamma}(t)\|_{L^p(\Omega)} \leq \frac{\tilde{C}_p(D_0)}{t^{1-\frac{1}{p}}},$
- (ii) $\|\tilde{\rho}(t)\|_{L^p(\Omega)} \leq \frac{K_p(D_0)}{t^{1-\frac{1}{p}}},$
- (iii) $\|\omega_\theta(t)\|_{L^p(\Omega)} \lesssim \frac{\tilde{C}_p(D_0)+K_p(D_0)}{t^{1-\frac{1}{p}}},$

where

$$D_0 = \|(\omega_0, \rho_0)\|_{L^1(\Omega) \times L^1(\mathbb{R}^3)} \quad (\text{III.107})$$

and

$$\sup_{p \in [1, \infty]} \tilde{C}_p(s) \stackrel{\text{def}}{=} \tilde{C}_\infty(s) < \infty, \quad \tilde{C}_p(s) \rightarrow 0, \quad \text{as } s \uparrow 0, \quad \forall p \in [1, \infty]. \quad (\text{III.108})$$

Proof

Let us point out that (iii) is a consequence of (i) and (ii). Thus, we shall focus ourselves to prove (i) and (ii).

(i) Due to the similarity of the equation of $\tilde{\Gamma}$ and the one of ω_θ for the Navier-Stokes (NS_μ) treated in [56], we follow the approach stated in Proposition 5.3 in [56]. The key point consists to employ the following estimate,

$$\left\| \frac{v^r}{r} \right\|_{L^\infty(\Omega)} \lesssim \frac{1}{t} \|(\omega_0, r\rho_0)\|_{L^1(\Omega)}. \quad (\text{III.109})$$

Indeed, Proposition 2.6 in [56] gives

$$\left\| \frac{v^r}{r} \right\|_{L^\infty(\Omega)} \lesssim \|\omega_\theta\|_{L^1(\Omega)}^{\frac{1}{3}} \left\| \frac{\omega_\theta}{r} \right\|_{L^\infty(\Omega)}^{\frac{2}{3}},$$

by using the fact that

$$\left\| \frac{\omega_\theta}{r} \right\|_{L^\infty(\Omega)} = \left\| \frac{\omega_\theta}{r} \right\|_{L^\infty(\mathbb{R}^3)}$$

combined with $\frac{\omega_\theta}{r} = \frac{\rho}{2} + \Gamma$, together with Propositions III.4.3 and III.4.4, lead to (III.109). So, the inequality (i) follows then by exploring (III.109) and repeating the outlines of the proof of Proposition 5.3 from [56], the details are left to the reader. We should only mention that the constant \tilde{C}_p in our proposition is the same as the one from proposition 5.3 in [56], which guaranties (III.108).

(ii) The estimate obviously holds for $p = 1$, whereas in the rest of the proof we shall deal with $p > 3$. The case $p \in (1, 3]$ follows by interpolation.

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We multiply the $\tilde{\rho}$ -equation by $|\tilde{\rho}|^{p-1}$, after some integration by parts we obtain

$$\frac{1}{p} \frac{d}{dt} \|\tilde{\rho}(t)\|_{L^p(\Omega)}^p \leq -4 \frac{(p-1)}{p^2} \int_{\Omega} |\nabla(|\tilde{\rho}|^{\frac{p}{2}})|^2 drdz + \left| - \int_{\Omega} \operatorname{div}_{\star}(v\tilde{\rho}) |\tilde{\rho}|^{p-1} drdz - \int_{\Omega} \partial_r \rho |\tilde{\rho}|^{p-1} drdz \right|. \quad (\text{III.110})$$

On the one hand, a straightforward computation gives

$$- \int_{\Omega} \operatorname{div}_{\star}(v\tilde{\rho}) |\tilde{\rho}|^{p-1} drdz = \left(1 - \frac{1}{p}\right) \int_{\Omega} \frac{v^r}{r} |\tilde{\rho}|^p drdz,$$

then (III.109) provides

$$- \int_{\Omega} \operatorname{div}_{\star}(v\tilde{\rho}) |\tilde{\rho}|^{p-1} drdz \leq CD_0 \left(1 - \frac{1}{p}\right) t^{-1} \int_{\Omega} |\tilde{\rho}|^p drdz, \quad (\text{III.111})$$

where D_0 is given by (III.107).

On the other hand, the fact $-\partial_r \rho = \frac{-\partial_r \tilde{\rho}}{r} + \frac{\tilde{\rho}}{r^2}$ yields

$$- \int_{\Omega} \partial_r \rho |\tilde{\rho}|^{p-1} drdz = \left(1 - \frac{1}{p}\right) \int_{\Omega} \frac{|\tilde{\rho}|^p}{r^2} drdz.$$

Next, let us write

$$\int_{\Omega} \frac{|\tilde{\rho}|^p}{r^2} drdz = \mathcal{I}_1 + \mathcal{I}_2, \quad (\text{III.112})$$

with

$$\mathcal{I}_1 \stackrel{\text{def}}{=} \int_{\Omega} \frac{|\tilde{\rho}|^p}{r^2} \mathbf{1}_{\{r \leq t^{1/2}\}} drdz, \quad \mathcal{I}_2 \stackrel{\text{def}}{=} \int_{\Omega} \frac{|\tilde{\rho}|^p}{r^2} \mathbf{1}_{\{r > t^{1/2}\}}(r, z) drdz.$$

For $p > 3$ we have,

$$\mathcal{I}_1 = \int_{\Omega} r^{p-3} |\rho|^p \mathbf{1}_{\{r \leq t^{1/2}\}} r drdz \leq t^{\frac{p-3}{2}} \|\rho\|_{L^p(\mathbb{R}^3)}^p.$$

So, by virtue of Proposition III.4.4, we infer that

$$\mathcal{I}_1 \leq C_p^p t^{-p} G_0^p, \quad (\text{III.113})$$

where $G_0 \stackrel{\text{def}}{=} \|\rho_0\|_{L^1(\mathbb{R}^3)}$, and C_p is the constant given by Proposition III.4.4.

For the term \mathcal{I}_2 an easy computation yields

$$\mathcal{I}_1 \leq t^{-1} \int_{\Omega} |\tilde{\rho}|^p drdz. \quad (\text{III.114})$$

Therefore, (III.113) and (III.114) give rise to

$$- \int_{\Omega} \partial_r \rho |\tilde{\rho}|^{p-1} drdz \leq \frac{p-1}{p} \left(C_p^p t^{-p} G_0^p + t^{-1} \int_{\Omega} |\tilde{\rho}|^p drdz \right). \quad (\text{III.115})$$

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Finally, Nash's inequality allows us to write

$$\int_{\Omega} |\tilde{\rho}|^p drdz \lesssim \left(\int_{\Omega} |\nabla(|\tilde{\rho}|^{\frac{p}{2}})|^2 drdz \right)^{\frac{1}{2}} \left(\int_{\Omega} |\tilde{\rho}|^{\frac{p}{2}} drdz \right). \quad (\text{III.116})$$

Since $p > 3$, so we have $1 < \frac{p}{2} < p$, and then by interpolation method, it happens

$$\left(\int_{\Omega} |\tilde{\rho}|^{\frac{p}{2}} drdz \right) \lesssim \left(\int_{\Omega} |\tilde{\rho}| drdz \right)^{\frac{p}{2(p-1)}} \left(\int_{\Omega} |\tilde{\rho}|^p drdz \right)^{\frac{p-2}{2(p-1)}}.$$

Plugging the last inequality in (III.116), it holds

$$\left(\int_{\Omega} |\tilde{\rho}|^p drdz \right)^{\frac{p}{2(p-1)}} \lesssim \left(\int_{\Omega} |\nabla(|\tilde{\rho}|^{\frac{p}{2}})|^2 \right)^{\frac{1}{2}} \left(\int_{\Omega} |\tilde{\rho}| drdz \right)^{\frac{p}{2(p-1)}}.$$

Since the inequality we aim to prove holds for $p = 1$, accordingly

$$CG_0^{-\frac{p}{p-1}} \left(\int_{\Omega} |\tilde{\rho}|^p drdz \right)^{\frac{p}{p-1}} \leq \left(\int_{\Omega} |\nabla(|\tilde{\rho}|^{\frac{p}{2}})|^2 drdz \right). \quad (\text{III.117})$$

Thus, by gathering (III.111), (III.115) and (III.117) and insert them in (III.110), it happens

$$f'(t) \lesssim (p-1) \left(-\frac{C}{p} G_0^{-\frac{p}{p-1}} (f(t))^{\frac{p}{p-1}} + (CD_0 + 1)t^{-1}f(t) + C_p^p t^{-p} G_0^p \right), \quad (\text{III.118})$$

where, $f(t) \stackrel{\text{def}}{=} \int_{\Omega} |\tilde{\rho}(t)|^p drdz$.

We recall that one may deduce from Proposition III.3.2, for all $p \in [1, \infty]$

$$f(t) \leq e_p(D_0)^p t^{-(p-1)}, \quad \forall 0 < t < T^*, \quad (\text{III.119})$$

for some $e_p(D_0) > 0$.

In a first step, we will show that $f(t)$ is finite for all $t > 0$, then we prove that the decay property (III.119) holds as well for all $t > 0$, for a suitable non negative constant $K_p(G_0)$. Indeed, the first step is easy, one should remark that (III.118) implies

$$f'(t) \lesssim (p-1) \left((CD_0 + 1)t^{-1}f(t) + C_p^p t^{-p} G_0^p \right).$$

Via, Gronwall inequality on (t_0, t) , for some $0 < t_0 < T^*$, we get for all $t > t_0$

$$f(t) \leq (f(t_0) + C_p^p t_0^{1-p}) \left(\frac{t}{t_0} \right)^{(p-1)(CD_0+1)}, \quad (\text{III.120})$$

which ensures that $f(t)$ is finite for all $t > 0$.

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Now, let us denote

$$\tilde{T} \stackrel{\text{def}}{=} \sup \{t > 0 : f(t) < K_p^p(D_0)t^{-(p-1)}\}, \quad (\text{III.121})$$

where $K_p(D_0)$ will be chosen later, and we will prove that (III.119) holds as well for all $t \in [\tilde{T}, \tilde{T} + \varepsilon]$, for some $\varepsilon > 0$, this should be enough to contradict the fact that $\tilde{T} < \infty$, and we shall conclude then that (III.119) is true for all $t > 0$. If \tilde{T} is finite then we deduce

$$f(\tilde{T}) = K_p^p(D_0)\tilde{T}^{-(p-1)}. \quad (\text{III.122})$$

Now, define g by

$$g(t) \stackrel{\text{def}}{=} f(t) - K_p^p(D_0)t^{-(p-1)}.$$

By virtue of (III.118) and (III.122), we find out that

$$g'(\tilde{T}) \leq \tilde{T}^{-p}(p-1) \left(-\frac{C}{p} G_0^{-\frac{p}{p-1}} (K_p(D_0))^{\frac{p^2}{p-1}} + K_p^p(D_0) + \Sigma_p(D_0) \right), \quad (\text{III.123})$$

where $\Sigma_p(D_0) = CD_0 + 1 + C_p^p D_0^p$. Since $\frac{p^2}{p-1} > p$, then if we choose $K_p^p(D_0)$ large enough, in terms of $\Sigma_p(D_0)$ and G_0 , we may conclude that

$$g'(\tilde{T}) < 0,$$

which in particular gives, for $\varepsilon \ll 1$

$$g(\tilde{T} + \varepsilon) \leq g(\tilde{T}) = 0.$$

This means that (III.119) holds for $t = \tilde{T} + \varepsilon$, which contradicts the definition of \tilde{T} . The choice of $K_p(D_0)$ can be made as

$$K_p(D_0) = \max \left\{ \left(C^{-1} p G_0^{\frac{p}{p-1}} (CD_0 + 2 + C_p^p D_0^p) \right)^{\frac{1}{p}}, 1, e_p(D_0) \right\}, \quad (\text{III.124})$$

where, $e_p(D_0)$ is the constant from (III.119). Thus, we end with, for all $p > 3$

$$\|\tilde{\rho}(t)\|_{L^p(\Omega)} \leq K_p(D_0)t^{-(1-\frac{1}{p})}. \quad (\text{III.125})$$

By denoting

$$K_\infty(D_0) \stackrel{\text{def}}{=} \lim_{p \rightarrow \infty} \max \left\{ \left(C^{-1} p G_0^{\frac{p}{p-1}} (CD_0 + 2 + C_\infty^p D_0^p) \right)^{\frac{1}{p}}, 1, e_p(D_0) \right\} = \max \{1 + C_\infty D_0, e_\infty(D_0)\}.$$

From proposition III.4.4, C_∞ is finite, hence $K_\infty(D_0)$ is also finite, thus by letting $p \rightarrow \infty$ in (III.125), we end up with

$$\|\tilde{\rho}(t)\|_{L^\infty(\Omega)} \leq K_\infty(D_0)t^{-1}.$$

□

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Remark III.4.1. As pointed out for the Navier-Stokes equations in Remark 5.4 from [56], for the global existence part in our case, we only need to mention that due to Proposition III.4.5 (resp. Proposition III.4.4) the $L^p(\Omega)$ norms of $\omega_\theta(t)$ and $r\rho(t)$ (resp. the $L^p(\mathbb{R}^3)$ of $\rho(t)$) can not blow-up in finite time, hence in view of remark III.3.2, it turns out that any constructed solution in the previous section is global for positive time, in addition of that, all the assertions (III.10)–(III.13) follow as a consequence of propositions III.4.4 and III.4.5.

CHAPTER IV

G.W-P of the axisymmetric Boussinesq system with no diffusion in the ρ -equation

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IV.1 Introduction

IV.1.1 Set-up of the problem

In this Chapter we shall be dealing with two cases: the first one consists in treating the 3D Navier-Stokes equation with general external force with certain regularity, and the second one is about the case where this external force is a scalar quantity, appearing in one direction, transported by the flow of the velocity field v , namely the Boussinesq equation. The axisymmetric Boussinesq system in three dimensions of space with zero diffusivity is a singular perturbation at level zero of the Navier-Stokes equations. Such system reads as follows

$$\begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla p = \rho \vec{e}_3, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \partial_t \rho + v \cdot \nabla \rho = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} v = 0, \\ (v, \rho)|_{t=0} = (v_0, \rho_0). \end{cases} \quad (\mathbf{B}_\mu)$$

Above, $v(t, x) \in \mathbb{R}^3$ refers to the velocity vector field localized in $x \in \mathbb{R}^3$ at a time t , $p(t, x) \in \mathbb{R}$ is the force of internal pressure which acts to enforce the incompressibility constraint $\operatorname{div} v = 0$ and it may be determined by a type of Poisson's equation and $\rho(t, x) \in \mathbb{R}_+$ stands either the temperature in the context of thermal convection or the mass density in the modeling of geophysical fluids. The condition $\operatorname{div} v = 0$, means that the volume of the fluid elements does not change (iso-volume) over time and $\mu > 0$ is well-known as the 'kinematic' viscosity's parameter of the fluid.

The key assumption of the Boussinesq system (\mathbf{B}_μ) is that the variations in density are small and their effect is neglected everywhere except in the buoyancy term $\rho \vec{e}_3$ that driving the fluid motion in the direction $\vec{e}_3 = (0, 0, 1)$.

The Boussinesq system occurs principally in the dynamics of geophysical fluids which illustrates many models coming from atmospheric or oceanographic turbulence where rotation and stratification play an important role, see e.g. [107].

Mathematically, this model is essentially limited to the case of dimension two due to a formal resemblance (in the case $\mu = 0$) with three-dimensional axisymmetric swirling flows. It can be shown that the fact that the solution develops singularities at time t is related at the simultaneous blow-up of $\nabla \rho$ and the vorticity in $L_t^1 L^\infty$, see [119]. Unfortunately, determining whether these quantities actually blow-up seems at least as difficult as addressing the similar problem for the system of Euler incompressible in dimension three of spaces.

For better understanding the analysis of (\mathbf{B}_μ), we come first to the particular case which we assume that the density is constant. Thus, we obtain the classical Navier-Stokes equations

$$\begin{cases} \partial_t v + v \cdot \nabla v - \mu \Delta v + \nabla p = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} v = 0, \\ v|_{t=0} = v_0. \end{cases} \quad (\text{NS}_\mu)$$

The significant leap was J. Leray's paper [96] in the thirties of last century where succeeded to build a weak family of solutions to (NS_μ) globally in time in the energy space for any dimension, via compactness method. Even though, the uniqueness issue is known only in dimension two. Afterwards, H. Fujita and T. Kato formulated in [45] a mild-solutions for (NS_μ) locally in time for initial data in critical Sobolev spaces $\dot{H}^{\frac{N}{2}-1}(\mathbb{R}^N)$ which respect the so-called invariant by scaling. A similar topic are developed in several functional spaces alike $L^N(\mathbb{R}^N)$, $\dot{B}_{p,\infty}^{-1+\frac{N}{p}}$ and BMO^{-1} , for other connected subjects we refer to [56, 77, 80, 86, 89, 94, 113, 109]. It should be noted that these kinds of solutions are globally in time apart from small initial data with respect to the viscosity parameter, except in dimension two. Thus, in general setting the global well-posedness for (NS_μ) is a till now an open problem in PDEs. For this purpose, it is legitimate to search the "best" conditions that guarantee the existence and uniqueness of a solution for (NS_μ) . This latter gave the opportunity to many authors to require that the velocity enjoying a geometric condition like *axisymmetric without swirl*. Namely, the velocity vector field can be splitted in cylindrical basis of \mathbb{R}^3 in the following way.

$$v(t, x) = v^r(t, r, z)\vec{e}_r + v^z(t, r, z)\vec{e}_z,$$

where for every $x = (x_1, x_2, z) \in \mathbb{R}^3$ we have

$$x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad r \geq 0, \quad 0 \leq \theta < 2\pi.$$

Above, the triplet $(\vec{e}_r, \vec{e}_\theta, \vec{e}_z)$ represents the usual frame of unit vectors in the radial, azimuthal and vertical directions with the notation

$$\vec{e}_r = \left(\frac{x_1}{r}, \frac{x_2}{r}, 0 \right), \quad \vec{e}_\theta = \left(-\frac{x_2}{r}, \frac{x_1}{r}, 0 \right), \quad \vec{e}_z = (0, 0, 1).$$

Since, the vorticity in \mathbb{R}^3 is defined by $\omega = \operatorname{curl} v = \nabla \times v$, so if we apply the curl operator to (NS_μ) we obtain that the vorticity takes the form $\omega \stackrel{\text{def}}{=} \omega_\theta \vec{e}_\theta$ with $\omega_\theta = \partial_z v^r - \partial_r v^z$ solves the following nonlinear parabolic equation

$$\partial_t \omega_\theta + (v \cdot \nabla) \omega_\theta - \mu \left(\Delta \omega_\theta - \frac{\omega_\theta}{r^2} \right) = \frac{v^r}{r} \omega_\theta, \quad (\text{IV.1})$$

with the notation $v \cdot \nabla = v^r \partial_r + v^z \partial_z$ and $\Delta = \partial_r^2 + \frac{\partial_r}{r} + \partial_z^2$. Deposit $\zeta = \frac{\omega_\theta}{r}$, a straightforward computations lead to

$$\begin{cases} \partial_t \zeta + v \cdot \nabla \zeta - \mu \left(\Delta + \frac{2}{r} \partial_r \right) \zeta = 0, \\ \zeta|_{t=0} = \zeta_0. \end{cases} \quad (\text{IV.2})$$

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The fact that, the dissipative operator $(\Delta + \frac{2}{r}\partial_r)$ has a good sign, consequently, we may estimate L^p -norms of ζ globally in time, that is

$$\|\zeta(t)\|_{L^p} \leq \|\zeta_0\|_{L^p}, \quad p \in [1, \infty], \quad t \geq 0. \quad (\text{IV.3})$$

In fact, M. Ukhovskii and V. Yudovich [116] independently O. Ladyzhenskaya [91] showed that the bound (IV.3) is considered as a bulwark to prohibit the formation of singularities in finite time for axisymmetric flows without swirl. More precisely, global existence and uniqueness were established for axisymmetric initial data $v_0 \in H^1$ and $\omega_0, \frac{\omega_0}{r} \in L^2 \cap L^\infty$. Lately, this latter was improved by S. Leonardi, J. Målek, J. Necās and M. Pokorný [95] for $v_0 \in H^2$ and external force $f \in L^2_{loc}(\mathbb{R}_+; H^2)$ in the sense that $v_0 \in H^2$ regains that $\omega_0, \frac{\omega_0}{r} \in L^2 \cap L^\infty$. Fairly recent, H. Abidi [3] relaxed the last result by assuming that v_0 is in the critical Sobolev space $H^{\frac{1}{2}}$ and $f \in L^2_{loc}(\mathbb{R}_+; H^\beta)$ with $\beta > \frac{1}{4}$. For critical regularities, that is $v_0 \in B_{p,1}^{-1+\frac{3}{p}}$, with $1 \leq p \leq \infty$ and $\frac{\omega_0}{r} \in L^{3,1}$, the second author and Hmidi investigated in [77] that (NS_μ) admits a unique global solution uniformly with respect to the viscosity in the absence of Beale-Kato-Majda criterion, see [9]. Furthermore, they also provided the rate of convergence whenever the viscosity goes to zero.

For the Boussinesq system (B_μ) , an intensive attention has been gained to the local/global well-posedness problem. In particular, in two dimensions of space and its dissipative counterpart phenomena for the density, refer to [4, 17, 21, 22, 23, 35, 64, 70, 71, 72, 76, 78, 79, 41, 98, 100, 119, 121] for a more complete history and references. For three-dimensional, the global regularity issue for the system (B_μ) in various spaces has received a considerable attention and widely studied in the last two decades. Worth mentioning, R. Danchin and M. Paicu showed in [36] that (B_μ) is well-posed in time for Leray's and Fujita-Kato's solutions in any dimension. Next, in the axisymmetric case, H. Abidi, T. Hmidi and S. Keraani have established in [5] that (B_μ) possesses a unique global solution by rewriting it under vorticity-density formulation.

$$\begin{cases} \partial_t \omega_\theta + v \cdot \nabla \omega_\theta - (\Delta - \frac{1}{r^2}) \omega_\theta = \frac{v^r}{r} \omega_\theta - \partial_r \rho, \\ \partial_t \rho + v \cdot \nabla \rho = 0, \\ (\omega_\theta, \rho)|_{t=0} = (\omega_0, \rho_0), \end{cases} \quad (\text{IV.4})$$

Consequently, the quantity $\zeta = \frac{\omega_\theta}{r}$ solves the equation

$$\partial_t \zeta + v \cdot \nabla \zeta - (\Delta + \frac{2}{r} \partial_r) \zeta = -\frac{\partial_r \rho}{r}. \quad (\text{IV.5})$$

They assumed that $v_0 \in H^1(\mathbb{R}^3)$, $\xi_0 \in L^2(\mathbb{R}^3)$, $\rho_0 \in L^2 \cap L^\infty$ with $\text{supp } \rho_0 \cap (Oz) = \emptyset$ and $P_z(\text{supp } \rho_0)$ is a compact set in \mathbb{R}^3 especially for dismissing the singularity $\frac{\partial_r \rho}{r}$, with P_z being the orthogonal projector over (Oz) . Those results were enhanced later by T. Hmidi and F. Rousset in [74] by removing the assumption on the support of density. Their paradigm requires to assume that $(v_0, \rho_0) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3) \cap B_{3,1}^0$ and $\frac{\omega_0}{r} \in L^2(\mathbb{R}^3)$ and couples the two-equations of (IV.4) by introducing a new unknown which called *coupled function*.

The overall aim here is to show first that (NS_μ) admits a unique global solution once $v_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$ is an axisymmetric vector field and f in $L_{loc}^\beta(\mathbb{R}_+, L^2(\mathbb{R}^3))$, with $\beta > 4$. Second, we exploit the previous result to prove that the system (B_μ) has a unique global solution whenever $v_0 \in H^{\frac{1}{2}}(\mathbb{R}^3) \cap \dot{B}_{3,1}^0(\mathbb{R}^3)$ and $\rho_0 \in L^2(\mathbb{R}^3) \cap \dot{B}_{3,1}^0(\mathbb{R}^3)$. This kind of result is considered as novel in the sense that interpolating the critical regularities following R. Danchin and M. Paicu [36] and the axisymmetric constraint of T. Hmidi and F. Rousset [74].

IV.1.2 Statement of the main results

Throughout this Chapter, we denote by C a positive constant which may be different in each occurrence, but it does not depend on the initial data. We shall sometimes alternatively use the notation $X \lesssim Y$ for an inequality of type $X \leq CY$ with C independent of X and Y . The notation C_0 means that the constant depends on the involved norms of the initial data.

In this Chapter, we will also be using some homogeneous Besov spaces of the form $\dot{B}_{p,r}^s(\mathbb{R}^3)$, we refer to the Appendix part for the definition and more details.

For $s \in \mathbb{R}$, the standard Sobolev spaces denoted by $H^s(\mathbb{R}^3)$ is defined as the set of all tempered distributions over \mathbb{R}^3 be such that

$$\|v\|_{H^s}^2 = \int_{\mathbb{R}^3} (1 + |\xi|^2)^s |\widehat{u}(\xi)|^2 d\xi < \infty.$$

Likewise, the homogeneous standard Sobolev spaces designated by $\dot{H}^s(\mathbb{R}^3)$ are defined as the set of all tempered distributions over \mathbb{R}^3 be such that $\widehat{u} \in L_{loc}^1$ and

$$\|v\|_{\dot{H}^s}^2 = \int_{\mathbb{R}^3} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi < \infty.$$

The scalar product in H^s and \dot{H}^s are denoted respective by $\langle \cdot | \cdot \rangle_s$ and $\langle \cdot | \cdot \rangle_s$, whereas the case $s = 0$ will be simply denoted by $\langle \cdot | \cdot \rangle$.

About Navier-Stokes equations with external force.

The incompressible Navier-Stokes equations with the external force in the whole space with the viscosity $\mu = 1$, still noted (NS) reads as follows.

$$\begin{cases} \partial_t v + v \cdot \nabla v - \Delta v + \nabla p = f & \text{if } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} v = 0, \\ v|_{t=0} = v_0. \end{cases} \quad (\text{NS})$$

If X is a spacial space, and $X^{(1)}$ is the space of tempered distributions U such that ∇U belongs to X , then we denote the energy space associated to X by

$$E_T(X) \stackrel{\text{def}}{=} L_T^\infty(X) \cap L_T^2(X^{(1)}).$$

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The energy inequality associated to (NS) reads for all $T > 0$ as follows

$$\|v\|_{L_T^\infty(L^2)} + \|\nabla v\|_{L_T^2(L^2)} \lesssim \|v_0\|_{L^2} + \|f\|_{L_T^2(\dot{H}^{-1})}, \quad \text{if } f \in L_T^2(\dot{H}^{-1}).$$

and

$$\|v\|_{L_T^\infty(L^2)} + \|\nabla v\|_{L_T^2(L^2)} \lesssim \|v_0\|_{L^2} + (1 + \sqrt{T})\|f\|_{L_T^2(H^{-1})}, \quad \text{if } f \in L_T^2(H^{-1}).$$

The scenario of mild solutions will be done by rewriting (NS) in terms of a fixed point problem for the heat semi-group, and thus obtain a unique solution v which belongs to a functional space, for example $\dot{H}^s(\mathbb{R}^N)$, such that the linear term $(\partial_t - \Delta)v$ and the nonlinear term $\mathbf{P}\text{div}(v \otimes v)$ have the same regularity, with \mathbf{P} designates the Leray's projector which acts on the divergence free vector field. This involves the scaling invariance for (NS). Indeed, if we denote by v the solution of (NS) with data v_0 and f . Thus, for $\lambda > 0$ the vector field

$$v_\lambda : (t, x) \mapsto \lambda v(\lambda^2 t, \lambda x)$$

with data

$$v_{0,\lambda} : x \mapsto \lambda v_0(\lambda x), \quad f_\lambda : (t, x) \mapsto \lambda^3 f(\lambda^2 t, \lambda x)$$

is also a solution of (NS).

It is simple to check that the Sobolev regularities according to Fujita-Kato for $N = 3$ are $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ for $v_{0,\lambda}$, $L^2(\mathbb{R}_+; \dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))$ for f_λ and $C(\mathbb{R}_+; \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbb{R}_+; \dot{H}^{\frac{3}{2}})$ for v_λ . Consequently, Fujita-Kato [45] succeed to recover (NS) globally in time in the following way.

Proposition IV.1.1. *Let $v_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $f \in L_T^2(\dot{H}^{-\frac{1}{2}}(\mathbb{R}^3))$, there exists some $c_0 > 0$ such that if*

$$\|v_0\|_{\dot{H}^{\frac{1}{2}}} + \|f\|_{L_T^2(\dot{H}^{-\frac{1}{2}})} < c_0.$$

Then (NS) has a unique global solution satisfying for all $T > 0$,

$$\|v\|_{L_T^\infty(\dot{H}^{\frac{1}{2}})} + \|\nabla v\|_{L_T^2(\dot{H}^{\frac{1}{2}})} \lesssim \|v_0\|_{\dot{H}^{\frac{1}{2}}} + \|f\|_{L_T^2(\dot{H}^{-\frac{1}{2}})}.$$

Without smallness condition, H. Abidi [3] developed another result of global wellposedness in time for (NS_μ) , by interpolating the critical space of Fujita-Kato and the axisymmetric condition of Målek- Necăs- Pokorný. Especially he proved the following theorem.

Theorem IV.1.1. *Let $v_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$ be an axisymmetric divergence free vector field vector without swirl and $f \in L_{loc}^2(\mathbb{R}_+; H^\beta(\mathbb{R}^3))$ be an axisymmetric vector field, for some $\beta > \frac{1}{4}$. Then (NS) has a unique global axisymmetric satisfying*

$$v \in C(\mathbb{R}_+; H^{\frac{1}{2}}) \cap L_{loc}^2(\mathbb{R}_+; H^{\frac{3}{2}}).$$

The approach suggested by H. Abidi to prove Theorem IV.1.1 is deeply based on the argument introduced by C. P. Calderón [14] to show that the Navier-Stokes equations

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admit a solution in L^p for $2 < p < N$ with $N = 3, 4, \dots$ and further performed by I. Gallagher and F. Planchon [54] with the purpose to study the global existence for the wave equations. This argument consists to split the data v_0 into two regular parts, the first one is more regular, denoted by $v_{0,\ell}$ and supported spectrally in the ball of radius N_0 , with N_0 being a large real number, the other one is less regular but small denoted by $v_{0,h}$. Likewise, for the external $f \equiv f_\ell + f_h$, with the subscripts h and ℓ refer respectively the high and low frequencies. More precisely, we write.

$$\begin{aligned} v_{0,\ell} &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{|\xi| \leq N_0} \widehat{v}_0), & v_{0,h} &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{|\xi| \geq N_0} \widehat{v}). \\ f_\ell &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{|\xi| \leq N_0} \widehat{f}), & f_h &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbf{1}_{|\xi| \geq N_0} \widehat{f}). \end{aligned}$$

Since v_0 and f are axisymmetric, it is clear that the functions $v_{0,\ell}, v_{0,h}, f_\ell$ and f_h are also because the cut-off operator doesn't disturb this structure.

Compared to Proposition IV.1.1 it seems that Theorem IV.1.1 is not optimal in terms of the required regularity on f , whereas the optimal regularity on f would be something like $f \in L_T^2(H^{-\frac{1}{2}})$. For the Boussinesq system, the approach used in [3] isn't applicable directly if $f \in L_T^2(H^\beta)$, with $\beta \leq \frac{1}{4}$, instead of that, we will relax the regularity with respect to β by means of asking for some more integrability in time. So, the general idea is as follows: for $f \in L_{loc}^2(\mathbb{R}_+; L^2)$ and $\alpha \leq 0$, we have

$$\|f_h\|_{L_T^2(H^\alpha)} \leq \varepsilon_{N_0, \alpha},$$

with

$$\varepsilon_{N_0, \alpha} = \begin{cases} N_0^\alpha \|f\|_{L_T^2(L^2)} & \text{if } \alpha < 0 \\ \left(\int_0^T \|\mathbf{1}_{|\xi| \geq N_0} \widehat{f}(\tau, \xi)\|_{L^2(\mathbb{R}^3)}^2 d\tau \right)^{\frac{1}{2}} & \text{if } \alpha = 0. \end{cases}$$

If N_0 is sufficiently large, then $\varepsilon_{N_0, \alpha}$ is small as much as we want, and eventually the $L_T^2(H^\alpha)$ -norm of f_h as well, for all $\alpha \in [-\frac{1}{2}, 0]$. Then, for the data f_h we define v_h as the solution of the following Navier-Stokes equations

$$\begin{cases} \partial_t v_h + v_h \cdot \nabla v_h - \Delta v_h + \nabla p_h = f_h \\ \operatorname{div} v_h = 0 \\ v_h|_{t=0} = 0. \end{cases} \quad (\text{NS}_h)$$

In the spirit of Proposition IV.1.1 for (NS_h) , we will prove the following result.

Proposition IV.1.2. *There exists $N_0 > 0$ large enough and ε small as much as we want in terms of N_0 for which (NS_h) has a unique global solution v_h in $E_T(\dot{H}^{\frac{1}{2}}) \cap E_T(\dot{H}^1)$, with*

$$\|v_h\|_{E_T(\dot{H}^{\frac{1}{2}}) \cap E_T(\dot{H}^1)} \lesssim \varepsilon.$$

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Next, we demonstrate that (NS) admits a unique global solution v of the form $v = v_\ell + v_h$ with v_ℓ satisfying the following modified system

$$\begin{cases} \partial_t v_\ell + v_\ell \cdot \nabla v_\ell - \Delta v_\ell + \nabla p_\ell = f_\ell + F_{v_h}(v_\ell) \\ \operatorname{div} v_\ell = 0 \\ v_{\ell, t=0} = v_0, \end{cases} \quad (\text{NS}_\ell)$$

where the operator F_a is given by

$$F_a(b) \stackrel{\text{def}}{=} -a \cdot \nabla b - b \cdot \nabla a \quad (\text{IV.6})$$

Unless $F_{v_h}(v_\ell) = 0$, it is not clear how to solve (NS_ℓ) by means of the previous proposition with an external force term as a sum of f_ℓ in $L^2(H^1)$ and a linear operator F_{v_h} . In fact, Proposition IV.1.2 is not enough to establish the desired global control in time of the H^1 -norm of the modified system (NS_ℓ), the issue is related to the fact that having a $L_T^\infty(\dot{H}^{\frac{3}{2}})$ estimate of the small solution (v_h in our context) is crucial to deal with the modified system. To remedy this, we will have to demand more regularity on v_h which comes from additional conditions on the external force f . We recall that in this work we will later treat the Boussinesq equations as well, so we want to keep the regularity zero with respect to the spacial variable, whereas we will add some integrability conditions with respect to time. Hence, we shall prove the following proposition.

Proposition IV.1.3. *Let v_h be the solution obtained by Proposition IV.1.2, if in addition f is in $L_T^\beta(L^2)$ for some $\beta > 4$. Then v_h is actually in $L_T^\infty(\dot{H}^{\frac{3}{2}})$, with*

$$\|v_h\|_{L_T^\infty(\dot{H}^{\frac{3}{2}})} \lesssim \varepsilon.$$

After this extensive explanation and a stack of important results, now we are ready to state our first main result which deals by establishing the following version of Theorem IV.1.1.

Theorem IV.1.2. *Let $v_0 \in H^{\frac{1}{2}}(\mathbb{R}^3)$ be an axisymmetric divergence free vector field vector without swirl and $f \in L_{loc}^\beta(\mathbb{R}_+; L^2(\mathbb{R}^3))$ be an axisymmetric vector field vector, for some $\beta > 4$. Then (NS) admits a unique global axisymmetric solution such that*

$$v \in C(\mathbb{R}_+; H^{\frac{1}{2}}) \cap L_{loc}^2(\mathbb{R}_+; H^{\frac{3}{2}}).$$

Remark IV.1.3. Let us point out that, the condition $\beta > 4$ is essentially entailed to prove that the solution v_h of (NS_h) is in $L_T^\infty(\dot{H}^{\frac{3}{2}})$, which is really important to get the global bound of the modified system. Equivalently, any couple of real numbers $(\alpha, \beta) \in (-\frac{1}{2}, \infty) \times [2, \infty]$ such that the condition $f \in L_T^\beta(H^\alpha)$ allows to get a solution v_h to (NS_h) in $E_T(H^{\frac{1}{2}} \cap H^1) \cap L_T^\infty(\dot{H}^{\frac{3}{2}})$, is admissible ¹ to reinstate the condition $f \in L_T^\beta(L^2)$ in Theorem IV.1.2.

¹One may check that our proof works in particular if $f \in L_T^\beta(H^\alpha)$ such that $\alpha + \frac{1}{2} > \frac{2}{\beta}$ and $\beta \in [2, \infty]$, this condition comes from the proof of Proposition IV.1.3.

About the Boussinesq system

Now, let us move to the Boussinesq system (B_μ) where we set the viscosity $\mu = 1$ to simplify the presentation, the system often obtained still denoted by (B) and given by the following coupled equations.

$$\begin{cases} \partial_t v + v \cdot \nabla v - \Delta v + \nabla p = \rho \vec{e}_3, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \partial_t \rho + v \cdot \nabla \rho = 0, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} v = 0, \\ (v, \rho)|_{t=0} = (v_0, \rho_0). \end{cases} \quad (\text{B})$$

Let us recall that the system (B) respects the following scaling invariance

$$v_\lambda : (t, x) \mapsto \lambda v(\lambda^2 t, \lambda x), \quad \rho_\lambda : (t, x) \mapsto \lambda^3 \rho(\lambda^2 t, \lambda x)$$

with initial data $(v_{0,\lambda}, \rho_{0,\lambda})$ given by $v_{0,\lambda}(x) = \lambda v_0(\lambda x)$ and $\rho_{0,\lambda}(x) = \lambda^3 \rho_0(\lambda x)$. In other words, the critical spaces for the velocity are the same as for the Navier-Stokes system, and it is necessary to require two less derivatives on the density. A straightforward computation claims that $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $\dot{B}_{3,1}^0(\mathbb{R}^3)$ are critical for the velocity, whereas $L^1(\mathbb{R}^3)$ is critical but for the density.

The second main result of this work handles with the global well-posedness for (B) and reads as follows.

Theorem IV.1.4. *Let $v_0 \in H^{\frac{1}{2}}(\mathbb{R}^3) \cap \dot{B}_{3,1}^0(\mathbb{R}^3)$ be an axisymmetric divergence free vector field without swirl and $\rho_0 \in L^2(\mathbb{R}^3) \cap \dot{B}_{3,1}^0(\mathbb{R}^3)$ be a scalar axisymmetric function. Then (B) has a unique global solution satisfying for all $T > 0$*

$$(v, \rho) \in E_T(H^{\frac{1}{2}}) \cap L_T^\infty(\dot{B}_{3,1}^0) \cap L_T^1(\dot{B}_{3,1}^2) \times L_T^\infty(L^2 \cap \dot{B}_{3,1}^0).$$

A bunch of important remarks concerning the previous theorem are in order.

Remark IV.1.5. An axisymmetric scalar function means a function that depends only on the variable (r, z) but not on the angle variable θ in cylindrical coordinates. We check obviously that the axisymmetric structure is preserved through the time in the way that if (v_0, ρ_0) is axisymmetric without swirl, then the obtained solution is it also.

Remark IV.1.6. As aforementioned above $H^{\frac{1}{2}}(\mathbb{R}^3) \subset \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $\dot{B}_{3,1}^0(\mathbb{R}^3)$ are critical spaces with respect to the velocity, whereas $L^2(\mathbb{R}^3)$ and $\dot{B}_{3,1}^0(\mathbb{R}^3)$ are not critical ones for the density, as pointed out, for instance, in [36] it is not clear how to solve the equations in critical spaces with respect to the density without additional required regularity on the initial data, our choice of these additional conditions is highly inspired from [36, 74] where the L^2 condition on ρ_0 seems to be crucial to solve the momentum equation in $H^{\frac{1}{2}}$, and the condition $\dot{B}_{3,1}^0$ is essentially helpful for the uniqueness.

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Let us briefly discuss the proof of Theorem IV.1.4, first by describing the formal energy inequalities associated to our system. To do this, let $(v, \rho, \nabla p)$ be a regular solution for Boussinesq system which decreasing at infinity with initial data (v_0, ρ_0) . First of all, we recall that the velocity vector field is in divergence free, so

$$\|\rho(t)\|_{L^p} = \|\rho_0\|_{L^p}, \quad p \in [1, \infty], \quad t \geq 0.$$

In particular, for all $r \in [1, \infty]$ we have

$$\|\rho\|_{L_T^r(L^2)} = T^{\frac{1}{r}} \|\rho_0\|_{L^2}. \quad (\text{IV.7})$$

and thus

$$\|v\|_{L_T^\infty(L^2)} + \|\nabla v\|_{L_T^2(L^2)} \lesssim \|v_0\|_{L^2} + T^{\frac{1}{2}} \|\rho_0\|_{L^2}.$$

By virtue of Theorem 1.1 from [36], the assumption v_0, ρ_0 in $L^2(\mathbb{R}^3)$ ensures the existence of at least one global weak solution (v, ρ) , that is, in view of (IV.7), ρ is in $L_T^\beta(L^2)$, for $\beta > 4$. Consequently, in accordance with Theorem IV.1.2 we may define the unique global axisymmetric solution v to the equation (NS) with $f = \rho \vec{e}_3$. Next step consists to propagate the regularity alike $v \in L_t^1(Lip)$ which seems to be crucial, according to [36], to prove the uniqueness of the constructed solution for (B). But the lack of diffusion term in the density equation contributes some technical drawback. To circumvent these difficulties we will need to supply some additional summability condition at the level of the dyadic decomposition of the (v_0, ρ_0) , rather than being just in $B_{2,2}^{\frac{1}{2}} \times B_{2,2}^0$, namely we will need them to be in $B_{3,1}^0 \times B_{3,1}^0$. Theorem IV.1.4 is then a consequence of the following proposition.

