On existence and non degeneracy of travelling waves for the two dimensional Gross Pitaevskii equation

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Sur l'existence et la non dégénérescence d'ondes progressives dans l'équation de Gross-Pitaevskii en dimension deux

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Sur l'existence et la non dégénérescence d'ondes progressives dans l'équation de Gross-Pitaevskii en dimension deux

Dans cette thèse, nous nous intéressons aux ondes progressives dans l'équation de Gross-Pitaevskii $i\partial_t u + \Delta u = (|u|^2 - 1)u$ en dimension 2, avec la condition à l'infini $|u|(x) \to 1$ quand $|x| \to \infty$. Cette équation a fait l'objet d'une étude intensive, que ce soit en physique ou en mathématiques. Il s'agit d'un modèle pour les condensats de Bose-Einstein, et décrit entre autres le comportement de superfluides.

Nous regardons des questions liées au programme de recherche de Jones-Roberts, notamment sur l'existence et l'unicité d'une onde progressive qui est un minimiseur globale de l'énergie à moment fixé. Ces questions ont été abordées dans des travaux précédents en utilisant des méthodes variationnelles. On construit ici, par des méthodes perturbatives et pour des petites vitesses, une branche d'onde progressive régulière par rapport à la vitesse, qui est constituée de deux vortex éloignés l'un de l'autre. Grâce aux propriétés connues sur les vortex, on peut en déduire des propriétés qualitatives satisfaisantes sur cette branche, qui sont meilleurs que ce que l'on peut obtenir par des constructions variationnelles.

Ensuite, on s'intéresse à des propriétés de stabilité sur cette branche. On montre tout d'abord des résultats de coercivité, en améliorant pour cela les résultats de coercivité connus sur les vortex. On en déduit en particulier le noyau de l'opérateur linéarisé, un résultat de stabilité spectrale, ainsi que des résultats d'unicités locales dans l'espace d'énergie. On inverse aussi l'opérateur linéarisé près d'une onde progressive dans des espaces adaptés. Ces résultats sont une étape cruciale pour la compréhension de la stabilité de la branche, et pour démontrer l'unicité du minimiseur de l'énergie. Ces résultats peuvent aussi servir à comprendre l'interaction entre plusieurs ondes progressives dans un même milieu.

Mots clefs: EDP, Gross-Pitaevskii, Ondes progressives, Coercivité, Stabilité

On existence and non degeneracy of travelling waves for the two dimensional Gross-Pitaevskii equaation

In this thesis, we focus on the study of travelling waves in the Gross-Pitaevskii equation $i\partial_t u + \Delta u = (|u|^2 - 1)u$ in dimension 2, with the condition at infinity $|u|(x) \to 1$ when $|x| \to \infty$. This equation has been studied extensively, both in physical and mathematical works. It is a model for Bose-Einstein condensates, and describes the behaviour of superfluids.

We are interested in problems related to the research program of Jones-Roberts, in particular about the existence and unicity of a travelling wave, that minimise the energy at fixed momentum. These questions have been studied, in previous works, using variationnal methods. We construct here, using perturbative methods and for small speeds, a branch of travelling waves, smooth with respect to the speed, which behaves like two vortices far from each other. Using known properties of the vortices, we can deduce good qualitative properties on this branch, that are better than the ones obtained using variationnal methods.

Then, we study stability properties of this branch. First, we show coercivity results, improving for that the known coercivity results on the vortices. In particular, we deduce the kernel of the linearized operator, a result about spectral stability, and a local uniqueness result in the energy space. We also are able to invert the linearized operator near a travelling wave in adapted spaces. These results are a key step for the understanding of the stability of the branch, and to show the unicity of the minimiser of the energy. These results are also a first step in understanding the interaction between several travelling waves.

Keywords: PDE, Gross-Pitaevskii, Travelling waves, Coercivity, Stability

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1. Dois-je aussi ajouter un anti-remerciement pour avoir inventé et calculé la constante de Pacherie ?
2. Réfléchis bien, je t'assure, je ne suis pas mutant !
3. Oh, inscris-toi à l'escalade plutôt que de lire cette footnote !
4. La situation est arrivée beaucoup trop souvent...
5. Et lis la première lettre des cinq premières footnotes :)
6. On dit Alexiss du coup ? En tout cas, il y en a trois, et ils devraient pouvoir s'identifier grace aux footnotes.
7. Celui qui est l'unique personne dans mes trois groupes d'amis principaux à Nice : la danse, l'escalade et le labo.
8. Celui qui pense que je diagonalise des laplaciens, alors qu'il n'y a qu'un seul moment dans ma thèse où ca sert vite fait.
9. Celui qui a le plus de peluche des trois.
10. Tu sais que j'ai toujours le texte "Eliot est-il un ornithorynque" ?
11. Merci aussi de ne pas être morte pendant la rando dans le Mercantour. Les cadavres me mettent mal à l'aise.
12. C'est de ta faute s'il y a tant de footnotes, fallait pas me donner l'idée !
13. Cette phrase est fausse.
14. Pour être tout à fait honnête, ils ont fini leurs thèses avant moi.
15. Hey Paul ! Dis, j'ai un plan pour donner des khôlles à Nice pour les oraux blancs. Ca t'intéresse ?
16. L'homme qui est allé sur la lune en train.

