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THÈSE

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(sections 25 & 26 du CNU)

par

Jon Asier BÁRCENA PETISCO

intitulée

**Résultats sur la contrôlabilité à zéro de quelques
systèmes paraboliques et dispersifs.**

Soutenue publiquement le 3 septembre 2020 devant le jury composé de :

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Rapporteurs :	M. Marius TUCSNAK M. Enrique ZUAZUA
Examineurs :	Mme. Karine BEAUCHARD M. Franck BOYER M. Jean-Michel CORON M. Sylvain ERVEDOZA M. Enrique FERNÁNDEZ-CARA

*A mis padres, por estar siempre a mi lado,
pese a los kilómetros que a veces nos separan.*

*Science is a cooperative enterprise,
spanning the generations.
It's the passing of a torch
from teacher, to student, to teacher.
A community of minds reaching back to antiquity
and forward to the stars.*
Neil deGrasse Tyson

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Résumé

Dans ce mémoire on étudie, en adaptant les techniques de Fursikov-Imanuvilov, la contrôlabilité à zéro par l'intermédiaire de contrôles localisés à l'intérieur de quelques systèmes paraboliques et dispersifs. Plus précisément, dans le Chapitre 2 on démontre que, à l'aide d'une hypothèse géométrique, on peut contrôler à zéro un système pénalisé de Stokes dans un domaine $\Omega \subset \mathbb{R}^2$ à l'aide d'une force scalaire et avec le coût du contrôle uniforme par rapport au paramètre qui tend vers zéro. Dans le Chapitre 3 on étudie le coût du contrôle d'un système de Stokes avec des conditions aux limites de non-glissement, un terme de transport et un coefficient de viscosité qui tend vers 0. Plus concrètement, on montre que dans $(0, \pi)^2$ le coût décroît exponentiellement pour tout temps T assez grand, tandis que dans $(0, \pi)^3$ le coût croît exponentiellement pour tout temps $T > 0$ si le contrôle est localisé dans un ensemble inclus compactement dans $(0, \pi)^3$. Pour expliquer ce phénomène on utilise la décomposition spectrale de l'opérateur elliptique associé. Dans le Chapitre 4 on prouve la contrôlabilité à zéro de l'équation de la chaleur dans des domaines avec une partie cylindrique et limités par un graphe Lipschitz. Pour cela, on vaincre la difficulté de la non-existence des poids habituels et on obtient une inégalité de Carleman avec les propriétés d'absorption habituelles. Pour conclure, dans le Chapitre 5 on montre la contrôlabilité locale à zéro d'un système de KdV couplé qui modélise l'interaction de deux ondes solitaires internes dans un conduit et exercent le contrôle dans une unique composante.

Mots-clés : contrôlabilité à zéro, contrôlabilité uniforme, équation de la chaleur, inégalité de Carleman, systèmes de Korteweg-de Vries, systèmes de Stokes

Abstract

In this thesis we study, adapting the techniques of Fursikov-Imanuvilov, the null controllability via controls localized in the interior of various parabolic and dispersive systems. In particular, in Chapter 2 we prove that, considering a geometric hypothesis, we can control to zero a penalized Stokes system in a domain $\Omega \subset \mathbb{R}^2$ with a scalar force and with the cost of the control uniform with respect to the parameter which tends to zero. In Chapter 3 we study the cost of the control of a Stokes system with no-slip boundary conditions, a transport term and a coefficient of viscosity which decays to 0. In particular, we show that in $(0, \pi)^2$ the cost decays exponentially for all time T big enough, while in $(0, \pi)^3$ the cost explodes exponentially for all time $T > 0$ if the control is localized in a set compactly included in $(0, \pi)^3$. To explain this phenomenon we use the spectral decomposition of the associated elliptic operator. In Chapter 4 we prove the null controllability of the heat equation in domains with a cylindrical part and limited by a Lipschitz graph. With that purpose, we deal with the difficulty of the non-existence of the usual weights and we obtain a Carleman inequality with the usual absorption properties. To conclude, in Chapter 5 we prove the local null controllability of a KdV coupled system which models the interaction of two solitary internal waves in a stratified fluid and where the control acts in a unique component.

Keywords : Carleman inequalities, heat equation, Korteweg-de Vries systems, null controllability, Stokes systems, uniform controllability

Chapitre 1

Introduction

Il y a beaucoup de phénomènes naturels et humains qui peuvent être décrits avec des équations différentielles. De fait, elles sont utilisées pour modéliser, par exemple, des phénomènes issus de la physique (la chaleur, les ondes, la mécanique des fluides, l'élasticité, l'électromagnétisme, etc.), la biologie (les dynamiques des populations, les modèles proie-prédateur, la chimiotaxie, etc.), la médecine (la croissance des tumeurs, la tolérance aux antibiotiques, etc.) et l'économie (les jeux à champ moyen, etc.). Pour apprécier l'ampleur des phénomènes qu'on peut décrire avec des équations différentielles on peut consulter [1].

Au cours de cette thèse on travaille sur la résolution de problèmes de contrôlabilité des EDPs qui intéressent aux physiciens. Plus précisément, on travaille dans le Chapitre 2 avec un système qui approche numériquement le système de Stokes, dans le Chapitre 3 avec un système de Stokes qui possède un terme de transport et une viscosité évanescence, dans le Chapitre 4 avec l'équation de la chaleur lorsque le domaine n'est pas C^2 et dans le Chapitre 5 avec un système non-linéaire de deux équations de KdV (Korteweg-de Vries) couplées. On considère dans ce manuscrit surtout des contrôles qui agissent à l'intérieur.

Dans ce chapitre on présente dans la Section 1.1 les notions basiques de contrôlabilité, dans la Section 1.2 un résumé des travaux qui composent ce mémoire, et dans la Section 1.3 quelques perspectives et problèmes ouverts.

1.1 Introduction générale à la théorie de contrôle

On présente dans la Section 1.1.1 les notions générales de contrôlabilité et dans la Section 1.1.2 la méthode d'unicité de Hilbert (la HUM).

1.1.1 Les notions générales de contrôlabilité

En termes mathématiques, on considère dans ce manuscrit des EDP (équations aux dérivées partielles) qui peuvent être modélisées par :

$$\begin{cases} y_t + \mathcal{P}y = f1_\omega & \text{pour } t > 0, \\ y(0) = y^0, \end{cases} \quad (1.1)$$

où $t \mapsto y(t, \cdot) \in \mathcal{E}(\Omega)$ décrit l'état d'un système pour Ω un domaine (ouvert borné connexe de \mathbb{R}^d) et pour $\mathcal{E}(\Omega)$ un espace de Banach avec une variable spatiale sur Ω (normalement un espace de Lebesgue ou de Sobolev), \mathcal{P} est un opérateur différentiel (pas nécessairement linéaire) défini sur $(0, T) \times \Omega$ agissant sur la variable spatiale, $T > 0$ est le temps final de contrôle, $\omega \subset \Omega$ est un domaine (appelé *domaine de contrôle*) et f est un contrôle qui appartient à un ensemble $\mathcal{U} \subset (L^2((0, T) \times \omega))^N$, où N est le nombre de composantes de y . L'équation (1.1)₁ peut être satisfaite dans plusieurs espaces, comme dans $\mathcal{D}'((0, T) \times \Omega)$ ou dans l'espace de distributions de fonctions à divergence nulle (c'est le cas de Stokes, par exemple). On suppose aussi que (1.1) est bien posé dans $C^0([0, T]; \mathcal{E}(\Omega))$ par rapport à $y^0 \in \mathcal{E}(\Omega)$ et à $f \in \mathcal{U}$.

Le problème de la contrôlabilité consiste, pour une condition initiale y^0 donnée et un temps final $T > 0$, de trouver un contrôle $f \in \mathcal{U}$ qui mène la solution du système (1.1) vers une cible à l'instant T . Pour une présentation complète et plus générale, on peut consulter [36, Section 2.3]. On rappelle les différentes notions de contrôlabilité :

Définition 1.1 (Contrôlabilité exacte). On dit que le système de contrôle (1.1) est *exactement contrôlable* au temps T si pour tout $y^0, y^T \in \mathcal{E}(\Omega)$ il existe un contrôle $f \in \mathcal{U}(\Omega)$ tel que la solution de (1.1) satisfait $y(T, \cdot) = y^T$.

Définition 1.2 (Contrôlabilité approchée). On dit que le système de contrôle (1.1) est *approximativement contrôlable* au temps T si pour tout $y^0, y^T \in \mathcal{E}(\Omega)$ et tout $\varepsilon > 0$ il existe un contrôle $f \in \mathcal{U}(\Omega)$ tel que la solution de (1.1) satisfait $\|y(T, \cdot) - y^T\|_{\mathcal{E}(\Omega)} \leq \varepsilon$.

Définition 1.3 (Contrôlabilité aux trajectoires). On dit que le système de contrôle (1.1) est *contrôlable aux trajectoires* au temps T si pour tout $y^0 \in \mathcal{E}(\Omega)$ et pour tout $(\bar{y}^0, \bar{y}, \bar{f})$ solution de (1.1) (tel que $\bar{y} \in C^0([0, T]; \mathcal{E}(\Omega))$ et $\bar{f} \in \mathcal{U}$), il existe un contrôle $f \in \mathcal{U}(\Omega)$ tel que la solution de (1.1) satisfait $y(T, \cdot) = \bar{y}(T, \cdot)$.

Définition 1.4 (Contrôlabilité à zéro). On dit que le système de contrôle (1.1) est *contrôlable à zéro* au temps T si pour tout $y^0 \in \mathcal{E}(\Omega)$ il existe un contrôle $f \in \mathcal{U}(\Omega)$ tel que la solution de (1.1) satisfait $y(T, \cdot) = 0$.

Ces définitions sont de type *globales*, car elles ne restreignent pas la taille de la condition initiale. Hélas, lorsque \mathcal{P} n'est pas linéaire l'étude des propriétés globales du système de contrôle (1.1) peut devenir trop complexe, parfois on ne connaît même pas des méthodes pour les étudier. Par conséquent, on peut se contenter de manière provisoire d'obtenir des résultats *locaux*; c'est-à-dire, des résultats de contrôlabilité où y^0 est proche de la cible à l'instant initial.

La résolution de chaque problème de contrôlabilité dépend des propriétés particulières de l'équation sur laquelle on travaille. Par exemple, il est connu que la contrôlabilité exacte de l'équation du transport n'est pas assurée si le temps final T est petit puisque la vitesse de propagation est finie. Par contre, si ω est à côté de la partie de la frontière où le flux du transport entre, il est aussi bien connu qu'on a la contrôlabilité exacte pour T suffisamment grand. Également, pour les équations paraboliques, comme l'équation de la chaleur, il n'y a pas d'espoir d'atteindre la contrôlabilité exacte dû à l'effet régularisant, mais c'est habituel d'avoir de la contrôlabilité aux trajectoires et approchée pour tout temps $T > 0$.

Dans une grande partie de ce mémoire on se concentrera sur la contrôlabilité à zéro des systèmes paraboliques et dispersifs linéaires. On rappelle que pour ces systèmes la contrôlabilité à zéro implique la contrôlabilité aux trajectoires et la contrôlabilité approchée (voir, par exemple, [36]). En plus, même pour prouver la contrôlabilité à zéro locale d'un système non-linéaire on peut montrer d'abord la contrôlabilité du système linéarisé, et puis revenir au problème original à l'aide d'un argument d'inversion locale ou de point fixe.

Finalement, une question annexe très importante et sur laquelle on va travailler pendant cette thèse est, une fois assurée la contrôlabilité de (1.1), d'estimer le coût du contrôle. On rappelle que le coût du contrôle à zéro du système (1.1) est donné, en suivant la convention $\inf \emptyset = +\infty$, par :

$$K(\Omega, \omega, \mathcal{P}) := \sup_{y^0 \in \mathcal{E}(\Omega) \setminus \{0\}} \inf_{f \in \mathcal{U}: y(T, \cdot) = 0} \frac{\|f\|_{(L^2((0, T) \times \omega))^N}}{\|y^0\|_{\mathcal{E}(\Omega)}}, \quad (1.2)$$

où y est la solution de (1.1). Par exemple, c'est pertinent d'obtenir des estimations asymptotiques du coût de la contrôlabilité de (1.1) lorsque $T \rightarrow 0$. Également, si l'on a une suite d'opérateurs \mathcal{P}_ε on peut se demander sur l'évolution du coût de la contrôlabilité lorsque $\varepsilon \rightarrow 0$, ce qui est pertinent par exemple lorsque les systèmes avec les opérateurs \mathcal{P}_ε approchent numériquement le système avec l'opérateur \mathcal{P} .

Dans la partie suivante on présentera, de manière simple, la méthode utilisée dans le cadre des espaces de Hilbert : la HUM (“Hilbert Uniqueness Method”). C'est une méthode utilisée pour déterminer la contrôlabilité à zéro des systèmes linéaires et, en même temps, si le système est contrôlable, pour avoir des estimations sur le coût du contrôle.

1.1.2 La HUM et l'inégalité d'observabilité

Le but de cette partie est de présenter la HUM (“Hilbert Uniqueness Method”), qui est une méthode classique (voir [97, 82]) lorsque \mathcal{P} est linéaire pour construire le contrôle v de norme minimale tel que la solution y de (1.1) satisfait $y(T, \cdot) = 0$. On suppose que $\mathcal{E}(\Omega)$ est un sous-espace fermé de $(L^2(\Omega))^N$, où $N \in \mathbb{N}$, et qu'il existe un opérateur différentiel \mathcal{P}^* tel que le système :

$$\begin{cases} -\varphi_t + \mathcal{P}^* \varphi = 0 & \text{pour } t < T, \\ \varphi(T, \cdot) = \varphi^T, \end{cases} \quad (1.3)$$

est bien posé avec une solution dans $C^0([0, T]; \mathcal{E}(\Omega))$. On suppose aussi que pour tout $f \in \mathcal{U}$, et $y^0, \varphi^T \in \mathcal{E}(\Omega)$ on a :

$$\iint_{(0, T) \times \omega} f \cdot \varphi dx dt = \int_{\Omega} y(T, x) \cdot \varphi^T(x) dx - \int_{\Omega} y^0(x) \cdot \varphi(0, x) dx, \quad (1.4)$$

où y est la solution de (1.1) et φ est celle de (1.3). Pour simplifier le problème, on suppose que $\mathcal{U} = (L^2((0, T) \times \omega))^N$, même si l'on peut adapter cette méthode lorsque \mathcal{U} n'est qu'un sous-ensemble de $(L^2((0, T) \times \omega))^N$.

On remarque que si le système (1.1) est contrôlable à zéro avec $K < +\infty$ on peut estimer la norme de $\varphi(0, \cdot)$ à l'aide de la norme de φ dans $(0, T) \times \omega$. En fait, on considère dans (1.4) la valeur initiale $y^0 = \varphi(0, \cdot)$ et une suite de contrôles f_m qui mènent $\varphi(0, \cdot)$ à 0 tels que (voir (1.2)) :

$$\limsup_m \|f_m\|_{(L^2((0,T)\times\omega))^N} \leq K \|\varphi(0, \cdot)\|_{(L^2(\Omega))^N}. \quad (1.5)$$

Comme $y_m(T, \cdot) = 0$ on obtient de (1.4) pour tout $m \in \mathbb{N}$ l'égalité :

$$\int_{\Omega} |\varphi(0, x)|^2 dx = - \iint_{(0,T)\times\omega} f_m \cdot \varphi dx dt.$$

En conséquence, on obtient par Cauchy-Schwarz et par (1.5) (en prenant la limite lorsque $m \rightarrow \infty$) l'inégalité :

$$\int_{\Omega} |\varphi(0, x)|^2 dx \leq K \|\varphi(0, \cdot)\|_{(L^2(\Omega))^N} \|\varphi\|_{(L^2((0,T)\times\omega))^N},$$

ce qui implique :

$$\|\varphi(0, \cdot)\|_{(L^2(\Omega))^N} \leq K \|\varphi\|_{(L^2((0,T)\times\omega))^N}, \quad (1.6)$$

pour toute solution de (1.3) (avec $\varphi^T \in \mathcal{E}(\Omega)$).

L'inégalité (1.6) est appelée *inégalité d'observabilité*. On définit :

$$\tilde{K}(\Omega, \omega, \mathcal{P}) = \sup_{\varphi^T \in (L^2(\Omega))^N \setminus \{0\}} \frac{\|\varphi(0, \cdot)\|_{\mathcal{E}(\Omega)}}{\|\varphi\|_{(L^2((0,T)\times\omega))^N}}, \quad (1.7)$$

en attribuant la valeur $+\infty$ à l'indétermination $\frac{0}{0}$. On a obtenu dans le paragraphe précédant que :

$$\tilde{K} \leq K. \quad (1.8)$$

Grâce à la méthode HUM on peut être plus précis et prouver que :

$$K = \tilde{K}. \quad (1.9)$$

Si $\tilde{K} = +\infty$, (1.8) implique que $K = +\infty$, donc il suffit de considérer $\tilde{K} < +\infty$. Considérons l'ensemble $\mathcal{E}(\Omega)$ avec la forme bilinéaire :

$$\langle \tilde{\varphi}^T, \varphi^T \rangle \mapsto \iint_{(0,T)\times\omega} \tilde{\varphi} \cdot \varphi dx dt. \quad (1.10)$$

Comme $\tilde{K} < +\infty$, on a que $\langle \varphi^T, \varphi^T \rangle \neq 0$ pour tout $\varphi^T \neq 0$, et donc (1.10) est un produit scalaire. On déduit du théorème de représentation de Riesz et de la construction de la complétude de $\mathcal{E}(\Omega)$ l'existence d'une suite $\varphi_m^T \in \mathcal{E}(\Omega)$ telle que pour tout $\varphi^T \in \mathcal{E}(\Omega)$:

$$\lim_{m \rightarrow \infty} \iint_{(0,T) \times \omega} \varphi_m \cdot \varphi dxdt = - \int_{\Omega} y^0(x) \cdot \varphi(0, x) dx.$$

La suite φ_m est donc une suite de Cauchy dans $(L^2((0, T) \times \omega))^N$, et alors elle a une limite f . Grâce à (1.7) on a que $\varphi_m(0, \cdot)$ est une suite de Cauchy sur $\mathcal{E}(\Omega)$, et donc f a une trace sur $\mathcal{E}(\Omega)$. Alors, en prenant la limite sur :

$$\iint_{(0,T) \times \omega} f \cdot \varphi_m dxdt = - \int_{\Omega} y^0(x) \cdot \varphi_m(0, x) dx,$$

on obtient :

$$\|f\|_{(L^2((0,T) \times \omega))^N} \leq \tilde{K} \|y^0\|_{\mathcal{E}(\Omega)}.$$

En plus, la solution de (1.1) satisfait :

$$0 = \int_{\Omega} y(T, x) \cdot \varphi^T(x) dx \quad \forall \varphi^T \in \mathcal{E}(\Omega),$$

ce qui implique que $y(T, \cdot) = 0$ et que $K \leq \tilde{K}$, donc avec (1.8) on a (1.9).

Remarque 1.5. En fait, dans ‘‘HUM’’ il y a le mot ‘‘unicité’’ car, pour y^0 fixé, le contrôle f qu'on obtient avec cette méthode est l'unique contrôle avec la norme minimale sur $(L^2((0, T) \times \omega))^N$. En fait, si \tilde{f} mène y^0 à 0 on obtient par (1.4) que :

$$\iint_{(0,T) \times \omega} \tilde{f} \cdot \varphi dxdt = - \int_{\Omega} y^0(x) \cdot \varphi(0, x) dx \quad \forall \varphi^T \in \mathcal{E}(\Omega),$$

ce qui implique que :

$$\iint_{(0,T) \times \omega} (\tilde{f} - f) \cdot \varphi dxdt = 0 \quad \forall \varphi^T \in \mathcal{E}(\Omega). \quad (1.11)$$

Alors, en tenant compte du fait que f est la limite de quelques solutions de (1.3) par rapport à la norme $L^2(((0, T) \times \omega)^N)$, on déduit de (1.11) que $\tilde{f} - f \perp f$. Par conséquent, on a :

$$\|\tilde{f}\|_{(L^2((0,T) \times \omega))^N}^2 = \|\tilde{f} - f\|_{(L^2((0,T) \times \omega))^N}^2 + \|f\|_{(L^2((0,T) \times \omega))^N}^2,$$

ce qui implique que f est l'unique contrôle de norme minimale.

L'inégalité (1.6) est très importante car elle est utilisée pour démontrer la contrôlabilité de nombreuses EDP (consulter [36, Section 2] ou [109] pour lire comment elle est utilisée dans les EDP les plus classiques). En fait, cela nous permet d'avoir une méthode générale pour construire un contrôle en réduisant la preuve à la démonstration des inégalités. Par contre, il n'y a pas de méthode universelle pour démontrer l'existence de $K < +\infty$ tel qu'on a (1.6), mais cela dépend de chaque équation. Dans l'équation de la chaleur, comme c'est expliqué dans la Section 1.2.3, cela peut être démontré de plusieurs manières. Cependant, la méthode qui permet une généralisation à un plus grand nombre d'équations est celle de Fursikov-Imanuvilov (voir [57]), qui s'appuie sur les inégalités de Carleman avec des poids du type $e^{-h(x)t^{-m}(T-t)^{-m}}$, où $m \in \mathbb{R}^+$ et h est une fonction qui dépend de Ω et ω avec certaines propriétés géométriques. De fait, une grande partie de la thèse est basée sur les techniques de [57] et des poids de ce type.

1.2 Résultats principaux et plan de la thèse

Dans cette section on utilise Ω pour noter un domaine (ouvert borné connexe de \mathbb{R}^d). En plus, $Q := (0, T) \times \Omega$ (ou $Q := (0, T) \times (0, L)$ quand on travaille dans un intervalle de \mathbb{R}), $\Sigma := (0, T) \times \partial\Omega$ et n désigne le vecteur normal unitaire extérieur sur $\partial\Omega$. Également, on note $Q_\omega := (0, T) \times \omega$ pour tout ouvert ω et on utilise le gras pour désigner les espaces à valeurs vectorielles.

On commence par une présentation des résultats principaux obtenus pendant la réalisation de cette thèse :

- Dans le Chapitre 2 on étudie la contrôlabilité du système suivant :

$$\begin{cases} y_t^\varepsilon - \Delta y^\varepsilon + \nabla p^\varepsilon = f^\varepsilon 1_\omega e_1 & \text{dans } Q, \\ \varepsilon p^\varepsilon + \nabla \cdot y^\varepsilon = 0 & \text{dans } Q, \\ y^\varepsilon = 0 & \text{sur } \Sigma, \\ y^\varepsilon(0, \cdot) = y^0 & \text{sur } \Omega. \end{cases}$$

On prouve que si $\Omega \subset \mathbb{R}^2$ est un domaine régulier qui satisfait une hypothèse géométrique, le système est contrôlable à zéro avec un coût borné uniformément sur $(0, \varepsilon_0)$ pour ε_0 suffisamment petit. Pour le prouver, on utilise la stratégie de [38]. La

difficulté principale est que dans notre système le couplage est fait avec la dérivée croisée et pas avec une dérivée partielle d'ordre 1. Alors, on prouve que la dérivée croisée est coercitive si l'on considère une norme plus forte que L^2 pour la dérivée croisée, des conditions aux limites de Dirichlet, une estimation sur la trace au bord des dérivées d'ordre deux et une hypothèse géométrique sur le domaine. C'est un travail présenté dans [15].

- Dans le Chapitre 3 on étudie le coût de la contrôlabilité à zéro du système de Stokes suivant :

$$\begin{cases} y_t - \varepsilon \Delta y + \partial_{x_d} y + \nabla q = f 1_\omega & \text{dans } Q, \\ \nabla \cdot y = 0 & \text{dans } Q, \\ y \cdot n = 0, \quad (Dy \cdot n)_{\text{tg}} = 0 & \text{sur } \Sigma, \\ y(0, \cdot) = y^0 & \text{sur } \Omega. \end{cases}$$

Concernant, il est connu (voir [67]) que ce système est contrôlable à zéro, mais la dépendance du coût par rapport au coefficient de viscosité restait à préciser. On montre que dans $(0, \pi)^2$ le coût est exponentiellement petit par rapport à ε si T est suffisamment grand, et qu'il est exponentiellement grand par rapport à ε si T est petit et ω est inclus compactement dans $(0, \pi)^2$. En plus, on prouve que dans $(0, \pi)^3$ pour tout $T > 0$ si ω est inclus compactement dans $(0, \pi)^3$, le coût est exponentiellement grand par rapport à ε . Concernant les techniques utilisées, pour démontrer ces résultats on utilise les techniques introduites dans [37, 71], mais on utilise aussi une technique originale dans des problèmes de transport-diffusion avec une viscosité évanescence : étudier les fonctions propres de l'opérateur elliptique associé au système adjoint. En fait, la décomposition spectrale explique (voir Remarque 1.16 et Remarque 1.19) les causes des différences entre la dimension 2 et la dimension 3. Ce problème est traité dans [14].

- Dans le Chapitre 4 on prouve la contrôlabilité à zéro de l'équation de la chaleur dans une famille de domaines avec une partie cylindrique limitée par un graphe Lipschitz. Pour montrer cela, on montre une inégalité de Carleman qui présente les propriétés d'absorption habituelles en adaptant les preuves de [57, 74]. La difficulté principale est la non-existence des poids habituels pour les domaines C^2 . Pour vaincre cela, on utilise la structure cylindrique et on approxime le système adjoint par le même système posé sur des domaines réguliers. Pour construire ces domaines, on considère les courbes de niveau d'une régularisation de la fonction distance à la frontière. Une autre nouveauté de cet article est que c'est la première fois où les techniques de

Fursikov-Imanuvilov sont adaptées à des domaines où les solutions de l'équation de la chaleur avec condition aux limites de Dirichlet et avec données régulières qui sont à support compact n'appartiennent pas à $L^2(0, T; H^2(\Omega))$. C'est un travail présenté dans [13].

- Dans le Chapitre 5 on prouve la contrôlabilité locale à 0 du système de KdV couplé suivant :

$$\begin{cases} u_t + uu_x + u_{xxx} + av_{xxx} + k_1vv_x + k_2(uv)_x = \mathfrak{h}_1 & \text{dans } Q, \\ cv_t + rv_x + vv_x + abu_{xxx} + v_{xxx} + k_2buu_x + k_1b(uv)_x = \mathfrak{h}_2 + f1_\omega & \text{dans } Q, \\ u(\cdot, 0) = u(\cdot, L) = u_x(\cdot, L) = 0 & \text{sur } (0, T), \\ v(\cdot, 0) = v(\cdot, L) = v_x(\cdot, L) = 0 & \text{sur } (0, T), \\ u(0, \cdot) = u^0, \quad v(0, \cdot) = v^0 & \text{sur } (0, L), \end{cases}$$

où $k_1, k_2, r, a, b, c \in \mathbb{R}$, $a, b, c > 0$, $a^2b < 1$, $T > 0$, $L > 0$ et \mathfrak{h}_1 et \mathfrak{h}_2 sont des fonctions qui décroissent exponentiellement au temps T (voir (1.66)). La nouveauté principale est que c'est la première preuve sur la contrôlabilité à zéro d'un système avec deux composantes et un couplage d'ordre 3 par l'intermédiaire d'un contrôle qui agit sur une unique composante. Pour montrer cela, on prouve d'abord la contrôlabilité du système linéarisé à l'aide d'une inégalité de Carleman, et après on revient au problème de contrôle original par un argument d'inversion locale. Ce problème est traité dans [17].

Dans ce qui suit, on présente chaque chapitre de ce mémoire de façon détaillée.

1.2.1 Contrôlabilité à zéro d'un problème de Stokes pénalisé en dimension deux avec un contrôle scalaire (voir [15])

Dans cette partie on étudie la contrôlabilité d'un système qui approche numériquement le système de Stokes. Concrètement, on étudie la contrôlabilité à zéro de :

$$\begin{cases} y_t^\varepsilon - \Delta y^\varepsilon + \nabla p^\varepsilon = 0 & \text{dans } Q, \\ \varepsilon p^\varepsilon + \nabla \cdot y^\varepsilon = 0 & \text{dans } Q, \\ y^\varepsilon = 0 & \text{sur } \Sigma, \\ y^\varepsilon(0, \cdot) = y^0 & \text{sur } \Omega. \end{cases} \quad (1.12)$$

Le système (1.12) est utilisé pour approcher numériquement le système de Stokes vu que la condition d'incompressibilité peut causer des problèmes. Cette approximation a été introduite dans [102]. La contrôlabilité de (1.12) a été étudiée dans la littérature (voir [73, 9]), mais uniquement quand le contrôle appartient à $\mathbf{L}^2(Q_\omega)$. Nous nous intéressons à la contrôlabilité à zéro avec $N - 1$ contrôles scalaires, ce qui n'avait pas encore été étudié. Plus précisément, on étudie :

$$\begin{cases} y_t^\varepsilon - \Delta y^\varepsilon + \nabla p^\varepsilon = f^\varepsilon 1_\omega e_1 & \text{dans } Q, \\ \varepsilon p^\varepsilon + \nabla \cdot y^\varepsilon = 0 & \text{dans } Q, \\ y^\varepsilon = 0 & \text{sur } \Sigma, \\ y^\varepsilon(0, \cdot) = y^0 & \text{sur } \Omega, \end{cases} \quad (1.13)$$

où $\Omega \subset \mathbb{R}^2$ est un domaine, $e_1 := (1, 0)$ et $y^0 \in \mathbf{L}^2(\Omega)$.

Remarque 1.6. L'aspect numérique de (1.13) est intéressant pour les données initiales :

$$y^0 \in \mathcal{H}(\Omega) := \{u \in \mathbf{L}^2(\Omega) : \nabla \cdot u = 0 \text{ dans } \Omega \text{ et } u \cdot n = 0 \text{ sur } \partial\Omega\}.$$

Dans ces cas, si $(f^\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ est une suite de contrôles bornée dans $L^2(Q_\omega)$ telle que $y^\varepsilon(T, \cdot) = 0$ pour tout $\varepsilon \in (0, \varepsilon_0)$, il y a une sous-suite f^{ε_k} convergente faiblement vers $f \in L^2(Q_\omega)$, et donc, y^{ε_k} tend vers y , la solution du système de Stokes avec valeur initiale y^0 et contrôle $f 1_\omega e_1$, qui satisfait $y(T, \cdot) = 0$. Cependant, comme les solutions de (1.13) pour $y^0 \in \mathcal{H}(\Omega)$ n'appartiennent pas à $C^0([0, T]; \mathcal{H}(\Omega))$ mais à $C^0([0, T]; \mathbf{L}^2(\Omega))$, pour pouvoir appliquer la HUM (voir Section 1.1.2) on travaille avec toutes les valeurs initiales dans $\mathbf{L}^2(\Omega)$, même si les seules conditions initiales intéressantes sont celles dans $\mathcal{H}(\Omega)$.

D'abord, on prouve que pour tout $\varepsilon > 0$ il y a un losange $\Omega_\varepsilon \subset \mathbb{R}^2$ tel que (1.13) n'est pas contrôlable à zéro avec un seul contrôle scalaire. On prouve ce résultat en montrant que la propriété de continuation unique dans ces domaines Ω_ε est fautive, et pour montrer cela on fait une recherche systématique dans l'ensemble des fonctions à variables séparées. L'existence de ces domaines est une difficulté implicite importante car elle implique que si l'on prouve la contrôlabilité de (1.13) pour des domaines réguliers la preuve doit avoir des raisonnements qui ne soient pas applicables dans le cas du losange.

Pour traiter la difficulté implicite on formule l'hypothèse suivante :

Hypothèse 1.7. Soit $\Omega \subset \mathbb{R}^2$ un domaine C^2 dont la frontière $\partial\Omega$ est paramétrisée par des fonctions $\sigma^1, \dots, \sigma^k$ de longueur d'arc. Alors, pour tout $i \in \{1, \dots, k\}$ et pour tout θ tels que $(\sigma_1^i)'(\theta) = 0$ ou $(\sigma_2^i)'(\theta) = 0$, on a que $\kappa^i(\theta) \neq 0$.

Plus précisément, on prouve le résultat suivant :

Théorème 1.8. *Soient $\Omega \subset \mathbb{R}^2$ un domaine régulier satisfaisant l'Hypothèse 1.7 et $\omega \subset \Omega$ un domaine. Alors, il y a $\varepsilon_0 > 0$ tel que pour tout $T > 0$ il y a $C > 0$ tel que si $\varepsilon \in (0, \varepsilon_0)$ et $y^0 \in \mathbf{L}^2(\Omega)$, il y a une fonction scalaire $f^\varepsilon \in L^2(Q_\omega)$ satisfaisant :*

$$\|f^\varepsilon\|_{L^2(Q_\omega)} \leq C \|y^0\|_{\mathbf{L}^2(\Omega)},$$

et telle que la solution de (1.13) satisfait $y^\varepsilon(T, \cdot) = 0$.

Remarque 1.9. L'Hypothèse 1.7 est satisfaite pour tout domaine strictement convexe. En plus, par le théorème de Sard, pour tout domaine Ω et presque tout $\psi \in [0, 2\pi)$ le domaine $U_\psi(\Omega)$ satisfait l'Hypothèse 1.7, où U_ψ est la rotation de ψ radians. Par conséquent, compte tenu du fait que le système (1.12) est invariant par rotations, on sait que, au moins pour tout domaine régulier Ω le système (1.12) est contrôlable à zéro à l'aide d'un contrôle scalaire supporté dans ω pour presque toute direction (avec la mesure de Lebesgue sur $[0, \pi)$). Alors, le Théorème 1.8 est applicable à tout domaine régulier de \mathbb{R}^2 après une rotation.

La preuve du Théorème 1.8 repose sur la HUM. Notamment, on prouve l'inégalité d'observabilité du système adjoint de (1.13), qui est donné par :

$$\begin{cases} -\varphi_t^\varepsilon - \Delta\varphi^\varepsilon + \nabla\pi^\varepsilon = 0 & \text{dans } Q, \\ \varepsilon\pi^\varepsilon + \nabla \cdot \varphi^\varepsilon = 0 & \text{dans } Q, \\ \varphi^\varepsilon = 0 & \text{sur } \Sigma, \\ \varphi^\varepsilon(T, \cdot) = \varphi^T & \text{sur } \Omega. \end{cases} \quad (1.14)$$

Concrètement, on prouve l'inégalité de Carleman suivante :

Théorème 1.10. *Soient Ω un domaine régulier qui satisfait l'Hypothèse 1.7, $\omega \subset \Omega$ un ouvert et $m \geq 8$. Alors, il y a $\varepsilon_0 > 0$, $C > 0$ et $\lambda_0 \geq 1$ tels que si $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\varphi^T \in \mathbf{L}^2(\Omega)$, $\lambda \geq \lambda_0$, et $s \geq e^{C\lambda}(T^m + T^{2m})$, on a :*

$$s^{15}\lambda^{16} \iint_Q e^{-2s\alpha^*} (\xi^*)^{15} |\varphi^\varepsilon|^2 \leq Cs^{34}\lambda^{35} \iint_{Q_\omega} e^{-2s\alpha} \xi^{34} |\varphi_1^\varepsilon|^2, \quad (1.15)$$

où φ^ε est la solution de (1.14).

Les poids α et ξ sont des fonctions du type $(C - h(x))t^{-m}(T - t)^{-m}$ et $h(x)t^{-m}(T - t)^{-m}$ respectivement, où h est une fonction qui dépend de Ω et ω avec certaines propriétés géométriques. En plus, $\xi^*(t)$ est le minimum de $\xi(t, \cdot)$ sur $\bar{\Omega}$ et $\alpha^*(t)$ est le maximum de $\alpha(t, \cdot)$ sur $\bar{\Omega}$. On peut consulter (2.8) pour avoir la définition exacte des poids.

Pour prouver le Théorème 1.10 on suit la stratégie de [38]. D'abord, on trouve une inégalité de Carleman pour les solutions de (1.14) avec des conditions aux limites non-homogènes. Plus précisément, on prouve :

Proposition 1.11. *Soient Ω un domaine C^4 , $\tilde{\omega}$ un sous-domaine de Ω tel que $\bar{\omega}_0 \subset \tilde{\omega}$ et $m \geq 8$. Alors, il y a $\varepsilon_0 > 0$, $C > 0$ et $\lambda_0 \geq 1$ tels que si $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\varphi^T \in \mathbf{L}^2(\Omega)$, $h \in \mathbf{H}^{2,5/2}(\Sigma)$, $\lambda \geq \lambda_0$ et $s \geq e^{C\lambda}(T^m + T^{2m})$, on a :*

$$s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |\varphi^\varepsilon|^2 + s \lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla \varphi^\varepsilon|^2 \leq C \left(s^4 \lambda^5 \iint_{Q_{\tilde{\omega}}} e^{-2s\alpha} \xi^4 |\varphi^\varepsilon|^2 + (1 + T) \left(\|(s\xi^*)^{1/4+1/m} e^{-s\alpha^*} h\|_{\mathbf{H}^{1,1/2}(\Sigma)}^2 + \|(s\xi^*)^{-3/4} e^{-s\alpha^*} h\|_{\mathbf{H}^{2,5/2}(\Sigma)}^2 \right) \right), \quad (1.16)$$

où φ^ε est la solution du système :

$$\begin{cases} -\varphi_t^\varepsilon - \Delta \varphi^\varepsilon + \nabla \pi^\varepsilon = 0 & \text{dans } Q, \\ \varepsilon \pi^\varepsilon + \nabla \cdot \varphi^\varepsilon = 0 & \text{dans } Q, \\ \partial_n \varphi^\varepsilon - \pi^\varepsilon n = h & \text{sur } \Sigma, \\ \varphi^\varepsilon(T, \cdot) = \varphi^T & \text{sur } \Omega. \end{cases} \quad (1.17)$$

D'abord, on utilise que φ^ε satisfait l'équation de la chaleur avec terme source $(1 + \varepsilon)^{-1} \nabla \times (\nabla \times \varphi^\varepsilon)$. En plus, on utilise que le rotationnel de φ satisfait l'équation de la chaleur, ce qui nous permet d'appliquer l'inégalité de Carleman donnée dans [52] pour l'équation de la chaleur. Finalement, on utilise les résultats de régularité de (1.17) pour traiter les traces.

Maintenant on va traiter la différence principale avec [38], qui est que le couplage dans (1.14) est fait par la dérivée croisée et pas par un opérateur différentiel d'ordre 1. On doit donc prouver la coercivité de la dérivée croisée, ce qui est la nouveauté principale de cet article. On sait que la dérivée croisée n'est pas coercitive dans un domaine régulier quelconque, même avec des conditions aux limites de type Dirichlet (on peut considérer,

par exemple, le disque unité avec une fonction $1 - x_1^2 - x_2^2$). Par contre, on peut obtenir sa coercivité si l'on rajoute un terme de bord :

Théorème 1.12. *Soit Ω un domaine C^4 satisfaisant l'Hypothèse 1.7. Alors, pour tout $a_0 > 0$ suffisamment petit, il y a $C > 0$ tel que pour toute fonction $u \in H^4(\Omega) \cap H_0^1(\Omega)$ et pour tout $a \in [0, a_0]$ on a que :*

$$\|\partial_{x_1} u\|_{C^0(\bar{\Omega})} \leq C(\|\partial_{x_1 x_2}^2 u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}), \quad (1.18)$$

où :

$$L_a u = -a\partial_{x_1}^2 u - \partial_{x_2}^2 u. \quad (1.19)$$

On ne sait pas si le résultat (1.18) est optimal, mais on sait au moins que le terme au bord et les conditions de Dirchlet sont nécessaires. De même, concernant la norme de la dérivée croisée on peut dire que la norme $L^2(\Omega)$ de la dérivée croisée ne suffit pas :

Proposition 1.13. *Soit Ω un domaine C^4 avec une frontière caractérisée par l'équation :*

$$g(x_1) + h(x_2) = 0,$$

où $g, h \in C^4(\mathbb{R})$. Alors, il n'existe pas de $C > 0$ tel que pour tout $u \in H^4(\Omega) \cap H_0^1(\Omega)$ on ait :

$$\|u\|_{L^2(\Omega)} \leq C(\|\partial_{x_1 x_2}^2 u\|_{L^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \quad (1.20)$$

On prouve la Proposition 1.13 par contradiction en approximant le Laplacien de la fonction $g(x_1) + h(x_2)$.

On prouve d'abord le Théorème 1.12 dans le cas où Ω est strictement convexe en utilisant les conditions aux limites de Dirchlet et l'égalité (1.19). Avec ces équations, on obtient une EDO (équation différentiel ordinaire) sur la frontière, avec un terme qui s'écrit en fonction de $\partial_{x_1 x_2}^2 u$ et $L_a u$, après on définit une variable auxiliaire et on fait des estimations, et finalement on estime $\partial_{x_1} u$ en termes de la fonction auxiliaire. Un point important de la preuve consiste à démontrer que si l'on a une estimation de $\partial_{x_1} u$ en un point p on l'a sur la droite verticale qui contient ce point.

Pour prouver le Théorème 1.12 dans tout domaine Ω qui satisfait l'Hypothèse 1.7 on coupe le domaine dans un nombre fini de morceaux. Après, on donne la preuve dans chaque situation par récurrence (en considérant la borne gauche de chaque morceau) en adaptant la preuve des domaines strictement convexes à chaque situation particulière.

Soit maintenant φ^ε la solution de (1.14). En considérant les conditions de Dirichlet, (1.14)₁ et (1.14)₂ on obtient que sur Σ on a :

$$\begin{cases} -\partial_{x_1}^2 \varphi_1^\varepsilon - \frac{\varepsilon}{1+\varepsilon} \partial_{x_2}^2 \varphi_1^\varepsilon = \frac{1}{1+\varepsilon} \partial_{x_1 x_2}^2 \varphi_2^\varepsilon, \\ -\frac{\varepsilon}{1+\varepsilon} \partial_{x_2}^2 \varphi_2^\varepsilon - \partial_{x_2}^2 \varphi_2^\varepsilon = \frac{1}{1+\varepsilon} \partial_{x_1 x_2}^2 \varphi_1^\varepsilon. \end{cases}$$

Alors, en utilisant le Théorème 1.12, sa version symétrique et l'inégalité de Poincaré on obtient que si Ω satisfait l'Hypothèse 1.7 pour tout $t \in [0, T]$ et $\varepsilon \in (0, \varepsilon_0]$ on a :

$$\|\varphi^\varepsilon(t, \cdot)\|_{\mathbf{L}^2(\Omega)} \leq C \|\partial_{x_1 x_2}^2 \varphi^\varepsilon(t, \cdot)\|_{\mathbf{H}^2(\Omega)}. \quad (1.21)$$

Avec (1.21) et (1.16) on suit la méthode de [38] et on prouve (1.15). En fait, on estime les dérivées croisées en considérant encore plus de dérivées, après on estime les traces avec les résultats de régularité et finalement on estime les termes locaux avec l'inégalité de Cauchy-Schwarz.

Remarque 1.14. C'est dans la preuve du Théorème 1.12 où l'Hypothèse 1.7 est nécessaire. Pour la plupart des inégalités de Carleman on utilise des estimations paraboliques simples qui n'ont besoin d'aucune hypothèse géométrique autre que la régularité du domaine.

1.2.2 Contrôlabilité uniforme d'un problème de Stokes avec un terme de transport et une viscosité évanescence (voir [14])

Un sujet de recherche important en contrôlabilité est l'étude du coût des contrôles des EDP paraboliques ou dispersives avec une diffusion ou dispersion évanescence. Dans ce sens, il y a des travaux dans la littérature, par exemple, sur l'équation de la chaleur (voir [37, 71]) et sur l'équation de KdV (voir [62, 63, 24, 25]). Dans cette partie on étudie à quel point les résultats des documents cités précédemment sont généralisables au système de Stokes incompressible avec des conditions aux limites de non-glissement, qui fait partie des EDP paraboliques classiques. On rappelle que l'espace fonctionnel naturel pour modéliser les fluides incompressibles est :

$$\mathcal{H}(\Omega) := \{u \in \mathbf{L}^2(\Omega) : \nabla \cdot u = 0 \text{ dans } \Omega \text{ et } u \cdot n = 0 \text{ sur } \partial\Omega\}.$$

Dans cette partie on travaille dans $\Omega_d := (0, 1)^d$. Plus précisément, on étudie dans Ω_2 et Ω_3 la contrôlabilité à zéro du système suivant :

$$\begin{cases} y_t - \varepsilon \Delta y + \partial_{x_d} y + \nabla q = f 1_\omega & \text{dans } Q, \\ \nabla \cdot y = 0 & \text{dans } Q, \\ y \cdot n = 0, \quad (Dy \cdot n)_{\text{tg}} = 0 & \text{sur } \Sigma, \\ y(0, \cdot) = y^0 & \text{sur } \Omega. \end{cases} \quad (1.22)$$

On rappelle que le gradient symétrisé est donné par $Du := \frac{1}{2} (\partial_{x_i} u_j + \partial_{x_j} u_i)_{i,j}$. En plus, on considère les ensembles de contrôles admissibles suivantes :

$$\begin{cases} S_1(y^0) := \{f \in \mathbf{L}^2(Q_\omega) : \Phi_2(y^0, f)(T, \cdot) = 0\}, \\ S_2(y^0) := \{f_2 \in L^2(Q_\omega) : \Phi_2(y^0, (0, f_2))(T, \cdot) = 0\}, \\ S_3(y^0) := \{f \in \mathbf{L}^2(Q_\omega) : \Phi_3(y^0, f)(T, \cdot) = 0\}, \end{cases}$$

où $y^0 \in \mathcal{H}(\Omega_d)$, $f \in \mathbf{L}^2(Q_\omega)$ et $\Phi_d(y^0, f)$ est la solution de (1.22) dans Ω_d . Grâce à [67] on sait que S_1 et S_3 sont non-vides, tandis qu'on prouve que S_2 est non-vide dans la Section 3.3.1. Quant au coût du contrôle, on le mesure avec les normes habituelles :

$$\begin{cases} K_1(T, \varepsilon, \omega) := \sup_{y^0 \in \mathcal{H}(\Omega_2) \setminus \{0\}} \inf_{f \in S_1(y^0)} \frac{\|f\|_{\mathbf{L}^2(Q_\omega)}}{\|y^0\|_{\mathbf{L}^2(\Omega_2)}}, \\ K_2(T, \varepsilon, \omega) := \sup_{y^0 \in \mathcal{H}(\Omega_2) \setminus \{0\}} \inf_{f_2 \in S_2(y^0)} \frac{\|f_2\|_{L^2(Q_\omega)}}{\|y^0\|_{\mathbf{L}^2(\Omega_2)}}, \\ K_3(T, \varepsilon, \omega) := \sup_{y^0 \in \mathcal{H}(\Omega_3) \setminus \{0\}} \inf_{f \in S_3(y^0)} \frac{\|f\|_{\mathbf{L}^2(Q_\omega)}}{\|y^0\|_{\mathbf{L}^2(\Omega_3)}}. \end{cases} \quad (1.23)$$

Les résultats principaux de ce chapitre sont les suivants :

Theorem 1.15. *On a les résultats suivants pour le problème de contrôle (1.22) :*

1. *Soit $\omega \subset \Omega_2$ un domaine. Alors, il y a $c, C, T_0 > 0$ tels que, si $T > T_0$ et $\varepsilon \in (0, 1)$:*

$$K_2(T, \varepsilon, \omega) \leq C e^{-c\varepsilon^{-1}}.$$

2. *Soient $h \in (0, \pi)$ et $\omega \subset (0, \pi) \times (\pi - h, \pi)$ un domaine. Alors, pour tout $T \in (0, 2(\pi - h))$, il y a $c > 0$ tel que, si $\varepsilon \in (0, 1)$:*

$$K_1(T, \varepsilon, \omega) \geq c e^{c\varepsilon^{-1}}.$$

3. Soient $h \in (0, \pi)$ et $\omega \subset (0, \pi) \times (0, h)$ un domaine. Alors, pour tout $T \in (0, \pi - h)$, il y a $c > 0$ tel que, si $\varepsilon \in (0, 1)$:

$$K_1(T, \varepsilon, \omega) \geq ce^{c\varepsilon^{-1}}.$$

4. Soient $h \in (0, \pi)$ et $\omega \subset (0, \pi)^2 \times (\pi - h, \pi)$ un domaine. Alors, pour tout $T > 0$ il y a $c > 0$ tel que, si $\varepsilon \in (0, 1)$:

$$K_3(T, \varepsilon, \omega) \geq ce^{c\varepsilon^{-1}}. \quad (1.24)$$

Remarque 1.16. Les trois premières propriétés du Théorème 1.15 sont des propriétés analogues à celles présentes dans [37, 71, 62, 63, 24, 25]. Par contre, la quatrième propriété est différente et montre (en lisant sa preuve) que les conditions aux limites sont importantes pour avoir de la contrôlabilité avec un coût exponentiellement petit lorsque la viscosité tend vers 0 et le temps est suffisamment grand. La cause de la différence entre Ω_2 et Ω_3 est que la condition d'incompressibilité est plus restrictive en dimension 2 qu'en dimension 3 pour construire des solutions (car en dimension 2 c'est une équation pour deux composantes, tandis qu'en dimension 3 c'est une équation pour trois composantes).

Pour obtenir les résultats de contrôlabilité on travaille avec le système adjoint, qui est donné par :

$$\begin{cases} -\varphi_t - \varepsilon \Delta \varphi - \partial_{x_d} \varphi + \nabla p = 0 & \text{dans } Q, \\ \nabla \cdot \varphi = 0 & \text{dans } Q, \\ \varphi \cdot n = 0, \quad (2\varepsilon D\varphi \cdot n + \varphi n_d)_{\text{tg}} = 0 & \text{sur } \Sigma, \\ \varphi(T, \cdot) = \varphi^T & \text{sur } \Omega, \end{cases} \quad (1.25)$$

où d est la dimension de Ω . On obtient avec des techniques classiques (voir la HUM dans la Section 1.1.2) les égalités suivantes :

$$\begin{cases} [K_1(T, \varepsilon, \omega)]^2 = \sup_{\varphi^T \in \mathcal{H}(\Omega_2) \setminus \{0\}} \frac{\int_{\Omega_2} |\varphi(0, x)|^2 dx}{\iint_{Q_\omega} |\varphi|^2 dx dt}, \\ [K_2(T, \varepsilon, \omega)]^2 = \sup_{\varphi^T \in \mathcal{H}(\Omega_2) \setminus \{0\}} \frac{\int_{\Omega_2} |\varphi(0, x)|^2 dx}{\iint_{Q_\omega} |\varphi_2|^2 dx dt}, \\ [K_3(T, \varepsilon, \omega)]^2 = \sup_{\varphi^T \in \mathcal{H}(\Omega_3) \setminus \{0\}} \frac{\int_{\Omega_3} |\varphi(0, x)|^2 dx}{\iint_{Q_\omega} |\varphi|^2 dx dt}. \end{cases} \quad (1.26)$$

Remarque 1.17. Le système (1.25) a des propriétés particulières sur Ω_d . Le gradient de la pression de toute solution de (1.25) est nul, ce qui implique que les composantes de toute solution de (1.25) sont des solutions de l'équation de la chaleur.

Pour étudier les solutions de (1.25) on obtient la décomposition spectrale de l'opérateur elliptique associé à (1.25); c'est-à-dire, on obtient les solutions de :

$$\begin{cases} -\varepsilon\Delta u - \partial_{x_d}u + \nabla p = \lambda u & \text{dans } \Omega, \\ \nabla \cdot u = 0 & \text{dans } \Omega, \\ u \cdot n = 0, \quad (2\varepsilon Du \cdot n + un_d)_{\text{tg}} = 0 & \text{sur } \partial\Omega. \end{cases} \quad (1.27)$$

L'approche de faire une décomposition spectrale n'est pas nouvelle en contrôlabilité (voir, par exemple, [36]). Concernant les systèmes avec diffusion (ou dispersion) évanescence, cette approche a été utilisée dans [37, 88] mais seulement pour obtenir des bornes du temps T_0 à partir duquel le coût du contrôle décroît exponentiellement avec ε . Par contre, c'est la première fois que la décomposition spectrale a été utilisée pour prouver l'inégalité de dissipation.

Pour obtenir les fonctions propres dans Ω_2 on considère les fonctions à variables séparées compatibles avec les conditions aux limites de (1.27) :

$$SV(\Omega_2) := \left\{ \left(\int_0^{x_1} g_1(s) ds g_2(x_2), -g_1(x_1) \int_0^{x_2} g_2(s) ds \right) : g_1, g_2 \in L_0^2(0, \pi) \right\}.$$

On prouve d'abord l'égalité :

$$\overline{\text{span}(SV(\Omega_2))} = \mathcal{H}(\Omega_2). \quad (1.28)$$

Après, on prouve par une recherche systématique que les vecteurs propres sont :

$$u_m(x) := e^{-(2\varepsilon)^{-1}x_2} \left(\frac{\sin(m_1x_1)}{m_1} (2m_2\varepsilon \cos(m_2x_2) - \sin(m_2x_2)), -2\varepsilon \cos(m_1x_1) \sin(m_2x_2) \right),$$

où $m = (m_1, m_2) \in (\mathbb{N}_*)^2$, et que ses valeurs propres associées sont :

$$\lambda_m^\varepsilon := (m_1^2 + m_2^2)\varepsilon + \frac{1}{4\varepsilon}.$$

En fait, on a l'égalité :

$$\overline{\text{span}(u_m)} = \mathcal{H}(\Omega_2), \quad (1.29)$$

ce qui n'est pas un résultat classique car l'opérateur elliptique associé au système (1.27) n'est pas auto-adjoint. Pour prouver (1.29) on prouve d'abord que $SV(\Omega_2) \subset \overline{\text{span}(u_m)}$ en utilisant des bases trigonométriques connues de $L^2(0, \pi)$, ce qui, avec (1.28), implique (1.29).

Une fois les fonctions propres calculées, on cherche une suite v_m dans $\mathbf{L}^2(\Omega_2)$ telle que $\{u_m, v_m\}$ soit bi-orthogonal; c'est-à-dire, tel que :

$$\int_{\Omega_2} u_m(x) \cdot v_{m'}(x) dx = 1_{m=m'}. \quad (1.30)$$

On rappelle que comme l'opérateur elliptique associé au système (1.27) n'est pas auto-adjoint, u_m n'est pas orthogonal, donc ce n'est pas une question mineure. Avec une recherche systématique en considérant les bases trigonométriques de $L^2(0, \pi)$, on trouve que (1.30) est satisfait pour :

$$v_m(x) := \frac{2}{\pi^2} e^{(2\varepsilon)^{-1}x_2} \left(m_1 \sin(m_1 x_1) \frac{2m_2 \varepsilon \cos(m_2 x_2) - \sin(m_2 x_2)}{1 + 4m_2^2 \varepsilon^2}, -\frac{\cos(m_1 x_1) \sin(m_2 x_2)}{2\varepsilon} \right). \quad (1.31)$$

Alors, on est prêts à exprimer les solutions de (1.25) à l'aide de la décomposition spectrale :

Proposition 1.18. *Soit $\varphi^T \in \mathcal{H}(\Omega_2)$. On désigne pour $t \leq T$, $L_\varepsilon^T(t)\varphi^T$ la valeur dans $\mathcal{H}(\Omega_2)$ de la solution d'énergie du système (1.25) en Ω_2 au temps t . Alors, on a que :*

1. *Pour tout $t < T$ et $\varepsilon > 0$:*

$$L_\varepsilon^T(t)\varphi^T = \sum_{m \in (\mathbb{N}_*)^2} \langle \varphi^T, v_m \rangle_{\mathbf{L}^2(\Omega_2)} \exp \left[\left((m_1^2 + m_2^2)\varepsilon + \frac{1}{4\varepsilon} \right) (t - T) \right] u_m. \quad (1.32)$$

En particulier, la série du côté droit de (1.32) est bien définie et absolument convergente dans $\mathcal{H}(\Omega_2)$.

2. *Pour tout $\delta > 0$ il y a $C_\delta > 0$ tel que pour tout $T > 0$, $\varphi^T \in \mathcal{H}(\Omega_2)$, $\varepsilon \in (0, 1)$ et $s \leq T - 2\pi - \delta$:*

$$\|L_\varepsilon^T(s)\varphi^T\|_{\mathbf{L}^2(\Omega_2)} \leq C_\delta \exp \left[\frac{s - (T - 2\pi - \delta)}{4\varepsilon} \right] \|\varphi^T\|_{\mathbf{L}^2(\Omega_2)}. \quad (1.33)$$

La preuve de la Proposition 1.18 se fait en prouvant les estimations pour $\varphi^T \in \text{span}(u_m)$ et en considérant la continuité par rapport à la valeur initiale. Pour obtenir les estimations on utilise l'inégalité triangulaire et on borne la série par des intégrales faciles à calculer.

Avec la Proposition 1.18 et (1.26) on est prêts à prouver les trois premiers points du Théorème 1.15 :

- Pour le premier point on suit la stratégie de [71] : avec l'inégalité de dissipation (1.33) et une inégalité de Carleman qu'on prouve, on obtient que le coût du contrôle décroît exponentiellement avec ε . Pour prouver l'inégalité de Carleman on utilise que la deuxième composante des solutions de (1.25) satisfait l'équation de la chaleur (voir la Remarque 1.17) et après on obtient l'inégalité pour la première composante en utilisant la condition d'incompressibilité.
- Pour le deuxième point on considère $\varphi(t, x) = e^{-\lambda_{1,1}^\varepsilon(T-t)}u_{1,1}(x)$, qui est une solution de (1.25), et on fait des estimations assez simples.
- Pour le troisième point on adapte la technique de [71, Théorème 1]. Dans [71] les auteurs utilisent le fait qu'une solution à valeur initiale positive reste positive, propriété qu'on n'a pas pour les solutions de (1.25). Par contre, on vérifie que si l'on prend une valeur initiale φ^T telle que $\text{supp}(\varphi^T) \subset (0, \pi) \times (\pi - \delta, \pi)$, il existe $c > 0$ tel que pour tout $\varepsilon \in (0, 1)$ on a :

$$\|\varphi^\varepsilon(0, \cdot)\|_{\mathbf{L}^2(\Omega)} \geq c,$$

inégalité qu'on montre par contradiction. Ensuite, on obtient des estimations de φ_2 sur Q_ω à l'aide d'une inégalité d'Agmon comme dans [71] en utilisant que φ_2 est une solution de la chaleur (voir Remarque 1.17). Pour conclure, on utilise des estimations de régularité parabolique pour avoir l'estimation de $\partial_{x_2}\varphi_2$, et donc avec la condition d'incompressibilité et Poincaré on obtient des estimations de φ_1 .

Finalement, pour prouver le quatrième point du Théorème 1.15 on cherche des solutions de (1.27) dans Ω_3 . De nouveau, on considère les fonctions à variables séparées compatibles avec les conditions aux limites de (1.27) :

$$\begin{aligned} SV(\Omega_3) &:= SV_1(\Omega_3) \cup SV_2(\Omega_3) \cup SV_3(\Omega_3) := \\ &\left\{ \left(0, g_1(x_1) \int_0^{x_2} g_2(s) ds g_3(x_3), -g_1(x_1)g_2(x_2) \int_0^{x_3} g_3(s) ds \right) : g_1 \in L^2(0, \pi); g_2, g_3 \in L_0^2(0, \pi) \right\} \\ &\cup \left\{ \left(-\int_0^{x_1} g_1(s) ds g_2(x_2)g_3(x_3), 0, g_1(x_1)g_2(x_2) \int_0^{x_3} g_3(s) ds \right) : g_2 \in L^2(0, \pi); g_1, g_3 \in L_0^2(0, \pi) \right\} \\ &\cup \left\{ \left(\int_0^{x_1} g_1(s) ds g_2(x_2)g_3(x_3), -g_1(x_1) \int_0^{x_2} g_2(s) ds g_3(x_3), 0 \right) : g_3 \in L^2(0, \pi); g_1, g_2 \in L_0^2(0, \pi) \right\}. \end{aligned} \tag{1.34}$$

Remarque 1.19. La différence principale entre $SV(\Omega_2)$ et $SV(\Omega_3)$ est que dans $SV(\Omega_3)$ il y a des fonctions auxiliaires dont la moyenne n'est pas nulle. Comme indiqué dans la

Remarque 1.17, la raison est que l'équation d'incompressibilité est plus restrictive dans \mathbb{R}^2 que dans \mathbb{R}^3 .

On peut prouver assez facilement que :

$$u(x) = \frac{2e^{-\varepsilon^{-1}x_3}}{\pi\sqrt{\varepsilon(1-e^{-2\pi\varepsilon^{-1}})}} (\sin(x_1)\cos(x_2), -\cos(x_1)\sin(x_2), 0),$$

est une solution de (1.27) dont valeur propre est $\lambda = 2\varepsilon$. Alors, on conclut la preuve de (1.24) en considérant (1.26)₃ et la fonction $\varphi(t, x) := u(x)e^{-2\varepsilon(T-t)}$, qui est une solution de (1.25).

1.2.3 Contrôlabilité à zéro de l'équation de la chaleur dans des pseudo-cylindres avec un contrôle qui agit à l'intérieur (voir [13])

La contrôlabilité à zéro de l'équation de la chaleur avec des contrôles internes est un des problèmes historiques en contrôlabilité des équations paraboliques. Pour prouver la contrôlabilité à zéro de cette équation il y a trois méthodes :

- Utiliser les estimations d'observabilité de l'équation des ondes (voir, par exemple, [96]).
- Prouver des inégalités spectrales et après considérer les propriétés des fonctions analytiques (voir, par exemple, [79]).
- Prouver des inégalités de Carleman avec des poids spécifiques (voir, par exemple, [57]). Cette dernière méthode est connue comme la méthode de *Fursikov-Imanuvilov*.

Chaque technique a ses points forts : la première permet d'obtenir des estimations très précises, la deuxième permet d'étudier la contrôlabilité quand le domaine de contrôle est un ensemble mesurable quelconque (de mesure non-nulle), et la troisième permet d'obtenir des estimations qui sont utiles pour prouver la contrôlabilité à zéro d'autres systèmes de contrôle paraboliques. Même si la technique de Fursikov-Imanuvilov répond à la question de la contrôlabilité à zéro pour des domaines C^2 , même si cette technique a été adaptée au produit tensoriel de domaines C^2 (voir, par exemple, [64]), et même si, récemment, cette question a été aussi répondue avec les inégalités spectrales et les propriétés de fonctions analytiques pour quelques domaines Lipschitz (voir [7]), la question de la contrôlabilité à zéro de l'équation de la chaleur dans tout domaine Lipschitz reste ouverte.

Dans cette partie on prouve la contrôlabilité à zéro dans les domaines suivants :

Définition 1.20. On dit que $\Omega \subset \mathbb{R}^{d+1}$ ($d \geq 1$) est un *pseudo-cylindre* s'il y a $\mathfrak{B} \subset \mathbb{R}^d$ un domaine C^2 , U la composition d'une rotation et une translation et $H : \overline{\mathfrak{B}} \mapsto \mathbb{R}^+$ une fonction Lipschitz satisfaisant $\inf_{\mathfrak{B}} H > 0$ tels que :

$$\Omega = U(\{(x, z) : x \in \mathfrak{B}, z \in (0, H(x))\}). \quad (1.35)$$

En plus, on désigne sa partie cylindrique par :

$$\mathfrak{C} := U\left(\mathfrak{B} \times \left(0, \inf_{\mathfrak{B}} H\right)\right). \quad (1.36)$$

Si $U = I$ (l'endomorphisme de l'identité), on dit que le pseudo-cylindre est *orienté canoniquement*.

Plus précisément, on prouve :

Théorème 1.21. *Soient Ω un pseudo-cylindre et $\omega \subset \Omega$ un sous-domaine. Alors, il y a $C > 0$ tel que si $T > 0$, $A \in (L^\infty(Q))^{d+1}$, $a \in L^\infty(Q)$ et $y^0 \in L^2(\Omega)$, il y a un contrôle $f \in L^2(Q_\omega)$ tel que la solution du système :*

$$\begin{cases} y_t - \Delta y + A \cdot \nabla y + ay = f1_\omega & \text{dans } Q, \\ y = 0 & \text{sur } \Sigma, \\ y(0, \cdot) = y^0 & \text{sur } \Omega, \end{cases}$$

satisfait $y(T, \cdot) = 0$, et tel que le contrôle satisfait l'estimation :

$$\|f\|_{L^2(Q_\omega)} \leq Ce^{C\kappa(T,a,A)} \|y^0\|_{L^2(\Omega)},$$

avec :

$$\kappa(T, a, A) := 1 + T^{-1} + T\|a\|_{L^\infty(Q)} + \|a\|_{L^\infty(Q)}^{2/3} + (1 + T)\|A\|_{(L^\infty(Q))^{d+1}}^2.$$

Remarque 1.22. On peut appliquer les techniques de [56] pour voir que la contrôlabilité à zéro avec le coût $Ce^{C\kappa(T,a,A)}$ implique la contrôlabilité aux trajectoires (et donc approchée) du système non-linéaire de l'équation de la chaleur si la non-linéarité est d'ordre 0 et suffisamment faible.

Remarque 1.23. On prouve que les pseudo-cylindres ne sont pas contenus dans les domaines considérés dans [7], donc c'est une contribution dans ce sens. En plus, notre résultat permet d'étudier, entre autres, la contrôlabilité de l'équation de la chaleur non-linéaire. Finalement, dans [7] les auteurs utilisent des inégalités spectrales et les propriétés des fonctions analytiques, tandis que nous, on le fait en adaptant la méthode de Fursikov-Imanuvilov.

Il suffit de prouver le Théorème 1.21 pour les pseudo-cylindres orientés canoniquement car le laplacien est invariant par rotations. Alors, on prouve le Théorème 1.21 dans ces cas en adaptant la méthode de Fursikov-Imanuvilov. En fait, cette méthode a deux difficultés à vaincre pour être généralisable aux ensembles Lipschitz. La première, et la plus importante, est le fait que la construction des fonctions auxiliaires proposée dans [57] ne marche que pour les domaines C^2 . En fait, on a besoin d'une fonction η telle que :

$$\eta \in C^2(\overline{\Omega}), \quad \partial_n \eta \leq 0 \text{ sur } \partial\Omega, \quad \inf_{\Omega \setminus \omega} |\nabla \eta| > 0. \quad (1.37)$$

Pour vaincre cette difficulté on construit une fonction auxiliaire en considérant la structure des domaines lorsque $\omega \subset \mathfrak{C}$ et on transmet l'information d'un domaine de contrôle à l'autre lorsque $\omega \subset \Omega \setminus \mathfrak{C}$.

La deuxième difficulté est le fait que les solutions à données régulières (et à support compact) de l'équation de la chaleur n'appartiennent pas toujours à $L^2(0, T; H^2(\Omega))$. Pour la résoudre on approche les domaines avec des domaines réguliers d'une façon compatible avec la fonction auxiliaire. Plus précisément :

Définition 1.24. Soient Ω un domaine, $\omega \subset \Omega$ un sous-domaine et η une fonction satisfaisant :

$$\eta \in C^2(\overline{\Omega}), \quad \inf_{\Omega \setminus \omega} |\nabla \eta| > 0. \quad (1.38)$$

On dit que $\tilde{\Omega}$ est un ouvert compatible avec Ω , ω et η si :

$$\tilde{\Omega} \text{ est un ouvert Lipschitz tel que } \omega \subset \tilde{\Omega} \subset \Omega, \quad (1.39)$$

$$\{u \in H_0^1(\tilde{\Omega}) : \Delta u \in L^2(\tilde{\Omega})\} \subset H^2(\tilde{\Omega}) \text{ continûment,} \quad (1.40)$$

$$\partial_{\tilde{n}} \eta \leq 0 \text{ sur } \partial\tilde{\Omega}. \quad (1.41)$$

Alors, on considère les poids :

$$\alpha := \frac{e^{\lambda \frac{2m+1}{2m} k} - e^{\lambda(k+\eta)}}{t^m(T-t)^m}, \quad \xi := \frac{e^{\lambda(k+\eta)}}{t^m(T-t)^m}, \quad (1.42)$$

où $m \geq 1$, η est une fonction auxiliaire qui peut avoir des valeurs négatives et $k > 2(m+1)\|\eta\|_{L^\infty(\Omega)}$. On prouve que :

Proposition 1.25. Soient Ω un domaine Lipschitz, $\omega \subset \Omega$ un sous-domaine, η une fonction satisfaisant (1.38) et $m \geq 1$. Supposons qu'il y a $\varepsilon_0 > 0$ et $(\Omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ satisfaisant

(1.39)-(1.41) et tels que :

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{\Omega \setminus \Omega_\varepsilon} d(\cdot, \partial\Omega) = 0. \quad (1.43)$$

Alors, il y a $C > 0$ tel que pour tout $T > 0$, $g \in L^2(Q)$, $G \in (L^2(Q))^d$, $u^T \in L^2(\Omega)$, $k > 2(m+1)\|\eta\|_{L^\infty(\Omega)}$, et pour les poids définis dans (1.42), on a que :

$$\begin{aligned} & s^3 \lambda^4 \iint_Q e^{-2s\alpha\xi^3} |u|^2 + s\lambda^2 \iint_Q e^{-2s\alpha\xi} |\nabla u|^2 \\ & \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha\xi^3} |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 + s^2 \lambda^2 \iint_Q e^{-2s\alpha\xi^2} |G|^2 \right), \end{aligned} \quad (1.44)$$

pour tout $\lambda \geq C$, $s \geq C(T^m + T^{2m})$ et où u est la solution de :

$$\begin{cases} -u_t - \Delta u = g + \nabla \cdot G & \text{dans } Q, \\ u = 0 & \text{sur } \Sigma, \\ u(T, \cdot) = u^T & \text{sur } \Omega. \end{cases} \quad (1.45)$$

La preuve de la Proposition 1.25 utilise les techniques habituelles (voir [57, 52, 74]) et le lemme suivant :

Lemme 1.26. Soient Ω un domaine Lipschitz, $\Omega_\varepsilon \subset \Omega$ des domaines Lipschitz satisfaisant (1.43), $u^0 \in L^2(\Omega)$ et $g \in L^2(Q)$. Alors,

$$\mathbb{S}(\Omega_\varepsilon, u^0 1_{\Omega_\varepsilon}, g 1_{\Omega_\varepsilon}) 1_{\Omega_\varepsilon} \rightarrow_{L^2(Q)} \mathbb{S}(\Omega, u^0, g), \quad (1.46)$$

où :

$$\mathbb{S}(\Omega, u^0, g)(t, x),$$

est l'unique solution dans $L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$ de :

$$\begin{cases} u_t - \Delta u = g & \text{dans } Q, \\ u = 0 & \text{sur } \Sigma, \\ u(0, \cdot) = u^0 & \text{sur } \Omega. \end{cases} \quad (1.47)$$

On prouve le Lemme 1.26 en considérant la formulation variationnelle de (1.47).

On peut construire assez facilement la fonction auxiliaire η lorsque Ω est un pseudo-cylindre canoniquement orienté et $\omega \subset \mathfrak{C}$. On peut supposer que :

$$\omega = B(\bar{x}, r) \times (\bar{z} - r, \bar{z} + r) \subset\subset \mathfrak{C}, \quad (1.48)$$

pour $r > 0$ et $(\bar{x}, \bar{z}) \in \Omega$ tels que :

$$\bar{z} < \inf_{\mathfrak{B}} H. \quad (1.49)$$

Alors, on considère :

$$\eta(x, z) := \eta_{\mathfrak{B}}(x) - \mathfrak{c}(z - \bar{z})^2, \quad (1.50)$$

où $\eta_{\mathfrak{B}} \in C^2(\overline{\mathfrak{B}})$ satisfait :

$$\eta_{\mathfrak{B}} = 0 \text{ sur } \partial\mathfrak{B}, \quad \eta_{\mathfrak{B}} > 0 \text{ dans } \mathfrak{B}, \quad \inf_{\mathfrak{B} \setminus B(\bar{x}, r)} |\nabla \eta_{\mathfrak{B}}| > 0, \quad (1.51)$$

et où \mathfrak{c} est une constante suffisamment grande. La fonction $\eta_{\mathfrak{B}}$ existe car \mathfrak{B} est C^2 (voir [57]).

