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# On a Hamiltonian regularisation and regularity of entropy solutions of some nonlinear hyperbolic equations

Billel Guelmame

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Billel Guelmame. On a Hamiltonian regularisation and regularity of entropy solutions of some nonlinear hyperbolic equations. Analysis of PDEs [math.AP]. Université Côte d'Azur, 2020. English. NNT : 2020COAZ4053 . tel-03177654

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$$\rho \left( \frac{\partial v}{\partial t} + v \cdot \nabla v \right) = -\nabla p + \nabla \cdot T + f$$

$$e^{i\pi} + 1 = 0$$

# THÈSE DE DOCTORAT

SUR UNE RÉGULARISATION HAMILTONIENNE ET LA  
RÉGULARITÉ DES SOLUTIONS ENTROPIQUES DE CERTAINES  
ÉQUATIONS HYPERBOLIQUES NON LINÉAIRES

**BILLEL GUELMAME**

LABORATOIRE DE MATHÉMATIQUES J.A. DIEUDONNÉ (LJAD)

PRÉSENTÉE EN VUE DE L'OBTENTION DU GRADE DE

**Docteur en Sciences de l'Université Côte d'Azur**

DISCIPLINE : MATHÉMATIQUES

THÈSE DIRIGÉE PAR : DIDIER CLAMOND ET STÉPHANE JUNCA

SOUTENUE LE 23 SEPTEMBRE 2020

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**Sur une régularisation hamiltonienne et la  
régularité des solutions entropiques de certaines équations  
hyperboliques non linéaires**

---

**On a Hamiltonian regularisation and regularity of entropy  
solutions of some nonlinear hyperbolic equations**

Thèse de doctorat  
soutenue le 23 septembre 2020  
par Billel Guelmame.

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# Sur une régularisation hamiltonienne et la régularité des solutions entropiques de certaines équations hyperboliques non linéaires

## Résumé

Dans cette thèse, nous étudions certaines régularisations conservatives et non dispersives pour des lois de conservation. Ces régularisations sont obtenues en s'inspirant de celle du système de Saint-Venant introduite par Clamond et Dutykh [2017]. Nous étudions également la régularité, dans des espaces  $BV$  généralisés, des solutions entropiques de certaines équations hyperboliques non linéaires.

Dans la première partie, nous obtenons et étudions une régularisation appropriée de l'équation de Burgers inviscide, ainsi que sa généralisation aux lois de conservation scalaires. Nous prouvons que cette généralisation est localement bien posée pour les solutions régulières. Nous montrons aussi l'existence globale des solutions qui satisfont une inégalité d'Oleinik pour des flux uniformément convexes. Lorsque le paramètre de régularisation  $\ell$  tend vers zéro, nous prouvons que ces solutions convergent, pour une sous-suite, vers les solutions de la loi de conservation scalaire originale, au moins pour un petit intervalle de temps.

Nous généralisons également les équations Saint-Venant régularisées afin d'obtenir une régularisation du système d'Euler barotrope, ainsi qu'une régularisation du système de Saint-Venant avec fond variable. Nous montrons que ces deux systèmes sont bien posés localement dans  $H^s$ , avec  $s \geq 2$ .

Dans la deuxième partie, nous démontrons un effet régularisant, sur les conditions initiales, des lois de conservation scalaires pour un flux lipschitzien strictement convexe, ainsi que pour des équations scalaires avec un terme source linéaire. Dans certains cas, nous donnons une borne de l'effet régularisant.

Enfin, nous prouvons l'existence globale des solutions entropiques d'une classe de système triangulaire ayant une équation de transport dans  $BV^s \times L^\infty$ , où  $s > 1/3$ .

**Mots clés :** EDPs hyperboliques non linéaires ; régularisation hamiltonienne ; inégalité d'Oleinik ; solutions entropiques ; équation de Burgers ; équations de Saint-Venant ; équations d'Euler ; équations de Hunter–Saxton ; systèmes triangulaires ; espaces  $BV$  fractionnaires.

# On a Hamiltonian regularisation and regularity of entropy solutions of some nonlinear hyperbolic equations

## Abstract

In this thesis, we study some non-dispersive conservative regularisations for the scalar conservation laws and also for the barotropic Euler system. Those regularisations are obtained inspired by the regularised Saint-Venant system introduced by Clamond and Dutykh [2017]. We also study the regularity, in generalised  $BV$  spaces, of the entropy solutions of some nonlinear hyperbolic equations.

In the first part, we obtain and study a suitable regularisation of the inviscid Burgers equation, as well as its generalisation to scalar conservation laws. We prove that this regularisation is locally well-posedness for smooth solutions. We also prove the global existence of solutions that satisfy a one-sided Oleinik inequality for uniformly convex fluxes. When the regularising parameter  $\ell$  goes to zero, we prove that the solutions converge, up to a subsequence, to the solutions of the original scalar conservation law, at least for a short time.

We also generalise the regularised Saint-Venant equations to obtain a regularisation of the barotropic Euler system, and the Saint-Venant system with uneven bottom. We prove that both systems are locally well-posed in  $H^s$ , with  $s \geq 2$ .

In the second part, we prove a regularising effect, on the initial data, of scalar conservation laws with Lipschitz strictly convex flux, and of scalar equations with a linear source term. For some cases, we give a limit of the regularising effect.

Finally, we prove the global existence of entropy solutions of a class of triangular systems involving a transport equation in  $BV^s \times L^\infty$ , where  $s > 1/3$ .

**Key words:** Nonlinear hyperbolic PDEs; Hamiltonian regularisation; Oleinik inequality; entropy solutions; Burgers equation; Saint-Venant equations; Euler equations; Hunter–Saxton equations; triangular systems; fractional  $BV$  spaces.





*To my beloved mother Rebaia and to the soul of my father Ali.*

# Acknowledgments

*“If I have seen further, it is by standing on the shoulders of giants.”*

Isaac Newton [60]

Undoubtedly, this work would never be that way without the support and the advice of my supervisors Pr. Didier Clamond & Pr. Stéphane Junca. I would like to thank you both for your unlimited help during more than three years, from the beginning of my master thesis until the end of this thesis. Thank you for your availability and for proposing such a rich subject. Thank you also for your accuracy and precision that I used to be scared of! Now, I do understand how valuable your rigorous remarks were. Thank you for making me learning a lot from you and from your large physical and mathematical cultures.

I would like to thank my collaborators Pr. C Bourdarias, Pr. A-P Choudhury, Pr. S Ghoshal, A Jana and Pr. R-L Pego for their work and remarks. I also would like to express my sincere thanks to Pr. Denys Dutykh for his help and advice concerning numerical analysis.

Also, many thanks to my thesis monitoring committee: Pr. Laurent Monasse and Pr. Francesca Rapetti for their kindness and helpful discussions.

My gratitude and many thanks go to Pr. Adrian Constantin and Pr. Philippe G LeFloch for being reporters of my thesis, and also to Pr. Isabelle Gallagher, Pr. Sergey Gavriluk, Pr. Paola Goatin and Pr. Boris Haspot for accepting to be a part of the jury. I'm particularly grateful to Pr. Boris Haspot for all his support, remarks and advice.

I would like also to thank my professors Brahim Benzeghli, Ahmed Zerrouk Mokrane, Amar Youkana, Abdelghani Zeghib and Mohamed Zerguine for their continuous support and encouragements.

My thanks also goes to all my friends and colleagues, especially, Ahmed, Alexis, Dahmane, Eliot, Gaëtan, Haroune, Kevin, Léo, Mehdi, Mhammed, Victor, Yash and Zakaria. Thank you for your support and your useful discussions.

An enormous gratitude and thanks go to my mother Rebaia and my sisters and brothers, especially, Amel and Salah Eddine. Thank you for all your help and your unconditional support. I also would like to thank my uncles Djamel and Lakhdar for their help, support and advice.

Finally, I would like to thank the painter of Figure 1.3, my dear wife Chahrazad. Thank you for your moral support and your patience. Without you, this thesis would be finished a long time ago. Or maybe not! So, thank you for making me “sometimes” forget and “sometimes” remember that I have a thesis to write.

Thanks

شكرا

Merci

Спасибо

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धन्यवाद

Muṭumiri

Grazie

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# Chapter 1

## Introduction

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Hyperbolic partial differential equations are used to describe many phenomena, for example in fluid mechanics and traffic flows [71]. Nonlinear hyperbolic equations behave “in general” in an interesting way, where their solutions can lose [44] and gain [48, 65] regularity over time in two different senses:

1. **Shocks in finite time**, “*loss of regularity*” [44]: Even for smooth (compactly supported  $C^\infty$ ) initial data, the derivative of the solution may blow-up in finite time, and a discontinuous shock can be created.

After the breaking-down time, classical (strong) solutions with classical derivatives can not exist. In order to obtain global (in time) solutions, weak solutions must be defined, where the derivatives are defined in the sense of distributions. Weak solutions do exist globally in time, but no uniqueness is insured, in general. Many “entropy” conditions on the solution have been imposed in the literature in order to obtain the uniqueness (for example Lax [48], Oleinik [65] and Kruzkov [46]) for the scalar case. However, for systems, the entropy condition is not enough to obtain the uniqueness of the solution [25]. For some hyperbolic equations, the entropy solutions enjoy an interesting regularising effect that is directly related to the well-known one-sided Oleinik inequality [65]

2. **Regularising effect**, “*gain of regularity*” [48, 65] Even for rough ( $L^\infty$ ) initial data, the entropy solution becomes instantaneously in a generalised bounded variation  $BV_{loc}$  space.

The transport equation “with a constant velocity” is the simplest hyperbolic equation, its solutions being nothing but a translation of the initial data. Thus, there is no gain or loss of the regularity. The two interesting behaviors presented above (loss and gain of the regularity) are related to the nonlinearity of the equations. Let us consider the simplest nonlinear hyperbolic equation, which is the inviscid Burgers equation

$$u_t + u u_x = 0, \tag{1.1}$$

where subscripts denote partial derivatives. The equation (1.1) can be seen as a transport

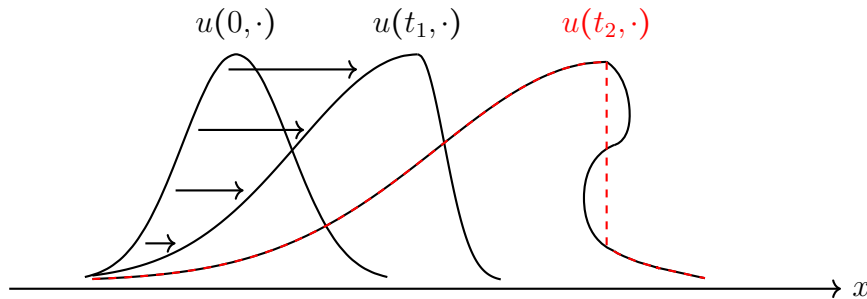


Figure 1.1 – Propagation of the initial data over the time.

equation, where the velocity is the value of  $u$  itself. That means, if we take a Gaussian-like

initial data (see Figure 1.1), the data will be propagated with maximum speed on the top and a minimum speed at the bottom. It is clear that on the increasing (with respect to  $x$ ) part of the wave, the slope becomes smaller over time. On that part of the wave, the solution “gains” some regularity. On another hand, we see that on the decreasing part of the wave, the slope becomes stronger over time, and a shock is developed in finite time which leads to a “loss” of regularity.

Letting  $v$  takes the values of  $u_x$  on the characteristics with the velocity  $u$ . The Riccati equation of (1.1) follows

$$v_t(t, \xi) = -v(t, \xi)^2, \quad \implies \quad v(t, \xi) = \frac{1}{1/v_0(\xi) + t}, \quad (1.2)$$

Both properties presented above can be obtained “formally” for the Burgers equation from the Riccati equation (1.2) as

- If  $v_0(\xi) < 0$  at least for one  $\xi \in \mathbb{R}$ , then  $\lim_{t \rightarrow -1/v_0(\xi)} v(t, \xi) = -\infty$ .
- The solutions of (1.1) satisfy the Oleinik inequality

$$u_x(t, x) \leq 1/t \quad \forall (t, x) \in ]0, +\infty[ \times \mathbb{R}, \quad (1.3)$$

which implies, with the maximum principle, that the solutions of the inviscid Burgers equation belong to the  $BV_{loc}$  space.

In summary, the space  $BV_{loc} \cap L^\infty$  is a suitable space for the entropy solutions of the Burgers equation. If the initial datum  $u_0$  belongs to  $L^\infty$ , the entropy solution gains instantaneously some regularity and belongs to  $BV_{loc} \cap L^\infty$ . For a non-trivial initial datum in  $C_c^\infty$ , the solution breaks-down in a finite time in the sens that the solution is not continuous anymore, despite that  $u \in BV_{loc} \cap L^\infty$ .

The same phenomenon can be observed in the unidimensional nonlinear long waves propagating in shallow water. Let  $u(t, x)$  be the averaged horizontal velocity, the equations

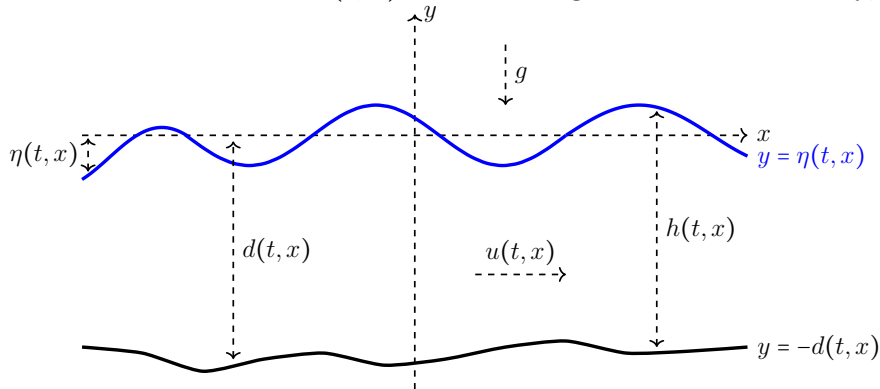


Figure 1.2 – Fluid domain.

$y = \eta(t, x)$  and  $y = -d(t, x)$  are respectively the equations of the free surface and of the



Figure 1.3 – Breaking wave.

bottom. Let also  $h(t, x) = d(t, x) + \eta(t, x)$  be the total depth and  $g$  be the gravitational acceleration (see Figure 1.2).

Even for a moving bottom, by using a change of variables, we can assume that the spacial mean water depth  $\bar{d}$  is a constant in time. In that case, the gravitational acceleration  $g = g(t)$  is a function of time. The conservation of the mass and momentum lead to the Saint-Venant system

$$h_t + [hu]_x = 0, \quad [hu]_t + [hu^2 + \frac{1}{2}gh^2]_x = gh d_x. \quad (1.4)$$

Note that the total water depth  $h$  is an univalued function of time and space. In general, this is not the case, and waves on the form of Figure 1.3 can appear. In that case, the model of Saint-Venant is no longer valid (do not describe exactly the movement of the wave), and a shock appears as in Figure 1.1.

In this thesis, we deal with both properties of hyperbolic equations presented above. For the gain of the regularity, we study the regularising effect of some hyperbolic equations in generalised  $BV^\Phi$  spaces, and we study the optimality of the chosen space. For the problem of the shocks and the loss of the regularity, we study some non-dispersive and non-dissipative regularisations of some hyperbolic equations.

## 1.1 Non-dispersive Hamiltonian regularisations of hyperbolic conservation laws

In order to prevent shocks from appearing in finite time, terms on the form  $\varepsilon F$  can be added to the classical hyperbolic equations, where  $\varepsilon$  is a “small” positive parameter and  $F$  is a function depending on the solution and its derivatives. The new equations are called *regularisations* and  $F$  should be chosen to obtain good properties. Several regularisations

have been proposed and studied in the literature, for example adding an artificial viscosity [8, 24, 69]. Global smooth solutions of this regularisation exist in the scalar case, and the solution converges to the unique solution of the classical equation. This regularisation is diffusive and the energy is a decreasing function of time which causes problems for the long time study. Another regularisation, that has been proposed and studied, is obtained adding a dispersion to the equation [49, 50, 51]. This dispersion causes high oscillations and, when  $\varepsilon \rightarrow 0$ , the solutions do not always converge to the entropy solution of the classical equations. Another regularisation, is obtained adding both viscosity and dispersion in [39, 47]. Also, Leray-type regularisations have been studied in [4, 5, 6, 7, 14, 61, 62], but the energy is not conserved and, when  $\varepsilon \rightarrow 0$ , the limit solution does not “in general” satisfy the Lax entropy condition [5].

Clamond and Dutykh [21] have proposed the regularised Saint-Venant (rSV) system for constant depth  $d$

$$h_t + [hu]_x = 0, \quad (1.5a)$$

$$[hu]_t + \left[ hu^2 + \frac{1}{2}gh^2 + \varepsilon\mathcal{R} \right]_x = 0, \quad (1.5b)$$

$$\mathcal{R} \stackrel{\text{def}}{=} h^3(-u_{tx} - uu_{xx} + u_x^2) - gh^2(hh_{xx} + \frac{1}{2}h_x^2), \quad (1.5c)$$

where  $h$  is the total water depth of the fluid and  $g$  is the gravitational acceleration. The regularised system (1.5) can be compared with the “dispersive” Serre–Green–Naghdi system with capillarity that can be obtained by replacing the  $\varepsilon\mathcal{R}$  in (1.5) by

$$\mathcal{R}_{SGN} \stackrel{\text{def}}{=} \frac{1}{3}h^3(-u_{tx} - uu_{xx} + u_x^2) - \tau(hh_{xx} - \frac{1}{2}h_x^2),$$

where  $\tau$  is a constant (the surface tension coefficient divided by the density). The rSV system is Galilean invariant, enjoys Hamiltonian and Lagrangian structures, it is linearly dispersionless and it satisfies formally the conservation of the energy

$$\begin{aligned} & \frac{1}{2} \left[ hu^2 + \varepsilon h^3 u_x^2 + gh^2 + \varepsilon gh^2 h_x^2 \right]_t + \\ & \left[ \left( \frac{1}{2}hu^2 + gh^2 + \frac{1}{2}\varepsilon h^3 u_x^2 + \frac{1}{2}\varepsilon gh^2 h_x^2 + \varepsilon\mathcal{R} \right) u + \varepsilon gh^3 h_x u_x \right]_x = 0. \end{aligned} \quad (1.6)$$

Weak travelling waves of the rSV system have been studied in [68]. Local well-posedness of the rSV system in  $H^s$ , with  $s > 3/2$ , and the existence of blow-up phenomena have been proved in [55]. The existence of global (in time) solutions, and the study of the limit  $\varepsilon \rightarrow 0$  remain open questions.

Inspired by rSV, several regularisations of hyperbolic equations are proposed in this thesis.

### 1.1.1 A regularised scalar conservation laws

We introduced in this thesis the Hamiltonian regularisation of scalar conservation laws

$$u_t + f(u)_x = \ell^2 \left[ u_{xxt} + f'(u)u_{xxx} + 2f''(u)u_x u_{xx} + \frac{1}{2}f'''(u)u_x^3 \right], \quad (1.7)$$

where  $\ell \geq 0$  is the length scale of the regularisation. Applying  $(1 - \ell^2 \partial_x^2)^{-1}$ , one obtains

$$u_t + f(u)_x + \frac{1}{2} \ell^2 (1 - \ell^2 \partial_x^2)^{-1} [f''(u) u_x^2]_x = 0. \quad (1.8)$$

For the special flux  $f(u) = u^2/2$ , the equation (1.7) should be compared with the dispersionless Camassa–Holm (CH) equation [15]

$$u_t + 3u u_x = u_{xxt} + u u_{xxx} + 2u_x u_{xx}. \quad (1.9)$$

We prove in this thesis that the equation (1.8) is locally well-posed in  $H^s$  with  $s > 3/2$ . In general, singularities appear in finite time, and smooth solutions can not hold for all time. Inspired by [12] and using an equivalent semi-linear system, we prove the global existence of conservative weak solutions of (1.8). Following [13], the existence of another type of global weak solutions called “dissipative” is proved for uniformly convex fluxes ( $f''(x) \geq C > 0$ ). Those dissipative solutions enjoy a uniform (on  $\ell$ ) Oleinik inequality for all time  $t > 0$ . Using a compactness argument, we also prove that when  $\ell \rightarrow 0$  (Resp.  $\ell \rightarrow +\infty$ ), the dissipative solutions of (1.8) converge “up to a subsequence” to  $u^0$  (Resp.  $u^\infty$ ). At least before the appearance of the singularities,  $u^0$  satisfies the classical scalar conservation law and  $u^\infty$  satisfies the generalised Hunter–Saxton equation

$$[u_t + f(u)_x]_x = \frac{1}{2} u_x^2 f''(u). \quad (1.10)$$

### 1.1.2 A regularised barotropic Euler system

By modifying the Lagrangian of the classical barotropic Euler system, we generalise the rSV system (1.5) as

$$\rho_t + [\rho u]_x = 0, \quad (1.11a)$$

$$[\rho u]_t + [\rho u^2 + \rho \mathcal{V}' - \mathcal{V} + \epsilon \mathcal{R}]_x = 0, \quad (1.11b)$$

$$\mathcal{R} \stackrel{\text{def}}{=} (\rho^2 \mathcal{A}')' u_x^2 - 2\rho \mathcal{A}' [u_t + u u_x + P_x/\rho]_x + (\rho \mathcal{V}''/\mathcal{A}')' \mathcal{A}_x^2, \quad (1.11c)$$

where  $\rho(t, x) > 0$  and  $u(t, x)$  denote the density and the velocity, respectively, primes denote derivatives with respect to  $\rho$ ,  $\mathcal{V}''(\rho) = P'(\rho)/\rho$  with  $P(\rho)$  is the pressure and  $\mathcal{A}$  is a smooth increasing function of  $\rho$ . The regularised barotropic Euler system (1.11), (rbE) enjoys all the properties of rSV and generalises it in two ways: (i) considering barotropic Euler system; (ii) introducing a family of regularisations (involving an arbitrary function  $\mathcal{A}(\rho)$ ).

Using the Sturm–Liouville operator

$$\mathcal{L}_\rho \stackrel{\text{def}}{=} \rho - 2\epsilon \partial_x \rho \mathcal{A}' \partial_x, \quad (1.12)$$

the system (1.11) can be written as

$$\rho_t + [\rho u]_x = 0, \quad (1.13a)$$

$$u_t + u u_x + P_x/\rho = -\epsilon \mathcal{L}_\rho^{-1} \partial_x \left\{ (\rho^2 \mathcal{A}')' u_x^2 + (\rho \mathcal{V}''/\mathcal{A}')' \mathcal{A}^{\prime 2} \rho_x^2 \right\}. \quad (1.13b)$$

Taking formally  $\varepsilon \rightarrow \infty$  in (1.13) one obtains the generalised two-component Hunter–Saxton system

$$\rho_t + [\rho u]_x = 0, \quad (1.14a)$$

$$u_t + uu_x + P_x/\rho = \partial_x^{-1} \left\{ \left( 1 + \frac{\rho \mathcal{A}''}{2 \mathcal{A}'} \right) u_x^2 + \left( \frac{(\rho \mathcal{V}'')'}{2 \rho} - \frac{\mathcal{V}'' \mathcal{A}''}{2 \mathcal{A}'} \right) \rho_x^2 \right\}. \quad (1.14b)$$

Following [42, 55, 56], we prove that if the initial data satisfies  $\rho_0 \geq \rho^* > 0$ , then the system (1.13) is locally well-posed in  $H^s$  with  $s \geq 2$ . Moreover, we prove that if the initial data is periodic and satisfies  $\rho_0 \geq \rho^* > 0$ , then the generalised two-component Hunter–Saxton system (1.14) is also locally well-posed in  $H^s([0, 1])$ .

### 1.1.3 A regularised Saint-Venant system with uneven bottom

In [22], the rSV system has been generalised to regularise the Saint-Venant system with uneven bottom. We also generalise the latter system involving the increasing regularisation function  $\mathcal{A}$  of  $h$  as

$$h_t + [hu]_x = 0, \quad (1.15a)$$

$$u_t + uu_x + g\eta_x = -\varepsilon \mathcal{L}_h^{-1} \partial_x \left\{ (h^2 \mathcal{A}')' u_x^2 - g (h \mathcal{A}'' - \mathcal{A}') (\eta_x^2 + 2\eta_x d_x) \right\} + 2\varepsilon g \mathcal{L}_h^{-1} \{ \mathcal{A}' \eta_x d_{xx} \}, \quad (1.15b)$$

where  $d = d(t, x)$  is the depth and  $\eta = h - d$ .

We show that if  $(\eta_0, u_0) \in H^s$  for  $s \geq 2$ ,  $h_0 \geq h^* > 0$ ,  $g \in C^1$  and  $d \in C^1([0, +\infty), H^{s+1})$  then the system (1.15) is locally well-posed in  $H^s$ .

## 1.2 Regularising effect for hyperbolic conservation laws

It is known from Lax [48] and Oleinik [65] that if the flux  $f$  is uniformly convex — i.e., if  $f''(u) \geq C > 0$  — then the entropy solutions of the scalar conservation law

$$u_t + f(u)_x = 0, \quad (1.16)$$

belong to the space of bounded variation  $BV_{loc}$  for all  $t > 0$  and any initial data in  $L^\infty$ . For flatter fluxes ( $f(u) = |u|^3$ ,  $f(u) = u^4$  for example), there exist some initial data  $u_0 \in L^\infty$  such that the corresponding entropy solution does not belong to  $BV_{loc}$  [20]. Other regularising effect results have been proved for hyperbolic equations in fractional Sobolev spaces  $W^{s,p}$  [43]. However, for hyperbolic conservation laws, and due to the Rankine–Hugoniot condition, it is convenient to work with solutions that admit traces everywhere, which is not the case for Sobolev spaces  $W^{s,p}$  with  $sp < 1$ . In order to obtain a suitable space for the hyperbolic conservation laws, the  $BV$  space has been generalised as  $BV^s$  with  $s \in ]0, 1]$  is the regularity, and  $BV^\Phi$ , where  $\Phi$  is a positive convex function such that

$BV^\Phi \equiv BV^s$  if  $\Phi(h) = h^{1/s}$  and  $BV^1 \equiv BV$ . It turns out that the  $BV^\Phi$  space admits traces everywhere [59], hence it seems a suitable space for hyperbolic equations.

The regularising effect proved by Lax and Oleinik has been generalised for smooth strictly convex fluxes with the power law degeneracy

$$\frac{|f'(u) - f'(v)|}{|u - v|^p} \geq c_0 > 0 \quad \text{for } u \neq v \text{ and } p \geq 1 \quad (1.17)$$

in  $BV_{loc}^{1/p}$  for all  $t > 0$  [10, 16]. For a  $C^1$  strictly convex flux, the entropy solution belongs to  $BV_{loc}^\Phi$  where  $\Phi$  is a convex function depending on the nonlinearity of the flux  $f$  [19, 17]. For smooth non-convex fluxes, the result remains true, and the solution  $u$  belongs to some  $BV_{loc}^\Phi$  [57]. For convex fluxes, a key point to prove those regularising effect is to use the one-sided Oleinik inequality

$$f'(u(t, x)) - f'(u(t, y)) \leq (x - y)/t \quad \text{for almost all } t > 0, x > y, \quad (1.18)$$

which implies, with the maximum principle, that  $f' \circ u \in BV_{loc}$ . Then, with a suitable choice of  $\Phi$ , it follows that  $u \in BV^\Phi$ . The optimality of the results above has not been proven in general. However, in some cases, some initial data have been constructed to show the optimality of the chosen  $BV^s$  for a short interval of time (before the interaction of waves).

In the second part of this thesis, we study the regularising effect of some scalar conservation/balance laws. We also study the existence of global entropy solutions of a class of triangular system involving a linear transport equation with a discontinuous velocity.

### 1.2.1 Scalar hyperbolic equation

We generalise the regularising effect proven in [16] in two ways.

#### Scalar conservation laws with a Lipschitz convex flux

As mentioned above, the regularising effect is related to the Oleinik inequality (1.18) if the flux  $f$  is  $C^1$ . If the flux  $f$  is not  $C^1$ ,  $f'$  is not defined everywhere. For the strictly convex flux  $f(u) = u^2 + |u|$ , we show that there exists an initial datum such that for any choice of  $f'(0) \in [-1, 1]$ , the entropy solution of (1.16) fails to satisfy the Oleinik inequality (1.18). Thus, the Oleinik inequality (1.18) cannot be used directly to obtain the regularising effect for non  $C^1$  fluxes.

For any strictly convex Lipschitz flux, we use a modified wave front tracking algorithm to show that the solution  $u(t, \cdot)$  belongs to  $BV_{loc}^\Phi$  for all  $t > 0$ , where  $\Phi$  depends on the non-linearity of  $f$ . Since the Oleinik inequality (1.18) is not valid, the question: *Can we define  $f'$  everywhere such that  $f' \circ u \in BV_{loc}$ ?* remains open.



## Scalar balance laws with a linear source term

For  $\alpha \in L^\infty([0, +\infty))$ , we consider the scalar balance law

$$u_t + f(u)_x = \alpha(t)u, \quad (1.19)$$

where  $f$  is a  $C^1$  strictly convex flux satisfying the power law degeneracy (1.17). Using some estimates on the generalised Lax–Oleinik formula given in [3], we prove that the entropy solution of (1.19) belongs to  $BV_{loc}^s$  with  $s = p^{-1}$ , where  $p \geq 1$  is obtained from (1.17).

We also prove that for a class of fluxes containing  $f(u) = |u|^{p+1}/(p+1)$ , the previous result is optimal for all time, i.e., there exists a compactly supported initial datum  $u_0 \in L^\infty$  such that for all  $t > 0$  and  $\varepsilon > 0$  we have

$$u(t, \cdot) \in BV^s \quad \text{and} \quad u(t, \cdot) \notin BV^{s+\varepsilon} \quad \text{where } s = p^{-1}. \quad (1.20)$$

## 1.2.2 A class of triangular systems

We consider the triangular system on the form

$$u_t + f(u)_x = 0, \quad u(0, x) = u_0(x) \quad (1.21)$$

$$v_t + (g(u)v)_x = 0, \quad v(0, x) = v_0(x). \quad (1.22)$$

This system is a coupling between a scalar conservation law with a linear transport equation with discontinuous (in general) velocity  $g(u)$ . This system has been studied for the case  $f' = g$  in [52], where measure solutions exist for  $v$ . Considering the strictly hyperbolic condition

$$f'(u) > g(u) \quad \forall u \in [-M, M], \quad M \stackrel{\text{def}}{=} \|u_0\|_{L^\infty}, \quad (1.23)$$

for large data, the value of  $f'(u_1)$  may coincide with the value of  $g(u_2)$  for some  $u_1, u_2 \in [-M, M]$ , which can also lead to the appearance of measure solutions [38]. In order to avoid measure solutions, we consider the uniform strictly hyperbolic condition

$$\inf_{|u| \leq M} f'(u) > \sup_{|u| \leq M} g(u), \quad (1.24)$$

where  $f \in C^4$  having at most a finite number of inflection points, and  $g \in C^3$ , we show that for any  $s > 1/3$ , if  $(u_0, v_0) \in BV^s \times L^\infty$ , then there exists a global weak entropy  $(u, v) \in L_t^\infty([0, +\infty), BV_x^s \times L_x^\infty)$ . We also prove that this result remains true for  $s = 1/3$  if the flux  $f$  is convex.

The optimality of the above result is also proven in the following sense. For  $s < 1/3$ , there exist  $f \in C^4$ ,  $g \in C^3$ ,  $u_0 \in BV^s - BV^{1/3}$  and  $v_0 \in L^\infty$  such that the norm  $\|v\|_{L^\infty}$  blows up at  $t = 0^+$ .

## 1.3 Thesis overview

During this thesis, the following papers and preprints have been written by the author and his collaborators:

- [35] Global weak solutions of a Hamiltonian regularised Burgers equation.  
*Submitted.*
- [32] On a Hamiltonian regularization of scalar conservation laws.  
*Submitted.*
- [33] Hamiltonian regularisation of the unidimensional barotropic Euler equations.  
*Submitted.*
- [34] Local well-posedness of a Hamiltonian regularisation of the Saint-Venant system with uneven bottom.  
*Submitted.*
- [36] Regularizing effect for conservation laws with a Lipschitz convex flux.  
*Commun. Math. Sci. 17 (2019) 2223–2238.*
- [9] Entropy solutions in  $BV^s$  for a class of triangular systems involving a transport equation.  
*Submitted.*
- [29] Optimal regularity for all time for entropy solutions of conservation laws in  $BV^s$ .  
*Nonlinear Differ. Equ. Appl. 27, 46 (2020).*

In the remainder of this thesis, we outline the main results of these papers. It is organised as follows.

- In Chapter 2, we study the local well-posedness of a regularisation of scalar conservation laws and we prove the global existence of two types of weak solutions. A study of the limiting cases is given as well. This chapter is based on the papers [35, 32].
- In Chapter 3, we study the local well-posedness of a regularisation of barotropic Euler, Saint-Venant and a two-component Hunter–Saxton system. This chapter is based on the papers [33, 34].
- In Chapter 4, we study some regularising effects for hyperbolic equations. A study of global existence of solutions of a triangular system in  $BV^s$  is given as well. This chapter is based on the papers [36, 9, 29].
- In Appendices A, B, C, D, E, F and G, we attach the references [35, 32, 33, 34, 36, 9, 29] respectively. Notations may change from an appendix to another.

# Chapter 2

## A Hamiltonian regularisation of scalar conservation laws

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This chapter is based on the papers [35, 32] that are attached in Appendices A and B, respectively.

The aim of this chapter is to introduce a suitable regularisation (i.e., with similar characteristics than (1.5)) of scalar conservation laws. We note first that the rSV equations yield the momentum equation

$$u_t + u u_x + g h_x + \epsilon (h \mathcal{R}_x + 2 \mathcal{R} h_x) = 0. \quad (2.1)$$

When  $h$  is constant, this equation (with the definition of  $\mathcal{R}$ ) becomes

$$u_t + u u_x = \ell^2 [u_{xxt} + u u_{xxx} - u_x u_{xx}], \quad (2.2)$$

with  $\ell \stackrel{\text{def}}{=} h\sqrt{\epsilon}$  is a constant characterising the “length” of the regularisation. The equation (2.2) is a regularisation of the inviscid Burgers equation. Another heuristic derivation of a suitable regularisation inviscid Burgers equation starts from the dispersionless Camassa–Holm (CH) equation [15]

$$u_t + 3u u_x = \ell^2 [u_{xxt} + u u_{xxx} + 2u_x u_{xx}]. \quad (2.3)$$

This equation is not Galilean invariant due to the coefficient 3 in the left-hand side of (2.3). Galilean invariance is an absolutely crucial feature for broad physically sound applications. Therefore, a regularisation of the Burgers equation is obtained substituting 1 for 3 in (2.3), i.e., we consider

$$u_t + u u_x = \ell^2 [u_{xxt} + u u_{xxx} + 2u_x u_{xx}], \quad (2.4)$$

that should be compared with (2.2). These heuristic derivation then suggest a family of regularised Burgers equations of parameter  $b$  (rB $_b$ )

$$u_t + u u_x = \ell^2 [u_{xxt} + u u_{xxx} + b u_x u_{xx}]. \quad (2.5)$$

This chapter is devoted to the study of the rB $_b$  equation, in particular the rB $_2$  that turns out to be the best element of the rB $_b$  family, as we shall see.

## 2.1 Regularised inviscid Burgers equation

As mentioned above, the classical inviscid Burgers equation

$$u_t + \frac{1}{2} [u^2]_x = 0, \quad (2.6)$$

can, *a priori*, be suitably regularised as the two-parameter equation

$$u_t + \frac{1}{2} [u^2]_x = \ell^2 [u_{xxt} + u u_{xxx} + b u_x u_{xx}], \quad (2.7)$$

where  $\ell$  and  $b$  are real parameters at our disposal. Equation (2.7) is called here the regularised Burgers equation (rB $_b$ ) equation.  $\ell$  has the dimension of a length that characterises the “thickness” of a regularised shock,  $b$  is a dimensionless parameter characterising the

energy source (or sink); for  $b = 2$  the energy is conserved. Indeed, the equation (2.7) can be rewritten in conservative form

$$\left[ u - \ell^2 u_{xx} \right]_t + \left[ \frac{1}{2} u^2 - \ell^2 u u_{xx} - \frac{1}{2} (b-1) \ell^2 u_x^2 \right]_x = 0, \quad (2.8)$$

and, multiplying by  $u$ , an energy equation is obtained as

$$\left[ \frac{1}{2} u^2 - \ell^2 u u_{xx} - \frac{1}{2} \ell^2 u_x^2 \right]_t + \left[ \frac{1}{3} u^3 - \ell^2 u^2 u_{xx} + \ell^2 u_x u_t \right]_x = (b-2) \ell^2 u u_x u_{xx}, \quad (2.9)$$

or as

$$\left[ \frac{1}{2} u^2 + \frac{1}{2} \ell^2 u_x^2 \right]_t + \left[ \frac{1}{3} u^3 - \ell^2 u^2 u_{xx} - \ell^2 u u_{xt} \right]_x = (b-2) \ell^2 u u_x u_{xx}, \quad (2.10)$$

Therefore, this energy equation can be written in conservative form only if  $b = 2$  when  $\ell \neq 0$ .

### 2.1.1 Alternative formulations

Introducing the momentum  $m = \mathcal{L}\{u\} \stackrel{\text{def}}{=} u - \ell^2 u_{xx}$ , the rB<sub>b</sub> equation can be rewritten

$$m_t + u m_x + b(m-u) u_x = 0. \quad (2.11)$$

The equation (2.7) can also be rewritten

$$\mathcal{L}\left\{ u_t + \frac{1}{2} [u^2]_x \right\} + \frac{1}{2} (3-b) \ell^2 [u_x^2]_x = 0, \quad (2.12)$$

The relation  $m = \mathcal{L}\{u\} = u - \ell^2 u_{xx}$  (defining the momentum  $m$ ) can be inverted as

$$u = \mathcal{G}\{m\}, \quad \mathcal{G} \stackrel{\text{def}}{=} \mathcal{L}^{-1} = (1 - \ell^2 \partial_x^2)^{-1}, \quad (2.13)$$

$\mathcal{G}$  being a positive-definite self-adjoint autonomous pseudo-differential operator acting on a pure frequency as

$$\mathcal{G}\{e^{ikx}\} = (1 + (k\ell)^2)^{-1} e^{ikx}. \quad (2.14)$$

Applying the operator  $\mathcal{G}$  to (2.12), the equation becomes

$$u_t + \frac{1}{2} [u^2]_x + \frac{1}{2} (3-b) \ell^2 (1 - \ell^2 \partial_x^2)^{-1} [u_x^2]_x = 0. \quad (2.15)$$

This form is more tractable for numerical computations. The pseudo-differential operator  $\mathcal{G}$  can be rewritten as a convolution integral, the equation (2.15) becoming

$$u_t + \left[ \frac{1}{2} u^2 + \frac{1}{2} (3-b) \ell^2 \mathfrak{G} * u_x^2 \right]_x = 0, \quad \mathfrak{G} \stackrel{\text{def}}{=} (2\ell)^{-1} \exp(-|x|/\ell), \quad (2.16)$$

where  $\mathfrak{G} * f \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \mathfrak{G}(x-y) f(y) dy = \int_{-\infty}^{\infty} \mathfrak{G}(y) f(x-y) dy$  for any function  $f$ . In the form (2.16), the rB<sub>b</sub> equation should be more tractable for proving global existence of weak solutions, following the methodology developed in [12, 13, 23, 72].

Following Bressan and Constantin [12], a (scaled) pressure-like term  $P$  is introduced as

$$P \stackrel{\text{def}}{=} \frac{1}{2} \mathfrak{G} * u_x^2 \quad \Rightarrow \quad P - \ell^2 P_{xx} = \frac{1}{2} u_x^2. \quad (2.17)$$

Differentiating (2.16) with respect of  $x$ , one obtains

$$u_{xt} + u u_{xx} + (b-3)P + \frac{1}{2}(b-1)u_x^2 = 0. \quad (2.18)$$

Adding (2.16) times  $u$  and (2.18) times  $\ell^2 u_x$ , we obtain the energy equation

$$\left[ \frac{1}{2}u^2 + \frac{1}{2}\ell^2 u_x^2 \right]_t + \left[ \frac{1}{3}u^3 + \frac{1}{2}\ell^2 u u_x^2 + (b-3)\ell^2 u P \right]_x = \frac{1}{2}(b-2)u_x^3, \quad (2.19)$$

to be compared with (2.10) (note that the temporal derivative inside the  $[\dots]_x$  bracket of the energy equation (2.10) is not present in the formulation (2.19)).

## 2.1.2 Connection with some classical equations

After differentiation with respect of  $x$ , (2.15) becomes

$$[u_t + u u_x]_x + \frac{1}{2}(3-b)\ell^2 \partial_x^2 \mathcal{G}\{u_x^2\} = 0. \quad (2.20)$$

For high frequencies, this equation can be approximate (because then  $\ell^2 \partial_x^2 \mathcal{G} \sim -\text{id}$ ) by

$$[u_t + u u_x]_x = \frac{1}{2}(3-b)u_x^2, \quad (2.21)$$

that, if  $b = 2$ , is the Hunter–Saxton (HS) equation [40, 41]. The HS equation being integrable, the question whether or not  $\text{rB}_2$  is integrable is posed.

The  $\text{rB}_b$  equation written in the form (2.11) should be compared with the equation

$$m_t + u m_x + b m u_x = 0, \quad (2.22)$$

that appeared before in the literature [26]. With  $b = 0$ , (2.22) is a regularised Burgers equation studied by Bhat and Fetecau [4, 5]. For  $b = 2$  and  $b = 3$  one obtains, respectively, the dispersionless Camassa–Holm (CH) and Degasperis–Procesi (DP) equations [15, 27]. The cases  $b = 2$  and  $b = 3$  are the only ones for which (2.22) possesses an infinite hierarchy of local higher symmetries [58].

A major difference between  $\text{rB}_b$  and (2.22) is, if  $u$  denotes a velocity field, that  $\text{rB}_b$  is Galilean invariant for all  $b$ , while (2.22) is Galilean invariant only if  $b = 0$ . This Galilean invariance is fundamental for many physical applications. Also, note that equation (2.22) can be rewritten as

$$u_t + \left[ \frac{1}{2}u^2 + \frac{1}{2}(3-b)\ell^2 \mathfrak{G} * u_x^2 + \frac{1}{2}b\mathfrak{G} * u^2 \right]_x = 0, \quad (2.23)$$

to be compared with (2.16), and as

$$u_t + (1+b)u u_x = \ell^2 [u_{xxt} + u u_{xxx} + b u_x u_{xx}], \quad (2.24)$$

to be compared with (2.7).

$\text{rB}_b$  is a special case of the three-parameter equation [28, #4.193]

$$u_t - \ell^2 u_{xxt} = a u u_x + b u_x u_{xx} + c u u_{xxx}, \quad (2.25)$$

whose some special solutions are given in [67, §13.5.2–9]. Special cases include the dispersionless CH and DP equations for, respectively,  $\{a = -3; b = 2; c = 1\}$  and  $\{a = -4; b = 3; c = 1\}$ , as well as their one-parameter generalisation (2.24) when  $a = -1 - b$  and  $c = 1$ . These special cases all satisfy the relation  $a + b + c = 0$  that has been investigated in the literature. The  $\text{rB}_b$  family of equations is also a special case of (2.25) when  $c = -a = 1$ , which does not satisfy the relation  $a + b + c = 0$  if  $b \neq 0$ . Degasperis and Procesi [27] found that there are only three equations satisfying the asymptotic integrability condition within the family (2.25): the KdV, CH and DP equations; therefore,  $\text{rB}_2$  does not satisfy the asymptotic integrability condition.

Novikov [63] classified some integrable equations of the form

$$(1 - \ell^2 \partial_x^2) u_t = F(u, u_x, u_{xx}, \dots), \quad (2.26)$$

where  $F$  is a homogenous polynomial, quadratic or cubic, in  $u$  and its  $x$ -derivatives. The  $\text{rB}_b$  equation belongs to the quadratic differential class (2.26) since it can be rewritten

$$(1 - \ell^2 \partial_x^2) u_t = \ell^2 u u_{xxx} + b \ell^2 u_x u_{xx} - u u_x. \quad (2.27)$$

For this quadratic differential class, Novikov [63, eq. (9)] considered the peculiar family (in our notations)

$$\begin{aligned} (1 - \ell^2 \partial_x^2) u_t = & c_1 u u_x + \ell [c_2 u u_{x^2} + c_3 u_x^2] + \ell^2 [c_4 u u_{x^3} + c_5 u_x u_{x^2}] \\ & + \ell^3 [c_6 u u_{x^4} + c_7 u_x u_{x^3} + c_8 u_{x^2}^2] \\ & + \ell^4 [c_9 u u_{x^5} + c_{10} u_x u_{x^4} + c_{11} u_{x^2} u_{x^3}], \end{aligned} \quad (2.28)$$

the  $c_i$  being constants. Equation (2.27) is a special case of (2.28) when  $c_4 = -c_1 = 1$ ,  $c_5 = b$  and  $c_2 = c_3 = c_{i>5} = 0$ . In his Theorem 3, Novikov [63] gives explicitly the ten equations of type (2.28) possessing an infinite hierarchy of quasi-local higher symmetries, under the condition  $c_2 \neq 0$  or  $c_6 \neq 0$  or  $c_9 \neq 0$  or  $c_4 \neq -c_1$ . The  $\text{rB}_b$  equation does not fulfil this condition and it is therefore not part of the list provided by Novikov [63].

Traveling peakons and cuspons for a two-parameter generalisation of CH, including  $\text{rB}_2$  as a special case, has been studied by Li and Qiao [54]. Apparently, the  $\text{rB}_2$  equation has not been studied much further in the litterature. From now on, we consider only the case  $b = 2$  and  $\text{rB}_2$  is replaced by  $\text{rB}$  for the sake of simplicity.

### 2.1.3 Variational formulations

It appeared above that the case  $b = 2$  is of special interest. Indeed, it also implies the existence of a variational principle, as shown here.

The classical inviscid Burgers equation can be obtained as the Euler–Lagrange of the functional  $\mathfrak{J}_0 = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \mathcal{L}_0(\phi) dx dt$  with the Lagrangian density

$$\mathcal{L}_0 \stackrel{\text{def}}{=} \frac{1}{2} \phi_x \phi_t + \frac{1}{6} \phi_x^3, \quad (2.29)$$

where  $\phi$  is a velocity potential, i.e.,  $u = \phi_x$ . Modifying the Lagrangian density (2.29) as

$$\mathcal{L}_\ell \stackrel{\text{def}}{=} \frac{1}{2} \phi_x \phi_t + \frac{1}{6} \phi_x^3 + \frac{1}{2} \ell^2 [\phi_x \phi_{xx}^2 - \phi_{xxx} \phi_t], \quad (2.30)$$

the rB equation is obtained at once as the Euler–Lagrange equation for (2.30).

The rB equation has also a Hamiltonian structure with the Hamiltonian operator and functional

$$\mathcal{D}_1 \stackrel{\text{def}}{=} \mathcal{L} \partial_x = \partial_x - \ell^2 \partial_x^3, \quad (2.31)$$

$$\mathfrak{H}_1 \stackrel{\text{def}}{=} \int \left[ \frac{1}{6} (\mathcal{G}\{m\})^3 + \frac{1}{2} \ell^2 \mathcal{G}\{m\} (\mathcal{G}\{m_x\})^2 \right] dx, \quad (2.32)$$

thence

$$\begin{aligned} m_t &= -\mathcal{D}_1 \delta_m \mathfrak{H}_1 \\ &= -\mathcal{D}_1 \left[ \frac{1}{2} \mathcal{G}\{(\mathcal{G}\{m\})^2\} + \frac{1}{2} \ell^2 \mathcal{G}\{(\mathcal{G}\{m_x\})^2\} - \ell^2 \partial_x \mathcal{G}\{\mathcal{G}\{m\} \mathcal{G}\{m_x\}\} \right] \\ &= -\partial_x \left[ \frac{1}{2} (\mathcal{G}\{m\})^2 + \frac{1}{2} \ell^2 (\mathcal{G}\{m_x\})^2 - \ell^2 \partial_x \{\mathcal{G}\{m\} \mathcal{G}\{m_x\}\} \right] \\ &= -\partial_x \left[ \frac{1}{2} (\mathcal{G}\{m\})^2 - \frac{1}{2} \ell^2 (\mathcal{G}\{m_x\})^2 - \ell^2 \mathcal{G}\{m\} \mathcal{G}\{m_{xx}\} \right], \end{aligned} \quad (2.33)$$

that, substituting  $u$  for  $\mathcal{G}\{m\}$ , is identical to the rB equation in its form (2.8).

The Hamiltonian formulation above is for the momentum  $m$ . We can also look for a Hamiltonian formalism for  $u$ . A Hamiltonian candidate is obtained substituting  $u$  for  $\mathcal{G}\{m\}$  in  $\mathfrak{H}_1$  and

$$\mathcal{D}'_1 \stackrel{\text{def}}{=} \mathcal{G} \partial_x, \quad \mathfrak{H}'_1 \stackrel{\text{def}}{=} \int \left[ \frac{1}{6} u^3 + \frac{1}{2} \ell^2 u u_x^2 \right] dx, \quad (2.34)$$

thence

$$\begin{aligned} u_t &= -\mathcal{D}'_1 \delta_u \mathfrak{H}'_1 = -\mathcal{D}'_1 \left[ \frac{1}{2} u^2 + \frac{1}{2} \ell^2 u_x^2 - \ell^3 \partial_x (u u_x) \right] \\ &= -\mathcal{G} \left\{ \left[ \frac{1}{2} u^2 - \frac{1}{2} \ell^2 u_x^2 - \ell^2 u u_{xx} \right]_x \right\}, \end{aligned} \quad (2.35)$$

that is the rB equation in its form (2.8) after applying the  $\mathcal{G}$  operator.

The rB equation has also (at least) another pseudo-Hamiltonian structures, which can be obtained with the Hamiltonian functional

$$\mathfrak{H}_2 \stackrel{\text{def}}{=} \frac{1}{2} \int \left[ (\mathcal{G}\{m\})^2 + \ell^2 (\mathcal{G}\{m_x\})^2 \right] dx = \frac{1}{2} \int \left[ u^2 + \ell^2 u_x^2 \right] dx, \quad (2.36)$$

and the operator

$$\mathcal{D}_2 \stackrel{\text{def}}{=} \left( m - \frac{2}{3} \mathcal{G}\{m\} \right) \partial_x + \partial_x \left( m - \frac{2}{3} \mathcal{G}\{m\} \right) = \left( m - \frac{2}{3} u \right) \partial_x + \partial_x \left( m - \frac{2}{3} u \right), \quad (2.37)$$

so the equation of motion is given by

$$\begin{aligned} m_t &= -\mathcal{D}_2 \delta_m \mathfrak{H}_2 = -\mathcal{D}_2 \left[ \mathcal{G}\{\mathcal{G}\{m\}\} - \ell^2 \mathcal{G}\{\partial_x \mathcal{G}\{m_x\}\} \right] = \mathcal{D}_2 [\mathcal{G}\{m\}] = -\mathcal{D}_2 [u] \\ &= -\left( m - \frac{2}{3} u \right) u_x - \partial_x \left[ m u - \frac{2}{3} u^2 \right] = -2m u_x - u m_x + 2u u_x, \end{aligned} \quad (2.38)$$

that is the rB equation in its form (2.11), where the operator  $\mathcal{D}_2$  is not an Hamiltonian operator [66].



## 2.2 Traveling waves of permanent form

In this section, we consider traveling waves of permanent form, i.e., we seek for solutions of the form  $u = u(x - ct)$ . Since the rB equation is Galilean invariant, the work is better done in the frame of reference moving with the wave where the motion is steady, i.e., we look for solutions such that  $u = u(x)$ .

The motion being steady, the momentum flux  $S$  and the energy flux  $F$  are constant, the equations (2.8) and (2.10) giving

$$S \stackrel{\text{def}}{=} \frac{1}{2} u^2 - \ell^2 u u_{xx} - \frac{1}{2} \ell^2 u_x^2, \quad (2.39)$$

$$F \stackrel{\text{def}}{=} \frac{1}{3} u^3 - \ell^2 u^2 u_{xx}. \quad (2.40)$$

Eliminating  $u_{xx}$  between (2.39) and (2.40), one obtains the first-order differential equation

$$3\ell^2 u u_x^2 = 6F - 6Su + u^3, \quad (2.41)$$

that can be solved analytically in terms of the Jacobian elliptic functions [1], yielding periodic and aperiodic solutions. In this thesis, we are concerned only by aperiodic (weak) solutions.

### 2.2.1 Local analysis of weak solutions

Consider the possibility of a singularity at  $x = 0$ , that can exist only if  $u(0) = 0$ .

If  $u(0) = 0$  the equation (2.41) yields

$$\ell^2 u u_x^2 \sim 2F \quad \text{as } x \rightarrow 0, \quad F \neq 0, \quad (2.42)$$

$$\ell^2 u_x^2 \sim -2S \quad \text{as } x \rightarrow 0, \quad F = 0, \quad S < 0, \quad (2.43)$$

implying that (as  $x \rightarrow 0$ )

$$|u| \sim \sqrt{|2S|} |x| / \ell \quad \text{if } F = 0 \quad \text{and } S < 0, \quad (2.44)$$

$$|u| \sim \sqrt[3]{2F} (3|x|/2\ell)^{2/3} \quad \text{if } F \neq 0. \quad (2.45)$$

### 2.2.2 Peakon-like solution

With zero energy flux (i.e.,  $F = 0$ ), we have the formal solution

$$u(x) = \begin{cases} \exp(\pm(x - x_0)/\sqrt{3}\ell) & \text{if } S = 0, \\ \sqrt{6S} \cosh((x - x_0)/\sqrt{3}\ell) & \text{if } S > 0, \\ \pm\sqrt{-6S} \sinh((x - x_0)/\sqrt{3}\ell) & \text{if } S < 0, \end{cases} \quad (2.46)$$

where  $x_0$  is an arbitrary phase shift. From these formal solutions we have the peculiar peakon-like wave of the form

$$u(x) = u_0 \exp(-|x|/\sqrt{3}\ell). \quad (2.47)$$

### 2.2.3 Shock wave

Consider now the more realistic case  $u \rightarrow u_{\pm}$  as  $x \rightarrow \pm\infty$  and weak solutions conserving the momentum flux, so  $2S = u_+^2 = u_-^2$  (implying that  $|u_-| = |u_+|$ ). Inspired by Pu et al [68], we look for smooth solutions for  $x \lesssim 0$ , continuous at  $x = 0$  and that do not necessarily conserve the energy, so the energy flux is piecewise constant. For  $x > 0$ , then  $F = F_+ = u_+^3/3$ , the equation (2.41) becomes

$$\ell^2 u_x^2 = \frac{u^3 - 3u_+^2 u + 2u_+^3}{3u} = \frac{(u - u_+)^2 (u + 2u_+)}{3u}. \quad (2.48)$$

For  $x < 0$ , then  $F = F_- = u_-^3/3$ , the equation (2.41) yields (2.48) with  $u_-$  substituted for  $u_+$ .

We consider the case  $u_+ < 0 < u_-$ , which corresponds to an entropic shock for the classical Burgers equation. The equation (2.48) forces  $u$  to be in  $]u_+, 0[$  because  $u \rightarrow u_+$  when  $x \rightarrow +\infty$ . Since  $u$  is a smooth function for  $x > 0$ , then (2.48) implies

$$\ell u_x = -\sqrt{\frac{(u - u_+)^2 (u + 2u_+)}{3u}}. \quad (2.49)$$

Solving the following equation for  $u$  (with  $x > 0$ )

$$\int_0^u \sqrt{\frac{3v}{(v - u_+)^2 (v + 2u_+)}} dv = -\frac{x}{\ell}, \quad (2.50)$$

we get a smooth decreasing function  $\eta_+$  such that  $u(x) = \eta_+(x/\ell)$  and  $\eta_+(0) = 0$ . Performing the same computations for  $x < 0$ , we get a smooth decreasing function  $\eta_-$  such that  $u(x) = \eta_-(x/\ell)$  and  $\eta_-(0) = 0$ . We can check easily that

$$\eta_{\pm}(x) \sim \sqrt[3]{3/2} u_{\pm} |x|^{2/3} \quad \text{as } x \rightarrow 0. \quad (2.51)$$

In summary the solution  $u$  is given by

$$u(x) = \begin{cases} \eta_+(x/\ell) & x \geq 0, \\ \eta_-(x/\ell) & x \leq 0. \end{cases} \quad (2.52)$$

Note that, since the rB equation is Galilean invariant, traveling waves in any frame of reference can be found using the change of variables (Galilei transform)

$$x \mapsto x - ct, \quad u \mapsto u - c. \quad (2.53)$$

Note also that if  $u_+ = u_-$ , the solution can be found with the same method yielding  $\eta_+(x) = \eta_-(-x)$ , so the solution is an even function.

For non smooth solutions of the rB equations, the energy equation (2.10) is no longer satisfied. We show below that for traveling waves corresponds to (2.52), satisfies an energy equation on the form

$$\left[ \frac{1}{2} u^2 + \frac{1}{2} \ell^2 u_x^2 \right]_t + \left[ \frac{1}{3} u^3 - \ell^2 u^2 u_{xx} - \ell^2 u u_{xt} \right]_x = \frac{1}{3} (u_+^3 - u_-^3) \delta_0(x - ct). \quad (2.54)$$

The proof will be done for  $c = 0$ , and the general result will be given by the change of variables (2.53).

For  $u$  given by (2.52), the solution is smooth for  $x \lesssim 0$ , then, it satisfies

$$\frac{1}{3} u^3 - \ell^2 u^2 u_{xx} = \begin{cases} F_+ & x > 0, \\ F_- & x < 0. \end{cases} \quad (2.55)$$

When  $x \rightarrow \pm\infty$ , we find that  $F_{\pm} = \frac{1}{3} u_{\pm}^3$ , then

$$\left[ \frac{1}{3} u^3 - \ell^2 u^2 u_{xx} \right]_x = \frac{1}{3} (u_+^3 - u_-^3) \delta_0. \quad (2.56)$$

**Remark 2.2.1.**

- Notice that  $\frac{1}{3}(u_+^3 - u_-^3) < 0$ , and the loss of the energy does not depend on  $\ell$ , and it is exactly the loss of energy for shocks in the classical Burgers equation.
- In the case  $u_+ = u_-$ , the energy is conserved and (2.10) holds.

Now, solutions in the parametric form  $\{x(\xi), u(\xi)\}$  will be given. Let

$$x(\xi) = \begin{cases} \int_0^{\xi} \sqrt{|u(\eta)/u_+|} \, d\eta, & \xi > 0, \\ \int_0^{\xi} \sqrt{|u(\eta)/u_-|} \, d\eta, & \xi < 0, \end{cases} \quad (2.57)$$

the choice of  $x(\xi)$  is devoted to keep  $x(0) = 0$ ,  $x \sim \xi$  when  $\xi \rightarrow \infty$ .

The equations (2.48) can be rewritten as

$$\ell^2 u_{\pm} u_{\xi}^2 = (2/3) u_{\pm}^3 - u_{\pm}^2 u + \frac{1}{3} u^3, \quad (2.58)$$

which has solutions on the form

$$u(\xi) = \begin{cases} u_+ - \frac{3 u_+}{\cosh^2(\frac{\xi}{2\ell} + \beta)}, & \xi > 0, \\ u_- - \frac{3 u_-}{\cosh^2(\frac{\xi}{2\ell} - \beta)}, & \xi < 0, \end{cases} \quad (2.59)$$

with  $\beta = \ln(\sqrt{2} + \sqrt{3})$ . It is easy to check that if  $\xi \sim 0$ , then

$$u \sim \sqrt{2/3} u_{\pm} |\xi|/\ell, \quad x_{\xi} = \sqrt{u/u_{\pm}} \sim (2/3)^{1/4} \sqrt{|\xi|/\ell}, \quad x \sim \pm (2/3)^{5/4} |\xi|^{3/2} / \sqrt{\ell}, \quad (2.60)$$

which implies

$$u \sim (2/3)^{-1/3} u_{\pm} \left| \frac{x}{\ell} \right|^{2/3},$$

this is exactly the result in (2.45) for  $F = u_{\pm}^3/3$ .

**Remark 2.2.2.** The solutions given by (2.52) can be seen as a regularization of the Riemann problem of the classical Burgers equation, and we can check easily that when  $\ell \rightarrow 0$  this sequence converges to a weak solution of the Burgers equation.

## 2.3 Regularised scalar conservation laws and main results

### 2.3.1 Regularised scalar conservation laws

Inspired by the rB equation introduced above and modifying the Lagrangian density, we can introduce the regularised scalar conservation law

$$u_t + f(u)_x = \ell^2 \left[ u_{xxt} + f'(u) u_{xxx} + 2f''(u) u_x u_{xx} + \frac{1}{2} f'''(u) u_x^3 \right], \quad (2.61)$$

where  $f$  is the flux. The equation (2.61) can be written in the form

$$u_t + \left[ f(u) + \frac{1}{2} \ell^2 \mathfrak{G} * f''(u) u_x^2 \right]_x = 0, \quad \mathfrak{G} \stackrel{\text{def}}{=} (2\ell)^{-1} \exp(-|x|/\ell), \quad (2.62)$$

and satisfies the energy conservation

$$\left[ \frac{1}{2} u^2 + \frac{1}{2} \ell^2 u_x^2 \right]_t + \left[ K(u) + \frac{1}{2} \ell^2 f'(u) u_x^2 + \ell^2 u P \right]_x = 0, \quad (2.63)$$

with  $K'(u) = u f'(u)$  and

$$P = \frac{1}{2} \mathfrak{G} * f''(u) u_x^2. \quad (2.64)$$

### 2.3.2 Local well-posedness and breaking-down

Using Kato's theorem [45], we prove the following theorem:

**Theorem 2.3.1.** [*Local existence of smooth solutions*] *Let  $f \in \mathcal{C}^{m+3}(\mathbb{R})$ ,  $m \geq 2$  and let  $u_0 \in H^s(\mathbb{R})$  with  $3/2 < s \leq m$ , then there exist a maximal time  $T > 0$  that does not depend on  $s$  and a unique solution  $u$  of (2.62) that depends continuously on  $u_0$ , such that  $u \in \mathcal{C}([0, T], H^s(\mathbb{R})) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}))$ . Moreover, if  $T < +\infty$ , then  $\limsup_{t \rightarrow T} \|u(t, \cdot)\|_{H^s} = +\infty$ .*

Differentiating (2.62) with respect to  $x$ , we obtain

$$u_{xt} + f'(u) u_{xx} + \frac{1}{2} f''(u) u_x^2 + P = 0. \quad (2.65)$$

Following the characteristics with the velocity  $f'(u)$ , we obtain a Riccati-type equation. For uniformly convex fluxes ( $f''(u) \geq C > 0$ ), the Riccati-type equation can be used to prove that non-trivial solutions of (2.62) develop singularities in finite time.

**Theorem 2.3.2.** *Let  $f \in \mathcal{C}^{m+3}(\mathbb{R})$  such that  $\tilde{C} \geq f''(u) \geq C > 0$ , let also be  $u_0 \in H^s(\mathbb{R})$  with  $2 \leq s \leq m$ . If  $u_0$  is not the zero function, then*

$$1/(\tilde{C} \sup |u'_0|) \leq T \leq -1/(C \inf u'_0). \quad (2.66)$$

The Riccati-type equation implies also that before the breaking-down, the solution satisfies the Oleinik inequality

$$u_x(t, x) \leq \frac{1}{Ct/2 + 1/M} \leq M, \quad (2.67)$$

where  $M \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} u'_0(x)$ .

### 2.3.3 Global existence of conservative and dissipative solutions

Since the  $H^s$  norm blows-up in finite time, for  $s \geq 2$ . We use the energy conservation (2.63) to obtain global solutions of (2.62) in  $H^1$ . For that purpose, and following [12], we define the independent variable  $\xi \in \mathbb{R}$ , the characteristics  $y$  with the velocity  $f'(u)$  and

$$v \stackrel{\text{def}}{=} 2 \arctan(u_x), \quad q \stackrel{\text{def}}{=} (1 + u_x^2) y_\xi. \quad (2.68)$$

The equation (2.62) can be transformed to the equivalent semi-linear system

$$y_t = f'(u), \quad (2.69a)$$

$$u_t = -\ell^2 P_x, \quad (2.69b)$$

$$v_t = -P(1 + \cos(v)) - f''(u) \sin^2(v/2), \quad (2.69c)$$

$$q_t = q \left( \frac{f''(u)}{2} - P \right) \sin(v). \quad (2.69d)$$

The latter system can be used to obtain global solutions of (2.62):

**Theorem 2.3.3. [Global existence of conservative solutions]** *Let  $u_0 \in H^1(\mathbb{R})$  and  $f \in \mathcal{C}^3(\mathbb{R})$ , then there exists a global weak solution  $u$  of the equation (2.62), such that  $\forall T > 0$ ,  $u \in \text{Lip}([0, T], L^2(\mathbb{R}))$  and*

$$\int_{\mathbb{R}} (u^2(t) + \ell^2 u_x(t)^2) dx = \int_{\mathbb{R}} (u_0^2 + \ell^2 u_0'^2) dx, \quad \text{for a.e. } t \in \mathbb{R}. \quad (2.70)$$

*This solution  $u$  is called a conservative solution. Moreover, if  $\|u_{0,n} - u_0\|_{H^1} \rightarrow 0$ . Then,  $u_n$  converges uniformly to  $u$ , for all  $t, x$  in any bounded set.*

Note that  $v_0 = 2 \arctan(u_0') \in ]-\pi, \pi[$ , but for uniformly convex fluxes ( $f''(u) \geq C > 0$ ), the equation (2.69c) implies that the value of  $v$  can cross  $-\pi$ . In that case, and since  $u_x = \tan(v/2)$ , the value of  $u_x$  can jump from  $-\infty$  to  $+\infty$ , which yields to the loss of the Oleinik inequality (2.67). In order to obtain solutions that satisfy the Oleinik inequality (2.67) for all time, following [13], the semi-linear system (2.69) is replaced by

$$y_t = f'(u), \quad (2.71a)$$

$$u_t = -\ell^2 P_x, \quad (2.71b)$$

$$v_t = \begin{cases} -P(1 + \cos v) - f''(u) \sin^2(v/2), & v > -\pi, \\ 0, & v \leq -\pi, \end{cases} \quad (2.71c)$$

$$q_t = \begin{cases} q \left( \frac{f''(u)}{2} - P \right) \sin(v), & v > -\pi \\ 0, & v \leq -\pi. \end{cases} \quad (2.71d)$$

The system (2.71) is used to obtain the global existence of weak dissipative solutions of (2.62):

**Theorem 2.3.4.** [*Global existence of dissipative solutions*] Let  $u_0 \in H^1(\mathbb{R})$  and  $f \in \mathcal{C}^3(\mathbb{R})$  such that  $f''(u) \geq C > 0$ , then there exists a global weak solution  $u \in \text{Lip}([0, T], L^2(\mathbb{R}))$  of the equation (2.62), satisfying

$$\int_{\mathbb{R}} (u^2(t) + \ell^2 u_x(t)^2) dx \leq \int_{\mathbb{R}} (u_0^2 + \ell^2 u_0'^2) dx, \quad \forall t \in \mathbb{R}, \quad (2.72)$$

and, for  $M = \sup_{x \in \mathbb{R}} u_0'(x) \in \mathbb{R}^+ \cup \{+\infty\}$  we have

$$u_x(t, x) \leq \frac{1}{Ct/2 + 1/M} \quad \forall t > 0. \quad (2.73)$$

This solution is called a dissipative solution.

### 2.3.4 On a generalised Hunter–Saxton equation

Taking “formally”  $\ell \rightarrow \infty$ , in (2.62), we obtain the generalised Hunter–Saxton equation

$$u_t + f(u)_x = \frac{1}{4} \left( \int_{-\infty}^x - \int_x^{+\infty} \right) u_x^2 f''(u) dx. \quad (2.74)$$

Differentiating (2.74) w.r.t  $x$  one obtains

$$[u_t + f(u)_x]_x = \frac{1}{2} u_x^2 f''(u), \quad (2.75)$$

multiplying by  $u$ , we obtain the conservation of the energy

$$[u_x^2]_t + [f'(u) u_x^2]_x = 0. \quad (2.76)$$

Following [12, 13] (as in Section 2.3.3), we prove the global existence of conservative and dissipative solutions of (2.74):

**Theorem 2.3.5.** Let  $u_0 \in \dot{H}^1 \cap L^\infty$  and  $f \in \mathcal{C}^3(\mathbb{R})$  such that  $f^{(3)}$  is bounded, then

- There exists a global weak solution  $u$  (called conservative) of (2.74), satisfying  $u \in \text{Lip}([0, T], L_{loc}^2(\mathbb{R}))$  for all  $T > 0$  and

$$\int_{\mathbb{R}} u_x(t)^2 dx = \int_{\mathbb{R}} u_0'^2 dx \quad \text{for almost all } t \in \mathbb{R}. \quad (2.77)$$

- If the flux is uniformly convex  $f''(u) \geq C > 0$ , then there also exists a global weak solution  $u$  (called dissipative) of the equation (2.74), satisfying  $u \in \text{Lip}([0, T], L_{loc}^2(\mathbb{R}))$  for all  $T > 0$  and

$$\int_{\mathbb{R}} u_x(t)^2 dx \leq \int_{\mathbb{R}} u_0'^2 dx \quad \text{for almost all } t \in \mathbb{R}. \quad (2.78)$$

Moreover, for  $M = \sup_{x \in \mathbb{R}} u_0'(x) \in \mathbb{R}^+ \cup \{+\infty\}$  the dissipative solution satisfies the Oleinik inequality

$$u_x(t, x) \leq \frac{1}{Ct/2 + 1/M} \quad \forall t > 0. \quad (2.79)$$

### 2.3.5 The limiting cases $\ell \rightarrow 0$ and $\ell \rightarrow \infty$ for dissipative solutions

In order to study the limiting cases  $\ell \rightarrow 0$  and  $\ell \rightarrow \infty$ , we use that the dissipative solutions satisfy (uniformly on  $\ell$ ) the Oleinik inequality (2.73). This inequality is an important tool to prove that if  $\tilde{C} \geq f''(u) \geq C > 0$ ,  $u'_0 \in L^1(\mathbb{R})$  and  $u'_0(x) \leq M < +\infty$ , then

$$\text{TV } u(t) = \|u_x(t)\|_{L^1} \leq \|u'_0\|_{L^1} (C M t/2 + 1)^{2\tilde{C}/C}. \quad (2.80)$$

This total variation estimate can be used to show that for all  $T > 0$ ,  $\mathcal{I} \Subset \mathbb{R}$ , there exists a sub-sequence of  $u$  that converges to  $u^0 \in L^\infty([0, T], L^1(\mathcal{I})) \cap L^\infty([0, T], BV(\mathbb{R}))$  (respectively  $u^\infty \in \mathcal{C}([0, T], L^1(\mathcal{I})) \cap L^\infty([0, T], BV(\mathbb{R}))$ ) when  $\ell \rightarrow 0$  (respectively  $\ell \rightarrow \infty$ ). The functions  $u^0$  and  $u^\infty$  satisfy the Oleinik inequality (2.73) in  $\mathcal{D}'(\mathbb{R})$  and

$$u_t^0 + f(u^0)_x = -\mu_x, \quad [u_t^\infty + f(u^\infty)_x]_x = \nu, \quad (2.81)$$

where  $\mu, \nu \in L^\infty([0, T], \mathcal{M}^1)$  are non-negative Radon measures. At least before the appearance of the singularities,  $\mu = 0$  and  $\nu = (u_x^\infty)^2 f''(u^\infty)/2$  — i.e.,  $u^0$  is the smooth solution of the scalar conservation law and  $u^\infty$  is a solution of the generalised Hunter–Saxton equation (2.75).

### 2.3.6 Optimality of the space $\dot{H}_{loc}^1$

As mentioned above, the space  $H^s$  with  $s > 3/2$  is not a good space to obtain global (in time) solutions. Due to the energy equation (2.63), the space  $H^1$  is a good space. We prove for quadratic fluxes that the derivative  $u_x$  cannot stay for all time in any space on the form  $\{v, \int v^2 |\log |v||^\varepsilon dx < +\infty\}$ . In other words:

**Theorem 2.3.6.** *Let  $f(u) = u^2/2$ ,  $\varepsilon > 0$  and  $g(v) = |\log |v||^\varepsilon$ , then there exist  $u_0 \in H^1 \cap W^{1, \infty}$ ,  $T > 0$  and a compact set  $\mathcal{K}$ , such that there exists a solution  $u$  of (2.63) satisfying*

$$\int_{\mathbb{R}} u'_0(x)^2 g(u'_0(x)) dx < +\infty, \quad \int_{\mathcal{K}} u_x(T, x)^2 g(u_x(T, x)) dx = +\infty. \quad (2.82)$$

# Chapter 3

## Hamiltonian regularisations some hyperbolic systems

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This chapter is devoted to generalise the regularised Saint-Venant equations (1.5) in two directions: (i) regularising the barotropic Euler system as (1.11); (ii) obtaining the regularised shallow water system with uneven bottom (1.15).

This chapter is based on the papers [33, 34] that are attached in Appendices C and D, respectively.

## 3.1 A Hamiltonian regularisations of barotropic Euler

The classical barotropic Euler system can be obtained from deriving the Lagrangian density

$$\mathcal{L}_0 \stackrel{\text{def}}{=} \frac{1}{2} \rho u^2 - \mathcal{V}(\rho) + \{ \rho_t + [\rho u]_x \} \phi, \quad (3.1)$$

which is the difference between the kinetic and the potential energies plus a constraint that is added to obtain the conservation of the mass. The classical barotropic Euler system conserves “formally” the  $L^2$ -like energy

$$\left[ \frac{1}{2} \rho u^2 + \mathcal{V} \right]_t + \left[ \left( \frac{1}{2} \rho u^2 + \rho \mathcal{V}' \right) u \right]_x = 0. \quad (3.2)$$

The  $L^2$  norm is not enough to ensure the continuity of the solutions. In order to obtain a regularisation of the barotropic Euler system that admits continuous solutions, we modify the Lagrangian density (3.2) as

$$\mathcal{L}_\epsilon \stackrel{\text{def}}{=} \frac{1}{2} \rho u^2 + \epsilon \mathcal{A}' \rho u_x^2 - \mathcal{V} - \epsilon \mathcal{B}' \mathcal{V}'' \rho_x^2 + \{ \rho_t + [\rho u]_x \} \phi, \quad (3.3)$$

where  $\mathcal{A}$  and  $\mathcal{B}$  are smooth functions of  $\rho$ . Note that the total energy of the regularised equation is

$$\mathcal{H}_\epsilon \stackrel{\text{def}}{=} \frac{1}{2} \rho u^2 + \epsilon \mathcal{A}' \rho u_x^2 + \mathcal{V} + \epsilon \mathcal{B}' \mathcal{V}'' \rho_x^2, \quad (3.4)$$

which is equivalent to the  $H^1$  norm if  $P', \mathcal{A}', \mathcal{B}', \rho > 0$ . It turns out that, the regularised equation is dispersionless if and only if (see Section C.3.2)  $\mathcal{A}(\rho) = \mathcal{B}(\rho)$ . From now on, we only consider the non dispersive case.

### 3.1.1 Equations of motion

The Euler–Lagrange equations for the Lagrangian density (3.3) yield

$$\delta \phi: \quad 0 = \rho_t + [\rho u]_x, \quad (3.5)$$

$$\delta u: \quad 0 = \rho u - 2\epsilon [\mathcal{A}' \rho u_x]_x - \rho \phi_x, \quad (3.6)$$

$$\begin{aligned} \delta \rho: \quad 0 = & \frac{1}{2} u^2 + \epsilon (\mathcal{A}' + \mathcal{A}'' \rho) u_x^2 - \mathcal{V}' + \epsilon (\mathcal{A}'' \mathcal{V}'' + \mathcal{A}' \mathcal{V}''') \rho_x^2 \\ & + 2\epsilon \mathcal{A}' \mathcal{V}'' \rho_{xx} - \phi_t - u \phi_x, \end{aligned} \quad (3.7)$$

thence

$$\phi_x = u - 2\epsilon \rho^{-1} [\mathcal{A}' \rho u_x]_x, \quad (3.8)$$

$$\begin{aligned} \phi_t = & -\frac{1}{2} u^2 + \epsilon (\mathcal{A}' + \mathcal{A}'' \rho) u_x^2 - \mathcal{V}' + \epsilon (\mathcal{A}'' \mathcal{V}'' + \mathcal{A}' \mathcal{V}''') \rho_x^2 \\ & + 2\epsilon \mathcal{A}' \mathcal{V}'' \rho_{xx} + 2\epsilon u \rho^{-1} [\mathcal{A}' \rho u_x]_x, \end{aligned} \quad (3.9)$$

which yield to the equations

$$u_t + u u_x + \varpi_x + \epsilon \rho^{-1} \mathcal{R}_x = 0, \quad (3.10)$$

$$[\rho u]_t + [\rho u^2 + \rho \mathcal{V}' - \mathcal{V} + \epsilon \mathcal{R}]_x = 0, \quad (3.11)$$

$$m_t + \left[ u m + \rho \mathcal{V}' - \mathcal{V} - \epsilon (\rho^2 \mathcal{A}')' u_x^2 - 2 \epsilon \rho \mathcal{A}' \varpi_{xx} + \epsilon (\rho \mathcal{V}'' / \mathcal{A}')' \mathcal{A}_x^2 \right]_x = 0, \quad (3.12)$$

$$\left[ \frac{1}{2} \rho u^2 + \epsilon \rho \mathcal{A}' u_x^2 + \mathcal{V} + \epsilon \mathcal{A}' \mathcal{V}'' \rho_x^2 \right]_t + \left[ \left( \frac{1}{2} \rho u^2 + \rho \mathcal{V}' + \epsilon \rho \mathcal{A}' u_x^2 + \epsilon \mathcal{A}' \mathcal{V}'' \rho_x^2 + \epsilon \mathcal{R} \right) u + 2 \epsilon \rho \mathcal{A}' \mathcal{V}'' \rho_x u_x \right]_x = 0, \quad (3.13)$$

where  $\varpi'(\rho) = P'(\rho)/\rho$  and

$$\mathcal{R} \stackrel{\text{def}}{=} (\rho^2 \mathcal{A}')' u_x^2 - 2 \rho \mathcal{A}' [u_t + u u_x + \varpi_x]_x + (\rho \mathcal{V}'' / \mathcal{A}')' \mathcal{A}_x^2, \quad (3.14)$$

$$m \stackrel{\text{def}}{=} \rho u - 2 \epsilon [\rho \mathcal{A}' u_x]_x. \quad (3.15)$$

Introducing the linear Sturm–Liouville operator  $\mathcal{L}_\rho \stackrel{\text{def}}{=} \rho - 2 \epsilon \partial_x \rho \mathcal{A}' \partial_x$ , if  $\rho, \mathcal{A}' > 0$  the operator  $\mathcal{L}_\rho$  is invertible and the equation (3.10) can be replaced by

$$u_t + u u_x + \varpi_x = - \epsilon \mathcal{L}_\rho^{-1} \partial_x \left\{ (\rho^2 \mathcal{A}')' u_x^2 + (\rho \mathcal{V}'' / \mathcal{A}')' \mathcal{A}_x^2 \right\}. \quad (3.16)$$

### 3.1.2 Hamiltonian formulation

Let be the momentum  $m \stackrel{\text{def}}{=} \mathcal{E}_u \{ \mathcal{H}_\epsilon \} = \rho u - 2 \epsilon [\rho \mathcal{A}' u_x]_x$  where  $\mathcal{E}_u$  is the Euler–Lagrange operator with respect of the variable  $u$ .  $m$  and  $u$  are related via the, linear non-autonomous self-adjoint definite-positive (because  $\rho \geq 0$  and  $\epsilon \geq 0$ ), Sturm–Liouville operator  $\mathcal{L}_\rho \stackrel{\text{def}}{=} \rho - 2 \epsilon \partial_x \rho \mathcal{A}' \partial_x$ , i.e.,  $m = \mathcal{L}_\rho \{ u \}$  that can be inverted as  $u = \mathcal{G}_\rho \{ m \}$  with  $\mathcal{G}_\rho \stackrel{\text{def}}{=} \mathcal{L}_\rho^{-1}$ . Note that, integrating by parts, the density kinetic energy can be rewritten

$$\mathcal{H}_\epsilon = \frac{1}{2} u \mathcal{L}_\rho \{ u \} + \epsilon u \partial_x \rho \mathcal{A}' u_x + \epsilon \mathcal{A}' \rho u_x^2 = \frac{1}{2} m \mathcal{G}_\rho \{ m \} + \text{B.T.}, \quad (3.17)$$

where ‘B.T.’ means ‘boundary terms’ (i.e., terms of the form  $[\dots]_t + [\dots]_x$  whose exact expressions do not matter).