Table of contents

Chapter 1
General introduction and presentation of the results

In this chapter, we summarize the known results on travelling waves in the Gross-Pitaevskii equation in dimension 2, including the results of this thesis. After an introduction on the Gross-Pitaevskii equation, we present, in section 1.2, previous results on this problem and on vortices. This gives an overview of the field, but also gives theorems that will be used in the proofs of the new results. Then, we present the main new results in sections 1.3 to 1.5. We will also give a sketch of the proofs, and provide some context and applications. The full proofs of these results compose the remaining chapters. Some related open problems are given at the end of this chapter, in section 1.6.

1.1 Presentation of the Gross-Pitaevskii equation

We are interested in the Gross-Pitaevskii equation in dimension 2:

$$\begin{cases} i\partial_t u + \Delta u = (|u|^2 - 1)u \text{ in } \mathbb{R}^2 \\ |u|(x) \to 1 \text{ as } |x| \to \infty. \end{cases}$$

It is a physical model for Bose-Einstein condensate (see [16], [39]). It also describes the behaviour of superfluids, as for instance a thin liquid helium film. This equation is closely related to the Ginzburg-Landau equation and superconductivity problems. It is associated with the Ginzburg-Landau energy

$$E(v) := \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^2} (1 - |v|^2)^2.$$

We are interested in the qualitative description of solutions for the Gross-Pitaevskii equation (we refer to [4], [14], [24] for the question of long time existence). It has some particular stationnary solutions, named vortices, that play the role of solitons. They solve the equation

$$(S)\begin{cases} \Delta u + (1 - |u|^2)u = 0 \text{ in } \mathbb{R}^2 \\ |u|(x) \to 1 \text{ as } |x| \to \infty. \end{cases}$$

The stationnary problem (S) is in itself an interesting one, as it is a particular case of (GP). Stationnary vortices solutions with radial symmetry (of the form $V_n(x) = \rho_n(r)e^{in\theta}$) have been constructed (see [7]), and the uniqueness of these solutions (up to a translation and a shift of phase) with degree one at infinity have been shown [38]. We are interested in several questions about (GP) and the vortices. Can we find particular solutions of the Gross-Pitaevskii equation that behave like multi vortices ? Are these solutions stable ? Can we understand the long time behaviour of solutions that are close to a multi vortex solution ?

Here, we will focus on the study of travelling waves in (GP). They are, in a sense, the most simple type of solutions after the stationnary ones. In particular, many conjectures exist on them in the physical litterature. We refer mainly to the series of works from Jones, Putterman and Roberts ([17], [26], [27] and references therein). In this both physical and numerical study, it is conjectured that travelling waves can only have speed between 0 and $\sqrt{2}$ (this limit being the speed of sound in the model). Furthermore, they predicted the existence of a particular branch of travelling waves on the full range of possible speeds, which is a global (or at least local) minimizer of the energy (at fixed either speed or momentum). This branch behaves, in the limit $c \to \sqrt{2}$, up to a rescale, to a solution of Kadomtsev-Petviashvili KP$-I$ (see [5]), and in the limit $c \to 0$, as two vortices, of degree $+1$ and -1, at a distance of order $2/c$ from each other (see [4]). We also refer to [31] for the construction of other travelling waves in (GP), and [29] for similar equations.

Many mathematical results have been proven in this direction. Non existence for supersonic speeds ($c \geqslant \sqrt{2}$) has been rigorously justified (see [18] and [20]), and this Jones-Roberts branch has been constructed using energy methods. We refer for instance to the partial construction of the branch by minimizing the energy at fixed small speed in [4], or at fixed momentum, in [6]. More recently, a method construction has been given for almost all subsonic speeds, in [2]. A main open problem is to show that all these constructions give in fact the same branch.

These methods of construction have been extended to other but similar problems. There are existence results on the Gross-Pitaevskii equation, but in other dimensions (see [32]), or with different nonlinearities ([8]).

These constructions by energetic methods give solutions that locally minimize the energy, but the conjectured structure in term of vortices in the limit $c \to 0$ remains unclear. This structure is visible in some sense in [4], but it is not clear for instance that the constructed travelling waves form a branch (namely, that $c \to Q_c$, where Q_c is the travelling wave of speed c, is continuous in some sense for these constructions).

We can therefore look for another way of constructing these travelling waves, which will make the branch structure clearer, rather than having properties on the energy of the solution. For that, perturbative methods are more adapated than energy ones. With this method of construction, and some known properties of vortices (in particular [10]), it allows a more precise study of this branch, and its stabilty.