On a le résultat suivant :

Proposition 1.27. *Soit Ω un pseudo-cylindre orienté canoniquement, ω donné par (1.48) et $\eta_{\mathfrak{B}} \in C^2(\overline{\mathfrak{B}})$ satisfaisant (1.51). Alors, il y a $\mathfrak{c}_0(\Omega, \omega, \eta_{\mathfrak{B}}) > 0$ tel que si η est donné par (1.50) avec $\mathfrak{c} \geq \mathfrak{c}_0$, il y a $\varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}}, \mathfrak{c}) > 0$ et des domaines $(\Omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ satisfaisant (1.39)-(1.41) et (1.46).*

Pour construire les domaines Ω_ε on considère les sous-domaines limités par des courbes de niveau de régularisations de la fonction $(x, z) \mapsto d((x, z), \partial\Omega)$. Après, on prouve (1.39)-(1.41) avec l'égalité de la courbe de niveau correspondante, et en distinguant des cas selon les différentes parties de la frontière (si l'on est près de $\mathfrak{B} \times \{0\}$, de $\{(x, z) : x \in \partial\mathfrak{B}, z \in (0, H(x))\}$ ou de $\{(x, H(x)) : x \in \mathfrak{B}\}$, ce qui est défini d'une façon rigoureuse).

Pour conclure le cas $\omega \subset \mathfrak{C}$, en considérant la Proposition 1.25 et la Proposition 1.27, on prouve assez facilement le résultat suivant :

Proposition 1.28. *Soient Ω un pseudo-cylindre orienté canoniquement, ω donné par (1.48) et $m \geq 1$. Alors, il y a $\mathfrak{c}_0(\Omega, \omega, \eta_{\mathfrak{B}}) > 0$ tel que si η est donné par (1.50) avec $\mathfrak{c} \geq \mathfrak{c}_0$, il y a $C > 0$ tel que pour tout $T > 0$, $A \in (L^\infty(Q))^{d+1}$, $a \in L^\infty(Q)$, $g \in L^2(Q)$,*

$G \in (L^2(Q))^{d+1}$, $u^T \in L^2(\Omega)$, $k > 2(m+1)\|\eta\|_{L^\infty(\Omega)}$, et pour les poids définis par (1.42), on a que :

$$\begin{aligned} & s^3 \lambda^4 \iint_Q e^{-2s\alpha\xi^3} |u|^2 + s\lambda^2 \iint_Q e^{-2s\alpha\xi} |\nabla u|^2 \\ & \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha\xi^3} |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 + s^2 \lambda^2 \iint_Q e^{-2s\alpha\xi^2} |G|^2 \right), \end{aligned} \quad (1.52)$$

pour tout $\lambda \geq C$,

$$s \geq C \left(T^m + T^{2m} (1 + \|a\|_{L^\infty(Q)}^{2/3} + \|A\|_{(L^\infty(Q))^{d+1}}^2) \right), \quad (1.53)$$

et pour u la solution de :

$$\begin{cases} -u_t - \Delta u - \nabla \cdot (Au) + au = g + \nabla \cdot G & \text{dans } Q, \\ u = 0 & \text{sur } \Sigma, \\ u(T, \cdot) = u^T & \text{sur } \Omega. \end{cases} \quad (1.54)$$

On rappelle que la Proposition 1.28 implique le Théorème 1.21 pour tout $\omega \subset \mathfrak{C}$ (voir, par exemple, [57, 56]).

Finalement, on prouve le cas $\omega \subset \Omega \setminus \bar{\mathfrak{C}}$. Dans ce cas, on peut supposer que :

$$\omega = B(\tilde{x}, r) \times (\tilde{z} - r, \tilde{z} + r) \subset\subset \Omega \setminus \bar{\mathfrak{C}}, \quad (1.55)$$

pour $(\tilde{x}, \tilde{z}) \in \Omega \setminus \bar{\mathfrak{C}}$ tels que :

$$\tilde{z} > \inf_{\mathfrak{B}} H,$$

et pour :

$$r \in \left(0, \inf_{\mathfrak{B}} \frac{H}{4} \right), \quad (1.56)$$

tel que :

$$\tilde{\Omega} := B(\tilde{x}, 3r) \times (0, \tilde{z} + 3r) \subset \Omega. \quad (1.57)$$

L'idée de la preuve est d'utiliser la Proposition 1.28 et de traiter notre problème comme un problème de transmission d'une estimation d'un domaine de contrôle à un autre. En fait, on définit :

$$\omega^* := B(\bar{x}, r) \times (\bar{z} - r, \bar{z} + r), \quad (1.58)$$

pour $\bar{x} := \tilde{x}$ et $\bar{z} := 3r$. Par conséquent, si l'on considère aussi (1.56) on obtient que :

$$\omega^* = B(\tilde{x}, r) \times (2r, 4r) \subset\subset \tilde{\Omega} \cap \mathfrak{E}.$$

Alors, par la Proposition 1.28 on a pour toute solution de (1.53) :

$$\begin{aligned} & s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u|^2 + s\lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla u|^2 \\ & \leq C \left(s^3 \lambda^4 \iint_{Q_{\omega^*}} e^{-2s\alpha} \xi^3 |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 + s^2 \lambda^2 \iint_Q e^{-2s\alpha} \xi^2 |G|^2 \right). \end{aligned} \quad (1.59)$$

Ensuite, on considère une fonction auxiliaire $\tilde{\eta}$ satisfaisant (1.37) sur $(\tilde{\Omega}, \omega)$ et telle que :

$$\begin{cases} \eta \leq \tilde{\eta} \text{ dans } \omega^*, \\ \tilde{\eta} \leq \eta \text{ dans } \tilde{\Omega} \setminus (B(\tilde{x}, 2r) \times (r, \tilde{z} + 2r)). \end{cases} \quad (1.60)$$

En plus, on considère une fonction positive régulière χ telle que :

$$\text{supp}(\chi) \subset\subset \tilde{\Omega}, \quad \chi = 1 \text{ dans } B(\tilde{x}, 2r) \times (r, \tilde{z} + 2r). \quad (1.61)$$

Pour transmettre l'information on utilise l'inégalité de Carleman suivante :

$$\begin{aligned} & s^3 \lambda^4 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} (\tilde{\xi})^3 |\chi u|^2 + s\lambda^2 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} \tilde{\xi} |\nabla(\chi u)|^2 \leq C \left(s^3 \lambda^4 \iint_{Q_{\omega}} e^{-2s\tilde{\alpha}} (\tilde{\xi})^3 |\chi u|^2 \right. \\ & \quad + \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} | -2\nabla\chi \cdot \nabla u - \Delta\chi u - (\nabla\chi \cdot A)u - \nabla\chi \cdot G + \chi g - a\chi u|^2 \\ & \quad \left. + s^2 \lambda^2 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} (\tilde{\xi})^2 (|\chi G - A\chi u|^2) \right). \end{aligned} \quad (1.62)$$

On prouve (1.62) en considérant le système satisfait par χu et que $\tilde{\Omega}$ est un produit tensoriel de domaines C^2 , donc satisfait (1.39)-(1.41) (voir, par exemple, [64]).

Si l'on considère (1.59), (1.62) et (1.60) avec des estimations et absorptions habituelles on obtient le résultat suivant :

Proposition 1.29. *Soient Ω un pseudo-cylindre orienté canoniquement, ω donné par (1.55), $\tilde{\Omega}$ donné par (1.57), η donné par (1.50) (avec $\bar{x} := \tilde{x}$ et $\bar{z} := 3r$), $\tilde{\eta}$ qui satisfait (1.37) sur $(\tilde{\Omega}, \omega)$ et (1.60), χ satisfaisant (1.61), et $m \geq 1$. Alors, il y a une constante*

$C > 0$ telle que si $T > 0$, $A \in (L^\infty(Q))^{d+1}$, $a \in L^\infty(Q)$, $g \in L^2(Q)$, $G \in (L^2(Q))^{d+1}$, $u^T \in L^2(\Omega)$, $k > 2(m+1) \max\{\|\tilde{\eta}\|_{L^\infty(\Omega)}, \|\eta\|_{L^\infty(\Omega)}\}$, $\lambda \geq C$, et s satisfait (1.53), on a :

$$\begin{aligned} & s^3 \lambda^4 \iint_Q e^{-2s\alpha\xi^3} |u|^2 + s\lambda^2 \iint_Q e^{-2s\alpha\xi} |\nabla u|^2 \\ & \quad + s^3 \lambda^4 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} (\tilde{\xi})^3 |\chi u|^2 + s\lambda^2 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} \tilde{\xi} |\nabla(\chi u)|^2 \\ & \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\tilde{\alpha}} (\tilde{\xi})^3 |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 + \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} |\chi g|^2 \right. \\ & \quad \left. + s^2 \lambda^2 \iint_Q e^{-2s\alpha} \xi^2 |G|^2 + s^2 \lambda^2 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} (\tilde{\xi})^2 |\chi G|^2 \right), \end{aligned}$$

où les poids sont définis dans (1.42) et u est la solution de (1.54).

On rappelle que la Proposition 1.29 implique le Théorème 1.21 lorsque $\omega \subset \Omega \setminus \mathfrak{C}$ (voir, par exemple, [57, 56]), ce qui conclut la preuve du Théorème 1.21.

1.2.4 Contrôlabilité à zéro locale d'un modèle d'interaction forte entre deux ondes internes solitaires (voir [17])

L'équation de Korteweg-de Vries (KdV en abrégé) est un modèle mathématique pour les vagues dans un canal peu profond. Elle est donnée par :

$$y_t + y_{xxx} + yy_x = 0, \quad \text{pour } t > 0, \quad x \in I, \quad (1.63)$$

où $I \subset \mathbb{R}$ est un intervalle (fini ou infini) et $y(t, x)$ est l'amplitude de l'eau au temps t au point x . Évidemment, il faut ajouter la valeur initiale et les conditions aux limites. On peut consulter [106] pour comprendre sa déduction et l'interprétation d'un point de vue physique.

Dans cette partie on s'intéresse aux propriétés de contrôle de la version de (1.63) suivante, qui modélise les interactions fortes de deux ondes de gravité qui se propagent sur

la direction horizontale :

$$\left\{ \begin{array}{ll} u_t + uu_x + u_{xxx} + av_{xxx} + k_1vv_x + k_2(uv)_x = \mathfrak{h}_1 & \text{dans } Q, \\ cv_t + rv_x + vv_x + abu_{xxx} + v_{xxx} + k_2buu_x + k_1b(uv)_x = \mathfrak{h}_2 + f1_\omega & \text{dans } Q, \\ u(\cdot, 0) = u(\cdot, L) = u_x(\cdot, L) = 0 & \text{sur } (0, T), \\ v(\cdot, 0) = v(\cdot, L) = v_x(\cdot, L) = 0 & \text{sur } (0, T), \\ u(0, \cdot) = u^0, \quad v(0, \cdot) = v^0 & \text{sur } (0, L), \end{array} \right. \quad (1.64)$$

où $u^0, v^0 \in L^2(0, L)$, $\mathfrak{h}_1, \mathfrak{h}_2$ sont deux fonctions exponentiellement petites près de T (voir (1.66)) et $a, b, c, r, k_1, k_2 \in \mathbb{R}$ tels que :

$$a, b, c > 0 \quad \text{et} \quad a^2b < 1. \quad (1.65)$$

On peut consulter [58] et [11] pour comprendre sa déduction et son interprétation d'un point de vue physique.

La plupart des résultats sur la contrôlabilité de (1.63) (et de ses versions linéarisées) traitent la contrôlabilité frontière. Les premiers résultats datent de 1997 (voir [94]). Pour un résumé des résultats de contrôlabilité sur l'équation de KdV avec des contrôles qui agissent sur la frontière on peut consulter [95]. Quant à la contrôlabilité de (1.63) avec des contrôles qui agissent à l'intérieur, c'est un sujet de recherche relativement récent (voir [22]).

En ce qui concerne les systèmes du type (1.64), il n'a été étudié dans la littérature qu'avec des contrôles qui agissent sur le bord (voir [18, 19, 29, 90]). Dans cette partie, on s'intéresse à la contrôlabilité de (1.64), où le contrôle agit à l'intérieur et uniquement sur la deuxième composante. Plus précisément, on prouve le résultat suivant :

Théorème 1.30. *Soient $L > 0$, $T > 0$ et supposons que (1.65) est satisfait. Alors, il y a $C > 0$ tel que pour tout $\mathfrak{h}_1, \mathfrak{h}_2$ satisfaisant*

$$e^{\frac{C}{(T-t)^{12}}}(\mathfrak{h}_1, \mathfrak{h}_2) \in L^2(0, T; \mathbf{H}^{-1}(0, L)), \quad (1.66)$$

il y a $\delta > 0$ tel que pour tout $\|U^0\| < \delta$ il y a $f \in L^2(Q_\omega)$ et $U \in C([0, T]; \mathbf{L}^2(0, L))$ satisfaisant que (U, f) est une solution de (1.64) et $U(T, \cdot) = 0$.

Dans cette partie on utilise la notation $U := (u, v)$ et $U^0 := (u^0, v^0)$.

La preuve du Théorème 1.30 repose sur un théorème d'inversion locale (version surjective) et sur un résultat de contrôlabilité à zéro pour le système linéarisé autour de 0 avec un terme source :

$$\begin{cases} \mathcal{L}U = (h_1, h_2 + f1_\omega) & \text{dans } Q, \\ U(\cdot, 0) = U(\cdot, L) = U_x(\cdot, L) = 0 & \text{sur } (0, T), \\ U(0, \cdot) = U^0 & \text{sur } (0, L), \end{cases} \quad (1.67)$$

où :

$$\mathcal{L}U := (u_t + u_{xxx} + av_{xxx}, cv_t + rv_x + abu_{xxx} + v_{xxx}). \quad (1.68)$$

Pour prouver la contrôlabilité de (1.67) on utilise la HUM et on prouve une inégalité de Carleman pour le système adjoint de (1.67), qui est donné par :

$$\begin{cases} \mathcal{L}^*\Phi = G & \text{dans } Q, \\ \Phi(\cdot, 0) = \Phi(\cdot, L) = \Phi_x(\cdot, 0) = 0 & \text{sur } (0, T), \\ \Phi(T, \cdot) = \Phi^T & \text{sur } (0, L), \end{cases} \quad (1.69)$$

où $\Phi := (\varphi, \phi)$, $\Phi^T = (\varphi^T, \phi^T) \in \mathbf{L}^2(0, L)$, $G \in \mathbf{L}^2(Q)$ et :

$$\mathcal{L}^*\Phi := (-\varphi_t - \varphi_{xxx} - ab\phi_{xxx}, -c\phi_t - r\phi_x - a\varphi_{xxx} - \phi_{xxx}). \quad (1.70)$$

Plus précisément, on montre :

Proposition 1.31. *Soient $L > 0$, $\omega \subset (0, L)$ un ouvert, $m \geq 12$ et supposons que (1.65) est satisfait. Alors, il y a $C > 0$ tel que pour tout $\lambda \geq C$, $s \geq C(T^m + T^{3m})$, $\Phi^T \in \mathbf{L}^2(0, L)$ et $G \in \mathbf{L}^2(Q)$ on a l'estimation :*

$$s^6 \lambda^6 \int_0^T e^{-2s\alpha^* - 10s\beta^*} (\xi^*)^6 \|\Phi\|_{\mathbf{H}^2(0, L)}^2 \leq C \left(\iint_{Q_\omega} e^{-s\alpha^* - 10s\beta^*} \phi^2 + \iint_Q e^{-8s\beta^*} |G|^2 \right), \quad (1.71)$$

où Φ est la solution d'énergie de (1.69).

De nouveau, les poids α^* et ξ^* sont des fonctions du type $Ct^{-m}(T-t)^{-m}$. Également, β^* est une fonction du type $C(T-t)^{-m}$. Pour avoir la définition exacte des poids on peut consulter (5.37) et (5.40).

Pour prouver la Proposition 1.31 on utilise la stratégie de [23]. Cette stratégie consiste de façon globale à décomposer les solutions du système linéarisé autour de 0 multipliées par des poids qui décroissent exponentiellement en 0 et en T . On les décompose en une première solution avec la source à laquelle on applique des estimations de régularité et en une deuxième solution régulière à laquelle on applique des estimations de Carleman comme dans [38].

Pour pouvoir utiliser cette méthode, il faut avoir des résultats de régularité de (1.69). Pour cela, on combine les stratégies présentées dans [62] et dans [90] et on prouve :

Proposition 1.32. *Soit $L > 0$ et supposons que (1.65) est satisfait. Alors :*

1. *Pour tout $T > 0$, $\Phi^T \in \mathbf{L}^2(0, L)$ et $G \in L^2(0, T; \mathbf{H}^{-1}(0, L))$ le système (1.69) admet une unique solution :*

$$\Phi \in C([0, T]; \mathbf{L}^2(0, L)) \cap L^2(0, T; \mathbf{H}^1(0, L)).$$

En plus, il y a $C > 0$ indépendant de T , Φ^T et G tels que :

$$\|\Phi\|_{C([0, T]; \mathbf{L}^2(0, L)) \cap L^2(0, T; \mathbf{H}^1(0, L))} \leq C(1 + T)(\|\Phi^T\|_{\mathbf{L}^2(0, L)} + \|G\|_{L^2(0, T; \mathbf{H}^{-1}(0, L))}).$$

2. *Il y a $C > 0$ tel que pour tout $T > 0$ et $G \in \mathbf{L}^2(Q)$ on a :*

$$\Phi \in L^2(0, T; \mathbf{H}^2(0, L)) \cap H^1(0, T; \mathbf{H}^{-1}(0, L))$$

satisfaisant l'estimation :

$$\|\Phi\|_{L^2(0, T; \mathbf{H}^2(0, L)) \cap H^1(0, T; \mathbf{H}^{-1}(0, L))} \leq C(1 + T)\|G\|_{\mathbf{L}^2(Q)},$$

pour Φ la solution d'énergie de (1.69) avec $\Phi^T = 0$.

3. *Il y a $C > 0$ tel que pour tout $T > 0$ et $G \in L^2(0, T; \mathbf{H}_0^1(0, L))$ on a $\Phi \in \mathbf{H}^{1,3}(Q)$ satisfaisant l'estimation :*

$$\|\Phi\|_{\mathbf{H}^{1,3}(Q)} \leq C(1 + T^3)\|G\|_{L^2(0, T; \mathbf{H}_0^1(0, L))},$$

pour Φ la solution d'énergie de (1.69) avec $\Phi^T = 0$.

En plus, on généralise ces résultats à des normes d'ordre supérieur par récurrence.

Il y a deux différences importantes entre notre preuve et celle de [23]. La première, lorsque l'on décompose (1.69) en une partie régulière et une partie avec la force comme dans [23, Section 2.1], il ne suffit pas de décomposer les solutions multipliées par les poids en 2 termes à cause des théorèmes de régularité, mais il faut les décomposer (au moins) en 3 termes. Ceci entraîne des changements comme utiliser des poids qui ne s'annulent pas en 0. La deuxième, pour traiter le couplage d'ordre 3 on obtient une équation d'ordre 6 sur ϕ , équation qu'on décompose dans deux équations de KdV linéarisées.

Une fois que (1.71) a été établi, on déduit l'inégalité d'observabilité suivante avec des poids qui ne s'annulent pas en $t = 0$:

$$\int_0^T e^{-14s\beta^*} \|\Phi\|_{\mathbf{H}_0^1(0,L)}^2 + \|e^{-7s\beta^*} \Phi\|_{C([0,T];\mathbf{L}^2(0,L))}^2 \leq C \left(\iint_{Q_\omega} e^{-10s\beta^*} \phi^2 + \iint_Q e^{-8s\beta^*} |G|^2 \right). \quad (1.72)$$

On cherche des solutions de (1.67) dans l'espace E de couples (U, f) tels que :

- $U \in L^2(e^{4s\beta^*}(0, T); \mathbf{L}^2(0, L))$,
- $U \in L^2(e^{\frac{7}{2}s\beta^*}(0, T); \mathbf{H}_0^1(0, L)) \cap C(e^{\frac{7}{2}s\beta^*}[0, T]; \mathbf{L}^2(0, L))$,
- $f1_\omega \in L^2(e^{5s\beta^*}(0, T); L^2(0, L))$,
- $\mathcal{L}U - (0, f1_\omega) \in L^2(e^{7s\beta^*}(0, T); \mathbf{H}^{-1}(0, L))$.

Concrètement, on a :

Proposition 1.33. *Soient $U^0 \in \mathbf{L}^2(0, L)$, $h_1, h_2 \in L^2(e^{7s\beta^*}(0, T); H^{-1}(0, L))$ et supposons que (1.65) est satisfait. Alors, il y a une solution de (1.67) qui satisfait $(U, f) \in E$.*

Remarque 1.34. Si $(U, f) \in E$, on a que $U(T, \cdot) = 0$.

La preuve de la Proposition 1.33 suit les arguments de [57]. On considère P_0 le sous-espace de $\mathbf{C}^3([0, T] \times [0, L])$ des fonctions Φ telles que :

$$\Phi(\cdot, 0) = \Phi(\cdot, L) = \Phi_x(\cdot, 0) = 0 \quad \text{sur } [0, T],$$

et on définit sur P_0 :

$$\begin{cases} \mathbf{a}(\hat{\Phi}, \Phi) := \iint_{Q_\omega} e^{-10s\beta^*} \hat{\phi}\phi + \iint_Q e^{-8s\beta^*} (\mathcal{L}^*\hat{\Phi}) \cdot (\mathcal{L}^*\Phi), \\ \ell(\Phi) := \int_0^L U^0 \cdot \Phi(0, x) dx + \int_0^T \langle (h_1, h_2), \Phi \rangle_{\mathbf{H}^{-1}(0,L) \times \mathbf{H}_0^1(0,L)}. \end{cases}$$

On montre à l'aide de (1.72) que \mathbf{a} est un produit scalaire et que ℓ est continue sur (P_0, \mathbf{a}) . Donc, si P est la fermeture de P_0 par rapport à \mathbf{a} , \mathbf{a} et ℓ admettent un unique prolongement continu sur P . En utilisant le théorème de représentation de Riesz on obtient que le problème variationnel suivant :

$$\mathbf{a}(\hat{\Phi}, \Phi) = \ell(\Phi), \quad \forall \Phi \in P,$$

admet une unique solution $\hat{\Phi} \in P$. Alors, on prouve que :

$$U = e^{-8s\beta^*} \mathcal{L}^* \hat{\Phi}, \quad f = -e^{-10s\beta^*} \hat{\phi} 1_\omega,$$

est une solution de (1.67) qui appartient à E .

Finalement, on considère $\mathbf{h}_1, \mathbf{h}_2$ comme énoncés dans (1.66), avec C suffisamment grand tels que $\mathbf{h}_1, \mathbf{h}_2 \in L^2(e^{7s\beta^*}(0, T); \mathbf{H}^{-1}(0, L))$. Soit :

$$\mathcal{G} : E \rightarrow L^2(e^{7s\beta^*}(0, T); \mathbf{H}^{-1}(0, L)) \times \mathbf{L}^2(0, L),$$

l'opérateur défini par :

$$\begin{aligned} \mathcal{G}(U, f) := & (u_t + uu_x + u_{xxx} + av_{xxx} + k_1vv_x + k_2(uv)_x - \mathbf{h}_1, \\ & cv_t + rv_x + vv_x + abu_{xxx} + v_{xxx} + k_2buu_x + k_1b(uv)_x - \mathbf{h}_2 - f1_\omega, u(0, \cdot), v(0, \cdot)). \end{aligned} \quad (1.73)$$

Pour finir la preuve du théorème 1.30 il suffit d'utiliser le théorème d'inversion locale qui dit que si \mathcal{G} est C^1 et si $D\mathcal{G}[0]$ est surjective, alors il y a $\delta > 0$ tel que si $\|U^0\| < \delta$ il y a $(U, f) \in E$ tel que :

$$\mathcal{G}(U, f) = (0_{\mathbb{R}^2}, U^0).$$

1.3 Un travail complémentaire et des problèmes ouverts

On finit ce chapitre introducteur en faisant une brève présentation d'un travail complémentaire concernant le coût de la contrôlabilité de l'équation de la chaleur avec des conditions aux limites mixtes et une diffusion évanescence, ainsi qu'en énonçant quelques perspectives et problèmes qui demeurent ouverts.

1.3.1 Travail complémentaire : étude du coût de la contrôlabilité d'une équation parabolique d'ordre deux avec diffusion évanescence et un terme de transport (voir [16])

Un travail complémentaire réalisé pendant la thèse est l'étude du coût de la contrôlabilité à zéro de l'équation de la chaleur avec diffusion évanescence, un terme de transport et des conditions aux limites mixtes, de type Fourier et de type Dirichlet. Plus précisément, on étudie la contrôlabilité du système suivant :

$$\begin{cases} y_t - \varepsilon \Delta y + \partial_{x_1} y = 1_\omega f, & \text{dans } (0, T) \times \Omega, \\ \partial_n y + ay = 0, & \text{sur } (0, T) \times \Gamma, \\ y = 0, & \text{sur } (0, T) \times \Gamma^*, \\ y(0, \cdot) = y^0, & \text{sur } \Omega, \end{cases} \quad (1.74)$$

où $\Gamma \subset \partial\Omega$ est relativement ouvert, $\Gamma^* := \partial\Omega \setminus \Gamma$ et $a(x, \varepsilon)$ est une fonction telle que $a(\cdot, \varepsilon) \in L^\infty(\Gamma)$ pour tout $\varepsilon \in (0, \varepsilon_0)$. Quant au coût du contrôle, on le définit par :

$$K(\Omega, \omega, T, \varepsilon) := \sup_{y^0 \in L^2(\Omega) \setminus \{0\}} \inf_{f \in S(y^0)} \frac{\|f\|_{L^2(Q_\omega)}}{\|y^0\|_{L^2(\Omega)}},$$

C'est un travail présenté dans [16].

La motivation de cette étude est de mieux comprendre les résultats du Chapitre 3 (voir [14]). De fait, les résultats obtenus au Chapitre 3 suggèrent que le comportement du coût du contrôle par rapport à ε dépend des conditions aux limites, et avec ce travail on vérifie que cette conjecture est vraie :

- On prouve que, sous certaines conditions, le coût décroît exponentiellement :

Theorem 1.35. *Soient Ω un domaine C^2 , $\omega \subset \Omega$ un sous-domaine et supposons que (Γ, a) satisfait $(a + (2\varepsilon)^{-1}n_1)1_\Gamma \geq 0$. Alors, il y a $T_0, c, C > 0$ tels que pour tout $\varepsilon \in (0, \varepsilon_0)$ et $T \geq T_0$ on obtient que :*

$$K(\Omega, \omega, T, \varepsilon) \leq Ce^{-c\varepsilon^{-1}}.$$

Les hypothèses du Théorème 1.35 incluent des systèmes avec des conditions aux limites qui sont presque de Dirichlet et avec des conditions aux limites mixtes qui uniquement ont des conditions de Robin (ou de Neumann) sur la partie du bord où le flux du transport sort.

- On prouve qu'avec des conditions aux limites de Neumann au moins dans la partie du bord où le flux du transport entre, le coût du contrôle peut croître exponentiellement :

Theorem 1.36. *Soient Ω un domaine, $\Gamma = \partial\Omega$, $a = 0$ et $\omega \subset \Omega$ un sous-domaine compactement inclus. Alors, pour tout $T > 0$ il y a $c > 0$ tel que pour tout $\varepsilon > 0$ on a que :*

$$K(\Omega, \omega, T, \varepsilon) \geq ce^{c\varepsilon^{-1}}.$$

Theorem 1.37. *Soient $L, h > 0$, $\Omega = (-L, 0)$, $\Gamma = \{-L\}$, $a = 0$ et $\omega \subset (-L+h, 0)$ un sous-domaine. Alors, pour tout $T > 0$ il y a $c, \varepsilon_0 > 0$ tels que pour tout $\varepsilon \in (0, \varepsilon_0)$ on a l'estimation :*

$$K(\Omega, \omega, T, \varepsilon) \geq ce^{c\varepsilon^{-1}}.$$

- On étudie le coût de la contrôlabilité avec des conditions aux limites de Neumann lorsque le domaine de contrôle est le domaine entier (lorsque $\omega = \Omega$) :

Theorem 1.38. *Soient $L > 0$, $\Omega = (-L, 0)$, $\Gamma = \{-L, 0\}$ et $a = 0$. Alors, il y a $c, C > 0$ tels que pour tout $T \geq 4L$ on a que :*

$$cT^{-1/2} \leq \liminf_{\varepsilon \rightarrow 0} K(\Omega, \Omega, T, \varepsilon) \leq \limsup_{\varepsilon \rightarrow 0} K(\Omega, \Omega, T, \varepsilon) \leq CT^{-1/2}.$$

Pour prouver les Théorèmes 1.35, 1.36, 1.37 et 1.38 on utilise la méthode de HUM. Plus précisément, on adapte la méthode de [14] et on se concentre dans la décomposition spectrale de l'opérateur elliptique associé au système adjoint de (1.74). En plus, dans le Carleman on adapte les preuves de [57] et [52] au système adjoint de (1.74) en vainquant les difficultés des conditions aux limites et de la manque de régularité de ces solutions (compte tenu du fait qu'on doit faire des estimations précises par rapport à ε).

1.3.2 Perspectives et problèmes ouverts

Dans cette partie, on fait quelques commentaires et on présente quelques problèmes ouverts.

Par rapport au Chapitre 2, on remarque que les problèmes suivants restent ouverts :

- Déterminer si pour tout domaine Ω qui satisfait l'Hypothèse 1.7, le système de Navier-Stokes avec la pénalisation $(1.13)_2$ est localement contrôlable à zéro avec un contrôle scalaire.
- Déterminer si l'hypothèse géométrique du Théorème 1.8 (respectivement du Théorème 1.12) est nécessaire pour $\varepsilon > 0$ (respectivement pour $a \geq 0$) suffisamment petit.
- Déterminer si le système pénalisé de Stokes en dimension 3 avec un contrôle qui agit sur deux composantes est contrôlable à zéro.
- Déterminer la contrôlabilité du système de Stokes (et de Navier-Stokes) avec d'autres conditions de compressibilité, comme $\nabla \cdot y = -\varepsilon p_t$ ou $\nabla \cdot y = \varepsilon \Delta p$ (voir [98]).

Par rapport au Chapitre 3, on peut mentionner :

- Les techniques qu'on utilise pour prouver le Théorème 1.15 sont très spécifiques. De fait, on ne peut pas généraliser la preuve à d'autres domaines ou avec d'autres conditions aux limites. Comme le cas qu'on a étudié est un cas particulier, les résultats obtenus dans le Théorème 1.15 risquent de ne pas pouvoir être généralisés. Cela serait intéressant d'étudier le coût de la contrôlabilité dans d'autres domaines et avec d'autres conditions aux limites. Par exemple, le problème (1.22) mais avec des conditions aux limites de type Dirichlet reste ouvert.
- Déterminer le T_0 optimal à partir duquel le coût de la contrôlabilité en dimension 2 décroît de façon exponentielle.

Par rapport au Chapitre 4, on remarque que les problèmes suivants restent ouverts :

- Déterminer la contrôlabilité à zéro de l'équation de la chaleur pour tout domaine Ω Lipschitz.
- Déterminer la contrôlabilité à zéro de l'équation de la chaleur dans des pseudo-cylindres avec d'autres conditions aux limites, par exemple des conditions aux limites de type Neumann.
- Déterminer la contrôlabilité à zéro de l'équation de la chaleur semi-linéaire dans des pseudo-cylindres lorsque la non-linéarité dépend du gradient de la solution.
- Établir la contrôlabilité à zéro d'autres systèmes paraboliques lorsque le domaine n'est pas C^2 ni un produit tensoriel de domaines C^2 ; par exemple, le système de Stokes.

Par rapport au Chapitre 5, on peut mentionner :

- Déterminer la contrôlabilité aux trajectoires du système (1.64).
- Déterminer la contrôlabilité locale à zéro de (1.64) avec une force agissant sur la frontière et sur une seule composante.

Chapitre 2

Null controllability of a penalized Stokes problem in dimension two with one scalar control

In this chapter we consider a penalized Stokes system defined in a regular domain $\Omega \subset \mathbb{R}^2$ and with Dirichlet boundary conditions. We prove that our system is null controllable using a scalar control localized in an open subset inside Ω and whose cost is bounded uniformly with respect to the parameter that converges to 0. This chapter is included in [15].

2.1 Introduction

2.1.1 Main results

Let $T > 0$ and $\Omega \subset \mathbb{R}^2$ a regular domain. Throughout this paper we use the term “domain” to refer to a bounded connected non-empty open set. As usual, we denote $Q := (0, T) \times \Omega$ and $\Sigma := (0, T) \times \partial\Omega$. In this paper we work on the Stokes penalized system

with Dirichlet boundary conditions. This system is given by the following equations :

$$\begin{cases} v_t^\varepsilon - \Delta v^\varepsilon + \nabla q^\varepsilon = f & \text{in } Q, \\ \varepsilon q^\varepsilon + \nabla \cdot v^\varepsilon = 0 & \text{in } Q, \\ v^\varepsilon = 0 & \text{on } \Sigma, \\ v^\varepsilon(0, \cdot) = v^0 & \text{in } \Omega. \end{cases} \quad (2.1)$$

Here $f : Q \rightarrow \mathbb{R}^2$ is a source term, $v^0 : \Omega \rightarrow \mathbb{R}^2$ is an initial condition and $\varepsilon > 0$. This system approximates the classical Stokes problem, which is given by the following equations :

$$\begin{cases} v_t - \Delta v + \nabla q = f & \text{in } Q, \\ \nabla \cdot v = 0 & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(0, \cdot) = v^0 & \text{in } \Omega. \end{cases} \quad (2.2)$$

The main objective of this paper is to prove that system (2.1) is null controllable with a one-dimensional control whose cost is uniformly bounded with respect to ε . We prove it for almost every direction, being these directions different for each Ω . In addition, if Ω is strictly convex we prove it for all the directions.

First of all, we state what hypothesis Ω must satisfy to be controllable by a force parallel to $e_1 := (1, 0)$. In order to do so, if $\Omega \subset \mathbb{R}^2$ is a C^2 domain, we use the standard convention to parametrize $\partial\Omega$: we denote the arc-length parametrization of each connected component of $\partial\Omega$ by $\sigma^i = (\sigma_1^i, \sigma_2^i)$ and the signed curvature of $\partial\Omega$ on the point $\sigma^i(\theta)$ by $\kappa^i(\theta)$. In both terms the superscript i is omitted if $\partial\Omega$ is connected. We suppose that each component is parametrized in the standard way ; that is, for $U(x, y) := (-y, x)$ (U is the rotation of 90 degrees to the left) and for all $p = \sigma^i(\theta) \in \partial\Omega$, there is $\delta_0(p) > 0$ such that if $\delta \in (0, \delta_0(p))$, then $p + \delta U((\sigma^i)'(\theta)) \in \Omega$.

Remark 2.1. Since the σ^i are arc-length, we have the well-known equalities :

$$\kappa^i = (\sigma_2^i)''(\sigma_1^i)' - (\sigma_1^i)''(\sigma_2^i)' = \frac{(\sigma_2^i)''}{(\sigma_1^i)'} = -\frac{(\sigma_1^i)''}{(\sigma_2^i)'}. \quad (2.3)$$

Hypothesis 2.1. Let $\Omega \subset \mathbb{R}^2$ be a C^2 domain of boundary $\partial\Omega$ parametrized by functions $\sigma^1, \dots, \sigma^k$ as explained in the previous paragraph. Then, for any $i \in \{1, \dots, k\}$ and for any θ such that $(\sigma_1^i)'(\theta) = 0$ or $(\sigma_2^i)'(\theta) = 0$, we have that $\kappa^i(\theta) \neq 0$.

Remark 2.2. Hypothesis 2.1 means that if Ω is a C^2 domain, then on all the points of the boundary of horizontal or vertical tangent line the curvature is not null. We use it to avoid pathologies near those points, since in that case we do not know how to proceed.

Hypothesis 2.1 is not restrictive at all, thanks to the following lemma, which we prove at the beginning of Subsection 2.4.1 :

Lemma 2.3. *Let Ω be a C^2 domain. Then, there is an orthogonal \mathbb{R}^2 -endomorphism U such that the domain $\tilde{\Omega} := U(\Omega)$ satisfies Hypothesis 2.1. In fact, if we denote U_ψ the endomorphism characterized by $e_1 := (1, 0) \mapsto (\cos(\psi), \sin(\psi))$ and $e_2 := (0, 1) \mapsto (-\sin(\psi), \cos(\psi))$, then, for almost every ψ in $[-\pi, \pi]$, $U_\psi(\Omega)$ satisfies Hypothesis 2.1.*

With Lemma 2.3 in mind, we state one of the main results of this paper :

Theorem 2.4. *Let $\Omega \subset \mathbb{R}^2$ be a regular domain that satisfies Hypothesis 2.1 and let $\omega \subset \Omega$ be a non-empty open set. Then, there is $\varepsilon_0 > 0$ such that for all $T > 0$ there is $C > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ and $y^0 \in \mathbf{L}^2(\Omega)$, there is a scalar-valued function $f^\varepsilon \in L^2((0, T) \times \omega)$ satisfying :*

$$\|f^\varepsilon\|_{L^2((0, T) \times \omega)} \leq C \|y^0\|_{\mathbf{L}^2(\Omega)},$$

and such that the solution of the following system :

$$\begin{cases} y_t^\varepsilon - \Delta y^\varepsilon + \nabla p^\varepsilon = f^\varepsilon 1_\omega e_1 & \text{in } Q, \\ \varepsilon p^\varepsilon + \nabla \cdot y^\varepsilon = 0 & \text{in } Q, \\ y^\varepsilon = 0 & \text{on } \Sigma, \\ y^\varepsilon(0, \cdot) = y^0 & \text{in } \Omega, \end{cases} \quad (2.4)$$

satisfies $y^\varepsilon(T, \cdot) = 0$.

As usual, in Theorem 2.4 and throughout this paper \mathbf{L}^p and \mathbf{H}^s denote respectively the Lebesgue and Sobolev spaces of vector-valued functions.

Remark 2.5. From Lemma 2.3 and since system (2.1) is invariant with respect to rotations, we actually have for almost all directions $e^\theta := (\sin(\theta), \cos(\theta))$ that there is $\varepsilon_0 > 0$ such that for all $T > 0$ there is $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ system (2.1) is null controllable with a force $f^\varepsilon 1_\omega e^\theta$ satisfying :

$$\|f^\varepsilon\|_{L^2((0, T) \times \omega)} \leq C \|y^0\|_{\mathbf{L}^2(\Omega)}.$$

Remark 2.6. A natural question that may arise is the relation between the control problem (2.4) and the control problem :

$$\begin{cases} y_t - \Delta y + \nabla p = f 1_\omega e_1 & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y^0 & \text{in } \Omega, \end{cases} \quad (2.5)$$

for $f \in L^2((0, T) \times \omega)$ and for $y^0 \in \mathcal{H}(\Omega)$ (the subspace of $\mathbf{L}^2(\Omega)$ of functions of null divergence and null normal trace). Using weak compactness, we have that there is a sequence f^{ε_k} which converges weakly in $L^2((0, T) \times \omega)$ to some function f^0 . Moreover, using the techniques presented in the proof of [102, Theorem I.2], we have that $y^{\varepsilon_k}(t, \cdot)$ converges in the $\mathbf{H}^{-1}(\Omega)$ -norm for all $t \in [0, T]$ to $y(t, \cdot)$ (the solution of (2.5)). In particular, since $y^{\varepsilon_k}(T, \cdot) = 0$, we have that $y(T, \cdot) = 0$. Consequently, this provides an alternative way of proving the well-known result that the system (2.5) is null controllable with a one-dimensional control supported in any regular domain (see [38]). In that sense, an interesting problem that remains open is if the control of minimal L^2 -norm of (2.4) converges to the control of minimal L^2 -norm of (2.5).

In order to prove the null controllability of (2.4), we consider as usual its adjoint system :

$$\begin{cases} -\varphi_t^\varepsilon - \Delta \varphi^\varepsilon + \nabla \pi^\varepsilon = 0 & \text{in } Q, \\ \varepsilon \pi^\varepsilon + \nabla \cdot \varphi^\varepsilon = 0 & \text{in } Q, \\ \varphi^\varepsilon = 0 & \text{on } \Sigma, \\ \varphi^\varepsilon(T, \cdot) = \varphi^T & \text{in } \Omega. \end{cases} \quad (2.6)$$

Indeed, it is a well-known result (see [97, 82]) that the existence in (2.4) of a control f^ε bounded uniformly in ε is equivalent to proving that there is $\varepsilon_0 > 0$ such that for all $T > 0$ there is $C > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ and $\varphi^T \in \mathbf{L}^2(\Omega)$ we have :

$$\int_{\Omega} |\varphi^\varepsilon(0, \cdot)|^2 \leq C \iint_{(0, T) \times \omega} |\varphi_1^\varepsilon|^2, \quad (2.7)$$

for φ^ε the solution of (2.6). The proof of Theorem 2.4 is thus reduced to the proof of the observability inequality (2.7), and we focus on it from now on.

In order to prove estimate (2.7) we prove a Carleman inequality. Before presenting it, let us define the weights we use throughout the paper as follows :

$$\begin{aligned}\alpha(t, x) &= \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0}}{(t(T-t))^m}, & \xi(t, x) &= \frac{e^{\lambda\eta^0}}{(t(T-t))^m}, \\ \alpha^*(t) &= \max_{x \in \bar{\Omega}} \alpha(t, x), & \xi^*(t) &= \min_{x \in \bar{\Omega}} \xi(t, x).\end{aligned}\tag{2.8}$$

These weights are classical and widely used in the literature. Here, λ and m are positive real numbers to be fixed later on. Moreover, $\eta^0 \in C^4(\bar{\Omega})$ is a fixed function that satisfies $\eta^0 > 0$ in Ω , $\eta^0 = 0$ on $\partial\Omega$, and $|\nabla\eta^0| > 0$ in $\bar{\Omega} \setminus \omega_0$ for some non-empty domain ω_0 compactly included in ω . We know that such function η^0 exists as long as $\Omega \in C^4$. Indeed, a classical proof of the existence of such function is given in [57], whereas an alternative recent proof is given in [73, Lemma 2.1].

Theorem 2.7. *Let Ω be a regular domain that satisfies Hypothesis 2.1, let $\omega \subset \Omega$ be an open set, and let $m \geq 8$. Then, there is $\varepsilon_0 > 0$, $C > 0$ and $\lambda_0 \geq 1$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\lambda \geq \lambda_0$, and $s \geq e^{C\lambda}(T^m + T^{2m})$, we have :*

$$s^{15}\lambda^{16} \iint_Q e^{-2s\alpha^*} (\xi^*)^{15} |\varphi^\varepsilon|^2 \leq Cs^{34}\lambda^{35} \iint_{(0,T) \times \omega} e^{-2s\alpha} \xi^{34} |\varphi_1^\varepsilon|^2,\tag{2.9}$$

for any φ^ε regular solution of (2.6) and for the weights defined in (2.8).

Proving (2.7) from (2.9) is mainly done by an energy estimate on $\chi\varphi^\varepsilon$, for $\chi \geq 0$ a regular cut-off function such that $\chi = 1$ in $[0, T/2]$ and $\chi = 0$ in $[3T/4, T]$. This is a classic and easy procedure, so it is omitted in this paper.

We remark that in (2.6) we can simplify the pressure and get the following equations for φ^ε :

$$\begin{cases} -\frac{\varepsilon}{1+\varepsilon}\partial_t\varphi_1^\varepsilon - \partial_{xx}\varphi_1^\varepsilon - \frac{\varepsilon}{1+\varepsilon}\partial_{yy}\varphi_1^\varepsilon = \frac{1}{1+\varepsilon}\partial_{xy}\varphi_2^\varepsilon, \\ -\frac{\varepsilon}{1+\varepsilon}\partial_t\varphi_2^\varepsilon - \frac{\varepsilon}{1+\varepsilon}\partial_{xx}\varphi_2^\varepsilon - \partial_{yy}\varphi_2^\varepsilon = \frac{1}{1+\varepsilon}\partial_{xy}\varphi_1^\varepsilon. \end{cases}\tag{2.10}$$

Using standard methods for coupled parabolic equations we know that the important term is $\partial_{xy}\varphi_2^\varepsilon$, which appears in the equation satisfied by φ_1^ε . The main difficulty on proving (2.9) is that $\|\partial_{xy} \cdot\|_{L^2(\Omega)}$ is not necessarily a norm (see Remark 2.31 and Proposition 2.32 below). However, if we add some information on the first and second order derivatives of $\partial_{xy}\varphi^\varepsilon$ and on the boundary, we do have a norm. In that sense, we consider the following operator :

$$L_a u = -a\partial_{xx}u - \partial_{yy}u.\tag{2.11}$$

Theorem 2.8. *Let Ω be a C^4 domain that satisfies Hypothesis 2.1. Then, for $a_0 > 0$ small enough, there is $C > 0$ such that for any function $u \in H^4(\Omega) \cap H_0^1(\Omega)$ and for any $a \in (0, a_0]$ we have that :*

$$\|\partial_x u\|_{C^0(\bar{\Omega})} \leq C(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \quad (2.12)$$

Remark 2.9. By continuity, (2.12) remains true for $a = 0$.

Remark 2.10. Thanks to Poincaré inequality, there is $C > 0$ and $a_0 > 0$ such that for all $a \in [0, a_0]$ and for all $u \in H^4(\Omega) \cap H_0^1(\Omega)$, we have :

$$\|u\|_{C^0(\bar{\Omega})} \leq C(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \quad (2.13)$$

Remark 2.11. By symmetry, we get an analogous estimate for $\partial_y u$ if instead of L_a we have the operator :

$$\tilde{L}_a u := -\partial_{xx} u - a\partial_{yy} u. \quad (2.14)$$

Indeed, in that case, there is $C > 0$ and $a_0 > 0$ such that for all $a \in [0, a_0]$ and for any function $u \in H^4(\Omega) \cap H_0^1(\Omega)$, we have :

$$\|\partial_y u\|_{C^0(\bar{\Omega})} \leq C(\|\partial_{xy} u\|_{H^2(\Omega)} + \|\tilde{L}_a u\|_{H^1(\partial\Omega)}).$$

In particular, in this case we also have an estimate of $\|u\|_{C^0(\bar{\Omega})}$ similar to (2.13).

The reason why (2.12) is useful to prove (2.9) is the following one :

Remark 2.12. Let us consider φ^ε a solution of (2.6). Using (2.10) and the Dirichlet boundary conditions, we have that, on $\partial\Omega$, for all $t \in [0, T)$:

$$\begin{cases} -\partial_{xx}\varphi_1^\varepsilon - \frac{\varepsilon}{1+\varepsilon}\partial_{yy}\varphi_1^\varepsilon = \frac{1}{1+\varepsilon}\partial_{xy}\varphi_2^\varepsilon, \\ -\frac{\varepsilon}{1+\varepsilon}\partial_{xx}\varphi_2^\varepsilon - \partial_{yy}\varphi_2^\varepsilon = \frac{1}{1+\varepsilon}\partial_{xy}\varphi_1^\varepsilon. \end{cases}$$

Thus, by Remark 2.10 and Remark 2.11, we get that there is $C > 0$ and $\varepsilon_0 > 0$ such that for all $t \in [0, T)$ and $\varepsilon \in (0, \varepsilon_0]$:

$$\|\varphi^\varepsilon(t, \cdot)\|_{\mathbf{L}^2(\Omega)} \leq C\|\partial_{xy}\varphi^\varepsilon(t, \cdot)\|_{\mathbf{H}^2(\Omega)}. \quad (2.15)$$

Finally, let us make some remarks about possible extensions of the work :

Remark 2.13. The case of Theorem 2.4 and Theorem 2.7 for $\Omega \subset \mathbb{R}^3$ is left for future research. The main complication that arises is to prove an analogous result to Theorem 2.8 because there is a larger variety of domains in \mathbb{R}^3 than in \mathbb{R}^2 . Indeed, in \mathbb{R}^3 there is one curvature for each direction.

Remark 2.14. The construction provided in Section 2.2.1 for Lipschitz domains in which (2.4) is not null-controllable just gives one problematic ε for each Ω . Thus, it is an open problem to know if for all Lipschitz domain $\Omega \subset \mathbb{R}^2$ there is ε_0 small enough such that if $\varepsilon \in (0, \varepsilon_0)$ system (2.4) is null controllable.

2.1.2 Historical background

Getting an approximation of the Stokes and the Navier-Stokes systems by approximating the incompressibility condition by a term involving the pressure was made for the first time in [102], where the author considered the almost incompressible Navier-Stokes system. Many other ways of approximating the Navier-Stokes equations have been presented throughout the years. In the survey [98] the author presents different ways of approximating the Navier-Stokes system through the incompressibility condition and compares them. Moreover, there are physical systems which satisfy in some ways the property of being almost incompressible, as shown in [100] and the references therein.

The interior null controllability of system (2.1) was first proved in [73, Section 4] with a control bounded uniformly with respect to ε , for ε small enough. Then, in [9], this same property is proved with an additional first order term. Moreover, in [9] the author also proves the local controllability to trajectories of the penalized Navier-Stokes system uniformly on ε for ε small enough.

There is an extensive literature on controllability of partial differential equations uniformly with respect to a vanishing parameter. For a transport equation with a small diffusion term, see [37] (see also [59] and [86]). The case of the KdV equation is treated in [62], [24] and [25], while a chemotaxis system is presented in [31].

As for the restriction of having controls with a reduced number of components, it is not new in the Navier-Stokes mathematical context. This same property has already been proved for the Stokes problem (2.2) in [38]. Consequently, in this paper we prove that a system which approximates the Stokes system conserves that property after choosing a valid reference system. Moreover, controllability results with controls having one null component have been proved for other systems : for instance, the local null controllability of the Navier-Stokes system (see [23]), the local controllability to the trajectories of the

Navier-Stokes and the Boussineq system when the domain “touches” the boundary (see [55]), or the existence of insensitizing controls (see [68, 26]). Similarly, the approximate controllability of the Stokes system in a cylindrical domain with a control having two null components is proved in [85]. Finally, the local null controllability of the Navier-Stokes system in dimension three with one scalar control is proved in [39].

Outside the Navier-Stokes context, there is a huge literature on controllability results with controls having a reduced number of components. For instance, the null controllability in the context of linear thermoelasticity (see [80]), the existence of insensitizing controls for the heat equation (see [42]), the controllability to trajectories in phase-field models (see [4]), the controllability in cascade-like systems (see [65]) and the controllability in reaction-diffusion systems (see [3]). For more results on the controllability of parabolic systems with a reduced number of control, see the survey [5] and the references therein.

Remark 2.15. The main difference of the problem we consider in this paper with respect to the above cited papers is the coupling. Indeed, in all the papers cited above (and in the literature as far as we know) the coupling is constituted by a zero, first or second order term which induces a norm in the subset of $H^2(\Omega)$ which satisfies their respective boundary conditions. However, in our situation this is clearly not the case since the differential operator ∂_{xy} with Dirichlet boundary conditions does not induce a norm for some Ω (see Remark 2.31 below).

The rest of the paper is organized as follows : in Section 2.2 we present some analytical results ; in Section 2.3 we prove Theorem 2.8 when Ω is strictly convex ; in Section 2.4 we prove Lemma 2.3 and Theorem 2.8 ; and in Section 2.5 we end the proof of Theorem 2.7. Finally, in the Appendix we prove some technical results stated in Section 2.2.

2.2 Some previous and intermediary results

In this section we present some results that are either interesting for understanding the problem or needed later. The section is split in three parts : first, in Subsection 2.2.1 we prove that there is a domain Ω which is not C^2 and where we do not even have approximate null controllability ; then, in Subsection 2.2.2 we present several results on

Cauchy problems and a classical result on linear ordinary differential equations; finally, in Subsection 2.2.3 we present some Carleman estimates.

2.2.1 A negative controllability result

In this subsection we provide a counterexample on null controllability with one component of (2.4) when Ω is not C^2 , even if $\omega = \Omega$. With that purpose, we show that we do not even have the approximate null controllability.

Definition 2.16. We recall that system (2.4) is approximately null controllable if for all $y^0 \in \mathbf{L}^2(\Omega)$ and for all $\eta > 0$ there is a function $f^\varepsilon \in L^2((0, T) \times \omega)$ such that the solution y^ε of (2.4) satisfies $\|y^\varepsilon(T, \cdot)\|_{\mathbf{L}^2(\Omega)} \leq \eta$.

Proposition 2.17. *Let $\varepsilon > 0$. Then, there is $\Omega \subset \mathbb{R}^2$ a domain that is just Lipschitz such that, even for $\omega = \Omega$, (2.4) is not approximately null controllable.*

In order to prove Proposition 2.17 we use the technique presented in [85, Section 3]. We give the proof for completeness, as it is an easy consequence of the results presented in [108] and [101]. Indeed, the Stokes penalized system and the Lamé system has the same elliptic operator after an easy reparametrization, being the first one its parabolic version (that is, the associated abstract heat equation) and the other one the hyperbolic version (that is, the associated abstract wave equation).

Proof of Proposition 2.17. It is a classical result that system (2.4) is not approximately null controllable if there is $\varphi^\varepsilon \neq 0$ solution of (2.6) such that $\varphi_1^\varepsilon = 0$ in $(0, T) \times \omega$. Suppose that the following scalar-valued system has a nonzero solution u^ε :

$$\begin{cases} -\partial_{xx}u^\varepsilon - \frac{1+\varepsilon}{\varepsilon}\partial_{yy}u^\varepsilon = \lambda u^\varepsilon & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \\ \partial_{xy}u^\varepsilon = 0 & \text{in } \Omega. \end{cases} \quad (2.16)$$

Then, $\varphi^\varepsilon(t, (x, y)) := (0, e^{\lambda t}u^\varepsilon(x, y))$ is a non-trivial solution of (2.6) which satisfies $\varphi_1^\varepsilon = 0$. Consequently, it suffices to find a domain Ω with a nonzero solution of (2.16).

The third equation of (2.16) is satisfied if $u^\varepsilon(x, y) = f(x) + g(y)$ for any $f, g \in C^2(\mathbb{R}; \mathbb{R})$. Moreover, u^ε satisfies the first equation of (2.16) if :

$$-f''(x) - \lambda f(x) = \frac{1+\varepsilon}{\varepsilon}g''(y) + \lambda g(y). \quad (2.17)$$

Since both sides of equation (2.17) depend on independent variables, they must be constant. So we have to solve an ordinary differential equation with constant coefficients. We can suppose that they are equal to 0. Otherwise, if they are equal to some other value α , we have that $\bar{f} := f + \frac{\alpha}{\lambda}$ and $\bar{g} := g - \frac{\alpha}{\lambda}$ are solutions for the case $\alpha = 0$ such that $u(x, y) = \bar{f}(x) + \bar{g}(y)$.

The solutions (f, g) of the system :

$$\begin{cases} f''(x) + \lambda f(x) = 0, \\ \frac{1+\varepsilon}{\varepsilon} g''(y) + \lambda g(y) = 0, \end{cases}$$

are exponential, affine or trigonometric functions, depending on the value of λ . Since we need u^ε to be null on a bounded boundary, then necessarily they must be trigonometric; that is, $\lambda > 0$. In fact, the first equation of (2.16) is satisfied by :

$$u^\varepsilon(x, y) = \sin\left(\sqrt{\lambda}x\right) - \sin\left(\sqrt{\frac{\varepsilon\lambda}{1+\varepsilon}}y\right). \quad (2.18)$$

Finally, we have to consider that the function u^ε given in (2.18) is null on the lines :

$$\begin{cases} x = \sqrt{\frac{\varepsilon}{1+\varepsilon}}y, \\ x = \sqrt{\frac{\varepsilon}{1+\varepsilon}}y + \frac{2\pi}{\sqrt{\lambda}}, \\ x = -\sqrt{\frac{\varepsilon}{1+\varepsilon}}y + \frac{\pi}{\sqrt{\lambda}}, \\ x = -\sqrt{\frac{\varepsilon}{1+\varepsilon}}y - \frac{\pi}{\sqrt{\lambda}}. \end{cases} \quad (2.19)$$

Consequently, the function u^ε given in (2.18) is a solution of (2.16) in $\mathcal{R}_{\varepsilon,\lambda}$, for $\mathcal{R}_{\varepsilon,\lambda}$ the domain limited by (2.19), which is a rhombus. \square

Remark 2.18. With this method we can find for $\mathcal{R}_{\varepsilon,\lambda}$ (for ε and λ fixed) a sequence of eigenfunctions which satisfy (2.16). The sequence is given by :

$$u_n^\varepsilon(x, y) = \sin\left((2n+1)\sqrt{\lambda}x\right) - \sin\left((2n+1)\sqrt{\frac{\varepsilon\lambda}{1+\varepsilon}}y\right), \quad n \in \mathbb{N},$$

and their respective eigenvalues are $\lambda_n = (2n+1)^2\lambda$.

Remark 2.19. The “reason” why unique continuation fails is that $\partial\mathcal{R}_{\varepsilon,\lambda}$ contains lines of a specific slope. Indeed, if for ε fixed we try to replicate the proof of Theorem 2.8 for the

rhombus $\mathcal{R}_{\varepsilon,\lambda}$ we find out that the information that (2.6) provides on the boundary is equivalent to the information provided by the Dirichlet condition. More precisely, for the case of $\Omega = \mathcal{R}_{\varepsilon,\lambda}$ we are blocked in (2.38) below, since we have $\kappa = 0$ and $(\sigma'_1)^2 - \frac{\varepsilon}{1+\varepsilon}(\sigma'_2)^2 = 0$. This problem somehow persists when we try to generalize the proof to regular domains and the only solution we have found is to exclude a few directions (a subset of $[0, 2\pi)$ of null measure).

2.2.2 Results on Cauchy problems

In this subsection we present some results about the Stokes penalized problem : first with Dirichlet boundary conditions and then with Neumann boundary conditions. We also present a classical estimate about a linear differential equation. But before, we recall the definition of the interpolation spaces, for $p, q \geq 0$:

$$\begin{aligned} H^{p,q}(Q) &:= H^p(0, T; L^2(\Omega)) \cap L^2(0, T; H^q(\Omega)), \\ H^{p,q}(\Sigma) &:= H^p(0, T; L^2(\partial\Omega)) \cap L^2(0, T; H^q(\partial\Omega)). \end{aligned}$$

Lemma 2.20. *Let $i \in \mathbb{N}$, $\Omega \in C^{2i}$. Then, there is $\varepsilon_0 > 0$ and $C > 0$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $v^0 = 0$ and $f \in \mathbf{H}^{i-1, 2i-2}(Q)$ satisfying $\partial_t^m f(t, \cdot) = 0$ for all $m \in \mathbb{N} \cap [0, i-2]$, we have that the solution v^ε of (2.1) satisfies $v^\varepsilon \in \mathbf{H}^{i, 2i}(Q)$ with the estimate :*

$$\|v^\varepsilon\|_{\mathbf{H}^{i, 2i}(Q)} + \varepsilon^{-1} \|\nabla \cdot v^\varepsilon\|_{H^{i-1, 2i-1}(Q)} \leq C \|f\|_{\mathbf{H}^{i-1, 2i-2}(Q)}. \quad (2.20)$$

The proof of Lemma 2.20 is mainly by induction. The base case ($i = 0$) can be proved by Galerkin method (we just have to replicate the method in [49, Chapter 7.1] and see that the constants are independent of ε). As for the inductive case, we get the regularity in time by considering that v_t^ε is a solution of (2.1) with $(f, 0)$ replaced by $(f_t, 0)$ and using again the Galerkin method. Moreover, we get the regularity in space by using the estimate for the steady Stokes problem given in [103, Proposition I.2.2].

Let us now state the Stokes penalized system with non-homogeneous Neumann boundary conditions :

$$\begin{cases} v_t^\varepsilon - \Delta v^\varepsilon + \nabla q^\varepsilon = f & \text{in } Q, \\ \varepsilon q^\varepsilon + \nabla \cdot v^\varepsilon = 0 & \text{in } Q, \\ \partial_n v^\varepsilon - q^\varepsilon n = h & \text{on } \Sigma, \\ v^\varepsilon(0, \cdot) = v^0 & \text{in } \Omega. \end{cases} \quad (2.21)$$

We have the following regularity and existence results, which are proved in Annex 2.A :

Lemma 2.21. *Let $\Omega \in C^2$. Then, there is $\varepsilon_0 > 0$ and $C > 0$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $v^0 \in \mathbf{H}^1(\Omega)$, $f \in \mathbf{L}^2(Q)$ and $h \in \mathbf{H}^{1,1/2}(\Sigma)$, system (2.21) has a unique solution :*

$$(v^\varepsilon, q^\varepsilon) \in \mathbf{H}^{1,2}(Q) \times H^{0,1}(Q).$$

In addition, that solution satisfies the estimate :

$$\begin{aligned} \|v^\varepsilon\|_{\mathbf{H}^{1,2}(Q)} + \|q^\varepsilon\|_{H^{0,1}(Q)} &\leq C\sqrt{1+T}(\|f\|_{\mathbf{L}^2(Q)} + \|h\|_{\mathbf{H}^{1,1/2}(\Sigma)}) \\ &+ C \left(\|v^0\|_{\mathbf{H}^1(\Omega)} + \left\| \frac{\nabla \cdot v^0}{\varepsilon} \right\|_{\mathbf{L}^2(\Omega)} + \|h(0, \cdot)\|_{\mathbf{L}^2(\Sigma)} + \|h(T, \cdot)\|_{\mathbf{L}^2(\Sigma)} \right). \end{aligned} \quad (2.22)$$

Remark 2.22. It is not necessary to assume that ε is small enough if we just want to prove existence and uniqueness of the energy solution of (2.21). Indeed, we prove collaterally that for all $\varepsilon \in \mathbb{R}^+$, (2.21) has a solution in $\mathbf{H}^{1,1}(Q)$ and that the norm $\|v^\varepsilon\|_{\mathbf{H}^{1,1}(Q)}$ can be estimated by the right-hand side of (2.22) for a constant C independent of ε (see (2.77) below).

Lemma 2.23. *Let $\Omega \in C^4$. Then, there is $\varepsilon_0 > 0$ and $C > 0$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $v^0 = 0$, $f \in \mathbf{H}^{1,2}(Q)$ satisfies $f(0, \cdot) = 0$ and $h \in \mathbf{H}^{2,5/2}(\Sigma)$ satisfies :*

$$h(0, \cdot) = 0, \quad h(T, \cdot) = 0, \quad \partial_t h(0, \cdot) = 0 \quad \text{and} \quad \partial_t h(T, \cdot) = 0,$$

we have that the solution v^ε of (2.21) belongs to $\mathbf{H}^{2,4}(Q)$ with the estimate :

$$\|v^\varepsilon\|_{\mathbf{H}^{2,4}(Q)} \leq C\sqrt{1+T} (\|f\|_{\mathbf{H}^{1,2}(Q)} + \|h\|_{\mathbf{H}^{2,5/2}(\Sigma)}). \quad (2.23)$$

Remark 2.24. These results are not optimal in terms of the regularity imposed on h , but they are enough for our purpose.

Finally, we recall the following classical estimate for a linear ordinary differential equation :

Lemma 2.25. *Let $T > 0$ and let x be the solution in $C^0([0, T])$ of the following ordinary differential equation :*

$$\begin{cases} a(t)x(t) + x'(t) = g(t) & t \in (0, T), \\ x(0) = x_0, \end{cases}$$

for $x_0 \in \mathbb{R}$, $a \in L^1(0, T)$ and $g \in L^1(0, T)$. Then, we have the estimate :

$$\|x\|_{C^0([0, T])} \leq (|x_0| + \|g\|_{L^1(0, T)})e^{\|a\|_{L^1(0, T)}} \quad (2.24)$$

2.2.3 Results about Carleman estimates

In this subsection we present some Carleman estimates that are needed later. We first state a Carleman estimate which concerns a parabolic equation with non-homogeneous Neumann boundary conditions. More precisely, we consider the following system :

$$\begin{cases} -\delta\varphi_t - \Delta\varphi = f & \text{in } Q, \\ \partial_n\varphi = h & \text{on } \Sigma, \\ \varphi(T, \cdot) = \varphi^T & \text{in } \Omega, \end{cases} \quad (2.25)$$

for $\varphi^T \in L^2(\Omega)$, $f \in L^2(Q)$ and $\delta \in (0, 1]$.

Lemma 2.26. *Let Ω be a C^4 domain, let $\hat{\omega}$ be an open set included in Ω such that $\overline{\omega_0} \subset \hat{\omega}$, let $m \geq 1$ and let $r \in \mathbb{R}$. Then, there is $C > 0$ and $\lambda_0 \geq 1$ such that if $T > 0$, $\delta \in (0, 1]$, $\varphi^T \in L^2(\Omega)$, $f \in L^2(Q)$, $h \in L^2(\Sigma)$, $\lambda \geq \lambda_0$ and $s \geq e^{C\lambda}(T^m + T^{2m})$ we have :*

$$\begin{aligned} & s^{3+r}\lambda^{4+r} \iint_Q e^{-2s\alpha\xi^{3+r}}|\varphi|^2 + s^{1+r}\lambda^{2+r} \iint_Q e^{-2s\alpha\xi^{1+r}}|\nabla\varphi|^2 \\ & \leq C \left(s^r\lambda^r \iint_Q e^{-2s\alpha\xi^r}|f|^2 + s^{3+r}\lambda^{4+r} \iint_{(0,T)\times\hat{\omega}} e^{-2s\alpha\xi^{3+r}}|\varphi|^2 + s^{1+r}\lambda^{1+r} \iint_{\Sigma} e^{-2s\alpha\xi^{1+r}}|h|^2 \right), \end{aligned} \quad (2.26)$$

for φ is the solution of (2.25).

The case $m = 1$, $\delta = 1$ and $r = 0$ of Lemma 2.26 is proved in [52, Theorem 1]. We get the case for $m \geq 1$ and $r \in \mathbb{R}$ repeating all the steps in [52] and we get uniformity on δ following the steps of, for instance, [73].

Next, we also need the following elliptic inequality, whose proof can be found in [38, Lemma 3] :

Lemma 2.27. *Let Ω be a C^4 domain, let $m \geq 1$ and let $r \in \mathbb{R}$. Then, there is $C > 0$ and $\lambda_0 \geq 1$ such that if $T > 0$, $\lambda \geq \lambda_0$, $s \geq CT^{2m}$ and $u \in L^2(0, T; H^1(\Omega))$, we have :*

$$s^{2+r}\lambda^{3+r} \iint_Q e^{-2s\alpha\xi^{2+r}}|u|^2 \leq C \left(s^r\lambda^{1+r} \iint_Q e^{-2s\alpha\xi^r}|\nabla u|^2 + s^{2+r}\lambda^{3+r} \iint_{(0,T)\times\omega_0} e^{-2s\alpha\xi^{2+r}}|u|^2 \right). \quad (2.27)$$

Finally, we need a Carleman inequality for the backwards solution of (2.21) (see (2.31) below). For a simpler statement of the Carleman inequality, we define the weights :

$$\eta(t) := (s\xi^*(t))^{1/4+1/m}e^{-s\alpha^*(t)}, \quad (2.28)$$

and

$$\tilde{\eta}(t) := (s\xi^*(t))^{-3/4}e^{-s\alpha^*(t)}. \quad (2.29)$$

Proposition 2.28. *Let Ω be a C^4 domain, let $\tilde{\omega}$ be an open subset Ω such that $\overline{\omega_0} \subset \tilde{\omega}$ and let $m \geq 8$. Then, there is $\varepsilon_0 > 0$, $C > 0$ and $\lambda_0 \geq 1$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\varphi^T \in \mathbf{L}^2(\Omega)$, $h \in \mathbf{H}^{2,5/2}(\Sigma)$, $\lambda \geq \lambda_0$ and $s \geq e^{C\lambda}(T^m + T^{2m})$, we have :*

$$\begin{aligned} & s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\varphi^\varepsilon|^2 + s\lambda^2 \iint_Q e^{-2s\alpha}\xi|\nabla\varphi^\varepsilon|^2 \\ & \leq C \left(s^4\lambda^5 \iint_{(0,T)\times\tilde{\omega}} e^{-2s\alpha}\xi^4|\varphi^\varepsilon|^2 + (1+T) \left(\|\eta h\|_{\mathbf{H}^{1,1/2}(\Sigma)}^2 + \|\tilde{\eta}h\|_{\mathbf{H}^{2,5/2}(\Sigma)}^2 \right) \right), \end{aligned} \quad (2.30)$$

for φ^ε the solution of the following system :

$$\begin{cases} -\varphi_t^\varepsilon - \Delta\varphi^\varepsilon + \nabla\pi^\varepsilon = 0 & \text{in } Q, \\ \varepsilon\pi^\varepsilon + \nabla \cdot \varphi^\varepsilon = 0 & \text{in } Q, \\ \partial_n\varphi^\varepsilon - \pi^\varepsilon n = h & \text{on } \Sigma, \\ \varphi^\varepsilon(T, \cdot) = \varphi^T & \text{in } \Omega. \end{cases} \quad (2.31)$$

The proof of this Carleman estimate is presented in Annex 2.B.

Remark 2.29. Proposition 2.28 with $h = 0$ implies that there is $\varepsilon_0 > 0$ and $C > 0$ such that for all $T > 0$, $\varepsilon \in (0, \varepsilon_0]$ and $y^0 \in \mathbf{L}^2(\Omega)$, there is $f^\varepsilon \in \mathbf{L}^2((0, T) \times \tilde{\omega})$ such that

$$\|f^\varepsilon\|_{\mathbf{L}^2((0,T)\times\tilde{\omega})} \leq C\|y^0\|_{\mathbf{L}^2(\Omega)};$$

and such that the solution of :

$$\begin{cases} y_t^\varepsilon - \Delta y^\varepsilon + \nabla p^\varepsilon = f^\varepsilon 1_{\tilde{\omega}} & \text{in } Q, \\ \varepsilon p^\varepsilon + \nabla \cdot y^\varepsilon = 0 & \text{in } Q, \\ \partial_n y^\varepsilon - p^\varepsilon n = 0 & \text{on } \Sigma, \\ y^\varepsilon(0, \cdot) = y^0 & \text{in } \Omega, \end{cases}$$

satisfies $y^\varepsilon(T, \cdot) = 0$.

Up to our knowledge the result presented in Remark 2.29 is new.

2.3 Proof and optimality of Theorem 2.8 when Ω is strictly convex

In Section 2.3 we first give some remarks about how much Theorem 2.8 can be improved and we then prove Theorem 2.8 when Ω is strictly convex. The proof is simpler, clearer and more explicit than when we are in a general domain. We recall that Ω strictly convex means that its boundary consists of one connected component and that :

$$\min_{\theta \in [0, |\partial\Omega|]} \kappa(\theta) > 0. \quad (2.32)$$

Moreover, we remark that a strictly convex domain always satisfies Hypothesis 2.1.

When Ω is a strictly convex domain, we do not need a_0 to be small. We state this in the following proposition :

Proposition 2.30. *Let Ω be a strictly convex C^4 domain. Then, there is $C > 0$ such that for all $a \in (0, 1]$ and for any real valued function $u \in H^4(\Omega) \cap H_0^1(\Omega)$, we have :*

$$\|\partial_x u\|_{C^0(\bar{\Omega})} \leq C (\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \quad (2.33)$$

Throughout this section we prove Proposition 2.30, which automatically implies Theorem 2.8 if Ω is strictly convex.

Remark 2.31. Estimate (2.33) is false if we remove the term $\|L_a u\|_{H^1(\partial\Omega)}$. Indeed, we just have to consider the domain $\Omega = \{(x, y) : x^2 + y^2 \leq 1\}$ and the function $u(x, y) = 1 - x^2 - y^2$.

Although it might be possible that the spaces we give on the right of (2.33) are not optimal, the statement is false if we replace $\|\partial_{xy} u\|_{H^2(\Omega)}$ by $\|\partial_{xy} u\|_{L^2(\Omega)}$ in the right-hand side of (2.33), even if we replace $\|\partial_x u\|_{C^0(\bar{\Omega})}$ by $\|u\|_{L^2(\Omega)}$ in the left-hand side of (2.33).