Proposition IV.1.4. *Let $T > 0$ and (v, ρ) be a solution to (B) on $(0, T)$ in $E_T(\dot{H}^{\frac{1}{2}}) \times L_T^\infty(L^2)$. If in addition $(v_0, \rho_0) \in \dot{B}_{3,1}^0 \times \dot{B}_{3,1}^0$, then*

$$(v, \rho) \in L_T^\infty(\dot{B}_{3,1}^0) \cap L_T^1(\dot{B}_{3,1}^2) \times L_T^\infty(\dot{B}_{3,1}^0)$$

and (v, ρ) is actually the unique solution to (B) on $(0, T)$.

IV.2 About the axisymmetric Navier-Stokes equations with external force term

IV.2.1 Some key estimates in Sobolev and Besov spaces

To state the proof of the main results, we first need to analyze some classical estimates in Sobolev and Besov spaces that will be more useful in the proof of our main theorems, we begin with proving the following proposition

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Proposition IV.2.1. *For $s \in [\frac{1}{2}, \frac{3}{2})$, there exists $C > 0$ such that for a, b and c are regular enough, the following assertions hold.*

$$\langle a \cdot \nabla c | b \rangle_s \leq C \|a\|_{\dot{H}^{\frac{1}{2}}}^2 \|\nabla c\|_{\dot{H}^s}^2 + \frac{1}{100} \|\nabla b\|_{\dot{H}^s}. \quad (\text{IV.8})$$

$$\langle a \cdot \nabla b | b \rangle_s \leq C \|a\|_{\dot{H}^{\frac{1}{2}}} \|\nabla b\|_{\dot{H}^s}^2 \quad (\text{IV.9})$$

$$\langle b \cdot \nabla a | b \rangle_s \leq \|\nabla a\|_{\dot{H}^{\frac{1}{2}}}^2 \|b\|_{\dot{H}^s}^2 + \frac{1}{100} \|\nabla b\|_{\dot{H}^s}^2. \quad (\text{IV.10})$$

$$\langle a \cdot \nabla b | b \rangle_s \leq C \|a\|_{\dot{H}^1}^4 \|b\|_{\dot{H}^s}^2 + \frac{1}{100} \|\nabla b\|_{\dot{H}^s}^2. \quad (\text{IV.11})$$

For $s = \frac{3}{2}$, we have

$$\langle \operatorname{div}(a \otimes b) | b \rangle_{\frac{3}{2}} + \langle \operatorname{div}(b \otimes a) | b \rangle_{\frac{3}{2}} \leq C (\|a\|_{\dot{H}^1}^2 \|b\|_{\dot{H}^2}^2 + \|b\|_{\dot{H}^1}^2 \|a\|_{\dot{H}^2}^2) + \frac{1}{100} \|\nabla b\|_{\dot{H}^{\frac{3}{2}}}^2. \quad (\text{IV.12})$$

And for $s = 2$, we have

$$\langle a \cdot \nabla b | c \rangle_2 \lesssim \|a\|_{\dot{H}^1} (\|\nabla b\|_{\dot{H}^{\frac{1}{2}}} + \|\nabla b\|_{\dot{H}^2}) \|c\|_{\dot{H}^3} \quad (\text{IV.13})$$

$$\langle a \cdot \nabla b | c \rangle_2 \lesssim (\|a\|_{\dot{H}^{\frac{1}{2}}} + \|a\|_{\dot{H}^2}) \|\nabla b\|_{\dot{H}^1} \|c\|_{\dot{H}^3} \quad (\text{IV.14})$$

Proof

We restrict ourselves to establish (IV.8). Estimates (IV.9) and (IV.10) will be done by the same method. By definition we have

$$\begin{aligned} \langle a \cdot \nabla c | b \rangle_s^2 &\leq \int_{\mathbb{R}^3} |\xi|^{2s} |(\widehat{a \cdot \nabla c})(\xi)| |\widehat{b}(\xi)| d\xi \\ &\leq \left(\int_{\mathbb{R}^3} |\xi|^{2(s-1)} |\widehat{a \cdot \nabla c}(\xi)|^2 d\xi \right)^{1/2} \left(\int_{\mathbb{R}^3} |\xi|^{2(s+1)} |\widehat{b}(\xi)|^2 d\xi \right)^{1/2} \\ &\leq \|a \cdot \nabla c\|_{\dot{H}^{s-1}} \|b\|_{\dot{H}^{s+1}}. \end{aligned}$$

On the other hand, the product law in dimension three $\dot{H}^s \cdot \dot{H}^{\frac{1}{2}} \subset \dot{H}^{s-1}$, for $s \in [\frac{1}{2}, \frac{3}{2})$, gives

$$\|a \cdot \nabla c\|_{\dot{H}^{s-1}} \leq C \|a\|_{\dot{H}^{\frac{1}{2}}} \|\nabla c\|_{\dot{H}^s}.$$

Putting together the last two estimates to obtain the scalar product $\langle \cdot | \cdot \rangle_s$ in \dot{H}^s ,

$$\langle a \cdot \nabla c | b \rangle_s \leq C \|a\|_{\dot{H}^{\frac{1}{2}}} \|\nabla c\|_{\dot{H}^s} \|\nabla b\|_{\dot{H}^s}.$$

So, Young's inequality $AB \leq CA^2 + \frac{1}{100}B^2$ gives the desired estimate.

The proof of (IV.11) is ensued from the following product law,

$$\dot{H}^{s-\frac{1}{2}} \cdot \dot{H}^1 \subset \dot{H}^{s-1} \quad \text{for all } s \in [\frac{1}{2}, \frac{3}{2}).$$

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Indeed, the scalar product in \dot{H}^s together with the product law above give rise to

$$\langle a \cdot \nabla b | b \rangle_s \lesssim \|a\|_{\dot{H}^1} \|\nabla b\|_{\dot{H}^{s-\frac{1}{2}}} \|b\|_{\dot{H}^{s+1}}$$

Next, in view of the following interpolation inequality in general case

$$\|f\|_{\dot{H}^{s+\frac{1}{2}}} \lesssim \|f\|_{\dot{H}^s}^{\frac{1}{2}} \|f\|_{\dot{H}^{s+1}}^{\frac{1}{2}}$$

and a new use of Young inequality $AB \leq CA^4 + \frac{1}{100}B^{\frac{4}{3}}$ implies

$$\langle a \cdot \nabla b | b \rangle_s \leq C \|a\|_{\dot{H}^1}^4 \|b\|_{\dot{H}^s}^2 + \frac{1}{100} \|\nabla b\|_{\dot{H}^s}^2.$$

To establish (IV.12) for $s = \frac{3}{2}$, we apply Bony's decomposition

$$a \otimes b = T_a b + T_b a + R(a, b).$$

The two first terms can be estimated along the same lines as follows

$$\begin{aligned} \|\Delta_j T_a b\|_{L^2} &\lesssim \|S_{j-1} a\|_{L^\infty} \|\Delta_j b\|_{L^2} \\ &\lesssim c_j 2^{-\frac{3}{2}j} \|a\|_{\dot{B}_{\infty, \infty}^{-\frac{1}{2}}} \|b\|_{\dot{H}^2}, \quad \text{with } \sum_{j \in \mathbb{Z}} c_j^2 \leq 1. \end{aligned}$$

Hence the Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow \dot{B}_{\infty, \infty}^{-\frac{1}{2}}(\mathbb{R}^3)$ gives the desired estimate for $T_a b$.

For the remainder term, Bernstein inequality and the definition of Besov spaces allow to conclude

$$\begin{aligned} \|\Delta_j R(a, b)\|_{L^2} &\lesssim 2^{\frac{3}{2}j} \sum_{k \geq j+N_0} \|\tilde{\Delta}_k a\|_{L^2} \|\Delta_k b\|_{L^2} \\ &\lesssim 2^{\frac{3}{2}j} \sum_{k \geq j+N_0} c_k 2^{-3k} \|a\|_{\dot{H}^1} \|b\|_{\dot{B}_{2, \infty}^2} \\ &\lesssim 2^{-\frac{3}{2}j} \sum_{k \geq j+N_0} c_k 2^{3(j-k)} \|a\|_{\dot{H}^1} \|b\|_{\dot{H}^2} \end{aligned}$$

thus, by using Hölder inequality, we finally obtain

$$\|\Delta_j R(a, b)\|_{L^2} \lesssim 2^{-\frac{3}{2}j} c_j \|a\|_{\dot{H}^1} \|b\|_{\dot{H}^2}.$$

Finally, we only prove (IV.13) because (IV.14) will be done by the same fashion and we left it to the reader. The definition of the scalar product in \dot{H}^2 gives

$$\langle a \cdot \nabla b | c \rangle_2 \lesssim \|a \cdot \nabla b\|_{\dot{H}^1} \|c\|_{\dot{H}^3}.$$

Then, by using the following three-dimensional law product, see [8]

$$\dot{H}^1 \cdot \dot{B}_{2,1}^{\frac{3}{2}} \subset \dot{H}^1$$

we infer that

$$\langle a \cdot \nabla b | c \rangle_2 \lesssim \|a\|_{\dot{H}^1} \|\nabla b\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \|c\|_{\dot{H}^3}$$

combined with the following interpolation estimate, the proof of which can be found again in [8],

$$\|f\|_{\dot{B}_{2,1}^{\frac{3}{2}}} \lesssim \|f\|_{\dot{H}^{\frac{1}{2}}} + \|f\|_{\dot{H}^2},$$

Proposition IV.2.1 is then proved. \square

The next lemma will be used to obtain the bound $L_T^\infty(\dot{H}^{\frac{3}{2}})$ of the small solution v_h .

Lemma IV.2.1. *For a regular enough divergence free vector fields u, v and $\beta \in [1, \infty]$, the following estimate holds.*

$$\|\operatorname{div}(u \otimes v)\|_{\tilde{L}_T^\beta(\dot{B}_{2,\infty}^0)} \lesssim \|u\|_{\tilde{L}_T^\infty(\dot{B}_{2,\infty}^{\frac{1}{2}})} \|v\|_{\tilde{L}_T^\beta(\dot{B}_{2,\infty}^2)} + \|v\|_{\tilde{L}_T^\infty(\dot{B}_{2,\infty}^{\frac{1}{2}})} \|u\|_{\tilde{L}_T^\beta(\dot{B}_{2,\infty}^2)}. \quad (\text{IV.15})$$

Proof

The proof is classical and relies on continuity properties of Bony's decomposition. To make the presentation more reputable, we omit the tensor product and estimate the formal product $\partial_j(uv)$. Hence we need to prove

$$\|uv\|_{\tilde{L}_T^\beta(\dot{B}_{2,\infty}^1)} \lesssim \|u\|_{\tilde{L}_T^\infty(\dot{B}_{2,\infty}^{\frac{1}{2}})} \|v\|_{\tilde{L}_T^\beta(\dot{B}_{2,\infty}^2)} + \|v\|_{\tilde{L}_T^\infty(\dot{B}_{2,\infty}^{\frac{1}{2}})} \|u\|_{\tilde{L}_T^\beta(\dot{B}_{2,\infty}^2)}.$$

Thanks to Bony's decomposition, we write

$$uv = T_u v + T_v u + R(u, v),$$

By changing the positions of u and v , the first two terms on the r.h.s can be dealt with similarly as follows

$$\begin{aligned} \|\Delta_k T_u v\|_{L_T^\beta(L^2)} &\lesssim \|S_{k-1} u\|_{L_T^\infty(L^\infty)} \|\Delta_k v\|_{L_T^\beta(L^2)} \\ &\lesssim 2^{-k} \|u\|_{\tilde{L}_T^\infty(\dot{B}_{\infty,\infty}^{-1})} \|v\|_{\tilde{L}_T^\beta(\dot{B}_{2,\infty}^2)} \end{aligned}$$

By Sobolev embedding we have

$$\|T_u v\|_{\tilde{L}_T^\beta(\dot{B}_{2,\infty}^1)} \lesssim \|u\|_{\tilde{L}_T^\infty(\dot{B}_{2,\infty}^{\frac{1}{2}})} \|v\|_{\tilde{L}_T^\beta(\dot{B}_{2,\infty}^2)}.$$

For the reminder term, the same analysis gives

$$\begin{aligned} \|\Delta_k R(u, v)\|_{L_T^\beta(L^2)} &\lesssim \sum_{m>k-N_0} \|\tilde{\Delta}_m u\|_{L_T^\infty(L^\infty)} \|\Delta_m v\|_{L_T^\beta(L^2)} \\ &\lesssim 2^{-k} \sum_{m>k-N_0} 2^{k-m} \|u\|_{\tilde{L}_T^\infty(\dot{B}_{2,\infty}^{1/2})} \|v\|_{\tilde{L}_T^\beta(\dot{B}_{2,\infty}^2)} \end{aligned}$$

thus, by Holder inequality we obtain

$$\|\Delta_k R(u, v)\|_{L_T^\beta(L^2)} \lesssim 2^{-k} \|u\|_{\tilde{L}_T^\infty(\dot{B}_{2,\infty}^{1/2})} \|v\|_{\tilde{L}_T^\beta(\dot{B}_{2,\infty}^2)}.$$

Lemma IV.2.1 is then proved. \square

IV.2.2 Global estimates for the small solution part (Proof of Propositions IV.1.2 and IV.1.3)

Proof of Proposition IV.1.2 Let us begin with establishing the $E_T(\dot{H}^{\frac{1}{2}})$ bound for v_h . Doing so, we take the inner product in $\dot{H}^{\frac{1}{2}}$ with v_h solution of (NS_h) to obtain

$$\frac{1}{2} \frac{d}{dt} \|v_h(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|\nabla v_h(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \leq \left| \langle v_h \cdot \nabla v_h | v_h \rangle_{\dot{H}^{\frac{1}{2}}} \right| + \|f_h\|_{\dot{H}^{-\frac{1}{2}}} \|\nabla v_h\|_{\dot{H}^{\frac{1}{2}}}.$$

By virtue of (IV.9) in Proposition IV.2.1 and Young's inequality, we infer that

$$\frac{1}{2} \frac{d}{dt} \|v_h(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|\nabla v_h(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \leq C \|v_h(t)\|_{\dot{H}^{\frac{1}{2}}} \|\nabla v_h(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + C \|f_h(t)\|_{\dot{H}^{-\frac{1}{2}}}^2 + \frac{1}{4} \|\nabla v_h(t)\|_{\dot{H}^{\frac{1}{2}}}^2.$$

Now, let

$$\tilde{T} = \sup \left\{ \tau > 0 : \|v_h(\tau)\|_{\dot{H}^{\frac{1}{2}}} < 1/4C \right\}.$$

The definition of f_h provides that for all $t < \tilde{T}$,

$$\|v_h(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|\nabla v_h(t)\|_{L_t^2(\dot{H}^{\frac{1}{2}})}^2 \leq C \varepsilon_{N_0, -\frac{1}{2}}^2.$$

We choose N_0 large enough such that $C \varepsilon_{N_0, -\frac{1}{2}}^2 < 1/(8C)^2$, hence for all $t < \tilde{T}$

$$\|v_h(t)\|_{\dot{H}^{\frac{1}{2}}} < 1/8C. \tag{IV.16}$$

On the one hand, we assume that $\tilde{T} < \infty$, so the definition of \tilde{T} and solution's continuity yield

$$\|v_h(\tilde{T})\|_{\dot{H}^{\frac{1}{2}}} = 1/4C.$$

Hence, letting t goes to \tilde{T} in (IV.16), we end up with

$$1/4C = \|v_h(\tilde{T})\|_{\dot{H}^{\frac{1}{2}}} \leq 1/8C.$$

This contradicts the fact that \tilde{T} is finite, and thus $\|v_h(t)\|_{\dot{H}^{\frac{1}{2}}}$ remains bounded for all $t > 0$ and can not blow-up in finite time. In particular, one concludes that for all $T > 0$

$$\|v_h\|_{E_T(\dot{H}^{\frac{1}{2}})} \lesssim \varepsilon. \tag{IV.17}$$

Let us now prove the same estimate for $\|v_h\|_{E_T(\dot{H}^1)}$. Similarly, as above, by virtue of (IV.9) stated in Proposition IV.2.1, we write

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|v_h(t)\|_{\dot{H}^1}^2 + \|\nabla v_h(t)\|_{\dot{H}^1}^2 &\leq C \|v_h(t)\|_{\dot{H}^{\frac{1}{2}}} \|\nabla v_h(t)\|_{\dot{H}^1}^2 + C \|f_h(t)\|_{L^2}^2 + \frac{1}{4} \|\nabla v_h(t)\|_{\dot{H}^1}^2 \\ &\leq C \|f_h(t)\|_{L^2}^2 + \frac{1}{2} \|\nabla v_h(t)\|_{\dot{H}^1}^2. \end{aligned}$$

Hence

$$\|v_h\|_{E_T(\dot{H}^1)} \lesssim \varepsilon_{N_0,0} \lesssim \varepsilon. \quad (\text{IV.18})$$

Grouping (IV.17) and (IV.18), we end up with

$$\|v_h\|_{E_T(\dot{H}^{\frac{1}{2}}) \cap E_T(\dot{H}^1)} \lesssim \varepsilon. \quad (\text{IV.19})$$

For the uniqueness topic, it is well-known to hold in these kinds of spaces, hence the details are left to the reader. The proof of Proposition IV.1.2 is now achieved. \square

Proof of Proposition IV.1.3 The perception of the proof will be done in the following way. First, we start by establishing that $v_h \in L_T^\infty(B_{2,\infty}^{2-\frac{2}{\beta}})$. Thereafter, remarking that for $\beta > 4$ then $2 - \frac{2}{\beta} > \frac{3}{2}$, so we conclude by making use of the interpolation argument that

$$L_T^\infty(H^1) \cap L_T^\infty(B_{2,\infty}^{2-\frac{2}{\beta}}) \hookrightarrow L_T^\infty(B_{2,1}^{\frac{3}{2}}) \hookrightarrow L_T^\infty(H^{\frac{3}{2}}).$$

So, to prove that v_h lies in $L_T^\infty(B_{2,\infty}^{2-\frac{2}{\beta}})$, we first write down the equivalent Duhamel's formula associated to system (NS_h)

$$v_h(t, \cdot) = - \int_0^t \mathbb{S}(t-\tau) \mathbf{P} \operatorname{div}(v_h \otimes v_h)(\tau, \cdot) d\tau - \int_0^t \mathbb{S}(t-\tau) \mathbf{P} f_h(\tau, \cdot) d\tau,$$

where \mathbb{S} represents the heat semi-group, defined by $\mathbb{S}(t) \stackrel{\text{def}}{=} e^{t\Delta}$.

Next, we apply the Lemma A.2.1 from the Appendix for $\kappa_1 = a$, $\kappa_2 = \beta$ and $p = 2$ to write

$$\left\| \int_0^\infty \mathbf{1}_{\tau \leq \cdot} \mathbb{S}(\cdot - \tau) \Delta_q g(s) \right\|_{L_T^a(L^2)} \lesssim 2^{2j(-1+\frac{1}{\beta}-\frac{1}{a})} \|\Delta_q g\|_{L_T^\beta(L^2)}, \quad \text{for all } a \in [\beta, \infty]. \quad (\text{IV.20})$$

Accordingly, the fixed-point method for the above Duhamel's formula can be done in the following functional space.

$$\mathcal{X}_T \stackrel{\text{def}}{=} \{g \in C((0, T); L^2) : \|g\|_{\mathcal{X}_T} < \infty\},$$

equipped with the norm

$$\|g\|_{\mathcal{X}_T} \stackrel{\text{def}}{=} \|g\|_{\tilde{L}_T^\infty(\dot{B}_{2,\infty}^{2-\frac{2}{\beta}})} + \|g\|_{\tilde{L}_T^\beta(\dot{B}_{2,\infty}^2)}.$$

The external term can be estimated by applying (IV.20), while the bilinear form can be carried out by employing (IV.15) in Lemma IV.2.1, we conclude that

$$\|v_h\|_{\mathcal{X}_T} \leq C \|v_h\|_{\tilde{L}_T^\infty(\dot{B}_{2,\infty}^{\frac{1}{2}})} \|v_h\|_{\mathcal{X}_T} + C \|f_h\|_{\tilde{L}_T^\beta(\dot{B}_{2,\infty}^0)}. \quad (\text{IV.21})$$

The fact that $L^2 = \dot{B}_{2,2}^0 \hookrightarrow \dot{B}_{2,\infty}^0$ leads to

$$\|f_h\|_{\tilde{L}_T^\beta(\dot{B}_{2,\infty}^0)} \leq \|f_h\|_{L_T^\beta(L^2)}.$$

Similarly, on account of Proposition IV.1.2, one may write

$$\|v_h\|_{\tilde{L}_T^\infty(\dot{B}_{2,\infty}^{\frac{1}{2}})} \leq \|v_h\|_{L_T^\infty(\dot{H}^{\frac{1}{2}})} \leq \varepsilon/C,$$

plugging these into (IV.21) gives

$$\|v_h\|_{\mathcal{X}_T} \lesssim \|f_h\|_{L_T^\beta(L^2)}.$$

By definition of f_h , for N_0 large enough we obtain the desired estimate. \square

IV.2.3 Global estimates for the modified system and proof of the main Theorem IV.1.2

Before to lay down the proof of Theorem IV.1.2 we begin by proving the following result.

Theorem IV.2.2. *Let v_0 be an axisymmetric divergence free vector field in $H^2(\mathbb{R}^3)$ and v_h the unique global solution of (NS_h) given by Propositions IV.1.2 and IV.1.3. Then (NS_ℓ) has a unique solution in $E_T(H^1)$ for all $T > 0$.*

The proof of Theorem IV.2.2 will be orchestrate, first by showing that (NS_ℓ) has a unique local solution in $E_{T^*}(H^s)$, for all $s \in [\frac{1}{2}, 2]$, with T^* can be considered as the maximal lifespan in H^1 . Second, we explore the axisymmetric structure to control the H^1 -norm over $(0, T^*]$, which allows to extend the solution beyond T^* . This contradicts the fact that T^* is maximal.

We start by proving the following proposition.

Proposition IV.2.2. *Let v_0 be a divergence free vector field in $H^2(\mathbb{R}^3)$ and v_h the unique global solution of (NS_h) constructed in Propositions IV.1.2 and IV.1.3. Then, there exists a maximal lifespan $T^* > 0$ such that (NS_ℓ) has a unique solution in $E_{T^*}(\dot{H}^s)$ for all $s \in [\frac{1}{2}, 2]$.*

Proof

We will only show the existence part and we skip the uniqueness, since this latter is well-known to hold in such spaces (the linear part with respect to v_ℓ can be dealt with as in the existence part). Define the free solution as

$$v^\ell(t, \cdot) \stackrel{def}{=} e^{t\Delta} v_0(\cdot)$$

and we look for a solution v^ℓ to (NS_ℓ) of the form $v^\ell = v^\ell + w^\ell$ with w^ℓ solving

$$\begin{cases} \partial_t w^\ell + w^\ell \cdot \nabla w^\ell - \Delta w^\ell + \nabla p^\ell = f_\ell + F_{v_h}(w^\ell) + F_{v^\ell}(w^\ell) - v^\ell \cdot \nabla v^\ell \\ \operatorname{div} v = 0 \\ w^\ell|_{t=0} = 0, \end{cases} \quad (NS^\ell)$$

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with $F_a(b) = -a \cdot \nabla b - b \cdot \nabla a$.

Let us point out first that, for all $s \in [\frac{1}{2}, 2]$ and $p \in [2, \infty]$, we have

$$\|v^l\|_{L_T^p(\dot{H}^{s+\frac{2}{p}})} \lesssim \|v_0\|_{\dot{H}^s}.$$

We only restrict ourselves to outline some a priori estimates while the construction of a solution to (NS^ℓ) , can be done by employing the well-known Friedrichs method based on formal calculation that we will prove here. For more details about this method, we refer the reader to [8].

We proceed in four steps.

• *1st Step:* $s = \frac{1}{2}$. Taking the scalar product in $\dot{H}^{\frac{1}{2}}$ for (NS^ℓ) with w^ℓ . Thus, (IV.8) in Proposition IV.2.1 gives

$$\begin{aligned} \frac{1}{2} \|w^\ell(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \|\nabla w^\ell\|_{L_t^2(\dot{H}^{\frac{1}{2}})}^2 &\leq \left(C \|w^\ell\|_{L_t^\infty(\dot{H}^{\frac{1}{2}})} + C \|v_h\|_{L_t^\infty(\dot{H}^{\frac{1}{2}})} + 1/4 \right) \|\nabla w^\ell\|_{L_t^2(\dot{H}^{\frac{1}{2}})}^2 \\ &\quad + C \|w^\ell\|_{L_t^\infty(\dot{H}^{\frac{1}{2}})}^2 \|\nabla v_h\|_{L_t^2(\dot{H}^{\frac{1}{2}})}^2 + C \delta(t), \end{aligned} \quad (IV.22)$$

with

$$\delta(t) \stackrel{def}{=} \|v^l\|_{L_t^\infty(\dot{H}^{\frac{1}{2}})}^2 \|\nabla v^l\|_{L_t^2(\dot{H}^{\frac{1}{2}})}^2 + \|v^l\|_{L_t^4(\dot{H}^1)}^4 \|w^\ell\|_{L_t^\infty(\dot{H}^{\frac{1}{2}})}^2 + \|f_\ell\|_{L_t^2(\dot{H}^{-\frac{1}{2}})}^2.$$

According to Proposition IV.1.2, we may choose N_0 large enough so that

$$\|v_h\|_{L_t^\infty(\dot{H}^{\frac{1}{2}})} + \|\nabla v_h\|_{L_t^2(\dot{H}^{\frac{1}{2}})} \leq \min \{1/4C, 1/4\sqrt{C}\},$$

this gives in particular

$$C \|w^\ell\|_{L_t^\infty(\dot{H}^{\frac{1}{2}})}^2 \|\nabla v_h\|_{L_t^2(\dot{H}^{\frac{1}{2}})}^2 \leq \frac{1}{4} \|w^\ell\|_{L_t^\infty(\dot{H}^{\frac{1}{2}})}^2.$$

By denoting

$$T_1 \stackrel{def}{=} \sup \left\{ \tau > 0 : C \|w^\ell\|_{L_{\tau^+}^\infty(\dot{H}^{\frac{1}{2}})} < 1/4 \right\}.$$

Inequality (IV.22) gives, for all $t < T_1$

$$\frac{1}{4} \|w^\ell(t)\|_{\dot{H}^{\frac{1}{2}}}^2 \leq C \delta(t)$$

and seen that $\|\nabla v^l(\cdot)\|_{\dot{H}^{\frac{1}{2}}} \in L^2(\mathbb{R}_+)$ and $\|v^l(\cdot)\|_{\dot{H}^1} \in L^4(\mathbb{R}_+)$, one has

$$\delta(t) \longrightarrow 0, \quad t \rightarrow 0^+.$$

Hence, if we denote T_2 the first $t > 0$ such that $\delta(t) < 1/64C^2$ for all $t < T_2$. Setting $T^* \stackrel{def}{=} \min\{T_1, T_2\}$, by usual continuity arguments, we conclude that, for all $t \in [0, T^*)$, the $E_t(\dot{H}^{\frac{1}{2}})$ norm of w^ℓ remains bounded for all $t < T^*$, more precisely we have

$$\|w^\ell\|_{L_t^\infty(\dot{H}^{\frac{1}{2}})} < 1/4C.$$

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- *2nd Step:* $s \in (\frac{1}{2}, \frac{3}{2})$. Again the scalar product in \dot{H}^s for (NS^ℓ) with w^ℓ and (IV.9) and (IV.10) of Proposition IV.2.1 allow us to get

$$\begin{aligned} \frac{1}{2} \|w^\ell(t)\|_{\dot{H}^s}^2 + \|\nabla w^\ell\|_{L_t^2(\dot{H}^s)}^2 &\leq \left(C \|w^\ell\|_{L_t^\infty(\dot{H}^{\frac{1}{2}})} + C \|v_h\|_{L_t^\infty(\dot{H}^{\frac{1}{2}})} + 1/4 \right) \|\nabla w^\ell\|_{L_t^2(\dot{H}^s)}^2 \\ &\quad + C \|w^\ell\|_{L_t^\infty(\dot{H}^s)}^2 \left(\|\nabla v_h\|_{L_t^2(\dot{H}^{\frac{1}{2}})}^2 + \|v^l\|_{L_t^4(\dot{H}^1)}^4 \right) + C \delta_s(t), \end{aligned}$$

with

$$\delta_s(t) \stackrel{def}{=} \|w^\ell\|_{L_t^4(\dot{H}^1)}^2 \|v^l\|_{L_t^4(\dot{H}^{s+\frac{1}{2}})}^2 + \|f_\ell\|_{L_t^2(\dot{H}^{s-1})}^2 < \infty, \quad \forall t < T^*.$$

We may suppose that, for t small enough (we keep the same notation for the maximal time T^*)

$$\|u^l\|_{L_t^4(\dot{H}^1)}^4 < 1/4C$$

to end with the following estimate, for all $t < T^*$

$$\|w^\ell(t)\|_{\dot{H}^s}^2 + \|\nabla w^\ell\|_{L_t^2(\dot{H}^s)}^2 \lesssim \delta_s(t) < \infty.$$

- *3rd Step:* $s = \frac{3}{2}$. By similar arguments and by making use of inequality (IV.12), it holds

$$\frac{1}{2} \|w^\ell(t)\|_{\dot{H}^{\frac{3}{2}}}^2 + \|\nabla w^\ell\|_{L_t^2(\dot{H}^{\frac{3}{2}})}^2 \leq C \tilde{\delta}(t) + \frac{1}{4} \|\nabla w^\ell\|_{L_t^2(\dot{H}^s)}^2,$$

with

$$\begin{aligned} \tilde{\delta}(t) \stackrel{def}{=} &\|v^l\|_{L_t^\infty(\dot{H}^1)}^2 \|v^l\|_{L_t^2(\dot{H}^2)}^2 + \|w^\ell\|_{L_t^\infty(\dot{H}^1)}^2 \|w^\ell\|_{L_t^2(\dot{H}^2)}^2 + \|v^l\|_{L_t^\infty(\dot{H}^1)}^2 \|w^\ell\|_{L_t^2(\dot{H}^2)}^2 \\ &+ \|w^\ell\|_{L_t^\infty(\dot{H}^1)}^2 \|v^l\|_{L_t^2(\dot{H}^2)}^2 + \|v_h\|_{L_t^\infty(\dot{H}^1)}^2 \|w^\ell\|_{L_t^2(\dot{H}^2)}^2 + \|w^\ell\|_{L_t^\infty(\dot{H}^1)}^2 \|v_h\|_{L_t^2(\dot{H}^2)}^2 + \|f_\ell\|_{L_t^2(\dot{H}^{\frac{1}{2}})}^2. \end{aligned}$$

One deduces for all $t < T^*$ that

$$\|w^\ell(t)\|_{\dot{H}^{\frac{3}{2}}}^2 + \|\nabla w^\ell\|_{L_t^2(\dot{H}^{\frac{3}{2}})}^2 \lesssim \tilde{\delta}(t) < \infty.$$

- *4th Step* $s = 2$. By using estimates (IV.13) and (IV.14), similar calculations to the previous steps lead to

$$\begin{aligned} \frac{1}{2} \|w^\ell(t)\|_{\dot{H}^2}^2 + \|\nabla w^\ell(t)\|_{L_t^2(\dot{H}^2)}^2 &\leq C \bar{\delta}(t) + C \int_0^t \left(\|\nabla v_h(\tau)\|_{\dot{H}^1}^2 \|w^\ell(\tau)\|_{\dot{H}^2}^2 \right) d\tau \\ &\quad + \frac{1}{4} \|\nabla w^\ell(t)\|_{L_t^2(\dot{H}^2)}^2 \end{aligned}$$

with

$$\begin{aligned} \bar{\delta}(t) \stackrel{def}{=} &\left(\|v_h\|_{L_t^4(\dot{H}^1)}^2 + \|w^\ell\|_{L_t^4(\dot{H}^1)}^2 + \|v^l\|_{L_t^4(\dot{H}^1)}^2 \right) \left(\|\nabla w^\ell\|_{L_t^2(\dot{H}^{\frac{1}{2}})}^2 + \|\nabla w^\ell\|_{L_t^2(\dot{H}^2)}^2 \right) \\ &+ \left(\|w^\ell\|_{L_t^4(\dot{H}^1)}^2 + \|w^\ell\|_{L_t^4(\dot{H}^1)}^2 \right) \left(\|\nabla v^l\|_{L_t^2(\dot{H}^{\frac{1}{2}})}^2 + \|\nabla v^l\|_{L_t^2(\dot{H}^2)}^2 \right) \\ &+ \|\nabla v_h\|_{L_t^2(\dot{H}^1)}^2 \|\nabla w^\ell\|_{L_t^\infty(\dot{H}^{\frac{1}{2}})}^2 + \|f_\ell\|_{L_t^2(\dot{H}^1)}^2, \end{aligned}$$

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according to the previous steps $\bar{\delta}(t)$ is finite for all $t < T^*$, Gronwall lemma then insures

$$\|w^\ell(t)\|_{\dot{H}^2}^2 + \|\nabla w^\ell(t)\|_{L_t^2(\dot{H}^2)}^2 \lesssim \bar{\delta}(t) \exp(C\|v_h\|_{E_t(\dot{H}^1)}^2) < \infty, \quad \forall t < T^*$$

This completes the proof of Proposition IV.2.2. \square

Proof of Theorem IV.2.2 To prove Theorem IV.2.2, we consider the unique local solution v_ℓ to (NS $_\ell$) given by Proposition IV.2.2 in $E_T(H^s)$ for all $T < T^*$ and $s \in [\frac{1}{2}, 2]$, with T^* can be considered as the maximal lifespan in H^1 , and we show now how to benefit of the axisymmetric structure to control the H^1 norm on $[T^* - \eta, T^*]$ for some $\eta \ll 1$, which allows to extend the solution, and contradicts the fact that T^* is maximal.

The key idea to prove the H^1 global bound of v_ℓ is to explore the axisymmetric structure to get a uniform bound of $\alpha \stackrel{def}{=} \frac{\nabla \times v_\ell}{r}$ in $E_T(L^2)$ for all $T > 0$. Indeed, the solution $v_\ell \in E_T(H^2)$, given by Proposition IV.2.2, for all $T < T^*$, combined with the Biot-Savart law, (see Lemma 2.2 from [3]) ensure that α lies in $E_T(L^2)$, for all $T < T^*$. Now, we state the proof of the H^1 -global bound of v_ℓ in the spirit [3]. For the rest of the proof we will agree the following notations.

$$\Gamma_h \stackrel{def}{=} (\nabla \times v_h) \vec{e}_\theta, \quad \Gamma_\ell \stackrel{def}{=} (\nabla \times v_\ell) \vec{e}_\theta, \quad g_\ell \stackrel{def}{=} (\nabla \times f_\ell) \vec{e}_\theta.$$

Recall that $v = v_h + v_\ell$ and it is clear that Γ_ℓ solves

$$\partial_t \Gamma_\ell + \left(v^r \partial_r + v^z \partial_z - \frac{v^r}{r} \right) \Gamma_\ell - \left(\partial_r^2 + \partial_z^2 + \frac{\partial_r}{r} - \frac{1}{r^2} \right) \Gamma_\ell = g_\ell - v_\ell^r \partial_r \Gamma_h - v_\ell^z \partial_z \Gamma_h + \frac{v_\ell^r}{r} \Gamma_h, \quad (\text{IV.23})$$

with (v_h^r, v_h^z) and (v_ℓ^r, v_ℓ^z) refer respectively to the components of v_h and v_ℓ .

Taking the L^2 -inner product of (IV.23) by Γ_ℓ , integrating by parts, the incompressibility condition $\text{div } v_\ell = \partial_r v_\ell^r + \partial_z v_\ell^z + \frac{v_\ell^r}{r} = 0$ and $\text{div } v_h = \partial_r v_h^r + \partial_z v_h^z + \frac{v_h^r}{r} = 0$ yield

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Gamma_\ell\|_{L^2}^2 + \|D\Gamma_\ell\|_{L^2}^2 + \left\| \frac{\Gamma_\ell}{r} \right\|_{L^2}^2 &= \int_{I(\pi, \mathbb{R}, \mathbb{R}_+)} \left(g_\ell \Gamma_\ell + \frac{v^r}{r} (\Gamma_\ell)^2 + (v_\ell^r \partial_r \Gamma_\ell + v_\ell^z \partial_z \Gamma_\ell) \Gamma_h \right. \\ &\quad \left. + \frac{v_\ell^r}{r} \Gamma_h \Gamma_\ell \right) r dr dz d\theta \end{aligned}$$

Via, Cauchy-Schwartz and Hölder's inequalities, we end up with

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\Gamma_\ell\|_{L^2}^2 + \|D\Gamma_\ell\|_{L^2}^2 + \left\| \frac{\Gamma_\ell}{r} \right\|_{L^2}^2 &\leq \|g_\ell\|_{\dot{H}^{-1}} \|\Gamma_\ell\|_{\dot{H}^1} + \left\| \frac{\Gamma_\ell}{r} \right\|_{L^2} \|v^r\|_{L^6} \|\Gamma_\ell\|_{L^3} \\ &\quad + \|v_\ell\|_{L^6} \|D\Gamma_\ell\|_{L^2} \|\Gamma_h\|_{L^3} + \left\| \frac{\Gamma_\ell}{r} \right\|_{L^2} \|v_\ell\|_{L^6} \|\Gamma_h\|_{L^3}, \end{aligned}$$

In accordance with the following Sobolev embedding $\|g_\ell\|_{\dot{H}^{-1}} \lesssim \|f_\ell\|_{L^2}$, $\|\Gamma_\ell\|_{\dot{H}^1} \approx \|D\Gamma_\ell\|_{L^2}$ and $\|v_\ell\|_{L^6} \lesssim \|\nabla v\|_{L^2}$ combined with Young inequality give rise to,

$$\frac{1}{2} \frac{d}{dt} \|\Gamma_\ell\|_{L^2}^2 + \|D\Gamma_\ell\|_{L^2}^2 + \left\| \frac{\Gamma_\ell}{r} \right\|_{L^2}^2 \lesssim \|f_\ell\|_{L^2}^2 + \left\| \frac{\Gamma_\ell}{r} \right\|_{L^2}^{\frac{4}{3}} \|\nabla v\|_{L^2}^{\frac{4}{3}} \|\Gamma_\ell\|_{L^2}^{\frac{2}{3}} + \|\nabla v_\ell\|_{L^2}^2 \|\Gamma_h\|_{L^3}^2.$$

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After an integration in time and Young inequality with respect to time, it holds

$$\begin{aligned} \|\Gamma_\ell(t)\|_{L^2}^2 + \|D\Gamma_\ell\|_{L_t^2(L^2)}^2 + \left\| \frac{\Gamma_\ell}{r} \right\|_{L_t^2(L^2)}^2 &\lesssim \|\Gamma_{0,\ell}\|_{L^2}^2 + \|f_\ell\|_{L_t^2(L^2)}^2 \\ &+ \left\| \frac{\Gamma_\ell}{r} \right\|_{L_t^\infty(L^2)}^{\frac{4}{3}} \|\nabla v\|_{L_t^2(L^2)}^{\frac{4}{3}} \|\Gamma_\ell\|_{L_t^2(L^2)}^{\frac{2}{3}} + \int_0^t \|\nabla v_\ell(\tau)\|_{L^2}^2 \|\Gamma_h(\tau)\|_{L^3}^2 d\tau. \end{aligned} \quad (\text{IV.24})$$

Now, we treat $\alpha \stackrel{def}{=} \frac{\Gamma_\ell}{r}$ which is governs the equation

$$\partial_t \alpha + (v^r \partial_r + v^z \partial_z) \alpha - \left(\partial_r^2 + \partial_z^2 + \frac{3}{r} \partial_r \right) \alpha = \frac{g_\ell}{r} - \frac{v_\ell^r}{r} \partial_r \Gamma_h - \frac{v_\ell^z}{r} \partial_z \Gamma_h + \frac{v_\ell^r}{r} \frac{\Gamma_h}{r}. \quad (\text{IV.25})$$

To do so, taking the L^2 -inner product of (IV.25) with α and integrating over \mathbb{R}^3 , one has

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\alpha\|_{L^2}^2 + \int_{\mathbb{R}^3} (v^r \partial_r \alpha + v^z \partial_z \alpha) \alpha dx - \int_{\mathbb{R}^3} \left(\partial_r^2 \alpha + \partial_z^2 \alpha + \frac{3}{r} \partial_r \alpha \right) \alpha dx & \quad (\text{IV.26}) \\ \leq \left| \int_{I(\pi, \mathbb{R}, \mathbb{R}_+)} \left(\frac{g_\ell}{r} - \frac{v_\ell^r}{r} \partial_r \Gamma_h - \frac{v_\ell^z}{r} \partial_z \Gamma_h + \frac{v_\ell^r}{r} \frac{\Gamma_h}{r} \right) \alpha r dr dz d\theta \right|. \end{aligned}$$

So, for the second term on the l.h.s we have

$$\begin{aligned} \int_{\mathbb{R}^3} (v^r \partial_r \alpha + v^z \partial_z \alpha) \alpha dx &= \int_{I(\pi, \mathbb{R}, \mathbb{R}_+)} (v^r \partial_r \alpha + v^z \partial_z \alpha) \alpha r dr dz d\theta \\ &= -\pi \int_{\mathbb{R} \times \mathbb{R}_+} \left(\partial_r v^r + \partial_z v^z + \frac{v^r}{r} \right) \alpha^2 r dr dz = 0, \end{aligned}$$

where we have used the fact $\text{div } v = \partial_r v^r + \partial_z v^z + \frac{v^r}{r} = 0$ and $v^r(0, \cdot) = 0$ because v is axisymmetric.