We give a few notations, that hold in all the chapters. We denote, for functions $f, g \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{C})$ such that $\mathfrak{Re}(f\bar{g}) \in L^1(\mathbb{R}^2, \mathbb{C})$, the quantity

$$\langle f, g \rangle := \int_{\mathbb{R}^2} \mathfrak{Re}(f\bar{g}),$$

even if $f, g \notin L^2(\mathbb{R}^2, \mathbb{C})$. We also use the notation $B(x, r)$ to define the closed ball in $\mathbb{R}^2$ of center $x \in \mathbb{R}^2$ and radius $r > 0$ for the Euclidean norm. We define, between two vectors $X = (X_1, X_2)$, $Y = (Y_1, Y_2) \in \mathbb{C}^2$, the quantity

$$X.Y := X_1 Y_1 + X_2 Y_2.$$

1.2 Previous results

We recall the Gross-Pitaevskii equation in dimension 2 (for $u : \mathbb{R}_t \times \mathbb{R}^2_x \to \mathbb{C}$)

$$(\text{GP})(u) := i\partial_t u + \Delta u - (|u|^2 - 1)u = 0.$$

The condition at infinity for (GP) will be

$$|u| \to 1 \quad \text{as} \quad |x| \to +\infty,$$

and it is associated with the Ginzburg-Landau energy

$$E(v) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla v|^2 + \frac{1}{4}\int_{\mathbb{R}^2} (1 - |v|^2)^2.$$

The Gross-Pitaevskii equation can be seen as a nonlinear Schrödinger equation, with a non trivial condition at infinity and a nonlinearity adapted to this condition. However, the condition $|u| \to 1$ as $|x| \to +\infty$ allows solutions to have a non trivial behaviour at infinity (behaving like $e^{i\theta}$ for instance), and thus the equation is not simply solved by the sum of a constant and a solution of a nonlinear Schrödinger equation going to 0 at infinity. An exemple such solutions are vortices.

1.2.1 Vortices in Gross-Pitaevskii

1.2.1.1 Existence and decay properties

Vortices are some particular stationnary solutions of the Gross-Pitaevskii equation. They solve

$$(S) \begin{cases} \Delta u + (1 - |u|^2)u = 0 \text{ in } \mathbb{R}^2 \\ |u|(x) \to 1 \text{ as } |x| \to \infty, \end{cases}$$

and are of the form $u(x)=\rho_n(r)e^{in\theta}$, where $n\in\mathbb{Z}^*$, (r,θ) are the polar coordinates of $x\in\mathbb{R}^2$, and ρ_n is a real-valued function. For $n=\pm1$, existence of such functions, and some of their properties, are listed in the following result.

Lemma 1.2.1. ([7] and [25]) *A vortex centered around 0, $V_1(x)=\rho_1(r)e^{i\theta}$, verifies $V_1(0)=0$, $E(V_1)=+\infty$ and there exist constants $K,\kappa>0$ such that*

$$\forall r>0, 0<\rho_1(r)<1, \rho_1(r)\sim_{r\to0}\kappa r, \rho_1'(r)\sim_{r\to0}\kappa$$

$$\rho_1'(r)>0; \rho_1'(r)=O_{r\to\infty}\left(\frac{1}{r^3}\right), |\rho''(r)|+|\rho'''(r)|\leqslant K,$$