Proposition 2.32. *Let Ω a C^4 domain with a boundary characterized by an equation of the following kind :*

$$g(x) + h(y) = 0,$$

for $g, h \in C^4(\mathbb{R})$. Then, there is no $C > 0$ such that for any $u \in H^4(\Omega) \cap H_0^1(\Omega)$, we have :

$$\|u\|_{L^2(\Omega)} \leq C (\|\partial_{xy} u\|_{L^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \quad (2.34)$$

Remark 2.33. The hypothesis of Proposition 2.32 includes circles, ellipses and p -norm spheres (for $p \geq 2$) among others.

Proof. [Proof of Proposition 2.32] We prove this assertion by contradiction. Let us suppose that there is $C > 0$ such that, for any $u \in H^4(\Omega) \cap H_0^1(\Omega)$, we have (2.34). Let us consider the function $w(x, y) := g(x) + h(y)$, for g and h the functions stated in the hypothesis of Proposition 2.32. Then, $w \in H^2(\Omega) \cap H_0^1(\Omega)$ and $\partial_{xy}w = 0$.

We consider $f_n \in \mathcal{D}(\Omega)$ a sequence such that $f_n \rightarrow L_a w$ in $L^2(\Omega)$ and we define u_n as the solution of the equations :

$$\begin{cases} L_a u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

By usual theorems on elliptic regularity we have $u_n \in H^4(\Omega) \cap H_0^1(\Omega)$. Thus, applying (2.34) and since f_n vanishes on the boundary, we get that :

$$\|u_n\|_{L^2(\Omega)} \leq C \|\partial_{xy}u_n\|_{L^2(\Omega)}. \quad (2.35)$$

Moreover, by continuity with respect to the force in the elliptic problem, we have that $u_n \rightarrow u$ in $H^2(\Omega)$, where u is the solution of

$$\begin{cases} L_a u = L_a w & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Consequently, $u_n \rightarrow w$ in $H^2(\Omega)$.

Thus, if we take limits in (2.35) we get :

$$\|w\|_{L^2(\Omega)} \leq C \|\partial_{xy}w\|_{L^2(\Omega)} = 0,$$

which is absurd. □

Proof. [Proof of Proposition 2.30] In order to make the proof more understandable we split it in three steps : first, we obtain a differential equation on the boundary in terms of $\partial_{xy}u$ and $L_a u$; then, we define an auxiliary function and perform estimates on it; finally, we estimate $\partial_x u$ in terms of the auxiliary function.

Step 1 : Getting an equation on the boundary.

In order to get a differential equation on the boundary, we consider that because of the Dirichlet boundary condition u satisfies the equation :

$$u(\sigma_1(\theta), \sigma_2(\theta)) = 0, \quad \forall \theta \in [0, |\partial\Omega|].$$

If we differentiate this, we have :

$$\sigma'_1 \partial_x u + \sigma'_2 \partial_y u = 0, \quad \forall \theta \in [0, |\partial\Omega|]. \quad (2.36)$$

Moreover, if we differentiate (2.36), we get :

$$\sigma''_1 \partial_x u + \sigma''_2 \partial_y u + (\sigma'_1)^2 \partial_{xx} u + (\sigma'_2)^2 \partial_{yy} u + 2\sigma'_1 \sigma'_2 \partial_{xy} u = 0, \quad \forall \theta \in [0, |\partial\Omega|]. \quad (2.37)$$

The idea is to get an equality from (2.37) in which we only have $\partial_x u$, $\partial_{xx} u$, $\partial_{xy} u$ and $L_a u$. In order to get it, we multiply (2.37) by σ'_2 and use (2.36) and (2.11). We get that :

$$(\sigma''_1 \sigma'_2 - \sigma''_2 \sigma'_1) \partial_x u + \sigma'_2 ((\sigma'_1)^2 - a(\sigma'_2)^2) \partial_{xx} u = -2\sigma'_1 (\sigma'_2)^2 \partial_{xy} u + (\sigma'_2)^3 L_a u, \quad \forall \theta \in [0, |\partial\Omega|].$$

Recalling (2.3) we can rewrite the previous equation as follows :

$$-\kappa \partial_x u + \sigma'_2 ((\sigma'_1)^2 - a(\sigma'_2)^2) \partial_{xx} u = -2\sigma'_1 (\sigma'_2)^2 \partial_{xy} u + (\sigma'_2)^3 L_a u, \quad \forall \theta \in [0, |\partial\Omega|]. \quad (2.38)$$

Thanks to (2.32), we can divide (2.38) by κ :

$$-\partial_x u + \frac{\sigma'_2}{\kappa} ((\sigma'_1)^2 - a(\sigma'_2)^2) \partial_{xx} u = -\frac{2\sigma'_1 (\sigma'_2)^2}{\kappa} \partial_{xy} u + \frac{(\sigma'_2)^3}{\kappa} L_a u, \quad \forall \theta \in [0, |\partial\Omega|]. \quad (2.39)$$

In order to shorten this expression, we introduce the following notation :

$$A(\theta) := \frac{\sigma'_2(\theta)}{\kappa(\theta)} ((\sigma'_1(\theta))^2 - a(\sigma'_2(\theta))^2) = \frac{\sigma'_2(\theta)}{\kappa(\theta)} (1 - (a+1)(\sigma'_2(\theta))^2). \quad (2.40)$$

Thus, (2.39) turns into :

$$-\partial_x u + A \partial_{xx} u = -\frac{2\sigma'_1 (\sigma'_2)^2}{\kappa} \partial_{xy} u + \frac{(\sigma'_2)^3}{\kappa} L_a u \quad \forall \theta \in [0, |\partial\Omega|]. \quad (2.41)$$

Step 2 : Defining an auxiliary function.

We now consider the lower part of the boundary :

$$\Gamma := \{(\sigma_1(\theta), \sigma_2(\theta)) : \sigma'_1(\theta) \geq 0\}.$$

We can extend the functions κ , σ'_i and σ'_{ii} ($i = 1, 2$) to Ω . In order to do so, we define $\Theta_h(x)$ as the only value $\theta \in [0, |\partial\Omega|]$ such that $\sigma_1(\theta) = x$ and $\sigma(\theta) \in \Gamma$. We consider the following auxiliary function in $\bar{\Omega}$:

$$g(x, y) := -\partial_x u(x, y) + A(\Theta_h(x))\partial_{xx}u(x, y). \quad (2.42)$$

Besides, for any set $S \subset \mathbb{R}^2$, we define :

$$\begin{aligned} \mathcal{O}(S) &:= (S + \mathbb{R}e_2) \cap \Omega, \\ P_h(S) &:= \sigma(\Theta_h(S)). \end{aligned}$$

We estimate g and $\partial_x g$ on horizontal segments ; that is, on segments of the type :

$$l := [x_l, x_r] \times \{y\} \subset \bar{\Omega}. \quad (2.43)$$

First, we estimate the L^1 -norm of $\partial_x g$ on any horizontal segment $l \subset \bar{\Omega}$. We consider the following equality :

$$\partial_x g(x, y) = \partial_x g(\sigma(\Theta_h(x))) + \int_{\sigma_2(\Theta_h(x))}^y \partial_{xy}g(x, z)dz, \quad \forall (x, y) \in \bar{\Omega}. \quad (2.44)$$

The second term in this equality is clearly estimated in $L^1(l, dx)$ -norm by $\|\partial_{xy}u\|_{H^2(\Omega)}$ (see (2.42)). In order to estimate the first term in the previous equality, we differentiate (2.41) in the direction θ :

$$\sigma'_1 \partial_x g = -\sigma'_2 \partial_y g + \partial_\theta \left(-\frac{2\sigma'_1(\sigma'_2)^2}{\kappa} \partial_{xy}u + \frac{(\sigma'_2)^3}{\kappa} L_a u \right), \quad \theta \in [\tilde{\theta}, \hat{\theta}].$$

Since $\sigma_1(\Theta_h(x)) = x$, we have that :

$$\Theta'_h(x) = \frac{1}{\sigma'_1(\Theta_h(x))}.$$

So, if we combine this with the fact that $|\sigma'| = 1$, with the fact that $\Theta_h(\sigma_1(\theta)) = \theta$, and recalling the notation presented in (2.43), we have that :

$$\begin{aligned} & \int_l \left(\frac{1}{|\sigma'_1|} \left| -\sigma'_2((\partial_y g) \circ \sigma) + \partial_\theta \left(-\frac{2\sigma'_1(\sigma'_2)^2}{\kappa}((\partial_{xy} u) \circ \sigma) + \frac{(\sigma'_2)^3}{\kappa}((L_a u) \circ \sigma) \right) \right| \right) (\Theta_h(x)) dx \\ &= \int_l \left| \frac{d(\sigma(\Theta_h(x)))}{dx} \right| \left| -\sigma'_2 \partial_y g - \partial_\theta \left(\frac{2\sigma'_1(\sigma'_2)^2}{\kappa} \right) \partial_{xy} u - \frac{2\sigma'_1(\sigma'_2)^2}{\kappa} (\sigma'_1 \partial_x + \sigma'_2 \partial_y) \partial_{xy} u \right. \\ &\quad \left. + \partial_\theta \left(\frac{(\sigma'_2)^3}{\kappa} \right) L_a u + \frac{(\sigma'_2)^3}{\kappa} (\sigma'_1 \partial_x + \sigma'_2 \partial_y) L_a u \right| (\sigma(\Theta_h(x))) dx \\ &= \int_{P_h(l)} \left| -\sigma'_2 \partial_y g - \partial_\theta \left(\frac{2\sigma'_1(\sigma'_2)^2}{\kappa} \right) \partial_{xy} u - \frac{2\sigma'_1(\sigma'_2)^2}{\kappa} (\sigma'_1 \partial_x + \sigma'_2 \partial_y) \partial_{xy} u \right. \\ &\quad \left. + \partial_\theta \left(\frac{(\sigma'_2)^3}{\kappa} \right) L_a u + \frac{(\sigma'_2)^3}{\kappa} (\sigma'_1 \partial_x + \sigma'_2 \partial_y) L_a u \right|, \end{aligned}$$

for the geometric functions in the second and third integral above evaluated in $\Theta_h(x)$, when we are in a point $p = (x, y) \in \Gamma$. Thus, recalling that $\partial_y g = -\partial_{xy} u + A \partial_{xxy} u$ and recalling (2.44) we get that :

$$\|\partial_x g\|_{L^1(l, dx)} \leq C (\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \quad (2.45)$$

Next, we estimate the C^0 -norm of g on any horizontal segment $l \subset \bar{\Omega}$. Indeed, if we use (2.39), (2.42) and the formula :

$$g(x, y) = g(\sigma(\Theta_h(x))) + \int_{\sigma_2(\Theta_h(x))}^y \partial_y g(x, z) dz, \quad \forall (x, y) \in \bar{\Omega},$$

we get, for some $p \in l$, an estimate of $g(p)$ in terms of $\|\partial_{xy} u\|_{H^2(\Omega)}$ and $\|L_a u\|_{H^1(\partial\Omega)}$ with a constant depending on l . So, if we also consider (2.45), we get that :

$$\|g\|_{C^0(l)} \leq C(l) (\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \quad (2.46)$$

Step 3 : Getting the information from the ordinary differential equation (2.42).

We split Ω in different subsets depending on the sign of $A(\Theta_h(x))$. With that purpose, we denote θ_0 , θ_a^\pm and θ_1^\pm the values such that :

$$\begin{cases} \sigma'(\theta_0) = e_1, \\ \sigma'(\theta_a^\pm) = \left(\sqrt{\frac{a}{a+1}}, \pm \sqrt{\frac{1}{a+1}} \right), \\ \sigma'(\theta_1^\pm) = \pm e_2. \end{cases} \quad (2.47)$$

Since the domain is strictly convex, all θ_i are uniquely determined for a fixed. We set :

$$x_0 := \sigma_1(\theta_0); x_a^\pm := \sigma_1(\theta_a^\pm) \text{ and } x_1^\pm := \sigma_1(\theta_1^\pm).$$

Because of (2.40) we have that $A(\theta_0) = 0$ and $A(\theta_a^\pm) = 0$. By (2.42) and (2.46), this implies that we can estimate $\partial_x u$ in the vertical segments given by $\{x = x_0\}$ and $\{x = x_a^\pm\}$. More generally, if we have an estimate of $|\partial_x u(x^*, y^*)|$, we have an estimate of $|\partial_x u(x^*, y)|$ for all y such that $(x^*, y) \in \Omega$ because $\|\partial_{xy} u\|_{C^0(\bar{\Omega})}$ can be estimated by $\|\partial_{xy} u\|_{H^2(\Omega)}$. Consequently, we just have to transmit horizontally the punctual estimates of $|\partial_x u|$. In order to do so, in the rest of the proof we fix some appropriate horizontal segments and regard equality (2.42) as an ordinary differential equation.

We first prove estimate (2.33) in $\Omega \cap \{x \geq x_0\}$. We consider a sequence of values $s_0 < \dots < s_n$ such that $s_0 := x_0$, $s_n := x_1^+$ and such that for any $i \in \{1, \dots, n\}$ there exists an horizontal segment $l_i \subset \bar{\Omega}$ such that the abscissa of its left endpoint (respectively its right endpoint) is s_{i-1} (respectively s_i). Because we are in a regular bounded convex domain, all this can be done (see Figure 2.1 for one such example). We denote by y_i the second coordinate of any point of l_i . Since the segments l_1, \dots, l_n do not depend on a , we have, due to (2.46), the following estimate in those segments with a constant that only depends on Ω :

$$\|g\|_{C^0(\cup_{i=1}^n l_i)} \leq C(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \quad (2.48)$$

Let us set $j_a \in \{1, \dots, n\}$ such that $\theta_a^+ \in \Theta_h(l_{j_a})$. If there are two such segments, we choose the one on the right. We first get an estimate in $\mathcal{O}(\{x \geq x_a^+\})$, where we have :

$$A(\Theta_h(x)) \leq 0 \quad (2.49)$$

(see (2.40)). For that purpose, we define $l_a^r := l_{j_a} \cap \mathcal{O}(\{x \geq x_a^+\})$.

First, considering that $\partial_x u(x_a^+, y) = -g(x_a^+, y)$ and (2.48), we obtain the following estimate :

$$|\partial_x u(x_a^+, y_{j_a})| \leq C(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \quad (2.50)$$

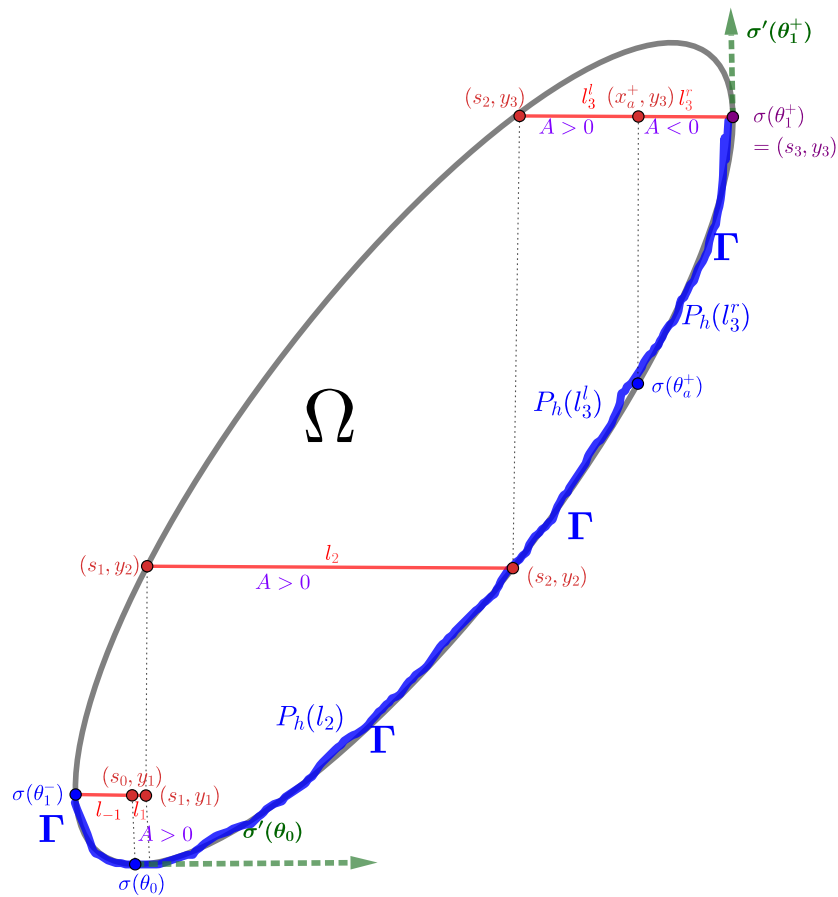


FIGURE 2.1 – An illustration of the strictly convex case

Next, we have that, for any $z \in [x_a^+, s_{j_a}]$:

$$\begin{aligned} - \int_{x_a^+}^z (x - z - 2)(\partial_x u \partial_{xx} u)(x, y_{j_a}) dx &= \frac{1}{2} \int_{x_a^+}^z |\partial_x u(x, y_{j_a})|^2 dx \\ &+ \frac{(x_a^+ - z - 2)}{2} |\partial_x u(x_a^+, y_{j_a})|^2 - (-1) |\partial_x u(z, y_{j_a})|^2. \end{aligned} \quad (2.51)$$

Thus, considering (2.42), (2.50) and (2.51), we have that for a constant C independent of z :

$$\begin{aligned} \frac{1}{2} \int_{x_a^+}^z |\partial_x u(x, y_{j_a})|^2 dx + \int_{x_a^+}^z (x - z - 2) A(\Theta_h(x)) |\partial_{xx} u(x, y_{j_a})|^2 dx + |\partial_x u(z, y_{j_a})|^2 \\ \leq \int_{x_a^+}^z (x - z - 2)(g \partial_{xx} u)(x, y_{j_a}) dx + C \left(\|\partial_{xy} u\|_{H^2(\Omega)}^2 + \|L_a u\|_{H^1(\partial\Omega)}^2 \right). \end{aligned} \quad (2.52)$$

So, the last term to be estimated in (2.52) is the one with g . In order to do that, we integrate by parts :

$$\begin{aligned} \int_{x_a^+}^z (x - z - 2)(g \partial_{xx} u)(x, y_{j_a}) dx &= - \int_{x_a^+}^z (x - z - 2)(\partial_x g \partial_x u)(x, y_{j_a}) dx \\ &- \int_{x_a^+}^z (g \partial_x u)(x, y_{j_a}) dx - (x_a^+ - z + 2)g(x_a^+, y_{j_a}) \partial_x u(x_a^+, y_{j_a}) - 2g(z, y_{j_a}) \partial_x u(z, y_{j_a}) \\ &\leq C \left(\|\partial_x g\|_{L^1(I_a^r, dx)} + \|g\|_{C^0(I_a^r)} \right) \|\partial_x u\|_{C^0(I_a^r)}. \end{aligned} \quad (2.53)$$

Using that $ab \leq \frac{1}{4\eta} a^2 + \eta b^2$ and estimating the norms of g by (2.45) and (2.48), we obtain the following from (2.52) :

$$\int_{x_a^+}^z |\partial_x u(x, y_{j_a})|^2 dx + |\partial_x u(z, y_{j_a})|^2 \leq C_\eta \left(\|\partial_{xy} u\|_{H^2(\Omega)}^2 + \|L_a u\|_{H^1(\partial\Omega)}^2 \right) + \eta \|\partial_x u\|_{C^0(I_a^r)}^2.$$

Since z is arbitrary, we deduce that :

$$\|\partial_x u\|_{C^0(I_a^r)} \leq C_\eta \left(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)} \right) + \eta \|\partial_x u\|_{C^0(I_a^r)}.$$

Thus, taking η small enough, we can absorb the last term on the right-hand side. Moreover, as $\|\partial_{xy} u\|_{C^0(\overline{\Omega})}$ is estimated in terms of $\|\partial_{xy} u\|_{H^2(\Omega)}$, we get :

$$\|\partial_x u\|_{C^0(\overline{\mathcal{O}(I_a^r)})} \leq C \left(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)} \right). \quad (2.54)$$

This method also works for l_{j_a+1}, \dots, l_n because the trace of $\partial_x u$ on the left of these segments is obtained first by (2.54) and then inductively. Indeed, in l_i it suffices to multiply

at both sides of the identity (2.42) by $(x - s_i - 2)\partial_{xx}u$ and integrate by parts as above. Therefore, we can get inductively the estimate :

$$\|\partial_x u\|_{C^0(\overline{\mathcal{O}(l_a^r)})} + \sum_{i=j_a+1}^n \|\partial_x u\|_{C^0(\overline{\mathcal{O}(l_i)})} \leq C (\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}),$$

which implies that :

$$\|\partial_x u\|_{C^0(\overline{\mathcal{O}(\{x \geq x_a^+\})})} \leq C (\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}).$$

As for getting the estimate in $\mathcal{O}(\{x \in [x_0, x_a^+]\})$, we can obtain it analogously as before : first in $l_a^l := l_{j_a} \cap \mathcal{O}(\{x \leq x_a^+\})$, then inductively in l_{j_a-1}, \dots, l_1 . The only difference is that this time we multiply by $x + x_a^+ + 2$ and we spread the estimates to the left. The reason for these changes is that for $x \in [x_0, x_a^+]$, $A(\Theta_h(x))$ is positive instead of negative (see (2.40)).

Finally, it is quite clear that we can get the estimate in a similar way for $\Omega \cap \{x < x_0\}$. Thus, (2.33) is true. \square

2.4 Proof of Theorem 2.8 and some geometrical results

In this section we present the proof of Theorem 2.8 as well as some strongly related results. First, in Subsection 2.4.1 we prove Lemma 2.3. Second, in Subsections 2.4.2 and 2.4.3 we state and prove some geometrical consequences of Hypothesis 2.1. Finally, in Subsection 2.4.4 we prove Theorem 2.8, using, among others, Section 2.3.

We recall that some of the notation used in this section has been introduced above Hypothesis 2.1.

2.4.1 Proof of Lemma 2.3

Lemma 2.3 is a consequence of Sard's Theorem :

Theorem 2.34 (Sard's Theorem). *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ a C^1 function. Let*

$$X := \{x \in \mathbb{R} : f'(x) = 0\}.$$

Then, $f(X)$ has zero measure.

Proof. [Proof of Lemma 2.3] In order to apply Sard's Theorem, we consider the functions :

$$(\sigma_1^1)', \dots, (\sigma_1^k)', (\sigma_2^1)', \dots, (\sigma_2^k)'$$

Let us denote $Z \subset [-1, 1]$ the set of values such that if $(\sigma_1^i)'(\theta) = z$ or if $(\sigma_1^i)'(\theta) = -z$ or if $(\sigma_2^i)'(\theta) = z$ or if $(\sigma_2^i)'(\theta) = -z$; then, $\kappa^i(\theta) \neq 0$. Since $\kappa^i(\theta) = 0$ implies that $(\sigma_1^i)''(\theta) = 0$ and $(\sigma_2^i)''(\theta) = 0$ (see (2.3)), thanks to Sard's Theorem (and to the fact that a finite union of null-measure sets and translations of those sets are still of null measure) we get that the measure of $[-1, 1] \setminus Z$ is 0.

Let us consider $\psi \in \sin^{-1}(Z)$. Then, we have that $\partial(U_\psi(\Omega)) = U_\psi(\partial\Omega)$ and it can be parametrized by $\tilde{\sigma}^i := U_\psi(\sigma^i)$. Moreover, we have $(U_\psi(\sigma^i))' = U_\psi((\sigma^i)')$. Consequently, $(\tilde{\sigma}^i)' = \pm e_2$ if and only if $(\sigma^i)'(\theta) = U(-\psi)(0, \pm 1)$; that is, if and only if $(\sigma^i)'(\theta) = \pm(\sin(\psi), \cos(\psi))$. Similarly, $(\tilde{\sigma}^i)' = \pm e_1$ if and only if $(\sigma^i)'(\theta) = \pm(\cos(\psi), -\sin(\psi))$. Furthermore, if $\tilde{\kappa}^i(\theta)$ is by definition the curvature in $\tilde{\sigma}^i(\theta)$, we have, by the non-variation of the curvature by rotations, $\tilde{\kappa}^i(\theta) = \kappa^i(\theta)$. Thus, by the definition of Z , taking $z = \sin \psi$, we have that all the points of $\partial(U_\psi(\Omega))$ with horizontal or vertical tangent vector have a non zero curvature.

Finally, the measure of $\mathbb{R} \setminus \sin^{-1}(Z)$ is null because, for all $k \in \mathbb{Z}$, the sinus is a diffeomorphism from $(\pi(k - 1/2), \pi(k + 1/2))$ to $(-1, 1)$ (and because a countable union of sets of null measure has null measure). \square

2.4.2 Geometrical consequences of Hypothesis 2.1

In order to prove Theorem 2.8, we need to define equivalent notions to the ones presented in the convex case (see Section 2.3).

Definition 2.35. We define Γ as the subset of $\partial\Omega$ such that $p = \sigma^i(\theta) \in \Gamma$ if and only if at least one of the following properties is satisfied :

- $\exists \delta_0(p) > 0 : \forall \delta \in (0, \delta_0(p)), p + \delta e_2 \in \overline{\Omega}$,
- $(\sigma^i)'(\theta) = \pm e_2$.

When Ω is convex we have that Γ is the bottom of Ω . Moreover, for an illustration on what Γ may look like in a non-convex domain, we can regard Figure 2.2 below.

Remark 2.36. The relative boundary of Γ is given by points of tangent vectors $\pm e_2$. Indeed, the components of $\partial\Omega$ are closed curves; thus, by regularity, having a vertical tangent vector is the only possibility.

Definition 2.37. Let $(x, y) \in \overline{\Omega}$. We define :

$$\mathbb{P}_h(x, y) := (x, y) - \lambda e_2 \text{ such that } \lambda := \min\{\lambda \in \mathbb{R}^+ : (x, y) - \lambda e_2 \in \Gamma\}.$$

We remark that when Ω is convex \mathbb{P}_h represents the vertical projection on Γ , is continuous and does not depend on y . In the general situation, though, there is a dependence on y and \mathbb{P}_h is not continuous when Γ is not connected (see B^1 in Figure 2.2 below). Yet, we can define an application P_h that coincides with the one given in Section 2.3 when Ω is convex :

Definition 2.38. Let $l_i = [x_l^i, x_r^i] \times \{y^i\} \subset \overline{\Omega}$ a segment. Then,

$$P_h(l_i) := \overline{\mathbb{P}_h([x_l^i, x_r^i] \times \{y^i\})}.$$

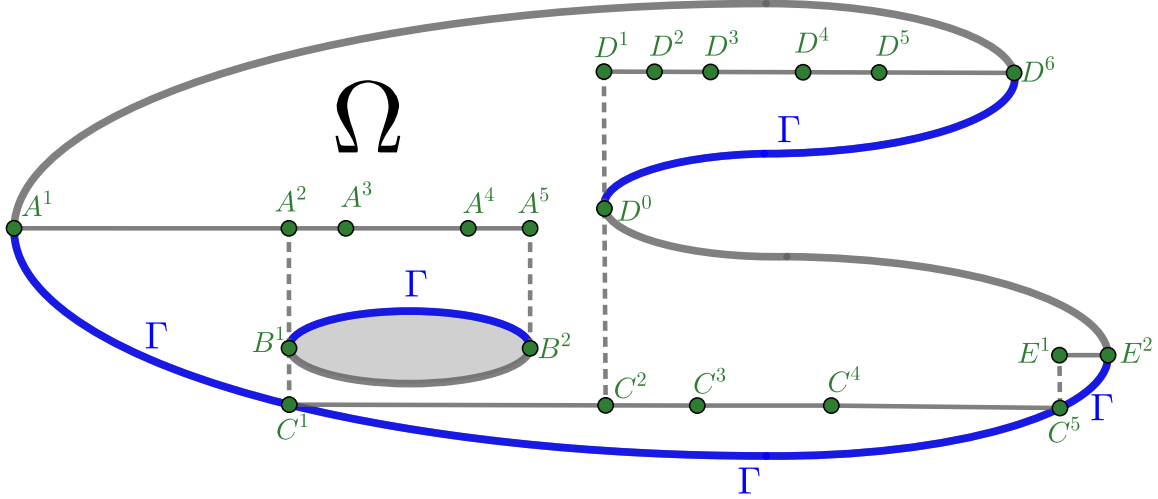
Moreover, we see that Hypothesis 2.1 implies the existence of segments like in the case of a convex domain :

Lemma 2.39. *Let Ω be a domain that satisfies Hypothesis 2.1. Then, there is a subset $S \subset \overline{\Omega}$ such that :*

- S is a finite union of horizontal segments $l_i := [x_l^i, x_r^i] \times \{y^i\}$.
- $\mathbb{P}_h(S) = \Gamma$.
- \mathbb{P}_h is continuous in the relative interior of each segment l_i .

Example 2.40. In Figure 2.2 S is given by the segments : $[A^1, A^2]$, $[A^2, A^5]$, $[C^1, C^5]$, $[D^1, D^6]$ and $[E^1, E^2]$.

The proof of Lemma 2.39 is postponed to Section 2.4.3. We first prove some geometrical results :

FIGURE 2.2 – An illustration of S in a non-convex domain

Lemma 2.41. *Let Ω be a domain that satisfies Hypothesis 2.1. We have :*

1. *If $(\sigma_1^i)'(\theta) = 0$ or if $(\sigma_2^i)'(\theta) = 0$, then, for some $\delta(\theta) > 0$, κ^i does not change of sign in $(\theta - \delta(\theta), \theta + \delta(\theta))$.*
2. *The number of points in $\partial\Omega$ with tangent vectors $\pm e_1$ or $\pm e_2$ is finite.*
3. *Given any $c \in \mathbb{R}$, the number of points in $\partial\Omega \cap \{x = c\}$ or in $\partial\Omega \cap \{y = c\}$ is finite.*
4. *Given any $c \in \mathbb{R}$, there is $\delta(c) > 0$ such that :*

- *We have*

$$([c - \delta(c), c + \delta(c)] \times \mathbb{R}) \cap \partial\Omega = \bigcup_{p=\sigma^{ip}(\theta_p) \in \partial\Omega \cap \{x=c\}} \sigma^{ip}(I_p),$$

for $I_p = (\theta_p^1, \theta_p^2)$, for some $\theta_p^1 < \theta_p < \theta_p^2$.

- *In the set*

$$(([c - \delta(c), c + \delta(c)] \setminus \{c\}) \times \mathbb{R}) \cap \partial\Omega,$$

we do not have $p = \sigma^i(\theta)$ with $(\sigma^i)'(\theta) = \pm e_2$.

5. *There is some $\eta > 0$ such that for all points $p = \sigma^i(\theta_p) \in \partial\Omega$ with $(\sigma^i)'(\theta_p) = \pm e_1$, there exists a neighbourhood $V_p = \sigma^i(I_p) \subset \partial\Omega$ ($I_p = (\theta_p^1, \theta_p^2)$, for some $\theta_p^1 < \theta_p < \theta_p^2$) such that $\sigma_2^i(\theta_p^1) = \sigma_2^i(\theta_p^2)$ and such that $|\kappa^i| > \eta$.*
6. *There exists $a_0 > 0$ small enough such that, for all $a \in (0, a_0)$, for each point $p = \sigma^i(\theta) \in \partial\Omega$ with $(\sigma^i)'(\theta) = \pm e_2$ there is a neighbourhood $U_p \subset \partial\Omega$ which*

has exactly a point of tangent vector $\pm \left(\sqrt{\frac{a}{1+a}}, \sqrt{\frac{1}{1+a}} \right)$ and exactly another one of tangent vector $\pm \left(\sqrt{\frac{a}{1+a}}, -\sqrt{\frac{1}{1+a}} \right)$. Reciprocally, if $p_a = \sigma^i(\theta^a) \in \partial\Omega$ satisfies $(\sigma^i)'(\theta^a) = \left(\pm \sqrt{\frac{a}{1+a}}, \pm \sqrt{\frac{1}{1+a}} \right)$, then $p_a \in U_p$, for U_p one of the above defined neighbourhoods. Finally, we can suppose that for some $\eta > 0$, $|\kappa^i| > \eta$ on those neighbourhoods.

Proof. [Proof of Lemma 2.41] Firstly, implication 1 is an easy consequence of Ω being at least C^2 .

Secondly, we prove implication 2 for points of tangent vector $\pm e_2$ by contradiction. If they are not finite, by (pre-)compactness and regularity of Ω , there is a point $p = \sigma^i(\theta)$ and $\theta_n \rightarrow \theta$ such that $(\sigma_1^i)'(\theta_n) = 0$. Obviously $(\sigma_1^i)'(\theta) = 0$. But, because of the regularity of Ω , we also have $(\sigma_1^i)''(\theta) = 0$. Indeed,

$$(\sigma_1^i)''(\theta) = \lim_{s \rightarrow \theta} \frac{(\sigma_1^i)'(s) - (\sigma_1^i)'(\theta)}{s - \theta} = \lim_{n \rightarrow \infty} \frac{(\sigma_1^i)'(\theta_n) - (\sigma_1^i)'(\theta)}{\theta_n - \theta} = 0. \quad (2.55)$$

Thus, we get by (2.3) that $\kappa^i(\theta) = 0$, which contradicts Hypothesis 2.1. The proof for points with tangent vector $\pm e_1$ is analogous.

Thirdly, given any line $x = c$, we prove by contradiction that there is a finite number of points in $\partial\Omega \cap \{x = c\}$. Indeed, if we have an infinite number of points, by regularity and compactness we can write a sequence of distinct elements as $\sigma^i(\theta_n)$ with $\theta_n \rightarrow \theta$. Since $\sigma_1^i(\theta_n) = c$, by an equality similar to (2.55), $(\sigma_1^i)'(\theta) = 0$. Since $\kappa^i(\theta) \neq 0$ by Hypothesis 2.1, $\sigma^i(\theta_n)$ cannot be in $\{x = c\}$ for n large enough, contradicting the choice of θ_n . The proof for $y = c$ is analogous.

Fourthly, statement 4 is a consequence of assertion 2. Indeed, the only possibility is that there is an infinite number of curves of $\partial\Omega$ that approach the line $x = c$ and then move away (like a parabola). But this implies that there is an infinite number of points of tangent vector $\pm e_2$, which contradicts statement 2.

Fifthly, assertion 5 is an easy consequence of statements 1 and 2 and of picking the neighbourhoods small enough.

Finally, statement 6 is a consequence of assertion 2. Indeed, we consider U_{p_j} some neighbourhoods of p_j , for p_j the points of tangent vector $\pm e_2$. We have the bound :

$$\inf_i \inf_{\theta: \sigma^i(\theta) \notin U_{p_j} \forall j} |(\sigma_1^i)'(\theta)| > 0.$$

Moreover, by making them smaller if necessary, we have due to Hypothesis 2.1 and the continuity of the κ^i :

$$\inf_i \inf_{\theta: \exists j: \sigma^i(\theta) \in U_{p_j}} |\kappa^i(\theta)| > 0.$$

Consequently, we can fix some smaller neighbourhoods U_{p_j} and the parameters a_0 and η . □

2.4.3 Proof of Lemma 2.39 and some remarks

In this subsection we first present the proof of Lemma 2.39; and then, we state some direct consequences.

First of all, we define some useful notation :

Definition 2.42. Let $\Omega \subset \mathbb{R}^2$ be a domain and $x \in \mathbb{R}$. We define :

$$\Omega_x := \Omega \cap ((-\infty, x) \times \mathbb{R}).$$

Now we are ready to present the proof :

Proof. [Proof of Lemma 2.39] Without loss of generality we can suppose that :

$$0 = \min\{x : \exists y \text{ with } (x, y) \in \overline{\Omega}\}.$$

We consider :

$$I = \{x \in \mathbb{R}^+ : \forall s \in [0, x] \exists S_s \subset \overline{\Omega}_s :$$

S_s satisfies the conclusion of Lemma 2.39 with Γ replaced by $\Gamma \cap \overline{\Omega}_s\}$.

First, we remark that $0 \in I$ because $\Omega_0 = \emptyset$. Thus, in order to prove the result it suffices to show that if $[0, c) \subset I$, we have $c \in I$ and that if $c \in I$, we have $(c, c + \delta(c)) \subset I$ for $\delta(c) > 0$ sufficiently small.

Next, let us show that $[0, c) \subset I$ implies $c \in I$. We consider $\delta(c)$ given by statement 4 in Lemma 2.41. In $\overline{\Omega_{c-\delta(c)}}$ we already have the segments $S_{c-\delta(c)}$. Thus, it suffices to define in $\overline{\Omega_c \setminus \Omega_{c-\delta(c)}}$ a finite number of segments slightly above the curves of Γ , which can be done by regularity and pre-compactness. If there is some point $p = \sigma^i(\theta) \in \Gamma$ with $x = c$ such that $(\sigma^i)'(\theta) = \pm e_2$ we might have to start a segment in p and spread it to the left (see the segment $[E^2, E^1]$ in Figure 2.2). So we have $c \in I$.

Finally, let us show that $c \in I$ implies $c + \delta(c) \in I$, for $\delta(c)$ sufficiently small. Again, we pick $\delta(c)$ small enough to satisfy the conclusion of statement 4 in Lemma 4. We already have S_c , so now we consider the possible situation for the points in $\Gamma \cap \{x = c\}$ (which are finite by statement 3 in Lemma 2.41) :

- For the points in $\{x = c\} \cap \Gamma$ which do not have as tangent vector $\pm e_2$, we just have to extend the segment of S_c with right endpoint in $x = c$ to the right of $\overline{\Omega_c}$, unless some segment has a right endpoint which belongs to $\partial\Omega$. In that case, we just have to start a new segment whose left endpoint can be joint by a vertical segment inside $\overline{\Omega}$ with the right endpoint of the former one (see the dashed segments near C^5 in Figure 2.2).
- As for the points in $\{x = c\} \cap \Gamma$ of tangent vector $\pm e_2$, we might not be allowed to extend the segment because \mathbb{P}_h becomes discontinuous and we need to start two new segments (for instance, in A^2 in Figure 2.2 we must stop and start two new segments : one with left endpoint A^2 , the other one with left endpoint C^1). It might also be the case that we need to start a new segment above the point (see near D^1 in Figure 2.2).

Summing up, since all this happens for a finite number of situations, for $\delta(c)$ sufficiently small we have $(c, c + \delta(c)) \subset I$. \square

Remark 2.43. Because of the conclusion of Lemma 2.39, the left endpoint of each segment l_i is either a point $p = \sigma^i(\theta) \in \Gamma$ with $(\sigma^i)'(\theta) = \pm e_2$ and $\kappa^i(\theta) > 0$ (the case of A^1 in Figure 2.2) or it can be joined by a vertical segment (including degenerated segments) inside Ω with some other segment l_j such that $x_l^j < x_l^i \leq x_r^j$ (the case of A^2 , C^1 , D^1 and E^1 in Figure 2.2).

Remark 2.44. Another easy consequence of Lemma 2.39 is that, if Ω satisfies Hypothesis 2.1, since $\mathbb{P}_h(S) = \Gamma$, for all $p \in \overline{\Omega}$ there is $\lambda \in \mathbb{R}$ such that $[p, p + \lambda e_2] \subset \overline{\Omega}$ and such

that $p + \lambda e_2 \in S$. Consequently, since $\|\partial_{xy}u\|_{C^0(\bar{\Omega})}$ is estimated by $\|\partial_{xy}u\|_{H^2(\Omega)}$, it suffices to get an estimate of $\|\partial_x u\|_{C^0(S)}$ to prove Theorem 2.8.

Remark 2.45. Given any segment l_i as defined in Lemma 2.39, because of its third property, it makes sense to define $\Theta_h^i(x)$, $A_i(x)$ and $g_i(x, y)$ as in the convex case (see Section 2.3) by looking at $P_h(l_i)$. Indeed, they make sense in the domain limited superiorly by l_i and inferiorly by $P_h(l_i)$. Moreover, we recall that :

$$g_i(x, y) := -\partial_x u(x, y) + A_i(\Theta_h^i(x))\partial_{xx}u(x, y). \quad (2.56)$$

We also remark that if $\min_{l_i} |\kappa(\Theta_h^i(x))| > 0$, we can prove the following estimate as in the convex case (see (2.45) and (2.46)) for a constant C depending only on S :

$$\|g_i\|_{C^0(l_i)} + \|\partial_x g_i\|_{L^1(l_i, dx)} \leq C(\|\partial_{xy}u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \quad (2.57)$$

2.4.4 Proof of Theorem 2.8

In order to prove Theorem 2.8 we get an estimate in each segment l_i given by Lemma 2.39. Indeed, we prove inductively that :

$$\|\partial_x u\|_{C^0(l_i)} \leq C(\|\partial_{xy}u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \quad (2.58)$$

By inductively, we mean that (2.58) is proved for any other segment l_j such that $x_l^j < x_l^i$ (see Lemma 2.39 for the notation).

First, for getting a pointwise estimate on x_l^i , we consider the two situations given in Remark 2.43. In the first case, because of statement 6 on Lemma 2.41, we can get an estimate on x_l^i by $\|\partial_{xy}u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}$ as in the convex case (see Section 2.3). In the second case, by the induction hypothesis and the fact that $\|\partial_{xy}u\|_{C^0(\bar{\Omega})}$ can be estimated by $\|\partial_{xy}u\|_{H^2(\Omega)}$, we get the estimate on x_l^i by $\|\partial_{xy}u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}$. So, in both cases we have :

$$|\partial_x u(x_l^i, y^i)| \leq C (\|\partial_{xy}u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \quad (2.59)$$

So, once we have (2.59), we have to propagate the estimate in l_i . Indeed, we can split, extend and move the segments l_i (see the points $A^3, A^4, C^3, C^4, D^2, D^3, D^4$ in Figure 2.2, which allow us to split their respective segment into smaller ones) so that we only have one of the four following possibilities for $P_h(l_i)$:

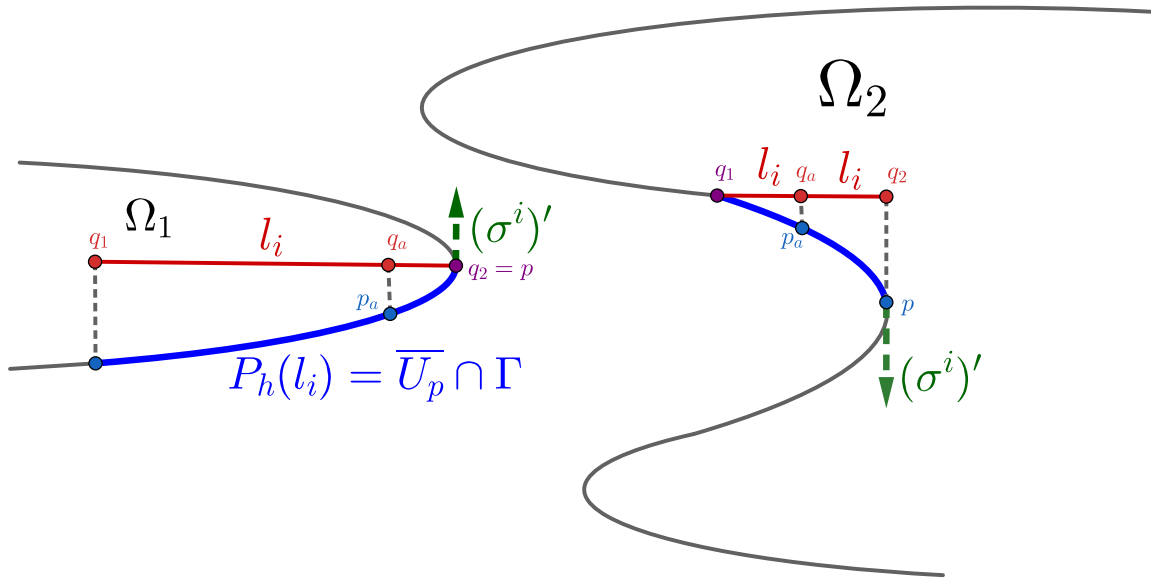


FIGURE 2.3 – Case 1 of the proof of Theorem 2.8

1. $P_h(l_i)$ is the intersection of Γ with one of the neighbourhoods $\overline{U_p}$ (see statement 6 in Lemma 2.41).
2. $P_h(l_i)$ has null intersection with all the neighbourhoods U_p and V_p .
3. $P_h(l_i)$ is one of the neighbourhoods $\overline{V_p}$ (see statement 5 in Lemma 2.41) which has a positive curvature.
4. $P_h(l_i)$ is one of the neighbourhoods $\overline{V_p}$ (see statement 5 in Lemma 2.41) which has a negative curvature.

- Let us first deal with the case 1. We recall that by Remark 2.45, functions like g_i or Θ_h^i or A_i make sense in l_i . We start in the subcase in which $(\sigma_1^i)'(\Theta_h^i(x_r^i)) = 0$. In Figure 2.3, Ω_1 illustrates an example in which the surface is locally convex and Ω_2 an example in which it is locally concave. We remark that, in the concave case, the ratio $\frac{(\sigma_2^i)'}{\kappa^i}$ has the same sign as in the convex case (by the criteria fixed above Hypothesis 2.1, $(\sigma_2^i)' < 0$, $(\sigma_1^i)' \geq 0$ and $\kappa^i < 0$ in that part of Γ). Analogously to the convex case (see Section 2.3), we define $p_a \in V_q$ as the point such that $A_i(\Theta_h(p_a)) = 0$. Moreover, q_a is the point in l_i such that $\mathbb{P}_h(q_a) = p_a$. Thus, in both cases, we can multiply (2.56) by $x + C$ for C large enough in the segment $[q_1, q_a]$ and by $x - C$

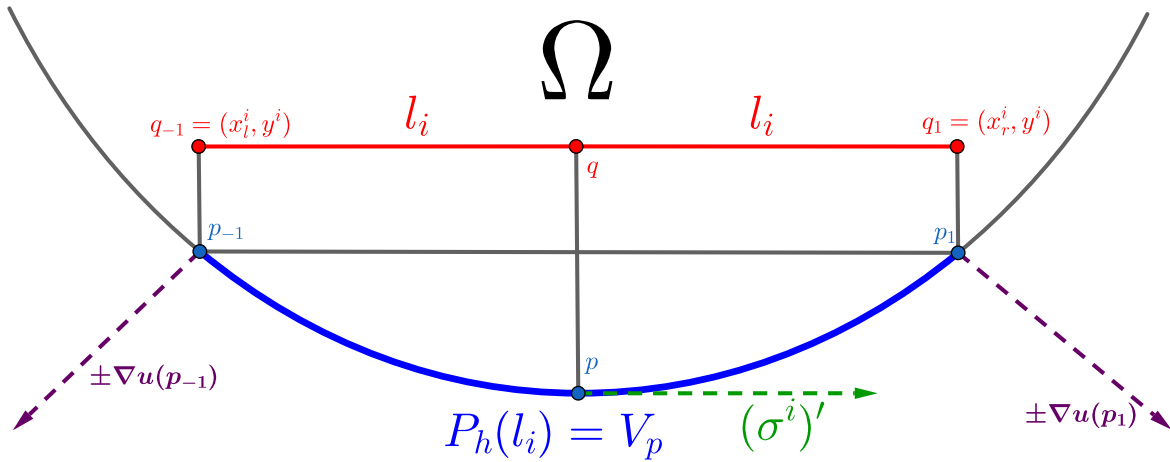


FIGURE 2.4 – Case 3 of the proof of Theorem 2.8

for C large enough in $[q_a, q_2]$ and then follow the procedure of the convex case. The case in which $(\sigma_1^i)'(\Theta_h^i(x_i^i)) = 0$ is analogous. Consequently, we have (2.58) in the case 1.

- If $P_h(l_i)$ belongs neither to U_p nor to V_p (situation 2), we have, by statement 6 in Lemma 2.41, that there is $\eta > 0$ such that :

$$|A_i(\Theta_h^i(l_i))| > \eta$$

(see Remark 2.45 or (2.40) for definition of A_i). Consequently, we can divide the equation (2.56) by A_i and hope to have uniform estimates. With that purpose, we define :

$$\tilde{g}_i(x, y) := -\frac{1}{A_i(\Theta_h^i(x))} \partial_x u(x, y) + \partial_{xx} u(x, y). \quad (2.60)$$

In that sense, we can get the same estimate for \tilde{g}_i as the one for g_i in (2.57) by following similar steps. Thus, seeing (2.60) as an ordinary linear differential equation in l_i whose initial value is in (x_i^i, y^i) , we get (2.58) by Lemma 2.25.

- As for the case 3 (see Figure 2.4 for the notation), we mainly replicate the method of the previous section. In this paragraph and in the following one, we do all the estimates by $\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}$ even if we do not explicitly write it. We already have by (2.59) a pointwise estimate in q_{-1} , so we can prove (2.58) in $[q_{-1}, q]$ following the method of the convex case. As for $[q, q_1]$, to replicate the method of the convex case, we need an estimate of $|\partial_x u(q_1)|$. In order to get it, first we use

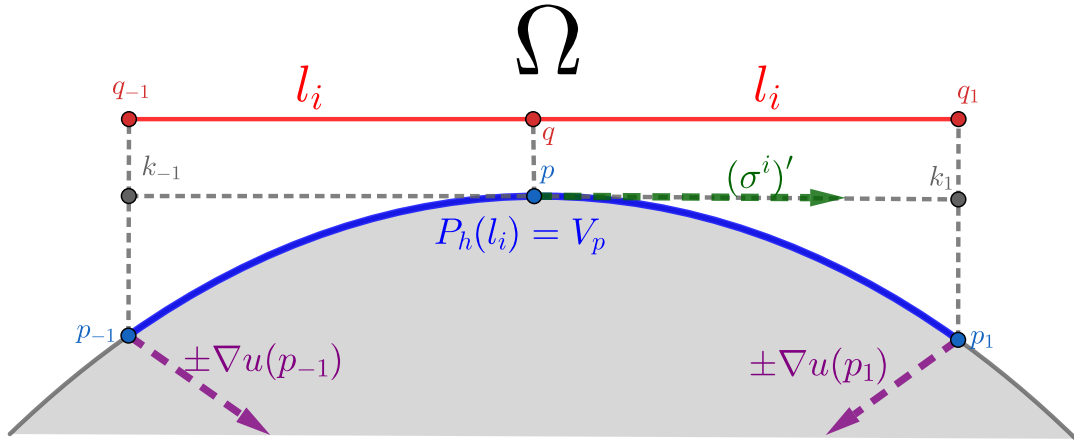


FIGURE 2.5 – Case 4 of the proof of Theorem 2.8

the estimate of $|\partial_x u(q_{-1})|$ in (2.59) to get an estimate of $|\partial_x u(p_{-1})|$. Then, using the Dirichlet boundary condition, we get an estimate of $|\partial_y u(p_{-1})|$. Next, with the estimate of $\|\partial_{xy} u\|_{C^0([p_{-1}, p_1])}$, we have an estimate of $|\partial_y u(p_1)|$. To continue with, again by Dirichlet conditions, we have an estimate of $|\partial_x u(p_1)|$. Finally, by estimating $\|\partial_{xy} u\|_{C^0([p_1, q_1])}$, we have an estimate of $|\partial_x u(q_1)|$, so we can replicate the method of the previous section to prove (2.58) in $[q, q_1]$.

- The last case is that $P_h(l_i)$ is locally concave (see Figure 2.5 for the notation). The technique is the same one as in the case 3 (recall that the ratio $\frac{(\sigma_2^i)'}{\kappa^i}$ and consequently A^i have the same sign as in the convex case), but this time we face the extra difficulty of showing that having the estimate of $|\partial_y u(p_{-1})|$ implies having the estimate of $|\partial_y u(p_1)|$. In order to do it, we consider that in each component of $\partial\Omega$ we have the following equality :

$$a\kappa^i \partial_y u - (\sigma_1^i)'(1 - (a+1)((\sigma_2^i)')^2) \partial_{yy} u = ((\sigma_1^i)')^3 L_a u - 2a((\sigma_1^i)')^2 (\sigma_2^i)' \partial_{xy} u.$$

This equality can be obtained similarly to (2.38). So, similarly as in Remark 2.45, we define locally at the left of $\sigma^i((\theta_p^1, \theta_p))$ (see statement 5 in Lemma 2.41 for the notation) a function $\Theta_v^i(y)$ and a function :

$$\hat{g}(x, y) := a\kappa^i (\Theta_v^i(y)) \partial_y u(x, y) - ((\sigma_1^i)'(1 - (a+1)((\sigma_2^i)')^2)) (\Theta_v^i(y)) \partial_{yy} u(x, y). \quad (2.61)$$

Moreover, we can obtain as (2.57) that (see Figure 2.5 for the notation) :

$$\|\hat{g}\|_{L^1([q_{-1}, k_{-1}], dy)} \leq C(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}).$$

So, by seeing (2.61) as an ordinary differential equation whose initial data is p_{-1} , because there is $\eta > 0$ such that $(\sigma_1^i)'(1 - (a+1)((\sigma_2^i)')^2) \geq \eta$ in V_p , we can use Lemma 2.25 to get an estimate of $|\partial_y u(k_{-1})|$. Then, because $\|\partial_{xy} u\|_{C^0([k_{-1}, k_1])}$ can be estimated, we have an estimate of $|\partial_y u(k_1)|$. Finally, we can propagate the pointwise estimate of $|\partial_y u|$ in the segment $[k_1, p_1]$ in an analogous way as in the segment $[p_{-1}, k_{-1}]$. Consequently, we have the estimate of $|\partial_y u(p_1)|$ and we can replicate the method of the case 3 to obtain (2.58).

Finally, by Remark 2.39, since we have (2.58) for all segments in S , we have (2.12).

2.5 Proof of Theorem 2.7

For the proof of this theorem we define a subdomain $\tilde{\omega}$ compactly contained in ω such that $\overline{\omega_0} \subset \tilde{\omega}$. Moreover, we consider a cut-off function $\chi \geq 0$ satisfying $\text{supp}(\chi) \subset \omega$ and $\chi = 1$ in $\tilde{\omega}$. In addition, we define D^i as the tensor which contains all the derivatives of order i with respect to the x and y variable. In order to clarify the proof we divide it in several steps.

Step 1 : Estimates of the crossed derivative.

First of all, we consider estimate (2.15) squared, multiplied by $(s\xi^*)^{15}\lambda^{16}e^{-2s\alpha^*}$ and integrated in time. If we also bound the weights (see (2.8)), we get that :

$$s^{15}\lambda^{16} \iint_Q e^{-2s\alpha^*} (\xi^*)^{15} |\varphi^\varepsilon|^2 \leq C s^{15}\lambda^{16} \sum_{i=0}^2 \iint_Q e^{-2s\alpha} \xi^{15} |D^i \partial_{xy} \varphi^\varepsilon|^2. \quad (2.62)$$

Next, we apply the elliptic estimate (2.27) to $D^i \partial_{xy} \varphi^\varepsilon$ for $i = 0, \dots, 7$ (we take $D^0 \partial_{xy} \varphi^\varepsilon := \partial_{xy} \varphi^\varepsilon$). We get that :

$$\begin{aligned} \sum_{i=0}^7 s^{19-2i} \lambda^{20-2i} \iint_Q e^{-2s\alpha} \xi^{19-2i} |D^i \partial_{xy} \varphi^\varepsilon|^2 &\leq C \left(s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |D^8 \partial_{xy} \varphi^\varepsilon|^2 \right. \\ &\quad \left. + \sum_{i=0}^7 s^{19-2i} \lambda^{20-2i} \iint_{(0,T) \times \omega_0} e^{-2s\alpha} \xi^{19-2i} |D^i \partial_{xy} \varphi^\varepsilon|^2 \right). \end{aligned} \quad (2.63)$$

Moreover, since under our hypothesis $1 \leq Cs\xi$, we combine that fact with (2.63) and (2.62), and we get that :

$$\begin{aligned} & s^{15}\lambda^{16} \iint_Q e^{-2s\alpha^*} (\xi^*)^{15} |\varphi^\varepsilon|^2 + \sum_{i=0}^7 s^{19-2i} \lambda^{20-2i} \iint_Q e^{-2s\alpha} \xi^{19-2i} |D^i \partial_{xy} \varphi^\varepsilon|^2 \\ & \leq C \left(s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |D^8 \partial_{xy} \varphi^\varepsilon|^2 + \sum_{i=0}^7 s^{19-2i} \lambda^{20-2i} \iint_{(0,T) \times \omega_0} e^{-2s\alpha} \xi^{19-2i} |D^i \partial_{xy} \varphi^\varepsilon|^2 \right). \end{aligned} \quad (2.64)$$

To continue with, we deal with each term of $D^8 \partial_{xy} \varphi^\varepsilon$. In order to do so, we use Proposition 2.28 (on each term of $D^8 \partial_{xy} \varphi^\varepsilon$) and get that :

$$\begin{aligned} & s\lambda^2 \iint_Q e^{-2s\alpha} \xi |D^9 \partial_{xy} \varphi^\varepsilon|^2 + s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |D^8 \partial_{xy} \varphi^\varepsilon|^2 \\ & \leq C \left(s^4 \lambda^5 \iint_{(0,T) \times \tilde{\omega}} e^{-2s\alpha} \xi^4 |D^8 \partial_{xy} \varphi^\varepsilon|^2 + (1+T) \left(\|\eta h\|_{\mathbf{H}^{1,1/2}(\Sigma)}^2 + \|\tilde{\eta} h\|_{\mathbf{H}^{2,5/2}(\Sigma)}^2 \right) \right), \end{aligned} \quad (2.65)$$

for $h := \partial_n D^8 \varphi^\varepsilon + \varepsilon^{-1} \nabla \cdot D^8 \varphi^\varepsilon$.

Remark 2.46. It is well-known since [38] that by taking enough derivatives we can absorb the trace. Indeed, each time we use (2.27), the weight is, up to a constant, divided by $s^2 \lambda^2 \xi^2$. Moreover, with the weights α^* and ξ^* (the weights on the boundary), we can formally “remove a derivative” by multiplying the weight by $C(s\xi^*)^{1+1/m}$ (this is rigorously done in the next step). Consequently, it is clear that we can absorb the trace using Proposition 2.28 by taking enough derivatives. In our case, if we take less derivatives, what happens when we remove the derivatives of the trace is that we get something with a weight larger than $s^{15} \lambda^{16} e^{-2s\alpha^*} (\xi^*)^{15}$, which cannot be absorbed by the left-hand side of (2.62).

Step 2 : Absorbing the trace terms.

Let us start absorbing $(1+T) \|\eta h\|_{\mathbf{H}^{1,1/2}(\Sigma)}^2$. We consider the continuous injections :

$$H^{6,12}(Q) \subset H^1(0, T; H^{10}(\Omega)) \text{ and } H^{5,11}(Q) \subset H^1(0, T; H^{8+4/5}(\Omega)).$$

Consequently, we have that :

$$\begin{aligned} \|\eta h\|_{\mathbf{H}^{1,1/2}(\Sigma)} &\leq C \left(\|\eta \varphi^\varepsilon\|_{H^1(0,T;\mathbf{H}^{10}(\Omega))} + \varepsilon^{-1} \|\nabla \cdot (\eta \varphi^\varepsilon)\|_{\mathbf{H}^{0,9}(Q)} \right. \\ &\quad \left. + \varepsilon^{-1} \|\nabla \cdot (\eta \varphi^\varepsilon)\|_{H^1(0,T;\mathbf{H}^{8+4/5}(\Omega))} \right) \leq C \left(\|\eta \varphi^\varepsilon\|_{\mathbf{H}^{6,12}(Q)} + \varepsilon^{-1} \|\nabla \cdot (\eta \varphi^\varepsilon)\|_{\mathbf{H}^{5,11}(Q)} \right). \end{aligned} \quad (2.66)$$

We recall that :

$$(t, x) \mapsto \eta(T-t)\varphi^\varepsilon(T-t, x)$$

is a solution of (2.1) with null initial value and force $-\eta'(T-t)\varphi^\varepsilon(T-t, x)$. Thus, applying estimate (2.20) with $i = 6$ to (2.66), we have that :

$$\|\eta \varphi^\varepsilon\|_{\mathbf{H}^{6,12}(Q)} + \varepsilon^{-1} \|\nabla \cdot (\eta \varphi^\varepsilon)\|_{\mathbf{H}^{5,11}(Q)} \leq C \|\eta' \varphi^\varepsilon\|_{\mathbf{H}^{5,10}(Q)}.$$

If we repeat this reasoning five times, we get that :

$$\|\eta h\|_{\mathbf{H}^{1,1/2}(\Sigma)} \leq C \|\eta^{vi}\varphi^\varepsilon\|_{\mathbf{L}^2(Q)}.$$

We have that, if $s \geq e^{C\lambda}(T^m + T^{2m})$ and $m \geq 8$ (see (2.28)) :

$$(1 + T^{1/2})|\eta^{vi}| \leq C(s\xi^*)^{6+1/4+7/m} e^{-s\alpha^*} \leq C(s\xi^*)^{15/2} e^{-s\alpha^*}.$$

Consequently, if we also have $\lambda \geq \lambda_0$, we can absorb the term $(1 + T)\|\eta h\|_{\mathbf{H}^{1,1/2}(\Sigma)}$ by the left-hand side of (2.64).

Finally, we have to absorb $(1 + T^{1/2})\|\tilde{\eta} h\|_{\mathbf{H}^{2,5/2}(\Sigma)}$. In order to do so, we recall that :

$$H^{7,14}(Q) \subset H^2(0, T; H^{10}(\Omega)) \text{ and } H^{6,13}(Q) \subset H^2(0, T; H^{8+2/3}(\Omega)).$$

Thus, we have that :

$$\begin{aligned} \|\tilde{\eta} h\|_{\mathbf{H}^{2,5/2}(\Sigma)} &\leq C \left(\|\tilde{\eta} \varphi^\varepsilon\|_{\mathbf{H}^{0,12}(Q)} + \|\tilde{\eta} \varphi^\varepsilon\|_{H^2(0,T;\mathbf{H}^{10}(\Omega))} + \varepsilon^{-1} \|\nabla \cdot (\tilde{\eta} \varphi^\varepsilon)\|_{\mathbf{H}^{0,11}(Q)} \right. \\ &\quad \left. + \varepsilon^{-1} \|\nabla \cdot (\tilde{\eta} \varphi^\varepsilon)\|_{H^2(0,T;\mathbf{H}^{8+2/3}(\Omega))} \right) \leq C \left(\|\tilde{\eta} \varphi^\varepsilon\|_{\mathbf{H}^{7,14}(Q)} + \varepsilon^{-1} \|\nabla \cdot (\tilde{\eta} \varphi^\varepsilon)\|_{\mathbf{H}^{6,13}(Q)} \right). \end{aligned}$$

Consequently, using estimate (2.20) seven times, we get that :

$$\|\tilde{\eta} h\|_{\mathbf{H}^{2,5/2}(\Sigma)} \leq \|\tilde{\eta}^{vii}\varphi^\varepsilon\|_{\mathbf{L}^2(Q)};$$

which is a term that can be absorbed by the left-hand side of (2.64) if $\lambda \geq \lambda_0$, $s \geq e^{C\lambda}(T^m + T^{2m})$ and $m \geq 8$, because under those hypothesis (see (2.29)) :

$$(1 + T^{1/2})|\tilde{\eta}^{vii}| \leq C(s\xi^*)^{6+1/4+7/m} e^{-s\alpha^*} \leq C(s\xi^*)^{15/2} e^{-s\alpha^*}. \quad (2.67)$$

Summing up, if we combine (2.64) and (2.65), and then do the corresponding absorptions, we have the estimate :

$$\begin{aligned}
& s^{15}\lambda^{16} \iint_Q e^{-2s\alpha^*} (\xi^*)^{15} |\varphi^\varepsilon|^2 + \sum_{i=0}^9 s^{19-2i} \lambda^{20-2i} \iint_Q e^{-2s\alpha} \xi^{19-2i} |D^i \partial_{xy} \varphi^\varepsilon|^2 \\
& \leq C \left(\sum_{i=0}^7 s^{19-2i} \lambda^{20-2i} \iint_{(0,T) \times \omega_0} e^{-2s\alpha} \xi^{19-2i} |D^i \partial_{xy} \varphi^\varepsilon|^2 + s^4 \lambda^5 \iint_{(0,T) \times \tilde{\omega}} e^{-2s\alpha} \xi^4 |D^8 \partial_{xy} \varphi^\varepsilon|^2 \right). \tag{2.68}
\end{aligned}$$

Step 3 : Bounding the local terms.

In order to bound the local terms, we start estimating everything by a local term of $\partial_{xy} \varphi^\varepsilon$. We do it with the usual technique : we bound each $1_{\tilde{\omega}}$ by χ (which is defined at the beginning of this section) to a sufficiently high power, we integrate by parts and we use properly weighted Cauchy-Schwarz inequalities. After all this process, we get from (2.68) that :

$$\begin{aligned}
& s^{15}\lambda^{16} \iint_Q e^{-2s\alpha^*} (\xi^*)^{15} |\varphi^\varepsilon|^2 + \sum_{i=0}^9 s^{19-2i} \lambda^{20-2i} \iint_Q e^{-2s\alpha} \xi^{19-2i} |D^i \partial_{xy} \varphi^\varepsilon|^2 \\
& + \sum_{i=1}^8 s^{28-3i} \lambda^{29-3i} \iint_{(0,T) \times \omega} \chi^{4+2i} e^{-2s\alpha} \xi^{28-3i} |D^i \partial_{xy} \varphi^\varepsilon|^2 \leq C s^{28} \lambda^{29} \iint_{(0,T) \times \omega} \chi^4 e^{-2s\alpha} \xi^{28} |\partial_{xy} \varphi^\varepsilon|^2. \tag{2.69}
\end{aligned}$$

Indeed, when $i = 1, \dots, 8$, we have that :

$$\begin{aligned}
& s^{28-3i} \lambda^{29-3i} \iint_{(0,T) \times \omega} \chi^{4+2i} e^{-2s\alpha} \xi^{28-3i} |D^i \partial_{xy} \varphi^\varepsilon|^2 \\
& = s^{28-3i} \lambda^{29-3i} \iint_{(0,T) \times \omega} D(\chi^{4+2i} e^{-2s\alpha} \xi^{28-3i} D^i \partial_{xy} \varphi^\varepsilon) \cdot D^{i-1} \partial_{xy} \varphi^\varepsilon \\
& \leq C_\delta s^{28-3(i-1)} \lambda^{29-3(i-1)} \iint_{(0,T) \times \omega} \chi^{4+2(i-1)} e^{-2s\alpha} \xi^{28-3(i-1)} |D^{i-1} \partial_{xy} \varphi^\varepsilon|^2 \\
& \quad + \delta \left(s^{28-3i} \lambda^{29-3i} \iint_{(0,T) \times \omega} \chi^{4+2i} e^{-2s\alpha} \xi^{28-3i} |D^i \partial_{xy} \varphi^\varepsilon|^2 \right. \\
& \quad \left. + s^{28-3(i+1)} \lambda^{29-3(i+1)} \iint_{(0,T) \times \omega} \chi^{4+2(i+1)} e^{-2s\alpha} \xi^{28-3(i+1)} |D^{i+1} \partial_{xy} \varphi^\varepsilon|^2 \right). \tag{2.70}
\end{aligned}$$

The exponents of s , ξ and λ in (2.69) might look strange. The reason is that we have $s^4\lambda^5$ in the last local term of (2.68) instead of $s^3\lambda^4$, which is the usual term.

In order to get in the right-hand side of (2.69) only a weighted local L^2 -norm of φ_1^ε we must treat $\partial_{xy}\varphi_1^\varepsilon$ and $\partial_{xy}\varphi_2^\varepsilon$ differently. As for $\partial_{xy}\varphi_1^\varepsilon$, we can deal with it quite easily. Indeed, when we integrate by parts twice, we have that :

$$\begin{aligned} s^{28}\lambda^{29} \iint_{(0,T)\times\omega} \chi^4 e^{-2s\alpha\xi^{28}} |\partial_{xy}\varphi_1^\varepsilon|^2 &= s^{28}\lambda^{29} \iint_{(0,T)\times\omega} \partial_{xy} (\chi^4 e^{-2s\alpha\xi^{28}}) \partial_{xy}\varphi_1^\varepsilon \varphi_1^\varepsilon \\ &+ s^{28}\lambda^{29} \iint_{(0,T)\times\omega} \partial_x (\chi^4 e^{-2s\alpha\xi^{28}}) \partial_{xyy}\varphi_1^\varepsilon \varphi_1^\varepsilon + s^{28}\lambda^{29} \iint_{(0,T)\times\omega} \partial_y (\chi^4 e^{-2s\alpha\xi^{28}}) \partial_{xxy}\varphi_1^\varepsilon \varphi_1^\varepsilon \\ &+ s^{28}\lambda^{29} \iint_{(0,T)\times\omega} \chi^4 e^{-2s\alpha\xi^{28}} \partial_{xxyy}\varphi_1^\varepsilon \varphi_1^\varepsilon. \end{aligned} \quad (2.71)$$

We can deal with all the term of (2.71) as usual. In the end, for $\delta > 0$ as small as needed, after an absorption, we get that :

$$\begin{aligned} s^{28}\lambda^{29} \iint_{(0,T)\times\omega} \chi^4 e^{-2s\alpha\xi^{28}} |\partial_{xy}\varphi_1^\varepsilon|^2 &\leq C_\delta s^{34}\lambda^{35} \iint_{(0,T)\times\omega} e^{-2s\alpha\xi^{34}} |\varphi_1^\varepsilon|^2 \\ &+ \delta \left(s^{22}\lambda^{23} \iint_{(0,T)\times\omega} \chi^8 e^{-2s\alpha\xi^{22}} |\partial_{xxyy}\varphi_1^\varepsilon|^2 + s^{24}\lambda^{25} \iint_{(0,T)\times\omega} \chi^6 e^{-2s\alpha\xi^{24}} |D^1\partial_{xy}\varphi_1^\varepsilon|^2 \right). \end{aligned} \quad (2.72)$$

Finally, we have to estimate the term of $\partial_{xy}\varphi_2^\varepsilon$. Indeed, by (2.6), we find out that :

$$\begin{aligned} s^{28}\lambda^{29} \iint_{(0,T)\times\omega} \chi^4 e^{-2s\alpha\xi^{28}} |\partial_{xy}\varphi_2^\varepsilon|^2 \\ = s^{28}\lambda^{29} \iint_{(0,T)\times\omega} \chi^4 e^{-2s\alpha\xi^{28}} \partial_{xy}\varphi_2^\varepsilon (-\varepsilon\partial_t\varphi_1^\varepsilon - (1+\varepsilon)\partial_{xx}\varphi_1^\varepsilon - \varepsilon\partial_{yy}\varphi_1^\varepsilon). \end{aligned}$$

We can deal with the term in the right-hand side integrating by parts in space and time and using weighted Cauchy-Schwarz inequalities. In order to deal with the term of $\varepsilon\partial_{txy}\varphi_2^\varepsilon$ that appears after the integration by parts, we have to consider that :

$$\varepsilon\partial_{txy}\varphi_2^\varepsilon = -(\varepsilon\partial_{xxy}\varphi_2^\varepsilon + (1+\varepsilon)\partial_{xyy}\varphi_2^\varepsilon + \partial_{xxy}\varphi_1^\varepsilon).$$

Consequently, we get that, after an absorption :

$$s^{28}\lambda^{29} \iint_{(0,T)\times\omega} \chi^4 e^{-2s\alpha\xi^{28}} |\partial_{xy}\varphi_2^\varepsilon|^2 \leq C_\delta s^{34}\lambda^{35} \iint_{(0,T)\times\omega} e^{-2s\alpha\xi^{34}} |\varphi_1^\varepsilon|^2 + \delta \left(s^{22}\lambda^{23} \iint_{(0,T)\times\omega} \chi^8 e^{-2s\alpha\xi^{22}} |D^2\partial_{xy}\varphi^\varepsilon|^2 + s^{24}\lambda^{25} \iint_{(0,T)\times\omega} \chi^6 e^{-2s\alpha\xi^{24}} |D^1\partial_{xy}\varphi^\varepsilon|^2 \right). \quad (2.73)$$

Summing up, if $m \geq 8$, $\lambda \geq \lambda_0$, and $s \geq e^{C\lambda}(T^m + T^{2m})$, combining (2.69), (2.72) and (2.73) we get (2.9).