Expressing the Hamiltonian functional density (3.4) as function of  $\rho$  and  $m$ , i.e.

$$\mathcal{H}_\epsilon = \frac{1}{2} m \mathcal{G}_\rho \{ m \} + \mathcal{V} + \epsilon \mathcal{A}' \mathcal{V}'' \rho_x^2 + \text{B.T.}, \quad (3.18)$$

we have

$$\mathcal{E}_m \{ \mathcal{H}_\epsilon \} = \mathcal{G}_\rho \{ m \} = u, \quad (3.19)$$

$$\mathcal{E}_\rho \{ \mathcal{H}_\epsilon \} = \mathcal{V}' - \epsilon (\mathcal{A}' \mathcal{V}'' )' \rho_x^2 - 2 \epsilon \mathcal{A}' \mathcal{V}'' \rho_{xx} - \frac{1}{2} u^2 - \epsilon (\rho \mathcal{A}')' u_x^2. \quad (3.20)$$

The derivation of (3.19) is straightforward because  $\mathcal{G}_\rho$  is self-adjoint, but the derivation of (3.20) is more involved. The latter is obtained exploiting the relations

$$\begin{aligned} \mathcal{L}_{\rho+\delta\rho} &= \rho + \delta\rho - 2 \epsilon \partial_x (\rho + \delta\rho) \mathcal{A}' (\rho + \delta\rho) \partial_x \\ &= \mathcal{L}_\rho + \delta\rho - 2 \epsilon \partial_x (\rho \mathcal{A}'' + \mathcal{A}') \delta\rho \partial_x + \text{O}((\delta\rho)^2) \\ &= \mathcal{L}_\rho [1 + \mathcal{G}_\rho \delta\rho - 2 \epsilon \mathcal{G}_\rho \partial_x (\rho \mathcal{A}'' + \mathcal{A}') \delta\rho \partial_x] + \text{O}((\delta\rho)^2), \end{aligned}$$

thence, inverting this relation,

$$\begin{aligned}\mathcal{G}_{\rho+\delta\rho} &= [1 + \mathcal{G}_\rho \delta\rho - 2\epsilon \mathcal{G}_\rho \partial_x (\rho \mathcal{A}'' + \mathcal{A}') \delta\rho \partial_x]^{-1} \mathcal{G}_\rho + O((\delta\rho)^2) \\ &= \mathcal{G}_\rho - \mathcal{G}_\rho \delta\rho \mathcal{G}_\rho + 2\epsilon \mathcal{G}_\rho \partial_x (\rho \mathcal{A}'' + \mathcal{A}') \delta\rho \partial_x \mathcal{G}_\rho + O((\delta\rho)^2).\end{aligned}$$

Thus, for the kinetic energy functional  $\mathfrak{K}[\rho, m] = \int \frac{1}{2} m \mathcal{G}_h \{m\} dx$  and with  $\delta_\rho \mathfrak{K} \stackrel{\text{def}}{=} \mathfrak{K}[\rho + \delta\rho, m] - \mathfrak{K}[\rho, m]$ , we have

$$\begin{aligned}\delta_\rho \mathfrak{K} &= -\frac{1}{2} \int m \mathcal{G}_\rho \{ \delta\rho u \} dx + \epsilon \int m \mathcal{G}_\rho \partial_x \{ (\rho \mathcal{A}'' + \mathcal{A}') \delta\rho u_x \} dx + O((\delta\rho)^2) \\ &= -\frac{1}{2} \int u \delta\rho u dx - \epsilon \int u_x (\rho \mathcal{A}'' + \mathcal{A}') \delta\rho u_x dx,\end{aligned}\tag{3.21}$$

where we have exploited the self- and anti- adjointness of, respectively,  $\mathcal{G}_\rho$  and  $\mathcal{G}_\rho \partial_x$ . It follows immediately that  $\mathcal{E}_\rho \{ \mathcal{K} \} = -\frac{1}{2} u^2 - \epsilon (\rho \mathcal{A}'' + \mathcal{A}') u_x^2$  and (3.20) is subsequently obtained at once.

The Hamiltonian structure is then

$$\partial_t \begin{pmatrix} \rho \\ m \end{pmatrix} = -\mathbb{J} \cdot \begin{pmatrix} \mathcal{E}_\rho \{ \mathcal{H}_\epsilon \} \\ \mathcal{E}_m \{ \mathcal{H}_\epsilon \} \end{pmatrix} = - \begin{bmatrix} 0 & \partial_x \rho \\ \rho \partial_x & m \partial_x + \partial_x m \end{bmatrix} \cdot \begin{pmatrix} \mathcal{E}_\rho \{ \mathcal{H}_\epsilon \} \\ \mathcal{E}_m \{ \mathcal{H}_\epsilon \} \end{pmatrix},\tag{3.22}$$

yielding the equations (3.5) and (3.12). It should be noted that  $\mathbb{J}$  being skew-symmetric and satisfying the Jacobi identity [64], it is a proper Hamiltonian (Lie–Poisson) operator.

### 3.1.3 Steady motions

We seek in this section for travelling waves moving with a constant speed. Due to the Galilean invariance of rbE, it suffices to look for solutions independent of the time  $t$ . The mass conversation then yields

$$u = I / \rho,\tag{3.23}$$

where  $I$  is an integration constant. From (3.11) and (3.13), the mean (constant) momentum and energy fluxes are respectively

$$S = \rho u^2 + \rho \mathcal{V}' - \mathcal{V} + \epsilon \mathcal{R},\tag{3.24}$$

$$F = \left( \frac{1}{2} \rho u^2 + \rho \mathcal{V}' + \epsilon \rho \mathcal{A}' u_x^2 + \epsilon \mathcal{A}' \mathcal{V}'' \rho_x^2 + \epsilon \mathcal{R} \right) u + 2\epsilon \rho \mathcal{A}' \mathcal{V}'' \rho_x u_x,\tag{3.25}$$

thence — eliminating  $\mathcal{R}$  and using (3.23) — the ordinary differential equation

$$\frac{2\epsilon \mathcal{A}'}{\rho^2} \left( \frac{d\rho}{dx} \right)^2 = \frac{I^2 - 2S\rho + 2(F/I)\rho^2 - 2\rho \mathcal{V}}{I^2 - \rho^3 \mathcal{V}''}.\tag{3.26}$$

Considering equilibrium states in the far field — i.e.,  $\rho \rightarrow \rho_\pm$  and  $u \rightarrow u_\pm$  as  $x \rightarrow \pm\infty$ ,  $\rho_\pm$  and  $u_\pm$  being constants — we have  $\mathcal{R} \rightarrow 0$  and the fluxes in the far field are

$$I_\pm \stackrel{\text{def}}{=} \rho_\pm u_\pm, \quad S_\pm \stackrel{\text{def}}{=} \rho_\pm u_\pm^2 + \rho_\pm \mathcal{V}'_\pm - \mathcal{V}_\pm, \quad F_\pm \stackrel{\text{def}}{=} \frac{1}{2} \rho_\pm u_\pm^3 + \rho_\pm \mathcal{V}'_\pm u_\pm.\tag{3.27}$$

For regular solutions, the fluxes of mass, momentum and energy are constants, so  $I_+ = I_- = I$ ,  $S_+ = S_- = S$  and  $F_+ = F_- = F$ . For weak solutions, however, we assume that only the mass and momentum are conserved (i.e.,  $I_+ = I_- = I$  and  $S_+ = S_- = S$ ), some energy being lost at the singularity (shock) so  $F_+ \neq F_-$ .

Let assume that we have a (weak) steady solution with far field conditions (3.27) and with, possibly, only one singularity at  $x = 0$  where the density is assumed on the form

$$\rho = \bar{\rho} + \varrho_{\pm} |x|^{\alpha} + o(|x|^{\alpha}), \quad (3.28)$$

where  $\alpha > 0$  is a constant to be found. The plus and minus subscripts in  $\varrho$  denote  $x > 0$  and  $x < 0$ , respectively. We show that (see Section C.3.8) the only possibility is  $\alpha = 2/3$ . This is similar to the weak steady solution of the regularised Burgers equation given in Section 2.2.

### 3.1.4 Local well-posedness of the regularised barotropic Euler system

We assume in this section that for  $\rho > 0$  we have  $\mathcal{A}'(\rho) > 0$  and  $P(\rho) > 0$ . In order to obtain the local well-posedness of rbE, we first prove that the operator  $\mathcal{L}_{\rho}$  is bijective from  $H^2$  to  $L^2$  and both of  $\mathcal{L}_{\rho}^{-1}$ ,  $\mathcal{L}_{\rho}^{-1} \partial_x$  are continuous from  $H^s$  to  $H^{s+1}$  for  $s \geq 0$ . Then, we write the system (3.5), (3.16) in the form

$$W_t + B(W)W_x = F(W), \quad W(0, x) = W_0(x), \quad (3.29)$$

where  $W \stackrel{\text{def}}{=}} (\tilde{\rho}, u)^T$  and

$$B(W) \stackrel{\text{def}}{=} \begin{pmatrix} u & \rho \\ \varpi' & u \end{pmatrix}, \quad F(W) \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ -\epsilon \mathcal{L}_{\rho}^{-1} \partial_x \{ (\rho^2 \mathcal{A}')' u_x^2 + (\rho \mathcal{V}'' / \mathcal{A}')' \mathcal{A}'^2 \rho_x^2 \} \end{pmatrix}.$$

Then, an iteration scheme is used by solving the linear symmetrisable hyperbolic system

$$\partial_t W^{n+1} + B(W^n) \partial_x W^{n+1} = F(W^n), \quad W^n(0, x) = (\tilde{\rho}_0(x), u_0(x))^T. \quad (3.30)$$

Using some classical energy estimates, one shows that for  $s \geq 2$ , the quantity  $\|W^n\|_{H^s}$  is uniformly bounded in  $L^{\infty}([0, T], H^s)$  and  $\rho^n \geq \rho^*$  for all  $n \in \mathbb{N}$ . Taking the limit  $n \rightarrow +\infty$ , we obtain:

**Theorem 3.1.1.** *Let  $\tilde{m} \geq s \geq 2$ ,  $\tilde{m}$  be an integer,  $P, \mathcal{A} \in C^{\tilde{m}+4}([0, +\infty[)$  such that  $P'(\rho) > 0$ ,  $\mathcal{A}'(\rho) > 0$  for  $\rho > 0$ . Let also  $W_0 = (\tilde{\rho}_0, u_0)^T \in H^s$  satisfying  $\inf_{x \in \mathbb{R}} \rho_0(x) > \rho^*$ , then there exist  $T > 0$  and a unique solution  $W \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$  of (3.5), (3.16) satisfying the non-emptiness condition  $\inf_{x \in \mathbb{R}} \rho(t, x) > 0$ , and the conservation of the energy*

$$\frac{d}{dt} \int_{\mathbb{R}} \left( \frac{1}{2} \rho u^2 + \epsilon \rho \mathcal{A}' u_x^2 + \mathcal{V} + \epsilon \mathcal{A}' \mathcal{V}'' \rho_x^2 \right) dx = 0. \quad (3.31)$$

Moreover, if the maximal existence time  $T_{max} < +\infty$ , then

$$\lim_{t \rightarrow T_{max}} \|W_x\|_{L^{\infty}} = +\infty. \quad (3.32)$$

## 3.2 On a generalised Hunter–Saxton system

At high frequencies, the operator  $\partial_x \mathcal{L}_\rho^{-1} \partial_x$  behaves like  $-\epsilon^{-1} (\rho \mathcal{A}')^{-1}/2$ . Differentiating with respect of  $x$  the equation (3.16) and considering the high-frequency approximation, the rbE system becomes

$$\rho_t + [\rho u]_x = 0, \quad (3.33)$$

$$[u_t + u u_x + \varpi_x]_x = \left(1 + \frac{\rho \mathcal{A}''}{2 \mathcal{A}'}\right) u_x^2 + \left(\frac{(\rho \mathcal{V}'')'}{2 \rho} - \frac{\mathcal{V}'' \mathcal{A}''}{2 \mathcal{A}'}\right) \rho_x^2, \quad (3.34)$$

that is a two-component generalisation of the Hunter–Saxton equation. Smooth solutions of (3.33), (3.34) satisfy the energy equation

$$[\rho \mathcal{A}' u_x^2 + \mathcal{A}' \mathcal{V}'' \rho_x^2]_t + [(\rho \mathcal{A}' u_x^2 + \mathcal{A}' \mathcal{V}'' \rho_x^2)u + 2 \rho \mathcal{A}' \mathcal{V}'' \rho_x u_x]_x = 0. \quad (3.35)$$

Integrating (3.34) with respect to  $x$ , we obtain

$$\rho_t + [\rho u]_x = 0, \quad (3.36)$$

$$u_t + u u_x + \varpi_x = \partial_x^{-1} \left\{ \left(1 + \frac{\rho \mathcal{A}''}{2 \mathcal{A}'}\right) u_x^2 + \left(\frac{(\rho \mathcal{V}'')'}{2 \rho} - \frac{\mathcal{V}'' \mathcal{A}''}{2 \mathcal{A}'}\right) \rho_x^2 \right\} + g(t), \quad (3.37)$$

where  $(\partial_x^{-1} f)(x) \stackrel{\text{def}}{=} \int_0^x f(y) dy$  and  $g(t) = u_t(t, 0) + u(t, 0)u_x(t, 0) + \varpi'(\rho(t, 0))\rho_x(t, 0)$ . Following the previous section, we prove

**Theorem 3.2.1.** *Let  $\tilde{m} \geq s \geq 2$ ,  $P, \mathcal{A} \in C^{\tilde{m}+4}([0, +\infty[)$  such that  $P'(\rho) > 0$ ,  $\mathcal{A}'(\rho) > 0$  for  $\rho > 0$ . Let also  $W_0 \in H^s([0, 1])$  be a periodic initial data satisfying  $\inf_{x \in [0, 1]} \rho_0(x) > \rho^*$  and  $g \in C([0, +\infty[)$ , then there exist  $T > 0$  and a unique periodic solution  $W \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$  of (3.36), (3.37) satisfying the non-emptiness condition  $\inf_{x \in [0, 1]} \rho(t, x) > 0$  and the conservation of the energy*

$$\frac{d}{dt} \int_0^1 (\rho \mathcal{A}' u_x^2 + \mathcal{A}' \mathcal{V}'' \rho_x^2) dx = 0. \quad (3.38)$$

Moreover, the maximal existence time  $T_{max} < +\infty$ , then

$$\lim_{t \rightarrow T_{max}} \|W_x\|_{L^\infty} = +\infty. \quad (3.39)$$

## 3.3 A Hamiltonian regularisations of Saint-Venant systems with uneven bottom

The regularised Saint-Venant system (1.5) has been generalised in [22] to regularise the Saint-Venant system with uneven bottom. Inspired by Section 3.1, we generalise the system introduced in [22] and we obtain

$$h_t + (h u)_x = 0, \quad (3.40)$$

$$[h u]_t + \left[ h u^2 + \frac{1}{2} g h^2 + \epsilon \mathcal{R} \right]_x = 2 \epsilon g \mathcal{A}' \eta_x d_{xx} + g h d_x, \quad (3.41)$$

where  $d$  is the depth and

$$\mathcal{R} \stackrel{\text{def}}{=} (h^2 \mathcal{A}')' u_x^2 - 2h \mathcal{A}' [u_t + u u_x + g \eta_x]_x - g (h \mathcal{A}'' - \mathcal{A}') (\eta_x^2 + 2\eta_x d_x).$$

Smooth solutions satisfy the energy equation

$$\begin{aligned} & \left[ \frac{1}{2} h u^2 + \epsilon h \mathcal{A}' u_x^2 + \frac{1}{2} g \eta^2 + \epsilon g \mathcal{A}' \eta_x^2 \right]_t \\ & + \left[ \left( \frac{1}{2} h u^2 + g h \eta + \epsilon h \mathcal{A}' u_x^2 + \epsilon g \mathcal{A}' \eta_x^2 + \epsilon \mathcal{R} \right) u + 2\epsilon g h \mathcal{A}' \eta_x u_x \right]_x \\ & = \frac{1}{2} \dot{g} (\eta^2 + 2\epsilon \mathcal{A}' \eta_x^2) - g \eta d_t - 2g \mathcal{A}' \eta_x d_{xt}, \end{aligned} \quad (3.42)$$

where  $\dot{g} = dg/dt$ . Using  $\mathcal{L}_h \stackrel{\text{def}}{=} h - 2\epsilon \partial_x h \mathcal{A}' \partial_x$ , we obtain

$$\begin{aligned} u_t + u u_x + g \eta_x + \epsilon \mathcal{L}_h^{-1} \partial_x \left\{ (h^2 \mathcal{A}')' u_x^2 - g (h \mathcal{A}'' - \mathcal{A}') (\eta_x^2 + 2\eta_x d_x) \right\} \\ = 2\epsilon g \mathcal{L}_h^{-1} \{ \mathcal{A}' \eta_x d_{xx} \}, \end{aligned} \quad (3.43)$$

and

$$\mathcal{R} = \mathcal{J}_h \left\{ (h^2 \mathcal{A}')' u_x^2 - g (h \mathcal{A}'' - \mathcal{A}') (\eta_x^2 + 2\eta_x d_x) \right\} + 2g \tilde{\mathcal{J}}_h \{ \mathcal{A}' \eta_x d_{xx} \}, \quad (3.44)$$

where

$$\begin{aligned} \mathcal{J}_h & \stackrel{\text{def}}{=} \partial_x^{-1} h \mathcal{L}_h^{-1} \partial_x = 1 + 2\epsilon h \mathcal{A}' \partial_x \mathcal{L}_h^{-1} \partial_x, \\ \tilde{\mathcal{J}}_h & \stackrel{\text{def}}{=} \partial_x^{-1} (1 - h \mathcal{L}_h^{-1}) = -2\epsilon h \mathcal{A}' \partial_x \mathcal{L}_h^{-1}. \end{aligned}$$

Let  $\bar{d}$  be the average depth, following [55], we prove the local (in time) well-posedness of the system (3.40), (3.43):

**Theorem 3.3.1.** *Let  $\tilde{m} \geq s \geq 2$ ,  $\mathcal{A} \in C^{\tilde{m}+4}([0, +\infty[)$  such that  $\mathcal{A}'(h) > 0$  for  $h > 0$ . Let  $0 < g \in C^1([0, +\infty[)$ ,  $d - \bar{d} \in C([0, +\infty], H^{s+1}) \cap C^1([0, +\infty], H^s)$  and let  $W_0 = (\eta_0, u_0)^T \in H^s$  satisfying  $\inf_{x \in \mathbb{R}} h_0(x) > h^{**}$ , then there exist  $T > 0$  and a unique solution  $W = (\eta, u) \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$  of (3.40), (3.43) satisfying the non-zero depth condition  $\inf_{x \in \mathbb{R}} h(t, x) > 0$ . Moreover, if the maximal time of existence  $T_{max} < +\infty$ , then*

$$\lim_{t \rightarrow T_{max}} \|W\|_{H^s} = +\infty \quad \text{or} \quad \inf_{(t,x) \in [0, T_{max}] \times \mathbb{R}} h(t, x) = 0. \quad (3.45)$$

Using the energy equation (3.42) and some estimates, we can improve the blow-up criteria (3.45) as

**Theorem 3.3.2.** *For any interval  $[0, T] \subset [0, T_{max}[$ , there exists  $C > 0$ , such that  $\forall t \in [0, T]$  we have*

$$\tilde{E}(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \left[ \frac{1}{2} h u^2 + \epsilon h \mathcal{A}' u_x^2 + \frac{1}{2} g \eta^2 + \epsilon g \mathcal{A}' \eta_x^2 \right] dx \leq C. \quad (3.46)$$

Moreover, if  $T_{max} < +\infty$ , then

$$\lim_{t \rightarrow T_{max}} \|W_x\|_{L^\infty} = +\infty. \quad (3.47)$$

# Chapter 4

## Regularising effect for some hyperbolic equations

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This chapter is based on the papers [36, 9, 29] that are attached in Appendices E, F and G, respectively.

Lax and Oleinik have proved independently in [48, 65] that the entropy solution of the scalar conservation law

$$u_t + f(u)_x = 0, \quad u(0, x) = u_0(x), \quad (4.1)$$

belongs to the space of bounded variation  $BV_{loc}$  for all  $t > 0$ , even if the initial datum  $u_0$  is only in  $L^\infty$  if the flux  $f$  is uniformly convex — i.e.,  $\exists C > 0, f''(u) \geq C$  —. This result have been generalised for  $C^1$  strictly convex fluxes, where the entropy solution belongs to some generalised  $BV_{loc}^\Phi$  (a definition is given below) space, where  $\Phi$  is a convex function depending on the non-linearity of the flux  $f$ . All those results are related to the well known Oleinik inequality

$$f'(u(t, x)) - f'(u(t, y)) \leq (x - y)/t \quad \text{for almost all } t > 0, x > y. \quad (4.2)$$

The  $BV^\Phi$  space is defined as follows

**Definition 4.0.1.** *Let  $\Phi$  be a convex function such that  $\Phi(0) = 0$  and  $\Phi(h) > 0$  for  $h > 0$ , the total  $\Phi$ -variation of  $v$  on  $K \subset \mathbb{R}$  is*

$$\text{TV}^\Phi v \{K\} = \sup_{p \in \mathcal{P}} \sum_{i=2}^n \Phi(|v(x_i) - v(x_{i-1})|) \quad (4.3)$$

where  $\mathcal{P} = \{\{x_1, \dots, x_n\}, x_1 < \dots < x_n\}$  is the set of all subdivisions of  $K$ . The space  $BV^\Phi$  is defined by  $BV^\Phi = \{v, \exists \lambda > 0, \text{TV}^\Phi(\lambda v) < \infty\}$ .

**Remark 4.0.1.** *if  $\Phi(h) = h^{1/s}$  for  $0 < s \leq 1$ , the  $BV^\Phi$  space is called  $BV^s$ . Moreover, if  $s = 1$ , we obtain the classical  $BV$  space.*

The regularising effect proved by Lax [48] and Oleinik [65] has been generalised in [10, 16, 43, 57, 19] for strictly convex  $C^1$  fluxes. The main result in [10] is that if the flux  $f$  is strictly convex and satisfies

$$\frac{|f'(u) - f'(v)|}{|u - v|^p} \geq c_0 > 0 \quad \text{for } u \neq v \text{ and } p \geq 1, \quad (4.4)$$

then the entropy solution  $u$  of (4.1) belongs to  $BV_{loc}^s$  for  $s = p^{-1}$ . The optimal of the regularising effect has been studied in [18, 17].

## 4.1 Regularising effect for hyperbolic equations in the $BV^\Phi$ space

In this section, we generalise the regularising effects introduced above in two directions.



### 4.1.1 Regularising effect for scalar conservation laws with a Lipschitz convex flux

If the flux  $f$  is strictly convex but not in  $C^1$ , the velocity  $a \stackrel{\text{def}}{=} f'$  is not continuous, thus,  $f'(u(t, x))$  is not well defined, which means that the Oleinik inequality (4.2) cannot be used. Defining  $\bar{a}(x) = \lambda a^+(x) + (1 - \lambda) a^-(x)$ , where  $a^-$  and  $a^+$  are the left and right limits of  $a$  respectively. We construct an example where the inequality (4.2) is lost for all  $\lambda \in [0, 1]$ .

In order to obtain a regularising effect of entropy solutions of (4.1) with only Lipschitz and convex flux, we define the function  $\Phi$  as the convex upper envelope of the generalised inverse of the modulus of continuity of the generalised inverse of the velocity, i.e.

$$\Phi \stackrel{\text{def}}{=} \text{upper convex envelope of } (\omega[a^{-1}])^{-1}, \quad (4.5)$$

where the generalised inverse of a non decreasing function  $g$  is

$$g^{-1}(y) \stackrel{\text{def}}{=} \inf\{x \in \mathbb{R}, y \leq g(x)\}, \quad (4.6)$$

**Remark 4.1.1.** *If the flux  $f(u) = |u|^{p+1}/(p+1)$  for  $p \geq 1$ , then  $\Phi(h) = h^p$ , which corresponds to the result given in [10].*

Then, we use a wave front tracking algorithm, which consists of approximating the initial datum  $u_0$  with a sequence of piecewise constant initial data  $u_{0,m}$  and approximating the flux  $f$  with a piecewise linear fluxes  $f_\varepsilon$  and solving the problem

$$u_t + f_\varepsilon(u)_x = 0, \quad u(0, x) = u_{0,m}. \quad (4.7)$$

Since the initial datum  $u_{0,m}$  is piecewise constant, we start by solving Riemann problems, and we deal with the interaction of waves in order to obtain a solution of (4.7). Then, some uniform estimates on a total variation of a modified velocity is used to obtain the compactness. Taking the limit  $(\varepsilon, m) \rightarrow (0, \infty)$  we can prove the following theorem

**Theorem 4.1.1. [Regularising effect in  $BV^\Phi$ ]** *Let  $f$  be a strictly convex flux on  $\mathbb{R}$ ,  $u_0 \in L^\infty$  and  $u$  being the unique entropy solution of (4.1), then  $u(t, \cdot) \in BV_{\text{loc}}^\Phi$ , i.e., for all  $[\alpha, \beta] \subset \mathbb{R}$ ,*

$$\text{TV}^{\Phi^+} u(t, \cdot)\{[\alpha, \beta]\} \leq (\beta - \alpha) t^{-1}, \quad (4.8)$$

$$\text{TV}^\Phi u(t, \cdot)\{[\alpha, \beta]\} \leq 2(\|a(u_0)\|_\infty + (\beta - \alpha) t^{-1}), \quad (4.9)$$

Moreover, if  $u_0$  is compactly supported, then there exists  $C > 0$  such that

$$\text{TV}^\Phi u(t, \cdot)\{\mathbb{R}\} \leq C(1 + t^{-1}). \quad (4.10)$$

The inequality (4.8) is a generalised Oleinik inequality.

### 4.1.2 Regularising effect for a class of scalar balance laws

In this section, we obtain a regularising effect for the scalar balance law

$$u_t + f(u)_x = \alpha(t) u, \quad u(0, x) = u_0(x), \quad (4.11)$$

where  $\alpha \in L^\infty([0, +\infty[)$  and  $f$  is a  $C^1$  strictly convex flux satisfying the power law condition (4.4). Adimurthi et al have proved in [3] that if the flux  $f$  is  $C^1$ , strictly convex and satisfy the super-linear growth

$$\lim_{|u| \rightarrow +\infty} \frac{f(u)}{|u|} = \infty, \quad (4.12)$$

then for all  $t > 0$ , there exists a unique increasing function  $\Psi(t, \cdot)$  satisfying

$$x = \int_0^t (f'(\Psi(t, x) e^{\beta(\theta)})) d\theta, \quad \beta(\theta) \stackrel{\text{def}}{=} \int_0^\theta \alpha(\theta') d\theta'. \quad (4.13)$$

The entropy solution of the balance law (4.11) can be obtained by the Lax–Oleinik formula

$$u(t, x) = e^{\beta(t)} \Psi(t, x - y(t, x)), \quad (4.14)$$

where  $x \mapsto y(t, x)$  is a non-decreasing function, such that for all  $T > 0$  there exists  $C(T) > 0$  and

$$|y(t, x) - x| \leq C(T) t \quad \forall t \in [0, T]. \quad (4.15)$$

We show that if the flux  $f$  is  $C^1$ , strictly convex and satisfies (4.4), then for any  $x_1, x_2 \in \mathbb{R}$  we have

$$|\Psi(t, x_1) - \Psi(t, x_2)| \leq \left( \frac{|x_1 - x_2|}{c_0 \gamma(t)} \right)^{\frac{1}{p}}, \quad \gamma(t) \stackrel{\text{def}}{=} \int_0^t e^{p\beta(\theta)} d\theta. \quad (4.16)$$

From (4.14), (4.15) and (4.16) we obtain

**Theorem 4.1.2.** *Let  $f \in C^1$  be a convex flux satisfying the power-law condition (4.4) and super linear growth condition (4.12). Let  $\alpha \in L^\infty$ . Let  $u_0 \in L^\infty$  and  $u$  be the entropy solution of the initial value problem for the balance law (4.11), then*

$$u(t, \cdot) \in BV_{loc}^s \text{ for } s = p^{-1} \text{ and } \forall t > 0. \quad (4.17)$$

### 4.1.3 Optimality for a class of unidimensional scalar balance laws for all time

Inspired by [2], we construct a solution of (4.11) to show the optimality for all time (after wave interactions) of the result given in Theorem 4.1.2 for a class of strictly convex fluxes

**Theorem 4.1.3.** *Let  $f \in C^1$  be a convex flux satisfying (4.12),  $f(0) = f'(0) = 0$  and there exist  $p \geq 1, r, C > 0$  such that*

$$0 \leq f'(a) - f'(b) \leq c(a - b)^p \quad \forall b \in ]-r, 0[ \text{ and } a \in ]0, r[. \quad (4.18)$$

*Then, there exists a compactly supported  $u_0 \in L^\infty$ , such that the entropy solution  $u$  of (4.11) satisfies*

$$u(t, \cdot) \notin BV_{loc}^s \text{ for all } t > 0 \text{ and } s > p^{-1} \quad (4.19)$$

**Remark 4.1.2.** Note that the flux  $f(u) = |u|^{p+1}/(p+1)$ ,  $p \geq 1$  satisfies the conditions (4.4), (4.12) and (4.18). In this case, Theorem 4.1.3 shows the optimality of the result proven in Theorem 4.1.2.

## 4.2 Existence of entropy solutions in $BV^s$ for a class of triangular systems involving a transport equation

We consider in this section the triangular system

$$u_t + f(u)_x = 0, \quad u(0, x) = u_0(x) \quad (4.20)$$

$$v_t + (g(u)v)_x = 0, \quad v(0, x) = v_0(x). \quad (4.21)$$

In order to solve the system above, one can solve the first equation, then the linear transport equation follows. Solving a transport equation with a discontinuous velocity can lead to the appearance of measure solutions [38]. We do not use the triangular point of view, and we treat (4.20), (4.21) as a coupled system.

In order to avoid measure solutions, we consider the uniformly strict hyperbolicity condition

$$\inf_{|u| \leq M} f'(u) > \sup_{|u| \leq M} g(u), \quad M \stackrel{\text{def}}{=} \|u_0\|_\infty. \quad (4.22)$$

If  $f$  and  $g$  satisfy (4.22), we use a wave front tracking algorithm to prove the following theorem

**Theorem 4.2.1.** Let  $f \in C^4$  and  $g \in C^3$  satisfying (4.22) and  $f$  has at most a finite number of inflection points. Let also  $(u_0, v_0) \in BV^s \times L^\infty$  for some  $s > 1/3$ , then there exists an entropy solution  $(u, v) \in L^\infty([0, +\infty), BV^s) \times L^\infty([0, +\infty) \times \mathbb{R})$  of (4.20), (4.21). In addition, if  $\inf_x v_0 > 0$  then  $\inf_{t,x} v > 0$ . Moreover, if the flux  $f$  is convex, then the result remains true for  $s = 1/3$ .

Even in the very favorable context of the uniform strict hyperbolicity condition (4.22), and for such a simple triangular system (4.20), (4.21) with the Burgers flux, if  $s < 1/3$  the previous wave front tracking algorithm blows-up instantaneously:

**Theorem 4.2.2.** For  $s < 1/3$ , there exist  $u_0 \in BV^s \setminus BV^{1/3}$ ,  $v_0 \in L^\infty$ ,  $f$  and  $g$  satisfying (4.22) such that the solution obtained by the same construction used in Theorem 4.2.1 blows-up at time  $t = 0^+$ .

Using Theorem 4.2.1 and Theorem 4.1.3 we deduce the optimal regularity

**Corollary 4.2.1.** Let  $f \in C^4$  is a convex flux satisfying the conditions of Theorem 4.1.3 and let  $g \in C^3$  such that (4.22) holds. Let also  $s = \max\{1/p, 1/3\}$  then there exists  $u_0$  such that for all  $v_0 \in L^\infty$  the system (4.20), (4.21) admits a global entropy solution  $u \in BV^s$ ,  $v \in L^\infty$  and  $TV^{s+\varepsilon}u(t, \cdot) = +\infty$  for all  $t > 0, \varepsilon > 0$ .

# Chapter 5

## Conclusion and perspectives

Many interesting questions remain open, we mention few of them.

- **The regularised rarefaction wave.** In [68] and Section 2.2, traveling waves have been studied which correspond to shock waves of the Riemann problem (see Remark 2.2.2). The approximation of the rarefaction wave should also be studied. If the latest exists, it would be an example of a smooth solution that does not blow-up in finite time.
- **Uniqueness of conservative and dissipative solutions.** In [11], Bressan et al have proved the uniqueness of conservative solutions of the Camassa-Holm equation. It would be interesting to study the uniqueness of the conservative or the dissipative solutions of the regularised scalar conservation law (rSCL) (2.62).
- **The limiting cases  $\varepsilon, \ell \rightarrow 0$  and  $\varepsilon, \ell \rightarrow \infty$ .** We have proved that the dissipative solutions of the regularised scalar conservation law (2.62) converge “up to a subsequence” to  $u^0$  (respectively  $u^\infty$ ) when  $\ell \rightarrow 0$  (respectively  $\ell \rightarrow \infty$ ). Before the appearance of singularities,  $u^0$  and  $u^\infty$  are respectively solutions of the classical scalar conservation law and the generalised Hunter–Saxton equation (1.10). Does this hold even after the breaking-down time?

Also, a study of the limiting cases  $\varepsilon \rightarrow 0/\infty$  of the regularised barotropic Euler system (1.11) should be studied.

- **Numerical analysis.** The study of the limiting cases mentioned above is quite complicated, especially for systems. Developing some numerical codes can be useful to understand those limiting cases.
- **Other regularisations.** The regularisations studied in this thesis are non-dispersive and conserve an  $H^1$ -like energy for smooth solutions. Also, traveling waves that correspond to entropy shocks dissipate the same energy as the original equations. It would be interesting to generalise those regularisations in the directions:
  - Regularising other interesting hyperbolic systems.

- Regularising hyperbolic equations in the multidimensional case.
- The rB equation (2.16) (for  $b = 2$ ) formally conserves an  $H^1$  energy. The rB equation can be seen as an “interpolation” between the Burgers equation ( $\ell = 0$ ) that conserves the  $L^2$  energy and the Hunter–Saxton equation ( $\ell = \infty$ ) that conserves the  $\dot{H}^1$  energy. Theorem 2.3.2 and Theorem 2.3.6 show that the solutions of rB are not more regular than  $H^1$  in general. It would be interesting to obtain some suitable equations that conserve the  $\dot{H}^m$  with  $m \geq 1$  and interpolate them with the Burgers equation to obtain other regularisations with smoother solutions.
- **Proof of the blow-up for other systems.** Theorem 2.3.2 shows that there exist some initial data where the corresponding solution of the rSCL blows-up in finite time. A similar result have been proved in [55] for the more complicated rSV system (1.5). Following [55], can we prove a blow-up result for the regularised barotropic Euler system (1.11) or for the Serre–Green–Naghdi system?
- **Global existence in  $H^1$ .** Due to the conservation of the  $H^1$ -like energy of the regularisations studied in this thesis. Two possibilities that may work to obtain global existence of solutions in  $H^1$ :
  - Using vanishing viscosity as in [31].
  - Using a semi-linear equivalent system as in [12, 13, 30, 70].
- **Generalised Hunter–Saxton system.** We have proved in this thesis the local well-posedness of the periodic generalised two-component Hunter–Saxton system (1.14). The latter system deserves to be studied further.
- **Lax–Oleinik formula for more balance laws.** In [3], a Lax–Oleinik formula is given of the balance law (1.19) with a linear source term. Can we obtain a generalised Lax–Oleinik formula and/or prove a regularising effect for more balance laws?
- **Global existence in  $BV^s$ .** Can the proof of the global existence of solutions of
  - the triangular system (1.21), (1.22) given in Appendix F,
  - the 2x2 nonlinear hyperbolic systems with one genuinely nonlinear field and one linearly degenerate field given in [37],
be generalised for 2x2 systems with two genuinely nonlinear fields or for other 3x3 systems?
- **Other regularising effects.** It would be interesting to obtain some regularising effects for
  - other non-linear hyperbolic systems,
  - for the multidimensional case, as in [53].

These interesting questions deserve to be studied in the future.

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## Appendices: Attached papers



# Appendix A

## Global weak solutions of a Hamiltonian regularised Burgers equation

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**Abstract:** Inspired by a recent nondispersive conservative regularisation of the shallow water equations, a similar regularisation is proposed and studied here for the inviscid Burgers equation. The regularised equation is parametrised by a positive number  $\ell$ , the inviscid Burgers equation corresponding to  $\ell = 0$  and the Hunter–Saxton equation being formally obtained letting  $\ell \rightarrow \infty$ . The breakdown of local smooth solutions is demonstrated. The existence of two types of global weak solutions, conserving or dissipating an  $H^1$  energy, is also studied. The built *dissipative* solution satisfies (uniformly with respect to  $\ell$ ) an Oleinik inequality, as do entropy solutions of the inviscid Burgers equation. The limit (up to a subsequence) of the dissipative solution when  $\ell \rightarrow 0$  (respectively  $\ell \rightarrow \infty$ ) satisfies the Burgers (resp. Hunter–Saxton) equation forced by an unknown remaining term. At least before the appearance of singularities, the limit satisfies the Burgers (resp. Hunter–Saxton) equation.

**AMS Classification:** 35B65; 35B44; 35Q35; 35L67.

**Key words:** Inviscid Burgers equation; regularisation; Hamiltonian; conservative and dissipative solutions; Oleinik inequality.

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## A.1 Introduction

The dispersionless shallow water equations, also called the Saint-Venant equations, admit shock-wave solutions. Recently, a Hamiltonian regularisation of this system (rSV), has been proposed which approximates these discontinuous waves by less singular ones [10, 11]. The rSV system can be written

$$h_t + [hu]_x = 0, \tag{A.1a}$$

$$[hu]_t + [hu^2 + \frac{1}{2}gh^2 + \epsilon \mathcal{R}h^2]_x = 0, \tag{A.1b}$$

$$\mathcal{R} \stackrel{\text{def}}{=} h(u_x^2 - u_{xt} - uu_{xx}) - g(hh_{xx} + \frac{1}{2}h_x^2), \tag{A.1c}$$

where  $\epsilon$  is a small positive parameter,  $h$  is the total water depth and  $u$  is the velocity. The classical Saint-Venant equations can be obtained letting  $\epsilon \rightarrow 0$ . This regularisation is Galilean invariant, non-dispersive, non-diffusive, and conserves energy for regular solutions. It also admits regularised shock-wave weak solutions which have the same wave speed and which dissipate energy at the same rate as shocks in the classical Saint-Venant (cSV) equations, [18].

Some mathematical results on rSV were obtained by Pu et al. [18] and Liu et al. [17], but several natural questions remain open, such as the existence of global weak solutions. In the present work we consider such questions for an analogous but simpler model equation, namely a Hamiltonian regularisation of the inviscid Burgers equation  $u_t + uu_x = 0$ . Indeed, the Burgers equation being scalar, it is more tractable than the rSV system of equations.

Inspired by the rSV and the dispersionless Camassa-Holm [8] equations, in Section A.2 we derive a regularised Burgers equation (rB) in the form

$$u_t + uu_x = \ell^2(u_{txx} + 2u_x u_{xx} + uu_{xxx}), \tag{A.2}$$

where  $\ell \geq 0$  is a parameter. The main purpose of the present paper is to study the existence of local smooth solutions, the blow-up time, global weak solutions of the regularised Burgers equation (A.2), and also to study the limiting cases  $\ell \rightarrow 0$  and  $\ell \rightarrow +\infty$ .

Some mathematical results on (A.2) are already known. For a generalisation of the Camassa-Holm equation [21], including rB as special case, the existence of local smooth solutions has been proved ([21], see also Theorem A.3.1 below). The existence of global weak solutions in  $H^1$  has also been proved using vanishing viscosity [20, 9]. Note that

Bressan and Constantin [6, 7] have proved global weak solutions of the Camassa–Holm equation (A.8) in  $H^1$ , using an equivalent semi-linear system.

In this paper, we rewrite (A.2) into an equivalent system, analogous to the treatment of the Camassa–Holm equation in [6, 7], without asking the initial datum to be in  $H^1$ . We prove the existence of a so-called *conservative* [6] global weak solution (Theorem A.4.1). This conservative solution conserves the energy, but it does not satisfy the Oleinik inequality, which is an important condition for entropic shock waves. To avoid this problem, the equivalent system is slightly modified in order to obtain another type of solutions called *dissipative* [7] (see Theorem A.4.2 below), which satisfies an Oleinik inequality given in (A.85). This inequality is well known to obtain uniqueness for entropy solutions of the Burgers equation. The dissipative solutions of rB can also be called “entropy solutions” because they satisfy an Oleinik inequality. However, the uniqueness for the dissipative solutions of rB remains an open problem.

The equivalent system, and the Oleinik inequality, are used to obtain a uniform BV estimate independent of the parameter  $\ell$  for the dissipative solutions (Lemma A.6.1), which is a key point to prove the compactness when  $\ell \rightarrow 0$  and  $\ell \rightarrow +\infty$ . When  $\ell \rightarrow 0$ , a dissipative solution converges (up to a subsequence) to a function  $u$  that satisfies the Burgers equation with a remaining term (A.104) (see Theorem A.6.1 below). If the remaining term is zero, then the entropy solutions of Burgers is recovered. We prove that this term is zero for smooth solutions of Burgers equations (see Proposition A.6.1). However, the disappearance of the remaining term in general remains an open problem. Similar results are obtained when  $\ell \rightarrow +\infty$ , where the limit is a solution of the Hunter–Saxton equation, at least before the appearance of singularities<sup>1</sup> (Theorem A.6.2 and Proposition A.6.2 below).

This paper is organised as follow. A heuristic derivation of the rB equation is given in section A.2. Section A.3 is devoted to study the existence of local smooth solutions. In Section A.4, proofs of existence of both global conservative and dissipative solutions are given. Section A.6 studies the limiting cases  $\ell \rightarrow 0$  and  $\ell \rightarrow +\infty$  for dissipative solutions. The optimality of the requirement that  $u_x \in L^2_{loc}$  for weak solutions is shown in section A.7, where we prove in particular that when a smooth solution breaks down,  $u_x$  may blow up in  $L^p_{loc}$  for all  $p > 2$ .

## A.2 Heuristic derivation of a regularised Burgers equation

In order to introduce a suitable regularisation of the inviscid Burgers equation with similar features as the rSV system (A.1), we note first that the rSV equations yield the momentum equation

$$u_t + u u_x + g h_x + \epsilon (h \mathcal{R}_x + 2 \mathcal{R} h_x) = 0. \quad (\text{A.3})$$

---

<sup>1</sup>“Singularity” is used here to describe the blow-up of derivatives which corresponds to shocks of the classical Burgers equation. Contrary to the Burgers case, the solutions of rB remains continuous at the singularities.



When  $h$  is constant, this equation (with the definition of  $\mathcal{R}$ ) becomes

$$u_t + u u_x = \ell^2 [u_{xxt} - u_x u_{xx} + u u_{xxx}], \quad (\text{A.4})$$

where  $\ell \stackrel{\text{def}}{=} h\sqrt{\epsilon} \geq 0$  is a constant characterising a length scale for the regularisation.

After the change of independent variables  $(t, x) \rightarrow (t/\ell, x/\ell)$ , which leaves the inviscid Burgers equation invariant, equation (A.4) becomes

$$u_t + u u_x = u_{xxt} - u_x u_{xx} + u u_{xxx}. \quad (\text{A.5})$$

Equation (A.5) belongs to the three-parameter family (for  $a, b, c \in \mathbb{R}$ ) of non-dispersive equations

$$u_t - u_{xxt} = a u u_x + b u_x u_{xx} + c u u_{xxx}. \quad (\text{A.6})$$

In this family, we look for an equation that has Galilean invariance and conservation of energy (at least for smooth solutions). A famous equation in the family (A.6) is the dispersionless Camassa–Holm (CH) equation [8]

$$u_t + 3u u_x = u_{xxt} + 2u_x u_{xx} + u u_{xxx}. \quad (\text{A.7})$$

This can be rewritten, by applying the inverse of the Helmholtz operator  $1 - \partial_x^2$ , as

$$u_t + u u_x + [\mathfrak{G} * (\frac{1}{2} u_x^2 + u^2)]_x = 0, \quad \mathfrak{G}(x) \stackrel{\text{def}}{=} \frac{1}{2} \exp(-|x|), \quad (\text{A.8})$$

where  $*$  denotes the convolution product. The family (A.6) covers other equations, such as the Degasperis–Procesi (DP) equations [12] and the Benjamin–Bona–Mahony (BBM) equation [1]. It is well known that the Camassa–Holm conserves the  $H^1$  energy [8], but is not Galilean invariant.

In order to obtain a Galilean invariant regularisation of the Burgers equation, one must take  $c = -a = 1$  in (A.6). The special case  $b = 0$  was studied by Bhat and Fetecau [2, 3, 4], who proved the existence of the solution and the convergence to weak solutions of the Burgers equation in the limit corresponding to  $\ell \rightarrow 0$ . The limit fails to satisfy the entropy condition for the Riemann problem with  $u_{\text{left}} < u_{\text{right}}$  [4]. For this regularisation, no energy conservation equation is known.

In the present paper, we consider  $c = -a = 1$  (to ensure Galilean invariance, as in [2]) and, in order to maintain conservation of the  $H^1$  norm at least for smooth solutions, we take  $b = 2$  (as in the Camassa–Holm equation). With this done, equation (A.6) becomes

$$u_t + u u_x = u_{txx} + 2u_x u_{xx} + u u_{xxx}, \quad (\text{A.9})$$

Introducing the scaling  $(t, x) \mapsto (\ell t, \ell x)$ , we obtain

$$u_t + u u_x = \ell^2 (u_{txx} + 2u_x u_{xx} + u u_{xxx}), \quad (\text{A.10})$$

that is a formal approximation of the Burgers equation for small  $\ell$ . Equation (A.10) is the regularised Burgers (rB) equation studied in this paper.

It can be shown that the equation (A.10) has Hamiltonian and Lagrangian structure (we omit the details), and that smooth solutions satisfy the following energy conservation law:

$$\left[ \frac{1}{2} u^2 + \frac{1}{2} \ell^2 u_x^2 \right]_t + \left[ \frac{1}{3} u^3 - \ell^2 u^2 u_{xx} - \ell^2 u u_{xt} \right]_x = 0. \quad (\text{A.11})$$

Applying the inverse Helmholtz operator  $(1 - \ell^2 \partial_x^2)^{-1}$ , the rB equation (A.10) can be rewritten

$$u_t + u u_x + \ell^2 P_x = 0, \quad P \stackrel{\text{def}}{=} \mathfrak{G} * \frac{1}{2} u_x^2 \geq 0, \quad \mathfrak{G} \stackrel{\text{def}}{=} (2\ell)^{-1} \exp(-|x|/\ell), \quad (\text{A.12})$$

to be compared with the Camassa–Holm equation in its form (A.8). Differentiating (A.12) with respect to  $x$ , and using that  $P - \ell^2 P_{xx} = \frac{1}{2} u_x^2$ , one obtains

$$\left[ u_t + u u_x \right]_x + P = \frac{1}{2} u_x^2. \quad (\text{A.13})$$

Note that  $P$  goes formally to zero as  $\ell \rightarrow +\infty$ , whence one obtains the Hunter–Saxton (HS) equation [14, 15]

$$\left[ u_t + u u_x \right]_x = \frac{1}{2} u_x^2. \quad (\text{A.14})$$

Note also that by taking  $\ell \rightarrow +\infty$  formally in (A.10), we obtain the derivative of (A.14) with respect of  $x$ .

In this section, we have heuristically derived a regularised Burgers equation, by imposing the important physical requirements of Galilean invariance and energy conservation. We have also related this equation with well-known equations. In the rest of the paper, we perform a rigorous mathematical investigation of solutions of this regularised Burgers equation.

### A.3 Existence and breakdown of smooth solutions

This section is devoted to show the local existence and breakdown of smooth solutions for the Cauchy problem (A.12) with  $u(0, x) = u_0(x)$ . The form (A.12) of the regularised Burgers equation is more convenient for studying smooth solutions than (A.10), because it involves fewer derivatives.

Usually, one needs an equation for  $u_x$  to study the life span of smooth solutions. Equation (A.13) can be written

$$u_{xt} + \frac{1}{2} u_x^2 + u u_{xx} + P = 0. \quad (\text{A.15})$$

Multiplying (A.12) by  $u$  and multiplying (A.15) by  $\ell^2 u_x$ , we obtain

$$\left[ \frac{1}{2} u^2 \right]_t + \left[ \frac{1}{3} u^3 + \ell^2 u P \right]_x = \ell^2 u_x P, \quad (\text{A.16})$$

$$\left[ \frac{1}{2} \ell^2 u_x^2 \right]_t + \left[ \frac{1}{2} \ell^2 u u_x^2 \right]_x = -\ell^2 u_x P, \quad (\text{A.17})$$

which imply an energy conservation law for smooth solutions; i.e., we have the (conservative) energy equation

$$\left[ \frac{1}{2} u^2 + \frac{1}{2} \ell^2 u_x^2 \right]_t + \left[ \frac{1}{3} u^3 + \ell^2 u P + \frac{1}{2} \ell^2 u u_x^2 \right]_x = 0. \quad (\text{A.18})$$

For a class of equations including rB as special case, Yin [22, 21] has proven the following local existence result.

**Theorem A.3.1** (Yin [22, 21]). *For an initial datum  $u_0 \in H^s(\mathbb{R})$  with  $s > 3/2$ , there exists a maximal time  $T^* > 0$  (independent of  $s$ ) and a unique solution  $u \in \mathcal{C}([0, T^*[, H^s)$  of (A.12) such that (blow-up criterium)*

$$T^* < +\infty \quad \Longrightarrow \quad \lim_{t \uparrow T^*} \|u(t, \cdot)\|_{H^s} = +\infty. \quad (\text{A.19})$$

Moreover, if  $s \geq 3$ , then

$$T^* < +\infty \quad \Longrightarrow \quad \lim_{t \uparrow T^*} \inf_{x \in \mathbb{R}} u_x(t, x) = -\infty. \quad (\text{A.20})$$

Furthermore, the solution given in this theorem satisfies the Oleinik inequality:

**Proposition A.3.1. (Oleinik inequality)** *Let  $u_0 \in H^s(\mathbb{R})$  with  $s \geq 2$  and let  $M = \sup_{x \in \mathbb{R}} u'_0(x)$ . Then, for all  $t \in [0, T^*[$  the solution given in Theorem A.3.1 satisfies*

$$u_x(t, x) \leq \frac{2M}{Mt+2} \leq M. \quad (\text{A.21})$$

*Proof.* Let  $x_0 \in \mathbb{R}$  and let the characteristic  $\eta(t, x_0)$  be defined as the solution of the Cauchy problem  $\eta_t(t, x_0) = u(t, \eta(t, x_0))$ , with the initial datum  $\eta(0, x_0) = x_0$ . With  $H(t, x_0) \stackrel{\text{def}}{=} u_x(t, \eta(t, x_0))$ , the equation (A.15) can be rewritten

$$H_t + \frac{1}{2} H^2 + P = 0. \quad (\text{A.22})$$

Since  $P \geq 0$ , it follows that  $H_t \leq -\frac{1}{2} H^2$  which implies that  $H(t, x_0) \leq \frac{2H(0, x_0)}{H(0, x_0)t+2} \leq \frac{2M}{Mt+2}$ .  $\square$

**Remark A.3.1.** *The Oleinik inequality (A.21) is valid only when the solution  $u$  is smooth. In Theorem A.4.2 below, we show that this inequality holds for all times also for a certain type of weak solutions (called dissipative) such that  $u \in H^1$  (and, possibly, for  $M = +\infty$ ).*

Unfortunately, the solution given in Theorem A.3.1 does not exist globally in time for all non trivial initial data [21]. Since Yin [21] studied a general family of equations including rB, his result is not optimal for rB. In the following proposition, this result is improved with a shorter proof.

**Proposition A.3.2. (An upper bound on the blow-up time)** *Let  $u_0 \in H^s(\mathbb{R})$  with  $s \geq 2$ . If there exists  $x_0 \in \mathbb{R}$  such that  $u'_0(x_0) < 0$ , then  $T^* \leq -2/\inf u'_0$ .*

*Proof.* From the proof of the previous proposition, we have

$$H(t, x_0) \leq \frac{2H(0, x_0)}{tH(0, x_0) + 2}, \quad t < T^*. \quad (\text{A.23})$$

If  $T^* > -2/\inf u'_0$  then  $H(0, x_0) < 0$  implies

$$\lim_{t \rightarrow -2/H(0, x_0)} H(t, x_0) = -\infty,$$

this contradicts  $u \in \mathcal{C}([0, T^*[, H^s)$ .  $\square$

A uniform (with respect to  $\ell$ ) lower bound on  $T^*$  is needed, in order to prove in section A.6 below the convergence of smooth solutions (see Proposition A.6.1 and Proposition A.6.2).

**Theorem A.3.2. (A lower bound on the blow-up time)** *Let  $u_0$  in  $H^s$  be non-trivial with  $s \geq 2$  and let*

$$m(t) \stackrel{\text{def}}{=} \inf_{x \in \mathbb{R}} u_x(t, x) < 0 < M(t) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} u_x(t, x), \quad t < T^*.$$

If  $|m(0)| \geq M(0)$  then

$$-1/\inf u'_0 \leq T^* \leq -2/\inf u'_0. \quad (\text{A.24})$$

If  $|m(0)| < M(0)$  then, there exists  $t^*$  such that  $0 < t^* \leq -m(0)^{-1} - M(0)^{-1}$  and  $m(t^*) = -M(t^*)$ . Therefore

$$t^* + 1/\sup u'_0 \leq T^* \leq -2/\inf u'_0. \quad (\text{A.25})$$

**Remark A.3.2.** *Note that the blow-up time  $T^*$  is uniformly (with respect to  $\ell$ ) bounded from below by  $1/\sup |u'_0|$ .*

*Proof.* Since  $u \in H^s$ ,  $u_x \rightarrow 0$  when  $x$  goes to  $\pm\infty$ , and  $u_x$  is not the zero function, so  $m(t) = \min_{x \in \mathbb{R}} H(t, x) < 0 < M(t) = \max_{x \in \mathbb{R}} H(t, x)$ . The equation (A.22) implies that  $m$  and  $M$  are decreasing in time, so  $|m| = -m$  is increasing. So, if  $|m(t_0)| \geq M(t_0)$ , then for all  $t > t_0$  we have  $|m(t)| \geq M(t)$ .

The inequality (A.23) shows that for  $t < T^*$

$$0 < M(t) \leq \frac{2M(0)}{M(0)t + 2}, \quad m(t) \leq \frac{2m(0)}{m(0)t + 2} < 0, \quad (\text{A.26})$$

which implies that, if  $|m(0)| < M(0)$ , there exists  $t^* \leq -(m(0) + M(0))/(m(0)M(0))$  such that  $|m(t^*)| = M(t^*)$ .

If  $\delta > 0$  is small enough, since the function  $H(t + \delta, \cdot)$  has a minimum, then there exists  $x_\delta$  such that  $m(t + \delta) = H(t + \delta, x_\delta)$ . Inspired by Junca and Lombard [16] one gets

$$\begin{aligned} m(t + \delta) &= H(t + \delta, x_\delta) = H(t, x_\delta) + \int_t^{t+\delta} H_t(s, x_\delta) ds \\ &\geq m(t) - \int_t^{t+\delta} \left( \frac{1}{2} H(s, x_\delta)^2 + P(s, x_\delta) \right) ds. \end{aligned} \quad (\text{A.27})$$

Since  $m(\cdot) < 0$  and  $\delta$  is arbitrary small, we have  $m(s) \leq H(s, x_\delta) \leq 0$  then  $m(s)^2 \geq H(s, x_\delta)^2$ , implying that

$$\frac{m(t + \delta) - m(t)}{\delta} \geq -\frac{1}{\delta} \int_t^{t+\delta} \left( \frac{1}{2} m(s)^2 + \sup_{x \in \mathbb{R}} P(s, x) \right) ds. \quad (\text{A.28})$$

Defining the generalised derivative

$$\dot{m}(t) \stackrel{\text{def}}{=} \liminf_{\delta \rightarrow 0^+} \frac{m(t + \delta) - m(t)}{\delta}, \quad (\text{A.29})$$

one can show that

$$\dot{m}(t) + \frac{1}{2} m(t)^2 \geq -\sup_{x \in \mathbb{R}} P(t, x). \quad (\text{A.30})$$

Using the definition of  $P$  from (A.12) and using that  $\|\mathfrak{G}\|_1 = 1$ , one obtains

$$\sup_{x \in \mathbb{R}} P(t, x) \leq \frac{1}{2} \|u_x(t, \cdot)\|_\infty^2 \leq \max\left\{\frac{1}{2} M(t)^2, \frac{1}{2} m(t)^2\right\}. \quad (\text{A.31})$$

and the Riccati-like inequality (A.30) becomes

$$\dot{m}(t) + m(t)^2 \geq 0 \quad t > t^*. \quad (\text{A.32})$$

Then  $T^* - t^* \geq -1/m(t^*) = 1/M(t^*)$ , and with (A.26), one obtains

$$T^* \geq t^* + 1/\sup u'_0. \quad \square$$

## A.4 Global weak solutions

Note that Proposition A.3.2 shows that, for  $s \geq 2$ , we have  $\lim_{t \uparrow T^*} \inf_{x \in \mathbb{R}} u_x(t, x) = -\infty$  which implies that

$$\lim_{t \uparrow T^*} \|u(t, \cdot)\|_{H^s} = +\infty.$$

Hence the space  $H^s$  with  $s \geq 2$  is not the right space in order to obtain the global existence of the solution.

Bressan and Constantin [6, 7] have proved the existence of two types of global solutions for the Camassa–Holm equation (A.8) in  $H^1$ . Using the formal energy equation (A.18), a similar proof (of global existence of conservative and dissipative solutions in  $H^1$ ) for rB can be done following [6, 7]. Another proof of existence of a dissipative solution, using the vanishing viscosity method, is given by Chen and Tian [9], Xin and Zhang [20].

In this paper, the existence theorem will be developed for solutions not vanishing as  $|x| \rightarrow \infty$ . Note that a major difference between the rB (A.12) and the Camassa–Holm (A.8) equations is that  $u^2$  does not appear in the non-local term of rB. This allows us to get global existence for rB without asking  $u$  to be in  $L^2(\mathbb{R})$ . Moreover, in Theorem A.7.1 below, we show that asking  $u_x \in L^2$  is optimal.

These remarks lead us to assert in the following the existence of two types of solutions of rB: conservative and dissipative.

### A.4.1 Global existence of conservative solutions

We start this subsection by defining a conservative solution.

**Definition A.4.1.** *A function  $u$  is called a conservative solution of rB if*

- *The function  $u$  belongs to  $\text{Lip}([0, T], L^2_{loc})$  and  $u_x \in L^\infty([0, T], L^2)$  for all  $T > 0$ .*
- *$u$  satisfies the equation (A.12), with an initial data  $u(0, x) = u_0(x)$ .*

- $u$  satisfies (A.18) in the sense of distributions.

It means that it is a weak solution conserving the energy, as smooth solutions.

**Remark A.4.1.** *The regularity  $u_x \in L^\infty([0, T], L^2)$  ensures that (A.16) is satisfied. Thence, the equalities (A.17) and (A.18) are equivalent.*

Introducing the homogeneous Sobolev space  $\dot{H}^1(\mathbb{R}) = \{f : \|f'\|_2 < +\infty\}$ , we can state the theorem:

**Theorem A.4.1.** *Let  $u_0 \in \dot{H}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ . If there exists a Lipschitz function  $\phi$  such that  $\phi' \in L^1(\mathbb{R})$  with  $u_0 - \phi \in H^1(\mathbb{R})$ , then there exists a global conservative solution  $u$  of (A.12), such that  $u(t, \cdot) - \phi \in H^1(\mathbb{R})$  for all  $t > 0$ . In addition, for all  $T > 0$*

$$\liminf_{t \uparrow T} \inf_{x \in \mathbb{R}} u_x(t, x) = -\infty \quad \implies \quad \limsup_{t \downarrow T} \sup_{x \in \mathbb{R}} u_x(t, x) = +\infty, \quad (\text{A.33})$$

and if  $u_0 \in H^1$ , then for all  $t > 0$

$$\int_{\mathbb{R}} [u(t, x)^2 + \ell^2 u_x(t, x)^2] dx = \int_{\mathbb{R}} [u_0(x)^2 + \ell^2 u_0'(x)^2] dx. \quad (\text{A.34})$$

**Remark A.4.2.** *This theorem covers also some solutions that do not have a limit when  $|x| \rightarrow \infty$ , such as  $\phi(x) = u_0(x) = \cos \ln(x^2 + 1)$ .*

**Remark A.4.3.** *Note that (A.33) implies that the Oleinik inequality (A.21) cannot hold after the appearance of singularities.*

In the special case  $u_0 \in H^1(\mathbb{R})$ , the proof can be done following Bressan and Constantin [6]. In the general case, the energy is modified as

$$E(t) = \int_{\mathbb{R}} [u(t, x) - \phi(x)]^2 + \ell^2 u_x(t, x)^2 dx, \quad (\text{A.35})$$

and the proof is done in steps as follows.

**Step 1: Formal energy estimate on the  $x$ -variable.** Let  $\tilde{u}(t, x) = u(t, x) - \phi(x)$ . The equation (A.12) can be rewritten

$$u_t + u u_x + \ell^2 P_x = \tilde{u}_t + u u_x + \ell^2 P_x = 0. \quad (\text{A.36})$$

Multiplying (A.36) by  $\tilde{u}$ , one gets

$$\left[ \frac{1}{2} \tilde{u}^2 \right]_t + \left[ \frac{1}{3} u^3 - \frac{1}{2} \phi u^2 \right]_x + \frac{1}{2} \phi_x u^2 + \ell^2 u P_x - \ell^2 \phi P_x = 0. \quad (\text{A.37})$$

Adding (A.37) and (A.17), we obtain

$$\frac{1}{2} \left[ \tilde{u}^2 + \ell^2 u_x^2 \right]_t + \left[ \frac{1}{3} u^3 + \frac{1}{6} \phi^3 - \frac{1}{2} \phi u^2 + \frac{1}{2} \ell^2 u u_x^2 + \ell^2 u P \right]_x = \ell^2 \phi P_x - \frac{1}{2} \phi_x (\tilde{u}^2 + 2\phi \tilde{u}). \quad (\text{A.38})$$

Integrating over the real line, one gets (exploiting the triangular inequality)

$$\frac{1}{2} E'(t) \leq \int_{\mathbb{R}} \left( \ell^2 |\phi P_x| + \frac{1}{2} |\phi'| (2\tilde{u}^2 + \phi^2) \right) dx. \quad (\text{A.39})$$

The Young inequality implies that

$$\|P(t)\|_p \leq \frac{1}{2\ell^2} \|\mathfrak{G}\|_p E(t) \quad \forall p \in [1, \infty], \quad (\text{A.40a})$$

$$\|P_x(t)\|_p \leq \frac{1}{2\ell^3} \|\mathfrak{G}\|_p E(t) \quad \forall p \in [1, \infty]. \quad (\text{A.40b})$$

Using (A.39) and (A.40b), we obtain

$$E'(t) \leq (\ell^{-1} \|\phi\|_\infty + 2 \|\phi'\|_\infty) E(t) + \|\phi\|_\infty^2 \|\phi'\|_1. \quad (\text{A.41})$$

Then the Gronwall lemma ensures that  $E(t)$  does not blow up in finite time.

**Step 2: Equivalent system.** As in [6], let  $\xi \in \mathbb{R}$  and let  $y_0(\xi)$  be defined by

$$\int_0^{y_0(\xi)} (1 + u_0'^2) dx = \xi, \quad (\text{A.42})$$

and let  $y(t, \xi)$  be the function<sup>2</sup> defined by the equation

$$y_t(t, \xi) = u(t, y(t, \xi)), \quad y(0, \xi) = y_0(\xi). \quad (\text{A.43})$$

Let also  $v = v(t, \xi)$  and  $q = q(t, \xi)$  be defined as

$$v \stackrel{\text{def}}{=} 2 \arctan(u_x), \quad q \stackrel{\text{def}}{=} (1 + u_x^2) y_\xi, \quad (\text{A.44})$$

where  $u_x(t, \xi) = u_x(t, y(t, \xi))$ . Notice that

$$\frac{1}{1 + u_x^2} = \cos^2\left(\frac{v}{2}\right), \quad \frac{u_x}{1 + u_x^2} = \frac{\sin(v)}{2}, \quad \frac{u_x^2}{1 + u_x^2} = \sin^2\left(\frac{v}{2}\right), \quad \frac{\partial y}{\partial \xi} = q \cos^2\left(\frac{v}{2}\right). \quad (\text{A.45})$$

Integrating the last equality in (A.45), one obtains

$$y(t, \xi') - y(t, \xi) = \int_\xi^{\xi'} q(t, s) \cos^2\left(\frac{v(t, s)}{2}\right) ds. \quad (\text{A.46})$$

Using (A.45) and the change of variables  $x = y(t, \xi')$ , (A.46),  $P$  and  $P_x$  can be written in the new variables as

$$\begin{aligned} P(t, \xi) &= \frac{1}{4\ell} \int_{\mathbb{R}} \exp\left(\frac{-|y(t, \xi) - x|}{\ell}\right) u_x^2(t, x) dx \\ &= \frac{1}{4\ell} \int_{\mathbb{R}} \exp\left(-\frac{1}{\ell} \left| \int_\xi^{\xi'} q(t, s) \cos^2\left(\frac{v(t, s)}{2}\right) ds \right| \right) q(t, \xi') \sin^2\left(\frac{v(t, \xi')}{2}\right) d\xi', \end{aligned} \quad (\text{A.47})$$

$$\begin{aligned} P_x(t, \xi) &= \frac{1}{4\ell^2} \left( \int_{y(t, \xi)}^{+\infty} - \int_{-\infty}^{y(t, \xi)} \right) \exp(-|y(t, \xi) - x|/\ell) u_x^2(t, x) dx \\ &= \left( \int_\xi^{+\infty} - \int_{-\infty}^\xi \right) \exp\left(-\left| \int_\xi^{\xi'} q(t, s) \cos^2\left(\frac{v(t, s)}{2}\right) \frac{ds}{\ell} \right| \right) q(t, \xi') \sin^2\left(\frac{v(t, \xi')}{2}\right) \frac{d\xi'}{4\ell^2}. \end{aligned} \quad (\text{A.48})$$

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<sup>2</sup>It will turn out that  $y(t, \xi)$  is the characteristic of rB corresponding to  $y_0(\xi)$ , with speed  $u(t, y(t, \xi))$ .

Then, a system equivalent to the rB equation is given by

$$y_t = u, \quad y(0, \xi) = y_0(\xi), \quad (\text{A.49a})$$

$$u_t = -\ell^2 P_x, \quad u(0, \xi) = u_0(y_0(\xi)), \quad (\text{A.49b})$$

$$v_t = -P(1 + \cos(v)) - \sin^2(v/2), \quad v(0, \xi) = 2 \arctan(u'_0(y_0(\xi))), \quad (\text{A.49c})$$

$$q_t = q\left(\frac{1}{2} - P\right) \sin(v), \quad q(0, \xi) = 1. \quad (\text{A.49d})$$

In order to prove Theorem A.4.1, we prove first the global existence of the solution of the initial-value problem (A.49), then we infer that this solution yields a conservative solution of rB.

**Step 3: Local existence of the new system.** Our goal is to prove that the system of equations (A.49) is locally well-posed. The proof given in [6] for the Camassa–Holm equation is slightly simplified here.

We first solve a coupled 2x2 subsystem instead of a 3x3 subsystem in [6]. Let  $u_0$  be a function such that  $u_0 - \phi \in H^1$ , then  $y_0$  is well defined in (A.42). Note that the right-hand side of (A.49) does not depend on  $y$ . Since  $P$  and  $P_x$  depend only on  $v$  and  $q$ , the right-hand sides of equations (A.49b), (A.49c) and (A.49d) do not depend on  $u$ . Also, the equations (A.49c) and (A.49d) are coupled. Thus, we are left to show that the system of two equations

$$v_t = -P(1 + \cos v) - \sin^2 \frac{v}{2}, \quad v(0, \xi) = v_0(\xi) = 2 \arctan u'_0(y_0(\xi)), \quad (\text{A.50a})$$

$$q_t = q\left(\frac{1}{2} - P\right) \sin v, \quad q(0, \xi) = q_0(\xi) = 1. \quad (\text{A.50b})$$

is well defined in the space  $X \stackrel{\text{def}}{=} \mathcal{C}([0, T], L^\infty(\mathbb{R}, \mathbb{R}^2))$ .

Let  $U = (v, q)$ , and let  $\mathfrak{D} \subset X$  be the closed set satisfying  $U(0, \xi) = U_0(\xi)$  and

$$1/C \leq q(t, \xi) \leq C \quad \forall (t, \xi) \in [0, T] \times \mathbb{R}, \quad (\text{A.51a})$$

$$\left| \left\{ \xi, \sin^2 \frac{v(t, \xi)}{2} \geq \frac{1}{2} \right\} \right| \leq C \quad \forall t \in [0, T], \quad (\text{A.51b})$$

where  $C > 0$  is a constant. Then, for  $\xi_1 < \xi_2$ , we get from the equations (A.51)

$$\int_{\xi_1}^{\xi_2} q(\xi) \cos^2 \frac{v(\xi)}{2} d\xi \geq \int_{\left\{ \xi \in [\xi_1, \xi_2], \sin^2 \frac{v(t, \xi)}{2} \leq \frac{1}{2} \right\}} \frac{C^{-1}}{2} d\xi \geq \left[ \frac{\xi_2 - \xi_1}{2} - \frac{C}{2} \right] C^{-1}. \quad (\text{A.52})$$

Let  $\Gamma$  be defined as

$$\Gamma(\zeta) = \min \left\{ 1, \exp \left( \frac{1}{2\ell} - \frac{|\zeta|}{2\ell} C^{-1} \right) \right\}. \quad (\text{A.53})$$

Then, for  $(v, q) \in \mathfrak{D}$ , the exponential terms in (A.47) and (A.48) are smaller than  $\Gamma(\xi - \xi')$ .

Let  $P(\xi, v, q)$  be defined by (A.47). If  $(v, q) \in \mathfrak{D}$  then, using Young inequality,  $\partial_v P$  and  $\partial_q P$  are bounded, i.e., for  $\{U, \tilde{U}\} \in \mathfrak{D}$  we have

$$\|P(\xi, U) - P(\xi, \tilde{U})\|_X \lesssim \|U - \tilde{U}\|_X, \quad (\text{A.54})$$



where the symbol  $\lesssim$  means “less or equal” with a constant depending only on  $C$  and  $\ell$ . Then, for  $T$  small enough, the Picard operator

$$(\mathcal{P}(U))(t, \xi) \stackrel{\text{def}}{=} U_0 + \int_0^t \left( -(1 + \cos v)P - \sin^2 \frac{v}{2}, q(\frac{1}{2} - P) \sin v \right) d\tau, \quad (\text{A.55})$$

is a contraction from  $\mathfrak{D}$  to  $\mathfrak{D}$ . The local existence of the solution of the Cauchy problem (A.50) follows at once.

**Step 4: Global existence for the equivalent system.** After proving the local existence of the solution of system (A.50), an estimate of the quantity

$$\|q(t)\|_\infty + \|1/q(t)\|_\infty + \|\sin^2(v(t)/2)\|_1 + \|v(t)\|_\infty, \quad (\text{A.56})$$

is needed to ensure that the solutions exist for all time. Let  $u$  be defined as

$$u(t, \xi) \stackrel{\text{def}}{=} u_0(y_0(\xi)) - \int_0^t \ell^2 P_x(s, \xi) ds, \quad (\text{A.57})$$

and let  $y$  be the family of characteristics

$$y(t, \xi) \stackrel{\text{def}}{=} y_0(\xi) + \int_0^t u(s, \xi) ds, \quad (\text{A.58})$$

and, finally, let  $\phi(t, \xi) \stackrel{\text{def}}{=} \phi(y(t, \xi))$ . Our task here is to show that the modified energy

$$\tilde{E}(t) = \int_{\mathbb{R}} \left[ (u - \phi)^2 \cos^2 \frac{v}{2} + \ell^2 \sin^2 \frac{v}{2} \right] q d\xi \quad (\text{A.59})$$

does not blow-up in finite time.

The system (A.49) implies that

$$\left( q \cos^2 \frac{v}{2} \right)_t = \frac{1}{2} q \sin v, \quad \left( q \sin^2 \frac{v}{2} \right)_t = q_t - \frac{1}{2} q \sin v = -qP \sin v, \quad (\text{A.60})$$

while the equations (A.47) and (A.48) imply that

$$P_\xi = q P_x \cos^2 \frac{v}{2}, \quad \ell^2 (P_x)_\xi = q P \cos^2 \frac{v}{2} - \frac{1}{2} q \sin^2 \frac{v}{2}. \quad (\text{A.61})$$

From (A.49), (A.57) and (A.61), we have

$$(u_\xi - \frac{1}{2} q \sin v)_t = 0,$$

and, for  $t = 0$ , we have from (A.44) and (A.45)

$$u_\xi - \frac{1}{2} q \sin v = u_x \frac{\partial y}{\partial \xi} - \frac{1}{2} \sin v = 0.$$

Thus, as long as the solution of (A.49) is defined, the equality

$$u_\xi = \frac{1}{2} q \sin v \quad (\text{A.62})$$

holds. Therefore, the equations (A.60), (A.61), (A.62) yield

$$\left[ \left( u^2 \cos^2 \frac{v}{2} + \ell^2 \sin^2 \frac{v}{2} \right) q \right]_t + \left[ 2 \ell^2 u P - \frac{1}{3} u^3 \right]_\xi = 0, \quad (\text{A.63})$$

which expresses conservation of energy in the  $(t, \xi)$ -variables when  $u_+ = u_- = 0$ , i.e., for  $\phi = 0$ .

From (A.60), (A.62) and (A.58), we have

$$\left( q \cos^2 \frac{v}{2} \right)_t = u_\xi = \left( \frac{\partial y}{\partial \xi} \right)_t,$$

implying that the equality

$$\frac{\partial y}{\partial \xi} = q \cos^2 \frac{v}{2}, \quad (\text{A.64})$$

holds for the  $(t, \xi)$ -variables (note that the equality is true for  $t = 0$  from (A.45)). Then, using (A.58) and (A.64), we get

$$\phi_t(t, \xi) = \frac{d}{dt} \phi(y(t, \xi)) = u \phi', \quad \phi_\xi(t, \xi) = q \cos^2 \left( \frac{v}{2} \right) \phi', \quad (\text{A.65})$$

so, using (A.57), (A.49), (A.65) and (A.60), we obtain

$$\begin{aligned} & \left[ (\phi^2 - 2u\phi) q \cos^2 \frac{v}{2} \right]_t + \left[ \frac{1}{2} \phi u^2 + \frac{1}{2} \phi (u - \phi)^2 - \frac{1}{6} \phi^3 \right]_\xi = \\ & \quad 2 \ell^2 P_x \phi q \cos^2 \frac{v}{2} - 2u \phi_\xi (u - \phi) + \frac{1}{2} \phi_\xi u^2 + (\phi^2 - 2u\phi) u_\xi \\ & \quad + \phi u u_\xi + \frac{1}{2} \phi_\xi (u - \phi)^2 + \phi (u - \phi)_\xi (u - \phi) - \frac{1}{2} \phi^2 \phi_\xi \\ & = -\phi_\xi (u - \phi)^2 + 2 \ell^2 P_x \phi q \cos^2 \frac{v}{2} - 2 \phi \phi_\xi (u - \phi). \end{aligned} \quad (\text{A.66})$$

Adding (A.63) and (A.66), with the trivial relation  $2\phi(u - \phi) \leq \phi^2 + (u - \phi)^2$ , then integrating the result with respect of  $\xi$ , we get

$$\tilde{E}'(t) \leq \int_{\mathbb{R}} \left( 2 \ell^2 |\phi P_x| q \cos^2 \frac{v}{2} + |\phi_\xi| (2(u - \phi)^2 + \phi^2) \right) d\xi. \quad (\text{A.67})$$

Using (A.64) and (A.65) with the change of variables  $x = y(t, \xi)$ , then exploiting (A.40), one obtains

$$\begin{aligned} \tilde{E}'(t) & \leq \int_{\{\xi, \cos v \neq -1\}} \left( 2 \ell^2 |\phi P_x| + |\phi'| (2(u - \phi)^2 + \phi^2) \right) q \cos^2 \frac{v}{2} d\xi \\ & = \int_{\mathbb{R}} \left( 2 \ell^2 |\phi P_x| + |\phi'| (2(u - \phi)^2 + \phi^2) \right) dx \\ & \leq \left( \ell^{-1} \|\phi\|_\infty + 2 \|\phi'\|_\infty \right) E(t) + \|\phi\|_\infty^2 \|\phi'\|_1, \end{aligned}$$

where  $P_x$  in the second equation is defined as  $P_x = \frac{1}{2} \mathfrak{G}_x * u_x^2$ .

From (A.35) and (A.59), and using the change of variables  $x = y(t, \xi)$ , one can show easily that

$$E(t) = \int_{\{\xi, \cos v \neq -1\}} \left[ (u - \phi)^2 \cos^2 \frac{v}{2} + \ell^2 \sin^2 \frac{v}{2} \right] q \, d\xi \leq \tilde{E}(t). \quad (\text{A.68})$$

Thence, the uniform estimate of  $\tilde{E}(t)$  on any bounded interval  $[0, T]$  follows by using Gronwall lemma.

We can show now that the quantity (A.56) does not blow up in finite time. Using Young inequality, (A.47), (A.48) and (A.53), one obtains

$$\|P(t)\|_p \leq \frac{1}{4\ell^3} \|\Gamma\|_p \tilde{E}(t) \quad \forall p \in [1, \infty], \quad (\text{A.69a})$$

$$\|P_x(t)\|_p \leq \frac{1}{4\ell^4} \|\Gamma\|_p \tilde{E}(t) \quad \forall p \in [1, \infty]. \quad (\text{A.69b})$$

The inequalities (A.69) are the identical estimates as (A.40), but in the  $(t, \xi)$ -variables. Using (A.49d) and (A.69), we get

$$|q_t| \leq \left( \frac{1}{2} + \frac{1}{4\ell^3} E(t) \right) q, \quad (\text{A.70})$$

implying that  $\|q(t)\|_\infty + \|1/q(t)\|_\infty$  does not blow-up in finite time. The equation (A.49c) and (A.69) imply that  $\|v(t)\|_\infty$  remains bounded on any finite interval  $[0, T]$ . Also, the boundedness of the energy  $\tilde{E}(t)$  and  $\|1/q(t)\|_\infty$  implies that  $\|\cos^2(v(t)/2)\|_1$  remains bounded on any interval  $[0, T]$ . This completes the proof of the global existence.