$$1-|V_1(x)|=\frac{1}{2r^2}+O_{r\to\infty}\left(\frac{1}{r^3}\right),$$

$$|\nabla V_1|\leqslant\frac{K}{1+r}, |\nabla^2 V_1|\leqslant\frac{K}{(1+r)^2}$$

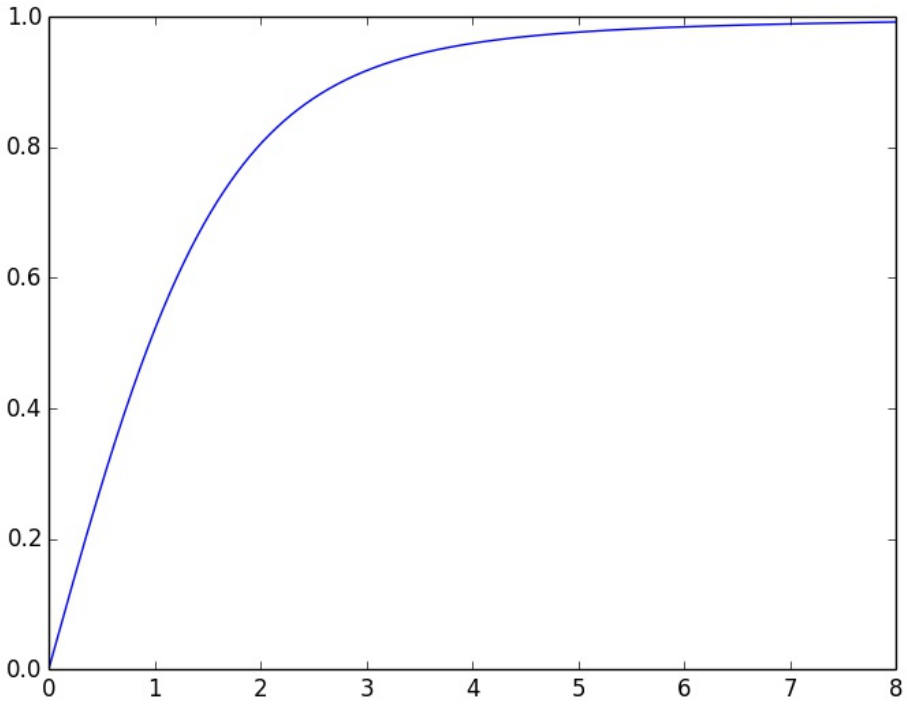
and

$$\nabla V_1(x)=i\,V_1(x)\frac{x^\perp}{r^2}+O_{r\to\infty}\left(\frac{1}{r^3}\right),$$

where $x^\perp:=(-x_2,x_1)$, $x=re^{i\theta}\in\mathbb{R}^2$. Furthermore, similar properties holds for V_{-1}, since

$$V_{-1}(x)=\overline{V_1(x)}.$$

Here is a graph of $\rho_1(r)$ for $r\in[0,8]$.



We refer to [25] for the existence and similar properties for vortices of other degrees. Still from [25], it is possible to compute asymptotics at all order of $\rho_n(r)$ for $r\to0$ and $r\to\infty$. Furthermore, by the invariances of (GP), we have that

$$\left\{V_{\pm1}(x-X)e^{i\gamma}, X\in\mathbb{R}^2, \gamma\in\mathbb{R}\right\}$$

are solutions of the problem (S).

Since $\rho_1(r) = 0$ if and only if $r = 0$, we define the center of a vortex by being the only point where the function is 0. Remark that, up to a shift of phase, a vortex in this manifold is completely defined by its degree (± 1) and its center.

It has been shown that these particular solutions of (S) are the only ones with degree ± 1:

Theorem 1.2.2. ([38]) *If u is a solution of $\Delta u + (1-|u|^2)u = 0$ in $\mathbb{R}^2$ with*

$$\int_{\mathbb{R}^2} (1-|u|^2)^2 < +\infty$$

and $u - e^{\pm i\theta} \to 0$ at infinity, then there exists $X \in \mathbb{R}^2$ such that

$$u = V_{\pm 1}(. - X).$$

This result will be not used in itself in the study of travelling waves in (GP). However, it shows that the vortices have a special role in this problem.

From Lemma 1.2.1, the energy of vortices is infinite. Despite that, they will play a role in the construction of finite energy travelling waves. Their energy is infinite because of their behaviour at infinity (the degree is not zero), but a multi vortex solution with a sum of degrees equal to 0 is, at least formally, of finite energy.

About vortices of degrees $n \geqslant 2$, few properties are known. We refer to [40] for some numerical results.

1.2.1.2 Coercivity results on vortices of degree ± 1

For a vortex $V_{\pm 1}$ centered at 0, we define the quadratic form, formally defined by the second variation of the energy E around $V_{\pm 1}$:

$$B_{V_{\pm 1}}(\varphi) := \int_{\mathbb{R}^2} |\nabla \varphi|^2 - (1-|V_{\pm 1}|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{V_{\pm 1}}\varphi),$$

for functions $\varphi \in H_{V_1}$, the associated energy space:

$$H_{V_{\pm 1}} := \left\{ \varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C}),\ \int_{\mathbb{R}^2} |\nabla \varphi|^2 + (1-|V_{\pm 1}|^2)|\varphi|^2 + \mathfrak{Re}^2(\overline{V_{\pm 1}}\varphi) < +\infty \right\}.$$

Remark by Lemma 1.2.1 that $1 - |V_{\pm 1}|^2 > 0$.

Theorem 1.2.3. (Theorem 1.1 of [10]) *For $\varphi \in H_{V_{\pm 1}}$,*

$$B_{V_{\pm 1}}(\varphi) \geqslant 0$$

and if $B_{V_{\pm 1}}(\varphi) = 0$, then

$$\varphi \in \mathrm{Span}_{\mathbb{R}}(\partial_{x_1} V_{\pm 1}, \partial_{x_2} V_{\pm 1}).$$

We will use this result, but also elements of its proof (in [10]) in the proofs of the new results. In summary, vortices of degree ± 1 are well understood, despite some difficulties. They have infinite energy, slow decays in position, and a weak coercivity result. This will make the construction by perturbative method difficult, but allow a rich dynamic. We refer to [41] (in particular, Corollary 3.3) and [37] for more related coercivity or stability results.

1.2.2 Travelling waves in Gross-Pitaevskii

Travelling wave solutions of (GP), i.e. solutions of the form

$$\mathfrak{u}(t,x) = v(x_1, x_2 + ct)$$

with $c \in \mathbb{R}$ (the equation is invariant by rotation, we therefore choose, without loss of generality, that the travelling wave moves in the direction $-\vec{e}_2$), solve the equation

$$0 = (\mathrm{TW}_c)(v) := -ic\partial_{x_2} v - \Delta v - (1-|v|^2)v.$$

1.2.2.1 Momentum and range of possible speeds

As in similar physical problems, we want to define the momentum of such a solution, but it is not easy since a travelling wave does not go at 0 at infinity. We refer to [32] for a definition in dimension $n \geqslant 3$.