2.A Existence, uniqueness and regularity of (2.21)

In this section we first prove Lemma 2.21 and then prove Lemma 2.23. Our proofs are classical, since they use Galerkin method and elliptic estimates (see Lemma 2.47 below). We follow the steps of [49, Chapter 7.1], but we do the necessary adaptations due to the different boundary conditions.

Proof of Lemma 2.21 : uniqueness. In order to prove the uniqueness, we just have to show that for $f = 0$, $h = 0$ and $v^0 = 0$, the unique solution is $v^\varepsilon = 0$. Indeed, by multiplying by v^ε the first equation of (2.21) and by integrating in $(0, t) \times \Omega$ (by parts), we have, for all $t \in [0, T]$:

$$\int_{\Omega} \frac{|v^\varepsilon(t, \cdot)|^2}{2} + \iint_{[0,t]\times\Omega} |\nabla v^\varepsilon|^2 + \varepsilon \iint_{[0,t]\times\Omega} |q^\varepsilon|^2 = 0,$$

which implies that $v^\varepsilon = 0$.

Proof of Lemma 2.21 : existence. As for the existence, we consider the Galerkin method. It is well-know that there is a set of eigenvalues $\{\lambda_i\}_{i \in \mathbb{N}} \rightarrow +\infty$ and a set of $\mathbf{L}^2(\Omega)$ -orthonormal and $\mathbf{H}^1(\Omega)$ -orthogonal eigenvectors w_i such that $\{w_i\}_{i \in \mathbb{N}} \subset \mathbf{H}^2(\Omega)$ and that $-\Delta w_i = \lambda_i w_i$. In that sense, for $u \in \mathbf{L}^2(\Omega)$, we denote $\mathbb{P}_n u$ the orthonormal projection of u into $\langle w_1, \dots, w_n \rangle$. We consider the Galerkin sub-problems, for $n \in \mathbb{N}$, $t \in [0, T]$:

$$\int_{\Omega} \partial_t v_n^\varepsilon(t, \cdot) \cdot w_i + \int_{\Omega} \nabla v_n^\varepsilon(t, \cdot) : \nabla w_i + \int_{\Omega} \frac{(\nabla \cdot v_n^\varepsilon(t, \cdot))(\nabla \cdot w_i)}{\varepsilon} = \int_{\Omega} f(t, \cdot) \cdot w_i + \int_{\partial\Omega} h(t, \cdot) \cdot w_i, \quad (2.74)$$

for all $i = 1, \dots, n$. We look for a solution which belongs to $\mathbf{C}^1([0, T]; \langle w_1, \dots, w_n \rangle)$; that is, we look for $a_{i,n}^\varepsilon \in C^1([0, T])$ such that $a_{i,n}^\varepsilon(0) = \langle v^0, w_i \rangle_{\mathbf{L}^2(\Omega)}$ and that $v_n^\varepsilon(t, x) := \sum_{i=1}^n a_{i,n}^\varepsilon(t) w_i(x)$ is a solution of (2.74).

Energy estimates. It is easy to see that each set of components $(a_{i,n}^\varepsilon)_{i=1}^n$ is the solution of a linear ordinary differential equation of n equations and n unknowns. Therefore, system (2.74) together with the initial condition has a well-defined solution. Moreover, adding up (2.74) multiplied by the coefficients $a_{i,n}^\varepsilon$ and integrating in time, we get that, for any $t \in [0, T]$, provided that v_n^ε is defined :

$$\int_{\Omega} \frac{|v_n^\varepsilon|^2(t, \cdot)}{2} + \iint_{[0,t] \times \Omega} |\nabla v_n^\varepsilon|^2 + \iint_{[0,t] \times \Omega} \frac{(\nabla \cdot v_n^\varepsilon)^2}{\varepsilon} = \iint_{[0,t] \times \Omega} f \cdot v_n^\varepsilon + \iint_{[0,t] \times \partial\Omega} h \cdot v_n^\varepsilon + \int_{\Omega} \frac{|\mathbb{P}_n v_0|^2}{2}.$$

We have to consider that :

$$\iint_{[0,t] \times \Omega} f \cdot v_n^\varepsilon \leq C(1+T) \iint_{[0,t] \times \Omega} |f|^2 + \frac{1}{4(1+T)} \iint_{[0,t] \times \Omega} |v_n^\varepsilon|^2.$$

Moreover, we have that :

$$\iint_{[0,t] \times \partial\Omega} h \cdot v_n^\varepsilon \leq C(1+T) \iint_{[0,t] \times \partial\Omega} |h|^2 + \frac{1}{4(1+T)} \iint_{[0,t] \times \Omega} (|v_n^\varepsilon|^2 + |\nabla v_n^\varepsilon|^2).$$

Consequently, due to Gronwall's inequality and usual absorptions, we get that :

$$\begin{aligned} & \int_{\Omega} |v_n^\varepsilon|^2(t, \cdot) + \iint_{[0,t] \times \Omega} |\nabla v_n^\varepsilon|^2 + \iint_{[0,t] \times \Omega} \frac{(\nabla \cdot v_n^\varepsilon)^2}{\varepsilon} \\ & \leq C \left((1+T) \iint_{[0,t] \times \Omega} |f|^2 + (1+T) \iint_{[0,t] \times \partial\Omega} |h|^2 + \int_{\Omega} |\mathbb{P}_n v_0|^2 \right). \end{aligned} \quad (2.75)$$

Thanks to estimate (2.75), all the solutions $(v_n^\varepsilon)_{n \in \mathbb{N}}$ are bounded uniformly in $C^0([0, t]; \mathbf{L}^2(\Omega)) \cap L^2(0, t; \mathbf{H}^1(\Omega))$. This implies, due to extension theorems related with the Cauchy-Lipschitz systems, that all the v_n^ε are defined in $[0, T]$ and that we can take a weak limit in $L^2(0, T; \mathbf{H}^1(\Omega))$.

Estimates on $\mathbf{H}^{1,1}(Q)$. Next, in order to take limits in (2.74), we need to prove that the v_n^ε are also uniformly bounded in $H^1(0, T; \mathbf{L}^2(\Omega))$. By multiplying (2.74) by $(a_{i,n}^\varepsilon)'$, adding all up and integrating in time, we have that :

$$\iint_Q |\partial_t v_n^\varepsilon|^2 + \int_0^T \frac{\partial_t}{2} \int_\Omega |\nabla v_n^\varepsilon|^2 + \int_0^T \frac{\partial_t}{2} \int_\Omega \frac{(\nabla \cdot v_n^\varepsilon)^2}{\varepsilon} = \iint_Q f \cdot \partial_t v_n^\varepsilon + \iint_\Sigma h \cdot \partial_t v_n^\varepsilon.$$

In order to deal with the term $\iint_\Sigma h \cdot \partial_t v_n^\varepsilon$ we have to integrate by parts in time. Then, using also (2.75), we get for a constant C that does not depend on n :

$$\begin{aligned} \|v_n^\varepsilon\|_{\mathbf{H}^{1,1}(Q)} &\leq C\sqrt{1+T}(\|f\|_{\mathbf{L}^2(Q)} + \|h\|_{\mathbf{H}^{1,1/2}(\Sigma)}) + \\ &C \left(\|v^0\|_{\mathbf{H}^1(\Omega)} + \left\| \frac{\nabla \cdot v^0}{\varepsilon} \right\|_{\mathbf{L}^2(\Omega)} + \|h(0, \cdot)\|_{\mathbf{L}^2(\Sigma)} + \|h(T, \cdot)\|_{\mathbf{L}^2(\Sigma)} \right). \end{aligned} \quad (2.76)$$

So, up to extracting a subsequence, we have that $(v_n^\varepsilon)_{n \in \mathbb{N}}$ converges weakly in $\mathbf{H}^{1,1}(Q)$ to some function v^ε which satisfies :

$$\begin{aligned} \|v^\varepsilon\|_{\mathbf{H}^{1,1}(Q)} &\leq C\sqrt{1+T}(\|f\|_{\mathbf{L}^2(Q)} + \|h\|_{\mathbf{H}^{1,1/2}(\Sigma)}) + \\ &C \left(\|v^0\|_{\mathbf{H}^1(\Omega)} + \left\| \frac{\nabla \cdot v^0}{\varepsilon} \right\|_{\mathbf{L}^2(\Omega)} + \|h(0, \cdot)\|_{\mathbf{L}^2(\Sigma)} + \|h(T, \cdot)\|_{\mathbf{L}^2(\Sigma)} \right). \end{aligned} \quad (2.77)$$

Thus, we can take limits in (2.74). Indeed, we have for every $i \in \mathbb{N}$, as functions of $L^2(0, T)$:

$$\int_\Omega v_t^\varepsilon \cdot w_i + \int_\Omega \nabla v^\varepsilon : \nabla w_i + \int_\Omega \frac{(\nabla \cdot v^\varepsilon)(\nabla \cdot w_i)}{\varepsilon} = \int_\Omega f \cdot w_i + \int_{\partial\Omega} h \cdot w_i. \quad (2.78)$$

We recall that $H^1(0, T; L^2(\Omega))$ is compactly embedded in $C^0([0, T]; H^{-1}(\Omega))$. Thus, weak convergence in $H^1(0, T; L^2(\Omega))$ implies strong convergence in $C^0([0, T]; H^{-1}(\Omega))$, so $v^\varepsilon(0, \cdot) = v^0$.

Estimates on $\mathbf{H}^{1,2}(Q)$. In order to prove that the solution is in $L^2(0, T; \mathbf{H}^2(\Omega))$, we use that it satisfies for any $i \in \mathbb{N}$, as functions of $L^2(0, T)$, that :

$$\int_\Omega \nabla v^\varepsilon : \nabla w_i + \int_\Omega \frac{(\nabla \cdot v^\varepsilon)(\nabla \cdot w_i)}{\varepsilon} = \int_\Omega (f - v_t^\varepsilon) \cdot w_i + \int_{\partial\Omega} h \cdot w_i. \quad (2.79)$$

So, we use the following lemma, whose proof can be found in [12, Theorem IV.7.1] :

Lemma 2.47. *Let us consider $\Omega \in C^2$ and the system :*

$$\begin{cases} -\Delta u + \nabla g = f_1 & \text{in } \Omega, \\ \nabla \cdot u = f_2 & \text{in } \Omega, \\ \partial_n u - gn = f_3 & \text{on } \partial\Omega, \\ \int_{\Omega} u = 0, \end{cases} \quad (2.80)$$

for $f_1 \in \mathbf{L}^2(\Omega)$, $f_2 \in L^2(\Omega)$ and $f_3 \in \mathbf{H}^{1/2}(\partial\Omega)$. Then, if we have as a vector equation :

$$\int_{\Omega} f_1 + \int_{\partial\Omega} f_3 = 0, \quad (2.81)$$

the solution (u, g) of (2.80) is unique and

$$\|D^2 u\|_{\mathbf{L}^2(\Omega)} + \|g\|_{H^1(\Omega)} \leq C (\|f_1\|_{\mathbf{L}^2(\Omega)} + \|f_2\|_{H^1(\Omega)} + \|f_3\|_{\mathbf{H}^{1/2}(\partial\Omega)}).$$

In order to apply Lemma 2.47 it suffices to take $f_1(t, \cdot) := f(t, \cdot) - v_t^\varepsilon(t, \cdot)$, $f_2(t, \cdot) := \nabla \cdot v^\varepsilon(t, \cdot)$ and $f_3(t, \cdot) := h(t, \cdot)$. In addition to that, (2.81) is satisfied because :

$$\int_{\Omega} v_t^\varepsilon = \int_{\Omega} (\Delta v^\varepsilon - \nabla q) + \int_{\Omega} f = \int_{\partial\Omega} (\partial_n v^\varepsilon - qn) + \int_{\Omega} f = \int_{\partial\Omega} h + \int_{\Omega} f.$$

Thus, since :

$$u(t, \cdot) = v^\varepsilon(t, \cdot) - \frac{1}{|\Omega|} \int_{\Omega} v^\varepsilon(t, \cdot) \quad \text{and} \quad g(t, \cdot) = \frac{\nabla \cdot v^\varepsilon}{\varepsilon}(t, \cdot) = q^\varepsilon(t, \cdot),$$

if we combine (2.77) and Lemma 2.47, remarking that $D^2 u = D^2 v^\varepsilon$, we get the estimate :

$$\begin{aligned} \|v^\varepsilon\|_{\mathbf{H}^{1,2}(Q)} + \|q^\varepsilon\|_{H^{0,1}(Q)} &\leq C\sqrt{1+T} (\|f\|_{\mathbf{L}^2(Q)} + \|h\|_{\mathbf{H}^{1,1/2}(\Sigma)}) \\ &+ C \left(\|v^0\|_{\mathbf{H}^1(\Omega)} + \|\nabla \cdot v^\varepsilon\|_{H^{0,1}(Q)} + \left\| \frac{\nabla \cdot v^0}{\varepsilon} \right\|_{\mathbf{L}^2(\Omega)} + \|h(0, \cdot)\|_{\mathbf{L}^2(\Sigma)} + \|h(T, \cdot)\|_{\mathbf{L}^2(\Sigma)} \right). \end{aligned} \quad (2.82)$$

This expression can be simplified since for ε small enough we can absorb the term $\|\nabla \cdot v^\varepsilon\|_{H^{0,1}(Q)}$ by $\|q^\varepsilon\|_{H^{0,1}(Q)}$. So estimate (2.22) is established.

Proof of Lemma 2.23. As for the proof of Lemma 2.23, it consists of repeating the Galerkin method for v_t^ε , since v_t^ε is a solution of (2.21) with $(f, h, 0)$ replaced by $(f_t, h_t, 0)$. Indeed, we first get an estimate for each $\partial_t v_n^\varepsilon$ and then pass to the limit. Finally, we use a more complete version of Lemma 2.47, which can be found in [12, Theorem IV.7.1].

2.B Proof of Proposition 2.28

Throughout this proof we consider $\hat{\omega}$ some open subdomain of Ω compactly contained in $\tilde{\omega}$ such that $\overline{\omega_0} \subset \hat{\omega}$ (see Proposition 2.28 for the definition of $\tilde{\omega}$). In order to make the reading of the proof more comfortable we split it in several steps : first, we bound left of (2.30) by a trace and a local term with the help of the rotational ; then, we deal with the trace and local terms as usual.

Step 1 : Bounding by a trace and a local term.

To begin with, we have that $\nabla \times \varphi^\varepsilon$ is a solution of the heat equation, since $\nabla \times (\nabla \pi^\varepsilon) = 0$. So, using Lemma 2.26 for $r = -1$ and $\delta = 1$, we get that if $\lambda \geq \lambda_0$ and $s \geq e^{C\lambda}(T^m + T^{2m})$:

$$\begin{aligned} & s^2 \lambda^3 \iint_Q e^{-2s\alpha} \xi^2 |\nabla \times \varphi^\varepsilon|^2 + \lambda \iint_Q e^{-2s\alpha} |\nabla(\nabla \times \varphi^\varepsilon)|^2 \\ & \leq C \left(\iint_\Sigma e^{-2s\alpha} |\partial_n(\nabla \times \varphi^\varepsilon)|^2 + s^2 \lambda^3 \iint_{(0,T) \times \hat{\omega}} e^{-2s\alpha} \xi^2 |\nabla \times \varphi^\varepsilon|^2 \right). \end{aligned} \quad (2.83)$$

Next, we consider that the divergence satisfies :

$$\nabla(\nabla \cdot \varphi^\varepsilon) = \Delta \varphi^\varepsilon + \nabla \times (\nabla \times \varphi^\varepsilon).$$

This implies that φ^ε satisfies :

$$-\frac{\varepsilon}{1+\varepsilon} \partial_t \varphi^\varepsilon - \Delta \varphi^\varepsilon = \frac{1}{1+\varepsilon} (\nabla \times (\nabla \times \varphi^\varepsilon)).$$

Thus, using again Lemma 2.26 for $\hat{\omega}$ defined as before, $r = 0$ and now $\delta = \frac{\varepsilon}{1+\varepsilon}$, we get that if $\lambda \geq \lambda_0$ and $s \geq e^{C\lambda}(T^m + T^{2m})$:

$$\begin{aligned} & s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |\varphi^\varepsilon|^2 + s \lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla \varphi^\varepsilon|^2 \\ & \leq C \left(s^3 \lambda^4 \iint_{(0,T) \times \hat{\omega}} e^{-2s\alpha} \xi^3 |\varphi^\varepsilon|^2 + \iint_Q e^{-2s\alpha} |\nabla(\nabla \times \varphi^\varepsilon)|^2 + s \lambda \iint_\Sigma e^{-2s\alpha} \xi |\partial_n \varphi^\varepsilon|^2 \right). \end{aligned} \quad (2.84)$$

Next, we remark that the term of $\nabla(\nabla \times \varphi^\varepsilon)$ on the right-hand side of (2.84) can be

absorbed by the left-hand side of (2.83) for $\lambda \geq \lambda_0$. Thus, we have that :

$$\begin{aligned}
& s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |\varphi^\varepsilon|^2 + s \lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla \varphi^\varepsilon|^2 + s^2 \lambda^3 \iint_Q e^{-2s\alpha} \xi^2 |\nabla \times \varphi^\varepsilon|^2 \\
& \quad + \lambda \iint_Q e^{-2s\alpha} |\nabla(\nabla \times \varphi^\varepsilon)|^2 \leq C \left(s^3 \lambda^4 \iint_{(0,T) \times \hat{\omega}} e^{-2s\alpha} \xi^3 |\varphi^\varepsilon|^2 \right. \\
& \quad \left. + s^2 \lambda^3 \iint_{(0,T) \times \hat{\omega}} e^{-2s\alpha} \xi^2 |\nabla \times \varphi^\varepsilon|^2 + s \lambda \iint_\Sigma e^{-2s\alpha} \xi |\partial_n \varphi^\varepsilon|^2 + \iint_\Sigma e^{-2s\alpha} |\partial_n(\nabla \times \varphi^\varepsilon)|^2 \right).
\end{aligned} \tag{2.85}$$

Step 2 : Absorption of the trace.

In this step we absorb the traces with the estimates established in Lemma 2.21. We recall that on $\partial\Omega$: $\alpha = \alpha^*$ and $\xi = \xi^*$.

Let us first bound the third integral on the right-hand side of (2.85). First, we consider that, integrating by parts :

$$s \lambda \iint_\Sigma e^{-2s\alpha^*} \xi^* |\partial_n \varphi^\varepsilon|^2 \leq C \|(s\xi^*)^{5/4-1/m} \lambda^2 e^{-s\alpha^*} \varphi^\varepsilon\|_{\mathbf{L}^2(Q)}^{1/2} \|(s\xi^*)^{1/4+1/m} e^{-s\alpha^*} \varphi^\varepsilon\|_{\mathbf{H}^{0,2}(Q)}^{3/2}.$$

Using Young's inequality we get that :

$$s \lambda \iint_\Sigma e^{-2s\alpha^*} \xi^* |\partial_n \varphi^\varepsilon|^2 \leq C \left(\|(s\xi^*)^{5/4-1/m} \lambda^2 e^{-s\alpha^*} \varphi^\varepsilon\|_{\mathbf{L}^2(Q)}^2 + \|(s\xi^*)^{1/4+1/m} e^{-s\alpha^*} \varphi^\varepsilon\|_{\mathbf{H}^{0,2}(Q)}^2 \right). \tag{2.86}$$

We can absorb the first term on the right-hand side of (2.86) by the left-hand side of (2.85) by taking $s \geq CT^{2m}$ and $\lambda \geq 1$.

We can bound the fourth integral at the right-hand side of (2.85) similarly. Indeed, integrating by parts, we get that, if $s \geq CT^{2m}$:

$$\begin{aligned}
\iint_\Sigma e^{-2s\alpha^*} |\partial_n(\nabla \times \varphi^\varepsilon)|^2 & \leq C \|(s\xi^*)^{1/4} e^{-s\alpha^*} \varphi^\varepsilon\|_{\mathbf{H}^{0,2}(Q)}^{3/2} \|(s\xi^*)^{-3/4} e^{-s\alpha^*} \varphi^\varepsilon\|_{\mathbf{H}^{0,4}(Q)}^{1/2} \\
& \leq C \left(\|(s\xi^*)^{1/4} e^{-s\alpha^*} \varphi^\varepsilon\|_{\mathbf{H}^{0,2}(Q)}^2 + \|(s\xi^*)^{-3/4} e^{-s\alpha^*} \varphi^\varepsilon\|_{\mathbf{H}^{0,4}(Q)}^2 \right).
\end{aligned} \tag{2.87}$$

So, we first deal with the term $\|\eta\varphi^\varepsilon\|_{\mathbf{H}^{1,2}(Q)}^2$ (see (2.28) for the definition of η). We remark that

$$(t, x) \mapsto \eta(T-t)\varphi^\varepsilon(T-t, x)$$

is a solution of (2.21) with null initial value, force $-\eta'(T-t)\varphi^\varepsilon(T-t, x)$ and boundary Neumann term $\eta(T-t)h(T-t, x)$. Consequently, because of (2.22), we get that :

$$\|\eta\varphi^\varepsilon\|_{\mathbf{H}^{1,2}(Q)}^2 \leq C(1+T) \left(\|\eta'\varphi^\varepsilon\|_{\mathbf{L}^2(Q)}^2 + \|\eta h\|_{\mathbf{H}^{1,1/2}(\Sigma)}^2 \right). \quad (2.88)$$

Moreover, the term of $\eta'\varphi^\varepsilon$ can be absorbed by the left of (2.85) if $m \geq 8$, $\lambda \geq \lambda_0$, and $s \geq e^{C\lambda}(T^m + T^{2m})$, since in that case :

$$(1+T^{1/2})|\eta'| \leq C(s\xi^*)^{1+1/4+2/m}e^{-s\alpha^*} \leq C(s\xi^*)^{3/2}e^{-s\alpha^*}. \quad (2.89)$$

Let us now estimate the term $\|\tilde{\eta}\varphi^\varepsilon\|_{\mathbf{H}^{0,4}(Q)}^2$ (see (2.29) for the definition of $\tilde{\eta}$). We have that

$$(t, x) \mapsto \tilde{\eta}(T-t)\varphi^\varepsilon(T-t, x)$$

is a solution of (2.21) with null initial value, force $-\tilde{\eta}'(T-t)\varphi^\varepsilon(T-t, x)$ and boundary Neumann term $\tilde{\eta}(T-t)h(T-t, x)$. Consequently, if we use (2.23), we get that :

$$\|\tilde{\eta}\varphi^\varepsilon\|_{\mathbf{H}^{0,4}(Q)}^2 \leq C(1+T) \left(\|\tilde{\eta}'\varphi^\varepsilon\|_{\mathbf{H}^{1,2}(Q)}^2 + \|\tilde{\eta}h\|_{\mathbf{H}^{2,5/2}(\Sigma)}^2 \right). \quad (2.90)$$

Let us now estimate the first norm at the right-hand side of (2.90). To begin with, since $m \geq 8$ and $s \geq e^{C\lambda}(T^m + T^{2m})$ imply that $(1+T^{1/2})|\tilde{\eta}'| \leq C\eta$, we have that :

$$(1+T)\|\tilde{\eta}'\varphi^\varepsilon\|_{\mathbf{H}^{0,2}(Q)}^2 \leq C\|\eta\varphi^\varepsilon\|_{\mathbf{H}^{0,2}(Q)}^2, \quad (2.91)$$

which is estimated by the left-hand side of (2.88). To continue with, we have that, if $m \geq 8$ and $s \geq e^{C\lambda}(T^m + T^{2m})$:

$$(1+T)\|\tilde{\eta}''\varphi^\varepsilon\|_{\mathbf{L}^2(Q)}^2 \leq C\|(s\xi^*)^{1+1/4+2/m}e^{-s\alpha^*}\varphi^\varepsilon\|_{\mathbf{L}^2(Q)}^2 \leq C\|(s\xi^*)^{3/2}e^{-s\alpha^*}\varphi^\varepsilon\|_{\mathbf{L}^2(Q)}^2, \quad (2.92)$$

a term which can be absorbed by the left-hand side of (2.85) for λ large enough. Finally, we have that, if $m \geq 8$ and $s \geq e^{C\lambda}(T^m + T^{2m})$:

$$(1+T^{1/2})|\tilde{\eta}'\varphi_t| \leq |\eta\varphi_t^\varepsilon| \leq |(\eta\varphi^\varepsilon)_t| + |\eta'\varphi^\varepsilon|,$$

which implies that :

$$(1+T)\|\tilde{\eta}'\varphi_t^\varepsilon\|_{\mathbf{L}^2(Q)}^2 \leq C(\|\eta\varphi^\varepsilon\|_{\mathbf{H}^{1,0}(Q)}^2 + \|\eta'\varphi^\varepsilon\|_{\mathbf{L}^2(Q)}^2), \quad (2.93)$$

terms which can be estimated by the left-hand side of (2.88) and (2.85) respectively.

Summing up, if we combine (2.85)-(2.93) we get that, if $m \geq 8$, $\lambda \geq \lambda_0$ and $s \geq e^{C\lambda}(T^m + T^{2m})$:

$$\begin{aligned} & s^3 \lambda^4 \iint_Q e^{-2s\alpha\xi^3} |\varphi^\varepsilon|^2 + s\lambda^2 \iint_Q e^{-2s\alpha\xi} |\nabla \varphi^\varepsilon|^2 + s^2 \lambda^3 \iint_Q e^{-2s\alpha\xi^2} |\nabla \times \varphi^\varepsilon|^2 \\ & + \lambda \iint_Q e^{-2s\alpha} |\nabla(\nabla \times \varphi^\varepsilon)|^2 \leq C \left(s^3 \lambda^4 \iint_{(0,T) \times \hat{\omega}} e^{-2s\alpha\xi^3} |\varphi^\varepsilon|^2 \right. \\ & \left. + s^2 \lambda^3 \iint_{(0,T) \times \hat{\omega}} e^{-2s\alpha\xi^2} |\nabla \times \varphi^\varepsilon|^2 + (1+T) \left(\|\eta h\|_{\mathbf{H}^{1,1/2}(\Sigma)}^2 + \|\tilde{\eta} h\|_{\mathbf{H}^{2,5/2}(\Sigma)}^2 \right) \right). \end{aligned}$$

Finally, we remove the derivative from the local terms. We do it with the usual localizing techniques : we multiply by a cut-off function χ , integrate by parts and use Cauchy-Schwarz weighted inequalities. So, if $\lambda \geq \lambda_0$, $s \geq e^{C\lambda}(T^m + T^{2m})$ and $m \geq 8$, we get estimate (2.30).

Chapitre 3

Uniform controllability of a Stokes problem with a transport term in the zero-diffusion limit

In this chapter we consider a Stokes system with no-slip boundary conditions. The main results concern the behaviour of the cost of null controllability with respect to the diffusion coefficient when the control acts in the interior. In particular, we prove in $(0, \pi)^2$ that for a sufficiently large time the cost decays exponentially as the diffusion coefficient vanishes, whereas in $(0, \pi)^3$ we prove that for most of the control domains and for any time $T > 0$ the cost explodes exponentially as the diffusion coefficient vanishes. This chapter is included in [14].

3.1 Introduction

3.1.1 Main results

Throughout the paper we are interested in understanding the following control problem :

$$\begin{cases} y_t - \varepsilon \Delta y + \partial_{x_d} y + \nabla q = f 1_\omega & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y \cdot n = 0, \quad (Dy \cdot n)_{\text{tg}} = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y^0 & \text{on } \Omega. \end{cases} \quad (3.1)$$

As usual, here and throughout the paper, $T > 0$, $\Omega \subset \mathbb{R}^d$ and $\omega \subset \Omega$ are domains, $Q := (0, T) \times \Omega$, $Q_\omega := (0, T) \times \omega$, $\Sigma := (0, T) \times \partial\Omega$, $Du := \frac{1}{2} (\partial_{x_i} u_j + \partial_{x_j} u_i)_{i,j}$ for all $u \in \mathbf{L}^2(\Omega)$ (in the distributional sense), n denotes the normal vector pointing outwards $\partial\Omega$, “ ∂_n ” denotes the normal outward partial derivative on $\partial\Omega$, and $v_{\text{tg}} := v - (v \cdot n)n$. The bold notation is used to denote vectorial spaces. As for initial data, y^0 , we take it in $\mathcal{H}(\Omega)$, which is defined by :

$$\mathcal{H}(\Omega) := \{u \in \mathbf{L}^2(\Omega) : \nabla \cdot u = 0 \text{ in } \Omega \text{ and } u \cdot n = 0 \text{ on } \partial\Omega\}.$$

We recall that we can define a normal trace which belongs to $H^{-1/2}(\partial\Omega)$ in the (closed) subspace of $\mathbf{L}^2(\Omega)$ of functions that have a divergence in $L^2(\Omega)$. As for the control force, f , we want it at least in $\mathbf{L}^2(Q_\omega)$ and, if possible, of null first component. Finally, throughout the document $\Omega_d := (0, \pi)^d$.

Let us denote the sets of admissible controls as :

$$\begin{cases} S_1(y^0) := \{f \in \mathbf{L}^2(Q_\omega) : \Phi_2(y^0, f)(T, \cdot) = 0\}, \\ S_2(y^0) := \{f_2 \in L^2(Q_\omega) : \Phi_2(y^0, (0, f_2))(T, \cdot) = 0\}, \\ S_3(y^0) := \{f \in \mathbf{L}^2(Q_\omega) : \Phi_3(y^0, f)(T, \cdot) = 0\}, \end{cases}$$

for $y^0 \in \mathcal{H}(\Omega_d)$, for $f \in \mathbf{L}^2(Q_\omega)$ and for $\Phi_d(y^0, f)$ the solution of (3.1) in Ω_d . Thanks to [67], we know that S_1 and S_3 are non-empty, whereas we obtain in Section 3.3.1 that S_2 is non-empty by proving the corresponding observability estimate. As for the cost of the

control, we measure it with the usual norms. In fact, we denote :

$$\begin{cases} K_1(T, \varepsilon, \omega) := \sup_{y^0 \in \mathcal{H}(\Omega_2) \setminus \{0\}} \inf_{f \in S_1(y^0)} \frac{\|f\|_{\mathbf{L}^2(Q_\omega)}}{\|y^0\|_{\mathbf{L}^2(\Omega_2)}}, \\ K_2(T, \varepsilon, \omega) := \sup_{y^0 \in \mathcal{H}(\Omega_2) \setminus \{0\}} \inf_{f_2 \in S_2(y^0)} \frac{\|f_2\|_{L^2(Q_\omega)}}{\|y^0\|_{\mathbf{L}^2(\Omega_2)}}, \\ K_3(T, \varepsilon, \omega) := \sup_{y^0 \in \mathcal{H}(\Omega_3) \setminus \{0\}} \inf_{f \in S_3(y^0)} \frac{\|f\|_{\mathbf{L}^2(Q_\omega)}}{\|y^0\|_{\mathbf{L}^2(\Omega_3)}}. \end{cases} \quad (3.2)$$

The main results that we prove in the paper are the following ones :

Theorem 3.1. *We have the following results for the control system (3.1) :*

1. *Let $\omega \subset \Omega_2$ a domain. Then, there are some $c, C, T_0 > 0$ such that, if $T > T_0$ and $\varepsilon \in (0, 1)$:*

$$K_2(T, \varepsilon, \omega) \leq Ce^{-c\varepsilon^{-1}}.$$

2. *Let $h \in (0, \pi)$ and $\omega \subset (0, \pi) \times (\pi - h, \pi)$ a domain. Then, for any $T \in (0, 2(\pi - h))$, there is $c > 0$ such that, if $\varepsilon \in (0, 1)$:*

$$K_1(T, \varepsilon, \omega) \geq ce^{c\varepsilon^{-1}}.$$

3. *Let $h \in (0, \pi)$ and $\omega \subset (0, \pi) \times (0, h)$ a domain. Then, for any $T \in (0, \pi - h)$, there is $c > 0$ such that, if $\varepsilon \in (0, 1)$:*

$$K_1(T, \varepsilon, \omega) \geq ce^{c\varepsilon^{-1}}.$$

4. *Let $h \in (0, \pi)$ and $\omega \subset (0, \pi)^2 \times (\pi - h, \pi)$ a domain. Then, for any $T > 0$ there is $c > 0$ such that, if $\varepsilon \in (0, 1)$:*

$$K_3(T, \varepsilon, \omega) \geq ce^{c\varepsilon^{-1}}.$$

Remark 3.2. The results in dimension 2 are not surprising as they are similar to those in [37, 71, 60, 87, 88]. The result in dimension 3 (see Item 4), however, is surprising because it holds for all $T > 0$ and for a wide range of control domains (in particular, for all the control domains compactly supported in Ω_3). This result is explained because we have much more freedom to construct eigenfunctions of the adjoint operator (see (3.5) and Remark 3.21 in Section 3.4 below). Indeed, we can construct in dimension 3 a continuous family of eigenfunctions whose respective eigenvalues do not explode when $\varepsilon \rightarrow 0^+$. The existence of a system with this property is a novelty with respect to the existing literature.

Remark 3.3. The analogous results to the ones stated in Theorem 3.1 and in the lemmas and propositions throughout the paper are true for any rectangle or cuboids and with any velocity parallel to their edges. Indeed, all the operations can be replicated under those assumptions. The reason of restricting to $(0, \pi)^2$ or $(0, \pi)^3$ is to reduce the number of operations so that the reader can focus on the main ideas of the proof.

In order to estimate the cost of the control, we study the adjoint system :

$$\begin{cases} -\varphi_t - \varepsilon \Delta \varphi - \partial_{x_d} \varphi + \nabla p = 0 & \text{in } Q, \\ \nabla \cdot \varphi = 0 & \text{in } Q, \\ \varphi \cdot n = 0, \quad (2\varepsilon D\varphi \cdot n + \varphi n_d)_{\text{tg}} = 0 & \text{on } \Sigma, \\ \varphi(T, \cdot) = \varphi^T & \text{on } \Omega, \end{cases} \quad (3.3)$$

for d the dimension of Ω . We recall the following classical result (see, for instance, [97, 82]) :

$$\begin{cases} [K_1(T, \varepsilon, \omega)]^2 = \sup_{\varphi^T \in \mathcal{H}(\Omega_2) \setminus \{0\}} \frac{\int_{\Omega_2} |\varphi(0, x)|^2 dx}{\iint_{Q_\omega} |\varphi|^2 dx dt}, \\ [K_2(T, \varepsilon, \omega)]^2 = \sup_{\varphi^T \in \mathcal{H}(\Omega_2) \setminus \{0\}} \frac{\int_{\Omega_2} |\varphi(0, x)|^2 dx}{\iint_{Q_\omega} |\varphi_2|^2 dx dt}, \\ [K_3(T, \varepsilon, \omega)]^2 = \sup_{\varphi^T \in \mathcal{H}(\Omega_3) \setminus \{0\}} \frac{\int_{\Omega_3} |\varphi(0, x)|^2 dx}{\iint_{Q_\omega} |\varphi|^2 dx dt}. \end{cases} \quad (3.4)$$

While working in Ω_2 we use the following notation :

$$\Sigma_{lr} := (0, T) \times \{0, \pi\} \times (0, \pi), \quad \text{and} \quad \Sigma_{bt} := (0, T) \times (0, \pi) \times \{0, \pi\}.$$

In order to understand the (energy) solutions of (3.3), we do a spectral decomposition. In particular, we look for the solutions of :

$$\begin{cases} -\varepsilon \Delta u - \partial_{x_d} u + \nabla p = \lambda u & \text{in } \Omega, \\ \nabla \cdot u = 0 & \text{in } \Omega, \\ u \cdot n = 0, \quad (2\varepsilon Du \cdot n + un_d)_{\text{tg}} = 0 & \text{on } \partial\Omega, \end{cases} \quad (3.5)$$

for d the dimension of Ω .

Remark 3.4. One interesting property of system (3.3) considered in Ω_2 is that, as long as $\varphi^T \in \mathcal{H}(\Omega_2)$, the gradient of the pressure of the solution of (3.3) is null for any given time. Indeed, by taking the divergence in (3.3)₁ we obtain that $-\Delta p = 0$, so the result is

proved if we show that $\partial_n p = 0$ on Σ . For that purpose, we multiply (3.3)₁ by n . First, $\varphi_t \cdot n = 0$ as $\varphi \cdot n = 0$. Second, we have that :

$$-(\varepsilon \Delta \varphi + \partial_{x_2} \varphi) \cdot n 1_{\Sigma_{lr}} = -(\varepsilon \partial_{x_1}^2 \varphi_1 + \varepsilon \partial_{x_2}^2 \varphi_1 + \partial_{x_2} \varphi_1) n_1 1_{\Sigma_{lr}} = \varepsilon \partial_{x_2} \partial_{x_1} \varphi_2 n_1 1_{\Sigma_{lr}} = 0.$$

We have used on the previous equality that $n_2 = 0$ on Σ_{lr} , (3.3)₂ and that $\varphi_1 = \partial_{x_1} \varphi_2 = 0$ on Σ_{lr} because of (3.3)₃. And finally, we have that :

$$-(\varepsilon \Delta \varphi + \partial_{x_2} \varphi) \cdot n 1_{\Sigma_{bt}} = -(\varepsilon \partial_{x_1}^2 \varphi_2 + \varepsilon \partial_{x_2}^2 \varphi_2 + \partial_{x_2} \varphi_2) n_2 1_{\Sigma_{bt}} = (\varepsilon \partial_{x_1} \partial_{x_2} \varphi_1 + \partial_{x_1} \varphi_1) n_2 1_{\Sigma_{bt}} = 0.$$

We have used that $n_1 = 0$ on Σ_{bt} , (3.3)₂ and that $\varphi_2 = \varepsilon \partial_{x_2} \varphi_1 + \varphi_1 = 0$ on Σ_{bt} because of (3.3)₃.

Remark 3.5. An immediate consequence of Remark 3.4 is that if $\varphi^T \in \mathcal{H}(\Omega_2)$, φ_1 (the first component of the solution φ of (3.3) in Ω_2) satisfies :

$$\begin{cases} -z_t - \varepsilon \Delta z - \partial_{x_2} z = 0 & \text{in } Q, \\ z = 0 & \text{on } \Sigma_{lr}, \\ \varepsilon \partial_{x_2} z + z = 0 & \text{on } \Sigma_{bt}, \\ z(T, \cdot) = z^T & \text{on } \Omega_2, \end{cases} \quad (3.6)$$

for $z^T = \varphi_1^T$. In addition, φ_2 satisfies :

$$\begin{cases} -z_t - \varepsilon \Delta z - \partial_{x_2} z = 0 & \text{in } Q, \\ \partial_{x_1} z = \partial_n z = 0 & \text{on } \Sigma_{lr}, \\ z = 0 & \text{on } \Sigma_{bt}, \\ z(T, \cdot) = z^T & \text{on } \Omega_2, \end{cases} \quad (3.7)$$

for $z^T = \varphi_2^T$.

Throughout the paper we use c and C to denote strictly positive constants, which might be different each time. We denote the small constants by c and the large ones by C .

3.1.2 Historical background

The first control system with a small diffusion and a transport term that was analysed was the heat equation in dimension 1 with Dirichlet boundary conditions in [37].

Afterwards, the same problem but in any dimension and with any speed belonging to $W^{1,+\infty}(\mathbb{R}^+ \times \Omega)$ was studied in [71]. More recently, better approximations of the optimal time in which the cost of the control decays have been given : the upper bound was improved in [60, 87] (in the first one through complex analysis and in the second one by transforming the original equation into the pure heat equation), and the lower bound was improved in [88, 76] through complex analysis and properties of the entire functions in the first one and spectral analysis on the second. As for similar results, work has been done in the the Burgers equation (see [61]), in the KdV equation (see [62, 63, 24, 25]) and in an artificial advection-diffusion problem (see [33, 34]). As for the Stokes system with small diffusion and a transport term, this is the first time that such a system has been studied and is, indeed, one of the contributions of the paper.

The study of control problems associated to Stokes systems with Navier-slip boundary condition (see (3.1)) is not new in the literature : the existence of d dimensional controls leading to null controllability was proved in [67], while the existence of $d - 1$ dimensional controls has recently been proved in [72]. The need of working in a specific domain to simplify the problem is not new either. For example, recently, in [40], they prove that the Navier-Stokes is (globally) null controllable in any rectangle with boundary controls at two opposing edges and with a phantom force (see also [70] for a weaker result in cuboids).

Finally, the idea of using spectral decomposition is not new in Control Theory (see, for instance, [36]). Indeed, for the heat and the Stokes context alone, there are many documents which deal with eigenfunctions of the elliptic operator, for proving the existence of some control (see, for example, [79, 85, 32]), for estimating the cost of the control (see, for instance, [46] and, very recently, [77]) , and for giving negative answer to the existence of a control (see, for example, [85]). As for a system with small diffusion and a transport term, a spectral decomposition indirectly appears in [37, 88], when getting lower bounds for the optimal time T_0 in which the cost of the control decays exponentially with ε . However, as far as the author knows, this is the first time that a spectral decomposition has been used to get the dissipation estimate in a transport-diffusion system.

The rest of the paper is organized as follows : in Section 3.2 we study systems (3.3) and (3.5) for Ω_2 , in Section 3.3 we study the cost of the control for Ω_2 , in Section 3.4 we study the control problem (3.1) and its adjoint system for Ω_3 , in Section 3.5 we do some further

comments about the techniques and point out some open problems, and in Appendix 3.A we prove a technical result.

3.2 Spectral decomposition of $\mathcal{H}((0, \pi)^2)$ and some immediate consequences

In this section we look for solutions of (3.5) in Ω_2 and use them to extract information about the solutions of (3.3) in Ω_2 . In particular, in Section 3.2.1 we provide the proof of a technical result, in Section 3.2.2 we get some eigenfunctions (solutions of (3.5)) which form a total set in $\mathcal{H}(\Omega_2)$, and in Section 3.2.3 we use those eigenfunctions to extract information about the solutions of (3.3). We avoid using Remark 3.4 and Remark 3.5 to have a proof that can be generalized to other Stokes systems and because it does not spare us many calculations.

As for the notation, given any real Banach space $(V, \|\cdot\|)$ and $S \subset V$, we recall that $\text{span}(S)$ denotes the set of all linear combinations of the elements of S . Moreover, we recall that S is a *total set* if $\overline{\text{span}(S)} = V$. Also,

$$l^2 := \left\{ (a_k)_{k \in \mathbb{N}_*} : a_k \in \mathbb{R} \text{ and } \sum_{k \in \mathbb{N}_*} a_k^2 < +\infty \right\}.$$

Besides, for any bounded domain $\Theta \subset \mathbb{R}^d$, we denote :

$$L_0^2(\Theta) := \left\{ g \in L^2(\Theta) : \int_{\Theta} g(x) dx = 0 \right\}.$$

Moreover, we use $m = (m_1, m_2) \in (\mathbb{N}_*)^2$ to index the eigenfunctions. Finally, we denote the set of functions with separated variables having null normal trace and null divergence as :

$$SV(\Omega_2) := \left\{ \left(\int_0^{x_1} g_1(s) ds g_2(x_2), -g_1(x_1) \int_0^{x_2} g_2(s) ds \right) : g_1, g_2 \in L_0^2(0, \pi) \right\}. \quad (3.8)$$

3.2.1 Some functional analysis results

In order to look for eigenfunctions, we look among those which have a simple structure ; in particular, among those which belong to $SV(\Omega_2)$. Indeed, by construction, $SV(\Omega_2) \subset \mathcal{H}(\Omega_2)$. Moreover, the interest of $SV(\Omega_2)$ is that it is a total set :

Proposition 3.6. *We have :*

$$\overline{\text{span}(SV(\Omega_2))} = \mathcal{H}(\Omega_2).$$

Remark 3.7. This result is not surprising since it is well-known that functions with separated variables form a total set in $L^2(\Omega_2)$.

In order to prove Proposition 3.6, we first need some technical results whose proofs, up to the author's knowledge, do not appear explicitly in any book or paper :

Lemma 3.8. *Let $A_1, A_2, B_1, B_2 \in \mathbb{R}$. Let $\Theta := (A_1, A_2) \times (B_1, B_2)$, $p \in (1, +\infty)$ and $h \in L^p(\Theta)$ a function such that :*

$$\int_{\Theta} h(x_1, x_2)g_1(x_1)g_2(x_2)dx = 0, \quad (3.9)$$

for all $g_1 \in L_0^\infty(A_1, A_2)$ and for all $g_2 \in L_0^\infty(B_1, B_2)$. Then, there are $h_1 \in L^p(A_1, A_2)$ and $h_2 \in L^p(B_1, B_2)$ such that :

$$h(x_1, x_2) = h_1(x_1) + h_2(x_2) \text{ almost everywhere in } \Theta,$$

and such that :

$$\int_{A_1}^{A_2} h_1(s)ds = 0. \quad (3.10)$$

For the proof, we follow the classical scheme of first proving a regular version of the lemma, and then generalizing it to weaker spaces thanks to mollifiers.

Lemma 3.9. *Let $A_1, A_2, B_1, B_2 \in \mathbb{R}$. Let $\Theta := (A_1, A_2) \times (B_1, B_2)$, $N \geq 0$ and $h \in C^N(\overline{\Theta})$ a function such that :*

$$\int_{\Theta} h(x_1, x_2)g_1(x_1)g_2(x_2)dx = 0, \quad (3.11)$$

for all $g_1 \in L_0^\infty(A_1, A_2)$ and for all $g_2 \in L_0^\infty(B_1, B_2)$. Then, there are $h_1 \in C^N([A_1, A_2])$ and $h_2 \in C^N([B_1, B_2])$ such that :

$$h(x_1, x_2) = h_1(x_1) + h_2(x_2) \text{ in } \overline{\Theta},$$

and such that :

$$\int_{A_1}^{A_2} h_1(s)ds = 0. \quad (3.12)$$

We remark that (3.10) and (3.12) are required to have some continuity in the decomposition (h_1, h_2) . Moreover, they also provide uniqueness.

Proof. [Proof of Lemma 3.9] Since the Lebesgue measure works well with translations and dilations, it suffices to prove the result for $\Theta = (0, 1)^2$. In particular, it suffices to prove that for all $x_1, x_2, \tilde{x}_1, \tilde{x}_2 \in (0, 1)$ such that $x_1 \neq \tilde{x}_1$ and $x_2 \neq \tilde{x}_2$:

$$h(x_1, x_2) + h(\tilde{x}_1, \tilde{x}_2) = h(x_1, \tilde{x}_2) + h(\tilde{x}_1, x_2). \quad (3.13)$$

Indeed, by continuity, (3.13) extends to all $x_1, x_2, \tilde{x}_1, \tilde{x}_2 \in [0, 1]$. Consequently, if we consider $(\tilde{x}_1, \tilde{x}_2) = (0, 0)$, we get that :

$$h(x_1, x_2) = h(x_1, 0) + h(0, x_2) - h(0, 0).$$

Let us suppose that there exist $a_1, a_2, b_1, b_2 \in (0, 1)$ such that $a_1 \neq a_2$ and $b_1 \neq b_2$ and such that (3.13) is not true. By symmetry, we can suppose that :

$$h(a_1, b_1) + h(a_2, b_2) > h(a_1, b_2) + h(a_2, b_1).$$

Then, for some $\delta > 0$ small enough (in particular smaller than $a_1, a_2, b_1, b_2, 1 - a_1, 1 - a_2, 1 - b_1, 1 - b_2$), by the continuity of h , we have for all $s = (s_1, s_2) \in [-\delta, \delta]^2$ the inequality :

$$h(a_1 + s_1, b_1 + s_2) + h(a_2 + s_1, b_2 + s_2) > h(a_1 + s_1, b_2 + s_2) + h(a_2 + s_1, b_1 + s_2).$$

Let us define :

$$g_1 := \mathbf{1}_{(a_1 - \delta, a_1 + \delta)} - \mathbf{1}_{(a_2 - \delta, a_2 + \delta)},$$

and :

$$g_2 := \mathbf{1}_{(b_1 - \delta, b_1 + \delta)} - \mathbf{1}_{(b_2 - \delta, b_2 + \delta)}.$$

Then, we have that $g_1, g_2 \in L_0^\infty(0, 1)$, but :

$$\begin{aligned} \int_{\Theta} h(x_1, x_2) g_1(x_1) g_2(x_2) dx &= \int_{[-\delta, \delta]^2} \left(h(a_1 + s_1, b_1 + s_2) + h(a_2 + s_1, b_2 + s_2) \right. \\ &\quad \left. - h(a_1 + s_1, b_2 + s_2) - h(a_2 + s_1, b_1 + s_2) \right) ds_1 ds_2 > 0, \end{aligned}$$

which contradicts (3.11). □

For the following proof we use the notation $I_k := (\frac{1}{k}, 1 - \frac{1}{k})$ and $I_k^2 := I_k \times I_k$, for k sufficiently large.

Proof. [Proof of Lemma 3.8] Again, since the Lebesgue measure works well with translations and dilations, it suffices to prove the result for $\Theta = (0, 1)^2$. In particular, in order to prove the existence of such h_1, h_2 , it suffices to prove that, for all $k \in \mathbb{N}_*$, there is $h_{1,k} \in L_0^p(I_k)$ and $h_{2,k} \in L^p(I_k)$ with $\|h_{1,k}\|_{L^p(I_k)}$ and $\|h_{2,k}\|_{L^p(I_k)}$ bounded uniformly, such that :

$$h(x_1, x_2) = h_{1,k}(x_1) + h_{2,k}(x_2) \text{ almost everywhere in } I_k^2. \quad (3.14)$$

Indeed, we just have to take weak limits (up to extracting a subsequence) at both sides of :

$$h(x_1, x_2)1_{I_k^2}(x_1, x_2) = h_{1,k}(x_1)1_{I_k}(x_1) + h_{2,k}(x_2)1_{I_k}(x_2),$$

and use that the unit ball of $L^p(0, 1)$ is weakly compact.

In order to prove the existence of $h_{1,k}$ and $h_{2,k}$, let us consider ξ an even mollifier (a positive $C^\infty(\mathbb{R})$ function such that $\|\xi\|_{L^1(\mathbb{R})} = 1$) supported in $[-1, 1]$. We denote $\xi_\varepsilon(s) := \varepsilon^{-1}\xi(\varepsilon^{-1}s)$. Let us now consider $g_1, g_2 \in L_0^\infty(I_k)$ (and null outside I_k). We have, for all $\varepsilon < k^{-1}$, by Fubini : $g_1 \star \xi_\varepsilon, g_2 \star \xi_\varepsilon \in L_0^\infty(0, 1)$. Thus, by (3.9), Fubini, the even symmetry of ξ , and the support of g_1 and g_2 , we have that :

$$\begin{aligned} 0 &= \int_{\Theta} h(x_1, x_2) \left(\int_0^1 g_1(s_1)\xi_\varepsilon(x_1 - s_1)ds_1 \right) \left(\int_0^1 g_2(s_2)\xi_\varepsilon(x_2 - s_2)ds_2 \right) dx \\ &= \int_{\Theta} g_1(s_1)g_2(s_2) \left(\int_{\Theta} h(x_1, x_2)\xi_\varepsilon(s_1 - x_1)\xi_\varepsilon(s_2 - x_2)dx \right) ds_1 ds_2 \\ &= \int_{I_k^2} g_1(s_1)g_2(s_2)(h \star (\xi_\varepsilon \otimes \xi_\varepsilon))(s_1, s_2)ds_1 ds_2. \end{aligned} \quad (3.15)$$

Let us denote $h_\varepsilon := h \star (\xi_\varepsilon \otimes \xi_\varepsilon)$. Since ξ is a mollifier, $h_\varepsilon \in C^\infty(\overline{I_k^2})$ and :

$$h_\varepsilon \rightarrow h \text{ in } L^p(I_k^2). \quad (3.16)$$

Moreover, since identity (3.15) is satisfied for any g_1, g_2 in $L_0^\infty(I_k)$, we have that, because of Lemma 3.9, the existence of $h_{1,\varepsilon}$ and $h_{2,\varepsilon}$ in $C^\infty(\overline{I_k})$ such that :

$$h_\varepsilon(x_1, x_2) = h_{1,\varepsilon}(x_1) + h_{2,\varepsilon}(x_2) \text{ in } I_k^2,$$

and :

$$\int_{I_k} h_{1,\varepsilon}(s)ds = 0. \quad (3.17)$$

Because of (3.16) and the justifications given in the previous paragraph it suffices to prove that $\|h_{1,\varepsilon}\|_{L^p(I_k)}$ and $\|h_{2,\varepsilon}\|_{L^p(I_k)}$ are uniformly bounded. In the case of $h_{2,\varepsilon}$, using (3.17), we get that :

$$\left(\frac{k-2}{k}\right) h_{2,\varepsilon}(x_2) = \int_{I_k} h_\varepsilon(x_1, x_2) dx_1 \text{ in } I_k.$$

Thus, by Fubini, Hölder and Young, we get for $k \geq 4$ the estimate :

$$\|h_{2,\varepsilon}\|_{L^p(I_k)} \leq 2\|h_\varepsilon\|_{L^p(I_k^2)} \leq 2\|h\|_{L^p(\Theta)}. \quad (3.18)$$

As for $h_{1,\varepsilon}$, since :

$$\left(\frac{k-2}{k}\right) h_{1,\varepsilon}(x_1) = \int_{I_k} (h_\varepsilon(x_1, x_2) - h_{2,\varepsilon}(x_2)) dx_2,$$

by the triangular inequality, Fubini, Hölder, Young and (3.18), we get for $k \geq 4$ the estimate :

$$\|h_{1,\varepsilon}\|_{L^p(I_k)} \leq 6\|h\|_{L^p(\Theta)}.$$

□

Now we are ready to prove Proposition 3.6 :

Proof. [Proof of proposition 3.6] Since $\overline{\text{span}(SV(\Omega_2))}$ is a closed vectorial subspace of $\mathcal{H}(\Omega_2)$, it suffices to prove that :

$$\overline{\text{span}(SV(\Omega_2))}^\perp = \{0\}. \quad (3.19)$$

So, let us consider $u \in \mathcal{H}(\Omega_2)$ such that :

$$\int_{\Omega_2} u \cdot v dx = 0,$$

for any $v \in SV(\Omega_2)$. Recalling (3.8) and with an integration by parts, we have that, for any $g_1, g_2 \in L_0^2(0, \pi)$:

$$\begin{aligned} 0 &= \int_{\Omega_2} u_1(x_1, x_2) \left(\int_0^{x_1} g_1(s) ds \right) g_2(x_2) dx - \int_{\Omega_2} u_2(x_1, x_2) g_1(x_1) \left(\int_0^{x_2} g_2(s) ds \right) dx \\ &= \int_{\Omega_2} \left(- \int_0^{x_1} u_1(s, x_2) ds + \int_0^{x_2} u_2(x_1, s) ds \right) g_1(x_1) g_2(x_2) dx. \end{aligned} \quad (3.20)$$

Thus, using Lemma 3.8 for $p = 2$ and $\Theta = \Omega_2$, we have that there are $h_1, h_2 \in L^2(0, \pi)$ such that :

$$- \int_0^{x_1} u_1(s, x_2) ds + \int_0^{x_2} u_2(x_1, s) ds = h_1(x_1) + h_2(x_2).$$

If we differentiate with respect to $\partial_{x_1 x_2}$, we have that :

$$0 = -\partial_{x_2} u_1(x_1, x_2) + \partial_{x_1} u_2(x_1, x_2) = \nabla \times u.$$

In addition, since $u \in \mathcal{H}(\Omega_2)$, we also have that $\nabla \cdot u = 0$ and $u \cdot n = 0$. Consequently, since Ω_2 is Lipschitz and simply connected, we have that $u = 0$ (see, for instance, [12, Lemma IV.4.6]). \square

3.2.2 A total set of $\mathcal{H}(\Omega_2)$ formed by solutions of (3.5)

Throughout this section we focus on getting all the solutions of (3.5) (in Ω_2). In particular, we first get the candidates and then prove that they form a total set :

Proposition 3.10. *For all $m = (m_1, m_2) \in (\mathbb{N}_*)^2$ let us set :*

$$u_m(x) := \left(\int_0^{x_1} g_{1,m_1}(s) ds g_{2,m_2}(x_2), -g_{1,m_1}(x_1) \int_0^{x_2} g_{2,m_2}(s) ds \right), \quad (3.21)$$

for

$$g_{1,m_1}(s) := \cos(m_1 s), \quad g_{2,m_2}(s) := (2m_2 \varepsilon \cos(m_2 s) - \sin(m_2 s)) e^{-(2\varepsilon)^{-1} s}. \quad (3.22)$$

Then, u_m is a solution of the system (3.5) (in Ω_2) such that :

$$\|u_m\|_{\mathbf{L}^2(\Omega_2)} = \sqrt{\frac{(1 - e^{-\pi\varepsilon^{-1}}) m_2^2 \pi \varepsilon^3 (1 + 4\varepsilon^2 (m_1^2 + m_2^2))}{m_1^2 + 4m_1^2 m_2^2 \varepsilon^2}}, \quad (3.23)$$

whose associated pressure is constant and whose associated eigenvalue is :

$$\lambda_m = (m_1^2 + m_2^2) \varepsilon + \frac{1}{4\varepsilon}. \quad (3.24)$$

Remark 3.11. We have the equality :

$$u_m(x) = e^{-(2\varepsilon)^{-1} x_2} \left(\frac{\sin(m_1 x_1)}{m_1} (2m_2 \varepsilon \cos(m_2 x_2) - \sin(m_2 x_2)), -2\varepsilon \cos(m_1 x_1) \sin(m_2 x_2) \right). \quad (3.25)$$

Moreover, using that $m_1, m_2 \geq 1$ and $\varepsilon \in (0, 1)$, we obtain from (3.25) the estimate :

$$\|u_m\|_{\mathbf{L}^\infty(\Omega_2)} \leq C(m_2 \varepsilon + 1). \quad (3.26)$$

Proof. We can check directly the conclusions of Proposition 3.10 for the functions given in (3.25). In addition, we can get all the eigenfunctions of the elliptic system associated to (3.7) and (3.6) and find the pairs of compatibility. However, we show an “intuitive” way to get the u_m to provide a better insight to the reader which can help him/her to understand what happens in other Stokes systems. With that purpose, we look for eigenfunctions in $SV(\Omega_2)$ (see (3.8)) such that $g_1, g_2 \in C^\infty([0, \pi])$. We can prove easily that in that case the condition (3.5)₃ is equivalent to (if $g_1 \neq 0$ and $g_2 \neq 0$) :

$$g_1'(0) = g_1'(\pi) = \varepsilon g_2'(0) + g_2(0) = \varepsilon g_2'(\pi) + g_2(\pi) = 0. \quad (3.27)$$

If we apply the divergence operator to (3.5)₁, we obtain that $\Delta p = 0$. In particular, we can apply the Laplacian operator to (3.5)₁ to get possible solutions. Indeed, since :

$$\begin{aligned} \Delta u_1 &= g_1'(x_1)g_2(x_2) + \int_0^{x_1} g_1(s)ds g_2''(x_2), \\ \partial_{x_2} \Delta u_1 &= g_1'(x_1)g_2'(x_2) + \int_0^{x_1} g_1(s)ds g_2'''(x_2), \\ \Delta^2 u_1 &= g_1'''(x_1)g_2(x_2) + 2g_1'(x_1)g_2''(x_2) + \int_0^{x_1} g_1(s)ds g_2^{iv}(x_2); \end{aligned}$$

we have that :

$$\begin{aligned} -\varepsilon g_1'''(x_1)g_2(x_2) - 2\varepsilon g_1'(x_1)g_2''(x_2) - \varepsilon \int_0^{x_1} g_1(s)ds g_2^{iv}(x_2) - g_1'(x_1)g_2'(x_2) \\ - \int_0^{x_1} g_1(s)ds g_2'''(x_2) - \lambda g_1'(x_1)g_2(x_2) - \lambda \int_0^{x_1} g_1(s)ds g_2''(x_2) = 0. \end{aligned}$$

We have that $g_{1,m_1} \in L_0^2(0, \pi)$ (see (3.22)), that g_{1,m_1} satisfies the first two conditions of (3.27), and that g_{1,m_1} is an eigenfunction of the Laplacian ; so it is a reasonable function to do an attempt. Under that choice, the functions $g_2 \in L_0^2(0, \pi)$ have to satisfy the two last conditions of (3.27) and the ODE :

$$-\varepsilon m_1^4 g_2 + 2\varepsilon m_1^2 g_2'' - \varepsilon g_2^{iv} + m_1^2 g_2' - g_2''' + \lambda m_1^2 g_2 - \lambda g_2'' = 0. \quad (3.28)$$

In order to solve (3.28) we calculate the roots of its characteristic polynomial :

$$-\varepsilon(r^2 - m_1^2)^2 - r(r^2 - m_1^2) - \lambda(r^2 - m_1^2) = 0. \quad (3.29)$$

Let us focus on the roots of :

$$r^2 + \frac{r}{\varepsilon} + \frac{\lambda}{\varepsilon} - m_1^2 = 0, \quad (3.30)$$

which are given by :

$$r = \frac{-\frac{1}{\varepsilon} \pm \sqrt{\frac{1}{\varepsilon^2} - 4\frac{\lambda}{\varepsilon} + 4m_1^2}}{2}.$$

These roots are complex if λ is sufficiently large with respect to ε^{-1} . We then have that the associated solutions for λ large enough are :

$$g_{2,m_2}(s) := (A \cos(m_2 s) + B \sin(m_2 s))e^{-(2\varepsilon)^{-1}s}, \quad (3.31)$$

where $m_2 > 0$ is defined by :

$$m_2 := m_2(\varepsilon, \lambda, m_1) := \sqrt{\frac{\lambda}{\varepsilon} - \frac{1}{4\varepsilon^2} - m_1^2}. \quad (3.32)$$

Consequently, the g_{2,m_2} that we get are solutions of :

$$-\varepsilon g_2'' - g_2' = (\lambda - \varepsilon m_1^2)g_2. \quad (3.33)$$

We are going to see that there are some $g_{2,m_2} \in L_0^2(0, \pi)$ that also satisfy the last two conditions of (3.27). Indeed, we have that :

$$g_{2,m_2}'(s) = \left[\left(-\frac{A}{2\varepsilon} + m_2 B \right) \cos(m_2 s) + \left(-\frac{B}{2\varepsilon} - A m_2 \right) \sin(m_2 s) \right] e^{-(2\varepsilon)^{-1}s}.$$

From $g_{2,m_2}(0) = -\varepsilon g_{2,m_2}'(0)$, we obtain that :

$$A = -2m_2 \varepsilon B. \quad (3.34)$$

Moreover, from $g_{2,m_2}(\pi) = -\varepsilon g_{2,m_2}'(\pi)$ and (3.34) we have that :

$$-2m_2 \varepsilon B \cos(m_2 \pi) + B \sin(m_2 \pi) = -2m_2 \varepsilon B \cos(m_2 \pi) + \left(\frac{1}{2} - 2m_2^2 \varepsilon^2 \right) B \sin(m_2 \pi).$$

Consequently,

$$B \sin(m_2 \pi) = -4m_2^2 \varepsilon^2 B \sin(m_2 \pi).$$

Hence, we have either $m_2 \in \mathbb{N}_*$ or $B = 0$, which by (3.34) implies that $g_{2,m_2} = 0$, which just gives the null solution. Consequently, the possible values for g_{2,m_2} are (we have taken $B = -1$) :

$$g_{2,m_2}(s) = (2m_2 \varepsilon \cos(m_2 s) - \sin(m_2 s))e^{-(2\varepsilon)^{-1}s} = \frac{d \left(2\varepsilon \sin(m_2 s) e^{-(2\varepsilon)^{-1}s} \right)}{ds}, \quad \forall m_2 \in \mathbb{N}_*, \quad (3.35)$$

which clearly belong to $L_0^2(0, \pi)$. Thus, our candidates are those defined in (3.22) and λ_m satisfies (3.24) because of (3.32).

In order to see that u_m satisfies the first equation of (3.5)₁ with a constant pressure for any given time, we have to use (3.21), that g_{2,m_2} is a solution of (3.33) and that g_{1,m_1} is a solution of $-\varepsilon g''_{1,m_1} = \varepsilon m_1^2 g_{1,m_1}$. Moreover, in order to see that they satisfy the second equation of (3.5)₁, we consider (3.33) and that $\varepsilon g'_{2,m_2}(0) + g_{2,m_2}(0) = 0$, which imply that :

$$(\lambda - \varepsilon m_1^2) \int_0^s g_{2,m_2}(s') ds' = -\varepsilon g'_{2,m_2} - g_{2,m_2} = -\varepsilon \frac{d^2 \left(\int_0^s g_{2,m_2}(s') ds' \right)}{ds^2} - \frac{d \left(\int_0^s g_{2,m_2}(s') ds' \right)}{ds}.$$

Finally, let us prove (3.23) using (3.25). We recall that, if $a, b \in \mathbb{R}$ such that $a \neq 0$ or $b \neq 0$:

$$\begin{cases} \int_0^s \sin(as') e^{bs'} ds' = \frac{e^{bs}(b \sin(as) - a \cos(as)) + a}{a^2 + b^2}, \\ \int_0^s \cos(as') e^{bs'} ds' = \frac{e^{bs}(b \cos(as) + a \sin(as)) - b}{a^2 + b^2}. \end{cases} \quad (3.36)$$

In addition, recalling that $m_1 \in \mathbb{N}_*$, we have the identity :

$$\int_0^\pi \sin^2(m_1 x_1) dx_1 = \frac{\pi}{2}. \quad (3.37)$$

Similarly, we have that :

$$\begin{aligned} & (2m_2 \varepsilon \cos(m_2 x_2) - \sin(m_2 x_2))^2 \\ &= 2m_2^2 \varepsilon^2 (1 + \cos(2m_2 x_2)) - 2m_2 \varepsilon \sin(2m_2 x_2) + \frac{(1 - \cos(2m_2 x_2))}{2} \\ &= \frac{4m_2^2 \varepsilon^2 + 1}{2} + \frac{4m_2^2 \varepsilon^2 - 1}{2} \cos(2m_2 x_2) - 2m_2 \varepsilon \sin(2m_2 x_2). \end{aligned} \quad (3.38)$$

Consequently, considering that $m_2 \in \mathbb{N}_*$, (3.36) and (3.38) we obtain the equality :

$$\begin{aligned} & \int_0^\pi (2m_2 \varepsilon \cos(m_2 x_2) - \sin(m_2 x_2))^2 e^{-\varepsilon^{-1} x_2} dx_2 \\ &= \int_0^\pi \left(\frac{4m_2^2 \varepsilon^2 + 1}{2} + \frac{4m_2^2 \varepsilon^2 - 1}{2} \cos(2m_2 x_2) - 2m_2 \varepsilon \sin(2m_2 x_2) \right) e^{-\varepsilon^{-1} x_2} dx_2 \\ &= \varepsilon \frac{4m_2^2 \varepsilon^2 + 1}{2} (1 - e^{-\pi \varepsilon^{-1}}) + \frac{4m_2^2 \varepsilon^2 - 1}{2} \frac{\varepsilon^{-1} (1 - e^{-\pi \varepsilon^{-1}})}{4m_2^2 + \varepsilon^{-2}} - 2m_2 \varepsilon \frac{2m_2 (1 - e^{-\pi \varepsilon^{-1}})}{4m_2^2 + \varepsilon^{-2}} \\ &= \frac{(4m_2^2 \varepsilon^3 + \varepsilon)(4m_2^2 \varepsilon^2 + 1) + 4m_2^2 \varepsilon^3 - \varepsilon - 8m_2^2 \varepsilon^3}{8m_2^2 \varepsilon^2 + 2} (1 - e^{-\pi \varepsilon^{-1}}) = 2m_2^2 \varepsilon^3 (1 - e^{-\pi \varepsilon^{-1}}). \end{aligned} \quad (3.39)$$

Moreover, since $m_1 \in \mathbb{N}$ we have the identity :

$$\int_0^\pi \cos^2(m_1 x_1) dx_1 = \frac{\pi}{2}. \quad (3.40)$$

In addition, from $m_2 \in \mathbb{N}_*$ we obtain the equality :

$$\begin{aligned} \int_0^\pi \sin^2(m_2 x_2) e^{-\varepsilon^{-1} x_2} dx_2 &= \int_0^\pi \frac{1 - \cos(2m_2 x_2)}{2} e^{-\varepsilon^{-1} x_2} = \frac{1}{2} \left(\varepsilon - \frac{\varepsilon}{4m_2^2 \varepsilon^2 + 1} \right) (1 - e^{-\pi \varepsilon^{-1}}) \\ &= \frac{2m_2^2 \varepsilon^3}{4m_2^2 \varepsilon^2 + 1} (1 - e^{-\pi \varepsilon^{-1}}). \end{aligned} \quad (3.41)$$

So, combining (3.25), (3.37), (3.39), (3.40) and (3.41) we obtain that :

$$\begin{aligned} \|u_m\|_{\mathbf{L}^2(\Omega)}^2 &= \frac{\pi}{m_1^2} m_2^2 \varepsilon^3 (1 - e^{-\pi \varepsilon^{-1}}) + \frac{4\pi m_2^2 \varepsilon^5}{4m_2^2 \varepsilon^2 + 1} (1 - e^{-\pi \varepsilon^{-1}}) \\ &= (1 - e^{-\pi \varepsilon^{-1}}) \pi \frac{4m_2^4 \varepsilon^5 + m_2^2 \varepsilon^3 + 4m_2^2 m_1^2 \varepsilon^5}{4m_2^2 m_1^2 \varepsilon^2 + m_1^2} = \frac{(1 - e^{-\pi \varepsilon^{-1}}) m_2^2 \pi \varepsilon^3 (1 + 4\varepsilon^2 (m_1^2 + m_2^2))}{m_1^2 + 4m_1^2 m_2^2 \varepsilon^2}, \end{aligned}$$

which implies (3.23). \square

Proposition 3.12. *The set $\{u_m\}_{m \in (\mathbb{N}_*)^2}$ is a total set.*

Proof. Thanks to Proposition 3.6, it suffices to prove that :

$$SV(\Omega_2) \subset \overline{\text{span}\{u_m\}}. \quad (3.42)$$

We prove this inclusion constructively; that is, writing an element of $SV(\Omega_2)$ as a limit of linear combinations of the eigenfunctions u_m . We consider a function $u \in SV(\Omega_2)$ (see (3.8)).

Let us recall that $\left\{ \frac{\cos(ks)}{\sqrt{2^{-1}\pi}} \right\}_{k \in \mathbb{N}_*} \cup \{\pi^{-1}\}$ is an orthonormal basis of $L^2(0, \pi)$ (the basis obtained by the diagonalization of the Laplacian with Neumann boundary conditions in $(0, \pi)$). So, since $g_1 \in L_0^2(0, \pi)$, we have that, for some $a_k \in \ell^2$:

$$g_1(s) = \sum_{k \geq 1} a_k \frac{\cos(ks)}{\sqrt{2^{-1}\pi}}. \quad (3.43)$$

That series must be understood as a limit in $L^2(0, \pi)$.

Next, we remark that :

$$\left\{ \frac{\gamma_k^\varepsilon}{\sqrt{2^{-1}\pi} \sqrt{1 + 4k^2 \varepsilon^2}} \right\}_{k \geq 1} \cup \left\{ \frac{e^{-(2\varepsilon)^{-1}s}}{\sqrt{\varepsilon(1 - e^{-\pi/\varepsilon})}} \right\}, \quad (3.44)$$

for :

$$\gamma_k^\varepsilon(s) := 2k\varepsilon \cos(ks) - \sin(ks), \quad (3.45)$$

is the orthonormal basis obtained by diagonalization of the Laplacian in $(0, \pi)$ with Robin boundary conditions :

$$\begin{cases} -2\varepsilon g'(0) = g(0), \\ -2\varepsilon g'(\pi) = g(\pi). \end{cases} \quad (3.46)$$

Indeed, we have that the Laplacian is self-adjoint in the subspace of $H^2(\Omega_2)$ that satisfies (3.46). Thus, we have, for some $b_k \in \ell^2$:

$$g_2(s) = \left(g_2(s) e^{(2\varepsilon)^{-1}s} \right) e^{-(2\varepsilon)^{-1}s} = \sum_{k \geq 1} b_k \frac{\gamma_k^\varepsilon(s) e^{-(2\varepsilon)^{-1}s}}{\sqrt{2^{-1}\pi} \sqrt{1 + 4k^2\varepsilon^2}}. \quad (3.47)$$

We have used that $g_2 \in L^2_0(0, \pi)$ implies that $g_2 e^{(2\varepsilon)^{-1}s}$ is orthogonal to $e^{-(2\varepsilon)^{-1}s}$. Moreover, we have to understand the series as a limit in $L^2(0, \pi)$, not as a pointwise one. In that sense, we can place the exponential into the series because the multiplication of the functions of $L^2(0, \pi)$ by an element of $L^\infty(0, \pi)$ is a continuous linear form.