For the second term, we use the fact $\Delta = \partial_r^2 + \partial_z^2 + \frac{1}{r} \partial_r$ to obtain

$$\begin{aligned} - \int_{\mathbb{R}^3} \left(\partial_r^2 \alpha + \partial_z^2 \alpha + \frac{3}{r} \partial_r \alpha \right) \alpha dx &= - \int_{\mathbb{R}^3} (\Delta \alpha) \alpha dx - \int_{\mathbb{R}^3} \frac{2}{r} (\partial_r \alpha) \alpha dx \\ &= \|\nabla \alpha\|_{L^2}^2 + 2\pi \int_{\mathbb{R}} \alpha^2(t, 0, z) dz \geq \|\nabla \alpha\|_{L^2}^2. \end{aligned}$$

Collecting the last two estimates and inserting them in (IV.26), it follows

$$\frac{1}{2} \frac{d}{dt} \|\alpha\|_{L^2}^2 + \|D\alpha(t)\|_{L^2}^2 \leq \left| \int_{I(\pi, \mathbb{R}, \mathbb{R}_+)} \left(\frac{g_\ell}{r} - \frac{v_\ell^r}{r} \partial_r \Gamma_h - \frac{v_\ell^z}{r} \partial_z \Gamma_h + \frac{v_\ell^r}{r} \frac{\Gamma_h}{r} \right) \alpha r dr dz d\theta \right|. \quad (\text{IV.27})$$

Now, for the second term on the r.h.s, a straightforward computation yields

$$-\frac{v_\ell^r}{r} \partial_r \Gamma_h - \frac{v_\ell^z}{r} \partial_z \Gamma_h + \frac{v_\ell^r}{r} \frac{\Gamma_h}{r} = -v_\ell^r \partial_r \left(\frac{\Gamma_h}{r} \right) - v_\ell^z \partial_r \left(\frac{\Gamma_h}{r} \right),$$

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so, a new use of integration by parts, the fact $\operatorname{div} v_\ell = 0$ and $v_\ell^r(0, \cdot) = 0$ we further get

$$- \int_{I(\pi, \mathbb{R}, \mathbb{R}_+)} \left(\frac{v_\ell^r}{r} \partial_r \Gamma_h + \frac{v_\ell^z}{r} \partial_z \Gamma_h - \frac{v_\ell^r}{r} \frac{\Gamma_h}{r} \right) \alpha r dr dz d\theta = \int_{I(\pi, \mathbb{R}, \mathbb{R}_+)} \left(v_\ell^r \partial_r \alpha + v_\ell^z \partial_z \alpha \right) \frac{\Gamma_h}{r} r dr dz d\theta.$$

Gathering the last two and plug them in (IV.27), thus Cauchy-Schwartz's and Hölder's inequalities lead to

$$\frac{1}{2} \frac{d}{dt} \|\alpha\|_{L^2}^2 + \|D\alpha\|_{L^2}^2 \leq \left\| \frac{g_\ell}{r} \right\|_{L^2} \|\alpha\|_{L^2} + \|D\alpha\|_{L^2} \left\| \frac{v_\ell}{r} \right\|_{L^6} \|\Gamma_h\|_{L^3}$$

Integrating in time over $(0, t)$, then in view of Young inequality, one has

$$\|\alpha(t)\|_{L^2} + \|D\alpha\|_{L_t^2(L^2)} \lesssim \|\alpha_0\|_{L^2} + \left\| \frac{g_\ell}{r} \right\|_{L_t^1(L^2)} + \left(\int_0^t \left\| \frac{v_\ell(\tau)}{r} \right\|_{L^6}^2 \|\Gamma_h(\tau)\|_{L^3}^2 d\tau \right)^{\frac{1}{2}}, \quad (\text{IV.28})$$

For the first term on the r.h.s., Hardy inequality, Sobolev embedding $\dot{H}^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$ and (A.17) provide that

$$\begin{aligned} \left\| \frac{v_\ell}{r} \right\|_{L^6} &\lesssim \|\partial_r v_\ell\|_{L^6} \lesssim \|\nabla v_\ell\|_{\dot{H}^1} \\ &\lesssim \|D\Gamma_\ell\|_{L^2} + \left\| \frac{\Gamma_\ell}{r} \right\|_{L^2}, \end{aligned}$$

whereas, concerning the second one we have

$$\left\| \frac{g_\ell}{r} \right\|_{L_t^1(L^2)} \lesssim \|f_\ell\|_{L_t^1(H^2)}.$$

Plugging these into (IV.28), it happens

$$\|\alpha(t)\|_{L^2} + \|D\alpha\|_{L_t^2(L^2)} \lesssim \|v_0\|_{H^2} + \|f_\ell\|_{L_t^1(H^2)} + \left(\int_0^t \left(\|D\Gamma_\ell\|_{L^2} + \left\| \frac{\Gamma_\ell}{r} \right\|_{L^2} \right)^2 \|\Gamma_h(\tau)\|_{L^3}^2 d\tau \right)^{\frac{1}{2}} \quad (\text{IV.29})$$

Let us denote

$$\Pi(T_0) \stackrel{\text{def}}{=} \sup_{t \in [0, T_0]} \|\Gamma_\ell(t)\|_{L^2} + \|D\Gamma_\ell\|_{L_t^2(L^2)} + \left\| \frac{\Gamma_\ell}{r} \right\|_{L_t^2(L^2)}.$$

In accordance with Sobolev embedding and Proposition IV.1.3, one obtains

$$\|\Gamma_h\|_{L_t^\infty(L^3)} \lesssim \|v_h\|_{L_t^\infty(H^{\frac{3}{2}})} \lesssim \varepsilon$$

Inserting this latest in (IV.29), then for all $t \in (0, T_0)$, we find

$$\|\alpha(t)\|_{L^2} + \|D\alpha\|_{L_t^2(L^2)} \lesssim \|v_0\|_{H^2} + \|f_\ell\|_{L_t^1(H^2)} + \varepsilon \Pi(T_0) \quad (\text{IV.30})$$

Substituting (IV.30) into (IV.24), we infer that

$$\begin{aligned} \Pi(T_0) &\lesssim \|v_0\|_{H^1} + \|f_\ell\|_{L^2_{T_0}(L^2)} + \Pi(T_0)\|\Gamma_h\|_{L^2_{T_0}(L^3)} + \|\alpha\|_{L^\infty_{T_0}(L^2)}\|\nabla v\|_{L^2_{T_0}(L^2)}\|\Gamma_\ell\|_{L^2_{T_0}(L^2)} \\ &\lesssim \|v_0\|_{H^1} + \|f_\ell\|_{L^2_{T_0}(L^2)} + \Pi(T_0)\|\Gamma_h\|_{L^2_{T_0}(L^3)} + \left(\|v_0\|_{H^2} + \|f_\ell\|_{L^1_t(H^2)} \right. \\ &\quad \left. + \varepsilon\Pi(T_0) \right)^{\frac{2}{3}} \times \|\nabla v\|_{L^2_{T_0}(L^2)}\|\nabla v_\ell\|_{L^2_{T_0}(L^2)}. \end{aligned}$$

Again, Sobolev embedding and Proposition IV.1.2 yield

$$\|\Gamma_h\|_{L^2_{T_0}(L^3)} \lesssim \|v_h\|_{L^2_{T_0}(H^{\frac{3}{2}})} \lesssim \varepsilon.$$

Now, we choose $N_0 \gg 1$ such that $\varepsilon \ll 1$, to end with

$$\Pi(T_0) \lesssim \|v_0\|_{H^1} + \|v_0\|_{H^2} + \|f_\ell\|_{L^2_{T_0}(L^2)} + \|f_\ell\|_{L^1_{T_0}(H^2)} + \|Dv\|_{L^2_{T_0}(L^2)}^3.$$

The left hand-side of the last inequality is finite for all finite $T_0 > 0$, more precisely we have

$$\Pi(T_0) \lesssim \|v_0\|_{H^1} + \|v_0\|_{H^2} + (1 + T_0^{\frac{1}{2}}N_0^2)\|f\|_{L^2_{T_0}(L^2)} + (\|v_0\|_L^2 + (1 + \sqrt{T_0})\|f\|_{L^2_{T_0}(H^{-1})})^3 < \infty.$$

The last inequality provides the desired $E_T(H^1)$, for all finite $T > 0$. Theorem IV.2.2 is then proved. \square

Remark IV.2.3. The $L_T^\beta(L^2)$ -norm of f does not explicitly appear in the last inequality above, but according to the proof of Proposition IV.1.3, the choice of N_0 is really related to fact that $\|f\|_{L_T^\beta(L^2)}$ is finite.

Proof of Theorem IV.1.2 In order to derive Theorem IV.1.2 from Theorem IV.2.2, we will proceed by the following argument. For v_0 in $H^{\frac{1}{2}}$, we construct a unique solution $v = v_h + v_\ell$, with v_h is in $E_t(H^{\frac{1}{2}})$ for all $t > 0$, and v_ℓ solves (NS $_\ell$) in $E_T(H^{\frac{1}{2}})$, with $T < T^*$ the maximal lifespan of existence in $H^{\frac{1}{2}}$. Since $v_\ell \in L_T^2(H^{\frac{3}{2}})$, there exists some $t_0 \in (0, T^*)$ such that $v_\ell(t_0) \in H^{\frac{3}{2}}$, then Proposition IV.2.2 enables to solve (NS $_\ell$) again on (t_0, T^*) with initial data $v_\ell(t_0)$. So, due to the uniqueness of the solution, we infer that $v_\ell \in L^\infty((t_0, T^*); H^{\frac{3}{2}}) \cap L^2((t_0, T^*); H^{\frac{5}{2}})$. By the same processes there exists $t_1 \in (t_0, T^*)$ such that $v_\ell(t_1) \in H^2$. Hence, if we take $v_\ell(t_1)$ as the new initial data we succeed to construct a unique solution given by Theorem IV.2.2 on (t_1, T) for all $T > t_1$, we denote this solution by \tilde{v}_ℓ . The uniqueness property of the Navier-Stokes equations in $E_T(\dot{H}^{\frac{1}{2}})$ guarantees that $v_\ell = \tilde{v}_\ell$ on $[t_1, T^*)$ and Theorem IV.1.2 follows. \square

IV.3 Global wellposedness of the Axisymmetric Boussinesq system (Proof of Theorem IV.1.4)

The goal of this section is to show how to derive Theorem IV.1.4 from Theorem IV.1.2.

Proof of Theorem IV.1.4 Since $(v_0, \rho_0) \in H^{\frac{1}{2}}(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$, then in accordance of Theorem 1.1 in [36] we can construct at least one global solution (v, ρ) to (B) which satisfies the energy inequalities. Next, for $\rho \in L^\infty(L^2) \subset L_{loc}^\beta(L^2)$ for $\beta > 4$, by virtue of Theorem IV.1.2, we associate to this ρ , the unique global solution to (NS) in $E_T(H^{\frac{1}{2}})$, for all $T > 0$. The final step consists at proving that the solution (v, ρ) is the unique one. Doing so, we will need some additional regularity on v and ρ , therefore the conclusion of the proof is a direct application of Proposition IV.1.4 that we prove below. \square

Proof of Proposition IV.1.4 We will only show how to propagate the regularity in Proposition IV.1.4, the uniqueness part is well known to hold in these spaces, see for instance Theorem 1.3 in [36].

We intend to extend the proof of the previous proposition to more general class of system than (B). Such system takes the form

$$\begin{cases} \partial_t v - \mu \Delta v + \nabla p = Q(v, v) + \rho \vec{e}_3 & \text{if } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \partial_t \rho + v \cdot \nabla \rho = 0 & \text{if } (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \operatorname{div} v = 0, \\ (v, \rho)|_{t=0} = (v_0, \rho_0). \end{cases} \quad (\text{B}_{\mu, Q})$$

where

$$(Q(v, v))^j = \sum_{i=1}^3 q_{i,j}(D)(v^i v^j)$$

and $\{q_{i,j}\}_{1 \leq i,j \leq 3}$ are Fourier multipliers of order 1.

Thanks to Proposition A.3.1 with $f \equiv 0$, we have the following estimate

$$\|\rho\|_{\tilde{L}_t^\infty(\dot{B}_{3,1}^0)} \lesssim \|\rho_0\|_{\dot{B}_{3,1}^0} \left(1 + \int_0^t \|\nabla v(\tau)\|_{\dot{B}_{\infty,1}^0} d\tau \right). \quad (\text{IV.31})$$

Therefore, the control of ρ in $L_T^\infty(\dot{B}_{3,1}^0)$ requires a control of v in $L_T^1(\dot{B}_{\infty,1}^1)$, which can be done, due to sobolev embedding, if we know how to control v in $L_T^1(B_{3,1}^2)$. To do so, we consider the Duhamel's formula associated to the velocity equation

$$v(t, \cdot) = \mathbb{S}(t)v_0(\cdot) - \int_0^t \mathbb{S}(t-\tau) \mathbf{P}Q(v, v)(\tau, \cdot) d\tau - \int_0^t \mathbb{S}(t-\tau) \mathbf{P}\rho \vec{e}_3(\tau, \cdot) d\tau.$$

To make the presentation simple enough, we will use in the rest of the proof the following notation

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$$\tilde{\mathcal{E}}_T(\dot{B}_{3,1}^0) \stackrel{def}{=} \tilde{L}_T^\infty(\dot{B}_{3,1}^0) \cap \tilde{L}_T^1(\dot{B}_{3,1}^2).$$

Owing to the continuity property of the heat semi-group $\mathbf{S}(\cdot)$, we have

$$\|v\|_{\tilde{\mathcal{E}}_t(\dot{B}_{3,1}^0)} \lesssim \|v_0\|_{\dot{B}_{3,1}^0} + \|Q(v, v)\|_{\tilde{L}_t^1(\dot{B}_{3,1}^0)} + \|\rho\|_{\tilde{L}_t^1(\dot{B}_{3,1}^0)}. \quad (\text{IV.32})$$

In fact, if we deal directly with $\tilde{L}_T^1(\dot{B}_{3,1}^2)$ estimate we will end with the following issue. By exploring the following estimate the proof of which is an easy application of para-product law,

$$\|Q(v, v)\|_{\tilde{L}_t^1(\dot{B}_{3,1}^0)} \lesssim \|(v, v)\|_{\tilde{L}_t^1(\dot{B}_{3,1}^1)} \lesssim \|v\|_{L_t^\infty(\dot{B}_{\infty, \infty}^{-1})} \|v\|_{L_t^1(\dot{B}_{3,1}^2)},$$

by denoting $\mathcal{V}(t) \stackrel{def}{=} \|v\|_{L_t^1(\dot{B}_{3,1}^2)}$ and $K(t) \stackrel{def}{=} \|v\|_{L_t^\infty(H^{\frac{1}{2}})}$ and on account of (IV.31), one has

$$\mathcal{V}(t) \lesssim C_0 + K(t)\mathcal{V}(t) + \int_0^t C_0 \mathcal{V}(\tau) d\tau. \quad (\text{IV.33})$$

Remark that the above estimate doesn't provide a global control of $\mathcal{V}(t)$ for all $t \leq T < \infty$. The difference factor between our context and that of [74] is the following: in [74], the authors dealt with an initial data v_0 in $H^1(\mathbb{R}^3)$ which gives a solution v in $L_T^2(H^2)$ and eventually by interpolation, one obtains in their context that v belongs to $L_T^2(\dot{B}_{2,1}^{\frac{3}{2}})$. Or, this latter space in an algebra, whereupon the following estimate holds

$$\begin{aligned} \|Q(v, v)\|_{\tilde{L}_t^1(\dot{B}_{3,1}^0)} &\lesssim \|v\|_{L_t^2(\dot{B}_{2,1}^{\frac{3}{2}})}^2 \\ &\leq C_0 e^{C_0 t}. \end{aligned} \quad (\text{IV.34})$$

Therefore in [74], instead of (IV.33), they have

$$\mathcal{V}(t) \lesssim C_0 + C_0 e^{C_0 t} + \int_0^t C_0 \mathcal{V}(\tau) d\tau.$$

which is sufficient to control \mathcal{V} . Contrary, in our case we deal only with initial data in $H^{\frac{1}{2}}$, so the maximal gain of regularity is $L_T^2(H^{\frac{3}{2}})$. Consequently, we are below to the regularity threshold of Hmidi-Rousset's approach[74] which in turn becomes not available in our case. To remedy this latter, we must propose another argument, in particular to control the non-linear term without using the $L_t^2(H^{\frac{3}{2}})$ norm when we deal with the low frequencies in the Bony's decomposition of $v \otimes v$. The first step consists to establish a control of the velocity v in $\tilde{L}_t^{\frac{4}{3}}(\dot{B}_{3,4}^{\frac{3}{3}}) = L_t^{\frac{4}{3}}(\dot{B}_{3,4/3}^{\frac{3}{3}})$ in terms of the $L_t^4(\dot{H}^1)$ and $L_T^2(\dot{H}^{\frac{3}{2}})$ norms of v , which are already controlled, for all $t \leq T$, since $v \in E_T(H^{\frac{1}{2}})$. In the next step,

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in order to control $\|v\|_{L_t^1(\dot{B}_{3,1}^2)}$, we estimate the bi-linear term in terms of the $L_t^{\frac{4}{3}}(\dot{B}_{3,4/3}^{\frac{3}{2}})$ and the $L_t^4(\dot{H}^1)$ norms of v , more details are given below.

Taking advantage of the continuity property of $\mathbb{S}(\tau)$ we have for all $r \geq 1$

$$\|\mathbb{S}(\cdot)v_0\|_{\tilde{L}_t^{\frac{4}{3}}(\dot{B}_{3,r}^{\frac{3}{2}})} \lesssim \|v_0\|_{\dot{B}_{3,1}^0}.$$

In particular, for $r = \frac{4}{3}$, we obtain

$$\|\mathbb{S}(\cdot)v_0\|_{\tilde{L}_t^{\frac{4}{3}}(\dot{B}_{3,4/3}^{\frac{3}{2}})} \lesssim \|v_0\|_{\dot{B}_{3,1}^0}.$$

Defining

$$\mathcal{K}g(t, \cdot) \stackrel{def}{=} \int_0^t \mathbb{S}(t-\tau)g(\tau, \cdot) d\tau.$$

The following inequality

$$\|\mathcal{K}g\|_{L_t^{\frac{4}{3}}(\dot{B}_{3,\frac{4}{3}}^{\frac{3}{2}})} \lesssim \|g\|_{L_t^{\frac{4}{3}}(\dot{B}_{3,4/3}^{\frac{3}{2}-2})}$$

together with Sobolev embedding $\dot{B}_{2,1}^0 \hookrightarrow \dot{B}_{3,4/3}^{\frac{3}{2}-2}$ and the estimate

$$\|Q(v, v)\|_{L_t^{\frac{4}{3}}(\dot{B}_{2,1}^0)} \lesssim \|v\|_{L_t^4(\dot{H}^1)} \|v\|_{L_t^2(\dot{H}^{\frac{3}{2}})},$$

lead to

$$\|v\|_{\tilde{L}_t^{\frac{4}{3}}(\dot{B}_{3,\frac{4}{3}}^{\frac{3}{2}})} \lesssim \|v_0\|_{\dot{B}_{3,1}^0} + C\|v\|_{L_t^4(\dot{H}^1)} \|u\|_{L_t^2(\dot{H}^{\frac{3}{2}})} + \|\rho\|_{L_t^1(\dot{B}_{3,1}^0)}$$

By means of (IV.31), it follows

$$\|v\|_{\tilde{L}_t^{\frac{4}{3}}(\dot{B}_{3,\frac{4}{3}}^{\frac{3}{2}})} \lesssim \|v_0\|_{\dot{B}_{3,1}^0} + C\|v\|_{L_t^4(\dot{H}^1)} \|u\|_{L_t^2(\dot{H}^{\frac{3}{2}})} + \|\rho_0\|_{\dot{B}_{3,1}^0} \left(t + \int_0^t \mathcal{V}(\tau) d\tau \right). \quad (\text{IV.35})$$

Finally, we come back to estimate $\mathcal{V}(t) = \|v\|_{\tilde{L}_t^1(\dot{B}_{3,1}^2)}$, to do so, we summarize from (IV.32), and instead of using (IV.34), we use the following estimates, which we will prove at the end of this section

$$\begin{aligned} \|Q(v \otimes v)\|_{L_t^1(\dot{B}_{3,1}^0)} &\lesssim \|v\|_{L_t^4(\dot{H}^1)} \|v\|_{L_t^{\frac{4}{3}}(\dot{B}_{3,2}^{\frac{3}{2}})} \\ &\lesssim \|v\|_{L_t^4(\dot{H}^1)} \|v\|_{L_t^{\frac{4}{3}}(\dot{B}_{3,\frac{4}{3}}^{\frac{3}{2}})} \end{aligned} \quad (\text{IV.36})$$

with the help of (IV.35) and the embedding $E_T(\dot{H}^{\frac{1}{2}}) \hookrightarrow L_T^4(\dot{H}^1)$, it happens

$$\|Q(v, v)\|_{L^1(\dot{B}_{3,1}^0)} \lesssim \|v\|_{E_t(\dot{H}^{\frac{1}{2}})} \left(\|v_0\|_{\dot{B}_{3,1}^0} + C\|v\|_{E_t(\dot{H}^{\frac{1}{2}})}^2 + \|\rho_0\|_{\dot{B}_{3,1}^0} \left(t + \int_0^t \mathcal{V}(\tau) d\tau \right) \right).$$

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By setting

$$\mathcal{A}(t) \stackrel{def}{=} \|v_0\|_{\dot{B}_{3,1}^0} + t\|\rho_0\|_{\dot{B}_{3,1}^0} + \|v\|_{E_t(\dot{H}^{\frac{1}{2}})} \left(\|v_0\|_{\dot{B}_{3,1}^0} + C\|v\|_{E_t(\dot{H}^{\frac{1}{2}})}^2 + t\|\rho_0\|_{\dot{B}_{3,1}^0} \right),$$

and

$$\mathcal{B}(t) \stackrel{def}{=} \|\rho_0\|_{\dot{B}_{3,1}^0} \left(1 + \|v\|_{E_t(\dot{H}^{\frac{1}{2}})} \right),$$

which are finite for all $t < \infty$. Therefore, one concludes from (IV.32) that

$$\mathcal{V}(t) \lesssim \mathcal{A}(t) + \mathcal{B}(t) \int_0^t \mathcal{V}(\tau) d\tau,$$

Gronwall's estimate yields

$$\mathcal{V}(t) \lesssim \mathcal{A}(t)e^{t\mathcal{B}(t)}, \quad \text{for all } t < \infty. \quad (\text{IV.37})$$

Once, we have established the bound of $\mathcal{V}(t)$ for all $t > 0$, we can control v in $\tilde{\mathcal{E}}_t(\dot{B}_{3,1}^0)$ and ρ in $\tilde{L}_t^\infty(\dot{B}_{3,1}^0)$ by substituting (IV.37) in (IV.32) and (IV.31). Proposition IV.1.4 is then proved.

For the sake of completeness, we briefly outline the proof of (IV.36). For this aim, we employ the following law product in three dimensions of space

$$(\dot{H}^1 \cap \dot{B}_{3,2}^{\frac{3}{2}}) \cdot (\dot{H}^1 \cap \dot{B}_{3,2}^{\frac{3}{2}}) \subset \dot{B}_{3,1}^1.$$

Indeed, Bony's decomposition enables us to write

$$uv = T_u v + T_v u + R(u, v).$$

For $T_u v$, we have

$$\begin{aligned} \|\dot{\Delta}_j(T_u v)\|_{L^3} &\lesssim \|\dot{S}_{j-1} u\|_{L^\infty} \|\dot{\Delta}_j v\|_{L^3} \\ &\lesssim c_j^2 2^{-j} \|u\|_{\dot{B}_{\infty,2}^{-\frac{1}{2}}} \|v\|_{\dot{B}_{3,2}^{\frac{3}{2}}}, \end{aligned}$$

with $\sum_{j \in \mathbb{Z}} c_j^2 \leq 1$. Whence, the 3D Sobolev embedding $\dot{H}^1 \hookrightarrow \dot{B}_{\infty,2}^{-\frac{1}{2}}$ gives

$$\|T_u v\|_{\dot{B}_{3,1}^1} \lesssim \|u\|_{\dot{H}^1} \|v\|_{\dot{B}_{3,2}^{\frac{3}{2}}}.$$

Similarly, by exchanging the positions of u and v in the above estimates, we infer that

$$\|T_v u\|_{\dot{B}_{3,1}^1} \lesssim \|v\|_{\dot{H}^1} \|u\|_{\dot{B}_{3,2}^{\frac{3}{2}}}.$$

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Finally, for the remainder term, we proceed as follows

$$\begin{aligned}
\|\dot{\Delta}_j(R(u, v))\|_{L^3} &\lesssim \sum_{k \geq j - N_0} \|\tilde{\Delta}_k u\|_{L^\infty} \|\dot{\Delta}_k v\|_{L^3} \\
&\lesssim 2^{-j} \left(\sum_{k \geq j - N_0} 2^{j-k} C_k^2 \right) \|u\|_{\dot{B}_{\infty,2}^{-\frac{1}{2}}} \|v\|_{\dot{B}_{3,2}^{\frac{3}{2}}}, \\
&\lesssim 2^{-j} d_j \|u\|_{\dot{H}^1} \|v\|_{\dot{B}_{3,2}^{\frac{3}{2}}}, \quad \sum_{j \in \mathbb{Z}} d_j \leq 1.
\end{aligned}$$

□

Remark IV.3.1. The uniqueness part of Proposition IV.1.4 can be established as well for the general system $(\mathbf{B}_{\mu, Q})$ by following the proof's approach of Theorem 1.3 from [36], and by estimating the difference between two solutions still noted by $(\delta v, \delta \rho)$ in $\dot{B}_{3,1}^{-1} \times \dot{B}_{3,1}^{-1}$, one may notice that the proof of the uniqueness in Theorem 1.3 in [36] doesn't use at any step the special structure of the bi-linear term.

CHAPTER V

Uniqueness result for the Boussinesq system with horizontal dissipations

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V.1 Introduction and statement of the main results

In this Chapter we consider the following Navier-Stokes-Boussinesq equations with horizontal dissipation

$$\left\{ \begin{array}{l} (\partial_t + u \cdot \nabla)u - \Delta_h u + \nabla P = \rho e_3, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ (\partial_t + u \cdot \nabla)\rho - \Delta_h \rho = 0 \\ \operatorname{div} u = 0 \\ (u, \rho)|_{t=0} = (u_0, \rho_0), \end{array} \right. \quad (NSB_h)$$

where $\Delta_h \stackrel{def}{=} \partial_1^2 + \partial_2^2$ denotes the horizontal laplacian and $e_3 = (0, 0, 1)^T$ is the third vector of the canonical basis of \mathbb{R}^3 .

The unknowns of the system are $u = (u^1, u^2, u^3)$, ρ and P which represent respectively: the velocity, the density and the pressure of the fluid.

In the following we will say that (u, ρ) is a solution to (NSB_h) if it is a weak solution in the classical sense (see for instance [28] pages 123,132 and 204). We recall also that from a solution (u, ρ) we may use a result of De Rham in order to recover a pressure P (which depends on u and ρ) and to obtain a distributional solution (u, ρ, P) of the system (NSB_h) .

Note that in (NSB_h) the diffusion only occurs in the horizontal direction. This is a natural assumption for several cases of interest in geophysical fluids flows (see [107]). However $-\Delta_h$ is a less regularizing operator than the laplacian $-\Delta$ and we cannot expect a better theory than for the classical Navier-Stokes-Boussinesq equations:

$$\left\{ \begin{array}{l} (\partial_t + u \cdot \nabla)u - \Delta u + \nabla P = \rho e_3, \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ (\partial_t + u \cdot \nabla)\rho - \Delta \rho = 0, \\ \operatorname{div} u = 0, \\ (u, \rho)|_{t=0} = (u_0, \rho_0). \end{array} \right. \quad (NSB)$$

In particular, the question of the global well-posedness of (NSB) and consequently of (NSB_h) remains largely open, but recently the system (NSB_h) has received a lot of attention from mathematicians (see for instance [101, 2, 120]) and significant progress in its analysis have been made. See also [65, 68, 72, 82] for more related results for other models of the Boussinesq system.

Note again that (NSB_h) involves the operator $-\Delta_h$ which smooth only along the horizontal variables. Hence, we need to estimate differently the horizontal and the vertical directions, and the natural functional setting for the analysis involves some anisotropic Sobolev and Besov spaces. The definitions of these spaces and some of their important properties are recalled in the next section.

In order to analyse (NSB_h) it is useful to forget its second equation for a while, and to

consider first the Navier-Stokes equations with horizontal laplacian:

$$\begin{cases} (\partial_t + u \cdot \nabla)u - \Delta_h u + \nabla P = 0, & \text{in } \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \quad (NS_h)$$

Several interesting studies for this last system were done. In [28], the authors proved the local existence and the global one for small data in $H^{0,s}$ for some $s > \frac{1}{2}$. The proof of the existence part in [28] uses deeply the structure of the equation and the fact that u is a divergence free vector field. The key point used in their estimates is related to the fact¹ that $H^{\frac{1}{2}}(\mathbb{R}^2) \hookrightarrow L^4(\mathbb{R}^2)$ and that $H^s(\mathbb{R})$ is an algebra. Hence, it is easy to deal with the term $u^h \cdot \nabla_h u$ by using some product rules in the well-chosen spaces. Next, after using the divergence free condition together with some Littlewood-Paley stuffs in a clever way they were able to treat the term $u^3 \partial_3 u$ with the same argument.

Always in [28], the authors proved also a uniqueness result (but only for $s > \frac{3}{2}$, because of the term $w^3 \partial_3 u$) by establishing a H^{0,s_0} -energy estimate for a difference between two solutions $w = u - v$, where $s_0 \in]\frac{1}{2}, s]$. Later, in [85], D.Iftimie had overcome the difficulty by remarking that it is sufficient to estimate w in $H^{-\frac{1}{2}}$ with respect to the vertical variable, and this only requires an $H^{\frac{1}{2}}$ regularity for u in the vertical direction. Then he proved a uniqueness result for any $s > \frac{1}{2}$, and the gap between existence and uniqueness was closed.

To do something similar with system (NSB_h) , we begin by estimating the horizontal terms (terms which contain only horizontal derivatives) by using some product rules in the adequate Besov and Sobolev spaces. For the vertical terms (terms which contain only vertical derivatives) we follow in general the idea in [28] in order to transform them into terms similar to the horizontal ones by using the divergence free condition. Hence, for $s \in]\frac{1}{2}, 1]$, we first propose to estimate the difference between two solutions $w = u - v$ in $H^{0,s-1}$ instead of $H^{0,-\frac{1}{2}}$ providing that the solution u already exists in the $H^{0,s}$ energy-space (see Appendix for a proof of an existence result). For the second equation, denoting the difference between two solutions $\theta = \rho_1 - \rho_2$ we remark that:

- The function ρ only appears in the third equation of u (the equation for the component u^3). Hence, a priori, we only need to estimate ρ in the H^{s-1} -norm with respect to the vertical variable.
- In order to deal with the term $u^h \cdot \nabla_h \theta$, we must estimate θ with respect to the vertical variable in some space $H^{-\alpha}$, with $\alpha \geq 0$ and such that $H^s(\mathbb{R}) \times H^{-\alpha}(\mathbb{R})$ holds to be a subspace of $H^{-\alpha}(\mathbb{R})$. In fact, Lemma V.2.1 below says that the minimum index $-\alpha$ that can be chosen is $-\alpha = -s$.
- For the term $w^h \cdot \nabla_h \rho$, if we consider that ρ lies in some H^β -space, with respect to the vertical variable, then a direct application of the product rules shows that we

¹Recall that this argument permits to prove the uniqueness of weak-solution for the classical Navier-Stokes problem in dimension two.

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need $\beta \geq 1 - s$. Moreover, because the system is hyperbolic in the vertical direction, we expect the loss of one derivative.

Hence, we will estimate vertically ρ in H^{1-s} and θ in H^{-s} .

For the critical case where $s = \frac{1}{2}$, in [106] M.Paicu proved a uniqueness² result for (NS_h) in $L_T^\infty(H^{0,\frac{1}{2}}) \cap L_T^2(H^{1,\frac{1}{2}})$. It is clear that such a space falls to be embedded in L^∞ in the vertical direction which is the major problem that prevents using similar arguments to those in the case where $s > \frac{1}{2}$. In order to prove the uniqueness, the author in [106] established a double logarithm estimate (see (V.6)) and concluded by using the Osgood's Lemma. We will provide some details to adapte the idea of [106] and apply it to (NSB_h) .

Our main result is the following:

Theorem V.1.1. (Uniqueness)

Let $s \in [\frac{1}{2}, 1]$ and $(u, \rho), (v, \eta)$ be two solutions for system (NSB_h) in

$$L_{loc}^\infty(\mathbb{R}_+; H^{0,s}) \cap L_{loc}^2(\mathbb{R}_+; H^{1,s}) \times L_{loc}^\infty(\mathbb{R}_+; H^{0,1-s}) \cap L_{loc}^2(\mathbb{R}_+; H^{1,1-s}).$$

Then $(u, \rho) = (v, \eta)$.

As an interesting consequence, we can improve the results of global well-posedness in the case of axisymmetric initial data established in [101].

Let us first recall some basic notions: We say that a vector field u is axisymmetric if it satisfies

$$\mathcal{R}_{-\alpha}(u(\mathcal{R}_\alpha(x))) = u(x), \quad \forall \alpha \in [0, 2\pi], \quad \forall x \in \mathbb{R}^3,$$

where \mathcal{R}_α denotes the rotation of axis (Oz) and with angle α . Moreover, an axisymmetric vector field u is called without swirl if it has the form:

$$u(x) = u^r(r, z)e_r + u^z(r, z)e_z, \quad x = (x_1, x_2, x_3), \quad r = \sqrt{x_1^2 + x_2^2} \text{ and } z = x_3.$$

We say that a scalar function f is axisymmetric, if the vector field $x \mapsto f(x)e_z$ is axisymmetric. We also denote by $\omega = \text{curl } u$ the vorticity of u . Then we will prove:

Theorem V.1.2. (Global well-posedness)

Let $u_0 \in H^1(\mathbb{R}^3)$ be an axisymmetric divergence free vector field without swirl such that $\frac{\omega_0}{r} \in L^2$ and let $\rho_0 \in L^2$ be an axisymmetric function. Then there exists a unique global solution (u, ρ) of the system (NSB_h) . Moreover, we have:

$$\begin{aligned} u &\in \mathcal{C}(\mathbb{R}_+; H^1) \cap L_{loc}^2(\mathbb{R}_+; H^{1,1} \cap H^{2,0}), \quad \frac{\omega}{r} \in L_{loc}^\infty(\mathbb{R}_+; L^2) \cap L_{loc}^2(\mathbb{R}_+; H^{1,0}), \\ \rho &\in \mathcal{C}(\mathbb{R}_+; L^2) \cap L_{loc}^2(\mathbb{R}_+; H^{1,0}). \end{aligned}$$

²We should mention that the existence of solution in such scaling-invariant space is still an open problem even for the classical Navier-Stokes system.

Note that in Theorem V.1.2, we only assume that $(u_0, \rho_0) \in H^1 \times L^2$ whereas in [101] the authors consider a stronger condition. Namely, in addition of the hypothesis of Theorem V.1.2 they assume that $(\nabla \times u_0, \rho_0)$ is in $H^{0,1} \times H^{0,1}$ or in $L^\infty \times H^{0,1}$. In both works the key point consists to establish an uniqueness result: it is the Theorem V.1.1 for us, whereas in [101] the authors assume a strong initial condition in order to obtain some double exponential control in time for the gradient of u .

The last result we prove in this Chapter is a sort of a local wellposedness of NSB_h , where the uniqueness of the solution is a direct application of Theorem V.1.1, namely we will prove

Theorem V.1.3. (Local well-posedness)

Let $s \in [\frac{1}{2}, 1]$, $\delta \in [0, s]$ and $(u_0, \rho_0) \in H^{0,s} \times H^{0,\delta}$. We have:

- There exists $C_s > 0$ such that if

$$\|u_0\|_{0,s}^2 + T \|\rho_0\|_{L^2} \left(\|u_0\|_{L^2} + \|\rho_0\|_{L^2} \left(1 + \frac{T}{2}\right) \right) < C_s^2.$$

then (NSB_h) has at least one solution (u, ρ) in $\mathcal{X}^{s,\delta}(T)$, where:

$$\mathcal{X}^{s,\delta}(T) \stackrel{\text{def}}{=} L_T^\infty H^{0,s} \cap L_T^2 H^{1,s} \times L_T^\infty H^{0,\delta} \cap L_T^2 H^{1,\delta}.$$

- The solution is unique if $\delta \geq 1 - s$.

Remark V.1.1. Notice that, according to Theorem V.1.3, the construction of solutions to (NSB_h) (at least locally in time) does not require high regularity for the initial density ρ_0 . This is in fact possible due to the high vertical regularity of the velocity which allows us to rigorously justify the a priori estimates (as in the Friedrich's method explained in the proof of Theorem V.1.2).

Remark V.1.2. The constant C_s in Theorem V.1.3 tends in fact to zero when s tends to $\frac{1}{2}$. In this case, our a priori estimates fall, and hence the existence of solutions in the case $s = \frac{1}{2}$ remains open, even for the classical Navier-Stokes equations.

Remark V.1.3. In the case where $s = \frac{1}{2}$, one may prove a local well-posedness result if the initial data (u_0, ρ_0) is in $(L_h^2(B_{2,1}^{\frac{1}{2}})_v)^2$. See for instance [106] where this is established for (NS_h) .

Throughout this chapter we write $\mathbb{R}^3 = \mathbb{R}_h^2 \times \mathbb{R}_v$ and for any vector $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$, we will denote the two first components by ξ_h and the last one by ξ_v , that is to say: $\xi = (\xi_1, \xi_2, \xi_3) \stackrel{\text{def}}{=} (\xi_h, \xi_v)$. Similarly, for any vector field $X = (X^1, X^2, X^3)$ we will write $X = (X^h, X^v)$ with the meaning that $X^h = (X^1, X^2)$ and $X^v = X^3$.

We will also use the notations

$$H_h^s = H^s(\mathbb{R}_h^2), \quad H_v^s = H^s(\mathbb{R}_v), \quad L_v^p(H_h^s) = L^p(\mathbb{R}_v; H_h^s) \quad \text{and} \quad L_T^r L_h^p L_v^q = L^r(0, T; L^p(\mathbb{R}_h^2; L^q(\mathbb{R}_v))).$$

Recall that (NSB_h) involves the operator $-\Delta_h$ which only regularizes along the horizontal direction. Hence the regularity of the functions along the vertical variable must be measured differently than the horizontal ones, that is we need to work in anisotropic function spaces, we refer to the Appendix part of this thesis for the definition and more details about these spaces.

This Chapter is organized as follows: in section 2, we establish several a priori estimates which are then used in section 3 to prove our theorems. Finally in section 4 we shall prove Theorem V.1.1, then we outline the proofs of Theorems V.1.2 and V.1.3, where the uniqueness part is an application of Theorem V.1.1.

V.2 A priori estimates and crucial lemmata

In this section we establish the main a priori estimates required to prove our theorems. But before that, let us prove two lemmata which serves the purpose of this section. The following lemma is a sort of product laws in non-homogeneous Sobolev spaces

Lemma V.2.1. *Let $\sigma, \sigma', s, s_0 \in \mathbb{R}$ verifying $\sigma, \sigma' < 1$, $\sigma + \sigma' > 0$, $s_0 > \frac{1}{2}$, $s \leq s_0$ and $s + s_0 \geq 0$ then there exists a constant $C = C(\sigma, \sigma', s, s_0)$ such that:*

$$\|ab\|_{H^{\sigma+\sigma'-1}, s} \leq C \|a\|_{H^{\sigma, s}} \|b\|_{H^{\sigma', s_0}}, \quad \forall a, b \in \mathcal{S}.$$

Proof

Remark first that because $\|ab\|_{H^{\sigma+\sigma'-1}, s} = \left\| \|ab\|_{H^s_v} \right\|_{H^{\sigma+\sigma'-1}_h}$, we have only to prove that:

$$H^s(\mathbb{R}) \cdot H^{s_0}(\mathbb{R}) \subset H^s(\mathbb{R}). \tag{V.1}$$

Indeed, by using (V.1) together with the usual product rules with respect to the horizontal variables, the desired result follows (see for instance [8]). Note also that when $s_0 > s > \frac{1}{2}$, the inclusion (V.1) is trivial since in this case the space H^s is an algebra and clearly $H^{s_0} \hookrightarrow H^s$.

It remains then only to prove (V.1) in the situation $s_0 > \frac{1}{2} \geq s$ and $s + s_0 \geq 0$. In order to do this, we use the Bony's decomposition in the vertical variable: $ab = T_a^v b + T_b^v a + R^v(a, b)$. For the first term, let us consider the two cases: $s < \frac{1}{2}$ and $s = \frac{1}{2}$.