Theorem 1.2.4. ([8]) *A travelling wave with finite energy converges to a constant at infinity in position. Up to a shift of phase, we can therefore suppose that a travelling wave of finite energy solves the problem*

$$(\mathrm{TW}_c^1) \begin{cases} -ic\partial_{x_2} v - \Delta v - (1-|v|^2)v = 0 \text{ in } \mathbb{R}^2 \\ v(x) \to 1 \text{ as } |x| \to \infty. \end{cases}$$

Then, the quantity

$$\vec{P}(v) := \langle i\nabla v, v-1\rangle$$

is well defined, and is the momentum of the solution. Furthermore, for $\psi \in H^1(\mathbb{R}^2, \mathbb{C})$,

$$\langle i\partial_{x_2}\psi, v-1\rangle = -\langle i\psi, \partial_{x_2} v\rangle.$$

Remark that $\vec{P}(v) \in \mathbb{R}^2$, and we denote $P_1(v)$ and $P_2(v)$ its two components. We now focus on the possible speeds of a travelling wave. It has been first conjectured in physics, then shown rigorously, that there are no travelling waves at sonic or supersonic speed:

Theorem 1.2.5. ([18] and [20]) *If $|c| \geqslant \sqrt{2}$, the only solution with finite energy of (TW_c^1) is the constant 1.*

In (GP), up to the physical rescaling, $\sqrt{2}$ is the speed of sound. It is fully expected that all the speeds in $]0, \sqrt{2}[$ are reached by a travelling wave. We now give some precise existence results.

1.2.2.2 Existence results for travelling waves

Travelling waves have been constructed using energy methods. The idea is to look for a minimizer of the energy at fixed momentum and using a mountain pass argument. Such constructions have been done in different regimes. First, for small speeds, with the apparition of a two vortices structure.

Theorem 1.2.6. ([4]) *There exists some constant $c_0 > 0$ such that, for $0 < c < c_0$, there exists a non constant solution v of (TW_c^1) with finite energy. Moreover, there exists $\Lambda_0, \Lambda_1 > 0$ such that*

$$2\pi|\log c| + \Lambda_0 \leqslant E(v) \leqslant 2\pi|\log c| + \Lambda_1.$$

This function v is smooth, and there exists $0 < \varepsilon_1 < \varepsilon_2 < 1$, $\rho \in [c^{-\varepsilon_1}, c^{\varepsilon_2}]$, exactly two points a_1, $a_2 \in \mathbb{R}^2$ such that

$$|v(x)| \geqslant 1/2 \quad \text{on} \quad \mathbb{R}^2 \backslash \bigcup_{i=1}^{2} B(a_i, \rho),$$

$$\deg(a_i) = (-1)^i$$

and

$$c||a_1 - a_2| - 2| + |a_{1,1} - a_{2,1}| = o_{c\to 0}(1).$$

We see that the solution cancels only in two regions, separated by a distance of order $2/c$, and the degrees are ± 1. This is the only construction by energy method where this structure has been shown.

Another way to construct solutions of (TW_c^1) is to fix the momentum, and minimize locally the energy. This can be done for any momentum $P_2(v)>0$. Remark that large momentum yields small speeds in dimension 2, and small momentum speeds close to $\sqrt{2}$ (still in dimension 2).

Theorem 1.2.7. ([6]) *Let $p>0$. There exists a non constant finite energy solution v_p to (TW_c^1) for a speed $c=c(p)$, such that $P_1(v_p)=0, P_2(v_p)=p$. This function is solution to the minimization problem*

$$E(v_p)=\inf\big\{E(v), v\in W(\mathbb{R}^2), P_1(v_p)=0, P_2(v)=p\big\},$$

where $W(\mathbb{R}^2)=\{1\}+V(\mathbb{R}^2)$ and

$$V(\mathbb{R}^2)=\big\{v:\mathbb{R}^2\mapsto\mathbb{C}, \nabla v, \mathfrak{Re}(v)\in L^2(\mathbb{R}^2), \mathfrak{Im}(v)\in L^4(\mathbb{R}^2), \nabla\mathfrak{Re}(v)\in L^{4/3}(\mathbb{R}^2)\big\}.$$

Here, the speed appears as a Lagrange multiplier. The method of construction used here has also been improved and extended for other nonlinearities in [8]. There, they show in addition some precompactness and orbital stability results. We state here the results in the case of the Gross-Pitaevskii equation in dimension 2.