**Step 5: Global existence of a conservative solution.** Here, we show that the global solution of the equivalent system (A.49) yields a global solution of the rB equation.

Let  $u$  and  $y$  be defined by (A.57) and (A.58), respectively. We claim that the solution of rB can be written as

$$u(t, x) = u(t, \xi), \quad y(t, \xi) = x. \quad (\text{A.71})$$

Using (A.62), (A.65) and the change of variables  $x = y(t, \xi)$  with (A.64), one obtains

$$\begin{aligned} |u(t, \xi) - \phi(t, \xi)|^2 &\leq 2 \int_{\mathbb{R}} |u - \phi| |u_\xi - \phi_\xi| \, d\xi \\ &\leq 2 \int_{\mathbb{R}} |u - \phi| q \left( \sin \frac{v}{2} \cos \frac{v}{2} + \phi' \cos^2 \frac{v}{2} \right) \, d\xi \\ &\leq 2 E(t) + \|\phi'\|_2^2, \end{aligned}$$

implying that  $\|u(t)\|_\infty$  is uniformly bounded on any bounded interval  $[0, T]$ . Therefore, from (A.58), we get

$$y_0(\xi) - \|u(t)\|_\infty t \leq y(t, \xi) \leq y_0(\xi) + \|u(t)\|_\infty t, \quad (\text{A.72})$$

and thus

$$\lim_{\xi \rightarrow \pm\infty} y_0(t, \xi) = \pm\infty. \quad (\text{A.73})$$

The equation (A.64) implies that the mapping  $\xi \mapsto y(t, \xi)$  is non-decreasing and, if for  $\xi < \xi'$  we have  $y(t, \xi) = y(t, \xi')$ , then  $\sin(v) = 2 \cos(v/2) \sin(v/2) = 0$  between  $\xi$  and  $\xi'$  (see eq. A.64). Integrating (A.62) with respect to  $\xi$ , one obtains that  $u(t, \xi) = u(t, \xi')$ , so  $u$  is well-defined in (A.71).

Proceeding as in [6, section 4], we can prove that for each interval  $[t_1, t_2]$  there exists a constant  $C = C(\ell, t_2)$  such that,  $\forall t \in [t_1, t_2 - h]$ ,

$$\int_{\mathbb{R}} |u(t+h, x) - u(t, x)|^2 dx \leq C h^2, \quad (\text{A.74})$$

and then  $u$  satisfies

$$\frac{d}{dt} u(t, y(t, \xi)) = -P_x(t, \xi). \quad (\text{A.75})$$

The inequality (A.74) implies that  $u$  belongs to  $Lip([0, T], L_{loc}^2)$ . Straightforward calculations show that, for  $x = y(t, \xi)$  and for  $\cos(v(t, \xi)) \neq 1$ , we have

$$u_x(t, x) = \tan\left(\frac{v(t, \xi)}{2}\right) = \frac{\sin(v(t, \xi))}{1 + \cos(v(t, \xi))}. \quad (\text{A.76})$$

Using the change of variables  $x = y(t, \xi)$  with (A.64), one can show that  $u$  is a global solution of rB.

In order to prove (A.17), let  $\psi$  be a test function and let  $\tilde{\psi}(t, \xi) = \psi(t, y(t, \xi))$ . Multiplying (A.60b) by  $\tilde{\psi}$  and integrating the result with respect to  $\xi$ , one obtains

$$\begin{aligned} 0 &= \int_0^{+\infty} \int_{\mathbb{R}} [(q \sin^2 v/2)_t + q P \sin v] \tilde{\psi} dt d\xi, \\ &= \int_0^{+\infty} \int_{\mathbb{R}} [-\tilde{\psi}_t q \sin^2 v/2 + \tilde{\psi} q P \sin v] dt d\xi + \int_{\mathbb{R}} \tilde{\psi}(0, x) \sin^2 v(0, \xi)/2 d\xi, \\ &= \int \int_{\{\cos v > -1\}} [-\tilde{\psi}_t q \sin^2 v/2 + \tilde{\psi} q P \sin v] dt d\xi + \int_{\{v_0 > -\pi\}} \tilde{\psi}(0, x) \sin^2 v(0, \xi)/2 d\xi, \\ &\quad + \int \int_{\{\cos v = -1\}} -q \tilde{\psi}_t dt d\xi + \int_{\{v_0 = -\pi\}} \tilde{\psi}(0, x) d\xi. \end{aligned} \quad (\text{A.77})$$

It is clear from (A.49c) that

$$|\{\xi, \cos v(t, \xi) = -1\}| = 0 \quad \text{for almost all } t \geq 0. \quad (\text{A.78})$$

Using that  $\tilde{\psi}_t = \psi_t + u \psi_x$  and the change of variables  $x = y(t, \xi)$ , the equation (A.17) follows in the sense of distributions.

Finally, let  $u_0 \in H^1$ . The equation (A.63) implies that

$$\frac{d}{dt} \int_{\mathbb{R}} \left( u^2 \cos^2 \frac{v(t, \xi)}{2} + \ell^2 \sin^2 \frac{v(t, \xi)}{2} \right) q(t, \xi) d\xi = 0, \quad (\text{A.79})$$

hence  $\tilde{E}(t) = \tilde{E}(0)$ . In addition, using the change of variables  $x = y(t, \xi)$  with (A.64) and (A.76), one obtains

$$\int_{\mathbb{R}} u(t, x)^2 + \ell^2 u_x(t, x)^2 dx = \int_{\{\xi, \cos v(t, \xi) > -1\}} \left( u^2 \cos^2 \frac{v(t, \xi)}{2} + \ell^2 \sin^2 \frac{v(t, \xi)}{2} \right) q(t, \xi) d\xi. \quad (\text{A.80})$$

Using (A.78), the conservation of the energy (A.34) follows.

We end this demonstration with the proof of the property (A.33). The equation (A.49c) implies that  $v$  is decreasing in time. Further, if  $v(T, \xi) = -\pi$  (corresponding to an infinite value of  $u_x$ , see (A.76) above) then  $v_t(T, \xi) = -1$ , meaning that the value of  $v(t, \xi)$  crosses  $-\pi$  and  $v(t, \xi) < -\pi$  for all  $t > T$ . Then, (A.33) follows using (A.76).  $\square$

## A.4.2 Global existence of dissipative solutions

We start this subsection by defining *dissipative* solutions, this kind of solutions being very important for applications. We note in passing that when  $\ell$  goes to zero, we expect to recover the entropy solution of the Burgers equation. However, in Section A.6, we show that the limit (up to a subsequence) is a solution of the Burgers equation with a remaining forcing term.

**Definition A.4.2.** *A function  $u$  is called a dissipative solution of rB if*

- *The function  $u$  belongs to  $\text{Lip}([0, T], L^2_{loc}) \cap L^\infty([0, T], \dot{H}^1)$  for all  $T > 0$ ;*
- *$u$  satisfies the equation (A.12), with an initial data  $u(0, x) = u_0(x)$ ;*
- *$u$  satisfies the inequality*

$$\left[ \frac{1}{2} u^2 + \frac{1}{2} \ell^2 u_x^2 \right]_t + \left[ \frac{1}{3} u^3 + \ell^2 u P + \frac{1}{2} \ell^2 u u_x^2 \right]_x \leq 0, \quad (\text{A.81})$$

*in the sense of distributions.*

- *There exists a constant  $C$  such that  $u$  satisfies the Oleinik inequality*

$$u_x(t, x) \leq C/t \quad \forall t, x.$$

**Remark A.4.4.** *Following [7], we construct in Theorem A.4.2 a dissipative solution of rB with  $C = 2$ . The entropy solutions of the classical Burgers equation satisfy the Oleinik inequality with  $C = 1$ .*

As mentioned above, when  $v$  crosses the value  $-\pi$ ,  $u_x$  jumps from  $-\infty$  to  $+\infty$ , which means that the Oleinik inequality cannot be satisfied. Thus, to enforce the Oleinik inequality, the value of  $v$  is not allowed to leave the interval  $[-\pi, \pi[$ . For that purpose, the system (A.50) is modified (as in [7]) to become

$$u_t = -\ell^2 P_x, \quad (\text{A.82a})$$

$$v_t = \begin{cases} -P(1 + \cos v) - \sin^2(v/2), & v > -\pi, \\ 0 & v \leq -\pi, \end{cases} \quad (\text{A.82b})$$

$$q_t = \begin{cases} q(\frac{1}{2} - P) \sin(v), & v > -\pi \\ 0 & v \leq -\pi. \end{cases} \quad (\text{A.82c})$$

and  $P$  and  $P_x$  are also modified as

$$P(t, \xi) = \frac{1}{4\ell} \int_{\mathbb{R}} \exp \left\{ -\frac{1}{\ell} \left| \int_{\xi}^{\xi'} \bar{q}(t, s) \cos^2 \frac{v(t, s)}{2} ds \right| \right\} \bar{q}(t, \xi') \sin^2 \frac{v(t, \xi')}{2} d\xi', \quad (\text{A.83})$$

$$P_x(t, \xi) = \frac{1}{4\ell^2} \left( \int_{\xi}^{+\infty} - \int_{-\infty}^{\xi} \right) \exp \left\{ -\frac{1}{\ell} \left| \int_{\xi}^{\xi'} \bar{q}(t, s) \cos^2 \frac{v(t, s)}{2} ds \right| \right\} \bar{q}(t, \xi') \sin^2 \frac{v(t, \xi')}{2} d\xi', \quad (\text{A.84})$$

where  $\bar{q}(t, \xi) = q(t, \xi)$  if  $v(t, \xi) > -\pi$  and  $\bar{q}(t, \xi) = 0$  if  $v(t, \xi) \leq -\pi$ . The system (A.82) is the key tool to prove the following theorem.

**Theorem A.4.2.** *Let  $u_0 \in \dot{H}^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ . If there exist a Lipschitz function  $\phi$  such that  $\phi' \in L^1(\mathbb{R})$  and with  $u_0 - \phi \in H^1(\mathbb{R})$ , then there exists a global dissipative solution  $u$  of the equation (A.12), such that  $u(t, \cdot) - \phi \in H^1(\mathbb{R})$  for all  $t > 0$ . In addition, for all  $t > 0$*

$$u_x(t, x) \leq 2/t \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \quad (\text{A.85})$$

and if  $u_0 \in H^1$ , then for all  $t > 0$

$$\int_{\mathbb{R}} [u(t, x)^2 + \ell^2 u_x(t, x)^2] dx \leq \int_{\mathbb{R}} [u_0(x)^2 + \ell^2 u_0'(x)^2] dx. \quad (\text{A.86})$$

**Remark A.4.5.** *Due to the loss of the Oleinik inequality (cf. Remark A.4.3), the system (A.49) is slightly modified to (A.82) in order to obtain dissipative solutions of rB that satisfies the Oleinik inequality (A.85).*

**Remark A.4.6.** *In general, if the initial datum satisfies  $u_0' \leq M \in \mathbb{R} \cup \{+\infty\}$ , then the Oleinik inequality (A.85) can be improved as*

$$u_x(t, x) \leq 2M / (Mt + 2) \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}, \quad (\text{A.87})$$

as shown in (A.97) below.

The idea of the proof is similar to Theorem A.4.1 above and it is done in several steps:

**Step 1: Existence of a solution.** As in the proof of Theorem A.4.1, it suffices to show that (A.82b) and (A.82c) are locally well posed in the domain  $\mathfrak{D} \subset X$ ,  $\mathfrak{D}$  being defined below and  $X \stackrel{\text{def}}{=} \mathcal{C}([0, T], L^\infty(\mathbb{R}, \mathbb{R}^2))$ .

Note that if  $v$  is near  $-\pi$  the right-hand side of (A.82b) is discontinuous. To avoid this discontinuity, the system (A.82) is replaced, as in [7], by

$$U_t(t, \xi) = F(U(t, \xi)) + G(\xi, U(t, \cdot)), \quad U = (v, q), \quad (\text{A.88})$$

with

$$F(U) \stackrel{\text{def}}{=} \begin{cases} (-\sin^2 \frac{v}{2}, \frac{1}{2} q \sin v) & v > -\pi \\ (-1, 0) & v \leq -\pi. \end{cases}, \quad G(U) \stackrel{\text{def}}{=} \begin{cases} (-P(1 + \cos v), -Pq \sin v) & v > -\pi \\ (0, 0) & v \leq -\pi. \end{cases}.$$

Note also that, as long as the solution to (A.88) is well defined, replacing  $v$  by  $\max\{-\pi, v\}$  gives a solution of the equations (A.82b) and (A.82c). In the rest of this step, our aim is to show that the system (A.88) is locally well-posed. Let  $\delta \in ]0, \frac{2\pi}{3}]$  and let  $\Lambda$  be defined by

$$\Lambda \stackrel{\text{def}}{=} \left\{ \xi, v_0(\xi) \in ]-\pi, \delta - \pi] \right\}. \quad (\text{A.89})$$

The equation (A.88) implies that, if  $v \in ]-\pi, \delta - \pi] \subset ]-\pi, -\frac{\pi}{3}]$ , then  $v_t \leq -\frac{1}{2}$ . Let  $\mathfrak{D} \subset X$  satisfy  $U(0, \xi) = U_0(\xi)$  and

$$1/C \leq q(t, \xi) \leq C \quad \forall (t, \xi) \in [0, T] \times \mathbb{R}, \quad (\text{A.90a})$$

$$\left| \left\{ \xi, \sin^2(v(t, \xi)/2) \geq \frac{1}{2} \right\} \right| \leq C \quad \forall t \in [0, T], \quad (\text{A.90b})$$

$$\|U(t) - U(s)\|_\infty \leq C|t - s| \quad \forall t, s \in [0, T], \quad (\text{A.90c})$$

$$v(t, \xi) - v(s, \xi) \leq -\frac{t-s}{2} \quad \forall \xi \in \Lambda, 0 \leq s \leq t \leq T, . \quad (\text{A.90d})$$

Taking  $(v, q) \in \mathfrak{D}$  and using (A.69), one gets that the right-hand sides of (A.82b) and (A.82c) are bounded. However, the inequality (A.54) is no longer true and we have instead

$$\|P(U) - P(\tilde{U})\|_\infty \lesssim \|U - \tilde{U}\|_\infty + \left| \left\{ \xi, (v(\xi) + \pi)(\tilde{v}(\xi) + \pi) < 0 \right\} \right|, \quad (\text{A.91})$$

which implies that

$$\|F(U) - F(\tilde{U})\|_\infty \lesssim \|U - \tilde{U}\|_\infty, \quad (\text{A.92})$$

$$\|G(U) - G(\tilde{U})\|_\infty \lesssim \|U - \tilde{U}\|_\infty + \left| \left\{ \xi, (v(\xi) + \pi)(\tilde{v}(\xi) + \pi) < 0 \right\} \right|. \quad (\text{A.93})$$

In order to estimate the second term of the right-hand side of the last equation, the crossing time  $\tau$  is defined as

$$\tau(\xi) \stackrel{\text{def}}{=} \sup \{t \in [0, T], v(t, \xi) > -\pi\}. \quad (\text{A.94})$$

Note that the equation (A.90c) implies that  $|v(t, \xi) - v_0(\xi)| \leq Ct$ . So, if  $\xi \notin \Lambda$  then

$$\min \{\tau(\xi), \tilde{\tau}(\xi)\} \geq \delta/C.$$

Taking  $T$  small enough ( $T < \delta/C$ ) and using (A.90d), one obtains

$$\begin{aligned} \int_0^T \left| \left\{ \xi, (v(\tau, \xi) + \pi)(\tilde{v}(\tau, \xi) + \pi) < 0 \right\} \right| d\tau &\leq \int_\Lambda |\tau(\xi) - \tilde{\tau}(\xi)| d\xi \\ &\leq 2|\Lambda| \|U - \tilde{U}\|_\infty. \end{aligned}$$

Now, the Picard operator

$$(\mathcal{P}(U))(t, \xi) = U_0 + \int_0^t [F(U) + G(U)] d\tau, \quad (\text{A.95})$$

satisfies

$$\|\mathcal{P}(U) - \mathcal{P}(\tilde{U})\|_\infty \leq K(T + |\Lambda|) \|U - \tilde{U}\|_\infty, \quad (\text{A.96})$$

where  $K$  depends only on  $C$  and  $\ell$ . Since  $\sin^2 \frac{v_0}{2} \in L^1$ , by choosing  $\delta > 0$  small enough, one can make  $|\Lambda|$  arbitrary small. Choosing also  $T$  small enough, one obtains the local existence of the solution of the system (A.88), yielding a solution of (A.82). The rest of the proof of the existence can be done following the proof of Theorem A.4.1.

**Step 2: Oleinik inequality and the dissipation of the energy.** The equation (A.82b) implies that if  $v(0, \xi) \leq 0$ , then for all  $t \geq 0$   $v(t, \xi)$  remains in  $[-\pi, 0]$ . If  $v_0(\xi) \in ]0, \pi[$  then, as long as  $v$  is positive, the following inequality holds

$$\left[ \arctan \frac{v}{2} \right]_t \leq -\frac{1}{2} \arctan^2 \frac{v}{2}.$$

This implies that, if  $\arctan \frac{v_0(\xi)}{2} \leq M$ , then

$$u_x = \arctan \frac{v(t, \xi)}{2} \leq \frac{2M}{Mt + 2}. \quad (\text{A.97})$$

The Oleinik inequality (A.85) follows taking  $M = +\infty$  and using (A.76).

In order to prove the dissipation of the energy (A.81), let  $\psi$  be a non-negative test function, then we follow the same computations in the proof of Theorem A.4.1. Since (A.78) is no longer true for the system (A.82), one can obtain from (A.77) that

$$\begin{aligned} & \iint_{[0, +\infty[ \times \mathbb{R}} [-u_x^2 \psi_t - uu_x^2 \psi_x + u_x P] dt dx - \int_{\mathbb{R}} u_0^2(x) \psi(0, x) dx \\ &= - \int_{\{\tau(\xi) < +\infty\}} q(\tau(\xi), \xi) \tilde{\psi}(\tau(\xi), \xi) d\xi \leq 0, \end{aligned}$$

where  $\tau(\xi)$  is the crossing time defined as  $\tau(\xi) \stackrel{\text{def}}{=} \sup \{t \geq 0, v(t, \xi) > -\pi\}$ . Since (A.16) is satisfied (see Remark A.4.1), the dissipation of the energy (A.81) follows.

If  $u_0 \in H^1$ , as in the last step of the proof of Theorem A.4.1, one can show that (A.79) and (A.80) hold for the solution of (A.82), while the measure in (A.78) is not always zero. Then, the dissipation of the energy (A.86) follows.  $\square$

## A.5 Comments on the conservative and dissipative solutions

As shown above, the major difference between the conservative system (A.49) and the dissipative system (A.82) is that the system (A.49) allows  $v$  to cross the value  $-\pi$ , causing a jump of  $u_x$  from  $-\infty$  to  $+\infty$  (see eq. (A.76)), which implies (A.33), thence the loss of the Oleinik inequality (Remark A.4.3). But, the value  $v = -\pi$  is a barrier that cannot be crossed for the system (A.82). It follows that if  $v(t, \xi_0) = -\pi$  at a time  $t$ , then  $v(\tau, \xi_0) = -\pi$  for all times  $\tau \geq t$  (see figure A.1). This property is important to obtain the Oleinik inequality (A.85), which yields the dissipation of the energy (A.86).



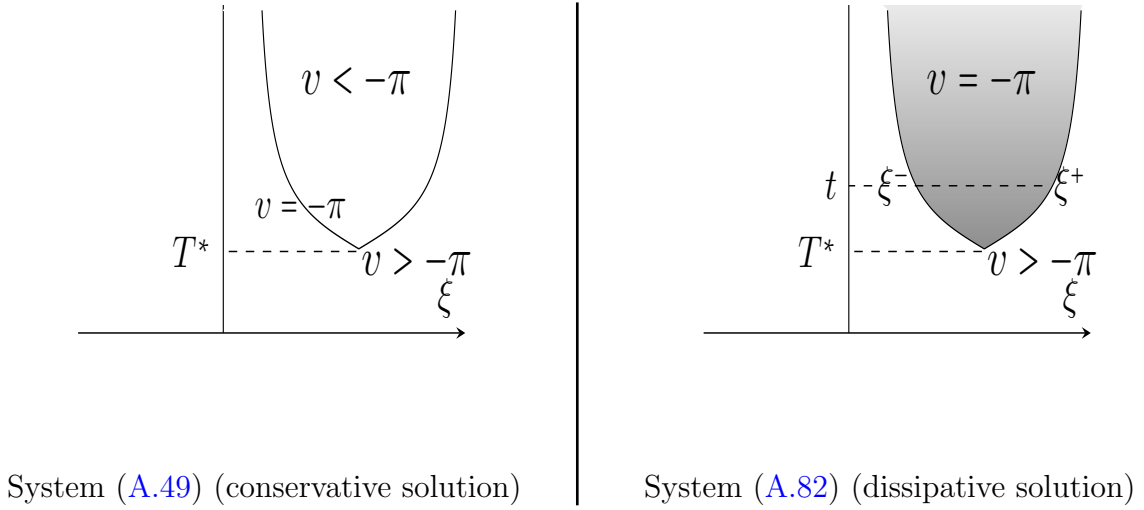


Figure A.1 – Regions where  $v = -\pi$ .

The figure A.1 shows the domains where  $v = -\pi$  for the systems (A.49) and (A.82).

Note that the dissipative solution of rB has similar properties (A.85) and (A.86) as the entropy solution of the classical inviscid Burgers equation that are

$$u_x(t, x) \leq 1/t, \quad \|u\|_2 \leq \|u_0\|_2. \quad (\text{A.98})$$

Because of this similarity, the dissipative solutions of rB are more likely to converge to the entropic solution of the Burgers equation when  $\ell \rightarrow 0$ , but this result remains to be proven.

## A.6 The limiting cases $\ell \rightarrow 0$ and $\ell \rightarrow +\infty$ for dissipative solutions

Taking formally  $\ell = 0$ , the rB equation becomes the classical Burgers equation, and letting  $\ell \rightarrow +\infty$  it becomes the Hunter–Saxton equation. In this section, we study the compactness of the dissipative solutions when taking  $\ell \rightarrow 0$  and  $\ell \rightarrow +\infty$ .

Let the initial datum  $u_0$  be taken in  $H^1$ , with  $u'_0 \in L^1$  and  $M \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} u'_0(x) < +\infty$ . Let also  $u^\ell$  be the dissipative solution of the rB equation given in Theorem A.4.2. In order to take the limit, an estimate on the total variation of  $u^\ell$  is needed. For that purpose, the following Lemma is given

**Lemma A.6.1.** *[BV estimate] If  $u'_0$  satisfies the conditions of Theorem A.4.2 with  $u'_0 \in L^1$  and  $u'_0(x) \leq M \forall x$ , then for all  $t \in \mathbb{R}^+$*

$$\text{TV}u^\ell(t, \cdot) = \|u_x^\ell(t, \cdot)\|_1 \leq \|u'_0\|_1 \left( \frac{Mt + 2}{2} \right)^2. \quad (\text{A.99})$$

*Proof.* We commence this proof by a formal computation on  $\|u_x^\ell(t, \cdot)\|_1$ . Multiplying (A.15) by  $s \stackrel{\text{def}}{=} \text{sgn}(u_x^\ell)$  we have  $|u_x^\ell|_t + (u^\ell u_x^\ell)_x s = (u_x^{\ell 2}/2 - P)s$ . Due to Saks' lemma [19], the integral  $\int_{\mathbb{R}} (u^\ell u_x^\ell)_x s dx$  equals to zero. Using that  $P - u_x^{\ell 2}/2 = \ell^2 P_{xx}$  and the Oleinik inequality (A.85) one obtains

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} |u_x^\ell| dx &= \int_{\mathbb{R}} \left(\frac{1}{2} u_x^{\ell 2} - P\right) s dx \\
&= \int_{\{u_x^\ell \geq 0\}} \left(\frac{1}{2} u_x^{\ell 2} - P\right) dx + \int_{\{u_x^\ell < 0\}} \left(P - \frac{1}{2} u_x^{\ell 2}\right) dx \\
&= 2 \int_{\{u_x^\ell \geq 0\}} \left(\frac{1}{2} u_x^{\ell 2} - P\right) dx + \ell^2 \int_{\mathbb{R}} P_{xx} dx \\
&\leq \frac{2M}{Mt + 2} \int_{\mathbb{R}} |u_x^\ell| dx.
\end{aligned} \tag{A.100}$$

Thus, the result follows by Gronwall's lemma.

The Saks lemma is used for smooth solutions. If  $u$  is not smooth enough, the same estimates can be done on the  $\xi$ -variable in the system (A.82). For  $v \in ]-\pi, \pi[$ , the equation (A.62) implies

$$\tilde{s} \stackrel{\text{def}}{=} \text{sgn}(u_\xi^\ell) = \text{sgn}(\sin(v^\ell)) = \text{sgn}(\sin(\frac{v^\ell}{2})), \quad \cos(\frac{v^\ell}{2}) \geq 0. \tag{A.101}$$

Note that  $\tan(v^\ell/2) \leq \tan(v_0/2) = 2M/(Mt + 2)$  from (A.97). Differentiating (A.82a) w.r.t  $\xi$ , multiplying by  $\tilde{s}$  — and using (A.61), (A.62) and  $\sin v = 2 \sin(v/2) \cos(v/2)$  — one gets

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} |u_\xi^\ell| d\xi &= -\ell^2 \int_{\mathbb{R}} \tilde{s} (P_x)_\xi d\xi \\
&= -\ell^2 \int_{\{\tilde{s} > 0\}} (P_x)_\xi d\xi + \ell^2 \int_{\{\tilde{s} < 0\}} (P_x)_\xi d\xi \\
&= -2\ell^2 \int_{\{\tilde{s} > 0\}} (P_x)_\xi d\xi + \ell^2 \int_{\mathbb{R}} (P_x)_\xi d\xi \\
&= -2 \int_{\{\tilde{s} > 0\}} \left( q^\ell P \cos^2 \frac{v^\ell}{2} - \frac{1}{2} q^\ell \sin^2 \frac{v^\ell}{2} \right) d\xi \\
&\leq \int_{\{\tilde{s} > 0\}} q^\ell \sin \frac{v^\ell}{2} \cos \frac{v^\ell}{2} \tan \frac{v^\ell}{2} d\xi \\
&\leq \frac{2M}{Mt + 2} \int_{\mathbb{R}} |u_\xi^\ell| d\xi.
\end{aligned} \tag{A.102}$$

Gronwall lemma then implies that

$$\|u_\xi\|_1 \leq \|(u_0)_\xi\|_1 \left( \frac{Mt + 2}{2} \right)^2. \tag{A.103}$$

Note that the last inequality is on the  $\xi$ -variable. Using that the application  $\xi \mapsto y(t, \xi)$  is not decreasing for all  $t$  and using that  $\text{TV} f = \|f'\|_1$  for smooth solutions ( $f \in W_{loc}^{1,1}$ ), the result follows.  $\square$

### A.6.1 The limiting case $\ell \rightarrow 0$

The goal of this subsection is to show that when  $\ell \rightarrow 0$ , the dissipative solution  $u^\ell$  converges (up to a subsequence) to a function  $u$  satisfying the Burgers equation with a source term:

$$u_t + \frac{1}{2} [u^2]_x = -\mu_x, \quad (\text{A.104})$$

where  $\mu$  is a measure such that  $0 \leq \mu \in L^\infty([0, +\infty[, \mathcal{M}^1)$ . In Proposition A.6.1 below, we show that the measure  $\mu$  is zero before the appearance of singularities. The question whether or not  $\mu$  is zero after singularities is open, in general. The following theorem can be stated

**Theorem A.6.1.** *Let  $u_0 \in H^1$ , such that  $u'_0 \in L^1$  and  $u'_0(x) \leq M \forall x$ , then there exists  $u \in L^\infty([0, T], BV(\mathbb{R}))$  for all  $T > 0$ , such that there exists a subsequence of  $u^\ell$  (also noted  $u^\ell$ ) and for all interval  $\mathcal{I} \subset \mathbb{R}$  we have*

$$u^\ell \xrightarrow{\ell \rightarrow 0} u \text{ in } \mathcal{C}([0, T], L^1(\mathcal{I})), \quad (\text{A.105})$$

and  $u$  satisfies the equation (A.104). Moreover,  $u$  satisfies the Oleinik inequality

$$u_x(t, x) \leq \frac{2M}{Mt + 2} \quad \text{in } \mathcal{D}'(\mathbb{R}). \quad (\text{A.106})$$

**Remark A.6.1.** *If  $\mu = 0$  then, due to the Oleinik inequality,  $u$  is the unique entropy solution of the Burgers equation.*

In order to prove Theorem A.6.1, the following definition and lemma are needed: Let  $\mathcal{I} \subset \mathbb{R}$  be a bounded interval and let

$$W(\mathcal{I}) \stackrel{\text{def}}{=} \{f \in \mathcal{D}'(\mathcal{I}), \exists F \in L^1(\mathcal{I}) \text{ such that } F' = f\}, \quad (\text{A.107})$$

where the norm of the space  $W(\mathcal{I})$  is given by

$$\|f\|_{W(\mathcal{I})} \stackrel{\text{def}}{=} \inf_{c \in \mathbb{R}} \|F + c\|_{L^1(\mathcal{I})} = \min_{c \in \mathbb{R}} \|F + c\|_{L^1(\mathcal{I})}. \quad (\text{A.108})$$

**Lemma A.6.2.** *The space  $W(\mathcal{I})$  is a Banach space and the embedding*

$$L^1(\mathcal{I}) \hookrightarrow W(\mathcal{I}), \quad (\text{A.109})$$

*is continuous.*

*Proof.* Let  $(f_n)_{n \in \mathbb{N}}$  be a Cauchy sequence in  $W(\mathcal{I})$  and let  $F_n$  be a primitive of  $f_n$ . From the definition of the norm (A.108), there exists a constant  $c_n$  such that  $(\tilde{F}_n - c_n)_{n \in \mathbb{N}}$  (where  $\tilde{F}_n = F_n + c_n$ ) is a Cauchy sequence in  $L^1(\mathcal{I})$ . Let  $\tilde{F}$  be the limit of  $\tilde{F}_n$  in  $L^1(\mathcal{I})$ . Then

$$\|f_n - \tilde{F}'\|_{W(\mathcal{I})} \leq \|\tilde{F}_n - \tilde{F}\|_{L^1(\mathcal{I})}, \quad (\text{A.110})$$

implying that  $W(\mathcal{I})$  is a Banach space. Now, the continuous embedding can be proved.

If  $f \in L^1(\mathcal{I})$ , then  $F(x) - F(a) = \int_a^x f(y) \, dy$  for almost all  $x, a \in \mathcal{I}$ . Therefore,

$$\|f\|_{W(\mathcal{I})} \leq \int_{\mathcal{I}} |F(x) - F(a)| \, dx \leq |\mathcal{I}| \int_{\mathcal{I}} |f(y)| \, dy, \quad (\text{A.111})$$

which ends the proof.  $\square$

The previous lemma and Helly's selection theorem imply that

$$W^{1,1}(\mathcal{I}) \hookrightarrow L^1(\mathcal{I}) \hookrightarrow W(\mathcal{I}), \quad (\text{A.112})$$

where the first embedding is compact and the second is continuous.

**Proof of Theorem A.6.1:** Let the compact set  $[0, T] \times \mathcal{I} \subset \mathbb{R}^+ \times \mathbb{R}$ . Supposing that  $\ell \leq 1$  then, from (A.86), the dissipative solutions of rB satisfies

$$\|u^\ell\|_2^2 \leq \|u_0\|_{H^1}^2, \quad \ell^2 \|P\|_1 = \frac{1}{2} \ell^2 \|u_x^\ell\|_2^2 \leq \frac{1}{2} \|u_0\|_{H^1}^2, \quad (\text{A.113})$$

implying that  $u^\ell$  is uniformly bounded on  $L^\infty([0, T], L^2(\mathbb{R}))$ . Subsequently, it is also uniformly bounded on  $L^\infty([0, T], L^1(\mathcal{I}))$ . Because Lemma A.6.1 yields that  $u^\ell$  is bounded on  $L^\infty([0, T], W^{1,1}(\mathcal{I}))$ , and the equation (A.113) implies that  $\frac{1}{2}u^{\ell^2} + \ell^2 P$  is uniformly bounded on  $L^\infty([0, T], L^1(\mathcal{I}))$ , then since  $u_t^\ell = -(\frac{1}{2}u^{\ell^2} + \ell^2 P)_x$ , (A.108) implies that  $u_t^\ell$  is bounded on  $L^\infty([0, T], W(\mathcal{I}))$ . Then, using the Aubin theorem, the compactness follows.

The quantity  $\ell^2 P$  is non-negative and bounded in  $L^\infty([0, +\infty[, L^1(\mathbb{R}))$ , implying the existence of a non-negative measure  $\mu \in L^\infty([0, +\infty[, \mathcal{M}^1(\mathbb{R}))$  such that  $\ell^2 P$  converges (up to a subsequence) weakly to  $\mu$ . The equation (A.104) follows taking the limit  $\ell \rightarrow 0$  in the weak formulation of (A.12). Finally, taking the limit in the weak formulation of (A.87), we can prove that  $u_x(t, x) \leq 2M/(Mt + 2)$ .  $\square$

The question whether or not  $\mu = 0$  is open. The following proposition shows that when  $\ell \rightarrow 0$  for smooth solutions (i.e., before appearance of singularities),  $u^\ell$  converges to the unique solution  $u$  of the classical Burgers equation.

**Proposition A.6.1.** *If  $u_0$  is in  $H^s \cap BV$  with  $s \geq 3$ , then for  $t < 1/\sup_x |u'_0(x)|$  we have*

$$\mu = 0. \quad (\text{A.114})$$

*Proof.* From Theorem A.3.2 and Remark A.3.2, we can find a uniform upper bound on  $u_x^\ell$  in the space  $L^\infty([0, T], L^\infty(\mathbb{R}))$  with  $T < 1/\sup_x |u'_0(x)|$ , which implies that  $\ell^2 P \rightarrow 0$ .  $\square$

## A.6.2 The limiting case $\ell \rightarrow +\infty$

The goal of this subsection is to show that, when  $\ell \rightarrow +\infty$ , the dissipative solution  $u^\ell$  converges (up to a subsequence) to a function  $u$  that satisfies:

$$\left[ u_t + \frac{1}{2}(u^2)_x \right]_x = \nu, \quad (\text{A.115})$$

where  $0 \leq \nu \in L^\infty([0, +\infty[, \mathcal{M}^1)$ . In Proposition A.6.2 below, we show that before the appearance of singularities, the measure  $\nu = u_x^2/2$ . The question whether or not  $\nu = u_x^2/2$  in general is posed. We have the following theorem:

**Theorem A.6.2.** Let  $u_0 \in H^1$  such that  $u_0' \in L^1$  and  $u_0'(x) \leq M \forall x$ , then there exists  $u \in L^\infty([0, T], BV(\mathbb{R}))$  for all  $T > 0$ , such that there exists a subsequence of  $u^\ell$  (noted also  $u^\ell$ ) and for all interval  $\mathcal{I} \subset \mathbb{R}$  we have

$$u^\ell \xrightarrow{\ell \rightarrow +\infty} u \text{ in } \mathcal{C}([0, T], L^1(\mathcal{I})), \quad (\text{A.116})$$

and  $u$  satisfies the equation (A.115). Moreover,  $u$  satisfies the Oleinik inequality

$$u_x(t, x) \leq \frac{2M}{Mt + 2} \quad \text{in } \mathcal{D}'(\mathbb{R}). \quad (\text{A.117})$$

**Remark A.6.2.** If  $\nu = \frac{1}{2}u_x^2$  then  $u$  is a dissipative solution of the Hunter–Saxton equation [5].

*Proof.* Let the compact set  $[0, T] \times \mathcal{I} \subset \mathbb{R}^+ \times \mathbb{R}$ . Supposing that  $\ell \geq 1$  then, from (A.86), the dissipative solution of rB satisfies

$$\|P\|_1 = \frac{1}{2} \|u_x^\ell\|_2^2 \leq \frac{1}{2} \|u_0\|_{H^1}^2. \quad (\text{A.118})$$

Using Lemma A.6.1, one gets that  $u^\ell$  is bounded in  $L^\infty([0, T] \times \mathbb{R})$  and

$$\int_{\mathbb{R}} |u^\ell(t, x+h) - u^\ell(t, x)| dx \leq \|u_0'\|_1 \left(\frac{MT+2}{2}\right)^2 |h|. \quad (\text{A.119})$$

Integrating (A.12) between  $t_1$  and  $t_2$ , one obtains

$$u^\ell(t_1, x) - u^\ell(t_2, x) = \int_{t_1}^{t_2} (u^\ell u_x^\ell + \ell^2 P_x) dt. \quad (\text{A.120})$$

Using Lemma A.6.1, inequality (A.118) and

$$\|P_x\|_\infty \leq \frac{1}{4} \ell^{-2} \|u_x^\ell\|_2^2,$$

we can show that there exists  $B = B(T, \mathcal{I})$  such that

$$\int_{\mathcal{I}} |u^\ell(t_2, x) - u^\ell(t_1, x)| dx \leq B |t_2 - t_1|. \quad (\text{A.121})$$

The compactness follows using Theorem A.8 in [13].

The quantity  $\frac{1}{2} u_x^2$  is non-negative and bounded in  $L^\infty([0, +\infty[, L^1(\mathbb{R}))$ , which implies that there exists a non-negative measure  $\nu \in L^\infty([0, +\infty[, \mathcal{M}^1(\mathbb{R}))$  such that  $P$  converges (up to a subsequence) weakly to  $\nu$ . The equation (A.115) follows by taking the limit  $\ell \rightarrow +\infty$ , in the weak formulation of (A.15). Finally, taking the limit in the weak formulation of (A.87), we can prove that  $u_x(t, x) \leq \frac{2M}{Mt+2}$ .  $\square$

The question whether or not the equality always holds  $\nu = u_x^2/2$  is open. The following proposition shows that, when  $\ell \rightarrow +\infty$  for smooth solutions (before appearance of singularities),  $u^\ell$  converges to a dissipative solution  $u$  of the Hunter–Saxton equation [5].

**Proposition A.6.2.** If  $u_0$  is in  $H^s \cap BV$  with  $s \geq 3$ , then for  $t < 1/\sup_x |u_0'(x)|$  we have

$$\nu = \frac{1}{2} u_x^2. \quad (\text{A.122})$$

*Proof.* From Theorem A.3.2 and Remark A.3.2, we can find a uniform upper bound on  $u_x^\ell$  in the space  $L^\infty([0, T], L^\infty(\mathbb{R}))$  with  $T < 1/\sup_x |u_0'(x)|$ , which implies that the convergence  $u_x^\ell$  to  $u_x$  is strong. Thus,  $u_x^{\ell^2} \rightarrow u_x^2$ .  $\square$

## A.7 Optimality of the $\dot{H}_{loc}^1$ space

In the previous sections (see Proposition A.3.2, Theorem A.3.2 and Theorem A.4.2), we have shown, on one side, that even if the initial datum  $u_0$  is smooth, there exists a blow-up in finite time  $T^* > 0$  such that

$$\inf_{x \in \mathbb{R}} u_x(t, x) > -\infty \quad \forall t < T^*, \quad \inf_{x \in \mathbb{R}} u_x(T^*, x) = -\infty. \quad (\text{A.123})$$

On the other side, the Oleinik inequality (A.85) shows that, even if the initial datum is not Lipschitz, the derivative of the solution becomes instantly bounded from above, i.e.

$$\sup_{x \in \mathbb{R}} u'_0(x) = +\infty, \quad \sup_{x \in \mathbb{R}} u_x(t, x) < +\infty \quad \forall t > 0. \quad (\text{A.124})$$

**Remark A.7.1.** *If the derivative of the initial datum is bounded from below and not from above, it will be instantly bounded from both sides<sup>3</sup> and, after  $T^*$ , it will be bounded from above and not from below.*

This remark is important to prove that the space  $\dot{H}_{loc}^1$  is the best space to obtain global (in time) solutions, the optimality being in the following sense.

**Theorem A.7.1.** *Let  $\delta > 0$  and  $g(h) \stackrel{\text{def}}{=} [\ln|h|]^\delta$ , then there exist  $u_0 \in H^1 \cap W^{1,\infty}$ ,  $T > 0$  and a compact set  $\mathcal{K}$ , such that there exists a solution  $u$  of (A.12) satisfying*

$$\int_{\mathbb{R}} u'_0(x)^2 g(u'_0(x)) dx < +\infty, \quad \int_{\mathcal{K}} u_x(T, x)^2 g(u_x(T, x)) dx = +\infty. \quad (\text{A.125})$$

Thus, we cannot expect that  $u$  belongs to  $W^{1,p}$  for  $p > 2$  for all time. In other words, the space  $H^1 = W^{1,2}$  is optimal for the equation (A.12).

Before proving Theorem A.7.1, let  $u_0 \in H^s$  with  $s$  big enough, and let  $u$  be a solution of rB with  $u(0, x) = u_0(x)$ . The main quantity is the following integral

$$\int_{\mathcal{K}} u_x^2(T, x) g(u_x(T, x)) dx, \quad (\text{A.126})$$

where  $T > 0$  and  $\mathcal{K}$  is a compact set. Using the change of variable  $x = y(T, \xi)$ , one gets

$$\int_{\mathcal{K}} u_x^2(T, x) g(u_x(T, x)) dx = \int_{\mathcal{K}'} q \sin^2(v/2) g(\tan(v/2)) d\xi, \quad (\text{A.127})$$

where  $\mathcal{K}'$  is another compact set. From previous sections, the quantity  $q$  is always bounded, which implies that if  $g$  is bounded then (A.126) is bounded. If  $g$  is not bounded (see Theorem A.7.1), then the quantity (A.126) depends on the behaviour of the derivative  $u_x$  at time  $T$ . The proof of Theorem A.7.1 is done by building  $u(T, \cdot)$ , such that the quantity (A.126) is infinite. Then, we use a backward system to go back in time and find a Lipschitz initial datum.

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<sup>3</sup>Note that the gain of regularity (A.124) is instantaneous, while the loss of regularity (A.123) needs some time.

**Proof of Theorem A.7.1:**

Let  $g(h) \stackrel{\text{def}}{=} [\ln|h|]^\delta$  for  $\delta > 0$  and let  $\bar{u}$  be a compactly supported odd function such that  $\bar{u} \in \mathcal{C}^\infty(\mathbb{R}/\{0\})$  and for all  $x \in ]0, \frac{1}{2}[$  we have

$$\bar{u}'(x) \stackrel{\text{def}}{=} -\frac{1}{\sqrt{x}} (-\ln(x))^{-\frac{1+\delta}{2}}.$$

It is clear that  $\bar{u} \in H^1(\mathbb{R})$  and

$$\int_{\mathcal{V}(0)} \bar{u}'(x)^2 g(\bar{u}'(x)) dx = +\infty, \quad \bar{u}'(x) \leq C, \quad (\text{A.128})$$

where  $\mathcal{V}(0)$  denotes a neighbourhood of 0.

The idea of the proof is to use a backward (in time) system such that  $u(T, x) = \bar{u}(x)$ . The initial datum  $u_0$  is the unknown. To simplify the presentation, the conservative system (A.49) is used. With this system, we will obtain a local (in time) Oleinik inequality, which is enough for our construction. A similar proof can be used with the dissipative system (A.50) with a global Oleinik inequality. The built solution in the interval  $[0, T[$  is Lipschitz, so both systems (A.49), (A.50) yield the same solution.

In order to build  $u_0$ , we use the forward existence proof given in Section A.4. One can use the change of variable  $t \rightarrow -t$ . The conservative system (A.49) becomes then

$$y_t = -u, \quad y(-T, \xi) = \bar{y}(\xi), \quad (\text{A.129a})$$

$$u_t = \ell^2 P_x, \quad u(-T, \xi) = \bar{u}(\bar{y}(\xi)), \quad (\text{A.129b})$$

$$v_t = P(1 + \cos(v)) + \sin^2(v/2), \quad v(-T, \xi) = 2 \arctan(\bar{u}'(\bar{y}(\xi))), \quad (\text{A.129c})$$

$$q_t = -q\left(\frac{1}{2} - P\right) \sin(v), \quad q(-T, \xi) = 1, \quad (\text{A.129d})$$

where  $t \in [-T, 0]$  and  $\bar{y}$  is defined as in (A.42), replacing  $u_0$  by  $\bar{u}$ .

The proof of a local existence of solutions can be done as in Section A.4. Due to the change of variable  $t \rightarrow -t$ , the Oleinik inequality becomes

$$u_x(t, x) \geq -2/(t+T) \quad (\text{A.130})$$

for  $t > -T$  and  $t$  close enough to  $-T$ . The proof of this Oleinik inequality proceeds as in Section A.4 using the equation (A.129c), which implies that the derivative of the solution is bounded from below. As in Remark A.7.1, since  $\bar{u}' = u_x(-T, \cdot) \leq C$ , the derivative of the solution remains bounded from above for  $t > -T$  and  $t$  close enough to  $-T$ . Taking  $T > 0$  small so the solution is Lipschitz until  $t = 0$ , and thus

$$\int_{\mathbb{R}} u_x(0, x)^2 g(u_x(0, x)') dx < +\infty.$$

The result follows directly by using the change of variable  $t \rightarrow -t$ . □

**Remark A.7.2.** 1. *The optimality given in Theorem A.7.1 is also true for the Camassa–Holm equation.*

2. *Xin and Zhang [20] have proved that the Camassa–Holm equation admits dissipative solutions that satisfy*

$$\int_0^T \int_{|x| \leq R} |u_x(t, x)|^p dx dt < +\infty \quad \forall T > 0, R > 0, p < 3. \quad (\text{A.131})$$

*This result can also be proven for the rB equation.*

3. *Theorem A.7.1 does not contradict with (A.131). Theorem A.7.1 shows that the function*

$$t \mapsto \int_{|x| \leq R} |u_x(t, x)|^p dx$$

*does not necessarily belong to  $L_{loc}^\infty([0, +\infty))$ . However, the inequality (A.131) shows that this function belongs to  $L_{loc}^1([0, +\infty))$  if  $p < 3$ .*

## A.8 Conclusion

In this paper, we have studied a regularisation of the inviscid Burgers equation (A.12). For a smooth initial datum, the regularised equation (A.12) has a unique smooth solution locally in time. After the blow-up time, the solution is no longer unique, nor smooth. At least two types of solutions exist: conservative and dissipative solutions. We find that the built dissipative solutions are more interesting because they satisfy an Oleinik inequality (A.85), which plays an important role in showing that solutions converge (up to a subsequence) when  $\ell \rightarrow 0$  and when  $\ell \rightarrow \infty$  ( $\ell$  the regularising positive parameter). Before the appearance of singularities, the limit when  $\ell \rightarrow 0$  (respectively  $\ell \rightarrow \infty$ ) is a smooth solution of the inviscid Burgers (resp. the Hunter–Saxton) equation. After the breakdown time, it remains open to determine whether the Burgers (resp. the Hunter–Saxton) equation holds in the limit without a remaining forcing term.

## Acknowledgments

This material is based upon work supported by the National Science Foundation under Grant No. DMS 1812609 (RLP).



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# Appendix B

## On a Hamiltonian regularization of scalar conservation laws

Guelmame, B.

**Abstract:** In this paper, we study a regularization of a scalar conservation law (SCL), which is obtained by modifying the Lagrangian of SCL. This regularization is parameterized by  $\ell$  and conserves formally an  $H^1$ -like energy. Proof of the existence of local smooth solutions are given in this paper. In addition, we prove the existence of global weak solutions satisfying a uniform (on  $\ell$ ) one-sided Oleinik inequality for this regularization, and also for a generalized Hunter–Saxton equation. Moreover, when  $\ell \rightarrow 0$  (resp.  $\ell \rightarrow \infty$ ), we prove that the solutions of the regularized equation converge, up to a subsequence, to  $u^0$  (resp.  $u^\infty$ ) a solution of the SCL (resp. a generalized Hunter–Saxton equation), at least before the appearance of singularities.

**AMS Classification:** 35L65; 35B65; 35B44; 35L67; 37K05.

**Key words:** Scalar conservation laws; nonlinear evolution equation; generalized Hunter–Saxton equation; regularization; Hamiltonian; conservative and dissipative solutions; Oleinik inequality.

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## B.1 Introduction

Hyperbolic conservation laws (the inviscid Burgers equation and the shallow-water system for example) are known to develop discontinuous shocks in finite time, even if the initial datum is a  $\mathcal{C}^\infty$  function. Those shocks are problematic for numerical simulations and also for theory. To avoid those shocks, usually small dissipation and/or dispersion terms can be added to the equation [1, 3, 2, 18, 19, 20, 12, 25]. In [9], Clamond and Dutykh have proposed a non-dispersive regularization of the shallow-water equation that conserves an  $H^1$ -like energy for smooth solutions. This regularization has been studied after in [21, 23]. Recently, a similar regularization of the inviscid Burgers equation

$$u_t + u u_x = \ell^2 [u_{txx} + 2u_x u_{xx} + u u_{xxx}], \quad (\text{B.1})$$

was proposed in [11], where  $\ell$  is a positive parameter. This equation appeared before in the literature as a particular case of a generalized Camassa–Holm equation [8], an existence of local (in time) smooth solutions was given in [28, 27]. A proof of existence of global weak solutions and a study of the limiting cases  $\ell \rightarrow 0, +\infty$  was done in [11]. The so-called dissipative solution of (B.1) satisfies (uniformly on  $\ell$ ) the one-sided Oleinik inequality

$$u_x(t, x) \leq 2/t, \quad \forall t > 0. \quad (\text{B.2})$$

Due to the  $H^1$ -like energy equation, the solutions of (B.1) remain continuous for all time ( $\ell > 0$ ), which is not the case for the inviscid Burgers equation ( $\ell = 0$ ). A natural question is: Can we approximate the shocks of other interesting hyperbolic partial differential equations conserving the same properties? For example, the general scalar conservation law

$$u_t + f(u)_x = 0, \quad (\text{B.3})$$

some triangular systems, the isentropic Euler equations?

The equation (B.3) can be used for traffic flow models and other physical phenomena [26]. The aim of this paper is to derive, introduce and study the equation

$$u_t + f(u)_x = \ell^2 [u_{xxt} + f'(u) u_{xxx} + 2f''(u) u_x u_{xx} + \frac{1}{2} f'''(u) u_x^3], \quad (\text{B.4})$$

which regularizes the scalar conservation law (B.3). Note that for  $f(u) = u^2/2$ , we obtain the regularization of the inviscid Burgers equation (B.1). The equation (B.4) conserves an energy for smooth solutions, it also has Hamiltonian and Lagrangian structures. The existence of global solutions of (B.4) has been proven for strictly convex fluxes  $f$  in [13]. Using Kato's Theorem for quasi-linear hyperbolic equations [15], we prove in this paper for general fluxes that if the initial datum  $u_0$  belongs to some space  $H^s$  with  $s > 3/2$ , then there exists a local (in time) unique smooth solution. On the contrary of the Burgers

case, the velocity  $f'(u)$  being nonlinear adds some difficulties to prove the estimates in Kato's Theorem (see Lemma B.3.3). An existence of blowing-up solutions in finite time is also proven, where we give some estimates of the blow-up time. In order to prove the existence of global solutions, a change of variables is used inspired by Bressan and Constantin [6] yielding to an equivalent semi-linear nonlocal system. The latest system is used to obtain global weak solutions that conserves an  $H^1$ -like energy for general (not necessarily convex) smooth fluxes, but do not satisfy the Oleinik inequality. Inspired by [7], another type of global weak solutions for uniformly convex fluxes ( $f''(u) \geq C > 0$ ) is obtained, those solutions dissipate the energy on the singularities and satisfy a one-sided Oleinik inequality (see (B.47) below). Using the Oleinik inequality, we prove also that when  $\ell \rightarrow 0$ , (respectively  $\ell \rightarrow \infty$ ) the dissipative solution converges up to a subsequence to a function  $u^0$  (respectively  $u^\infty$ ). We also prove that before the appearance of the singularities,  $u^0$  is the classical solution of (B.3) and  $u^\infty$  is a solution of the generalized Hunter–Saxton equation

$$[u_t + f(u)_x]_x = \frac{1}{2} u_x^2 f''(u). \quad (\text{B.5})$$

A proof of the existence of dissipative solutions of the classical Hunter–Saxton equation ( $f(u) = u^2/2$ ) can be found in [5]. For non-quadratic fluxes, the right-hand side of (B.5) depends on  $u$ , thence the proof given in [5] must be modified. In this paper, a Hamiltonian structure and a proof of global existence of both conservative and dissipative solutions of the equation (B.5) are given.

The paper is organized as follows. In Section B.2, an energy equation and Hamiltonian, Lagrangian structures of the equation (B.4) are provided. Section B.3 is devoted to prove the local existence of smooth solutions. Some estimates on the blow-up time of the smooth solutions are studies in Section B.4. In Section B.5 and Section B.6, existence of two types of global weak solutions of (B.4) and (B.5) respectively are proven. The limiting cases  $\ell \rightarrow 0$  and  $\ell \rightarrow \infty$  are studied in Section B.7.

## B.2 Modeling and properties

Inspired by [11], the scalar conservation law (B.3) can be suitable regularized as

$$u_t + f(u)_x = \ell^2 [u_{xxt} + f'(u) u_{xxx} + 2f''(u) u_x u_{xx} + \frac{1}{2} f'''(u) u_x^3], \quad (\text{B.6})$$

where  $\ell > 0$ . The equation (B.6) can be written in the conservative form

$$[u - \ell^2 u_{xx}]_t + [f(u) - \ell^2 f'(u) u_{xx} - \frac{1}{2} \ell^2 f''(u) u_x^2]_x = 0. \quad (\text{B.7})$$

Multiplying (B.7) by  $u$  we obtain the energy equation for smooth solutions

$$\left[ \frac{1}{2} u^2 + \frac{1}{2} \ell^2 u_x^2 \right]_t + \left[ K(u) - \ell^2 K'(u) u_{xx} + \frac{1}{2} \ell^2 (u f''(u) - f'(u)) u_x^2 - \ell^2 u u_{xt} \right]_x = 0, \quad (\text{B.8})$$

where  $K'(u) = u f'(u)$ .

Introducing the momentum  $m \stackrel{\text{def}}{=} u - \ell^2 u_{xx}$ , the equation (B.7) can be rewritten as

$$m_t + f'(u)m_x + 2(m-u)u_x f''(u) - \frac{1}{2} \ell^2 f'''(u) u_x^3 = 0, \quad (\text{B.9})$$

or as

$$(1 - \ell^2 \partial_x^2) \{ u_t + f(u)_x \} + \frac{1}{2} \ell^2 [ f''(u) u_x^2 ]_x = 0. \quad (\text{B.10})$$

Applying the operator  $(1 - \ell^2 \partial_x^2)^{-1}$  to (B.10), the equation becomes

$$u_t + f(u)_x + \frac{1}{2} \ell^2 (1 - \ell^2 \partial_x^2)^{-1} [ f''(u) u_x^2 ]_x = 0. \quad (\text{B.11})$$

This form is more tractable for numerical computations and also for proving the well-posedness of the equation (see Section B.3 and Section B.5 below).

The classical conservation law (B.3) can be obtained as the Euler–Lagrange of the functional  $\mathfrak{J}_0 = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \mathcal{L}_0(\phi) dx dt$  with the Lagrangian density

$$\mathcal{L}_0 \stackrel{\text{def}}{=} \frac{1}{2} \phi_x \phi_t + F(\phi_x), \quad (\text{B.12})$$

where  $\phi$  is a velocity potential, i.e.,  $u = \phi_x$  and  $F'(u) = f(u)$ . The Euler–Lagrange equation for this functional yields to (B.3) at once.

The equation (B.6) can be obtained as the Euler–Lagrange equation of the Lagrangian density

$$\mathcal{L}_\ell \stackrel{\text{def}}{=} \frac{1}{2} \phi_x \phi_t + F(\phi_x) - \frac{1}{2} \ell^2 \phi_x [\phi_{tx} + f(\phi_x)_x]_x, \quad (\text{B.13})$$

this Lagrangian density is equivalent to

$$\tilde{\mathcal{L}}_\ell \stackrel{\text{def}}{=} \frac{1}{2} \phi_x \phi_t + F(\phi_x) + \frac{1}{2} \ell^2 [ f'(\phi_x) \phi_{xx}^2 - \phi_{xxx} \phi_t ]. \quad (\text{B.14})$$

A Hamiltonian structure also exists for the equation (B.6), that can be obtained with the Hamiltonian operator and functional

$$\mathcal{D} \stackrel{\text{def}}{=} (1 - \ell^2 \partial_x^2)^{-1} \partial_x, \quad (\text{B.15})$$

$$\mathfrak{H} \stackrel{\text{def}}{=} \int [ F(u) + \frac{1}{2} \ell^2 f'(u) u_x^2 ] dx, \quad (\text{B.16})$$

so the equation of motion is given by

$$u_t = -\mathcal{D} \delta_u \mathfrak{H}, \quad (\text{B.17})$$

which is the equation (B.7), where the operator  $\mathcal{D}$  is a Hamiltonian operator [22].

### B.3 Existence of local smooth solutions of the regularized conservation laws

This section is devoted to study the local (in time) well-posedness of the Cauchy problem

$$u_t + [ f(u) + \frac{1}{2} \ell^2 \mathfrak{G} * f''(u) u_x^2 ]_x = 0, \quad \mathfrak{G} \stackrel{\text{def}}{=} (2\ell)^{-1} \exp(-|x|/\ell), \quad (\text{B.18})$$

with  $u(0, x) = u_0(x)$ . Let be

$$P = \frac{1}{2} \mathfrak{G} * f''(u) u_x^2, \quad (\text{B.19})$$

notice that  $P - \ell^2 P_{xx} = f''(u) u_x^2 / 2$ . Differentiating (B.18) w.r.t  $x$  one obtains

$$u_{xt} + f'(u) u_{xx} + \frac{1}{2} f''(u) u_x^2 + P = 0, \quad (\text{B.20})$$

multiplying (B.18) by  $u$  and (B.20) by  $\ell^2 u_x$ , an energy equations for smooth solutions is obtained

$$\left[ \frac{1}{2} u^2 + \frac{1}{2} \ell^2 u_x^2 \right]_t + \left[ K(u) + \frac{1}{2} \ell^2 f'(u) u_x^2 + \ell^2 u P \right]_x = 0. \quad (\text{B.21})$$

Another conservative equation that corresponds to the Hamiltonian (B.16) can be obtained

$$\left[ F(u) + \frac{1}{2} \ell^2 u_x^2 f'(u) \right]_t + \left[ H(u) + \ell^2 f(u) P + \frac{1}{2} \ell^2 f'(u)^2 u_x^2 + \frac{1}{2} \ell^4 P^2 - \frac{1}{2} \ell^6 P_x^2 \right]_x = 0,$$

where  $H'(u) = f(u) f'(u)$ .

Let  $s \in \mathbb{R}$  and let be

$$H^s(\mathbb{R}) \stackrel{\text{def}}{=} \left\{ u \in \mathcal{S}'(\mathbb{R}), \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{u}(\xi)|^2 d\xi < +\infty \right\}. \quad (\text{B.22})$$

The norm of the space  $H^s$  is given by

$$\|u\|_{H^s}^2 \stackrel{\text{def}}{=} \int_{\mathbb{R}} (1 + \xi^2)^s |\hat{u}(\xi)|^2 d\xi = c \|\Lambda^s u\|_{L^2}^2,$$

where  $\Lambda^s = (1 - \partial_x^2)^{s/2}$  and  $c > 0$  is a constant depending only on the definition of the Fourier transform. Let  $[A, B] \stackrel{\text{def}}{=} AB - BA$  be the commutator of  $A$  and  $B$ . In order to prove the well-posedness of the equation (B.18), the following classical lemmas are needed:

**Lemma B.3.1.** ([15, 17]) *If  $r > 0$ , then there exists a constant  $c > 0$ , such that*

$$\|fg\|_{H^{-r}} \leq \|f\|_{L^\infty} \|g\|_{H^{-r}}, \quad (\text{B.23})$$

$$\|fg\|_{H^r} \leq c (\|f\|_{L^\infty} \|g\|_{H^r} + \|f\|_{H^r} \|g\|_{L^\infty}), \quad (\text{B.24})$$

$$\|[\Lambda^r, f]g\|_{L^2} \leq c (\|f_x\|_{L^\infty} \|g\|_{H^{r-1}} + \|f\|_{H^r} \|g\|_{L^\infty}). \quad (\text{B.25})$$

Let  $f, g$  be smooth functions and  $h \in H^r$ , then

$$|(fg, h)_{H^{-r}, H^r}| = \left| \int_{\mathbb{R}} f g h dx \right| \leq \|f\|_{L^\infty} \|g\|_{H^{-r}} \|h\|_{H^r}, \quad (\text{B.26})$$

which implies (B.23). A rigorous proof of (B.23) can be found in [15]. Inequalities (B.24) and (B.25) can be found in [17].

**Lemma B.3.2.** ([10]) *Let  $F \in \mathcal{C}^{m+2}$ , if  $1/2 < s \leq m$ , then there exist a continuous function  $\tilde{F}$ , such that*

$$\|F(u) - F(0)\|_{H^s} \leq \tilde{F}(\|u\|_{L^\infty}) \|u\|_{H^s}. \quad (\text{B.27})$$

*If  $3/2 < s \leq m$ , then there exist a continuous function  $\bar{F}$  such that*

$$\|F(u) - F(v)\|_{H^s} \leq \bar{F}(\|u\|_{H^s}, \|v\|_{H^s}) \|u - v\|_{H^s}. \quad (\text{B.28})$$



*Proof.* The proof of (B.27) can be found in [10]. The inequality (B.28) is also proven in [10], a shorter proof is given here.

Let  $G'(u) \stackrel{\text{def}}{=} F'(u) - F'(0)$ , using (B.24) and (B.27) we obtain that for all  $s > 1/2$

$$\begin{aligned}
\|F(u) - F(v)\|_{H^s} &\lesssim \|F'(u)u_x - F'(v)v_x\|_{H^{s-1}} + \|F(u) - F(v)\|_{L^2} \\
&= \|F'(u)u_x - F'(u)v_x + G'(u)v_x - G'(v)v_x\|_{H^{s-1}} + \|F(u) - F(v)\|_{L^2} \\
&\lesssim \|F'(u)\|_{L^\infty} \|u_x - v_x\|_{H^{s-1}} + \|G'(u)\|_{H^{s-1}} \|u_x - v_x\|_{L^\infty} + \|u - v\|_{L^2} \\
&\quad + \|v_x\|_{L^\infty} \|F'(u) - F'(v)\|_{H^{s-1}} + \|v_x\|_{H^{s-1}} \|F'(u) - F'(v)\|_{L^\infty} \\
&\lesssim \|u - v\|_{H^s} + \|F'(u) - F'(v)\|_{H^{s-1}}, \tag{B.29}
\end{aligned}$$

where  $\lesssim$  means  $\leq c$  with  $c = c(F, \|u\|_{H^s}, \|v\|_{H^s})$  is a positive constant.

In order to prove (B.28), we suppose at first that  $s = n \in \mathbb{N}$  and the proof will be done by induction. It is clear that

$$\|F(u) - F(v)\|_{L^2} \leq \|F''\|_{L^\infty} \|u - v\|_{L^2}.$$

The equation (B.29) shows that if (B.28) is true for an integer  $0 \leq k \leq n - 1$ , then it remains true for  $k + 1$ , which ends the proof of (B.28) for an integer  $s$ .

If  $s$  is not an integer, using (B.29) and using (B.28) for  $\lfloor s \rfloor$  one obtains

$$\begin{aligned}
\|F(u) - F(v)\|_{H^s} &\lesssim \|u - v\|_{H^s} + \|F'(u) - F'(v)\|_{H^{s-1}} \\
&\lesssim \|u - v\|_{H^s} + \|F'(u) - F'(v)\|_{H^{\lfloor s \rfloor}} \\
&\lesssim \|u - v\|_{H^s} + \|u - v\|_{H^{\lfloor s \rfloor}} \\
&\lesssim \|u - v\|_{H^s}. \quad \square
\end{aligned}$$

Now, the conditions given in [15] to obtain the local well-posedness of

$$u_t + A(u)u = F(u), \tag{B.30}$$

are recalled:

(X)  $X$  and  $Y$  are reflexive Banach spaces.  $Y$  is dense and continuously embedded in  $X$ . There exists an isomorphism  $S$  from  $Y$  to  $X$ .

(A1) Let  $W = B_Y(y_0, R_0)$  be an open ball in  $Y$ , there exist  $\beta \in \mathbb{R}$ , such that for all  $t > 0$  and  $y \in W$

$$\|e^{-tA(y)}\|_X \leq e^{\beta t}.$$

(A2) If  $y \in W$ , then  $B(y) \stackrel{\text{def}}{=} SA(y)S^{-1} - A(y)$  is bounded on  $X$ .

(A3) For  $y, z \in W$ , we have  $A(y)$  is bounded from  $Y$  to  $X$  and

$$\|A(y) - A(z)\|_{Y, X} \lesssim \|y - z\|_X.$$

(A4) For all  $y \in W$ , we have  $A(y)y_0 \in Y$ .

(A5) For all  $y, z \in W$ , we have

$$\|B(y) - B(z)\|_X \lesssim \|y - z\|_Y.$$

(F1)  $F$  is a bounded function from  $W$  to  $Y$  and

$$\|F(y) - F(x)\|_X \lesssim \|y - x\|_X \quad \forall y, z \in W.$$

(F2)

$$\|F(y) - F(x)\|_Y \lesssim \|y - x\|_Y \quad \forall y, z \in W.$$

Now, the local well-posedness of (B.18) can be stated:

**Theorem B.3.1.** [*Local existence of smooth solutions*] Let  $f \in \mathcal{C}^{m+3}(\mathbb{R})$ ,  $m \geq 2$  and let  $u_0 \in H^s(\mathbb{R})$  with  $3/2 < s \leq m$ , then there exist a maximal time  $T > 0$  that does not depend on  $s$  and a unique solution  $u$  of (B.18) that depends continuously on  $u_0$ , such that  $u \in \mathcal{C}([0, T], H^s(\mathbb{R})) \cap \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}))$ . Moreover, if  $T < +\infty$ , then  $\limsup_{t \rightarrow T} \|u(t, \cdot)\|_{H^s} = +\infty$ .

The proof is based on Kato's existence theorem of quasi-linear equations [15]. The following definitions are used in order to prove Theorem B.3.1

$$X = L^2(\mathbb{R}), \quad Y = H^s(\mathbb{R}), \quad W = B_Y(0, R_0) \subset Y.$$

$$A(y) = f'(y) \partial_x, \quad F(y) = -\frac{1}{2} \ell^2 \partial_x (1 - \ell^2 \partial_x^2)^{-1} (f'(y)_x y_x), \quad y \in Y.$$

$$S = \Lambda^s = (1 - \partial_x^2)^{s/2}.$$

$$B(y) = [\Lambda^s, A(y)] \Lambda^{-s} = [\Lambda^s, f'(y) - f'(0)] \partial_x \Lambda^{-s}.$$

Then, the equation (B.18) can be written as (B.30). We start by proving the following lemma:

**Lemma B.3.3.** Under the assumptions of Theorem B.3.1, the conditions (X), (A1) to (A5), (F1) and (F2) are satisfied.

*Proof.* (A1) can be proved easily by following the proof of Lemma 2 in [24]. (X) and (A4) are trivial. Let  $w \in X = L^2(\mathbb{R})$ ,  $v \in Y = H^s(\mathbb{R})$  and  $y, z \in W$ .

(A2) Using (B.25) and (B.27) one obtains

$$\begin{aligned} \|B(y) w\|_{L^2} &\lesssim \|f''(y) y_x\|_{L^\infty} \|w_x\|_{H^{-1}} + \|f'(y) - f'(0)\|_{H^s} \|\Lambda^{-s} w_x\|_\infty, \\ &\lesssim \|f''(y)\|_\infty \|y\|_{H^s} \|w\|_{L^2} + \|y\|_{H^s} \|w\|_{L^2}, \\ &\lesssim \|w\|_{L^2}. \end{aligned}$$

(A3) It is clear that

$$\|A(y)v\|_{L^2} = \|f'(y)v_x\|_{L^2} \lesssim \|f'(y)\|_{L^\infty} \|v\|_{H^s} \lesssim \|v\|_{H^s}, \quad (\text{B.31})$$

then  $A(y) \in B(H^s, L^2)$ . The Lipschitz-continuity of  $A(y)$  can be done as the following

$$\begin{aligned} \|(A(y) - A(z))v\|_{L^2} &= \|(f'(y) - f'(z))v_x\|_{L^2} \\ &\lesssim \|f''(y)\|_{L^\infty} \|y - z\|_{L^2} \|v_x\|_{L^\infty} \\ &\lesssim \|y - z\|_{L^2} \|v\|_{H^s}. \end{aligned}$$

(A5) Using (B.25) and (B.28) one obtains

$$\begin{aligned} \|(B(y) - B(z))w\|_{L^2} &= \|[\Lambda^s, f'(y) - f'(z)] \partial_x \Lambda^{-s} w\|_{L^2} \\ &\lesssim \|f'(y) - f'(z)\|_{W^{1,\infty}} \|w_x\|_{H^{-1}} + \|f'(y) - f'(z)\|_{H^s} \|\Lambda^{-s} w_x\|_{L^\infty} \\ &\lesssim \|y - z\|_{H^s} \|w\|_{L^2}. \end{aligned}$$

(F1) Using the definition of  $F$  and the inequality (B.23) we get

$$\begin{aligned} \|F(y) - F(z)\|_{L^2} &\lesssim \|y_x (f'(y) - f'(z))_x + f'(z)_x (y - z)_x\|_{H^{-1}} \\ &\lesssim \|y_x\|_{L^\infty} \|f'(y) - f'(z)\|_{L^2} + \|f''(z) z_x\|_{L^\infty} \|y - z\|_{L^2} \\ &\lesssim \|y - z\|_{L^2}. \end{aligned}$$

(F2) Using (B.24), (B.27) and (B.28) one obtains

$$\begin{aligned} \|F(y) - F(z)\|_{H^s} &\lesssim \|y_x (f'(y) - f'(z))_x + f'(z)_x (y - z)_x\|_{H^{s-1}} \\ &\lesssim \|y_x\|_{L^\infty} \|f'(y) - f'(z)\|_{H^s} + \|y\|_{H^s} \|f''(y) y_x - f''(z) z_x\|_{L^\infty} \\ &\quad + \|f''(z) z_x\|_{L^\infty} \|y - z\|_{H^s} + \|f'(z)\|_{H^s} \|y_x - z_x\|_{L^\infty} \\ &\lesssim \|y - z\|_{H^s}. \quad \square \end{aligned}$$

**Proof of Theorem B.3.1.** Theorem 6, Theorem 7 in [15] and Lemma B.3.3 assure the existence of a unique solution  $u$  that depends continuously on the initial datum, such that  $u \in \mathcal{C}([0, T], H^s(\mathbb{R})) \cap \mathcal{C}^1([0, T], L^2(\mathbb{R}))$ . Using (B.18) one obtains that  $u \in \mathcal{C}^1([0, T], H^{s-1}(\mathbb{R}))$ .

The proof that  $T$  may be chosen independent on  $s$ , can be done by following [16].  $\square$

For uniformly convex fluxes, the solution given in Theorem B.3.1 satisfies the Oleinik inequality:

**Proposition B.3.1. [Oleinik inequality]** *Let  $f \in \mathcal{C}^{m+3}(\mathbb{R})$ ,  $m \geq 2$  and let  $u_0 \in H^s(\mathbb{R})$  with  $2 \leq s \leq m$ . If  $f''(u) \geq C > 0$ , then for all  $t \in [0, T]$  the solution given in Theorem B.3.1 satisfies the Oleinik inequality*

$$u_x(t, x) \leq \frac{1}{Ct/2 + 1/M} \leq M, \quad (\text{B.32})$$

where  $M \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} u'_0(x)$ .

*Proof.* Let  $x_0 \in \mathbb{R}$  and let  $y$  defined as the unique solution of the Cauchy problem

$$y_t(t, x_0) = f'(u(t, y(t, x_0))), \quad y(0, x_0) = x_0. \quad (\text{B.33})$$

Let  $H(t, x_0) \stackrel{\text{def}}{=} u_x(t, y(t, x_0))$ . Using that  $P \geq 0$ , the equation (B.20) implies that

$$H_t + \frac{1}{2} C H^2 \leq H_t + \frac{1}{2} f''(u) H^2 = -P \leq 0, \quad (\text{B.34})$$

implying that

$$H(t, x_0) \leq \frac{1}{Ct/2 + 1/H(0, x_0)}. \quad (\text{B.35})$$

The Oleinik inequality (B.32) follows directly from (B.35).  $\square$

## B.4 Blow-up of smooth solutions

This section is devoted to prove that the solutions given in Theorem B.3.1 blow-up in finite time for uniformly convex fluxes. Estimates on the blow-up time  $T$  of those solutions are also given. We start by the following proposition:

**Proposition B.4.1.** *[An upper bound of the blow-up time] Let  $f \in \mathcal{C}^{m+3}(\mathbb{R})$ , such that  $f''(u) \geq C > 0$  and  $m \geq 2$ . Let also  $u_0 \in H^s(\mathbb{R})$  with  $2 \leq s \leq m$ . If  $u'_0(x_0) < 0$  for some  $x_0 \in \mathbb{R}$ , then*

$$T \leq \frac{-2}{C \inf_{x \in \mathbb{R}} u'_0(x)}.$$

*Proof.* The inequality (B.35) implies that

$$\lim_{t \rightarrow -2/(CH(0, x_0))} H(t, x_0) = -\infty,$$

implying

$$\lim_{t \rightarrow -2/(CH(0, x_0))} \|u\|_{H^s} \gtrsim \lim_{t \rightarrow -2/(CH(0, x_0))} \|u_x\|_{L^\infty} = +\infty. \quad \square$$

It is clear that if  $\|u_x\|_{L^\infty}$  blows-up then  $\|u\|_{H^s}$  blows-up also. The converse is not true in general. The following Lemma shows that if  $u$  is a solution of (B.18), then the converse is true:

**Lemma B.4.1.** *Let  $f \in \mathcal{C}^{m+3}(\mathbb{R})$ ,  $m \geq 2$  and let  $u_0 \in H^s(\mathbb{R})$  with  $3/2 < s \leq m$ . If  $\|u_x\|_{L^\infty}$  is bounded for  $t \in [0, T[$ , then  $\|u\|_{H^s}$  remains bounded for  $t \in [0, T[$ .*

The proof of Lemma B.4.1 can be done by following Theorem 3.1 in [27]. The following theorem is devoted to improve the blow-up criteria in Theorem B.3.1.

**Theorem B.4.1.** *[The blow-up criteria] Let  $f \in \mathcal{C}^{m+3}(\mathbb{R})$ , such that  $f''(u) \geq C > 0$ ,  $m \geq 2$ . Let also  $u_0 \in H^s(\mathbb{R})$  with  $2 \leq s \leq m$ , then*

$$T < +\infty \implies \liminf_{t \rightarrow T} \inf_{x \in \mathbb{R}} u_x(t, x) = -\infty.$$

The proof of the latest Theorem follows directly from Proposition B.3.1, Lemma B.4.1 and the blow-up criteria given in Theorem B.3.1.  $\square$

Note that if  $u_0 \in H^s(\mathbb{R})$  with  $s \geq 2$ , the equation (B.21) implies that the energy

$$E(t) = \int_{\mathbb{R}} (u^2 + \ell^2 u_x^2) dx, \quad (\text{B.36})$$

is invariant. Then, the solutions remains bounded

$$\|u\|_{L^\infty} \leq \|u\|_{H^1} \leq \alpha_\ell \sqrt{E(t)} = \alpha_\ell \sqrt{E(0)}. \quad (\text{B.37})$$

Let

$$\tilde{C} = \sup_{|u| \leq \alpha_\ell \sqrt{E(0)}} f''(u), \quad (\text{B.38})$$

a lower bound of the blow-up time can also given by

**Theorem B.4.2.** *[A lower bound of the blow-up time] Let  $f \in C^{m+3}(\mathbb{R})$  such that  $f''(u) \geq C > 0$ , let also be  $u_0 \in H^s(\mathbb{R})$  with  $2 \leq s \leq m$ . If  $u_0$  is not the zero function, then*

$$1/(\tilde{C} \sup |u'_0|) \leq T. \quad (\text{B.39})$$

*Proof.* Let be

$$m(t) \stackrel{\text{def}}{=} \inf_{x \in \mathbb{R}} u_x(t, x) < 0 < M(t) \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}} u_x(t, x), \quad t < T.$$

The equation (B.34) implies that the functions  $m(\cdot)$  and  $M(\cdot)$  are decreasing in time. We consider the three cases:

- $M(t) > -m(t)$  for all  $t > 0$ . This implies that

$$0 < -m(0) \leq -m(t) = |m(t)| < M(t) = |M(t)| \leq M(0).$$

which implies with Lemma B.4.1 that  $T = +\infty$ .

- There exists  $t_0 \geq 0$ , such that  $-m(t_0) = M(t_0)$ . Using the same argument as in [11] one can show that

$$\dot{m} + \tilde{C} m^2 \geq 0, \quad \forall t \geq t_0, \quad (\text{B.40})$$

where  $\dot{m} \stackrel{\text{def}}{=} \liminf_{\delta > 0, \delta \rightarrow 0} \frac{m(t+\delta) - m(t)}{\delta}$  is the generalized derivative of  $m$ . One can easily shows that

$$m(t) \geq \frac{m(t_0)}{1 + m(t_0) \tilde{C} (t - t_0)},$$

then

$$T \geq t_0 - \frac{1}{\tilde{C} m(t_0)} = t_0 + \frac{1}{\tilde{C} M(t_0)} \geq \frac{1}{\tilde{C} \sup_{x \in \mathbb{R}} |u'_0(x)|}.$$

- $M(0) \leq -m(0)$ . This case can be done as the previous one.  $\square$

**Remark B.4.1.** Note that  $\tilde{C}$  depends on  $\ell$ , then the estimate given in Theorem B.4.2 depends also on  $\ell$ . In Section B.7 below, uniform (on  $\ell$ ) estimates are needed, then the flux  $f$  is assumed to satisfy  $f''(u) \leq \tilde{C}$ , where  $\tilde{C}$  is a fixed constant and not the constant defined in (B.38).

## B.5 Existence of global weak solutions of the regularized conservation laws

As shown in Proposition B.4.1 above, the solutions given in Theorem B.3.1 do not hold for all time due to the blow-up of  $\|u\|_{H^s}$  for  $s \geq 2$ . In order to obtain global solutions of (B.18), one needs to look for weaker solutions in a bigger space. Thanks to the energy equation (B.21), the space  $H^1$  is a natural candidate to obtain global solutions.