The case $s < \frac{1}{2}$: By using the embedding $H^s(\mathbb{R}) \hookrightarrow B_{\infty, 2}^{s-\frac{1}{2}}(\mathbb{R})$, together with (A.5) we obtain for any $q \geq -1$:

$$\begin{aligned} \|\Delta_q^v(T_a^v b)\|_{L^2(\mathbb{R})} &\lesssim \|S_{q-1}^v a\|_{L^\infty(\mathbb{R})} \|\Delta_q^v b\|_{(L^2\mathbb{R})} \lesssim c_q^2 2^{-q(s-\frac{1}{2})} 2^{-qs_0} \|a\|_{B_{\infty, 2}^{s-\frac{1}{2}}(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})} \\ &\lesssim c_q^2 \max\{1, 2^{s_0-\frac{1}{2}}\} 2^{-qs} \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})} \lesssim c_q^2 2^{-qs} \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}. \end{aligned}$$

It follows that

$$\|T_a^v b\|_{H^s(\mathbb{R})} \lesssim \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}. \tag{V.2}$$

The case $s = \frac{1}{2}$. We use the following estimate:

$$\|S_{q-1}^v a\|_{L^\infty(\mathbb{R})} \leq \sum_{-1 \leq j \leq q} 2^{\frac{j}{2}} \|\Delta_j^v a\|_{L^2(\mathbb{R})} \lesssim \sqrt{q} \|a\|_{H^{\frac{1}{2}}(\mathbb{R})},$$

in order to obtain:

$$\|\Delta_q^v(T_a^v b)\|_{L^2(\mathbb{R})} \lesssim \|S_{q-1}^v a\|_{L^\infty(\mathbb{R})} \|\Delta_q^v b\|_{L^2(\mathbb{R})} \lesssim c_q \sqrt{q} 2^{-q(s_0 - \frac{1}{2})} 2^{-\frac{q}{2}} \|a\|_{H^{\frac{1}{2}}(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}.$$

Seen that $\forall \varepsilon > 0$, there exists $C_\varepsilon > 0$ such that for all $q \in \mathbb{R}^+$: $\sqrt{q} 2^{-q\varepsilon} \leq C_\varepsilon$, we infer that:

$$\|T_a^v b\|_{H^{\frac{1}{2}}(\mathbb{R})} \lesssim \|a\|_{H^{\frac{1}{2}}(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})},$$

and (V.2) follows for all $s \leq \frac{1}{2} < s_0$. Moreover, by using the embedding $H^{s_0}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ together with the estimate:

$$\|\Delta_q^v(T_b^v a)\|_{L^2(\mathbb{R})} \lesssim \|S_{q-1}^v b\|_{L^\infty(\mathbb{R})} \|\Delta_q^v a\|_{L^2(\mathbb{R})} \lesssim \|b\|_{L^\infty(\mathbb{R})} c_q 2^{-qs} \|a\|_{H^s(\mathbb{R})},$$

we obtain:

$$\|T_b^v a\|_{H^s(\mathbb{R})} \lesssim \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}.$$

For the reminder term, if $s + s_0 > 0$, then applying Lemma A.2.2 together with (A.5) gives:

$$\begin{aligned} \|\Delta_q^v(R(a, b))\|_{L^2(\mathbb{R})} &\lesssim 2^{\frac{q}{2}} \sum_{j \geq q - N_0} \|\Delta_j^v a\|_{L^2(\mathbb{R})} \|\tilde{\Delta}_j^v b\|_{L^2(\mathbb{R})}, \\ &\lesssim 2^{\frac{q}{2}} \sum_{j \geq q - N_0} (c_j^2 2^{-j(s+s_0)}) \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}, \end{aligned}$$

where

$$\tilde{\Delta}_q^v \stackrel{def}{=} \sum_{i \in \{-1, 0, 1\}} \Delta_{q+i}^v.$$

Consequently, for any $q \geq -1$ we get:

$$\begin{aligned} 2^{qs} \|\Delta_q^v(R(a, b))\|_{L^2(\mathbb{R})} &\lesssim 2^{-q(s_0 - \frac{1}{2})} \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})} \sum_{j \geq q - N_0} c_j^2 2^{-(j-q)(s+s_0)} \\ &\lesssim c_q 2^{-q(s_0 - \frac{1}{2})} \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}. \end{aligned}$$

It is then easy to show that $\|R(a, b)\|_{H^s(\mathbb{R})} \lesssim \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}$.

If $s + s_0 = 0$, then along the same lines we can prove that:

$$\|R(a, b)\|_{B_{2,\infty}^{-\frac{1}{2}}(\mathbb{R})} \lesssim \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})}.$$

The last step consists to use the following inequality by taking $a = -\frac{1}{2}$ and $\varepsilon = s_0 - \frac{1}{2}$:

$$\|f\|_{B_{2,1}^{a-\varepsilon}} = \sum_{k \geq -1} 2^{k(a-\varepsilon)} \|\Delta_k f\|_{L^2} \leq C(\varepsilon) \|f\|_{B_{2,\infty}^a}, \quad \forall a \in \mathbb{R}, \varepsilon > 0. \quad (\text{V.3})$$

We get:

$$\|R(a, b)\|_{H^{-s_0}(\mathbb{R})} = \|R(a, b)\|_{H^s(\mathbb{R})} \lesssim \|a\|_{H^s(\mathbb{R})} \|b\|_{H^{s_0}(\mathbb{R})},$$

which ends the proof. \square

Another important result is the following commutator-type estimate:

Lemma V.2.2. *Let u, f be regular where u is a divergence free vector field in \mathbb{R}^3 . We have:*

$$\|[\Delta_q^v, S_{j-1}^v u^3(\cdot, x_3)]f\|_{L^2 H_h^{-\frac{1}{2}}} \lesssim 2^{-q} \|S_{j-1}^v \nabla_h u(\cdot, x_3)\|_{L^\infty L_h^2} \|f\|_{L^2 H_h^{\frac{1}{2}}}.$$

Proof

The proof is essentially based on the fact that $-\partial_3 u^3 = \nabla_h \cdot u^h$ and the following usual commutator estimate used with respect to the vertical variable:

$$\|[\Delta_j, a]b\|_{L^r} \lesssim 2^{-j} \|\nabla a\|_{L^p} \|b\|_{L^q}, \quad \text{with } \frac{1}{r} = \frac{1}{p} + \frac{1}{q}. \quad (\text{V.4})$$

For the proof of estimates of type (V.4) one may see for example [8], and for a detailed proof of Lemma V.2.2, one may see [28]. \square

Let $\langle f, g \rangle \stackrel{\text{def}}{=} \langle f, g \rangle_{L^2(\mathbb{R}^3)}$ be the usual L^2 -scalar product, and $\langle f, g \rangle_{\alpha, \beta}$ denotes the scalar product between f and g in $H^{\alpha, \beta}(\mathbb{R}^3)$. In order to simplify the redaction, we introduce the following notations:

$$\begin{aligned} L_1 &\stackrel{\text{def}}{=} \sum_{q \geq -1} 2^{2q(s-1)} \langle \Delta_q^v(u^h \cdot \nabla_h w), \Delta_q^v w \rangle, & L_2 &\stackrel{\text{def}}{=} \sum_{q \geq -1} 2^{2q(s-1)} \langle \Delta_q^v(u^3 \partial_3 w), \Delta_q^v w \rangle, \\ L_3 &\stackrel{\text{def}}{=} \sum_{q \geq -1} 2^{2q(s-1)} \langle \Delta_q^v(w^h \cdot \nabla_h v), \Delta_q^v w \rangle, & L_4 &\stackrel{\text{def}}{=} \sum_{q \geq -1} 2^{2q(s-1)} \langle \Delta_q^v(w^3 \partial_3 v), \Delta_q^v w \rangle, \\ L_5 &\stackrel{\text{def}}{=} \sum_{q \geq -1} 2^{-2qs} \langle \Delta_q^v(u^h \cdot \nabla_h \theta), \Delta_q^v \theta \rangle, & L_6 &\stackrel{\text{def}}{=} \sum_{q \geq -1} 2^{-2qs} \langle \Delta_q^v(u^3 \partial_3 \theta), \Delta_q^v \theta \rangle, \\ L_7 &\stackrel{\text{def}}{=} \sum_{q \geq -1} 2^{-2qs} \langle \Delta_q^v(w^h \cdot \nabla_h \eta), \Delta_q^v \theta \rangle, & L_8 &\stackrel{\text{def}}{=} \sum_{q \geq -1} 2^{-2qs} \langle \Delta_q^v(w^3 \partial_3 \eta), \Delta_q^v \theta \rangle, \\ L_9 &\stackrel{\text{def}}{=} \sum_{q \geq -1} 2^{2q(s-1)} \langle \Delta_q^v \theta, \Delta_q^v(w^3) \rangle. \end{aligned}$$

We shall prove:

Proposition V.2.1. *Let $s \in]\frac{1}{2}, 1]$. Then for $u, v, w, \rho, \eta, \theta$ verifying:*

$$\begin{aligned} u, v, \nabla_h u, \nabla_h v &\in H^{0, s}, & \rho, \eta, \nabla_h \rho, \nabla_h \eta &\in H^{0, 1-s}, \\ w, \nabla_h w &\in H^{0, s-1}, & \theta, \nabla_h \theta &\in H^{0, -s}, \\ \operatorname{div} u &= \operatorname{div} v = \operatorname{div} w = 0, \end{aligned}$$

we have:

$$\begin{aligned}
 L_1 &\lesssim \|u\|_{\frac{1}{2},s} \|\nabla_h w\|_{0,s-1} \|w\|_{\frac{1}{2},s-1}, & L_2 &\lesssim \|\nabla_h u\|_{0,s} \|w\|_{\frac{1}{2},s-1}^2, \\
 L_3 &\lesssim \|\nabla_h v\|_{0,s} \|w\|_{\frac{1}{2},s-1}^2, & L_4 &\lesssim \|v\|_{\frac{1}{2},s} (\|w\|_{0,s-1} + \|\nabla_h w\|_{0,s-1}) \|w\|_{\frac{1}{2},s-1}, \\
 L_5 &\lesssim \|u\|_{\frac{1}{2},s} \|\nabla_h \theta\|_{0,-s} \|\theta\|_{\frac{1}{2},-s}, & L_6 &\lesssim \|\nabla_h u\|_{0,s} \|\theta\|_{\frac{1}{2},-s}^2, \\
 L_7 &\lesssim \|\nabla_h \eta\|_{0,1-s} \|w\|_{\frac{1}{2},s-1} \|\theta\|_{\frac{1}{2},-s}, & L_8 &\lesssim \|\eta\|_{\frac{1}{2},1-s} (\|w\|_{0,s-1} + \|\nabla_h w\|_{0,s-1}) \|\theta\|_{\frac{1}{2},-s}, \\
 L_9 &\lesssim \|\theta\|_{0,-s} (\|\nabla_h w\|_{0,s-1} + \|w\|_{0,s-1}).
 \end{aligned}$$

Proof

In the following we denote by c_q some constant $c_q \stackrel{def}{=} c_q(u, v, w, \theta, \rho, t)$ with $\sum_{q \geq -1} c_q^2 \leq 1$

which comes from the fact (A.4) or (A.5). This constant is allowed to differ from one line to another.

• **L_1 estimate**

Since $s + (s - 1) > 0$, by using product Lemma V.2.1 between $H^{\frac{1}{2},s}$ and $H^{0,s-1}$, we obtain

$$\begin{aligned}
 L_1 &= \langle u^h \cdot \nabla_h w, w \rangle_{0,s-1} \leq \|u^h \cdot \nabla_h w\|_{-\frac{1}{2},s-1} \|w\|_{\frac{1}{2},s-1} \\
 &\lesssim \|u\|_{\frac{1}{2},s} \|\nabla_h w\|_{0,s-1} \|w\|_{\frac{1}{2},s-1}.
 \end{aligned}$$

• **L_2 estimate**

We write $L_2 = L_2^{(1)} + L_2^{(2)} + L_2^{(3)}$ where

$$\begin{aligned}
 L_2^{(1)} &\stackrel{def}{=} \sum_{q \geq -1} 2^{2q(s-1)} \sum_{|j-q| \leq N_0} \langle \Delta_q^v (\Delta_j^v (u^3) S_{j-1}^v (\partial_3 w)), \Delta_q^v w \rangle, \\
 L_2^{(2)} &\stackrel{def}{=} \sum_{q \geq -1} 2^{2q(s-1)} \sum_{|j-q| \leq N_0} \langle \Delta_q^v (S_{j-1}^v (u^3) \Delta_j^v (\partial_3 w)), \Delta_q^v w \rangle, \\
 L_2^{(3)} &\stackrel{def}{=} \sum_{q \geq -1} 2^{2q(s-1)} \sum_{i \in \{0, -1, 1\}} \sum_{j \geq q - N_0} \langle \Delta_q^v (\Delta_{j+1}^v (u^3) \Delta_j^v (\partial_3 w)), \Delta_q^v w \rangle.
 \end{aligned}$$

Then, by using the embedding of $H^{\frac{1}{2}}(\mathbb{R}_h^2)$ in $L^4(\mathbb{R}_h^2)$, Bernstein Lemma for the vertical

variable together with statement (A.4), we obtain

$$\begin{aligned}
 L_2^{(1)} &\lesssim \|w\|_{\frac{1}{2},s-1} \sum_{q \geq -1} c_q 2^{q(s-1)} 2^{q/2} \sum_{|j-q| \leq N_0} \|S_{j-1}^v w\|_{L_h^4 L_v^2} \|\Delta_j^v \nabla_h u\|_{L^2} \\
 &\lesssim \|w\|_{\frac{1}{2},s-1} \|w\|_{\frac{1}{2},s-1} \|\nabla_h u\|_{0,s} \sum_{q \geq -1} c_q 2^{q(s-\frac{1}{2})} \sum_{|j-q| \leq N_0} c_j^2 2^{j(1-s)} 2^{-sj} \\
 &\lesssim \|w\|_{\frac{1}{2},s-1} \|w\|_{\frac{1}{2},s-1} \|\nabla_h u\|_{0,s} \sum_{q \geq -1} c_q 2^{q(s-\frac{1}{2})} 2^{q(1-s)} 2^{-qs} \\
 &\lesssim \|w\|_{\frac{1}{2},s-1} \|w\|_{\frac{1}{2},s-1} \|\nabla_h u\|_{0,s} \sum_{q \geq -1} c_q 2^{-q(s-\frac{1}{2})} \\
 &\lesssim \|w\|_{\frac{1}{2},s-1} \|w\|_{\frac{1}{2},s-1} \|\nabla_h u\|_{0,s}.
 \end{aligned}$$

To estimate $L_2^{(2)}$, we consider the decomposition used in [28], by writing $L_2^{(2)} = A_1 + A_2 + A_3$, with

$$\begin{aligned}
 A_1 &\stackrel{def}{=} \sum_{q \geq -1} 2^{2q(s-1)} \langle S_q^v(u^3) \Delta_q^v(\partial_3 w), \Delta_q^v w \rangle, \\
 A_2 &\stackrel{def}{=} \sum_{q \geq -1} 2^{2q(s-1)} \sum_{|j-q| \leq N_0} \langle (S_q^v - S_{j-1}^v)(u^3) \Delta_j^v(\partial_3 w), \Delta_q^v w \rangle, \\
 A_3 &\stackrel{def}{=} \sum_{q \geq -1} 2^{2q(s-1)} \sum_{|j-q| \leq N_0} \langle [\Delta_q^v, S_{j-1}^v(u^3)] \Delta_j^v(\partial_3 w), \Delta_q^v w \rangle,
 \end{aligned}$$

where $[\Delta_q^v, S_{j-1}^v(u^3)]$ denotes the commutator between Δ_q^v and $S_{j-1}^v(u^3)$. After integration by parts we obtain

$$\begin{aligned}
 A_1 &= -\frac{1}{2} \sum_{q \geq -1} 2^{2q(s-1)} \langle S_q^v(\partial_3 u^3) \Delta_q^v(w), \Delta_q^v w \rangle \\
 &= \frac{1}{2} \sum_{q \geq -1} 2^{2q(s-1)} \langle S_q^v(\nabla_h \cdot u^h) \Delta_q^v(w), \Delta_q^v w \rangle \\
 &\lesssim \|w\|_{\frac{1}{2},s-1} \sum_{q \geq -1} c_q 2^{-q(s-1)} \|\Delta_q^v w\|_{L_h^4 L_v^2} \|S_q^v(\nabla_h u)\|_{L_h^2 L_v^\infty} \\
 &\lesssim \|w\|_{\frac{1}{2},s-1}^2 \|\nabla_h u\|_{0,s}.
 \end{aligned}$$

In order to estimate A_2 , we remark first that $S_q^v - S_{j-1}^v$ is supported away from 0 in Fourier

side, that is we can use Lemma A.2.2 to estimate A_2 just like $L_2^{(1)}$. Indeed

$$\begin{aligned}
 A_2 &\lesssim \|w\|_{\frac{1}{2},s-1} \sum_{q \geq -1} c_q 2^{q(s-1)} \sum_{|j-q| \leq N_0} \left\| (S_q^v - S_{j-1}^v) \partial_3 u^3 \right\|_{L_h^2 L_v^\infty} \left\| \Delta_j^v w \right\|_{L_h^4 L_v^2} \\
 &\lesssim \|w\|_{\frac{1}{2},s-1} \sum_{q \geq -1} c_q 2^{q(s-1)} \sum_{|j-q| \leq N_0} \left\| (S_q^v - S_{j-1}^v) \nabla u^h \right\|_{L_h^2 L_v^\infty} \left\| \Delta_j^v w \right\|_{L_h^4 L_v^2} \\
 &\lesssim \|w\|_{\frac{1}{2},s-1}^2 \|\nabla_h u\|_{0,s} \sum_{|i| \leq N_0} \sum_{q \geq -1} c_q 2^{q(s-1)} c_{j+i} 2^{(q+i)(1-s)} \\
 &\lesssim \|w\|_{\frac{1}{2},s-1}^2 \|\nabla_h u\|_{0,s}.
 \end{aligned}$$

Finally, for A_3 we use the commutator estimate proved in Lemma V.2.2 to obtain

$$\begin{aligned}
 A_3 &\lesssim \|w\|_{\frac{1}{2},s-1} \sum_{q \geq -1} c_q 2^{q(s-1)} \sum_{|j-q| \leq N_0} \left\| [\Delta_q^v, S_{j-1}^v(u^3)] \Delta_j^v(\partial_3 w) \right\|_{L^2(\mathbb{R}_{x_3}; H^{-\frac{1}{2}}(\mathbb{R}^2))} \\
 &\lesssim \|w\|_{\frac{1}{2},s-1} \sum_{q \geq -1} c_q 2^{q(s-1)} \sum_{|j-q| \leq N_0} \left\| S_{j-1}^v \nabla_h u(\cdot, x_3) \right\|_{L_h^2 L_v^\infty} \left\| \Delta_j^v w \right\|_{\frac{1}{2},0} \\
 &\lesssim \|w\|_{\frac{1}{2},s-1}^2 \|\nabla_h u\|_{0,s} \sum_{i \in \{0, -1, 1\}} \sum_{q \geq -1} c_q c_{q+i} \\
 &\lesssim \|w\|_{\frac{1}{2},s-1}^2 \|\nabla_h u\|_{0,s}.
 \end{aligned}$$

Ditto for the last term in this part, using the fact that $\partial_3 u^3 = -\nabla_h \cdot u^h$, it happens

$$\begin{aligned}
 L_2^{(3)} &\lesssim \|w\|_{\frac{1}{2},s-1} \sum_{q \geq -1} c_q 2^{q(s-1)} 2^{q/2} \sum_{i \in \{0, -1, 1\}} \sum_{j \geq q - N_0} \left\| \Delta_{j+i}^v w \right\|_{L_h^4 L_v^2} \left\| \Delta_j^v \nabla_h u \right\|_{L^2} \\
 &\lesssim \|w\|_{\frac{1}{2},s-1} \|w\|_{\frac{1}{2},s-1} \|\nabla_h u\|_{0,s} \sum_{q \geq -1} c_q 2^{q(s-\frac{1}{2})} \sum_{i \in \{0, -1, 1\}} \sum_{j \geq q - N_0} c_j c_{j+i} 2^{j(1-s)} 2^{-sj} \\
 &\lesssim \|w\|_{\frac{1}{2},s-1}^2 \|\nabla_h u\|_{0,s} \sum_{q \geq -1} c_q 2^{q(s-\frac{1}{2})} 2^{q(1-2s)} \sum_{i \in \{0, -1, 1\}} \sum_{j \geq q - N_0} c_j c_{j+i},
 \end{aligned}$$

where we used the fact that $s \in]\frac{1}{2}, 1]$ that is $1 - 2s < 0$. We obtain finally

$$\begin{aligned}
 L_2^{(3)} &\lesssim \|w\|_{\frac{1}{2},s-1}^2 \|\nabla_h u\|_{0,s} \sum_{q \geq -1} c_q 2^{-q(s-\frac{1}{2})} \\
 &\lesssim \|w\|_{\frac{1}{2},s-1}^2 \|\nabla_h u\|_{0,s}.
 \end{aligned}$$

Remark In the case where $s = 1$ we do not have to deal with $L_1 + L_2$ which is equal to 0 because of the identity $\langle u \cdot \nabla w, w \rangle = 0$.

• **L_3 estimate**

By using product Lemma V.2.1 between $H^{0,s}$ and $H^{\frac{1}{2},s-1}$ we obtain

$$\begin{aligned}
 L_3 &= \langle w^h \nabla_h v, w \rangle_{0,s-1} \leq \left\| w^h \nabla_h v \right\|_{-\frac{1}{2},s-1} \|w\|_{\frac{1}{2},s-1} \\
 &\lesssim \|\nabla_h v\|_{0,s} \|w\|_{\frac{1}{2},s-1}^2.
 \end{aligned}$$

• L_4 estimate

We write $L_4 = L_4^{(1)} + L_4^{(2)}$, where

$$L_4^{(1)} \stackrel{def}{=} \sum_{q \geq -1} 2^{2q(s-1)} \sum_{j \geq q-N_0} \langle \Delta_q^v (\Delta_j^v w^3 S_{j+2}^v (\partial_3 v)), \Delta_q^v w \rangle,$$

$$L_4^{(2)} \stackrel{def}{=} \sum_{q \geq -1} 2^{2q(s-1)} \sum_{|j-q| \leq N_0} \langle \Delta_q^v (S_{j-1}^v (w^3) \Delta_j^v \partial_3 v), \Delta_q^v w \rangle.$$

Hence, by using again Lemma V.2.1, we infer that

$$L_4^{(1)} \leq \|w\|_{\frac{1}{2}, s-1} \sum_{q \geq -1} c_q 2^{q(s-\frac{1}{2})} \sum_{j \geq q-N_0} \|\Delta_j^v w^3\|_{L^2} \|S_{j+2}^v (\partial_3 v)\|_{L_h^4 L_v^2}.$$

For $s \neq 1$, after certain calculations, we get

$$\begin{aligned} \|S_{j+2}^v (\partial_3 v)\|_{L_h^4 L_v^2} &\leq \sum_{m \leq j+1} 2^{m(1-s)} 2^{sm} \|\Delta_m^v v\|_{L_h^4 L_v^2} \\ &\leq \left(\sum_{m \leq j+1} 2^{2m(1-s)} \right)^{\frac{1}{2}} \|v\|_{\frac{1}{2}, s} \\ &\lesssim 2^{j(1-s)} \|v\|_{\frac{1}{2}, s}. \end{aligned}$$

Thus, by using Lemma A.2.2 together with the previous estimate and the divergence free condition on w , we find

$$\begin{aligned} L_4^{(1)} &\lesssim \|w\|_{\frac{1}{2}, s-1} \|v\|_{\frac{1}{2}, s} \sum_{q \geq -1} c_q 2^{q(s-\frac{1}{2})} \sum_{j \geq q-N_0} 2^{-js} \|\Delta_j^v \nabla_h w\|_{L^2} \\ &\lesssim \|w\|_{\frac{1}{2}, s-1} \|v\|_{\frac{1}{2}, s} \|\nabla_h w\|_{0, s-1} \sum_{q \geq -1} c_q 2^{q(s-\frac{1}{2})} \sum_{j \geq q-N_0} c_j 2^{j(1-2s)} \\ &\lesssim \|w\|_{\frac{1}{2}, s-1} \|v\|_{\frac{1}{2}, s} \|\nabla_h w\|_{0, s-1} \sum_{q \geq -1} c_q 2^{q(s-\frac{1}{2})} \left(\sum_{j \geq q-N_0} 2^{2j(1-2s)} \right)^{\frac{1}{2}} \|c_j\|_{l^2(\mathbb{N} \cup \{-1\})} \\ &\lesssim \|w\|_{\frac{1}{2}, s-1} \|v\|_{\frac{1}{2}, s} \|\nabla_h w\|_{0, s-1} \sum_{q \geq -1} c_q 2^{q(s-\frac{1}{2})} 2^{q(1-2s)} \\ &\lesssim \|w\|_{\frac{1}{2}, s-1} \|v\|_{\frac{1}{2}, s} \|\nabla_h w\|_{0, s-1} \sum_{q \geq -1} c_q 2^{q(\frac{1}{2}-s)} \\ &\lesssim \|w\|_{\frac{1}{2}, s-1} \|v\|_{\frac{1}{2}, s} \|\nabla_h w\|_{0, s-1}. \end{aligned}$$

For the second term we proceed as follows

$$\begin{aligned}
 L_4^{(2)} &\lesssim \|w\|_{\frac{1}{2},s-1} \sum_{q \geq -1} c_q 2^{q(s-1)} 2^q \sum_{|j-q| \leq N_0} 2^{j-q} \|S_{j-1}^v w^3\|_{L_h^2 L_v^\infty} \|\Delta_j^v v\|_{L_h^4 L_v^2} \\
 &\lesssim \|w\|_{\frac{1}{2},s-1} \|w^3\|_{0,s} \sum_{|j| \leq N_0} \sum_{q \geq -1} c_q 2^{q(s-1)} 2^q \|\Delta_{j+q}^v v\|_{L_h^4 L_v^2} \\
 &\lesssim \|v\|_{\frac{1}{2},s} \|w\|_{\frac{1}{2},s-1} \|w^3\|_{0,s} \sum_{|i| \leq N_0} \sum_{q \geq -1} c_q c_{q+i} \\
 &\lesssim \|v\|_{\frac{1}{2},s} \|w\|_{\frac{1}{2},s-1} \|w^3\|_{0,s}.
 \end{aligned}$$

In order to close the estimates of $L_4^{(2)}$ we remark that, for any $s \in [\frac{1}{2}, 1]$, we have

$$\begin{aligned}
 \|w^3\|_{0,s} &\leq \|w^3\|_{0,s-1} + \|\partial_3 w^3\|_{0,s-1} \\
 &\leq \|w^3\|_{0,s-1} + \|\nabla_h w\|_{0,s-1}.
 \end{aligned}$$

In the case where $s = 1$, note that the estimate can be obtained easily, by using product rules and the previous inequality, as the following

$$\begin{aligned}
 \langle w^3 \partial_3 v, w \rangle_{L^2} &\leq \|w^3 \partial_3 v\|_{-\frac{1}{2},0} \|w\|_{\frac{1}{2},0} \\
 &\lesssim \|w^3\|_{0,1} \|\partial_3 v\|_{\frac{1}{2},0} \|w\|_{\frac{1}{2},0} \\
 &\lesssim \|v\|_{\frac{1}{2},1} (\|w\|_{L^2} + \|\nabla_h w\|_{L^2}) \|w\|_{\frac{1}{2},0}.
 \end{aligned}$$

• **L_5 estimate**

In order to estimate this term we proceed by duality by inferring firstly that

$$L_5 \leq \|u^h \nabla_h \theta\|_{-\frac{1}{2},-s} \|\theta\|_{\frac{1}{2},-s}.$$

Moreover, Lemma V.2.1 gives

$$\|u^h \nabla_h \theta\|_{-\frac{1}{2},-s} \lesssim \|u^h\|_{\frac{1}{2},s} \|\nabla_h \theta\|_{0,-s}.$$

It follows that

$$L_5 \lesssim \|u^h\|_{\frac{1}{2},s} \|\nabla_h \theta\|_{0,-s} \|\theta\|_{\frac{1}{2},-s}.$$

• **L_6 estimate**

We use the Bony's decomposition $L_6 = L_6^{(1)} + L_6^{(2)} + L_6^{(3)}$, where

$$\begin{aligned}
 L_6^{(1)} &\stackrel{def}{=} \sum_{q \geq -1} 2^{2q(-s)} \sum_{|j-q| \leq N_0} \langle \Delta_q^v (\Delta_j^v (u^3) S_{j-1}^v (\partial_3 \theta)), \Delta_q^v \theta \rangle, \\
 L_6^{(2)} &\stackrel{def}{=} \sum_{q \geq -1} 2^{2q(-s)} \sum_{|j-q| \leq N_0} \langle \Delta_q^v (S_{j-1}^v (u^3) \Delta_j^v (\partial_3 \theta)), \Delta_q^v \theta \rangle, \\
 L_6^{(3)} &\stackrel{def}{=} \sum_{q \geq -1} 2^{2q(-s)} \sum_{i \in \{0, -1, 1\}} \sum_{j \geq q - N_0} \langle \Delta_q^v (\Delta_{j+1}^v (u^3) \Delta_j^v (\partial_3 \theta)), \Delta_q^v \theta \rangle.
 \end{aligned}$$

CHAPTER V. BOUSSINESQ – HORIZONTAL DISSIPATION

For the first term, we use the Bernstein Lemma together with usual Sobolev embedding and the free divergence condition to obtain

$$\begin{aligned}
 L_6^{(1)} &\leq C \|\theta\|_{\frac{1}{2}, -s} \sum_{q \geq -1} c_q 2^{-qs} 2^{q/2} \sum_{|j-q| \leq N_0} \|S_{j-1}^v \theta\|_{L_h^4 L_v^2} \|\Delta_j^v \nabla_h \cdot u^h\|_{L^2} \\
 &\lesssim \|\theta\|_{\frac{1}{2}, -s} \|\theta\|_{\frac{1}{2}, s-1} \|\nabla_h u\|_{0, s} \sum_{q \geq -1} c_q 2^{q(\frac{1}{2}-s)} \sum_{|j-q| \leq N_0} c_j^2 \\
 &\lesssim \|\theta\|_{\frac{1}{2}, -s}^2 \|\nabla_h u\|_{0, s}.
 \end{aligned}$$

For $L_6^{(2)}$, we follow the same decomposition used for $L_2^{(2)}$, so we write $L_6^{(2)} = B_1 + B_2 + B_3$, where

$$\begin{aligned}
 B_1 &\stackrel{def}{=} \sum_{q \geq -1} 2^{-2qs} \langle S_q^v(u^3) \Delta_q^v(\partial_3 \theta), \Delta_q^v \theta \rangle, \\
 B_2 &\stackrel{def}{=} \sum_{q \geq -1} 2^{-2qs} \sum_{|j-q| \leq N_0} \langle (S_q^v - S_{j-1}^v)(u^3) \Delta_j^v(\partial_3 \theta), \Delta_q^v \theta \rangle, \\
 B_3 &\stackrel{def}{=} \sum_{q \geq -1} 2^{-2qs} \sum_{|j-q| \leq N_0} \langle [\Delta_q^v, S_{j-1}^v(u^3)] \Delta_j^v(\partial_3 \theta), \Delta_q^v \theta \rangle.
 \end{aligned}$$

After integration by parts, we obtain

$$\begin{aligned}
 B_1 &= \frac{1}{2} \sum_{q \geq -1} 2^{-2qs} \langle S_q^v(\partial_3 u^3) \Delta_q^v(\theta), \Delta_q^v \theta \rangle \\
 &= \frac{1}{2} \sum_{q \geq -1} 2^{-2qs} \langle S_q^v(\nabla_h u^h) \Delta_q^v(\theta), \Delta_q^v \theta \rangle \\
 &\lesssim \|\theta\|_{\frac{1}{2}, -s} \sum_{q \geq -1} c_q 2^{-qs} \|\Delta_q^v \theta\|_{L_h^4 L_v^2} \|S_q^v(\nabla_h u)\|_{L_h^2 L_v^\infty} \\
 &\lesssim \|\theta\|_{\frac{1}{2}, s-1}^2 \|\nabla_h u\|_{0, s}.
 \end{aligned}$$

To estimate B_2 we remark first that $S_q^v - S_{j-1}^v$ is supported away from 0 in Fourier side, that is we can use Lemma A.2.2 and estimate B_2 as A_2 , indeed

$$\begin{aligned}
 B_2 &\lesssim \|\theta\|_{\frac{1}{2}, -s} \sum_{q \geq -1} c_q 2^{-qs} \sum_{|j-q| \leq N_0} \|(S_q^v - S_{j-1}^v) \partial_3 u^3\|_{L_h^2 L_v^\infty} \|\Delta_j^v \theta\|_{L_h^4 L_v^2} \\
 &\lesssim \|\theta\|_{\frac{1}{2}, -s} \sum_{q \geq -1} c_q 2^{-qs} \sum_{|j-q| \leq N_0} \|(S_q^v - S_{j-1}^v) \nabla_h u^h\|_{L_h^2 L_v^\infty} \|\Delta_j^v \theta\|_{L_h^4 L_v^2} \\
 &\lesssim \|\theta\|_{\frac{1}{2}, -s}^2 \|\nabla_h u\|_{0, s} \sum_{|i| \leq N_0} \sum_{q \geq -1} c_q c_{q+i} \\
 &\lesssim \|\theta\|_{\frac{1}{2}, -s}^2 \|\nabla_h u\|_{0, s}.
 \end{aligned}$$

Finally, for B_3 we use the commutator estimate proved in Lemma V.2.2 to obtain

$$\begin{aligned}
 B_3 &\lesssim \|\theta\|_{\frac{1}{2},-s} \sum_{q \geq -1} c_q 2^{-qs} \sum_{|j-q| \leq N_0} \left\| [\Delta_q^v, S_{j-1}^v(u^3)] \Delta_j^v(\partial_3 \theta) \right\|_{L^2(\mathbb{R}_{x_3}; H^{-\frac{1}{2}}(\mathbb{R}^2))} \\
 &\lesssim \|\theta\|_{\frac{1}{2},-s} \sum_{q \geq -1} c_q 2^{-qs} \sum_{|j-q| \leq N_0} \left\| S_{j-1}^v \nabla_h u(\cdot, x_3) \right\|_{L_h^2 L_v^\infty} \left\| \Delta_j^v \theta \right\|_{\frac{1}{2},0} \\
 &\lesssim \|\theta\|_{\frac{1}{2},-s}^2 \|\nabla_h u\|_{0,s} \sum_{i \in \{0,-1,1\}} \sum_{q \geq -1} c_q c_{q+i} \\
 &\lesssim \|\theta\|_{\frac{1}{2},-s}^2 \|\nabla_h u\|_{0,s}.
 \end{aligned}$$

For $L_6^{(3)}$, by using the same arguments we find

$$\begin{aligned}
 L_6^{(3)} &\lesssim \|\theta\|_{\frac{1}{2},-s} \sum_{q \geq -1} c_q 2^{-qs} 2^{q/2} \sum_{i \in \{0,-1,1\}} \sum_{j \geq q-N_0} \left\| \Delta_{j+i}^v \theta \right\|_{L_h^4 L_v^2} \left\| \Delta_j^v \nabla_h \cdot u^h \right\|_{L^2} \\
 &\lesssim \|\theta\|_{\frac{1}{2},-s} \|\theta\|_{\frac{1}{2},-s} \|\nabla_h u\|_{0,s} \sum_{q \geq -1} c_q 2^{-q(s-\frac{1}{2})} \sum_{i \in \{0,-1,1\}} \sum_{j \geq q-N_0} c_j c_{j+i} \\
 &\lesssim \|\theta\|_{\frac{1}{2},-s}^2 \|\nabla_h u\|_{0,s}.
 \end{aligned}$$

• **L_7 estimate**

This term can be estimated by using the following property based on product rules in dimension one together with inequality (V.3)

$$H^{s-1}(\mathbb{R}) \cdot H^{1-s}(\mathbb{R}) \subset B_{2,\infty}^{-\frac{1}{2}}(\mathbb{R}) \hookrightarrow H^{-s}.$$

Indeed, based on the Bony's decomposition with respect to the vertical variable we write $L_7 = L_7^{(1)} + L_7^{(2)} + L_7^{(3)}$, where

$$\begin{aligned}
 L_7^{(1)} &\stackrel{def}{=} \sum_{q \geq -1} 2^{-2qs} \sum_{|j-q| \leq N_0} \langle \Delta_q^v(S_{j-1}^v(w^h) \Delta_j^v(\nabla_h \eta)), \Delta_q^v \theta \rangle, \\
 L_7^{(2)} &\stackrel{def}{=} \sum_{q \geq -1} 2^{-2qs} \sum_{|j-q| \leq N_0} \langle \Delta_q^v(\Delta_j^v(w^h) S_{j-1}^v(\nabla_h \eta)), \Delta_q^v \theta \rangle, \\
 L_7^{(3)} &\stackrel{def}{=} \sum_{q \geq -1} 2^{-2qs} \sum_{i \in \{0,-1,1\}} \sum_{j \geq q-N_0} \langle \Delta_q^v(\Delta_j^v(w^h) \Delta_{j+i}^v(\nabla_h \eta)), \Delta_q^v \theta \rangle.
 \end{aligned}$$

By using inequality (A.5), similar arguments give then

$$\begin{aligned}
 L_7^{(1)} &\leq \|\theta\|_{\frac{1}{2},-s} \sum_{q \geq -1} c_q 2^{-q(s-\frac{1}{2})} \sum_{|j-q| \leq N_0} \left\| S_{j-1}^v w \right\|_{L_h^4 L_v^2} \left\| \Delta_j^v(\nabla_h \eta) \right\|_{L^2} \\
 &\leq \|\nabla_h \eta\|_{0,1-s} \|w\|_{\frac{1}{2},s-1} \|\theta\|_{\frac{1}{2},-s} \sum_{q \geq -1} c_q 2^{-q(s-\frac{1}{2})} \sum_{|j-q| \leq N_0} c_j^2 \\
 &\lesssim \|\nabla_h \eta\|_{0,1-s} \|w\|_{\frac{1}{2},s-1} \|\theta\|_{\frac{1}{2},-s}.
 \end{aligned}$$

For the second term we proceed as follows

$$\begin{aligned}
 L_7^{(2)} &\leq \|\theta\|_{\frac{1}{2}, -s} \sum_{q \geq -1} c_q 2^{-q(s-\frac{1}{2})} \sum_{|j-q| \leq N_0} 2^{\frac{1}{2}(j-q)} (\|\Delta_j^v w\|_{L_h^4 L_v^2} 2^{j(s-1)}) (2^{j(1-s-\frac{1}{2})} \|S_{j-1}^v(\nabla_h \eta)\|_{L_h^2 L_v^\infty}) \\
 &\leq \|\nabla_h \eta\|_{B_{\infty,2}^{0,1-s-\frac{1}{2}}} \|w\|_{\frac{1}{2}, s-1} \|\theta\|_{\frac{1}{2}, -s} \sum_{q \geq -1} c_q 2^{-q(s-\frac{1}{2})} \sum_{|j-q| \leq N_0} c_j^2 \\
 &\lesssim \|\nabla_h \eta\|_{0,1-s} \|w\|_{\frac{1}{2}, s-1} \|\theta\|_{\frac{1}{2}, -s},
 \end{aligned}$$

where we used the embedding $H^{0,1-s} \hookrightarrow B_{\infty,2}^{0,1-s-\frac{1}{2}}$ and the fact that $1-s-\frac{1}{2} < 0$. For the last term we proceed as follows

$$\begin{aligned}
 L_7^{(3)} &\leq \|\theta\|_{\frac{1}{2}, -s} \sum_{q \geq -1} c_q 2^{-q(s-\frac{1}{2})} \sum_{i \in \{0, -1, 1\}} \sum_{j \geq q-N_0} \|\Delta_j^v w\|_{L_h^4 L_v^2} \|\Delta_{j+i}^v(\nabla_h \eta)\|_{L^2} \\
 &\leq \|\nabla_h \eta\|_{0,1-s} \|w\|_{\frac{1}{2}, s-1} \|\theta\|_{\frac{1}{2}, -s} \sum_{i \in \{0, -1, 1\}} \sum_{q \geq -1} c_q 2^{-q(s-\frac{1}{2})} \sum_{j \geq q-N_0} c_j c_{j+i} \\
 &\lesssim \|\nabla_h \eta\|_{0,1-s} \|w\|_{\frac{1}{2}, s-1} \|\theta\|_{\frac{1}{2}, -s}.
 \end{aligned}$$

• **L_8 estimate**

We write $L_8 = L_8^{(1)} + L_8^{(2)} + L_8^{(3)}$, where

$$L_8^{(1)} \stackrel{def}{=} \sum_{q \geq -1} 2^{-2qs} \sum_{|j-q| \leq N_0} \langle \Delta_q^v(\Delta_j^v w^3 S_{j-1}^v(\partial_3 \eta)), \Delta_q^v \theta \rangle,$$

$$L_8^{(2)} \stackrel{def}{=} \sum_{q \geq -1} 2^{-2qs} \sum_{|j-q| \leq N_0} \langle \Delta_q^v(S_{j-1}^v(w^3) \Delta_j^v \partial_3 \eta), \Delta_q^v \theta \rangle,$$

$$L_8^{(3)} \stackrel{def}{=} \sum_{q \geq -1} 2^{-2qs} \sum_{i \in \{0, -1, 1\}} \sum_{j \geq q-N_0} \langle \Delta_q^v(\Delta_j^v w^3 \Delta_{j+i}^v(\partial_3 \eta)), \Delta_q^v \theta \rangle.$$

Then, for the first term, we have

$$L_8^{(1)} \leq \|\theta\|_{\frac{1}{2}, -s} \sum_{q \geq -1} c_q 2^{-qs} \sum_{|j-q| \leq N_0} \|\Delta_j^v w^3\|_{L^2} \|S_{j-1}^v(\partial_3 \eta)\|_{L_h^4 L_v^\infty}.$$

Now we use

$$\begin{aligned}
 \|S_{j-1}^v(\partial_3 \eta)\|_{L_h^4 L_v^\infty} &\leq 2^j c_j 2^{-j(1-s-\frac{1}{2})} \|\eta\|_{H_h^{\frac{1}{2}}(B_{\infty,2}^{1-s-\frac{1}{2}})_v} \\
 &\lesssim 2^j c_j 2^{-j(1-s-\frac{1}{2})} \|\eta\|_{H^{\frac{1}{2}, 1-s}},
 \end{aligned}$$

and

$$\|\Delta_j^v w^3\|_{L^2} \leq 2^{-j} c_j 2^{-j(s-1)} \|\nabla_h w\|_{H^{0, s-1}}.$$

Therefore we find

$$\begin{aligned} L_8^{(1)} &\leq \|\theta\|_{\frac{1}{2}, -s} \|\nabla_h w\|_{H^{0, s-1}} \|\eta\|_{H^{\frac{1}{2}, 1-s}} \sum_{q \geq -1} c_q 2^{q(\frac{1}{2}-s)} \sum_{|j-q| \leq N_0} c_j^2 2^{\frac{1}{2}(j-q)} \\ &\lesssim \|\theta\|_{\frac{1}{2}, -s} \|\nabla_h w\|_{H^{0, s-1}} \|\eta\|_{H^{\frac{1}{2}, 1-s}}. \end{aligned}$$

Next, for the second term

$$\begin{aligned} L_8^{(2)} &\leq \|\theta\|_{\frac{1}{2}, -s} \sum_{q \geq -1} c_q \sum_{|j-q| \leq N_0} 2^{-(q-j)s} \|S_{j-1}^v w^3\|_{L_h^2 L_v^\infty} 2^{j(1-s)} \|\Delta_j^v(\eta)\|_{L_h^4 L_v^2} \\ &\leq \|\theta\|_{\frac{1}{2}, -s} \|w^3\|_{0, s} \|\eta\|_{\frac{1}{2}, 1-s} \sum_{q \geq -1} c_q \sum_{|j-q| \leq N_0} c_j \\ &\leq \|\theta\|_{\frac{1}{2}, -s} \|w^3\|_{0, s} \|\eta\|_{\frac{1}{2}, 1-s} \sum_{|j| \leq N_0} \sum_{q \geq -1} c_q c_{j+q} \\ &\lesssim \|\theta\|_{\frac{1}{2}, -s} (\|w^3\|_{0, s-1} + \|\nabla_h w\|_{0, s-1}) \|\eta\|_{\frac{1}{2}, 1-s}. \end{aligned}$$

For $L_8^{(3)}$ we have

$$\begin{aligned} L_8^{(3)} &\leq \|\theta\|_{\frac{1}{2}, -s} \sum_{q \geq -1} c_q 2^{-qs} 2^{q/2} \sum_{i \in 0, -1, 1} \sum_{j \geq q - N_0} \|\Delta_j^v w^3\|_{L^2} \|\Delta_{j+i}^v(\partial_3 \eta)\|_{L_h^4 L_v^2} \\ &\leq \|\theta\|_{\frac{1}{2}, -s} \sum_{q \geq -1} c_q 2^{q(\frac{1}{2}-s)} \sum_{i \in 0, -1, 1} \sum_{j \geq q - N_0} \|\Delta_j^v \nabla_h w\|_{L^2} \|\Delta_{j+i}^v \eta\|_{L_h^4 L_v^2} \\ &\leq \|\theta\|_{\frac{1}{2}, -s} \sum_{q \geq -1} c_q 2^{q(\frac{1}{2}-s)} \sum_{i \in 0, -1, 1} \sum_{j \geq q - N_0} 2^{j(s-1)} \|\Delta_j^v \nabla_h w\|_{L^2} 2^{j(1-s)} \|\Delta_{j+i}^v \eta\|_{L_h^4 L_v^2} \\ &\lesssim \|\theta\|_{\frac{1}{2}, -s} \|\nabla_h w\|_{0, s-1} \|\eta\|_{\frac{1}{2}, 1-s}. \end{aligned}$$

Finally, for the last term, we use the fact that $s \leq 1$ which implies that $2s - 1 \leq 1$, and that w is a free divergence vector field to infer that

$$\begin{aligned} L_9 &= 2^{-2(s-1)} \langle S_0^v \theta, S_0^v w^3 \rangle + \sum_{q \geq 0} 2^{2q(s-1)} \langle \Delta_q^v \theta, \Delta_q^v w^3 \rangle \\ &\lesssim \|S_0^v \theta\|_{L^2} \|S_0^v w\|_{L^2} + \sum_{q \geq 0} 2^{-qs} \|\Delta_q^v \theta\|_{L^2} 2^{q(s-1)} 2^{q(2s-1)} \|\Delta_q^v w^3\|_{L^2} \\ &\lesssim \|S_0^v \theta\|_{L^2} \|S_0^v w\|_{L^2} + \sum_{q \geq 0} 2^{-qs} \|\Delta_q^v \theta\|_{L^2} 2^{q(s-1)} \|\Delta_q^v \nabla_h \cdot w^h\|_{L^2} \\ &\lesssim \|\theta\|_{0, -s} (\|\nabla_h w\|_{0, s-1} + \|w\|_{0, s-1}). \end{aligned}$$

□

In the case where $s = \frac{1}{2}$, the estimates are more delicate since the space $H^{\frac{1}{2}}(\mathbb{R})$ is not an algebra. In the present paper, and in this particular case, we will just take up again the

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reasoning due to M. Paicu in [106], and use his estimates to treat all the equations in the same energy space of $H^{0,-\frac{1}{2}}$.