Theorem 1.2.8. ([8]) *For $q\geqslant 0$, let*

$$E_{\min}(q)=\inf\big\{E(\psi), |\psi|-1\in L^2(\mathbb{R}^2), \nabla\psi\in L^2(\mathbb{R}^2), P_2(\psi)=q\big\}.$$

Then:

(i) The function $E_{\min}$ is concave, increasing on $[0,\infty[$, $E_{\min}(q)\leqslant\sqrt{2}q$ for any $q\geqslant 0$, the right derivative of $E_{\min}$ at 0 is $\sqrt{2}$, $E_{\min}(q)\to\infty$ and $\frac{E_{\min}(q)}{q}\to 0$ as $q\to\infty$.

(ii) Let $q_0=\inf\big\{q>0,\ E_{\min}(q)<\sqrt{2}q\big\}$. For any $q>q_0$, all sequences $(\psi_n)_{n\geqslant 1}\subset\{\psi, |\psi|-1\in L^2(\mathbb{R}^2), \nabla\psi\in L^2(\mathbb{R}^2)\}$ satisfying $P_2(\psi_n)\to q$ and $E(\psi_n)\to E_{\min}(q)$ are precompact for the semi distance $d_0(\psi_1,\psi_2)=\|\nabla\psi_1-\nabla\psi_2\|_{L^2(\mathbb{R}^2)}+\||\psi_1|-|\psi_2|\|_{L^2(\mathbb{R}^2)}$ (modulo translations).

The set $S_q=\{\psi, |\psi|-1\in L^2(\mathbb{R}^2), \nabla\psi\in L^2(\mathbb{R}^2), P_2(\psi)=q, E(\psi)=E_{\min}(q)\}$ is not empty and is orbitally stable (for the semi distance d_0 by the flow associated to $i\partial_t\Phi+\Delta\Phi-(|\Phi|^2-1)\Phi=0$)

(iii) Any $\psi_q\in S_q$ is a travelling wave for the Gross-Pitaevskii equation, of speed $c(\psi_q)\in[d^+E_{\min}(q),\ d^-E_{\min}(q)]$, where we denote by d^- and d^+ the left and right deriatives. We have $c(\psi_q)\to 0$ as $q\to\infty$.

More recently, another construction has been done for almost all speeds in $\left]0,\sqrt{2}\right[$.

Theorem 1.2.9. ([2]) *There exists a subset $S\subset\left]0,\sqrt{2}\right[$ of full measure such that, for any $c\in S$, there exists a non constant finite energy solution v_c of (TW_c^1).*

Furthermore, for any $c_0\in\left]0,\sqrt{2}\right[$, there exists $K(c_0)>0$ such that

$$0<(E-cP_2)(v_c)\leqslant K(c_0)$$

for all $c\in S$, $c\geqslant c_0$.

It is still an open problem to show that all of these constructions yield the same solution.

1.2.2.3 Qualitative properties of travelling waves

We present here some qualitative properties of travelling waves, assuming the existence. In particular, their asymptotics development at infinity in position have been computed.

Theorem 1.2.10. ([13]) *For Q_c a solution of (TW_c^1) with finite energy,*

$$\|Q_c\|_{L^\infty(\mathbb{R}^2)}\leqslant\sqrt{1+\frac{c^2}{4}}.$$

Theorem 1.2.11. ([21], Theorems 1 and 2, Propositions 5 and 7) *For Q_c a solution of (TW_c^1) with finite energy, writing $Q_c(x)=|Q_c|(x)e^{i\theta(x)}$, for $x\in\mathbb{R}^2$, $x=R\sigma$, $R>0$, $\sigma=(\sigma_1,\sigma_2)\in\mathbb{S}^1$, where $\mathbb{S}^1$ is the unit circle,*

$$R(Q_c(R\sigma)-1)=\frac{-i\alpha_c\sigma_2}{\left(1-\frac{c^2}{2}+\frac{c^2\sigma_2^2}{2}\right)}+o^c_{R\to\infty}(1),$$

$$R^2(1-|Q_c|^2)(R\sigma)=\alpha_c c\left(\frac{1}{\left(1-\frac{c^2}{2}+\frac{c^2\sigma_2^2}{2}\right)}-\frac{2\sigma_2^2}{\left(1-\frac{c^2}{2}+\frac{c^2\sigma_2^2}{2}\right)^2}\right)+o^c_{R\to\infty}(1)$$

in $C^1(\mathbb{S}^1)$, *and*

$$R\,\theta(R\sigma)=\frac{-\alpha_c\sigma_2}{\left(1-\frac{c^2}{2}+\frac{c^2\sigma_2^2}{2}\right)}+o^c_{R\to\infty}(1)$$

in $C^2(\mathbb{S}^1)$, *with* $\alpha_c:=\frac{c\,E(Q_c)+\left(2-\frac{c^2}{2}\right)P_2(Q_c)}{2\pi\sqrt{1-\frac{c^2}{2}}}$.

Here, $o^c_{R\to\infty}(1)$ denotes a quantity going to 0 when $R\to\infty$ for a fixed $c>0$.

This thesis aims to improve qualitative results on the branch in the limit $c\to 0$. We want to improve the smoothness of the branch with respect to the speed, and we want to study the structure in term of vortices. In particular, in none of the three constructed branch is it proven that the branch is even continuous with respect to the speed, and the vortices structure can be made clearer.