Inspired by [6, 7] (see also [11]), let  $\xi \in \mathbb{R}$  and let  $y_0(\xi)$  defined by

$$\int_0^{y_0(\xi)} (1 + u_0'^2) dx = \xi. \quad (\text{B.41})$$

Let also  $y$  be the characteristic starting from  $y_0$  with velocity  $f'(u)$ . The quantities  $v, q, P$  and  $P_x$  are defined as the following

$$v \stackrel{\text{def}}{=} 2 \arctan(u_x), \quad q \stackrel{\text{def}}{=} (1 + u_x^2) y_\xi. \quad (\text{B.42})$$

$$P(t, \xi) = \frac{1}{4\ell} \int_{\mathbb{R}} \exp\left(-\frac{1}{\ell} \left| \int_{\xi}^{\xi'} q(t, s) \cos^2\left(\frac{v(t, s)}{2}\right) ds \right| \right) q(t, \xi') \sin^2\left(\frac{v(t, \xi')}{2}\right) f''(u(t, \xi')) d\xi',$$

$$P_x(t, \xi) = \left( \int_{\xi}^{+\infty} - \int_{-\infty}^{\xi} \right) \exp\left(-\left| \int_{\xi}^{\xi'} q(t, s) \cos^2\left(\frac{v(t, s)}{2}\right) \frac{ds}{\ell} \right| \right) q(t, \xi') \sin^2\left(\frac{v(t, \xi')}{2}\right) f''(u(t, \xi')) \frac{d\xi'}{4\ell^2}.$$

Then, the equation (B.18) can be transformed to the equivalent system

$$y_t = f'(u), \quad y(0, \xi) = y_0(\xi), \quad (\text{B.43a})$$

$$u_t = -\ell^2 P_x, \quad u(0, \xi) = u_0(y_0(\xi)), \quad (\text{B.43b})$$

$$v_t = -P(1 + \cos(v)) - f''(u) \sin^2(v/2), \quad v(0, \xi) = 2 \arctan(u_0'(y_0(\xi))), \quad (\text{B.43c})$$

$$q_t = q \left( \frac{f''(u)}{2} - P \right) \sin(v), \quad q(0, \xi) = 1, \quad (\text{B.43d})$$

which can be used to prove the following theorem

**Theorem B.5.1. [Global existence of conservative solutions]** *Let  $u_0 \in H^1(\mathbb{R})$  and  $f \in C^3(\mathbb{R})$ , then there exists a global weak solution  $u$  of the equation (B.18), such that  $\forall T > 0$ ,  $u \in Lip([0, T], L^2(\mathbb{R}))$  and*

$$\int_{\mathbb{R}} (u^2(t) + \ell^2 u_x(t)^2) dx = \int_{\mathbb{R}} (u_0^2 + \ell^2 u_0'^2) dx, \quad \text{for a.e. } t \in \mathbb{R}. \quad (\text{B.44})$$

*This solution  $u$  is called a conservative solution. Moreover, if  $\|u_{0,n} - u_0\|_{H^1} \rightarrow 0$ . Then,  $u_n$  converges uniformly to  $u$ , for all  $t, x$  in any bounded set.*

Note that  $v_0 \in [-\pi, \pi]$ , but for  $t > 0$  the value of  $v$  is allowed not to be in  $[-\pi, \pi]$ . When  $v$  crossed the value  $\pm\pi$ , the value of  $u_x$  jumps from  $\pm\infty$  to  $\mp\infty$ , which implies that the Oleinik inequality

$$f'(u)_x \lesssim t^{-1},$$

can not be satisfied if the sign of  $f''(u)$  is constant. In order to obtain a solutions satisfying the Oleinik inequality, we suppose that  $f''(u) \geq C > 0$  and the system (B.43) is modified as

$$y_t = f'(u), \quad (\text{B.45a})$$

$$u_t = -\ell^2 P_x, \quad (\text{B.45b})$$

$$v_t = \begin{cases} -P(1 + \cos v) - f''(u) \sin^2(v/2), & v > -\pi, \\ 0, & v \leq -\pi, \end{cases} \quad (\text{B.45c})$$

$$q_t = \begin{cases} q \left( \frac{f''(u)}{2} - P \right) \sin(v), & v > -\pi \\ 0, & v \leq -\pi. \end{cases} \quad (\text{B.45d})$$

$P$  and  $P_x$  are also modified as

$$P(t, \xi) = \frac{1}{4\ell} \int_{\mathbb{R}} \exp \left\{ -\frac{1}{\ell} \left| \int_{\xi}^{\xi'} \bar{q}(t, s) \cos^2 \frac{v(t, s)}{2} ds \right| \right\} \bar{q}(t, \xi') \sin^2 \frac{v(t, \xi')}{2} f''(u(t, \xi')) d\xi',$$

$$P_x(t, \xi) = \frac{1}{4\ell^2} \left( \int_{\xi}^{+\infty} - \int_{-\infty}^{\xi} \right) \exp \left\{ -\frac{1}{\ell} \left| \int_{\xi}^{\xi'} \bar{q}(t, s) \cos^2 \frac{v(t, s)}{2} ds \right| \right\} \bar{q}(t, \xi') \sin^2 \frac{v(t, \xi')}{2} f''(u(t, \xi')) d\xi',$$

where  $\bar{q}(t, \xi) = q(t, \xi)$  if  $v(t, \xi) > -\pi$  and  $\bar{q}(t, \xi) = 0$  if  $v(t, \xi) \leq -\pi$ . Following [7, 11] one can proves the following result

**Theorem B.5.2. [Global existence of dissipative solutions]** *Let  $u_0 \in H^1(\mathbb{R})$  and  $f \in \mathcal{C}^3(\mathbb{R})$  such that  $f''(u) \geq C > 0$ , then there exists a global weak solution  $u$  of the equation (B.18), satisfying  $u \in Lip([0, T], L^2(\mathbb{R}))$  for all  $T > 0$  and*

$$\int_{\mathbb{R}} (u^2(t) + \ell^2 u_x(t)^2) dx \leq \int_{\mathbb{R}} (u_0^2 + \ell^2 u_0'^2) dx, \quad \forall t \in \mathbb{R}. \quad (\text{B.46})$$

*This solution is called a dissipative solution. Moreover, for  $M = \sup_{x \in \mathbb{R}} u_0'(x) \in \mathbb{R}^+ \cup \{+\infty\}$  we have*

$$u_x(t, x) \leq \frac{1}{Ct/2 + 1/M}. \quad (\text{B.47})$$

The Oleinik inequality (B.47) is a cornerstone of scalar conservation laws. In this paper, this inequality plays an important role to study the limiting cases in Section B.7. Thence, we are more interested by the dissipative solutions in this paper, so the proof of Theorem B.5.1 is omitted, which can be done following [6, 11] and also the following proof:

*Proof of Theorem B.5.2: Step 1: Local existence for the equivalent system.* In order to prove Theorem B.5.2, we will prove the well-posedness of the system (B.45), and then we prove that the solution of (B.45) yields to a dissipative solution of (B.18). Since the right hand sides of (B.45b), (B.45c) and (B.45d) do not depend of  $y$ , it suffices to show that the system of three equations (B.45b), (B.45c), (B.45d) is well posed. Due to the

discontinuity of the write hand of (B.45c), we consider the following system

$$u_t = -\ell^2 P_x, \quad (\text{B.48a})$$

$$v_t = \begin{cases} -P(1 + \cos v) - f''(u) \sin^2(v/2), & v > -\pi, \\ -f''(u), & v \leq -\pi, \end{cases} \quad (\text{B.48b})$$

$$q_t = \begin{cases} q \left( \frac{f''(u)}{2} - P \right) \sin(v), & v > -\pi, \\ 0, & v \leq -\pi. \end{cases} \quad (\text{B.48c})$$

It is clear that if  $(u, v, q)$  is a solution of (B.48), we can obtain a solution of (B.45) by replacing  $v$  with  $\max\{v, -\pi\}$ . The system (B.48) can be written as

$$U_t(t, \xi) = F(U(t, \xi)) + G(\xi, U(t, \cdot)), \quad U = (u, v, q), \quad (\text{B.49})$$

such that

$$F(U) = \begin{cases} (0, -f''(u) \sin^2 \frac{v}{2}, \frac{1}{2} f''(u) q \sin v) & v > -\pi, \\ (0, -f''(u), 0) & v \leq -\pi, \end{cases}$$

$$G(U) = \begin{cases} (-\ell^2 P_x, -P(1 + \cos v), -P q \sin v) & v > -\pi, \\ (-\ell^2 P_x, 0, 0) & v \leq -\pi. \end{cases}$$

Our aim now is to prove local existence of solutions of (B.49). Let  $\delta \in ]0, \frac{2\pi}{3}]$  and let  $\Lambda$  defined by

$$\Lambda \stackrel{\text{def}}{=} \{ \xi, v_0(\xi) \in ]-\pi, \delta - \pi] \}. \quad (\text{B.50})$$

Note that (B.48b) implies that if  $v \in ]-\pi, \delta - \pi] \subset ]-\pi, -\frac{\pi}{3}]$ , then  $v_t \leq -C/2$ . Let  $X \stackrel{\text{def}}{=} \mathcal{C}([0, T], L^\infty(\mathbb{R}, \mathbb{R}^3))$ ,  $c > 0$  and let  $\mathfrak{D} \subset X$  satisfying  $U(0, \xi) = U_0(\xi)$  and

$$1/c \leq q(t, \xi) \leq c \quad \forall (t, \xi) \in [0, T] \times \mathbb{R}, \quad (\text{B.51a})$$

$$\{ \xi, \sin^2(v(t, \xi)/2) \geq \frac{1}{2} \} \leq c \quad \forall t \in [0, T], \quad (\text{B.51b})$$

$$\|U(t) - U(s)\|_\infty \leq c|t - s| \quad \forall t, s \in [0, T], \quad (\text{B.51c})$$

$$v(t, \xi) - v(s, \xi) \leq -c \frac{t-s}{2} \quad \forall 0 \leq s \leq t \leq T, \xi \in \Lambda. \quad (\text{B.51d})$$

The equation (B.51b) implies that if  $\xi_1 < \xi_2$ , then

$$\int_{\xi_1}^{\xi_2} q(\xi) \cos^2 \frac{v(\xi)}{2} d\xi \geq \int_{\{ \xi \in [\xi_1, \xi_2], \sin^2 \frac{v(t, \xi)}{2} \leq \frac{1}{2} \}} \frac{c^{-1}}{2} d\xi \geq \left[ \frac{\xi_2 - \xi_1}{2} - \frac{c}{2} \right] c^{-1}, \quad (\text{B.52})$$

implying that the term  $\exp \left\{ -\frac{1}{\ell} \left| \int_{\xi}^{\xi'} \bar{q}(t, s) \cos^2 \frac{v(t, s)}{2} ds \right| \right\}$  in the definition of  $P$  and  $P_x$  decays exponentially when  $|\xi - \xi'| \rightarrow +\infty$ . Defining  $\Gamma(\zeta) = \min \left\{ 1, \exp \left( \frac{1}{2\ell} - \frac{|\zeta|}{2\ell} C^{-1} \right) \right\}$ , then Young inequality implies that

$$\|P\|_{L^\infty}, \|P_x\|_{L^\infty} \lesssim \|\Gamma\|_{L^1} \|f''(u)\|_{L^\infty}. \quad (\text{B.53})$$



Taking  $(u, v, q) \in \mathfrak{D}$  and using (B.53) one can show that

$$\|P(U) - P(\tilde{U})\|_{L^\infty}, \|P_x(U) - P_x(\tilde{U})\|_{L^\infty} \lesssim \|U - \tilde{U}\|_{L^\infty} + |\{\xi, (v(\xi) + \pi)(\tilde{v}(\xi) + \pi) < 0\}|,$$

and

$$\|F(U) - F(\tilde{U})\|_{L^\infty} \lesssim \|U - \tilde{U}\|_{L^\infty}, \quad (\text{B.54})$$

$$\|G(U) - G(\tilde{U})\|_{L^\infty} \lesssim \|U - \tilde{U}\|_{L^\infty} + \left| \{\xi, (v(\xi) + \pi)(\tilde{v}(\xi) + \pi) < 0\} \right|. \quad (\text{B.55})$$

Now, we need to estimate the term  $|\{\xi, (v(\xi) + \pi)(\tilde{v}(\xi) + \pi) < 0\}|$ . For that purpose, let the crossing time defined as

$$\tau(\xi) \stackrel{\text{def}}{=} \sup \{t \in [0, T], v(t, \xi) > -\pi\}. \quad (\text{B.56})$$

The equation (B.51c) implies  $|v(t, \xi) - v_0(\xi)| \leq Ct$ , then if  $\xi \notin \Lambda$ , i.e.  $v_0(\xi) > \delta - \pi$  we get

$$\min \{\tau(\xi), \tilde{\tau}(\xi)\} \geq \delta / C.$$

Let be  $T < \delta / C$ , the equation (B.51d) implies

$$\begin{aligned} \int_0^T \left| \{\xi, (v(\tau, \xi) + \pi)(\tilde{v}(\tau, \xi) + \pi) < 0\} \right| d\tau &\leq \int_\Lambda |\tau(\xi) - \tilde{\tau}(\xi)| d\xi \\ &\leq 2|\Lambda| \|U - \tilde{U}\|_{L^\infty} / c. \end{aligned}$$

The Picard operator

$$(\mathcal{P}(U))(t, \xi) = U_0 + \int_0^t [F(U) + G(U)] d\tau, \quad (\text{B.57})$$

then satisfies

$$\|\mathcal{P}(U) - \mathcal{P}(\tilde{U})\|_{L^\infty} \leq \tilde{K} (T + |\Lambda|) \|U - \tilde{U}\|_{L^\infty}, \quad (\text{B.58})$$

where  $\tilde{K}$  depends only on  $C$  and  $\ell$ . The function  $\sin^2 \frac{v_0}{2}$  belongs to  $L^1$ , then  $\delta > 0$  can be chosen such that  $|\Lambda|$  is arbitrary small. Choosing also  $T$  small enough one obtains the local existence of a solution of the system (B.48). Replacing  $v$  by  $\max\{v, -\pi\}$  we obtain a solution of (B.45).

**Step 2: Global existence.** The aim of this step is to show that the solution given in the previous step holds globally in time. For that purpose, we need to show that the quantity

$$\|q\|_{L^\infty} + \|1/q\|_{L^\infty} + \|\sin^2 v/2\|_{L^1} + \|v\|_{L^\infty} + \|u\|_{L^\infty}, \quad (\text{B.59})$$

does not blow-up in finite time. Following [7, 11] we can easily show the equalities

$$(q \cos^2 v/2)_t = \frac{1}{2} q f''(u) \sin v, \quad (\text{B.60a})$$

$$(q \sin^2 v/2)_t = -q P \sin v, \quad (\text{B.60b})$$

$$P_\xi = \begin{cases} q P_x \cos^2 v/2, & v > -\pi, \\ 0, & v \leq \pi, \end{cases} \quad (\text{B.60c})$$

$$\ell^2 (P_x)_\xi = \begin{cases} q (P \cos^2 v/2 - \frac{1}{2} f''(u) \sin^2 v/2), & v > -\pi, \\ 0, & v \leq \pi, \end{cases} \quad (\text{B.60d})$$

$$\left(\frac{1}{2} q \sin v\right)_t = \begin{cases} -q (P \cos^2 v/2 - \frac{1}{2} f''(u) \sin^2 v/2), & v > -\pi, \\ 0, & v \leq \pi, \end{cases} \quad (\text{B.60e})$$

$$u_\xi = \frac{1}{2} q \sin v, \quad (\text{B.60f})$$

$$y_\xi = q \cos^2 v/2. \quad (\text{B.60g})$$

Defining  $H(u) \stackrel{\text{def}}{=} \int_0^u w^2 f''(w) dw$ , we can deduce the energy equation

$$\left[ q u^2 \cos^2 v/2 + \ell^2 q \sin^2 v/2 \right]_t + \left[ H(u) - 2 \ell^2 u P \right]_x = 0. \quad (\text{B.61})$$

Integrating over  $\xi$  one obtains the conservation of the energy

$$\frac{d}{dt} \int_{\mathbb{R}} (u^2 \cos^2 v/2 + \ell^2 \sin^2 v/2) q d\xi = 0. \quad (\text{B.62})$$

Then we have

$$\begin{aligned} \|u\|_{L^\infty}^2 &\leq 2 \int_{\mathbb{R}} |u u_\xi| d\xi = \int_{\mathbb{R}} |u q \sin v| d\xi = 2 \int_{\mathbb{R}} |u \sqrt{q} \cos v/2 \sqrt{q} \sin v/2| d\xi \\ &\leq \int_{\mathbb{R}} (u^2 \cos^2 v/2 + \sin^2 v/2) q d\xi, \end{aligned}$$

implying that  $\|u\|_{L^\infty}$  is bounded for all time  $t > 0$ . The equation (B.53) implies that  $\|P\|_{L^\infty} + \|P_x\|_{L^\infty}$  is bounded for all  $t > 0$ . The equation (B.48c) implies that  $|q_t|/q \leq \tilde{C}/2 + \|P\|_{L^\infty}$ , which implies with Gronwall lemma that  $\|q\|_{L^\infty} + \|1/q\|_{L^\infty}$  remains bounded in all interval  $[0, T]$ . Using that  $q$  is far from zero and using the conservation of energy

$$\tilde{E}(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} (u^2 \cos^2 v/2 + \ell^2 \sin^2 v/2) q d\xi = \tilde{E}(0), \quad (\text{B.63})$$

one obtains that  $\|\sin^2 v/2\|_{L^1}$  does not blow-up in finite time, which finishes the proof of the global existence.

Following [7, 11] and using the change of variables  $x = y(t, \xi)$ , one can show that the solution of (B.18) can be obtained as

$$u(t, x) = u(t, \xi), \quad y(t, \xi) = x, \quad (\text{B.64})$$

which belongs to  $\mathcal{C}([0, T], L_x^2(\mathbb{R}))$  and its derivative can be obtained if  $v(t, \xi) \neq -\pi$  as

$$u_x(t, x) = \tan\left(\frac{v(t, \xi)}{2}\right) = \frac{\sin(v(t, \xi))}{1 + \cos(v(t, \xi))}. \quad (\text{B.65})$$

**Step 3: Dissipation of the energy and Oleinik inequality.** Let  $\xi \in \mathbb{R}$ , if  $v(t_0, \xi) \leq 0$  then  $v(t, \xi) \in [-\pi, 0]$  for all  $t \geq t_0$ . If  $v(0, \xi) > 0$  then the equation (B.45c) implies that

$$\left(\arctan \frac{v}{2}\right)_t \leq -\frac{C}{2} \arctan^2 \frac{v}{2}.$$

Since  $\arctan \frac{v_0(\xi)}{2} \leq M$ , we have

$$u_x = \arctan \frac{v(t, \xi)}{2} \leq \frac{1}{Ct/2 + 1/M}. \quad (\text{B.66})$$

To prove the dissipation of the energy (B.46), we use the change of variables  $x = y(t, \xi)$ , using also (B.36) and (B.63) one can show that

$$\begin{aligned} E(t) &= \int_{\mathbb{R}} (u^2 + \ell^2 u_x^2) dx = \int_{\{\xi, v(t, \xi) > -\pi\}} (u^2 \cos^2 v/2 + \ell^2 \sin^2 v/2) q d\xi \\ &\leq \int_{\mathbb{R}} (u^2 \cos^2 v/2 + \ell^2 \sin^2 v/2) q d\xi = \tilde{E}(0) = E(0). \quad \square \end{aligned}$$

Due to the Oleinik inequality, we will be only interested by the dissipative solutions given in Theorem B.5.2. As mentioned in the introduction above, taking formally  $\ell \rightarrow \infty$  we obtain the generalized Hunter–Saxton equation (B.5). The next section is devoted to study the global well-posedness of the equation (B.5).

## B.6 On a generalized Hunter–Saxton equation

Note that taking  $\ell \rightarrow \infty$  in (B.18), we have  $\ell^2 \mathfrak{G}_x \rightarrow -\text{sgn}(x)/2$ . Then, we formally obtain the generalized Hunter–Saxton (gHS) equation

$$u_t + f(u)_x = \frac{1}{4} \left( \int_{-\infty}^x - \int_x^{+\infty} \right) u_x^2 f''(u) dx. \quad (\text{B.67})$$

Differentiating w.r.t  $x$  we obtain

$$[u_t + f(u)_x]_x = \frac{1}{2} u_x^2 f''(u). \quad (\text{B.68})$$

a second differentiation gives

$$u_{xxt} + f'(u) u_{xxx} + 2 f''(u) u_x u_{xx} + \frac{1}{2} f'''(u) u_x^3 = 0. \quad (\text{B.69})$$

Since  $P \rightarrow 0$  when  $\ell \rightarrow \infty$ , note that the equation (B.68) (resp. (B.69)) can also be obtained by taking formally  $\ell \rightarrow \infty$  in (B.20) (resp. (B.4)). Multiplying (B.68) by  $u_x$  we obtain the conservation of the  $\dot{H}^1$  energy

$$[u_x^2]_t + [f'(u) u_x^2]_x = 0, \quad (\text{B.70})$$

for smooth solutions.

The generalized Hunter–Saxton equation has the Hamiltonian structure

$$u_t = -\mathcal{D}_1 \delta_u \mathfrak{H}_1, \quad (\text{B.71})$$

where

$$\mathfrak{H}_1 \stackrel{\text{def}}{=} \frac{1}{2} \int f'(u) u_x^2 dx, \quad \mathcal{D}_1 u = \partial_x^{-1} u \stackrel{\text{def}}{=} \frac{1}{2} \left( \int_{-\infty}^x - \int_x^{+\infty} \right) u dx.$$

The gHS equation can also be obtained by the Hamiltonian  $\mathfrak{H}_2 \stackrel{\text{def}}{=} \frac{1}{2} \int u_x^2 dx$  and the operator

$$\mathcal{D}_2 \stackrel{\text{def}}{=} \partial_x^{-2} \left[ \left( \frac{f'(u)}{u} u_{xx} + \frac{f''(u)u - f'(u)}{2u^2} u_x^2 \right) \partial_x + \partial_x \left( \frac{f'(u)}{u} u_{xx} + \frac{f''(u)u - f'(u)}{2u^2} u_x^2 \right) \right] \partial_x^{-2}.$$

If  $f(u) = u^2/2$ , we obtain the bi-Hamiltonian Hunter–Saxton equation [14] and

$$\mathcal{D}_2 = \partial_x^{-2} [u_{xx} \partial_x + \partial_x u_{xx}] \partial_x^{-2} = u_x \partial_x^{-2} - \partial_x^{-2} u_x.$$

For general fluxes, the operator  $\mathcal{D}_2$  is not a Poisson operator [22], so the integrability of the generalized Hunter–Saxton equation remains an open question.

The goal of this section is to prove the existence of global weak solutions of the generalized Hunter–Saxton equation (B.67). The classical Hunter–Saxton equation can be obtained by taking  $f(u) = u^2/2$ , a proof of global dissipative solutions of the classical Hunter–Saxton equation can be found in [5]. If the flux  $f$  is quadratic, the right hand side of (B.67) depends only on  $u_x$  and not on  $u$  which makes the proof easier. The main result of this section is the following theorem:

**Theorem B.6.1.** *Let  $u_0 \in \dot{H}^1 \cap L^\infty$  and  $f \in \mathcal{C}^3(\mathbb{R})$  such that  $f^{(3)}$  is bounded, then*

- *There exists a global weak solution  $u$  (called conservative) of (B.67), satisfying  $u \in \text{Lip}([0, T], L_{loc}^2(\mathbb{R}))$  for all  $T > 0$  and*

$$\int_{\mathbb{R}} u_x(t)^2 dx = \int_{\mathbb{R}} u_0^2 dx \quad \text{for almost all } t \in \mathbb{R}. \quad (\text{B.72})$$

- *If the flux is uniformly convex  $f''(u) \geq C > 0$ , then there also exists a global weak solution  $u$  (called dissipative) of the equation (B.67), satisfying  $u \in \text{Lip}([0, T], L_{loc}^2(\mathbb{R}))$  for all  $T > 0$  and*

$$\int_{\mathbb{R}} u_x(t)^2 dx \leq \int_{\mathbb{R}} u_0^2 dx \quad \text{for almost all } t \in \mathbb{R}. \quad (\text{B.73})$$

Moreover, for  $M = \sup_{x \in \mathbb{R}} u_0'(x) \in \mathbb{R}^+ \cup \{+\infty\}$  the dissipative solution satisfies the Oleinik inequality

$$u_x(t, x) \leq \frac{1}{Ct/2 + 1/M}. \quad (\text{B.74})$$

**Remark B.6.1.** *The condition “ $f^{(3)}$  is bounded” is important to show that the solution is globally well-defined. Otherwise, local (in time) weak solutions exist even if  $f^{(3)}$  is not bounded.*

The proof of the existence of the conservative solutions in Theorem B.6.1 can be done following [6, 11]. To prove the existence of dissipative solutions, one follows the proof of Theorem B.5.2 with small modifications to obtain the equivalent system

$$y_t = f'(u), \quad (\text{B.75a})$$

$$u_t = -Q, \quad (\text{B.75b})$$

$$v_t = \begin{cases} -f''(u) \sin^2(v/2), & v > -\pi, \\ 0, & v \leq -\pi, \end{cases} \quad (\text{B.75c})$$

$$q_t = \begin{cases} \frac{1}{2} q f''(u) \sin(v), & v > -\pi \\ 0, & v \leq -\pi, \end{cases} \quad (\text{B.75d})$$

where  $Q$  is defined as

$$Q(t, \xi) = \frac{1}{4} \left( \int_{\xi}^{+\infty} - \int_{-\infty}^{\xi} \right) \bar{q}(t, \xi') \sin^2 \frac{v(t, \xi')}{2} f''(u(t, \xi')) d\xi', \quad (\text{B.76})$$

where  $\bar{q}(t, \xi) = q(t, \xi)$  if  $v(t, \xi) > -\pi$  and  $\bar{q}(t, \xi) = 0$  if  $v(t, \xi) \leq -\pi$ .

**Remark B.6.2.** *If the flux  $f$  is quadratic, the condition  $u_0 \in L^\infty$  can be removed (see [5]). Indeed, in this case the right-hand sides of (B.75b), (B.75c) and (B.75d) do not depend on  $u$ , so to prove that the system (B.75) is well-posed it suffices to prove the well-posedness of (B.75c) and (B.75d).*

On the contrary of the definition of  $P$  and  $P_x$  in the previous section, the exponential term does not appear in the definition of  $Q$  in (B.76), which means that the inequality (B.53) can not be used. To avoid this problem, the existence for the equivalent system (B.75b), (B.75c) and (B.75d) is done in the space  $X \stackrel{\text{def}}{=} \mathcal{C}([0, T], L^\infty \times (L^\infty \cap L^2) \times L^\infty)$  and the inequality (B.53) can be replaced then by

$$\|Q\|_{L^\infty} \lesssim \|q \sin^2 v/2\|_{L^1} \|f''(u)\|_{L^\infty} \lesssim \|q\|_{L^\infty} \|f''(u)\|_{L^\infty} \|v\|_{L^2}. \quad (\text{B.77})$$

Since  $f^{(3)}$  is bounded, then

$$\|f''(u)\|_{L^\infty} \leq \|f^{(3)}\|_{L^\infty} \|u\|_{L^\infty} + |f''(0)|.$$

The rest of the proofs of local existence, dissipation of the energy and the Oleinik inequality are the same as the previous section, so they are omitted. In order to prove that the solution holds for all time, one can show that the quantity

$$\|q\|_{L^\infty} + \|1/q\|_{L^\infty} + \|v\|_{L^2} + \|v\|_{L^\infty} + \|u\|_{L^\infty}, \quad (\text{B.78})$$

does not blow-up in finite time.  $\square$

In next section, we study the convergence of the dissipative solutions given in Theorem B.5.2 (when  $\ell \rightarrow 0$  and when  $\ell \rightarrow \infty$ ) using a  $BV$  estimate that is based on the Oleinik inequality (B.47).

## B.7 The limiting cases $\ell \rightarrow 0$ and $\ell \rightarrow \infty$

In this section we study the convergence of the dissipative solutions of (B.18) when  $\ell \rightarrow 0$  and when  $\ell \rightarrow \infty$ . For that purpose we denote  $u^\ell$  the dissipative solution given in Theorem B.5.2 and we start by the uniform (on  $\ell$ ) BV estimate of the solution

**Lemma B.7.1. [BV estimate]** *Let  $f \in \mathcal{C}^3$  such that  $\tilde{C} \geq f''(u) \geq C > 0$ , let also  $u_0 \in H^1(\mathbb{R})$  such that  $u'_0 \in L^1(\mathbb{R})$  and  $u'_0(x) \leq M < +\infty$ , then*

$$\text{TV } u^\ell(t) = \|u_x^\ell(t)\|_{L^1} \leq \|u'_0\|_{L^1} (CMt/2 + 1)^{2\tilde{C}/C}. \quad (\text{B.79})$$

*Proof.* Let be  $s \stackrel{\text{def}}{=} \text{sgn}(u_\xi^\ell) = \text{sgn}(v^\ell)$ . Differentiating (B.45b) w.r.t  $\xi$ , multiplying by  $s$ , integrating over  $\xi$ , using the Oleinik inequality (B.47) and (B.60) one obtains

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} |u_\xi^\ell| d\xi &= -\ell^2 \int_{\mathbb{R}} (P_x)_\xi s d\xi, \\ &= -\ell^2 \int_{\{s \geq 0\}} (P_x)_\xi d\xi + \ell^2 \int_{\{s < 0\}} (P_x)_\xi d\xi, \\ &= -2\ell^2 \int_{\{s \geq 0\}} (P_x)_\xi d\xi + \ell^2 \int_{\mathbb{R}} (P_x)_\xi d\xi, \\ &= \int_{\{s \geq 0, v^\ell > -\pi\}} q (f''(u^\ell) \sin^2 v^\ell / 2 - 2P \cos^2 v^\ell / 2) d\xi, \\ &\leq \int_{\{s \geq 0, v^\ell > -\pi\}} \frac{1}{2} q f''(u^\ell) \sin v^\ell \tan v^\ell / 2 d\xi, \\ &\leq \frac{\tilde{C}}{Ct/2 + 1/M} \int_{\mathbb{R}} |u_\xi^\ell| d\xi. \end{aligned}$$

Using Gronwall lemma, we deduce that  $\text{TV}_\xi u^\ell(t) = \|u_\xi^\ell\|_{L^1} \leq \|u'_0\|_{L^1} (CMt/2 + 1)^{2\tilde{C}/C}$ , where  $\text{TV}_\xi$  is the total variation with respect to  $\xi$ . Since the characteristics  $y(t, \cdot)$  are monotonic, then (B.79) follows.  $\square$

Following [11], one can easily prove the following theorem:

**Theorem B.7.1.** *Let  $f \in \mathcal{C}^3$  such that  $\tilde{C} \geq f''(u) \geq C > 0$ , let also  $u_0 \in H^1(\mathbb{R})$  such that  $u'_0 \in L^1(\mathbb{R})$  and  $u'_0(x) \leq M < +\infty$ . Then, there exist*

- $u^0 \in L^\infty([0, T], L^1(\mathcal{I})) \cap L^\infty([0, T], BV(\mathbb{R}))$ ,
- $u^\infty \in \mathcal{C}([0, T], L^1(\mathcal{I})) \cap L^\infty([0, T], BV(\mathbb{R}))$ ,
- $\mu, \nu \in L^\infty([0, T], \mathcal{M}^1)$  non-negative Radon measures,
- a sub-sequence of  $(u^\ell)$ , noted also  $(u^\ell)$

for all  $T > 0$ ,  $\mathcal{I} \Subset \mathbb{R}$ , such that

- $u^0$  and  $u^\infty$  satisfy the Oleinik inequalities

$$u_x^0(t, x) \leq \frac{1}{Ct/2 + 1/M}, \quad u_x^\infty(t, x) \leq \frac{1}{Ct/2 + 1/M} \quad \text{in } \mathcal{D}'(\mathbb{R}), \quad (\text{B.80})$$

- we have the convergences

$$u^\ell \xrightarrow{\ell \rightarrow 0} u^0 \text{ in } L^\infty([0, T], L^1(\mathcal{I})), \quad u^\ell \xrightarrow{\ell \rightarrow \infty} u^\infty \text{ in } \mathcal{C}([0, T], L^1(\mathcal{I})),$$

- $u^0$  and  $\mu$  satisfy the equation

$$u_t^0 + f(u^0)_x = -\mu_x, \quad (\text{B.81})$$

- $u^\infty$  and  $\nu$  satisfy the equation

$$[u_t^\infty + f(u^\infty)_x]_x = \nu. \quad (\text{B.82})$$

**Remark B.7.1.**

- If  $\mu = 0$ , then  $u^0$  is the entropy solution of the scalar conservation law (B.3).
- If  $\nu = \frac{1}{2}(u_x^\infty)^2 f''(u^\infty)$ , then  $u^\infty$  is a dissipative solution of the generalized Hunter–Saxton equation (B.68).

*Proof.* We first study the case  $\ell \rightarrow 0$ , so we suppose that  $\ell \leq 1$ . Following [11] we define the Banach space  $W(\mathcal{I}) \stackrel{\text{def}}{=} \{f \in \mathcal{D}'(\mathcal{I}), \exists F \in L^1(\mathcal{I}) \text{ such that } F' = f\}$ , with the norm  $\|f\|_{W(\mathcal{I})} \stackrel{\text{def}}{=} \inf_{c \in \mathbb{R}} \|F + c\|_{L^1(\mathcal{I})} = \min_{c \in \mathbb{R}} \|F + c\|_{L^1(\mathcal{I})}$ . Then

$$W^{1,1}(\mathcal{I}) \hookrightarrow L^1(\mathcal{I}) \hookrightarrow W(\mathcal{I}), \quad (\text{B.83})$$

where the first embedding is compact and the second is continuous. Using the dissipation of the energy (B.46) and (B.79) we obtain that  $(u^\ell)^2, u^\ell$  and  $\ell^2 P$  are bounded in  $L^\infty([0, T], L^1(\mathcal{I}))$ . Then, the inequality

$$|f(u) - f(0)| \leq \tilde{C} u^2 + |f'(0)| |u|,$$

with (B.18) imply that  $u_t^\ell$  is bounded in  $L^\infty([0, T], W(\mathcal{I}))$ . Aubin theorem implies the compactness of the sequence  $(u^\ell)$ . Since  $\ell^2 P$  is bounded in  $L^\infty([0, T], L^1)$ , then there exists a non negative Radon measure  $\mu$  such that  $\ell^2 P$  converges weakly to  $\mu$ . Taking  $\ell \rightarrow 0$  in the weak formulation of (B.18) we obtain (B.81).

The proof of the case  $\ell \rightarrow \infty$  can be done following the proof of Theorem 6 in [11].  $\square$

Since we suppose that  $f''(u) \leq \tilde{C}$ , the lower bound of the interaction time and of the quantity  $\|u_x\|_{L^\infty}$  given in Theorem B.4.2 is uniform on  $\ell$ , which implies that  $\ell^2 P \rightarrow 0$  when  $\ell \rightarrow 0$ . Thence, one can easily prove the following proposition:

**Proposition B.7.1.** *Under the conditions of Theorem B.4.2, if  $f''(u) \leq \tilde{C}$  and  $TVu_0 < +\infty$ , then  $\forall t < \left(\tilde{C} \sup_{x \in \mathbb{R}} |u'_0(x)|\right)^{-1}$  we have*

$$\mu(t) = 0, \quad \nu(t) = \frac{1}{2} (u_x^\infty)^2 f''(u^\infty). \quad (\text{B.84})$$

This proposition shows that before the appearance of the singularities, the limit  $u^0$  (respectively  $u^\infty$ ) is the entropy solution of the scalar conservation law (B.3) (respectively a dissipative solution of the generalized Hunter–Saxton equation (B.68)).

## B.8 Conclusion

In this paper, the classical scalar conservation law (SCL) (B.3) is regularized by the equation (B.18) that is parameterized by  $\ell$ . This regularization is derived modifying the Lagrangian of (B.3). In this paper, we prove the local (in time) existence of smooth solutions, blow-up and global existence of weak solutions of the regularized equation. A new generalized Hunter–Saxton (gHS) equation has been introduced, studied and a proof of existence of global weak solutions is presented. In the last section, we have proved that the dissipative solutions converge up to a sub-sequence to a solution of SCL when  $\ell \rightarrow 0$  and a solution of gHS when  $\ell \rightarrow \infty$  before the appearance of the singularities. Several questions are still open, and deserve to be studied:

- We have shown in this paper that the regularization (B.18) and the gHS (B.5) are Hamiltonian equations. Do bi-Hamiltonian structures exist for those equations?
- Can the uniqueness of the global weak solutions be obtained? following [4] for example.
- Do the solutions of the regularized equation converge to the expected solution — the equation (B.84) holds — even after the appearance of the singularities?
- Is it possible to regularize other interesting hyperbolic systems conserving the same properties given in this paper and in [9, 21, 23, 11]?

## Acknowledgment

This work has been done while the author is a Ph.D. student at the University Côte d’Azur. The author thanks his supervisors Didier Clamond & Stephane Junca for their advice and remarks.



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# Appendix C

## Hamiltonian regularisation of the unidimensional barotropic Euler equations

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**Abstract:** Recently, a Hamiltonian regularised shallow water (Saint-Venant) system has been introduced by Clamond and Dutykh [13]. This system is Galilean invariant, linearly non-dispersive and conserves formally an  $H^1$ -like energy. In this paper, we generalise this regularisation for the barotropic Euler system preserving the same properties. We prove the local (in time) well-posedness of the regularised barotropic Euler system and a periodic generalised two-component Hunter–Saxton system. We also show for both systems that if singularities appear in finite time, they are necessary in the first derivatives.

**AMS Classification:** 35Q35; 35L65; 37K05; 35B65; 76B15.

**Key words:** Barotropic fluids; Euler system; nonlinear hyperbolic systems; generalised two-component Hunter–Saxton system; regularisation; Hamiltonian; energy conservation; dispersionless.

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## C.1 Introduction

The barotropic Euler system is a quasilinear system of partial differential equations that can be used to describe many phenomena in fluid mechanics. Denoting the time and the spacial coordinate by the independent variables  $t$  and  $x$ , respectively, and denoting the density, the velocity and the pressure by the dependent variables  $\rho(t, x) > 0$ ,  $u(t, x)$  and  $P(\rho)$ , respectively, the conservation of mass and momentum yield

$$\rho_t + [\rho u]_x = 0, \quad u_t + u u_x + P_x/\rho = 0, \quad (\text{C.1})$$

where subscripts denote partial derivatives. If the pressure  $P$  is an increasing function of  $\rho$ , the system (C.1) is hyperbolic and, even for smooth initial data, the solutions may develop shocks in finite time. In order to avoid those shocks, several regularisations have been proposed, for example by adding a “small” artificial viscosity and/or dispersive terms [7, 16, 24, 31, 33, 34, 35, 47]. The artificial viscosity leads to a loss of the energy everywhere and the dispersive terms lead to high oscillations which cause problems in numerical computations. Other regularisations of Leray-type (for Burgers equation, isentropic Euler system and others) have been proposed and studied in [3, 4, 5, 6, 11, 41, 42]. Those regularisations do not conserve the energy and the limit solution fails (in general) to satisfy the Lax entropy condition [4].

Modifying the dispersion of the Serre–Green–Naghdi system, Clamond and Dutykh [13] proposed the dispersionless regularised Saint-Venant (shallow water) system

$$h_t + [hu]_x = 0, \quad (\text{C.2a})$$

$$[hu]_t + \left[ hu^2 + \frac{1}{2}gh^2 + \epsilon\mathcal{R} \right]_x = 0, \quad (\text{C.2b})$$

$$\mathcal{R} \stackrel{\text{def}}{=} 2h^3u_x^2 - h^3[u_t + uu_x + gh_x]_x - \frac{1}{2}gh^2h_x^2, \quad (\text{C.2c})$$

where  $h$  is the total water depth of the fluid,  $g$  is the gravitational acceleration and  $\epsilon > 0$  is a dimensionless parameter. This Hamiltonian regularisation conserves an  $H^1$ -like energy for smooth solutions and it has the same shock speed as the classical Saint-Venant system. Weak singular shocks of (C.2) have been studied by Pu et al. [45]. Also, local (in time) well-posedness and existence of blowing-up solutions using Riccati-type equations have been proved in [36]. The global well-posedness and a mathematical study of the case  $\epsilon \rightarrow 0$  remains open problems. Recently, inspired by [13] and with the same properties as (C.2), a similar regularisation has been proposed for the inviscid Burgers equation in [23] and for general scalar conservation laws in [22], where solutions exist globally (in time) in  $H^1$ , those solutions converging to solutions of the classical equation when  $\epsilon \rightarrow 0$  at least for a short time [22, 23]. The regularised Saint-Venant system (C.2) has been also generalised for shallow water equations with uneven bottom [14].

The classical Saint-Venant system (letting  $\epsilon \rightarrow 0$  in (C.2)) is, formally, a special case of the barotropic Euler system (C.1) such that  $\rho \equiv h$  and  $P(\rho) \equiv gh^2/2$  (i.e., isentropic Euler equation with  $\gamma = 2$ , see Section C.2.5). The aim of this paper is to generalise the system (C.2) to regularise the barotropic Euler system (C.1) as in [13], that is preserving the same properties. In Section C.3 below, modifying the Lagrangian of (C.1), we obtain the regularised barotropic Euler (rbE) system

$$\rho_t + [\rho u]_x = 0, \quad (\text{C.3a})$$

$$[\rho u]_t + \left[ \rho u^2 + \rho\mathcal{V}' - \mathcal{V} + \epsilon\mathcal{R} \right]_x = 0, \quad (\text{C.3b})$$

$$\mathcal{R} \stackrel{\text{def}}{=} (\rho^2\mathcal{A}')' u_x^2 - 2\rho\mathcal{A}'[u_t + uu_x + \varpi_x]_x + (\rho\mathcal{V}''/\mathcal{A}')' \mathcal{A}_x^2, \quad (\text{C.3c})$$

where primes denote derivatives with respect to  $\rho$ ,  $\mathcal{V}''(\rho) = P'(\rho)/\rho$  and  $\mathcal{A}$  is a smooth increasing function of  $\rho$ . We show in this paper that the system (C.3) is non-dispersive, non-diffusive, it has a Hamiltonian structure, it has the same shock speed as (C.1) and, for all  $\mathcal{A}$  smooth increasing function of  $\rho$ , smooth solutions of (C.3) conserve an  $H^1$ -like energy. A study of steady solutions of rbE has been also done, which covers the traveling waves due to the Galilean invariance of rbE.

Introducing the linear Sturm–Liouville operator

$$\mathcal{L}_\rho \stackrel{\text{def}}{=} \rho - 2\epsilon\partial_x\rho\mathcal{A}'\partial_x, \quad (\text{C.4})$$

and applying  $\mathcal{L}_\rho^{-1}$  on (C.3b) (the invertibility of  $\mathcal{L}_\rho$  is insured by Lemma C.4.3 below), the system (C.3) becomes

$$\rho_t + [\rho u]_x = 0, \quad (\text{C.5a})$$

$$u_t + uu_x + P_x/\rho = -\epsilon\mathcal{L}_\rho^{-1}\partial_x \left\{ (\rho^2\mathcal{A}')' u_x^2 + (\rho\mathcal{V}''/\mathcal{A}')' \mathcal{A}'^2 \rho_x^2 \right\}. \quad (\text{C.5b})$$

The reason of applying  $\mathcal{L}_\rho^{-1}$  is to remove the derivative with respect to  $t$  and the high-order derivatives with respect to  $x$  appearing in (C.3c). The form (C.5) is then more convenient to obtain the local well-posedness of rbE. Following [1, 2, 27, 36, 39], we prove that if the initial data is an  $H^s$  perturbation of a constant state (with  $s \geq 2$ , and  $\rho \geq \rho^* > 0$ ), then (C.5) is locally well-posed. The same proof is used to prove the local existence of periodic solutions of the generalised two-component Hunter–Saxton system

$$\rho_t + [\rho u]_x = 0, \quad (\text{C.6a})$$

$$u_t + u u_x + P_x/\rho = \partial_x^{-1} \left\{ \left( 1 + \frac{\rho \mathcal{A}''}{2 \mathcal{A}'} \right) u_x^2 + \left( \frac{(\rho \mathcal{V}'')'}{2 \rho} - \frac{\mathcal{V}'' \mathcal{A}''}{2 \mathcal{A}'} \right) \rho_x^2 \right\}, \quad (\text{C.6b})$$

that can be obtained by formally taking  $\varepsilon \rightarrow \infty$  in (C.5).

This paper is divided on two parts. A first part (sections C.2, C.3) presents the physical motivations of the regularised barotropic Euler system and its properties. A second part (sections C.4, C.5) consist on mathematical proofs for existence results. Shortly speaking, the first part is more physical and the second one is more mathematical. More specifically, the content of the paper is organised as follows. In Section C.2, we recall some classical properties of the barotropic Euler system (C.1). Section C.3 is devoted to derive the regularised system (C.3), study its properties and steady motions. In Section C.4, we prove the local well-posedness and a blow-up criteria of (C.5). In Section C.5, the generalised Hunter–Saxton system (C.6) is introduced, and a well-posedness theorem is given. A special choice of the regularising function  $\mathcal{A}$ , with some interesting properties, is briefly discussed in Section C.6.

## C.2 Equations for barotropic perfect fluids

Let us recall the conservation of mass and momentum for perfect fluids in Eulerian description of motion

$$\rho_t + [\rho u]_x = 0, \quad (\text{C.7a})$$

$$u_t + u u_x + P_x/\rho = 0. \quad (\text{C.7b})$$

Note that (conservative) body forces, if present, are incorporated into the definition of the pressure  $P$ . In the special case of barotropic motions [46] — i.e., when  $\rho = \rho(P)$  or  $P = P(\rho)$  — it is convenient to introduce the so-called *specific enthalpy* [18, §3.3]  $\varpi$  such that

$$\varpi = \int \frac{dP}{\rho(P)} = \int \frac{dP(\rho)}{d\rho} \frac{d\rho}{\rho} \quad \Rightarrow \quad \rho d\varpi = dP, \quad \partial_x \varpi = \frac{\partial_x P}{\rho}. \quad (\text{C.8})$$

$\varpi$  being an antiderivative of  $1/\rho(P)$ , it is defined modulo an additional arbitrary integration constant, so the value of  $\varpi$  can be freely chosen on a given isobaric surface  $P = \text{constant}$  (thus providing gauge condition for the specific enthalpy). The relation (C.8) can also be

written in the reciprocal form  $P(\varpi) = \int \rho(\varpi) d\varpi$ , thence  $P$  is a known function of  $\varpi$  and, obviously,  $P = \rho\varpi$  if the density is constant. The speed of sound  $c_s$  is defined by

$$c_s \stackrel{\text{def}}{=} [d\rho/dP]^{-\frac{1}{2}} = [\rho^{-1} d\rho/d\varpi]^{-\frac{1}{2}}. \quad (\text{C.9})$$

From this definition we have  $\rho d\varpi = c_s^2 d\rho$ , thence with the mass conservation (C.7a)

$$D_t \varpi = \rho^{-1} c_s^2 D_t \rho = -c_s^2 u_x, \quad (\text{C.10})$$

where  $D_t \stackrel{\text{def}}{=} \partial_t + u\partial_x$  is the temporal derivative following the motion. The relation (C.10) is of special interest when  $c_s$  is constant. Many equations of state for compressible fluids can be found in the literature [49]. Isentropic motions are of special interest so their equation of state is given in Section C.2.5.

### C.2.1 Cauchy–Lagrange equation

For barotropic fluids, the momentum equation (C.7b) becomes

$$u_t + u u_x + \varpi_x = 0, \quad (\text{C.11})$$

and introducing a velocity potential  $\phi$  such that  $u = \phi_x$ , the equation (C.11) is integrated into a Cauchy–Lagrange equation

$$\phi_t + \frac{1}{2} \phi_x^2 + \varpi = K(t) \equiv 0, \quad (\text{C.12})$$

where  $K(t)$  is an integration constant that can be set to zero without loose of generality (gauge condition for the velocity potential).

### C.2.2 Conservation laws

For regular solutions, secondary conservation laws can be derived from (C.7), e.g.,

$$[\rho u]_t + [\rho u^2 + \rho \mathcal{V}' - \mathcal{V}]_x = 0, \quad (\text{C.13})$$

$$[u]_t + [\frac{1}{2} u^2 + \mathcal{V}']_x = 0, \quad (\text{C.14})$$

$$[\frac{1}{2} \rho u^2 + \mathcal{V}]_t + [(\frac{1}{2} \rho u^2 + \rho \mathcal{V}') u]_x = 0, \quad (\text{C.15})$$

$$[\frac{1}{6} \rho u^3 + \mathcal{V} u]_t + [\frac{1}{2} (\frac{1}{3} \rho u^2 + \rho \mathcal{V}' + \mathcal{V}) u^2 + \mathcal{W}_1]_x = 0, \quad (\text{C.16})$$

$$[\frac{1}{24} \rho u^4 + \frac{1}{2} \mathcal{V} u^2 + \mathcal{W}_3]_t + [\frac{1}{3} (\frac{1}{8} \rho u^2 + \frac{1}{2} \rho \mathcal{V}' + \mathcal{V}) u^3 + \mathcal{W}_2 u]_x = 0, \quad (\text{C.17})$$

where

$$\mathcal{V} \stackrel{\text{def}}{=} \int \varpi d\rho, \quad \mathcal{W}_1 \stackrel{\text{def}}{=} \int \mathcal{V} \mathcal{V}'' d\rho, \quad \mathcal{W}_2 \stackrel{\text{def}}{=} \rho \int \frac{\mathcal{V} \mathcal{V}''}{\rho} d\rho, \quad \mathcal{W}_3 \stackrel{\text{def}}{=} \int \frac{\mathcal{W}_2}{\rho} d\rho.$$

Actually, since the barotropic Euler system is a  $2 \times 2$  strictly hyperbolic system, an infinite number of conservation laws can be derived [17, 32].



### C.2.3 Jump conditions

The Euler equations admit weak solutions. For discontinuous  $\rho$  and  $u$ , the Rankine–Hugoniot conditions for the mass and momentum conservation are

$$(u - \dot{s}) \llbracket \rho \rrbracket + \rho \llbracket u \rrbracket = 0, \quad (u - \dot{s}) \llbracket u \rrbracket + \varpi' \llbracket \rho \rrbracket = 0, \quad (\text{C.18})$$

where  $\varpi' = d\varpi/d\rho$ ,  $\dot{s} \stackrel{\text{def}}{=} ds/dt$  is the speed of the shock located at  $x = s(t)$  and  $\llbracket f \rrbracket \stackrel{\text{def}}{=} f(x = s^+) - f(x = s^-)$  denotes the jump across the shock for any function  $f$ . The Rankine–Hugoniot conditions (C.18) yield at once the shock speed

$$\dot{s}(t) = u \pm \sqrt{\rho \varpi'} \quad \text{at} \quad x = s(t). \quad (\text{C.19})$$

A goal of the present work is to derive a regularisation of the Euler equation that preserves exactly this shock speed.

### C.2.4 Variational formulations

An interesting feature of the equations above is that they can be derived from a variational principle. Indeed, the (so-called action) functional  $S = \int_{t_1}^{t_2} \int_{x_1}^{x_2} \mathcal{L} \, dx \, dt$  with the Lagrangian density

$$\mathcal{L} \stackrel{\text{def}}{=} \rho \phi_t + \frac{1}{2} \rho \phi_x^2 + \mathcal{V}(\rho), \quad (\text{C.20})$$

where  $\mathcal{V}$  is the density of potential energy defined by

$$\mathcal{V}(\rho) \stackrel{\text{def}}{=} \int \varpi(\rho) \, d\rho, \quad (\text{C.21})$$

provided that an equation of state  $\varpi(\rho)$  (such as (C.31) given in Section C.2.5), is substituted into the right-hand side of (C.21). Since  $\varpi(\bar{\rho}) = 0$  with (C.31) ( $\bar{\rho}$  a constant state of reference),  $\mathcal{V}$  is such that  $\mathcal{V}'(\bar{\rho}) = 0$ . Note that  $\mathcal{V}$  can also be kept explicitly into the Lagrangian if the equation of state is added via a Lagrange multiplier  $\lambda$ , i.e., considering the Lagrangian density

$$\mathcal{L}' \stackrel{\text{def}}{=} \mathcal{L} + \left\{ \mathcal{V}(\rho) - \int \varpi(\rho) \, d\rho \right\} \lambda. \quad (\text{C.22})$$

This is of no interest here, however, so we do not consider this generalisation, for simplicity. The Euler–Lagrange equations for the Lagrangian density (C.20) yield

$$\delta\phi: \quad 0 = \rho_t + [\rho \phi_x]_x, \quad (\text{C.23})$$

$$\delta\rho: \quad 0 = \phi_t + \frac{1}{2} \phi_x^2 + \mathcal{V}'(\rho), \quad (\text{C.24})$$

so the equations of motion (C.7a) and (C.12) are recovered.

An alternative variational formulation is obtained from the Hamilton principle yielding the Lagrangian density

$$\mathcal{L}_0 \stackrel{\text{def}}{=} \frac{1}{2} \rho u^2 - \mathcal{V}(\rho) + \{ \rho_t + [\rho u]_x \} \phi, \quad (\text{C.25})$$

that is the kinetic minus potential energies plus a constraint enforcing the mass conservation. The Euler–Lagrange equations for (C.25) yield

$$\delta\phi: 0 = \rho_t + [\rho u]_x, \quad (\text{C.26})$$

$$\delta u: 0 = u - \phi_x, \quad (\text{C.27})$$

$$\delta\rho: 0 = \frac{1}{2}u^2 - \mathcal{V}'(\rho) - \phi_t - u\phi_x, \quad (\text{C.28})$$

and, substituting  $\varpi$  for  $\mathcal{V}'$ , the barotropic equations (C.7a) and (C.11) are recovered.

The two variational principles above differ by boundary terms only, i.e.,

$$\mathcal{L} + \mathcal{L}_0 = [\rho\phi]_t + [\rho u\phi]_x + \frac{1}{2}\rho(u - \phi_x)^2, \quad (\text{C.29})$$

so the right-hand side yields only boundary terms since  $u = \phi_x$ . As advocated by Clamond and Dutykh [12], the variational Hamilton principle is more useful for practical applications; this point is illustrated in the section C.3 below.

## C.2.5 Isentropic flows

Isentropic motions obey the equation of state

$$\rho/\bar{\rho} = (P/\bar{P})^{1/\gamma}, \quad P/\bar{P} = (\rho/\bar{\rho})^\gamma, \quad (\text{C.30})$$

where  $\bar{\rho}$  and  $\bar{P}$  are positive constants characterising the fluid physical properties at the rest state, and  $\gamma \stackrel{\text{def}}{=} C_p/C_v$  is the (constant) ratio of specific heats  $C_p$  and  $C_v$ . It should be noted that the constitutive relation (C.30) gauges the pressure field, so zero pressure level can no longer be chosen arbitrarily without loss of generality. For these isentropic motions, we have if  $\gamma \neq 1$  (taking  $\varpi(\bar{P}) = 0$ )

$$\varpi = \int \left(\frac{\bar{P}}{P}\right)^{\frac{1}{\gamma}} \frac{dP}{\bar{\rho}} = \bar{\omega} \frac{(P/\bar{P})^{\frac{\gamma-1}{\gamma}} - 1}{\gamma - 1} = \bar{\omega} \frac{(\rho/\bar{\rho})^{\gamma-1} - 1}{\gamma - 1}, \quad (\text{C.31})$$

$$\frac{\mathcal{V}}{\bar{P}} = \frac{\gamma}{\gamma - 1} \left[ \frac{1}{\gamma} \left(\frac{\rho}{\bar{\rho}}\right)^\gamma - \frac{\rho}{\bar{\rho}} \right] = \frac{\gamma}{\gamma - 1} \left[ \frac{1}{\gamma} \frac{P}{\bar{P}} - \frac{\rho}{\bar{\rho}} \right], \quad (\text{C.32})$$

where  $\bar{\omega} \stackrel{\text{def}}{=} \gamma\bar{P}/\bar{\rho}$ , thence

$$P/\bar{P} = (c_s^2/\bar{\omega})^{\gamma/(\gamma-1)}, \quad c_s^2/\bar{\omega} = 1 + (\gamma - 1)(\varpi/\bar{\omega}). \quad (\text{C.33})$$

In the limiting case  $\gamma \rightarrow 1$  (isothermal motions), these relations become

$$c_s^2 = \bar{\omega} = \frac{\bar{P}}{\bar{\rho}}, \quad \frac{\rho}{\bar{\rho}} = \frac{P}{\bar{P}} = \exp\left(\frac{\varpi}{\bar{\omega}}\right), \quad \frac{\mathcal{V}}{\bar{P}} = \frac{\rho}{\bar{\rho}} \log\left|\frac{\rho}{\bar{\rho}}\right| - \frac{\rho}{\bar{\rho}}, \quad (\text{C.34})$$

so the speed of sound is constant while the density is not. The special case  $\gamma = 1$  is relevant for applications in oceanography because for seawater (at salinity 35 g/kg and atmospheric pressure)  $\gamma \approx 1.0004$  at  $0^\circ\text{C}$  and  $\gamma \approx 1.0106$  at  $20^\circ\text{C}$  [30].

## C.3 Regularised barotropic flows

Here, we seek for a regularisation of the barotropic Euler equations. We give some heuristic arguments for the derivation of such models.

### C.3.1 Modified Lagrangian

Following the regularisation of Clamond and Dutykh [13] for the Saint-Venant shallow water equations, we seek for a regularisation of the barotropic Euler equation modifying the Lagrangian density as

$$\overline{\mathcal{L}}_\epsilon \stackrel{\text{def}}{=} \mathcal{L}_0 + \epsilon \mathcal{A}(\rho) [u_t + u u_x]_x + \epsilon \mathcal{B}(\rho) [\mathcal{V}''(\rho) \rho_x]_x, \quad (\text{C.35})$$

where  $\epsilon \geq 0$  is a real parameter at our disposal and  $\mathcal{A}$  and  $\mathcal{B}$  are functions of  $\rho$  to be chosen later with suitable properties.

Note that we could also seek for modifications separating  $u_t$  and  $u u_x$  in the additional terms — i.e., replacing  $\epsilon \mathcal{A}(\rho) [u_t + u u_x]_x$  by  $\epsilon \mathcal{A}(\rho) u_{xt} + \epsilon \mathcal{C}(\rho) [u u_x]_x$  — but that would break the Galilean invariance. So,  $\mathcal{C} = \mathcal{A}$  is the only physically admissible possibilities.

Exploiting the relations

$$\mathcal{A}(\rho) [u_t + u u_x]_x = [\mathcal{A}(\rho) u_x]_t + [\mathcal{A}(\rho) u u_x]_x + \mathcal{A}'(\rho) \rho u_x^2, \quad (\text{C.36})$$

$$\mathcal{B}(\rho) [\mathcal{V}''(\rho) \rho_x]_x = [\mathcal{B}(\rho) \mathcal{V}''(\rho) \rho_x]_x - \mathcal{B}'(\rho) \mathcal{V}''(\rho) \rho_x^2, \quad (\text{C.37})$$

we derive the equivalent simplified Lagrangian density

$$\mathcal{L}_\epsilon \stackrel{\text{def}}{=} \frac{1}{2} \rho u^2 + \epsilon \mathcal{A}' \rho u_x^2 - \mathcal{V} - \epsilon \mathcal{B}' \mathcal{V}'' \rho_x^2 + \{\rho_t + [\rho u]_x\} \phi. \quad (\text{C.38})$$

The functionals given by  $\mathcal{L}_\epsilon$  and  $\overline{\mathcal{L}}_\epsilon$  differing only by boundary terms (i.e.,  $\mathcal{L}_\epsilon - \overline{\mathcal{L}}_\epsilon = [\dots]_t + [\dots]_x$ ), they yield the same equations of motion.

From (C.38), the regularised kinetic and potential energy densities, respectively  $\mathcal{H}_\epsilon$  and  $\mathcal{V}_\epsilon$ , are

$$\mathcal{H}_\epsilon \stackrel{\text{def}}{=} \frac{1}{2} \rho u^2 + \epsilon \mathcal{A}' \rho u_x^2, \quad \mathcal{V}_\epsilon \stackrel{\text{def}}{=} \mathcal{V} + \epsilon \mathcal{B}' \mathcal{V}'' \rho_x^2. \quad (\text{C.39})$$

The total energy is then

$$\mathcal{H}_\epsilon \stackrel{\text{def}}{=} \frac{1}{2} \rho u^2 + \epsilon \mathcal{A}' \rho u_x^2 + \mathcal{V} + \epsilon \mathcal{B}' \mathcal{V}'' \rho_x^2. \quad (\text{C.40})$$

Note that these energies are positive for all  $\epsilon \geq 0$  if  $\mathcal{A}$ ,  $\mathcal{B}$  and  $\mathcal{V}'$  are increasing functions of  $\rho$ .

### C.3.2 Linearised equations

Here, we consider small perturbations around the rest state  $\rho = \bar{\rho}$ ,  $u = 0$  and  $\phi = 0$ ,  $\bar{\rho}$  being a positive constant. Introducing  $\rho = \bar{\rho} + \tilde{\rho}$ ,  $u \approx \tilde{u}$ ,  $\phi \approx \tilde{\phi}$ , and  $f_0 \stackrel{\text{def}}{=} f(\bar{\rho})$  for any function  $f$ ,

the tilde quantities being assumed small, an approximation of  $\mathcal{L}_\epsilon$  up to the second-order is

$$\widetilde{\mathcal{L}}_\epsilon = \frac{1}{2} \bar{\rho} \tilde{u}^2 + \epsilon \mathcal{A}'_0 \bar{\rho} \tilde{u}_x^2 - \mathcal{V}_0 - \frac{1}{2} \mathcal{V}_0'' \tilde{\rho}^2 - \epsilon \mathcal{B}'_0 \mathcal{V}_0'' \tilde{\rho}_x^2 + (\tilde{\rho}_t + \bar{\rho} \tilde{u}_x) \tilde{\phi}. \quad (\text{C.41})$$

The Euler–Lagrange (linear) equations for this approximate Lagrangian are

$$\delta \tilde{\phi}: \quad 0 = \tilde{\rho}_t + \bar{\rho} \tilde{u}_x, \quad (\text{C.42})$$

$$\delta \tilde{u}: \quad 0 = \tilde{u} - 2\epsilon \mathcal{A}'_0 \tilde{u}_{xx} - \tilde{\phi}_x, \quad (\text{C.43})$$

$$\delta \tilde{\rho}: \quad 0 = \mathcal{V}_0'' \tilde{\rho} - 2\epsilon \mathcal{B}'_0 \mathcal{V}_0'' \tilde{\rho}_{xx} + \tilde{\phi}_t. \quad (\text{C.44})$$

Looking for traveling waves of the form  $\tilde{\rho} = R \cos(kx - \omega t)$ ,  $\tilde{u} = U \cos(kx - \omega t)$  and  $\tilde{\phi} = \Phi \sin(kx - \omega t)$ , the equations (C.42)–(C.44) yield  $\Phi = (1 + 2\epsilon k^2 \mathcal{A}'_0)U/k$ ,  $U = \omega R/k\bar{\rho}$  and the dispersion relation

$$\frac{\omega^2}{k^2} = \bar{\rho} \mathcal{V}_0'' \frac{1 + 2\epsilon k^2 \mathcal{B}'_0}{1 + 2\epsilon k^2 \mathcal{A}'_0}. \quad (\text{C.45})$$

If  $\epsilon = 0$  the wave is dispersionless, i.e., the phase velocity  $c \stackrel{\text{def}}{=} \omega/k$  is independent of the wave number  $k$ . If  $\epsilon > 0$ , the wave is dispersionless if  $\mathcal{B}'_0 = \mathcal{A}'_0$ . This condition should be satisfied for all  $\bar{\rho}$  and for all possible (barotropic) equation of state. Thus, we should take

$$\mathcal{B}(\rho) = \mathcal{A}(\rho). \quad (\text{C.46})$$

Hereafter, we consider only the special case (C.46) because we are only interested by non-dispersive regularisations of the barotropic Euler equations.

### C.3.3 Equations of motion

With (C.46), the Euler–Lagrange equations for the Lagrangian density (C.38) yield

$$\delta \phi: \quad 0 = \rho_t + [\rho u]_x, \quad (\text{C.47})$$

$$\delta u: \quad 0 = \rho u - 2\epsilon [\mathcal{A}' \rho u_x]_x - \rho \phi_x, \quad (\text{C.48})$$

$$\begin{aligned} \delta \rho: \quad 0 = & \frac{1}{2} u^2 + \epsilon (\mathcal{A}' + \mathcal{A}'' \rho) u_x^2 - \mathcal{V}' + \epsilon (\mathcal{A}'' \mathcal{V}'' + \mathcal{A}' \mathcal{V}''') \rho_x^2 \\ & + 2\epsilon \mathcal{A}' \mathcal{V}'' \rho_{xx} - \phi_t - u \phi_x, \end{aligned} \quad (\text{C.49})$$

thence

$$\phi_x = u - 2\epsilon \rho^{-1} [\mathcal{A}' \rho u_x]_x, \quad (\text{C.50})$$

$$\begin{aligned} \phi_t = & -\frac{1}{2} u^2 + \epsilon (\mathcal{A}' + \mathcal{A}'' \rho) u_x^2 - \mathcal{V}' + \epsilon (\mathcal{A}'' \mathcal{V}'' + \mathcal{A}' \mathcal{V}''') \rho_x^2 \\ & + 2\epsilon \mathcal{A}' \mathcal{V}'' \rho_{xx} + 2\epsilon u \rho^{-1} [\mathcal{A}' \rho u_x]_x. \end{aligned} \quad (\text{C.51})$$

Eliminating  $\phi$  between these last two relations one obtains

$$\begin{aligned} 0 = & \partial_t \left\{ u - 2\epsilon \rho^{-1} [\mathcal{A}' \rho u_x]_x \right\} + \partial_x \left\{ \frac{1}{2} u^2 + \mathcal{V}' - 2\epsilon \mathcal{A}' \mathcal{V}'' \rho_{xx} \right. \\ & \left. - \epsilon (\mathcal{A}' + \mathcal{A}'' \rho) u_x^2 - \epsilon (\mathcal{A}'' \mathcal{V}'' + \mathcal{A}' \mathcal{V}''') \rho_x^2 - 2\epsilon u \rho^{-1} [\mathcal{A}' \rho u_x]_x \right\}. \end{aligned} \quad (\text{C.52})$$

The equations (C.47) and (C.52) form the regularised Euler equations for barotropic motions studied in the present paper.

### C.3.4 Secondary equations

From the regularised barotropic Euler (rbE) equations (C.47) and (C.52), several secondary equations can be derived; in particular:

$$u_t + u u_x + \varpi_x + \epsilon \rho^{-1} \mathcal{R}_x = 0, \quad (\text{C.53})$$

$$[\rho u]_t + [\rho u^2 + \rho \mathcal{V}' - \mathcal{V} + \epsilon \mathcal{R}]_x = 0, \quad (\text{C.54})$$

$$m_t + \left[ u m + \rho \mathcal{V}' - \mathcal{V} - \epsilon (\rho^2 \mathcal{A}')' u_x^2 - 2 \epsilon \rho \mathcal{A}' \varpi_{xx} + \epsilon (\rho \mathcal{V}'' / \mathcal{A}')' \mathcal{A}_x^2 \right]_x = 0, \quad (\text{C.55})$$

$$\left[ \frac{1}{2} \rho u^2 + \epsilon \rho \mathcal{A}' u_x^2 + \mathcal{V} + \epsilon \mathcal{A}' \mathcal{V}'' \rho_x^2 \right]_t + \left[ \left( \frac{1}{2} \rho u^2 + \rho \mathcal{V}' + \epsilon \rho \mathcal{A}' u_x^2 + \epsilon \mathcal{A}' \mathcal{V}'' \rho_x^2 + \epsilon \mathcal{R} \right) u + 2 \epsilon \rho \mathcal{A}' \mathcal{V}'' \rho_x u_x \right]_x = 0, \quad (\text{C.56})$$

where

$$\mathcal{R} \stackrel{\text{def}}{=} (\rho^2 \mathcal{A}')' u_x^2 - 2 \rho \mathcal{A}' [u_t + u u_x + \varpi_x]_x + (\rho \mathcal{V}'' / \mathcal{A}')' \mathcal{A}_x^2, \quad (\text{C.57})$$

$$m \stackrel{\text{def}}{=} \rho u - 2 \epsilon [\rho \mathcal{A}' u_x]_x. \quad (\text{C.58})$$

Introducing the linear Sturm–Liouville operator  $\mathcal{L}_\rho \stackrel{\text{def}}{=} \rho - 2 \epsilon \partial_x \rho \mathcal{A}' \partial_x$ , the equation (C.53) multiplied by  $\rho$  becomes

$$\mathcal{L}_\rho \{ u_t + u u_x + \varpi_x \} + \epsilon \left[ (\rho^2 \mathcal{A}')' u_x^2 + (\rho \mathcal{V}'' / \mathcal{A}')' \mathcal{A}_x^2 \right]_x = 0, \quad (\text{C.59})$$

or, inverting the operator,

$$u_t + u u_x + \varpi_x = - \epsilon \mathcal{G}_\rho \partial_x \left\{ (\rho^2 \mathcal{A}')' u_x^2 + (\rho \mathcal{V}'' / \mathcal{A}')' \mathcal{A}_x^2 \right\}, \quad (\text{C.60})$$

where  $\mathcal{G}_\rho = \mathcal{L}_\rho^{-1}$ . The operator  $\mathcal{G}_\rho \partial_x$  acting on high frequencies like a first-order anti-derivative, it has a smoothing effect. However, this equation is in a non-conservative form. A conservative variant is obtained multiplying (C.60) by  $\rho$  and exploiting the mass conservation, hence

$$[\rho u]_t + \left[ \rho u^2 + \rho \mathcal{V}' - \mathcal{V} + \epsilon \mathcal{J}_\rho \left\{ (\rho^2 \mathcal{A}')' u_x^2 + (\rho \mathcal{V}'' / \mathcal{A}')' \mathcal{A}_x^2 \right\} \right]_x = 0, \quad (\text{C.61})$$

with the operator

$$\begin{aligned} \mathcal{J}_\rho &\stackrel{\text{def}}{=} \partial_x^{-1} \rho \mathcal{G}_\rho \partial_x = \partial_x^{-1} \rho \left[ 1 - 2 \epsilon \rho^{-1} \partial_x \rho \mathcal{A}' \partial_x \right]^{-1} \rho^{-1} \partial_x \\ &= \left[ 1 - 2 \epsilon \rho \mathcal{A}' \partial_x \rho^{-1} \partial_x \right]^{-1} = 1 + 2 \epsilon \rho \mathcal{A}' \partial_x \mathcal{G}_\rho \partial_x. \end{aligned} \quad (\text{C.62})$$

Comparing (C.61) with (C.54), one obtains at once an alternative expression for the regularising term

$$\mathcal{R} = \mathcal{J}_\rho \left\{ (\rho^2 \mathcal{A}')' u_x^2 + (\rho \mathcal{V}'' / \mathcal{A}')' \mathcal{A}_x^2 \right\}. \quad (\text{C.63})$$

While the definition (C.57) of  $\mathcal{R}$  involves second-order spacial derivatives, the alternative form (C.63) shows actually that  $\mathcal{R}$  behaves at high frequencies somehow like zeroth-order derivatives. Moreover, since the relations (C.57) and (C.63) are identical, we obtain yet another form of the momentum equation

$$2\rho\mathcal{A}'[u_t + uu_x + \varpi_x]_x + (\mathcal{J}_\rho - \mathcal{I})\left\{(\rho^2\mathcal{A}')'u_x^2 + (\rho\mathcal{V}''/\mathcal{A}')'\mathcal{A}_x^2\right\} = 0, \quad (\text{C.64})$$

where  $\mathcal{I}$  is the identity operator. Note that applying the operator  $\partial_x^{-1}(2\rho\mathcal{A}')^{-1}$ , the equation (C.64) can be rewritten

$$u_t + uu_x + \varpi_x + \partial_x^{-1}(2\rho\mathcal{A}')^{-1}(\mathcal{I} - \mathcal{J}_\rho^{-1})\{\mathcal{R}\} = 0, \quad (\text{C.65})$$

and with

$$\mathcal{I} - \mathcal{J}_\rho^{-1} = \mathcal{I} - \partial_x^{-1}\rho[1 - 2\epsilon\rho^{-1}\partial_x\rho\mathcal{A}'\partial_x]\rho^{-1}\partial_x = 2\epsilon\rho\mathcal{A}'\partial_x\rho^{-1}\partial_x, \quad (\text{C.66})$$

one gets the equation (C.53), as it should be.

### C.3.5 Rankine–Hugoniot conditions

Here, we assume that  $\rho_x$  and  $u_x$  are both continuous if  $\epsilon > 0$  and that discontinuities (if any) occur only in  $\rho_{xx}$  and  $u_{xx}$ . Differentiating twice with respect of  $x$  the mass conservation (C.47), the jump condition of the resulting equation is

$$(u - \dot{s})[\rho_{xx}] + \rho[u_{xx}] = 0, \quad (\text{C.67})$$

while the jump condition for (C.52) is (provided that  $\epsilon$  and  $\mathcal{A}'$  are not zero)

$$(u - \dot{s})[u_{xx}] + \mathcal{V}''[\rho_{xx}] = 0. \quad (\text{C.68})$$

Thus, the speed of the regularised shock is identical to the original one, whatever the function  $\mathcal{A}' \neq 0$  is. Therefore, a suitable choice for the function  $\mathcal{A}$  cannot be determined by this consideration.

### C.3.6 Hamiltonian formulation

Introducing the momentum  $m \stackrel{\text{def}}{=} \rho u - 2\epsilon[\rho\mathcal{A}'u_x]_x$  and the Hamiltonian functional density

$$\mathcal{H}_\epsilon(\rho, m) \stackrel{\text{def}}{=} \frac{1}{2}m\mathcal{G}_\rho\{m\} + \mathcal{V} + \epsilon\mathcal{A}'\mathcal{V}''\rho_x^2, \quad (\text{C.69})$$

we have

$$\mathcal{E}_m\{\mathcal{H}_\epsilon\} = \mathcal{G}_\rho\{m\} = u, \quad (\text{C.70})$$

$$\mathcal{E}_\rho\{\mathcal{H}_\epsilon\} = \mathcal{V}' - \epsilon(\mathcal{A}'\mathcal{V}''')'\rho_x^2 - 2\epsilon\mathcal{A}'\mathcal{V}''\rho_{xx} - \frac{1}{2}u^2 - \epsilon(\rho\mathcal{A}')'u_x^2, \quad (\text{C.71})$$

where  $\mathcal{E}_m$  and  $\mathcal{E}_\rho$  are the Euler–Lagrange operators with respect of  $m$  and  $\rho$ . The rbE equations have then the Hamiltonian structure

$$\partial_t \begin{pmatrix} \rho \\ m \end{pmatrix} = -\mathbb{J} \cdot \begin{pmatrix} \mathcal{E}_\rho\{\mathcal{H}_\epsilon\} \\ \mathcal{E}_m\{\mathcal{H}_\epsilon\} \end{pmatrix}, \quad \mathbb{J} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & \partial_x \rho \\ \rho \partial_x & m \partial_x + \partial_x m \end{bmatrix}, \quad (\text{C.72})$$

yielding the equations (C.47) and (C.55). It should be noted that  $\mathbb{J}$  being skew-symmetric and satisfying the Jacobi identity [43], it is a proper Hamiltonian (Lie–Poisson) operator.

### C.3.7 Steady motions

We seek here for solutions independent of the time  $t$ , i.e., we look for travelling waves of permanent form observed in the frame of reference moving with the wave (note that the rbE equations are Galilean invariant). For such flows, the mass conservation yields

$$u = I/\rho, \quad (\text{C.73})$$

where  $I$  is an integration constant (the mean impulse). From the relations (C.54) and (C.56), the mean (constant) momentum and energy fluxes are respectively

$$S = \rho u^2 + \rho \mathcal{V}' - \mathcal{V} + \epsilon \mathcal{R}, \quad (\text{C.74})$$

$$F = \left( \frac{1}{2} \rho u^2 + \rho \mathcal{V}' + \epsilon \rho \mathcal{A}' u_x^2 + \epsilon \mathcal{A}' \mathcal{V}'' \rho_x^2 + \epsilon \mathcal{R} \right) u + 2 \epsilon \rho \mathcal{A}' \mathcal{V}'' \rho_x u_x, \quad (\text{C.75})$$

thence — eliminating  $\mathcal{R}$  and using (C.73) — the ordinary differential equation

$$\frac{2 \epsilon \mathcal{A}' \left( \frac{d\rho}{dx} \right)^2}{\rho^2} = \frac{I^2 - 2S\rho + 2(F/I)\rho^2 - 2\rho \mathcal{V}}{I^2 - \rho^3 \mathcal{V}''}. \quad (\text{C.76})$$

Considering equilibrium states in the far field — i.e.,  $\rho \rightarrow \rho_{\pm}$  and  $u \rightarrow u_{\pm}$  as  $x \rightarrow \pm\infty$ ,  $\rho_{\pm}$  and  $u_{\pm}$  being constants — we have  $\mathcal{R} \rightarrow 0$  and the fluxes in the far field are

$$I_{\pm} \stackrel{\text{def}}{=} \rho_{\pm} u_{\pm}, \quad S_{\pm} \stackrel{\text{def}}{=} \rho_{\pm} u_{\pm}^2 + \rho_{\pm} \mathcal{V}'_{\pm} - \mathcal{V}_{\pm}, \quad F_{\pm} \stackrel{\text{def}}{=} \frac{1}{2} \rho_{\pm} u_{\pm}^3 + \rho_{\pm} \mathcal{V}'_{\pm} u_{\pm}. \quad (\text{C.77})$$

For regular solutions, the fluxes of mass, momentum and energy are constants, so  $I_+ = I_- = I$ ,  $S_+ = S_- = S$  and  $F_+ = F_- = F$ . For weak solutions, however, we assume that only the mass and momentum are conserved (i.e.,  $I_+ = I_- = I$  and  $S_+ = S_- = S$ ), some energy being lost at the singularity (shock) so  $F_+ \neq F_-$ .

It should be noted that  $\mathcal{A}$  does not appear in the relations (C.77). The role of  $\mathcal{A}$  is to control the singularity at the shock. So, *a priori*, a local analysis of a shock is necessary to obtain further informations on  $\mathcal{A}$ .

### C.3.8 Local analysis of steady solution

Let assume that we have a (weak) steady solution with far field conditions (C.77) and with, possibly, only one singularity at  $x = 0$  where the density is assumed on the form

$$\rho = \bar{\rho} + \varrho_{\pm} |x|^{\alpha} + o(|x|^{\alpha}), \quad (\text{C.78})$$

where  $\alpha > 0$  is a constant to be found. The plus and minus subscripts in  $\varrho$  denote  $x > 0$  and  $x < 0$ , respectively. With  $f_0 \stackrel{\text{def}}{=} f(\bar{\rho})$  for any function  $f$ , the constant mass flux (C.73) yields

$$u = \frac{I}{\bar{\rho}} \left( 1 - \frac{\varrho_{\pm}}{\bar{\rho}} |x|^{\alpha} \right) + o(|x|^{\alpha}), \quad (\text{C.79})$$

thence

$$\mathcal{R} = 2\alpha(\alpha - 1) \varrho_{\pm} \bar{\rho}^{-2} \mathcal{A}'_0 \left( I^2 - \bar{\rho}^3 \mathcal{V}''_0 \right) |x|^{\alpha-2} + o(|x|^{\alpha-2}), \quad (\text{C.80})$$

and the ODE (C.76) yields

$$\frac{2\epsilon\mathcal{A}'_0\alpha^2\varrho_{\pm}^2}{\bar{\rho}^2}|x|^{2\alpha-2} + o(|x|^{2\alpha-2}) = \frac{I^2 - 2S\bar{\rho} + 2(F_{\pm}/I)\bar{\rho}^2 - 2\bar{\rho}\mathcal{V}_0 + O(|x|^{\alpha})}{I^2 - \bar{\rho}^3\mathcal{V}_0'' - (3\bar{\rho}^2\mathcal{V}_0'' + \bar{\rho}^3\mathcal{V}_0''')\varrho_{\pm}|x|^{\alpha} + o(|x|^{\alpha})}. \quad (\text{C.81})$$

From (C.80) there are three (necessary) possibilities to obtain admissible solutions:  $\alpha = 1$  or  $\alpha > 1$  or  $I^2 = \bar{\rho}^3\mathcal{V}_0''$ .

If  $I^2 \neq \bar{\rho}^3\mathcal{V}_0''$ , the expansions (C.78) and (C.79) substituted into (C.74) and (C.75) show that  $S$  and  $F$  cannot be constant. Therefore, there are no solutions behaving like (C.78) if  $I^2 \neq \bar{\rho}^3\mathcal{V}_0''$ .