More precisely one may prove the following proposition:

Proposition V.2.2. *Let u, v, ρ and η be in space $L_T^\infty(H^{0,\frac{1}{2}})$ with $\nabla_h u, \nabla_h v, \nabla_h \rho, \nabla_h \eta$ in $L_T^2(H^{0,\frac{1}{2}})$ and u, v two divergence free vector fields. Let w, θ be in $L_T^\infty(H^{0,\frac{1}{2}})$ with $\nabla_h w$ and $\nabla_h \theta$ in $L_T^\infty(H^{0,\frac{1}{2}})$ solution to the following equations*

$$\begin{cases} \partial_t w + u \cdot \nabla w - \Delta_h w + \nabla \varpi = \theta e_3 - w \cdot \nabla v, \\ \partial_t \theta + u \cdot \nabla \theta - \Delta_h \theta = -w \cdot \nabla \eta, \\ \operatorname{div} w = 0. \end{cases}, \quad (\text{V.5})$$

Let $\chi(t) \stackrel{\text{def}}{=} \|w(t)\|_{0,-\frac{1}{2}}^2 + \|\theta(t)\|_{0,-\frac{1}{2}}^2$. If $\chi(t) \leq e^{-2}$, for all $0 < t < T$, then we have:

$$\frac{d}{dt} \chi(t) \leq C f(t) \chi(t) (1 - \ln \chi(t)) \ln(1 - \ln \chi(t)), \quad (\text{V.6})$$

where f is a locally integrable function depending on the norms of $u, v, w, \rho, \eta, \theta$ in $H^{0,\frac{1}{2}} \cap H^{1,\frac{1}{2}}$.

Proof

As mentioned before, the strategy is similar to the case $s > \frac{1}{2}$ but it needs more attention because of the lack of the embedding $H^{\frac{1}{2}}(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. The proof we propose here is a direct application of the estimates proved in [106]. In particular, note that the terms $u \cdot \nabla w$ and $u \cdot \nabla \theta$ (resp. $w \cdot \nabla v$ and $w \cdot \nabla \eta$) can be treated along the same way since we assume here that u and ρ (resp. v and η) have the same regularity. Some details are given below.

Let us recall the following estimate from the proof of Lemma 4.2 from [106]:

$$\begin{aligned} \sum_{q \geq -1} 2^{-q} \left(\langle \Delta_q^v(u \cdot \nabla w), \Delta_q^v w \rangle + \langle \Delta_q^v(w \cdot \nabla v), \Delta_q^v w \rangle \right) &\leq \frac{1}{50} \|\nabla_h w\|_{H^{0,-\frac{1}{2}}}^2 \\ &+ C f_1(t) \|w\|_{H^{0,-\frac{1}{2}}}^2 (1 - \|w\|_{H^{0,-\frac{1}{2}}}^2) \ln(1 - \ln \|w\|_{H^{0,-\frac{1}{2}}}^2), \end{aligned} \quad (\text{V.7})$$

where

$$f_1 \stackrel{\text{def}}{=} (1 + \|u\|_{1,\frac{1}{2}}^2 + \|v\|_{1,\frac{1}{2}}^2 + \|w\|_{1,\frac{1}{2}}^2) \times (1 + \|\nabla_h u\|_{1,\frac{1}{2}}^2 + \|\nabla_h v\|_{1,\frac{1}{2}}^2 + \|\nabla_h w\|_{1,\frac{1}{2}}^2).$$

Similar arguments can be used to establish the following estimate for the the second equation in (V.5)

$$\begin{aligned} \sum_{q \geq -1} 2^{-q} \left(\langle \Delta_q^v(u \cdot \nabla \theta), \Delta_q^v \theta \rangle + \langle \Delta_q^v(\theta \cdot \nabla \eta), \Delta_q^v \theta \rangle \right) &\leq \frac{1}{50} \|\nabla_h \theta\|_{H^{0,-\frac{1}{2}}}^2 \\ &+ C f_2(t) \|\theta\|_{H^{0,-\frac{1}{2}}}^2 (1 - \|\theta\|_{H^{0,-\frac{1}{2}}}^2) \ln(1 - \ln \|\theta\|_{H^{0,-\frac{1}{2}}}^2), \end{aligned} \quad (\text{V.8})$$

where

$$f_2 \stackrel{\text{def}}{=} (1 + \|u\|_{1,\frac{1}{2}}^2 + \|\eta\|_{1,\frac{1}{2}}^2 + \|\theta\|_{1,\frac{1}{2}}^2) \times (1 + \|\nabla_h u\|_{1,\frac{1}{2}}^2 + \|\nabla_h \eta\|_{1,\frac{1}{2}}^2 + \|\nabla_h \theta\|_{1,\frac{1}{2}}^2).$$

Both estimates (V.7) and (V.8) hold under the assumption $\chi(t) = \|w(t)\|_{0,-\frac{1}{2}}^2 + \|\theta(t)\|_{0,-\frac{1}{2}}^2 \leq e^{-2}$. Finally, the estimate of θe_3 is the following

$$\sum_{q \geq -1} 2^{-q} \langle \Delta_q^v \theta e_3, \Delta_q^v w \rangle \leq \|w\|_{H^{0,-\frac{1}{2}}} \|\theta\|_{H^{0,-\frac{1}{2}}} \leq \|w\|_{H^{0,-\frac{1}{2}}}^2 + \|\theta\|_{H^{0,-\frac{1}{2}}}^2. \quad (\text{V.9})$$

On the other hand, for $0 < x \ll 1$, the function $x \mapsto x(1 - \ln(x))\ln(1 - \ln(x))$ is non-decreasing. It is then easy to deduce (V.6) from (V.7), (V.8) and (V.9) by setting

$$f(t) \stackrel{\text{def}}{=} f_1(t) + f_2(t) + 1.$$

□

V.3 Proof of the main theorems

V.3.1 Proof of the uniqueness result “Theorem V.1.1”

In this subsection, we aim at showing how one can benefit of the a priori estimates proved in the section 1 in order to obtain the uniqueness result stated in Theorem V.1.1

Proof of theorem V.1.1 Let $(u, \rho, P), (v, \eta, \Pi)$ be two solutions for system (NSB_h) , and $w \stackrel{\text{def}}{=} u - v, \theta \stackrel{\text{def}}{=} \rho - \eta, \varpi \stackrel{\text{def}}{=} P - \Pi$ denote the difference functions. Then (w, θ, ϖ) satisfies

$$\begin{cases} (\partial_t + u \cdot \nabla)w - \Delta_h w + \nabla \varpi = \theta e_3 - w \cdot \nabla v, \\ (\partial_t + u \cdot \nabla)\theta - \Delta_h \theta = -w \cdot \nabla \eta, \\ \operatorname{div} w = 0, \\ w|_{t=0} = \theta|_{t=0} = 0. \end{cases} \quad (\mathcal{Q})$$

• **The case:** $s \neq \frac{1}{2}$

In the sequel, the constant C denotes a universal constant which is allowed to differ from line to line.

Recall first that, by interpolation, we have:

$$\|w\|_{\frac{1}{2},s-1} \lesssim \|w\|_{0,s-1}^{\frac{1}{2}} \|\nabla_h w\|_{0,s-1}^{\frac{1}{2}},$$

$$\|\theta\|_{\frac{1}{2},-s} \lesssim \|\theta\|_{0,-s}^{\frac{1}{2}} \|\nabla_h \theta\|_{0,-s}^{\frac{1}{2}}.$$

By using the above interpolation result with the estimates from Proposition (V.2.1), we infer that

$$L_1 + L_5 \lesssim \|u\|_{\frac{1}{2},s} \|\nabla_h w\|_{0,s-1}^{\frac{3}{2}} \|w\|_{0,s-1}^{\frac{1}{2}}, \quad (\text{V.10})$$

$$\sum_{i \in \{2,3,6,7\}} L_i \lesssim (\|\nabla_h u\|_{0,s} + \|\nabla_h v\|_{0,s} + \|\nabla_h \eta\|_{0,1-s}) (\|w\|_{0,s-1} + \|\theta\|_{0,-s}) \\ \times (\|\nabla_h w\|_{0,s-1} + \|\nabla_h \theta\|_{0,-s}), \quad (\text{V.11})$$

$$L_4 + L_8 \lesssim (\|v\|_{\frac{1}{2},s} + \|\eta\|_{\frac{1}{2},1-s}) \left[(\|w\|_{0,s-1} + \|\theta\|_{0,-s})^{\frac{3}{2}} (\|\nabla_h w\|_{0,s-1} + \|\nabla_h \theta\|_{0,s-1})^{\frac{1}{2}} \right. \\ \left. + (\|w\|_{0,s-1} + \|\theta\|_{0,-s})^{\frac{1}{2}} (\|\nabla_h w\|_{0,s-1} + \|\nabla_h \theta\|_{0,s-1})^{\frac{3}{2}} \right]. \quad (\text{V.12})$$

Finally, we recall the estimate of L_9

$$L_9 \lesssim \|\theta\|_{0,-s} \|\nabla_h w\|_{0,s-1} + \|\theta\|_{0,-s} \|w\|_{0,s-1}. \quad (\text{V.13})$$

An easy consequence of the Young inequality tells that, for any non negative real numbers α, β, A, B, D with $\alpha + \beta = 2$, we have

$$DA^\alpha B^\beta \leq \frac{1}{100} B^2 + CA^2 D^{2/\alpha}.$$

By a suitable choice of α, β in each one of the estimates (V.10), (V.11), (V.12) and (V.13), we infer that

$$\sum_{i=1}^9 L_i \leq \frac{1}{2} (\|\nabla_h w\|_{0,s-1}^2 + \|\nabla_h \theta\|_{0,-s}^2) + Cf(t) (\|w\|_{0,s-1}^2 + \|\theta\|_{0,-s}^2), \quad (\text{V.14})$$

where f is a function, locally integrable in time³, given by:

$$f = \|u\|_{\frac{1}{2},s}^4 + \|\nabla_h u\|_{0,s}^2 + \|\nabla_h v\|_{0,s}^2 + \|\nabla_h \eta\|_{0,1-s}^2 + \|v\|_{\frac{1}{2},s}^{\frac{4}{3}} + \|\eta\|_{\frac{1}{2},1-s}^{\frac{4}{3}} + \|v\|_{\frac{1}{2},s}^4 + \|\eta\|_{\frac{1}{2},s}^4 + 1.$$

On the other hand, by applying the operator Δ_q^v to (Q), and by summing with respect to $q \in \mathbb{N} \cup \{-1\}$, then (V.14) leads to

$$\|w(t)\|_{0,s-1}^2 + \|\theta(t)\|_{0,-s}^2 \leq C \int_0^t f(\tau) (\|w(\tau)\|_{0,s-1}^2 + \|\theta(\tau)\|_{0,-s}^2) d\tau.$$

Finally, we can conclude by using Gronwall's Lemma.

• **The case:** $s = \frac{1}{2}$

The uniqueness in this case can be deduced by applying the Osgood's Lemma to the estimate (V.6) given in Proposition V.2.2. \square

³Remark that the assumption on u, v being $L_T^\infty(H^{0,s}) \cap L_T^2(H^{1,s})$ is enough to ensure, by interpolation argument, that u and v belong to $L_T^4(H^{\frac{1}{2},s})$.

V.3.2 Global wellposedness in the axisymmetric case “Theorem V.1.2”

This subsection is devoted to a direct application of theorem V.1.1 to improve the minimum required regularity on the initial data stated in [101] in order to uniquely construct a unique axisymmetric solution of (NSB_h) .

Proof of Theorem V.1.2 As aforementioned, the uniqueness part in Theorem V.1.2 is a direct consequence of Theorem V.1.1 when we take $s = 1$. Then, we only have to prove the existence of a global solution (u, ρ) for (NSB_h) . The arguments given hereafter, based on the Friedrichs method, are very classical. (see for instance [8, 106, 36] for more details)

For $n \in \mathbb{N}$, we consider the following approximate system

$$\begin{cases} \partial_t u_n + \mathbb{E}_n(u_n \cdot \nabla u_n) - \Delta_h u_n + \nabla P_n = \rho_n e_3, \\ \partial_t \rho_n + \mathbb{E}_n(u_n \cdot \nabla \rho_n) - \Delta_h \rho_n = 0, \\ \operatorname{div} u_n = 0, \\ P_n = \mathbb{E}_n \sum_{k,j} (-\Delta)^{-1} \partial_j \partial_k (u_n^j u_n^k), \\ (u_n, \rho_n)|_{t=0} = (\mathbb{E}_n u_0, \mathbb{E}_n \rho_0), \end{cases} \quad (B_n)$$

where \mathbb{E}_n denotes the cut-off operator defined on $L^2(\mathbb{R}^3)$ by $\mathbb{E}_n u \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\mathbb{1}_{B(0,n)} \hat{u})$. It is then easy to see by using a fixed point argument that there exists some $T_n > 0$ for which (B_n) admits a unique solution $(u_n, \rho_n) \in \mathcal{C}^\infty([0, T_n[, \mathcal{L}_n^{2,\sigma})$, where

$$\mathcal{L}_n^{2,\sigma} \stackrel{\text{def}}{=} L_n^{2,\sigma} \times L_n^2(\mathbb{R}^3),$$

$$L_n^{2,\sigma} \stackrel{\text{def}}{=} \{v \in (L^2(\mathbb{R}^3))^3 : \operatorname{div} v = 0 \text{ and } \operatorname{Supp}(\hat{v}) \subset B(0, n)\},$$

$$L_n^2(\mathbb{R}^3) \stackrel{\text{def}}{=} \{\rho \in L^2(\mathbb{R}^3) : \operatorname{Supp}(\hat{\rho}) \subset B(0, n)\}.$$

Moreover, because u_n and ρ_n are regular, we can multiply the first equation in (B_n) by u_n and the second one by ρ_n . Then, for $t \in [0, T_n[$, after integrating the corresponding terms over $[0, t[\times \mathbb{R}^3$, we obtain the classical uniform L^2 -energy bounds

$$\|u_n(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla_h u_n(\tau)\|_{L^2}^2 d\tau \leq 2(\|u_0\|_{L^2}^2 + t^2 \|\rho_0\|_{L^2}^2), \quad (V.15)$$

$$\|\rho_n(t)\|_{L^2}^2 + 2 \int_0^t \|\nabla_h \rho_n(\tau)\|_{L^2}^2 d\tau \leq \|\rho_0\|_{L^2}^2. \quad (V.16)$$

Hence, (u_n, ρ_n) is a global solution, that is for any $T > 0$, $(u_n, \rho_n) \in \mathcal{C}^\infty([0, T[, \mathcal{L}_n^{2,\sigma})$ and it satisfies (B_n) on $[0, T[\times \mathbb{R}^3$. Moreover, we may extract a sub-sequence, still denoted (u_n, ρ_n) , such that

$$(u_n, \rho_n) \xrightarrow{*} (u, \rho) \quad \text{in } L_T^\infty L^2 \cap L_T^2 H^{1,0}.$$

However, in order to pass to the limit in the non-linear terms, we will need some strong convergence property. To this end we can use the Proposition 3.2 established in [101]. We obtain

$$\|u_n(t)\|_{H^1}^2 + \int_0^t \|\nabla_h u_n(\tau)\|_{H^1}^2 d\tau \leq C_0 e^{C_0 t}.$$

It follows that u_n is uniformly bounded in $L_T^\infty H^1 \cap L_T^2(H^{1,1} \cap H^{2,0})$.

Assume temporarily that

$$(L_T^\infty L^2 \cap L_T^2 H^{1,0}) \cap (L_T^\infty H^{0,1} \cap L_T^2 H^{1,1}) \hookrightarrow L_T^4 L_v^2(L_h^4) \cap L_T^4 L_v^\infty(L_h^4). \quad (\text{V.17})$$

We infer that

- (u_n) is bounded in $L_T^4 L_v^2(L_h^4) \cap L_T^4 L_v^\infty(L_h^4)$.
- (ρ_n) is bounded in $L_T^\infty L^2 \cap L_T^2 H^{1,0} \hookrightarrow L_T^4 L_v^2(L_h^4)$.

Hence, ∇P_n , $\text{div}(u_n \rho_n)$ and $\text{div}(u_n \otimes u_n)$ are bounded in $L_T^2 H^{-1}$, and this gives a bound for $\partial_t u_n$ and $\partial_t \rho_n$ in $L_T^2 H^{-1}$. We can then use Aubin-Lions Theorem in order to extract a new sub-sequence (still denoted (u_n, ρ_n)) that strongly converges to (u, ρ) in $L_T^2 H_{loc}^{\frac{1}{2}} \times L_T^2(H_{loc}^{-\frac{1}{2}})$. Now, we can pass to the limit $n \rightarrow \infty$ in all the terms in (B_n) , and we show that (u, ρ) is a solution to (NSB_h) .

In order to be more convenient, we briefly outline some details about how to pass to the limit in the non linear term in the ρ -equation: Let ψ, ζ be smooth functions in $C_c^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$, supported respectively in $K \Subset \mathbb{R}^+ \times \mathbb{R}^3$, and $\tilde{K} \Subset \mathbb{R}^+ \times \mathbb{R}^3$, with K being strictly included in \tilde{K} and $\zeta|_K = 1$, we claim

$$\lim_{n \rightarrow \infty} \int_{(0,T) \times \mathbb{R}^3} u_n \rho_n \cdot \nabla \psi = \int_{(0,T) \times \mathbb{R}^3} u \rho \cdot \nabla \psi. \quad (\text{V.18})$$

According to the properties of ψ and ζ we have

$$\int_{(0,T) \times \mathbb{R}^3} u_n \rho_n \cdot \nabla \psi = \int_{\tilde{K}} (\zeta u_n) (\rho_n \cdot \nabla \psi).$$

We recall that, from the uniform bounds proved above, we have for $s > \frac{1}{2}$

- (u_n) is bounded in $L_T^2(H^s)$ and $(\partial_t u_n)$ is bounded in $L_T^2(H^{-1})$.
- (ρ_n) is bounded in $L_T^2(L^2)$ and $(\partial_t \rho_n)$ is bounded in $L_T^2(H^{-1})$.

hence, by simple product laws we obtain the same bounds for (ζu_n) and $(\rho_n \cdot \nabla \psi)$ as (u_n) and (ρ_n) respectively. Therefore, by combining the Aubin-Lions Theorem, Theorem 2.94 from [8]⁴ and the weak compactness property of the Hilbert spaces, we infer that

⁴Theorem 2.94 from [8] gives in fact a more general result in the Besov context, what we would apply here is a particular case which says that the multiplication operator by a smooth function is a compact operator from H^s into $H^{s-\varepsilon}$, for any $s \in \mathbb{R}$ and $\varepsilon > 0$.

- (ζu_n) converges strongly to (ζu) in $L_T^2(H^{\frac{1}{2}})$,
- $(\rho_n \cdot \nabla \psi)$ converges strongly to $(\rho \cdot \nabla \psi)$ in $L_T^2(H^{-\frac{1}{2}})$.

This should be enough to justify assertion (V.18).

To complete our proof, we shall justify (V.17). To this end, we will prove a more general inequality: Let $f \in \mathcal{S}'(\mathbb{R}^3)$ and $s > \frac{1}{2}$ (note that in inequality (V.17) s is equal to 1). By using Lemma A.2.2, we obtain for some non-negative number N to be fixed later

$$\begin{aligned} \|f\|_{L_h^4(L_v^\infty)} &\leq \sum_{k \geq -1} \sum_{j \geq -1} \|\Delta_k^h \Delta_j^v f\|_{L_h^4(L_v^\infty)} \\ &\leq \sum_{N \geq k \geq -1} \sum_{j \geq -1} 2^{\frac{k}{2}} 2^{j(\frac{1}{2}-s)} 2^{js} \|\Delta_k^h \Delta_j^v f\|_{L^2} + \sum_{k \geq N+1} \sum_{j \geq -1} 2^{\frac{-k}{2}} 2^{j(\frac{1}{2}-s)} 2^{js} \|\nabla_h \Delta_k^h \Delta_j^v f\|_{L^2}. \end{aligned}$$

Hence, because $s > \frac{1}{2}$ and $j \geq -1$, the Cauchy-Schwarz inequality gives

$$\|f\|_{L_h^4(L_v^\infty)} \lesssim 2^{\frac{N}{2}} \|f\|_{L_h^2(H_v^s)} + 2^{-\frac{N}{2}} \|\nabla_h f\|_{L_h^2(H_v^s)}.$$

Therefore, by choosing N such that $2^N = \frac{\|\nabla_h f\|_{L_h^2(H_v^s)}}{\|f\|_{L_h^2(H_v^s)}}$, we obtain

$$\|f\|_{L_h^4(L_v^\infty)} \lesssim \|f\|_{L_h^2(H_v^s)}^{\frac{1}{2}} \|\nabla_h f\|_{L_h^2(H_v^s)}^{\frac{1}{2}}.$$

On the other hand, the Minkowski inequality ensures that $L_h^4(L_v^\infty) \hookrightarrow L_v^\infty(L_h^4)$. This is sufficient to conclude the proof of (V.17). \square

V.3.3 Local wellposedness result in the general case

In this subsection we prove the result of wellposedness for (NSB_h) given by Theorem V.1.3 under some smallness conditions on T , the norm of u_0 in $H^{0,s}(\mathbb{R}^3)$ and the norm of ρ in $L^2(\mathbb{R}^3)$. This result may not be optimal in this direction, but it gives the existence of solutions in a some new⁵ situations where the proof of the uniqueness part is reduced to an application of Theorem V.1.1.

The proof of the existence part in Theorem V.1.3 is based on the following Lemma:

Lemma V.3.1. *Let $s \in [\frac{1}{2}, 1]$, $\delta \in [0, s]$, then for all regulars vector fields a, b with $\operatorname{div} a = 0$, we have*

$$|\langle a \cdot \nabla b, b \rangle_{0,\delta}| \lesssim \|b\|_{\frac{1}{2},\delta} \left(\|a\|_{1,s} \|b\|_{\frac{1}{2},\delta} + \|a\|_{\frac{1}{2},s} \|b\|_{1,\delta} \right). \quad (\text{V.19})$$

$$|\langle \rho, a^3 \rangle_{0,s}| \leq \frac{1}{4} \|\rho\|_{L^2} \|a\|_{L^2} + \|\rho\|_{L^2} \|\nabla_h a\|_{0,s}. \quad (\text{V.20})$$

⁵We have already treated in Theorem V.1.2 some particular situations with axisymmetric initial data.

Proof

In order to prove (V.19), we follow the same approach as in [28], so we write

$$\langle a \cdot \nabla b, b \rangle_{0,\delta} = \langle a^h \cdot \nabla_h b, b \rangle_{0,\delta} + \langle a^3 \partial_3 b, b \rangle_{0,\delta}. \quad (\text{V.21})$$

We remark next that Lemma V.2.1 gives $H^{\frac{1}{2},s} \times H^{0,\delta} \hookrightarrow H^{-\frac{1}{2},\delta}$, which implies

$$\begin{aligned} |\langle a^h \cdot \nabla_h b, b \rangle_{0,\delta}| &\lesssim \|b\|_{\frac{1}{2},\delta} \|a^h \cdot \nabla_h b\|_{-\frac{1}{2},\delta} \\ &\lesssim \|b\|_{\frac{1}{2},\delta} \|a\|_{\frac{1}{2},s} \|b\|_{1,\delta}. \end{aligned}$$

It remains now only to estimate the second term in the right hand side of (V.21). Indeed, Bony's decomposition tells that

$$\Delta_q^v(a^3 \partial_3 b) = \Delta_q^v \left(\sum_{k \geq q-N_0} S_{k+2}(\partial_3 b) \Delta_k^v a^3 + \sum_{|q-k| \leq N_0} S_{k-1} a^3 \Delta_k^v b \right). \quad (\text{V.22})$$

Note then firstly that

$$\begin{aligned} \|S_{k+2}(\partial_3 b)\|_{L_h^4 L_v^2} &\leq \sum_{m \leq k+1} 2^{m(1-\delta)} 2^{\delta m} \|\Delta_m^v b\|_{L_h^4 L_v^2} \leq \left(\sum_{m \leq k+1} 2^{2m(1-\delta)} \right)^{\frac{1}{2}} \|b\|_{\frac{1}{2},\delta} \\ &\lesssim 2^{k(1-\delta)} \|b\|_{\frac{1}{2},\delta}. \end{aligned}$$

Moreover, by using the fact that $\partial_3 a^3 = -\nabla_h \cdot u^h$ together with Lemma A.2.2, we obtain

$$\|\Delta_k^v a^3\|_{L^2} \leq C 2^{-k} \|\Delta_k^v \nabla_h \cdot u^h\|_{L^2}.$$

This implies

$$\begin{aligned} \left| \left\langle \Delta_q^v \sum_{k \geq q-N_0} S_{k+2}(\partial_3 b) \Delta_k^v a^3, \Delta_q^v b \right\rangle \right| &\lesssim \|\Delta_q^v b\|_{L_h^4 L_v^2} 2^{q/2} \sum_{k \geq q-N_0} \|S_{k+2}(\partial_3 b)\|_{L_h^4 L_v^2} \|\Delta_k^v a^3\|_{L^2} \\ &\lesssim \|\Delta_q^v b\|_{L_h^4 L_v^2} 2^{q/2} \sum_{k \geq q-N_0} 2^{-k\delta} \|b\|_{\frac{1}{2},\delta} \|\Delta_k^v \nabla_h u\|_{L^2} \\ &\lesssim \|b\|_{\frac{1}{2},\delta} \|u\|_{1,s} 2^{-q\delta} C_q \|b\|_{\frac{1}{2},\delta} 2^{q/2} \sum_{k \geq q-N_0} 2^{-k(\delta+s)} C_k \\ &\lesssim \|b\|_{\frac{1}{2},\delta}^2 \|u\|_{1,s} 2^{-2q\delta} C_q 2^{q(\frac{1}{2}-s)}. \end{aligned}$$

Hence, thanks to the assumption $s > \frac{1}{2}$, we infer that

$$\left| \left\langle \sum_{k \geq -1} S_{k+2}(\partial_3 b) \Delta_k^v a^3, b \right\rangle_{0,\delta} \right| \lesssim \|b\|_{\frac{1}{2},\delta}^2 \|u\|_{1,s}.$$

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In order to estimate the second term in the right hand side of (V.22), we use the decomposition proposed in [28]

$$\begin{aligned} \langle \Delta_q^v \sum_{|q-k| \leq N_0} S_{k-1} a^3 \Delta_k^v b, \Delta_q^v b \rangle &= \langle S_q(a^3) \partial_3 \Delta_q^v b, \Delta_q^v b \rangle + \langle \sum_{|q-k| \leq N_0} [\Delta_q^v, S_{k-1} a^3] \partial_3 \Delta_k^v b, \Delta_q^v b \rangle \\ &+ \langle \sum_{|q-k| \leq N_0} (S_q - S_{k-1}) a^3 \Delta_k^v \partial_3 b, \Delta_q^v b \rangle \stackrel{def}{=} J_1^q + J_2^q + J_3^q. \end{aligned}$$

By using an integration by parts, we obtain

$$J_1^q = \langle S_q(a^3) \partial_3 \Delta_q^v b, \Delta_q^v b \rangle = \frac{1}{2} \langle S_q(\partial_3 a^3) \Delta_q^v b, \Delta_q^v b \rangle,$$

which shows that this term can be estimated in the same way as $\langle \nabla_h a \cdot b, b \rangle$ due to the divergence free condition. We get

$$\left| \sum_{q \geq -1} 2^{2q\delta} \langle S_q(a^3) \partial_3 \Delta_q^v b, \Delta_q^v b \rangle \right| \lesssim \|a\|_{1,s} \|b\|_{\frac{1}{2},\delta}^2.$$

Next from Lemma V.2.2, it easily follows

$$\begin{aligned} \left| \sum_{q \geq -1} 2^{2q\delta} J_2^q \right| &= \left| \sum_{q \geq -1} 2^{2q\delta} \left\langle \sum_{|q-k| \leq N_0} [\Delta_q^v, S_{k-1} a^3] \partial_3 \Delta_k^v b, \Delta_q^v b \right\rangle \right| \lesssim \|a\|_{1,s} \sum_{q \geq -1} 2^{2q\delta} \|\Delta_q^v b\|_{\frac{1}{2},0} \\ &\lesssim \|a\|_{1,s} \|b\|_{\frac{1}{2},\delta}^2. \end{aligned}$$

Finally, to estimate J_3^q , we use the fact that the support of $S_q^v - S_{k-1}^v$ is far from zero in Fourier side. This gives in particular

$$\|(S_q^v - S_{k-1}^v) a^3\|_{L_h^2(L_v^\infty)} \lesssim 2^{-k} \|(S_q^v - S_{k-1}^v) \partial_3 a^3\|_{L_h^2(L_v^\infty)}.$$

Thus, we obtain

$$J_3^q \lesssim \sum_{|q-k| \leq N_0} 2^{q-k} \|(S_q^v - S_{k-1}^v) \partial_3 a^3\|_{L_h^2(L_v^\infty)} \|\Delta_k^v b\|_{L_h^4(L_v^2)} \|\Delta_q^v b\|_{L_h^4(L_v^2)}.$$

By using the fact that $\operatorname{div} a = 0$ and $s > \frac{1}{2}$, we end up with

$$\left| \sum_{q \geq -1} 2^{2q\delta} J_3^q \right| \lesssim \|a\|_{1,s} \|b\|_{\frac{1}{2},\delta}^2.$$

This ends the proof of (V.19).

In order to prove (V.20), we use again the fact that $\operatorname{div} a = 0$ together with $s - 1 \leq 0$, to

obtain

$$\begin{aligned}
 |\langle \rho, a^3 \rangle_{0,s}| &= |2^{-2s} \langle S_0^v \rho, S_0^v a^3 \rangle + \sum_{q \geq 0} 2^{2qs} \langle \Delta_q^v \rho, \Delta_q^v a^3 \rangle| \\
 &\leq \frac{1}{4} \|S_0^v \rho\|_{L^2} \|S_0^v a\|_{L^2} + \sum_{q \geq 0} \|\Delta_q^v \rho\|_{L^2} 2^{q(s-1)} 2^{qs} \|\Delta_q^v \partial_3 a^3\|_{L^2} \\
 &\leq \frac{1}{4} \|S_0^v \rho\|_{L^2} \|S_0^v a\|_{L^2} + \sum_{q \geq 0} \|\Delta_q^v \rho\|_{L^2} 2^{qs} \|\Delta_q^v \nabla_h a\|_{L^2} \\
 &\leq \frac{1}{4} \|\rho\|_{L^2} \|a\|_{L^2} + \|\rho\|_{L^2} \|\nabla_h a\|_{0,s}.
 \end{aligned}$$

This concludes the proof of Lemma V.3.1 □

Now we are in position to prove Theorem V.1.3

Proof of Theorem V.1.3 The uniqueness part of Theorem V.1.3 is a direct consequence of Theorem V.1.1. Hence, it remains only to prove the existence part. This can be done in a similar way than explained in the proof of Theorem V.1.2, and in particular the construction of the approximate sequence (u_n, ρ_n) does not add any difficulty. However, the uniform bounds must now be obtained in some new adequate norms. For simplicity, in the following we will drop the index of the approximate sequence.

We apply Δ_q^v in both equations for u and ρ from (NSB_h) , then we multiply the first equation by $\Delta_q^v u$, the second one by $\Delta_q^v \rho$ and we sum over $q \geq -1$ to obtain

$$\frac{d}{dt} \|u(t)\|_{0,s}^2 + \|\nabla_h u(t)\|_{0,s}^2 \leq |\langle u \cdot \nabla u, u \rangle_{0,s}| + |\langle \rho, u^3 \rangle_{0,s}|. \quad (\text{V.23})$$

$$\frac{d}{dt} \|\rho(t)\|_{0,\delta}^2 + \|\nabla_h \rho(t)\|_{0,\delta}^2 \leq |\langle u \cdot \nabla \rho, \rho \rangle_{0,\delta}|. \quad (\text{V.24})$$

We prove firstly an uniform bound for u by using the L^2 -energy estimate of ρ and u . Indeed, by taking $a = b = u$ in Lemma V.3.1, then (V.23) gives

$$\frac{d}{dt} \|u\|_{0,s}^2 + 2 \|\nabla_h u\|_{0,s}^2 \leq \|\rho\|_{L^2} \|u\|_{L^2} + \|\nabla_h u\|_{0,s}^2 + \|\rho\|_{L^2}^2 + \tilde{C}_s \|u\|_{0,s} \|\nabla_h u\|_{0,s}^2, \quad (\text{V.25})$$

where \tilde{C}_s is a non negative constant that depends on s . Let us assume that

$$\|u_0\|_{0,s}^2 + T \|\rho_0\|_{L^2} \left(\|u_0\|_{L^2} + \|\rho_0\|_{L^2} \left(1 + \frac{T}{2}\right) \right) < C_s^2 < \frac{1}{4\tilde{C}_s^2}, \quad (\text{V.26})$$

which ensures that $\|u_0\|_{0,s} < C_s < \frac{1}{2\tilde{C}_s}$.

Let us now assume that there exists $T^{max} \in (0, T)$ satisfying

$$T^{max} \stackrel{def}{=} \inf\{t \in [0, T] : \|u(t)\|_{0,s} = C_s\}.$$

CHAPTER V. BOUSSINESQ – HORIZONTAL DISSIPATION

It follows that, for all $t \in [0, T^{max})$: $\|u(t)\|_{0,s} < C_s < \frac{1}{2C_s}$.

By using this last inequality in (V.25), and by integration on $[0, t)$, we obtain for all $t \in [0, T^{max})$

$$\|u(t)\|_{0,s}^2 + \frac{1}{2} \int_0^t \|\nabla_h u(\tau)\|_{0,s}^2 d\tau \leq \|u_0\|_{0,s}^2 + \int_0^t (\|\rho(\tau)\|_{L^2}^2 + \|u(\tau)\|_{L^2} \|\rho(\tau)\|_{L^2}) d\tau.$$

Hence, by using the L^2 energy estimate for ρ and u given by (V.15) and (V.16), we infer that

$$\|u(t)\|_{0,s}^2 \leq \|u_0\|_{0,s}^2 + t \|\rho_0\|_{L^2} \left(\|u_0\|_{L^2} + \|\rho_0\|_{L^2} \left(1 + \frac{t}{2}\right) \right), \quad \forall t \in [0, T^{max}),$$

then by using (V.26) and passing to the limit $t \rightarrow T^{max}$, we obtain

$$C_s^2 \leq \|u_0\|_{0,s}^2 + T \|\rho_0\|_{L^2} \left(\|u_0\|_{L^2} + \|\rho_0\|_{L^2} \left(1 + \frac{T}{2}\right) \right) < C_s^2,$$

which contradicts the existence of T^{max} and gives for all $t \in [0, T]$: $\|u(t)\|_{0,s} < C_s$. Therefore we have proved an uniform bound of u in $L_T^\infty H^{0,s}$. Plugging this bound into (V.25) gives a bound for u in $L_T^2 H^{1,s}$. Note that by using an argument of interpolation, this also gives a bound of u in $L_T^4 H^{\frac{1}{2},s}$.

Next, we can estimate the right hand side of (V.24) by using Lemma V.3.1 with $a = u$ and $b = \rho$. We obtain after some calculations

$$\frac{d}{dt} \|\rho(t)\|_{0,\delta}^2 + 2 \|\nabla_h \rho(\tau)\|_{0,\delta}^2 d\tau \leq \|\nabla_h \rho\|_{0,\delta}^2 + C_s A(t) \|\rho(t)\|_{0,\delta}^2,$$

where

$$A(\cdot) = \|\nabla_h u(\cdot)\|_{0,s}^2 + \|u(\cdot)\|_{\frac{1}{2},s}^4 \in L^1((0, T)).$$

Finally, by applying Gronwall's Lemma we obtain the adequate bound for ρ . □

Part III

About the Hall-MHD system

CHAPTER VI

Well-posedness and large time behavior of the electron inertia Hall-MHD system

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VI.1 Introduction

VI.1.1 Presentation of the model

In this Chapter we consider the incompressible 3D electron inertia Hall-MHD equations derived from the two fluid model of ion and electron

$$\left\{ \begin{array}{l} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla P = j \times [B - \delta(1 - \delta)\lambda^2 \Delta B], \\ \partial_t B' - \eta \Delta B' + (1 - \delta)\lambda^2 \mu_e \Delta^2 B = \nabla \times \{ [u - (1 - \delta)\lambda j] \times B' \} - \lambda \mu_e \Delta(\nabla \times u), \\ j = \frac{c}{4\pi} \nabla \times B, \\ B' = B - \delta(1 - \delta)\lambda^2 \Delta^2 B - \delta \lambda (\nabla \times u), \\ \operatorname{div} u = \operatorname{div} B = 0 \\ (u|_{t=0}, B|_{t=0}) = (u_0, B_0). \end{array} \right. \quad (1)$$

Here, u is the hydrodynamic velocity, B the magnetic field, the scalar function P denotes the pressure which can be recovered, due the incompressibility condition, from the relation $-\Delta P = \nabla \cdot (u \cdot \nabla u - j \times [B - \delta(1 - \delta)\lambda^2 \Delta B])$. c is the speed of light and j denotes the electric current density. μ_e is the kinematic viscosity of the electron fluid, and if we denote by μ_i that of the ion fluid, then the kinematic viscosity μ will be $\mu_i + \mu_e$. The very small parameter δ is given by $\delta = m_e/M$, where $M = m_e + m_i$, which is the sum of the mass of an ion and of an electron.

If we denote by e , n , L_0 the charge, the number density and the length scale respectively, and if we define $w_M \stackrel{def}{=} (4\pi e^2 n/M)^{\frac{1}{2}}$, then λ will be $\lambda = c/w_M L_0$.