1.3 Smooth branch of travelling waves for small speed

The result presented in this section have been submitted for publication as a paper in collaboration with David Chiron. We refer to Theorem 1.3.1 for the main result. A sketch of its proof is given in subsection 1.3.2. Chapter 2 is devoted to the full proof of this theorem.

1.3.1 Construction of the branch

The main result of this subsection is the construction of a branch of solution by perturbation of the product of two vortices for any small speed $c>0$, and the fact that this branch of solution is C^1 with respect to the speed.

Theorem 1.3.1. *There exists $c_0>0$ a small constant such that, for any $0<c\leqslant c_0$, there exists a solution of* (TW_c) *of the form*

$$Q_c=V_1\big(.-d_c\vec{e_1}\big)V_{-1}(.+d_c\vec{e_1})+\Gamma_{c,d_c},$$

where $d_c=\frac{1+o_{c\to 0}(1)}{c}$ is a continuous function of c. This solution has finite energy $(E(Q_c)<+\infty)$ and $Q_c\to 1$ when $|x|\to+\infty$.

Furthermore, for all $+\infty\geqslant p>2$, there exists $c_0(p)>0$ such that if $c<c_0(p)$, for the norm

$$\|h\|_{X_p}:=\|h\|_{L^p(\mathbb{R}^2)}+\|\nabla h\|_{L^{p-1}(\mathbb{R}^2)}$$

and the space $X_p:=\left\{f\in L^p(\mathbb{R}^2),\nabla f\in L^{p-1}(\mathbb{R}^2)\right\}$, one has

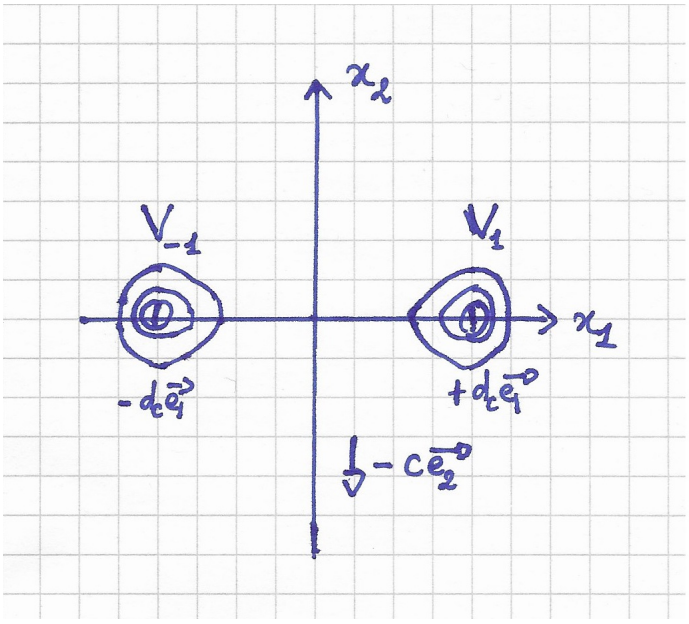
$$\|\Gamma_{c,d_c}\|_{X_p}=o_{c\to 0}(1).$$

In addition,

$$c\mapsto Q_c-1\in C^1(]0,c_0(p)[,X_p),$$

with the estimate (for $\nu(c)=\frac{1+o_{c\to 0}(1)}{c^2}$)

$$\|\partial_c Q_c+\nu(c)\partial_d(V_1(.-d\vec{e_1})V_{-1}(.+d\vec{e_1}))_{|\,d=d_c}\|_p=o_{c\to 0}\left(\frac{1}{c^2}\right).$$



In this representation of Q_c, the lines around $\pm d_c\vec{e}_1$ represent equivalues for $|Q_c|$.

Here, we use an implicit function argument to construct the solution, using technics developped in [11] or [29] for instance, displaying a clear understanding of the shape of the solution (see Lemma 2.2.8 for instance). We show in addition that the constructed branch is C^1, which is, to the best of our knowledge, the first result of this kind in dimension larger than one.

The formal method for this kind of construction is well known. Namely, it is a Lyapunov-Schmidt reduction in weighed L^∞ spaces. It has been done for instance rigorously in a bounded domain for the Ginzburg Landau equation ([11]). One of the difficulties here is to find the right functional setting to construct the C^1 branch, in particular with regards to the transport term $ic\partial_{x_2}v$. On the contrary of what is claimed in [29], the transport term can not be treated perturbatively. This is why we use another functional setting than [29] or [31] (see Remark 2.1.11 for more details)