Next, we recall that the operator \int_0^s is a continuous endomorphism in $L^2(0, \pi)$. Thus, we have that :

$$\int_0^s g_1(s') ds' = \sum_{k \geq 1} a_k \frac{\sin(ks)}{k\sqrt{2^{-1}\pi}}. \quad (3.48)$$

Similarly, we have that :

$$\int_0^s g_2(s') ds' = \sum_{k \geq 1} b_k \frac{2\varepsilon \sin(ks) e^{-(2\varepsilon)^{-1}s}}{\sqrt{2^{-1}\pi} \sqrt{1 + 4k^2\varepsilon^2}}. \quad (3.49)$$

The limits in (3.48) and (3.49) are both pointwise and in $L^2(0, \pi)$.

We finally recall that the tensor product of two convergent sequences of $L^2(0, \pi)$ converges in $L^2(\Omega_2)$ to the tensor product of their respective limits. Thus, we have that (3.43), (3.47), (3.48) and (3.49) imply :

$$\int_0^{x_1} g_1(s) ds g_2(x_2) = \lim_{k \rightarrow \infty} \sum_{m_1, m_2 \geq 1}^k \frac{2a_{m_1} b_{m_2}}{\pi \sqrt{1 + 4m_2^2\varepsilon^2}} \frac{\sin(m_1 x_1)}{m_1} \gamma_{m_2}^\varepsilon(x_2) e^{-(2\varepsilon)^{-1}x_2},$$

and that :

$$-g_1(x_1) \int_0^{x_2} g_2(s) ds = \lim_{k \rightarrow \infty} \sum_{m_1, m_2 \geq 1}^k \frac{2a_{m_1} b_{m_2}}{\pi \sqrt{1 + 4m_2^2\varepsilon^2}} \left(-2\varepsilon \cos(m_1 x_1) \sin(m_2 x_2) e^{-(2\varepsilon)^{-1}x_2} \right).$$

Consequently, we have that :

$$u = \lim_{k \rightarrow \infty} \sum_{m_1, m_2 \geq 1}^k \frac{2a_{m_1} b_{m_2}}{\pi \sqrt{1 + 4m_2^2 \varepsilon^2}} u_m,$$

which proves (3.42). \square

3.2.3 Properties of the solution of (3.3) in Ω_2

Since our operator is not self-adjoint, the total set $\{u_m\}$ given in (3.25) does not have to be orthogonal (and it is not). In addition, it is not immediate to write an element of $\mathcal{H}(\Omega_2)$ as a series of elements proportional to those in $\{u_m\}$. Indeed, there are total sets in l^2 (for instance $\{e_1\} \cup \{e_1 + e_k\}_{k \geq 2}$) such that it is not possible to write any element of l^2 (for instance $(k^{-1})_{k \geq 1}$) as a series of elements proportional to those in the total sets. In our particular situation we do not even know if this is possible (probably yes), but we will prove that at least we can express the solutions of (3.3) in Ω_2 with the help of $\{u_m\}$.

The next step is given any $u \in \text{span}\{u_m\}$, to get its coordinates in our generating system. In order to do so, we get a set of functions $v_m \in \mathbf{L}^2(\Omega_2)$ such that $\{(u_m, v_m)\}$ is a bi-orthogonal system ; that is, such that :

$$\int_{\Omega_2} u_m(x) \cdot v_{m'}(x) dx = 1_{m=m'}. \quad (3.50)$$

Usually, we can get those functions v_m by diagonalization of the adjoint operator, but it is easier to do a systematic search in $\mathbf{L}^2(\Omega_2)$. In that sense we define (recall (3.45)) :

$$v_m(x) := \frac{2}{\pi^2} e^{(2\varepsilon)^{-1}x_2} \left(m_1 \sin(m_1 x_1) \frac{\gamma_{m_2}^\varepsilon(x_2)}{1 + 4m_2^2 \varepsilon^2}, -\frac{\cos(m_1 x_1) \sin(m_2 x_2)}{2\varepsilon} \right). \quad (3.51)$$

We can easily verify (3.50) because $\left\{ \sqrt{2\pi^{-1}} \sin(ks) \right\}_{k \in \mathbb{N}_*}$, $\left\{ \sqrt{2\pi^{-1}} \cos(ks) \right\}_{k \in \mathbb{N}_*} \cup \{\pi^{-1}\}$ and (3.44) are Hilbert basis in $L^2(0, \pi)$. We remark that, for a constant C that does not depend on m_1 , m_2 or ε :

$$\|v_m\|_{\mathbf{L}^2(\Omega_2)} \leq C \left(m_1 + \frac{1}{\varepsilon} \right) e^{(2\varepsilon)^{-1}\pi}. \quad (3.52)$$

Now we are ready to get the solutions of (3.3) through a spectral decomposition.

Proposition 3.13. *Let $\varphi^T \in \mathcal{H}(\Omega_2)$. Let us denote, for $t \leq T$, $L_\varepsilon^T(t)\varphi^T$ the value in $\mathcal{H}(\Omega_2)$ of the energy solution of system (3.3) in Ω_2 on time t . Then, we have that :*

1. *For all $t < T$ and $\varepsilon > 0$:*

$$L_\varepsilon^T(t)\varphi^T = \sum_{m \in (\mathbb{N}_*)^2} \langle \varphi^T, v_m \rangle_{\mathbf{L}^2(\Omega_2)} \exp \left[\left((m_1^2 + m_2^2)\varepsilon + \frac{1}{4\varepsilon} \right) (t - T) \right] u_m. \quad (3.53)$$

In particular, the series of the right hand-side of (3.53) is well-defined and absolutely convergent in $\mathcal{H}(\Omega_2)$.

2. *For all $\delta > 0$ there is $C_\delta > 0$ such that for all $T > 0$, $\varphi^T \in \mathcal{H}(\Omega_2)$, $\varepsilon \in (0, 1)$ and $s \leq T - 2\pi - \delta$:*

$$\|L_\varepsilon^T(s)\varphi^T\|_{\mathbf{L}^2(\Omega_2)} \leq C_\delta \exp \left[\frac{s - (T - 2\pi - \delta)}{4\varepsilon} \right] \|\varphi^T\|_{\mathbf{L}^2(\Omega_2)}. \quad (3.54)$$

Throughout the proof we denote $C_{\varepsilon,t}$ a generic constant which might be different each time and which just depends on ε and t .

Proof. First of all, we have that (3.53) is true if $\varphi^T \in \text{span}\{u_m\}$ as a consequence of Proposition 3.10 (see (3.24)) and (3.50). Moreover, for $t > 0$ and $\varepsilon > 0$ fixed, $L_\varepsilon^T(t)$ is a continuous endomorphism on $\mathcal{H}(\Omega_2)$. Consequently in order to prove Item 1, it suffices to prove that the series of the right-hand side of (3.53) is well-defined, absolutely convergent and continuous. We shall prove the three properties at once.

Recalling estimate (3.26), estimate (3.52) and that xe^{-x^2} is bounded in \mathbb{R}^+ , we have that, if $\varepsilon \in (0, 1)$:

$$\begin{aligned} & \sum_{m \in (\mathbb{N}_*)^2} |\langle \varphi^T, v_m \rangle_{\mathbf{L}^2(\Omega_2)}| \exp \left[\left((m_1^2 + m_2^2)\varepsilon + \frac{1}{4\varepsilon} \right) (t - T) \right] \|u_m\|_{\mathbf{L}^2(\Omega_2)} \\ & \leq C_{\varepsilon,t} \|\varphi^T\|_{\mathbf{L}^2(\Omega_2)} \sum_{m \in (\mathbb{N}_*)^2} m_1 m_2 \exp \left((m_1^2 + m_2^2)\varepsilon(t - T) \right) \\ & \leq C_{\varepsilon,t} \|\varphi^T\|_{\mathbf{L}^2(\Omega_2)} \sum_{m \in (\mathbb{N}_*)^2} \exp \left((m_1^2 + m_2^2) \frac{\varepsilon(t - T)}{2} \right). \end{aligned} \quad (3.55)$$

The series in the right-hand side of (3.55) is convergent because, in $(\mathbb{R}^+)^2$:

$$\begin{aligned} & \exp \left((m_1^2 + m_2^2) \frac{\varepsilon(t - T)}{2} \right) 1_{[m_1-1, m_1] \times [m_2-1, m_2]}(x_1, x_2) \\ & \leq \exp \left((x_1^2 + x_2^2) \frac{\varepsilon(t - T)}{2} \right) 1_{[m_1-1, m_1] \times [m_2-1, m_2]}(x_1, x_2), \end{aligned}$$

which implies that :

$$\sum_{m \in (\mathbb{N}_*)^2} \exp\left((m_1^2 + m_2^2) \frac{\varepsilon(t-T)}{2}\right) \leq \int_{(\mathbb{R}^+)^2} \exp\left((x_1^2 + x_2^2) \frac{\varepsilon(t-T)}{2}\right) dx = \frac{2}{\varepsilon(T-t)} \frac{\pi}{4}. \quad (3.56)$$

Combining (3.55) and (3.56), we have that :

$$\sum_{m \in (\mathbb{N}_*)^2} |\langle \varphi^T, v_m \rangle_{\mathbf{L}^2(\Omega_2)}| \exp\left[\left((m_1^2 + m_2^2)\varepsilon + \frac{1}{4\varepsilon}\right)(t-T)\right] \|u_m\|_{\mathbf{L}^2(\Omega_2)} \leq C_{\varepsilon, t} \|\varphi^T\|_{\mathbf{L}^2(\Omega_2)}, \quad (3.57)$$

which implies that the series in the right-hand side of (3.53) is absolutely convergent and thus well-defined. Moreover, since the right-hand side of (3.53) is linear, (3.57) implies that the right-hand side of (3.53) is continuous in $\mathcal{H}(\Omega_2)$.

Finally, let us prove (3.54). Indeed, using (3.26), the triangular inequality, and Cauchy-Schwarz, we have that, for all $\varepsilon \in (0, 1)$:

$$\|L_\varepsilon^T(s)\varphi^T\|_{\mathbf{L}^2(\Omega_2)} \leq C \|\varphi^T\|_{\mathbf{L}^2(\Omega_2)} \sum_{m \in (\mathbb{N}_*)^2} m_2 \|v_m\|_{\mathbf{L}^2(\Omega_2)} \exp\left[\left((m_1^2 + m_2^2)\varepsilon + \frac{1}{4\varepsilon}\right)(s-T)\right]. \quad (3.58)$$

Next, using (3.52) and that xe^{-x^2} is bounded in \mathbb{R}^+ , if $s \leq T - 2$ and if $\varepsilon \in (0, 1)$, we have that (3.58) turns into :

$$\|L_\varepsilon^T(s)\varphi^T\|_{\mathbf{L}^2(\Omega_2)} \leq \frac{C \|\varphi^T\|_{\mathbf{L}^2(\Omega_2)}}{\varepsilon^2} \left[\sum_{m \in (\mathbb{N}_*)^2} \exp(-(m_1^2 + m_2^2)\varepsilon) \right] \exp\left(\frac{s - (T - 2\pi)}{4\varepsilon}\right). \quad (3.59)$$

Using (3.56) for $t = T - 2$, we have that there is $C_\delta > 0$ such that for all $\varepsilon \in (0, 1)$:

$$\frac{1}{\varepsilon^2} \sum_{m \in (\mathbb{N}_*)^2} \exp(-(m_1^2 + m_2^2)\varepsilon) \leq \frac{\pi}{4\varepsilon^3} \leq C_\delta e^{\delta(4\varepsilon)^{-1}}.$$

Consequently, (3.59) implies (3.54). \square

Remark 3.14. Let $T > 0$ and $\varphi^T \in \mathcal{H}(\Omega_2)$. Then, the application :

$$\mathbb{R}_*^+ \ni \varepsilon \mapsto \varphi^\varepsilon(0, \cdot) \in \mathcal{H}(\Omega_2)$$

(for φ^ε the solution of (3.3) in Ω_2 and initial value φ^T) is continuous. To prove the continuity, we fix $\varepsilon_0 > 0$ and $\varrho > 0$ and verify that if $\delta > 0$ is small enough, we have for all $\varepsilon \in (\varepsilon_0 - \delta, \varepsilon_0 + \delta)$ the following inequality :

$$\|\varphi^\varepsilon(0, \cdot) - \varphi^{\varepsilon_0}(0, \cdot)\|_{\mathbf{L}^2(\Omega_2)} \leq \varrho. \quad (3.60)$$

First, using (3.26), (3.52) and that $se^{-s^2} \in L^\infty(\mathbb{R}^+, ds)$ we obtain that for all $\varepsilon \in (\varepsilon_0/2, 3\varepsilon_0/2)$ and $M \in \mathbb{N}$:

$$\begin{aligned} & \left\| \sum_{m_1+m_2 \geq M} \langle \varphi^T, v_m \rangle_{\mathbf{L}^2(\Omega_2)} \exp \left[- \left((m_1^2 + m_2^2)\varepsilon + \frac{1}{4\varepsilon} \right) T \right] u_m \right\|_{\mathbf{L}^2(\Omega_2)} \\ & \leq C(T, \varphi^T) \left(1 + \frac{1}{\varepsilon^2} \right) \exp \left(\frac{-T + 2\pi}{4\varepsilon} \right) \left[\sum_{m_1+m_2 \geq M} \exp \left(- (m_1^2 + m_2^2) \frac{\varepsilon T}{2} \right) \right] \\ & \leq C(T, \varphi^T, \varepsilon_0) \left[\sum_{m_1+m_2 \geq M} \exp \left(- (m_1^2 + m_2^2) \frac{\varepsilon_0 T}{4} \right) \right]. \end{aligned} \quad (3.61)$$

Consequently, since the series in the right-hand of (3.61) is convergent, there is $M(T, \varphi^T, \varepsilon_0)$ large enough such that for all $\varepsilon \in (\varepsilon_0/2, 3\varepsilon_0/2)$:

$$\left\| \sum_{m_1+m_2 \geq M} \langle \varphi^T, v_m \rangle_{\mathbf{L}^2(\Omega_2)} \exp \left[- \left((m_1^2 + m_2^2)\varepsilon + \frac{1}{4\varepsilon} \right) T \right] u_m \right\|_{\mathbf{L}^2(\Omega_2)} \leq \frac{\rho}{3}. \quad (3.62)$$

So, using (3.53) and the triangular inequality, we obtain for all $\varepsilon \in (\varepsilon_0/2, 3\varepsilon_0/2)$ the estimate :

$$\begin{aligned} \|\varphi^\varepsilon(0, \cdot) - \varphi^{\varepsilon_0}(0, \cdot)\|_{\mathbf{L}^2(\Omega_2)} & \leq \frac{2\rho}{3} + \left\| \sum_{m_1+m_2 < M} \langle \varphi^T, v_m \rangle_{\mathbf{L}^2(\Omega_2)} \left(\exp \left[- \left((m_1^2 + m_2^2)\varepsilon + \frac{1}{4\varepsilon} \right) T \right] \right. \right. \\ & \quad \left. \left. - \exp \left[- \left((m_1^2 + m_2^2)\varepsilon_0 + \frac{1}{4\varepsilon_0} \right) T \right] \right) u_m \right\|_{\mathbf{L}^2(\Omega_2)}. \end{aligned} \quad (3.63)$$

Finally, considering that the sum in the right-hand side of (3.63) is finite and its terms are continuous with respect to ε , we have (3.60) for all $\varepsilon \in (\varepsilon_0 - \delta, \varepsilon_0 + \delta)$ if δ is small enough.

3.3 The cost of the control in $(0, \pi)^2$

In this section we prove the first three items of Theorem 3.1. In the three proofs we use the equivalent definitions of the costs given in (3.4). In addition, for the proof of Item 2 we use the eigenfunctions given in (3.25), whereas for the other two proofs we use Remark 3.5 and then inspire in [71]. The three proofs are independent.

3.3.1 Proof of Item 1 of Theorem 3.1

In order to prove that the cost of the control decays, we first state a Carleman inequality for system (3.7), then use that the divergence is null, and finally apply usual parabolic estimates. We recall that $\omega \subset \Omega_2$ is any arbitrary subdomain.

We consider ω_0 an open ball centered at some point $\bar{x} = (\bar{x}_1, \bar{x}_2)$ and whose closure is contained in ω . We consider the auxiliary functions :

$$\eta^\pm(x) := (2\pi^2)^{-1} (\pm G(x_1) - (x_2 - \bar{x}_2)^2),$$

for G a regular (positive) function such that $G(0) = G(\pi) = 0$, $G'' < 0$, $G'(\bar{x}_1) = 0$ and $\|G\|_{L^\infty(0,\pi)} \leq \pi^2$. Finally, let us consider the following classical weights, for $\tilde{T} > 0$:

$$\begin{aligned} \alpha_\pm(t, x) &:= \frac{e^{8\lambda} - e^{\lambda(\eta^\pm(x)+6)}}{t(\tilde{T}-t)}, & \xi(t, x) &:= \frac{e^{\lambda(\eta^\pm(x)+6)}}{t(\tilde{T}-t)}, \\ \alpha_\pm^*(t) &= \max_{x \in \tilde{\Omega}} \alpha_\pm(t, x), & \xi_\pm^*(t) &= \min_{x \in \tilde{\Omega}} \xi_\pm(t, x). \end{aligned} \quad (3.64)$$

Remark 3.15. By the choice of G we have that $\|\eta^\pm\|_{L^\infty(\Omega_2)} \leq 1$, the weights α^\pm are positive. This way of defining the weights is classical (see, for instance, [57] and [36]).

Proposition 3.16. *There is a constant $C > 0$ such that for any domain $\omega \subset \Omega_2$, $\tilde{T} > 0$ and $z^{\tilde{T}} \in L^2(\Omega_2)$ we have that :*

$$\begin{aligned} \tau \lambda^2 \iint_{(0, \tilde{T}) \times \Omega_2} e^{-2\tau\alpha_+} \xi_+ |\nabla z|^2 dx dt + \tau^3 \lambda^4 \iint_{(0, \tilde{T}) \times \Omega_2} e^{-2\tau\alpha_+} \xi_+^3 |z|^2 dx dt \\ \leq C \tau^3 \lambda^4 \iint_{(0, \tilde{T}) \times \omega} e^{-2\tau\alpha_+} \xi_+^3 |z|^2 dx dt, \end{aligned} \quad (3.65)$$

for z the solution of (3.7) (in $(0, \tilde{T})$ instead of in $(0, T)$), for the weights defined in (3.64) and for any $\varepsilon \in (0, 1)$, $\lambda \geq C$ and $\tau \geq C(\tilde{T} + \tilde{T}^2)\varepsilon^{-1}$.

The proof of Proposition 3.16 is essentially a combination of the proofs given in [71, Proposition 1] and [52, Lemma 1]. Since the ideas of the proof are not original and since the proof is rather long, we just sketch it in Appendix 3.A.

A straight consequence of Proposition 3.16 and Remark 3.5 is that there is $C > 0$ such that for any $\varphi^T \in \mathbf{L}^2(\Omega_2)$, if φ denotes the solution of (3.3) in Ω_2 , for any $\lambda \geq C$, for any $\tau \geq C\varepsilon^{-1}$, and for the weights (3.64) with $\tilde{T} = 1$, we have that :

$$\begin{aligned} \tau\lambda^2 \iint_{(0,1)\times\Omega_2} e^{-2\tau\alpha_+ \xi_+} |\nabla\varphi_2(t+T-1, x)|^2 dxdt + \tau^3\lambda^4 \iint_{(0,1)\times\Omega_2} e^{-2\tau\alpha_+ \xi_+^3} |\varphi_2(t+T-1, x)|^2 dxdt \\ \leq C\tau^3\lambda^4 \iint_{(0,1)\times\omega} e^{-2\tau\alpha_+ \xi_+^3} |\varphi_2(t+T-1, x)|^2 dxdt. \end{aligned} \quad (3.66)$$

Moreover, because of (3.6)₂, we have by Poincaré inequality (and Fubini) that for all $t < T$:

$$\begin{aligned} \tau\lambda^2 \int_{\Omega_2} e^{-2\tau\alpha_+^*(t)} \xi_+^*(t) |\varphi_1|^2(t, x) dx = \tau\lambda^2 e^{-2\tau\alpha_+^*(t)} \xi_+^*(t) \int_0^\pi \|\varphi_1(t, \cdot, x_2)\|_{L^2(0, \pi; dx_1)}^2 dx_2 \\ \leq C\tau\lambda^2 e^{-2\tau\alpha_+^*(t)} \xi_+^*(t) \int_0^\pi \|\partial_{x_1}\varphi_1(t, \cdot, x_2)\|_{L^2(0, \pi; dx_1)}^2 dx_2 \\ = C\tau\lambda^2 \int_{\Omega_2} e^{-2\tau\alpha_+^*(t)} \xi_+^*(t) |\partial_{x_1}\varphi_1(t, x)|^2 dx. \end{aligned} \quad (3.67)$$

Thus, since $\nabla \cdot \varphi = 0$, combining (3.66) with (3.67), we have for $\lambda \geq C$, $\tau \geq C\varepsilon^{-1}$ the estimate :

$$\tau\lambda^2 \iint_{(0,1)\times\Omega_2} e^{-2\tau\alpha_+^* \xi_+^*} |\varphi(t+T-1, x)|^2 dxdt \leq C\tau^3\lambda^4 \iint_{(0,1)\times\omega} e^{-2\tau\alpha_+ \xi_+^3} |\varphi_2(t+T-1, x)|^2 dxdt. \quad (3.68)$$

So, fixing λ large enough, and $\tau = \tau_0\varepsilon^{-1}$ for τ_0 large enough, we have that (3.68) implies that :

$$\|\varphi\|_{\mathbf{L}^2((T-2/3, T-1/3)\times\Omega_2)} \leq Ce^{C\varepsilon^{-1}} \|\varphi_2\|_{L^2((T-1, T)\times\omega)}. \quad (3.69)$$

Let $T > 7$, using Item 2 of Proposition 3.13 there is $C > 0$ such that for $\delta = 7 - 2\pi - 2/3$, for all $t' \in (T - 2/3, T - 1/3)$ and for $s = 0$, we have that :

$$\|\varphi(0, \cdot)\|_{\mathbf{L}^2(\Omega_2)} \leq C \exp\left[\frac{7 - 2/3 - t'}{4\varepsilon}\right] \|\varphi(t', \cdot)\|_{\mathbf{L}^2(\Omega_2)} \leq C \exp\left[\frac{7 - T}{4\varepsilon}\right] \|\varphi(t', \cdot)\|_{\mathbf{L}^2(\Omega_2)}. \quad (3.70)$$

So, combining (3.69) and (3.70), we get that, for any $T > 7$:

$$\|\varphi(0, \cdot)\|_{\mathbf{L}^2(\Omega_2)} \leq C \exp\left[\frac{7 - T}{4\varepsilon}\right] \|\varphi\|_{\mathbf{L}^2((T-2/3, T-1/3)\times\Omega_2)} \leq C \exp\left[\frac{C - T}{4\varepsilon}\right] \|\varphi_2\|_{L^2((T-1, T)\times\omega)}.$$

In particular, if T_0 is sufficiently large, Item 1 of Theorem 3.1 is true.

3.3.2 Proof of Item 2 of Theorem 3.1

Let us fix $h \in (0, \pi)$ and a control domain $\omega \subset (0, \pi) \times (\pi - h, \pi)$. We prove that, for $T \in (0, 2(\pi - h))$ fixed, the cost of the control is at least exponentially large with respect to ε^{-1} . We recall that :

$$u(x) := ce^{-(2\varepsilon)^{-1}x_2} (\sin(x_1)(2\varepsilon \cos(x_2) - \sin(x_2)), -2\varepsilon \cos(x_1) \sin(x_2)),$$

for :

$$c := \sqrt{\frac{1 + 4\varepsilon^2}{(1 - e^{-\pi\varepsilon^{-1}})\pi\varepsilon^3(1 + 8\varepsilon^2)}},$$

is a solution of (3.5) for $\lambda_\varepsilon := 2\varepsilon + \frac{1}{4\varepsilon}$ and such that $\|u\|_{\mathbf{L}^2(\Omega_2)} = 1$ (see Proposition 3.10). We remark that since $\omega \subset (0, \pi) \times (\pi - h, \pi)$ we have for all $\delta > 0$ and $\varepsilon \in (0, 1)$ the estimate :

$$\|u\|_{\mathbf{L}^\infty(\omega)} \leq C\varepsilon^{-3/2}e^{-(2\varepsilon)^{-1}(\pi-h)} \leq C_\delta e^{-(2\varepsilon)^{-1}(\pi-h-\delta)}. \quad (3.71)$$

Indeed, we have that $\varepsilon^{-3} \leq C_\delta e^{\delta\varepsilon^{-1}}$ for all $\delta > 0$ and $\varepsilon \in \mathbb{R}^+$.

For proving (3.71) we have calculated the maximum of $e^{-2\varepsilon^{-1}x_2}$ in $(0, \pi) \times (\pi - h, \pi)$. Moreover, we have that :

$$\varphi(t, x) := u(x)e^{-\lambda_\varepsilon(T-t)},$$

is a solution of (3.3) in Ω_2 for $\varphi^T = u$. On the one hand, we have that :

$$\int_{\Omega_2} |\varphi(0, x)|^2 dx = e^{-2\lambda_\varepsilon T}. \quad (3.72)$$

On the other hand, using (3.71) and bounding $e^{-2\lambda_\varepsilon(T-t)}$ by 1 we obtain the estimate :

$$\iint_{Q_\omega} |\varphi|^2 dxdt = \iint_{Q_\omega} |u(x)|^2 e^{-2\lambda_\varepsilon(T-t)} dxdt \leq \|u\|_{\mathbf{L}^\infty(\omega)}^2 \iint_{Q_\omega} dxdt \leq C_\delta \exp \left[\frac{h - \pi + \delta}{\varepsilon} \right]. \quad (3.73)$$

So, combining (3.4), (3.72) and (3.73) we get that :

$$[K_1(T, \varepsilon, \omega)]^2 \geq c_\delta \exp \left[\frac{\pi - h - \delta - 4\varepsilon^2 T - T/2}{\varepsilon} \right],$$

which implies Item 2 of Theorem 3.1 by choosing $\delta \in (0, \pi - h - T/2)$.

3.3.3 Proof of Item 3 of Theorem 3.1

Let $h \in (0, \pi)$, a control domain $\omega \subset (0, \pi) \times (0, h)$ and $T \in (0, \pi - h)$ fixed. In order to prove that the cost of the control blows out exponentially with respect to ε , we shall adapt the proof given in [71, Theorem 1]. Indeed, unlike in [71], φ is a vectorial function, the boundary conditions are not purely Dirichlet, and we cannot pick in (3.3) an initial value φ^T such that $\int_{\Omega} \varphi_2^T > 0$ because for all $u \in \mathcal{H}(\Omega)$ we have that $\int_{\Omega} u_1 = \int_{\Omega} u_2 = 0$.

To make the proof more clear, we split it with the help of two lemmas :

Lemma 3.17. *Let $h \in (0, \pi)$, $T \in (0, \pi - h)$ and $\delta \in (0, \pi - T - h)$. Let $\varphi^T \in \mathcal{H}(\Omega_2) \cap C^\infty(\overline{\Omega_2}) \setminus \{0\}$ such that $\text{supp}(\varphi^T) \subset (0, \pi) \times (\pi - \delta, \pi)$ and let φ^ε be the solution of (3.3) in Ω_2 and with initial value φ^T . Then, there is $c > 0$ such that, for all $\varepsilon \in (0, 1)$:*

$$\|\varphi^\varepsilon(0, \cdot)\|_{\mathbf{L}^2(\Omega)} \geq c. \quad (3.74)$$

Remark 3.18. For any $\delta > 0$ there are functions φ^T which satisfy the hypothesis of Lemma 3.17. For example, if $\chi_\delta \in C^\infty([0, \pi])$ is a function supported in $(\pi - \delta, \pi)$ of null mean, we can consider :

$$\varphi^T(x) := \left(\int_0^{x_1} \chi_\delta(s) ds \chi_\delta(x_2), -\chi_\delta(x_1) \int_0^{x_2} \chi_\delta(s) ds \right). \quad (3.75)$$

Consequently, neither Lemma 3.17 nor Lemma 3.19 below are empty results.

Proof. We prove (3.74) by contradiction. First of all, we remark that, thanks to forward uniqueness of (3.3) (which can be proved by contradiction with backward uniqueness and (3.53)), $\|\varphi^\varepsilon(0, \cdot)\|_{\mathbf{L}^2(\Omega)} > 0$ for all $\varepsilon \in (0, 1]$. In addition, because of Remark 3.14, the only problematic situation arises if there is $\varepsilon_k \rightarrow 0$ such that $\|\varphi^{\varepsilon_k}(0, \cdot)\|_{\mathbf{L}^2(\Omega)} \rightarrow 0$. Let us suppose the existence of such sequence ε_k and derive a contradiction.

We have that $\{\varphi_2^{\varepsilon} : \varepsilon \in (0, 1)\}$ is bounded in $L^2(Q)$ (see Remark 3.5); thus, we can suppose that $\varphi_2^{\varepsilon_k}$ converges weakly in $L^2(Q)$ to some function γ . Let us consider $\psi \in \mathcal{D}([0, T] \times \Omega_2)$. Using Remark 3.5 we have that :

$$0 = \iint_Q \varphi_2^{\varepsilon_k} (\partial_t \psi - \varepsilon_k \Delta \psi + \partial_{x_2} \psi) dx dt + \langle \varphi_2^{\varepsilon_k}(0, \cdot), \psi(0, \cdot) \rangle_{L^2(\Omega_2)} - \langle \varphi_2^T, \psi(T, \cdot) \rangle_{L^2(\Omega_2)}. \quad (3.76)$$

By taking the weak limit of $\varphi_2^{\varepsilon_k}$, the strong limit of $\varepsilon_k \Delta \psi$ (to 0) and the strong limit of $\varphi_2^{\varepsilon_k}(0, \cdot)$ (to 0), we get from (3.76) that :

$$0 = \iint_Q \gamma(\partial_t \psi + \partial_{x_2} \psi) dt dx - \langle \varphi_2^T, \psi(T, \cdot) \rangle_{L^2(\Omega_2)}. \quad (3.77)$$

We have that equation (3.77) is true in particular for $\psi(t, x) = \varphi_2^T(x_1, x_2 + T - t)$, which belongs to $\mathcal{D}([0, T] \times \Omega)$. Consequently, we get that $\|\varphi_2^T\|_{L^2(\Omega_2)} = 0$. Since $\varphi^T \in \mathcal{H}(\Omega_2) \cap \mathbf{C}^1(\overline{\Omega_2})$ this implies that $\varphi_1^T = 0$, contradicting the fact that $\varphi^T \neq 0$. \square

Lemma 3.19. *Let $h \in (0, \pi)$, $T \in (0, \pi - h)$ and $\delta \in (0, \pi - T - h)$. Then, if $\varphi^T \in \mathcal{H}(\Omega_2)$ such that $\text{supp}(\varphi^T) \subset (0, \pi) \times (\pi - \delta, \pi)$ there is $c > 0$ such that for all $\varepsilon \in (0, 1)$:*

$$\iint_{Q_\omega} |\varphi_2^\varepsilon|^2 dx dt \leq c e^{-c\varepsilon^{-1}}, \quad (3.78)$$

for φ^ε the solution of (3.3) in Ω_2 and for initial value φ^T .

For the following proof we fix a parameter $\tilde{\delta} > 0$ so that $T \in \left(0, \frac{\pi - h - \tilde{\delta}}{1 + \tilde{\delta}}\right)$.

Proof. First of all, we recall that φ_2^ε satisfies (3.7) for $z^T := \varphi_2^T$. Multiplying (3.7)₁ by

$$2e^{2(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + (\tilde{\delta})^2)(T - t))\varepsilon^{-1}} \varphi_2^\varepsilon,$$

and integrating in Ω_2 we deduce that for all $t \in [0, T]$ we have the following equality :

$$-\frac{d}{dt} \left(\int_{\Omega_2} \left| e^{(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + (\tilde{\delta})^2)(T - t))\varepsilon^{-1}} \varphi_2^\varepsilon \right|^2 dx \right) + 2\varepsilon \int_{\Omega_2} \left| \nabla \left(e^{(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + (\tilde{\delta})^2)(T - t))\varepsilon^{-1}} \varphi_2^\varepsilon \right) \right|^2 dx = 0. \quad (3.79)$$

Indeed, first, we have the equality :

$$\begin{aligned} -2 \int_{\Omega_2} e^{2(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + (\tilde{\delta})^2)(T - t))\varepsilon^{-1}} \varphi_2^\varepsilon \partial_t \varphi_2^\varepsilon dx &= -\frac{d}{dt} \left(\int_{\Omega_2} \left| e^{(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + (\tilde{\delta})^2)(T - t))\varepsilon^{-1}} \varphi_2^\varepsilon \right|^2 dx \right) \\ &+ 2(\tilde{\delta} + (\tilde{\delta})^2)\varepsilon^{-1} \int_{\Omega_2} e^{2(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + (\tilde{\delta})^2)(T - t))\varepsilon^{-1}} |\varphi_2^\varepsilon|^2 dx. \end{aligned} \quad (3.80)$$

Second, since φ_2^ε satisfies (3.7)₂ and (3.7)₃ we have that :

$$\begin{aligned} &-2\varepsilon \int_{\Omega_2} e^{2(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + (\tilde{\delta})^2)(T - t))\varepsilon^{-1}} \varphi_2^\varepsilon \Delta \varphi_2^\varepsilon dx \\ &= 2\varepsilon \int_{\Omega_2} e^{2(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + (\tilde{\delta})^2)(T - t))\varepsilon^{-1}} |\nabla \varphi_2^\varepsilon|^2 dx - 4\tilde{\delta} \int_{\Omega_2} e^{2(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + (\tilde{\delta})^2)(T - t))\varepsilon^{-1}} \partial_{x_2} \varphi_2^\varepsilon \varphi_2^\varepsilon dx \\ &= 2\varepsilon \int_{\Omega_2} \left| \nabla \left(e^{(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + (\tilde{\delta})^2)(T - t))\varepsilon^{-1}} \varphi_2^\varepsilon \right) \right|^2 dx - 2(\tilde{\delta})^2 \varepsilon^{-1} \int_{\Omega_2} e^{2(\tilde{\delta}(\pi - x_2) - (\tilde{\delta} + (\tilde{\delta})^2)(T - t))\varepsilon^{-1}} |\varphi_2^\varepsilon|^2 dx. \end{aligned} \quad (3.81)$$

Thirdly, since φ_2^ε satisfies (3.7)₃ we have the equality :

$$-2 \int_{\Omega_2} e^{2(\tilde{\delta}(\pi-x_2)-(\tilde{\delta}+(\tilde{\delta})^2)(T-t))\varepsilon^{-1}} \varphi_2^\varepsilon \partial_{x_2} \varphi_2^\varepsilon dx = -2\tilde{\delta}\varepsilon^{-1} \int_{\Omega_2} e^{2(\tilde{\delta}(\pi-x_2)-(\tilde{\delta}+(\tilde{\delta})^2)(T-t))\varepsilon^{-1}} |\varphi_2^\varepsilon|^2 dx. \quad (3.82)$$

Consequently, considering (3.7)₁ and (3.80)-(3.82) we obtain (3.79).

So, we obtain from (3.79) the following Agmon identity :

$$\begin{aligned} & \int_0^T \int_{\Omega_2} \left| e^{(\tilde{\delta}(\pi-x_2)-(\tilde{\delta}+(\tilde{\delta})^2)(T-t))\varepsilon^{-1}} \varphi_2^\varepsilon \right|^2 dx dt \\ +2\varepsilon \int_0^T \int_t^T \int_{\Omega_2} \left| \nabla \left(e^{(\tilde{\delta}(\pi-x_2)-(\tilde{\delta}+(\tilde{\delta})^2)(T-s))\varepsilon^{-1}} \varphi_2^\varepsilon(s, x) \right) \right|^2 dx ds dt &= T \int_{\Omega_2} \left| e^{\tilde{\delta}(\pi-x_2)\varepsilon^{-1}} \varphi_2^T \right|^2 dx. \end{aligned} \quad (3.83)$$

On the one hand, using that the support of φ^T is included in $(0, \pi) \times (\pi - \delta, \pi)$, we have that :

$$T \int_{\Omega_2} \left| e^{\tilde{\delta}(\pi-x_2)\varepsilon^{-1}} \varphi_2^T \right|^2 dx \leq \pi e^{2\tilde{\delta}\delta\varepsilon^{-1}} \int_{\Omega_2} |\varphi_2^T|^2 dx. \quad (3.84)$$

On the other hand, recalling that $\omega \subset (0, \pi) \times (0, h)$, we find that :

$$e^{2\tilde{\delta}(\pi-h-T-\tilde{\delta}T)\varepsilon^{-1}} \iint_{Q_\omega} |\varphi_2^\varepsilon|^2 dx dt \leq \int_0^T \int_{\Omega_2} \left| e^{(\tilde{\delta}(\pi-x_2)-(\tilde{\delta}+(\tilde{\delta})^2)(T-t))\varepsilon^{-1}} \varphi_2^\varepsilon \right|^2 dx dt. \quad (3.85)$$

Combining (3.83), (3.84) and (3.85), we have that :

$$\iint_{Q_\omega} |\varphi_2^\varepsilon|^2 dx dt \leq \pi \exp \left[\frac{2\tilde{\delta}(T + \tilde{\delta}T + h + \delta - \pi)}{\varepsilon} \right] \|\varphi_2^T\|_{L^2(\Omega)}^2; \quad (3.86)$$

that is, we get the decay of the L^2 norm of φ_2 in the control domain. \square

Remark 3.20. So far, combining (3.4), Lemma 3.17, Lemma 3.19 and Remark 3.18, we have that for a given $h \in (0, \pi)$, for all $\omega \subset (0, T) \times (0, h)$, and for all $T \in (0, \pi - h)$, there is $c > 0$ such that for all $\varepsilon \in (0, 1)$:

$$K_2(T, \varepsilon, \omega) \geq ce^{c\varepsilon^{-1}}.$$

Now we have the tools to end the proof of Item 3 of Theorem 3.1. For that proof, we denote, for $h \in (0, \pi)$ and $T \in (0, \pi - h)$ both fixed parameters :

$$\delta := (\pi - h - T)/3, \quad \omega_1 := (0, \pi) \times (0, h), \quad \omega_2 := (0, \pi) \times (0, h + \delta). \quad (3.87)$$

Proof. [End of the proof of Item 3 of Theorem 3.1] Let φ^ε be the solution of (3.3) in Ω_2 of initial value (3.75) and with δ as in (3.87). Thanks to (3.4), Lemma 3.17 and Lemma 3.19, the only thing left to prove is that $\iint_{Q_\omega} |\varphi_1^\varepsilon|^2 dxdt$ decays exponentially. For that purpose, we get an estimate of $\partial_{x_2}\varphi_2^\varepsilon$ in $L^2(Q_{\omega_1})$.

Let us fix $\theta(s)$ a positive regular scalar cut-off function supported in $(-\infty, h + \delta)$ and such that $\theta = 1$ in $(-\infty, h]$. We define :

$$\psi^\varepsilon(t, x) := \theta(x_2)\varphi_2^\varepsilon(t, x). \quad (3.88)$$

Recalling the support of φ^T (see (3.75)), we have that ψ^ε satisfies :

$$\begin{cases} -\partial_t \psi^\varepsilon - \varepsilon \Delta \psi^\varepsilon - \partial_{x_2} \psi^\varepsilon = \varepsilon \theta''(x_2) \varphi_2^\varepsilon - 2\varepsilon \partial_{x_2}(\theta'(x_2) \varphi_2^\varepsilon) - \theta'(x_2) \varphi_2^\varepsilon & \text{in } Q, \\ \psi^\varepsilon = 0 & \text{on } \Sigma_{b,t}, \\ \partial_{x_1} \psi^\varepsilon = 0 & \text{on } \Sigma_{l,r}, \\ \psi^\varepsilon(T, \cdot) = 0 & \text{on } \Omega_2. \end{cases} \quad (3.89)$$

In particular, if we multiply (3.89) by ψ^ε and we integrate by parts, it is not difficult to deduce that :

$$\iint_Q |\nabla \psi^\varepsilon|^2 dxdt \leq \frac{C}{\varepsilon} \iint_{Q_{\omega_2}} |\varphi_2^\varepsilon|^2 dxdt. \quad (3.90)$$

Using Lemma 3.19 for $h' = h + \delta$, $T' = T$ and $\delta' = \delta$ and using that for all $\tilde{\delta} > 0$ there is $C_{\tilde{\delta}} > 0$ such that $\varepsilon^{-1} \leq C_{\tilde{\delta}} e^{\tilde{\delta}\varepsilon^{-1}}$ in \mathbb{R}^+ , we get from (3.90) that :

$$\iint_{Q_{\omega_1}} |\partial_{x_2} \varphi_2^\varepsilon|^2 dxdt \leq C e^{-c\varepsilon^{-1}}.$$

In order to conclude, using that the divergence of φ^ε is null, we get that :

$$\iint_{Q_{\omega_1}} |\partial_{x_1} \varphi_1^\varepsilon|^2 dxdt \leq C e^{-c\varepsilon^{-1}}.$$

In addition, since φ_1^ε is null on $x_1 = 0$ (see (3.6)), we have, thanks to Poincaré inequality :

$$\iint_{Q_\omega} |\varphi_1^\varepsilon|^2 dxdt \leq \iint_{Q_{\omega_1}} |\varphi_1^\varepsilon|^2 dxdt \leq C e^{-c\varepsilon^{-1}}. \quad (3.91)$$

So, combining (3.4), (3.74), (3.78) and (3.91) we end the proof of Item 3 of Theorem 3.1. \square

3.4 The control problem (3.1) in $(0, \pi)^3$

In this section we first give some solutions of (3.5) in Ω_3 and then, we prove Item 4 of Theorem 3.1.

3.4.1 Brief study of the spectral decomposition of $\mathcal{H}((0, \pi)^3)$

In order to do the spectral decomposition, we consider the set of separated variables with null divergence, null normal trace, and a null component :

$$\begin{aligned} SV(\Omega_3) &:= SV_1(\Omega_3) \cup SV_2(\Omega_3) \cup SV_3(\Omega_3) := \\ &\left\{ \left(0, g_1(x_1) \int_0^{x_2} g_2(s) ds g_3(x_3), -g_1(x_1) g_2(x_2) \int_0^{x_3} g_3(s) ds \right) : g_1 \in L^2(0, \pi); g_2, g_3 \in L_0^2(0, \pi) \right\} \\ &\cup \left\{ \left(-\int_0^{x_1} g_1(s) ds g_2(x_2) g_3(x_3), 0, g_1(x_1) g_2(x_2) \int_0^{x_3} g_3(s) ds \right) : g_2 \in L^2(0, \pi); g_1, g_3 \in L_0^2(0, \pi) \right\} \\ &\cup \left\{ \left(\int_0^{x_1} g_1(s) ds g_2(x_2) g_3(x_3), -g_1(x_1) \int_0^{x_2} g_2(s) ds g_3(x_3), 0 \right) : g_3 \in L^2(0, \pi); g_1, g_2 \in L_0^2(0, \pi) \right\}. \end{aligned}$$

Remark 3.21. Unlike in $SV(\Omega_2)$, now the mean of one of the three auxiliary functions is not necessarily 0. This difference is crucial in explaining why the solutions of (3.3) behaves differently in Ω_3 . Indeed, this difference allows us to construct eigenfunctions whose associated eigenvalue does not explode with ε (see (3.93)), which is something that we cannot do in Ω_2 .

In order to look for eigenfunctions in $SV(\Omega_3)$, we remark that if g_1, g_2, g_3 are regular functions, the boundary conditions of (3.3) translates into :

$$g_1'(0) = g_1'(\pi) = g_2'(0) = g_2'(\pi) = \varepsilon g_3'(0) + g_3(0) = \varepsilon g_3'(\pi) + g_3(\pi) = 0. \quad (3.92)$$

Let us focus on $SV_3(\Omega_3)$. We can again consider for $m_1, m_2 \in \mathbb{N}_*$, $g_{1,m_1}(s) := \cos(m_1 s)$ and $g_{2,m_2}(s) := \cos(m_2 s)$. As for g_3 , we can look at the solutions of :

$$g'' + \frac{g'}{\varepsilon} + \left(\frac{\lambda}{\varepsilon} - m_1^2 - m_2^2 \right) g = 0,$$

an equation that can be obtained as in the proof of Proposition 3.10. Since the mean of g_3 is not necessarily 0 and since $\int_0^s g_3(s') ds'$ does not have to be an eigenfunction of

$-\varepsilon\Delta + \partial_{x_3}$ (because the third component of an element of $SV_3(\Omega_3)$ is null), we can consider small values of λ . In particular we get for any $m_1, m_2 \in \mathbb{N}_*$ the following eigenfunction :

$$\left(\frac{1}{m_1} \sin(m_1 x_1) \cos(m_2 x_2) e^{-\varepsilon^{-1} x_3}, -\frac{1}{m_2} \cos(m_1 x_1) \sin(m_2 x_2) e^{-\varepsilon^{-1} x_3}, 0 \right), \quad (3.93)$$

whose associated eigenvalues is :

$$\lambda_{m_1, m_2}^\varepsilon := \varepsilon(m_1^2 + m_2^2), \quad (3.94)$$

and whose associated pressure term is constant.

Remark 3.22. We can get for dimension 3 similar results to those of Remark 3.4, Proposition 3.6, Proposition 3.10 and Proposition 3.12, with the only difference that some eigenvalues do not explode with ε . In that sense, we can get similar results as Item 1 of Proposition 3.13, but we cannot get a general dissipation result as the one in Item 2 of Proposition 3.13.

3.4.2 Proof of Item 4 of Theorem 3.1

For this proof we consider the function :

$$u(x) := \frac{2e^{-\varepsilon^{-1} x_3}}{\pi \sqrt{\varepsilon} (1 - e^{-2\pi\varepsilon^{-1}})} (\sin(x_1) \cos(x_2), -\cos(x_1) \sin(x_2), 0),$$

which by (3.93) and (3.94) (we have taken $m_1 = m_2 = 1$) is a solution of (3.5) in Ω_3 for $\lambda = 2\varepsilon$ and a pressure which is constant. Moreover, we can easily check that $\|u\|_{\mathbf{L}^2(\Omega)} = 1$ and that $\varphi(t, x) := u(x) e^{-2\varepsilon(T-t)}$ is a solution of (3.3) in Ω_3 . In addition, since $\omega \subset (0, \pi)^2 \times (\pi - h, \pi)$ we can bound $e^{-\varepsilon^{-1} x_3}$ by $e^{-(\pi-h)\varepsilon^{-1}}$ and obtain for all $\delta > 0$ and $\varepsilon \in (0, 1)$ that :

$$\|u\|_{\mathbf{L}^\infty(\omega)} \leq C\varepsilon^{-1/2} e^{-(\pi-h)\varepsilon^{-1}} \leq C_\delta e^{-\varepsilon^{-1}(\pi-h-\delta)}. \quad (3.95)$$

Indeed, we have that $\varepsilon^{-1/2} \leq C_\delta e^{\delta\varepsilon^{-1}}$ for all $\delta > 0$ and $\varepsilon \in \mathbb{R}^+$.

On the one hand, we have that :

$$\int_{\Omega_3} |\varphi(0, x)|^2 dx = e^{-4\varepsilon T}. \quad (3.96)$$

On the other hand, using (3.95) and bounding $e^{-2\varepsilon(T-t)}$ by 1 we obtain that :

$$\begin{aligned} \iint_{Q_\omega} |\varphi|^2 dxdt &= \iint_{Q_\omega} |u(x)|^2 e^{-4\varepsilon(T-t)} dxdt \leq \|u\|_{\mathbf{L}^\infty(\omega)}^2 \iint_{Q_\omega} dxdt \\ &\leq TC_\delta \exp\left[\frac{2(-\pi + h + \delta)}{\varepsilon}\right]. \end{aligned} \quad (3.97)$$

So, if we combine (3.96) and (3.97), we have that for all $\delta \in (0, \pi - h)$ (remember (3.4) and that $\varepsilon \in (0, 1)$) :

$$[K_3(T, \varepsilon, \omega)]^2 \geq \frac{\int_{\Omega_3} |\varphi(0, x)|^2 dx}{\iint_{Q_\omega} |\varphi|^2 dxdt} \geq c_\delta \frac{e^{-4T}}{T} \exp\left[\frac{2(\pi - h - \delta)}{\varepsilon}\right];$$

that is, for T fixed the cost of the control explodes exponentially when $\varepsilon \rightarrow 0^+$.

Remark 3.23. A difference between dimension 3 and dimension 2 can be seen by comparing (3.72) and (3.96). Indeed, for T fixed (3.72) decays with ε , whereas (3.96) does not.

3.5 Further comments and open problems

- **About the boundary control.** The case where we have a control on the boundary of (3.1) (as proposed in [35]) instead of in the interior remains an open problem.

- **The spectral method in the transport-diffusion elemental equation.** Let us consider the control problem studied in [37, 71] :

$$\begin{cases} y_t + \mathcal{L}^\varepsilon y = 1_\omega f & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y^0 & \text{on } \Omega, \end{cases} \quad (3.98)$$

for $\mathcal{L}^\varepsilon := -\varepsilon\Delta + \partial_{x_d}$, for Ω any open regular domain. In order to prove the dissipation result for the adjoint system, we can use a spectral decomposition. In particular, if $\{w_m\}$ is the spectral basis when we diagonalize the Dirichlet Laplacian in Ω , of eigenvalues λ_m , for $\tilde{w}_m := w_m e^{(2\varepsilon)^{-1}x_d}$ we have that $\{\tilde{w}_m\}$ is a linearly independent total set such that $\mathcal{L}^\varepsilon \tilde{w}_m = (\varepsilon\lambda_m + (4\varepsilon)^{-1})\tilde{w}_m$, and such that each \tilde{w}_m satisfies Dirichlet boundary conditions. Consequently, we can replicate the procedure of Section 3.2.1 and get the dissipation result, with an accurate result (see (3.54)), which looking at the eigenvalues

clearly is the optimal one. As for the optimal dissipation rate, there is another proof in [37, Lemma 4] which uses a representation theorem and a comparison theorem.

• **The control problem (3.1) when we change the boundary conditions.** We can attempt for $\Omega = (0, \pi)^2$ and $d = 2$ to replicate this method to get similar results with a boundary condition of the type :

$$(-2\varepsilon Du \cdot n + \gamma n_2 u)_{\text{tg}} = 0, \quad (3.99)$$

for $\gamma \in \mathbb{R}$ (it is specially interesting the case $\gamma = 1/2$, when the operator is self-adjoint). Even if it actually produces eigenfunctions, it does it by taking into account the other two roots of (3.29); that is, $r = \pm m_1$. In particular, we get eigenfunctions with g_{2,m_2} of the type (see (3.31)) :

$$A \cos(m_2 \pi s) e^{(2\varepsilon)^{-1}s} + B \sin(m_2 \pi s) e^{(2\varepsilon)^{-1}s} + C e^{m_1 s} + D e^{-m_1 s},$$

and with m_2 not necessarily in \mathbb{N}_* (and not necessarily easy to calculate explicitly). The root $r = m_1$ does not depend on ε or, more astonishingly, on λ , because :

$$u(x_1, x_2) := (\sin(m_1 x_1) e^{m_1 x_2}, -\cos(m_1 x_1) e^{m_1 x_2}) = \nabla \left(\frac{-\cos(m_1 x_1) e^{m_1 x_2}}{m_1} \right).$$

Therefore, that term is an eigenfunction independent of m_2 or ε because of the help of the pressure term. A similar thing happens with the term of $D e^{-m_1 s}$. All this, of course, adds some difficulties in proving that they form a total set (if they actually do), difficulties which we have been unable to overcome. Therefore, getting the eigenfunctions of the Stokes system with a boundary condition of the type (3.99) (together with $u \cdot n = 0$) can be considered for future work.

As for the control problem (3.1) with Dirichlet boundary conditions, replicating the method of this paper has several problems. For instance, in dimension 2, using the notation of (3.8), we now have the boundary conditions :

$$g_1(0) = g_1(\pi) = g_2(0) = g_2(\pi) = 0.$$

So, the analogous choice for g_{1,m_1} would be $\sin(m_1 s)$ (see (3.22)₁); but, for m_1 odd it does not belong to $L_0^2(0, \pi)$. Moreover, for any $m_1 \in \mathbb{N}_*$, $\int_0^\pi \sin(m_1 s') ds' = m_1^{-1}(1 - \cos(m_1 \pi))$, which is not an eigenfunction of the Laplacian. Consequently, it is very likely that another method must be used to get the cost of the control.

• **The control problem (3.1) in other domains.** It is clear that working in $(0, \pi)^2$ or $(0, \pi)^3$ is really helpful, not only for the dissipation, but for the cost of the control as well. Indeed, we have used several times in Section 3.3 that the solutions of (3.3) are solutions of the heat equation. Consequently, it would be interesting to know what results can be obtained in other domains.

• **The cost of the control in dimension 3 if ω is near $x_3 = 0$.** It is an open question if $K_3(T, \varepsilon, \omega)$ (see (3.2)) also explodes with ε for large times T if there is $\bar{x} \in (0, \pi)^2 \times \{0\}$ and $\delta > 0$ such that $B(\bar{x}, \delta) \cap \{x_3 > 0\} \subset \omega$.

• **The optimal T_0 in which the cost of the control decays with ε .** The optimal T_0 in which the cost of the control decays with ε is an open problem (see Item 1 of Theorem 3.1). Thanks to Item 2 and Item 3 of Theorem 3.1 we can get a lower bound. Moreover, we could have got an upper bound by doing all the operations of the proof of the Carleman (Proposition 3.16) explicitly as in [37], but it would be far from optimal, so is not worth the effort. A more subtle technique would be to transform the problem in a fast-control problem as in [87]; but, on the one hand, when we multiply by a weight which depends on a spatial variable, the fact that the divergence is null is lost; and, on the other hand, the fact that fast controls have a cost of $e^{CT^{-1}}$ (without specifying the C) in the Stokes system is just a recent result (see [30] and [32]). In addition, this problem is probably more difficult than getting the optimal T_0 for (3.98), which is still an open problem, even in dimension 1.

• **Relation of the cost of control with a force of $d - 1$ components and with a force of d components.** An interesting question is to see if the fact that $K_1(T, \varepsilon, \omega)$ decays or explodes with ε is equivalent to the fact that $K_2(T, \varepsilon, \omega)$ decays or explodes with ε (see (3.2) for the definitions).

3.A Sketch of the proof of Proposition 3.16

We recall that $\Omega_2 := (0, \pi)^2$, ω is any open domain contained in Ω_2 , z is the solution of (3.7) (in $(0, \tilde{T})$ instead of in $(0, T)$) and the weights are given in (3.64). In addition,

in this section $Q := (0, \tilde{T}) \times \Omega_2$, $Q_\omega := (0, \tilde{T}) \times \omega$, $\Sigma_{l,r} := (0, \tilde{T}) \times \{0, \pi\} \times (0, \pi)$ and $\Sigma_{b,t} := (0, \tilde{T}) \times (0, \pi) \times \{0, \pi\}$.

We first remark that the auxiliary functions η^\pm satisfy :

$$\begin{aligned} \eta^+ &= \eta^-, \quad \partial_n \eta^+ + \partial_n \eta^- = 0, \quad |\nabla \eta^+| = |\nabla \eta^-| \text{ on } \{0, \pi\} \times (0, \pi), \\ \partial_n \eta^\pm &< 0 \text{ on } (0, \pi) \times \{0, \pi\}. \end{aligned} \quad (3.100)$$

In addition, there is $\delta > 0$ a constant that just depends on the control domain and fixed for this section such that :

$$|\nabla \eta^\pm| \geq \delta > 0 \text{ in } \overline{\Omega_2} \setminus \omega_0. \quad (3.101)$$

As for the weights, we have the usual bounds (see Remark 3.15) :

$$|\partial_{x_i} \alpha_\pm| = |\partial_{x_i} \xi_\pm| \leq C \lambda \xi_\pm, \quad |\partial_t \alpha_\pm| \leq C \tilde{T} \xi_\pm^2, \quad |\partial_{t^2}^2 \alpha_\pm| \leq C \xi_\pm^2 (1 + \tilde{T}^2 \xi_\pm). \quad (3.102)$$

In order to prove the Carleman inequality, as explained above, there is nothing original in the proof, since we do a mixture of the scheme of [71, Proposition 1] and [52, Lemma1]. In particular, we also consider the change of variables :

$$\psi^\pm := e^{-s\alpha_\pm} z. \quad (3.103)$$

We remark that :

$$\alpha_+ = \alpha_-, \quad \xi_+ = \xi_-, \quad \psi^+ = \psi^- \text{ on } \Sigma_{l,r}. \quad (3.104)$$

Moreover, ψ has Dirichlet boundary conditions on Σ_{bt} . As for $\Sigma_{l,r}$, since z has Neumann boundary conditions we have the equality :

$$\partial_n \psi^\pm = s \lambda \xi_\pm \partial_n \eta^\pm \psi^\pm. \quad (3.105)$$

In addition, combining (3.100)₁, (3.104) and (3.105), we have the equality :

$$|\nabla \psi^+| = |\nabla \psi^-| \text{ on } \Sigma_{l,r}. \quad (3.106)$$

We finally remark that :

$$L_1^\pm \psi^\pm + L_2^\pm \psi^\pm = L_3^\pm \psi^\pm, \quad (3.107)$$

for :

$$\begin{cases} L_1^\pm \psi^\pm := & -2\varepsilon\tau\lambda^2 |\nabla \eta^\pm|^2 \xi_\pm \psi^\pm - 2\varepsilon\tau\lambda \xi_\pm \nabla \eta^\pm \cdot \nabla \psi^\pm + \partial_{x_2} \psi^\pm + \psi_t^\pm, \\ L_2^\pm \psi^\pm := & \varepsilon\tau^2\lambda^2 |\nabla \eta^\pm|^2 \xi_\pm^2 \psi^\pm + \varepsilon\Delta \psi^\pm + \tau\partial_t(\alpha_\pm) \psi^\pm - \tau\lambda \partial_{x_2} \eta^\pm \xi_\pm \psi^\pm, \\ L_3^\pm \psi^\pm := & \varepsilon\tau\Delta \eta^\pm \xi_\pm \psi^\pm - \varepsilon\tau\lambda^2 |\nabla \eta^\pm|^2 \xi_\pm \psi^\pm. \end{cases} \quad (3.108)$$

As usual, we denote $(L_i^\pm \psi)_j$ the j -th term of $L_i^\pm \psi$ and calculate the product :

$$\langle L_1^\pm \psi^\pm, L_2^\pm \psi^\pm \rangle_{L^2(Q)}.$$

Most of the operations here are repetitions of [71, Proposition 1] with the exception of what we do with the boundary term when integrating by parts (and with the difference that z is a solution of the direct equation in [71]). Thus, we skip some operations.

To begin with, we have for $\lambda \geq C$ and $\tau \geq C\tilde{T}^2$ (see (3.101) for the definition of δ) the estimate :

$$\begin{aligned} \langle (L_1^\pm \psi^\pm)_1 + (L_1^\pm \psi^\pm)_2, (L_2^\pm \psi^\pm)_1 \rangle_{L^2(Q)} &= \varepsilon^2 \tau^3 \lambda^4 \iint_Q |\nabla \eta^\pm|^4 \xi_\pm^3 |\psi^\pm|^2 dx dt \\ &+ O\left(\varepsilon^2 s^2 \lambda^3 \iint_Q \xi_\pm^3 |\psi^\pm|^2 dx dt\right) - \varepsilon^2 \tau^3 \lambda^3 \iint_{\Sigma_{lr}} |\nabla \eta^\pm|^2 \xi_\pm^3 \partial_n \eta^\pm |\psi^\pm|^2 dx dt \\ &\geq \frac{3\delta^4}{4} \varepsilon^2 \tau^3 \lambda^4 \iint_Q \xi_\pm^3 |\psi^\pm|^2 dx dt - \delta^4 \varepsilon^2 \tau^3 \lambda^4 \iint_{Q_{\omega_0}} \xi_\pm^3 |\psi^\pm|^2 dx dt \\ &\quad - \varepsilon^2 \tau^3 \lambda^3 \iint_{\Sigma_{lr}} |\nabla \eta^\pm|^2 \xi_\pm^3 \partial_n \eta^\pm |\psi^\pm|^2 dx dt. \end{aligned} \quad (3.109)$$

Moreover, after an integration by parts, considering that $\psi^\pm = 0$ on Σ_{bt} and (3.102), we obtain for $\lambda \geq 1$ and $\tau \geq C(\tilde{T} + \tilde{T}^2)$ the bound :

$$\langle (L_1^\pm \psi^\pm)_3 + (L_1^\pm \psi^\pm)_4, (L_2^\pm \psi^\pm)_1 \rangle_{L^2(Q)} = O\left(\varepsilon \tau^2 \lambda^3 \iint_Q \xi_\pm^2 |\psi^\pm|^2 dx dt + \varepsilon \tilde{T} \tau^2 \lambda^2 \iint_Q \xi_\pm^3 |\psi^\pm|^2 dx dt\right). \quad (3.110)$$

Summing up, thanks to (3.109), (3.110), (3.100), (3.101) and (3.104), we have for $\lambda \geq C$ and $\tau \geq C(\tilde{T} + \tilde{T}^2)\varepsilon^{-1}$:

$$\sum_{i \in \{+, -\}} \langle L_1^i \psi^i, (L_2^i \psi^i)_1 \rangle_{L^2(Q)} \geq \sum_{i \in \{+, -\}} \frac{\delta^4}{2} \varepsilon^2 \tau^3 \lambda^4 \iint_Q \xi_i^3 |\psi^i|^2 dx dt - \delta^4 \varepsilon^2 \tau^3 \lambda^4 \iint_{Q_{\omega_0}} \xi_i^3 |\psi^i|^2 dx dt. \quad (3.111)$$

To continue with, we have that, integrating by parts, with Cauchy-Schwarz inequality and by (3.100) :

$$\begin{aligned} &\langle (L_1^\pm \psi^\pm)_1, (L_2^\pm \psi^\pm)_2 \rangle_{L^2(Q)} \\ &= 2\varepsilon^2 \tau \lambda^2 \iint_Q |\nabla \eta^\pm|^2 \xi_\pm |\nabla \psi^\pm|^2 dx dt - 2\varepsilon^2 \tau \lambda^2 \iint_{\Sigma_{lr}} |\nabla \eta^\pm|^2 \xi_\pm \partial_n \psi^\pm \psi^\pm dx dt \\ &\quad + O\left(\varepsilon^2 \tau^2 \lambda^4 \iint_Q \xi_\pm^2 |\psi^\pm|^2 dx dt + \varepsilon^2 \iint_Q (\tau \xi + \lambda^2) |\nabla \psi^\pm|^2 dx dt\right). \end{aligned} \quad (3.112)$$

Next, we have again by integration by parts that :

$$\begin{aligned} \langle (L_1^\pm \psi^\pm)_2, (L_2^\pm \psi^\pm)_2 \rangle_{L^2(Q)} &= -2\varepsilon^2 \tau \lambda \iint_{\Sigma_{bt}} \partial_n \eta^\pm \xi_\pm |\partial_n \psi^\pm|^2 dx_1 dt \\ &- 2\varepsilon^2 \tau \lambda \iint_{\Sigma_{lr}} (\partial_{tg} \eta^\pm \partial_{tg} \psi^\pm + \partial_n \eta^\pm \partial_n \psi^\pm) \xi_\pm \partial_n \psi^\pm dx_2 dt + O\left(\varepsilon^2 \tau \lambda \iint_Q \xi_\pm |\nabla \psi^\pm|^2 dx dt\right) \\ &+ 2\varepsilon^2 \tau \lambda^2 \iint_Q \xi_\pm |\nabla \eta^\pm \cdot \nabla \psi^\pm|^2 dx dt + \varepsilon^2 \tau \lambda \iint_Q \xi_\pm \nabla \eta^\pm \cdot \nabla |\nabla \psi^\pm|^2 dx dt. \end{aligned} \quad (3.113)$$

As for the term of the gradient, we have the equality :

$$\begin{aligned} \varepsilon^2 \tau \lambda \iint_Q \xi_\pm \nabla \eta^\pm \cdot \nabla |\nabla \psi^\pm|^2 dx dt &= \\ &- \varepsilon^2 \tau \lambda^2 \iint_Q |\nabla \eta^\pm|^2 \xi_\pm |\nabla \psi^\pm|^2 dx dy + O\left(\varepsilon^2 \tau \lambda \iint_Q \xi_\pm |\nabla \psi^\pm|^2 dx dt\right) \\ &+ \varepsilon^2 \tau \lambda \iint_{\Sigma_{bt}} \partial_n \eta^\pm \xi_\pm |\partial_n \psi^\pm|^2 dx_1 dt + \varepsilon^2 \tau \lambda \iint_{\Sigma_{lr}} \partial_n \eta^\pm \xi_\pm |\nabla \psi^\pm|^2 dx_2 dt. \end{aligned} \quad (3.114)$$

Next, we have that :

$$\langle (L_1^\pm \psi^\pm)_3, (L_2^\pm \psi^\pm)_2 \rangle_{L^2(Q)} = O\left(\varepsilon \iint_{\Sigma_{bt}} |\partial_n \psi^\pm|^2 dx_1 dt\right) + \varepsilon \iint_{\Sigma_{lr}} \partial_{x_2} \psi^\pm \partial_n \psi^\pm dx_2 dt. \quad (3.115)$$

To continue with, we have the equality :

$$\langle (L_1^\pm \psi^\pm)_4, (L_2^\pm \psi^\pm)_2 \rangle_{L^2(Q)} = \varepsilon \iint_{\Sigma_{lr}} \psi_t^\pm \partial_n \psi^\pm dx_2 dt. \quad (3.116)$$

So, before adding up, because of (3.100)₂, we have for $\lambda \geq C$ and $\tau \geq C(\tilde{T} + \tilde{T}^2)\varepsilon^{-1}$ the bound :

$$\begin{aligned} -2\varepsilon^2 \tau \lambda \iint_{\Sigma_{bt}} \partial_n \eta^\pm \xi_\pm |\partial_n \psi^\pm|^2 dx_1 dt + \varepsilon^2 \tau \lambda \iint_{\Sigma_{bt}} \partial_n \eta^\pm \xi_\pm |\partial_n \psi^\pm|^2 dx_1 dt \\ + O\left(\varepsilon \iint_{\Sigma_{bt}} |\partial_n \psi^\pm|^2 dx_1 dt\right) \geq 0. \end{aligned} \quad (3.117)$$

Summing up, if we consider (3.111)-(3.117), (3.100), (3.101) and (3.104), we get that :

$$\begin{aligned} \sum_{i \in \{+, -\}} \langle L_1^i \psi^i, (L_2^i \psi^i)_2 \rangle_{L^2(Q)} &\geq \sum_{i \in \{+, -\}} O\left(\varepsilon^2 \tau^2 \lambda^4 \iint_Q \xi_i^2 |\psi^i|^2 dx dt\right) \\ &+ \sum_{i \in \{+, -\}} \frac{\delta^2}{2} \varepsilon^2 \tau \lambda^2 \iint_Q \xi_i |\nabla \psi^i|^2 dx dt - \delta^2 \varepsilon^2 \tau \lambda^2 \iint_{Q_{\omega_0}} \xi_i |\nabla \psi^i|^2 dx dt. \end{aligned} \quad (3.118)$$

As for the rest of the terms, it is easy to verify for $\lambda \geq 1$ and $\tau \geq C(\tilde{T} + \tilde{T}^2)\varepsilon^{-1}$ the bound :

$$\sum_{i \in \{+, -\}} \langle L_1^i \psi^i, (L_2^i \psi^i)_3 + (L_2^i \psi^i)_4 \rangle_{L^2(Q)} = \sum_{i \in \{+, -\}} O\left(\varepsilon^2 \tau^3 \lambda^3 \iint_Q \xi_i^3 |\psi^i|^2 dxdt\right). \quad (3.119)$$

So, if we add (3.112), (3.118) and (3.119), we get after absorptions for $\lambda \geq C$ and $\tau \geq C(\tilde{T} + \tilde{T}^2)\varepsilon^{-1}$ that :

$$\begin{aligned} 2 \sum_{i \in \{+, -\}} \langle L_1^i \psi^i, L_2^i \psi^i \rangle_{L^2(Q)} + 2\delta^4 \varepsilon^2 \tau^3 \lambda^4 \iint_{Q_{\omega_0}} \xi_i^3 |\psi^i|^2 dxdt + 2\delta^2 \varepsilon^2 \tau \lambda^2 \iint_{Q_{\omega_0}} \xi_i |\nabla \psi^i|^2 dxdt \\ \geq \sum_{i \in \{+, -\}} \frac{\delta^4}{2} \varepsilon^2 \tau^3 \lambda^4 \iint_Q \xi_i^3 |\psi^i|^2 dxdt + \frac{\delta^2}{2} \varepsilon^2 \tau \lambda^2 \iint_Q \xi_i |\nabla \psi^i|^2 dxdt. \end{aligned}$$

So, considering (3.107) and (3.108)₃ we find after some easy absorptions for $\lambda \geq C$ and $\tau \geq C(\tilde{T} + \tilde{T}^2)\varepsilon^{-1}$ the estimate :

$$\begin{aligned} \sum_{i \in \{+, -\}} \|L_1^i \psi^i\|_{L^2(Q)}^2 + \|L_1^i \psi^i\|_{L^2(Q)}^2 + \frac{\delta^4}{4} \varepsilon^2 \tau^3 \lambda^4 \iint_Q \xi_i^3 |\psi^i|^2 dxdt + \frac{\delta^2}{2} \varepsilon^2 \tau \lambda^2 \iint_Q \xi_i |\nabla \psi^i|^2 dxdt \\ \leq \sum_{i \in \{+, -\}} 2\delta^4 \varepsilon^2 \tau^3 \lambda^4 \iint_{Q_{\omega_0}} \xi_i^3 |\psi^i|^2 dxdt + 2\delta^2 \varepsilon^2 \tau \lambda^2 \iint_{Q_{\omega_0}} \xi_i |\nabla \psi^i|^2 dxdt. \end{aligned}$$

Moreover, if we consider in (3.108) the equations of $\Delta \psi^\pm$ and $\partial_t \psi^\pm$, we have, after usual absorptions, for $\lambda \geq C$ and $\tau \geq C(\tilde{T} + \tilde{T}^2)\varepsilon^{-1}$, the inequality :

$$\begin{aligned} \sum_{i \in \{+, -\}} \tau^{-1} \iint_Q \xi^{-1} (\varepsilon^{-2} |\psi_t^i|^2 + |\Delta \psi^i|^2) dxdt \\ + \frac{\delta^4}{16} \tau^3 \lambda^4 \iint_Q \xi_i^3 |\psi^i|^2 dxdt + \frac{\delta^2}{8} \tau \lambda^2 \iint_Q \xi_i |\nabla \psi^i|^2 dxdt \\ \leq \sum_{i \in \{+, -\}} 2\delta^4 \tau^3 \lambda^4 \iint_{Q_{\omega_0}} \xi_i^3 |\psi^i|^2 dxdt + 2\delta^2 \tau \lambda^2 \iint_{Q_{\omega_0}} \xi_i |\nabla \psi^i|^2 dxdt. \quad (3.120) \end{aligned}$$

From (3.120) it is well-known how to obtain (3.65) (see, for instance, [57] and [52]).

Chapitre 4

Null controllability of the heat equation in pseudo-cylinders by an internal control

In this chapter we prove the null controllability of the heat equation in domains with a cylindrical part and limited by a Lipschitz graph. The proof consists mainly on getting a Carleman estimate which presents the usual absorption properties. The main difficulty we face is the loss of existence of the usual weighted function in C^2 smooth domains. In order to deal with this, we use its cylindrical structure and approximate the system by the same system stated in regular domains. Finally, we show some applications like the controllability of the semi-linear heat equation in those domains. This chapter is presented in [13].