As pointed out in the introduction of [46], the system above can be seen as a full two-fluids MHD description of a completely ionized hydrogen plasma, retaining the effects of the Hall current, electron pressure and electron inertia. For more details about the derivation of the system, we refer the reader to [1, 46].

One may notice that if we neglect the presence of the electrons in the equations, that is if we set $(\delta, \mu_e) = (0, 0)$ in (1), then it becomes the so-called Hall-MHD system as follows

$$\left\{ \begin{array}{l} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla P = j \times B, \\ \partial_t B - \eta \Delta B = \nabla \times \{ (u - \lambda j) \times B \}, \\ j = \frac{c}{4\pi} \nabla \times B, \\ \operatorname{div} u = \operatorname{div} B = 0, \\ (u|_{t=0}, B|_{t=0}) = (u_0, B_0). \end{array} \right. \quad (\text{Hall.MHD})$$

Hence, at least formally, the solution (u, B) to (Hall.MHD) can be seen as the limit of the solution of (1) when (δ, μ_e) tends to $(0, 0)$.

Several works have been devoted to the study of the systems (1), (Hall.MHD) and to the classical MHD as well ¹. One may see [46, 38, 39, 61, 62, 6, 24, 118] for more details.

¹The term $\lambda \nabla \times (j \times B)$ is known in the literature as the hall-term. The classical MHD system is then the case where $\lambda = 0$.

CHAPTER VI. W.P AND LONG TIME BEHAVIOR – HALL-MHD

The main purpose of this Chapter is to deal with the system (1). It is easy to check, by making use of the equation satisfied by $(\nabla \times u)$, that (1) can be simplified into the following system

$$\left\{ \begin{array}{l} \partial_t u + u \cdot \nabla u - \Delta u + \nabla P = (\nabla \times B) \times H \\ \partial_t H - \Delta H + 2\nabla \times ((\nabla \times B) \times H) = \nabla \times (u \times H) + \nabla \times ((\nabla \times B) \times (\nabla \times u)) \\ H = (Id - \Delta)B \\ div u = div B = 0 \\ (u|_{t=0}, B|_{t=0}) = (u_0, B_0), \end{array} \right. \quad (\widetilde{MHD})$$

where we omit the constants which will play no significant role in the proofs of our results. Hence, all the constants in our results below will be modulo μ, η, δ and λ .

In a first stage, we aim at generalizing some of the results in [46] from the Hilbert spaces $\dot{H}^s(\mathbb{R}^3)$ to the general Besov spaces of the form $\dot{B}_{p,r}^s(\mathbb{R}^3)$. To the best of our knowledge, the wellposedness of (1) in general *critical* Besov spaces has not been proved before this work². In the recent work [46], the authors consider initial data in spaces of the type $H^s(\mathbb{R}^3)$. It is then interesting to check if similar results can be proved as well in the context of general Besov spaces having the same scaling, which is the issue treated by our Theorem VI.1.1 below.

If we try to deal with (\widetilde{MHD}) as it is, the structure of some of the nonlinear terms will prevent us from establishing the wellposedness for all ³ $p \in [1, \infty[$. The issue will be in fact at the level of estimating the remainder terms in Bony's decomposition. Also, it is worth noting that (\widetilde{MHD}) does not have a scaling invariance structure as the classical Navier-Stokes equations or the classical MHD equations.

We recall that, the following spaces

$$\dot{B}_{m,n}^{\frac{3}{m}-1}(\mathbb{R}^3) \hookrightarrow \dot{H}^{\frac{1}{2}}(\mathbb{R}^3) \hookrightarrow \dot{B}_{p,r}^{\frac{3}{p}-1}(\mathbb{R}^3) \hookrightarrow \dot{B}_{q,r'}^{\frac{3}{q}-1}(\mathbb{R}^3) \hookrightarrow \dot{B}_{\infty,\infty}^{-1}(\mathbb{R}^3),$$

for all m, n, p, q, r, r' satisfying

$$m \leq 2 \leq p \leq q \leq \infty, \quad n \leq 2 \leq r \leq r' \leq \infty,$$

are critical for the 3D Navier-Stokes equations. On the other hand, the spaces

$$\dot{B}_{m,n}^{\frac{3}{m}}(\mathbb{R}^3) \hookrightarrow \dot{H}^{\frac{3}{2}}(\mathbb{R}^3) \hookrightarrow \dot{B}_{p,r}^{\frac{3}{p}}(\mathbb{R}^3) \hookrightarrow \dot{B}_{q,r'}^{\frac{3}{q}}(\mathbb{R}^3) \hookrightarrow \dot{B}_{\infty,\infty}^0(\mathbb{R}^3)$$

are critical for the following 3D system

$$\partial_t B - \Delta B = -\nabla \times ((\nabla \times B) \times B).$$

Based on this observation, one may ask the following question:

²We will explain in the sequel what do we mean by critical spaces to (1).

³ p refers here to the integrability with respect to space variables in the Besov spaces $\dot{B}_{p,r}^s$.

Question: Can we solve (1) with initial data (u_0, B_0) small enough in the above spaces?

As mentioned in [46] for (*Hall.MHD*), the difficulty in establishing the wellposedness result under the condition that $\|u_0\|_{\dot{H}^{\frac{1}{2}}} + \|B\|_{\dot{H}^{\frac{3}{2}}}$ being small enough comes from the nonlinear term $\nabla \times (u \times B)$. Up to our knowledge, all the existing results require additional smallness assumptions on $\|u_0\|_{\dot{H}^{\frac{1}{2}+\varepsilon}} + \|B_0\|_{\dot{H}^{\frac{3}{2}}}$ as in [118], or $\|u_0\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} + \|B_0\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} + \|\nabla \times B_0\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}}$ as in [38, 39].

In our analysis, we find that (*MHD*) is easier to deal with due to the flexibility of the operator $(Id - \Delta)^{-1}$ that appears in the Duhamel formula⁴ of (*MHD*). This remarkable feature is a key ingredient in our proofs, and it is in fact one of the reasons why, in our Theorem VI.1.2 below, we can omit the smallness condition on $\|B_0\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}}$ modulo an additional condition on the maximal time of existence. This provides a partial answer to the question above in the case of the Electron Inertial Hall-MHD.

On the other hand, to give a sense to the critical spaces in the case of (*Hall.MHD*), the authors in [38] considered the following “equivalent” system

$$\begin{cases} \partial_t u + u \cdot \nabla u - \mu \Delta u + \nabla P = j \times B, \\ \partial_t B - \eta \Delta B = \nabla \times \{(u - \lambda j) \times B\}, \\ \partial_t j - \eta \Delta j = \nabla \times (\nabla \times \{(u - \lambda j) \times curl^{-1} j\}), \\ j = \nabla \times B, \\ div u = div B = 0, \\ (u|_{t=0}, B|_{t=0}) = (u_0, B_0), \end{cases} \quad (\widetilde{Hall.MHD})$$

where the third equation is obtained from the second one by applying the curl operator. Above, the $curl^{-1}$ is defined as

$$curl^{-1} j \stackrel{def}{=} \mathcal{F}^{-1} \left(\frac{i\xi \times \hat{j}}{|\xi|^2} \right).$$

The system (*Hall.MHD*) has then a *scaling invariance*, which is exactly the same as that of the 3D Navier-Stokes equations and the classical MHD system.

Hence, taking into account the fact that (*Hall.MHD*) can be seen as the limit system of (1) when (δ, μ_e) vanishes, a space will be called critical to (1) if it is critical to (*Hall.MHD*) in the sense explained above and in [38].

We conclude this subsection by pointing out an interesting issue: One may notice that in our first result below for (1) (see Theorem VI.1.1) we will be considering data $u_0, B_0, \nabla \times B_0$ in $\dot{B}_{p,r}^{\frac{3}{p}-1}$, where r can take any value in $[1, \infty]$. On the other hand, for (*Hall.MHD*) and as mentioned in [38], proving the wellposedness in such spaces remains an open question for the time being, except for the case $p = 2$ or $r = 1$. As explained in [38], one may benefit

⁴See next subsection for the Duhamel formula associated to (*MHD*), and Proposition A.2.2 to understand what we mean by the flexibility of this operator $(Id - \Delta)^{-1}$.

from some cancellations in the case $p = 2$ to get ride of the most nonlinear term in the equation of J . Whereas, in the case $p \neq 2$, the index r is needed to be equal to 1 because, at some point, they need to estimate a product of two functions in the critical Besov space $\dot{B}_{p,r}^{\frac{3}{p}}$.⁵ This can be completely avoided in our case, thanks again to the flexibility of the operator $(Id - \Delta)^{-1}$.

We recall that, in the present work, we are not interested in studying the asymptotic limit of the solutions to (1) when (δ, μ_e) vanish. But to be more precise and rigorous, we should point out that, if we keep all the constants in our analysis, the operator $(Id - \Delta)^{-1}$ will be in fact $(Id - \delta(1 - \delta)\lambda\Delta)^{-1}$, which means that, when δ vanishes, we do not believe that we can give an answer to the open question in [38] by studying the limit (as (δ, μ_e) vanish) of our solution given by Theorem VI.1.1.

VI.1.2 Reformulation of the equations

We recall again that, for simplification, we will prove our theorems for (\widetilde{MHD}) instead of (1), whereas we should point out that all our results below hold as well for (1).

Let us then first rewrite the system (\widetilde{MHD}) in an appropriate form. To do so, we recall some vectorial concepts.

For U, V two divergence-free vector fields on \mathbb{R}^3 , we have

$$\nabla \times (U \times V) = V \cdot \nabla U - U \cdot \nabla V, \quad (\text{VI.1})$$

$$(\nabla \times U) \times U = U \cdot \nabla U - \frac{1}{2} \nabla |U|^2, \quad (\text{VI.2})$$

$$V \times (\nabla \times U) + U \times (\nabla \times V) = -\nabla \times (U \times V) - 2U \cdot \nabla V + \nabla(U \cdot V), \quad (\text{VI.3})$$

$$\nabla \times (\nabla \times U) = -\Delta U. \quad (\text{VI.4})$$

If we denote $J \stackrel{\text{def}}{=} \nabla \times B$, then according to VI.2 and VI.4, we obtain

$$(\nabla \times B) \times (B - \Delta B) = B \cdot \nabla B - J \cdot \nabla J + \nabla \left(\frac{|B|^2 - |J|^2}{2} \right).$$

On the other hand, we have

$$u \times H + (\nabla \times B) \times (\nabla \times u) = u \times B + u \times (\nabla \times J) + J \times (\nabla \times u).$$

Thus according to (VI.3), we infer that

$$u \times H + (\nabla \times B) \times (\nabla \times u) = u \times B - \nabla \times (J \times u) - 2J \cdot \nabla u + \nabla(J \cdot u). \quad (\text{VI.5})$$

⁵ Fore more details on that we suggest to see in particular the estimates (3.5) and (3.6) in [38].

Therefore, (\widetilde{MHD}) can be written as follows

$$\begin{cases} \partial_t u - \Delta u = B \cdot \nabla B - J \cdot \nabla J - u \cdot \nabla u + \nabla \tilde{P} \\ \partial_t B - \Delta B = (Id - \Delta)^{-1} \nabla \times \Theta \\ \partial_t J - \Delta J = -\Delta (Id - \Delta)^{-1} \Theta + \nabla (Id - \Delta)^{-1} (\nabla \cdot \Theta) \\ \operatorname{div} u = \operatorname{div} B = \operatorname{div} J = 0 \\ (u, B, J)|_{t=0} = (u_0, B_0, \nabla \times B_0), \end{cases} \quad (S_1)$$

where,

$$\tilde{P} \stackrel{def}{=} -p + \frac{|B|^2 - |J|^2}{2},$$

and

$$\Theta \stackrel{def}{=} u \times B - 2B \cdot \nabla B - J \cdot \nabla J - \nabla \times (J \times u) - 2J \cdot \nabla u + \nabla (J \cdot u).$$

One may notice that, in the expression of Θ above, the term $u \times B$ is not at the same level of scaling compared to the rest of quantities, besides that the non-homogeneous behavior of the operator $(Id - \Delta)^{-1}$ destroys any chances for (S_1) to have a scaling invariant structure as the classical Navier-Stokes has for example, or as in the case of the system ($\widetilde{Hall.MHD}$) studied in [38].

The justification of the choice of the functional framework in which we do our analysis is then made by working in critical spaces as explained in the previous subsection.

Let us now write down Duhamel's formula corresponding to (S_1) . Let \mathbb{P} be the Leray projector, we write

$$\begin{aligned} \mathcal{Q}(u, v) &\stackrel{def}{=} \nabla \cdot (u \otimes v), \\ \mathcal{P}(u, v) &\stackrel{def}{=} u \times v, \\ \mathcal{R}(u, v) &\stackrel{def}{=} \nabla \times (u \times v), \end{aligned}$$

for a three dimensional vectors $K = (K_1, K_2, K_3)$, $L = (L_1, L_2, L_3)$, we define

$$\begin{aligned} \Gamma(K, L) &\stackrel{def}{=} \begin{pmatrix} \mathcal{Q}(K_2, L_2) - \mathcal{Q}(K_1, L_1) - \mathcal{Q}(K_3, L_3) \\ \mathcal{P}(K_1, L_2) - \mathcal{R}(K_3, L_1) - \mathcal{Q}(K_3, L_3) - 2\mathcal{Q}(K_2, L_2) - 2\mathcal{Q}(K_3, L_1) \\ \mathcal{P}(K_1, L_2) - \mathcal{R}(K_3, L_1) - \mathcal{Q}(K_3, L_3) - 2\mathcal{Q}(K_2, L_2) - 2\mathcal{Q}(K_3, L_1) \end{pmatrix}, \\ \Omega(K, L) &\stackrel{def}{=} \mathbb{P} \begin{pmatrix} \Gamma_1(K, L) \\ (Id - \Delta)^{-1} \nabla \times \Gamma_2(K, L) \\ -\Delta (Id - \Delta)^{-1} \Gamma_3(K, L) \end{pmatrix}, \end{aligned} \quad (VI.6)$$

and

$$\zeta(K, L) \stackrel{def}{=} \mathcal{K} \Omega(K, L), \quad (VI.7)$$

where

$$\mathcal{K}\varphi(t, \cdot) = \int_0^t e^{(t-s)\Delta} \varphi(s, \cdot) ds.$$

Finally, if we write $\mathcal{U} \stackrel{def}{=} (u, B, J)$, then (S_1) is equivalent to

$$\begin{cases} \partial_t \mathcal{U} - \Delta \mathcal{U} = \zeta(\mathcal{U}, \mathcal{U}) \\ \operatorname{div} u = \operatorname{div} B = \operatorname{div} J = 0 \\ \mathcal{U}|_{t=0} = \mathcal{U}_0 \end{cases} \quad (S_\zeta)$$

Remark VI.1.1. *As mentioned in a paper of I.Gallagher, D.Iftime and F.Planchon [49] for the classical Navier Stokes equations, the theory of weak solutions to the Navier Stokes equations is related to the special structure of the equation, namely to the energy inequality, while the Kato's approach is more general and can be applied to more general parabolic or dispersive equations, this work is an example among many. The main issue here consists in writing the equations in an appropriate form in order to be able to adapt the techniques used for the classical Navier-Stokes [8, 49, 96, 37].*

VI.1.3 Statement of the main results

Before stating our results, let us fix some notations which will be of a constant use in this Chapter, we refer to the Appendix for more details about the functional spaces we use

- For A, B two real quantities, $A \lesssim B$ means that $A \leq cB$, for some $c > 0$ independent of A and B .
- $(c_{j,r})_{j \in \mathbb{Z}}$ will be a sequence satisfying $\sum_{j \in \mathbb{Z}} c_{j,r}^r \leq 1$. This sequence is allowed to differ from line to line. Let us point out that, due to the embedding $\ell^r(\mathbb{Z}) \in \ell^\infty(\mathbb{Z})$, we will often use the inequality $c_{j,r}^2 \leq c_{j,r}$.
- The \widehat{L}^p norm of u is given by

$$\|u\|_{\widehat{L}^p} \stackrel{def}{=} \|\widehat{u}\|_{L^{p'}},$$

where p' is the usual conjugate of p , and \widehat{u} denotes the Fourier transform of u .

- We use the notation

$$\mathcal{L}(\dot{B}_{p,r}^{s_p}) \stackrel{def}{=} \bigcap_{\rho \in [1, \infty]} \tilde{L}^\rho(\mathbb{R}^+; \dot{B}_{p,r}^{s_p + \frac{2}{\rho}}),$$

and for $T > 0$,

$$\mathcal{L}_T(\dot{B}_{p,r}^{s_p}) \stackrel{def}{=} \bigcap_{\rho \in [1, \infty]} \tilde{L}^\rho([0, T]; \dot{B}_{p,r}^{s_p + \frac{2}{\rho}})$$

- Finally, we point out that in this Chapter, we will be working only in the homogeneous version of the spaces defined in Appendix A, especially the Besov spaces. However to simplify the presentation, the notation Δ_q in this Chapter refers to the homogeneous dyadic blocs, i.e $\Delta_q \approx \widehat{\Delta}_q$.

Let us now define what we mean by a solution to (S_ζ) in this Chapter.

Definition VI.1.1. Let $T > 0$, and \mathcal{U}_0 be given in some Banach space \mathcal{X} , we say that \mathcal{U} is a solution to (S_ζ) on $(0, T)$ if $\mathcal{U} \in L_{loc}^\infty([0, T]; \mathcal{X})$ and satisfies, for a.e $t \in [0, T]$

$$\mathcal{U}(t, \cdot) = e^{t\Delta}\mathcal{U}_0(\cdot) + \zeta(\mathcal{U}, \mathcal{U}), \quad \text{in } \mathcal{X}$$

The authors in [46] proved the wellposedness of (\widetilde{MHD}) under the condition that $\|u_0\|_{H^{\frac{1}{2}}} + \|B_0\|_{H^{\frac{1}{2}}} + \|B_0\|_{H^{\frac{3}{2}}}$ is small enough. Our first result consists in generalizing this last one to the Besov context, it reads as follows.

Theorem VI.1.1. Let $p \in [1, \infty)$, $r \in [1, \infty]$ and $\mathcal{U}_0 = (u_0, B_0, \nabla \times B_0)$ be in $\dot{B}_{p,r}^{\frac{3}{p}-1}(\mathbb{R}^3)$. There exists $c_0 > 0$ such that, if

$$\|\mathcal{U}_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}} < c_0,$$

then (S_ζ) has a unique global solution \mathcal{U} in $\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})$, with

$$\|\mathcal{U}\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})} < 2c_0.$$

Remark VI.1.2. One may show that the solution, in the case $r < \infty$, is continuous in time with value in $\dot{B}_{p,r}^{\frac{3}{p}-1}$, while in the case $r = \infty$ it is just weakly-continuous in time.

Remark VI.1.3. One may prove a local in time wellposedness for large initial data, by slightly modifying the proof of theorem VI.1.1, we will give some details about that in corollary VI.3.

In terms of the required regularity, in Theorem VI.1.1 we ask for the initial data of B to be in $\dot{B}_{p,r}^{\frac{3}{p}-1}(\mathbb{R}^3) \cap \dot{B}_{p,r}^{\frac{3}{p}}(\mathbb{R}^3)$.

It is worth noting that, it is because of the two non linear terms $u \times B$ in the equation of B , and $B \cdot \nabla B$ in the equation of u , that we do not know how to prove an analogous result to Theorem VI.1.1, starting from initial data B_0 only in $\dot{B}_{p,r}^{\frac{3}{p}}$.

However, in the case $r = 1$, we will prove that a small enough “compared to the maximal time of existence T^* ” initial data B_0 in $\dot{B}_{p,1}^{\frac{3}{p}}$ should generate a unique solution, at least up to time T^* . More precisely, we will prove

Theorem VI.1.2. Let $T > 0$, $p \in [1, \infty)$ and (u_0, B_0) be two divergence-free vector fields in $\dot{B}_{p,1}^{\frac{3}{p}-1}(\mathbb{R}^3) \times \dot{B}_{p,1}^{\frac{3}{p}}(\mathbb{R}^3)$, there exists $c_0 > 0$ such that if

$$\|u_0\|_{\dot{B}_{p,1}^{\frac{3}{p}-1}} + (2 + T) \|B_0\|_{\dot{B}_{p,1}^{\frac{3}{p}}} < c_0,$$

then (S_ζ) has a unique solution (u, B) on $(0, T)$ with

$$\|u\|_{\mathcal{L}_T(\dot{B}_{p,1}^{\frac{3}{p}-1})} + (2 + T) \|B\|_{\mathcal{L}_T(\dot{B}_{p,1}^{\frac{3}{p}})} < 2c_0.$$

The question of the behavior, for large time, of the solution obtained in Theorem VI.1.1 can be established along “approximately” the same lines as those used, for instance, for the 3D Navier-Stokes equations in [49]. That is to say, it should be possible to prove that $\|\mathcal{U}(t)\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}}$ tends to zero as t tends to infinity.

It is also well known for the Navier Stokes equations (see the appendix of [49]), that for an initial data $u_0 \in \dot{B}_{p,r}^{\frac{3}{p}-1}(\mathbb{R}^3)$, $p \in (3, \infty)$, the L^∞ norm of the velocity also decays to zero at infinity. More precisely it is controlled by $Ct^{-\frac{1}{2}}$.

The proof of this result relies on the fact that the bi-linear operator in Duhamel’s formula acts well on the Kato’s space, together with the fact that we can iterate, as much as we want, the solution u in Duhamel’s formula in order to obtain a solution of the form of a sum of some N multi-linear terms of $e^{t\Delta}u_0$, and a more regular remainder term r_{N+1} which is unique in $L_t^\infty(L^3)$.

A priori, this approach should work as well in our case, but we will not enter into these details in this thesis.

In contrast to that, we will treat the case of initial data in the Herz-space $\widehat{B}_{p,r}^{\frac{3}{p}-1}(\mathbb{R}^3)$ ⁶, and we will give some details in the case $r = \infty$ as an example. More precisely, we will prove

Theorem VI.1.3. *Let $p \in (3, \infty)$ and u_0, B_0 be two divergence free vector fields. There exists $c_0 > 0$ such that if*

$$\|e^{t\Delta}u_0\|_{\widehat{K}_p^{1-\frac{3}{p}}} + \|e^{t\Delta}B_0\|_{\widehat{K}_p^{1-\frac{3}{p}}} + \|e^{t\Delta}(\nabla \times B_0)\|_{\widehat{K}_p^{1-\frac{3}{p}}} < c_0,$$

then (S_ζ) has a unique global solution $\mathcal{U} = (u, B, \nabla \times B)$ in $\widehat{K}_p^{1-\frac{3}{p}}$ satisfying

$$\|\mathcal{U}(t, \cdot)\|_{L^p} \lesssim t^{-\frac{1}{2}(1-\frac{3}{p})}.$$

Remark VI.1.4. *By virtue of the characterization (A.8), we can show that the solution obtained in Theorem VI.1.3 is also in $\mathcal{L}(\widehat{B}_{p,\infty}^{\frac{3}{p}-1})$, by following approximately the same steps in the proof of Theorems VI.1.1 and VI.1.3. In fact for $U_0 \in \widehat{B}_{p,r}^{\frac{3}{p}-1}(\mathbb{R}^3)$, small enough, we may construct a unique global solution in $\mathcal{L}(\widehat{B}_{p,r}^{\frac{3}{p}-1})$ by proceeding as in the proofs we will show in the next section. The details are left to the reader.*

The rest of this chapter is organized as follows: In section 2, we will prove the wellposedness of the system in study in the context of Besov spaces as stated in Theorems VI.1.1 and VI.1.2. Then, in section 3, we provide the details of the proof of the wellposedness in the Kato-Herz space and the decay property described in Theorem VI.1.3.

⁶see Appendix A for the definition and some properties of such spaces.

VI.2 Wellposedness for small initial data in Besov spaces

VI.2.1 About the Cauchy problem for (u_0, B_0) in $\dot{B}_{p,r}^{\frac{3}{p}-1} \times \dot{B}_{p,r}^{\frac{3}{p}-1} \cap \dot{B}_{p,r}^{\frac{3}{p}}$

In this subsection we aim at proving Theorem VI.1.1, the proof of which is mainly based on the following proposition.

Proposition VI.2.1. *Let (p, r) be in $[1, \infty)^2$, u, v in $\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})$, $\mathcal{Q}, \mathcal{R}, \mathcal{P}$ be given as in the introduction. We have*

$$\begin{aligned} \|\mathcal{Q}(u, v)\|_{\tilde{L}^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} &\lesssim \|u\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})} \|v\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})}, \\ \|\mathcal{R}(u, v)\|_{\tilde{L}^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} &\lesssim \|u\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})} \|v\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})}, \\ \|\mathcal{P}(u, v)\|_{\tilde{L}^1(\dot{B}_{p,r}^{\frac{3}{p}})} &\lesssim \|u\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})} \|v\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})}. \end{aligned}$$

Proof

We will focus on the last inequality. The first two ones will follow by noticing that \mathcal{Q} and \mathcal{R} can be written as $\mathcal{D}^1(\mathcal{P})$, where \mathcal{D}^1 is a Fourier-multiplier of order 1.

We consider the Bony's decomposition described in the Appendix, to write

$$uv = T_u v + T_v u + R(u, v).$$

For the first term, we have

$$\|\Delta_j T_u v\|_{L^1 L_x^p} \lesssim \sum_{k \sim j} \|S_{k-1} u\|_{L^4 L_x^\infty} \|\Delta_k v\|_{L^{\frac{4}{3}} L_x^p}.$$

Using proposition A.2.1, we infer that

$$\begin{aligned} \|\Delta_j T_u v\|_{L^1 L_x^p} &\lesssim \sum_{k \sim j} c_{k,r}^2 2^{-k \frac{3}{p}} \|u\|_{\tilde{L}^4(\dot{B}_{\infty,r}^{-\frac{1}{2}})} \|v\|_{\tilde{L}^{\frac{4}{3}}(\dot{B}_{p,r}^{\frac{3}{p}+\frac{1}{2}})} \\ &\lesssim c_{j,r} 2^{-j \frac{3}{p}} \|u\|_{\tilde{L}^4(\dot{B}_{p,r}^{\frac{3}{p}-\frac{1}{2}})} \|v\|_{\tilde{L}^{\frac{4}{3}}(\dot{B}_{p,r}^{\frac{3}{p}+\frac{1}{2}})}. \end{aligned}$$

$T_v u$ enjoys the same estimate: by commuting u and v in the previous one, we obtain then

$$\|\Delta_j T_v u\|_{L^1 L_x^p} \lesssim c_{j,r} 2^{-j \frac{3}{p}} \|v\|_{\tilde{L}^4(\dot{B}_{p,r}^{\frac{3}{p}-\frac{1}{2}})} \|u\|_{\tilde{L}^{\frac{4}{3}}(\dot{B}_{p,r}^{\frac{3}{p}+\frac{1}{2}})}.$$

Finally, the remainder term, can be dealt with along the same lines, we infer that

$$\begin{aligned} \|\Delta_j R(u, v)\|_{L^1(L_x^p)} &\lesssim \sum_{k \geq j+N_0} \|\tilde{\Delta}_k u\|_{L^4(L_x^\infty)} \|\Delta_k v\|_{L^{\frac{4}{3}}(L_x^p)} \\ &\lesssim 2^{-j \frac{3}{p}} \sum_{k \geq j+N_0} 2^{(k-j) \frac{3}{p}} c_{k,r}^2 \|u\|_{\tilde{L}^4(\dot{B}_{\infty,r}^{-\frac{1}{2}})} \|v\|_{\tilde{L}^{\frac{4}{3}}(\dot{B}_{p,r}^{\frac{3}{p}+\frac{1}{2}})} \\ &\lesssim c_{j,r} 2^{-j \frac{3}{p}} \|u\|_{\tilde{L}^4(\dot{B}_{p,r}^{\frac{3}{p}-\frac{1}{2}})} \|v\|_{\tilde{L}^{\frac{4}{3}}(\dot{B}_{p,r}^{\frac{3}{p}+\frac{1}{2}})}. \end{aligned}$$

Thanks to the embedding $\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}}) \hookrightarrow \tilde{L}^{\frac{4}{3}}(\dot{B}_{p,r}^{\frac{3}{p}+\frac{1}{2}}) \cap \tilde{L}^4(\dot{B}_{p,r}^{\frac{3}{p}-\frac{1}{2}})$, then we obtain

$$\|uv\|_{\tilde{L}^1(\dot{B}_{p,r}^{\frac{3}{p}})} \lesssim \|u\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})} \|v\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})} \quad (\text{VI.8})$$

It follows that

$$\|\mathcal{P}(u, v)\|_{\tilde{L}^1(\dot{B}_{p,r}^{\frac{3}{p}})} \approx \|uv\|_{\tilde{L}^1(\dot{B}_{p,r}^{\frac{3}{p}})} \lesssim \|u\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})} \|v\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})}, \quad (\text{VI.9})$$

and

$$\|\mathcal{Q}(u, v)\|_{\tilde{L}^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} \approx \|\mathcal{R}(u, v)\|_{\tilde{L}^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} \lesssim \|u\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})} \|v\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})}. \quad (\text{VI.10})$$

Proposition VI.2.1 is proved. \square

Remark VI.2.1. We can replace $\tilde{L}^1(\cdot)$ and $\mathcal{L}(\cdot)$ in the previous proposition, respectively, by $\tilde{L}_T^1(\cdot)$ and $\mathcal{L}_T(\cdot)$, for $T > 0$.

In order to prove Theorem VI.1.1, we will use the following the abstract Banach fixed point theorem stated in the following lemma. The reader can see Lemma 4 in [15] for more details

Lemma VI.2.1. Let \mathcal{X} be an abstract Banach space with norm $\|\cdot\|$, let ζ be a bi-linear operator mapping $\mathcal{X} \times \mathcal{X}$ into \mathcal{X} satisfying

$$\forall x_1, x_2 \in \mathcal{X}, \quad \|\zeta(x_1, x_2)\| \leq \eta \|x_1\| \|x_2\|, \quad \text{for some } \eta > 0,$$

then for all $y \in \mathcal{X}$ such that

$$\|y\| < \frac{1}{4\eta}$$

the equation

$$x = y + \zeta(x, x)$$

has a solution $x \in \mathcal{B}_{\mathcal{X}}(0, 2\|y\|)$.

This solution is the unique one in the ball $\mathcal{B}_{\mathcal{X}}(0, \frac{1}{2\eta})$

Proof of Theorem VI.1.1 In order to apply Lemma VI.2.1, all we need to show is that

$$\|\zeta(\mathcal{U}, \mathcal{V})\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})} \lesssim \|\mathcal{U}\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})} \|\mathcal{V}\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})}$$

$$\|e^{t\Delta}\mathcal{U}_0\|_{\mathcal{L}(\dot{B}_{p,r}^{\frac{3}{p}-1})} \lesssim \|\mathcal{U}_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}}.$$

The second inequality follows directly from inequality (A.12) from Lemma A.2.5 in Appendix, and the first one follows by combining Proposition VI.2.1, Proposition A.2.2 and inequality (A.13) from Lemma A.2.5. Indeed, $\zeta = (\zeta_1, \zeta_2, \zeta_3)^T$ contains in each component the bi-linear operators $\mathcal{Q}, \mathcal{P}, \mathcal{R}$:

- For \mathcal{Q} and \mathcal{R} :
 - in ζ_1 we apply directly Proposition VI.2.1,
 - in ζ_2 we apply Proposition VI.2.1 and inequality (A.9) for $\rho = 1$ and $k = 1$
 - in ζ_2 we apply Proposition VI.2.1 and inequality (A.10) for $\rho = 1$ and $k = 2$
- For \mathcal{P}
 - in ζ_2 we apply Proposition VI.2.1 and inequality (A.9) for $\rho = 1$ and $k = 0$
 - in ζ_2 we apply Proposition VI.2.1 and inequality (A.10) for $\rho = 1$ and $k = 1$

This ends the proof of Theorem VI.1.1. □

As a corollary, one may replace the smallness condition on the initial data in Theorem VI.1.1, by another one on the maximal time of existence, namely we can prove:

Corollary VI.3. *Let $p \in [1, \infty)$, $r \in [1, \infty]$, and $\mathcal{U}_0 = (u_0, B_0, \nabla \times B_0)$, be in $\dot{B}_{p,r}^{\frac{3}{p}-1}(\mathbb{R}^3)$. There exists $T^* > 0$ and a unique solution \mathcal{U} to (\mathcal{S}_ζ) in $\mathcal{L}_T(\dot{B}_{p,r}^{\frac{3}{p}-1})$, for all $T < T^*$.*

Proof

We split the solution \mathcal{U} into a sum

$$\mathcal{U} = \mathcal{V} + \mathcal{W},$$

where \mathcal{V} is given by

$$\mathcal{V}(t, \cdot) \stackrel{def}{=} e^{t\Delta} \mathcal{U}_0.$$

then it remains to solve, by fixed point argument, the equation on \mathcal{W}

$$\mathcal{W} = 2\zeta(\mathcal{V}, \mathcal{V}) + 2\zeta(\mathcal{W}, \mathcal{W}) + \zeta(\mathcal{V}, \mathcal{W}) + \zeta(\mathcal{W}, \mathcal{V}).$$

Let us point out that, according to the previous calculations, we have

$$\|\Omega(\mathcal{W}, \mathcal{W})\|_{\tilde{L}_T^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} \leq \gamma \|\mathcal{W}\|_{\mathcal{L}_T(\dot{B}_{p,r}^{\frac{3}{p}-1})}^2$$

for some $\gamma > 0$, where Ω is given by (VI.6). We also have

$$\|\zeta(\mathcal{V}, \mathcal{V})\|_{\mathcal{L}_T(\dot{B}_{p,r}^{\frac{3}{p}-1})} = \|\mathcal{K}\Omega(\mathcal{V}, \mathcal{V})\|_{\mathcal{L}_T(\dot{B}_{p,r}^{\frac{3}{p}-1})} \lesssim \|\Omega(\mathcal{V}, \mathcal{V})\|_{\tilde{L}_T^1(\dot{B}_{p,r}^{\frac{3}{p}-1})}.$$

On the other hand we know that $\Omega(\mathcal{V}, \mathcal{V}) \in \tilde{L}_T^1(\dot{B}_{p,r}^{\frac{3}{p}-1})$ from the estimates of Proposition VI.2.1, and Lemma A.2.5, which gives in particular

$$\|\Omega(\mathcal{V}, \mathcal{V})\|_{\tilde{L}_T^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} \lesssim \|\mathcal{V}\|_{\mathcal{L}_T(\dot{B}_{p,r}^{\frac{3}{p}-1})}^2 \lesssim \|\mathcal{V}_0\|_{\dot{B}_{p,r}^{\frac{3}{p}-1}}^2 < \infty.$$

For the linear term on \mathcal{W} , by virtue of Lemma A.2.5 and the proof of Proposition VI.2.1, where we showed that we can obtain the estimates of \mathcal{Q} , \mathcal{P} and \mathcal{R} , by using only the norm of V in $\mathcal{Z}_T \stackrel{def}{=} \tilde{L}_T^4(\dot{B}_p^{\frac{3}{p}-\frac{1}{2}}) \cap \tilde{L}_T^{\frac{4}{3}}(\dot{B}_p^{\frac{3}{p}+\frac{1}{2}})$, we infer that

$$\|\zeta(\mathcal{V}, \mathcal{W})\|_{\mathcal{L}_T(\dot{B}_{p,r}^{\frac{3}{p}-1})} + \|\zeta(\mathcal{W}, \mathcal{V})\|_{\mathcal{L}_T(\dot{B}_{p,r}^{\frac{3}{p}-1})} \leq C \|\mathcal{V}\|_{\mathcal{Z}_T} \|\mathcal{W}\|_{\mathcal{L}_T(\dot{B}_{p,r}^{\frac{3}{p}-1})},$$

for some universal constant $C > 0$. We chose T_1 such that, for all $T < T_1$

$$C \|\mathcal{V}\|_{\mathcal{Z}_T} < \lambda < 1,$$

we can then chose $T^* \leq T_1$ small enough so that, for all $T < T^* \leq T_1$, we have

$$\|\Omega(\mathcal{V}, \mathcal{V})\|_{\tilde{L}_T^1(\dot{B}_{p,r}^{\frac{3}{p}-1})} < \varepsilon_1 \leq \frac{(1-\lambda)^2}{4\gamma}$$

The result follows by a direct application of Lemma A.3.1. \square

VI.3.1 Local wellposedness for initial data (u_0, B_0) in $\dot{B}_{p,1}^{\frac{3}{p}-1} \times \dot{B}_{p,1}^{\frac{3}{p}}$

We present in this subsection the outlines of the proof of Theorem VI.1.2. The following a priori estimates will play a crucial role to reach our purpose

Proposition VI.3.1. *Let p be in $[1, \infty)$, $T > 0$, u in $\mathcal{L}_T(\dot{B}_{p,1}^{\frac{3}{p}-1})$ and w, z be in $\mathcal{L}_T(\dot{B}_{p,1}^{\frac{3}{p}})$, then we have*

$$\|\mathcal{Q}(w, z)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}-1})} \lesssim T \|w\|_{\mathcal{L}_T(\dot{B}_{p,1}^{\frac{3}{p}})} \|z\|_{\mathcal{L}_T(\dot{B}_{p,1}^{\frac{3}{p}})} \quad (\text{VI.11})$$

$$\|\mathcal{Q}(w, z)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}+1})} \lesssim \|w\|_{\mathcal{L}_T(\dot{B}_{p,1}^{\frac{3}{p}})} \|z\|_{\mathcal{L}_T(\dot{B}_{p,1}^{\frac{3}{p}})} \quad (\text{VI.12})$$

$$\|\mathcal{P}(u, w)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}+1})} \lesssim \|u\|_{\mathcal{L}_T(\dot{B}_{p,1}^{\frac{3}{p}-1})} \|w\|_{\mathcal{L}_T(\dot{B}_{p,1}^{\frac{3}{p}})} \quad (\text{VI.13})$$

$$\|\mathcal{R}(u, w)\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}})} \lesssim \|u\|_{\mathcal{L}_T(\dot{B}_{p,1}^{\frac{3}{p}-1})} \|w\|_{\mathcal{L}_T(\dot{B}_{p,1}^{\frac{3}{p}})} \quad (\text{VI.14})$$

The proof does not work for $r > 1$, as the embedding $\dot{B}_{p,r}^{\frac{3}{p}}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ fails to be true unless when $r = 1$. In this part of the thesis, we will denote $d_j \stackrel{def}{=} c_{j,1}$ being a sequence in $\ell^1(\mathbb{Z})$.

Proof of Proposition VI.3.1

Proof of (VI.11): Inequality (VI.11) follows directly from the fact that $\tilde{L}^\infty(\dot{B}_{p,1}^{\frac{3}{p}})$ is an algebra and the (local in time) embedding

$$\|a\|_{\tilde{L}^1(\dot{B}_{p,1}^{\frac{3}{p}})} \leq T \|a\|_{\tilde{L}^\infty(\dot{B}_{p,1}^{\frac{3}{p}})}.$$

Proof of (VI.12):

According to Bony's decomposition, we have

$$wz = T_w z + T_z w + R(w, z).$$

We then show how to estimate the first and the third term. We have

$$\begin{aligned} \|\Delta_j T_w z\|_{L_T^1 L^p} &\lesssim \|S_{j-1} w\|_{L_T^\infty L^\infty} \|\Delta_j z\|_{L_T^1 L^p} \\ &\lesssim d_j 2^{-j(\frac{3}{p}+2)} \|w\|_{L_T^\infty(L^\infty)} \|w\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}+2})}. \end{aligned}$$

We deduce from the embedding $\dot{B}_{p,1}^{\frac{3}{p}}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, together with Minkowski's inequality, give

$$\|\Delta_j T_w z\|_{L_T^1 L^p} \lesssim d_j 2^{-j(\frac{3}{p}+2)} \|w\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{3}{p}})} \|w\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}+2})}.$$

For the remainder term, we proceed as follows

$$\begin{aligned} \|\Delta_j R(w, z)\|_{L_T^1 L^p} &\lesssim \sum_{k \geq j+N_0} \left\| \tilde{\Delta}_k w \right\|_{L_T^1 L^p} \|\Delta_k z\|_{L^\infty L^\infty} \\ &\lesssim 2^{-j(\frac{3}{p}+2)} \sum_{k \geq j+N_0} d_k 2^{(j-k)(\frac{3}{p}+2)} \|w\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}+2})} \|z\|_{\tilde{L}^\infty(\dot{B}_{\infty,\infty}^0)} \\ &\lesssim 2^{-j(\frac{3}{p}+2)} d_j \|w\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}+2})} \|z\|_{\tilde{L}^\infty(\dot{B}_{p,1}^{\frac{3}{p}})}. \end{aligned}$$

Inequality (VI.12) follows.

proof of (VI.13): Let us point out again that, due to the fact that $\mathcal{D}^1(\mathcal{P}) \approx \mathcal{R}$, (VI.13) and (VI.14) can be proved along the same way, we will thus concentrate on the proof of (VI.13).