If  $I^2 = \bar{\rho}^3\mathcal{V}_0''$ , the equation (C.81) implies that  $\alpha = 2/3$  if  $\mathcal{V}_0 \neq I^2/2\bar{\rho} - S + (F_{\pm}/I)\bar{\rho}$  and  $\alpha = 1$  if  $\mathcal{V}_0 = I^2/2\bar{\rho} - S + (F_{\pm}/I)\bar{\rho}$ . The latter case does not yield constant  $S$  and  $F$ , so it must be rejected. Finally, the only possibility is  $\alpha = 2/3$  and (C.81) gives

$$\frac{8\epsilon\mathcal{A}'_0\varrho_{\pm}^2}{9\bar{\rho}^3} = -\frac{I^2 - 2S\bar{\rho} + 2(F_{\pm}/I)\bar{\rho}^2 - 2\bar{\rho}\mathcal{V}_0}{3I^2 + \bar{\rho}^4\mathcal{V}_0''}. \quad (\text{C.82})$$

In summary, the local analysis does not give hints for a suitable choice of  $\mathcal{A}$ . However, as in [44], we found the interesting feature that stationary weak solutions have universal singularities as  $|x|^{2/3}$ , whatever the potential  $\mathcal{V}$  is and for all possible regularising functions  $\mathcal{A}$ . Note that the analysis above does not rule out the possibility of different type of singularities such as  $|x|^{\alpha}(\log|x|)^{\beta}$ .

## C.4 Local well-posedness of the regularised barotropic Euler system

The aim of this section is to prove the local well-posedness of the regularised barotropic Euler system introduced in Section C.3.

Let  $P = P(\rho)$  denotes the pressure, and let  $\rho = \tilde{\rho} + \bar{\rho}$ , where  $\bar{\rho}$  is a positive constant. Let also

$$\varpi(\rho) \stackrel{\text{def}}{=} \int_{\bar{\rho}}^{\rho} \frac{P'(\alpha)}{\alpha} d\alpha, \quad \mathcal{V} \stackrel{\text{def}}{=} \int_{\bar{\rho}}^{\rho} \varpi(\alpha) d\alpha, \quad (\text{C.83})$$

where the prime denotes the derivative with respect to  $\rho$ .

Recalling the operator  $\mathcal{L}_{\rho} \stackrel{\text{def}}{=} \rho - 2\epsilon\partial_x\rho\mathcal{A}'\partial_x$  and the system (C.47), (C.60)

$$\rho_t + [\rho u]_x = 0, \quad (\text{C.84})$$

$$u_t + uu_x + \varpi_x = -\epsilon\mathcal{L}_{\rho}^{-1}\partial_x \left\{ (\rho^2\mathcal{A}')' u_x^2 + (\rho\mathcal{V}''/\mathcal{A}')' \mathcal{A}'^2 \rho_x^2 \right\}, \quad (\text{C.85})$$

where smooth solutions of (C.84), (C.85) satisfy the energy equation (C.56) with  $\mathcal{R}$  is defined in (C.63). The goal of this section is to prove the following theorem

**Theorem C.4.1.** *Let  $\tilde{m} \geq s \geq 2$ ,  $\tilde{m}$  be an integer,  $P, \mathcal{A} \in C^{\tilde{m}+4}(]0, +\infty[)$  such that  $P'(\rho) > 0$ ,  $\mathcal{A}'(\rho) > 0$  for  $\rho > 0$ . Let also  $W_0 = (\tilde{\rho}_0, u_0)^{\top} \in H^s$  satisfying  $\inf_{x \in \mathbb{R}} \rho_0(x) > \rho^*$ ,*



then there exist  $T > 0$  and a unique solution  $W \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$  of (C.84), (C.85) satisfying the non-emptiness condition  $\inf_{x \in \mathbb{R}} \rho(t, x) > 0$ , and the conservation of the energy

$$\frac{d}{dt} \int_{\mathbb{R}} \left( \frac{1}{2} \rho u^2 + \epsilon \rho \mathcal{A}' u_x^2 + \mathcal{V} + \epsilon \mathcal{A}' \mathcal{V}'' \rho_x^2 \right) dx = 0. \quad (\text{C.86})$$

Moreover, if the maximal existence time  $T_{max} < +\infty$ , then

$$\lim_{t \rightarrow T_{max}} \|W_x\|_{L^\infty} = +\infty. \quad (\text{C.87})$$

**Remark C.4.1.** The solution given in the previous theorem depends continuously on the initial data in the sense: If  $W_0, \tilde{W}_0 \in H^s$ , such that  $\rho_0, \tilde{\rho}_0 \geq \rho^*$ , then there exists a constant  $C(\|\tilde{W}\|_{L^\infty([0, T], H^s)}, \|W\|_{L^\infty([0, T], H^s)}) > 0$ , such that

$$\|W - \tilde{W}\|_{L^\infty([0, T], H^{s-1})} \leq C \|W_0 - \tilde{W}_0\|_{H^s}. \quad (\text{C.88})$$

**Remark C.4.2.** Theorem C.4.1 holds also for periodic domains.

**Remark C.4.3.** Note that if  $\rho \in [\rho_{\inf}, \rho_{\sup}] \subset ]0, +\infty[$ , then  $0 < \alpha \leq P'(\rho)/\rho \leq \beta < +\infty$ . This implies with the definition (C.83) that  $\alpha(\rho - \bar{\rho})^2 \leq \mathcal{V} \leq \beta(\rho - \bar{\rho})^2$ . Then, the conserved energy (C.86) is equivalent to the  $H^1$  norm of  $(\tilde{\rho}, u)$ .

## C.4.1 Preliminary results

Let  $\Lambda$  be defined such that  $\widehat{\Lambda}f = (1 + \xi^2)^{\frac{1}{2}} \hat{f}$ . In order to prove Theorem C.4.1, we recall the classical lemmas.

**Lemma C.4.1.** ([29]) Let  $[A, B] \stackrel{\text{def}}{=} AB - BA$  be the commutator of the operators  $A$  and  $B$ . If  $r \geq 0$ , then

$$\|fg\|_{H^r} \lesssim \|f\|_{L^\infty} \|g\|_{H^r} + \|f\|_{H^r} \|g\|_{L^\infty}, \quad (\text{C.89})$$

$$\|[\Lambda^r, f]g\|_{L^2} \lesssim \|f_x\|_{L^\infty} \|g\|_{H^{r-1}} + \|f\|_{H^r} \|g\|_{L^\infty}. \quad (\text{C.90})$$

**Lemma C.4.2.** ([15]) Let  $F \in C^{\tilde{m}+2}(\mathbb{R})$  with  $F(0) = 0$  and  $0 \leq s \leq \tilde{m}$ , then there exists a continuous function  $\tilde{F}$ , such that for all  $f \in H^s \cap W^{1, \infty}$  we have

$$\|F(f)\|_{H^s} \leq \tilde{F}(\|f\|_{W^{1, \infty}}) \|f\|_{H^s}. \quad (\text{C.91})$$

In the following lemma, we prove the invertibility of the operator  $\mathcal{L}_\rho$  (C.4) and we obtain some estimates satisfied by  $\mathcal{L}_\rho^{-1}$ .

**Lemma C.4.3.** Let  $0 < \rho_{\inf} \leq \rho \in W^{1, \infty}$  and  $\mathcal{A} \in C^2(]0, +\infty[)$  satisfying  $\mathcal{A}' > 0$ , then the operator  $\mathcal{L}_\rho$  is an isomorphism from  $H^2$  to  $L^2$  and

1. If  $0 \leq s \leq \tilde{m} \in \mathbb{N}$  and  $\mathcal{A} \in C^{\tilde{m}+3}(]0, +\infty[)$ , then

$$\|\mathcal{L}_\rho^{-1} \partial_x \psi\|_{H^{s+1}} \lesssim \|\psi\|_{H^s} + \|\rho - \bar{\rho}\|_{H^s} \|\mathcal{L}_\rho^{-1} \partial_x \psi\|_{W^{1, \infty}}, \quad (\text{C.92a})$$

$$\|\mathcal{L}_\rho^{-1} \phi\|_{H^{s+1}} \lesssim \|\phi\|_{H^s} + \|\rho - \bar{\rho}\|_{H^s} \|\mathcal{L}_\rho^{-1} \phi\|_{W^{1, \infty}}. \quad (\text{C.92b})$$

2. If  $0 \leq s \leq \tilde{m} \in \mathbb{N}$  and  $\mathcal{A} \in C^{\tilde{m}+3}(]0, +\infty[)$ , then

$$\|\mathcal{L}_\rho^{-1} \partial_x \psi\|_{H^{s+1}} \lesssim \|\psi\|_{H^s} (1 + \|\rho - \bar{\rho}\|_{H^s}), \quad (\text{C.93a})$$

$$\|\mathcal{L}_\rho^{-1} \phi\|_{H^{s+1}} \lesssim \|\phi\|_{H^s} (1 + \|\rho - \bar{\rho}\|_{H^s}), \quad (\text{C.93b})$$

3. If  $\phi \in C_{\text{lim}} \stackrel{\text{def}}{=} \{f \in C, f(\pm\infty) \in \mathbb{R}\}$ , then  $\mathcal{L}_\rho^{-1} \phi$  is well defined and

$$\|\mathcal{L}_\rho^{-1} \phi\|_{W^{2,\infty}} \lesssim \|\phi\|_{L^\infty}. \quad (\text{C.94})$$

4. If  $\psi \in C_{\text{lim}} \cap L^1$ , then

$$\|\mathcal{L}_\rho^{-1} \partial_x \psi\|_{W^{1,\infty}} \lesssim \|\psi\|_{L^\infty} + \|\psi\|_{L^1}. \quad (\text{C.95})$$

All the constants depend on  $s, \varepsilon, \rho_{\text{inf}}, \|\rho - \bar{\rho}\|_{W^{1,\infty}}$  and not on  $\|\rho - \bar{\rho}\|_{H^s}$ .

The previous lemma is proven in [36] for the special case  $\mathcal{A}(\rho) = \rho^3/6$ . Here, the same proof is followed

*Proof. Step 0:* In the first step, we prove, using the Lax–Milgram theorem, that  $\mathcal{L}_\rho$  is an isomorphism from  $H^2$  to  $L^2$ , let the bi-linear function  $a$  from  $H^1 \times H^1$  to  $\mathbb{R}$  such that

$$a(u, v) \stackrel{\text{def}}{=} (\rho u, v) + 2\varepsilon (\rho \mathcal{A}' u_x, v_x).$$

Using that  $\rho$  is bounded and far from zero, one can easily show that the function  $a$  is continuous and coercive, then Lax–Milgram theorem shows that there exists a continuous bijection  $J$  between  $H^1$  and  $H^{-1}$ , such that for all  $u, v \in H^1$  we have

$$a(u, v) = (Ju, v)_{H^{-1} \times H^1}.$$

If  $Ju \in L^2$ , and integration by parts shows that  $2\varepsilon (\rho \mathcal{A}' u_x)_x = \rho u - Ju \in L^2$  and  $J = \mathcal{L}_\rho$ , this implies that  $u \in H^2$  which finishes the proof that  $\mathcal{L}_\rho$  is an isomorphism from  $H^2$  to  $L^2$ .

**Step 1:** Let  $\mathcal{L}_\rho u = \phi + \psi_x$ , then

$$\begin{aligned} \|u\|_{H^1}^2 &= (u, u) + (u_x, u_x) \\ &\lesssim (\rho u, u) + 2\varepsilon (\rho \mathcal{A}' u_x, u_x) \\ &= (\mathcal{L}_\rho u, u) = (\phi, u) - (\psi, u_x) \\ &\lesssim \|u\|_{H^1} (\|\phi\|_{L^2} + \|\psi\|_{L^2}), \end{aligned}$$

which implies that

$$\|u\|_{H^1} \lesssim \|\phi\|_{L^2} + \|\psi\|_{L^2}. \quad (\text{C.96})$$

Using the Young inequality  $ab \leq \frac{1}{2\alpha} a^2 + \frac{\alpha}{2} b^2$  with  $\alpha > 0$  we obtain

$$\begin{aligned} \|u_x\|_{H^1}^2 &= (u_x, u_x) + (u_{xx}, u_{xx}) \\ &\lesssim (\rho u_x, u_x) + 2\varepsilon (\rho \mathcal{A}' u_{xx}, u_{xx}) \\ &= -(\rho u, u_{xx}) - (\rho_x u, u_x) + 2\varepsilon ((\rho \mathcal{A}' u_x)_x - (\rho \mathcal{A}')_x u_x, u_{xx}) \\ &= -(\mathcal{L}_\rho u, u_{xx}) - (\rho_x u, u_x) - 2\varepsilon ((\rho \mathcal{A}')_x u_x, u_{xx}) \\ &\lesssim \alpha \|u_{xx}\|_{L^2}^2 + \frac{1}{\alpha} (\|\mathcal{L}_\rho u\|_{L^2}^2 + \|u_x\|_{L^2}^2) + \|u\|_{H^1}^2. \end{aligned}$$

Taking  $\alpha > 0$  small enough we obtain that

$$\|u_x\|_{H^1}^2 \lesssim \|\mathcal{L}_\rho u\|_{L^2}^2 + \|u\|_{H^1}^2.$$

then

$$\|u_x\|_{H^1} \lesssim \|\mathcal{L}_\rho u\|_{L^2} + \|u\|_{H^1}.$$

Taking  $\phi = 0$  (respectively  $\psi = 0$ ) and using (C.96), we obtain

$$\|\mathcal{L}_\rho^{-1} \partial_x \psi\|_{H^1} \lesssim \|\psi\|_{H^1} \quad \|\mathcal{L}_\rho^{-1} \phi\|_{H^{s+1}} \lesssim \|\phi\|_{L^2}.$$

An interpolation with (C.96) implies that

$$\|\mathcal{L}_\rho^{-1} \partial_x \psi\|_{H^{s+1}} \lesssim \|\psi\|_{H^s}, \quad \|\mathcal{L}_\rho^{-1} \phi\|_{H^{s+1}} \lesssim \|\phi\|_{H^s} \quad \forall s \in [0, 1]. \quad (\text{C.97})$$

Let now  $s > 0$ , and let  $\mathcal{L}_\rho u = \phi + \psi_x$ , we then have

$$\mathcal{L}_\rho \Lambda^s u = [\rho, \Lambda^s] u + \Lambda^s \phi + \partial_x \{-2\varepsilon [\rho \mathcal{A}', \Lambda^s] u_x + \Lambda^s \psi\}.$$

Defining  $\tilde{u} = \Lambda^s u$ ,  $\tilde{\phi} = [\rho, \Lambda^s] u + \Lambda^s \phi$  and  $\tilde{\psi} = -2\varepsilon [\rho \mathcal{A}', \Lambda^s] u_x + \Lambda^s \psi$  and using (C.96), (C.90), (C.91) we obtain

$$\begin{aligned} \|\Lambda^s u\|_{H^1} &\lesssim \|[\rho, \Lambda^s] u\|_{L^2} + \|[\rho \mathcal{A}', \Lambda^s] u_x\|_{L^2} + \|\phi\|_{H^s} + \|\psi\|_{H^s} \\ &\lesssim \|\Lambda^{s-1} u\|_{H^1} + \|\rho - \bar{\rho}\|_{H^s} \|u\|_{W^{1,\infty}} + \|\phi\|_{H^s} + \|\psi\|_{H^s}. \end{aligned}$$

Then, by induction (on  $s$ ) and using (C.97) one obtains that (C.92) holds for all  $s \geq 0$ .

**Step 2:** If  $s \leq 1$ , then (C.93) follows directly from (C.97). If  $s > 1$ , using the embedding  $H^1 \hookrightarrow L^\infty$ , (C.92) and (C.97) for  $s = 1$  we obtain (C.93).

**Step 3:** Let  $C_0 \stackrel{\text{def}}{=} \{f \in C, f(\pm\infty) = 0\}$ , using that  $L^2 \cap C_0$  is dense in  $C_0$  one can define  $\mathcal{L}_\rho^{-1}$  on  $C_0$ . If  $\phi$  is in  $C_{\text{lim}}$ , we use the change of functions (see Lemma 4.4 in [36])

$$\phi_0(x) \stackrel{\text{def}}{=} \phi(x) - \frac{1}{\bar{\rho}} \mathcal{L}_\rho \left( \phi(-\infty) + (\phi(+\infty) - \phi(-\infty)) \frac{e^x}{1+e^x} \right) \in C_0,$$

the operator  $\mathcal{L}_\rho^{-1}$  can be defined as

$$\mathcal{L}_\rho^{-1} \phi \stackrel{\text{def}}{=} \mathcal{L}_\rho^{-1} \phi_0 + \frac{1}{\bar{\rho}} \left( \phi(-\infty) + (\phi(+\infty) - \phi(-\infty)) \frac{e^x}{1+e^x} \right). \quad (\text{C.98})$$

In order to prove (C.94), let  $\phi = \mathcal{L}_\rho u$ , using the variable

$$z \stackrel{\text{def}}{=} \int \frac{dx}{2\rho(x) \mathcal{A}'(\rho(x))}, \quad (\text{C.99})$$

we obtain that

$$\phi = \rho u - \frac{\varepsilon}{2\rho \mathcal{A}'} u_{zz}. \quad (\text{C.100})$$

The classical maximum principle equations implies that  $\|u\|_{L^\infty} \leq \|\phi\|_{L^\infty}/\rho_{\text{inf}}$ , which implies with (C.100) that  $\|u_{zz}\|_{L^\infty} \lesssim \|\phi\|_{L^\infty}$ , then the Landau–Kolmogorov inequality (see Lemma 4.3 in [36] for example) implies that  $\|u_z\|_{L^\infty} \lesssim \|\phi\|_{L^\infty}$ . The last inequality with the change of variables (C.99) imply that  $\|u_x\|_{L^\infty} \lesssim \|\phi\|_{L^\infty}$ , using that  $2\varepsilon\rho\mathcal{A}'u_{xx} = \rho u - 2\varepsilon(\rho\mathcal{A}')_x u_x - \phi$  we obtain (C.94).

**Step 4:** Note that

$$\mathcal{L}_\rho \int_{-\infty}^x \frac{\psi}{\rho\mathcal{A}'} dy = \rho \int_{-\infty}^x \frac{\psi}{\rho\mathcal{A}'} dy - 2\varepsilon\psi_x.$$

Applying  $\mathcal{L}_\rho^{-1}$  and  $\partial_x$  one obtains

$$\begin{aligned} 2\varepsilon\mathcal{L}_\rho^{-1}\partial_x\psi &= \mathcal{L}_\rho^{-1}\left(\rho \int_{-\infty}^x \frac{\psi}{\rho\mathcal{A}'} dy\right) - \int_{-\infty}^x \frac{\psi}{\rho\mathcal{A}'} dy, \\ 2\varepsilon\partial_x\mathcal{L}_\rho^{-1}\partial_x\psi &= \partial_x\mathcal{L}_\rho^{-1}\left(\rho \int_{-\infty}^x \frac{\psi}{\rho\mathcal{A}'} dy\right) - \frac{\psi}{\rho\mathcal{A}'}. \end{aligned}$$

The last two inequalities with (C.94) imply (C.95).  $\square$

## C.4.2 Iteration scheme and energy estimate

Defining  $W \stackrel{\text{def}}{=} (\tilde{\rho}, u)^\top$  and

$$B(W) \stackrel{\text{def}}{=} \begin{pmatrix} u & \rho \\ \varpi' & u \end{pmatrix}, \quad F(W) \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ -\varepsilon\mathcal{L}_\rho^{-1}\partial_x\{(\rho^2\mathcal{A}')'u_x^2 + (\rho\mathcal{V}''/\mathcal{A}')'\mathcal{A}'^2\rho_x^2\} \end{pmatrix},$$

the system (C.84), (C.85) becomes

$$W_t + B(W)W_x = F(W), \quad W(0, x) = W_0(x). \quad (\text{C.101})$$

The proof of the local existence of (C.101) is based on solving the linear hyperbolic system

$$\partial_t W^{n+1} + B(W^n)\partial_x W^{n+1} = F(W^n), \quad W^n(0, x) = (\tilde{\rho}_0(x), u_0(x))^\top, \quad (\text{C.102})$$

for all  $n \geq 0$ , where  $W^0(t, x) = (\tilde{\rho}_0(x), u_0(x))^\top$ . Then, uniform (on  $n$ ) estimates of an energy that is equivalent to the  $H^s$  norm will be given. Taking the limit  $n \rightarrow \infty$ , we obtain a solution of (C.101). Since  $\varpi' > 0$ , the system (C.102) is hyperbolic; which is an important point to solve each iteration in (C.102).

For the sake of simplicity, let be  $\underline{W} = W^n$  (known on every step of the iteration) and let  $W = W^{n+1}$  be the solution of the linear system

$$W_t + B(\underline{W})\partial_x W_x = F(\underline{W}), \quad W(0, x) = (\tilde{\rho}_0(x), u_0(x))^\top. \quad (\text{C.103})$$

Note that a symmetriser of  $\underline{B} = B(\underline{W})$  is

$$\underline{A} = A(\underline{W}) \stackrel{\text{def}}{=} \begin{pmatrix} \varpi' & 0 \\ 0 & \rho \end{pmatrix}. \quad (\text{C.104})$$

Let the energy of (C.103) be defined as

$$E(W) \stackrel{\text{def}}{=} (\Lambda^s W, \underline{A} \Lambda^s W), \quad (\text{C.105})$$

where  $(\cdot, \cdot)$  is the scalar product in  $L^2$ . Since the matrix  $\underline{A} \underline{B}$  is symmetric, a helpful feature for the energy estimates below, it justifies the use of  $\underline{A}$  in the definition of the energy  $E(W)$ . Note that if  $\rho$  is bounded and far from zero, then  $E(W)$  is equivalent to  $\|W\|_{H^s}^2$ . In order to prove Theorem C.4.1, the following energy estimate is needed.

**Theorem C.4.2.** *Let  $W = (\tilde{\rho}, u)^\top$ ,  $\rho = \tilde{\rho} + \tilde{\rho}$ ,  $s \geq 2$  and  $\rho^*, R > 0$  then there exist  $K, T > 0$  such that: if the initial data  $(\tilde{\rho}_0, u_0) \in H^s$  satisfy*

$$\inf_{x \in \mathbb{R}} \rho_0(x) > \rho^*, \quad E(W_0) < R, \quad (\text{C.106})$$

and  $\underline{W} \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$ , satisfying for all  $t \in [0, T]$

$$\underline{\rho} \geq \rho^*, \quad \|\underline{W}_t\|_{H^{s-1}} \leq K, \quad E(\underline{W}) \leq R, \quad (\text{C.107})$$

then there exists a unique  $W \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$  a solution of (C.103) such that

$$\rho \geq \rho^*, \quad \|W_t\|_{H^{s-1}} \leq K, \quad E(W) \leq R. \quad (\text{C.108})$$

*Proof.* For the existence of  $W$  see Appendix A in [27]. Defining  $\underline{F} \stackrel{\text{def}}{=} F(\underline{W})$  and using (C.103) we obtain

$$\begin{aligned} E(W)_t &= 2 (\Lambda^s W_t, \underline{A} \Lambda^s W) + (\Lambda^s W, \underline{A}_t \Lambda^s W) \\ &= -2 (\Lambda^s \underline{B} W_x, \underline{A} \Lambda^s W) - 2 (\Lambda^s \underline{F}, \underline{A} \Lambda^s W) + (\Lambda^s W, \underline{A}_t \Lambda^s W) \\ &= -2 ([\Lambda^s, \underline{B}] W_x, \underline{A} \Lambda^s W) - 2 (\underline{B} \Lambda^s W_x, \underline{A} \Lambda^s W) \\ &\quad - 2 (\Lambda^s \underline{F}, \underline{A} \Lambda^s W) + (\Lambda^s W, \underline{A}_t \Lambda^s W) \\ &=: I + II + III + IV, \end{aligned} \quad (\text{C.109})$$

Now, some bounds of the four terms will be given. Note that

$$\begin{aligned} -\frac{1}{2} I &= ([\Lambda^s, \underline{u}] \tilde{\rho}_x, \underline{\varpi}' \Lambda^s \tilde{\rho}) + ([\Lambda^s, \underline{\rho}] u_x, \underline{\varpi}' \Lambda^s \tilde{\rho}) + ([\Lambda^s, \underline{u}] u_x, \underline{\rho} \Lambda^s u) \\ &\quad + ([\Lambda^s, \underline{\varpi}'] \tilde{\rho}_x, \underline{\rho} \Lambda^s u). \end{aligned}$$

Using (C.90) and (C.91) we obtain

$$\begin{aligned} |([\Lambda^s, \underline{u}] \tilde{\rho}_x, \underline{\varpi}' \Lambda^s \tilde{\rho})| &\leq \|[\Lambda^s, \underline{u}] \tilde{\rho}_x\|_{L^2} \|\underline{\varpi}' \Lambda^s \tilde{\rho}\|_{L^2} \\ &\lesssim (\|\underline{u}\|_{L^\infty} \|\tilde{\rho}_x\|_{H^{s-1}} + \|\underline{u}\|_{H^s} \|\tilde{\rho}_x\|_{L^\infty}) \|\underline{\varpi}'\|_{L^\infty} \|\Lambda^s \tilde{\rho}\|_{L^2} \\ &\lesssim \|W\|_{H^s}^2 \lesssim E(W). \end{aligned}$$

All the terms of  $I$  can be studied by the same way to obtain

$$|I| \lesssim E(W). \quad (\text{C.110})$$

Using that  $\underline{A}$  and  $\underline{AB}$  are symmetric, an integration by parts yield to

$$|II| = |(\Lambda^s W, (\underline{AB})_x \Lambda^s W)| \leq \|(\underline{AB})_x\|_{L^\infty} \|W\|_{H^s} \lesssim \|W\|_{H^s} \lesssim E(W). \quad (\text{C.111})$$

Using the Young inequality  $2ab \leq a^2 + b^2$  one obtains

$$|III| \leq \|\underline{A}\|_{L^\infty} (\|\underline{F}\|_{H^s}^2 + \|W\|_{H^s}^2)$$

From the inequality (C.93) we have

$$\|\underline{F}\|_{H^s} \lesssim \left\| \left( \underline{\rho}^2 \underline{\mathcal{A}}' \right)' \underline{u}_x^2 + \left( \underline{\rho} \underline{\mathcal{V}}'' / \underline{\mathcal{A}}' \right)' \underline{\mathcal{A}}'^2 \underline{\rho}_x^2 \right\|_{H^{s-1}},$$

which implies with (C.89) and (C.91) that  $\|\underline{F}\|_{H^s}$  is bounded, then

$$|III| \lesssim E(W) + 1. \quad (\text{C.112})$$

Note that  $\|\tilde{\underline{\rho}}_t\|_{L^\infty} \leq \|\tilde{\underline{\rho}}_t\|_{H^{s-1}} \leq K$ , then

$$|IV| \leq \|\underline{A}_t\|_{L^\infty} \|W\|_{H^s}^2 \lesssim K E(W) \quad (\text{C.113})$$

The system (C.103) implies

$$\begin{aligned} \|W_t\|_{H^{s-1}} &= \|B(W) W_x + \underline{F}\|_{H^{s-1}} \\ &\lesssim \|B(W)\|_{L^\infty} \|W_x\|_{H^{s-1}} + \|B(W)\|_{H^{s-1}} \|W_x\|_{L^\infty} + \|\underline{F}\|_{H^s} \\ &\lesssim E(W) + 1. \end{aligned} \quad (\text{C.114})$$

All the constants hidden in "  $\lesssim$  " do not depend on  $K$  and  $W$ . Using (C.110), (C.111), (C.112) and (C.113) we obtain that

$$\partial_t E(W) \leq C(K+1)[E(W) + 1], \quad (\text{C.115})$$

which implies with Gronwall lemma that

$$E(W) \leq [E(W_0) + 1] e^{C(K+1)t} - 1. \quad (\text{C.116})$$

Since  $E(W_0) < R$ , choosing first  $K > 0$  and then  $T > 0$  such that

$$C(R + 1) \leq K, \quad [E(W_0) + 1] e^{C(K+1)T} - 1 \leq R \quad (\text{C.117})$$

we obtain with (C.114) and (C.116) that  $\|W_t\|_{H^{s-1}} \leq K$  and  $E(W) \leq R$ . Since  $\|\rho_t\|_{L^\infty} \lesssim \|W\|_{H^{s-1}} \leq K$  and  $\rho_0 > \rho^*$  then taking  $T$  small enough we have  $\rho \geq \rho^*$ .  $\square$

### C.4.3 Proof of Theorem C.4.1

Theorem C.4.2 shows that if the initial data satisfy (C.106), then the sequence  $(W^n)_{n \in \mathbb{N}}$  exists, it is uniformly bounded in  $C([0, T], H^s) \cap C^1([0, T], H^{s-1})$  and satisfies  $\rho^n \geq \rho^*$ . Using classical arguments of Sobolev spaces one can prove that there exists  $W \in C([0, T], H^s)$  such that  $W^n$  converges “up to a sub-sequence” to  $W$  in  $C([0, T], H^{s'})$  for all  $s' \in [0, s[$ . Before taking the limit  $n \rightarrow \infty$  in (C.102), we will verify that if  $W^n$  converges, then  $W^{n+1}$  converges too and towards the same limit. For that purpose, let

$$\tilde{E}_n \stackrel{\text{def}}{=} (\Lambda^{s-1}(W^{n+1} - W^n), A_n \Lambda^{s-1}(W^{n+1} - W^n)), \quad (\text{C.118})$$

using estimates as in the proof of Theorem C.4.2 (see also [39, 44] for more details) one can prove that for  $T > 0$  small enough, we obtain that  $\tilde{E}_{n+1} \leq \tilde{E}_n/2$ , which implies that  $\|W^{n+1} - W^n\|_{H^{s-1}} \rightarrow 0$ . Taking  $n \rightarrow \infty$  in the weak formulation of (C.102), we obtain that  $W$  is a weak solution of (C.101). Using that  $W \in C([0, T], H^s)$  and (C.101) we deduce that  $W$  is a strong solution and  $W \in C^1([0, T], H^{s-1})$ .

In order to prove the blow-up criteria (C.87), we suppose that  $\|W_x\|_{L^\infty}$  is bounded and we prove that  $\rho$  is far from zero and  $\|W\|_{H^s}$  is bounded on any bounded time interval  $[0, T]$ . Using the characteristics

$$\chi(0, x) = x, \quad \chi_t(t, x) = u(t, \chi(t, x)),$$

the conservation of the mass (C.84) becomes

$$\frac{d}{dt} \rho + u_x \rho = 0, \quad \implies \quad \rho_0(x) e^{-t \|u_x\|_{L^\infty}} \leq \rho(t, x) \leq \rho_0(x) e^{t \|u_x\|_{L^\infty}},$$

which implies that  $\rho$  is bounded and far from zero. The conservation of the energy (C.86) with the Sobolev embedding  $H^1 \hookrightarrow L^\infty$  imply that  $\|W\|_{L^\infty}$  is bounded.

Now, we will use that  $\rho$  is far from zero and the boundness of  $\|W\|_{W^{1,\infty}}$  to prove that  $\|W\|_{H^s}$  is also bounded. As in the proof of Theorem C.4.2, let

$$A(W) \stackrel{\text{def}}{=} \begin{pmatrix} \varpi' & 0 \\ 0 & \rho \end{pmatrix}, \quad B(W) \stackrel{\text{def}}{=} \begin{pmatrix} u & \rho \\ \varpi' & u \end{pmatrix}, \quad \tilde{E}(W) \stackrel{\text{def}}{=} (\Lambda^s W, A \Lambda^s W),$$

$$F(W) \stackrel{\text{def}}{=} \begin{pmatrix} 0 \\ -\epsilon \mathcal{L}_\rho^{-1} \partial_x \{ (\rho^2 \mathcal{A}')' u_x^2 + (\rho \mathcal{V}'' / \mathcal{A}')' \mathcal{A}'^2 \rho_x^2 \} \end{pmatrix},$$

the system (C.84), (C.85) then becomes

$$W_t + B(W) W_x = F(W). \quad (\text{C.119})$$

As in (C.109), we have

$$\begin{aligned} \tilde{E}(W)_t &= -2 ([\Lambda^s, B] W_x, A \Lambda^s W) - 2 (B \Lambda^s W_x, A \Lambda^s W) \\ &\quad - 2 (\Lambda^s F, A \Lambda^s W) + (\Lambda^s W, A_t \Lambda^s W) \\ &=: I + II + III + IV. \end{aligned} \quad (\text{C.120})$$

Note that

$$-\frac{1}{2}I = ([\Lambda^s, u] \tilde{\rho}_x, \varpi' \tilde{\rho}) + ([\Lambda^s, \rho - \bar{\rho}] u_x, \varpi' \tilde{\rho}) + ([\Lambda^s, \varpi'(\rho) - \varpi(\bar{\rho})] \tilde{\rho}_x, \rho u) + ([\Lambda^s, u] u_x, \rho u).$$

Using (C.90) and (C.91) we have

$$|([\Lambda^s, u] \tilde{\rho}_x, \varpi' \tilde{\rho})| \lesssim (\|\tilde{\rho}_x\|_{H^{s-1}} + \|u\|_{H^s}) \|\tilde{\rho}\|_{H^s},$$

where the multiplicative constant depend on  $\|W\|_{W^{1,\infty}}$ . Doing the same for all the terms we obtain that

$$|I| \lesssim \|W\|_{H^s}^2 \lesssim \tilde{E}(W). \quad (\text{C.121})$$

As in (C.111), we obtain

$$|II| \lesssim \tilde{E}(W). \quad (\text{C.122})$$

To estimate  $III$ , we use (C.92) and (C.95) to obtain that

$$\|F\|_{H^s} \lesssim \|\psi\|_{H^{s-1}} + \|\tilde{\rho}\|_{H^{s-1}} (\|\psi\|_{L^\infty} + \|\psi\|_{L^1}), \quad (\text{C.123})$$

where  $\psi = (\rho^2 \mathcal{A}')' u_x^2 + (\rho \mathcal{V}'' / \mathcal{A}')' \mathcal{A}'^2 \rho_x^2$ . Using (C.89), (C.91) one obtain that  $\|\psi\|_{H^{s-1}} \lesssim \|W\|_{H^s}$ . Due to the conservation of the energy (C.86), the quantity  $\|W\|_{H^1}$  is bounded, then  $\|\psi\|_{L^1}$  is also bounded. Using that  $\|W\|_{W^{1,\infty}}$  is bounded, we obtain that  $\|\psi\|_{L^\infty}$  is also bounded. The inequality (C.123) then becomes

$$\|F\|_{H^s} \lesssim \|W\|_{H^s},$$

which implies that

$$|III| \lesssim \|W\|_{H^s}^2 \lesssim \tilde{E}(W). \quad (\text{C.124})$$

The conservation of the mass (C.84) implies that  $\|\rho_t\|_{L^\infty} = \|(\rho u)_x\|_{L^\infty}$  which is bounded. Then

$$|IV| \lesssim \|W\|_{H^s}^2 \lesssim \tilde{E}(W). \quad (\text{C.125})$$

The equations (C.121), (C.122), (C.124), (C.125) and (C.120) imply that

$$\tilde{E}(W)_t \lesssim \tilde{E}(W). \quad (\text{C.126})$$

Gronwall's lemma implies that  $\tilde{E}(W)$  does not blow-up in finite time. This ends the proof of the blow-up criteria (C.87).  $\square$

## C.5 A generalised two-component Hunter–Saxton system

We have introduced the Sturm–Louville operator  $\mathcal{L}_\rho = \rho - 2\epsilon \partial_x \rho \mathcal{A}' \partial_x$  and its inverse  $\mathcal{G}_\rho = [1 - 2\epsilon \rho^{-1} \partial_x \rho \mathcal{A}' \partial_x]^{-1} \rho^{-1}$ . At high frequencies, the operator  $\partial_x \mathcal{G}_\rho \partial_x$  obviously behaves like

$$\partial_x \mathcal{G}_\rho \partial_x \sim -\frac{1}{2} \epsilon^{-1} (\rho \mathcal{A}')^{-1}. \quad (\text{C.127})$$



Thus, differentiating with respect of  $x$  the equation (C.60) and considering the high-frequency approximation (C.127), the rbE equations become the system of equations

$$\rho_t + [\rho u]_x = 0, \quad (\text{C.128})$$

$$[u_t + u u_x + \varpi_x]_x = \left(1 + \frac{\rho \mathcal{A}''}{2 \mathcal{A}'}\right) u_x^2 + \left(\frac{(\rho \mathcal{V}''')'}{2 \rho} - \frac{\mathcal{V}'' \mathcal{A}''}{2 \mathcal{A}'}\right) \rho_x^2, \quad (\text{C.129})$$

that is a two-component generalisation of the Hunter–Saxton equation [25]. Smooth solutions of (C.128), (C.129) satisfy the energy equation

$$[\rho \mathcal{A}' u_x^2 + \mathcal{A}' \mathcal{V}'' \rho_x^2]_t + [(\rho \mathcal{A}' u_x^2 + \mathcal{A}' \mathcal{V}'' \rho_x^2)u + 2 \rho \mathcal{A}' \mathcal{V}'' \rho_x u_x]_x = 0. \quad (\text{C.130})$$

There are several generalisations of the Hunter–Saxton equation in the literature, including two-component generalisations. The generalisation (C.128)–(C.129) is apparently new and it deserves to be studied since it is a simpler system than rbE, being somehow an asymptotic approximation.

This generalised Hunter–Saxton (gHS) system of equations is to rbE what the Hunter–Saxton equation [25] is to the dispersionless Camassa–Holm equation [10], i.e., a “high frequency limit”. Since the Hunter–Saxton equation is integrable [26], it is of interest to check if this property is shared with the gHS. It should be noted that equation (C.129) corresponds to  $\mathcal{R} = 0$ , as easily seen considering (C.57). From a physical viewpoint, the Euler equations describe the “outer” part of a shock, while the gHS equations describe its “inner” structure; the rbE equations being an unification of these two (outer and inner) systems.

Integrating (C.129) with respect to  $x$ , we obtain

$$\rho_t + [\rho u]_x = 0, \quad (\text{C.131})$$

$$u_t + u u_x + \varpi_x = \partial_x^{-1} \left\{ \left(1 + \frac{\rho \mathcal{A}''}{2 \mathcal{A}'}\right) u_x^2 + \left(\frac{(\rho \mathcal{V}''')'}{2 \rho} - \frac{\mathcal{V}'' \mathcal{A}''}{2 \mathcal{A}'}\right) \rho_x^2 \right\} + g(t), \quad (\text{C.132})$$

where  $(\partial_x^{-1} f)(x) \stackrel{\text{def}}{=} \int_0^x f(y) dy$  and  $g(t) = u_t(t, 0) + u(t, 0)u_x(t, 0) + \varpi'(t, 0)\rho_x(t, 0)$ .

In the case  $\varpi' \equiv 0$ , the proof of local well-posedness of (C.131), (C.132) can be done by using Kato’s theorem [28] as in [37, 38, 40, 50]. Following the proof of Theorem C.4.1 and using the inequality

$$\|\partial_x^{-1} f\|_{H^{s+1}([0,1])} \lesssim \|f\|_{H^s([0,1])} \quad \forall s \geq 0, \quad (\text{C.133})$$

one can prove the following theorem

**Theorem C.5.1.** *Let  $\tilde{m} \geq s \geq 2$ ,  $P, \mathcal{A} \in C^{\tilde{m}+4}([0, +\infty[)$  such that  $P'(\rho) > 0$ ,  $\mathcal{A}'(\rho) > 0$  for  $\rho > 0$ . Let also  $W_0 \in H^s([0, 1])$  be a periodic initial data satisfying  $\inf_{x \in [0,1]} \rho_0(x) > \rho^*$  and  $g \in C([0, +\infty[)$ , then there exist  $T > 0$  and a unique periodic solution  $W \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$  of (C.131), (C.132) satisfying the non-emptiness condition  $\inf_{x \in [0,1]} \rho(t, x) > 0$  and the conservation of the energy*

$$\frac{d}{dt} \int_0^1 (\rho \mathcal{A}' u_x^2 + \mathcal{A}' \mathcal{V}'' \rho_x^2) dx = 0. \quad (\text{C.134})$$

Moreover, the maximal existence time  $T_{max} < +\infty$ , then

$$\lim_{t \rightarrow T_{max}} \|W_x\|_{L^\infty} = +\infty. \quad (\text{C.135})$$

**Remark C.5.1.** The system (C.131), (C.132) do not change if  $\mathcal{A}$  is replaced by  $-\mathcal{A}$ . Then, the result of Theorem C.5.1 holds in the case  $\mathcal{A}'(\rho) < 0$ .

## C.6 Remarks on a special regularisation

As proved in [36], the solutions given by Theorem C.4.1 do not hold for all time in general. An inspiring way to obtain global (in time) weak solutions, is to use an equivalent semi-linear system of ordinary differential equations as in [8, 9, 20, 48]. In this case, the lemma C.4.3 is not enough, and an explicit formula of the operator  $\mathcal{L}_\rho^{-1}$  is needed.

At this stage, the regularising factor  $\mathcal{A}$  can be chosen freely, provided that  $\mathcal{A}' > 0$ . Here, we investigate further the special choice

$$\mathcal{A}(\rho) = -A\bar{\rho}/\rho, \quad (\text{C.136})$$

where  $A > 0$  is a constant. We show in this section that with this special choice of  $\mathcal{A}$ , a formula of  $\mathcal{L}_\rho^{-1}$  can be obtained, and the rbE system can be simplified.

The Sturm–Liouville operator becomes

$$\mathcal{L}_\rho = \rho - 2\epsilon A\bar{\rho}\partial_x\rho^{-1}\partial_x = \rho[1 - 2\epsilon A\bar{\rho}\rho^{-1}\partial_x\rho^{-1}\partial_x], \quad (\text{C.137})$$

so its inverse is

$$\mathcal{G}_\rho = [1 - 2\epsilon A\bar{\rho}\rho^{-1}\partial_x\rho^{-1}\partial_x]^{-1}\rho^{-1}. \quad (\text{C.138})$$

Similarly, the operator  $\mathcal{J}_\rho$  becomes

$$\mathcal{J}_\rho = \partial_x^{-1}\rho\mathcal{G}_\rho\partial_x = [1 - 2\epsilon A\bar{\rho}\rho^{-1}\partial_x\rho^{-1}\partial_x]^{-1}. \quad (\text{C.139})$$

This special choice for  $\mathcal{A}$  suggests the change of independent variables  $(t, x) \mapsto (\tau, \xi)$  with

$$\tau \stackrel{\text{def}}{=} t, \quad \xi \stackrel{\text{def}}{=} \int \rho(t, x) dx, \quad (\text{C.140})$$

i.e.,  $\xi$  is a density potential (defined modulo an arbitrary function of  $t$ ). After one spacial integration, the equation (C.47) for the mass conservation yields

$$\xi_t + u\xi_x = K(t) \equiv 0, \quad (\text{C.141})$$

$K(t)$  being an arbitrary function of  $t$  (an integration ‘constant’) that can be set to zero without loss of generality, thus providing a gauge condition for  $\xi$  (i.e.,  $\xi$  is no longer defined modulo an arbitrary function of  $t$ ). Thus, with this change of independent variables, the differentiation operators become

$$\partial_x \mapsto \rho\partial_\xi, \quad \mathcal{J}_\rho \mapsto [1 - 2\epsilon A\bar{\rho}\partial_\xi^2]^{-1}, \quad \mathcal{G}_\rho\partial_x \mapsto \mathcal{J}_\rho\partial_\xi, \quad (\text{C.142})$$

$$\partial_t \mapsto \partial_\tau - \rho u\partial_\xi, \quad \partial_t + u\partial_x \mapsto \partial_\tau, \quad (\text{C.143})$$

and the regularising term, together with (C.136), becomes

$$\mathcal{R} = A\bar{\rho}\mathfrak{J} * \{(\rho\mathcal{V}'''' + 3\mathcal{V}''')\rho_\xi^2\}, \quad \mathfrak{J}(\xi) \stackrel{\text{def}}{=} \frac{1}{2\sqrt{2\epsilon A\bar{\rho}}} \exp\left(\frac{-|\xi|}{\sqrt{2\epsilon A\bar{\rho}}}\right), \quad (\text{C.144})$$

where an asterisk denotes a convolution product, i.e.,  $\mathfrak{J}(\xi) * f(\xi) \stackrel{\text{def}}{=} \int_{-\infty}^{\infty} \mathfrak{J}(\xi - \bar{\rho})f(\bar{\rho})d\bar{\rho} = \int_{-\infty}^{\infty} \mathfrak{J}(\bar{\rho})f(\xi - \bar{\rho})d\bar{\rho}$  for any function  $f$ . Note that  $\mathfrak{J}$  is the pseudo-differential operator  $\mathcal{J}_\rho$  rewritten as an integral operator, because it is more convenient when applied to weakly regular functions.

With  $(\tau, \xi)$  as independent variables, the mass and momentum equations, respectively (C.47) and (C.53), become

$$\rho_\tau + \rho^2 u_\xi = 0, \quad u_\tau + [\rho\mathcal{V}' - \mathcal{V} + \epsilon\mathcal{R}]_\xi = 0, \quad (\text{C.145})$$

with  $\mathcal{R}$  given by (C.144). Denoting  $v \stackrel{\text{def}}{=} 1/\rho$  the specific volume, the system (C.145) becomes

$$v_\tau = u_\xi, \quad u_\tau = \left[ \frac{d(v\mathcal{V})}{dv} + \epsilon A\bar{\rho}\mathfrak{J} * \left\{ \frac{d^3(v\mathcal{V})}{dv^3} v_\xi^2 \right\} \right]_\xi. \quad (\text{C.146})$$

Eliminating  $u$  between these two relations, we obtain

$$v_{\tau\tau} - \left[ \frac{d(v\mathcal{V})}{dv} \right]_{\xi\xi} = \epsilon A\bar{\rho}\partial_\xi^2 \mathfrak{J} * \left\{ \frac{d^3(v\mathcal{V})}{dv^3} v_\xi^2 \right\}. \quad (\text{C.147})$$

At high frequencies, this partial differential equation is approximately

$$v_{\tau\tau} - \left[ \frac{d(v\mathcal{V})}{dv} \right]_{\xi\xi} \approx - \frac{d^3(v\mathcal{V})}{dv^3} \frac{v_\xi^2}{2}, \quad (\text{C.148})$$

that can be rewritten

$$v_{\tau\tau} - \frac{d^2(v\mathcal{V})}{dv^2} v_{\xi\xi} = \frac{d^3(v\mathcal{V})}{dv^3} \frac{v_\xi^2}{2}, \quad (\text{C.149})$$

that is a proper hyperbolic partial differential equation if

$$\frac{d^2(v\mathcal{V})}{dv^2} > 0. \quad (\text{C.150})$$

Introducing the velocity  $c(v) \stackrel{\text{def}}{=} \sqrt{d^2(v\mathcal{V})/dv^2}$ , the equation (C.149) is rewritten

$$v_{\tau\tau} - c(v)^2 v_{\xi\xi} = c(v) c'(v) v_\xi^2, \quad (\text{C.151})$$

an equation appearing in the theory of liquid crystals, for which smooth solution break down in finite time [19].

## C.7 Conclusion and perspectives

In this paper, we have introduced the regularised barotropic Euler system (C.5), inspired by the Hamiltonian regularisation of the shallow water (Saint-Venant) system with a constant depth introduced in [13]. The latter work is generalised in two ways: (i) considering a more general equation (i.e., barotropic Euler); (ii) introducing a family of regularisations (involving an arbitrary function  $\mathcal{A}(\rho)$ ).

For this system — and also for the periodic generalised two-component Hunter–Saxton system (C.6) — we prove the local (in time) well-posedness in  $H^s$  for  $s \geq 2$  and a blow-up criteria. As proven by Liu et al. [36], those solutions do not exist for all time, in general.

An interesting question that remains open is: *Due to the energy equations (C.56) and (C.130), do global weak solutions exist in  $H^1$  (or in  $\dot{H}^1$  for (C.6))?* Two possibilities, that have been used for the Camassa–Holm equation, may also work for the systems introduced in the present paper, i.e., using a vanishing viscosity [21] or using a semi-linear equivalent system [8, 9, 20, 48]. Another interesting problem is the study of the limiting cases  $\varepsilon \rightarrow 0$  and  $\varepsilon \rightarrow \infty$  as in [22, 23].

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# Appendix D

## Local well-posedness of a Hamiltonian regularisation of the Saint-Venant system with uneven bottom

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**Abstract:** In this paper, we generalise the non dispersive regularisation of the Saint-Venant system with uneven bottom introduced by Clamond et al [2]. We prove a the local well-posedness of the latter system and of a large class of  $2 \times 2$  symmetrisable hyperbolic systems involving non-local source terms.

**AMS Classification:** 35Q35; 35L65; 37K05; 35B65; 76B15.

**Key words:** Shallow water system; nonlinear hyperbolic systems; regularisation; Hamiltonian; energy conservation; dispersionless.

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### D.1 Introduction

Clamond and Dutykh [1] have proposed a non dispersive Hamiltonian regularisation of the Saint-Venant system with a constant bottom. This regularisation has been studied in [8, 9, 68]. Inspired by [1], some suitable regularisations have been proposed and studied,

for, the inviscid Burgers equation [6], scalar conservation laws [4], barotropic Euler system [5] and also for the Saint-Venant system with a moving bottom [2].

In this paper, we generalise the latter regularisation to obtain the family of regularisations of the Saint-Venant system with a moving bottom

$$h_t + [hu]_x = 0, \quad (\text{D.1a})$$

$$[hu]_t + \left[ hu^2 + \frac{1}{2}gh^2 + \epsilon\mathcal{R} \right]_x = 2\epsilon g \mathcal{A}' \eta_x d_{xx} + gh d_x, \quad (\text{D.1b})$$

$$\mathcal{R} \stackrel{\text{def}}{=} (h^2 \mathcal{A}')' u_x^2 - 2h \mathcal{A}' [u_t + uu_x + g\eta_x]_x - g(h \mathcal{A}'' - \mathcal{A}') (\eta_x^2 + 2\eta_x d_x). \quad (\text{D.1c})$$

Here,  $u = u(t, x)$  is the average horizontal velocity,  $h = h(t, x) \stackrel{\text{def}}{=} \eta(t, x) + d(t, x)$  denotes the high of the water,  $\mathcal{A}$  is an increasing smooth function of  $h$ , and  $d$  is the depth, we can assume, without losing generality, that the spacial depth average  $\bar{d}$  is a constant in time. In that case, the gravity acceleration  $g = g(t)$  may be a function of time.

When  $\mathcal{A}(h) = h^3/6$ , the system (D.1) matches the regularisation introduced in [2]. If  $\mathcal{A}'(h) > 0$ , introducing the invertible elliptic operator

$$\mathcal{L}_h \stackrel{\text{def}}{=} h - 2\epsilon \partial_x h \mathcal{A}' \partial_x, \quad (\text{D.2})$$

the system (D.1) can be written on the form

$$h_t + [hu]_x = 0, \quad (\text{D.3a})$$

$$u_t + uu_x + g\eta_x = -\epsilon \mathcal{L}_h^{-1} \partial_x \left\{ (h^2 \mathcal{A}')' u_x^2 - g(h \mathcal{A}'' - \mathcal{A}') (\eta_x^2 + 2\eta_x d_x) \right\} + 2\epsilon g \mathcal{L}_h^{-1} \{ \mathcal{A}' \eta_x d_{xx} \}. \quad (\text{D.3b})$$

Smooth solutions of (D.3) satisfy the energy equation

$$\begin{aligned} & \left[ \frac{1}{2} h u^2 + \epsilon h \mathcal{A}' u_x^2 + \frac{1}{2} g \eta^2 + \epsilon g \mathcal{A}' \eta_x^2 \right]_t \\ & + \left[ \left( \frac{1}{2} h u^2 + g h \eta + \epsilon h \mathcal{A}' u_x^2 + \epsilon g \mathcal{A}' \eta_x^2 + \epsilon \mathcal{R} \right) u + 2\epsilon g h \mathcal{A}' \eta_x u_x \right]_x \\ & = \frac{1}{2} \dot{g} (\eta^2 + 2\epsilon \mathcal{A}' \eta_x^2) - g \eta d_t - 2\epsilon g \mathcal{A}' \eta_x d_{xt}. \end{aligned} \quad (\text{D.4})$$

We prove in this paper the local (in time) well-posedness of a general  $2 \times 2$  symmetrisable hyperbolic system in  $H^s$  for a real number  $s \geq 2$ . Using some estimates of the operator  $\mathcal{L}_h^{-1}$ , we prove that the system (D.3) is locally well-posed. We also prove that if the  $L^\infty$  norm of the first derivatives remains bounded, the singularities cannot appear in finite time.

This paper is organised as follows. In Section D.2 we derive the system (D.1) and we state the main results. Section D.3 is devoted to prove the local well-posedness of a general system. The proof of the main results is given in Section D.4.

## D.2 Derivation of the regularised Saint-Venant system and main results

In [2], a Hamiltonian regularised 1D shallow water system with uneven bottom has been proposed by deriving a Lagrangian density. Following [5], we generalise the Lagrangian

density given in [2] as

$$\mathcal{L}_\epsilon \stackrel{\text{def}}{=} \frac{1}{2} h u^2 + \epsilon \mathcal{A}' h u_x^2 - \frac{1}{2} g \eta^2 - g \epsilon \mathcal{A}' \eta_x^2 + (h_t + [h u]_x) \phi. \quad (\text{D.5})$$

The Euler–Lagrange equations of (D.5) yield

$$\delta\phi: 0 = h_t + [h u]_x, \quad (\text{D.6})$$

$$\delta u: 0 = h u - 2\epsilon [\mathcal{A}' h u_x]_x - h \phi_x, \quad (\text{D.7})$$

$$\begin{aligned} \delta\eta: 0 = \frac{1}{2} u^2 + \epsilon (\mathcal{A}' + \mathcal{A}'' h) u_x^2 - g \eta + \epsilon g \mathcal{A}'' \eta_x (h_x + d_x) \\ + 2\epsilon g \mathcal{A}' \eta_{xx} - \phi_t - u \phi_x, \end{aligned} \quad (\text{D.8})$$

which yield, with  $\mathcal{R}$  is defined by (D.1c), to the equations

$$\begin{aligned} \partial_t \{ u - 2\epsilon h^{-1} [\mathcal{A}' h u_x]_x \} + \partial_x \{ \frac{1}{2} u^2 - \epsilon (\mathcal{A}' + \mathcal{A}'' h) u_x^2 \\ + g \eta - \epsilon g \mathcal{A}'' \eta_x (h_x + d_x) - 2\epsilon g \mathcal{A}' \eta_{xx} - 2\epsilon u h^{-1} [\mathcal{A}' h u_x]_x \} = 0, \end{aligned} \quad (\text{D.9})$$

$$u_t + u u_x + g \eta_x + \epsilon h^{-1} \mathcal{R}_x = 2\epsilon g h^{-1} \mathcal{A}' \eta_x d_{xx}, \quad (\text{D.10})$$

$$[h u]_t + [h u^2 + \frac{1}{2} g h^2 + \epsilon \mathcal{R}]_x = 2\epsilon g \mathcal{A}' \eta_x d_{xx} + g h d_x, \quad (\text{D.11})$$

$$\begin{aligned} [h u - 2\epsilon [h \mathcal{A}' u_x]_x]_t + \left[ u m + \frac{1}{2} g h^2 - \epsilon (h^2 \mathcal{A}')' u_x^2 - 2\epsilon g h \mathcal{A}' \eta_{xx} + \right. \\ \left. \epsilon g (\mathcal{A}' - h \mathcal{A}'') (\eta_x^2 + 2\eta_x d_x) \right]_x = 2\epsilon g \mathcal{A}' \eta_x d_{xx} + g h d_x. \end{aligned} \quad (\text{D.12})$$

Combining (D.3b) with (D.10) one can replace  $\mathcal{R}$  given in (D.1c) by

$$\mathcal{R} = \mathcal{J}_h \left\{ (h^2 \mathcal{A}')' u_x^2 - g (h \mathcal{A}''' - \mathcal{A}') (\eta_x^2 + 2\eta_x d_x) \right\} + 2g \tilde{\mathcal{J}}_h \{ \mathcal{A}' \eta_x d_{xx} \}, \quad (\text{D.13})$$

where

$$\begin{aligned} \mathcal{J}_h &\stackrel{\text{def}}{=} \partial_x^{-1} h \mathcal{L}_h^{-1} \partial_x = 1 + 2\epsilon h \mathcal{A}' \partial_x \mathcal{L}_h^{-1} \partial_x, \\ \tilde{\mathcal{J}}_h &\stackrel{\text{def}}{=} \partial_x^{-1} (1 - h \mathcal{L}_h^{-1}) = -2\epsilon h \mathcal{A}' \partial_x \mathcal{L}_h^{-1}. \end{aligned}$$

In order to state the main results of this paper, let  $d$  be a smooth function of  $t$  and  $x$  and

$$h \stackrel{\text{def}}{=} \eta + d, \quad \bar{d} \stackrel{\text{def}}{=} \lim_{|x| \rightarrow +\infty} d(t, x) > 0, \quad \text{and} \quad \inf_{(t,x) \in \mathbb{R}^+ \times \mathbb{R}} d(t, x) > 0, \quad (\text{D.14})$$

then

**Theorem D.2.1.** *Let  $\tilde{m} \geq s \geq 2$ ,  $\mathcal{A} \in C^{\tilde{m}+4}([0, +\infty[)$  such that  $\mathcal{A}'(h) > 0$  for  $h > 0$ . Let  $0 < g \in C^1([0, +\infty[)$ ,  $d - \bar{d} \in C([0, +\infty], H^{s+1}) \cap C^1([0, +\infty], H^s)$  and let  $W_0 = (\eta_0, u_0)^\top \in H^s$  satisfying  $\inf_{x \in \mathbb{R}} h_0(x) > h^*$ , then there exist  $T > 0$  and a unique solution  $W = (\eta, u) \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$  of (D.3) satisfying the non-zero depth condition  $\inf_{x \in \mathbb{R}} h(t, x) > 0$ . Moreover, if the maximal time of existence  $T_{max} < +\infty$ , then*

$$\lim_{t \rightarrow T_{max}} \|W\|_{H^s} = +\infty \quad \text{or} \quad \inf_{(t,x) \in [0, T_{max}] \times \mathbb{R}} h(t, x) = 0. \quad (\text{D.15})$$

A proof of local well-posedness of a large family of  $2 \times 2$  symmetrisable hyperbolic systems with non-local additional terms is given in the next section. Theorem D.2.1 follows then directly. Using the energy equation (D.4) and some estimates, we can improve the blow-up criteria (D.15) as

**Theorem D.2.2.** *For any interval  $[0, T] \subset [0, T_{max}[$ , there exists  $C > 0$ , such that  $\forall t \in [0, T]$  we have*

$$\mathcal{E}(t) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \left[ \frac{1}{2} h u^2 + \epsilon h \mathcal{A}' u_x^2 + \frac{1}{2} g \eta^2 + \epsilon g \mathcal{A}' \eta_x^2 \right] dx \leq C. \quad (\text{D.16})$$

Moreover, if  $T_{max} < +\infty$ , then

$$\lim_{t \rightarrow T_{max}} \|W_x\|_{L^\infty} = +\infty. \quad (\text{D.17})$$

### D.3 Local well-posedness of a general system

The aim of this section is to prove the local well-posedness of a class of systems with non-local operators in the  $H^s$  space with  $s > 3/2$ .

Let  $d$  be a smooth function such that (D.14) holds, let also  $N \geq 1$  be a natural number and  $G \stackrel{\text{def}}{=} (g_1, \dots, g_N)$  be a smooth function of  $t$  and  $x$ , possibly depending on  $d$ , such that

$$g_\infty(t) \stackrel{\text{def}}{=} g_1(t, \infty) = \lim_{|x| \rightarrow +\infty} g_1(t, x) > 0 \quad \text{and} \quad g_{\text{inf}} \stackrel{\text{def}}{=} \inf_{(t,x) \in \mathbb{R}^+ \times \mathbb{R}} g_1(t, x) > 0. \quad (\text{D.18})$$

Let  $f$  be a positive function of  $d$  and  $h$ , and let  $f_1, f_2$  be functions of  $d, h, u, \eta_x, u_x$  and  $G$ . Let also  $a, b, c, f_3, f_4$  be functions of  $d, h, u$  and  $G$ . We consider the symmetrisable hyperbolic system

$$\eta_t + a(d, h, u, G) \eta_x + b(d, h, u, G) u_x = \mathfrak{A}_1 f_1 + \mathfrak{A}_3 f_3, \quad (\text{D.19a})$$

$$u_t + g_1 f(d, h) b(d, h, u, G) \eta_x + c(d, h, u, G) u_x = \mathfrak{A}_2 f_2 + \mathfrak{A}_4 f_4, \quad (\text{D.19b})$$

where  $\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3$  and  $\mathfrak{A}_4$  are linear operators depending on  $h$  and  $u$ . Let  $s$  be a real number such that  $s > 3/2$ , in order to obtain the well-posedness of the system (D.19) in  $H^s$ , we define  $W \stackrel{\text{def}}{=} (\eta, u)^\top$ ,  $G_0 \stackrel{\text{def}}{=} (g_\infty, 0, \dots, 0)$  and

$$B(W) \stackrel{\text{def}}{=} \begin{pmatrix} a(d, h, u, G) & b(d, h, u, G) \\ g_1 f(d, h) b(d, h, u, G) & c(d, h, u, G) \end{pmatrix},$$

$$F(W) \stackrel{\text{def}}{=} \begin{pmatrix} \mathfrak{A}_1 f_1(d, h, u, h_x, u_x, G) + \mathfrak{A}_3 f_3(d, h, u, G) \\ \mathfrak{A}_2 f_2(d, h, u, h_x, u_x, G) + \mathfrak{A}_4 f_4(d, h, u, G) \end{pmatrix},$$

the system (D.19) can be written as

$$W_t + B(W)W_x = F(W), \quad W(0, x) = W_0(x). \quad (\text{D.20})$$

We assume that

(A1) For  $s \leq \tilde{m} \in \mathbb{N}$  we have

- $d - \bar{d}, g_1 - g_\infty, g_2, g_3, \dots, g_N \in C(\mathbb{R}^+, H^s)$  and  $d - \bar{d}, g_1 - g_\infty \in C^1(\mathbb{R}^+, H^{s-1})$ .
- $f \in C^{\tilde{m}+2}([0, +\infty[^2)$  and for all  $h_1, h_2 > 0$  we have  $f(h_1, h_2) > 0$ .
- $f_1, f_2 \in C^{\tilde{m}+2}([0, +\infty[^2 \times \mathbb{R}^3 \times ]0, +\infty[ \times \mathbb{R}^{N-1})$ .
- $a, b, c, f_3, f_4 \in C^{\tilde{m}+2}([0, +\infty[^2 \times \mathbb{R} \times ]0, +\infty[ \times \mathbb{R}^{N-1})$ .
- $f_1(\bar{d}, \bar{d}, 0, 0, 0, G_0) = f_2(\bar{d}, \bar{d}, 0, 0, 0, G_0) = f_3(\bar{d}, \bar{d}, 0, G_0) = f_4(\bar{d}, \bar{d}, 0, G_0) = 0$ .

(A2) For all  $r \in [s-1, s]$ , if  $\phi \in H^r$  and  $\psi \in H^{r-1}$ , then

$$\begin{aligned} \|\mathfrak{A}_1 \psi\|_{H^r} + \|\mathfrak{A}_2 \psi\|_{H^r} &\leq C(s, r, d, \|W\|_{H^r}) \|\psi\|_{H^{r-1}}, \\ \|\mathfrak{A}_3 \phi\|_{H^r} + \|\mathfrak{A}_4 \phi\|_{H^r} &\leq C(s, r, d, \|W\|_{H^r}) \|\phi\|_{H^r}. \end{aligned}$$

(A3) If  $\phi, W, \tilde{W} \in H^s$  and  $\psi \in H^{s-1}$ , then

$$\begin{aligned} \|(\mathfrak{A}_1(W) - \mathfrak{A}_1(\tilde{W}))\psi\|_{H^{s-1}} + \|(\mathfrak{A}_2(W) - \mathfrak{A}_2(\tilde{W}))\psi\|_{H^{s-1}} &\leq C \|W - \tilde{W}\|_{H^{s-1}}, \\ \|(\mathfrak{A}_3(W) - \mathfrak{A}_3(\tilde{W}))\phi\|_{H^{s-1}} + \|(\mathfrak{A}_4(W) - \mathfrak{A}_4(\tilde{W}))\phi\|_{H^{s-1}} &\leq C \|W - \tilde{W}\|_{H^{s-1}}, \end{aligned}$$

where  $C = C(s, d, \|W\|_{H^s}, \|\tilde{W}\|_{H^s}, \|\phi\|_{H^s}, \|\psi\|_{H^{s-1}})$ .

Note that if  $h$  is far from zero, then  $g_1 f(d, h)$  is positive and far from zero. Then, the system (D.19) is symmetrisable and hyperbolic. The main result of this section is the following theorem

**Theorem D.3.1.** *For  $s > 3/2$  and under the assumptions (A1), (A2) and (A3), if  $W_0 \in H^s$  satisfy the non-emptiness condition*

$$\inf_{x \in \mathbb{R}} h_0(x) = \inf_{x \in \mathbb{R}} (\eta_0(x) + d(0, x)) > 0, \quad (\text{D.21})$$

*then there exist  $T > 0$  and a unique solution  $W \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$  of the system (D.20). Moreover, if the maximal existence time  $T_{max} < +\infty$  then*

$$\inf_{(t,x) \in [0, T_{max}[ \times \mathbb{R}} h(t, x) = 0 \quad \text{or} \quad \lim_{t \rightarrow T_{max}} \|W\|_{H^s} = +\infty. \quad (\text{D.22})$$

**Remarks D.3.1.**

- *Theorem D.3.1 holds also for periodic domains.*

- The right-hand side of (D.20) can be replaced by a finite sum on the form

$$F(W) = \begin{pmatrix} \mathfrak{A}_1 f_1 + \mathfrak{A}_3 f_3 \\ \mathfrak{A}_2 f_2 + \mathfrak{A}_4 f_4 \end{pmatrix} + \begin{pmatrix} \mathfrak{B}_1 k_1 + \mathfrak{B}_3 k_3 \\ \mathfrak{B}_2 k_2 + \mathfrak{B}_4 k_4 \end{pmatrix} + \dots \quad (\text{D.23})$$

where the additional terms satisfy also the conditions (A1), (A2) and (A3).

- Under some additional assumptions, the blow-up criteria (D.22) can be improved (see Theorem D.2.2 for example).
- If for some  $2 \leq i \leq N$ , the function  $g_i$  appears only on  $f_1$  and  $f_2$ , then, due to (A2), the assumption  $g_i \in C(\mathbb{R}^+, H^s)$  can be replaced by  $g_i \in C(\mathbb{R}^+, H^{s-1})$ .

In order to prove the local well-posedness of (D.20), we consider

$$\partial_t W^{n+1} + B(W^n) \partial_x W^{n+1} = F(W^n), \quad W^n(0, x) = (\eta_0(x), u_0(x))^\top, \quad (\text{D.24})$$

where  $n \geq 0$  and  $W^0(t, x) = (\eta_0(x), u_0(x))^\top$ . The idea of the proof is to solve the linear system (D.24), then, taking the limit  $n \rightarrow \infty$  we obtain a solution of (D.20). Note that we have assumed that  $g_1$  and  $f$  are positive so  $g_1 f > 0$ , then the system (D.24) is hyperbolic; it is an important point to solve each iteration in (D.24).

Note that a symmetriser of the matrix  $B(W)$  is

$$A(W) \stackrel{\text{def}}{=} \begin{pmatrix} g_1 f(d, h) & 0 \\ 0 & 1 \end{pmatrix}. \quad (\text{D.25})$$

Let  $(\cdot, \cdot)$  be the scalar product in  $L^2$  and let the energy of (D.24) be defined as

$$E^n(0) \stackrel{\text{def}}{=} (\Lambda^s W_0, A_0 \Lambda^s W_0) \quad \forall n \geq 0, \quad E^n(t) \stackrel{\text{def}}{=} (\Lambda^s W^{n+1}, A^n \Lambda^s W^{n+1}) \quad \forall t > 0.$$

Note that if  $g_1 f$  is bounded and far from 0, then  $E^n(t)$  is equivalent to  $\|W^n\|_{H^s}$ . In order to prove Theorem D.3.1, the following results are needed.

**Theorem D.3.2.** *Let  $s > 3/2$ ,  $h^*$  and  $R > 0$  then there exist  $K, T > 0$  such that: if the initial data  $(\eta_0, h_0) \in H^s$  satisfy*

$$\inf_{x \in \mathbb{R}} h_0(x) > h^*, \quad E^n(0) < R, \quad (\text{D.26})$$

and  $W^n \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$ , satisfying for all  $t \in [0, T]$

$$h^n \geq h^*, \quad \|(W^n)_t\|_{H^{s-1}} \leq K, \quad E^n(t) \leq R, \quad (\text{D.27})$$

then there exists a unique  $W^{n+1} \in C([0, T], H^s) \cap C^1([0, T], H^{s-1})$  a solution of (D.24) such that

$$h^{n+1} \geq h^*, \quad \|(W^{n+1})_t\|_{H^{s-1}} \leq K, \quad E^{n+1}(t) \leq R. \quad (\text{D.28})$$

The proof of Theorem D.3.2 is classic, and can be done following [5, 8] and using the following lemmas.

Let  $\Lambda$  be defined such that  $\widehat{\Lambda f} = (1+\xi^2)^{\frac{1}{2}} \widehat{f}$ , and let  $[A, B] \stackrel{\text{def}}{=} AB - BA$  be the commutator of the operators  $A$  and  $B$ .

**Lemma D.3.1.** ([7]) *If  $r \geq 0$ , then*

$$\|fg\|_{H^r} \lesssim \|f\|_{L^\infty} \|g\|_{H^r} + \|f\|_{H^r} \|g\|_{L^\infty}, \quad (\text{D.29})$$

$$\|[\Lambda^r, f]g\|_{L^2} \lesssim \|f_x\|_{L^\infty} \|g\|_{H^{r-1}} + \|f\|_{H^r} \|g\|_{L^\infty}. \quad (\text{D.30})$$

**Lemma D.3.2.** *Let  $k \in \mathbb{N}^*$ ,  $F \in C^{m+2}(\mathbb{R}^k)$  with  $F(0, \dots, 0) = 0$  and  $0 \leq s \leq m$ , then there exists a continuous function  $\tilde{F}$ , such that for all  $f = (f_1, \dots, f_k) \in H^s \cap W^{1, \infty}$  we have*

$$\|F(f)\|_{H^s} \leq \tilde{F}(\|f\|_{W^{1, \infty}}) \|f\|_{H^s}. \quad (\text{D.31})$$

*Proof.* The case  $k = 1$  has been proved in [3]. Here we prove the inequality (D.31) by induction (on  $s$ ). Note that

$$\begin{aligned} F(f_1, \dots, f_k) &= F(0, f_2, \dots, f_k) + \int_0^{f_1} F_{f_1}(g_1, f_2, \dots, f_k) dg_1 \\ &= F(0, 0, f_3, \dots, f_k) + \int_0^{f_1} F_{f_1}(g_1, f_2, \dots, f_k) dg_1 + \int_0^{f_2} F_{f_2}(0, g_2, f_3, \dots, f_k) dg_2 \\ &\quad \dots \\ &= \int_0^{f_1} F_{f_1}(g_1, f_2, \dots, f_k) dg_1 + \dots + \int_0^{f_k} F_{f_k}(0, \dots, 0, g_k) dg_k. \end{aligned}$$

This implies that

$$\|F(f_1, \dots, f_k)\|_{L^2} \lesssim \|f\|_{L^2}, \quad (\text{D.32})$$

which is (D.31) for  $s = 0$ . For  $s \in ]0, 1[$ , let

$$\begin{aligned} &|F(f_1(x+y), \dots, f_k(x+y)) - F(f_1(x), \dots, f_k(x))| \\ &\leq |F(f_1(x+y), \dots, f_k(x+y)) - F(f_1(x), f_2(x+y), \dots, f_k(x+y))| \\ &\quad + |F(f_1(x), f_2(x+y), \dots, f_k(x+y)) - F(f_1(x), f_2(x), f_3(x+y), \dots, f_k(x+y))| \\ &\quad \dots \\ &\quad + |F(f_1(x), \dots, f_{k-1}(x), f_k(x+y)) - F(f_1(x), \dots, f_k(x))| \\ &\leq \sum_{i=1}^k |f_i(x+y) - f_i(x)| \|F_{f_i}\|_{L^\infty}. \end{aligned}$$

The last inequality with the definition of the  $H^s$  space for  $s \in ]0, 1[$

$$H^s \stackrel{\text{def}}{=} \left\{ f \in L^2, \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|f(x+y) - f(x)|^2}{|y|^{1+2s}} dx dy < +\infty \right\}$$



imply (D.31) for  $s \in ]0, 1[$ . For  $s \geq 1$ , the proof is done by induction, using (D.29) and (D.32) we obtain

$$\begin{aligned} \|F(f)\|_{H^s} &\lesssim \left\| \sum_{i=1}^k F_{f_i}(f) \partial_x f_i \right\|_{H^{s-1}} + \|F(f)\|_{L^2} \\ &\lesssim \|f\|_{H^s} + \sum_{i=1}^k \|F_{f_i}(f)\|_{H^{s-1}}. \end{aligned}$$

Using the induction and the last inequality, we obtain (D.31) for all  $s \geq 0$ .  $\square$

Now, we can prove Theorem D.3.1.

**Proof of Theorem D.3.1.** Using Theorem D.3.2, one obtains that  $(W^n)$  is uniformly bounded in  $C([0, T], H^s) \cap C^1([0, T], H^{s-1})$  and satisfies  $h^n \geq h^*$ . We define

$$\tilde{E}^{n+1}(t) \stackrel{\text{def}}{=} (\Lambda^{s-1}(W^{n+1} - W^n), A^n \Lambda^{s-1}(W^{n+1} - W^n)). \quad (\text{D.33})$$

Using (D.24) one obtains

$$\begin{aligned} \partial_t \tilde{E}^{n+1} &= 2 (\Lambda^{s-1}(F^n - F^{n-1} + (B^{n-1} - B^n) \partial_x W^n), A^n \Lambda^{s-1}(W^{n+1} - W^n)) \\ &\quad - 2 ([\Lambda^{s-1}, B^n](W^{n+1} - W^n)_x, A^n \Lambda^{s-1}(W^{n+1} - W^n)) \\ &\quad + (\Lambda^{s-1}((A^n B^n)_x(W^{n+1} - W^n)), \Lambda^{s-1}(W^{n+1} - W^n)) \\ &\quad + (\Lambda^{s-1}(W^{n+1} - W^n), (A^n)_t \Lambda^{s-1}(W^{n+1} - W^n)). \end{aligned} \quad (\text{D.34})$$

Using (A2), (A3), (D.29) and (D.31) one obtains

$$\|F^n - F^{n-1}\|_{H^{s-1}} + \|(B^{n-1} - B^n) \partial_x W^n\|_{H^{s-1}} \lesssim \|W^n - W^{n-1}\|_{H^{s-1}} \lesssim \sqrt{\tilde{E}^n}. \quad (\text{D.35})$$

Using (D.30) and (D.31) we obtain

$$\|[\Lambda^{s-1}, B^n](W^{n+1} - W^n)_x\|_{L^2} \lesssim \|W^{n+1} - W^n\|_{H^{s-1}} \lesssim \sqrt{\tilde{E}^{n+1}}. \quad (\text{D.36})$$

From (D.29) and (D.31), it follows that

$$\|(A^n B^n)_x(W^{n+1} - W^n)\|_{H^{s-1}} \lesssim \|W^{n+1} - W^n\|_{H^{s-1}} \lesssim \sqrt{\tilde{E}^{n+1}}. \quad (\text{D.37})$$

Combining the estimates above, we obtain that

$$\partial_t \tilde{E}^{n+1} \lesssim \tilde{E}^{n+1} + \tilde{E}^n. \quad (\text{D.38})$$

Using that  $\tilde{E}^n(0) = 0$ , we obtain

$$\tilde{E}^{n+1} \leq (e^{Ct} - 1) \tilde{E}^n \quad (\text{D.39})$$

Taking  $T > 0$  small enough, it follows that

$$\|W^{n+1} - W^n\|_{H^{s-1}} \lesssim \tilde{E}^{n+1} \leq \frac{1}{2} \tilde{E}^n \leq \frac{1}{2^n} \tilde{E}^1. \quad (\text{D.40})$$

Finally, taking the limit  $n \rightarrow \infty$  in the weak formulation of (D.24) and using (A3) we obtain a solutions of the system (D.19). This completes the proof of Theorem D.3.1.  $\square$

## D.4 Proof of Theorems D.2.1 and D.2.2

The system (D.3) is written on the form (D.19) by replacing the right-hand side of (D.19), as in (D.23), taking  $N = 4$  and

$$\begin{aligned} G(t, x) &= (g(t), d_x, d_{xx}, d_t) \quad a(d, h, u, g, d_x, d_{xx}, d_t) = c(d, h, u, g, d_x, d_{xx}, d_t) = u \\ b(d, h, u, g, d_x, d_{xx}, d_t) &= h, \quad f(d, h) = h^{-1}, \quad f_1 = f_4 = k_1 = k_3 = k_4 = 0, \\ f_2(d, h, u, h_x, u_x, g, d_x, d_{xx}, d_t) &= (h^2 \mathcal{A}')' u_x^2 - g(h \mathcal{A}'' - \mathcal{A}') (\eta_x^2 + 2\eta_x d_x) \\ f_3(d, h, u, g, d_x, d_{xx}, d_t) &= -d_t - u d_x, \quad k_2(d, h, u, h_x, u_x, g, d_x, d_{xx}, d_t) = 2g \mathcal{A}' \eta_x d_{xx}, \\ \mathfrak{A}_2 &= -\varepsilon \mathcal{L}_h^{-1} \partial_x, \quad \mathfrak{A}_3 = 1, \quad \mathfrak{B}_2 = \varepsilon \mathcal{L}_h^{-1}. \end{aligned}$$

Then, in order to prove Theorem D.2.1, the following lemma is needed.

**Lemma D.4.1.** ([5]) *Let  $0 < h_{\inf} \leq h \in W^{1,\infty}$  and  $\mathcal{A} \in C^2(\cdot]0, +\infty[)$  satisfying  $\mathcal{A}' > 0$ , then the operator  $\mathcal{L}_h$  is an isomorphism from  $H^2$  to  $L^2$  and*

1. *If  $0 \leq s \leq \tilde{m} \in \mathbb{N}$  and  $\mathcal{A} \in C^{\tilde{m}+3}(\cdot]0, +\infty[)$ , then*

$$\|\mathcal{L}_h^{-1} \psi\|_{H^{s+1}} + \|\mathcal{L}_h^{-1} \partial_x \psi\|_{H^{s+1}} \lesssim \|\psi\|_{H^s} (1 + \|h - \bar{d}\|_{H^s}). \quad (\text{D.41})$$

2. *If  $\phi \in C_{\lim} \stackrel{\text{def}}{=} \{f \in C, f(\pm\infty) \in \mathbb{R}\}$ , then  $\mathcal{L}_h^{-1} \phi$  is well defined and*

$$\|\mathcal{L}_h^{-1} \phi\|_{W^{2,\infty}} \lesssim \|\phi\|_{L^\infty}. \quad (\text{D.42})$$

3. *If  $\psi \in C_{\lim} \cap L^1$ , then*

$$\|\mathcal{L}_h^{-1} \partial_x \psi\|_{W^{1,\infty}} \lesssim \|\psi\|_{L^\infty} + \|\psi\|_{L^1}. \quad (\text{D.43})$$

All the constants depend on  $s, \varepsilon, h_{\inf}, \|h - \bar{d}\|_{W^{1,\infty}}$  and not on  $\|h - \bar{d}\|_{H^s}$ .

**Proof of Theorem D.2.1.** In order to prove Theorem D.2.1, it suffices to prove (A1)–(A3). The assumption (A1) is obviously satisfied and (A2) follows from Lemma D.4.1.