4.1 Introduction

The null controllability of the heat equation has been an interesting research topic in the last 40 years. Regarding this topic, in this paper we focus on domains; that is, on bounded connected non-empty open sets of \mathbb{R}^d . There are three main approaches in the literature :

- The first one uses the observability estimates of the abstract wave equation to prove the null controllability of the abstract heat equation. This method dates back to

1973 (see [96]) and recently, it has been used to get accurate bounds on the cost of the null controllability of the heat equation (see, for instance, [91], [92], [104] and [46]). As for other parabolic equations, it has also been used to get estimate of the Stokes equation (see, for instance, [30]).

- The second one is based on spectral inequalities and on properties of analytic functions. The first time this approach is used for the heat equation was in 1995 in [79], where the authors proved the controllability in C^∞ domains. More recently, the null controllability of the heat equation has been studied with these techniques when the control is localized in an arbitrary measurable set of positive measure (see, for instance, [105], [107] and [6]), which is one of the advantages of this method. In addition, these results have recently been extended to Lipschitz domains which are locally star-shaped (see [7]). As for other parabolic equations, this method has been used recently to study the controllability of the Stokes equation (see [32]) and higher order parabolic equations (see, for instance, [47] and [48]).
- The third one is based on multiplying the adjoint function by an exponential weight and using the PDE satisfied by the new function. Indeed, Fursikov and Imanuvilov proved with this method in 1996 the controllability of the heat equation in C^2 domains (see [57]) when the control domain is an open set. This method is really flexible and it is the base to study the controllability of countless parabolic differential equations, both linear and non-linear (see, for instance, [57]). As a continuation of the work made in [57], Carleman estimates of the heat equation have been established when the system is not regular, which proves the null controllability of some related linear and non-linear systems. For instance, in [74] and [53] the authors consider non-regular source terms. Moreover, in [52] and [73] the authors study some boundary conditions which imply that the solution is not in $L^2(0, T; H^2(\Omega))$. Additional examples are given in [45], [10] and [78], where the authors analyse parabolic equations in which the diffusion coefficient is not continuous. Finally, it is well-known that this method can be easily generalized to cubes or, more generally, to any cartesian product of C^2 domains (see, for instance, [64]).

The main objective of this paper is to extend the method of Fursikov and Imanuvilov to additional domains which are not C^2 when there is a lack of regularity caused by the domain and the control acts on an internal subdomain. In particular, we aim to get a Carleman inequality with a source term. It is a relevant problem because, unlike in [7],

this method allows us to consider lower order terms with coefficients that just belong to L^∞ , and hence treat non-linearities. We recall that the authors in [57] prove a Carleman inequality with the help of an auxiliary function η which satisfies :

$$\eta \in C^2(\overline{\Omega}), \quad \eta = 0 \text{ on } \partial\Omega, \quad \eta > 0 \text{ in } \Omega, \quad \inf_{\overline{\Omega} \setminus \omega} |\nabla \eta| > 0, \quad (4.1)$$

for ω the control domain. Since in a Lipschitz domain the heat equation has a unique energy solution, the existence of some Carleman estimate seems reasonable. However, two difficulties arise :

- The first and main difficulty is that the construction of the function η given in [57] does not work when Ω is not C^2 . Indeed, if Ω has corners and if ω is compactly included in Ω , the three conditions in (4.1) are incompatible. The only solution known so far for other domains (like cylinders) is to construct manually an auxiliary function which satisfies the following assumptions (which are verified by any η satisfying (4.1)) :

$$\eta \in C^2(\overline{\Omega}), \quad \partial_n \eta \leq 0 \text{ on } \partial\Omega, \quad \inf_{\overline{\Omega} \setminus \omega} |\nabla \eta| > 0, \quad (4.2)$$

for n the outward unit normal vector on $\partial\Omega$. In the case of cylinders, the construction of η is done for instance as a sum of functions of different variables.

- The second difficulty is that [57] uses that the Laplacian behaves well in Ω in the sense that $\mathcal{D}_\Omega(\Delta) \subset H^2(\Omega)$ continuously, for

$$\mathcal{D}_\Omega(\Delta) := \{w \in H_0^1(\Omega) : \Delta w \in L^2(\Omega)\},$$

omitting the Ω if it is clear. However, when Ω is Lipschitz this is not always true, since in a general Lipschitz domain the most we can ensure is that $\mathcal{D}(\Delta) \subset H^{3/2}(\Omega)$ continuously. Indeed, the fact that $\mathcal{D}(\Delta) \subset H^{3/2}(\Omega)$ continuously is proved in [75], and for all $\varepsilon > 0$ a counter-example of a domain Ω_ε for which there is $w \in H_0^1(\Omega_\varepsilon)$ such that $-\Delta w \in C^\infty(\overline{\Omega_\varepsilon})$ but $w \notin H^{3/2+\varepsilon}(\Omega)$ is given for instance in [66]. In particular, if u is a solution of the backwards heat equation with Dirichlet boundary conditions, initial value in $H_0^1(\Omega)$ and source term in $L^2((0, T) \times \Omega)$, the most that we can expect is that $u \in H^1(0, T; L^2(\Omega)) \cap L^2(0, T; \mathcal{D}(\Delta))$. However, while doing the Carleman in [57] the authors do some calculations with the second order derivatives, so they need that the regular solutions of the heat equation belong to $L^2(0, T; H^2(\Omega))$. One of the contributions of this paper on the Fursikov-Imanuvilov method is reducing the difficulty that the L^2 -norm of the Laplacian is not equivalent

to the H^2 -norm to the existence of a proper auxiliary function. We do it in Section 4.2 by approximating the domain with regular subdomains.

As a first step to prove the null controllability of the heat equation by an internal control in any Lipschitz domain, in this paper we prove the null controllability in pseudo-cylinders. By pseudo-cylinder we mean the following :

Definition 4.1. We say that $\Omega \subset \mathbb{R}^{d+1}$ ($d \geq 1$) is a *pseudo-cylinder* if there are $\mathfrak{B} \subset \mathbb{R}^d$ a C^2 domain, U the composition of a rotation and a translation and $H : \overline{\mathfrak{B}} \mapsto \mathbb{R}^+$ a Lipschitz function satisfying $\inf_{\mathfrak{B}} H > 0$ such that :

$$\Omega = U(\{(x, z) : x \in \mathfrak{B}, z \in (0, H(x))\}). \quad (4.3)$$

Moreover, we denote the cylindrical part by :

$$\mathfrak{C} := U\left(\mathfrak{B} \times \left(0, \inf_{\mathfrak{B}} H\right)\right). \quad (4.4)$$

If $U = I$ (the identity endomorphism), we say that the pseudo-cylinder is *canonically oriented*. In that case, we have :

$$\mathfrak{C} = \mathfrak{B} \times \left(0, \inf_{\mathfrak{B}} H\right). \quad (4.5)$$

Remark 4.2. If Ω is a pseudo-cylinder, we can split $\partial\Omega$ into three parts :

$$U(\mathfrak{B} \times \{0\}), \quad \mathfrak{L} := U(\{(x, z) : x \in \partial\mathfrak{B}, z \in (0, H(x))\}), \quad \mathfrak{T} := U(\{(x, H(x)) : x \in \mathfrak{B}\}).$$

If Ω is canonically oriented, those parts are given by :

$$\mathfrak{B} \times \{0\}, \quad \mathfrak{L} = \{(x, z) : x \in \partial\mathfrak{B}, z \in (0, H(x))\}, \quad \mathfrak{T} = \{(x, H(x)) : x \in \mathfrak{B}\}.$$

Example 4.3. See Figure 4.1 for a canonically-oriented pseudo-cylinder in \mathbb{R}^2 .

Remark 4.4. The pseudo-cylinders are relevant domains, for example, in fluid mechanics (see, for instance, [93] and [69]).

In this paper we solve the previously presented difficulties for pseudo-cylinders and get some Carleman estimates (see Proposition 4.28 and Proposition 4.32 below) which present the usual absorption properties. However, as we explain in Comment 5 of Section 4.3.3, our method does not work in every Lipschitz domain, so the general case remains an interesting open problem.

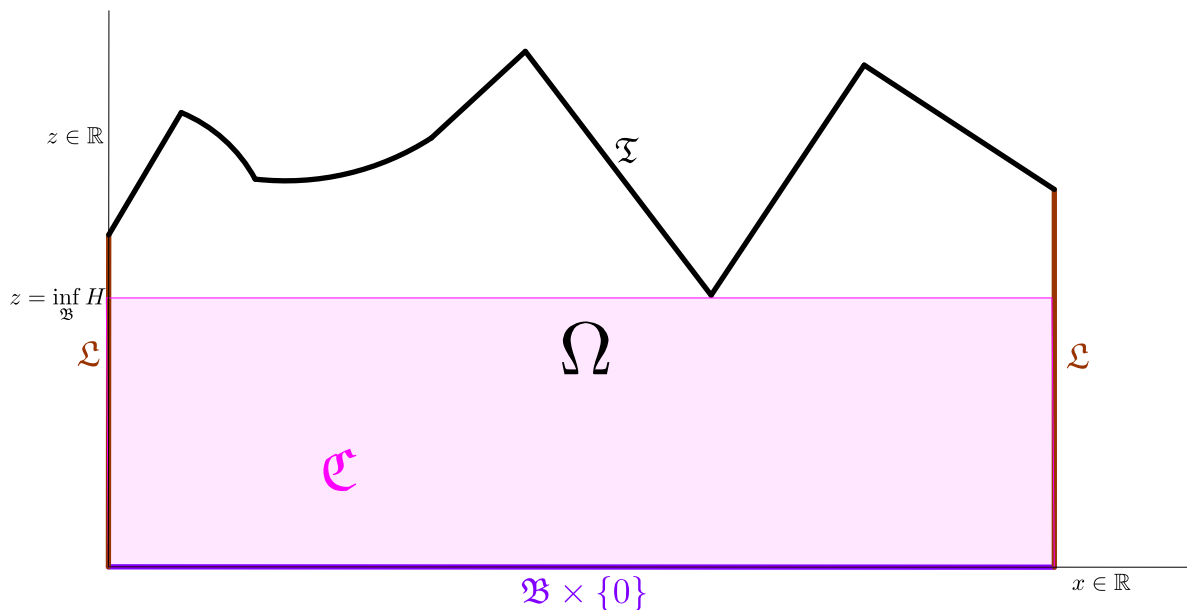


FIGURE 4.1 – A canonically oriented pseudo-cylinder.

Before presenting the main result of this paper we introduce the following notation :

Definition 4.5. Let Ω be any domain and let $\omega \subset \Omega$ be a subdomain. We define as usual $Q := (0, T) \times \Omega$, $Q_\omega := (0, T) \times \omega$, $\Sigma := (0, T) \times \partial\Omega$ and n the outward unit normal vector on $\partial\Omega$. Similarly, if the domain is denoted by $\tilde{\Omega}$, we define $\tilde{Q} := (0, T) \times \tilde{\Omega}$, $\tilde{\Sigma} := (0, T) \times \partial\tilde{\Omega}$ and \tilde{n} as the outward unit normal vector on $\partial\tilde{\Omega}$. Moreover, we denote by $\omega \subset\subset \Omega$ if ω is compactly included in Ω . Finally, in order to shorten the notation we omit the “dt”, “dx” and “dz” when we are integrating and the integrated variables can be easily deduced by looking at the integration domain.

Now, we may state the main result of this paper :

Theorem 4.6. *Let Ω be a pseudo-cylinder and $\omega \subset \Omega$ be a subdomain. Then, there is $C > 0$ such that if $T > 0$, $A \in (L^\infty(Q))^{d+1}$, $a \in L^\infty(Q)$, and $y^0 \in L^2(\Omega)$, there is a control $v \in L^2(Q_\omega)$ such that the solution of the system :*

$$\begin{cases} y_t - \Delta y + A \cdot \nabla y + ay = v1_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y^0 & \text{on } \Omega, \end{cases}$$

satisfies $y(T, \cdot) = 0$, and such that the control satisfies the estimate :

$$\|v\|_{L^2(Q_\omega)} \leq C e^{CK(T,a,A)} \|y^0\|_{L^2(\Omega)},$$

for :

$$K(T, a, A) := 1 + T^{-1} + T\|a\|_{L^\infty(Q)} + \|a\|_{L^\infty(Q)}^{2/3} + (1 + T)\|A\|_{(L^\infty(Q))^{d+1}}^2.$$

Remark 4.7. Since the Laplacian is invariant under rotations (and translations), it suffices to prove Theorem 4.6 when Ω is canonically oriented. Moreover, we remark that if we prove Theorem 4.6 for some control domain $\tilde{\omega}$, the conclusions of Theorem 4.6 are true for all $\omega \supset \tilde{\omega}$, as it suffices to consider $v = \tilde{v}1_{\tilde{\omega}}$, for \tilde{v} the control supported in $\tilde{\omega}$.

Using Theorem 4.6 we can get positive results about the controllability of the semi-linear heat equation when the non-linearity depends only on y (see Comment 6 on Section 4.3.3 for a remark about non-linearities depending on ∇y) :

$$\begin{cases} y_t - \Delta y + f(y) = v1_\omega & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0, \cdot) = y^0 & \text{on } \Omega. \end{cases} \quad (4.6)$$

Indeed, under weak non-linearities we have controllability to trajectories and approximate controllability :

Corollary 4.8. *Let Ω be a pseudo-cylinder, $\omega \subset \Omega$ be a subdomain, $T > 0$ and $y^* \in C([0, T]; L^2(\Omega))$ be a solution of (4.6) corresponding to some data $(y^0)^* \in L^2(\Omega)$ and $v^* \in L^\infty(Q_\omega)$. Assume that $f : \mathbb{R} \mapsto \mathbb{R}$ is locally Lipschitz and satisfies almost everywhere in \mathbb{R} :*

$$|f'(s)| \leq C(1 + |s|^{1+4/d}), \quad (4.7)$$

and :

$$\lim_{|s| \rightarrow \infty} \frac{f(s)}{|s| \log^{3/2}(1 + |s|)} = 0. \quad (4.8)$$

Then, for all $y^0 \in L^2(\Omega)$ there is $v \in L^\infty(Q_\omega)$ such that the solution of (4.6) belongs to $C([0, T]; L^2(\Omega))$ and satisfies $y(T, \cdot) = y^(T, \cdot)$.*

Corollary 4.9. *Let Ω be a pseudo-cylinder, $\omega \subset \Omega$ be a domain, $T > 0$ and let us assume that (4.6) has a solution $y^* \in C([0, T]; L^2(\Omega))$ corresponding to some initial state in $(y^0)^* \in L^2(\Omega)$ and to some control function $v^* \in L^\infty(Q_\omega)$. Assume that f is locally Lipschitz and satisfies (4.7) almost everywhere in \mathbb{R} and (4.8). Then, for all $y^1 \in L^2(\Omega)$ and all $\varepsilon > 0$ there is $v \in L^\infty(Q_\omega)$ such that (4.6) has a solution in $C([0, T]; L^2(\Omega))$ satisfying :*

$$\|y(T, \cdot) - y^1\| \leq \varepsilon.$$

In order to see that Theorem 4.6 implies Corollaries 4.8 and 4.9 it suffices to follow step by step the proof given in [56] for the case in which Ω is a C^2 domain. Indeed, in order to see that results on Sobolev spaces used in [56] remain true when Ω is just Lipschitz, the reader can consult for instance [2], [12] and [99]. In addition, we can find in [56] some facts about controllability of the semi-linear heat equation which remain true in pseudo-cylinders, like that (4.6) is not approximately controllable for $f(s) = \int_0^s \log^p(1 + \sigma)d\sigma$, $p > 2$ and $\omega \subset\subset \Omega$.

Remark 4.10. The results presented in this paper differ from [7] because of the following reasons :

- There are some pseudo-cylinders which are not locally star-shaped (see Appendix 4.B); thus, this is the first time that the null controllability of the heat equation is proved in these domains. In addition, there are locally star-shaped domains which are not pseudo-cylinders (for example, a disk or a pentagon). Of course, there also are pseudo-cylinders which are locally star-shaped (for instance, the rectangle).
- The coefficient of the lower order terms in Theorem 4.6 can depend on t and x , whereas in [7] the coefficients only depend on t .
- Thanks to the fact that we can establish controllability results for the heat equation with coefficients that depend on t and x , we can prove controllability results for the semi-linear heat equation, which is an open problem of [7].
- In [7] the control is localized in an arbitrary measurable subset of strictly positive measure, whereas in this paper the control is localized in open subsets.

In this document we consider the following weights defined in Q :

$$\alpha_\eta := \frac{e^{\lambda \frac{2m+1}{2m}k} - e^{\lambda(k+\eta)}}{t^m(T-t)^m}, \quad \xi_\eta := \frac{e^{\lambda(k+\eta)}}{t^m(T-t)^m}, \quad (4.9)$$

for $m \geq 1$ a real constant, η some auxiliary $C^2(\overline{\Omega})$ function (which, unlike it is usually done in the literature, may take negative values) and $k > 2(m+1)\|\eta\|_{L^\infty(\Omega)}$ a real constant, all of them to be fixed later. In (4.9) the subindex η might be omitted if it is clear. Moreover, if we denote some function by $\tilde{\eta}$, we use the notation $\tilde{\alpha}$ and $\tilde{\xi}$ to refer to $\alpha_{\tilde{\eta}}$ and $\xi_{\tilde{\eta}}$, respectively. We recall that we have the following estimates for the weights given in (4.9), for some $C > 0$ that is universal (independent of any variable like Ω , η , m , k , s , λ , ...):

$$|\xi_t| + |\alpha_t| \leq CmT\xi^{(m+1)/m}, \quad |\alpha_{tt}| \leq C(m\xi^{(m+1)/m} + m(m+1)T^2\xi^{(m+2)/m}). \quad (4.10)$$

Weights similar to (4.9) were first introduced in [57]. However, our version is more similar to the one presented in [54], though we have done slight modifications to also consider the situations in which η is not a positive function.

Remark 4.11. We need $k > 2(m+1)\|\eta\|_{L^\infty(\Omega)}$ to assure that the numerator of α is strictly positive, so that we can integrate by parts in the time variable with a null boundary term when proving the Carleman inequality. Moreover, we also need it to prove (4.10).

Remark 4.12. These weights, stated as in (4.9), have the advantage that can be compared easily when using two different auxiliary functions. In particular, if we have η_1 and η_2 defined in a canonically oriented pseudo-cylinder, for any $m \geq 1$, $k > 2(m+1) \max\{\|\eta_1\|_{L^\infty(\Omega)}, \|\eta_2\|_{L^\infty(\Omega)}\}$, $t \in (0, T)$ and $(x, z) \in \bar{\Omega}$, we find :

$$\xi_{\eta_1}(t, x, z) \leq \xi_{\eta_2}(t, x, z) \text{ if and only if } \eta_1(x, z) \leq \eta_2(x, z),$$

and :

$$-\alpha_{\eta_1}(t, x, z) \leq -\alpha_{\eta_2}(t, x, z) \text{ if and only if } \eta_1(x, z) \leq \eta_2(x, z).$$

Let us explain briefly how to solve the two problems presented at the beginning of this section on pseudo-cylinders :

- Concerning the construction of the auxiliary function, which is the main difficulty, we consider different approaches depending on the control domain. If $\omega \subset \Omega$ we use an approach consisting of expressing η as a sum of two different variables, whereas if $\omega \subset \Omega \setminus \mathfrak{C}$, we “transmit” the estimate from one control domain to another with a second auxiliary function. This is done in Section 4.3.
- As for dealing with the fact that $\mathcal{D}(\Omega) \not\subset H^2(\Omega)$, we just have to approximate Ω by C^2 domains which are compatible with η (once η is constructed). This is explained in Section 4.2 for Lipschitz domains.

The rest of the paper is organised as follows : in Section 4.2 we explain how to deal with Ω being just Lipschitz, in Section 4.3 we get the Carleman inequalities, in Appendix 4.A we prove the compatibility of the proposed auxiliary function with the domain and in Appendix 4.B we show that there are pseudo-cylinders which are not locally star-shaped.

4.2 A Carleman inequality for some Lipschitz domains

Let Ω be a Lipschitz domain. We define :

$$\mathcal{W}_\Omega := L^2(0, T; H_0^1(\Omega)) \cap H^1(0, T; H^{-1}(\Omega)),$$

omitting the subindex Ω if it is clear. In addition, we define :

Definition 4.13. Let Ω be a Lipschitz domain, let $u^0 \in L^2(\Omega)$ and let $g \in L^2(Q)$. Then, we denote by :

$$\mathbb{S}(\Omega, u^0, g)(t, x),$$

the only solution in \mathcal{W} of :

$$\begin{cases} u_t - \Delta u = g & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(0, \cdot) = u^0 & \text{on } \Omega. \end{cases} \quad (4.11)$$

Finally, we define the convergence of domains in the following way :

Definition 4.14. Let $\varepsilon_0 > 0$, Ω be a domain and $(\Omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ be domains such that $\Omega_\varepsilon \subset \Omega$ for all $\varepsilon \in (0, \varepsilon_0)$. We say that $\Omega_\varepsilon \rightarrow_d \Omega$ if :

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{\Omega \setminus \Omega_\varepsilon} d(\cdot, \partial\Omega) = 0. \quad (4.12)$$

Remark 4.15. If $\Omega_\varepsilon \rightarrow_d \Omega$, then Ω_ε converges to Ω in measure ; that is, $\|1_{\Omega \setminus \Omega_\varepsilon}\|_{L^1(\mathbb{R}^d)} \rightarrow 0$, because of the Dominated Convergence Theorem.

We have the following technical result :

Lemma 4.16. *Let Ω be a Lipschitz domain, $\Omega_\varepsilon \subset \Omega$ be some Lipschitz domains such that :*

$$\Omega_\varepsilon \rightarrow_d \Omega,$$

$u^0 \in L^2(\Omega)$ and $g \in L^2(Q)$. Then,

$$\mathbb{S}(\Omega_\varepsilon, u^0 1_{\Omega_\varepsilon}, g 1_{\Omega_\varepsilon}) 1_{\Omega_\varepsilon} \rightarrow_{L^2(Q)} \mathbb{S}(\Omega, u^0, g). \quad (4.13)$$

As far as we know, the proof of Lemma 4.16 is not available in the literature, so we give a proof, whose originality we do not claim. For the proof of Lemma 4.16 we need the following interpolation result, proved for instance in [84] and [82] :

Lemma 4.17. *Let Ω be a Lipschitz domain. Then,*

1. *The injection $\mathcal{W} \subset L^2(Q)$ is compact.*
2. *The injection $\mathcal{W} \subset C([0, T]; L^2(\Omega))$ is continuous.*
3. *For all $u, w \in \mathcal{W}$, we have $\langle u(t, \cdot), w(t, \cdot) \rangle_{L^2(\Omega)} \in W^{1,1}(0, T)$ and :*

$$\partial_t \left(\langle u(t, \cdot), w(t, \cdot) \rangle_{L^2(\Omega)} \right) = \langle \partial_t u(t, \cdot), w(t, \cdot) \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} + \langle \partial_t w(t, \cdot), u(t, \cdot) \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)}.$$

Proof. [Proof of Lemma 4.16] In order to prove Lemma 4.16 we recall that for any Lipschitz domain $\tilde{\Omega}$, any $\tilde{u}^0 \in L^2(\tilde{\Omega})$ and any $\tilde{g} \in L^2(Q)$, $\mathbb{S}(\tilde{\Omega}, \tilde{u}^0, \tilde{g})$ is characterized as the only element in $\mathcal{W}_{\tilde{\Omega}}$ such that :

$$\begin{cases} \int_0^T \left\langle \partial_t \mathbb{S}(\tilde{\Omega}, \tilde{u}^0, \tilde{g}), \phi \right\rangle_{H^{-1}(\tilde{\Omega}), H_0^1(\tilde{\Omega})} + \iint_{\tilde{Q}} \nabla \mathbb{S}(\tilde{\Omega}, \tilde{u}^0, \tilde{g}) \cdot \nabla \phi = \iint_{\tilde{Q}} g \phi & \forall \phi \in \mathcal{D}(\tilde{Q}), \\ \mathbb{S}(\tilde{\Omega}, \tilde{u}^0, \tilde{g})(0, \cdot) = \tilde{u}^0. \end{cases} \quad (4.14)$$

We define :

$$\mathbb{S}_0 := \mathbb{S}(\Omega, u^0, g), \quad \mathbb{S}_\varepsilon := \mathbb{S}(\Omega_\varepsilon, u^0 1_{\Omega_\varepsilon}, g 1_{\Omega_\varepsilon}) 1_{\Omega_\varepsilon} \quad \text{and} \quad Q_\varepsilon := (0, T) \times \Omega_\varepsilon,$$

for Ω, u^0, g and Ω_ε as stated in Lemma 4.16. We recall that, since $w \mapsto w 1_{\Omega_\varepsilon}$ is an isometric embedding from $\mathcal{W}_{\Omega_\varepsilon}$ to \mathcal{W}_Ω , we have $\mathbb{S}_\varepsilon \in \mathcal{W}_\Omega$ and the equality :

$$\|\mathbb{S}_\varepsilon\|_{\mathcal{W}_\Omega} = \|\mathbb{S}_\varepsilon\|_{\mathcal{W}_{\Omega_\varepsilon}}. \quad (4.15)$$

Using the density of $\mathcal{D}(Q_\varepsilon)$ in $\mathcal{W}_{\Omega_\varepsilon}$ we find the equality :

$$\|\mathbb{S}_\varepsilon(T, \cdot)\|_{L^2(\Omega_\varepsilon)}^2 + 2 \iint_{Q_\varepsilon} |\nabla \mathbb{S}_\varepsilon|^2 = 2 \iint_{Q_\varepsilon} g \mathbb{S}_\varepsilon + \|u^0\|_{L^2(\Omega_\varepsilon)}^2.$$

In particular, if we use the Poincaré inequality on Ω and Cauchy-Schwarz, we obtain for a constant $C(\text{diam}(\Omega)) > 0$:

$$\|\mathbb{S}_\varepsilon\|_{L^2(0, T; H_0^1(\Omega_\varepsilon))} \leq C (\|g\|_{L^2(Q_\varepsilon)} + \|u^0\|_{L^2(\Omega_\varepsilon)}) \leq C (\|g\|_{L^2(Q)} + \|u^0\|_{L^2(\Omega)}).$$

In addition, looking at (4.11)₁ we obtain that $\partial_t \mathbb{S}_\varepsilon \in L^2(0, T; H^{-1}(\Omega_\varepsilon))$ and the existence of a constant $C(\text{diam}(\Omega)) > 0$ such that :

$$\|\mathbb{S}_\varepsilon\|_{\mathcal{W}_{\Omega_\varepsilon}} \leq C (\|g\|_{L^2(Q)} + \|u^0\|_{L^2(Q)}).$$

Consequently, \mathbb{S}_ε is uniformly bounded in \mathcal{W}_Ω due to (4.15) ; thus, \mathbb{S}_ε has at least a weakly convergent sequence in \mathcal{W}_Ω . Moreover, thanks to Item 1 of Lemma 4.17, we have that the sequence is convergent in $L^2(Q)$.

In order to end the proof, it suffices to see that for all $u \in \mathcal{W}_\Omega$ such that there is a sequence $\mathbb{S}_{\varepsilon_i}$ satisfying that $\mathbb{S}_{\varepsilon_i} \rightharpoonup u$ in \mathcal{W}_Ω (with $\varepsilon_i \rightarrow 0$), then $u = \mathbb{S}_0$. To get the equality, we prove that u satisfies (4.14) for $(\tilde{\Omega}, \tilde{u}^0, \tilde{g}) = (\Omega, u^0, g)$:

- In order to prove the variational condition of (4.14), we pick $\phi \in \mathcal{D}(Q)$. We remark that $d(\text{supp}_x \phi, \partial\Omega) > 0$, for :

$$\text{supp}_x \phi := \overline{\{x : \exists t \in [0, T] : \phi(t, x) \neq 0\}}.$$

In particular, if i is sufficiently large, we have that $\text{supp}_x \phi \subset\subset \Omega_{\varepsilon_i}$. Consequently, by (4.14)₁ (for $(\tilde{\Omega}, \tilde{u}^0, \tilde{g}) = (\Omega_{\varepsilon_i}, u^0 1_{\Omega_{\varepsilon_i}}, g 1_{\Omega_{\varepsilon_i}})$) and by taking into account the support of ϕ , we obtain for i large enough the equality :

$$\int_0^T \langle \partial_t \mathbb{S}_{\varepsilon_i}, \phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \iint_Q \nabla \mathbb{S}_{\varepsilon_i} \cdot \nabla \phi = \iint_Q g \phi.$$

Thus, if we take the weak limit in \mathcal{W}_Ω , we get that :

$$\int_0^T \langle \partial_t u, \phi \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} + \iint_Q \nabla u \cdot \nabla \phi = \iint_Q g \phi.$$

Since ϕ is arbitrary, u satisfies (4.14)₁.

- As for the initial condition of (4.14), we recall that because of Item 2 of Lemma 4.17,

$$w \in \mathcal{W} \mapsto w(0, \cdot) \in L^2(\Omega),$$

is a continuous operator. Therefore, since the weak limit is preserved by linear continuous operators between Hilbert spaces, we have that $\mathbb{S}_{\varepsilon_i}(0, \cdot) \rightharpoonup u(0, \cdot)$ in $L^2(\Omega)$. Moreover, $\mathbb{S}_{\varepsilon_i}(0, \cdot) = u^0 1_{\Omega_{\varepsilon_i}} \rightarrow u^0$ in $L^2(\Omega)$ by Remark 4.15. Consequently, from the uniqueness of the weak limit we obtain that $u(0, \cdot) = u^0$. □

In order to prove a Carleman inequality for Ω , the approximation must be made not only by C^2 domains, but also these domains must be compatible with the function η in the following way :

Definition 4.18. Let Ω be a domain, let $\omega \subset \Omega$ be a subdomain and let η be a real valued function satisfying :

$$\eta \in C^2(\overline{\Omega}), \quad \inf_{\Omega \setminus \omega} |\nabla \eta| > 0. \quad (4.16)$$

We say that $\tilde{\Omega}$ is a compatible open set with respect to Ω , ω and η if we have that :

$$\tilde{\Omega} \text{ is a Lipschitz open set such that } \omega \subset \tilde{\Omega} \subset \Omega, \quad (4.17)$$

$$\mathcal{D}_{\tilde{\Omega}}(\Delta) \subset H^2(\tilde{\Omega}) \text{ continuously,} \quad (4.18)$$

$$\partial_{\tilde{n}}\eta \leq 0 \text{ on } \partial\tilde{\Omega}. \quad (4.19)$$

Remark 4.19. It is possible for η to take negative values. Indeed, in [57] they do not need that $\eta \geq 0$ for proving the controllability of the heat equation with Dirichlet boundary conditions. They need it, though, to prove the controllability of other systems, like the Stokes equation or the heat equation with Neumann boundary conditions.

Remark 4.20. One consequence of Definition 4.18 is that $\tilde{\Omega}$ is connected. Indeed, if $\tilde{\Omega}_i$ is a connected component, then $\omega \cap \tilde{\Omega}_i \neq \emptyset$. Otherwise, because of (4.19) and (4.16)₂ the maximum of η in $\tilde{\Omega}_i$ is not reached on the boundary (the tangential derivative of η is non-null), but again because of (4.16)₂ the maximum is not reached in the interior, getting an absurd. Consequently, all the connected components of $\tilde{\Omega}$ intersect ω . Since ω is connected, $\tilde{\Omega}$ has only one connected component, so $\tilde{\Omega}$ is connected.

Remark 4.21. Every C^2 domain satisfies (4.18). Moreover, there are some Lipschitz domains which also satisfy it, like $(0, 1)^d$ or, in general, any cartesian product of C^2 domains.

Let us now state the Carleman inequality for domains which satisfy (4.17)-(4.19) :

Lemma 4.22. *Let Ω be a Lipschitz domain, $\omega \subset \Omega$ be a subdomain, η be a function satisfying (4.16) and $m \geq 1$. Then, there is $C > 0$ such that for all $T > 0$, $\tilde{\Omega}$ satisfying (4.17)-(4.19), $g \in L^2(\tilde{Q})$, $u^T \in L^2(\tilde{\Omega})$, $k > 2(m+1)\|\eta\|_{L^\infty(\Omega)}$, and for the weights defined in (4.9), we have that :*

$$\begin{aligned} s^3 \lambda^4 \iint_{\tilde{Q}} e^{-2s\alpha\xi^3} |u|^2 + s\lambda^2 \iint_{\tilde{Q}} e^{-2s\alpha\xi} |\nabla u|^2 + s^{-1} \iint_{\tilde{Q}} e^{-2s\alpha\xi^{-1}} (|\Delta u|^2 + |u_t|^2) \\ \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha\xi^3} |u|^2 + \iint_{\tilde{Q}} e^{-2s\alpha} |g|^2 \right), \end{aligned} \quad (4.20)$$

for any $\lambda \geq C$, $s \geq C(T^m + T^{2m})$ and for u the solution of :

$$\begin{cases} -u_t - \Delta u = g & \text{in } \tilde{Q}, \\ u = 0 & \text{on } \tilde{\Sigma}, \\ u(T, \cdot) = u^T & \text{on } \tilde{\Omega}. \end{cases}$$

Proof. Lemma 4.22 is a direct consequence of the proof of the usual Carleman estimate, which can be found for instance in [57] and [54]. We remark that since $\inf_{\Omega \setminus \omega} |\nabla \eta| > 0$, there exists a domain $\omega_0 \subset\subset \omega$ which satisfies $\inf_{\Omega \setminus \omega_0} |\nabla \eta| > 0$. The only additional difficulty is to prove that the constant C is independent of $\tilde{\Omega}$. Indeed, the proofs presented in [57] and [54] show that C depends continuously only on :

- $\|\eta\|_{C^2(\bar{\Omega})}$ (which can be bounded by $\|\eta\|_{C^2(\bar{\Omega})}$),
- m (see (4.10)),
- $\inf_{\tilde{\Omega} \setminus \omega_0} |\nabla \eta|$ (which can be inferiorly bounded by $\inf_{\Omega \setminus \omega_0} |\nabla \eta| > 0$),
- ω and ω_0 .

□

Now we are ready to prove a Carleman inequality for domains that can be approximated properly by regular domains :

Lemma 4.23. *Let Ω be a Lipschitz domain, $\omega \subset \Omega$ be a subdomain, η be a function satisfying (4.16) and $m \geq 1$. Let us suppose that there are $\varepsilon_0 > 0$ and $(\Omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ satisfying (4.17)-(4.19) and such that $\Omega_\varepsilon \rightarrow_d \Omega$. Then, there is $C > 0$ such that for all $T > 0$, $g \in L^2(Q)$, $u^T \in L^2(\Omega)$, $k > 2(m+1)\|\eta\|_{L^\infty(\Omega)}$, and for the weights defined in (4.9), we have that :*

$$\begin{aligned} s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u|^2 + s \lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla u|^2 + s^{-1} \iint_Q e^{-2s\alpha} \xi^{-1} (|\Delta u|^2 + |u_t|^2) \\ \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha} \xi^3 |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 \right), \end{aligned} \quad (4.21)$$

for any $\lambda \geq C$, $s \geq C(T^m + T^{2m})$ and for u the solution of :

$$\begin{cases} -u_t - \Delta u = g & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(T, \cdot) = u^T & \text{on } \Omega. \end{cases} \quad (4.22)$$

In order to prove Lemma 4.23 we first get the estimate on the zero-order term, and then get the higher order terms as in [52] and [53].

Proof. By Lemma 4.22 we have that for all $m \geq 1$ there is $C > 0$ such that for all $T > 0$, $u^T \in L^2(\Omega)$, $g \in L^2(Q)$, and $\varepsilon \in (0, \varepsilon_0)$:

$$s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u^\varepsilon 1_{\Omega_\varepsilon}|^2 \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha} \xi^3 |u^\varepsilon|^2 + \iint_Q e^{-2s\alpha} |g 1_{\Omega_\varepsilon}|^2 \right), \quad (4.23)$$

for any $\lambda \geq C$, $s \geq C(T^m + T^{2m})$ and for :

$$u^\varepsilon(t, x) := \mathbb{S}(\Omega_\varepsilon, u^T 1_{\Omega_\varepsilon}, g(T - \cdot, \cdot) 1_{\Omega_\varepsilon})(T - t, x).$$

Considering Lemma 4.16, we can take the limit in (4.23) and get the estimate :

$$s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u|^2 \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha} \xi^3 |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 \right). \quad (4.24)$$

Next, multiplying (4.22)₁ by $s\lambda^2 e^{-2s\alpha} \xi u$ we get the term of ∇u . Indeed, integrating by parts we get that :

$$\begin{aligned} s\lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla u|^2 &= s\lambda^2 \iint_Q e^{-2s\alpha} g u - s\lambda^3 \iint_Q e^{-2s\alpha} \xi \nabla \eta \cdot \nabla u u \\ &+ 2s^2 \lambda^3 \iint_Q e^{-2s\alpha} \xi^2 \nabla \eta \cdot \nabla u u - \frac{s\lambda^2}{2} \iint_Q e^{-2s\alpha} \xi_t |u|^2 + s^2 \lambda^2 \iint_Q e^{-2s\alpha} \alpha_t \xi |u|^2. \end{aligned} \quad (4.25)$$

By doing weighted Cauchy-Schwarz estimates in (4.25) and considering (4.10) and (4.24) we get for any $\lambda \geq C$ and $s \geq C(T^m + T^{2m})$ that :

$$s\lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla u|^2 \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha} \xi^3 |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 \right). \quad (4.26)$$

To continue with, we assume that $u^T \in H_0^1(\Omega)$. Multiplying (4.22)₁ by $-s^{-1} e^{-2s\alpha} \xi^{-1} u_t$ we obtain that :

$$\begin{aligned} s^{-1} \iint_Q e^{-2s\alpha} \xi^{-1} |u_t|^2 \\ = -s^{-1} \iint_Q e^{-2s\alpha} \xi^{-1} g u_t - \frac{1}{2} s^{-1} \iint_Q (e^{-2s\alpha} \xi^{-1})_t |\nabla u|^2 + s^{-1} \iint_Q \nabla(e^{-2s\alpha} \xi^{-1}) \cdot \nabla u u_t. \end{aligned} \quad (4.27)$$

We have used that the solutions of the heat equation with initial value in $H_0^1(\Omega)$ and source term in $L^2(Q)$ belongs to the closure of $H^1(0, T; H_0^1(\Omega)) \cap L^2(0, T; \mathcal{D}(\Delta))$ with norm $H^1(0, T; L^2(\Omega)) \cap L^2(0, T; \mathcal{D}(\Delta))$ when Ω is Lipschitz. Using weighted Cauchy-Schwarz inequalities, (4.10) and (4.26) we find for any $\lambda \geq C$ and $s \geq C(T^m + T^{2m})$ that :

$$s^{-1} \iint_Q e^{-2s\alpha} \xi^{-1} |u_t|^2 \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha} \xi^3 |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 \right). \quad (4.28)$$

Finally, considering that $-\Delta u = u_t + g$ we obtain from (4.28) that :

$$s^{-1} \iint_Q e^{-2s\alpha} \xi^{-1} |\Delta u|^2 \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha} \xi^3 |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 \right). \quad (4.29)$$

So, combining (4.24), (4.26), (4.28) and (4.29) we obtain (4.21) for all $u^T \in H_0^1(\Omega)$. Finally, we obtain (4.21) for all $u^T \in L^2(\Omega)$ by density. \square

Finally, we remark that we can extend the result presented in Lemma 4.23 to situations in which there is a source term in $L^2(0, T; H^{-1}(\Omega))$ reproducing step by step the proof given in [74] (the reader can consult for instance [2] and [12] to see that the results of the Sobolev spaces used in [74] remain true when Ω is Lipschitz) :

Proposition 4.24. *Let Ω be a Lipschitz domain, $\omega \subset \Omega$ be a subdomain, η be a function satisfying (4.16) and $m \geq 1$. Let us suppose that there are $\varepsilon_0 > 0$ and $(\Omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ satisfying (4.17)-(4.19) and such that $\Omega_\varepsilon \rightarrow_d \Omega$. Then, there is $C > 0$ such that for all $T > 0$, $g \in L^2(Q)$, $G \in (L^2(Q))^d$, $u^T \in L^2(\Omega)$, $k > 2(m+1)\|\eta\|_{L^\infty(\Omega)}$, and for the weights defined in (4.9), we have that :*

$$\begin{aligned} & s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u|^2 + s \lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla u|^2 \\ & \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha} \xi^3 |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 + s^2 \lambda^2 \iint_Q e^{-2s\alpha} \xi^2 |G|^2 \right), \end{aligned} \quad (4.30)$$

for any $\lambda \geq C$, $s \geq C(T^m + T^{2m})$ and for u the solution of :

$$\begin{cases} -u_t - \Delta u = g + \nabla \cdot G & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(T, \cdot) = u^T & \text{on } \Omega. \end{cases} \quad (4.31)$$

Remark 4.25. As the source term in (4.31) is just in $L^2(0, T; H^{-1}(\Omega))$, we do not (and cannot) have neither the Laplacian nor the time derivative on the left-hand side of (4.30).

Remark 4.26. The constant C in Lemma 4.23 and Proposition 4.24 is independent of the sequence Ω_ε .

4.3 Proof of Theorem 4.6

This section is split in three parts : in Subsection 4.3.1 we prove Theorem 4.6 in the case $\omega \subset \mathfrak{C}$, in Subsection 4.3.2 we prove Theorem 4.6 in the case $\omega \subset \Omega \setminus \overline{\mathfrak{C}}$, and in

Subsection 4.3.3 we make some comments about some technical motivations and possible extensions.

4.3.1 The case $\omega \subset \mathfrak{C}$

Let Ω be a canonically oriented pseudo-cylinder (see Definition 4.1) and let $\omega \subset \mathfrak{C}$. By taking a smaller control domain if necessary, we can suppose that :

$$\omega = B(\bar{x}, r) \times (\bar{z} - r, \bar{z} + r) \subset\subset \mathfrak{C}, \quad (4.32)$$

for some $r > 0$ and some $(\bar{x}, \bar{z}) \in \Omega$ such that :

$$\bar{z} < \inf_{\mathfrak{B}} H. \quad (4.33)$$

We construct an auxiliary function η satisfying (4.16). Considering that \mathfrak{B} is C^2 and (4.1), there is $\eta_{\mathfrak{B}} \in C^2(\overline{\mathfrak{B}})$ satisfying :

$$\eta_{\mathfrak{B}} = 0 \text{ on } \partial\mathfrak{B}, \quad \eta_{\mathfrak{B}} > 0 \text{ in } \mathfrak{B}, \quad \inf_{\mathfrak{B} \setminus B(\bar{x}, r)} |\nabla \eta_{\mathfrak{B}}| > 0. \quad (4.34)$$

One important consequence of (4.34) is the inequality :

$$\sup_{\partial\mathfrak{B}} \partial_{n_{\mathfrak{B}}} \eta_{\mathfrak{B}} < 0. \quad (4.35)$$

Indeed, using (4.34)₁ we have that the tangential derivative of $\eta_{\mathfrak{B}}$ is null on $\partial\mathfrak{B}$. Thus, using (4.34)₂ and that $B(\bar{x}, r) \subset\subset \mathfrak{B}$, we obtain that $\partial_{n_{\mathfrak{B}}} \eta_{\mathfrak{B}}(x) \neq 0$ for all $x \in \partial\mathfrak{B}$. Moreover, from (4.34)₁ and (4.34)₂ we find that $\partial_{n_{\mathfrak{B}}} \eta_{\mathfrak{B}}(x) < 0$ for all $x \in \partial\mathfrak{B}$, which implies (4.35) by compactness.

So, we define in $\overline{\Omega}$:

$$\eta(x, z) := \eta_{\mathfrak{B}}(x) - \mathfrak{c}(z - \bar{z})^2, \quad (4.36)$$

for \mathfrak{c} a large positive constant to be fixed later on. It is evident that $\eta \in C^2(\overline{\Omega})$. As $\nabla \eta = (\nabla \eta_{\mathfrak{B}}, -2\mathfrak{c}(z - \bar{z}))$, it follows from (4.32) and (4.34)₃ the inequality :

$$\inf_{\Omega \setminus \omega} |\nabla \eta| > 0.$$

Example 4.27. See Figure 4.2 for an illustration of a control domain and of $\nabla \eta$ in the pseudo-cylinder introduced in Example 4.3.

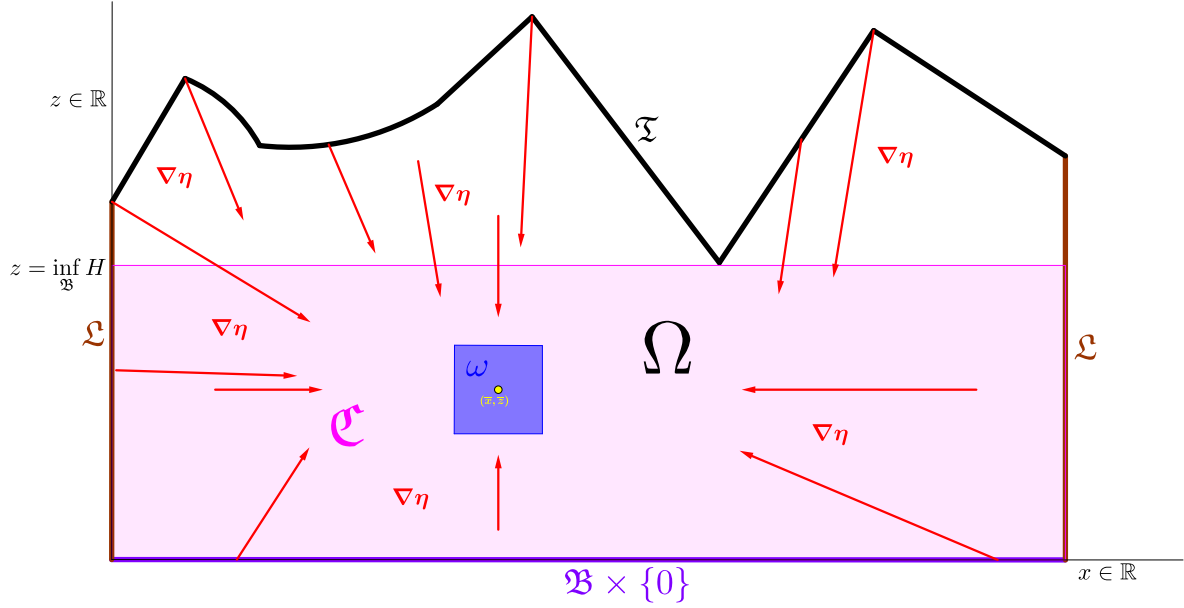


FIGURE 4.2 – A canonically oriented pseudo-cylinder with its auxiliary function.

With the above auxiliary function we obtain the following Carleman inequality :

Proposition 4.28. *Let Ω be a canonically oriented pseudo-cylinder, ω be given by (4.32) and $m \geq 1$. Then, there exists $\mathbf{c}_0(\Omega, \omega, \eta_{\mathfrak{B}}) > 0$ such that if η is defined as in (4.36) with $\mathbf{c} \geq \mathbf{c}_0$, there is $C > 0$ such that for all $T > 0$, $A \in (L^\infty(Q))^{d+1}$, $a \in L^\infty(Q)$, $g \in L^2(Q)$, $G \in (L^2(Q))^{d+1}$, $u^T \in L^2(\Omega)$, $k > 2(m+1)\|\eta\|_{L^\infty(\Omega)}$, and for the weights defined in (4.9), we have that :*

$$\begin{aligned} & s^3 \lambda^4 \iint_Q e^{-2s\alpha\xi^3} |u|^2 + s\lambda^2 \iint_Q e^{-2s\alpha\xi} |\nabla u|^2 \\ & \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha\xi^3} |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 + s^2 \lambda^2 \iint_Q e^{-2s\alpha\xi^2} |G|^2 \right), \end{aligned} \quad (4.37)$$

for any $\lambda \geq C$,

$$s \geq C \left(T^m + T^{2m} (1 + \|a\|_{L^\infty(Q)}^{2/3} + \|A\|_{(L^\infty(Q))^{d+1}}^2) \right), \quad (4.38)$$

and for u the solution of :

$$\begin{cases} -u_t - \Delta u - \nabla \cdot (Au) + au = g + \nabla \cdot G & \text{in } Q, \\ u = 0 & \text{on } \Sigma, \\ u(T, \cdot) = u^T & \text{on } \Omega. \end{cases} \quad (4.39)$$

In order to prove Proposition 4.28 we state the following technical result whose proof is postponed to Appendix 4.A.

Proposition 4.29. *Let Ω be a canonically oriented pseudo-cylinder, ω be given by (4.32) and $\eta_{\mathfrak{B}} \in C^2(\overline{\mathfrak{B}})$ satisfying (4.34). Then, there is $\mathfrak{c}_0(\Omega, \omega, \eta_{\mathfrak{B}}) > 0$ such that if η is given by (4.36) with $\mathfrak{c} \geq \mathfrak{c}_0$, there exist $\varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}}, \mathfrak{c}) > 0$ and some domains $(\Omega_\varepsilon)_{\varepsilon \in (0, \varepsilon_0)}$ satisfying (4.17)-(4.19) and :*

$$\Omega_\varepsilon \rightarrow_d \Omega. \quad (4.40)$$

Proof. [Proof of Proposition 4.28] Using Propositions 4.29 and 4.24, we obtain that for all $\lambda \geq C$ and $s \geq C(T^m + T^{2m})$:

$$\begin{aligned} & s^3 \lambda^4 \iint_Q e^{-2s\alpha \xi^3} |u|^2 + s \lambda^2 \iint_Q e^{-2s\alpha \xi} |\nabla u|^2 \\ & \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha \xi^3} |u|^2 + \iint_Q e^{-2s\alpha} (|g|^2 + |au|^2) + s^2 \lambda^2 \iint_Q e^{-2s\alpha \xi^2} (|G|^2 + |Au|^2) \right) \\ & \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\alpha \xi^3} |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 + s^2 \lambda^2 \iint_Q e^{-2s\alpha \xi^2} |G|^2 \right. \\ & \quad \left. + \|a\|_{L^\infty(Q)}^2 T^{6m} \iint_Q e^{-2s\alpha \xi^3} |u|^2 + s^2 \lambda^2 \|A\|_{L^\infty(Q)}^2 T^{2m} \iint_Q e^{-2s\alpha \xi^3} |u|^2 \right). \quad (4.41) \end{aligned}$$

Thus, taking $\lambda \geq C$ and s satisfying (4.38) we can absorb the the last two terms in the right-hand side of (4.41). \square

Remark 4.30. It is a classical result (see, for instance, [97], [82], [57] and [56]) that Proposition 4.28 (taking $m = 1$) implies that Theorem 4.6 is true for all canonically oriented pseudo-cylinders and all ω given by (4.32). By Remark 4.7 this implies Theorem 4.6 for all pseudo-cylinders and all $w \subset \mathfrak{C}$.

4.3.2 The case $\omega \subset \Omega \setminus \overline{\mathfrak{C}}$

Let Ω be a canonically oriented pseudo-cylinder and let $\omega \subset \Omega \setminus \overline{\mathfrak{C}}$. For this case we suppose, by making again the control domain smaller if necessary, that :

$$\omega = B(\tilde{x}, r) \times (\tilde{z} - r, \tilde{z} + r) \subset\subset \Omega \setminus \overline{\mathfrak{C}}, \quad (4.42)$$

for some $(\tilde{x}, \tilde{z}) \in \Omega \setminus \overline{\mathfrak{C}}$ such that :

$$\tilde{z} > \inf_{\mathfrak{B}} H,$$

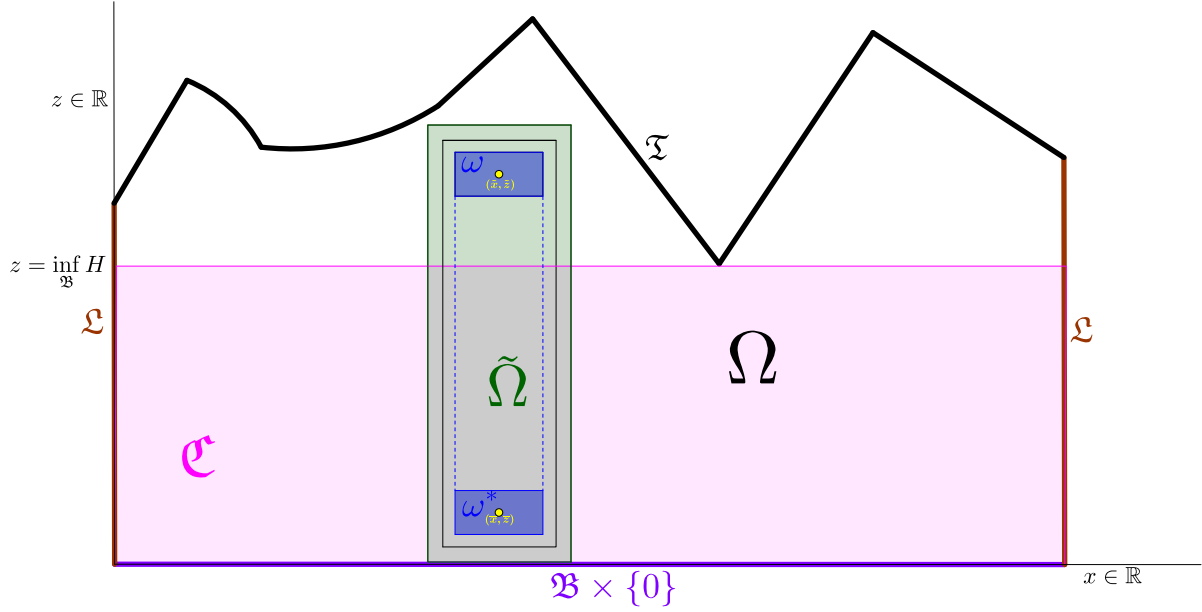


FIGURE 4.3 – A canonically oriented pseudo-cylinder with a control domain in $\Omega \setminus \bar{\mathcal{C}}$.

and some :

$$r \in \left(0, \inf_{\mathfrak{B}} \frac{H}{4}\right), \quad (4.43)$$

such that :

$$\tilde{\Omega} := B(\tilde{x}, 3r) \times (0, \tilde{z} + 3r) \subset \Omega. \quad (4.44)$$

In this case we cannot consider an auxiliary function similar to (4.36) because now we have that $\mathfrak{T} \cap \{(x, z) : z = \tilde{z}\} \neq \emptyset$. However, thanks to Proposition 4.28 our problem can be seen as a problem of transmitting the estimate from one control domain to another. Indeed, we define :

$$\omega^* := B(\bar{x}, r) \times (\bar{z} - r, \bar{z} + r), \quad (4.45)$$

for $\bar{x} := \tilde{x}$ and $\bar{z} := 3r$. Thus, considering also (4.43) we have that :

$$\omega^* = B(\tilde{x}, r) \times (2r, 4r) \subset\subset \tilde{\Omega} \cap \mathcal{C}.$$

Example 4.31. See Figure 4.3 for an illustration of the pseudo-cylinder given in Example 4.3 when $\omega \subset\subset \Omega \setminus \bar{\mathcal{C}}$.

We fix $m \geq 1$ and define η as in (4.36) with $\mathbf{c} \geq \mathbf{c}_0(\Omega, \omega^*, \eta_{\mathfrak{B}})$. We obtain by Proposition 4.28 a constant $C > 0$ such that for all $T > 0$, $A \in (L^\infty(Q))^{d+1}$, $a \in L^\infty(Q)$, $g \in L^2(Q)$,

$G \in (L^2(Q))^{d+1}$, $u^T \in L^2(\Omega)$, and $k > 2(m+1)\|\eta\|_{L^\infty(\Omega)}$ we have that :

$$\begin{aligned} & s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u|^2 + s \lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla u|^2 \\ & \leq C \left(s^3 \lambda^4 \iint_{Q_{\omega^*}} e^{-2s\alpha} \xi^3 |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 + s^2 \lambda^2 \iint_Q e^{-2s\alpha} \xi^2 |G|^2 \right), \end{aligned} \quad (4.46)$$

for $\lambda \geq C$, s satisfying (4.38) and u the solution of (4.39).

Next, we estimate the weighted $L^2(Q_{\omega^*})$ -norm by a weighted $L^2(Q_\omega)$ -norm. To do this, we consider a function whose maximum is in ω but which is large in ω^* . Specifically, we consider :

$$\tilde{\eta}(x, z) := -\tilde{\eta}_1(|x - \tilde{x}|) - \tilde{\eta}_2(z) + 2 \sup_{\Omega} |\eta|, \quad (4.47)$$

for $\tilde{\eta}_1$ a positive C^∞ function supported in $(r/2, +\infty)$ such that :

$$\begin{cases} \inf_{(r, +\infty)} \tilde{\eta}'_1 > 0, \\ \tilde{\eta}_1 < \sup_{\Omega} |\eta| / 2 \text{ in } (r/2, r), \\ \tilde{\eta}_1 > 3 \sup_{\Omega} |\eta| \text{ in } (2r, 3r), \end{cases} \quad (4.48)$$

and for $\tilde{\eta}_2$ a positive C^∞ function such that :

$$\begin{cases} \sup_{(0, \tilde{z}-r)} \tilde{\eta}'_2 < 0, \\ \inf_{(\tilde{z}+r, \tilde{z}+3r)} \tilde{\eta}'_2 > 0, \\ \tilde{\eta}_2 < \sup_{\Omega} |\eta| / 2 \text{ in } (2r, 4r), \\ \tilde{\eta}_2 > 3 \sup_{\Omega} |\eta| \text{ in } (0, r) \cup (\tilde{z} + 2r, \tilde{z} + 3r). \end{cases} \quad (4.49)$$

We remark that $\tilde{\eta} \in C^2(\tilde{\Omega})$ and satisfies :

$$\partial_n \tilde{\eta} < 0 \text{ on } \partial \tilde{\Omega}, \quad \inf_{\tilde{\Omega} \setminus \omega} |\nabla \tilde{\eta}| > 0. \quad (4.50)$$

Moreover, from (4.45), (4.48) and (4.49) we get that :

$$\begin{cases} \eta \leq \tilde{\eta} \text{ in } \omega^*, \\ \tilde{\eta} \leq \eta \text{ in } \tilde{\Omega} \setminus (B(\tilde{x}, 2r) \times (r, \tilde{z} + 2r)). \end{cases} \quad (4.51)$$

With the function $\tilde{\eta}$ we can define the weights $\tilde{\alpha}$ and $\tilde{\xi}$ as in (4.9). Let us fix :

$$k > 2(m+1) \max\{\|\tilde{\eta}\|_{L^\infty(\Omega)}, \|\eta\|_{L^\infty(\Omega)}\} = 2(m+1)\|\tilde{\eta}\|_{L^\infty(\Omega)}. \quad (4.52)$$

Thanks to (4.51) and Remark 4.12 we have that :

$$\begin{cases} \xi \leq \tilde{\xi}; & -\alpha \leq -\tilde{\alpha} & \text{in } Q_{\omega^*}, \\ \tilde{\xi} \leq \xi; & -\tilde{\alpha} \leq -\alpha & \text{in } (0, T) \times \left(\tilde{\Omega} \setminus (B(\tilde{x}, 2r) \times (r, \tilde{z} + 2r))\right). \end{cases} \quad (4.53)$$

Next, we consider that u is the solution of (4.39) and that $\chi(x, z)$ is a positive regular cut-off function such that :

$$\text{supp}(\chi) \subset\subset \tilde{\Omega}, \quad \chi = 1 \text{ in } B(\tilde{x}, 2r) \times (r, \tilde{z} + 2r). \quad (4.54)$$

Then, $v := \chi u$ is the solution of :

$$\begin{cases} -v_t - \Delta v - \nabla \cdot (Av) + av = F(\nabla \chi, u) + \chi g + \nabla \cdot (\chi G) & \text{in } \tilde{Q}, \\ v = 0 & \text{on } \tilde{\Sigma}, \\ v(T, \cdot) = \chi u^T & \text{on } \tilde{\Omega}, \end{cases} \quad (4.55)$$

for :

$$F(\nabla \chi, u) := -2\nabla \chi \cdot \nabla u - \Delta \chi u - (\nabla \chi \cdot A)u - \nabla \chi \cdot G. \quad (4.56)$$

Using Proposition 4.24 ($\tilde{\Omega}$ satisfies (4.18) by Remark 4.21 and (4.19) by (4.50), so it suffices to take $\Omega_\varepsilon = \tilde{\Omega}$), we have that there is $C(\tilde{\eta}, m) > 0$ such that if $A \in (L^\infty(Q))^{d+1}$, $a \in L^\infty(Q)$, $g \in L^2(Q)$, $G \in (L^2(Q))^{d+1}$, $u^T \in L^2(\Omega)$, $\lambda \geq C$ and $s \geq C(T^m + T^{2m})$:

$$\begin{aligned} s^3 \lambda^4 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} \left(\tilde{\xi}\right)^3 |v|^2 + s\lambda^2 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} \tilde{\xi} |\nabla v|^2 &\leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\tilde{\alpha}} \left(\tilde{\xi}\right)^3 |v|^2 \right. \\ &\left. + \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} |F(\nabla \chi, u) + \chi g - av|^2 + s^2 \lambda^2 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} \left(\tilde{\xi}\right)^2 (|\chi G - Av|^2) \right). \end{aligned} \quad (4.57)$$

Considering that $v := \chi u$ and (4.57), after some easy absorptions and bounds we obtain for $\lambda \geq C$ and s satisfying (4.38) that :

$$\begin{aligned} s^3 \lambda^4 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} \left(\tilde{\xi}\right)^3 |\chi u|^2 + s\lambda^2 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} \tilde{\xi} |\nabla(\chi u)|^2 &\leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\tilde{\alpha}} \left(\tilde{\xi}\right)^3 |\chi u|^2 \right. \\ &\left. + \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} (|\chi g|^2 + |F(\nabla \chi, u)|^2) + s^2 \lambda^2 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} \left(\tilde{\xi}\right)^2 |\chi G|^2 \right). \end{aligned} \quad (4.58)$$

To continue with, as $\chi = 1$ in ω and ω^* , $\chi \in \mathcal{D}(\tilde{\Omega})$, and we have (4.53)₁, we may combine (4.46) and (4.58) and get that :

$$\begin{aligned} & s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u|^2 + s \lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla u|^2 \\ & \quad + s^3 \lambda^4 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} (\tilde{\xi})^3 |\chi u|^2 + s \lambda^2 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} \tilde{\xi} |\nabla(\chi u)|^2 \\ & \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\tilde{\alpha}} (\tilde{\xi})^3 |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 + \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} (|\chi g|^2 + |F(\nabla \chi, u)|^2) \right. \\ & \quad \left. + s^2 \lambda^2 \iint_Q e^{-2s\alpha} \xi^2 |G|^2 + s^2 \lambda^2 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} (\tilde{\xi})^2 |\chi G|^2 \right). \end{aligned} \quad (4.59)$$

Next, we recall that $\chi \in C^2(\bar{\Omega})$, that :

$$\text{supp}(\Delta \chi), \text{supp}(\nabla \chi) \subset \tilde{\Omega} \setminus (B(\tilde{x}, 2r) \times (r, \tilde{z} + 2r)),$$

and (4.53)₂. These assertions imply that by taking $\lambda \geq C$ and s satisfying (4.38) the term of $F(\nabla \chi, u)$ in the right-hand side of (4.59) can be absorbed. Thus, we get :

$$\begin{aligned} & s^3 \lambda^4 \iint_Q e^{-2s\alpha} \xi^3 |u|^2 + s \lambda^2 \iint_Q e^{-2s\alpha} \xi |\nabla u|^2 \\ & \quad + s^3 \lambda^4 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} (\tilde{\xi})^3 |\chi u|^2 + s \lambda^2 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} \tilde{\xi} |\nabla(\chi u)|^2 \\ & \leq C \left(s^3 \lambda^4 \iint_{Q_\omega} e^{-2s\tilde{\alpha}} (\tilde{\xi})^3 |u|^2 + \iint_Q e^{-2s\alpha} |g|^2 + \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} |\chi g|^2 \right. \\ & \quad \left. + s^2 \lambda^2 \iint_Q e^{-2s\alpha} \xi^2 |G|^2 + s^2 \lambda^2 \iint_{\tilde{Q}} e^{-2s\tilde{\alpha}} (\tilde{\xi})^2 |\chi G|^2 \right). \end{aligned} \quad (4.60)$$

Finally, let us state what we have proved :

Proposition 4.32. *Let Ω be a canonically oriented pseudo-cylinder, ω be given by (4.42), $\tilde{\Omega}$ be given by (4.44), η be given by (4.36) (with $\bar{x} := \tilde{x}$ and $\bar{z} := 3r$), $\tilde{\eta}$ be given by (4.47), χ be given by (4.54) and $m \geq 1$. Then, there exists a constant $C > 0$ such that if $T > 0$, $A \in (L^\infty(Q))^{d+1}$, $a \in L^\infty(Q)$, $g \in L^2(Q)$, $G \in (L^2(Q))^{d+1}$, $u^T \in L^2(\Omega)$, k satisfies (4.52), $\lambda \geq C$ and s satisfies (4.38), we have (4.60) for the weights defined in (4.9) and u the solution of (4.39).*

Remark 4.33. It is a classical result (see, for instance, [97], [82], [57] and [56]) that Proposition 4.32 (taking $m = 1$) implies that Theorem 4.6 is true for all canonically oriented pseudo-cylinders and all ω given by (4.42). Together with Remarks 4.30 and 4.7, this result ends the proof of Theorem 4.6 (up to proving Proposition 4.29).

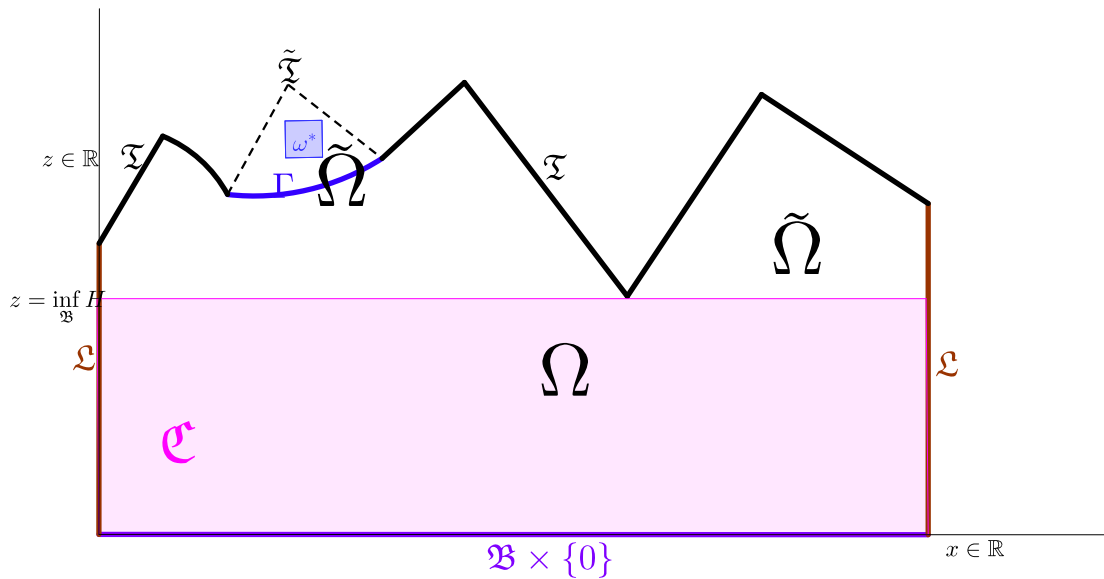


FIGURE 4.4 – A canonically oriented pseudo-cylinder with a control domain $\Gamma \subset \mathfrak{T}$.

4.3.3 Further comments

1. **Boundary controllability.** If $\Gamma \subset \mathfrak{T}$ (see Remark 4.2) is a relatively open connected subset we can easily prove the null controllability of the heat equation with a control supported in Γ . Indeed, it suffices to extend the domain over Γ as in Figure 4.4 to some other pseudo-cylinder $\tilde{\Omega}$, use the null controllability by means of an internal control of the heat equation with Dirichlet boundary condition, consider the restriction of the value in Γ and use the uniqueness of solutions by transposition of the heat equation. If $\Gamma \subset \mathfrak{L}$ or if $\Gamma \subset \mathfrak{B} \times \{0\}$ this method fails because the extended domain is not a pseudo-cylinder. Thus, for those situations the boundary controllability problem remains open. Moreover, it would also be interesting to obtain Carleman inequalities of the Fursikov-Imanuvilov's type for a control domain $\Gamma \subset \partial\Omega$, which has a local boundary term of $\partial_n u$ in the right-hand side of the Carleman estimate.
2. **The Carleman estimate when A is more regular and $G = 0$.** If we have that $\nabla \cdot A \in L^\infty(Q)$ and $G = 0$ we can add a term of a Laplacian and of a time derivative in the Carleman estimates obtained in Propositions 4.28 and 4.32 by using Lemma 4.23 instead of Proposition 4.24.
3. **Validity of the partial results in any pseudo-cylinder.** Even if we have stated

and proved the partial results (see Propositions 4.28, 4.29 and 4.32) in canonically oriented pseudo-cylinders, these results hold in any pseudo-cylinder since the Laplacian operator is invariant under rotations (and translations).

4. **The dependence of \mathbf{c} with respect to $\eta_{\mathfrak{B}}$.** It seems impossible to approximate a domain Ω by some subdomains Ω_ε satisfying (4.17)-(4.19) without having $\partial_n \eta \leq 0$ on $\partial\Omega$ (though this is an interesting open question). This implies that if Ω is a pseudo-cylinder, if ω is given by (4.32) and if η is the auxiliary function defined in (4.36), we need to take \mathbf{c} large enough with respect to $\eta_{\mathfrak{B}}$. Indeed, let us compute $\partial_n \eta$ and verify that it is negative on all the subsets of the boundary given by Remark 4.2.

- On $\mathfrak{B} \times \{0\}$ we have that $\partial_n \eta = -\partial_z \eta = -2\mathbf{c}\bar{z} < 0$.
- On \mathfrak{L} we have by (4.35) that $\partial_n \eta = \partial_{n_{\mathfrak{B}}} \eta_{\mathfrak{B}} < 0$.
- On \mathfrak{T} we find that :

$$n(x, H(x)) = \frac{1}{\sqrt{1 + |\nabla H(x)|^2}} (-\nabla H(x), 1).$$

Thus, on \mathfrak{T} the normal derivative is given by :

$$\partial_n \eta(x, H(x)) = \frac{-\nabla H(x) \cdot \nabla \eta_{\mathfrak{B}}(x) - 2\mathbf{c}(H(x) - \bar{z})}{\sqrt{1 + |\nabla H(x)|^2}}.$$

This implies that the condition :

$$\mathbf{c} \geq \frac{\|\nabla H\|_{L^\infty(\mathfrak{B})} \|\nabla \eta_{\mathfrak{B}}\|_{C(\overline{\mathfrak{B}})}}{2(\inf_{\mathfrak{B}} H(x) - \bar{z})}, \quad (4.61)$$

is necessary to ensure that $\partial_n \eta \leq 0$ on \mathfrak{T} (the denominator in (4.61) is not null by (4.33)).

5. **Problems when extending this method to Lipschitz domains.** A problem that remains open is the null controllability of the heat equation in any Lipschitz domain. A natural attempt following the Fursikov-Imanuvilov approach is to split the domain into a finite amount of pseudo-cylinders and a single compactly included C^2 subdomain, and then to apply a Carleman estimate in each subdomain (with the help of some cut-off functions). To that end, we need to absorb what is in a neighbourhood \mathfrak{L} (see Remark 4.2) with estimates in some other pseudo-cylinders and then use the same technique as Section 4.3.2 to pass from a finite number of control domains that are compactly contained in Ω to a single control domain. This

idea could work if we had a function η satisfying (4.16) which is much bigger in the interior of the cylinders than close to \mathfrak{L} . However, it seems incompatible with picking in (4.36) \mathfrak{c} large with respect to $\eta_{\mathfrak{B}}$.