We consider again the Bony's decomposition

$$uw = T_u w + T_w u + R(u, w).$$

For $T_u w$, we have

$$\begin{aligned} \|\Delta_j(T_u w)\|_{L_T^1 L^p} &\lesssim \|S_{j-1} u\|_{L_T^\infty L^\infty} \|\Delta_j w\|_{L_T^1 L^p} \\ &\lesssim d_j 2^{-j(\frac{3}{p}+1)} \|u\|_{\tilde{L}_T^\infty(\dot{B}_{\infty,\infty}^{-1})} \|w\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}+2})} \\ &\lesssim d_j 2^{-j(\frac{3}{p}+1)} \|u\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{3}{p}-1})} \|w\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}+2})}. \end{aligned}$$

For $T_w u$, by using the embedding $\dot{B}_{p,1}^{\frac{3}{p}}(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$, we infer that

$$\begin{aligned} \|\Delta_j(T_w u)\|_{L_T^1 L^p} &\lesssim \|S_{j-1} w\|_{L_T^\infty L^\infty} \|\Delta_j u\|_{L_T^1 L^p} \\ &\lesssim d_j 2^{-j(\frac{3}{p}+1)} \|w\|_{L_T^\infty(L^\infty)} \|u\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}+1})} \\ &\lesssim d_j 2^{-j(\frac{3}{p}+1)} \|w\|_{L_T^\infty(\dot{B}_{p,1}^{\frac{3}{p}})} \|u\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}+1})} \end{aligned}$$

Minkoski's inequality then gives

$$\|\Delta_j(T_w u)\|_{L_T^1 L^p} \lesssim d_j 2^{-j(\frac{3}{p}+1)} \|w\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{3}{p}})} \|u\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}+1})}.$$

For the remainder term, we have

$$\begin{aligned} \|\Delta_j R(u, w)\|_{L_T^1 L^p} &\lesssim \sum_{k \geq j+N_0} \left\| \tilde{\Delta}_k u \right\|_{L_T^1 L^p} \|\Delta_k w\|_{L_T^\infty L^\infty} \\ &\lesssim 2^{-j(\frac{3}{p}+1)} \sum_{k \geq j+N_0} d_k 2^{(j-k)(\frac{3}{p}+1)} \|u\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}+1})} \|w\|_{\tilde{L}_T^\infty(\dot{B}_{\infty,\infty}^0)} \\ &\lesssim 2^{-j(\frac{3}{p}+1)} d_j \|u\|_{\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}+1})} \|w\|_{\tilde{L}_T^\infty(\dot{B}_{p,1}^{\frac{3}{p}})}. \end{aligned}$$

This ends the proof of inequality (VI.13), and eventually (VI.14).
 Lemma VI.3.1 is then proved. □

The proof of Theorem VI.1.2 is based on the following variation of Lemma VI.2.1.

Lemma VI.3.1. *Let $\{A_i\}_{i \in \{1,2,3,4\}}$ be a set of bi-linear operators with*

$$\begin{aligned} \|A_1(x_1, x_2)\|_{\mathcal{X}} &\leq \eta_1 \|x_1\|_{\mathcal{X}} \|x_2\|_{\mathcal{X}} \\ \|A_2(y_1, y_2)\|_{\mathcal{X}} &\leq (1+T)\eta_2 \|y_1\|_{\mathcal{Y}} \|y_2\|_{\mathcal{Y}} \\ \|A_3(x_1, y_2)\|_{\mathcal{Y}} &\leq \eta_3 \|x_1\|_{\mathcal{X}} \|y_2\|_{\mathcal{Y}} \\ \|A_4(y_1, y_2)\|_{\mathcal{Y}} &\leq \eta_4 \|y_1\|_{\mathcal{Y}} \|y_2\|_{\mathcal{Y}} \end{aligned}$$

for some non negative $T, (\eta_i)_{i \in \{1,2,3,4\}}$.

Let $\eta \stackrel{\text{def}}{=} \max_{i \in \{1,2,3,4\}} \{\eta_i\}$. Then for all $(x_0, y_0) \in (\mathcal{X} \times \mathcal{Y})$ such that

$$\|x_0\|_{\mathcal{X}} + (2+T)\|y_0\|_{\mathcal{Y}} < \frac{1}{24\eta}, \tag{VI.15}$$

the system

$$\begin{cases} x = x_0 + A_1(x, x) + A_2(y, y) \\ y = y_0 + A_3(x, y) + A_4(y, y) \end{cases}$$

has a unique solution (x, y) in $\mathcal{X} \times \mathcal{Y}$, which also satisfies

$$\|x\|_{\mathcal{X}} + (2+T)\|y\|_{\mathcal{Y}} < \frac{1}{12\eta}.$$

Proof

Let us present briefly the outlines of the proof, the idea is classical:
 We define the sequence (x^n, y^n) by

$$\begin{cases} (x^0, y^0) = (x_0, y_0) \\ x^{n+1} = x_0 + A_1(x^n, x^n) + A_2(y^n, z^n) \\ y^{n+1} = y_0 + A_3(x^n, y^n) + A_4(y^n, y^n) \end{cases}$$

If we write $z^n \stackrel{def}{=} (1+T)y^n$, then the system above is equivalent to

$$(Seq) \begin{cases} (x^0, y^0, z^0) = (x_0, y_0, (1+T)y^0) \\ x^{n+1} = x_0 + A_1(x^n, x^n) + \tilde{A}_2(y^n, z^n) \\ y^{n+1} = y_0 + A_3(x^n, y^n) + A_4(y^n, y^n) \\ z^{n+1} = z_0 + A_3(x^n, z^n) + A_4(y^n, z^n) \end{cases}$$

with $\tilde{A}_2 = \frac{1}{1+T}A_2$ whose norm is less than η .

Let $\alpha \stackrel{def}{=} \|x_0\|_{\mathcal{X}} + \|y_0\|_{\mathcal{Y}} + \|z_0\|_{\mathcal{Y}} < \frac{1}{24\eta}$, we claim that (x^n, y^n, z^n) is a Cauchy (bounded) sequence in $\mathcal{B}_{\mathcal{X} \times \mathcal{Y} \times \mathcal{Y}}(0, 2\alpha)$.

By virtue of the definition of (x^n, y^n, z^n) and the continuity of A_i , we proceed by induction to obtain

$$\begin{cases} \|x^{n+1}\|_{\mathcal{X}} \leq \|x_0\|_{\mathcal{X}} + 8\eta\alpha^2 \\ \|y^{n+1}\|_{\mathcal{Y}} \leq \|y_0\|_{\mathcal{Y}} + 8\eta\alpha^2 \\ \|z^{n+1}\|_{\mathcal{Y}} \leq \|z_0\|_{\mathcal{Y}} + 8\eta\alpha^2 \end{cases}$$

which gives

$$\|x^{n+1}\|_{\mathcal{X}} + \|y^{n+1}\|_{\mathcal{Y}} + \|z^{n+1}\|_{\mathcal{Y}} < 2\alpha.$$

In order to prove that (x^n, y^n, z^n) is a Cauchy sequence, similar computation lead to

$$I_n \stackrel{def}{=} \|x^{n+1} - x^n\|_{\mathcal{X}} + \|y^{n+1} - y^n\|_{\mathcal{Y}} + \|z^{n+1} - z^n\|_{\mathcal{Y}} \leq (24\eta\alpha)I_{n-1}$$

This is enough to conclude the proof. \square

Proof of Theorem VI.1.2 In order to apply Lemma VI.3.1, let us rewrite the system (S_{ζ}) as follows. We define

$$\begin{pmatrix} \psi_1(x_1, x_2) \\ \psi_2(y_1, y_2) \\ \psi_3(x_1, y_1) \\ \psi_4(y_1, y_2) \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} -\mathcal{Q}(x_1, x_2) \\ \mathcal{Q}(y_1, y_2) - \mathcal{Q}(\nabla \times y_1, \nabla \times y_2) \\ \mathcal{P}(x_1, y_1) - \mathcal{R}(\nabla \times y_1, x_1) - 2\mathcal{Q}(\nabla \times y_1, x_1) \\ -2\mathcal{Q}(y_1, y_2) - \mathcal{Q}(\nabla \times y_1, \nabla \times y_2) \end{pmatrix},$$

so that

$$\varphi \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix} = \begin{pmatrix} \varphi_1(x_1, x_1) \\ \varphi_2(y_1, y_2) \\ \varphi_3(x_1, y_1) \\ \varphi_4(y_1, y_2) \end{pmatrix} \stackrel{def}{=} \begin{pmatrix} \psi_1(x_1, x_2) \\ \psi_2(y_1, y_2) \\ (Id - \Delta)^{-1} \nabla \times \psi_3(x_1, y_1) \\ (Id - \Delta)^{-1} \nabla \times \psi_4(y_1, y_2) \end{pmatrix},$$

and then, we define

$$\begin{pmatrix} A_1(x_1, x_2) \\ A_2(y_1, y_2) \\ A_3(x_1, y_1) \\ A_4(y_1, y_2) \end{pmatrix} \stackrel{def}{=} \mathcal{K}\mathbb{P}\varphi \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix},$$

with

$$\mathcal{K}\mathbb{P}\varphi(t, \cdot) = \int_0^t e^{(t-s)\Delta} \mathbb{P}\varphi(s, \cdot) ds.$$

Therefore, the system (S_ζ) is equivalent to the following one

$$\begin{cases} u(t, \cdot) = e^{t\Delta} u_0 + A_1(u, u) + A_2(B, B) \\ B(t, \cdot) = e^{t\Delta} B_0 + A_3(u, B) + A_4(B, B) \end{cases}$$

The proof of Theorem VI.1.2 can be reduced to a direct application of Lemma VI.3.1, thus all we need to show then is that $\{A_i\}_{i \in \{1,2,3,4\}}$ satisfies the hypothesis of Lemma VI.3.1 with $\mathcal{X} = \mathcal{L}(\dot{B}_{p,1}^{\frac{3}{p}-1})$ and $\mathcal{Y} = \mathcal{L}(\dot{B}_{p,1}^{\frac{3}{p}})$. To do so, according to Lemma A.2.5 we should estimate φ in

$$\tilde{L}_T^1(\dot{B}_{p,1}^{\frac{3}{p}-1} \times \dot{B}_{p,1}^{\frac{3}{p}-1} \times \dot{B}_{p,1}^{\frac{3}{p}} \times \dot{B}_{p,1}^{\frac{3}{p}}).$$

Now, each component of φ contains a combination of \mathcal{Q} , \mathcal{P} , and \mathcal{R} , we will thus show how to use Proposition VI.2.1, Proposition VI.3.1 and Lemma A.2.4 to deal with each one

- For \mathcal{Q}
 - in φ_1 , we apply Proposition VI.2.1.
 - in φ_2 , for $\mathcal{Q}(y_1, y_2)$ we apply inequality (VI.11) from Proposition VI.3.1, and for $\mathcal{Q}(\nabla \times y_1, \nabla \times y_2)$, we apply Proposition VI.2.1.
 - in φ_3 , we apply respectively Proposition VI.2.1, then inequality (A.9) from Proposition A.2.2 for $k = 2$.
 - in φ_4 ,
 - * for $\mathcal{Q}(y_1, y_2)$, we apply inequality (VI.12) from Proposition VI.3.1, then inequality (A.9) from Proposition A.2.2 for $k = 0$
 - * for $\mathcal{Q}(\nabla \times y_1, \nabla \times y_2)$, we apply Proposition VI.2.1, then inequality (A.9) from Proposition A.2.2 for $k = 2$.
- For \mathcal{P} , we apply inequality (VI.13) from Proposition VI.3.1, then inequality (A.9) from Proposition A.2.2 for $k = 0$.
- For \mathcal{R} , we apply inequality (VI.14) from Proposition VI.3.1, then inequality (A.9) from Proposition A.2.2 for $k = 1$.

□

This ends the proof of Theorem VI.1.2.

VI.4 Wellposedness for small initial data in the Kato-Herz spaces and large time behavior

In this subsection, we shall give some details about the wellposedness in the hat-Kato space $\widehat{K}_p^{1-\frac{3}{p}}$, and then we establish the decay property described in Theorem VI.1.3. It is all based on the following proposition

Proposition VI.4.1. *Let ζ be given by (VI.7). Then ζ maps $\widehat{K}_p^{1-\frac{3}{p}} \times \widehat{K}_p^{1-\frac{3}{p}}$ continuously into $\widehat{K}_p^{1-\frac{3}{p}}$. That is to say, there exists $\kappa > 0$ such that*

$$\|\zeta(L, M)\|_{\widehat{K}_p^{1-\frac{3}{p}}} \leq \kappa \|L\|_{\widehat{K}_p^{1-\frac{3}{p}}} \|M\|_{\widehat{K}_p^{1-\frac{3}{p}}},$$

for all $L, M \in \widehat{K}_p^{1-\frac{3}{p}}$.

Proof

By taking into account the inequality, for all $m \in [0, 2]$ and $|\xi| \sim 2^j$

$$\frac{1}{1 + |\xi|^2} \lesssim \frac{1}{2^{jm}}, \quad (\text{VI.16})$$

in order to prove the continuity property of ζ on $\widehat{K}_p^{1-\frac{3}{p}}$, we only need to show that, for all $t > 0$

$$t^{\frac{1}{2}(1-\frac{3}{p})} \int_0^t \left\| e^{-(t-s)|\cdot|^2} \cdot |(\widehat{L} \star \widehat{M})(s, \cdot)| \right\|_{L^{p'}} ds \lesssim \|L\|_{\widehat{K}_p^{1-\frac{3}{p}}} \|M\|_{\widehat{K}_p^{1-\frac{3}{p}}}, \quad (\text{VI.17})$$

that is, due to (VI.16), all the components of ζ can be dominated by a Gaussian multiplied by an order one Fourier-multiplier, as in the proof of Theorem VI.1.2.

Let us then give some details about the proof of (VI.17). By setting, for $p > 3$,

$$\frac{1}{r} \stackrel{\text{def}}{=} 1 - \frac{2}{p} \iff \frac{1}{p} + \frac{1}{r} = \frac{1}{p'} \iff \frac{1}{p'} + \frac{1}{p'} - \frac{1}{r} = 1,$$

by virtue of Holder inequality, we infer that

$$\int_0^t \left\| e^{-(t-s)|\cdot|^2} \cdot |(\widehat{L} \star \widehat{M})(s, \cdot)| \right\|_{L^{p'}} ds \lesssim \int_0^t \|\mathcal{G}(t-s, \cdot)\|_{L^p} \left\| \widehat{L}(s, \cdot) \right\|_{L^{p'}} \left\| \widehat{M}(s, \cdot) \right\|_{L^{p'}} ds,$$

where

$$\mathcal{G}(\tau, \xi) \stackrel{\text{def}}{=} e^{-\tau|\xi|^2} |\xi|.$$

By a change of variable in the L^p norm of $\mathcal{G}(t-s, \cdot)$, we obtain

$$\|\mathcal{G}(t-s, \cdot)\|_{L^p} \lesssim \frac{1}{(t-s)^{\frac{1}{2} + \frac{3}{2p}}} \|\mathcal{G}(1, \cdot)\|_{L^p}.$$

This gives

$$\begin{aligned} \int_0^t \left\| e^{-(t-s)|\cdot|^2} \cdot |(\widehat{L} \star \widehat{M})(s, \cdot)| \right\|_{L^{p'}} ds &\lesssim \|L\|_{\widehat{K}_p^{1-\frac{3}{p}}} \|M\|_{\widehat{K}_p^{1-\frac{3}{p}}} \int_0^t \frac{1}{(t-s)^{\frac{1}{2}+\frac{3}{2p}} s^{1-\frac{3}{p}}} ds, \\ &\lesssim t^{-\frac{1}{2}(1-\frac{3}{p})} \|L\|_{\widehat{K}_p^{1-\frac{3}{p}}} \|M\|_{\widehat{K}_p^{1-\frac{3}{p}}}. \end{aligned}$$

Inequality (VI.17) follows, and Proposition VI.4.1 is then proved. \square

Proof of theorem VI.1.3 According to proposition VI.4.1, we can apply the fixed point lemma VI.2.1, that is, if

$$\|e^{t\Delta} u_0\|_{\widehat{K}_p^{\frac{3}{p}-1}} + \|e^{t\Delta} B_0\|_{\widehat{K}_p^{\frac{3}{p}-1}} + \|e^{t\Delta} (\nabla \times B_0)\|_{\widehat{K}_p^{\frac{3}{p}-1}} < \frac{1}{4\kappa}$$

then we can construct a unique solution \mathcal{U} of (S_ζ) in $\widehat{K}_p^{\frac{3}{p}-1}$ with

$$\|\mathcal{U}\|_{\widehat{K}_p^{\frac{3}{p}-1}} < \frac{1}{2\kappa}.$$

By virtue of the continuity of the Fourier transform, from L^q into $L^{q'}$, for $q \in [1, 2]$, we infer that, for $p \in (3, \infty)$,

$$\|\mathcal{U}(t, \cdot)\|_{L^p} = \left\| \widehat{\mathcal{U}}(t, \cdot) \right\|_{L^p} \lesssim \left\| \widehat{\mathcal{U}}(t, \cdot) \right\|_{L^{p'}}$$

it follows then, for all $t > 0$

$$\|\mathcal{U}(t, \cdot)\|_{L^p} \lesssim \left\| \widehat{\mathcal{U}}(t, \cdot) \right\|_{L^{p'}} \lesssim t^{-\frac{1}{2}(1-\frac{3}{p})}.$$

Theorem VI.1.3 is then proved. \square

Part IV
Appendix

APPENDIX A

Harmonic analysis tool box and functional framework

In this Chapter, we present briefly some definitions, functional framework and some results which were of a constant use in the previous Chapters.

A.1 Littlewood-Paley theory and Functional spaces

In this section, we recall the basic concepts about the Littlewood-Paley theory and we define a bunch of functional spaces used in this thesis.

Let (ψ, φ) be a couple of smooth functions with value in $[0, 1]$ satisfying:

$$\begin{aligned} \text{Supp } \psi &\subset \{\xi \in \mathbb{R} : |\xi| \leq \frac{4}{3}\}, & \text{Supp } \varphi &\subset \{\xi \in \mathbb{R} : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\} \\ \psi(\xi) + \sum_{q \in \mathbb{N}} \varphi(2^{-q}\xi) &= 1 \quad \forall \xi \in \mathbb{R}, & \sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) &= 1 \quad \forall \xi \in \mathbb{R} \setminus \{0\}. \end{aligned}$$

For every $u \in S'(\mathbb{R}^3)$, define the cut-off or dyadic operators by

$$\Delta_{-1}u \stackrel{\text{def}}{=} \psi(D)u, \quad \Delta_q u \stackrel{\text{def}}{=} \varphi(2^{-q}D)u \quad \text{if } q \in \mathbb{N}, \quad S_q u \stackrel{\text{def}}{=} \sum_{j \leq q-1} \Delta_j u \quad \text{for } q \geq 0, \quad (\text{A.1})$$

where, $\varphi(\lambda D)$ is the Fourier multiplier by $\varphi(\lambda|\cdot|)$.

Besides, other nice properties Δ_q and S_q are listed in the following points. Namely, for $u, v \in S'(\mathbb{R}^3)$ we have

$$(i) \quad |p - q| \geq 2 \implies \Delta_p \Delta_q u \equiv 0,$$

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- (ii) $|p - q| \geq 4 \implies \Delta_q(S_{p-1}u\Delta_p v) \equiv 0$,
- (iii) $\Delta_q, S_q : L^p \rightarrow L^p$ uniformly with respect to q and p .
- (iv)

$$u = \sum_{q \geq -1} \Delta_q u.$$

Likewise the homogeneous operators $\dot{\Delta}_q$ and \dot{S}_q are defined by

$$\forall q \in \mathbb{Z} \quad \dot{\Delta}_q = \varphi(2^q D)u, \quad \dot{S}_q = \sum_{j \leq q-1} \dot{\Delta}_j v. \quad (\text{A.2})$$

Before we define the Besov spaces, let us introduce a subspace of the tempered distribution space $\mathcal{S}(\mathbb{R}^d)$. This subspace will be useful in the sequel.

Definition A.1.1. We denote by $\mathcal{S}'_h(\mathbb{R}^d)$ the space of tempered distributions u such that

$$\lim_{\lambda \rightarrow \infty} \|\theta(\lambda D)u\|_{L^\infty} = 0, \quad \forall \theta \in \mathcal{D}(\mathbb{R}^d).$$

Remark A.1.1. We can check that $u \in \mathcal{S}'_h(\mathbb{R}^d)$ if and only if we can find some $\theta \in \mathcal{D}(\mathbb{R}^d)$ satisfying the above equality and $\theta(0) \neq 0$.

Definition A.1.2. For $(s, p, r) \in \mathbb{R} \times [1, +\infty]^2$. The non-homogeneous Besov space $B_{p,r}^s$ (resp. the homogeneous Besov space $\dot{B}_{p,r}^s$) is the set of all tempered distributions $u \in \mathcal{S}'$ (resp. $u \in \mathcal{S}'_h(\mathbb{R}^d)$) such that

$$\begin{aligned} \|u\|_{B_{p,r}^s} &\stackrel{\text{def}}{=} \|2^{qs} \|\Delta_q u\|_{L^p}\|_{\ell^r(\mathbb{Z})} < \infty. \\ (\text{resp. } \|u\|_{\dot{B}_{p,r}^s} &\stackrel{\text{def}}{=} \left\| \left\| 2^{qs} \|\dot{\Delta}_q u\|_{L^p} \right\|_{\ell^r(\mathbb{Z})} < \infty \right), \end{aligned}$$

We point out that, for $s \in \mathbb{R}$, the spaces $H^s(\mathbb{R}^3)$ and $\dot{H}^s(\mathbb{R}^3)$ can be identified respectively to $B_{2,2}^s(\mathbb{R}^3)$ and $\dot{B}_{2,2}^s(\mathbb{R}^3)$ with equivalent norms.

The well known *Bony's* decomposition [11] enables us to split formally the product of two tempered distributions u and v into three pieces. Especially, we have

Definition A.1.3. For a given $u, v \in \mathcal{S}'$ we have

$$uv = T_u v + T_v u + R(u, v), \quad (\text{A.3})$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \quad R(u, v) = \sum_q \Delta_q u \tilde{\Delta}_q v \quad \text{and} \quad \tilde{\Delta}_q = \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$

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Remark A.1.2. *Similar decomposition is available for the homogeneous blocks $\dot{\Delta}_q$.*

The mixed space-time functional spaces are more useful in several parts of this thesis which motivates the following definition.

Definition A.1.4. *Let $T > 0$ and $(\gamma, p, r, s) \in [1, \infty]^3 \times \mathbb{R}$. We define the spaces $L_T^\gamma B_{p,r}^s$ and $\tilde{L}_T^\gamma B_{p,r}^s$ respectively by:*

$$L_T^\gamma B_{p,r}^s \stackrel{def}{=} \left\{ u : [0, T] \rightarrow S'; \|u\|_{L_T^\gamma B_{p,r}^s} = \left\| (2^{qs} \|\Delta_q u\|_{L^p})_{\ell^r} \right\|_{L_T^\gamma} < \infty \right\},$$

$$\tilde{L}_T^\gamma B_{p,r}^s \stackrel{def}{=} \left\{ u : [0, T] \rightarrow S'; \|u\|_{\tilde{L}_T^\gamma B_{p,r}^s} = (2^{qs} \|\Delta_q u\|_{L_T^\gamma L^p})_{\ell^r} < \infty \right\}.$$

Next, we recall the definition of Kato spaces, then we introduce the Kato-Herz and the Fourier-Herz spaces used in Theorem VI.1.3, for more details about the Fourier-Herz spaces the reader can see for instance [33].

Definition A.1.5 (Kato spaces). *Let p be in $[1, \infty]$, $\sigma \in \mathbb{R}_+^*$, we define the space $K_{p,r}^\sigma(T)$ (or simply $K_{p,r}^\sigma$ when $T = \infty$), as the space of functions u on $\mathbb{R}^+ \times \mathbb{R}^d$, such that*

$$\|u\|_{K_{p,r}^\sigma(T)} \stackrel{def}{=} \left\| t^{\frac{\sigma}{2}} \|u(t, \cdot)\|_{L^p(\mathbb{R}^d)} \right\|_{L^r((0,T); \frac{dt}{t})}.$$

In the case $r = \infty$, we simply denote $K_{p,\infty}^\sigma = K_p^\sigma$, such that

$$\|u\|_{K_p^\sigma(T)} \stackrel{def}{=} \sup_{t \in (0,T)} \{t^{\frac{\sigma}{2}} \|u(t, \cdot)\|_{L^p(\mathbb{R}^d)}\} < \infty.$$

Definition A.1.6 (Fourier-Herz and Kato-Herz spaces). *Let p, r be in $[1, \infty]$, $(s, \sigma) \in \mathbb{R}_+^* \times \mathbb{R}$, we define the Fourier-Herz space $\widehat{B}_{p,r}^s(\mathbb{R}^d)$ as the space of tempered distribution w on \mathbb{R}^d such that*

$$\|u\|_{\widehat{B}_{p,r}^s} \stackrel{def}{=} \left\| 2^{js} \left\| \widehat{\Delta_j} u \right\|_{L^{p'}(\mathbb{Z}_j^r)} \right\| < \infty,$$

and we define $\widehat{K}_{p,r}^\sigma(T)$ (or simply $\widehat{K}_{p,r}^\sigma$ when $T = \infty$), as the space of functions u on $\mathbb{R}^+ \times \mathbb{R}^d$, such that

$$\|u\|_{\widehat{K}_{p,r}^\sigma(T)} \stackrel{def}{=} \left\| t^{\frac{\sigma}{2}} \|\widehat{u}(t, \cdot)\|_{L^{p'}(\mathbb{R}^d)} \right\|_{L^r((0,T); \frac{dt}{t})}.$$

In the case $r = \infty$, we simply denote $\widehat{K}_{p,\infty}^\sigma = \widehat{K}_p^\sigma$, such that

$$\|u\|_{\widehat{K}_p^\sigma(T)} \stackrel{def}{=} \sup_{t \in (0,T)} \{t^{\frac{\sigma}{2}} \|\widehat{u}(t, \cdot)\|_{L^{p'}(\mathbb{R}^d)}\} < \infty.$$

Remark A.1.3. *In terms of the scaling, the Fourier-Herz space $\widehat{B}_{p,r}^s$ (resp. the Kato-Herz space $\widehat{K}_{p,r}^s$) has the same scale as the usual Besov $B_{p,r}^s$ (resp. the usual Kato $K_{p,r}^s$).*

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Let a be a tempered distribution in $S'(\mathbb{R}^3)$, (ψ, φ) by given as at the beginning of this section, $\lambda \in \mathbb{R}^+$ and $\xi = (\xi_h, \xi_v) \in \mathbb{R}^2 \times \mathbb{R}$, by denoting $\varphi(\lambda D_h)$ (resp. $\varphi(\lambda D_v)$) the Fourier multiplier by $\varphi(\lambda|\xi_h|)$ (resp. $\varphi(\lambda|\xi_v|)$) we define the non-homogeneous dyadic blocks Δ_q^h (resp. Δ_q^v) and the homogeneous ones $\dot{\Delta}_q^h$ (resp. $\dot{\Delta}_q^v$) by setting:

$$\begin{aligned} \Delta_q^h &\stackrel{def}{=} \begin{cases} \varphi(2^{-q}D_h)a, & \text{for } q \in \mathbb{N} \\ \psi(D_h)a, & \text{for } q = -1 \\ 0, & \text{for } q \leq -2 \end{cases} & \Delta_j^v &\stackrel{def}{=} \begin{cases} \varphi(2^{-q}D_v)a, & \text{for } j \in \mathbb{N} \\ \psi(D_v)a, & \text{for } j = -1 \\ 0, & \text{for } j \leq -2 \end{cases} \\ S_q^h &\stackrel{def}{=} \sum_{m < j} \Delta_m^v, \quad \forall q \in \mathbb{Z}, & S_j^v &\stackrel{def}{=} \sum_{m < j} \Delta_m^h, \quad \forall j \in \mathbb{Z} \\ \dot{\Delta}_q^h &\stackrel{def}{=} \varphi(2^{-q}D_h)a \quad \forall, \quad q \in \mathbb{Z}, & \dot{\Delta}_j^v &\stackrel{def}{=} \varphi(2^{-q}D_v)a \quad \forall, \quad j \in \mathbb{Z}, \\ \dot{S}_q^h &\stackrel{def}{=} \sum_{m < j} \dot{\Delta}_m^h, \quad \forall q \in \mathbb{Z}, & \dot{S}_j^v &\stackrel{def}{=} \sum_{m < j} \dot{\Delta}_m^v, \quad \forall j \in \mathbb{Z}. \end{aligned}$$

similarly to the classical case, we define the anisotropic Besov spaces as follows

Definition A.1.1. *Let s, t be two real numbers and let p, q_1, q_2 be in $[1, +\infty]$, we define the space $(B_{p, q_1}^t)_h (B_{p, q_2}^s)_v$ as the space of tempered distributions u such that*

$$\|u\|_{(B_{p, q_1}^t)_h (B_{p, q_2}^s)_v} := \left\| 2^{kt} 2^{js} \left\| \Delta_k^h \Delta_j^v u \right\|_{L^p} \right\|_{\ell_k^{q_1}(\mathbb{Z}; \ell_j^{q_2}(\mathbb{Z}))} < \infty.$$

Remark A.1.4. *We point out that*

- *The homogeneous anisotropic Besov spaces can be defined similarly by replacing the non-homogeneous blocks by the homogeneous ones.*
- *In the situation where $q_1 = q_2 = q$, we use the notation $B_{p, q}^{t, s} := (B_{p, q}^t)_h (B_{p, q}^s)_v$.*
- *In the case $p = q = 2$ the above spaces are nothing but the anisotropic Sobolev spaces, still denoted by $H^{t, s}$.*
- *If moreover $t = 0$ then we have:*

$$\|u\|_{H^{0, s}} \approx \left(\sum_{j \in \mathbb{Z}} 2^{2js} \left\| \Delta_j^v u \right\|_{L^2}^2 \right)^{\frac{1}{2}}$$

A.2 Various results

We begin this section by recalling the Bernstein lemma from [8]

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Lemma A.2.1 (Bernstein). *Let \mathcal{B} be a ball of \mathbb{R}^d , and \mathcal{C} be a ring of \mathbb{R}^d . Let also a be a tempered distribution and \hat{a} its Fourier transform. Then for $1 \leq p_2 \leq p_1 \leq \infty$ we have:*

$$\begin{aligned} \text{Supp } \hat{a} \subset 2^k \mathcal{B} &\implies \|\partial_x^\alpha a\|_{L^{p_1}} \lesssim 2^{k(|\alpha|+d(\frac{1}{p_2}-\frac{1}{p_1}))} \|a\|_{L^{p_2}} \\ \text{Supp } \hat{a} \subset 2^k \mathcal{C} &\implies \|a\|_{L^{p_1}} \lesssim 2^{-kN} \sup_{|\alpha|=N} \|\partial^\alpha a\|_{L^{p_1}} \end{aligned}$$

similarly, one may prove an analogue result in the anisotropic case

Lemma A.2.2. *Let \mathcal{B}_h (resp. \mathcal{B}_v) be a ball of \mathbb{R}_h^2 (resp. \mathbb{R}_v) and \mathcal{C}_h (resp. \mathcal{C}_v) a ring of \mathbb{R}_h^2 (resp. \mathbb{R}_v). Let also a be a tempered distribution and \hat{a} its Fourier transform. Then for $1 \leq p_2 \leq p_1 \leq \infty$ and $1 \leq q_2 \leq q_1 \leq \infty$ we have:*

$$\begin{aligned} \text{Supp } \hat{a} \subset 2^k \mathcal{B}_h &\implies \|\partial_{x_h}^\alpha a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{k(|\alpha|+2(\frac{1}{p_2}-\frac{1}{p_1}))} \|a\|_{L_h^{p_2}(L_v^{q_1})} \\ \text{Supp } \hat{a} \subset 2^l \mathcal{B}_v &\implies \|\partial_{x_3}^\beta a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{l(\beta+(\frac{1}{q_2}-\frac{1}{q_1}))} \|a\|_{L_h^{p_1}(L_v^{q_2})} \\ \text{Supp } \hat{a} \subset 2^k \mathcal{C}_h &\implies \|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-kN} \sup_{|\alpha|=N} \|\partial_{x_h}^\alpha a\|_{L_h^{p_1}(L_v^{q_1})} \\ \text{Supp } \hat{a} \subset 2^l \mathcal{C}_v &\implies \|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-lN} \|\partial_{x_3}^N a\|_{L_h^{p_1}(L_v^{q_1})} \end{aligned}$$

The above properties of the Littlewood-Paley theory allow to establish several important results on the products and the embedding between the Besov spaces, we begin by recalling an anisotropic product version in dimension three (see Lemma 4.5 from [30])

Lemma A.2.3. *Let $q \geq 1$, $p_1 \geq p_2 \geq 1$ with $\frac{1}{p_1} + \frac{1}{p_2} \leq 1$, and $s_1 < \frac{2}{p_1}$, $s_2 < \frac{2}{p_2}$ (resp. $s_1 \leq \frac{2}{p_1}$, $s_2 \leq \frac{2}{p_2}$ if $q = 1$) with $s_1 + s_2 > 0$. Let $\sigma_1 < \frac{1}{p_1}$, $\sigma_2 < \frac{1}{p_2}$ (resp. $\sigma_1 \leq \frac{1}{p_1}$, $\sigma_2 \leq \frac{1}{p_2}$ if $q = 1$) with $\sigma_1 + \sigma_2 > 0$. Then for a in $B_{p_1,q}^{s_1,\sigma_1}(\mathbb{R}^3)$ and b in $B_{p_2,q}^{s_2,\sigma_2}(\mathbb{R}^3)$, the product ab belongs to $B_{p_1,q}^{s_1+s_2-\frac{2}{p_2},\sigma_1+\sigma_2-\frac{1}{p_2}}(\mathbb{R}^3)$ and we have*

$$\|ab\|_{B_{p_1,q}^{s_1+s_2-\frac{2}{p_2},\sigma_1+\sigma_2-\frac{1}{p_2}}} \lesssim \|a\|_{B_{p_1,q}^{s_1,\sigma_1}} \|b\|_{B_{p_2,q}^{s_2,\sigma_2}}$$

In the next proposition we collect some very useful results related to some of the spaces defined above, see [8, 33, 15, 80, 41] for more details.

Proposition A.2.1. *Let (δ, s) be in $\mathbb{R} \times \mathbb{R}_-^*$, and p, r, ρ, m be in $[1, \infty]$,*

- *In the case of non positive regularity, one may replace, equivalently, $\dot{\Delta}_j$ in the definitions of the Besov space by \dot{S}_j , that is we have*

$$\|u\|_{\dot{B}_{p,r}^s} \approx \left\| 2^{js} \left\| \dot{S}_j u \right\|_{L^p} \right\|_{\ell_j^r(\mathbb{Z})}.$$

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- For $u \in \dot{B}_{p,r}^\delta$, there exists some sequence $(c_{j,r})_{r \in \mathbb{Z}}$ such that

$$\left\| \dot{\Delta}_j u \right\|_{L^p} \leq c_{j,r} 2^{-\delta j} \|u\|_{\dot{B}_{p,r}^\delta}, \text{ and } \sum_{j \in \mathbb{Z}} c_{j,r}^r \leq 1.$$

- similarly in the anostopic particular case, $u \in B_{p,2}^{0,s}$, we have

$$\forall \delta \in \mathbb{R}, \exists c_q = c_q(u) : \left\| \Delta_q^v u \right\|_{L^p} \leq c_q 2^{-\delta q} \|f\|_{B_{p,2}^{0,\delta}}, \text{ and } \sum_{q \geq -1} c_q^2 \leq 1, \quad (\text{A.4})$$

$$\forall s < 0, \exists \tilde{c}_q = \tilde{c}_q(u) : \left\| S_q^v u \right\|_{L^p} \leq \tilde{c}_q 2^{-sq} \|f\|_{B_{p,2}^{0,s}}, \text{ and } \sum_{q \geq -1} \tilde{c}_q^2 \leq 1. \quad (\text{A.5})$$

- According to Minkowski's inequality, we have

$$L_T^\rho(\dot{B}_{p,r}^s) \hookrightarrow \tilde{L}_T^\rho(\dot{B}_{p,r}^s) \text{ if } \rho \leq r, \quad \tilde{L}_T^\rho(\dot{B}_{p,r}^s) \hookrightarrow L_T^\rho(\dot{B}_{p,r}^s) \text{ if } r \leq \rho. \quad (\text{A.6})$$

- For $1 \leq p \leq q \leq \infty$, and $1 \leq r \leq m \leq \infty$, we have

$$\dot{B}_{p,r}^\delta(\mathbb{R}^d) \hookrightarrow \dot{B}_{q,m}^{\delta-d(\frac{1}{p}-\frac{1}{q})}(\mathbb{R}^d).$$

- In terms of Kato spaces (resp. Kato-Herz spaces), we have the following characterization of Besov spaces (resp. Fourier-Hezr spaces) of negative regularity $s < 0$

$$\|f\|_{\dot{B}_{p,r}^s} \approx \|e^{t\Delta} f\|_{K_{p,r}^{-s}}, \quad (\text{A.7})$$

$$\|f\|_{\hat{B}_{p,r}^s} \approx \|e^{t\Delta} f\|_{\hat{K}_{p,r}^{-s}}. \quad (\text{A.8})$$

Remark A.2.1. Except for the last point of the proposition above, the rest of the results listed above hold as well in the non-homogeneous case.

The following proposition, which has been used in the previous section, describes the continuity in Chemin-Lerner spaces of some Fourier multipliers

Proposition A.2.2. let s be a real number, (p, r) be in $[1, \infty]^2$, then we have, for all $f \in \tilde{L}_t^\rho(\dot{B}_{p,r}^s(\mathbb{R}^d))$, for all $k \in [0, 2]$

$$\|(Id - \Delta)^{-1} \nabla \times f\|_{\tilde{L}_t^\rho(\dot{B}_{p,r}^{s+k-1})} \lesssim \|f\|_{\tilde{L}_t^\rho(\dot{B}_{p,r}^s)} \quad (\text{A.9})$$

$$\|(Id - \Delta)^{-1} \Delta f\|_{\tilde{L}_t^\rho(\dot{B}_{p,r}^{s+k-2})} \lesssim \|f\|_{\tilde{L}_t^\rho(\dot{B}_{p,r}^s)} \quad (\text{A.10})$$

The proof of proposition A.2.2 is based on Bernstein lemma and the following one

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Lemma A.2.4. *Let \mathcal{C} be an annulus in \mathbb{R}^d , $m \in \mathbb{R}$, and k be the integer part of $1 + \frac{d}{2}$ ($k \stackrel{\text{def}}{=} [1 + \frac{d}{2}]$). Let σ be k -times differentiable function on \mathbb{R}^* such that for all $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq k$, there exists C_α satisfying*

$$\forall \xi \in \mathbb{R}^d, |\partial^\alpha \sigma(\xi)| \leq C_\alpha (1 + |\xi|^2)^m |\xi|^{-\alpha}$$

Then there exists $C > 0$ depends only on C_α such that for any $p \in [1, \infty]$ and $\lambda > 0$, we have, for any $u \in L^p$ satisfying $\text{supp}(\hat{u}) \subset \lambda\mathcal{C}$,

$$\|\sigma(D)u\|_{L^p} \lesssim C(1 + \lambda^2)^m \|u\|_{L^p} \text{ with } \sigma(D)u \stackrel{\text{def}}{=} \mathcal{F}^{-1}(\sigma\hat{u})$$

Proof

Following the proof of Lemma 2.2 from [8], seen that $\text{supp}(\hat{u}) \subset \lambda\mathcal{C}$, we can write

$$\begin{aligned} \sigma(D)u &= \lambda^d K_\lambda(\lambda \cdot) \star u \text{ with} \\ K_\lambda(x) &\stackrel{\text{def}}{=} (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i(x|\xi)} \tilde{\phi}(\xi) \sigma(\lambda\xi) d\xi \end{aligned}$$

for some smooth function $\tilde{\phi}$ supported in an annulus and having value 1 in \mathcal{C} . Let M be the integer part of $1 + \frac{d}{2}$. We have

$$\begin{aligned} (1 + |x|^2)^M |K_\lambda(x)| &\approx \left| \int_{\mathbb{R}^d} e^{i(x|\xi)} (Id - \Delta_\xi)^M \tilde{\phi}(\xi) \sigma(\lambda\xi) d\xi \right| \\ &= \left| \sum_{|\alpha|+|\beta| \leq 2M} c_{\alpha,\beta} \lambda^{|\beta|} \int_{\text{supp}(\tilde{\phi})} e^{i(x|\xi)} \partial^\alpha \tilde{\phi}(\xi) \partial^\beta \sigma(\lambda\xi) d\xi \right| \\ &\lesssim C(1 + \lambda^2)^m. \end{aligned}$$

As $2M > d$, we deduce that

$$\|K_\lambda\|_{L^1} \leq C(1 + \lambda^2)^m,$$

thus Young's inequality concludes the proof of the desired inequality. □

Proof of proposition A.2.2 According to Lemma A.2.4, we have

$$\begin{aligned} \left\| (Id - \Delta)^{-1} \nabla \times (\dot{\Delta}_j f) \right\|_{L_t^1 L_x^p} &\lesssim \frac{2^j}{1 + 2^{2j}} \left\| \dot{\Delta}_j f \right\|_{L_t^1 L_x^p}, \\ \left\| (Id - \Delta)^{-1} \Delta (\dot{\Delta}_j f) \right\|_{L_t^1 L_x^p} &\lesssim \frac{2^{2j}}{1 + 2^{2j}} \left\| \dot{\Delta}_j f \right\|_{L_t^1 L_x^p}, \end{aligned}$$

the result follows from the fact that, for all $k \in [0, 2]$ we have

$$2^{jk} \lesssim 1 + 2^{2j}.$$

Proposition A.2.2 is then proved. □

Let us now recall a result describing how the dyadic blocks act on the heat semi-group

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Lemma A.2.1. *There exists a positive constant C such that for $T \geq 0, q \in \mathbb{Z}$ and $p \in [1, \infty]$ we have*

$$\left\| \int_0^\infty \mathbf{1}_{\{\tau \leq \cdot\}} e^{\tau \Delta} \dot{\Delta}_q u(\tau) \right\|_{L_T^{\kappa_1}(L^p)} \leq C 2^{2q(-1+1/\kappa_2-1/\kappa_1)} \|\dot{\Delta}_q u\|_{L_T^{\kappa_2}(L^p)}, \quad \text{for all } \kappa_1 \in [\kappa_2, \infty]. \quad (\text{A.11})$$

Proof

Let $u \in L^p$ with $p \in [1, \infty]$. Since, for all $q \in \mathbb{Z}$, $\text{supp } \widehat{\dot{\Delta}_q u}$ is included in the annulus $\mathcal{A}(0, 2^q, 2^{q+1})$, then for $t \geq 0$ we infer that there exist two constants c and C such that

$$\|e^{t\Delta} \dot{\Delta}_q u(t)\|_{L^p} \leq C e^{-ct2^{2q}} \|\dot{\Delta}_q u(t)\|_{L^p}.$$

So, for all $\tau \in [0, t]$ we have

$$\left\| \int_0^\infty \mathbf{1}_{\{\tau \leq \cdot\}} e^{(\cdot-\tau)\Delta} \dot{\Delta}_q u(\tau) \right\|_{L^p} \leq C \int_0^\tau e^{-c(\tau-\tau')2^{2q}} \|\dot{\Delta}_q u(\tau')\|_{L^p} d\tau'.$$

Via, Young's inequality with respect to time, it follows for $\kappa_1 \in [\kappa_2, \infty]$ with $1/\kappa = 1 + 1/\kappa_1 - 1/\kappa_2$

$$\begin{aligned} \left\| \int_0^\infty \mathbf{1}_{\{\tau \leq \cdot\}} e^{(\cdot-\tau)\Delta} \dot{\Delta}_q u(\tau) \right\|_{L_T^{\kappa_1} L^p} &\leq C \left(\frac{1 - e^{-c\kappa_2 t 2^{2q}}}{c\kappa_2 2^{2q}} \right)^{1/\kappa} \|\dot{\Delta}_q u\|_{L_T^{\kappa_2} L^p} \\ &\lesssim 2^{2q(-1+1/\kappa_2-1/\kappa_1)} \|\dot{\Delta}_q u\|_{L_T^{\kappa_2} L^p}, \end{aligned}$$

this completes the proof. □

As a consequence of the lemma above, we deduce the following smoothing effect of the heat semi-group, one may see [8, 36] for more details.