In order to prove (A3), let  $W, \tilde{W}, \psi \in H^s$ , using Lemma D.4.1 and (D.29) we obtain

$$\begin{aligned} \|(\mathcal{L}_h^{-1} - \mathcal{L}_{\tilde{h}}^{-1}) \psi\|_{H^{s-1}} &= \|\mathcal{L}_h^{-1} (\mathcal{L}_{\tilde{h}} - \mathcal{L}_h) \mathcal{L}_{\tilde{h}}^{-1} \psi\|_{H^{s-1}} \\ &\lesssim \|(\mathcal{L}_{\tilde{h}} - \mathcal{L}_h) \mathcal{L}_{\tilde{h}}^{-1} \psi\|_{H^{s-2}} \\ &\lesssim \|h - \tilde{h}\|_{H^{s-1}} \leq \|W - \tilde{W}\|_{H^{s-1}}. \end{aligned}$$

where the constants depend on  $s, d, \|W\|_{H^s}, \|\tilde{W}\|_{H^s}, \|\psi\|_{H^{s-1}}$ . The same proof can be used for  $\mathfrak{A}_2$ .  $\square$

**Proof of Theorem D.2.2.** Using the characteristics

$$\chi(0, x) = x, \quad \chi_t(t, x) = u(t, \chi(t, x)),$$

the conservation of the mass (D.6) becomes

$$\frac{d}{dt} h + u_x h = 0, \quad \implies \quad h_0(x) e^{-t \|u_x\|_{L^\infty}} \leq h(t, x) \leq h_0(x) e^{t \|u_x\|_{L^\infty}}. \quad (\text{D.44})$$

In order to prove (D.16), the energy equation (D.4) implies that

$$\mathcal{E}'(t) \leq (|\dot{g}|/g + 1 + \varepsilon^{-1}) \mathcal{E}(t) + g \int_{\mathbb{R}} \left( \frac{1}{2} d_t^2 + \mathcal{A}' d_{xt}^2 \right) dx, \quad (\text{D.45})$$

since,  $h$  is bounded and far from 0,  $\mathcal{A}'$  is bounded, then (D.16) follows by Gronwall lemma.

In order to prove the blow-up criteria, we first suppose that  $\|W_x\|_{L^\infty}$  is bounded and we show that the scenario (D.15) is impossible. The equation (D.44) implies that  $h$  is bounded and far from 0. Using  $\|W\|_{L^\infty} \lesssim \|W\|_{H^1} \lesssim \mathcal{E}(t)$  one obtains that  $\|W\|_{W^{1,\infty}}$  is bounded on any interval  $[0, T]$ .

Using Lemma D.4.1, and doing some classical energy estimates (see [5, 8]), we can prove that  $\|W\|_{H^s}$  is also bounded. This ends the proof of Theorem D.2.2.  $\square$

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# Appendix E

## Regularizing effect for conservation laws with a Lipschitz convex flux

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**Abstract:** This paper studies the smoothing effect for entropy solutions of conservation laws with general nonlinear convex fluxes on  $\mathbb{R}$ . Beside convexity, no additional regularity is assumed on the flux. Thus, we generalize the well-known BV smoothing effect for  $C^2$  uniformly convex fluxes discovered independently by P. D. Lax [23] and O. Oleinik [26], while in the present paper the flux is only locally Lipschitz. Therefore, the wave velocity can be discontinuous and the one-sided Oleinik inequality is lost. This inequality is usually the fundamental tool to get a sharp regularizing effect for the entropy solution. We modify the wave velocity in order to get an Oleinik inequality useful for the wave front tracking algorithm. Then, we prove that the unique entropy solution belongs to a generalized BV space,  $BV^\Phi$ .

**AMS Classification:** 35L65, 35B65, 35L67, 26A45, (46E30).

**Key words:** Scalar conservation laws, entropy solution, strictly convex flux, discontinuous velocity, wave front tracking, smoothing effect,  $BV^\Phi$  spaces.

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## E.1 Introduction

This paper is about the regularization effect on the unique entropy solution of the scalar hyperbolic conservation law

$$u_t + f(u)_x = 0, \quad u(0, x) = u_0(x), \quad M = \|u_0\|_\infty. \quad (\text{E.1})$$

The initial datum  $u_0$  belongs to  $L^\infty(\mathbb{R}, \mathbb{R})$ . In (E.1),  $f$  is a nonlinear convex flux on the whole real line, thence  $f$  is Lipschitz on  $[-M, M]$ . The regularity of  $u$  for positive time  $t$  depends on the nonlinearity of  $f$  on  $[-M, M]$ . (For a linear flux, the solution is nothing but a translation of the initial datum with a constant speed, so no regularity is enforced by the equation (E.1).) To obtain a smoothing effect, the following Tartar condition [30] is needed:

$$\textit{There are no non-trivial interval where } f \textit{ is affine.} \quad (\text{E.2})$$

Here, the flux being nonlinear and convex on  $\mathbb{R}$ , it is strictly convex and thus it necessarily satisfies the condition (E.2).

In [23, 26], both Lax and Oleinik prove that for an uniform convex flux  $f$  such that  $f'' > c > 0$  for some constant  $c$  (as, e.g., for the Burgers equation), the solution  $u(t, \cdot)$  is in  $BV_{\text{loc}}$ , for all time  $t > 0$ . (Definitions of the various BV spaces, spaces of functions of bounded variation, can be found below and also in [1, 25].) This result is no longer true for flatter fluxes [12], such as  $f(u) = |u|^3$  and  $f(u) = u^4$ . The solution regularities in SBV and Sobolev spaces are obtained in [19, 22]. To obtain more information on the regularity of  $u$ , generalized BV spaces,  $BV^s$  and  $BV^\Phi$ , are needed. The regularity in those spaces implies the right regularity in Sobolev spaces, as well as the left and right traces for shock waves. For smooth fluxes with a polynomial degeneracy (e.g.,  $f(u) = |u|^3$ ,  $f(u) = u^4$ ), the solution  $u(t, \cdot)$  belongs to  $BV_{\text{loc}}^s$  in space for  $t > 0$  (see the last paragraph of section E.2 and [3, 7], and see [2, 24] for non-convex fluxes). This kind of regularity is still true for a  $C^1$  convex flux in a bigger generalized BV space,  $u(t, \cdot) \in BV_{\text{loc}}^\Phi$ ,  $t > 0$  with a convex function  $\Phi$  depending on the nonlinearity of  $f$  [10, 24].

In this paper, we show that this last result [10] remains true for all convex fluxes  $f$  on  $\mathbb{R}$  satisfying the condition (E.2), without requiring  $f$  to be in  $C^1$ . Such a flux can appear in applications, such as in traffic flow model [31] with a concave flux. If  $f$  is a strictly convex flux, then the (necessarily increasing) velocity

$$a(u) = f'(u)$$

exists almost everywhere. The set of discontinuities of  $a$  is countable, the left and right limits  $a^-(u) \leq a^+(u)$  existing everywhere. Thanks to the maximum principle, the entropy solution  $u$  takes values only in  $[-M, M]$ , hence  $a$  is bounded on  $[-M, M]$ . In the case of  $C^1$  convex fluxes, the simplest proof (see [7] after [26]) is based on the fundamental one-sided Oleinik inequality [20],

$$a(u(t, x)) - a(u(t, y)) \leq (x - y)/t \quad \text{a.e. } x > y, \quad (\text{E.3})$$

which implies that  $a(u)$  is a BV function and then  $u$  belongs to a  $BV^\Phi$ . Unfortunately, this inequality is no longer true for convex Lipschitz fluxes. Indeed, first,  $a(u)$  is not well defined because  $a$  is not continuous and, second, the Oleinik inequality is not true almost everywhere, as shown in Example E.3.1 below. To our knowledge, the loss of the Oleinik inequality appears in the classical literature of conservation laws only in [20], for a piecewise linear flux. Note that, though not always true, the Oleinik inequality is true on a large subset of  $\mathbb{R}^+ \times \mathbb{R}$ . We prove in this paper that this is enough to still obtain the smoothing effect in the right  $BV^\Phi$  space with a modified wave velocity and a wave front tracking algorithm for scalar conservation laws [16].

The paper is organized as follows. In Section E.2, the function  $\Phi$  is built to state the main theorem. The loss of the Oleinik inequality and the resulting difficulty to prove the main theorem E.2.1 is discussed in section E.3. Section E.4 studies the approximate Riemann problem and a modified Oleinik inequality. Section E.5 is devoted to obtaining a BV estimates on the modified velocity by the wave front tracking algorithm. The main result is proved in Section E.6.

## E.2 The main result

In this section, definitions of weak entropy solutions and  $BV^\Phi$  spaces are recalled and the function  $\Phi$  related to the smoothing effect is built. Then, the smoothing effect is stated in Theorem E.2.1.

**Definition E.2.1.**  *$u$  is called a weak solution of (E.1), if for all smooth functions  $\theta$  with a compact support, i.e., for  $\theta \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R})$*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^+} [u(t, x) \theta_t(t, x) + f(u(t, x)) \theta_x(t, x)] dt dx + \int_{\mathbb{R}} u_0(x) \theta(0, x) dx = 0. \quad (\text{E.4})$$

For a given  $u_0 \in L^\infty$ , the equation (E.4) has at least one weak solution [4, 15], the uniqueness being ensured by the Kruzkov entropy conditions:

**Definition E.2.2** (Kruzkov entropy solution). *A weak solution of (E.4) is called an entropy solution if for all positive  $\theta \in \mathcal{D}(\mathbb{R}^+ \times \mathbb{R})$  and for all convex functions  $\eta \in C^1$  and with  $F \stackrel{\text{def}}{=} \int f'(u) \eta'(u) du$  (primes denoting the derivatives), the following inequality holds:*

$$\int_{\mathbb{R}} \int_{\mathbb{R}^+} [\eta(u(t, x)) \theta_t(t, x) + F(u(t, x)) \theta_x(t, x)] dt dx \geq 0. \quad (\text{E.5})$$

In addition,  $u$  has to belong to  $C^0([0, +\infty[, L^1_{\text{loc}}(\mathbb{R}))$ .

The functional space  $BV^\Phi$  [25] is defined as follow.

**Definition E.2.3.** *Let  $\Phi$  be a convex function such that  $\Phi(0) = 0$  and  $\Phi(h) > 0$  for  $h > 0$ , the total  $\Phi$ -variation of  $v$  on  $K \subset \mathbb{R}$  is*

$$\text{TV}^\Phi v \{K\} = \sup_{p \in \mathcal{P}} \sum_{i=2}^n \Phi(|v(x_i) - v(x_{i-1})|) \quad (\text{E.6})$$



where  $\mathcal{P} = \{\{x_1, \dots, x_n\}, x_1 < \dots < x_n\}$  is the set of all subdivisions of  $K$ . The space  $BV^\Phi$  is defined by  $BV^\Phi = \{v, \exists \lambda > 0, TV^\Phi(\lambda v) < \infty\}$ .

The  $BV^\Phi$  space, a generalization of the BV space, is the space of functions with generalized bounded variations. Our goal here is to construct the best convex function  $\Phi$ , such that  $u$  is in  $BV_{loc}^\Phi$ , that means choosing  $\Phi$  to obtain the smallest space  $BV^\Phi$  in order to characterize the regularity of the entropy solution.

The one-sided Oleinik inequality is directly linked with the increasing variation of an entropy solution. Such variation is defined with  $y^+ = \max(y, 0)$  as follow with the same notations as in the previous definition [E.2.3](#).

$$TV^{\Phi^+} v \{K\} = \sup_{\mathcal{P} \in \mathcal{P}} \sum_{i=2}^n \Phi((v(x_i) - v(x_{i-1})))^+ \quad (\text{E.7})$$

In order to build the function  $\Phi$ , a definition of the generalized inverse of non decreasing functions is needed

**Definition E.2.4.** Let  $g$  be a non decreasing function from  $\mathbb{R}$  to  $\mathbb{R}$ , the generalized inverse of  $g$  is defined on  $g([-M, M])$  as following

$$g^{-1}(y) \stackrel{\text{def}}{=} \inf\{x \in \mathbb{R}, y \leq g(x)\}, \quad (\text{E.8})$$

**Remark E.2.1.** It is obvious from the definition that

$$(g^{-1} \circ g)(x) \leq x, \quad \forall x. \quad (\text{E.9})$$

The usual properties of the generalized inverse can be found in [\[17\]](#).

**Proposition E.2.1** (see proposition 2.3 in [\[17\]](#)).

1- If  $g$  is continuous then  $g^{-1}$  is a strictly increasing function and

$$(g \circ g^{-1})(y) = y, \quad \forall y \quad (\text{E.10})$$

2- If  $g$  is strictly increasing then  $g^{-1}$  is continuous and

$$(g^{-1} \circ g)(x) = x, \quad \forall x, y \quad (\text{E.11})$$

Now, the function  $\Phi$  will be built, as a consequence of the strict convexity of  $f$ ,  $a = f'$  is strictly increasing so its generalized inverse

$$b = a^{-1}, \quad (\text{E.12})$$

is continuous and non decreasing. The function  $b$  is constant on  $[a^-(u), a^+(u)]$  when  $a$  is discontinuous at  $u$ .

Let  $\omega[b]$  be the modulus of continuity of  $b$ , i.e.,

$$\omega[b](h) \stackrel{\text{def}}{=} \sup_{\substack{|x-y| \leq h \\ x, y \in a([-M, M])}} |b(x) - b(y)|, \quad (\text{E.13})$$

and let  $\phi$  be the generalized inverse of  $\omega[b]$ , i.e.,  $\phi(y) = \inf\{x \in \mathbb{R}, y \leq \omega[b](x)\}$ . We denote  $\Phi$  the convex upper envelope of  $\phi$ , that is related to the nonlinearity of the flux via the velocity. Let us write, in a concise way, the definition of  $\Phi$ . This function is the key ingredient to define a suitable functional space describing the regularity of entropy solutions.

**Definition E.2.5** (Choice of  $\Phi$ ).  *$\Phi$  is the convex upper envelope of the generalized inverse of the modulus of continuity of the generalized inverse of the velocity,*

$$\Phi \stackrel{\text{def}}{=} \text{upper convex envelope of } (\omega[a^{-1}])^{-1}. \quad (\text{E.14})$$

**Remark E.2.2.** *This definition generalizes the ones given in [3, 10] for a discontinuous velocity. This is the optimal choice for a flux with a power law degeneracy, as proved in e.g. [8, 18], for the convex power flux  $f(u) = |u|^{1+p}/(1+p)$ ,  $p \geq 1$ ,  $\Phi(u) = |u|^p = |a(u)|$ . Indeed, when the velocity is convex for  $u > 0$  and is an odd function, then  $\Phi(u) = |a(u)|$  [10].*

$C_w$  denoting the space of continuous functions with modulus of continuity  $w$ , the goal of this paper is to prove the theorem:

**Theorem E.2.1.** [**Regularizing effect in  $BV^\Phi$** ] *Let  $f$  be a strictly convex flux on  $\mathbb{R}$ ,  $u_0 \in L^\infty$  and  $u$  being the unique entropy solution of (E.1), then  $u(t, \cdot) \in BV_{\text{loc}}^\Phi$ , i.e., for all  $[\alpha, \beta] \subset \mathbb{R}$ ,*

$$\text{TV}^{\Phi^+} u(t, \cdot) \{[\alpha, \beta]\} \leq (\beta - \alpha) t^{-1}, \quad (\text{E.15})$$

$$\text{TV}^\Phi u(t, \cdot) \{[\alpha, \beta]\} \leq 2(\|a(u_0)\|_\infty + (\beta - \alpha) t^{-1}), \quad (\text{E.16})$$

Moreover, if  $u_0$  is compactly supported, then there exists  $C > 0$  such that

$$\text{TV}^\Phi u(t, \cdot) \{\mathbb{R}\} \leq C(1 + t^{-1}), \quad (\text{E.17})$$

and, in addition, for all  $\tau > 0$

$$u \in C_{w_\tau} \left( ]\tau, +\infty[, L_{\text{loc}}^1 \right) \quad \text{where} \quad w_\tau(y) = \Phi^{-1}(C(1 + \tau^{-1})y). \quad (\text{E.18})$$

Inequality (E.15) is the natural way to recover the one-sided Oleinik inequality. When  $\Phi$  is the identity function, so  $\text{TV}^\Phi u = \text{TV}u$ , the inequality (E.17) is the classical one for uniformly convex smooth flux. This regularity in time is proven in the last section of the present paper.

Theorem E.2.1 covers all previous results (with a different proof) on the smoothing effect for a strictly convex flux, the  $C^2$  case being considered in [3, 7, 9, 8, 22, 23, 24, 26]

and the  $C^1$  case being treated in [10]. All these proofs make use, directly or indirectly, of the Oleinik inequality (E.3). The proof of Theorem E.2.1 here is necessarily more complicated due to the loss of the Oleinik inequality. This crucial point is discussed in details in the next section.

### E.3 Notes on the Oleinik inequality for discontinuous wave speeds

The following example shows that the Oleinik inequality (E.3) is no longer true everywhere when  $a = f'$  is not continuous on  $[-M, M]$ . The Oleinik inequality requires the velocity is defined everywhere. For this purpose, the velocity can be defined everywhere as the mean of its left and right limits with a weight  $\lambda \in [0, 1]$ , i.e.,

$$\bar{a}(x) = \lambda a^+(x) + (1 - \lambda) a^-(x).$$

Now, a key result about the Oleinik inequality for the proof of the theorem is stated with this mean velocity.

Let us consider the Riemann problem, that is a Cauchy problem with a piecewise constant datum  $u_0(x) = u_l$  for  $x < 0$  and  $u_0(x) = u_r$  for  $x > 0$ . The Oleinik inequality is clearly true for a shock wave— i.e.,  $u_l > u_r$  —, but it is not always valid for a rarefaction wave — i.e.,  $u_l < u_r$  —.

**Proposition E.3.1** (One-sided Oleinik inequality). *For a Riemann problem producing a rarefaction wave if  $x/t, y/t \in \bar{a}([-M, M])$  then the Oleinik inequality holds*

$$\bar{a}(u(t, x)) - \bar{a}(u(t, y)) \leq (x - y)/t \quad \text{a.e. } x > y. \quad (\text{E.19})$$

The set  $\bar{a}([-M, M])$  is not an interval since  $a$  is not continuous. This is a reason for the loss of the Oleinik inequality. Moreover, the solution is constant where  $a$  is not defined, as shown in the example E.3.1 below. When the velocity is continuous this problem disappears, as proved in the next section.

**Remark E.3.1.** *The Oleinik inequality is true a.e. with a velocity chosen as*

$$a^-(u(t, x)) - a^+(u(t, y)) \leq (x - y)/t \quad \text{a.e. } x > y. \quad (\text{E.20})$$

*But, it is less useful to get the BV estimate for this velocity. This is the key point to prove the  $BV^\Phi$  regularity of the entropy solution. It should also be noticed that the Oleinik inequality (E.20) can be invalid if the signs are exchanged.*

From now on, we denote  $\xi = x/t$  and  $\eta = y/t$  for brevity.

**Example E.3.1.** *[The one-sided Oleinik inequality is not always valid] Consider*

$$f(u) = u^2 + |u|, \quad a(u) = 2u + \text{sign}(u) \quad (\text{E.21})$$

and  $u_0(x) = \text{sign}(x)$ . The entropy solution of (E.1) is  $u(t, x) = U(\xi)$  with

$$U(\xi) = \begin{cases} -1 & \xi \leq -3, \\ \frac{1}{2}(\xi + 1) & -3 \leq \xi \leq -1, \\ 0 & -1 \leq \xi \leq 1, \\ \frac{1}{2}(\xi - 1) & 1 \leq \xi \leq 3, \\ 1 & 3 \leq \xi. \end{cases}$$

Considering  $t > 0$ , the Oleinik inequality is not satisfied and

$$\bar{a}(u(t, x)) - \bar{a}(u(t, y)) > (x - y)/t$$

in the following cases

- if  $\lambda = 0$  and  $-1 < \eta < 1 < \xi < 3$ ;
- if  $\lambda = 1$  and  $-3 < \eta < -1 < \xi < 1$ ;
- if  $\lambda \in ]0, 1[$  and  $(2\lambda - 1 < \eta < 1 < \xi < 3$  or  $-3 < \eta < -1 < \xi < 1 - 2\lambda)$ .

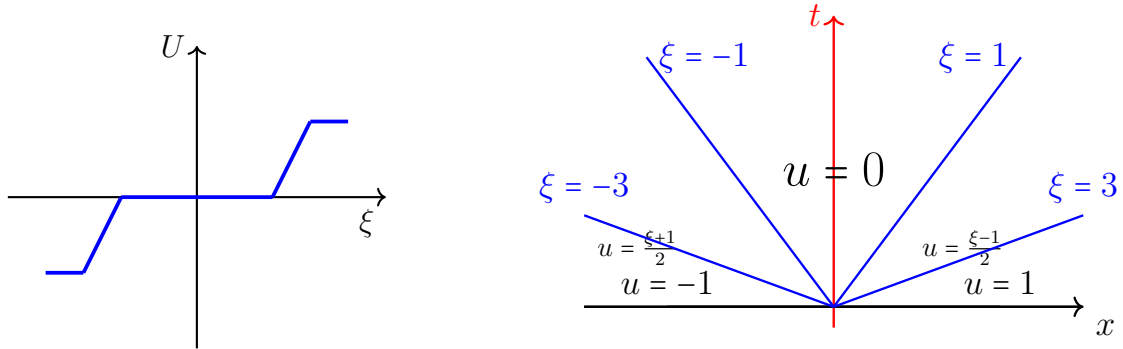


Figure E.1 – The solution of the Riemann problem.

**Remark E.3.2.** The function  $b$  is the generalized inverse of  $\bar{a}$  for all  $\lambda$ , and  $b(\bar{a}(u)) = u$ .

Hereinafter, we take  $\lambda = 1/2$  for the sake of simplicity, thus

$$\bar{a}(x) = \frac{1}{2} [a^+(x) + a^-(x)]. \quad (\text{E.22})$$

**Remark E.3.3.** If we change the flux  $f$  in the example E.3.1 by  $f(u) = u^2 + u + |u|$  and if  $\alpha < 0 < \beta$ , then  $\bar{a}(u(t, \beta)) - \bar{a}(u(t, \alpha)) \geq 1$ , where the inequality (E.15) remains true.

**Remark E.3.4.** The converse of Proposition E.3.1 is false in general. For instance, it is sufficient to take  $x, y$  in  $] -t, 0[$  or in  $] 0, t[$  in Example E.3.1.

At this stage, it is important to outline the main difficulties for proving Theorem E.2.1. These difficulties result from the discontinuity of the velocity  $a(u)$  (yielding the loss of the Oleinik inequality). For instance, consider the case when  $a(u)$  is discontinuous at  $u = u^\#$  with the jump

$$[[a]](u^\#) = a^+(u^\#) - a^-(u^\#) > 0.$$

Let  $u(t, x)$  be a rarefaction wave, non-decreasing solution of the Riemann problem with initial data  $u_0(x) = u_l$  for  $x < 0$  and  $u_0(x) = u_r$  for  $x > 0$  with  $u_l < u^\# < u_r$ . The solution  $u(t, x)$  is flat, with value  $u = u^\#$  on the interval  $x \in [a^-(u^\#) \times t; a^+(u^\#) \times t]$ , see figure E.1. The length of this interval, at time  $t$ , is exactly

$$\Delta x = t \times [[a]](u^\#).$$

At first sight, it seems a good case where the Oleinik inequality is actually an equality. However, it is not the case for the two following reasons.

First,  $u$  being constant on the flat part (with  $u \equiv u^\#$ ), there are of course no variations of  $u$  on this part, while there is a variation of  $a$  equals to the jump of  $a$  at  $u^\#$ . This shows that the variations of  $a$  are bad indicators of the variations of  $u$ . Usually, the total variation of  $u$  is controlled by the total variation of  $a(u)$  [7].

Second, since  $u$  is constant on this part, the shock wave penetrates the flat part, reducing the length of this part, i.e.

$$\Delta x < t \times [[a]](u^\#).<sup>1</sup>$$

In other words, the jump of  $a(u)$  at  $u^\#$  does not represent well the size of the flat part, which is problematic as already mentioned. As a consequence, the total variation of  $a(u)$  is not controlled in the present work. It is an important difference with the case of smooth fluxes, where  $a(u)$  is known to be in BV [13], at least  $C^2$ , with precise assumptions and counter-examples given in [24]. Here, assuming only that the convex flux is Lipschitz, it is not clear whether  $a(u)$  belongs to BV or not.

If the velocity has only one discontinuity, it is easy to overcome this difficulty. Consider

$$a(u) = \tilde{a}(u) + [[a(u^\#)]]H(u - u^\#),$$

where  $H$  is the Heaviside function; that is to say  $\tilde{a}$  is the continuous part of the velocity  $a(u)$ . The generalised inverses of  $a$  and  $\tilde{a}$  having the same modulus of continuity, they define the same space  $BV^\Phi$ . Moreover,  $\tilde{a}$  is easy to estimate in BV. Thus, the  $BV^\Phi$  regularity of  $u$  follows as in [7]. This simple case shows again that the variation of  $a$  through its discontinuity is useless to capture the regularity of  $u$ . Moreover, for the example E.3.1 above, the regularity of  $u$  is simply BV since  $\tilde{a}$  corresponds to a Burgers flux.

Removing the discontinuity of  $a$  can be done only if  $a$  has finitely many points of discontinuity. This is not possible in general. Consider the example with the following

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<sup>1</sup>Note that it is the converse of the Oleinik inequality.

velocity

$$a(u) = \sum_n 2^{-n} H(u - r_n),$$

where the sequence  $(r_n)$  takes all the values of rational numbers. This velocity corresponds to a strictly convex flux where the second derivatives have only an atomic part. Thus,  $u$  is regularised in some  $BV^\Phi$ . Removing all the discontinuities of  $a$  is not a good idea here since the corresponding flux is flat and does not correspond to any smoothing effect.

The way we solve this difficulty is to keep the Oleinik inequality by introducing a new velocity, called  $\chi$  below. The total variation of  $\chi$  can be estimated geometrically by the mean of characteristics through a wave front tracking algorithm. Moreover, the variation of  $\chi$  corresponds exactly to the generalised variation of the entropy solution  $u$ .

In order to prove Theorem E.2.1, the wave front tracking algorithm will be used. To do so,  $u_0$  is approximated by a sequence of step functions and thus the Riemann problem for each sequence can be solved [16, Lemma 3.1]. A Riemann problem with a discontinuous velocity is expounded in the next section.

## E.4 Riemann problem

This section is devoted to the Riemann problem. The shock wave is solved as usual, but solving the rarefaction wave is more complicated. For this purpose, the flux  $f$  is approached by piecewise quadratic  $\mathcal{C}^1$  fluxes, in order to show that, for a rarefaction wave, the solution is given by  $b(x/t)$ ; this is the classic formula for smooth fluxes where  $b$  is the inverse of the velocity. We extend this formula when  $b$  is the generalized inverse of a discontinuous velocity. The second part deals with a piecewise linear flux  $f_\varepsilon$  [16], which gives a modified Oleinik inequality up to a small error.

### E.4.1 The exact solution of the Riemann problem

The Riemann problem consists in solving  $u_t + f(u)_x = 0$  with the initial condition

$$u(0, x) = u_l \quad \text{for } x < 0 \quad \text{and} \quad u(0, x) = u_r \quad \text{for } x > 0. \quad (\text{E.23})$$

If  $u_l > u_r$ , the solution generates a shock with a speed given by the Rankine-Hugoniot relation  $s = \llbracket f(u) \rrbracket / \llbracket u \rrbracket$ , where  $\llbracket u \rrbracket = u_r - u_l$  is the jump of  $u$ . The entropy solution is

$$u(t, x) = u_l \quad \text{for } x < st \quad \text{and} \quad u(t, x) = u_r \quad \text{for } x > st.$$

The interesting case is  $u_l < u_r$  because the Oleinik inequality is not always true in this case. The solution has a non decreasing rarefaction wave between  $x = a^+(u_l)t$  and  $x = a^-(u_r)t$ , given by the following proposition:

**Proposition E.4.1.** *Let  $u$  be the entropy solution of the Riemann problem with  $u_l < u_r$ . For  $\xi = x/t$  the solution is*

$$u(t, x) = \begin{cases} u_l & \xi < a^-(u_l), \\ b(\xi) & a^-(u_l) < \xi < a^+(u_r), \\ u_r & \xi > a^+(u_r). \end{cases} \quad (\text{E.24})$$

**Remark E.4.1.** *A similar formula for systems with Lipschitz fluxes is given in [14, Th. 3.3, p. 279]. Another formula is proposed by Bressan in [4, Problem 3, p. 120]. In Bressan's book, the result is given for all Lipschitz fluxes. One can take the upper convex envelope of the flux instead of the flux in the same formula, thanks to the Oleinik criteria for entropy solutions with general fluxes. Moreover, if the flux is strictly convex, then the solution is defined everywhere by formula (E.24). Otherwise, it is defined a.e in [4]. To be self contained, Proposition E.4.1 has been added here with a short proof (see also [5, 6]).*

For  $n \in \mathbb{N}^*$ , let  $v_i = u_l + (i/n)(u_r - u_l)$  ( $i = 0, 1, \dots, n$ ) and let  $a_n$  be the sequence of functions such that  $\forall i$

$$a_n(v_i) = \bar{a}(v_i),$$

$a_n$  being linear on  $[v_i, v_{i+1}]$ , so  $f_n(u) = f(u_l) + \int_{u_l}^u a_n(v) dv$ . For proving Proposition E.4.1, we need Lemma E.4.1:

**Lemma E.4.1.** *For all  $(v, \xi) \in [-M, M] \times a([-M, M])$ , the sequences  $f_n(v)$  and  $b_n(\xi)$  converge, respectively, to  $f(v)$  and  $b(\xi)$ , and  $a_n(v) \rightarrow a(v)$  when  $n \rightarrow \infty$  if  $a$  is continuous at  $v$ . Moreover, the sequence  $(b_n)$  converges uniformly towards  $b$  on any bounded set.*

*Proof.* • For a given  $v$  such that  $a$  is continuous at  $v$ ,  $v_i = u_l + i(u_r - u_l)/n$  and  $v_{i+1}$  are chosen such that  $v \in [v_i, v_{i+1}]$  (note that  $v_i$  depends on  $n$ ). By definition,  $a_n(v) = n \frac{\bar{a}(v_{i+1}) - \bar{a}(v_i)}{u_r - u_l} (v - v_i) + \bar{a}(v_i)$ . Since  $|v - v_i| \leq \frac{u_r - u_l}{n}$  and since  $a$  is continuous at  $v$ , then  $\lim_{n \rightarrow \infty} a_n(v) = a(v)$ . Therefore,  $\lim_{n \rightarrow \infty} f_n(v) = f(v)$  follows at once from the definition of  $f_n$ .

- For  $n \in \mathbb{N}^*$  and  $\xi \in a([-M, M])$ , there exists  $i$  such that  $b(\xi) \in [v_i, v_{i+1}]$ . Moreover,  $b_n$  being linear on  $[a_n(v_i), a_n(v_{i+1})]$ ,  $|b_n(\xi) - b(\xi)| \leq v_{i+1} - v_i = \frac{u_r - u_l}{n}$ . In addition,  $b$  being continuous, a Dini's Lemma yields the uniform convergence of  $b_n$  to  $b$  as  $n \rightarrow \infty$ . □

We are now able to prove Proposition E.4.1.

*Proof of Proposition E.4.1:* By definition of  $a_n$ ,  $a_n(u_l) = \bar{a}(u_l)$  and  $a_n(u_r) = \bar{a}(u_r)$ . Since  $f_n \in \mathcal{C}^1$ , then, from [4, 15], the exact entropy solution of  $u_t + f_n(u)_x = 0$  with the initial condition (E.23) is

$$u_n(t, x) = \begin{cases} u_l & \xi < \bar{a}(u_l), \\ b_n(\xi) & \bar{a}(u_l) < \xi < \bar{a}(u_r), \\ u_r & \bar{a}(u_r) < \xi. \end{cases} \quad (\text{E.25})$$

Taking the limit  $n \rightarrow \infty$ , thanks to Lemma E.4.1,  $u$  has an explicit continuous formula. Thus,  $u$  is a weak entropy solution satisfying (E.4) and (E.5). Noting that  $b(\xi) = u_l \forall \xi \in [a^-(u_l), a^+(u_l)]$ ,  $\bar{a}(u_l)$  in (E.25) can be replaced by  $a^-(u_l)$  or by  $a^+(u_l)$ . Similarly,  $\bar{a}(u_r)$  can be replaced by  $a^-(u_r)$  or  $a^+(u_r)$ , thus concluding the proof.  $\square$

Finally, Proposition E.3.1 can be proved:

*Proof.* Since  $x/t, y/t \in \bar{a}([-M, M])$  and since  $b$  is the generalized inverse of  $a$ , then

$$\bar{a}(u(t, x)) = \bar{a}(b(x/t)) = x/t, \quad \bar{a}(u(t, y)) = \bar{a}(b(y/t)) = y/t.$$

Therefore, the Oleinik inequality is true.  $\square$

## E.4.2 Approximate Riemann solver

In Section E.5 below, the wave front tracking algorithm is used. Therefore, a suitable Oleinik inequality is needed for the approximate solutions. In order to get this inequality, the flux is replaced by a suitable piecewise linear approximation.

For this purpose, a piecewise constant approximation of the velocity is used as in [16]. A key point is to choose a discrete set of the value of the approximate solution  $u^\varepsilon$ , taking into account the discontinuities of the velocity. Let  $\varepsilon > 0$  and let

$$\mathfrak{B} = \{c_0 = -M, c_1, \dots, c_p = M\} \tag{E.26}$$

be a subdivision of the interval  $[-M, M]$  including a too large jump of the velocity. For this purpose the subdivision is chosen such that  $c_i < c_{i+1}$ ,  $\forall i$ , and

$$a^-(c_{i+1}) - a^+(c_i) \leq \frac{1}{4} \varepsilon. \tag{E.27}$$

That means that the variation of  $a$  is small on  $]c_i, c_{i+1}[$ ,  $\forall i$ , thus the big jumps of  $a$  are located at  $c_i$ .

**Remark E.4.2.** *Due to the jumps of the velocity  $a$  which are not expected on a uniform grid, the subdivision  $2^{-n}\mathbb{Z}$  cannot be used as in [4].*

**Remark E.4.3.**  $\frac{1}{2}(a^+(u) - a^-(u)) = \frac{1}{2}[[a]](u)$  can be bigger than  $\varepsilon/4$ , thus, the condition (E.27) cannot be replaced by  $\bar{a}(c_{i+1}) - \bar{a}(c_i) \leq \varepsilon/4$ . Notice that in this case  $u$  necessarily belongs to  $\mathfrak{B}$ , because the velocity has a big discontinuity at  $u$ .

**Remark E.4.4.** *The condition (E.27) is enough to show that if  $\varepsilon \rightarrow 0$ , then  $c_{i+1} - c_i \rightarrow 0$ ,  $\forall i$ , since*

$$c_{i+1} - c_i = b(a^-(c_{i+1})) - b(a^+(c_i)) \leq \omega[b](a^-(c_{i+1}) - a^+(c_i)) \leq \omega[b]\left(\frac{1}{4}\varepsilon\right). \tag{E.28}$$

*And  $\omega[b]$  is continuous at 0, thanks to the Heine theorem.*



As in [16], the flux  $f$  is approximated by a continuous and piecewise linear flux. For this purpose, the approximate flux  $f_\varepsilon$  is chosen as the continuous piecewise linear interpolation of  $f$  on the subdivision  $\mathfrak{B}$ ,  $f_\varepsilon(c_i) = f(c_i) \forall i$  and  $f_\varepsilon$  is linear on  $[c_i, c_{i+1}]$ . Its derivative  $a_\varepsilon = f'_\varepsilon$  is piecewise constant.

Now, the approximate Riemann solver is expounded. Let  $u_l, u_r \in \mathfrak{B}$ , and let  $u^\varepsilon$  be the entropy solution of the Riemann problem  $u_t + f_\varepsilon(u)_x = 0$ , with the initial data (E.23).

If  $u_l > u_r$ , as in the previous subsection, the solution generates a shock with the Rankine-Hungoniot relation  $s = \llbracket f_\varepsilon(u) \rrbracket / \llbracket u \rrbracket = \llbracket f(u) \rrbracket / \llbracket u \rrbracket$  (notice that  $f = f_\varepsilon$  on  $\mathfrak{B}$ ).

If  $u_r > u_l$ , let  $u_l = c_k, u_r = c_{k'}$ , for a fixed  $t > 0$ ,  $u^\varepsilon(t, \cdot)$  is non decreasing and piecewise constant. Defining

$$s_i = \frac{f_\varepsilon(c_i) - f_\varepsilon(c_{i-1})}{c_i - c_{i-1}} = \frac{f(c_i) - f(c_{i-1})}{c_i - c_{i-1}} = a_\varepsilon(c) \text{ on } (c_{i-1}, c_i). \quad (\text{E.29})$$

The solution is given by  $u^\varepsilon(t, x) = c_i$  for  $x/t \in ]s_i, s_{i+1}[$  as in [16]. The curves of discontinuity in this case are called contact discontinuities. In fact those curves represent an approximation of a rarefaction wave, so here we call them rarefaction curves.

Since  $s_i \leq \bar{a}(c_i) \leq s_{i+1}$ , then (E.29) implies

$$u(t, ts_i) = b(s_i) \leq b(\bar{a}(c_i)) = c_i \leq b(s_{i+1}) = u(t, ts_{i+1}). \quad (\text{E.30})$$

For  $i$  with  $k < i < k'$ , the equation  $u(t, \tilde{x}_i) = c_i$  has at least one solution since the exact solution  $u$  is non decreasing and continuous. Adding the condition  $\tilde{\xi}_i = \tilde{x}_i/t \in \bar{a}([-M, M])$ , this solution is unique, and  $\tilde{x}_i = \bar{a}(c_i)t$  so  $\tilde{\xi}_i = \bar{a}(c_i)$ .

Let  $\tilde{\xi}_k, \tilde{\xi}_k^+, \tilde{\xi}_{k'}, \tilde{\xi}_{k'}^-$  be defined as

- $\tilde{\xi}_k = \bar{a}(u_l)$ ,
- $\tilde{\xi}_k^+ = a_\varepsilon^+(u_l)$ ,
- $\tilde{\xi}_{k'} = \bar{a}(u_r)$ ,
- $\tilde{\xi}_{k'}^- = a_\varepsilon^-(u_r)$ .

Now,  $\tilde{x}_k, \tilde{x}_k^+, \tilde{x}_{k'}, \tilde{x}_{k'}^-$  are defined by the relation  $x = \xi t$ .

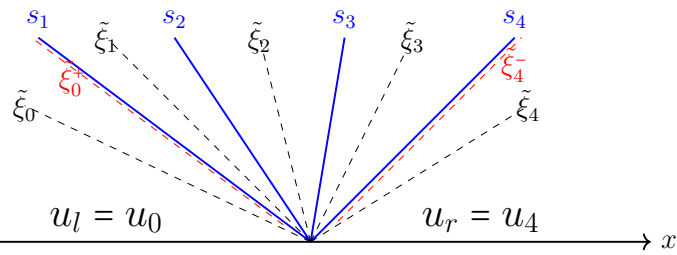


Figure E.2 – An example of a rarefaction wave with  $k = 0, k' = 4$ .

By construction, the approximate solution at the point  $(t, \tilde{x}_i)$  equals the exact solution at the same point, i.e.,  $u^\varepsilon(t, \tilde{x}_i) = u(t, \tilde{x}_i) \forall i = k+1, \dots, k'-1$ . Since  $\tilde{\xi}_i \in \bar{a}([-M, M])$ , then,  $\forall i = k+1, \dots, k'-2$ ,

$$\bar{a}(u^\varepsilon(t, \tilde{x}_{i+1})) - \bar{a}(u^\varepsilon(t, \tilde{x}_i)) = \bar{a}(b(\tilde{x}_{i+1}/t)) - \bar{a}(b(\tilde{x}_i/t)) = (\tilde{x}_{i+1} - \tilde{x}_i)/t.$$

Summing up for all  $i$ , since

$$a^-(u_{k'}) - a_\varepsilon^-(u_{k'}) \leq a^-(u_{k'}) - a^+(u_{k'-1}) \leq \varepsilon/4,$$

and

$$a_\varepsilon^+(u_k) - a^+(u_k) \leq a^-(u_{k+1}) - a^+(u_k) \leq \varepsilon/4,$$

the error is smaller than  $\varepsilon/4$  on each boundary term. Thus, the modified Oleinik inequality holds also on the whole interval as

$$\bar{a}(u^\varepsilon(t, \tilde{x}_{k'})) - \bar{a}(u^\varepsilon(t, \tilde{x}_k)) = (\tilde{x}_{k'} - \tilde{x}_k)/t \tag{E.31a}$$

$$a^-(u^\varepsilon(t, \tilde{x}_{k'}^-)) - \bar{a}(u^\varepsilon(t, \tilde{x}_k)) \leq (\tilde{x}_{k'}^- - \tilde{x}_k)/t + \varepsilon/4, \tag{E.31b}$$

$$\bar{a}(u^\varepsilon(t, \tilde{x}_{k'})) - a^+(u^\varepsilon(t, \tilde{x}_k^+)) \leq (\tilde{x}_{k'} - \tilde{x}_k^+)/t + \varepsilon/4, \tag{E.31c}$$

$$a^-(u^\varepsilon(t, \tilde{x}_{k'}^-)) - a^+(u^\varepsilon(t, \tilde{x}_k^+)) \leq (\tilde{x}_{k'}^- - \tilde{x}_k^+)/t + \varepsilon/2. \tag{E.31d}$$

The value of  $u^\varepsilon(t, \tilde{x}_{k'}^-)$  is considered on the right of the curve of discontinuity, and the value of  $u^\varepsilon(t, \tilde{x}_k^+)$  is considered on its left (see figure E.2), i.e.,  $u^\varepsilon(t, \tilde{x}_{k'}^-) = u_{k'} = u_r$ ,  $u^\varepsilon(t, \tilde{x}_k^+) = u_k = u_l$ . This proves the approximated Oleinik inequality for rarefaction waves.

Here, the rarefaction wave has been approached by a sequence of step functions and satisfies the approximate Oleinik inequality (E.31). Usually, the Oleinik inequality gives that  $a(u)$  is in BV and then  $u$  is in  $BV^\Phi$  [7]. Unfortunately,  $a(u)$  is not well defined. Moreover, the modified Oleinik inequality (E.31) does not imply that  $\bar{a}(u)$  is in BV. The next section is devoted to define another velocity  $\chi \cong a(u^\varepsilon)$  which can be controlled in BV with the wave front tracking algorithm and the restricted Oleinik inequality (E.31).

## E.5 Wave front tracking algorithm

This section deals with the BV estimate of a velocity  $\chi$  defined below. For that purpose, a  $BV^+$  estimate is used. With  $(x)^+ \stackrel{\text{def}}{=} \max\{x, 0\}$ , the  $BV^+$  space is defined by  $BV^+ \stackrel{\text{def}}{=} \{u, TV^+u < \infty\}$ ,  $TV^+$  being the *positive total variation*

$$TV^+v \stackrel{\text{def}}{=} \sup_{\mathcal{P} \in \mathcal{P}} \sum_{i=2}^n (v(x_i) - v(x_{i-1}))^+, \tag{E.32}$$

where  $\mathcal{P} = \{\{x_1, x_2, \dots, x_n\}, x_1 < \dots < x_n, 1 \leq n\}$  is the set of all subdivisions of  $\mathbb{R}$ .

The function  $u_0$  being bounded, we can assume that  $u_0$  has a compact support to prove Theorem E.2.1 (thanks to the finite speed of wave propagation).

Let  $A$  be positive such that  $\text{supp}(u_0) \subset [-A, A]$ . Let  $h = A/2m$  and  $x_i = -A + hi$  with  $i = 0, 1, \dots, m$ . The initial datum  $u_0$  is approached by a sequence of step functions  $(u_{0,m})_m$

taking values in  $\mathfrak{B}$  (see remark E.4.4) and constant in  $]x_i, x_{i+1}[$  as in [15, Chapter XIV]. Consider the initial value problem

$$u_t + f_\varepsilon(u)_x = 0, \quad u(0, x) = u_{0,m}. \quad (\text{E.33})$$

The entropy solution of (E.33),  $u_m^\varepsilon$ , is piecewise constant [4, 15]. The problem (E.33) requires to solve  $m + 1$  Riemann problems and the need to study the wave interactions.

Note that in the special case of figure E.2, if there is a shock on the right of the rarefaction wave, that has the values  $u_4$  on its left and  $u_5 < u_3$  on its right, then the distance between the shock and the rarefaction becomes very small. There, the total variation of  $\bar{a}(u)$  is bigger than  $[[a]](u_4)$ , that is problematic because the total variation of  $\bar{a}(u)$  can not be controlled by the distance between the rarefaction wave and the shock wave. To avoid this problem, the velocity on the part  $u = u_4$  is replaced by  $a^-(u_4)$  instead of  $\bar{a}(u_4)$ .

In the general case, a new velocity,  $\chi_m^\varepsilon$ , is introduced. This velocity is defined by removing the jumps of  $a(u)$  on the boundaries of the rarefaction wave, if this wave is close to a shock. Consider  $t > 0$  a fixed time and  $x \in \mathbb{R}$ . If there is a shock on the left of the point  $(t, x)$  and a rarefaction on its right, then

$$\chi_m^\varepsilon(t, x) = a^+(u_m^\varepsilon(t, x)).$$

If there is a shock on the right of the point  $(t, x)$  and a rarefaction of its left, then

$$\chi_m^\varepsilon(t, x) = a^-(u_m^\varepsilon(t, x)).$$

Otherwise

$$\chi_m^\varepsilon(t, x) = \bar{a}(u_m^\varepsilon(t, x)).$$

This definition avoids the problem mentioned above for the special case of figure E.2.

Note that in all the three cases, the solution  $u_m^\varepsilon$  can be obtained by  $u_m^\varepsilon(t, x) = b(\chi_m^\varepsilon(t, x))$ , which is a key point to take the limit in section E.6. Note also that if  $f \in \mathcal{C}^1$ , then  $\chi_m^\varepsilon = a(u_m^\varepsilon)$ .

The choice of the inequality (E.31) depends on the following cases:

- If two shocks appear on both sides of the rarefaction wave, then the inequality (E.31d) is used;
- If a shock appears only on the left of the rarefaction wave, then the inequality (E.31c) is used;
- If a shock appears only on the right of the rarefaction wave, then the inequality (E.31b) is used;
- Else, the inequality (E.31a) is used.

For  $t > 0$ , the positive total variation of  $\chi_m^\varepsilon$  over a rarefaction is smaller than the length of the rarefaction divided by  $t$ , plus a small error (E.31). For a shock, the positive total variation is equal to zero.

Here, the positive total variation is estimated after wave interactions, as in Bressan's book [4, Chap. 6, Prob. 6]. Let  $u_1, u_2$  and  $u_3$  be the values of  $u_m^\varepsilon$  from the left to the right. The speed of the left jump is  $s_1 = \frac{f_\varepsilon(u_2) - f_\varepsilon(u_1)}{u_2 - u_1}$  and the speed of the right jump is  $s_2 = \frac{f_\varepsilon(u_2) - f_\varepsilon(u_3)}{u_2 - u_3}$ . The rarefaction wave is replaced by a contact discontinuity since the flux is piecewise affine. All the possibilities for wave interactions are listed below.

- (SS) Shock–shock interaction:  $u_3 < u_2 < u_1$ . When two shocks collide they generate a new shock, and the positive total variation is always equal to zero.
- (RS) Rarefaction–shock interaction:  $u_3 \leq u_1 < u_2$ . the values  $u_1$  and  $u_2$  are consecutive in  $I$ , which implies that  $u_3 \leq u_1$ . We consider the two cases:

- $u_1 = u_3$ . After the interaction, the curves of discontinuity will disappear, and the positive total variation will be zero.
- $u_3 < u_1$ . After the interaction, a shock will appear (see figure E.3), the rarefaction fan will be smaller, and it will lose the curve of discontinuity on the right (between  $u_1$  and  $u_2$ ). The value of  $\chi_m^\varepsilon$  will be changed from  $\bar{a}(u_m^\varepsilon(u_1))$  to  $a^-(u_m^\varepsilon(u_1))$  (see the definition of  $\chi_m^\varepsilon$ ), and also the choice of the points will be changed from  $\tilde{\xi}_1$  to  $\tilde{\xi}_1^-$ , which makes the inequalities (E.31) hold after the interaction.

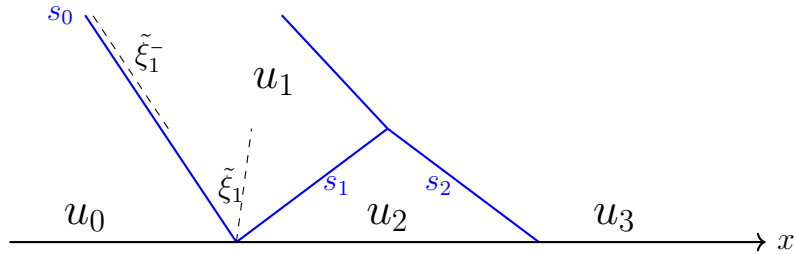


Figure E.3 – Interaction (RS).

- (SR) Shock–rarefaction interaction:  $u_2 < u_3 \leq u_1$ . This case can be treated exactly like the case Rarefaction–shock.
- (RR) Rarefaction–rarefaction interaction:  $u_1 < u_2 < u_3$ . Two rarefactions cannot collide (even the points  $\tilde{x}_i$ ). This case is impossible, because the convexity of  $f_\varepsilon$  implies that  $s_1 < s_2$ .

**Remark E.5.1.** In the case (RS) the new shock that appears can be very close to the rarefaction, which means that if the jump of the velocity  $a$  on  $u_1$  is big enough, then, the

positive total variation of  $\bar{a}(u_m^\varepsilon)$  cannot be controlled by the length of the rarefaction wave. That is the reason of using the function  $\chi_m^\varepsilon$ .

In summary, the positive total variation of  $\chi_m^\varepsilon$  and the number of rarefaction waves do not increase. Also,  $\text{TV}^+ \chi_m^\varepsilon(t, \cdot)$  is bounded by summing up the positive variation of all the rarefaction waves, thanks to the modified one-sided Oleinik inequality (E.31). For each rarefaction wave, this variation is related to the length of the interval at time  $t > 0$  up to a small error  $\varepsilon/2$ . The sum of all lengths of the intervals cannot exceed the size of the support of the solution. Since the number of rarefaction waves is less than  $m$ , there are only  $m$  error terms of size less than  $\varepsilon/2$ . Thence

$$\text{TV}^+ \chi_m^\varepsilon(t, \cdot) \leq L(t)/t + m\varepsilon/2, \quad (\text{E.34})$$

where  $L(t) = 2A(t)$  and  $\text{supp } u_m^\varepsilon(t, \cdot) \subset [-A(t), A(t)]$ . Recall that  $\text{BV}^+ \cap L^\infty = \text{BV}$  since

$$\text{TV} \chi \leq 2(\text{TV}^+ \chi + \|\chi\|_\infty).$$

The boundedness of the propagation velocity yields  $L(t) \leq 2A + 2t\|a(u)\|_\infty$ . Then, taking the constant  $C = C(u_0, f) = \max(4A, 6\|a(u)\|_\infty) > 0$ , which doesn't depend on  $\varepsilon$  and  $m$ , gives

$$\text{TV} \chi_m^\varepsilon(t, \cdot) \leq C(1 + 1/t) + m\varepsilon. \quad (\text{E.35})$$

## E.6 Compactness and regularity

This section is devoted to the proof of Theorem E.2.1. For this purpose, an uniform estimate of the velocity  $\chi_m^\varepsilon$  is obtained, which gives the compactness of the sequence  $(\chi_m^\varepsilon)$ . To take the limit in (E.35) as  $m \rightarrow \infty$  and  $\varepsilon \rightarrow 0$ , the parameter  $m$  can be chosen such that

$$\lim_{\varepsilon \rightarrow 0} m_\varepsilon \varepsilon = 0. \quad (\text{E.36})$$

We begin with an estimate of the velocity.

The BV estimate of the velocity  $\chi_m^\varepsilon$  proved in the previous section yields to  $\text{Lip}_t L_x^1$  estimates. First, let  $t$  a fixed time belonging to  $[T_1, T_2] \subset ]0, +\infty[$ . We have

$$\int_{\mathbb{R}} |\chi_m^\varepsilon(t, x+h) - \chi_m^\varepsilon(t, x)| dx \leq \text{TV} \chi_m^\varepsilon(t, \cdot) |h| \leq \left[ C \left( 1 + \frac{1}{T_1} \right) + m\varepsilon \right] |h|. \quad (\text{E.37})$$

Second, consider two different times  $T_1 < T_2$ .  $\chi_m^\varepsilon$  is piecewise constant and it has exactly the same curves of discontinuity of  $u_m^\varepsilon$ . These curves are Lipschitz and the speed of any curve cannot exceed  $k = \|a(u_0)\|_\infty$ . We suppose at first that there is no wave interaction between  $T_1$  and  $T_2$ . Then the domain  $[T_1, T_2] \times \mathbb{R}$  can be divided as in figure E.4.

Using (E.37) within each small rectangle, gives

$$\int_{\mathbb{R}} |\chi_m^\varepsilon(\tilde{T}_j, x) - \chi_m^\varepsilon(\tilde{T}_{j+1}, x)| dx \leq K \left[ C \left( 1 + \frac{1}{T_1} \right) + m\varepsilon \right] |\tilde{T}_{j+1} - \tilde{T}_j|.$$

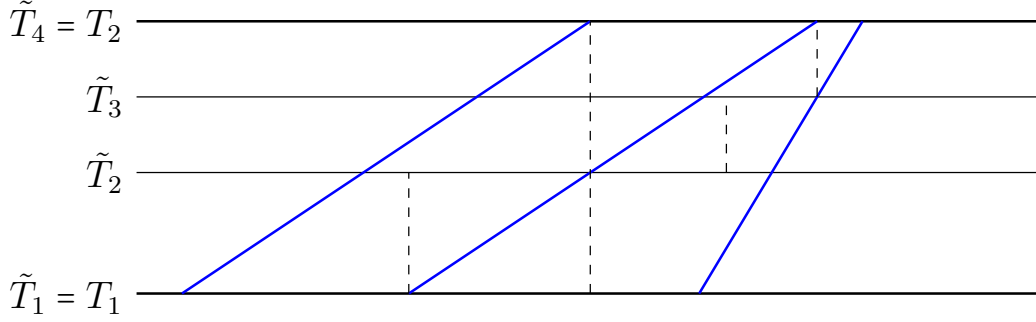


Figure E.4 – Decomposition of the domain, the blue lines are the curves of discontinuity.

In general, there are many interactions, so  $[T_1, T_2] = \cup_{j=0}^J [t_j, t_{j+1}]$ , where  $t_0 = T_1$ ,  $t_J = T_2$  and the points  $t_j$ ,  $j = 1, \dots, J-1$  are the instants of the interactions. Let  $0 < \delta < \frac{1}{2} \inf_j (t_{j+1} - t_j)$ . The inequality holds true for  $t \in [t_j + \delta, t_{j+1} - \delta]$ . Taking  $\delta \rightarrow 0$  and using that  $\chi_m^\varepsilon(\cdot, x)$  is continuous at  $t_j$  for almost all  $x$ , the inequality

$$\int_{\mathbb{R}} \left| \chi_m^\varepsilon(T_1, x) - \chi_m^\varepsilon(T_2, x) \right| dx \leq K \left[ C \left( 1 + \frac{1}{T_1} \right) + m \varepsilon \right] |T_1 - T_2| \quad (\text{E.38})$$

follows. Notice that in the estimates (E.37) and (E.38), the term  $m \varepsilon$  is bounded by a constant, thanks to (E.36). These two inequalities and the uniform boundedness of  $\chi_m^\varepsilon$  by  $k$  are the conditions of the classical compactness theorem A.8 in [21]. Hence, the sequence  $\chi_m^\varepsilon$  converges up to a sub-sequence (if necessary) to some function  $\chi$  in  $\mathcal{C}([T_1, T_2], L_{\text{loc}}^1)$

$$\chi = \lim_{\varepsilon \rightarrow 0} \chi_m^\varepsilon. \quad (\text{E.39})$$

Due to the lower semi-continuity of the total variation, we have  $\chi \in BV$  and Lipschitz in time with value in  $L_{\text{loc}}^1$  in space. Thus, for any  $0 < t$ ,  $\chi$  satisfies

$$\text{TV } \chi(t, \cdot) \leq C \left( 1 + \frac{1}{t} \right), \quad (\text{E.40})$$

$$\int_{\mathbb{R}} \left| \chi(T_1, x) - \chi(T_2, x) \right| dx \leq KC \left( 1 + \frac{1}{T_1} \right) |T_1 - T_2|. \quad (\text{E.41})$$

Using that  $u_m^\varepsilon = b(\chi_m^\varepsilon)$ , which also provides the compactness of the sequence of approximate solutions. Taking the limit  $m \rightarrow \infty$  in (E.4) and (E.5), gives that

$$u = b(\chi) \quad (\text{E.42})$$

is an entropy solution of (E.1). The main theorem of [11] (see also [27, 28]) ensures that the initial datum is recovered. Then,  $u$  is the unique Kruzkov entropy solution with the initial datum  $u_0$ .

**Remark E.6.1.** Equality (E.42) means that  $\chi = a(u)$  for a smooth velocity. Here, it is not necessarily true almost everywhere since the velocity  $a(u)$  can be discontinuous where  $u$  is constant.

Now, the  $BV^\Phi$  regularity of the entropy solution is proven. Let us first check that  $\Phi$  is positive. The function  $b$  is not constant on the whole interval  $[-M, M]$ . Then,  $\omega(h) > 0$  for  $h > 0$ . Thanks to Heine's theorem,  $\omega$  is continuous at 0, ensuring that  $\phi(y) > 0$  for  $y > 0$ .  $\Phi$  the convex envelope of  $\phi$  is then also strictly positive [10].

The  $BV^\Phi$  regularity of  $u$  is a direct consequence of the BV regularity of  $\chi$  and the definition of  $\Phi$ , which yield with (E.42), (E.13) and Remark E.2.1 to the following inequality for almost all  $t_1, t_2, x, y$

$$\begin{aligned} \Phi(|u(t_1, x) - u(t_2, y)|) &= \Phi(|b(\chi(t_1, x)) - b(\chi(t_2, y))|) \\ &\leq \phi(|b(\chi(t_1, x)) - b(\chi(t_2, y))|) \\ &\leq \phi(\omega(|\chi(t_1, x) - \chi(t_2, y)|)) \\ &\leq |\chi(t_1, x) - \chi(t_2, y)|. \end{aligned} \tag{E.43}$$

The BV regularity of  $\chi$  (E.40) and the inequality (E.43) show that  $u \in BV^\Phi$ . The  $\text{Lip}_t L_x^1$  regularity of  $\chi$  (E.41) and inequality (E.43) again imply that

$$\int_{\mathbb{R}} \Phi(|u(t_1, x) - u(t_2, x)|) dx \leq KC(1 + \tau^{-1})|t_1 - t_2|. \tag{E.44}$$

This is an estimate in the Orlicz space  $L^\Phi$ , i.e.,  $L^\Phi(\mathbb{R})$  denotes the set of measurable functions  $f$  such that  $\int_{\mathbb{R}} \Phi(|f(x)|) dx < \infty$  [29].

If  $\Phi(u) = |u|^p$  then the  $\text{Lip}^{\frac{1}{p}}([\tau, +\infty[, L_{\text{loc}}^p)$  estimate in [3] is recovered. In general, Jensen inequality gives  $u \in \mathcal{C}_w([\tau, +\infty[, L_{\text{loc}}^1)$ .

We now prove the inequality (E.15):

The proof of Theorem E.2.1 is done for  $u_0$  compactly supported. In the general case, to prove the inequality (E.15), the segment  $[\alpha, \beta]$  is divided by a subdivision  $\mathfrak{S} = \{\alpha_0 = \alpha, \alpha_1, \dots, \alpha_l = \beta\}$  in order to separate rarefaction waves and shocks. Inequality (E.31) implies that

$$a^-(u(t, \alpha_{i+1})) - a^+(u(t, \alpha_i)) \leq (\alpha_{i+1} - \alpha_i) t^{-1} + \varepsilon/4. \tag{E.45}$$

Replacing  $\chi(t_1, x) - \chi(t_2, y)$  by  $a^-(u(t_1, x)) - a^+(u(t_2, y))$  in the inequality (E.43), the result follows by restarting a similar proof, summing up for all  $i$  and taking the limit  $\varepsilon \rightarrow 0$ .  $\square$

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# Appendix F

## Entropy solutions in $BV^s$ for a class of triangular systems involving a transport equation

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**Abstract:** Strictly hyperbolic triangular systems with a decoupled nonlinear conservation law and a coupled “linear” transport equation with a discontinuous velocity are known to create measure solutions for the initial value problem [38]. Adding a uniform strictly hyperbolic assumption on such systems we are able to obtain bounded solutions in  $L^\infty$ . The Pressure Swing Adsorption process (PSA) [15] is an example coming from chemistry which, after a change of variables from Euler to Lagrange, has such a triangular structure [19]. Here, we provide global weak  $L^\infty$  entropy solutions in the framework of fractional  $BV$  spaces:  $BV^s$ ,  $1/3 < s < 1$ , when the zero set of the second derivative of the decoupled flux is locally finite.

**AMS Classification:** 35L65, 35L67, 35Q35, 26A45, 76N10.

**Key words:** Conservation laws; triangular systems; entropy solution; fractional  $BV$  spaces; linear transport; chromatography; Cauchy problem; non convex flux; regularity.

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## F.1 Introduction

We consider triangular systems of the form

$$\partial_t u + \partial_x f(u) = 0, \tag{F.1}$$

$$\partial_t v + \partial_x (a(u)v) = 0. \tag{F.2}$$

Here  $f$  is the scalar flux function (which we shall henceforth refer to as the flux) for the equation (F.1) and the function  $a(\cdot)$  denotes the velocity of the linear (with respect to  $v$ ) equation (F.2). The above system is complemented by a set of initial data

$$u(x, 0) = u_0(x), \tag{F.3}$$

$$v(x, 0) = v_0(x). \tag{F.4}$$

Such systems are already of mathematical interest due to the coupling of the theory of scalar conservation laws with the theory of transport equations. This system was studied in [49] in a non hyperbolic setting,  $f' = a$ , with measure solutions for  $v$ . It is classically used before the apparition of shock waves [5, 35]. Here, we study this in a strictly hyperbolic setting obtaining global weak entropic bounded solutions in  $L^\infty$  with a minimal fractional  $BV$  regularity for  $u_0$ .

Another class of triangular system is possible [40], for instance the  $2 \times 2$  system studied in [6]. The system (F.1), (F.2) appears in chemical engineering [34, 60, 61] for two chemical species [15, 19] after rewriting the system in Lagrangian coordinates [58, 66].

At the first sight, such a system seems easy to solve in the “triangular” manner, that is solve the first equation (F.1) to get  $u$  and then solve the linear equation (F.2) keeping  $u$  fixed. This method works well for smooth solutions [5, 35], but, when a shock wave appears in  $u$  the velocity  $a(u)$  becomes discontinuous. The theory of linear transport equations with discontinuous velocity is a delicate topic yielding measure solutions and a loss of uniqueness [10, 11, 12, 59].

In this paper, we propose a different approach to obtain global weak solutions bounded in  $L^\infty$ . A main idea is to consider the system (F.1), (F.2) not as a triangular system,

but, really, as a  $2 \times 2$  hyperbolic system. If  $f$  is nonlinear, one field is nonlinear and the other one is linearly degenerate. When  $f$  is uniformly convex it is a particular case of  $2 \times 2$  systems with one genuinely nonlinear field and a linearly degenerate one [39]. When the flux  $f$  is piecewise convex or concave, the system corresponds to the most important case in gas-liquid chromatography [15, 16, 34].

A linearly degenerate field is known to simplify the study of a  $2 \times 2$  system [57, 58]. Such systems are known to propagate one component with less regularity, see [27, 30, 37] where one component has a large total variation. In [18, 19], one component is only  $L^\infty$ . On the contrary such linearly degenerate field can also produce blow-up behaviors [17, 19, 55].

We show that the behavior of the proposed entropy solutions is linked to the fractional  $BV$  regularity of  $u$ , indeed,  $u_0 \in BV^s$ ,  $0 < s \leq 1$ . The  $BV^s$  framework seems optimal to study the optimal regularity of entropy solutions of scalar conservation laws [18, 52]. This framework and more references are recalled in Section F.1.4. For the scalar case, the theory works well for all  $s > 0$ . For the triangular system (F.1), (F.2), we prove that the regularity  $s = 1/3$  is critical for the existence of  $L^\infty$  entropy solutions. As a consequence, it proves that for general nonlinear  $2 \times 2$  systems with a genuinely nonlinear eigenvalue and a linearly degenerate one, the existence result obtained in [39] is optimal. The exponent  $s = 1/3$  is directly linked to the cubic estimate on the Lax curves [19, 48]. For nonconvex fields, it is known that the Lax curves are less regular [7, 3, 4, 50]. However, for the triangular system (F.1), (F.2), with a uniform hyperbolicity assumption,  $f \in C^4$ ,  $a \in C^3$ , we prove that the exponent  $s = 1/3$  is still critical for the existence theory.

The paper is organized as follows. The hyperbolicity of the triangular system and the key assumptions are given in Section F.1.1. The definitions of weak and entropy solutions are stated in Section F.1.2. Explicit occurrences of measure solutions are discussed in Section F.1.3. The  $BV^s$  framework is recalled in Section F.1.4. In Section F.2 the two main results are stated: existence for  $s > 1/3$ , blow-up for  $s < 1/3$ . In Section F.3 the Riemann invariants and the Riemann problem are studied. The Lax curves and the key cubic estimates are studied in Section F.4. The existence proof is done in Section F.5 and the blow-up in Section F.6.

### F.1.1 The hyperbolic triangular system

The system (F.1), (F.2) of conservation laws is clearly hyperbolic and can be rewritten, using the vectorial flux  $\mathbf{F}$ , as  $\partial_t \mathbf{U} + \partial_x \mathbf{F}(\mathbf{U}) = 0$ ,  $\mathbf{U} = (u, v)^\top$ . The matrix of the linearized system has a triangular structure,

$$D\mathbf{F}(\mathbf{U}) = \begin{pmatrix} f'(u) & 0 \\ a'(u)v & a(u) \end{pmatrix}.$$

This system has the unbounded invariant region  $[-M, M]_u \times \mathbb{R}_v$ , where  $M = \|u_0\|_\infty$ . In this paper, the system is assumed to be strictly hyperbolic through the condition

$$\forall u \in [-M, M], \quad f'(u) > a(u). \quad (\text{F.5})$$

Of course, the symmetric assumption:  $\forall u \in [-M, M], f'(u) < a(u)$ , yields a similar study. For large data, the strict hyperbolicity condition (F.5) has to be strengthened on the set  $[-M, M]$ , by the following uniformly strict hyperbolicity (USH) condition:

$$\inf_{|u| \leq M} f'(u) > \sup_{|u| \leq M} a(u). \quad (\text{F.6})$$

An interesting case is already when  $f$  is convex,  $f'' > 0$  everywhere, or concave  $f'' < 0$ . This case occurs for a chromatography system with a convex isotherm written in appropriate Lagrangian coordinates [19].

In this paper, the flux  $f(\cdot)$  belongs to  $C^4$  and the velocity  $a(\cdot)$  belongs to  $C^3$ . Moreover, the flux is locally piecewise convex or concave as a consequence of the following assumption,

$$\mathcal{Z} = \{u, f''(u) = 0, |u| \leq M\} \text{ is finite.} \quad (\text{F.7})$$

## F.1.2 Weak and entropy solutions

A weak solution of the system (F.1)-(F.2) satisfying the initial conditions (F.3)-(F.4) is defined as follows.

**Definition F.1.1** (Weak solutions). *The pair  $(u, v)$  is a weak solution of the system (F.1)-(F.2) with initial data (F.3)-(F.4) if for all compactly supported test functions  $\varphi, \psi \in C_c^1(\mathbb{R} \times [0, +\infty[, \mathbb{R})$ , the following integral identities hold:*

$$\int_0^{+\infty} \int_{\mathbb{R}} (u \partial_t \varphi + f(u) \partial_x \varphi) dx dt + \int_{\mathbb{R}} u_0(x) \varphi(x, 0) dx = 0, \quad (\text{F.8})$$

$$\int_0^{+\infty} \int_{\mathbb{R}} (v \partial_t \psi + a(u) v \partial_x \psi) dx dt + \int_{\mathbb{R}} v_0(x) \psi(x, 0) dx = 0. \quad (\text{F.9})$$

In a quite general framework, the following regularity is required for  $(u, v)$ :  $u \in L_{loc}^\infty$ ,  $v \in L_{loc}^1$  or more generally  $v$  is a measure. In the case when  $v$  is a measure, there are some issues in defining the product  $a(u) \times v$  [11]. We shall briefly touch upon the issue of measure solutions. But our main focus in this paper is on entropy solutions  $u, v \in L_{loc}^\infty$ .

We propose below a notion of entropy solutions for the system (F.1)-(F.2). As in [14, 19, 45], the entropy condition is only tested on the nonlinear component  $u$ . Contact discontinuity waves linked to a linearly degenerate field are well known not to affect the entropy inequality which remains an equality [29].

**Definition F.1.2** (entropy solutions). *The pair  $(u, v)$  is an entropy solution of the system (F.1)-(F.2) with initial data (F.3)-(F.4) if it is a weak solution and for all convex function  $\eta$  and all non-negative test functions  $\varphi \in C_c^1(\mathbb{R} \times ]0, +\infty[, \mathbb{R})$ , with  $q' = \eta' f'$ ,  $u$  satisfies the following inequality:*

$$\int_0^{+\infty} \int_{\mathbb{R}} (\eta(u) \partial_t \varphi + q(u) \partial_x \varphi) dx dt + \int_{\mathbb{R}} \eta(u_0(x)) \varphi(x, 0) dx \geq 0. \quad (\text{F.10})$$

Thus, it suffices only to have a weak solution of the system and the entropy solution of (F.1). We do not need to use another family of entropies associated to  $v$  [14, 19]. This is due to the linear degeneracy of the second equation. For some diagonal systems with linearly degenerate eigenvalues [45], such definition is enough to get uniqueness of entropy solutions.

Uniqueness of entropy solutions with a fixed initial data  $(u_0, v_0)$  is wrong without additive assumptions. It is due to the lack of uniqueness of weak solutions for the linear transport equation with a discontinuous velocity.

### F.1.3 Measure solutions

For triangular systems as discussed in this paper, one can have measure solutions. A prototypical example of such triangular systems is given by

$$\begin{aligned} \partial_t u + \partial_x \left[ \frac{u^2}{2} \right] &= 0, \\ \partial_t v + \partial_x [(u-1)v] &= 0, \end{aligned} \tag{F.11}$$

with  $f(u) = \frac{u^2}{2}$  and  $a(u) = u - 1$ . Clearly this satisfies the strict hyperbolicity condition (F.5) but not the more restrictive uniform strict hyperbolicity condition (F.6). In [38], it was shown that for strong shocks the solutions must be searched in the class of Radon measures. In particular, let us consider the Riemann problem with initial data

$$u(x, 0) = \begin{cases} u_l, & x < 0, \\ u_r, & x > 0, \end{cases} \quad v(x, 0) = \begin{cases} v_l, & x < 0, \\ v_r, & x > 0, \end{cases}$$

where  $u_l, u_r, v_l$  and  $v_r$  are constants. Then for  $u_l \geq u_r + 2$  (see Section 3, [38]), the system (F.11) has a solution given by

$$u(x, t) = \begin{cases} u_l, & x < \bar{u}t, \\ u_r, & x > \bar{u}t, \end{cases} \quad \bar{u} = \frac{u_l + u_r}{2},$$

which is the entropy solution of the Burgers equation satisfied by  $u$ , and

$$v(x, t) = \begin{cases} v_l, & x < \bar{u}t, \\ v_r, & x > \bar{u}t, \end{cases}$$

with a Dirac mass propagating with the  $\delta$ -shock in  $v$  on the line  $x = \bar{u}t$ , whose strength is given by

$$\int_{\bar{u}t^-}^{\bar{u}t^+} v(x, t) dx = \frac{t}{2} [v_r(u_l - u_r + 2) - v_l(u_r - u_l + 2)].$$

In other words,  $v$  is of the form

$$v(x, t) = v_l \chi_{x < \bar{u}t}(x, t) + v_r \chi_{x > \bar{u}t}(x, t) + \frac{t}{2} [v_r(u_l - u_r + 2) - v_l(u_r - u_l + 2)] \delta(x - \bar{u}t),$$



where  $\chi_A$  is the characteristic function of the set  $A$ .

Thus, corresponding to a strong shock in  $u$ ,  $v$  lies in the class of bounded Radon measures. Similar phenomenon is well known for transport equations [11] and is also widely observed in the case of non-strictly hyperbolic systems [20, 12].

The main approach in solving the system (F.11) in [38] was to observe that the first equation in  $u$  can be solved independently. The second equation can then be considered as a transport equation in  $v$  with a discontinuous coefficient  $u$  (see also [11]).

But this idea of looking at the two equations separately might not be a good one. Our aim in this article is to prove the existence of solutions to systems of type (F.11) in the class of fractional BV functions and hence one need not appeal to  $\delta$ -shock wave type solutions. Moreover, to obtain  $L^\infty$  solution with a wave front algorithm, we suppose that the system (F.1), (F.2) is a strictly hyperbolic system (F.5). Notice that we also need the uniform strict hyperbolicity condition (F.5), else we are not able to solve the Riemann problem appropriately.

For instance, consider the example with  $f'(u) = u$  and  $a(u) = u - d$ ,  $d > 0$  and  $M = 1$ . This system is strictly hyperbolic for  $(u, v) \in [-1, 1] \times \mathbb{R}$ . For  $d \leq 2$ , the uniform strict hyperbolicity condition (F.6) is not fulfilled. Consider for instance the Riemann problem with initial data  $u_0(x) = \pm 1$ ,  $\pm x > 0$  and  $v_0(x) \equiv 1$ . For  $d > 1$ , the Riemann problem can be solved by a contact discontinuity with speed  $a(1) = 1 - d < 0$  and a stationary shock wave with 0 speed. The intermediary constant state between the two waves is computed thanks to the Rankine-Hugoniot condition along the shock at  $x = 0$  which says that  $a(u)v$  is constant through the shock (since the shock speed is 0). The intermediary constant state is  $(u, v) = (-1, v^*)$  with

$$v^* = 1 \times \frac{a(1)}{a(-1)} = \frac{1-d}{-1-d} = \frac{d-1}{d+1} < 0.$$

For  $d = 1$  there is no room to put this intermediary state. It is even worse for  $0 < d < 1$ . The velocity  $a(u)$  becomes positive, so the contact discontinuity wave has to be on the right of the shock. But its speed  $a(u)$  corresponds to the left state. Thus, it is impossible to solve it by this way.

## Transport equations and $\delta$ -shock waves

The phenomenon of Dirac mass already appears for the linear transport equation with a discontinuous coefficient [11, 59].

$$\partial_t v + \partial_x(\alpha(x)v) = 0, \quad v(x, 0) = v_0(x). \quad (\text{F.12})$$

For instance, if  $\alpha(x) = -\text{sign}(x)$  and  $v_0(x) \equiv 1$  a Dirac mass appears immediately at  $x = 0$ :  $v(t, 0) = 2t\delta(x - 0)$ .

The point is that the linear transport equation (F.2) is the more difficult part of the triangular system (F.1)-(F.2). In this paper, a construction of  $L^\infty_{loc}$  weak solution is proposed.

It is tempting to solve the singular transport equation (F.12) with the triangular system (F.1)-(F.2), choosing a flux  $f$  and  $\alpha$  such that  $\alpha(x, t) = a(u(x, t))$  but it is impossible.

$u(x, t)$  has a stationary shock so  $0 \in f'([-M, M])$  and the uniform strict hyperbolicity is lost.

### F.1.4 $BV^s$ functions

For one dimensional scalar conservation laws, the spaces  $BV^s$  are known to give optimal results for weak entropy solutions [18, 24, 52, 53], first on the fractional Sobolev regularity [51, 43] and second on the structure of such solutions [1, 2, 8, 9, 31, 32]. Such results on the maximal regularity has been extended for some systems and the multidimensional case [25, 35, 44]. For less regular fluxes more generalized  $BV$  spaces are also used in [26, 52, 53, 36]. In this section basic facts on  $BV^s$  functions are recalled.

**Definition F.1.3.** [54] *A function  $u$  is said to be in  $BV^s(\mathbb{R})$  with  $0 < s \leq 1$  if  $TV^s u < +\infty$  where*

$$TV^s u := \sup_{n \in \mathbb{N}, x_1 < \dots < x_n} \sum_{i=1}^n |u(x_{i+1}) - u(x_i)|^{1/s}.$$

The  $BV^s$  semi-norm is defined by

$$|u|_{BV^s} := (TV^s u)^s$$

and a norm on this space is defined by

$$\|u\|_{BV^s} := \|u\|_{L^\infty} + |u|_{BV^s}.$$

Fractional  $BV$  functions have traces like  $BV$  functions. This is a fundamental property to define the Rankine-Hugoniot condition for shock waves. Moreover, this property is not true for the Sobolev functions in  $W^{s,1/s}$ , the Sobolev space nearest to  $BV^s$  [18].

**Theorem F.1.1.** [54] *For all  $s \in ]0, 1[$ ,  $BV^s$  functions are regulated functions.*

The fractional total variation only depends on the local extrema of the function and the order of this extrema.

**Lemma F.1.1.** [18, 39] *If  $u$  is a piecewise monotonous function and if its local extrema are located in the increasing sequence  $(x_i)_{i \in I}$ , then  $TV^s u$  only depends on the sequence  $(u(x_i))_{i \in I}$ . Moreover, there exists an ordered subset  $J$  of  $I$  such that*

$$TV^s u = \sum_{j \in J, j \neq \max J} |u(x_{suc(j)}) - u(x_j)|^{1/s},$$

where  $suc(j)$  denotes the successor of  $j$  in  $J$ ,  $suc(j) = \min\{k \in J, k > j\}$ .

Moreover, it is dangerous to refine the mesh to compute the fractional total variation [18, example 2.1], [22, 23]. Consider  $u(x) \equiv x$  on  $[0, R]$ ,  $p = 1/s > 1$ . Then  $TV^s u[0, R] = R^p$  but, when  $n \rightarrow +\infty$ ,

$$\sum_{i=1}^n |u(iR/n) - u((i-1)R/n)|^p = n(R/n)^p = R^p/n^{p-1} \rightarrow 0.$$

This property, which is not true in  $BV$ , is used later to prove the existence of weak solutions in  $BV^{1/3}$  for the triangular system (F.1), (F.2) when the flux  $f$  is convex.

## F.2 Main results

The main results are stated in the  $BV^s$  framework. The basic facts on this setting were recalled in Section F.1.4. The critical space for the existence theory is  $BV^{1/3}$  when the flux  $f$  is convex. For the nonconvex case, the existence is proved in the little smaller space than  $BV^{1/3}$  [18],

$$BV^{1/3+0} := \bigcup_{s>1/3} BV^s \not\subseteq BV^{1/3}.$$

**Theorem F.2.1** (Existence in  $BV^{1/3+0} \times L^\infty$ ). *The following assumptions are required on the flux  $f(\cdot)$  and the transport velocity  $a(\cdot)$  on  $[-M, M]$  where  $M = \|u_0\|_\infty$ :*

1.  $f'(\cdot)$  and  $a(\cdot)$  belong to  $C^3([-M, M], \mathbb{R})$ ,
2.  $f'(u) > a(u)$  and satisfy the uniform strict hyperbolicity assumption (USH) (F.6),
3.  $f(\cdot)$  has at most a finite number of inflections points (F.7).