6. **Semi-linear heat equation.** A problem that remains open is the controllability of the semi-linear heat equation stated in pseudo-cylinders with a non-linearity acting also in the first order term. Indeed, we cannot just follow the steps in [44] because they require for Ω to be C^2 when they prove that the Carleman estimate implies the null-controllability of the semi-linear heat equation (more specifically, they use that $\mathcal{D}(\Delta) \subset H^2(\Omega)$ continuously and the analogous results for L^p -spaces).

4.A Proof of Proposition 4.29

We recall that Ω is a canonically oriented pseudo-cylinder, ω is given by (4.32) and η is given by (4.36). Similarly, in this proof we use c and C to denote respectively small and large strictly positive constants that may be different each time and only depend on Ω , ω and $\eta_{\mathfrak{B}}$, and ε is taken as a small strictly positive parameter.

The structure of the proof is the following : first we present an auxiliary result from differential geometry, second we introduce the regularized measure, third we approximate H (see Definition 4.1) by smaller regular functions, fourth we introduce some domains Ω_ε and prove that they satisfy (4.40), fifth we prove that the domains Ω_ε satisfy (4.17) and (4.18) for all $\varepsilon \ll \varepsilon_0(\Omega, \omega)$, and finally we assume that $\mathfrak{c} \gg \mathfrak{c}_0(\Omega, \omega, \eta_{\mathfrak{B}})$ and $\varepsilon \ll \varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}}, \mathfrak{c})$ and prove (4.19). We recall that once the open sets Ω_ε satisfy (4.17)-(4.19), Remark 4.20 implies that they are connected.

In the proof we state that several things happen if $\mathfrak{c} \gg \mathfrak{c}_0(\Omega, \omega, \eta_{\mathfrak{B}})$ and $\varepsilon \ll \varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}}, \mathfrak{c})$. We do not calculate explicitly the value of $\mathfrak{c}_0(\Omega, \omega, \eta_{\mathfrak{B}})$ and of $\varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}}, \mathfrak{c})$ the same way we do not calculate explicitly the values of C or c . Indeed, note that the number of times that we require such a bound is finite (as \mathfrak{c}_0 and ε_0 do not depend on the specific (x, z) but at most on a finite division of subcases), so it suffices to consider the maximum of all the \mathfrak{c}_0 and the minimum of all the ε_0 .

Step 1 : an auxiliary result. In order to construct these domains, we need the following basic result from differential geometry, whose proof can be found for instance in [41] :

Lemma 4.34. *Let $\tilde{\Omega}$ be a domain and let $\Phi : \overline{\tilde{\Omega}} \mapsto \mathbb{R}$ be a C^2 function such that $\Phi \leq 0$ on $\partial\tilde{\Omega}$. Then, for all $\theta > 0$ satisfying $\Phi^{-1}(\theta, +\infty) \neq \emptyset$ and :*

$$\Phi^{-1}(\{\theta\}) \cap (\nabla\Phi)^{-1}(\{0\}) = \emptyset, \quad (4.62)$$

we have that $\Phi^{-1}(\theta, +\infty)$ is a C^2 open set. Moreover, the boundary of $\Phi^{-1}(\theta, +\infty)$ is given by $\Phi^{-1}(\{\theta\})$. Finally, the outward normal unit vector, which we denote by n_θ , satisfies on $\Phi^{-1}(\{\theta\})$:

$$n_\theta(x) = -\frac{\nabla\Phi(x)}{|\nabla\Phi(x)|}. \quad (4.63)$$

A first approach can be to use Lemma 4.34 on the distance function from (x, z) to $\partial\Omega$. Nonetheless, this is not possible because that function is not differentiable. On the one hand, even in regular domains $d(\cdot, \partial\Omega)$ may not be C^1 in the interior (for instance, in the disk the distance to the boundary is not differentiable at its center). On the other hand, near \mathfrak{I} the function $d(\cdot, \partial\Omega)$ is not C^1 when \mathfrak{I} is just Lipschitz. However, it is possible to construct a function which behaves similarly to the distance. In particular, to face the first problem we regularize the distance with the tools introduced in [81] and to face the second one we approximate H by regular functions.

Step 2 : the regularized measure. We define the function :

$$\mathfrak{d} : x \in \overline{\mathfrak{B}} \mapsto d(x, \partial\mathfrak{B}),$$

and the sets :

$$\mathfrak{B}_\varepsilon := \mathfrak{d}^{-1}(\varepsilon, +\infty). \quad (4.64)$$

Since \mathfrak{B} is C^2 , we have for some $\tilde{\varepsilon}(\Omega) > 0$ the following properties :

- \mathfrak{d} is a $W^{2,\infty}(\mathfrak{d}^{-1}([0, \tilde{\varepsilon}]))$ function such that $|\nabla\mathfrak{d}| = 1$ in $\mathfrak{d}^{-1}([0, \tilde{\varepsilon}])$.
- For all $x \in \mathfrak{d}^{-1}([0, \tilde{\varepsilon}])$ there is a unique point $P(x)$ such that :

$$d(x, P(x)) = \mathfrak{d}(x). \quad (4.65)$$

Moreover, the function $x \mapsto P(x)$ is continuous in $\mathfrak{d}^{-1}([0, \tilde{\varepsilon}])$.

The proof of these two assertions can be found for instance in [12, Section III.3]. Also, it can be proved easily that for all $\varepsilon \ll \varepsilon_0(\Omega)$ the set \mathfrak{B}_ε is a domain satisfying :

$$\partial\mathfrak{B}_\varepsilon = \mathfrak{d}^{-1}(\{\varepsilon\}).$$

In order to regularize \mathfrak{d} in the interior we use the function introduced in [81], whose properties can be consulted for instance in [12, Section III.3.3]. Indeed, we consider :

$$\rho : x \in \overline{\mathfrak{B}} \mapsto \mathbb{R},$$

which is defined as the only fixed point of $\tau \mapsto G(x, \tau)$, for :

$$G(x, \tau) := \int_{B(0,1)} \mathfrak{d} \left(x + \frac{\tau}{2} \tilde{x} \right) \varsigma(\tilde{x}) d\tilde{x},$$

for ς a mollifier in \mathbb{R}^d ; that is, a positive function belonging to $\mathcal{D}(B(0,1))$ such that :

$$\int_{B(0,1)} \varsigma(x) dx = 1.$$

We have that ρ satisfies the following properties, which are proved for instance in [12, Section III.3.3] :

$$\rho \in W^{2,\infty}(\mathfrak{B}) \cap C^\infty(\mathfrak{B}), \quad (4.66)$$

$$\rho \geq 0 \text{ in } \overline{\mathfrak{B}} \text{ and } \rho = 0 \text{ if and only if } x \in \partial\mathfrak{B}, \quad (4.67)$$

$$\frac{1}{2} \leq \frac{\mathfrak{d}(x)}{\rho(x)} \leq \frac{3}{2}, \quad \forall x \in \mathfrak{B}, \quad (4.68)$$

$$\nabla \rho(x) = -n_{\mathfrak{B}}(x), \quad \forall x \in \partial\mathfrak{B}. \quad (4.69)$$

Step 3 : approximating H by smaller regular functions. We consider again ς a mollifier in \mathbb{R}^d , and we define $\varsigma_\varepsilon := \varepsilon^{-d} \varsigma(\frac{\cdot}{\varepsilon})$, which is supported in $B(0, \varepsilon)$ and satisfies :

$$\int_{B(0,\varepsilon)} \varsigma_\varepsilon(x) dx = 1. \quad (4.70)$$

In addition, we define in \mathfrak{B}_ε the function $H_\varepsilon := (1 - \sqrt{\varepsilon}) H * \varsigma_\varepsilon$, which is well-defined because $x \in \mathfrak{B}_\varepsilon$ implies that $B(x, \varepsilon) \subset \mathfrak{B}$ (see (4.64)). Furthermore, if $x \in \mathfrak{B}_\varepsilon$:

$$\begin{aligned} H_\varepsilon(x) &= (1 - \sqrt{\varepsilon}) \int_{B(0,\varepsilon)} H(x - \tilde{x}) \varsigma_\varepsilon(\tilde{x}) d\tilde{x} \\ &\leq (1 - \sqrt{\varepsilon}) (H(x) + \varepsilon \|H\|_{W^{1,\infty}(\mathfrak{B})}) \int_{B(0,\varepsilon)} \varsigma_\varepsilon(\tilde{x}) d\tilde{x} \\ &= H(x) - \sqrt{\varepsilon} H(x) + \varepsilon \|H\|_{W^{1,\infty}(\mathfrak{B})} - \varepsilon^{3/2} \|H\|_{W^{1,\infty}(\mathfrak{B})}. \end{aligned} \quad (4.71)$$

This follows from (4.70) and from the fact that for all $\tilde{x} \in B(0, \varepsilon)$ we have that :

$$H(x - \tilde{x}) = H(x) - \int_0^1 \nabla H(x - \tau \tilde{x}) \cdot \tilde{x} d\tau \leq H(x) + |\tilde{x}| \|H\|_{W^{1,\infty}(\mathfrak{B})} \leq H(x) + \varepsilon \|H\|_{W^{1,\infty}(\mathfrak{B})}.$$

So, since $\inf_{\mathfrak{B}} H > 0$, from (4.71) we obtain for all $\varepsilon \ll \ll \varepsilon_0(\Omega)$ that :

$$H_\varepsilon \leq H \text{ in } \mathfrak{B}_\varepsilon. \quad (4.72)$$

Remark 4.35. We have included a term $1 - \sqrt{\varepsilon}$ in the definition of H_ε to have (4.72).

Similarly, we have for all $x \in \mathfrak{B}_\varepsilon$ the lower bound :

$$H_\varepsilon(x) \geq (1 - \sqrt{\varepsilon}) (H(x) - \varepsilon \|H\|_{W^{1,\infty}(\mathfrak{B})}). \quad (4.73)$$

Indeed, this follows from (4.70) and from the fact that for all $\tilde{x} \in B(0, \varepsilon)$ we have that :

$$H(x - \tilde{x}) = H(x) - \int_0^1 \nabla H(x - \tau \tilde{x}) \cdot \tilde{x} d\tau \geq H(x) - |\tilde{x}| \|H\|_{W^{1,\infty}(\mathfrak{B})} \geq H(x) - \varepsilon \|H\|_{W^{1,\infty}(\mathfrak{B})}.$$

One important consequence of (4.71) and (4.73) is the convergence :

$$\inf_{\mathfrak{B}_\varepsilon} H_\varepsilon \rightarrow \inf_{\mathfrak{B}} H. \quad (4.74)$$

A last important property of the functions H_ε is that they are uniformly Lipschitz for $\varepsilon \ll \ll \varepsilon_0(\Omega)$. Indeed, from Young's convolution inequality and (4.70) we obtain that :

$$\|\nabla H_\varepsilon\|_{L^\infty(\mathfrak{B}_\varepsilon)} = (1 - \sqrt{\varepsilon}) \|\zeta_\varepsilon * \nabla H\|_{L^\infty(\mathfrak{B}_\varepsilon)} \leq \|\nabla H\|_{L^\infty(\mathfrak{B})}. \quad (4.75)$$

Step 4 : approximating Ω by some regular domains Ω_ε . We define in $\overline{\Omega}$ for all ε satisfying that $\mathfrak{B}_{2\varepsilon} \neq \emptyset$ the following functions :

$$\mathfrak{D}_\varepsilon(x, z) := \chi_\varepsilon(x) \rho(x) (H_\varepsilon(x) - z) z, \quad \forall (x, z) \in \overline{\Omega}, \quad (4.76)$$

for χ_ε a positive $C^\infty(\mathbb{R}^d)$ cut-off function which takes values in $[0, 1]$, supported in \mathfrak{B}_ε and such that $\chi_\varepsilon = 1$ in $\overline{\mathfrak{B}_{2\varepsilon}}$ (for instance $1_{\mathfrak{B}_{(3/2)\varepsilon}} * \zeta_{\varepsilon/4}$).

Remark 4.36. Since $\chi_\varepsilon = 1$ in $\overline{\mathfrak{B}_{2\varepsilon}}$ we obtain from (4.76) that :

$$\nabla \mathfrak{D}_\varepsilon(x, z) = (z \rho(x) \nabla H_\varepsilon(x) + z (H_\varepsilon(x) - z) \nabla \rho(x), \rho(x) (H_\varepsilon(x) - 2z)), \quad \forall (x, z) \in \overline{\mathfrak{B}_{2\varepsilon}}. \quad (4.77)$$

We approximate Ω by :

$$\Omega_\varepsilon := \mathfrak{D}_\varepsilon^{-1}(\varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2, +\infty),$$

which is well-defined if $\varepsilon \ll \varepsilon_0(\Omega)$ (as in that case $\mathfrak{B}_{2\varepsilon} \neq \emptyset$). We first prove (4.40). For that, we fix $\varepsilon \ll \varepsilon_0(\Omega)$, we also fix $(x, z) \in \Omega \setminus \Omega_\varepsilon$ and we get a bound of $d((x, z), \partial\Omega)$. Since $(x, z) \in \Omega \setminus \Omega_\varepsilon$, we have that $\mathfrak{D}_\varepsilon(x, z) \leq \varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2$, which implies that :

$$\min\{\chi_\varepsilon(x), \rho(x), (H_\varepsilon(x) - z), z\} \leq \varepsilon^{1/4} \|H\|_{L^\infty(\mathfrak{B})}^2,$$

so we are in at least one of the following cases :

- If $\chi_\varepsilon(x) \leq \varepsilon^{1/4} \|H\|_{L^\infty(\mathfrak{B})}^2$, then from $\varepsilon \ll \varepsilon_0(\Omega)$ we have that $x \notin \mathfrak{B}_{2\varepsilon}$, which implies that :

$$d((x, z), \partial\Omega) \leq \mathfrak{d}(x) \leq 2\varepsilon. \quad (4.78)$$

- If $\rho(x) \leq \varepsilon^{1/4} \|H\|_{L^\infty(\mathfrak{B})}^2$, we have by (4.68) that :

$$d((x, z), \partial\Omega) \leq \mathfrak{d}(x) \leq \frac{3}{2} \varepsilon^{1/4} \|H\|_{L^\infty(\mathfrak{B})}^2. \quad (4.79)$$

- If $H_\varepsilon(x) - z \leq \varepsilon^{1/4} \|H\|_{L^\infty(\mathfrak{B})}^2$, using (4.71) we have that :

$$\begin{aligned} d((x, z), \partial\Omega) &\leq d((x, z), \mathfrak{T}) = H(x) - z = (H(x) - H_\varepsilon(x)) + H_\varepsilon(x) - z \\ &\leq \varepsilon^{1/4} \|H\|_{L^\infty(\mathfrak{B})}^2 + C\sqrt{\varepsilon}. \end{aligned} \quad (4.80)$$

- If $z \leq \varepsilon^{1/4} \|H\|_{L^\infty(\mathfrak{B})}^2$ we have that :

$$d((x, z), \partial\Omega) \leq d((x, z), \mathfrak{B} \times \{0\}) \leq \varepsilon^{1/4} \|H\|_{L^\infty(\mathfrak{B})}^2. \quad (4.81)$$

So, from (4.78)-(4.81) we obtain for all $\varepsilon \ll \varepsilon_0(\Omega)$ and $(x, z) \in \Omega \setminus \Omega_\varepsilon$ that :

$$d((x, z), \partial\Omega) \leq C\varepsilon^{1/4},$$

which implies (4.40).

Step 5 : proof of the regularity of the domains Ω_ε . Next, let us prove that if $\varepsilon \ll \varepsilon_0(\Omega, \omega)$ the open sets Ω_ε satisfy (4.17) and (4.18). Indeed, if $\varepsilon \ll \varepsilon_0(\Omega, \omega)$ we have by (4.40) that :

$$\sup_{\Omega \setminus \Omega_\varepsilon} d(\cdot, \partial\Omega) < d(\omega, \partial\Omega),$$

which implies that $\omega \subset \Omega_\varepsilon$. Consequently, we only have to prove that if $\varepsilon \ll \varepsilon_0(\Omega)$ the domains Ω_ε are C^2 (see Remark 4.21), which we will prove by using Lemma 4.34. So we fix $\varepsilon \ll \varepsilon_0(\Omega)$ and check that the domain Ω_ε satisfies the requirements of Lemma 4.34 :

First, it is clear that \mathfrak{D}_ε is a function which belongs to $C^2(\overline{\Omega})$ ($\rho \in C^2(\text{supp}(\chi_\varepsilon))$ by (4.66)). Moreover, it satisfies $\mathfrak{D}_\varepsilon \leq 0$ on $\partial\Omega$. Indeed, we consider (4.76) and the division of the boundary as stated in Remark 4.2 :

- If $(x, z) \in \mathfrak{B} \times \{0\}$, then $\mathfrak{D}_\varepsilon(x, z) = 0$.
- If $(x, z) \in \mathfrak{L}$, then $x \in \partial\mathfrak{B}$, so $\chi_\varepsilon(x) = 0$ (by the support of χ_ε), so $\mathfrak{D}_\varepsilon(x, z) = 0$.
- If $(x, z) \in \mathfrak{T}$, then $z = H(x)$. If $\mathfrak{D}_\varepsilon(x, z) \neq 0$ we have that $x \in \text{supp}(\chi_\varepsilon) \subset \mathfrak{B}_\varepsilon$, so $H_\varepsilon(x) \leq H(x)$ by (4.72), so $\mathfrak{D}_\varepsilon(x, z) < 0$; otherwise, $\mathfrak{D}_\varepsilon(x, z) = 0$. Consequently, if $(x, z) \in \mathfrak{T}$ we have $\mathfrak{D}_\varepsilon(x, z) \leq 0$.

Thus, to apply Lemma 4.34 it remains to check that :

$$\mathfrak{D}_\varepsilon^{-1}(\varepsilon\|H\|_{L^\infty(\Omega)}^2) \cap (\nabla\mathfrak{D}_\varepsilon)^{-1}(\{0\}) = \emptyset. \quad (4.82)$$

For that purpose we fix $(x, z) \in \Omega$ satisfying :

$$\mathfrak{D}_\varepsilon(x, z) = \varepsilon\|H\|_{L^\infty(\mathfrak{B})}^2, \quad \partial_z\mathfrak{D}_\varepsilon(x, z) = 0, \quad (4.83)$$

and prove that $\nabla_x\mathfrak{D}_\varepsilon(x, z) \neq 0$.

We first remark that from (4.83)₁ we have that :

$$x \in \overline{\mathfrak{B}_{2\varepsilon}}. \quad (4.84)$$

Indeed, from (4.72) we obtain that :

$$\chi_\varepsilon(x)(H_\varepsilon(x) - z)z \leq \frac{\|H\|_{L^\infty(\mathfrak{B})}^2}{4},$$

which implies by (4.83)₁ that :

$$\rho(x) \geq 4\varepsilon, \quad (4.85)$$

which implies using (4.68) that :

$$\mathfrak{d}(x) \geq 2\varepsilon,$$

that is, (4.84).

From (4.77) and (4.83)₂ we get that $z = H_\varepsilon(x)/2$. Hence, from (4.84) and (4.83)₁ we have the equality :

$$\rho(x) = \frac{4\varepsilon\|H\|_{L^\infty(\mathfrak{B})}^2}{H_\varepsilon^2(x)}, \quad (4.86)$$

which implies together with (4.77) the equality :

$$\nabla \mathfrak{D}_\varepsilon(x, z) = \left(\frac{2\varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2}{H_\varepsilon(x)} \nabla H_\varepsilon(x) + \frac{H_\varepsilon^2(x)}{4} \nabla \rho(x), 0 \right).$$

So, because of (4.74) and (4.75) the previous equality turns into :

$$\nabla \mathfrak{D}_\varepsilon(x, z) = \left(O(\varepsilon) + \frac{H_\varepsilon^2(x)}{4} \nabla \rho(x), 0 \right). \quad (4.87)$$

So, from (4.87), (4.86), $\varepsilon \ll \ll \varepsilon_0(\Omega)$, (4.74), (4.66), (4.68) and (4.69) we obtain that (4.83) implies that $\nabla_x \mathfrak{D}_\varepsilon(x, z) \neq 0$, which proves (4.82).

Consequently, thanks to Lemma 4.34, if $\varepsilon \ll \ll \varepsilon_0(\Omega)$, we have that Ω_ε is C^2 and :

$$\partial \Omega_\varepsilon = \mathfrak{D}_\varepsilon^{-1}(\varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2). \quad (4.88)$$

Remark 4.37. In order to prove (4.84) we have only used (4.83)₁ and that $\varepsilon \ll \ll \varepsilon_0(\Omega)$. Consequently, if $\varepsilon \ll \ll \varepsilon_0(\Omega)$, considering (4.88), (4.84) and that $\mathfrak{B}_{2\varepsilon} \subset \chi_\varepsilon^{-1}(\{1\})$ we find that :

$$\rho(x)(H_\varepsilon(x) - z)z = \varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2, \quad \forall (x, z) \in \partial \Omega_\varepsilon. \quad (4.89)$$

Similarly, if $\varepsilon \ll \ll \varepsilon_0(\Omega)$ and $(x, z) \in \partial \Omega_\varepsilon$, using (4.63), (4.77) and (4.84) we obtain that :

$$n_\varepsilon(x, z) = -\frac{\nabla \mathfrak{D}_\varepsilon(x, z)}{|\nabla \mathfrak{D}_\varepsilon(x, z)|} = \frac{(-z\rho(x)\nabla H_\varepsilon(x) - z(H_\varepsilon(x) - z)\nabla \rho(x), \rho(x)(2z - H_\varepsilon(x)))}{|(-z\rho(x)\nabla H_\varepsilon(x) - z(H_\varepsilon(x) - z)\nabla \rho(x), \rho(x)(2z - H_\varepsilon(x)))|}, \quad (4.90)$$

where n_ε denotes the outward unit normal vector on $\partial \Omega_\varepsilon$.

Step 6 : the domains Ω_ε satisfy (4.19). Considering (4.90) and (4.36), we have to prove that if $\mathfrak{c} \gg \gg \mathfrak{c}_0(\Omega, \omega, \eta_{\mathfrak{B}})$, $\varepsilon \ll \ll \varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}}, \mathfrak{c})$ and $(x, z) \in \partial \Omega_\varepsilon$, we have that :

$$-z\rho(x)\nabla H_\varepsilon(x) \cdot \nabla \eta_{\mathfrak{B}}(x) - z(H_\varepsilon(x) - z)\nabla \rho(x) \cdot \nabla \eta_{\mathfrak{B}}(x) - 2\rho(x)(2z - H_\varepsilon(x))\mathfrak{c}(z - \bar{z}) \leq 0. \quad (4.91)$$

Intuitively, the proof consists on using that $\partial \Omega_\varepsilon$ is near $\partial \Omega$ and that n_ε somehow approximates n . Indeed, we follow different (but a finite number of) approaches depending if we are close to $\mathfrak{B} \times \{0\}$, \mathfrak{L} or \mathfrak{T} (see Remark 4.2 for the notation). This is necessary because η and n_ε have different behaviours in different parts of the boundary (they do not present any symmetry), so considering only (4.85) does not suffice and we have to be more precise.

Rigorously, we fix first $\mathbf{c} \geq \mathbf{c}_0(\Omega, \omega, \eta_{\mathfrak{B}})$, second $\varepsilon \ll \varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}}, \mathbf{c})$ and third $(x, z) \in \partial\Omega_\varepsilon$, for \mathbf{c}_0 and ε_0 to be obtained during the proof. We prove (4.91) for those values. As stated before, we follow different (but a finite number of) approaches depending on (x, z) . A first partition of the boundary is the following :

$$\begin{cases} \partial\Omega_{\varepsilon,b} := \partial\Omega_\varepsilon \cap \{(x, z) : z \in (0, H_\varepsilon(x)/2]\}, \\ \partial\Omega_{\varepsilon,t} := \partial\Omega_\varepsilon \cap \{(x, z) : z \in (H_\varepsilon(x)/2, H_\varepsilon(x))\}. \end{cases} \quad (4.92)$$

Case 1 : if $(x, z) \in \partial\Omega_{\varepsilon,b}$. From $\varepsilon \ll \varepsilon_0(\Omega)$, (4.92)₁, (4.89), and (4.72) we obtain that :

$$\varepsilon \|H\|_{L^\infty(\mathfrak{B})} \leq z\rho(x) \leq \frac{2\varepsilon \|H\|_{L^\infty(\mathfrak{B})}^2}{\inf_{\mathfrak{B}_\varepsilon} H_\varepsilon}. \quad (4.93)$$

Thanks to (4.74), from $\varepsilon \ll \varepsilon_0(\Omega)$ we have the inequality :

$$\frac{1}{\inf_{\mathfrak{B}_\varepsilon} H_\varepsilon} \leq \frac{2}{\inf_{\mathfrak{B}} H};$$

so if we define :

$$\kappa_1 := \|H\|_{L^\infty(\mathfrak{B})}, \quad \kappa_2 := \frac{4\|H\|_{L^\infty(\mathfrak{B})}^2}{\inf_{\mathfrak{B}} H}, \quad (4.94)$$

we have that (4.93) and $\varepsilon \ll \varepsilon_0(\Omega)$ imply that :

$$\kappa_1\varepsilon \leq z\rho(x) \leq \kappa_2\varepsilon. \quad (4.95)$$

So, either $\rho(x) \leq \sqrt{\kappa_2\varepsilon}$ or $z \leq \sqrt{\kappa_2\varepsilon}$ is obtained. We study both cases separately, since in the first one (x, z) is near \mathfrak{L} , whereas in the second one (x, z) is near $\mathfrak{B} \times \{0\}$.

Case 1.1 : if $(x, z) \in \partial\Omega_{\varepsilon,b}$ and if $\rho(x) \leq \sqrt{\kappa_2\varepsilon}$. Due to (4.95) we have the bound :

$$z \geq \kappa_1 \sqrt{\kappa_2^{-1}\varepsilon}. \quad (4.96)$$

We first focus on the sign of :

$$-z\rho(x)\nabla H_\varepsilon(x) \cdot \nabla \eta_{\mathfrak{B}}(x) - z(H_\varepsilon(x) - z)\nabla \rho(x) \cdot \nabla \eta_{\mathfrak{B}}(x).$$

From (4.95) we obtain the estimate :

$$|z\rho(x)\nabla H_\varepsilon(x) \cdot \nabla \eta_{\mathfrak{B}}(x)| \leq C\varepsilon. \quad (4.97)$$

Moreover, we have the equality (see (4.65)) :

$$-z(H_\varepsilon(x) - z)\nabla\rho(x) = z(H_\varepsilon(x) - z)n_{\mathfrak{B}}(P(x)) + z(H_\varepsilon(x) - z)(-n_{\mathfrak{B}}(P(x)) - \nabla\rho(x)). \quad (4.98)$$

Recalling (4.65)-(4.69), from $\varepsilon \ll \varepsilon_0(\Omega)$ we obtain the estimate :

$$\begin{aligned} | -n_{\mathfrak{B}}(P(x)) - \nabla\rho(x) | &= | \nabla\rho(P(x)) - \nabla\rho(x) | \\ &\leq \|\rho\|_{W^{2,\infty}(\mathfrak{B})} |P(x) - x| = \|\rho\|_{W^{2,\infty}(\mathfrak{B})} \mathfrak{d}(x) \leq \frac{3}{2} \|\rho\|_{W^{2,\infty}(\mathfrak{B})} \rho(x). \end{aligned} \quad (4.99)$$

So, combining (4.99) and (4.95) we get the bound :

$$|z(H_\varepsilon(x) - z)(-n_{\mathfrak{B}}(P(x)) - \nabla\rho(x)) \cdot \nabla\eta_{\mathfrak{B}}(x)| \leq C|z\rho(x)| \leq C\varepsilon. \quad (4.100)$$

Finally, we have the equality :

$$\begin{aligned} z(H_\varepsilon(x) - z)n_{\mathfrak{B}}(P(x)) \cdot \nabla\eta_{\mathfrak{B}}(x) &= z(H_\varepsilon(x) - z)n_{\mathfrak{B}}(P(x)) \cdot \nabla\eta_{\mathfrak{B}}(P(x)) \\ &\quad + z(H_\varepsilon(x) - z)n_{\mathfrak{B}}(P(x)) \cdot (-\nabla\eta_{\mathfrak{B}}(P(x)) + \nabla\eta_{\mathfrak{B}}(x)). \end{aligned} \quad (4.101)$$

Since $\eta_{\mathfrak{B}} \in C^2(\overline{\mathfrak{B}})$, arguing similarly to (4.99) we obtain the estimate :

$$|z(H_\varepsilon(x) - z)n_{\mathfrak{B}}(P(x)) \cdot (-\nabla\eta_{\mathfrak{B}}(P(x)) + \nabla\eta_{\mathfrak{B}}(x))| \leq C\varepsilon. \quad (4.102)$$

Moreover, if we take into account (4.96), (4.92)₁, (4.74) and (4.35) we have the bound :

$$z(H_\varepsilon(x) - z)n_{\mathfrak{B}}(P(x)) \cdot \nabla\eta_{\mathfrak{B}}(P(x)) \leq -c\sqrt{\varepsilon}. \quad (4.103)$$

Consequently, if we combine (4.97)-(4.103), we find that :

$$-z\rho(x)\nabla H_\varepsilon(x) \cdot \nabla\eta_{\mathfrak{B}}(x) - z(H_\varepsilon(x) - z)\nabla\rho(x) \cdot \nabla\eta_{\mathfrak{B}}(x) \leq C\varepsilon - c\sqrt{\varepsilon}. \quad (4.104)$$

Let us now analyze the term :

$$-2\rho(x)(2z - H_\varepsilon(x))\mathfrak{c}(z - \bar{z}) = 2\rho(x)(H_\varepsilon(x) - 2z)\mathfrak{c}(z - \bar{z}).$$

- On the one hand, if $z \leq \bar{z}$ (see (4.32) for the definition of \bar{z}), we have that (see (4.92)₁) :

$$2\rho(x)(H_\varepsilon(x) - 2z)\mathfrak{c}(z - \bar{z}) \leq 0,$$

which, together with (4.104) implies (4.91) since $\varepsilon \ll \varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}})$.

- On the other hand, if $z \geq \bar{z}$, we get from (4.95) the inequality :

$$\rho(x) \leq C\varepsilon. \quad (4.105)$$

Thus, we obtain the estimate :

$$|2\rho(x)(H_\varepsilon(x) - 2z)\mathbf{c}(z - \bar{z})| \leq C\mathbf{c}\varepsilon. \quad (4.106)$$

Therefore, if we combine (4.104) and (4.106), we get the bound :

$$\begin{aligned} -z\rho(x)\nabla H_\varepsilon(x) \cdot \nabla \eta_{\mathfrak{B}}(x) - z(H_\varepsilon(x) - z)\nabla \rho(x) \cdot \nabla \eta_{\mathfrak{B}}(x) - 2\rho(x)(2z - H_\varepsilon(x))\mathbf{c}(z - \bar{z}) \\ \leq C(1 + \mathbf{c})\varepsilon - c\sqrt{\varepsilon}, \end{aligned}$$

which implies (4.91) since $\varepsilon \ll \varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}}, \mathbf{c})$.

Remark 4.38. Note that for some control domains there are $(x, z) \in \partial\Omega_{\varepsilon, b}$ satisfying $\rho(x) \leq \sqrt{\kappa_2\varepsilon}$ and (4.105). In fact, (4.105) provides us some additional information because $\sqrt{\varepsilon} \gg \varepsilon$ for ε small. Moreover, this does not suppose a contradiction with (4.85) because it just means that $C > 4$ in (4.105).

Case 1.2 : if $(x, z) \in \partial\Omega_{\varepsilon, b}$ and if $z \leq \sqrt{\kappa_2\varepsilon}$. We first remark that $\varepsilon \ll \varepsilon_0(\Omega, \omega)$ implies :

$$z \leq \sqrt{\kappa_2\varepsilon} \leq \frac{\bar{z}}{4} \leq \frac{\inf_{\mathfrak{B}} H}{4}. \quad (4.107)$$

This can be done because κ_1, κ_2 and \bar{z} are strictly positive constants that only depend on Ω and ω (see (4.94) and (4.32)). Moreover, we have from (4.95) that :

$$\rho(x) \geq \kappa_2\sqrt{\kappa_1^{-1}\varepsilon}. \quad (4.108)$$

If we use (4.95), (4.75), (4.72), that $z \leq \sqrt{\kappa_2\varepsilon}$, (4.66), that $\eta_{\mathfrak{B}} \in C^2(\overline{\mathfrak{B}})$, (4.108), (4.107) and (4.74), we obtain from $\varepsilon \ll \varepsilon_0(\Omega, \omega)$ the inequality :

$$\begin{aligned} -z\rho(x)\nabla H_\varepsilon(x) \cdot \nabla \eta_{\mathfrak{B}}(x) - z(H_\varepsilon(x) - z)\nabla \rho(x) \cdot \nabla \eta_{\mathfrak{B}}(x) - 2\rho(x)(2z - H_\varepsilon(x))\mathbf{c}(z - \bar{z}) \\ \leq C\varepsilon + C\varepsilon^{1/2} - c\varepsilon^{1/2}. \end{aligned}$$

Consequently, as $\mathbf{c} \gg \mathbf{c}_0(\Omega, \omega, \eta_{\mathfrak{B}})$ and $\varepsilon \ll \varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}}, \mathbf{c})$, we have (4.91).

Case 2 : if $(x, z) \in \partial\Omega_{\varepsilon,t}$. As for $\partial\Omega_{\varepsilon,t}$ the situation is very similar. Indeed, from (4.89) we find the bounds :

$$\kappa_1\varepsilon \leq (H_\varepsilon(x) - z)\rho(x) \leq \kappa_2\varepsilon, \quad (4.109)$$

for κ_1 and κ_2 defined in (4.94). As before, we distinguish the cases $\rho(x) \leq \sqrt{\kappa_2\varepsilon}$ and $H_\varepsilon(x) - z \leq \sqrt{\kappa_2\varepsilon}$. The first one concerns the points near \mathfrak{L} , whereas the second one concerns the points near \mathfrak{T} .

Case 2.1 : if $(x, z) \in \partial\Omega_{\varepsilon,t}$ and if $\rho(x) \leq \sqrt{\kappa_2\varepsilon}$. Due to (4.109) we have the bound :

$$H_\varepsilon(x) - z \geq \kappa_1\sqrt{\kappa_2^{-1}\varepsilon}. \quad (4.110)$$

Arguing as in the case 1.1, we get the estimate :

$$-z(H_\varepsilon(x) - z)\nabla\rho(x) \cdot \nabla\eta_{\mathfrak{B}}(x) \leq C\varepsilon - c\varepsilon^{1/2}. \quad (4.111)$$

In order to continue, as before, we make a distinction depending on how close z is to $H_\varepsilon(x)$:

- We start with the subcase :

$$z \geq \max\left\{\sqrt{\frac{\bar{z}}{\inf_{\mathfrak{B}} H}}, \frac{2}{3}\right\} H_\varepsilon(x). \quad (4.112)$$

We obtain from (4.33) and $\varepsilon \ll \varepsilon_0(\Omega, \omega)$ the inequality :

$$H_\varepsilon(x) \geq \sqrt[4]{\frac{\bar{z}}{\inf_{\mathfrak{B}} H}} H(x). \quad (4.113)$$

Thanks to (4.112), (4.113) and $\varepsilon \ll \varepsilon_0(\Omega, \omega)$, we have the lower bound :

$$z \geq \sqrt[4]{(\bar{z})^3 \inf_{\mathfrak{B}} H} = \sqrt[4]{\frac{\inf_{\mathfrak{B}} H}{\bar{z}}} \bar{z}. \quad (4.114)$$

So, combining (4.112), (4.74), (4.114) and (4.33) and that $\varepsilon \ll \varepsilon_0(\Omega, \omega)$, we obtain the inequality :

$$-2\rho(x)(2z - H_\varepsilon(x))\mathfrak{c}(z - \bar{z}) \leq -c\mathfrak{c}\rho(x). \quad (4.115)$$

Moreover, we clearly have the estimate :

$$-z\rho(x)\nabla H_\varepsilon(x) \cdot \nabla\eta_{\mathfrak{B}}(x) \leq C\rho(x). \quad (4.116)$$

Thus, we get from $\mathfrak{c} \gg \mathfrak{c}_0(\Omega, \omega, \eta_{\mathfrak{B}})$, $\rho \geq 0$, (4.115) and (4.116) the upper bound :

$$-z\rho(x)\nabla H_\varepsilon(x) \cdot \nabla\eta_{\mathfrak{B}}(x) - 2\rho(x)(2z - H_\varepsilon(x))\mathfrak{c}(z - \bar{z}) \leq 0. \quad (4.117)$$

Hence, combining (4.111) and (4.117) we get (4.91) from $\varepsilon \ll \varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}})$.

- Let us now suppose that :

$$z < \max \left\{ \sqrt{\frac{\bar{z}}{\inf_{\mathfrak{B}} H}}, \frac{2}{3} \right\} H_\varepsilon(x). \quad (4.118)$$

Considering (4.109) and (4.74) we have that :

$$\rho(x) < C\varepsilon. \quad (4.119)$$

Thus, we find the estimate :

$$-z\rho(x)\nabla H_\varepsilon(x) \cdot \nabla \eta_{\mathfrak{B}}(x) - 2\rho(x)(2z - H_\varepsilon(x))\mathbf{c}(z - \bar{z}) < C\mathbf{c}\varepsilon. \quad (4.120)$$

So, from (4.111), (4.120) and $\varepsilon \ll \ll \varepsilon_0(\Omega, \omega, \eta_{\mathfrak{B}}, \mathbf{c})$ we obtain (4.91).

Remark 4.39. Again, note that for some control domains there are $(x, z) \in \partial\Omega_{\varepsilon, t}$ satisfying $\rho(x) \leq \sqrt{\kappa_2\varepsilon}$ and (4.119). In fact, (4.119) provides us some additional information because $\sqrt{\varepsilon} \gg \gg \varepsilon$ for ε small. Moreover, this does not suppose a contradiction with (4.85) because it just means that $C > 4$ in (4.119).

Case 2.2 : if $(x, z) \in \partial\Omega_{\varepsilon, t}$ and if $H_\varepsilon(x) - z \leq \sqrt{\kappa_2\varepsilon}$. We have from (4.109) the inequality :

$$\rho(x) \geq \kappa_1 \sqrt{\kappa_2^{-1}\varepsilon}. \quad (4.121)$$

Moreover, from (4.75), $H_\varepsilon(x) - z \leq \sqrt{\kappa_2\varepsilon}$ and (4.121) we find the upper bound :

$$-z\rho(x)\nabla H_\varepsilon(x) \cdot \nabla \eta_{\mathfrak{B}}(x) - z(H_\varepsilon(x) - z)\nabla \rho(x) \cdot \nabla \eta_{\mathfrak{B}}(x) \leq C(\rho(x) + \varepsilon^{1/2}) \leq C\rho(x). \quad (4.122)$$

In addition, since $H_\varepsilon(x) - z \leq \sqrt{\kappa_2\varepsilon}$ and $\varepsilon \ll \ll \varepsilon_0(\Omega, \omega)$ we have that :

$$z > \max \left\{ \sqrt{\frac{\inf_{\mathfrak{B}_\varepsilon} H_\varepsilon}{\bar{z}}}, \frac{2}{3} H_\varepsilon(x) \right\}. \quad (4.123)$$

Indeed, because of (4.74), $H_\varepsilon(x) - z \leq \sqrt{\kappa_2\varepsilon}$ and $\varepsilon \ll \ll \varepsilon_0(\Omega)$ we have that :

$$z \geq H_\varepsilon(x) - \sqrt{\kappa_2\varepsilon} \geq H_\varepsilon(x) - \frac{1}{6} \inf_{\mathfrak{B}} H \geq H_\varepsilon(x) - \frac{1}{3} \inf_{\mathfrak{B}_\varepsilon} H_\varepsilon(x) \geq \frac{2}{3} H_\varepsilon(x).$$

Similarly, as a consequence of (4.74), (4.33) and $\varepsilon \ll \ll \varepsilon_0(\Omega, \omega)$ we have that :

$$\begin{aligned} z &\geq H_\varepsilon(x) - \sqrt{\kappa_2\varepsilon} \geq \inf_{\mathfrak{B}_\varepsilon} H_\varepsilon - \sqrt{\kappa_2\varepsilon} \geq \inf_{\mathfrak{B}_\varepsilon} H_\varepsilon - \frac{1}{2} \left(1 - \sqrt{\frac{\bar{z}}{\inf_{\mathfrak{B}} H}} \right) \inf_{\mathfrak{B}} H \\ &\geq \inf_{\mathfrak{B}_\varepsilon} H_\varepsilon - \left(1 - \sqrt{\frac{\bar{z}}{\inf_{\mathfrak{B}_\varepsilon} H_\varepsilon}} \right) \inf_{\mathfrak{B}_\varepsilon} H_\varepsilon = \sqrt{\frac{\bar{z}}{\inf_{\mathfrak{B}_\varepsilon} H_\varepsilon}} \inf_{\mathfrak{B}_\varepsilon} H_\varepsilon = \sqrt{\frac{\inf_{\mathfrak{B}_\varepsilon} H_\varepsilon}{\bar{z}}}. \end{aligned}$$

So, from (4.123), (4.74) and (4.33) we find that :

$$-2\rho(x)(2z - H_\varepsilon(x))\mathfrak{c}(z - \bar{z}) \leq -c\mathfrak{c}\rho(x). \quad (4.124)$$

Consequently, from (4.122), (4.124) and $\mathfrak{c} \gg \gg \gg \mathfrak{c}_0(\Omega, \omega, \eta_{\mathfrak{B}})$ we obtain (4.91).

4.B The existence of a pseudo-cylinder which is not locally star-shaped

We show in this section the existence of a pseudo-cylinder in \mathbb{R}^2 which is not locally star-shaped. The structure of this section is the following : first we recall the definition of locally star-shaped as stated in [7], but adapted to our notation, second we construct the pseudo-cylinder, and thirdly we prove that it is not locally star-shaped.

Definition 4.40. A Lipschitz domain Ω is *locally star-shaped* if for all $p \in \partial\Omega$ there is $r_p > 0$ and $q_p \in \Omega$ such that $|p - q_p| < r_p$ and $B(q_p, r_p) \cap \Omega$ is star-shaped with center in q_p .

Remark 4.41. As a consequence of [7, Remark 3.5] we can choose r_p as small as needed. This implies that a locally star-shaped domain can also be defined as a Lipschitz domain Ω such that for all $p \in \partial\Omega$ and all $r > 0$ there is some $\tilde{r} \in (0, r)$ and $q \in \Omega$ satisfying $|p - q| < \tilde{r}$ and that $B(q, \tilde{r}) \cap \Omega$ is star-shaped with center in q .

We define :

$$\Omega = \{(x, z) : x \in (0, 1), z \in (\tilde{H}(x), 3)\},$$

for

$$\tilde{H}(x) := 6 \int_0^x h(s) ds, \quad \text{for } h(x) := \sum_{i \in \mathbb{N}_*} 1_{[1/3^i, 2/3^i)}(x) - \sum_{i \in \mathbb{N}_*} 1_{[2/3^i, 1/3^{i-1})}(x).$$

We have that Ω clearly is a pseudo-cylinder (take $U(x, z) = (x, 3 - z)$, $\mathfrak{B} = (0, 1)$ and $H(x) = 3 - \tilde{H}(x)$ in Definition 4.1, as $\inf_{(0,1)} H(x) = H(2/3) = 3 - 2 = 1$). See Figure 4.5 for an approximate illustration of Ω near each point $(\frac{2}{3^{i+2}}, \frac{2}{3^{i+1}})$, for $i \in \mathbb{N}_*$.

Remark 4.42. \tilde{H} is a positive Lipschitz function which satisfies for all $i \in \mathbb{N}_*$ that $\tilde{H}(x) = -\frac{2}{3^{i-1}} + 6x$ in $[1/3^i, 2/3^i)$ and :

$$\tilde{H}(x) = \frac{2}{3^{i-2}} - 6x, \quad \forall x \in [2/3^i, 1/3^{i-1}). \quad (4.125)$$

Thus, its local minimums are taken at $x = 0$ and $x = 3^{-i}$ for all $i \in \mathbb{N}$ (with $\tilde{H}(3^{-i}) = 0$). In addition, its local maximums are taken at $x = 2 \cdot 3^{-i}$, for all $i \in \mathbb{N}_*$ (with $\tilde{H}(2 \cdot 3^{-i}) = 3 \cdot (2 \cdot 3^{-i})$). In fact, $\tilde{H}(x) \leq 3x$ for all $x \in [0, 1]$, being equal if and only if $x = 0$ or $x = 2 \cdot 3^{-i}$, for all $i \in \mathbb{N}_*$.

In order to prove that Ω is not locally star-shaped we show that for all $q \in B(0, 1/3) \cap \Omega$ there is a segment ℓ of length $|q|$ with one end in q and such that $\ell \cap \Omega$ is not connected. This shows that the alternative definition given in Remark 4.41 is not satisfied for $p = (0, 0)$ and $r = 1/3$. Hence, we fix $q = (x_q, z_q) \in B(0, 1/3) \cap \Omega$ and follow different approaches depending on q :

Case 1 : if $z_q < 3x_q$ (see q_1 in Figure 4.5). If $z_q < 3x_q$, we consider ($x_q > 0$ because $q \in \Omega$) :

$$\ell = \left\{ \left(x, \frac{z_q}{x_q} x \right) : x \in [0, x_q] \right\}.$$

Clearly, ℓ is of length $|q|$ and one of its ends is q . In addition, we have that ℓ leaves and enters Ω an infinite number of times, as we have for all $i \in \mathbb{N}_*$ that :

$$\frac{z_q}{x_q} \cdot (3^{-i}) > 0 = \tilde{H}(3^{-i}), \quad \frac{z_q}{x_q} \cdot (2 \cdot 3^{-i}) < 3 \cdot (2 \cdot 3^{-i}) = \tilde{H}(2 \cdot 3^{-i}),$$

so $\ell \cap \Omega$ is not connected.

Case 2 : if $z_q \geq 3x_q$ and if there is $i \in \mathbb{N}_$ such that $z_q \in [2/3^{i+1}, 2/3^i)$ and $z_q < \frac{2}{3^i} - 6x_q$ (see q_2 in Figure 4.5).* Under these hypotheses the point q is in the following triangle :

$$T_{1,i} := \left\{ (x, z) \in \mathbb{R}^2 : z \geq \frac{2}{3^{i+1}}, x > 0, z < \frac{2}{3^i} - 6x \right\}, \quad (4.126)$$

whose vertices are :

$$\left\{ \left(0, \frac{2}{3^{i+1}} \right), \left(\frac{2}{3^{i+2}}, \frac{2}{3^{i+1}} \right), \left(0, \frac{2}{3^i} \right) \right\}. \quad (4.127)$$

A consequence of (4.126) is that :

$$T_{1,i} \subset \left\{ (x, z) : x < \frac{2}{3^{i+2}} \right\}. \quad (4.128)$$

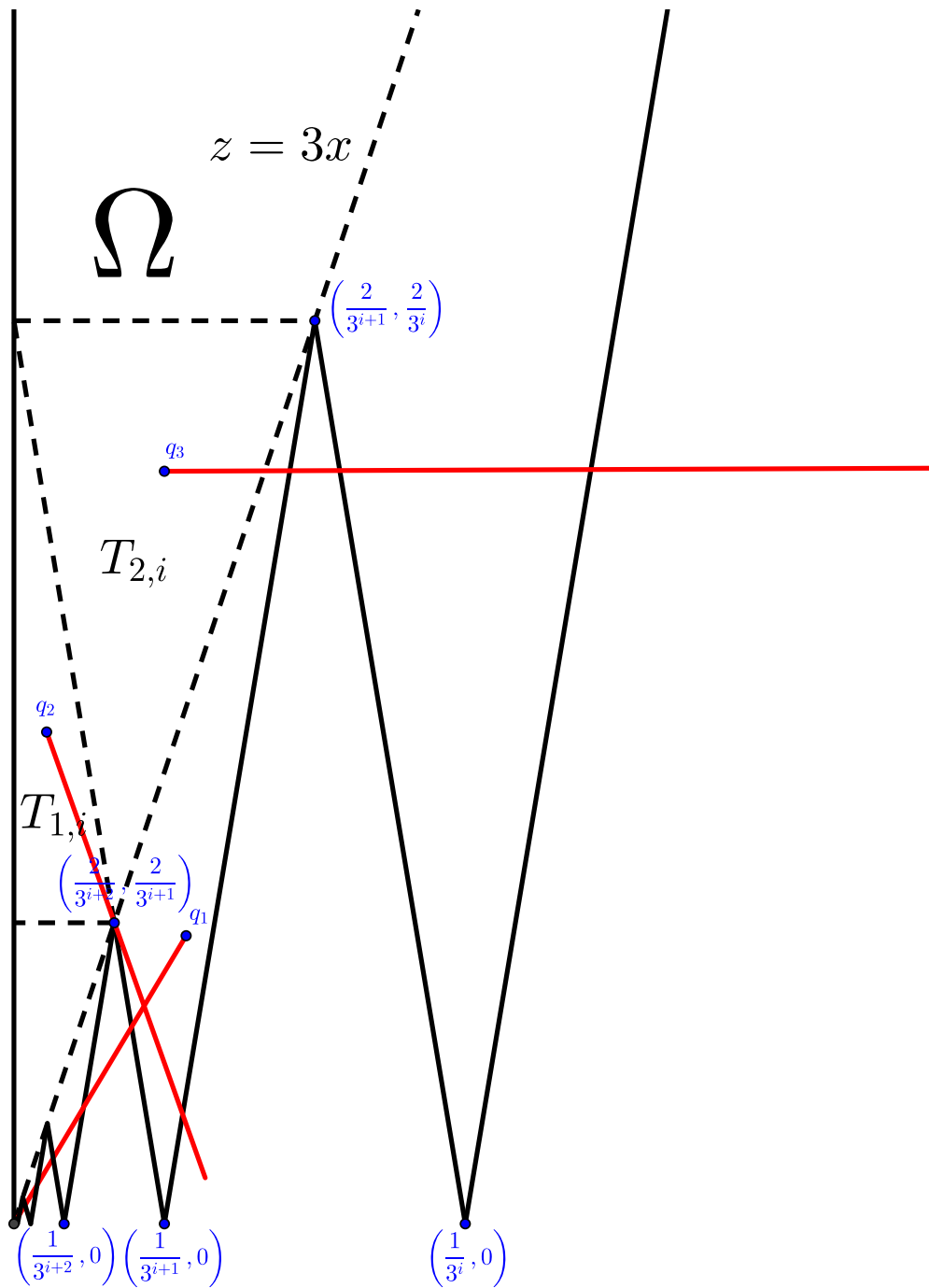


FIGURE 4.5 – A pseudo-cylinder which is not locally star-shaped (approximate figure).

Let us first prove that :

$$d\left(q, \left(\frac{2}{3^{i+2}}, \frac{2}{3^{i+1}}\right)\right) < |q|. \quad (4.129)$$

We remark that for all $(x, z) \in \overline{T_{1,i}}$ we have that :

$$\begin{aligned} d^2\left((x, z), \left(\frac{2}{3^{i+2}}, \frac{2}{3^{i+1}}\right)\right) &= \left(\frac{2}{3^{i+2}} - x\right)^2 + \left(z - \frac{2}{3^{i+1}}\right)^2 \\ &= x^2 + z^2 - \frac{4}{3^{i+2}}x - \frac{4}{3^{i+1}}z + \frac{4}{3^{2i+4}} + \frac{4}{3^{2i+2}} := |(x, z)|^2 + \mathcal{G}_1(x, z). \end{aligned} \quad (4.130)$$

Therefore, proving (4.129) reduces to finding the maximum of \mathcal{G}_1 on $\overline{T_{1,i}}$, which is a simplex. Since \mathcal{G}_1 is an affine function, it is well-known that the maximum is reached on the vertices, which in this case are given by (4.127). Since :

$$\mathcal{G}_1\left(0, \frac{2}{3^{i+1}}\right) = -\frac{32}{3^{2i+4}}, \quad \mathcal{G}_1\left(\frac{2}{3^{i+2}}, \frac{2}{3^{i+1}}\right) = -\frac{40}{3^{2i+4}}, \quad \mathcal{G}_1\left(0, \frac{2}{3^i}\right) = -\frac{176}{3^{2i+4}}. \quad (4.131)$$

we obtain from (4.130) and (4.131) that $d^2\left((x, z), \left(\frac{2}{3^{i+2}}, \frac{2}{3^{i+1}}\right)\right) < |(x, z)|^2$ for all $(x, z) \in T_{1,i}$, which in particular implies (4.129).

We define :

$$\ell := \{(x, g(x)) : |(x, g(x)) - q| \leq |q|, x \geq x_q\}, \quad \text{for } g(x) := -\frac{z_q - \frac{2}{3^{i+1}}}{\frac{2}{3^{i+2}} - x_q}(x - x_q) + z_q. \quad (4.132)$$

We have that g is well-defined because of (4.128). In fact, ℓ is the segment with one end in q and containing $\left(\frac{2}{3^{i+2}}, \frac{2}{3^{i+1}}\right)$ because of (4.129) and because $g\left(\frac{2}{3^{i+2}}\right) = \frac{2}{3^{i+1}}$. Moreover, we recall that :

$$\left(\frac{2}{3^{i+2}}, \frac{2}{3^{i+1}}\right) \subset \partial\Omega \subset \mathbb{R}^2 \setminus \Omega.$$

In addition, using (4.126)₃ and (4.128) we obtain that :

$$\frac{\frac{2}{3^{i+1}} - z_q}{\frac{2}{3^{i+2}} - x_q} > \frac{\frac{2}{3^{i+1}} - \frac{2}{3^i} + 6x_q}{\frac{2}{3^{i+2}} - x_q} = -\frac{\frac{4}{3^{i+1}} - 6x_q}{\frac{2}{3^{i+2}} - x_q} = -6. \quad (4.133)$$

Considering that in a neighbourhood of $\left(\frac{2}{3^{i+2}}, \frac{2}{3^{i+1}}\right)$ the set $\partial\Omega$ is a segment of slope -6 (see (4.125) for $i+2$ instead of i) and (4.133), we get by continuity for some $\varepsilon_q > 0$ small enough that :

$$\left(\frac{2}{3^{i+2}} + \varepsilon_q, g\left(\frac{2}{3^{i+1}} + \varepsilon_q\right)\right) \in \ell \cap \Omega.$$

Consequently, ℓ is a segment of length $|q|$, with one end in q and such that $\ell \cap \Omega$ is not connected.

Case 3 : if $z_q \geq 3x_q$ and if there is $i \in \mathbb{N}_*$ such that $z_q \in [2/3^{i+1}, 2/3^i)$ and $z_q \geq \frac{2}{3^i} - 6x_q$ (see q_3 in Figure 4.5). Under these hypotheses q is in the triangle :

$$T_{2,i} := \left\{ (x, z) : z < \frac{2}{3^i}, z \geq \frac{2}{3^i} - 6x, z \geq 3x \right\} \setminus \left\{ \left(\frac{2}{3^{i+1}}, \frac{2}{3^i} \right) \right\}, \quad (4.134)$$

whose vertices are given by :

$$\left\{ \left(0, \frac{2}{3^i} \right), \left(\frac{2}{3^{i+2}}, \frac{2}{3^{i+1}} \right), \left(\frac{2}{3^{i+1}}, \frac{2}{3^i} \right) \right\}.$$

It can be deduced from (4.134) that :

$$T_{2,i} \subset \left\{ (x, z) : z > \frac{2}{3^{i+1}} \right\}. \quad (4.135)$$

We remark that the horizontal distance of the points in the line $z = \frac{2}{3^i} - 6x$ with respect to the line $z = \frac{2}{3^{i+1}} - 6x$ is given by $\frac{2}{3^{i+1}}$ (both lines are parallel). This implies that the vertical distance of q to the line $z = \frac{2}{3^{i+1}} - 6x$ is bounded by $\frac{2}{3^{i+1}}$. Moreover, because of (4.135) we have that $|(x, z)| > \frac{2}{3^{i+1}}$ for all $(x, z) \in T_{2,i}$. Thus, the segment $\ell = \{(x, z_q) : x \in [x_q, x_q + |q|]\}$ is of length $|q|$, one of its ends is q , and satisfies that $\ell \cap \Omega$ is not connected, as $(\frac{2}{3^{i+1}}, z_q) \in \ell \cap (\mathbb{R}^2 \setminus \Omega)$ and $(\frac{1}{3^i} - \frac{z_q}{6} + \varepsilon_q, z_q) \in \ell \cap \Omega$ for some $\varepsilon_q > 0$ small enough (see (4.125) for $i + 1$ instead of i).

Remark 4.43. The construction given in this section can clearly be generalized to higher dimensions.

Chapitre 5

Local null controllability of a model system for strong interactions between internal solitary waves

In this chapter we prove the local null controllability property for a nonlinear coupled system of two Korteweg-de Vries equations posed on a bounded interval and with a source term decaying exponentially on $t = T$. The system was introduced by Gear and Grimshaw to model the interactions of two dimensional, long, internal gravity waves propagation in a stratified fluid. We address the controllability problem by means of a control supported on an interior open subset of the domain and acting on one equation only. The proof consists mainly on proving the controllability of the linearized system, which is done by getting a Carleman estimate for the adjoint system. While doing the Carleman we improve the techniques for dealing with the fact that the solutions of dispersive and parabolic equations with a source term in L^2 have a limited regularity. A local inversion theorem is applied to get the result for the nonlinear system. This chapter is included in [17].

5.1 Introduction

Nonlinear dispersive partial differential equations naturally appear as models describing wave phenomena in various branches of physics and engineering, such as quantum mechanics, nonlinear optics, plasma physics, water waves, and atmospheric sciences. They

have received wide attention from the applied science community due to their importance in applications and have also been studied extensively from the theoretical point of view, providing a framework for development of analytical ideas and tools.

In this context, the interaction of internal gravity waves propagating in one horizontal direction has been considered by Gear and Grimshaw [58]. More precisely, the authors derived a model to describe strong interactions of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion. It can be written as

$$\begin{aligned} u_t + uu_x + u_{xxx} + av_{xxx} + k_1vv_x + k_2(uv)_x &= 0, \\ cv_t + rv_x + vv_x + abu_{xxx} + v_{xxx} + k_2buv_x + k_1b(uv)_x &= 0, \end{aligned} \quad (5.1)$$

where a, b, c, r, k_1, k_2 are real parameters, the unknowns u and v are real valued functions of the variables x and t and subscripts indicate partial differentiation. Observe that (5.1) has the structure of a pair of Korteweg-de Vries (KdV) equations with both linear and nonlinear coupling terms.

We also refer to [11] for an extensive discussion on the physical relevance of the system in its full structure.

5.1.1 Setting of the problem

In this paper, we are concerned with the study of the controllability properties of the system (5.1), posed on a bounded interval $(0, L)$, with homogeneous boundary conditions. The main purpose is to steer the solutions of the system to the rest by means of a control supported on an interior open subset ω of $(0, L)$, acting on one equation only. As usual, we denote

$$\begin{aligned} Q &:= (0, T) \times (0, L), \quad Q_\omega := (0, T) \times \omega, \quad \text{for any open subset } \omega \text{ of } (0, L), \\ 1_\omega &, \text{ the characteristic function on } \omega. \end{aligned} \quad (5.2)$$

With the notation above, our attention is given to the following distributed control

system with initial data (u^0, v^0) , source term $(\mathfrak{h}_1, \mathfrak{h}_2)$ and control f :

$$\begin{cases} u_t + uu_x + u_{xxx} + av_{xxx} + k_1vv_x + k_2(uv)_x = \mathfrak{h}_1, & \text{in } Q, \\ cv_t + rv_x + vv_x + abu_{xxx} + v_{xxx} + k_2buu_x + k_1b(uv)_x = \mathfrak{h}_2 + f1_\omega, & \text{in } Q, \\ u(\cdot, 0) = u(\cdot, L) = u_x(\cdot, L) = 0, & \text{on } (0, T), \\ v(\cdot, 0) = v(\cdot, L) = v_x(\cdot, L) = 0, & \text{on } (0, T), \\ u(0, \cdot) = u^0, \quad v(0, \cdot) = v^0, & \text{on } (0, L). \end{cases} \quad (5.3)$$

Under these conditions, considerations will be given to the following null controllability problem :

Given $T > 0$, an initial state (u^0, v^0) in a certain space, and a source term $(\mathfrak{h}_1, \mathfrak{h}_2)$ in a certain space, can one find an appropriate control f , so that the system (5.3) admits a solution (u, v) which satisfies $(u(T, \cdot), v(T, \cdot)) = (0, 0)$?

Our main result gives a positive answer to this question. The proof combines the analysis of the linearized system and an inverse function theorem for the full system. In order to analyze the linearized system, we follow the classical duality approach [43, 83], which reduces the null controllability property to the proof of an *observability inequality* for the solutions of the corresponding adjoint system. Here, this is done by using Carleman estimates.

In order to overcome some technical difficulties that will become clear later, we assume that the coefficients a , b and c satisfy

$$a, b, c > 0 \quad \text{and} \quad a^2b < 1. \quad (5.4)$$

To our knowledge, this problem has not been addressed in the literature and the existing developments do not allow to give an immediate answer to it. Indeed, the controllability problem for system (5.1) was first solved when the control acts through the boundary conditions [18, 19, 29, 90]. The results are obtained combining the analysis of the linearized system and the Banach's fixed-point theorem. In order to study the linear problem, the authors follow closely the ideas introduced by Rosier [94] while studying the boundary controllability of the KdV equation (based on compactness arguments, a spectral analysis of the state operator and multipliers). But, being different from other systems, the length L of the spatial domain may play a crucial role in determining the controllability of the system, specially when some configurations of the controls input are allowed to be used.

This phenomenon, the so-called *critical length phenomenon*, was observed for the first time by Rosier in [94]. Roughly speaking, it was proved the existence of a finite dimensional subspace M of $L^2(0, L)$, which is not reachable by the KdV equation, when starting from the origin, if L belongs to a countable set of lengths.

More recently, by using the approach used in the present paper, the internal null controllability for the Hirota-Satsuma system was addressed in [28]. However, due to the complexity of the system (5.1), which has coupling constituted by third order terms, the control problem presents new difficulties we have to deal with. Then, we develop techniques applicable to this more complicated situation, which allow to slightly improve the result present in [28] and, in addition, can also be applied to other dispersive and parabolic systems.

To conclude, we can mention the work [20] where the authors consider the problem of controlling pointwise, by means of a time dependent Dirac measure supported at a given point, the linear system associated with (5.1) on the unit circle. The results are obtained by means of spectral analysis and Fourier expansion of the solutions.

5.1.2 Main results

Before stating our main result, we introduce the space

$$H^{p,p'}(Q) := H^p(0, T; L^2(0, L)) \cap L^2(0, T; H^{p'}(0, L)), \quad \text{for } p, p' \geq 0,$$

endowed with its natural norm, where Q was defined in (5.2). In addition, to shorten the notation we denote

- $U := (u, v)$, $U^0 := (u^0, v^0)$,
- $\mathbf{V} = V^2$, for V any Banach space (like L^2 , H_0^1 or $H^{p,p'}$),
- $\Phi := (\varphi, \phi)$ the adjoint variable,
- C (resp. q) a generic positive constant (resp. polynomial of positive coefficients) that may be different each time.

The main result that we prove in this paper concerns the local null controllability of (5.3) :

Theorem 5.1. *Let $L > 0$, $T > 0$ and assume that (5.4) is satisfied. Then, there is $C > 0$ such that for all $\mathfrak{h}_1, \mathfrak{h}_2$ satisfying*

$$e^{\frac{C}{(T-t)^{12}}}(\mathfrak{h}_1, \mathfrak{h}_2) \in L^2(0, T; \mathbf{H}^{-1}(0, L)),$$

there is $\delta > 0$ such that for all $\|U^0\| < \delta$ there is $f \in L^2(Q_\omega)$ and $U \in C([0, T]; \mathbf{L}^2(0, L))$ satisfying that (U, f) is a solution of (5.3) and $U(T, \cdot) = 0$.

As usual, we obtain the null controllability of (5.3) by proving the null controllability of the linearized system around 0. That system is given by

$$\begin{cases} \mathcal{L}U = (h_1, h_2 + f1_\omega), & \text{in } Q, \\ U(\cdot, 0) = U(\cdot, L) = U_x(\cdot, L) = 0, & \text{on } (0, T), \\ U(0, \cdot) = U^0, & \text{on } (0, L), \end{cases} \quad (5.5)$$

for $h_1, h_2 \in L^2(0, T; \mathbf{H}^{-1}(0, L))$ decaying exponentially at $t = T$ as specified in Section 5.4.1, $U^0 \in \mathbf{L}^2(0, L)$ and

$$\mathcal{L}U := (u_t + u_{xxx} + av_{xxx}, cv_t + rv_x + abu_{xxx} + v_{xxx}). \quad (5.6)$$

First, in Section 5.2 we study the main properties of the associated Cauchy problem, which is given by

$$\begin{cases} \mathcal{L}U = G, & \text{in } Q, \\ U(\cdot, 0) = U(\cdot, L) = U_x(\cdot, L) = 0, & \text{on } (0, T), \\ U(0, \cdot) = U^0, & \text{on } (0, L), \end{cases} \quad (5.7)$$

for $U^0 \in \mathbf{L}^2(0, L)$ and $G \in L^2(0, T; \mathbf{H}^{-1}(0, L))$. Afterwards, in order to show the null controllability of (5.5) we follow the classical duality approach [43, 83], which reduces the problem to prove an observability inequality for the solutions of the corresponding adjoint system, which is given by

$$\begin{cases} \mathcal{L}^*\Phi = G, & \text{in } Q, \\ \Phi(\cdot, 0) = \Phi(\cdot, L) = \Phi_x(\cdot, 0) = 0, & \text{on } (0, T), \\ \Phi(T, \cdot) = \Phi^T, & \text{on } (0, L), \end{cases} \quad (5.8)$$

for $\Phi^T \in \mathbf{L}^2(0, L)$, $G \in \mathbf{L}^2(Q)$ and

$$\mathcal{L}^*\Phi := (-\varphi_t - \varphi_{xxx} - ab\phi_{xxx}, -c\phi_t - r\phi_x - a\varphi_{xxx} - \phi_{xxx}). \quad (5.9)$$

In fact, we study the main properties of (5.8) in Section 5.2, and then we prove a Carleman estimate in Section 5.3.

One of the main contributions of this paper is to make it clear the difficulty of getting Carleman estimates for any well-posed dispersive or parabolic system with a force $G \in \mathbf{L}^2(Q)$ or $G = 0$. Indeed, to overcome the fact that solution Φ is not regular enough (because $G \in \mathbf{L}^2(Q)$) the proof of the Carleman estimate presents the novelty that we decompose Φ as the sum of the solution of three Cauchy problems. In the literature, the adjoint variable has been decomposed as the sum of the solution of two Cauchy problems (see, for instance, [23]), but for (5.8) we clearly need more. This implies some changes in the proof with respect to the existing results, which are explained in Section 5.3.2 and can be applied for many other dispersive and parabolic systems.

5.1.3 Historical background

There is a huge literature on controllability results with controls having a reduced number of components concerning parabolic equations (see the survey [5] and the references therein). For instance, there are a lot of results about Navier-Stokes-like systems, such as [15, 23, 26, 38, 39, 55, 68, 85]. It has also been studied, for example, the controllability in cascade-like systems [65, 27], the null controllability in the context of linear thermoelasticity [80], the controllability to trajectories in phase-field models [4], the existence of insensitizing controls for the heat equation [42], and the controllability in reaction-diffusion systems [3]. However, as for KdV-like systems, it is a very recent research topic which has been studied only in [20] and [28].

The main difference of the problem we consider in this paper with respect to the above cited papers, except [20], is the coupling. Indeed, in the papers cited above (and in the literature, as far as we know) the coupling is constituted by a zero, first or second order term. However, in this paper the coupling is constituted by a third order term. In this sense, the main contribution of this paper is in the technicalities of the Carleman, which provides a technique to move from the controllability of the dispersive linearized system with null force to the controllability of the linearized system with a force decaying in T (usually a necessary step to prove the local null controllability of some nonlinear systems).

In [20], the authors consider the problem of controlling pointwise, by means of a time dependent Dirac measure supported by a given point, the linearized system on the unit circle. More precisely, by means of spectral analysis and Fourier expansion they prove, under general assumptions on the physical parameters of the system, a pointwise observability inequality which leads to the pointwise controllability by using one control

function. It is also possible to find the sharp time of the controllability.

By contrast, the study of the boundary controllability properties is considerably more developed. The first result was obtained in [89], when the model is posed on a periodic domain and $r = 0$. In this case, a diagonalization of the main terms allows to decouple the corresponding linear system into two scalar KdV equations and to use the previous results available in the literature. Later on, in [90], the authors proved the local exact boundary controllability property for the nonlinear system (5.1) posed on a bounded interval. Their result was improved in [19], [29] and in [18], where the same boundary control problem was addressed with a different set of boundary conditions. In all those works, except in [29], the results were proved by applying the duality approach and some ideas introduced by Rosier in [94]. Then, it was discovered that whether the system is exactly controllable depends on the length of the spatial domain, i.e., if L does not lie in the so-called countable set of *critical lengths*. More precisely, the observability inequality is proved with the aid of a compactness argument that leads the issue to a nonstandard unique continuation principle for the eigenvalues of the state operator associated with the model. Then, the problem is solved by using the Paley-Wiener method introduced by Rosier, which allows the authors determine explicitly all the critical lengths for which the exact controllability fails for the linearized system. The same problem was addressed in [29] when only two controls act on the Neumann boundary conditions. In this case, the analysis of the linearized system is much more complicated. Therefore, the authors used a direct approach based on the multiplier technique that gives the observability inequality for small values of the length L and large time of control T .

Before closing this section, we mention the work [21], in which the boundary controllability problem was addressed for a Boussinesq system of KdV type.

The rest of the paper is organized as follows : in Section 5.2 we study the regularity estimates of some elliptic and evolution problems, in Section 5.3 we prove a Carleman inequality for (5.8), and finally in Section 5.4 we end the proof of Theorem 5.1.

5.2 Some elliptic and evolution problems

In this section we first prove a Poincaré inequality and secondly we prove some existence, uniqueness and regularity results for (5.7) and (5.8).

5.2.1 A Poincaré inequality

In this subsection we prove the following Poincaré inequality :

Proposition 5.2. *Let $L > 0$ and $i \in \mathbb{N}$. Then, there is $C > 0$ such that for all $w \in H^{2+i}(0, L)$ satisfying*

$$w(0) = w(L) = w'(0) = 0, \quad (5.10)$$

we have the estimate

$$\|w\|_{H^{2+i}(0,L)} \leq C \|w_{xxx}\|_{H^{i-1}(0,L)}. \quad (5.11)$$

Remark 5.3. By symmetry, for all $i \in \mathbb{N}$ we also have (5.11) for all $w \in H^{2+i}(0, L)$ satisfying

$$w(0) = w(L) = w'(L) = 0.$$

Proposition 5.2, for $i \geq 1$, is well-known and has been used in the literature (see for instance [62]). However, as far as we know, there is no reference for $i = 0$.