Lemma A.2.5. *Let $(s, p, r) \in \mathbb{R} \times [1, \infty]^2$ and the operators \mathcal{T} and \mathcal{T}_0 be given by*

$$\begin{aligned} (T_0 \mathcal{K}_0)(t, x) &\stackrel{\text{def}}{=} e^{t\Delta} \mathcal{K}_0(x), \\ T\mathcal{K}(t, x) &\stackrel{\text{def}}{=} \int_0^t e^{\Delta(t-s)} \mathcal{K}(s, x) ds, \end{aligned}$$

then,

$$\|T_0 \mathcal{K}_0\|_{\mathcal{L}(\dot{B}_{p,r}^s)} \lesssim \|\mathcal{K}_0\|_{\dot{B}_{p,r}^s}, \quad (\text{A.12})$$

and

$$\|T_0 \mathcal{K}_0\|_{\mathcal{L}(\dot{B}_{p,r}^s)} \lesssim \|\mathcal{K}\|_{\dot{L}^1(\dot{B}_{p,r}^s)}, \quad (\text{A.13})$$

where $\mathcal{L}(\dot{B}_{p,r}^s) = L^\infty(\dot{B}_{p,r}^s) \cap L^1(\dot{B}_{p,r}^{s+2})$.

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The theory of interpolation is well known to be very useful in the analysis of PDE's, here we present some lemmata the proofs of which are based on this theory.

Lemma A.2.6. (Lemma 4.3 from [30]) For any s positive, for all $(p, q) \in [1, \infty]$ and any $\theta \in]0, s[$, we have

$$\|f\|_{(\dot{B}_{p,q}^{s-\theta})_h(\dot{B}_{p,1}^\theta)_v} \lesssim \|f\|_{\dot{B}_{p,q}^s}.$$

Next, we recall Lemma A.2 from [29]

Lemma A.2.7. For any function a in the space $\dot{H}^{\frac{1}{2}+s}(\mathbb{R}^3)$ with $\frac{1}{2} \leq s < 1$, there holds

$$\|a\|_{L_v^\infty(\dot{H}_h^s)} \leq \sqrt{2} \|a\|_{\dot{H}^{\frac{1}{2}+s}(\mathbb{R}^3)}.$$

The following lemma shows the gain of summability with respect to the third index in Besov spaces, for functions belonging to $\dot{B}_{p,r}^{s_1} \cap \dot{B}_{p,r}^{s_2}$, with $s_1 < s_2$

Lemma A.2.8. Let $T > 0$, $s_1 < s < s_2$, $a, b, c \in [1, \infty]$ and $\varsigma \in (0, 1)$ such that $\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$ $s = \varsigma s_1 + (1 - \varsigma)s_2$. Then we have

$$\|u\|_{\tilde{L}_T^a B_{p,1}^s} \leq C \|u\|_{\tilde{L}_T^b B_{p,\infty}^{s_1}} \|u\|_{\tilde{L}_T^c B_{p,\infty}^{s_2}}^{1-\varsigma}. \quad (\text{A.14})$$

In order to give an idea of the proof of the three lemmas above, we chosed to prove the following lemma, used in Chapter 2 which is also a particular case of Lemma A.2.8 if we ignore the time variable

Lemma A.2.9. For all $p \in]2, \infty[$, there exists a constant $c_p > 0$, such that for all u in $L_T^\infty(L^2(\mathbb{R}^3)) \cap L_T^2(\dot{H}^1(\mathbb{R}^3))$ we have

$$\|u\|_{L_T^p(\dot{B}_{2,1}^{\frac{2}{p}}(\mathbb{R}^3))} \leq c_p \|u\|_{L_T^2(\dot{H}^1(\mathbb{R}^3))}^{\frac{2}{p}} \|u\|_{L_T^\infty(L^2(\mathbb{R}^3))}^{1-\frac{2}{p}}.$$

Proof The proof is classical, we proceed as the following:

Let $N(t) > 0$ to be fixed later, we use Lemma A.2.2 and Cauchy-Swartz inequality, to write

$$\begin{aligned} \sum_{j \in \mathbb{Z}} 2^{\frac{2}{p}j} \|\Delta_j u(t, \cdot)\|_{L^2} &= \sum_{j \leq N(t)} 2^{j\frac{2}{p}} \|\Delta_j u(t, \cdot)\|_{L^2} + \sum_{j > N(t)} 2^{j(\frac{2}{p}-1)} \|\Delta_j \nabla u(t, \cdot)\|_{L^2} \\ &\leq 2^{\frac{2}{p}N(t)} \|u(t, \cdot)\|_{L^2} + 2^{(\frac{2}{p}-1)N(t)} \|\nabla u(t, \cdot)\|_{L^2}. \end{aligned}$$

The choice of $N(t)$ such that

$$2^{N(t)} \stackrel{\text{def}}{=} \left(\frac{1 - \frac{2}{p}}{\frac{2}{p}} \right) \frac{\|\nabla u(t, \cdot)\|_{L^2}}{\|u(t, \cdot)\|_{L^2}}$$

gives

$$\|u(t, \cdot)\|_{\dot{B}_{2,1}^{\frac{2}{p}}(\mathbb{R}^3)} \leq c_p \|u(t, \cdot)\|_{\dot{H}^1(\mathbb{R}^3)}^{\frac{2}{p}} \|u(t, \cdot)\|_{L^2(\mathbb{R}^3)}^{1-\frac{2}{p}}. \quad (\text{A.15})$$

The lemma follows by taking the L^p norm in time. □

A.3 Other results

The following fixed point argument has been used to prove corollary VI.3, the proof of which can be found for instance in [15]

Lemma A.3.1. *Let X be a Banach space, L a linear operator from X to X , with norm equals to $\lambda < 1$, and let B be a bi-linear operator mapping from $X \times X$ in X , with norm $\|B\| = \gamma$, then for all $y \in X$ such that*

$$\|y\|_X < \frac{(1 - \lambda)^2}{4\gamma},$$

the equation

$$x = y + L(x) + B(x, x)$$

has a unique solution in the ball $B_X(0, \frac{1-\lambda}{2\gamma})$.

We recall also the so-called Osgood's Lemma (see for instance [8]):

Lemma A.3.2. Osgood's lemma

Let g be a measurable function from $[t_0, T]$ to $[0, a]$, γ a locally integrable function from $[t_0, T]$ to \mathbb{R}^+ and μ a continuous and non-decreasing function from $[0, a]$ to \mathbb{R}^+ . Assume that for some non-negative real number c , g satisfies:

$$g(t) \leq c + \int_{t_0}^t \gamma(\tau) \mu(g(\tau)) d\tau, \quad \text{a.e. } t \in [t_0, T],$$

then we have for a.e. $t \in [t_0, T]$:

$$c > 0 \implies -M(g(t)) + M(c) \leq \int_{t_0}^t \gamma(\tau) d\tau, \quad \text{where } M(x) = \int_x^a \frac{d\tau}{\mu(\tau)},$$

$$c = 0 \text{ and } \int_0^a \frac{d\tau}{\mu(\tau)} = \infty \implies g = 0.$$

The compactness property is very helpful and it is needed to construct the weak solutions as explained in some parts of this work. A very useful tool to guarantee this property in some special cases is the following result due to Aubin–Lions, for more details we suggest to see [97]

Theorem A.3.1. *Let X_0, X, X_1 be three Banach spaces such that*

$$X_0 \hookrightarrow X \hookrightarrow X_1,$$

we suppose also that the embedding $X_0 \hookrightarrow X$ is compact.

For $1 \leq p, q \leq \infty$, we define W to be

$$W \stackrel{\text{def}}{=} \{u \in L^p([0, T]; X_0) \mid \partial_t u \in L^q([0, T]; X_1)\}.$$

Then we have

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- If $p < \infty$ then the embedding of W in $L^p([0, T]; X_0)$ is compact.
- If $p = \infty$ and $q > 1$ then the embedding of W in $C([0, T]; X_0)$ is compact.

It is well known that the Sobolev embedding in the whole space \mathbb{R}^d fail to be compact, which means that the above theorem does not guarantee any compactness result if X_0, X and X_1 take the form of some $B_{p,q}^s(\mathbb{R}^d)$. However we can rely on the following result in the case of the whole space, we refer to Theorem 2.94 [8] for more details

Theorem A.3.2. *Let $s' < s$, then for all φ in $\mathcal{S}(\mathbb{R}^d)$, the multiplication by φ is a compact operator from $B_{p,\infty}^s(\mathbb{R}^d)$ to $B_{p,1}^{s'}(\mathbb{R}^d)$.*

Now, we recall some estimates between the velocity and its vorticity in the framework of axisymmetric geometry along the following lines which are strongly useful in section IV.2.

First, let us denote by v the velocity vector field and ω its vorticity, with $\omega \stackrel{def}{=} \text{curl} v = \omega_\theta \vec{e}_\theta$. Consequently, for $x \in \mathbb{R}^3$ with $r = \sqrt{x_1^2 + x_2^2} \geq 0$ we have $|\omega(x)| = |\omega_\theta(x)|$ and

$$|\nabla \omega_\theta| \approx |\partial_r \omega_\theta(x)| + |\partial_z \omega_\theta(x)| + \left| \frac{\omega_\theta(x)}{r} \right|.$$

So, some equivalent norms in terms of v and ω are given by the following. The proof can be found in [95].

Lemma A.3.1. *Let v be a smooth vector field, divergence free and axisymmetric. Then the following assertions hold.*

$$\|\omega\|_{L^p} \approx \|\nabla v\|_{L^p}, \quad p \in]1, \infty[. \quad (\text{A.16})$$

$$\|D\omega_\theta\|_{L^p} + \left\| \frac{\omega_\theta}{r} \right\|_{L^p} \approx \|\nabla^2 v\|_{L^p}, \quad p \in]1, \infty[. \quad (\text{A.17})$$

$$\|D^2 \omega_\theta\|_{L^2} + \left\| D \left(\frac{\omega_\theta}{r} \right) \right\|_{L^2} \leq C \|\nabla^3 v\|_{L^2}. \quad (\text{A.18})$$

The symbol \approx means the equivalence of two norms and by D we understand (∂_r, ∂_z) , while ∇ designates $(\partial_{x_1}, \partial_{x_2}, \partial_z)$.

The last result we recall in this work precises a refined estimates for the transport equation in Besov Spaces with Index 0, for more details one may see for instance [28]

Proposition A.3.1. *For $T > 0$, let v be a divergence free vector field such that ∇v belongs to $L_T^1([0, T]; L^\infty)$. Then for $\Theta_0 \in \dot{B}_{p,r}^0$, $f \in \tilde{L}_T^1(\dot{B}_{p,r}^0)$, any solution Θ to the following transport-diffusion equation.*

$$\begin{cases} \partial_t \Theta + v \cdot \nabla \Theta = f \\ \Theta_{t=0} = \Theta_0, \end{cases}$$

satisfies $\Theta \in \tilde{L}_T^\infty(\dot{B}_{p,r}^0)$, and for all $t \in [0, T]$

$$\|\Theta\|_{\tilde{L}_T^\infty(\dot{B}_{p,r}^0)} \leq C \left(\|\Theta_0\|_{\dot{B}_{p,r}^0} + \|f\|_{\tilde{L}_T^1(\dot{B}_{p,r}^0)} \right) \left(1 + \int_0^t \|\nabla v(\tau)\|_{\dot{B}_{\infty,1}^0} d\tau \right),$$

with $C > 0$ is a universal constant.

APPENDIX B

The Navier-Stokes equations

As one may notice, all the presented results in this work are essentially related to the classical Navier-Stokes equations, and all the models we dealt with have in their radical form the equation

$$\partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f, \quad (\text{B.1})$$

coupled with the incompressibility condition

$$\operatorname{div} u = 0, \quad (\text{B.2})$$

and the initial condition

$$u(0, \cdot) = u_0(\cdot). \quad (\text{B.3})$$

This part of the Appendix is then devoted to give an abstract to some of the important results dealing with (B.1)-(B.3), then we briefly mention a couple of interesting open questions which are related to some topics studied in this thesis.

It is worth noting that, due to (B.2), the pressure p is in general neglected in several studies since this latter can be recovered by solving the following elliptic equation (as long as u has already been recovered)

$$-\Delta p = -\operatorname{div} f + \operatorname{div}(u \cdot \nabla u). \quad (\text{B.4})$$

B.1 Basic properties

Before stating any results, we recall the two major facts of the 3D Navier-Stokes equations, we will assume $f = 0$ in the sequel.

APPENDIX B. THE NAVIER-STOKES EQUATIONS

Energy inequality

Any regular solution u to (B.1)-(B.3) should satisfy, for all positive time t

$$\frac{1}{2} \|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau = \frac{1}{2} \|u_0\|_{L^2}^2. \quad (\text{B.5})$$

Scaling invariance

The second major property of these equations, is the fact of respecting some sort of scaling in the following sense: if u is a solution to (B.1)-(B.3) on $[0, T] \times \mathbb{R}^3$, then, for all $\lambda > 0$, u_λ is also a solution to (B.1)-(B.3), by replacing (B.3) with

$$u_\lambda(\cdot) = u_{0,\lambda}(\cdot) \stackrel{\text{def}}{=} \lambda u_0(\lambda \cdot),$$

where u_λ is given by

$$u_\lambda(t, x) \stackrel{\text{def}}{=} \lambda u(\lambda^2 t, \lambda x).$$

According to this property, a functional space X is said to be critical if its norm is invariant by the above λ -transformation, that is if it satisfies

$$\|u_\lambda\|_X = \|u\|_X.$$

The following spaces are then critical regarding the above definition

$$\dot{B}_{\tilde{p},\tilde{q}}^{-1+\frac{3}{\tilde{p}}} \hookrightarrow \dot{H}^{\frac{1}{2}} \hookrightarrow \dot{B}_{p,q}^{-1+\frac{3}{p}} \hookrightarrow \mathcal{BMO}^{-1} \hookrightarrow \dot{B}_{\infty,\infty}^{-1}, \quad (\text{B.6})$$

for all $\tilde{p} \leq 2 \leq p < \infty$, and $\tilde{q} \leq 2 \leq q \leq \infty$.

B.2 What do we mean by a solution to the Navier-Stokes equations?

The notions of a solution to PDE's in general and for the Navier-Stokes equations in particular depends on the methods used to solve the problem in study. For assertions (B.1)-(B.3) the notion of a solution can be defined as the weak solution (turbulent solution) or the strong solution (mild solution), the two notions are equivalent if the solution is regular enough. We give in the sequel some more details

B.2.1 Weak and turbulent solutions to the Navier-Stokes equations

The notion of weak solutions to the Navier-Stokes equations has been introduced at the first time by J. Leray in 1934 in his famous paper [96], in particular this type of solutions satisfies the equations in the distribution sense. ¹

In the sequel, d will denote the spacial dimension such that $d \geq 2$ unless we precise its value.

Definition B.2.1 (Weak solution). *We say that a divergence free vector field $u \in L^2_{loc}([0, T] \times \mathbb{R}^d)$ is a weak solution to the 3D Navier-Stokes equations (B.1)-(B.3) if it satisfies*

$$\int_{\mathbb{R}^d} u(t, x) \cdot \psi(t, x) dx = \int_{\mathbb{R}^d} u_0(x) \cdot \psi(0, x) dx + \int_0^t \int_{\mathbb{R}^d} (u \cdot \Delta \psi + u \otimes u : \nabla \psi + u \cdot \partial_t \psi) dx d\tau, \quad (\text{B.7})$$

for all $t \in [0, T]$, and divergence free ψ in $C_c^\infty([0, T] \times \mathbb{R}^d)$.

Let us now define the turbulent solution of J. Leray

Definition B.2.2 (Turbulent solution). *Let $u_0 \in L^2(\mathbb{R}^d)$, we say that u is a turbulent solution to (B.1)-(B.3) on $[0, T]$ if u is a weak solution satisfying*

$$u \in L^\infty([0, T]; L^2(\mathbb{R}^d)) \cap L^2([0, T]; \dot{H}^1(\mathbb{R}^d)),$$

and for all $t \in [0, T]$

$$\frac{1}{2} \|u(t)\|_{L^2}^2 + \int_0^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \leq \|u_0\|_{L^2}^2.$$

B.2.2 Mild solutions to the Navier-Stokes equations

We end up this subsection by introducing the definition of mild solutions, which does not really use the special structure of the equations as the turbulent solutions do

Definition B.2.3 (Mild solution). *We say that u is a mild solution to the Navier-Stokes equations (B.1)-(B.3), if it satisfies its associated Duhamel formula. That is if u satisfies*

$$u(t, \cdot) = e^{t\Delta} u_0(\cdot) - \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u \otimes u)(\tau, \cdot) d\tau, \quad (\text{B.8})$$

where \mathbb{P} denotes the Leray projector which is given by

$$\mathbb{P} \stackrel{\text{def}}{=} id - \nabla \Delta^{-1} \operatorname{div}.$$

¹Notice that J. Leray was able to give a sense to the weak solutions even before L.Schwartz introduces the theory of distributions.

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The construction of Leray's solutions is essentially based on compactness arguments, the procedure takes in general the following form:

We begin by solving an approximate system based on regularizing the nonlinear term $u \cdot \nabla u$ in order to be able to solve the new system and to construct a sequence of unique and global solutions (denoted $(u_n)_{n \in \mathbb{N}}$) to the approximate system. Then, we prove global uniform bounds for u_n with respect to n in the energy space. Finally, by making use of an adequate compactness arguments we prove the existence of a weak limit of the sequence $(u_n)_{n \in \mathbb{N}}$, and we prove that this weak limit satisfies (B.7) together with the energy inequality (B.5). On the other hand the construction of mild solutions relies on fixed point arguments.

Remark B.2.1. *We often call a weak solution any solution given by compactness arguments (in particular Leray's solutions), and we call a strong solution any solution given by the fixed point arguments.*

Remark B.2.2. *Remark that Leray's solutions do not satisfy in general an **energy equality**, but just an **inequality**, this is in fact because of the lack of the strong compactness in infinite dimension spaces as the Lebesgue and Sobolev spaces.*

B.3 Some important results

This section is devoted to some technical results from the historical point of view of the Navier-Stokes equations, the first brilliant result of the mathematical study of the Navier-Stokes equations is due to J. Leray in his famous paper [96], in which he proved

Theorem B.3.1. *Let $u_0 \in L^2(\mathbb{R}^d)$ be a given divergence free vector field, then there exists at least one global turbulent solution for (B.1)-(B.3), defined for all $t > 0$.*

This was the first positive affirmation of a construction of solutions to the Navier-Stokes equations since the derivation of the equations in 1845. It confirms the existence of solutions that can be defined for large times. One may notice that the above theorem does not say anything about the uniqueness of such solutions, in fact the question of the uniqueness here stands to be open till today, unless in the case of dimension two, which was also proved by J. Leray. Let us go back to dimension three, since the uniqueness is not known whether it holds to be true or not, it is a worthy question to ask: Under what kind of hypothesis one can prove that the Leray's solutions are unique? In this direction, we may prove the so-called weak-strong uniqueness property of turbulent solutions

Proposition B.3.1. *Let u, v be two turbulent solutions associated to the same initial data $u \in L^2(\mathbb{R}^3)$, if in addition $v \in L^p([0, T]; L^q(\mathbb{R}^3))$, for some $p \in [2, \infty)$ with*

$$\frac{2}{p} + \frac{3}{q} = 1$$

then $u = v$.

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The previous proposition shows then the hope of reaching the wellposedness of the 3D Navier-Stokes equations if one may construct a little bit more regular solutions, this fact of matter inspires the following question: An additional regularity assumption on the initial data, can it provide the global wellposedness of these equations? A first partial answer was given by J. Leray as follows

Theorem B.3.2. *Let u_0 be a divergence free vector field in $H^1(\mathbb{R}^3)$, then there exists $T > 0$ and a unique solution to (B.1)-(B.3) in $C^0([0, T]; \dot{H}^1) \cap L^2([0, T]; \dot{H}^2)$. In addition of that, there exists $c > 0$, such that if*

$$\|u_0\|_{L^2} \|u_0\|_{\dot{H}^1} \leq c, \tag{B.9}$$

then $T = \infty$.

Theorem B.3.2 above, besides ensuring the uniqueness of the solution, it has two main features:

- One may notice that it provides a local wellposedness result for any initial data sufficiently regular (H^1 -initial data). On the other hand, a smallness condition on the size of the initial data is needed in order to guarantee the life of the solution up to $T = \infty$. Point out that, till today we do not know if the local solution given by the theorem above blows-up at some finite time T^* or it can be extended to infinity, also we do not know how to get rid of the smallness condition (B.9).
- The second remark that was the starting point for several research is the following: Assertion (B.9) is invariant under the scaling of the 3D Navier-Stokes equations, but it requires one gradient of the initial data to be in L^2 . Whereas such condition does not say anything about initial data like being in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, or critical data with different properties.²

It was then due to H.Fujita and T.Kato in [44] the answer of the local wellposedness for initial data being in the critical space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, the following theorem gives some more details (see also chapter 5 in [8] and chapter 15 in [94]).

Theorem B.3.3. *Let u_0 be a divergence free vector field in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, then there exists a unique maximal life time $T^* = T^*(u_0)$ and a unique u solution to (B.8) for which the following assertions hold*

- u has the following regularity, for all $t < T^*$

$$u \in C^0([0, T]; \dot{H}^{\frac{1}{2}}) \cap L^2([0, T], \dot{H}^{\frac{3}{2}}).$$

²We will go back to more detail about that in the comments on Theorem B.3.3, which traits the case of some initial data which is not concerned by assertion (B.9).

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- If $T^* < \infty$ then, for all $p \in [2, \infty)$

$$\int_0^{T^*} \|u(\tau)\|_{\dot{H}^{\frac{1}{2}+\frac{2}{p}}}^p d\tau = \infty. \quad (\text{B.10})$$

- There exists a universal constant $c > 0$ for which, if

$$\|u_0\|_{\dot{H}^{\frac{1}{2}}} \leq c, \quad (\text{B.11})$$

then $T^* = \infty$.

- The solutions are $\dot{H}^{\frac{1}{2}}$ -stable on $[0, t]$ provided that they belong to $L^p([0, t]; \dot{H}^{\frac{1}{2}+\frac{2}{p}})$, for some $p \in [2, \infty)$, more precisely we have, for any u, v two solutions to (B.1)-(B.3),

$$\begin{aligned} & \|u(t) - v(t)\|_{\dot{H}^{\frac{1}{2}}}^2 + \int_0^t \|u(\tau) - v(\tau)\|_{\dot{H}^{\frac{3}{2}}}^2 d\tau \\ & \leq \|u_0 - v_0\|_{\dot{H}^{\frac{1}{2}}}^2 \exp\left(C\left(\|u\|_{L^p([0,t]; \dot{H}^{\frac{1}{2}+\frac{2}{p}})}^p + \|v\|_{L^p([0,t]; \dot{H}^{\frac{1}{2}+\frac{2}{p}})}^p\right)\right). \end{aligned}$$

As aforementioned, the proof of Theorem B.3.3 relies on fixed point argument and the solutions obtained satisfy in a first time the integral form (B.8). In fact the proof of the above theorem provides more precise information concerning the size of the global solution regarding the smallness conditions (B.11): if the initial data is of size $c \ll 1$, then the solution is of size $2c$. In addition of that, we can prove that the map $t \mapsto \|u(t)\|_{\dot{H}^{\frac{1}{2}}}$ is decreasing. These two facts together ensures that the $\dot{H}^{\frac{1}{2}}$ -norm of $u(t, \cdot)$ should tend to 0 as t tends to infinity.

An interesting question is then the following: supposing that some $\dot{H}^{\frac{1}{2}}$ initial data u_0 (with arbitrary large size) generates a global solution to (B.1)-(B.3), does the scenario of small solutions shows up always at large time? A positive answer of that has been confirmed by I. Gallagher, D. Iftimie, and F. Planchon in [49], more precisely they proved

Theorem B.3.4. *Let u_0 be a divergence free vector field in $\dot{H}^{\frac{1}{2}}$ generating a global solution to (B.1)-(B.3) u in $C^0([0, T]; \dot{H}^{\frac{1}{2}}) \cap L^2([0, T], \dot{H}^{\frac{3}{2}})$, for all $T > 0$, then we have*

$$\lim_{t \rightarrow +\infty} \|u(t)\|_{\dot{H}^{\frac{1}{2}}} = 0. \quad (\text{B.12})$$

Unlike the proof of Theorem B.3.3, the proof of Theorem B.3.4 uses the structure of the equations, namely the energy inequality.

On the other hand, according to the embedding (B.6), Theorem B.3.3 does not say anything about small data in spaces larger than $\dot{H}^{\frac{1}{2}}$, a little bit after, in [87] T.Kato provided an answer in the case of initial data in L^3 by proving

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Theorem B.3.5. *Let u_0 be a divergence free vector field in L^3 , then there exists a unique maximum time of existence $T^* = T^*(u_0)$ and a unique solution u to (B.8) in the Kato space*

$$K_T \stackrel{\text{def}}{=} \left\{ u \in C((0, T); L^3) : \|u\|_{K_T} \stackrel{\text{def}}{=} \sup_{t \in (0, T)} t^{\frac{3}{2}(\frac{1}{3} - \frac{1}{p})} \|u(t, \cdot)\|_{L^p} \right\}, \quad \text{for } p \in (3, \infty)$$

If in addition, the L^3 -norm of u_0 is small enough then $T^ = \infty$.*

The strategy to prove the above theorem is similar to the one proving Theorem B.3.3, and it relies on finding a fix point for the functional F given by

$$F(u) = e^{t\Delta} u_0 + \mathcal{B}(u, u)$$

with

$$\mathcal{B}(u, v) \stackrel{\text{def}}{=} - \int_0^t e^{(t-\tau)\Delta} \mathbb{P} \operatorname{div}(u \otimes v) d\tau$$

A fact of matter is that F.Oru proved in [105] that \mathcal{B} is not continuous from $L^\infty(L^3) \times L^\infty(L^3)$ into $L^\infty(L^3)$, which means that searching of a solution in spaces like the Kato space K_T is necessary in the proof of Theorem B.3.5. However, for the uniqueness task Furioli, Lemarié-Rieusset and Terraneo showed in [47] that there is only one mild solution of the Navier-Stokes equations in $C([0, T]; L^3)$. That is the uniqueness of solutions to (B.8) holds in a space larger than K_T .

This is good in the point of view of the initial data that uniquely generates (at least local) solutions, but what about the initial data that behaves like $x \mapsto \frac{1}{|x|}$? this kind of initial data does not belong to any Lebesgue or Sobolev spaces, but it still respects the critical scaling of the initial data, besides of that it belongs to the critical spaces of the form $\dot{B}_{p, \infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$. In order to treat the case of the initial data belonging to larger spaces such $\dot{B}_{p, \infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$, $p \in [3, \infty)$, the fixed point argument shows up to be applicable again in the Besov-type spaces, and the affirmative answer comes then due to M.Cannone in [16] (for $p \in (3, 6]$) and F.Planchon in [110] (for $p \in [3, \infty)$). We resume these results and announce them in the following form (see also [26] for more details)

Theorem B.3.6. *Let u_0 be a divergence free vector field belonging to $\dot{B}_{p, \infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)$, for some $p \in [3, \infty)$, then there exists $T > 0$ and a unique u solution to (B.8) in*

$$\tilde{L}^\infty([0, T]; \dot{B}_{p, \infty}^{-1+\frac{3}{p}}(\mathbb{R}^3)) \cap \tilde{L}^1([0, T]; \dot{B}_{p, \infty}^{1+\frac{3}{p}}(\mathbb{R}^3)).$$

In addition of that, T can be chosen to be $+\infty$ if u_0 is small enough in $\dot{B}_{p, \infty}^{-1+\frac{3}{p}}$.

Above the space $\tilde{L}^p([0, T]; \dot{B}_{p, r}^s)$ refers to the so-called Chemin-Lerner space defined in the Appendix A.

A particularity of the above theorem is the ability to construct global solutions starting

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from initial data which is not necessary small in $\dot{H}^{\frac{1}{2}}$ or even in L^3 . The fundamental example of that is the data with a lot of oscillations in one direction. Indeed taking for instance $\psi \in \mathcal{S}(\mathbb{R}^3)$, and we define

$$u_0^\varepsilon \stackrel{def}{=} \sin\left(\frac{x_1}{\varepsilon}\right)(0, -\partial_3\psi(x), \partial_2\psi(x)),$$

we can check that the L^3 norm of u_0^ε has a size 1 while its norm in $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}$ is small for all $p > 3$ as long as ε is small enough.

Next, one may ask: what is then the largest critical space for the initial data that guarantees the global wellposedness (at least for small data)? According to (B.6) the space $\dot{B}_{\infty,\infty}^{-1}$ appears to be the largest one. Indeed Y. Meyer proved in [99] that any Banach space $X \subset \mathcal{S}(\mathbb{R}^d)$ which is invariant by translation and dilatation in the following sense

$$\|f\|_X = \lambda \|f(\lambda \cdot -a)\|_X, \quad \forall f \in X, a \in \mathbb{R}^d, \lambda > 0,$$

is continuously embedded in $\dot{B}_{\infty,\infty}^{-1}$. But unfortunately it has been proved (see [12] and [59]) that the Navier-Stokes equations are ill-posed in this space.

In the other hand, more recently H. Koch et D. Tataru proved in [89] the wellposedness of these equations in \mathcal{BMO}^{-1} defined by the set of tempered distribution u whose the following norm is finite

$$\|u\|_{\mathcal{BMO}^{-1}}^2 \stackrel{def}{=} \sup_{x_0 \in \mathbb{R}^3, R > 0} \frac{1}{|B(x_0, \sqrt{R})|} \int_{[0,R] \times B(x_0, \sqrt{R})} |e^{t\Delta} u|^2 dt dy.$$

Whereas the fixed point argument used to solve the equations has been done in the space X defined by its norm

$$\|u\|_X^2 \stackrel{def}{=} \sup_{t \geq 0} t \|u(t, \cdot)\|_{L^\infty}^2 + \sup_{x_0 \in \mathbb{R}^3, R > 0} \frac{1}{|B(x_0, \sqrt{R})|} \int_{[0,R] \times B(x_0, \sqrt{R})} |u(t, x)|^2 dt dy.$$

More precisely, H. Koch et D. Tataru ³ proved

Theorem B.3.7. *If the initial data u_0 is small enough in \mathcal{BMO}^{-1} , then (B.8) has a unique global solution in the space X defined above, and we have in addition $\|u\|_X \leq 2 \|u_0\|_{\mathcal{BMO}^{-1}}$.*

³one may see [7] for a second proof of Theorem B.3.7

APPENDIX C

Perspectives

Besides the millennium problem of the 3D Navier-Stokes equations, and despite the work that has been done, a lot of interesting open questions are waiting for answers, here we discuss some of them which are related to the main results of this thesis.

As it is well known, the global regularity question of the 3D Navier-Stokes equations is still open. However we could establish blow-up criterion as (B.10) for $p \in [2, +\infty)$. The limit case $p = +\infty$ is considerably hard to prove compared to the rest of cases, a first answer was due to L. Escauriaza, G. Seregin et V. Sverák in [42] where they proved that if the maximal life span of the solution T^* is finite, then $\limsup_{t \rightarrow T^*} \|u(t)\|_{L^3} = \infty$. This was extended ¹ to the general critical Besov spaces by I. Gallagher, G. Koch, F. Planchon in [52].

Recently, it has been shown that the divergence free condition, together with the structure of the equations can provide better regularity criterion, better with respect to the number of velocity's components that should satisfy conditions like (B.10). Indeed, several works have been done in this direction, but the first result involving only one component in scaling invariant class of spaces was due to J-Y. Chemin and P. Zhang in [31]. Followed by a result due to the same authors together with Z. Zhang in [32]. And more recently by J-Y. Chemin, I. Gallagher and P. Zhang in [29]. More details are already given in Chapter II of this thesis.

As mentioned above, the limit case $p = \infty$ is hard to prove, the authors in [29] proved some results concerning this case, and in the present work we slightly modified their proof to obtain similar result in another type of spaces (see Theorems II.1.1 and II.1.2 in Chapter II).

However we do not know if one component of the velocity being bounded in $L^\infty([0, T]; \mathcal{Y})$,

¹Also it has been extended to the full limit in time in $\dot{H}^{\frac{1}{2}}$ by G. Seregin in [114]

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for some critical space \mathcal{Y} being any one from (B.6), is sufficient to guarantee the smoothness of the solution until time T , or if a singularity can show up at some $T^* \leq T$.

Another interesting question in this direction, besides the limit case, up to my knowledge all the one component-blow-up criterion have been established in terms of the Sobolev norms in space variables. Is it then possible to prove a Serrin-type criterion in critical Lebesgue spaces under conditions involving only one component of the velocity?

Let us now move to another version of the Navier-Stokes equations. Recent works have been addressed to the study of the 3D Navier-Stokes equations with partial dissipation,

$$\partial_t u + u \cdot \nabla u - \Delta_h u + \nabla p = 0. \quad (\text{C.1})$$

And eventually some similar systems as the Boussinesq system with different partial dissipation. We may refer to Chapters V for more details and for the progress of studying such systems. Here we would like to briefly mention two open questions concerning (C.1):

- In [106] M. Paicu was able to prove the first result of global wellposedness for small data ² in the critical space $\mathcal{B}^{0, \frac{1}{2}}$ defined by its norm

$$\|u\|_{\mathcal{B}^{0, \frac{1}{2}}} \stackrel{def}{=} \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \|\Delta_\ell^v u\|_{L^2(\mathbb{R}^3)}, \quad (\text{C.2})$$

where Δ_ℓ^v stands to be the vertical version of the dyadic blocks defined in the Appendix A.

Remark first that this was a new result even for the classical Navier-Stokes equations.

Moreover, the uniqueness result that he proves in his paper requires the solution to be only in $L_T^\infty(H^{0, \frac{1}{2}}) \cap L_T^2(H^{1, \frac{1}{2}})$, which corresponds to the "horizontal"-energy space of $H^{0, \frac{1}{2}}$. However, because of the lack of the embedding of this latter in $L_h^2(L_v^\infty)$, which is really important in the proof of the existence part in [106], we do not know yet if initial data in $H^{0, \frac{1}{2}}$ can generate local solutions to (C.1) or no.

- Based on the results of [28, 106], several works have been done. For instance a case of initial data which is not necessary small in $\mathcal{B}^{0, \frac{1}{2}}$ has been treated in [31] by introducing the critical space $\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}$ defined by its norm

$$\|u\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}} \stackrel{def}{=} \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \left(\sum_{k \geq \ell-1} 2^{-k} \|\Delta_k^h \Delta_\ell^v u\|_{L_h^4(L_v^2)}^2 \right)^{\frac{1}{2}} + \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \|S_{j-1}^h \Delta_j^v u\|_{L^2}. \quad (\text{C.3})$$

²Up to my knowledge this was the first result that ensures the existence and the uniqueness in scaling invariant spaces, before that I. Gallagher, J-Y Chemin, B. Desjardins and E. Grenier proved in [28] a wellposedness result that requires the initial data to be in some sup-critical spaces, followed by the work of D.Iftime who improved the uniqueness result of [28] and ensured that it holds in almost critical spaces, this was before M.Paicu proved his result, more details can be found in Chapter V.

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The above norm is inspired by two facts:

The first one is when we look at the first part of the norm above, we remark that the horizontal derivatives control the vertical ones. If we ignore the convective term in (C.1), and if we assume that the solution preserves this structure, in this case will have $\Delta_h \approx \Delta$.

That is to say, any initial data that has the form of the first term on the r.h.s of (C.3) would generate a solution to

$$(\partial_t - \Delta_h)f = 0 \tag{C.4}$$

which gains some regularity with respect to the vertical variables despite the lack of the term $-\partial_{zz}$ in (C.4).

Secondly, the last term on the r.h.s of (C.3) considers the case where the vertical variables dominate the horizontal ones. Here as there is no gain of regularity with respect to the vertical variables in (C.1) the problem should be dealt with as hyperbolic system as well explained in [31]. The work of M.Paicu [106] shows up to be crucial in this step.

As pointed out above and in [31], one of the advantages of solving the equations in $\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}$ instead of $\mathcal{B}^{0, \frac{1}{2}}$ is to generate solutions from initial data like $\phi_\varepsilon \stackrel{def}{=} e^{ix_1/\varepsilon}\phi(x)$, $\phi \in \mathcal{S}(\mathbb{R}^3)$, which satisfies

$$\|\phi_\varepsilon\|_{\mathcal{B}^{0, \frac{1}{2}}} = \|\phi\|_{\mathcal{B}^{0, \frac{1}{2}}}, \quad \|\phi_\varepsilon\|_{\mathcal{B}_4^{-\frac{1}{2}, \frac{1}{2}}} \leq C_\phi \varepsilon^{\frac{1}{2}}.$$

Which means that this kind of initial data can not be treated by the result of [106] whereas [31] ensures the global wellposedness of (C.1) with initial data like ϕ_ε and $\varepsilon \ll 1$.

The outlines of the proof in [31] relies on searching for a solution of the form $u = u_F + w$, where

$$u_F \stackrel{def}{=} e^{t\Delta} u_{hh}, \quad u_{hh} \stackrel{def}{=} \sum_{k \geq \ell - 1} \Delta_k^h \Delta_\ell^v u_0,$$

and w belongs to an horizontal-energy-type space of $\mathcal{B}^{0, \frac{1}{2}}$. The equation of w is then a modified Navier-Stokes equation with source term $u_F \cdot \nabla u_F$. It is in fact because of this term, the maximal horizontal integrability chosen in the definition of the first term on r.h.s of the norm (C.3) is L_h^4 .

One may ask if we can generalize the result of [31] for small initial data belonging to

$$\|u\|_{\mathcal{B}_p^{\frac{2}{p}-1, \frac{1}{2}}} \stackrel{def}{=} \sum_{\ell \in \mathbb{Z}} 2^{\frac{\ell}{2}} \left(\sum_{k \geq \ell - 1} 2^{2k(\frac{2}{p}-1)} \|\Delta_k^h \Delta_\ell^v u\|_{L_h^p(L_v^2)}^2 \right)^{\frac{1}{2}} + \sum_{j \in \mathbb{Z}} 2^{\frac{j}{2}} \|S_{j-1}^h \Delta_j^v u\|_{L^2}. \tag{C.5}$$

A positive answer was given in [122], if in addition, the non linear term $u_F \cdot \nabla u_F$ belongs to $L_T^1(\mathcal{B}^{0, \frac{1}{2}})$. This additional condition is true for all data belonging to $\mathcal{B}_p^{\frac{2}{p}-1, \frac{1}{2}}$, and $p \in [2, 4]$. The choice of $p = 4$ in [31] is optimal regarding the idea they follow to prove their result (which is the same in [122]).

The question that we address here is whether or not can we get rid of the additional condition $u_F \cdot \nabla u_F \in L_T^1(\mathcal{B}^{0, \frac{1}{2}})$? The major problem is the following:

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Roughly speaking, we would like to solve a modified version of (C.1) on w with $u_F \cdot \nabla u_F$ as a source term. At some point this modified equation should be solved in $L_T^\infty(\mathcal{B}^{0, \frac{1}{2}}) \cap L_T^2(\mathcal{B}^{1, \frac{1}{2}})$, which justify the choice of the condition $u_F \cdot \nabla u_F \in L_T^1(\mathcal{B}^{0, \frac{1}{2}})$. The use of the L^2 norm in the last term on the r.h.s of (C.5) is crucial in order to follow the work of M.Paicu in [106], and to benefit of the structure of the non linear term $w \cdot \nabla w$.

That is to say, replacing L^2 in the last term on the r.h.s of (C.5) by some L^p , for $p \neq 2$, does not seem to be the key to solve the equations. Moreover, the choice of $p = 4$ in the space $\mathcal{B}_p^{\frac{2}{p}-1, \frac{1}{2}}$ seems to be optimal since having u_F in some $\mathcal{B}_p^{\frac{2}{p}-1, \frac{1}{2}}$ with respect to the spacial variables infers that the best low integrability that we can have for the non linear term $u_F \cdot \nabla u_F$ would be $\frac{p}{2}$. Which means that $p = 4$ looks like to be the best choice here.

Similar and more are the questions that we can address to the rest of the models studied in this thesis. For which we hope that we will be able to provide some answers in our future works.

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