*If  $(u_0, v_0)$  belongs to  $BV^s \times L^\infty$  and  $s > 1/3$ , then there exists an entropy solution  $(u, v)$  of system (F.1), (F.2), which belongs to  $L^\infty([0, +\infty)_t, BV^s(\mathbb{R}_x, \mathbb{R})) \times L^\infty((0, +\infty)_t \times \mathbb{R}_x, \mathbb{R})$ .*

*In addition, the positivity of the initial data is preserved, if  $\inf v_0 > 0$  then  $\inf v > 0$ .*

*Moreover, if the flux  $f$  is convex then the existence result remains true for  $s = 1/3$ .*

Such a result requires many comments since such systems are known to produce  $\delta$ -shocks instantly [64]. To have a bounded solution in  $L^\infty$ , we forget the triangular structure and do not solve the first equation (F.1) independently and then the second one (F.2). We consider the triangular system (F.1), (F.2) as a whole system. For a Riemann problem involving a shock wave we write the Rankine-Hugoniot condition for the whole system. The first equation (F.1) fixes the speed of the shock wave  $s = [f(u)]/[u]$  independently of  $v$  where  $[u]$  is the jump of  $u$ . Then we solve the second equation  $s[v] = [a(u)v]$  with  $s$  already fixed. This process is possible under the uniformly strictly hyperbolicity hypothesis (USH) (F.6), as explained later in Section F.3.2. Thus, (USH) is a key assumption to use such a process and to avoid a blow-up as in [17, 55]. The philosophy of using a whole system to solve a singular scalar equation is already in the smart paper [6]. For the gas-liquid chromatography system, (USH) is automatically fulfilled if the incoming gas-velocity is positive [14]. Notice that the positivity of  $v$  is preserved a.e. by Theorem F.2.1. Of course, by linearity of the equation (F.2) with respect to  $v$ , the negativity of the initial data is also preserved, if  $\sup v_0 < 0$  then  $\sup v < 0$ .

The triangular approach, that means solving first (F.1) with the unique Krushkov entropy solutions [47] and then (F.2), usually yields a measure solution  $v$ . It is the reason why we do not expect an uniqueness result. Of course, we have uniqueness for  $u$  but the problem of uniqueness remains for  $v$ . For a weakly coupled system with some linearly degenerate fields, the entropy condition only on the nonlinear field is enough to ensure the uniqueness in [45]. For the triangular system the coupling by the transport velocity is too

nonlinear and certainly needs another condition to enforce the uniqueness.

For the existence result, a Wave Front Tracking algorithm (WFT) is proposed following the approach of solving a Riemann problem for the whole system. As a matter of fact the approximate solution  $u$  satisfies the  $BV^s$  uniform bounds [18]. The difficult point is to bound  $v$  in  $L^\infty$ . Since the system is linear with respect to  $v$ , an  $L^\infty$  bound for  $v$  is enough to get an existence theorem as in [14].

Theorem (F.2.1) is optimal and, in general, we cannot reduce the  $BV^{1/3}$  regularity of  $u_0$ , else a blow-up can occur. For this purpose, we build a sharp example in a space very near to  $BV^{1/3}$ , namely,  $BV^{1/3-0}$

$$BV^{1/3} \not\subset BV^{1/3-0} := \bigcap_{s < 1/3} BV^s.$$

**Theorem F.2.2** (Blow-up in  $BV^{1/3-0}$ ). *There exist  $u_0 \notin BV^{1/3}$ , but  $u_0 \in BV^{1/3-0}$ ,  $v_0 \in L^\infty$ ,  $a$  and  $f$  such that the solution of the initial value problem provided by our construction has a blow-up immediately at time  $t = 0^+$ .*

There is no blow-up for  $u$  since the entropy solution  $u$  of the scalar conservation law satisfies the maximum principle. Indeed, only the function  $v$  has a blow-up at time  $t = 0^+$ . For the flux  $f$ , only a nonlinear flux is needed. We provide a simple example with a Burgers' flux  $f$  and a linear velocity  $a$ . Notice also that the blow-up depends only on the regularity of  $u_0$  and not of  $v_0$ .

Another way to present this blow-up is the following.

**Remark F.2.1.** *For the Burgers' flux  $f(u) = u^2/2$  and the velocity  $a(u) = u - 3$ , there exists a sequence of initial data  $(u_{0,n}, v_{0,n})$  in  $BV$  such that the system (F.1), (F.2) is uniformly strictly hyperbolic (USH) and, the associated solutions  $(u_n, v_n)$  provided by our wave front tracking algorithm satisfy*

$$\lim_{n \rightarrow +\infty} TV^{1/3} u_{0,n} = +\infty \tag{F.13}$$

$$\lim_{n \rightarrow +\infty} t_n = 0 \tag{F.14}$$

$$\lim_{n \rightarrow +\infty} \|v_n(\cdot, t_n)\|_\infty = +\infty. \tag{F.15}$$

As a consequence, it proves that for general nonlinear  $2 \times 2$  systems with a genuinely nonlinear eigenvalue and a linearly degenerate one the existence result proven in [39] is optimal.

### F.3 The uniformly strictly hyperbolic system

In this section, we get the Riemann invariant and solve the Riemann problem study for the triangular system (F.1)-(F.2).

### F.3.1 Riemann invariants

A  $2 \times 2$  strictly hyperbolic system admits, at least locally, a set of coordinates which diagonalizes the hyperbolic system for smooth solutions [63]. The knowledge of this coordinate system, given by the Riemann invariants is often useful in understanding the structure of the system. Next we study the Riemann invariants for the system (F.1)-(F.2).

The eigenvalues of the system (F.1)-(F.2) are:

$$\lambda_1 = f'(u) > a(u) = \lambda_2. \quad (\text{F.16})$$

Notice that the eigenvalues are functions of  $u$  only. Let  $r_1$  and  $r_2$  denote the corresponding right eigenvectors.

Clearly,  $u$  is a 2-Riemann invariant associated to the right eigenvector  $r_2 = (0, 1)^\top$  and satisfies

$$\partial_t u + f'(u)\partial_x u = 0.$$

A 1-Riemann invariant, which we denote as  $z(u, v)$ , corresponding to the right eigenvector  $r_1$ , can be computed in the following manner. We note that a right eigenvector of the matrix  $D\mathbf{F}(u)$  corresponding to the eigenvalue  $f'(u)$  is given by

$$r_1 = \begin{pmatrix} 1 \\ \frac{a'(u)v}{f'(u)-a(u)} \end{pmatrix}.$$

Then  $z$  satisfies

$$\frac{1}{v}\partial_u z = \frac{a'(u)}{a(u) - f'(u)}\partial_v z.$$

This can be solved using a separation of variables.

$$\partial_u z = \frac{a'(u)}{a(u) - f'(u)}, \quad \partial_v z = \frac{1}{v}.$$

For instance, a Riemann invariant is

$$z(u, v) = A(u) + \ln(|v|), \quad A'(u) = \frac{a'(u)}{a(u) - f'(u)}.$$

To avoid the singularity at  $v = 0$ , it suffices to take the exponential,

$$Z = \exp(z) = v \exp(A(u)), \quad (\text{F.17})$$

which satisfies the equation

$$\partial_t Z + a(u)\partial_x Z = 0.$$

$Z$  is exactly the Riemann invariant discovered by the first author in [13] for a chromatography system studied also in [14, 16, 17, 19].

### F.3.2 The Riemann problem

We study the Riemann problem for the system (F.1)-(F.2) with initial data:

$$u_0(x) = u_{\pm}, \pm x > 0, \quad (\text{F.18})$$

$$v_0(x) = v_{\pm}, \pm x > 0. \quad (\text{F.19})$$

A direct and somewhat naive approach is to solve the conservation law (F.1) first and then the second equation (F.2) using the solution  $u$ . In such an approach, one faces the difficulty of solving the linear transport equation with a discontinuous coefficient.

Instead, we consider the two equations together as a system. This is a key point as in [6]. The solution of the Riemann problem consists of two waves separated by an intermediary state  $(u_m, v_m)$  where  $u_m = u_-$  and  $v_m$  is unknown. In accordance with the labeling of the eigenvalues (F.16), a wave associated to  $u$  is called a 1-wave and a wave associated to  $Z$  is called a 2-wave.

1. A wave associated to the eigenvalue  $\lambda_1 = f'$ : this wave is a shock wave or a rarefaction wave if  $f$  is convex or concave. For a non convex flux  $f$ , this wave is a composite wave.
2. A linearly degenerate wave associated to the eigenvalue  $\lambda_2 = a$ : the speed of this 2-wave is  $a(u_-)$ .

The intermediate value  $v_m$  has to be computed through the 1-wave. Now, the various 1-waves that can occur are detailed. For this purpose, we consider Riemann problems yielding only a 1-wave.

#### Shock waves

Let us denote  $U = (u, v)$  and  $\tilde{U} = (u, Z)$ .  
The Rankine-Hugoniot condition gives

$$s[u] = [f(u)], \quad s[v] = [a(u)v], \quad (\text{F.20})$$

where  $s$  denotes the speed of the discontinuity (or the slope of the jump).

Thus, the slope of the jump is determined by  $u_{\pm}$  and the flux  $f$ ,

$$s = \frac{[f(u)]}{[u]}. \quad (\text{F.21})$$

Since entropy solutions are considered, the Oleinik-entropy condition [21] for (F.1) enforces the Lax-entropy conditions,

$$f'(u_-) \geq s \geq f'(u_+), \quad (\text{F.22})$$

whence

$$s \geq f'(u_+) > a(u_+). \quad (\text{F.23})$$

Using the second equation of (F.20), yields

$$(s - a(u_+))v_+ = (s - a(u_-))v_-, \quad (\text{F.24})$$

$$v_+ = v_- \frac{s - a(u_-)}{s - a(u_+)} := \mathcal{S}_-(u_+; U_-). \quad (\text{F.25})$$

The equation (F.25) can be interpreted in terms of the Lax shock curve. For a fixed  $U_- = (u_-, v_-)$ , the right hand side of (F.25) is only a function of  $u_+$  as  $s$  is given by (F.21) as a function of  $u_+$ . In the plane  $U = (u, v)$ , (F.25) describes the set of  $U_+$  such that the Riemann problem with initial data  $U_\pm$  is solved by a shock. On this curve, only  $U_+$  satisfying the Oleinik condition (F.22) are considered to allow an entropic shock.

Conversely, if  $U_+$  is fixed, the Lax shock curve is parametrized by  $u_-$  and reads

$$v_- = v_+ \frac{s - a(u_+)}{s - a(u_-)} := \mathcal{S}_+(u_-; U_+). \quad (\text{F.26})$$

Proceeding similarly with  $Z = v \exp(A(u))$ ,  $\tilde{U}_- = (u_-, Z_-)$  and keeping the notation  $\mathcal{S}$  in coordinates  $(u, Z)$  yields,

$$Z_+ = Z_- \frac{s - a(u_-)}{s - a(u_+)} \exp[A(u_+) - A(u_-)] := \mathcal{S}_-(u_+; \tilde{U}_-),$$

and

$$Z_- = Z_+ \frac{s - a(u_+)}{s - a(u_-)} \exp[A(u_-) - A(u_+)] := \mathcal{S}_+(u_-; \tilde{U}_+).$$

### 1-Contact discontinuity

This is a limiting case of the preceding one, when  $f'$  is constant on the interval  $[u_-, u_+]$ . The same formula for the shock wave follows.

$$s = f'(u_\pm) = f'(u) > a(u_\pm), \quad u \in [u_-, u_+], \quad (\text{F.27})$$

$$v_+ = v_- \frac{f'(u_-) - a(u_-)}{f'(u_+) - a(u_+)} = \mathcal{S}_-(u_+; U_-), \quad (\text{F.28})$$

$$Z_+ = Z_- \frac{f'(u_-) - a(u_-)}{f'(u_+) - a(u_+)} \exp[A(u_+) - A(u_-)] := \mathcal{S}_-(u_+; \tilde{U}_-). \quad (\text{F.29})$$

Such waves arise in the wave-front tracking (WFT) method when the flux is approximated by piecewise linear functions.

Let us suppose that  $f''(u) \neq 0$  in the interval  $(u_-, u_+) = \{(1 - \theta)u_- + \theta u_+, \theta \in [0, 1]\}$ . Indeed, [21, 29]:

- if  $f'' > 0$  then  $u_- < u_+$ ,
- if  $f'' < 0$  then  $u_- > u_+$ .

Since  $Z$  is a 1-Riemann invariant, the Lax rarefaction curve  $\mathcal{R}_-(\tilde{U}_-)$  is simply,

$$Z_+ = Z_-. \quad (\text{F.30})$$

Using the definition of  $Z$ , this implies  $v_+ \exp(A(u_+)) = v_- \exp(A(u_-))$  and thus the Lax rarefaction curve can be written explicitly,

$$v_+ = \mathcal{R}_-(u_+; U_-) = v_- \exp(A(u_-) - A(u_+)), \quad (\text{F.31})$$

$$v_- = \mathcal{R}_+(u_-; U_+) = v_+ \exp(A(u_+) - A(u_-)). \quad (\text{F.32})$$

Notice that  $v_+$  and  $v_-$  have the same signs. In particular, if  $v_- = 0$ ,  $v = 0$  is constant through the rarefaction wave.

## Composite waves

In general, the entropy solution  $u$  is a composite wave [14]. If  $f$  has a finite number  $N$  of inflection points, then there are at most  $N$  contact-shock waves [29, 50, 56]. The Lax curves associated to such waves are studied below in Section F.4.

## 2-Contact discontinuity

$u$  is a 2-Riemann invariant and hence is constant along the 2-contact discontinuity. Thus the Lax curve is simply a vertical line in the plane  $U = (u, v)$  or the plane  $\tilde{U} = (u, Z)$ ,

$$u_- = u_+. \quad (\text{F.33})$$

## F.4 The Lax curves

A fundamental theorem due to Lax [48] states that the shock curve and the rarefaction curve emanating from a constant state  $U_-$  in the plane  $(u, Z)$  or  $(u, v)$  have a contact of the second order for a genuinely nonlinear eigenvalue. This means that the shock curve can be replaced by the rarefaction curve up to an error of order  $[u]^3$  where  $[u] = u_+ - u_-$  [21, 29, 62]. For the triangular system, a genuinely nonlinear eigenvalue means  $f'' > 0$  (or  $f'' < 0$ ) everywhere. For nonconvex cases, typically  $f''$  locally has a finite number of roots where  $f''$  changes its sign and the Lax curves are less regular due to the occurrence of contact-shocks. Under a concave-convex assumption, which means here that  $f'''$  does not vanish, the regularity of the Lax curves is only piecewise  $C^2$  [50], see also [3, 4]. As a consequence, the error becomes of order  $[u]^2$  for the variation of  $Z$  through a contact-shock [50]. The situation is worse in general, the Lax curves are only Lipschitz [7, 42]. However, we prove that cubic estimates are still valid for the triangular system (F.1), (F.2). It is mainly due of the existence of Riemann invariant coordinates.

In this section, cubic estimates on the Lax curves for the triangular system in the plane  $(u, Z)$  are done.



Let  $(u_-, v_-)$ ,  $(u_+, v_+)$  be two constant states connected by a rarefaction or a shock wave. It is more convenient to use the Riemann invariant coordinates  $(u, Z)$ . The rarefaction curves in the plane  $(u, Z)$  are simply,

$$Z_- = Z_+, \quad (\text{F.34})$$

which means that in the  $(u, v)$  plane

$$v_- \exp(A(u_-)) = v_+ \exp(A(u_+)). \quad (\text{F.35})$$

For the shock curve, the Rankine-Hugoniot condition is written in the conservative variables  $(u, v)$ ,

$$s = \frac{[f(u)]}{[u]} = \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \quad (\text{F.36})$$

$$v_-(s - a(u_-)) = v_+(s - a(u_+)). \quad (\text{F.37})$$

Since  $f \in C^4$ ,  $s$  is a  $C^3$  function of its arguments. Moreover, fixing  $(u_+, v_+)$  and considering  $u_-$  as a variable, the Lax shock curve is  $C^3$  with respect to  $u_-$ ,

$$v_- = v_+ \frac{s - a(u_+)}{s - a(u_-)}. \quad (\text{F.38})$$

Indeed, the denominator never vanishes due to the uniform strictly hyperbolic assumption (USH) (F.6). The same regularity of the shock curve holds in the variables  $(u, Z)$

$$Z_- = Z_+ \frac{s - a(u_+)}{s - a(u_-)} \exp(A(u_-) - A(u_+)). \quad (\text{F.39})$$

Of course, the Lax rarefaction curve and the Lax shock curve has to be restricted on the subset satisfying entropy conditions. Nevertheless, we use these curves for all range of  $u_-$  in  $\mathbb{R}$  (at least for  $-M \leq u_- \leq M$ ) to obtain a generalized Lax cubic estimate for the nonconvex case.

#### F.4.1 The Lax cubic estimate on the Rankine-Hugoniot curve

The Lax cubic estimate [48] can be written as follows for a shock wave connecting  $(u_-, Z_-)$  to  $(u_+, Z_+)$  for the triangular system (F.1),(F.2), as soon as  $Z$  is bounded,

$$[Z] = \mathcal{O}([u])^3, \quad [Z] = Z_+ - Z_-, \quad [u] = u_+ - u_-. \quad (\text{F.40})$$

The Riemann invariant  $Z$  is constant along the rarefaction curves. The Lax' cubic estimate means that the shock curve and the rarefaction curve have a contact of order 2. The Lax' cubic estimate was written in a genuinely nonlinear framework [48]. This means that  $f''$  does not vanish. Indeed, the Lax' computations are still valid without this convex assumption and without only considering the entropic part of the Rankine-Hugoniot curve.

We are also used the non entropic part of the Rankine curve. It is a key point in this paper, used many times in this paper, first in the next section F.4.2 to get a cubic estimate for the Riemann problem for a non convex flux.

For the triangular system, the Rankine-Hugoniot curve is global and well defined thanks to the uniformly strictly hyperbolic assumption (F.6). For  $2 \times 2$  systems, there is not always such a global curve [46].

Now, to prove cubic estimates, we have to write the Rankine-Hugoniot curve. Here, we choose to write  $Z_-$  as a function of  $u_-$  when  $(u_+, Z_+)$  are fixed for the following reasons.

1. To solve the Riemann problem and compute the intermediary state  $v_m$  which corresponds to  $Z_m$  and here  $Z_-$ .
2. To check that the triangular system (F.1),(F.2) is not a Temple system [65].
3. To obtain the cubic estimate on the global Rankine-Hugoniot curve below.
4. To obtain the existence result, bounding  $Z$  along the 2-characteristics.
5. To build a blow-up, again computing  $Z$  from the right to the left on the 2-characteristics.

The Rankine-Hugoniot curve  $\mathcal{RH}_+$  when  $(\tilde{U}_+) = (u_+, Z_+)$  is fixed and  $Z_-$  is a function of  $u_-$  is then,

$$Z_- = Z_+ \frac{s - a(u_+)}{s - a(u_-)} \exp [A(u_-) - A(u_+)] = Z_+ r(u_-, u_+) := \mathcal{RH}_+(u_-; u_+, Z_+). \quad (\text{F.41})$$

The classic Lax' cubic estimate on the shock curve is generalized on the global Rankine-Hugoniot curve.

**Proposition F.4.1** (Cubic flatness of the global Rankine-Hugoniot curve). *If  $f'$  and  $a$  belong to  $C^3(\mathbb{R}, \mathbb{R})$  and satisfy the uniform strict hyperbolicity condition (F.6) then*

$$s = s(u_-, u_+) = \frac{[f]}{[u]} = \frac{f(u_+) - f(u_-)}{u_+ - u_-} \in C^3([-M, M]^2, \mathbb{R}),$$

$$r = r(u_-, u_+) = \frac{s - a(u_+)}{s - a(u_-)} \exp [A(u_-) - A(u_+)] \in C^3([-M, M]^2, \mathbb{R}),$$

$$r = 1 + \mathcal{O}(1) [u]^3 > 0, \quad \forall (u_-, u_+) \in [-M, M]^2, \quad (\text{F.42})$$

$$Z_- = \mathcal{O}(1) Z_+ [u]^3, \quad \forall (u_-, u_+, Z_+) \in [-M, M]^2 \times \mathbb{R}, \quad (\text{F.43})$$

where the constant  $\mathcal{O}(1)$  depends only on the derivatives of  $f'$  and  $a$  on  $[-M, M]$ . Moreover,  $Z_-$  has the same sign as  $Z_+$ , more precisely,

$$\begin{aligned} Z_+ = 0 &\Rightarrow Z_- = 0, \\ Z_+ \neq 0 &\Rightarrow Z_- Z_+ > 0. \end{aligned}$$

The cubic flatness of the Rankine-Hugoniot curve  $\mathcal{RH}_+$  is expressed in (F.43). It does not depend on the convexity of the fields, although in the classical textbooks [21, 30, 41, 50, 62, 63] some nonlinearity assumptions on the fields are given. The reason of these nonlinear assumptions on the textbooks is to introduce the rarefaction curve and the shock curve. Here, the global Rankine-Hugoniot defined for all  $u_- \in [-M, M)$  is the main subject without looking at the entropic parts of this curve. A careful reading of the classical proof of the cubic estimate in textbooks shows that it is a geometric property of the Rankine-Hugoniot curve itself, without referring at the entropic or nonlinearity assumptions. This geometric property is a consequence of the symmetry of the Rankine-Hugoniot condition with respect to  $U_-$  and  $U_+$  [62]. It is very important in this paper to prove the cubic estimate for the non entropic part of the Rankine-Hugoniot curve for two reasons.

1. The cubic estimates on the Lax' curve which is not piecewise  $C^3$  for nonconvex  $f$  [50] uses the global Rankine-Hugoniot curve in the next section F.4.2.
2. Rarefaction wave fans are replaced by weak non entropic jumps in the wave front tracking algorithm [21]. The error in the weak formulation has to be controlled by the cubic estimate to pass to the limit and get a weak solution of (F.1), (F.2).

The proof appears as a direct consequence of (F.42). An elementary and self-contained proof using only Taylor's expansions are proposed. A more tedious computation can give the more precise result.

$$r(u_-, u_+) = 1 + E[u]^3 + \mathcal{O}([u])^3, \quad (\text{F.44})$$

where  $E$  depends in a quite complicate way on the derivatives of  $f'$  and  $a$  at  $u = u_+$ . The computations of  $E$  is quite intricate and not useful here, except in the last section F.6 on the blow-up where a direct computation of  $E$  at  $u_+ = 0$  is given when  $f$  is quadratic and  $a$  is linear.

Now, Proposition F.4.1 is proven.

*Proof.* The positivity on  $r$  is a consequence of the assumption (F.6). This positivity implies that  $Z_-$  has the same sign as  $Z_+$ .

Many Taylor expansions are used to obtain (F.42).  $u_+$  is fixed and  $u_-$  is the variable near  $u_+$ . The notations  $a_- = a(u_-)$ ,  $a_+ = a(u_+)$  and so on are used to shorten the expressions.

$$\begin{aligned} a_- &= a_+ - a'_+[u] + (a''_+/2)[u]^2 + \mathcal{O}([u])^3, \\ s &= \frac{[f]}{[u]} = \frac{f_- - f_+}{-[u]} = f'_+ - (f''_+/2)[u] + (f'''_+/6)[u]^2 + \mathcal{O}([u])^3. \end{aligned}$$

The hyperbolic quantity  $h$  is used,

$$h := f' - a > 0. \quad (\text{F.45})$$

The sign of  $h$  is the consequence of the strict hyperbolicity assumption (F.6). We start with the fraction part of  $r$  and  $h_+ = f'_+ - a_+ \neq 0$ .

$$\frac{s - a_+}{s - a_-} = \frac{(f'_+ - a_+) - (f''_+/2)[u] + (f'''_+/6)[u]^2}{(f'_+ - a_+) - (f''_+/2 - a'_+)[u] + (f'''_+/6 - a''_+/2)[u]^2} + \mathcal{O}([u])^3 \quad (\text{F.46})$$

$$= 1 - \frac{a'_+}{h_+}[u] + \frac{h_+ a''_+ - a'_+ f''_+ + 2a'^2_+}{2h^2_+}[u]^2 + \mathcal{O}([u])^3 \quad (\text{F.47})$$

For the term  $\exp(-[A])$  in  $r$ , the Taylor expansion of  $A'_-$  is used at the first order.

$$\begin{aligned} -A'_- &= \frac{a'_-}{h_-} = \frac{a'_+ - a''_+[u]}{h_+ - h'_+[u]} + \mathcal{O}([u])^2 \\ &= \frac{a'_+}{h_+} + \frac{a'_+ h'_+ - a''_+ h_+}{h^2_+}[u] + \mathcal{O}([u])^2 \end{aligned}$$

Integrating with respect to  $u_-$  yields,

$$A_- - A_+ = \frac{a'_+}{h_+}[u] + \frac{a'_+ h'_+ - a''_+ h_+}{2h^2_+}[u]^2 + \mathcal{O}([u])^3$$

Since  $\exp(x) = 1 + x + x^2/2 + \mathcal{O}(x)^3$ , it yields

$$\exp[A_- - A_+] = 1 + \frac{a'_+}{h_+}[u] + \frac{a'_+ h'_+ - a''_+ h_+ + a'^2_+}{2h^2_+}[u]^2 + \mathcal{O}([u])^3 \quad (\text{F.48})$$

Now, multiplying (F.47) and (F.48) yields  $r = 1 + \mathcal{O}([u])^3$ .  $\square$

## F.4.2 Cubic estimates for the Riemann problem

Now, the intermediary state  $Z_m$  of a Riemann problem has to be estimated. For this purpose, the variation of  $Z$  along a composite wave is studied. When the flux is convex, Lax proved that the variation of  $Z$  is a cubic order of the variation of  $u$  [48]. For a non convex flux, it is well known that the Lax curve is less regular, piecewise  $C^2$  [50] or only Lipschitz [7]. However, we prove that for our triangular system we are able to keep a cubic order. This is mainly due to the existence of Riemann coordinates for  $2 \times 2$  systems and the cubic estimate for the global Rankine-Hugoniot locus, Proposition F.4.1. As a consequence, we can prove a similar estimate for the variation of  $Z$  over a composite 1-wave. This improves the well known square root estimate for concave-convex eigenvalues [50], which correspond to cubic degeneracies for  $f$ . That means that for the triangular system (F.1), (F.2) the estimate is as precise as for the convex case [48].

**Proposition F.4.2** (Variation of  $Z$  through a composite wave). *Let the states  $\tilde{U}_i = (u_i, Z_i)$ ,  $i = 1, \dots, m$ , where  $u_0 < u_1 < \dots < u_m$  ( $m \leq N_{infl} + 1$ ), comprise a composite 1-wave,  $Z_- = Z_0$  and  $Z_+ = Z_m$ , then, the total variation of  $Z$  through a 1-wave is,*

$$\|Z\|_\infty \leq |Z_+| \exp(\mathcal{O}(u_+ - u_-)^3), \quad (\text{F.49})$$

$$TVZ \leq \mathcal{O}(1)|Z_+||u_+ - u_-|^3. \quad (\text{F.50})$$

*Proof.* We note that  $Z$  is a 1-Riemann invariant and therefore it remains constant over a rarefaction wave. Moreover, if the states  $\tilde{U}_i$  and  $\tilde{U}_{i+1}$  are joined by a jump, by Proposition F.4.1 and the classic inequality  $1 + x \leq \exp(x)$  we have the estimate

$$|Z_i| = |Z_{i+1}r(u_i, u_{i+1})| = |Z_{i+1}|(1 + \mathcal{O}(u_i - u_{i+1})^3) \leq |Z_{i+1}|\exp(\mathcal{O}(u_i - u_{i+1})^3).$$

Summing up and noting that  $u_0, u_1, \dots, u_m$  are ordered, the estimate (F.49) follows,

$$\max_i |Z_i| \leq |Z_+|\exp(\mathcal{O}(u_+ - u_-)^3).$$

Now, the  $BV$  bound for  $Z$  through the composite wave is computed.

$$\begin{aligned} Z_i - Z_{i+1} &= Z_{i+1}(r(u_i, u_{i+1}) - 1) = Z_{i+1}\mathcal{O}(u_i - u_{i+1})^3 \\ TVZ &\leq \max_i |Z_i| \sum_i \mathcal{O}(u_i - u_{i+1})^3 \leq |Z_+|\mathcal{O}(u_+ - u_-)^3 \exp(\mathcal{O}(u_+ - u_-)^3). \end{aligned}$$

$u$  satisfies the maximum principle, so  $\exp(\mathcal{O}(u_+ - u_-)^3) = \mathcal{O}(1)$  which only depends on the  $L^\infty$  bound of the initial data,  $\|u_0\|_\infty$ , so the inequality (F.50) is proved.  $\square$

## F.5 Existence in $BV^s$

In this section, Theorem F.2.1 is proved using a simplified Wave Front Tracking (WFT) [21, 29, 41] algorithm for such a triangular system (F.1), (F.2). The  $BV^s$  estimates for  $u$  are a consequence of such estimates for scalar conservation laws [18]. The  $L^\infty$  bound for  $v$  and the proof that a weak solution is obtained for the triangular system are based on an approach using a  $BV^{1/3}$  regularity for  $u$ .

### F.5.1 The Wave Front Tracking algorithm

The (WFT) depends on an integer parameter  $\nu > 0$ . The approximate solutions will be denoted by  $u_\nu, v_\nu, Z_\nu$ . We use mostly the Riemann invariant coordinates  $(u_\nu, Z_\nu)$  except when passing to the limit in the weak formulation.

This algorithm is explained in many books [21, 29, 41] on hyperbolic systems. Taking advantage of the structure of the triangular system (F.1), (F.2), we will mix the (WFT) for the scalar case [28] and for systems [21, 41]. The principle is to work with piecewise constant approximations.

As in the scalar case [21, 41], the values of  $u_\nu$  is taken on a uniform grid parametrized by the integer  $\nu$ ,

$$u_\nu \in \nu^{-1}\mathbb{Z}. \tag{F.51}$$

$v_\nu$ , or equivalently  $Z_\nu$ , is not required to stay on the uniform grid,

$$v_\nu, Z_\nu \in \mathbb{R}, \tag{F.52}$$

because, we solve the exact Riemann problem to compute  $v_\nu$ , or equivalently  $Z_\nu$ . In this way, we can use the cubic estimate on the global Rankine-Hugoniot curves of Proposition F.4.1. A similar approach for a convex case was used in [39].

The initial data are approximated as follows. Let  $N_0 + 1$  be the number of constant states for the approximate initial data  $u_{0,\nu}, v_{0,\nu}$  and the corresponding  $Z_{0,\nu}$ .  $N_0$  is related to the uniform grid  $\nu^{-1}\mathbb{Z}$ . Later, to prove that  $(u_\nu, v_\nu)$  converges towards a weak solution, the following requirement is imposed on  $N_0$ ,

$$N_0 = N_{0,\nu} = \mathcal{O}(\nu). \quad (\text{F.53})$$

The approximate initial data can be chosen to satisfy the following uniform estimates with respect to  $\nu$  [18]:

$$\|u_{0,\nu}\|_\infty \leq \|u_0\|_\infty, \quad (\text{F.54})$$

$$TV^s u_{0,\nu} \leq TV^s u_0, \quad (\text{F.55})$$

$$\|v_{0,\nu}\|_\infty \leq \|v_0\|_\infty. \quad (\text{F.56})$$

Moreover,  $(u_{0,\nu}, v_{0,\nu}) \rightarrow (u_0, v_0)$  a.e. when  $\nu \rightarrow +\infty$  so, the previous inequalities become equalities at the limit  $\nu \rightarrow +\infty$ .

Now, at  $t = 0+$ ,  $N_0$  Riemann problem are solved. In general, the flux is non convex, so there are composite waves built with a succession of shock waves (or jump waves) and rarefaction waves. The shock waves are solved with the exact shock speed. The rarefaction waves are not piecewise constant, thus, they must be approximated by a series of small jumps. These jumps are non entropic shocks but are still weak solutions. The built  $u_\nu$  is a weak solution, but not an entropic one, to the scalar conservation law (F.1). Moreover, the entropy condition is recovered if the flux  $f$  is replaced by a piecewise-linear continuous flux [28]. The piecewise-linear continuous flux  $f_\nu$  coincides with  $f$  on the uniform grid [21, Ch. 6], [41, p. 70],

$$f_\nu(k/\nu) = f(k/\nu), \quad \forall k \in \mathbb{Z}. \quad (\text{F.57})$$

That means that  $u_\nu$  is the weak entropy solution of

$$\partial_t u_\nu + \partial_x f_\nu(u_\nu) = 0, \quad u_\nu(x, 0) = u_{0,\nu}(x). \quad (\text{F.58})$$

With strong compactness on  $u_\nu$ , the Kruzkov entropy solution is recovered [21, 41].

For  $v_\nu$  the situation is less simple than for  $u_\nu$ . There is an error in the weak formulation discussed in Section F.5.3. This is due to the approximation of rarefaction waves.

$$\partial_t v_\nu + \partial_x (a(u_\nu)v_\nu) = \text{Error}_\nu, \quad v_\nu(x, 0) = v_{0,\nu}(x). \quad (\text{F.59})$$

As in the case of systems of conservation laws, a consistency error remains in the weak formulation [21, 41].

At the same time, some Riemann problems interact and new Riemann problems have to be solved. We follow the approach of the scalar case, many interactions can appear at the same time. The detailed nonlinear interactions are given for the triangular system below. The process continues until the second time of interactions and so on.

Now, the well posedness of the WFT is proved. That means that the process can be continued for all time because there is only a finite number of interactions for all time. For  $u_\nu$  this is clear since this result is known for the scalar case. An explicit bound of the number of interactions is given in [41, p. 71-72]. The 2-waves, associated to the linearly degenerate eigenvalue  $a(u_\nu)$ , never interact together since they are contact discontinuities. Due to the transversality assumption (USH), a 2-wave can interact only once with a 1-wave and creates a new 2-wave (and not modifies the 1-wave). Thus, the number of such interactions and of 2-waves is finite. This proves that the WFT is well defined for all time.

Now, the approximate Riemann solver and the nonlinear interaction of waves are detailed.

### Approximate Riemann solver

In this short section, the approximate Riemann solver is detailed. The initial data are  $(u_-, Z_-)$ ,  $(u_+, Z_+)$ . For  $u_\nu$  the solution is a series of entropic jumps  $u_0 < u_1 < \dots < u_m$  for the piecewise linear flux  $f_\nu$ . For  $Z_\nu$ , there are many possibilities. We want to approximate the exact solution of the Riemann problem and keep the cubic estimates (Proposition F.4.2) which generalize the Lax cubic estimates for genuinely nonlinear waves. So, we use the exact solution of the Riemann problem and the exact Lax curve to determine  $Z_{m-1}, \dots, Z_1$ . Let  $u_0 = u_-$  and  $u_m = u_+$ . Then  $Z_i$ ,  $i = m, m-1, \dots, 2$  are built as follows:

- If the jump between  $u_{i-1}$  and  $u_i$  corresponds to an entropic jump for the exact flux  $f$  then  $Z_{i-1}$  is given by the Rankine-Hugoniot curve (F.41),

$$Z_{i-1} = \mathcal{RH}_+(u_{i-1}; u_i, Z_i). \quad (\text{F.60})$$

Thus, the jump between  $(u_{i-1}, Z_{i-1})$  and  $(u_i, Z_i)$  satisfies the Rankine-Hugoniot condition and there is no error in the weak formulation of the exact triangular system (F.1), (F.2).

- If the jump between  $u_{i-1}$  and  $u_i$  corresponds to a rarefaction for  $f$  then, necessarily,  $|u_i - u_{i-1}| = \nu^{-1}$  and we keep  $Z$  constant as for the exact solution,

$$Z_i = Z_{i-1}. \quad (\text{F.61})$$

The chosen approximate solution of the Riemann problem is a jump which does not satisfy the Rankine-Hugoniot condition. An error in the weak formulation of the exact system appears and it is controlled in a cubic way thanks to Proposition F.4.1.

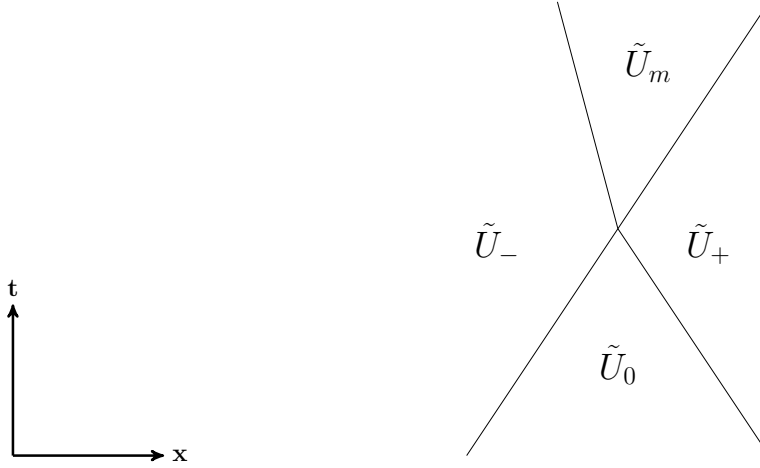


Figure F.1 – Nonlinear interaction of two waves. The 1-wave on the right crosses the interaction point with the same speed and the same value  $u_-$  on the left of the 1-wave ( $u_m = u_-$ ) and  $u_+$  on the right. On the other hand, the second wave is affected by the interaction: the speed of the 2-wave and a new value  $Z_m$  appears.

### Nonlinear wave interactions

We briefly describe the different possible nonlinear wave interactions and the details of the interactions are given in the  $(u, Z)$  plane.

In all the cases, we consider three states  $\tilde{U}_-, \tilde{U}_0, \tilde{U}_+$  before the interaction. Here  $\tilde{U}_0$  denotes the intermediary state which disappears after the interaction. The states  $\tilde{U}_-, \tilde{U}_0$  are connected by an elementary wave and similarly the states  $\tilde{U}_0, \tilde{U}_+$  are connected by an elementary wave. An elementary wave is either a 1-wave: shock or a small jump, or a 2-wave: a contact discontinuity. At a time of interaction  $t_{interact}$ , the Riemann problem is solved with a new intermediary state  $\tilde{U}_0^*$  (see Figure F.1).

We shall use the following notations.

- $S$  or  $S_1$  stands for a shock wave which is always a 1-wave.
- $R$  or  $R_1$  stands for a rarefaction wave which is always a 1-wave.
- $C_1$  or  $C_2$  stands for a contact discontinuity associated to  $\lambda_1$  or  $\lambda_2$ , a 1-wave or a 2-wave.  $C_1$  is considered as a degenerate shock  $S_1$  or as a non-entropic jump when it is used to approximate a rarefaction  $R_1$ .

The key point here is to understand the effect of the  $L^\infty$  norm of  $Z$  after the interaction. First of all, we note that there is no self interaction for the second family since there are only  $C_2$  waves. Also the case of interactions between the waves of the first family have already been well-studied (see [29],[21],[41]). We have seen in Proposition F.4.1 and Proposition F.4.2 that the change in  $L^\infty$  norm of  $Z$  is of the order of the cube of the change in  $u$  (or  $u_\nu$ ).

Finally, we consider the interaction of a 1-wave with a 2-wave.



1. In the case of an interaction of the form  $S_1 - C_2$  (which means that a 1-shock interacts with a 2-contact discontinuity), the outgoing wave is of the form  $C_2 - S_1$ . The shock continues with the same slope and the same value  $u_-, u_+$  and still satisfies the entropy condition (F.22). Thus, for  $u$  there is no change before and after the interaction. Roughly speaking, the interaction of a 1-shock with a 2-contact discontinuity is transparent for  $u$ . On the contrary, there is a change for  $Z$  and following [39], it can be shown that the change in  $L^\infty$  norm of  $Z$  is of the order of cube of the change in  $u$ .
2. The interaction  $C_1 - C_2$  (that is, when a 1-contact discontinuity interacts with a 2-contact discontinuity) generates outgoing waves of the form  $C_2 - C_1$ . This case can be dealt in a similar manner as in the case of  $S_1 - C_2$  and it follows that the change in  $L^\infty$  norm of  $Z$  is of the order of cube of the change in  $u$ .

## F.5.2 Uniform estimates

The  $BV^s$  estimates for  $u_\nu$  are already known since  $u$  is the entropy solution of the scalar conservation law (F.1). These estimates are recalled briefly in the first paragraph. The only difficulty in this section is to obtain the  $L^\infty$  estimates for  $v$ . For this purpose, we generalize the approach first used in [13, 14, 19] and recently in [39]. The approach consists in bounding  $v_\nu$ , indeed  $Z_\nu$ , along the 2-characteristics. For the chromatography system [14], the 2-characteristics are simply lines. In general, here, the 2-characteristics are piecewise lines. They are uniquely defined since the second eigenvalue  $a(u)$  is linearly degenerate and its integral curves are transverse to the discontinuity lines of the first field (assumption (USH)). The precise definition of such characteristics is given in the second paragraph. Then the estimate on  $Z_\nu$  along 2-characteristics, as in [13, 14, 19, 39], is given in the last paragraph using our generalized estimates on the Lax curves.

### $BV^s$ estimates for $u$

The  $TV^s$  decay is known for the Glimm scheme, the Godunov scheme and the Wave Front Tracking algorithm [18, 19]. Another argument is that  $u_\nu$  is also the exact entropy solution of the scalar conservation law (F.58) with the piecewise-constant initial data  $u_{0,\nu}$ , so the decay of  $TV^s u$  gives the uniform estimates with respect to  $\nu$ ,

$$TV^s u_\nu(t, \cdot) \leq TV^s u_{0,\nu} \leq TV^s u_0. \quad (\text{F.62})$$

### The 2-approximate characteristics

Essentially, for the WFT, an approximate  $i$ -characteristic is a continuous curve which is piecewise linear following the velocity  $\lambda_i$ ,  $i = 1, 2$ . Since the eigenvalues depend only on  $u$ , there is a problem to define an  $i$ -characteristic where  $u$  is not defined. For  $i = 2$ , there is no problem of uniqueness, since a 2-characteristic is always transverse to the discontinuity lines of  $u$ . Thus, the 2-characteristic crosses the  $u$  discontinuity with a kink.

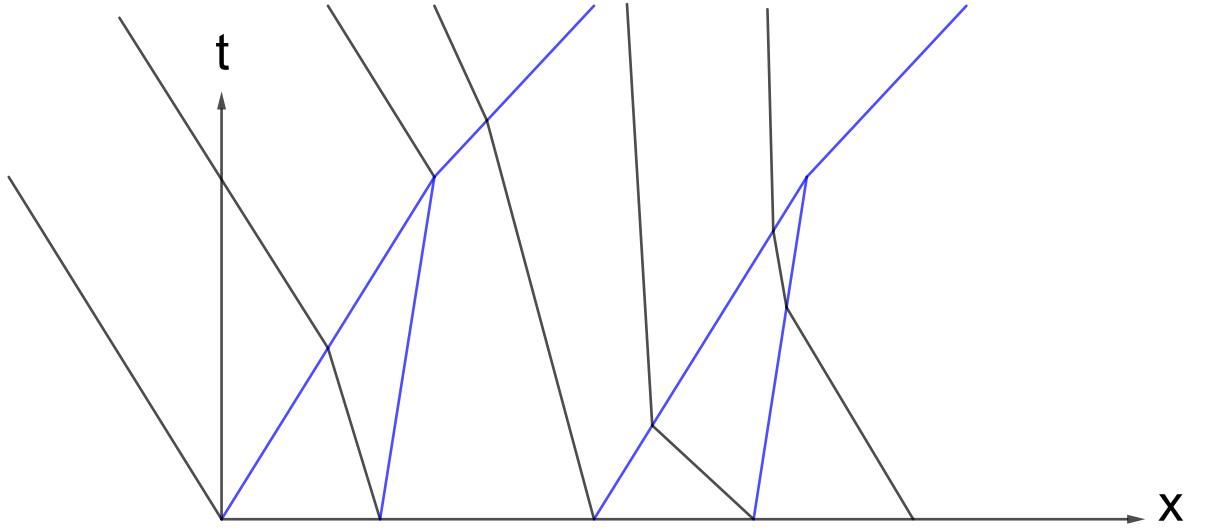


Figure F.2 – Wave front tracking algorithm. For the picture,  $f'(u) > 0 > a(u)$ , thus the 1-waves go to the right and the 2-waves to the left. The 1-wave are blue. Notice that these are not affected by the interaction with the 2-waves. The 2-waves are black. The 2-waves are affected by the interaction.

Let  $\gamma_\nu(x_0, t)$  be the forward generalized 2-characteristic starting at the point  $x_0$ , that is  $\gamma_\nu(x_0, t)$  is a solution of the differential inclusion

$$\frac{d}{dt}\gamma_\nu(x_0, t) \in [a(u_\nu(\gamma_\nu(x_0, t) - 0, t)), a(u_\nu(\gamma_\nu(x_0, t) + 0, t))], \quad \gamma_\nu(x_0, 0) = x_0.$$

For the wave front tracking, these 2-characteristics are uniquely determined and piecewise linear continuous curves thanks to the transversality assumption (USH) and satisfy the differential equation, except for a finite number of times which correspond to a jump of the piecewise constant function  $u_\nu$ ,

$$\frac{d}{dt}\gamma_\nu(x_0, t) = a(u_\nu(\gamma_\nu(x_0, t), t)), \quad \gamma_\nu(x_0, 0) = x_0. \quad (\text{F.63})$$

### $L^\infty$ estimates for $v$

The  $L^\infty$  estimate on  $v_\nu$  is first obtained on the approximate Riemann invariant  $Z_\nu$ .  $Z_\nu$  is easy to bound through a rarefaction wave as it is constant through it. However,  $Z_\nu$  is not constant through a shock wave. But, we know that the variation of  $Z_\nu$  is of order of the cube of the variation of  $u_\nu$ , Proposition F.4.2. When there is no shock, the simple curve to bound  $Z_\nu$  is the 2-characteristic. In this “smooth” case  $Z_\nu(\gamma_\nu(x_0), t) = Z_\nu(x_0, 0)$ .

Notice that  $Z_\nu$  is well defined on a 2-characteristic if the 2-characteristic does not contain a 2-wave where  $Z$  has a jump. Since the number of 2-wave fronts is finite, the number of such 2-characteristics is also finite. Hence, only 2-characteristics not touching a 2-wave is considered. This choice of 2-characteristics is enough to estimate  $\|Z_\nu\|_\infty$ .

Now, due to the transversality conditions, a 2-characteristic meets many 1-waves. A 1-wave is usually a composite wave and thanks to Proposition F.4.2 we have a cubic estimate for the  $L^\infty$  norm and the total variation of  $Z$  through a 1-wave. More precisely, if  $t_1$  is a time just before the 2-characteristic meets a 1-wave and  $t_2$  is the time just after the 2-characteristic crosses all the 1-wave, we have the following estimate,

$$\begin{aligned} \|Z_\nu\|_{\infty,(\gamma_n u(x_0,\cdot),\cdot)[t_1,t_2]} &\leq |Z_\nu(t_1)| \exp\left(\mathcal{O}\left(TV^{1/3}u_\nu(\gamma_n u(x_0,t_2),t_2)\right)\right), \\ TV Z_\nu(\gamma_n u(x_0,\cdot),\cdot)[t_1,t_2] &\leq \|Z_\nu\|_{\infty,(\gamma_n u(x_0,\cdot),\cdot)[t_1,t_2]} \mathcal{O}\left(TV^{1/3}u_\nu(\gamma_n u(x_0,\cdot),\cdot)[t_1,t_2]\right). \end{aligned}$$

That means that the  $L^\infty$  norm and the total variation of  $Z_\nu$  along the piece of curve  $\{(\gamma_n u(t,x_0),t), t \in [t_1,t_2]\}$  is controlled by  $TV^{1/3}u_\nu$  along the same piece of curve. Notice also that the sign of  $Z_\nu$  is constant along a 2-characteristic by Proposition F.43.

First, the  $L^\infty$  norm of  $Z$  is bounded on the 2-characteristic and, second, the estimation on  $TV Z_\nu$  follows. The total variation is additive and the fractional total variation is sub-additive [23], so adding all these estimates, on the whole 2-characteristic starting at  $x = x_0$   $\Gamma(x_0) = \{(\gamma_n u(x_0,t),t), t > 0\}$  we have the estimate,

$$\begin{aligned} \|Z\|_{\infty,\Gamma(x_0)} &\leq |Z_{0,\nu}(x_0)| \exp\left(\mathcal{O}\left(TV^{1/3}u_\nu[\Gamma(x_0)]\right)\right), \\ &\leq \|Z_0\|_\infty \exp\left(\mathcal{O}\left(TV^{1/3}u_\nu[\Gamma(x_0)]\right)\right), \\ TV Z_\nu[\Gamma(x_0)] &\leq \|Z_0\|_\infty \Psi\left(\mathcal{O}\left(TV^{1/3}u_\nu[\Gamma(x_0)]\right)\right), \\ \Psi(x) &= x \exp(x). \end{aligned}$$

$\Gamma(x_0)$  is the space like curve for the 1-characteristics of (F.58) so

$$TV^{1/3}u_\nu[\Gamma(x_0)] \leq TV^{1/3}u_{0,\nu} \leq TV^{1/3}u_0.$$

This yields an  $L^\infty$  and  $BV$  bound for  $Z_\nu$  along 2-characteristics uniform with respect to  $\nu$ ,

$$\|Z_\nu\|_\infty \leq \|Z_0\|_\infty \exp\left(\mathcal{O}(1)TV^{1/3}u_0\right), \quad (\text{F.64})$$

$$TV Z_\nu[\Gamma(x_0)] \leq \|Z_0\|_\infty \Psi\left(\mathcal{O}\left(TV^{1/3}u_0\right)\right). \quad (\text{F.65})$$

Notice also that the positivity is preserved. If  $\inf v_0 > 0$  then  $\inf Z_0 > 0$  and by similar arguments presented above  $\inf Z > \inf Z_0 \exp\left(\mathcal{O}(1)TV^{1/3}u_0\right) > 0$  where the constant  $\mathcal{O}(1)$  is negative. Finally, a  $L^\infty$  bound (respectively a positivity) for  $Z_\nu$  yields a  $L^\infty$  bound (respectively a positivity) for  $v_\nu$ .

We can say more about the stratified structure of  $Z$  along the 2-characteristics [16, 39]. However, for the triangular system, the  $L^\infty$  bound of  $Z_\nu$  and hence of  $v_\nu$  is enough to pass to the weak limit in (F.67) since the left hand side is linear with respect to  $v_\nu$ . The uniform  $BV$  bound on  $Z$  along the 2-characteristics can be used to recover a strong trace at  $t = 0$ , like in [14] (at  $x = 0$ ).

### F.5.3 Passage to the limit in the weak formulation

Passing to the strong-weak limit in the equation (F.2) which is linear with respect to  $v$  allows to get a weak solution. The error of consistency of the scheme has to be studied.

**Lemma F.5.1.** *The error of consistency  $E_\nu$  of the scheme is only on the transport equation (F.67) and satisfies,*

$$\partial_t u_\nu + \partial_x f_\nu(u_\nu) = 0, \quad (\text{F.66})$$

$$\partial_t v_\nu + \partial_x (a(u_\nu)v_\nu) = \text{div}_{x,t} E_\nu. \quad (\text{F.67})$$

Moreover, the following estimate holds,

$$E_\nu = \mathcal{O}(TV^{1/3}u_0) \in L^\infty((0, +\infty)_t, L^1(\mathbb{R}_x, \mathbb{R}^2)) \quad (\text{F.68})$$

This error  $E_\nu$  converges towards 0 if  $u_0$  belongs to  $BV^s$  with  $s > 1/3$  or, if the flux  $f$  is strictly convex and  $s \geq 1/3$ .

There is no error of consistency in the independant scalar equation for  $u_\nu$  (F.66) [28]. The consistency error is hidden in the flux  $f_\nu$ . If we replace the piecewise-linear flux  $f_\nu$  by the exact flux  $f$ , then the consistency error  $\partial_x(f(u_\nu) - f_\nu(u_\nu))$  appears at the right hand side of (F.66).

The convergence towards 0 of the error of consistency for systems is usually done in  $BV$ , [21, p. 126], [41, p. 305]. With less regularity in  $BV^s$ ,  $s < 1$ , new features in the estimate of the error of consistency occur.

*Proof.* The approximate solution is piecewise constant, presenting only contact discontinuities or shock waves for  $u_\nu$ . Since  $f_\nu = f$  on the grid  $\nu^{-1}\mathbb{Z}$  a weak jump solution of (F.66) is also a weak solution of the scalar conservation law with the exact flux  $f$ . Moreover, the approximate speed  $s_\nu$  equals the exact speed. If the jump is entropic for the exact flux  $f$ , then  $E_\nu = 0$  because  $v$  is chosen on the exact Lax curve. The problem of consistency occurs only when the jump is not entropic for the exact flux. That is the jump corresponds to a piecewise constant approximation of a rarefaction. In this case, let  $u_0$  be the left state,  $u_m$  the right one and  $u_i$  the intermediary states,  $i = 1, \dots, m-1$ . Then, due to the approximation of a rarefaction wave fan by many non entropic small jumps with the size  $\nu^{-1}$  we have  $|u_{i+1} - u_i| = \nu^{-1}$ . Localising the error at the front  $i$  between  $u_i$  and  $u_{i+1}$ ,  $E_\nu = \mathcal{O}(u_{i+1} - u_i)^3 = \mathcal{O}(\nu^{-3})$ , due to the error between the rarefaction curve and the Rankine-Hugoniot curve, Proposition F.4.1. Thus, adding this local error yields the estimate (F.68).

Now, if  $u_0 \in BV^s$  with  $s > 1/3$  then we can split the local error term with the notation  $p = 1/s = 3 - \eta$ ,  $\eta > 0$ .

$$|u_{i+1} - u_i|^3 = |u_{i+1} - u_i|^p |u_{i+1} - u_i|^\eta = |u_{i+1} - u_i|^p \nu^{-\eta}, \quad (\text{F.69})$$

$$\int_{\mathbb{R}} |E_\nu| dx = \mathcal{O}(1) \sum_{\text{rarefaction fronts}} |u_{i+1} - u_i|^p \nu^{-\eta} \leq \mathcal{O}(1) \nu^{-\eta} TV^s u_0. \quad (\text{F.70})$$

The sum is taken over all rarefaction wave fronts. Thus the error of consistency converges towards zero in  $L^1$  when  $\nu \rightarrow +\infty$ .

Now, consider the case  $s = 1/3$  when the exact flux is strictly convex. The number of rarefaction wave fans is not increasing and bounded by the number  $N_0$  of initial Riemann problems at  $t = 0$  in the WFT. A rarefaction wave fan with amplitude  $R$  is splitted in many fronts  $m/\nu = R$ , i.e.

$$m = \nu R.$$

The local estimate in  $L^1$  of the error over all the wave fan is, up to a constant,

$$\sum |u_{i+1} - u_i|^3 = m\nu^{-3} = R^3 m^{-2}. \quad (\text{F.71})$$

Enumerating the rarefaction wave fan with the amplitude  $R_i$  and its number of fronts  $m_i$  yields,

$$\begin{aligned} \int_{\mathbb{R}} |E_\nu| dx &\leq \mathcal{O}(1) \sum_{i \leq N_0} m_i \nu^{-3} \leq \nu^{-2} \mathcal{O}(1) \sum_{i \leq N_0} R_i \leq \mathcal{O}(1) \nu^{-2} N_0 \max_i R_i. \\ &\leq \mathcal{O}(1) \nu^{-2} N_0. \end{aligned} \quad (\text{F.72})$$

The amplitude of the rarefaction are bounded uniformly due to the previous  $L^\infty$  estimates. Thus, it suffices to take  $N_0 = N_0^\nu = \nu$  to have a vanishing error of consistency.  $\square$

## F.6 $L^\infty$ blow-up for $v$ when $u_0 \notin BV^{1/3}$

In this section, we provide a proof of Theorem F.2.2 by constructing an example of blow-up at  $t = 0+$  for the system (F.1)-(F.2). For this purpose, initial data  $u_0, Z_0$  satisfy

$$\begin{cases} u_0 \in BV^{1/3-0}(\mathbb{R}), \\ Z_0 \in L^\infty(\mathbb{R}). \end{cases}$$

That means that the  $u_0$  provided is in all  $BV^s$  for  $s < 1/3$ . Thus, the existence Theorem F.2.1 is optimal. Our idea of construction is motivated by similar examples studied in [1, 5, 26, 33, 35].

Notice that when  $v_0 \equiv 0$ , i.e.  $Z_0 \equiv 0$ , no blow-up occurs since  $(u, v) \equiv (u, 0)$  gives a global entropy solution where  $u$  the entropy solution associated to the  $L^\infty$  initial data  $u_0$ . Thus, in the following construction, we have to avoid the value 0 for  $Z$ .

Let us consider the system (F.1)-(F.2) with flux  $f(u) = \frac{u^2}{2}$ ,  $a(u) = u - 3$  and initial data  $u_0$  as described below and  $Z_0 \equiv 1$ .

Let  $x_0 = 0$  and  $x_n = 1 - \frac{1}{2^n}$ ,  $n \geq 1$ . Let  $B_n$  be chosen such that

$$x_n = x_{n-1} + 2B_n,$$

that is,  $B_n = \frac{1}{2^{n+1}}$ . Let  $b_n = \frac{1}{(n+8)^{\frac{1}{3}}}$  and we define

$$U_0(x, b, B) = b \mathbf{1}_{[0, B)}(x) - b \mathbf{1}_{[B, 2B)}(x).$$

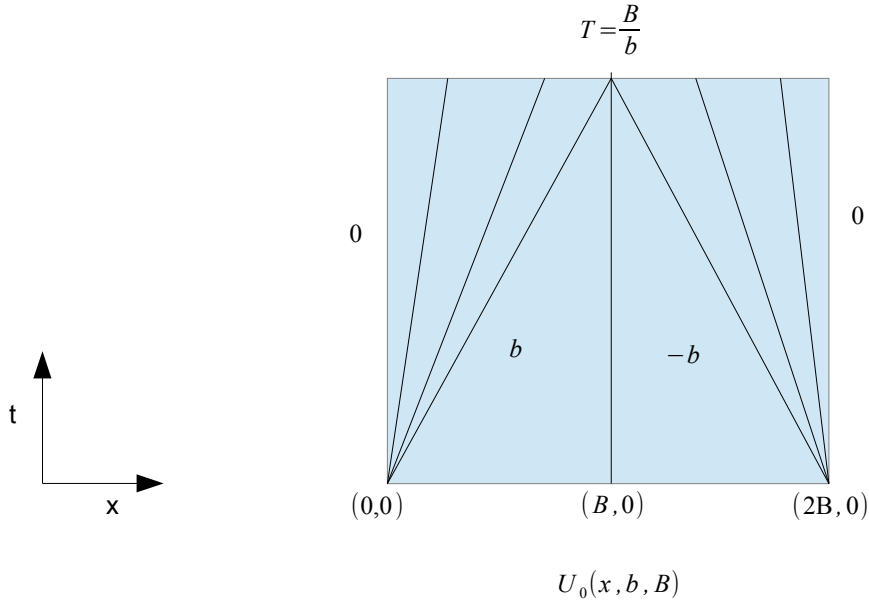


Figure F.3 – A typical building-block

Using this we define the initial data  $u_0$  as

$$u_0(x) = \sum_{n \geq 1} U_0(x - x_{n-1}, b_n, B_n).$$

The first interaction times  $T_n$  for the Riemann problems for the equation

$$\partial_t u + \partial_x f(u) = 0$$

with initial data  $u_0$  is given by

$$T_n = \frac{B_n}{b_n} = \frac{(n+8)^{\frac{1}{3}}}{2^{n+1}}.$$

Note that the first interaction time  $T_n$  satisfies the relation

$$T_n > 1 - x_n. \tag{F.73}$$

The initial data  $u_0$ , described above, clearly does not belong to  $BV^{\frac{1}{3}}(\mathbb{R})$ , but using Proposition 10 in [17], we can conclude that  $u_0 \in BV^{1/3-0}(\mathbb{R})$ .

Now as in the case of WFT described in the last section, we use the forward generalized characteristic for  $Z$ . Since  $a(u) = u - 3$  and the first interaction times of the Riemann problems for  $u$  satisfy (F.73), it follows that the forward generalized characteristic for  $Z$  starting at the point  $x_\infty := 1$  crosses infinitely many shocks before the first interaction of the waves in  $u$ .

As we have already seen from the nonlinear wave interactions, the  $L^\infty$  norm of  $Z$  does not change when it interacts with a 1-rarefaction wave.

Now, let us consider a left state  $u_-$  and a right state  $u_+$  connected by a 1-shock wave. In our example, we have  $u_- = b > 0$  and  $u_+ = -b < 0$ . Hence, the speed of the shock  $s = 0$ . Also by construction  $0 < b_n < \frac{1}{2} < 1$  and hence we assume that the prototype  $b$  satisfies the same.

Now

$$\begin{aligned}
\frac{Z_-}{Z_+} &= \frac{s - a(u_+)}{s - a(u_-)} \exp [A(u_-) - A(u_+)] \\
&= \frac{a(u_+)}{a(u_-)} \exp [A(u_-) - A(u_+)] \\
&= \frac{u_+ - 3}{u_- - 3} \exp [A(u_-) - A(u_+)] \\
&= \frac{3 + b}{3 - b} \exp [A(u_-) - A(u_+)].
\end{aligned} \tag{F.74}$$

Also

$$A'(u) = \frac{a'(u)}{a(u) - f'(u)} = \frac{1}{(u - 3) - u} = -\frac{1}{3} < 0,$$

and hence

$$A(u_-) - A(u_+) = \int_{u_+}^{u_-} A'(u) du = -\frac{1}{3}(u_- - u_+) = -\frac{2b}{3}.$$

Therefore from (F.74), we have

$$\frac{Z_-}{Z_+} = \frac{3 + b}{3 - b} e^{-\frac{2b}{3}}. \tag{F.75}$$

We show that for  $b$  positive small enough,

$$\frac{3 + b}{3 - b} e^{-\frac{2b}{3}} > 1. \tag{F.76}$$

and therefore  $Z_- > Z_+$ . Thus,  $Z$  increases in strength as the forward 2-generalized characteristic crosses a shock (from right to left).

Notice that we simply need that  $Z$  increases when  $b \sim 0$ . It is for very small oscillations that  $Z$  blows-up. Inequality (F.76) follows from a Taylor expansion up to the third order:

$$\begin{aligned}
(3 + b) \exp(-2b/3) &= (3 + b)(1 - 2b/3 + 1/2(2b/3)^2 - 1/6(2b/3)^3 + \mathcal{O}(b)^4) \\
&= 3 + (1 - 2)b + (-2/3 + 2/3)b^2 + (2/9 - 4/3^4)b^3 + \mathcal{O}(b)^4 \\
&= 3 - b + 14/81b^3 + \mathcal{O}(b)^4 \\
&> 3 - b
\end{aligned}$$

for  $b$  sufficiently small and hence  $Z_- > Z_+ > 0$ .

Moreover, Inequality (F.76) is valid for all  $b \in (0, 1)$ . To see this we consider

$$\frac{3 + b}{3 - b} e^{-\frac{2b}{3}} - 1 = \frac{(3 + b)e^{-\frac{2b}{3}} - (3 - b)}{3 - b}.$$

Define the function  $p(b) := (3+b)e^{-\frac{2b}{3}} - (3-b)$ . Then  $p(0) = 0$  and

$$\begin{aligned} p'(b) &= e^{-\frac{2b}{3}} + (3+b)\left(-\frac{2b}{3}\right)e^{-\frac{2b}{3}} + 1 \\ &= e^{-\frac{2b}{3}} + 1 - 2be^{-\frac{2b}{3}} - \frac{2b^2}{3}e^{-\frac{2b}{3}} \\ &= \underbrace{(1-2b)e^{-\frac{2b}{3}}}_{>0} + \underbrace{\left[1 - \frac{2b^2}{3}e^{-\frac{2b}{3}}\right]}_{:=g(b)} \end{aligned}$$

Now  $g(b) = 1 - \frac{2b^2}{3}e^{-\frac{2b}{3}}$  and so  $g(0) = 1$ .

And  $g'(b) = -\frac{4b}{3}e^{-\frac{2b}{3}} - \frac{2b^2}{3}\left(-\frac{2b}{3}\right)e^{-\frac{2b}{3}} = e^{-\frac{2b}{3}}\left[-\frac{4b}{3} + \frac{4b^3}{9}\right] < 0$ , as  $0 < b < 1$ . Also

$$g(1) = 1 - \frac{2}{3}e^{-\frac{2}{3}} = \frac{3-2e^{-\frac{2}{3}}}{3} > \frac{2-2e^{-\frac{2}{3}}}{3} = \frac{2}{3}(1-e^{-\frac{2}{3}}) > 0.$$

Thus,  $g(b) > 0$  for  $b \in (0, 1)$ . Using this, we see that

$$p'(b) > 0, \quad b \in (0, 1)$$

and hence  $p(b) > 0$ ,  $b \in (0, 1)$ .

Therefore Inequality (F.76) is satisfied for all  $b \in (0, 1)$  and so

$$Z_- > Z_+.$$

Thus, due to an interaction with a 1-shock wave, there is a change of order  $[u]^3$  in the  $L^\infty$  norm of  $Z$ . Since

$$|[u]^3| = \sum_{n \geq 1} \left( \frac{2}{(n+8)^{\frac{1}{3}}} \right)^3 = 8 \sum_{n \geq 1} \frac{1}{(n+8)} = +\infty,$$

we find that the  $L^\infty$  norm for  $Z$  blows up.

**Remark F.6.1.** We can also replace  $n+8$  by  $n$  in the construction above. This would mean that the interaction time for the first few waves in  $u$  would be before the forward generalized characteristic for  $Z$  reaches them, but the generalized characteristic would still have to cross infinitely many shock waves and hence our conclusion remains valid.

**Remark F.6.2.** Let  $\Gamma(x_0) = \{(\gamma(x_0, t), t), t \geq 0\}$  be the 2-characteristic issued from  $x = x_0$  at  $t = 0$ . The solution is well defined under  $\Gamma(1)$  that is on the set  $\{(x, t), x < \gamma(1, t), t \geq 0\}$ . But, over  $\Gamma(1)$ ,  $\{(x, t), 1 > x > \gamma(1, t), t > 0\}$ ,  $Z$  and  $v$  blow up,  $v = +\infty$ .

**Remark F.6.3.** This example does not contradict the Lax-Oleinik smoothing effect ([48]) as the blow-up for  $Z$  occurs only at  $t = 0$ , that is, there is an immediate blow-up. Such a blow-up is not possible for a time  $t_0 > 0$  due to the BV smoothing of  $u$ .



## Acknowledgments

A part of this work was done when the last author was visiting NISER, Bhubaneswar, India. We also thank the IFCAM project “ $BV^s$ , control & interfaces” which allowed the interaction between the Indian hyperbolic community with the French one. We also thank Boris Haspot for fruitful discussions.

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# Appendix G

## Optimal regularity for all time for entropy solutions of conservation laws in $BV^s$

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**Abstract:** This paper deals with the optimal regularity for entropy solutions of conservation laws. For this purpose, we use two key ingredients: (a) fine structure of entropy solutions and (b) fractional  $BV$  spaces. We show that optimality of the regularizing effect for the initial value problem from  $L^\infty$  to fractional Sobolev space and fractional  $BV$  spaces is valid for all time. Previously, such optimality was proven only for a finite time, before the nonlinear interaction of waves. Here for some well-chosen examples, the sharp regularity is obtained after the interaction of waves. Moreover, we prove sharp smoothing in  $BV^s$  for a convex scalar conservation law with a linear source term. Next, we provide an upper bound of the maximal smoothing effect for nonlinear scalar multi-dimensional conservation laws and some hyperbolic systems in one or multi-dimension.

**Key words:** Conservation laws, entropy solutions, shocks, smoothing effect, fractional  $BV$  spaces  $BV^s$ .

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## G.1 Introduction

For nonlinear conservation laws, it is known since Lax-Oleĭnik [36, 41] that the entropy solution can have a better regularity than the initial data for Burgers type fluxes. Such smoothing effect has been obtained in fractional Sobolev spaces [37] and recently in fractional  $BV$  space [13] for more general fluxes. The optimality of such regularization is largely open in general. For scalar 1-D conservation laws, there are some optimal results proven up to finite time [18, 25, 38]. The aim of this article is to obtain the same optimality for all time.

We start with the one-dimensional scalar conservation laws which reads as follow:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0 \quad \text{for } x \in \mathbb{R}, t > 0, \quad (\text{G.1})$$

$$u(x, 0) = u_0(x) \quad \text{for } x \in \mathbb{R}. \quad (\text{G.2})$$

The classical well-posedness theory for the Cauchy problem (G.1)–(G.2) is available for  $L^\infty$  and  $BV$  initial data [35, 36, 41].  $BV$ -regularizing effect on entropy solutions has been established in [36, 41] for uniformly convex fluxes. It is well know that if the flux function is not uniformly convex then in general, the entropy solution of (G.1) may not have a finite total variation, [4, 21]. It can be shown that in one dimension if  $f''$  vanishes at some point then there exists a class of initial data such that  $f$  can not regularize the corresponding entropy solution up to  $BV$  for all time [28]. Hence, to understand the optimal regularity of the entropy solution of (G.1), one works with more general space like fractional Sobolev space  $W^{s,p}$  and fractional  $BV$  spaces  $BV^s$ ,  $0 < s < 1$ ,  $1 \leq p$ .

The advantage of  $BV^s$  spaces is to recover the fractional Sobolev regularity  $W^{\sigma,p}$  for all  $\sigma < s$ ,  $1 \leq p < s^{-1}$  and to get the  $BV$  like trace properties of entropy solutions [23, 42, 43]. In one dimension, existence of the entropy solutions of (G.1) in  $BV^s$ , with  $BV^s$  data has been done in [13] and with  $L^\infty$  data in the same spaces in [13, 38, 39]. For non-convex fluxes a Lagrangian framework is used [8, 9]. For the scalar 1-D case, the  $BV^s$  smoothing effect corresponds to the optimal smoothing effect conjectured by Lions, Perthame and Tadmor in Sobolev spaces with the same fractional derivative  $s$  [37]. In multi-dimension, for a  $C^{2,\gamma}$  flux, it has been shown [28] that entropy solutions do not need to have fractional derivative  $s + \varepsilon$  for  $\varepsilon > 0$ . For multi-dimensional scalar conservation laws, regularizing effect in fractional Sobolev space was first studied in [37]. We refer [44] for the best known result in this direction and [31] for further improvement with some extra assumptions, see also [26] for such results with a source term. The proof of optimality of the exponent  $s > 0$  is limited to some one-dimensional scalar examples [18, 25] and before the nonlinear interaction of waves. It has been extended for the scalar multi-dimensional case in [20, 32] but not for all time. Recently, in [28] it has been shown that in multi-dimension for any  $C^2$  flux  $f$  there exists initial data  $u_0$  such that the corresponding entropy solution is not in  $BV$  for all time  $t > 0$ .

The present article resolves the following:

- In one dimension, the optimal smoothing effect in fractional  $BV$  spaces is known for the equation (G.1) in bounded strip of time  $(0, T)$  for  $T > 0$ , before the interactions

of waves [18]. So it is natural to ask the following question:

*Does there exist an entropy solution to (G.1) with compact support such that it does not belong to  $BV^{s+\varepsilon}$  for all  $t > 0$ ,  $\varepsilon > 0$ ?* (Q)

where  $s$  depends on the non-linearity of flux function. We first obtain an entropy solution to (G.1) for flux  $f(u) = (1+p)^{-1}|u|^{1+p}$  such that  $TV^{s+\varepsilon}(u(\cdot, t)) = \infty$ , for all  $\varepsilon > 0$ , for all  $t > 0$ , whereas  $TV^s(u(\cdot, t)) < \infty$  with  $s = p^{-1}$ . Later we generalize this result for a larger class of fluxes.

- We extend the above result to higher dimension under some smooth regularity assumption on the flux in section G.4.
- We are also able to answer the question (Q) for entropy solutions to balance laws which read as follow where  $\alpha \in L^\infty((0, +\infty), \mathbb{R})$ :

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \alpha(t)u \quad \text{for } x \in \mathbb{R}, t > 0, \quad (\text{G.3})$$

$$u(x, 0) = u_0(x) \quad \text{for } x \in \mathbb{R}. \quad (\text{G.4})$$

- Smoothing effect for balance laws of type (G.3) in fractional BV space is not known. Based on a recent Lax-Oleřnik type formula [5] we prove the  $BV^s$  regularizing effect for entropy solutions to (G.3) with a convex flux satisfying the  $p$ -degeneracy power law condition [13],

$$\frac{|f'(u) - f'(v)|}{|u - v|^p} \geq c_0 > 0 \quad \text{for } u \neq v \in [-M, M]. \quad (\text{G.5})$$

We recall that if  $f \in C^2([-M, M])$  then  $p \geq 1$  [13]. The exact power-law degeneracy is given by the infimum of  $p$  satisfying (G.5). When  $f$  is smooth, the infimum is a minimum [13].

- Finding an upper bound of the maximal regularity, even in a given strip  $(\mathbb{R} \times [0, T])$ , was unknown for some triangular systems of conservation laws e.g., a pressure swing adsorption system. We answer this question for a class of 1-D triangular systems and the multi-D Keyfitz-Kranzer system in section G.5 and G.6 respectively.

To provide an answer to the question (Q) we recall some of the previously constructed examples [2, 3, 4]. In Section G.3, Theorem G.3.1 provides the direct answer to (Q) for a power-law type flux function  $f(u) = (1+p)^{-1}|u|^{1+p}$ . We have discussed before that convex flux function with  $p$ -degeneracy (i.e., satisfying (G.5)) gives a regularizing effect in  $BV^s$  with  $s = 1/p$  [13]. We construct an entropy solution  $u$  to (G.1) such that  $TV^{s+\varepsilon}(u(\cdot, t)) = \infty$  for all  $t > 0$  and  $\varepsilon > 0$  with  $s = 1/p$ . Following the constructions in [3, 4] we build the entropy solution  $u$ , consisting infinitely many shock profiles in a compact interval. These shock profiles are named *Asymptotically Single Shock Packet (ASSP)* in [2]. Loosely speaking an ASSP is a solution with a special structure between two parallel lines in the half plane

$\mathbb{R}_x \times \mathbb{R}_t^+$  such that in large time only one shock curve appears between them. *ASSP* plays a role to describe structure and large time behaviour of the entropy solution to strictly convex flux [2]. For the more complex structures of solutions for non-convex fluxes we refer interested reader to [8, 9]. The construction is done in Section G.3. The building block of such solutions has a support in half strip  $[a, b] \times \mathbb{R}_+$  for some  $a < b$  and having an oscillation of amount  $\delta_n$  up to time  $t_n$ . Then we club all of these building-blocks to get a solution with the same regularity for all time. Similar type of constructions for a slightly different aspect have been used in [1, 27] to build non *BV* solutions of scalar conservation laws with discontinuous fluxes. A larger class of non-uniformly convex fluxes has been considered in [4] to build non-*BV* solutions for all time. Such a flux  $f$  satisfies,

$$0 < f'(a) - f'(b) \leq C(a - b)^q \text{ for all } b < \theta_f < a \quad (\text{G.6})$$

where  $q > 1$  and  $C > 0$ , which implies that  $f''(\theta_f) = 0$ . Condition (G.6) is about a minimal degeneracy condition of  $f'$  near the point where  $f''$  vanishes. We answer the question (Q) for this general class of convex functions satisfying (G.6).

The notion of *ASSP* has been generalized recently in [5] for balance laws of type (G.3). Based on a Lax-Oleñik type formula [5] for entropy solutions to (G.3), we are able to answer analogous version of (Q) for such a balance law with linear source term. In Section G.3 we provide a construction to show the optimality of the regularizing effect in balance law set up for power-law type flux functions. Like the case for  $\alpha \equiv 0$ , the constructed solution for balance law is a juxtaposition of infinitely many *ASSP*. Naturally, for such balance laws, the boundaries of *ASSP* are curves instead straight lines. Moreover, when  $\alpha \equiv 0$ , the case of conservation laws is recovered. We choose to answer to the question (Q) in this slightly more general setting.

In the remainder of the paper, Sections G.4, G.5 and G.6, the results obtained for the one dimensional scalar case are used to bound the maximal smoothing effect for solutions of three different problems, namely, scalar multidimensional equations, a class of triangular systems and a multidimensional system. For the multidimensional case, planar waves are used as in [20, 28, 32]. For a class of triangular systems involving a transport equation, the main problem is to keep the linear component bounded and not being a  $\delta$ - shock while the nonlinear component belongs to the critical *BV* space. For multi-dimensional Keyfitz-Kranzer system [34], it has been shown, [24] that small *TV* bound of initial data is not enough to get immediately a *BV* renormalized solution of the Keyfitz-Kranzer system. In this article we implement his construction to get a similar blow up in all  $BV^s$  spaces,  $s > 0$ .

## G.2 Fractional *BV* spaces, $BV^s$ , $0 < s \leq 1$

In this section, the definition of generalized  $BV(\mathbb{R})$  spaces are recalled [40]. Then the multi-D case is stated.

**Definition G.2.1** ( $BV^s(\mathbb{R}, \mathbb{R})$ ). *Let  $p = 1/s$ , the  $TV^s$  variation also called the total  $p$ -*

variation of any real function  $v$  is:

$$\mathrm{TV}^s v = \sup_{\{x_i\} \in \mathcal{P}} \sum_{i=2}^n |v(x_i) - v(x_{i-1})|^p \quad (\text{G.7})$$

where  $\mathcal{P} = \{\{x_1, \dots, x_n\}, x_1 < \dots < x_n, 2 \leq n \in \mathbb{N}\}$  is the set of subdivisions of  $\mathbb{R}$ .

The space  $BV^s(\mathbb{R}, \mathbb{R})$  is the subset of real functions such that,

$$BV^s(\mathbb{R}) = \{v, \mathrm{TV}^s(v) < \infty\}. \quad (\text{G.8})$$

Notice that  $BV^1 = BV$  and  $BV^s \subset L^\infty$  for all  $0 < s \leq 1$ . By convention, we set  $BV^0 = L^\infty$ . A similar definition can be used to defined  $BV^s(I, \mathbb{R})$  where  $I \subset \mathbb{R}$ , only considering the subdivisions of  $I$ . The fractional Sobolev space  $W^{s,p}$  can be defined as follows:

**Definition G.2.2.** Let  $\Omega \subset \mathbb{R}^N$  be open. Let  $s \in (0, 1)$  and  $p \in [1, \infty)$ . By  $W^{s,p}(\Omega)$  we denote the set of all  $u \in L^p(\Omega)$  such that

$$\frac{|u(x) - u(y)|}{|x - y|^{s + \frac{N}{p}}} \in L^p(\Omega \times \Omega). \quad (\text{G.9})$$

It is worth mentioning that  $BV^s$  does not coincide with fractional Sobolev space,  $W^{s,p}$  but it is closely related to  $W^{s,p}$  with the critical exponent  $p$  for the traces theory, that is,  $sp = 1$ . More precisely, for all  $\varepsilon > 0$ ,  $BV_{loc}^s \subset W_{loc}^{s-\varepsilon, 1/s} \subset W_{loc}^{s-\varepsilon, 1}$  [13]. All the examples valid for all times in this article present shocks, so are discontinuous and therefore never belong to  $W^{s+\varepsilon, 1/s}$ ,  $\forall \varepsilon > 0$ . Thus, a non  $BV^s$  regularity corresponds to a non Sobolev regularity with the same exponent up to any positive  $\varepsilon$ . The optimality can also be studied in  $BV^s$  and corresponds to the similar Sobolev optimaty. Notice also that the estimates in fractional  $BV$  spaces can be simpler than in fractional Sobolev spaces as in [17]. It is the reason why only result in  $BV^s$  are given in this paper.

Furthermore,  $BV^s$  regularity guarantees left and right traces like  $BV$  functions. That is why  $BV^s$  spaces seem more well fitted to study the regularity of the solutions of conservation laws than the corresponding Sobolev spaces.