In order to prove Proposition 5.2, we use the following interpolation result :

Lemma 5.4. *Let $L > 0$. Then, there is $C > 0$ such that for all $z \in H^1(0, L)$ we have that*

$$\|z_x\|_{L^2(0,L)} \leq C(\|z\|_{L^2(0,L)} + \|z_{xx}\|_{H^{-1}(0,L)}). \quad (5.12)$$

Proof. [Proof of Lemma 5.4] We fix $\chi \in \mathcal{D}(0, L)$ such that $\int_0^L \chi = 1$. Then, for any $\psi \in \mathcal{D}(0, L)$ we have the estimate below

$$\begin{aligned} \left| \int_0^L z_x \psi \right| &= \left| \int_0^L z_x \left(\int_0^x \left(\psi(s) - \chi(s) \int_0^L \psi(z) dz \right) ds \right)_x dx + \int_0^L z_x \chi \int_0^L \psi(s) ds dx \right| \\ &= \left| \left\langle -z_{xx}, \int_0^x \left(\psi(s) - \chi(s) \int_0^L \psi(z) dz \right) ds \right\rangle_{H^{-1}(0,L) \times H_0^1(0,L)} - \int_0^L z \chi_x \int_0^L \psi(s) ds \right| \\ &\leq C(\|z_{xx}\|_{H^{-1}(0,L)} + \|z\|_{L^2(0,L)}) \|\psi\|_{L^2(0,L)}. \quad (5.13) \end{aligned}$$

Consequently, (5.12) is obtained from Riesz theorem and the density of $\mathcal{D}(0, L)$ in $L^2(0, L)$. \square

Proof. [Proof of Proposition 5.2] It suffices to prove Proposition 5.2 for $i = 0$. Indeed, once that case is proved, we have the following estimate

$$\begin{aligned} \|w\|_{H^{2+i}(0,L)} &\leq C(\|w\|_{H^2(0,L)} + \|w_{xxx}\|_{H^{i-1}(0,L)}) \\ &\leq C(\|w_{xxx}\|_{H^{-1}(0,L)} + \|w_{xxx}\|_{H^{i-1}(0,L)}) \leq C\|w_{xxx}\|_{H^{i-1}(0,L)}. \end{aligned}$$

We prove the case $i = 0$ by contradiction.

If Proposition 5.2 is false for $i = 0$, there is a sequence $w^j \in H^2(0, L)$ satisfying (5.10) such that

$$\|w^j\|_{H^2(0,L)} = 1, \quad (5.14)$$

and

$$\|w^j_{xxx}\|_{H^{-1}(0,L)} \rightarrow 0. \quad (5.15)$$

By compactness, we can suppose that w^j is convergent in $H^{7/4}(0, L)$ to some function w . The convergence implies that w satisfies (5.10) and that $w_{xxx} = 0$ as a distribution. Hence, we can prove easily that $w = 0$. Consequently,

$$\|w^j\|_{H^1(0,L)} \rightarrow 0, \quad (5.16)$$

which together with Lemma 5.4 for $z = w_x$ and (5.15) implies that

$$\|w^j_{xx}\|_{L^2(0,L)} \rightarrow 0. \quad (5.17)$$

Thus, we find from (5.16) and (5.17) that $\|w^j\|_{H^2(0,L)} \rightarrow 0$, contradicting (5.14). \square

5.2.2 Existence, uniqueness and regularity properties of the solutions of some dissipative evolution problems

We have the following result for (5.7), which is a consequence of the results presented in [90, Section 2.1] and classical semigroup theory :

Lemma 5.5. *Let $L > 0$, $T > 0$ and assume that (5.4) is satisfied. Then, for any $U^0 \in \mathbf{L}^2(0, L)$ and $G \in L^1(0, T; \mathbf{L}^2(0, L))$ system (5.7) admits a unique solution*

$$U \in C([0, T]; \mathbf{L}^2(0, L)) \cap L^2(0, T; \mathbf{H}^1(0, L)).$$

Moreover, there is $C > 0$ independent of U^0 and G such that

$$\|U\|_{C([0,T];\mathbf{L}^2(0,L)) \cap L^2(0,T;\mathbf{H}^1(0,L))} \leq C(\|U^0\|_{\mathbf{L}^2(0,L)} + \|G\|_{L^1(0,T;\mathbf{L}^2(0,L))}).$$

In addition, if $U^0 \in \mathcal{D}(0, L)$ and $G \in \mathcal{D}(Q)$ we have that $U \in \mathbf{C}^\infty(\overline{Q})$.

In a similar way, we have an analogous result for the adjoint system :

Lemma 5.6. *Let $L > 0$, $T > 0$ and assume that (5.4) is satisfied. Then, for any $\Phi^T \in \mathbf{L}^2(0, L)$ and $G \in L^1(0, T; \mathbf{L}^2(0, L))$ system (5.8) admits a unique solution*

$$\Phi \in C([0, T]; \mathbf{L}^2(0, L)) \cap L^2(0, T; \mathbf{H}^1(0, L)).$$

Moreover, there is $C > 0$ independent of Φ^T and G such that

$$\|\Phi\|_{C([0, T]; \mathbf{L}^2(0, L)) \cap L^2(0, T; \mathbf{H}^1(0, L))} \leq C(\|\Phi^T\|_{\mathbf{L}^2(0, L)} + \|G\|_{L^1(0, T; \mathbf{L}^2(0, L))}).$$

In addition, if $\Phi^T \in \mathcal{D}(0, L)$ and $G \in \mathcal{D}(Q)$ we have that $\Phi \in \mathbf{C}^\infty(\overline{Q})$.

To continue with, we prove more accurate results about the regularity of the solutions of (5.8) :

Proposition 5.7. *Let $L > 0$ and assume that (5.4) is satisfied. Then :*

1. *For all $T > 0$, $\Phi^T \in \mathbf{L}^2(0, L)$ and $G \in L^2(0, T; \mathbf{H}^{-1}(0, L))$ system (5.8) admits a unique solution*

$$\Phi \in C([0, T]; \mathbf{L}^2(0, L)) \cap L^2(0, T; \mathbf{H}^1(0, L)).$$

Moreover, there is $C > 0$ independent of T , Φ^T and G such that

$$\|\Phi\|_{C([0, T]; \mathbf{L}^2(0, L)) \cap L^2(0, T; \mathbf{H}^1(0, L))} \leq C(1+T)(\|\Phi^T\|_{\mathbf{L}^2(0, L)} + \|G\|_{L^2(0, T; \mathbf{H}^{-1}(0, L))}). \quad (5.18)$$

2. *There is $C > 0$ such that for all $T > 0$ and $G \in \mathbf{L}^2(Q)$ we have*

$$\Phi \in L^2(0, T; \mathbf{H}^2(0, L)) \cap H^1(0, T; \mathbf{H}^{-1}(0, L))$$

satisfying the estimate

$$\|\Phi\|_{L^2(0, T; \mathbf{H}^2(0, L)) \cap H^1(0, T; \mathbf{H}^{-1}(0, L))} \leq C(1+T)\|G\|_{\mathbf{L}^2(Q)}, \quad (5.19)$$

for Φ the energy solution of (5.8) with initial value $\Phi^T = 0$.

3. *There is $C > 0$ such that for all $T > 0$ and $G \in L^2(0, T; \mathbf{H}_0^1(0, L))$ we have $\Phi \in \mathbf{H}^{1,3}(Q)$ satisfying the estimate*

$$\|\Phi\|_{\mathbf{H}^{1,3}(Q)} \leq C(1+T^3)\|G\|_{L^2(0, T; \mathbf{H}_0^1(0, L))}, \quad (5.20)$$

for Φ the energy solution of (5.8) with initial value $\Phi^T = 0$.

For the proof of Proposition 5.7 we need the following result, which is proved in [62] :

Lemma 5.8. *Let $L > 0$ and $k > 0$. Then, there is $C > 0$ such that for all $T > 0$ and $g \in L^2(Q)$ we have $w \in L^2(0, T; H^2(0, L))$ satisfying the estimate*

$$\|w\|_{L^2(0, T; H^2(0, L))} \leq C \|g\|_{L^2(Q)}, \quad (5.21)$$

for w the energy solution of

$$\begin{cases} -w_t - kw_{xxx} = g, & \text{in } Q, \\ w(\cdot, 0) = w(\cdot, L) = w_x(\cdot, 0) = 0, & \text{on } (0, T), \\ w(T, \cdot) = 0 & \text{on } (0, L). \end{cases}$$

Proof. [Proof of Proposition 5.7.] *Step 1 : some first facts and presentation of the proof.* The uniqueness result of statement 1 is a consequence of the linearity of the problem and of Lemma 5.6 applied to $\Phi^T = 0$ and $G = 0$. Thus, we just have to focus on the existence results and the regularity estimates (5.18)-(5.20). For that purpose, thanks to linearity and density results, it actually suffices to prove (5.18)-(5.20) when $\Phi^T \in \mathcal{D}(0, L)$ (or $\Phi^T = 0$) and $G \in \mathcal{D}(Q)$. We remind the reader that under those hypothesis thanks to Lemma 5.6 the energy solution of (5.8) belongs to $\mathbf{C}^\infty(\overline{Q})$. Hence, the rest of the proof is structured as follows : we first diagonalize (5.8) and secondly we prove the estimates (5.18)-(5.20).

Step 2 : diagonalization of (5.8). We first remark that (5.8) is given by

$$\begin{cases} -\Phi_t^* - M\Phi_{xxx}^* - \begin{bmatrix} 0 & 0 \\ 0 & \frac{r}{c} \end{bmatrix} \Phi_x^* = \begin{bmatrix} g_1 \\ \frac{g_2}{c} \end{bmatrix}, & \text{in } Q, \\ \Phi^*(\cdot, 0) = \Phi^*(\cdot, L) = \Phi_x^*(\cdot, 0) = 0, & \text{on } (0, T), \\ \Phi^*(T, \cdot) = (\Phi^T)^*, & \text{on } (0, L), \end{cases} \quad (5.22)$$

for

$$M := \begin{bmatrix} 1 & ab \\ \frac{a}{c} & \frac{1}{c} \end{bmatrix}, \quad \Phi^* := \begin{bmatrix} \varphi \\ \phi \end{bmatrix}, \quad (\Phi^T)^* := \begin{bmatrix} \varphi^T \\ \phi^T \end{bmatrix}.$$

We have that the eigenvalues of M are

$$\lambda_{\pm} := \frac{1 + \frac{1}{c} \pm \sqrt{\left(1 + \frac{1}{c}\right)^2 - 4\frac{1-a^2b}{c}}}{2} = \frac{1 + \frac{1}{c} \pm \sqrt{\left(1 - \frac{1}{c}\right)^2 + 4\frac{a^2b}{c}}}{2}, \quad (5.23)$$

which are strictly positive and different by (5.4). Thereby, there is a matrix P such that

$$PMP^{-1} = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}.$$

Then, if

$$\tilde{\Phi} := P\Phi^*, \quad (5.24)$$

we have that $\tilde{\Phi}$ satisfies the system

$$\begin{cases} -\tilde{\Phi}_t - \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} \tilde{\Phi}_{xxx} - \tilde{M}\tilde{\Phi}_x = \tilde{G}, & \text{in } Q, \\ \tilde{\Phi}(\cdot, 0) = \tilde{\Phi}(\cdot, L) = \tilde{\Phi}_x(\cdot, 0) = 0, & \text{on } (0, T), \\ \tilde{\Phi}(T, \cdot) = P(\Phi^T)^*, & \text{on } (0, L), \end{cases} \quad (5.25)$$

for

$$\tilde{M} := P \begin{bmatrix} 0 & 0 \\ 0 & \frac{r}{c} \end{bmatrix} P^{-1}, \quad \tilde{G} := P \begin{bmatrix} g_1 \\ \frac{g_2}{c} \end{bmatrix}. \quad (5.26)$$

Clearly, it suffices to get the estimates for system (5.25), which is easier because the coupling appears only on the first-order spatial term.

Step 3 : proof of (5.18). If we multiply (5.25) by $\tilde{\Phi}$, after integrating by parts and applying Cauchy-Schwarz inequality we obtain

$$\|\tilde{\Phi}\|_{C([0,T];L^2(0,L))}^2 \leq C_\delta T \|\tilde{G}\|_{L^2(0,T;\mathbf{H}^{-1}(0,L))}^2 + \frac{\delta}{T} \|\tilde{\Phi}\|_{L^2(0,T;\mathbf{H}_0^1(0,L))}^2. \quad (5.27)$$

Thus, if we multiply (5.25) by $(L+1-x)\tilde{\Phi}$, (5.18) is obtained by using classical estimates, integration by parts, (5.27) to deal with the first order term and (5.24).

Step 4 : proof of (5.19). We recall that from now on

$$\tilde{\Phi}^T = P(\Phi^T)^* = 0. \quad (5.28)$$

Thanks to (5.25) we know that system (5.8) behaves in the higher order terms as two independent KdV equations. In particular, by considering (5.28), (5.21) (g is taken as the components of $\tilde{G} + \tilde{M}\tilde{\Phi}_x$) and (5.18) we find that

$$\|\tilde{\Phi}\|_{L^2(0,T;\mathbf{H}^2(0,L))} \leq C(1+T)\|\tilde{G}\|_{L^2(Q)}. \quad (5.29)$$

Hence, we obtain from (5.24), (5.26) and (5.29) that

$$\|\Phi\|_{L^2(0,T;\mathbf{H}^2(0,L))} \leq C(1+T)\|G\|_{\mathbf{L}^2(Q)}. \quad (5.30)$$

We can easily get (5.19) from (5.30) and (5.8).

Step 5 : proof of (5.20). Considering (5.28), we remark that $\Psi := \tilde{\Phi}_{xxx}$ is the solution of the system

$$\begin{cases} \Psi_t + \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix} \Psi_{xxx} + \tilde{M}\Psi_x = \tilde{G}_{xxx}, & \text{in } Q, \\ \Psi(\cdot, 0) = -\hat{M}\tilde{M}\tilde{\Phi}_x(\cdot, 0) = 0, & \text{on } (0, T), \\ \Psi(\cdot, L) = -\hat{M}\tilde{M}\tilde{\Phi}_x(\cdot, L), & \text{on } (0, T), \\ \Psi_x(\cdot, 0) = -\hat{M}\tilde{M}\tilde{\Phi}_{xx}(\cdot, 0), & \text{on } (0, T), \\ \Psi(T, \cdot) = 0, & \text{on } (0, L), \end{cases}$$

for

$$\hat{M} := \begin{bmatrix} \lambda_+^{-1} & 0 \\ 0 & \lambda_-^{-1} \end{bmatrix}.$$

Using that $\tilde{G} \in L^2(0, T; \mathbf{H}_0^3(0, L))$ and considering that the solution can be defined by transposition as in [62, Proposition 1] we find the following estimate :

$$\|\Psi\|_{\mathbf{L}^2(Q)} \leq C(1+T)(\|\tilde{G}\|_{L^2(0,T;\mathbf{H}_0^3(0,L))} + \|\tilde{\Phi}_x(\cdot, L)\|_{\mathbf{L}^2(0,T)} + \|\tilde{\Phi}_{xx}(\cdot, 0)\|_{\mathbf{L}^2(0,T)}). \quad (5.31)$$

Recalling that $\Psi = \tilde{\Phi}_{xxx}$ and (5.29) we obtain from (5.31) that

$$\|\tilde{\Phi}\|_{L^2(0,T;\mathbf{H}^3(0,L))} \leq C((1+T^2)\|\tilde{G}\|_{L^2(0,T;\mathbf{H}_0^3(0,L))} + (1+T)\|\tilde{\Phi}_{xx}(\cdot, 0)\|_{\mathbf{L}^2(0,T)}). \quad (5.32)$$

Consequently, using the following trace estimate

$$\|\tilde{\Phi}_{xx}(\cdot, 0)\|_{\mathbf{L}^2(0,T)} \leq \frac{\delta}{1+T}\|\tilde{\Phi}\|_{L^2(0,T;\mathbf{H}^3(0,L))} + C_\delta(1+T)\|\tilde{\Phi}\|_{L^2(0,T;\mathbf{H}^2(0,L))},$$

we find from (5.24), (5.26) and (5.32) that

$$\|\Phi\|_{L^2(0,T;\mathbf{H}^3(0,L))} \leq C(1+T^3)\|G\|_{L^2(0,T;\mathbf{H}_0^1(0,L))}. \quad (5.33)$$

In addition, the \mathbf{L}^2 -norm of Φ_t is estimated from (5.8) and (5.33), and thus we obtain (5.20). \square

Two consequences of Proposition 5.7 are the following :

Corollary 5.9. *Let $L > 0$, $p \in \mathbb{N}_*$ and assume that (5.4) is satisfied. Then, there is a polynomial q such that for all $T > 0$ and*

$$G \in H^{p-1}(0, T; \mathbf{H}_0^1(0, L)) \cap L^2(0, T; \mathbf{H}^{3p-3}(0, L))$$

satisfying

$$\partial_t^i G(T, \cdot) = 0, \quad \forall i \in \{0, 1, \dots, p-2\},$$

we have that $\Phi \in \mathbf{H}^{p, 3p}(Q)$ and the following estimate holds

$$\|\Phi\|_{\mathbf{H}^{p, 3p}(Q)} \leq q(T) (\|G\|_{H^{p-1}(0, T; \mathbf{H}_0^1(0, L)) \cap L^2(0, T; \mathbf{H}^{3p-3}(0, L))}), \quad (5.34)$$

for Φ the energy solution of (5.8) with initial value $\Phi^T = 0$.

Corollary 5.10. *Let $L > 0$, $p \in \mathbb{N}_*$ and assume that (5.4) is satisfied. Then, there a polynomial q such that for all $T > 0$ and*

$$G \in H^p(0, T; \mathbf{H}^{-1}(0, L)) \cap L^2(0, T; \mathbf{H}^{3p-2}(0, L))$$

satisfying

$$\partial_t^i G(T, \cdot) = 0, \quad \forall i \in \{0, 1, \dots, p-1\},$$

we have that $\Phi \in H^p(0, T; \mathbf{H}^1(0, L)) \cap L^2(0, T; \mathbf{H}^{3p+1}(0, L))$ and the following estimate holds

$$\|\Phi\|_{H^p(0, T; \mathbf{H}^1(0, L)) \cap L^2(0, T; \mathbf{H}^{3p+1}(0, L))} \leq q(T) \|G\|_{H^p(0, T; \mathbf{H}^{-1}(0, L)) \cap L^2(0, T; \mathbf{H}^{3p-2}(0, L))}, \quad (5.35)$$

for Φ the energy solution of (5.8) with initial value $\Phi^T = 0$.

Corollary 5.9 is proved by induction, using the inductive hypothesis on Φ_t and diagonalising \mathcal{L}^* as in (5.25). The time estimate of Corollary 5.10 is proved by using (5.18) on $\partial_t^p \Phi$. The space estimate of Corollary 5.10 is proved by diagonalising \mathcal{L}^* as in (5.25) and induction : first by using (5.18) on Φ_t , and then by applying the inductive hypothesis on Φ_t . Indeed, we have the following continuous inclusions :

$$\begin{aligned} & H^p(0, T; \mathbf{H}^{-1}(0, L)) \cap L^2(0, T; \mathbf{H}^{3p-2}(0, L)) \\ & \subset H^1\left(0, T; \mathbf{H}^{3(p-1)-2\frac{p-1}{p}-\frac{1}{p}}(0, L)\right) \subset H^1\left(0, T; \mathbf{H}^{3(p-1)-2}(0, L)\right). \end{aligned}$$

Remark 5.11. We have for (5.7) regularity results analogous to those in Proposition 5.7, Corollary 5.9 and Corollary 5.10.

5.3 A Carleman inequality for the solutions of (5.8)

In order to shorten the notation, we define for a function $g \in C([0, L])$ and any subset $X \subset [0, L]$ the integral :

$$\int_X g := \sum_{x \in X} g(x).$$

The structure of this section is the following : first we present the weights, and secondly we state and prove the Carleman inequality.

5.3.1 Definition of the weights and an auxiliary result

Let $L > 0$ and $\omega \subset (0, L)$ be an open subdomain. We consider $\eta \in C^3([0, L])$ a function satisfying :

$$\|\eta\|_\infty = 1, \quad \inf_{(0,L) \setminus \omega} |\eta'| > 0, \quad \sup_{(0,L)} \eta'' < 0, \quad \eta(0) = \eta(L) = 0. \quad (5.36)$$

With that auxiliary function we define the following classical weights :

$$\begin{aligned} \alpha(t, x) &:= \frac{e^{\lambda(2m+2)} - e^{\lambda(2m+\eta(x))}}{t^m(T-t)^m}, & \xi(t, x) &:= \frac{e^{\lambda(2m+\eta(x))}}{t^m(T-t)^m}, \\ \alpha^*(t) &:= \sup_{x \in (0,L)} \alpha(t, x), & \xi^*(t) &:= \inf_{x \in (0,L)} \xi(t, x), \end{aligned} \quad (5.37)$$

for $m \in \mathbb{R}^+$ and $\lambda > 0$ to be fixed later. Similar weights were first defined in [57], but this version is taken from [54]. We remark that (5.36)₃ and (5.36)₄ imply the equalities :

$$\alpha^*(t) = \alpha(t, 0) = \alpha(t, L), \quad \xi^*(t) = \xi(t, 0) = \xi(t, L). \quad (5.38)$$

We recall that for all $i \in \mathbb{N}_*$ there is $C > 0$ such that if $\lambda \geq C$ we have the estimate :

$$|\partial_t^i \xi| + |\partial_t^i \alpha| \leq CT^i (\xi^*)^{1+i/m}. \quad (5.39)$$

Next, we define the weights :

$$\beta(t, x) := \frac{e^{\lambda(2m+2)} - e^{\lambda(2m+\eta(x))}}{(T^2 l(t/T))^m}, \quad \beta^*(t) := \sup_{x \in (0,L)} \beta(t, x), \quad (5.40)$$

for $l(t) \in C^\infty([0, 1])$ a positive function satisfying :

$$l(t) = t(1-t), \text{ in } [3/4, 1], \quad l(t) > t(1-t), \text{ in } [0, 3/4), \quad l'(t) < 0, \text{ in } [0, 1]. \quad (5.41)$$

An easy consequence of (5.41) is that

$$\beta^* \leq \alpha^* \text{ in } [0, T]. \quad (5.42)$$

As for these weights, we have the following bound for $\lambda \geq C$:

$$|\beta_t^*| = \left| mTl'(t/T) \frac{e^{\lambda(2m+2)} - e^{2m\lambda}}{(T^2l(t/T))^{m+1}} \right| \leq CT(\beta^*)^{1+1/m}. \quad (5.43)$$

In general, for all $i \in \mathbb{N}_*$ there is $C > 0$ such that if $\lambda \geq C$ we have the estimates :

$$|\partial_t^i \beta^*| \leq CT^i(\beta^*)^{1+i/m}, \quad |\partial_t^i \beta^*| \leq CT^i(\xi^*)^{1+i/m}. \quad (5.44)$$

Finally, for all $\delta > 0$ and $k \in \mathbb{R}^+$ there is $C > 0$ such that for all $s \in \mathbb{R}^+$:

$$(s\beta^*)^k \leq Ce^{\delta s\beta^*}. \quad (5.45)$$

Indeed, $x^k e^{-\delta x} \in L^\infty(\mathbb{R}^+, dx)$.

In order to prove a Carleman inequality for (5.8) we need the following technical result :

Proposition 5.12. *Let $L > 0$, $i \geq 0$, $j, k_1, k_2 \in \mathbb{R}$, $k_3 \in \mathbb{R} \setminus \{0\}$, $m \geq 1/2$ and $\omega \subset (0, L)$. There is $C > 0$ such that for all $\lambda \geq C$, $s \geq C(T^m + T^{2m})$, and $w \in H^{2,3}(Q)$ we have the estimate*

$$\begin{aligned} & s^{1+i}\lambda^{2+j} \iint_Q e^{-2s\alpha\xi^{1+i}} |w_{xx}|^2 + s^{3+i}\lambda^{4+j} \iint_Q e^{-2s\alpha\xi^{3+i}} |w_x|^2 + s^{5+i}\lambda^{6+j} \iint_Q e^{-2s\alpha\xi^{5+i}} w^2 \\ & \leq C \left(s^i \lambda^j \iint_Q e^{-2s\alpha\xi^i} \tilde{g}^2 + s^{1+i}\lambda^{2+j} \iint_{Q_\omega} e^{-2s\alpha\xi^{1+i}} |w_{xx}|^2 + s^{5+i}\lambda^{6+j} \iint_{Q_\omega} e^{-2s\alpha\xi^{5+i}} w^2 \right. \\ & \quad \left. + s^{-1+i}\lambda^{-1+j} \int_0^T \int_{\{0,L\}} e^{-2s\alpha\xi^{-1+i}} |w_t|^2 \right), \quad (5.46) \end{aligned}$$

for

$$\tilde{g} := w_t + k_1 w_x + k_2 w_{xx} + k_3 w_{xxx}. \quad (5.47)$$

Proposition 5.12 can be proved for $i = j = k_1 = k_2 = 0$ by following, step by step, with some small changes the proof presented in [22, Proposition 3.1] and in [8, Theorem 3.1] (which are based on the proof presented in [94]). In addition, we have for any $i \geq 0$ that $s^{i/2} e^{-s\alpha\xi^{i/2}} w \in H^{2,3}(Q)$, so we just have to replace w by $s^{i/2} \xi^{i/2} w$ in (5.46) and do the usual absorptions. As for j , it suffices to multiply (5.46) by λ^j at both sides of the inequality. Finally, we can prove it for any $k_1, k_2 \in \mathbb{R}$ by considering that $w_t + k_3 w_{xxx} = \tilde{g} - k_1 w_x - k_2 w_{xx}$, and then absorbing the first and second order term with the left-hand side of (5.46).

5.3.2 The Carleman estimate

In this section we prove the following Carleman estimate :

Proposition 5.13. *Let $L > 0$, $\omega \subset (0, L)$ be an open subset, $m \geq 12$ and assume that (5.4) is satisfied. There is $C > 0$ such that for all $\lambda \geq C$, $s \geq C(T^m + T^{3m})$, $\Phi^T \in \mathbf{L}^2(0, L)$ and $G \in \mathbf{L}^2(Q)$ we have the estimate*

$$s^6 \lambda^6 \int_0^T e^{-2s\alpha^* - 10s\beta^*} (\xi^*)^6 \|\Phi\|_{\mathbf{H}^2(0,L)}^2 \leq C \left(\iint_{Q_\omega} e^{-s\alpha^* - 10s\beta^*} \phi^2 + \iint_Q e^{-8s\beta^*} |G|^2 \right), \quad (5.48)$$

for Φ the energy solution of (5.8).

Remark 5.14. We have that $s \geq C(T^m + T^{3m})$ implies that $s \geq CT^{2m}$, which implies that $s\xi^* \geq 1$.

The structure of the proof is the following : first we reduce it to the case in which the solution is regular, second we decompose Φ as the sum of the solutions of three Cauchy problems, third we get regularity estimates for the first two, fourth we deduce a PDE that just contains the second component of the third auxiliary variable, fifth we get a Carleman estimate for the third auxiliary variable, sixth we estimate the trace terms, seventh we estimate the local terms, and finally we deduce a Carleman estimate just in terms of a local term of ϕ .

Remark 5.15. As explained in the introduction, the proof of Proposition 5.13 presents the novelty that we decompose Φ as the sum of the solutions of three Cauchy problems to overcome the fact that the adjoint variable is not regular enough. The main difference in considering three Cauchy problems instead of two (as it has been done in the literature) is that in many parts of the proof we need to use the weights (5.40) instead of (5.37) as a way to ensure that the solutions of those Cauchy problems belong to $\mathbf{L}^2(Q)$. In addition, there is a second novelty which is to use a factorization to simplify the sixth order differential equation that we get when we uncouple.

Proof. [Proof of Proposition 5.13] *Step 1 : reducing to regular solutions.* By linearity arguments, density arguments and Lemma 5.6 it suffices to prove (5.48) for $G \in \mathcal{D}(Q)$ and $\Phi^T \in \mathcal{D}(0, L)$. In those cases we have by Lemma 5.6 that $\Phi \in \mathbf{C}^\infty(\overline{Q})$. Thus, from now we suppose that $G \in \mathcal{D}(Q)$ and $\Phi^T \in \mathcal{D}(0, L)$.

Step 2 : Decomposition of Φ . We consider that

$$e^{-5s\beta^*} \Phi = \hat{\Phi} + \bar{\Phi} + \tilde{\Phi}, \quad (5.49)$$

for $\hat{\Phi} := (\hat{\varphi}, \hat{\phi})$ the solution of the system

$$\begin{cases} \mathcal{L}^* \hat{\Phi} = e^{-5s\beta^*} G, & \text{in } Q, \\ \hat{\Phi}(\cdot, 0) = \hat{\Phi}(\cdot, L) = \hat{\Phi}_x(\cdot, 0) = 0, & \text{on } (0, T), \\ \hat{\Phi}(T, \cdot) = 0, & \text{on } (0, L), \end{cases} \quad (5.50)$$

$\bar{\Phi} := (\bar{\varphi}, \bar{\phi})$ the solution of the system

$$\begin{cases} \mathcal{L}^* \bar{\Phi} = 5s\beta_t^* \hat{\Phi}, & \text{in } Q, \\ \bar{\Phi}(\cdot, 0) = \bar{\Phi}(\cdot, L) = \bar{\Phi}_x(\cdot, 0) = 0, & \text{on } (0, T), \\ \bar{\Phi}(T, \cdot) = 0, & \text{on } (0, L), \end{cases} \quad (5.51)$$

and $\tilde{\Phi} := (\tilde{\varphi}, \tilde{\phi})$ the solution of the system

$$\begin{cases} \mathcal{L}^* \tilde{\Phi} = 5s\beta_t^* (e^{-5s\beta^*} \Phi - \hat{\Phi}), & \text{in } Q, \\ \tilde{\Phi}(\cdot, 0) = \tilde{\Phi}(\cdot, L) = \tilde{\Phi}_x(\cdot, 0) = 0, & \text{on } (0, T), \\ \tilde{\Phi}(T, \cdot) = 0, & \text{on } (0, L). \end{cases} \quad (5.52)$$

We remark that because of (5.49), $\tilde{\Phi}$ satisfies :

$$\begin{cases} \mathcal{L}^* \tilde{\Phi} = 5s\beta_t^* (\bar{\Phi} + \tilde{\Phi}), & \text{in } Q, \\ \tilde{\Phi}(\cdot, 0) = \tilde{\Phi}(\cdot, L) = \tilde{\Phi}_x(\cdot, 0) = 0, & \text{on } (0, T), \\ \tilde{\Phi}(T, \cdot) = 0, & \text{on } (0, L). \end{cases} \quad (5.53)$$

Moreover, from Proposition 5.7, Corollary 5.9 and Corollary 5.10 we deduce that

$$\hat{\Phi}, \bar{\Phi}, \tilde{\Phi} \in \mathbf{C}^\infty(\bar{Q}).$$

Step 3 : estimates of the solutions of (5.50) and (5.51). First of all, we obtain from (5.19) that :

$$\|\hat{\Phi}\|_{L^2(0,T;\mathbf{H}^2(0,L))} \leq q(T) \|e^{-5s\beta^*} G\|_{\mathbf{L}^2(Q)}. \quad (5.54)$$

We recall that q stands for a generic polynomial of positive coefficient that might be different each time it appears.

Moreover, using (5.18) for system (5.50) and (5.41)₃, we have that for all $\delta > 0$ there is a polynomial q of positive coefficients such that, for all $t \in [0, T]$,

$$\|\hat{\Phi}(t, \cdot)\|_{\mathbf{L}^2(0,L)} \leq q(T-t) \|e^{-5s\beta^*} G\|_{\mathbf{L}^2((t,T) \times (0,L))} \leq q(T) e^{-\delta s\beta^*(t)} \|e^{-(5-\delta)s\beta^*} G\|_{\mathbf{L}^2(Q)}. \quad (5.55)$$

This allows us to estimate $\|s\beta_t^* \hat{\Phi}\|_{\mathbf{L}^2(Q)}$ and $\|(s\beta_t^*)^2 \hat{\Phi}\|_{\mathbf{L}^2(Q)}$ in terms of $\|e^{-(5-\delta)s\beta^*} G\|_{\mathbf{L}^2(Q)}$. Consequently, looking at the system satisfied by $\beta_t^* \hat{\Phi}$, from (5.19), (5.44)₁, (5.45) and (5.55) we get

$$\|s\beta_t^* \hat{\Phi}\|_{L^2(0,T;\mathbf{H}^2(0,L)) \cap H^1(0,T;\mathbf{H}^{-1}(0,L))} \leq q(T) \|e^{-(5-\delta)s\beta^*} G\|_{\mathbf{L}^2(Q)}. \quad (5.56)$$

Thus, using (5.35) for $p = 1$, from (5.56) it follows that

$$\|\bar{\Phi}\|_{H^1(0,T;\mathbf{H}^1(0,L)) \cap L^2(0,T;\mathbf{H}^4(0,L))} \leq q(T) \|e^{-(5-\delta)s\beta^*} G\|_{\mathbf{L}^2(Q)}. \quad (5.57)$$

Finally, since $\lambda \geq C$ and $s \geq C(T^m + T^{3m})$, from (5.45) and (5.41)₃ we obtain the estimate

$$T \leq C \frac{s^{1/m}}{T^2} \leq C(s\beta^*(0))^{1/m} \leq C e^{s\delta\beta^*(0)} \leq C e^{s\delta\beta^*(t)}, \quad \forall t \in [0, T]. \quad (5.58)$$

Consequently, taking $\delta > 0$ small enough and keeping the interesting terms, for $\lambda \geq C$ and $s \geq C(T^m + T^{3m})$ we have

$$\|\hat{\Phi}\|_{L^2(0,T;\mathbf{H}^2(0,L))} + \|\bar{\Phi}\|_{H^1(0,T;\mathbf{H}^1(0,L)) \cap L^2(0,T;\mathbf{H}^4(0,L))} \leq C \|e^{-4s\beta^*} G\|_{\mathbf{L}^2(Q)}. \quad (5.59)$$

Step 4 : uncoupling (5.53). We now get an equation containing just $\tilde{\phi}$. By applying the operator $-\partial_t - \partial_x^3$ to the second equation of (5.53)₁ and using the first equation of (5.53)₁, we obtain the PDE :

$$\begin{aligned} c\tilde{\phi}_{tt} + c\tilde{\phi}_{txxx} + r\tilde{\phi}_{tx} + r\partial_x^4 \tilde{\phi} - a^2 b \partial_x^6 \tilde{\phi} + \tilde{\phi}_{txxx} + \partial_x^6 \tilde{\phi} \\ = 5as\beta_t^* \partial_x^3 (\bar{\varphi} + \tilde{\varphi}) - 5s(\partial_t + \partial_x^3)(\beta_t^* (\bar{\phi} + \tilde{\phi})). \end{aligned} \quad (5.60)$$

With the objective of simplifying the left-hand side of (5.60), we show the existence of $\kappa_1, \kappa_2, \tilde{\kappa}_1, \tilde{\kappa}_2 \in \mathbb{R}$ such that the following equality is satisfied :

$$\begin{aligned} c(\partial_t + \tilde{\kappa}_1 \partial_x + \kappa_1 \partial_x^3)(\partial_t + \tilde{\kappa}_2 \partial_x + \kappa_2 \partial_x^3) \tilde{\phi} \\ = \tilde{\kappa}_1 \tilde{\kappa}_2 \partial_x^2 \tilde{\phi} - 5s\beta_t^* \partial_x^3 (\bar{\varphi} + \tilde{\varphi}) - 5s(\partial_t + \partial_x^3)(\beta_t^* (\bar{\phi} + \tilde{\phi})). \end{aligned} \quad (5.61)$$

The higher order terms form the system :

$$\begin{cases} \kappa_1 + \kappa_2 = 1 + \frac{1}{c}, \\ \kappa_1 \kappa_2 = \frac{1-a^2b}{c}. \end{cases} \quad (5.62)$$

Thus, we search for κ_1 such that :

$$\kappa_1^2 - \frac{c+1}{c}\kappa_1 + \frac{1-a^2b}{c} = 0. \quad (5.63)$$

The discriminant of (5.63) is given by :

$$\frac{(c+1)^2}{c^2} - 4\frac{1-a^2b}{c} = \frac{c^2 + 2c + 1 - 4c + 4a^2bc}{c^2} = \frac{(c-1)^2 + 4a^2bc}{c^2} > 0;$$

hence, equation (5.63) and system (5.62) have two real solutions. As for the coefficients of the terms of order 4, we find the system

$$\begin{cases} \tilde{\kappa}_1 + \tilde{\kappa}_2 = \frac{r}{c}, \\ \kappa_2 \tilde{\kappa}_1 + \kappa_1 \tilde{\kappa}_2 = \frac{r}{c}, \end{cases}$$

which clearly has at least a real solution $(\tilde{\kappa}_1, \tilde{\kappa}_2)$. Consequently, the decomposition (5.61) is possible. In particular, we define the auxiliary variable

$$\theta := (\partial_t + \tilde{\kappa}_2 \partial_x + \kappa_2 \partial_x^3) \tilde{\phi}. \quad (5.64)$$

Step 5 : a Carleman estimate for $\tilde{\Phi}$. We have the following Carleman estimates :

- If we use Proposition 5.12 on $\tilde{\phi}$ for $(k_1, k_2, k_3) = (\tilde{\kappa}_2, 0, \kappa_2)$ and consider the homogeneous Dirichlet boundary conditions, we obtain that :

$$\begin{aligned} & s^6 \lambda^7 \iint_Q e^{-2s\alpha \xi^6} |\tilde{\phi}_{xx}|^2 + s^8 \lambda^9 \iint_Q e^{-2s\alpha \xi^8} |\tilde{\phi}_x|^2 + s^{10} \lambda^{11} \iint_Q e^{-2s\alpha \xi^{10}} \tilde{\phi}^2 \\ & \leq C \left(s^5 \lambda^5 \iint_Q e^{-2s\alpha \xi^5} \theta^2 + s^6 \lambda^7 \iint_{Q_\omega} e^{-2s\alpha \xi^6} |\tilde{\phi}_{xx}|^2 + s^{10} \lambda^{11} \iint_{Q_\omega} e^{-2s\alpha \xi^{10}} \tilde{\phi}^2 \right). \end{aligned} \quad (5.65)$$

- If we use Proposition 5.12 on $\tilde{\phi}_x$ and recall the boundary conditions of $\tilde{\Phi}$, we get :

$$\begin{aligned} & s^4 \lambda^5 \iint_Q e^{-2s\alpha \xi^4} |\tilde{\phi}_{xxx}|^2 + s^6 \lambda^7 \iint_Q e^{-2s\alpha \xi^6} |\tilde{\phi}_{xx}|^2 + s^8 \lambda^9 \iint_Q e^{-2s\alpha \xi^8} |\tilde{\phi}_x|^2 \\ & \leq C \left(s^3 \lambda^3 \iint_Q e^{-2s\alpha \xi^3} |\theta_x|^2 + s^4 \lambda^5 \iint_{Q_\omega} e^{-2s\alpha \xi^2} |\tilde{\phi}_{xxx}|^2 + s^8 \lambda^9 \iint_{Q_\omega} e^{-2s\alpha \xi^8} |\tilde{\phi}_x|^2 \right. \\ & \quad \left. + s^2 \lambda^2 \int_0^T \int_{\{L\}} e^{-2s\alpha \xi^2} |\tilde{\phi}_{tx}|^2 \right). \end{aligned} \quad (5.66)$$

- If we use Proposition 5.12 on θ for $(k_1, k_2, k_3) = (\tilde{\kappa}_1, 0, \kappa_1)$, we get from $c > 0$, (5.64) and (5.61) the following estimate :

$$\begin{aligned} & s^2 \lambda^3 \iint_Q e^{-2s\alpha} \xi^2 |\theta_{xx}|^2 + s^4 \lambda^5 \iint_Q e^{-2s\alpha} \xi^4 |\theta_x|^2 + s^6 \lambda^7 \iint_Q e^{-2s\alpha} \xi^6 \theta^2 \\ & \leq C \left(s\lambda \iint_Q e^{-2s\alpha} \xi \left[\tilde{\kappa}_1 \tilde{\kappa}_2 \tilde{\phi}_{xx} - 5s\beta_t^* (\bar{\varphi}_{xxx} + \tilde{\varphi}_{xxx}) - 5s(\partial_t + \partial_x^3) \left(\beta_t^* (\bar{\phi} + \tilde{\phi}) \right) \right]^2 \right. \\ & \quad \left. + s^2 \lambda^3 \iint_{Q_\omega} e^{-2s\alpha} \xi^2 |\theta_{xx}|^2 + s^6 \lambda^7 \iint_{Q_\omega} e^{-2s\alpha} \xi^6 \theta^2 + \int_0^T \int_{\{0,L\}} e^{-2s\alpha} |\theta_t|^2 \right). \end{aligned} \quad (5.67)$$

- From (5.64) we have :

$$s^4 \lambda^5 \iint_Q e^{-2s\alpha} \xi^4 |\tilde{\phi}_t|^2 \leq C s^4 \lambda^5 \iint_Q e^{-2s\alpha} \xi^4 (|\tilde{\phi}_{xxx}|^2 + |\tilde{\phi}_x|^2 + \theta^2). \quad (5.68)$$

- If we consider the second equation of (5.53)₁, it follows that :

$$\begin{aligned} & s^4 \lambda^5 \iint_Q e^{-2s\alpha} \xi^4 |\tilde{\varphi}_{xxx}|^2 \\ & \leq C s^4 \lambda^5 \iint_Q e^{-2s\alpha} \xi^4 (|\tilde{\phi}_{xxx}|^2 + |\tilde{\phi}_x|^2 + |\tilde{\phi}_t|^2 + s^2 (\beta_t^*)^2 \bar{\phi}^2 + s^2 (\beta_t^*)^2 \tilde{\phi}^2). \end{aligned} \quad (5.69)$$

- Thanks to the second equation of (5.53)₁, (5.64) and Poincaré inequality (Proposition 5.2 for $i = 0$) we obtain :

$$\begin{aligned} & s^6 \lambda^6 \int_0^T e^{-2s\alpha^*} (\xi^*)^6 \|\tilde{\varphi}\|_{H^2(0,L)}^2 \\ & \leq C s^6 \lambda^6 \iint_Q e^{-2s\alpha} (\xi^*)^6 (|\tilde{\phi}_{xx}|^2 + |\tilde{\phi}_x|^2 + \tilde{\phi}^2 + \theta^2 + s^2 (\beta_t^*)^2 \bar{\phi}^2 + s^2 (\beta_t^*)^2 \tilde{\phi}^2). \end{aligned} \quad (5.70)$$

- Considering the first equation of (5.53)₁ the following holds :

$$\begin{aligned} & s^{7/2} \lambda^4 \iint_Q e^{-2s\alpha^*} (\xi^*)^{7/2} |\tilde{\varphi}_t|^2 \\ & \leq C s^{7/2} \lambda^4 \iint_Q e^{-2s\alpha^*} ((\xi^*)^{7/2} (|\tilde{\varphi}_{xxx}|^2 + |\tilde{\phi}_{xxx}|^2 + (5s\beta_t^*)^2 (\tilde{\varphi}^2 + \bar{\varphi}^2))). \end{aligned} \quad (5.71)$$

We recall that because of (5.44)₂ we have for $\lambda \geq C$ and $s \geq C(T^m + T^{3m})$ the estimate

$$(s\beta_t^*)^2 \leq C s^{2+2/m} (\xi^*)^{2+2/m}.$$

Thus, the zero-order term of $\tilde{\varphi}$ in the right-hand side of (5.71) can be absorbed by the left-hand side of (5.70) if $m \geq 4$, $\lambda \geq C$ and $s \geq C(T^m + T^{3m})$.

So, considering (5.44)₂, (5.59) and (5.65)-(5.71), after some easy absorptions and majorations we obtain for $m \geq 4$, $\lambda \geq C$ and $s \geq C(T^m + T^{3m})$ that :

$$\begin{aligned}
& s^4 \lambda^5 \iint_Q e^{-2s\alpha} \xi^4 |\tilde{\phi}_t|^2 + \sum_{i=0}^3 s^{10-2i} \lambda^{11-2i} \iint_Q e^{-2s\alpha} \xi^{10-2i} |\partial_x^i \tilde{\phi}|^2 \\
& + s^{7/2} \lambda^4 \iint_Q e^{-2s\alpha^*} (\xi^*)^{7/2} |\tilde{\varphi}_t|^2 + s^4 \lambda^5 \iint_Q e^{-2s\alpha} \xi^4 |\tilde{\varphi}_{xxx}|^2 + s^6 \lambda^6 \int_0^T e^{-2s\alpha^*} (\xi^*)^6 \|\tilde{\varphi}\|_{H^2(0,L)}^2 \\
& + \sum_{i=0}^2 s^{6-2i} \lambda^{7-2i} \iint_Q e^{-2s\alpha} \xi^{6-2i} |\partial_x^i \theta|^2 + \|\hat{\Phi}\|_{L^2(0,T;\mathbf{H}^2(0,L))}^2 + \|\bar{\Phi}\|_{H^1(0,T;\mathbf{H}^1(0,L)) \cap L^2(0,T;\mathbf{H}^4(0,L))}^2 \\
& \leq C \left(\sum_{i=0}^3 s^{10-2i} \lambda^{11-2i} \iint_{Q_\omega} e^{-2s\alpha} \xi^{10-2i} |\partial_x^i \tilde{\phi}|^2 + \sum_{i=0,2} s^{6-2i} \lambda^{7-2i} \iint_{Q_\omega} e^{-2s\alpha} \xi^{6-2i} |\partial_x^i \theta|^2 \right. \\
& \quad \left. + \int_0^T \int_{\{0,L\}} e^{-2s\alpha} |\theta_t|^2 + s^2 \lambda^2 \int_0^T \int_{\{L\}} e^{-2s\alpha} \xi^2 |\tilde{\phi}_{tx}|^2 + \iint_Q e^{-8s\beta^*} |G|^2 \right). \quad (5.72)
\end{aligned}$$

Step 6 : absorption of the trace terms. Let us first show how to absorb the trace term of ϕ . We recall (5.38). From trace interpolation estimates and Young inequality we get :

$$\begin{aligned}
& s^2 \lambda^2 \int_0^T \int_{\{L\}} e^{-2s\alpha} \xi^2 |\tilde{\phi}_{tx}|^2 \leq C s^2 \lambda^2 \int_0^T e^{-2s\alpha^*} (\xi^*)^2 \|\tilde{\phi}_t\|_{H^2(0,L)}^{3/2} \|\tilde{\phi}_t\|_{L^2(0,L)}^{1/2} \\
& \leq C \left(s^4 \lambda^4 \iint_Q e^{-2s\alpha^*} (\xi^*)^4 |\tilde{\phi}_t|^2 + s^{4/3} \lambda^{4/3} \int_0^T e^{-2s\alpha^*} (\xi^*)^{4/3} \|\tilde{\phi}_t\|_{H^2(0,L)}^2 \right). \quad (5.73)
\end{aligned}$$

It is clear that the first term in the right-hand side of (5.73) can be absorbed by the left-hand side of (5.72) for $\lambda \geq C$. As for the second term, we deduce that :

$$\begin{aligned}
& s^{4/3} \lambda^{4/3} \int_0^T e^{-2s\alpha^*} (\xi^*)^{4/3} \|\tilde{\phi}_t\|_{H^2(0,L)}^2 \\
& \leq C \int_0^T |(s^{2/3} \lambda^{2/3} e^{-s\alpha^*} (\xi^*)^{2/3})_t|^2 \|\tilde{\phi}\|_{H^2(0,L)}^2 + \|s^{2/3} \lambda^{2/3} e^{-s\alpha^*} (\xi^*)^{2/3} \tilde{\phi}\|_{H^1(0,T;H^2(0,L))}^2. \quad (5.74)
\end{aligned}$$

It can be proved easily that the first term in the right-hand side of (5.74) can be absorbed by the left-hand side of (5.72) using that $m \geq 4$. As for the second one, using (5.19) on Φ_t we have :

$$\begin{aligned}
& \|s^{2/3} \lambda^{2/3} e^{-s\alpha^*} (\xi^*)^{2/3} \tilde{\phi}\|_{H^1(0,T;H^2(0,L))}^2 \leq (1 + T^2) \left(\|(s^{2/3} \lambda^{2/3} e^{-s\alpha^*} (\xi^*)^{2/3})_t \tilde{\Phi}\|_{\mathbf{H}^{1,0}(Q)}^2 \right. \\
& \quad \left. + \|s^{2/3} \lambda^{2/3} e^{-s\alpha^*} (\xi^*)^{2/3} s \beta_t^* \tilde{\Phi}\|_{\mathbf{H}^{1,0}(Q)}^2 + \|s^{2/3} \lambda^{2/3} e^{-s\alpha^*} (\xi^*)^{2/3} s \beta_t^* \bar{\Phi}\|_{\mathbf{H}^{1,0}(Q)}^2 \right). \quad (5.75)
\end{aligned}$$

The third term in the right-hand side of (5.75) can be estimated by (5.59). Moreover, the first two terms in the right-hand side of (5.75) can be absorbed by the left-hand side of (5.72) if $m \geq 12$. Indeed, the most problematic term is

$$s^{10/3} \lambda^{4/3} \iint_Q e^{-2s\alpha^*} (|\alpha_t^*|^2 + |\beta_t^*|^2) (\xi^*)^{4/3} |\tilde{\Phi}_t|^2,$$

which can be absorbed by the left-hand side of (5.72) by considering (5.39), (5.44)₂ and that $CT^{2+2} = CT^4 \leq s^{2/m}$. The other terms can be estimated similarly.

Let us now estimate the trace term of θ . Because of (5.64) and the Dirichlet boundary conditions, we have that :

$$\theta_t = \tilde{\kappa}_2 \tilde{\phi}_{tx} + \kappa_2 \tilde{\phi}_{txxx}, \quad \text{on } \{0, L\} \times (0, T).$$

In addition, considering that the source term in (5.52) satisfies Dirichlet boundary conditions and that \mathcal{L}^* can be diagonalize as in (5.25), we obtain from (5.53) that there is $\kappa_3 \in \mathbb{R}$ such that

$$\theta_t(\cdot, 0) = 0, \quad \text{on } (0, T), \quad \theta_t(\cdot, L) = \kappa_3 \tilde{\phi}_{tx}(\cdot, L), \quad \text{on } (0, T).$$

Thus, by Remark 5.14 we obtain

$$\int_0^T \int_{\{0,L\}} e^{-2s\alpha} |\theta_t|^2 \leq Cs^2 \lambda^2 \int_0^T \int_{\{L\}} e^{-2s\alpha} \xi^2 |\tilde{\phi}_{tx}|^2,$$

a term which has already been treated.

Summing up, after the operations on this step and keeping the interesting terms in (5.72) we get if $m \geq 12$, $\lambda \geq C$ and $s \geq C(T^m + T^{3m})$ the following estimate :

$$\begin{aligned} & s^4 \lambda^5 \iint_Q e^{-2s\alpha} \xi^4 |\tilde{\phi}_t|^2 + \sum_{i=0}^3 s^{10-2i} \lambda^{11-2i} \iint_Q e^{-2s\alpha} \xi^{10-2i} |\partial_x^i \tilde{\phi}|^2 \\ & + s^{7/2} \lambda^4 \iint_Q e^{-2s\alpha^*} (\xi^*)^{7/2} |\tilde{\varphi}_t|^2 + s^4 \lambda^5 \iint_Q e^{-2s\alpha} \xi^4 |\tilde{\varphi}_{xxx}|^2 + s^6 \lambda^6 \int_0^T e^{-2s\alpha^*} (\xi^*)^6 \|\tilde{\varphi}\|_{H^2(0,L)}^2 \\ & \quad + \|\hat{\Phi}\|_{L^2(0,T;\mathbf{H}^2(0,L))}^2 + \|\bar{\Phi}\|_{H^1(0,T;\mathbf{H}^1(0,L)) \cap L^2(0,T;\mathbf{H}^4(0,L))}^2 \\ & \leq C \left(\sum_{i=0}^3 s^{10-2i} \lambda^{11-2i} \iint_{Q_\omega} e^{-2s\alpha} \xi^{10-2i} |\partial_x^i \tilde{\phi}|^2 + \sum_{i=0,2} s^{6-2i} \lambda^{7-2i} \iint_{Q_\omega} e^{-2s\alpha} \xi^{6-2i} |\partial_x^i \theta|^2 \right. \\ & \quad \left. + \iint_Q e^{-8s\beta^*} |G|^2 \right). \quad (5.76) \end{aligned}$$

Step 7 : absorption of the local terms. Using space interpolation and Young inequality, we obtain that :

$$\begin{aligned} \sum_{i=0}^3 s^{10-2i} \lambda^{11-2i} \iint_{Q_\omega} e^{-2s\alpha} \xi^{10-2i} |\partial_x^i \tilde{\phi}|^2 &\leq C \int_0^T e^{-\frac{15}{8}s\alpha^*} \|\tilde{\phi}\|_{H^3(\omega)}^2 \\ &\leq C \int_0^T e^{-\frac{15}{8}s\alpha^*} \|\tilde{\phi}\|_{L^2(\omega)}^{1/2} \|\tilde{\phi}\|_{H^4(\omega)}^{3/2} \leq C \left(\iint_{Q_\omega} e^{-s\alpha^*} \tilde{\phi}^2 + \|e^{-\frac{13}{12}s\alpha^*} \tilde{\phi}\|_{L^2(0,T;H^4(0,L))}^2 \right). \end{aligned} \quad (5.77)$$

The second term on the right hand-side of (5.77) can be bounded with (5.35) for $p = 1$:

$$\begin{aligned} \|e^{-\frac{13}{12}s\alpha^*} \tilde{\phi}\|_{L^2(0,T;H^4(0,L))}^2 &\leq C \left(\|(e^{-\frac{13}{12}s\alpha^*})_t \tilde{\Phi}\|_{H^1(0,T;\mathbf{H}^{-1}(0,L)) \cap L^2(0,T;\mathbf{H}^1(0,L))}^2 \right. \\ &\quad + \|s\beta_t^* e^{-\frac{13}{12}s\alpha^*} \tilde{\Phi}\|_{H^1(0,T;\mathbf{H}^{-1}(0,L)) \cap L^2(0,T;\mathbf{H}^1(0,L))}^2 \\ &\quad \left. + \|s\beta_t^* e^{-\frac{13}{12}s\alpha^*} \bar{\Phi}\|_{H^1(0,T;\mathbf{H}^{-1}(0,L)) \cap L^2(0,T;\mathbf{H}^1(0,L))}^2 \right). \end{aligned} \quad (5.78)$$

The third term in the right-hand side of (5.78) can be estimated with (5.59). As for the other two terms, they can be absorbed by the left-hand side of (5.76).

In order to study the local term of θ , we substitute θ by the right-hand side of (5.64). The spatial terms can be treated similarly as before. As for $\tilde{\phi}_t$ we obtain for $\lambda \geq C$ and $s \geq C(T^m + T^{3m})$ the estimate

$$\begin{aligned} \sum_{i=0,2} s^{6-2i} \lambda^{7-2i} \iint_{Q_\omega} e^{-2s\alpha} \xi^{6-2i} (\partial_x^i \partial_t \tilde{\phi})^2 &\leq C \iint_Q e^{-\frac{63}{32}s\alpha^*} \|\tilde{\phi}_t\|_{L^2(\omega)}^{2/3} \|\tilde{\phi}_t\|_{H^3(\omega)}^{4/3} \\ &\leq C \left(\iint_{Q_\omega} e^{-\frac{15}{8}s\alpha^*} \tilde{\phi}_t^2 + \int_0^T e^{-\frac{129}{64}s\alpha^*} \|\tilde{\phi}_t\|_{H^3(0,L)}^2 \right). \end{aligned} \quad (5.79)$$

We can treat the last term at the left hand-side of (5.79) as before thanks to interpolation results, Corollary 5.9 (for $p = 2$), Corollary 5.10 (for $p = 1$), (5.58) and (5.42). As for the first term in the right-hand side of (5.79), we can simply integrate by parts :

$$\begin{aligned} \iint_{Q_\omega} e^{-\frac{15}{8}s\alpha^*} |\tilde{\phi}_t|^2 &= \frac{1}{2} \iint_{Q_\omega} (e^{-\frac{15}{8}s\alpha^*})_{tt} \tilde{\phi}^2 - \iint_{Q_\omega} e^{-\frac{15}{8}s\alpha^*} \tilde{\phi}_{tt} \tilde{\phi} \\ &\leq C \left(\iint_{Q_\omega} e^{-s\alpha^*} \tilde{\phi}^2 + \iint_Q e^{-\frac{11}{4}s\alpha^*} |\tilde{\phi}_{tt}|^2 \right). \end{aligned} \quad (5.80)$$

The last term in the right-hand side of (5.80) can be estimated as before with the help of Corollary 5.9 (for $p = 2$), Corollary 5.10 (for $p = 1$), (5.58) and (5.42).

So, summing up and keeping the interesting terms, from (5.77) we get

$$s^6 \lambda^6 \int_0^T e^{-2s\alpha^*} (\xi^*)^6 \|\tilde{\Phi}\|_{\mathbf{H}^2(0,L)}^2 + \|\hat{\Phi}\|_{L^2(0,T;\mathbf{H}^2(0,L))}^2 + \|\bar{\Phi}\|_{L^2(0,T;\mathbf{H}^2(0,L))}^2 \leq C \left(\iint_{Q_\omega} e^{-s\alpha^*} \tilde{\phi}^2 + \iint_Q e^{-8s\beta^*} |G|^2 \right). \quad (5.81)$$

Step 8 : returning to the variable Φ . Considering (5.49) and (5.59), from (5.81) we deduce that

$$s^6 \lambda^6 \int_0^T e^{-2s\alpha^*} (\xi^*)^6 \|\tilde{\Phi}\|_{\mathbf{H}^2(0,L)}^2 + \|\hat{\Phi}\|_{L^2(0,T;\mathbf{H}^2(0,L))}^2 + \|\bar{\Phi}\|_{L^2(0,T;\mathbf{H}^2(0,L))}^2 \leq C \left(\iint_{Q_\omega} e^{-s\alpha^* - 10s\beta^*} \phi^2 + \iint_Q e^{-8s\beta^*} |G|^2 \right). \quad (5.82)$$

Therefore, from (5.82) and (5.49) we obtain (5.48). \square

Remark 5.16. It is clear that the technique for decomposing $e^{-5s\beta^*} \Phi$ as the solution of three Cauchy problem can be generalized to decomposing $e^{-5s\beta^*} \Phi$ as the solution of N Cauchy problems. This could be useful in situations in which we need to have higher regularity for the solutions of the last two Cauchy problems. It is thanks to this technique why in parabolic and dissipative linear systems with the usual regularity estimates (that is, which satisfy the analogue regularity estimates of either the heat equation or the KdV equation) the effort of getting a Carleman estimate with source term $G = 0$ or $G \in \mathbf{L}^2(Q)$ is very similar. Indeed, the careful reader can remark that in the proof of Proposition 5.13 the terms of $\bar{\Phi}$ has been absorbed easily because of the presence of an exponential weight, whereas it is in the terms of $\tilde{\Phi}$ where we have done more work.

5.4 The nonlinear control problem

In the following, we denote

$$L^r(\rho(t)(0, T); V) := \{w : \rho(t)w \in L^r(0, T; V)\},$$

for V a Banach space, ρ a measurable function and $r \in [1, +\infty]$. Moreover, we define

$$\|w\|_{L^r(\rho(t)(0,T);V)} := \|\rho(t)w\|_{L^r(0,T;V)}.$$

Similarly, we define

$$C(\rho(t)[0, T]; V) := \{w : \rho(t)w \in C([0, T]; V)\}.$$

In addition, in this section we fix (m, λ, s) so that Proposition 5.13 is satisfied and

$$|s\beta_t^*| \leq Ce^{\frac{1}{8}s\beta^*}. \quad (5.83)$$

In that sense, in the following the generic constant C may also depend on s, λ and T .

The structure of this section is the following : first we study the controllability of the linearized problem, second we end the prove of Theorem 5.1, and thirdly we make some remarks about similar control problems which can be studied by using the technique presented in this paper.

5.4.1 Null controllability of (5.5)

Let E be the couples (U, f) such that :

- $U \in L^2(e^{4s\beta^*}(0, T); \mathbf{L}^2(0, L))$,
- $U \in L^2(e^{\frac{7}{2}s\beta^*}(0, T); \mathbf{H}_0^1(0, L)) \cap C(e^{\frac{7}{2}s\beta^*}[0, T]; \mathbf{L}^2(0, L))$,
- $f1_\omega \in L^2(e^{5s\beta^*}(0, T); L^2(0, L))$,
- $\mathcal{L}U - (0, f1_\omega) \in L^2(e^{7s\beta^*}(0, T); \mathbf{H}^{-1}(0, L))$.

Then, we have the following controllability result :

Proposition 5.17. *Let $U^0 \in \mathbf{L}^2(0, L)$, $h_1, h_2 \in L^2(e^{7s\beta^*}(0, T); H^{-1}(0, L))$ and assume that (5.4) is satisfied. Then, there is a solution of (5.5) satisfying $(U, f) \in E$.*

Remark 5.18. If $(U, f) \in E$, we have that $U(T, \cdot) = 0$.

For the proof of Proposition 5.17 we follow the approach introduced in [57] which is based on duality. In particular, we first prove the following auxiliary result :

Lemma 5.19. *Let $L > 0$, $\omega \subset (0, L)$ be an open subset and assume that (5.4) is satisfied. There is $C > 0$ such that*

$$\int_0^T e^{-14s\beta^*} \|\Phi\|_{\mathbf{H}_0^1(0, L)}^2 + \|e^{-7s\beta^*} \Phi\|_{C([0, T]; \mathbf{L}^2(0, L))}^2 \leq C \left(\iint_{Q_\omega} e^{-10s\beta^*} \phi^2 + \iint_Q e^{-8s\beta^*} |G|^2 \right), \quad (5.84)$$

for Φ the energy solution of (5.8) and (m, λ, s) fixed in the beginning of this section.

Proof. [Proof of Lemma 5.19.] It is a straight consequence of (5.40), (5.41)₁, Proposition 5.7 applied to $e^{-7s\beta^*}\Phi$ in $[3T/4, T]$ and (5.48) that the following estimate holds :

$$\int_{3T/4}^T e^{-14s\beta^*} \|\Phi\|_{\mathbf{H}_0^1(0,L)}^2 + \|e^{-7s\beta^*}\Phi\|_{C([3T/4,T];\mathbf{L}^2(0,L))}^2 \leq C \left(\iint_{Q_\omega} e^{-10s\beta^*} \phi^2 + \iint_Q e^{-8s\beta^*} |G|^2 \right). \quad (5.85)$$

Let χ be a positive function supported in $[0, 7/8]$ such that $\chi = 1$ in $[0, 3/4]$. By considering (5.40), the problem satisfied by $\chi(t/T)\Phi$, (5.18) and (5.85) we easily obtain (5.84). \square

Proof. [Proof of Proposition 5.17] Let P_0 be the subspace of $\mathbf{C}^3([0, T] \times [0, L])$ of the functions Φ such that

$$\Phi(\cdot, 0) = \Phi(\cdot, L) = \Phi_x(\cdot, 0) = 0, \quad \text{on } [0, T].$$

Let $\mathbf{a} : P_0 \times P_0 \rightarrow \mathbb{R}$ be the bilinear form

$$\mathbf{a}(\hat{\Phi}, \Phi) = \iint_{Q_\omega} e^{-10s\beta^*} \hat{\phi}\phi + \iint_Q e^{-8s\beta^*} (\mathcal{L}^*\hat{\Phi}) \cdot (\mathcal{L}^*\Phi) \quad (5.86)$$

and $\ell : P_0 \rightarrow \mathbb{R}$ the linear form

$$\ell(\Phi) = \int_0^L U^0 \cdot \Phi(0, x) dx + \int_0^T \langle (h_1, h_2), \Phi \rangle_{\mathbf{H}^{-1}(0,L) \times \mathbf{H}_0^1(0,L)}. \quad (5.87)$$

Notice that the observability inequality (5.84) holds for every $\Phi \in P_0$ taking $G = \mathcal{L}^*\Phi$. Consequently, \mathbf{a} is a scalar product in P_0 and there is $C > 0$ such that for all $\Phi \in P_0$ we have that

$$|\ell(\Phi)| \leq C (\|e^{7s\beta^*}(h_1, h_2)\|_{L^2(0,T;\mathbf{H}^{-1}(0,L))} + \|(u_0, v_0)\|_{\mathbf{L}^2(0,L)}) \sqrt{\mathbf{a}(\Phi, \Phi)}.$$

In the following we denote P the completion of P_0 with respect to \mathbf{a} (and keep denoting \mathbf{a} and ℓ to their only continuous extensions). By Riesz theorem there is $\hat{\Phi} \in P$ such that

$$\mathbf{a}(\hat{\Phi}, \Phi) = \ell(\Phi), \quad \forall \Phi \in P.$$

We define (U, f) by

$$U := e^{-8s\beta^*} \mathcal{L}^*\hat{\Phi}, \quad f := -e^{-10s\beta^*} \hat{\phi}1_\omega. \quad (5.88)$$

It can be proved easily that U is the unique solution by transposition of

$$\begin{cases} \mathcal{L}U = (h_1, h_2 + f1_\omega), & \text{in } Q, \\ U(\cdot, 0) = U(\cdot, L) = U_x(\cdot, L) = 0, & \text{on } (0, T), \\ U(0, \cdot) = U^0, & \text{on } (0, L). \end{cases} \quad (5.89)$$

Indeed, let Φ be the solution of (5.8) for $G \in \mathcal{D}(Q)$ and for $\Phi^T = 0$. Then, we have by Lemma 5.6 that $\Phi \in \mathbf{C}^\infty(\overline{Q}) \cap P_0$, which implies that

$$\begin{aligned} \int_0^T \langle U, G \rangle_{\mathbf{H}^{-1}(0,L), \mathbf{H}_0^1(0,L)} &= \iint_Q e^{-8s\beta^*} (\mathcal{L}^* \hat{\Phi}) \cdot G = \iint_Q e^{-8s\beta^*} (\mathcal{L}^* \hat{\Phi}) \cdot (\mathcal{L}^* \Phi) \\ &= \mathbf{a}(\hat{\Phi}, \Phi) - \iint_{Q_\omega} e^{-10s\beta^*} \hat{\phi} \phi = \int_0^L U^0 \cdot \Phi(0, x) dx + \int_0^T \langle (h_1, h_2 + f1_\omega), \Phi \rangle_{\mathbf{H}^{-1}(0,L) \times \mathbf{H}_0^1(0,L)}. \end{aligned}$$

So we just have to use the density of $\mathcal{D}(Q)$ in $L^2(0, T; \mathbf{H}_0^1(0, L))$. Consequently, since $f, h_1, h_2 \in L^2(0, T; H^{-1}(0, L))$, we obtain from Remark 5.11 that

$$U \in L^2(0, T; \mathbf{H}_0^1(0, L)) \cap C([0, T]; \mathbf{L}^2(0, L)).$$

As for the estimates, from (5.88) we get

$$\iint_{Q_\omega} e^{10s\beta^*} |f|^2 + \iint_Q e^{8s\beta^*} |U|^2 < +\infty. \quad (5.90)$$

Moreover, since U is the solution by transposition of (5.89) we have that

$$\mathcal{L}U - (0, f1_\omega) = (h_1, h_2) \in L^2(e^{7s\beta^*}(0, T); \mathbf{H}^{-1}(0, L)). \quad (5.91)$$

Finally, we consider that $\tilde{U} = e^{\frac{7}{2}s\beta^*} U$ satisfies :

$$\begin{cases} \mathcal{L}\tilde{U} = \frac{7}{2}s\beta_t^* e^{\frac{7}{2}s\beta^*} U + e^{\frac{7}{2}s\beta^*} (h_1, h_2 + f1_\omega), & \text{in } Q, \\ \tilde{U}(\cdot, 0) = \tilde{U}(\cdot, L) = \tilde{U}_x(\cdot, L) = 0, & \text{on } (0, T), \\ \tilde{U}(0, \cdot) = e^{\frac{7}{2}s\beta^*(0)} U^0, & \text{in } (0, L). \end{cases}$$

Moreover, from $U \in L^2(e^{4s\beta^*}(0, T); \mathbf{H}^{-1}(0, L))$ and (5.83) we obtain

$$\frac{7}{2}s\beta_t^* e^{\frac{7}{2}s\beta^*} U + e^{\frac{7}{2}s\beta^*} (h_1, h_2 + f1_\omega) \in L^2(0, T; \mathbf{H}^{-1}(0, L)).$$

Consequently, from Remark 5.11 we find

$$U \in L^2(e^{\frac{7}{2}s\beta^*}(0, T); \mathbf{H}_0^1(0, L)) \cap C(e^{\frac{7}{2}s\beta^*}[0, T]; \mathbf{L}^2(0, L)). \quad (5.92)$$

Thereby, we obtain from (5.90)-(5.92) that $(U, f) \in E$, which along with being the solution of (5.89) completes the proof. \square

5.4.2 Proof of Theorem 5.1

We now prove the local null controllability of system (5.3) by a local inversion argument. We recall that the components of U are denoted by u and v . We consider $\mathfrak{h}_1, \mathfrak{h}_2$ as in the statement of Theorem 5.1, with C large enough so that $\mathfrak{h}_1, \mathfrak{h}_2 \in L^2(e^{7s\beta^*}(0, T); \mathbf{H}^{-1}(0, L))$. Let $\mathcal{G} : E \rightarrow L^2(e^{7s\beta^*}(0, T); \mathbf{H}^{-1}(0, L)) \times \mathbf{L}^2(0, L)$ be an operator defined by :

$$\begin{aligned} \mathcal{G}(U, f) := & (u_t + uu_x + u_{xxx} + av_{xxx} + k_1vv_x + k_2(uv)_x - \mathfrak{h}_1, \\ & cv_t + rv_x + vv_x + abu_{xxx} + v_{xxx} + k_2buu_x + k_1b(uv)_x - \mathfrak{h}_2 - f1_\omega, u(0, \cdot), v(0, \cdot)). \end{aligned} \quad (5.93)$$

We recall that if \mathcal{G} is C^1 and if $D\mathcal{G}[0]$ is surjective, there exists $\delta > 0$ such that if $\|U^0\| < \delta$ there exists $(U, f) \in E$ such that

$$\mathcal{G}(U, f) = (0_{\mathbb{R}^2}, U^0).$$

Hence, by Remark 5.18 we just have to check those two hypotheses to conclude the proof of Theorem 5.1 :

- In order to see that \mathcal{G} is C^1 , since \mathcal{G} is the sum of an affine form and some bilinear forms, it suffices to see that it is a bounded operator. The fact that the affine form is continuous is a straight consequence of the definition of E . As for the bilinear terms of (5.93), we can prove that they are bounded by considering that for all $w_1 \in L^\infty(e^{\frac{7}{2}s\beta^*}(0, T); L^2(0, L))$ and $w_2 \in L^2(e^{\frac{7}{2}s\beta^*}(0, T); H_0^1(0, L))$ we have the estimate

$$\begin{aligned} \|w_1 \partial_x w_2\|_{L^2(e^{7s\beta^*}(0, T); H^{-1}(0, L))} & \leq C \|w_1 \partial_x w_2\|_{L^2(e^{7s\beta^*}(0, T); L^1(0, L))} \\ & \leq C \|w_1\|_{L^\infty(e^{\frac{7}{2}s\beta^*}(0, T); L^2(0, L))} \|w_2\|_{L^2(e^{\frac{7}{2}s\beta^*}(0, T); H_0^1(0, L))}. \end{aligned}$$

- As for the surjectivity of $D\mathcal{G}[0]$, we remark that

$$D\mathcal{G}[0](U, f) = (\mathcal{L}U - (0, f1_\omega), U(0, \cdot));$$

thus, it is a consequence of Proposition 5.17.

5.4.3 Some additional remarks

The technique presented in this paper can be used to solve other controllability problems :

- We can prove the local null controllability of the following control system

$$\begin{cases} u_t + uu_x + u_{xxx} + av_{xxx} + k_1vv_x + k_2(uv)_x = \mathfrak{h}_1 + f1_\omega, & \text{in } Q, \\ cv_t + rv_x + vv_x + abu_{xxx} + v_{xxx} + k_2buu_x + k_1b(uv)_x = \mathfrak{h}_2, & \text{in } Q, \\ u(\cdot, 0) = u(\cdot, L) = u_x(\cdot, L) = 0, & \text{on } (0, T), \\ v(\cdot, 0) = v(\cdot, L) = v_x(\cdot, L) = 0, & \text{on } (0, T), \\ u(0, \cdot) = u^0, \quad v(0, \cdot) = v^0, & \text{on } (0, L), \end{cases} \quad (5.94)$$

assuming (5.4). Indeed, we can do that by replicating the proof of Theorem 5.1 step by step.

- We can slightly improve the result presented in [28] and show that the general Hirota-Satsuma system can be controlled with two forces supported in any two control domains (in [28] the authors need that one of the control domains touches the boundary). The proof is very similar to the one of Theorem 5.1, as we only have to do some minor adjustments.

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