To extend the definition of  $BV^s$  for the multi-D case, a theorem characterizing  $BV^s$  in 1-D is used. This theorem characterizes the space  $BV^s$  with the Holder space  $\mathrm{Lip}^s$  and the  $BV$  space. It is due to Michel Bruneau [16].

**Theorem G.2.1** (Bruneau, 1974). For any  $u \in BV^s$  there exists the following factorization by a  $s$ -Holder function and a  $BV$  function,

$$u \in BV^s \Leftrightarrow \exists L \in \mathrm{Lip}^s(\mathbb{R}, \mathbb{R}), \exists v \in BV(\mathbb{R}) \text{ s.t. } u = L \circ v.$$

That means that

$$BV^s(\mathbb{R}, \mathbb{R}) = \mathrm{Lip}^s(\mathbb{R}, \mathbb{R}) \circ BV(\mathbb{R}, \mathbb{R}).$$

In order to define  $BV^s(\mathbb{R}^m)$ , we recall the definition of  $BV(\mathbb{R}^m)$  for  $m \geq 1$

**Definition G.2.3** ( $BV(\mathbb{R}^m)$ ). A function  $u$  belongs to  $BV(\mathbb{R}^m)$  if there exists a Radon measure  $\mu$  such that

$$\int_{\mathbb{R}} u(x) \operatorname{div} \phi(x) dx = - \langle \mu, \phi \rangle \quad \forall \phi \in \mathcal{C}_c^1(\mathbb{R}^m).$$

Now, the following natural definition of  $BV^s(\mathbb{R}^m)$  is proposed for  $m \geq 1$ .

**Definition G.2.4** ( $BV^s(\mathbb{R}^m)$ ). A function  $u$  belongs to  $BV^s(\mathbb{R}^m)$  if there exists the following factorization by an  $s$ -Holder function  $L \in \operatorname{Lip}^s(\mathbb{R}, \mathbb{R})$  and a  $BV(\mathbb{R}^m, \mathbb{R})$  function  $v$  such that

$$u = L \circ v.$$

That means that

$$BV^s(\mathbb{R}^m, \mathbb{R}) = \operatorname{Lip}^s(\mathbb{R}, \mathbb{R}) \circ BV(\mathbb{R}^m, \mathbb{R}). \quad (\text{G.10})$$

This definition can be extended to  $BV_{loc}^s(\mathbb{R}^m)$  by:

$$BV_{loc}^s(\mathbb{R}^m, \mathbb{R}) = \operatorname{Lip}^s(\mathbb{R}, \mathbb{R}) \circ BV_{loc}(\mathbb{R}^m, \mathbb{R}). \quad (\text{G.11})$$

Notice that the Holder function has to be globally on  $\mathbb{R}$  an Holder function since  $BV(\mathbb{R}^m)$  is not a subset of  $L^\infty$  for  $m > 1$ .

This definition is valid for  $m = 1$  thanks to Bruneau's Theorem G.2.1. Moreover, a  $BV_{loc}^s(\mathbb{R})$  1-D function can be also considered as a  $BV_{loc}^s(\mathbb{R}^m)$  multi-D function by the following lemma.

**Lemma G.2.1.** Let  $\xi \in S^{m-1}$  and  $U(X) = u(\xi \cdot X)$ ,  $U \in BV_{loc}^s(\mathbb{R}^m)$  if and only if  $u \in BV_{loc}^s(\mathbb{R})$

*Proof.* From the Bruneau's Theorem G.2.1, slightly extended on bounded set,  $u(x) = L(v(x))$  where  $v \in BV_{loc}(\mathbb{R})$ . Let  $V(X)$  be  $v(\xi \cdot X)$ .  $V$  belongs to  $BV_{loc}(\mathbb{R}^m)$  [6, 29]. Thus  $U(X) = L(V(X))$  belongs to  $BV_{loc}^s(\mathbb{R}^m, \mathbb{R})$ . The converse is also clear.  $\square$

### G.3 Sharp regularity for scalar 1D entropy solutions

In this section, we will build some examples to show the optimality of smoothing effect in  $BV^s$  for all time. This regularity has been obtained in [13, 30, 17, 38, 39]. The optimality for all time is new. For that purpose, we consider the flux  $f(u) = |u|^{p+1}/(p+1)$  so  $f'(u) = u|u|^{p-1}$ . It is shown that for  $u_0 \in L^\infty$ , the solution becomes instantly in  $BV_{loc}^s$ , with  $s = p^{-1}$ . Theorem G.3.1 stated below shows that the regularizing in  $BV^s$  space is optimal for all time since there exist entropy solutions  $u$  such that  $u(\cdot, t) \notin BV^{s+\varepsilon}$  for all  $\varepsilon > 0$  and for all  $t > 0$ . The construction of this example is similar to the one done in [2] to show infinitely many shock profile occurrence in compact interval. Similar construction has been also used in [4] to show the existence of an entropy solution which does not belong to  $BV$  for all time. Here we use it to show the existence of an entropy solution which is exactly in  $BV^s$  with  $s = p^{-1}$  for all time  $t > 0$  with no more regularity.

**Theorem G.3.1.** *There exists compactly supported initial data  $u_0 \in L^\infty(\mathbb{R})$  such that the corresponding entropy solution  $u(\cdot, t) \in L^\infty(\mathbb{R} \times [0, \infty))$  of the scalar conservation law (G.1) with the flux  $f(u) = |u|^{p+1}/(p+1)$ ,  $p \geq 1$  satisfies for all  $t > 0$ , for all  $\varepsilon > 0$  with  $s = p^{-1}$ ,*

$$TV^s u(\cdot, t) < +\infty = TV^{s+\varepsilon} u(\cdot, t).$$

Theorem G.3.1 can be seen as a particular case (that is,  $\alpha \equiv 0$ ) of the following result stated in context of balance laws.

**Theorem G.3.2.** *There exists an initial data  $u_0 \in L^\infty(\mathbb{R})$  such that the corresponding entropy solution to balance law (G.3) with flux  $f(u) = (1+p)^{-1}|u|^{1+p}$  for  $p > 1$  and  $\alpha \in L^\infty(0, \infty)$  satisfies the following with  $s = p^{-1}$*

$$TV^s(u(\cdot, t)) < \infty = TV^{s+\varepsilon}(u(\cdot, t)) \text{ for all } t > 0 \text{ and } \varepsilon > 0. \quad (\text{G.12})$$

Theorem G.3.2 also states about the regularizing of entropy solution corresponding to a particular initial data  $u_0$  and flux  $f(u) = (p+1)^{-1}|u|^{p+1}$ . Next we will show that it is not restricted to a special choice of data and flux. If a flux satisfies a  $p$ -degeneracy condition like (G.5) then regularizing is valid for any  $L^\infty$  initial data. More precisely, we have the following

**Theorem G.3.3.** *Let  $f \in C^1(\mathbb{R})$  be a convex flux satisfying a power-law condition (G.5) and super linear growth condition (G.15). Let  $\alpha \in L^\infty(0, \infty)$ . Let  $u_0 \in L^\infty(\mathbb{R})$ . Let  $u$  be the entropy solution of the initial value problem for the balance law (G.3), with the initial data  $u_0$  (G.4), then*

$$u(\cdot, t) \in BV_{loc}^s(\mathbb{R}) \text{ for } s = \frac{1}{p} \text{ and } \forall t > 0. \quad (\text{G.13})$$

As we have discussed before for  $\alpha \equiv 0$  case, that for entropy solutions to (G.1), uniformly convex flux regularizes the solution in  $BV$  space [36, 41] and it fails once we drop the uniform convexity assumption on flux function [4, 21]. As a natural extension, one can ask for the regularizing effect for strictly convex fluxes and it has been shown in [13] that regularizing is valid in fractional  $BV$  space once the flux satisfying a  $p$ -degeneracy condition (G.5). For strictly convex Lipschitz flux, regularizing effect can be obtained in more general spaces like  $BV^\Phi$  with a special choice of  $\Phi$ , [30]. To prove  $TV^{s+\varepsilon}(u(\cdot, t)) = \infty$  for all  $t > 0$  we construct an entropy solution consisting  $ASSP$ 's (see [2] for more detail on  $ASSP$ ). The other part, that is,  $u \in BV^s$  follows from [13] in the case of Theorem G.3.1 that is, when  $u$  solves (G.1). But for balance law of type (G.3) no such result exists. It can be proved in a similar fashion as it was done in [13] for conservation laws. We first give a brief sketch of the proof for  $u \in BV^s$  where  $u$  is the entropy solution to balance law (G.3). In order to do that let us first recall some of the definitions and results from [5].

**Definition G.3.1.** *Let  $\alpha \in L^\infty(0, \infty)$ . Let  $\beta$  be primitive of  $\alpha$ , that is,*

$$\beta(t) = \int_0^t \alpha(\theta) d\theta. \quad (\text{G.14})$$

Suppose that the flux  $f$  is having super-linear growth, that is,

$$\lim_{|v| \rightarrow \infty} \frac{f(v)}{|v|} = \infty. \quad (\text{G.15})$$

We define  $\Psi : \mathbb{R} \times \mathbb{R}_+ \rightarrow \mathbb{R}$ -function as follows:

$$x = \int_0^t f'(\Psi(x, t)e^{\beta(\theta)}) d\theta \text{ for each } x \in \mathbb{R}. \quad (\text{G.16})$$

Note that  $\Psi$  in (G.16) is well-defined on  $\mathbb{R} \times \mathbb{R}_+$  due to super-linear growth (G.15) of  $f$  (see [5]). For  $\alpha \equiv 0$  and strictly convex  $C^1$  flux  $f$ , the  $\Psi$ -function is nothing but  $(f')^{-1}(x/t)$ . As it is observed in [5], the map  $x \mapsto \Psi(x, t)$  is increasing for strictly convex flux  $f$ .

**Proposition G.3.1.** ([5]) *Let  $\alpha \in L^\infty(\mathbb{R}_+)$  and the flux  $f \in C^1(\mathbb{R})$  satisfying (G.15). Let  $u$  be the entropy solution to (G.3) with initial data  $u_0 \in L^\infty(\mathbb{R})$ . Then  $u$  satisfies*

$$u(x, t) = e^{\beta(t)} \Psi(x - y(x, t), t) \text{ for all } x \in \mathbb{R}, t > 0 \quad (\text{G.17})$$

for some function  $y$  such that  $x \mapsto y(x, t)$  is non-decreasing and  $\beta$  is defined as in (G.14). Moreover, for each  $T > 0$  there exists a constant  $C(T)$  such that

$$|x - y(x, t)| \leq C(T)t \text{ for all } t \in [0, T]. \quad (\text{G.18})$$

Since  $\Psi$  is increasing in its first variable we have the following lemma.

**Lemma G.3.1.** *Let  $f \in C^1$  be a convex flux satisfying the super linear growth condition (G.15) and power-law condition (G.5) with  $p \geq 1$ . Then for any  $z_1, z_2 \in \mathbb{R}$  we have with  $s = p^{-1}$ ,*

$$|\Psi(z_1, t) - \Psi(z_2, t)| \leq \left( \frac{|z_1 - z_2|}{c_0 \gamma(t)} \right)^s, \quad (\text{G.19})$$

with

$$\gamma(t) := \int_0^t e^{p\beta(\theta)} d\theta \text{ where } \beta \text{ is defined as in (G.14)}. \quad (\text{G.20})$$

*Proof.* Fix two points  $z_1, z_2 \in \mathbb{R}$ . Without loss of generality, we can assume  $z_1 > z_2$ . Since  $\Psi$  is increasing in its first variable, we have

$$\Psi(z_1, t)e^{-\beta(t)} \geq \Psi(z_2, t)e^{-\beta(t)}. \quad (\text{G.21})$$

Since  $f'$  satisfies  $p$ -degeneracy condition (G.5) and  $f'$  is continuous, that means that  $f'$  is monotone. Assume that  $f'$  is increasing, so the absolute values are skipped in (G.5) and,

$$f'(a) - f'(b) \geq c_0(a - b)^p \text{ for } a \geq b. \quad (\text{G.22})$$

Therefore, by the definition (G.16) of  $\Psi$  we also have for  $z_1 > z_2$ ,

$$|z_1 - z_2| = z_1 - z_2 = \int_0^t f'(\Psi(z_1, t)e^{\beta(\theta)}) d\theta - \int_0^t f'(\Psi(z_2, t)e^{\beta(\theta)}) d\theta \quad (\text{G.23})$$

$$\geq \int_0^t c_0 (\Psi(z_1, t) - \Psi(z_2, t))^p e^{p\beta(\theta)} d\theta \quad (\text{G.24})$$

$$= c_0 |\Psi(z_1, t) - \Psi(z_2, t)|^p \int_0^t e^{p\beta(\theta)} d\theta. \quad (\text{G.25})$$

This proves the inequality (G.19).  $\square$

Now we are ready to prove the regularity result for entropy solution to balance laws (G.3).

*Proof of Theorem G.3.3.* Fix a partition between  $a = x_0 < x_1 < \dots < x_m = b$ . By Proposition G.3.1, we have

$$\sum_{k=1}^m |u(x_k, t) - u(x_{k-1}, t)|^p = e^{p\beta(t)} \sum_{k=1}^m |\Psi(x_k - y(x_k, t), t) - \Psi(x_{k-1} - y(x_{k-1}, t), t)|^p. \quad (\text{G.26})$$

By virtue of Lemma G.3.1 we have

$$\sum_{k=1}^m |u(x_k, t) - u(x_{k-1}, t)|^p \leq e^{p\beta(t)} (c_0 \gamma(t))^{-1} \sum_{k=1}^m |x_k - y(x_k, t) - x_{k-1} - y(x_{k-1}, t)|. \quad (\text{G.27})$$

Since  $x \mapsto y(x, t)$  is increasing for each fixed  $t$ , we have

$$\begin{aligned} \sum_{k=1}^m |u(x_k, t) - u(x_{k-1}, t)|^p &\leq e^{p\beta(t)} (c_0 \gamma(t))^{-1} [b - a + y(b, t) - y(a, t)] \\ &\leq e^{p\beta(t)} (c_0 \gamma(t))^{-1} [2(b - a) + 2C(T)t] \end{aligned} \quad (\text{G.28})$$

for all  $t \in (0, T)$ . The last line follows from the inequality (G.18). This completes the proof of Theorem G.3.3.  $\square$

Our next aim is to establish the optimality of Theorem G.3.3 for all time  $t > 0$  and for that we restrict our discussion for power-law type fluxes, more precisely,  $f(u) = (p + 1)^{-1}|u|^{p+1}$  for  $p > 1$ .

*Proof of Theorem G.3.2.* To set the path for constructing an entropy solution to (G.3) which does not belong to  $BV_{loc}^s(\mathbb{R})$  for  $s > 1/p$ , we first observe the structure of entropy solution to the following initial data.

$$u_0^n(x) = \begin{cases} 0 & \text{for } x < x_n - \Delta x_n, \\ \delta_n & \text{for } x_n - \Delta x_n < x < x_n, \\ -\delta_n & \text{for } x_n < x < x_n + \Delta x_n, \\ 0 & \text{for } x_n + \Delta x_n < x \end{cases} \quad (\text{G.29})$$



for  $\delta_n, \Delta x_n > 0$  and  $x_n \in \mathbb{R}$ . For  $f(u) = (p+1)^{-1}|u|^{p+1}$ ,  $\Psi$  has the following form

$$\Psi(x, t) = x|x|^{-\frac{p-1}{p}} \gamma(t)^{-\frac{1}{p}} \quad (\text{G.30})$$

where  $\gamma(t)$  is defined as in (G.20). With the help of results from [2, 5] we have the following observations

1. Consider a Riemann problem  $w_C^0$  defined as follows

$$w_C^0 = \begin{cases} w_- & \text{for } x < x_0, \\ w_+ & \text{for } x > x_0, \end{cases} \quad \text{where } w_- > w_+. \quad (\text{G.31})$$

The entropy solution,  $w_C$  to (G.3) corresponding to Riemann data  $w_C^0$  has the following form

$$w_C = \begin{cases} w_- e^{\beta(t)} & \text{for } x < x_0 + \lambda(t), \\ w_+ e^{\beta(t)} & \text{for } x > x_0 + \lambda(t), \end{cases} \quad (\text{G.32})$$

for  $t > 0$  where  $\lambda(t)$  is defined as follows:

$$\lambda(t) := \frac{1}{w_+ - w_-} \int_0^t [f(w_+ e^{\beta(\theta)}) - f(w_- e^{\beta(\theta)})] e^{-\beta(\theta)} d\theta. \quad (\text{G.33})$$

2. Next we consider a special data  $w_L^0$  defined as follows:

$$w_L^0(x) = \begin{cases} 0 & \text{for } x < x_L, \\ \delta_n & \text{for } x > x_L, \end{cases} \quad (\text{G.34})$$

where  $\delta_n > 0$ . Then entropy solution to (G.3) with initial data  $w_L^0$  will look like

$$w_L(x, t) = \begin{cases} 0 & \text{for } x < x_L, \\ \Psi(x - x_L, t) e^{\beta(t)} & \text{for } x_L \leq x \leq \zeta_L(t), \\ \delta_n e^{\beta(t)} & \text{for } x > \zeta_L(t), \end{cases} \quad (\text{G.35})$$

for  $t > 0$  where  $\zeta_L(t)$  are determined by

$$\Psi(\zeta_L(t) - x_L, t) = \delta_n. \quad (\text{G.36})$$

3. Now consider the following data

$$w_R^0(x) = \begin{cases} -\delta_n & \text{for } x < x_R, \\ 0 & \text{for } x > x_R, \end{cases} \quad (\text{G.37})$$

where  $\delta_n > 0$ . Then entropy solution to (G.3) will look like

$$w_R(x, t) = \begin{cases} -\delta_n e^{\beta(t)} & \text{for } x < \zeta_R(t), \\ \Psi(x - x_R, t) e^{\beta(t)} & \text{for } \zeta_R(t) \leq x \leq x_R, \\ 0 & \text{for } x > x_R, \end{cases} \quad \text{for } t > 0, \quad (\text{G.38})$$

where  $\zeta_R$  is determined by

$$\Psi(\zeta_R(t) - x_R, t) = -\delta_n. \quad (\text{G.39})$$

Let us set  $\bar{x}_0 := x_n$ ,  $x_L := x_n - \Delta x_n$  and  $x_R := x_n + \Delta x_n$ . Suppose the corresponding  $\zeta_L(t)$  and  $\zeta_R(t)$  intersect each other at  $(\tilde{x}_n, t_n)$  for the first time. From (G.35) and (G.38) we observe that  $x_n - \Delta x_n = x_L \leq \tilde{x}_n \leq x_R = x_n + \Delta x_n$ . By using (G.30), (G.36) and (G.39) we have

$$(\tilde{x}_n - x_n + \Delta x_n)^{\frac{1}{p}} \gamma(t_n)^{-\frac{1}{p}} = \delta_n = (x_n + \Delta x_n - \tilde{x}_n)^{\frac{1}{p}} \gamma(t_n)^{-\frac{1}{p}}. \quad (\text{G.40})$$

Hence, we get  $\tilde{x}_n = x_n$ . Subsequently, we obtain  $(\Delta x_n)^{\frac{1}{p}} \gamma(t_n)^{-\frac{1}{p}} = \delta_n$ . Recall definition of  $\gamma(t)$  as in (G.20). Therefore,  $t_n$  is determined as follows,

$$\int_0^{t_n} e^{p\beta(\theta)} d\theta = \frac{\Delta x_n}{\delta_n^p}. \quad (\text{G.41})$$

Suppose  $B^*$  is the integration of  $e^{\beta}$  over  $\mathbb{R}_+$ , that is,

$$B^* := \int_0^{\infty} e^{p\beta(\theta)} d\theta = \gamma(+\infty). \quad (\text{G.42})$$

Note that  $B^* \in (0, \infty]$ . For  $\alpha \equiv 0$  case, we have  $B^* = \infty$ . Next consider the case when  $B^* < \infty$  and

$$\frac{\Delta x_n}{\delta_n^p} \geq B^*. \quad (\text{G.43})$$

In this case, we have the following feature which does not arise for solutions of (G.1):

$$\zeta_L(t) < x_n < \zeta_R(t) \text{ for all } t \in (0, \infty). \quad (\text{G.44})$$

Summarizing these observations we have

- O1.** If  $\frac{\Delta x_n}{\delta_n^p} < B^*$  then (G.41) has a unique solution in  $(0, \infty)$ .
- O2.** If  $\frac{\Delta x_n}{\delta_n^p} \geq B^*$  then (G.41) has no solution in  $(0, \infty)$ . In this case, we set  $t_n = \infty$ , that is to say that  $\zeta_L$  and  $\zeta_R$  never meet with each other.

If  $t_n < \infty$  then note that for  $t > t_n$  we have

$$\Psi(x_n - x_L, t) = -\Psi(x_n - x_R, t). \quad (\text{G.45})$$

Therefore we have the following structure of entropy solution  $u$ , to (G.3) with initial data  $u_0$  as in (G.29):

1. For  $0 < t < t_n$  we have

$$u^n(x, t) = \begin{cases} 0 & \text{for } x < x_n - \Delta x_n, \\ \Psi(x - (x_n - \Delta x_n), t) e^{\beta(t)} & \text{for } x_n - \Delta x_n < x < \zeta_L(t), \\ \delta_n e^{\beta(t)} & \text{for } \zeta_L(t) < x < x_n, \\ -\delta_n e^{\beta(t)} & \text{for } x_n < x < \zeta_R(t), \\ \Psi(x - (x_n + \Delta x_n), t) e^{\beta(t)} & \text{for } \zeta_R(t) < x < x_n + \Delta x_n, \\ 0 & \text{for } x_n + \Delta x_n < x. \end{cases} \quad (\text{G.46})$$

2. For  $t > t_n$  we have

$$u^n(x, t) = \begin{cases} 0 & \text{for } x < x_n - \Delta x_n, \\ \Psi(x - (x_n - \Delta x_n), t)e^{\beta(t)} & \text{for } x_n - \Delta x_n < x < x_n, \\ \Psi(x - (x_n + \Delta x_n), t)e^{\beta(t)} & \text{for } x_n < x < x_n + \Delta x_n, \\ 0 & \text{for } x_n + \Delta x_n < x. \end{cases} \quad (\text{G.47})$$

Note that  $TV^{s+\varepsilon}(u^n(\cdot, t)) \geq (2\delta_n)^{\frac{1}{s+\varepsilon}} e^{\frac{\beta(t)}{s+\varepsilon}}$  for  $t \in [0, t_n]$ . From the above discussion we know that support of the entropy solution  $u_n(\cdot, t)$  lies in  $[x_n - \Delta x_n, x_n + \Delta x_n]$  for all time  $t > 0$ . We choose  $\Delta x_n = (n \log^2(n+1))^{-1}$  and  $\delta_n = (n \log^3(n+1))^{-\frac{1}{p}}$ . Subsequently, we have

$$\frac{\Delta x_n}{\delta_n^p} = \log(n+1) \rightarrow \infty \text{ as } n \rightarrow \infty. \quad (\text{G.48})$$

Since  $\sum_{n=1}^{\infty} \Delta x_n < \infty$  we can choose a sequence  $x_n$  such that  $x_n + \Delta x_n < x_{n+1} - \Delta x_{n+1} < x^* < \infty$  for all  $n \geq 1$ . Now we define an initial data  $u_0$  as follows

$$u_0 = \sum_{n=1}^{\infty} u_0^n. \quad (\text{G.49})$$

Note that by previous observation and choice of  $x_n$ , entropy solutions  $u_n$  has mutually disjoint support for all  $t > 0$ . Therefore, the entropy solution  $u$  of (G.3) corresponding to initial data  $u_0$  can be written as

$$u(x, t) = \sum_{n=1}^{\infty} u_n(x, t) \text{ for all } x \in \mathbb{R}, t > 0. \quad (\text{G.50})$$

Recall observations (O1.) and (O2.). Hence, for each fixed  $t \in (0, \infty)$  there exists an  $n_0$  such that  $t < t_n$  for all  $n \geq n_0$  due to (G.48). From definition (G.14) of  $\beta(t)$  we have  $\beta(t) \geq -t\|\alpha\|_{L^\infty(\mathbb{R}_+)}$  for all  $t \geq 0$ . Therefore, we have

$$TV^{s+\varepsilon}(u(\cdot, t)) \geq 2^{\frac{1}{s+\varepsilon}} e^{\frac{\beta(t)}{s+\varepsilon}} \sum_{n=n_0}^{\infty} \delta_n^{\frac{p}{1+p\varepsilon}} = 2^{\frac{1}{s+\varepsilon}} e^{\frac{-t}{s+\varepsilon}\|\alpha\|_{L^\infty(\mathbb{R}_+)}} \sum_{n=n_0}^{\infty} \frac{1}{(n \log^3(n+1))^{\frac{1}{1+p\varepsilon}}} = \infty. \quad (\text{G.51})$$

Note that the  $TV^s(u(\cdot, t)) < \infty$  for  $s = 1/p$  and  $t > 0$  due to Theorem G.3.3.  $\square$

Our next result upgrades Theorem G.3.1 for more general class of functions satisfying the following hypothesis:

(H-1)  $f \in C^1(\mathbb{R})$  is a strictly convex function such that  $f(0) = f'(0) = 0$ .

(H-2) There exist  $q > 1$ ,  $r > 0$  and  $C > 0$  such that

$$0 \leq f'(a) - f'(b) \leq C(a-b)^q \text{ for all } b \in (-r, 0) \text{ and } a \in (0, r). \quad (\text{G.52})$$

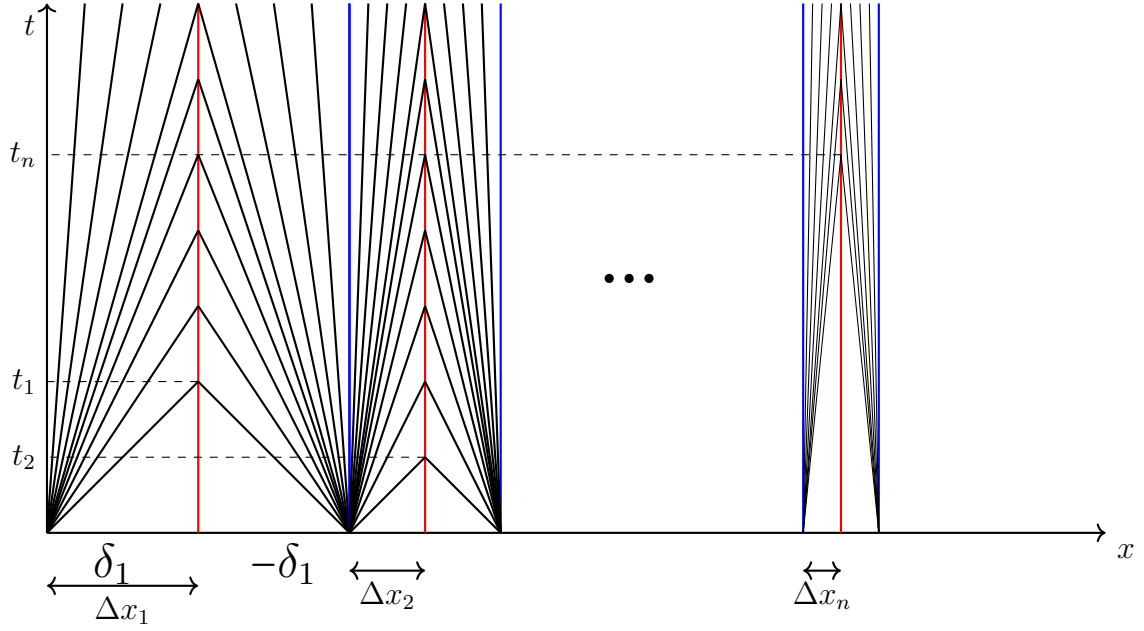


Figure G.1 – This picture illustrates the entropy solution constructed in Theorem G.3.1 for  $\alpha \equiv 0$  case. This construction and structure of entropy solution have been previously studied in [2, 3, 4].

**Remark G.3.1.** We do not lose generality by assuming that  $f(0) = f'(0) = 0$ , due to the change of variables  $x \mapsto x - f'(0)t$  and  $\tilde{f}(u) \stackrel{\text{def}}{=} f(u) - f(0) - f'(0)u$ .

The class of function satisfying (H-1) and (H-2) was previously considered in [4] to show non-BV propagation for all time  $t > 0$ . In this article, we will show the non- $BV^s$  propagation for same class of function in the context of balance laws (G.3).

**Theorem G.3.4.** Let  $f \in C^1(\mathbb{R})$  satisfying (H-1) and (H-2) along with super linear growth condition (G.15). Let  $\alpha \in L^\infty(0, \infty)$ . Then there exists a compact support initial data  $u_0 \in L^\infty(\mathbb{R})$  such that the corresponding entropy solution  $u$  of (G.3) satisfies the following:

$$u(\cdot, t) \notin BV_{loc}^s(\mathbb{R}) \text{ for all } s > 1/q, t > 0. \quad (\text{G.53})$$

**Remark G.3.2.** Note that Theorem G.3.4 is optimal for the class of fluxes satisfying (G.52). It is easy to verify that  $f(u) = (q+1)^{-1}|u|^{q+1}$  satisfies (G.52) and as we have seen in Theorem G.3.3,  $u(\cdot, t) \in BV^{1/q}$  for  $t > 0$ .

**Observation:** We want to make a remark that for convex  $f$  satisfying (H-1), we have  $\Psi(0, t) = 0$ . From the definition of  $\Psi$  we have

$$0 = \int_0^t f'(\Psi(0, t)e^{\beta(\theta)}) d\theta. \quad (\text{G.54})$$

Since  $f$  is a  $C^1$  strictly convex function,  $f'$  is increasing. Hence  $af'(a) > 0$  for any  $a \neq 0$  because  $f'(0) = 0$ . Suppose  $\Psi(0, t) > 0$  then

$$\int_0^t f'(\Psi(0, t)e^{\beta(\theta)}) d\theta > 0. \quad (\text{G.55})$$

Similarly, if  $\Psi(0, t) < 0$ , then we have

$$\int_0^t f'(\Psi(0, t)e^{\beta(\theta)}) d\theta < 0. \quad (\text{G.56})$$

Note that both (G.55) and (G.56) contradict with (G.54). Therefore we have  $\Psi(0, t) = 0$ . Note that  $\Psi$  is increasing in its first variable due to strict convexity assumption on  $f$ . Subsequently, we get  $x\Psi(x, t) > 0$  for any  $x \neq 0$ .

Before we give the main construction to prove Theorem G.3.4 we first recall some results from [4] and find structure of the entropy solution to the following data

$$u_{A,B}^0(x) = \begin{cases} 0 & \text{if } x \notin [A, B], \\ a_{A,B} & \text{if } x \in [A, \tau], \\ b_{A,B} & \text{if } x \in (\tau, B]. \end{cases} \quad (\text{G.57})$$

Next we make a choice for the pair  $(a_{A,B}, b_{A,B})$  depending on  $A, B$ . For that purpose we define  $G: \mathbb{R} \rightarrow \mathbb{R}$  as

$$G(a) = \int_0^{t_0} af'(ae^{\beta(\theta)}) - f(ae^{\beta(\theta)})e^{-\beta(\theta)} d\theta, \quad (\text{G.58})$$

where  $t_0 > 0$  is fixed. Now we claim that  $z \mapsto G(z)$  is increasing for  $z > 0$  and decreasing for  $z < 0$ . To see this consider  $a > a_1 > 0$ , then by Mean Value Theorem we have

$$\begin{aligned} & ae^{\beta(\theta)} f'(ae^{\beta(\theta)}) - a_1 e^{\beta(\theta)} f'(a_1 e^{\beta(\theta)}) - f(ae^{\beta(\theta)}) + f(a_1 e^{\beta(\theta)}) \\ &= ae^{\beta(\theta)} f'(ae^{\beta(\theta)}) - a_1 e^{\beta(\theta)} f'(a_1 e^{\beta(\theta)}) - f'(c^*)(a - a_1)e^{\beta(\theta)}, \end{aligned}$$

for some  $c^* \in (a_1 e^{\beta(\theta)}, ae^{\beta(\theta)})$ . Since  $f'$  is increasing, we get

$$\begin{aligned} & ae^{\beta(\theta)} f'(ae^{\beta(\theta)}) - a_1 e^{\beta(\theta)} f'(a_1 e^{\beta(\theta)}) - f(ae^{\beta(\theta)}) + f(a_1 e^{\beta(\theta)}) \\ & \geq ae^{\beta(\theta)} f'(ae^{\beta(\theta)}) - a_1 e^{\beta(\theta)} f'(a_1 e^{\beta(\theta)}) - f'(ae^{\beta(\theta)})(a - a_1)e^{\beta(\theta)} \\ &= a_1 e^{\beta(\theta)} [f'(ae^{\beta(\theta)}) - f'(a_1 e^{\beta(\theta)})] \\ & \geq 0. \end{aligned}$$

Hence, from (G.58) we obtain  $a \mapsto G(a)$  is increasing for  $a > 0$ . By a similar argument, we get  $b \mapsto G(b)$  is decreasing for  $b < 0$ . Therefore, we have that there exists  $r_0 > 0$  such that

for a given  $a \in (0, r_0)$  there is a  $b \in (-r, 0)$  satisfying  $G(a) = G(b)$ . Let us fix  $a_0 \in (0, r_0)$  and  $b_0 \in (-r, 0)$  such that  $G(a_0) = G(b_0)$ . Define  $F_- : [b_0, 0] \rightarrow \mathbb{R}$  and  $F_+ : [0, a_0] \rightarrow \mathbb{R}$  as follows

$$F_+(a) = \int_0^{t_0} f'(ae^{\beta(\theta)}) d\theta \text{ and } F_-(b) = - \int_0^{t_0} f'(be^{\beta(\theta)}) d\theta. \quad (\text{G.59})$$

Since  $f'$  is increasing and  $f'(0) = 0$ ,  $F_+$  is increasing and  $F_-$  is decreasing. We also have  $F_+(0) = F_-(0) = 0$ . Now we fix  $A, B$  such that

$$B - A \leq \min\{F_+(a_0), F_-(b_0)\}. \quad (\text{G.60})$$

We wish to find  $a_{A,B} \in [0, a_0], b_{A,B} \in [b_0, 0]$  such that

$$G(a_{A,B}) = G(b_{A,B}) \text{ and } B - A = F_+(a_{A,B}) + F_-(b_{A,B}). \quad (\text{G.61})$$

Since  $B - A \leq \min\{F_+(a_0), F_-(b_0)\}$ , by Intermediate Value Theorem there exist  $\bar{a} \in [0, a_0], \bar{b} \in [b_0, 0]$  such that  $B - A = F_+(\bar{a}) = F_-(\bar{b})$ . Define  $\lambda := \min\{G(\bar{a}), G(\bar{b})\}$ . Without loss of generality, suppose  $\lambda = G(\bar{a})$ . Then we set  $a_1 = \bar{a}$  and  $b_1 = 0$ . Hence  $B - A = F_+(a_1) + F_-(b_1)$ . Now we choose  $b_2 \in [b_0, 0]$  such that  $G(b_2) = G(a_1) \in [0, \lambda]$ . Note that  $b_2 \geq \bar{b}$  since  $G(x)$  is decreasing for  $x < 0$  and  $G(b_2) \leq \lambda \leq G(\bar{b})$ . Since  $F_-$  is decreasing, we get  $0 \leq F_-(b_2) \leq F_-(\bar{b}) = B - A$ . Now by Intermediate Value Theorem, we choose  $a_2 \in [0, \bar{a}]$  such that  $B - A = F_-(b_2) + F_+(a_2)$ . Having defined  $\{a_k\}_{1 \leq k \leq n} \subset [0, \bar{a}]$  and  $\{b_k\}_{1 \leq k \leq n} \subset [b_0, 0]$  such that  $B - A = F_-(b_n) + F_+(a_n)$  we choose  $b_{n+1} \in [b_0, 0]$  such that  $G(b_{n+1}) = G(a_n)$ . Note that the choice of  $b_{n+1}$  is guaranteed as  $0 \leq G(a_n) \leq \lambda \leq G(\bar{b})$ . Subsequently, we get  $0 \leq F_-(b_{n+1}) \leq B - A = F_+(a_n)$ . Now we choose  $a_{n+1} \in [0, \bar{a}]$  such that  $B - A = F_-(b_{n+1}) + F_+(a_{n+1})$ . Hence, by this inductive process we get  $\{a_n\}_{n \in \mathbb{N}} \subset [0, \bar{a}]$  and  $\{b_n\}_{n \in \mathbb{N}} \subset [b_0, 0]$ . Since both sequences are bounded, there is a subsequence  $n_k$  such that  $b_{n_k} \rightarrow b_{A,B}$  and  $a_{n_k} \rightarrow a_{A,B}$  as  $k \rightarrow \infty$  for some  $a_{A,B} \in [0, \bar{a}]$  and  $b_{A,B} \in [b_0, 0]$ . Since  $F_{\pm}, G$  are continuous functions, by passing to the limit we show that  $a_{A,B} \in [0, \bar{a}]$  and  $b_{A,B} \in [b_0, 0]$  satisfy (G.61).

Suppose  $a_{A,B}, b_{A,B}$  are satisfying (G.61). Now we can choose  $\tau$  as follows:

$$\tau + \int_0^{t_0} \frac{f(a_{A,B}e^{\beta(\theta)}) - f(b_{A,B}e^{\beta(\theta)})}{a_{A,B} - b_{A,B}} e^{-\beta(\theta)} d\theta = A + \int_0^{t_0} f'(a_{A,B}e^{\beta(\theta)}) d\theta. \quad (\text{G.62})$$

Since  $G(a_{A,B}) = G(b_{A,B})$  and  $B - A = F_-(b_{A,B}) + F_+(a_{A,B})$  we get

$$a_{A,B}(\tau - A) + b_{A,B}(B - \tau) = 0. \quad (\text{G.63})$$

Suppose  $u_{A,B}(x, t)$  is the entropy solution to (G.3) for initial data (G.57). Then as it has been discussed in [4, section 3],  $u_{A,B}$  enjoys the following structure up to time  $t_0$ :

$$u_{A,B}(x, t) = \begin{cases} 0 & \text{if } x \notin [\xi_-(t), \xi_+(t)], \\ \Psi(x - A, t)e^{\beta(t)} & \text{if } \xi_-(t) \leq x \leq \zeta_-(t), \\ a_{A,B}e^{\beta(t)} & \text{if } \zeta_-(t) < x < \zeta_0(t), \\ b_{A,B}e^{\beta(t)} & \text{if } \zeta_0(t) < x < \zeta_+(t), \\ \Psi(x - B, t)e^{\beta(t)} & \text{if } \zeta_+(t) \leq x \leq \xi_+(t), \end{cases} \quad (\text{G.64})$$

where the curves  $\xi_{\pm}, \zeta_{\pm}, \zeta_0$  are determined as follows

$$\Psi(\xi_-(t) - A, t) = 0 = \Psi(\xi_+(t) - B, t), \quad (\text{G.65})$$

$$\Psi(\zeta_-(t) - A, t) = a \text{ and } \Psi(\zeta_+(t) - B, t) = b, \quad (\text{G.66})$$

$$\zeta_0(t) = \frac{1}{a_{A,B} - b_{A,B}} \int_0^t (f(a_{A,B}e^{\beta(\theta)}) - f(b_{A,B}e^{\beta(\theta)})) e^{-\beta(\theta)} d\theta. \quad (\text{G.67})$$

Note that by hypothesis (H-1)  $\xi_-(t) = A$  and  $\xi_+(t) = B$ . By (G.62) we have that two curves  $\zeta_{\pm}$  meet with each other at point  $t_0$ . For  $t \in (t_0, t_0 + \Delta t)$  for small  $\Delta t > 0$ , the entropy solution  $u_{A,B}$  satisfies the following structure

$$u_{A,B}(x, t) = \begin{cases} 0 & \text{if } x \notin [A, B], \\ \Psi(x - A)e^{\beta(t)} & \text{if } A \leq x < \zeta_M(t), \\ \Psi(x - B)e^{\beta(t)} & \text{if } \zeta_M(t) < x \leq B, \end{cases} \quad (\text{G.68})$$

where  $\zeta_M(t)$  is the characteristic curve starting at the point  $(\tau, 0)$ . Next we claim that  $\zeta_M(t) \in (A, B)$  and the structure (G.68) continues to hold for all  $t > t_0$ . We can prove this in the same way as it was done for [4, Lemma 3.12]. Suppose the curve  $t \mapsto \zeta_M(t)$  intersects either  $x = A$  line or  $x = B$  line. Without loss of generality we assume that  $\zeta_M(t)$  first meets  $x = B$  line. Therefore there exists a time  $t_1 > 0$  such that at  $t = t_1$  we have  $\zeta_M(t_1) = B$  and  $A < \zeta_M(t) < B$  for  $0 \leq t < t_1$ . Therefore, (G.68) is valid up to time  $t_1$ . Consider  $\gamma_{\pm}$  defined as follows

$$\Psi(\gamma_-(t) - A, t) = \Psi(\zeta_M(t_1) - A, t_1) \text{ and } \gamma_+ = \xi_+. \quad (\text{G.69})$$

Since (G.68) is valid up to time  $t_1$ ,  $\gamma_{\pm}(t)$  are minimizing curve of the following value function

$$U(x, t) = \min \left\{ \int_0^{\gamma(0)} u_{A,B}^0(y) dy + \int_0^t e^{-\beta(\theta)} f^*(\dot{\gamma}(\theta)) d\theta; \gamma : [0, t] \rightarrow \mathbb{R}, \gamma(t) = x \right\} \quad (\text{G.70})$$

where  $f^*$  is the Legendre transform of  $f$ . By (G.62) we have

$$\int_A^B u_{A,B}^0 = 0. \quad (\text{G.71})$$

Therefore we have

$$\int_0^{t_1} e^{-\beta(\theta)} f^*(\dot{\gamma}_+(\theta)) d\theta = \int_0^{t_1} e^{-\beta(\theta)} f^*(\dot{\gamma}_-(\theta)) d\theta. \quad (\text{G.72})$$

By using the definition of  $\Psi$  we have

$$\gamma_-(t) - A = \int_0^t f'(\Psi(\gamma_-(t) - A, t)e^{\beta(\theta)}) d\theta = \int_0^t f'(\Psi(\zeta_M(t_1) - A, t_1)e^{\beta(\theta)}) d\theta. \quad (\text{G.73})$$

Differentiating (G.73) with respect to  $t$ , we obtain

$$\dot{\gamma}_-(\theta) = f'(\Psi(\zeta_M(t_1) - A, t_1)e^{\beta(\theta)}) \text{ for } \theta \in (0, t_1). \quad (\text{G.74})$$

Similarly, we have

$$\dot{\gamma}_+(\theta) = 0 \text{ for } \theta \in (0, t_1). \quad (\text{G.75})$$

Since  $f^* \geq 0$  we have  $f^*(\dot{\gamma}_-(\theta)) = 0$  for a.e.  $\theta \in [0, t_1]$ . Since 0 is unique minima of  $f^*$ , we have  $\dot{\gamma}_-(\theta) = 0$  a.e.  $\theta \in [0, t_1]$ . This gives a contradiction. Hence our claim is proved i.e.  $\zeta_M(t) \in (A, B)$  for all  $t \geq 0$ . Now we are ready to prove Theorem G.3.4.

*Proof of Theorem G.3.4:* Define  $A_n, B_n$  as follows:

$$A_n = x_n - \frac{1}{n(\log(n+1))^2} \text{ and } B_n = x_n + \frac{1}{n(\log(n+1))^2},$$

$$\text{where } x_n = 4 \sum_{k=1}^{n-1} \frac{1}{k(\log(k+1))^2} + \frac{2}{n(\log(n+1))^2}. \quad (\text{G.76})$$

From the choice of  $A_n, B_n$  and  $x_n$  it is clear that

$$B_{n-1} < A_n < B_n < A_{n+1} \text{ and } \lim_{n \rightarrow \infty} x_n = x_0 < \infty. \quad (\text{G.77})$$

Note that  $B_n - A_n = 2n^{-1}(\log(n+1))^{-2} \rightarrow 0$  as  $n \rightarrow \infty$ . Therefore, there exists  $n_0 \in \mathbb{N}$  such that  $A_n, B_n$  satisfy (G.60) for all  $n \geq n_0$ . By the previous observation, we find  $a_{A_n, B_n}, b_{A_n, B_n}$  satisfying (G.61) for  $B_n - A_n$ . Next we define initial data  $u_0$  as follows:

$$u_0(x) = \begin{cases} u_{A_n, B_n}^0 & \text{if } x \in [A_n, B_n] \text{ for } n \geq n_0, \\ 0 & \text{otherwise} \end{cases} \quad (\text{G.78})$$

where  $u_{A_n, B_n}^0$  is defined in (G.57). To simplify the notation we denote  $a_n = a_{A_n, B_n}$  and  $b_n = b_{A_n, B_n}$ . By using (H-2) in (G.61) we get

$$B_n - A_n \leq C(a_n - b_n)^q \int_0^{t_0} e^{q\beta(\theta)} d\theta. \quad (\text{G.79})$$

Therefore, we get

$$a_n - b_n \geq c_0^{-\frac{1}{q}} (B_n - A_n)^{\frac{1}{q}} \text{ where } c_0 = C \int_0^{t_0} e^{q\beta(\theta)} d\theta. \quad (\text{G.80})$$

From (G.77) it is clear that  $u_0$  has compact support in  $\mathbb{R}$ . By structure (G.64) and (G.68) we know that if  $u_{A_n, B_n}$  is the entropy solution to (G.3) for initial data  $u_{A_n, B_n}^0$  then the support of  $u_{A_n, B_n}$  lies in the strip  $[A_n, B_n] \times [0, \infty)$ . Therefore, if  $u(x, t)$  is the solution to (G.3) then  $u$  has the following structure

$$u(x, t) = \begin{cases} u_{A_n, B_n}(x, t) & \text{if } x \in [A_n, B_n] \text{ for } n \geq n_0, \\ 0 & \text{otherwise.} \end{cases} \quad (\text{G.81})$$



By  $\zeta_n$  we denote the curve  $\zeta_M$  appeared in the structure (G.68) of  $u_{A,B}$  corresponding to  $A = A_n, B = B_n$ . From (G.64) and (G.68) we obtain the following estimate for any  $t > 0$ ,

$$TV^s(u_{A_n, B_n}(\cdot, t))[A_n, B_n] \geq \min \left\{ (b_n - a_n)^{\frac{1}{s}} e^{\frac{\beta(t)}{s}}, |\Psi(\zeta_n(t) - A_n, t) - \Psi(\zeta_n(t) - B_n, t)|^{\frac{1}{s}} e^{\frac{\beta(t)}{s}} \right\}. \quad (\text{G.82})$$

From the definition of  $\Psi$  we have

$$\zeta_n(t) - A_n - \zeta_n(t) + B_n = \int_0^t (f'(\Psi(\zeta_n(t) - A_n, t)e^{\beta(\theta)}) - f'(\Psi(\zeta_n(t) - B_n, t)e^{\beta(\theta)})) d\theta. \quad (\text{G.83})$$

Note that  $\zeta_n(t) - A_n > 0 > \zeta_n(t) - B_n$ . Since  $\Psi$  is increasing in its first variable and  $\Psi(0, t) = 0$ , we have

$$\Psi(\zeta_n(t) - A_n, t) > 0 > \Psi(\zeta_n(t) - B_n, t). \quad (\text{G.84})$$

From the decay condition (G.52) we have

$$B_n - A_n \leq C \int_0^t (\Psi(\zeta_n(t) - A_n, t) - \Psi(\zeta_n(t) - B_n, t))^q e^{q\beta(\theta)} d\theta. \quad (\text{G.85})$$

Therefore we have

$$|\Psi(\zeta_n(t) - A_n, t) - \Psi(\zeta_n(t) - B_n, t)| \geq (B_n - A_n)^{\frac{1}{q}} \varrho(t)^{-\frac{1}{q}} \text{ where } \varrho(t) := C \int_0^t e^{q\beta(\theta)} d\theta. \quad (\text{G.86})$$

Combining (G.80), (G.82) and (G.86) we have

$$TV^s(u_{A_n, B_n}(\cdot, t))[A_n, B_n] \geq \min \left\{ c_0^{-\frac{1}{qs}} e^{\frac{1}{s}\beta(t)} (B_n - A_n)^{\frac{1}{qs}}, (B_n - A_n)^{\frac{1}{qs}} \varrho(t)^{-\frac{1}{qs}} e^{\frac{1}{s}\beta(t)} \right\}. \quad (\text{G.87})$$

Fix an  $s > q^{-1}$ . Then there exists  $\delta > 0$  such that  $s = q^{-1} + \delta$ . By our choice of  $A_n$  and  $B_n$  we have

$$B_n - A_n = \frac{2}{n(\log(n+1))^2}. \quad (\text{G.88})$$

Since  $s = (1/q) + \delta$  we have  $sq = 1 + q\delta$ . We observe that  $\beta(t) \geq -t\|\alpha\|_{L^\infty(\mathbb{R}_+)}$  and  $\varrho(t) \leq tC e^{q\|\alpha\|_{L^\infty(\mathbb{R}_+)}}$ . From definition of  $c_0$  we have  $c_0 \leq t_0 C e^{q\|\alpha\|_{L^\infty(\mathbb{R}_+)}}$ . Hence, we obtain

$$\begin{aligned} TV^s(u_{A_n, B_n}(\cdot, t))[A_n, B_n] &\geq 2^{\frac{1}{1+q\delta}} e^{\frac{1}{s}\beta(t)} \min \left\{ c_0^{-\frac{1}{qs}}, \varrho(t)^{-\frac{1}{qs}} \right\} n^{-\frac{1}{1+q\delta}} (\log(n+1))^{-\frac{2}{1+q\delta}} \\ &\geq 2^{\frac{1}{1+q\delta}} e^{-\frac{(t+1)}{s}\|\alpha\|_{L^\infty(\mathbb{R}_+)}} \min \left\{ (Ct_0)^{-\frac{1}{qs}}, (Ct)^{-\frac{1}{qs}} \right\} \frac{n^{-\frac{1}{1+q\delta}}}{(\log(n+1))^{\frac{2}{1+q\delta}}}. \end{aligned} \quad (\text{G.89})$$

Since  $[A_n, B_n], n \geq n_0$  are disjoint intervals we have

$$\begin{aligned} TV^s(u(\cdot, t)) &\geq \sum_{n \geq n_0} TV^s(u_{A_n, B_n}(\cdot, t))[A_n, B_n] \\ &\geq 2^{\frac{1}{1+q\delta}} e^{-\frac{(t+1)}{s}} \|\alpha\|_{L^\infty(\mathbb{R}_+)} \min \left\{ (Ct_0)^{-\frac{1}{qs}}, (Ct)^{-\frac{1}{qs}} \right\} \sum_{n \geq n_0} \frac{n^{-\frac{1}{1+q\delta}}}{(\log(n+1))^{\frac{2}{1+q\delta}}} = \infty. \end{aligned}$$

□

## G.4 The scalar multi-D case

In this section we deal with  $C^\infty$ -flux function for multi-D scalar conservation laws which reads as follows

$$\partial_t U + \operatorname{div}_X F(U) = 0, \quad U(X, 0) = U_0(X). \quad (\text{G.90})$$

Non-linearity of a multi-D smooth flux is defined as

**Definition G.4.1** (Nonlinear flux, [32]). *Let  $F$  belong to  $C^\infty([a, b], \mathbb{R}^m)$   $[a, b]$  and for each  $U \in [a, b]$ ,*

$$d_F := \sup_{U \in [a, b]} \inf \{ k \in \mathbb{N}; k \geq 1, \operatorname{span}(F''(U), \dots, F^{k+1}(U)) = \mathbb{R}^m \} \in \mathbb{N} \cup \{+\infty\}, \quad (\text{G.91})$$

If  $d_F < +\infty$  then  $F$  is called a nonlinear flux.

If  $d_F = m$  it is called a genuinely nonlinear flux.

It can be checked [32] that since  $[a, b]$  is compact  $d_F[\cdot]$  attains its maximum at some point  $\bar{U} \in [a, b]$ , so,  $d_F$  is well defined in  $\mathbb{N} \cup \{+\infty\}$ . By definition of  $d_F$ , at least  $m$  derivatives of  $F'$  are needed to span the  $m$ -dimensional space  $\mathbb{R}^m$  so  $d_F \geq m$ . If  $F$  is a linear flux,  $d_F = +\infty$ . Notice that for some exponentially flat fluxes, it is possible to have  $d_F = +\infty$  already in dimension one [19, 20]. In this case no  $BV^s$  smoothing effect is expected. Indeed, there is a low smoothing effect in a generalized  $BV$  space  $BV^\Phi$  [19, 38, 39].

For a  $C^\infty$  nonlinear flux  $F$ , the Lions, Perthame and Tadmor conjecture [37] can be reformulated as follows [32], any entropy solution of (G.90) such that  $U_0(\mathbb{R}^m) \subset [a, b]$  are regularized in  $W^{s,1}$  for all  $s < d_F^{-1}$  where  $d_F$  is the non-linearity index as in (G.91). The Lions, Perthame and Tadmor conjecture is still an open problem.

We prove the limitation of the regularizing effect for the class of  $C^\infty$  nonlinear fluxes  $F$  such that  $d_F$  is odd. The restriction of  $d_F$  for odd numbers is due to our previous explicit construction in one dimensional case of solution with the exact maximal regularity for all time only for convex fluxes. The existence of an entropy solution with the conjectured maximal regularity and not more is provided by a construction of a planar wave. This regularity is not improved for large time.

For a bounded strip of time the limitation of the smoothing effect for entropy solutions of multi-dimensional scalar conservation laws in Sobolev spaces has been already proven in [20, 32]. On one hand, the limitation for bounded time was due to the difficulty to

study in general behaviour of the solutions after interactions of waves as in [21, 22, 25]. On the other hand, multidimensional fractional  $BV$  spaces were not known at that time. Recently, in [28] it has been shown that given a  $C^2$  flux there exists an entropy solution in multi-D such that it is not in  $BV_{loc}$  for all time. Authors also prove that there exists an entropy solution which is not in  $W_{loc}^{s+\varepsilon,1}$ ,  $\forall \varepsilon > 0$  for all time with  $C^{2,\gamma}$  with  $d_F = 1/s$ .

The point in this section is to obtain the optimality for all time and in the multi-D  $BV^s$  framework. To get the optimality for the multi-D case, a planar wave is used as in [20, 28].

The flux being nonlinear [37] there exist a constant state  $\underline{U}$  and a direction  $\xi$  such that the flux reaches its degeneracy  $d_F$  near  $\underline{U}$  and following the direction  $\xi$  [32]. That simply means that the scalar flux  $f(u) = \xi \cdot F(\underline{U} + u)$  has an exact  $p$ -degeneracy (G.5) with the optimal  $p = d_F$ . Moreover, for smooth flux,  $p$  is an integer [32] bigger than the space dimension:  $m \leq p \in \mathbb{N}$ . That means that for small  $u$  the derivative of the flux  $f'$  has exactly a power-law behaviour like  $u^p$ . For  $p$  odd,  $f$  is locally convex (or concave) and the Theorem G.3.4 can be used. The result reads as follow.

**Corollary G.4.1.** *Let  $F$  be a  $C^\infty(\mathbb{R}, \mathbb{R}^m)$  flux with an odd exact degeneracy  $d_F = p$  on  $[-M, M]$  for some  $M > 0$  then there exists an entropy solution  $U$  of (G.90) such that  $\forall \varepsilon > 0, \forall t > 0, U(\cdot, t) \in BV_{loc}^s(\mathbb{R}^m, \mathbb{R})$  and  $U(\cdot, t) \notin BV_{loc}^{s+\varepsilon}(\mathbb{R}^m, \mathbb{R})$  where  $s = 1/p$ .*

**Remark G.4.1.** *The parity restriction of  $p$  should be neglected with an implicit and more complicated construction used in [28]. Such a solution does not have the same compact support forever.*

We just recall the main features of the proof in [20] for the optimality of the  $BV^s$  regularity for a bounded time and then using example of the section G.3 and Lemma G.2.1 the optimality for all time follows. First, take an example given in the proof of the Theorem G.3.1 with  $u_0(x)$  and  $u$  the corresponding entropy solution for the flux  $f$  and  $U_0(X) = \underline{U} + u_0(\xi \cdot X)$  then for all time [20],

$$U(X, t) = \underline{U} + u(\xi \cdot X, t).$$

The  $BV^s$  multi-D regularity of the entropy solution  $U$  is the consequence of 1-D optimality of  $u$  and Lemma G.2.1.

## G.5 A class of $2 \times 2$ triangular systems

Getting optimal  $BV^s$  solutions for general systems for all time is an open problem. Also, the existence of  $BV^s$  solutions for systems is in general open. There are some exceptions,  $BV^s$  solutions exist for a gas-chromatography system [14], a nonlinear acoustics model and also for diagonal systems [33]. However, the optimality of the regularity is not yet proven. In this section, we consider the first example. The gas-chromatography system is not a Temple system as the well-known chromatography system presented in Bressan's

book [15] for instance. Otherwise, this gas-chromatography system enjoys a nice property in Lagrangian variables [14], it has the triangular structure:

$$\partial_t u + \partial_x f(u) = 0 \quad \text{for } x \in \mathbb{R}, t > 0, \quad (\text{G.92})$$

$$\partial_t v + \partial_x (g(u)v) = 0 \quad \text{for } x \in \mathbb{R}, t > 0, \quad (\text{G.93})$$

$$(u, v)(x, 0) = (u_0, v_0)(x) \text{ for } x \in \mathbb{R}. \quad (\text{G.94})$$

At first sight, the system (G.92)–(G.93) seems easy to solve. First, one takes the entropy solution of the equation (G.92). Second, solve the linear transport equation with (G.93). But, the velocity of the transport equation is  $g(u)$  which can be discontinuous. For such equations, a Dirac mass can appear [10]. Thus, due to the transport equation, such systems are not easy to solve in general. The pressureless-gas dynamics system is an example of such problematic systems [11].

In this section we propose optimal  $BV^s$  solutions for two cases. First, a self contained construction for a finite time  $[0, T_0]$ ,  $T_0 > 0$  where the component  $v$  stays continuous. Second, using a recent result of global existence of bounded entropy solutions, we get, as a corollary, the optimality in  $BV^s$  for all time.

In the next theorem we construct a solution of the system (G.92)–(G.93) such that  $(u, v) \notin BV^{s+\varepsilon}(\mathbb{R} \times [0, T_0])$  for all  $\varepsilon > 0$  and for power-law type functions  $f$  and  $g$  satisfying the following relation,

$$g = h \circ f', \quad (\text{G.95})$$

where  $h$  is a Lipschitz function. We first build a continuous solution  $u$  to (G.92) and then solve (G.93) by using  $u$ . Similar line of thought has been previously instrumentalized in [7] to characterize the attainable set for triangular systems.

**Theorem G.5.1.** *Let  $T > 0$ . Let  $f(u) = |u|^{p+1}/(p+1)$ ,  $p \geq 1$ ,  $s = 1/p$  and  $g = h \circ f'$  where  $h$  is a Lipschitz function. Then there exist compactly supported initial data  $u_0, v_0 \in L^\infty(\mathbb{R})$  such that the corresponding entropy solution  $(u, v)$  of the triangular system (G.92)–(G.93) satisfies  $\forall t \in [0, T]$ ,*

$$u, v \in L^\infty([0, T] \times \mathbb{R}), TV^{s+\varepsilon}u(\cdot, t) = \infty \text{ for all } \varepsilon > 0 \text{ and } TV^{s'}v(\cdot, t) = \infty \text{ for all } s' \in (0, 1].$$

*Proof.* If  $f'(u(x, t))$  is Lipschitz in  $x$ -variable then we have  $g(u(x, t))$  is Lipschitz in  $x$ -variable by the choice of  $g$ . The construction is done in two steps

**Step 1:** *Construction of a continuous solution of (G.1) such that  $f'(u(x, t))$  is Lipschitz in the  $x$  variable:*

$$\text{Let } \Delta x_n := 1/(n \log^2(n+1)), t_n := \frac{\log(n+1)}{\log 2}(T+1) > T \text{ and let } \delta_n := \left(\frac{\Delta x_n}{t_n}\right)^{\frac{1}{p}}. \text{ We}$$

define  $x_1 = 0$  and  $x_n = 2 \sum_{m=1}^n \Delta x_m$  for  $n \geq 2$ . Next we consider the following initial data

$$w_0^n(x) = \begin{cases} \left(\frac{\Delta x_n - x}{t_n}\right)^s & \text{if } 0 < x \leq \Delta x_n, \\ -\left(\frac{x - \Delta x_n}{t_n}\right)^s & \text{if } \Delta x_n \leq x < 2 \Delta x_n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore  $\forall x \in \mathbb{R}, \forall t > 0$ , the entropy solution  $w_n$  of (G.92) with the flux  $f(u) = |u|^{p+1}/(p+1)$ , is given by

$$w_n(x, t) = \begin{cases} \left(\frac{x}{t}\right)^s & \text{if } 0 < x < \min\{\delta_n^p t, \Delta x_n\}, \\ \left(\frac{\Delta x_n - x}{t_n - t}\right)^s & \text{if } \delta_n^p t < x < \Delta x_n, \\ -\left(\frac{x - \Delta x_n}{t_n - t}\right)^s & \text{if } \Delta x_n < x < 2\Delta x_n - \delta_n^p t, \\ -\left(\frac{2\Delta x_n - x}{t}\right)^s & \text{if } \max\{2\Delta x_n - \delta_n^p t, \Delta x_n\} < x < 2\delta_n, \\ 0 & \text{otherwise.} \end{cases}$$

Let  $u_0(x) = \sum_n w_0^n(x - x_n)$  and let  $u$  be the entropy solution of (G.92) with the same flux  $f$ . Then one can show that  $u(x, t)$  is continuous function on  $\mathbb{R} \times (0, T]$  and  $f'(u(x, t))$  is Lipschitz in the  $x$  variable. We also have

$$TV^{s+\varepsilon}u(\cdot, t)\{[0, 2\Delta x_n]\} \geq 4(\Delta x_n/t_n)^{\frac{1}{1+p\varepsilon}} \text{ for } t_n > t, \varepsilon > 0.$$

**Step 2:** We devote this step to find the component  $v$  as in (G.93). In order to do that, it is enough to find a solution of

$$\left. \begin{aligned} \partial_t v + \partial_x(c(x, t)v) &= 0 & \text{for } (x, t) \in \mathbb{R} \times (0, T), \\ v(x, 0) &= v_0(x) & \text{for } x \in \mathbb{R} \end{aligned} \right\} \quad (\text{G.96})$$

where  $c(x, t) = g(u(x, t))$  and  $u$  is the entropy solution of (G.92). We observed that  $f'(u)(x, t)$  is Lipschitz in  $x$ -variable and so is  $c(x, t)$  thanks to (G.95). We can solve (G.96) by the method of characteristics and for that, we need to find the solution of the following Cauchy problem

$$\frac{d}{dt}X(t, x_0) = c(X(t, x_0), t), \quad X(0, x_0) = x_0, \quad (\text{G.97})$$

for each  $x_0 \in \mathbb{R}$ . By using the classical Cauchy–Lipschitz theorem we obtain a unique solution of (G.97). In this way, we construct a solution of (G.96) for  $L^\infty$  initial data  $v_0$ . Let  $v_0$  be defined as follows

$$v_0(x) = \begin{cases} -1 & \text{if } 2^{-2k} < x < 2^{-2k+1} \text{ for } k \geq 1, \\ 1 & \text{if } 2^{-2k-1} < x < 2^{-2k} \text{ for } k \geq 1, \\ 1 & \text{if } x > 1/2 \text{ or } x < 0. \end{cases} \quad (\text{G.98})$$

Consider the sequence  $\{y_n\}$  defined as  $y_n = (2^{-n} + 2^{-n+1})/2$  for  $n \geq 1$ . Now fix a  $t \in [0, T]$ . We define  $z_n = X(t, y_n)$ . Note that  $v(z_n, t) = v_0(y_n)$ . Let  $s' \in (0, 1]$ . By the choice of  $y_n$  we get

$$TV^{s'}v(\cdot, t) \geq \sum_{n=1}^{\infty} |v(z_n, t) - v(z_{n+1}, t)|^{1/s'} = \sum_{n=1}^{\infty} |v_0(y_n) - v_0(y_{n+1})|^{1/s'} = \infty.$$

Hence we obtain a solution  $(u, v)$  of (G.92) and (G.93) such that  $u \notin BV^{s+\varepsilon}$  for any  $\varepsilon > 0$  and  $v \notin BV^{s'}$  for all  $s' \in (0, 1]$ .  $\square$

Now, the optimality in  $BV^s$  for all time is presented. In a recent paper the existence of weak entropy solutions for such triangular system are obtained in [12] under the following assumptions for a convex flux.

(T-1) The flux  $f \in C^4$  is convex and  $g \in C^3$ .

(T-2) Initial data  $u_0$  belongs to  $BV^{1/3}$  and  $v_0$  to  $L^\infty$ .

(T-3) The system is uniformly strictly hyperbolic,

$$\inf_{|u| \leq M} f' > \sup_{|u| \leq M} g \text{ where } M := \|u_0\|_\infty.$$

Notice, that, in the Theorem G.5.1, the system is not assumed to be strictly hyperbolic, which  $h = id$  for instance. Here, the strict hyperbolicity is assumed. Moreover, a minimal regularity of the initial data  $u_0$  is needed to ensure the global existence in  $L^\infty$  of a solution  $(u, v)$ .

**Corollary G.5.1.** *Assume (T-1)–(T-3), and*

$$s = \max\left(\frac{1}{3}, \frac{1}{q}\right),$$

where  $q$  is the power of the degeneracy condition (G.6). There exists an initial data  $u_0$  such that, for all  $v_0 \in L^\infty$ , the triangular system admits a global solution  $u$  staying in  $BV^s$  for all time,  $v \in L^\infty([0, +\infty), \mathbb{R})$ , and,  $\forall \epsilon > 0, \forall t > 0, TV^{s+\epsilon}u(\cdot, t) = +\infty$ .

## G.6 The multi-D Keyfitz-Kranzer system

In this section, we show that even for data with small total variation, renormalized solution to the Keyfitz-Kranzer system may not be in  $BV^s$ . We use the example in [24]. We modified the renormalized solution considered in [24] to show that even if the data has arbitrary small TV the  $TV^s$  norm of the solution blows up. Here we mention the key points and the necessary changes. The rest follows from the analysis done in [24].

Consider the following system

$$\begin{aligned} \partial_t u + \operatorname{div}_z(h(|u|)u) &= 0 & \text{for } z \in \mathbb{R}^m, t > 0, \\ u(z, 0) &= u_0(z) & \text{for } z \in \mathbb{R}^m. \end{aligned} \tag{G.99}$$

where  $u : \mathbb{R}^m \times \mathbb{R}_+ \rightarrow \mathbb{R}^k$  and  $h \in C^1(\mathbb{R}, \mathbb{R}^m)$ . Suppose  $\eta := |u|$  solves the following in the sense of Kruřkov

$$\begin{aligned} \partial_t \eta + \operatorname{div}_z(h(\eta)\eta) &= 0 & \text{for } z \in \mathbb{R}^m, t > 0, \\ \eta(\cdot, 0) &= |u_0| & \text{for } z \in \mathbb{R}^m. \end{aligned} \tag{G.100}$$

Let  $\omega := u/|u|$  solves the following transport equation

$$\begin{aligned}\partial_t(\eta\omega) + \operatorname{div}_z(h(\eta)\eta\omega) &= 0 && \text{for } z \in \mathbb{R}^m, t > 0, \\ \omega(\cdot, 0) &= u_0/|u_0| && \text{for } z \in \mathbb{R}^m.\end{aligned}$$

We call  $u = \eta\omega$  as *renormalized entropy solution*. Note that the notion of renormalized entropy solution is different from the notion of standard entropy solution. Now we consider a special case of the system (G.99) with  $h = (g, 0, \dots, 0)$ . Then we have the following proposition

**Proposition G.6.1.** *Let  $h = (g, 0, \dots, 0)$  for  $g \in C^1(\mathbb{R})$ . Let  $k \geq 2, m \geq 2$ , and  $b \in \mathbb{R}^k \setminus \{0\}$  such that  $g'(|b|) \neq 0$ . Then there exists a sequence of initial data  $u_0^n : \mathbb{R}^m \rightarrow \mathbb{R}^k$  such that*

1.  $\|u_0^n - b\|_{BV(\mathbb{R}^m)} + \|u_0^n - b\|_\infty \rightarrow 0$  as  $n \rightarrow \infty$ ,
2.  $u_0^n = b$  on  $\mathbb{R}^m \setminus B_\lambda(0)$  for some  $\lambda > 0$  independent of  $n$ ,
3. if  $u^n$  is the renormalized entropy solution of (G.99) with initial data  $u_0^n$  then  $u^n(\cdot, t) \notin BV_{loc}^s$  for each  $n \in \mathbb{N}, t \in (0, 1)$  and  $s \in (0, 1)$ .

*Proof.* For the sake of simplicity, we prove Proposition G.6.1 for  $m = 2$  and  $k = 2$ . Suppose  $g'(|b|) = 1, g(|b|) = 0$ . Let  $\delta > 0$  be fixed and  $p = s^{-1} > 1$ . Let  $m_i = i^{p+p\delta}$  for all  $i \in \mathbb{N}$ . Then we have

$$\sum_i m_i 2^{-i} < +\infty. \quad (\text{G.101})$$

Let  $\varepsilon > 0$  be very small such that

- $g$  is injective on  $[|b| - 2\varepsilon, |b| + 2\varepsilon]$ ,
- $[-\varepsilon, \varepsilon] \subset g([|b| - 2\varepsilon, |b| + 2\varepsilon])$ .

Then for sufficiently large  $i$  we can choose  $r_i \in [-2\varepsilon, 2\varepsilon]$  such that  $g(|b| + r_i) = 2^{-i}$ . Note that for sufficiently large  $i$  we have  $r_i \leq 2^{-i+1}$ . We write  $\beta := b/|b|$ , and for each  $i$  we choose a  $\beta_i \in S^{k-1}$  such that  $|\beta - \beta_i| = i^{-1-\delta}$ . Consider

$$I_i = [2^{-i}, 2^{-i+1}), \quad (\text{G.102})$$

$$I_i^j = \left[ 2^{-i} + \frac{(j-1)2^{-i}}{m_i}, 2^{-i} + \frac{j2^{-i}}{m_i} \right) \text{ for } 1 \leq j \leq m_i. \quad (\text{G.103})$$

Define  $\phi_i : \mathbb{R}^2 \rightarrow S^{k-1}$  as follows

$$\phi_i(x, y) := \begin{cases} \beta_i & \text{when } y \in I_i \text{ and } [x2^i] \text{ is odd,} \\ \beta & \text{otherwise.} \end{cases} \quad (\text{G.104})$$

Also we define  $\Lambda_i : \mathbb{R}^2 \rightarrow \mathbb{R}$  as

$$\Lambda_i(x, y) := \begin{cases} r_i & \text{when } y \in I_i \text{ for even } j \text{ and } x \in [-M, M], \\ r_{i+1} & \text{when } y \in I_i \text{ for odd } j \text{ and } x \in [-M, M], \\ 0 & \text{otherwise,} \end{cases} \quad (\text{G.105})$$

where  $M$  is some positive real number bigger than 1 and it will be chosen later. Next we define

$$\eta_0^n := |b| + \sum_{i=n}^{\infty} \Lambda_i, \quad (\text{G.106})$$

$$\omega_0^n(x, y) := \begin{cases} \phi_i(x, y) & \text{when } y \in I_i \text{ for some } i \geq n \text{ and } x \in [-M, M], \\ \beta & \text{otherwise,} \end{cases} \quad (\text{G.107})$$

$$u_0^n := \eta_0^n \omega_0^n. \quad (\text{G.108})$$

Next we show the following two properties of  $\{u_0^n\}$  sequence.

$$\text{M.1 } \|u_0^n - b\|_{L^\infty(\mathbb{R}^2)} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

$$\text{M.2 } \|u_0^n - b\|_{BV(\mathbb{R}^2)} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

These have been shown in [24]. For the sake of completion, we briefly mention key steps. From (G.106)–(G.108) note that

$$\|u_0^n - b\|_{L^\infty(\mathbb{R}^2)} \leq |b| |\beta - \beta_n| + r_n \leq |b| n^{-1-\delta} + 2^{-n+1} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (\text{G.109})$$

We observe that  $\text{supp}(u_0^n) \subset [-M, M] \times [0, 1] \subset [-M, M]^2$  as  $M > 1$ . Note that  $\|\eta_0^n\|_{L^\infty(\mathbb{R}^2)} \leq |b| + 1$  and  $\|\omega_0^n\|_{L^\infty(\mathbb{R}^2)} \leq 2|\beta| + 1$  for all  $n \in \mathbb{N}$ . Therefore, to prove (M.2) it is enough to show that

$$\|\eta_0^n - |b|\|_{BV([-2M, 2M]^2)} \rightarrow 0 \text{ and } \|\omega_0^n - \beta\|_{BV([-2M, 2M]^2)} \rightarrow 0 \text{ as } n \rightarrow \infty. \quad (\text{G.110})$$

Similar to [24], we can show

$$\begin{aligned} \|\eta_0^n - |b|\|_{BV([-2M, 2M]^2)} &\leq 4M^2 \|u_0^n - b\|_{L^\infty(\mathbb{R}^2)} + \sum_{i \geq n} m_i 2^{-i} + (4M + 2)r_n, \\ \|\omega_0^n - \beta\|_{BV([-2M, 2M]^2)} &\leq 4M^2 \|\omega_0^n - \beta\|_{L^\infty(\mathbb{R}^2)} + 2M \sum_{i \geq n} 2^{-i} i^{-1-\delta} 2^i \\ &\quad + 2M \sum_{i \geq n} [i^{-1-\delta} + (i+1)^{-1-\delta}] + (4M + 2)r_n. \end{aligned}$$

Note that  $\|\omega_0^n - \beta\|_{L^\infty(\mathbb{R}^2)} \leq |\beta - \beta_n| \rightarrow 0$  as  $n \rightarrow \infty$ . Since  $\sum_{i \geq 1} m_i 2^{-i} < \infty$  and  $\sum_{i \geq 1} i^{-1-\delta} < \infty$ , we have

$$\sum_{i \geq n} m_i 2^{-i} \rightarrow 0, \quad \sum_{i \geq n} i^{-1-\delta} \rightarrow 0 \text{ and } \sum_{i \geq n} [i^{-1-\delta} + (i+1)^{-1-\delta}] \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, we obtain (G.110). Suppose  $u^n$  is the unique renormalized solution of (G.99). We have seen  $\eta^n$  is the unique solution to (G.100) with initial data  $\eta_0^n$ . Notice that  $\eta_0^n(\cdot, y)$  is constant on  $[-M, M]$  and by finite speed of conservation laws we get  $\eta^n(x, y, t) = \eta_0^n(x, y)$  if  $(x, y, t) \in \left\{ \sqrt{x^2 + y^2} \leq C(M - t) \right\}$  where  $C = C(\|\eta_0^n\|_\infty)$ . Note that for each  $R > 0$  we can choose  $M > 0$  large enough such that

$$\eta^n(x, y, t) = \eta_0^n(x, y) \text{ for } t \in [0, 1] \text{ and } (x, y) \in (-R, R) \times [0, 1]. \quad (\text{G.111})$$



We will choose  $R$  later. To analyze the angular part  $\omega^n := u^n/|u^n|$  we use the fact that  $\eta^n$  is a constant on the curve  $\Psi_n(\cdot, x, \cdot)$  where  $\Psi_n(\cdot, x, \cdot)$  satisfies

$$\frac{d}{dt}\Psi_n(t, x, y) = h(\eta^n(\Psi_n(t, x, y), t)), \quad (\text{G.112})$$

$$\Psi_n(0, x, y) = (x, y). \quad (\text{G.113})$$

We can choose  $R$  large enough so that for any  $(\tau, x_1, y_1) \in [0, 1]^3 \subset [0, 1] \times [-R, R] \times [0, 1]$ , the curve  $t \mapsto \Psi_n(t, x_0, y_0)$  lies on the plane  $y = y_1$  for  $t \in (0, \tau)$  and remains a straight line for  $t \in (0, \tau)$  (see [24] for more detailed discussion on this). As it has been observed in [24], choice of  $R$  can depend only on  $g$  and  $\|\rho_0^n\|_{L^\infty(\mathbb{R}^2)}$ . Since there exists a constant  $C > 0$  such that  $\|\rho_0^n\|_{L^\infty(\mathbb{R}^2)} \leq C$  for all  $n \geq 1$ , we conclude that choice of  $R$  does not depend on  $n$ . Once we fix the choice of  $R$ , we make the choice of  $M$ . By a similar discussion as in [24] we have the following,

- if  $\eta_0^n(x, y) = |b|$ , then  $\omega^n(x, y, t) = \omega_0^n(x, y)$ ,
- if  $\eta_0^n(x, y) = |b| + r_i$ , then  $\omega^n(x, y, t) = \omega_0^n(x - t2^{-i}, y)$ .

Therefore, for  $j \in \{1, \dots, m_i\}$ ,  $i \geq n$  and  $l \in \{1, \dots, 2^i - 1\}$  the function  $\omega^n(\cdot, \cdot, t)$  jumps on the segments

$$J_{j,i,l} := \left\{ y = 2^{-i} + \frac{j2^{-i}}{m_i}, x \in [l2^{-i}, (l+t)2^{-i}] \right\}.$$

For a fixed  $t > 0$ , suppose  $\omega^n(\cdot, \cdot, t) \in BV_{loc}^s(\mathbb{R}^2)$ . Then, by (G.11) there exists a  $W^n \in BV_{loc}(\mathbb{R}^2)$  and  $\pi^n \in Lip^s(\mathbb{R})$  such that  $\omega^n(x, y, t) = \pi^n \circ W^n(x, y)$ . Hence,  $|\omega^n(x_1, y_1, t) - \omega^n(x_2, y_2, t)| \leq C_1 |W^n(x_1, y_1) - W^n(x_2, y_2)|^s$  where  $C_1 = Lip^s(\pi^n)$ . Let  $p = s^{-1}$ . Then  $|W^n(x_1, y_1) - W^n(x_2, y_2)| \geq C_1^{-p} |\omega^n(x_1, y_1, t) - \omega^n(x_2, y_2, t)|^p$ . Now we observe that

$$V_i := \int_{J_{j,i,l}} |W^n(x, y_+) - W^n(x, y_-)| d\mathcal{H}^1(x) \geq C_1^{-p} t 2^{-i} |\beta - \beta_i|^p = 2^{-i} C_1^{-p} t i^{-p-p\delta},$$

where  $\mathcal{H}^1$  denotes the one dimensional Hausdorff measure. Therefore, we have

$$\|W^n(\cdot, \cdot)\|_{BV([-2M, 2M]^2)} \geq \sum_{i \geq n} \sum_{j=1}^{m_i-1} \sum_{l=1}^{2^i-1} V_i \geq \sum_{i \geq n} \sum_{j=1}^{m_i-1} \sum_{l=1}^{2^i-1} t 2^{-i} i^{-p-p\delta} \geq \frac{t}{2} \sum_{i \geq n} (m_i - 1) i^{-p-p\delta}.$$

Since  $m_i = i^{p+p\delta}$  we obtain  $\|W^n(\cdot, \cdot)\|_{BV([-2M, 2M]^2)} = \infty$ . This gives a contradiction. Hence,  $\omega^n(\cdot, \cdot, t) \notin BV_{loc}^s(\mathbb{R}^2)$ .  $\square$

## Acknowledgement

Authors thank the anonymous referee for valuable comments and suggestions. The first author would like to thank Inspire faculty-research grant DST/INSPIRE/04/2016/000237. Authors would like to thank the IFCAM project ‘‘Conservation laws:  $BV^s$ , interface and control’’. The first and third authors acknowledge the support of the Department of Atomic Energy, Government of India, under project no. 12-R&D-TFR-5.01-0520. The second and the fourth authors would like to thank TIFR-CAM for the hospitality.

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