1.3.2 Sketch of the proof of Theorem 1.3.1

As mentioned above, we look for an ansatz which is a perturbation of two vortices. Take $d = O_{c\to 0}\big(\frac{1}{c}\big)$ a large free parameter, η a smooth cutoff function such that $\eta = 1$ in $B(\pm d\vec{e}_1, 1)$ and 0 outside of $B(\pm d\vec{e}_1, 2)$, and an ansatz of the form

$$Q_c(x) = \eta(x)V(x)(1+\Psi(x)) + (1-\eta(x))V(x)e^{\Psi(x)},$$

with $V(x) = V_1(x - d\vec{e}_1)V_{-1}(x + d\vec{e}_1)$. Writing the perturbation as an exponential is well adapted to the problem. This can not be done near $\pm d\vec{e}_1$, since there, the product of the vortices V has zeros. This explains the shape of the ansatz, it is additive close to the center of the vortices, and exponential far from them. A similar decomposition was used in [11] and [29]. We look for a perturbation $\Psi = \Psi_1 + i\,\Psi_2 \in \mathcal{E}_{*,\sigma,d}$, small in the norm of this space, that is, for $0 < \sigma < 1$,

$$\begin{aligned}\|\Psi\|_{*,\sigma,d} &= \|V\Psi\|_{C^2(\{\tilde{r}\leqslant 3\})} + \|\tilde{r}^{1+\sigma}\Psi_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\nabla\Psi_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})} \\ &+ \|\tilde{r}^{\sigma}\Psi_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{1+\sigma}\nabla\Psi_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\nabla^2\Psi\|_{L^\infty(\{\tilde{r}\geqslant 2\})},\end{aligned}$$

where $\tilde{r} = \min\left(|x - d\vec{e}_1|, |x + d\vec{e}_1|\right)$ is the minimum to the distance to the two vortices. At the end of the proof, we will have that $\|\Psi\|_{*,\sigma,d} \leqslant K(\sigma,\sigma')c^{1-\sigma'}$ for any $0 < \sigma < \sigma' < 1$. We also suppose that the ansatz has two symmetries:

$$\forall x = (x_1, x_2) \in \mathbb{R}^2, \Psi(x_1, x_2) = \overline{\Psi(x_1, -x_2)} = \Psi(-x_1, x_2).$$

For now, the parameter d is free, it will help to cancel a Lagrange multiplier later on.

The equation on the perturbation is then written as (see Lemma 2.1.7)

$$\eta L(\Phi) + (1-\eta)VL'(\Psi) + E - ic\partial_{x_2}V + V(1-\eta)(-\nabla\Psi.\nabla\Psi + |V|^2 S(\Psi)) + R(\Psi),$$

where $\Phi = V\Psi$,

$$L'(\Psi) = -\Delta\Psi - 2\frac{\nabla V}{V}.\nabla\Psi + 2|V|^2\mathfrak{Re}(\Psi) - ic\partial_{x_2}\Psi,$$

$$L(\Phi) = -\Delta\Phi - (1-|V|^2)\Phi + 2\mathfrak{Re}(\bar{V}\Phi)V - ic\partial_{x_2}\Phi$$

are the linearized operator around V (with the exponential or additive perturbation respectively), $E - ic\partial_{x_2}V = -\Delta V - (1-|V|^2)V - ic\partial_{x_2}V = \mathrm{TW}_c(V)$ is the small source term, and $S(\Psi), R(\Psi)$ are nonlinear terms. The goal is to do a Lyapunov-Schmidt reduction to construct the function Ψ, and reduce the problem to a one dimensional one, on $d \in \mathbb{R}$.

For that, the first step, as for general method of construction by a perturbative method, is to invert the linearized operator L (or equivalently L') around V in the space $\mathcal{E}_{*,\sigma,d}$ (from another weighed L^∞ space $\mathcal{E}_{**,\sigma',d}$, see subsection 2.1.3). In the limit $c \to 0$ (thus $d \to \infty$), the function $V = V_1(.-d\vec{e}_1)V_{-1}(.+d\vec{e}_1)$ will behave like two decoupled vortices. The linearized operator around a single vortex is well understood (see Theorem 1.2.3), and has two zeros. We therefore expect four directions that might pose difficulties for the inversion (coming from the two translations for each vortices). With the two symmetries, there is only one direction left, and thus we will invert the operator with one orthogonality condition. This direction is $\partial_d V(x) = -\partial_{x_1}V_1(x-d\vec{e}_1)V_{-1}(x+d\vec{e}_1) + \partial_{x_1}V_{-1}(x-d\vec{e}_1)V_1(x+d\vec{e}_1)$, and will be dealt with by choosing the right value for the parameter d later on.

For the inversion, we start with an a priori estimate on the problem

$$L(\Phi) = Vh.$$

We want to show that if $\Phi = V\Psi$ and h satisfy this equation, with an orthogonality condition on Φ, then for small speeds, $\|\Psi\|_{*,\sigma,d} \leqslant K(\sigma,\sigma')\|h\|_{**,\sigma',d}$ (the norm $\|.\|_{**,\sigma',d}$ is a weighed L^∞ norm, as $\|.\|_{*,\sigma,d}$, see subsection 2.1.3, and $0<\sigma<\sigma'<1$). This is done by contradiction. Suppose that it does not hold. Then, there exist $c_n \to 0$ (thus $d_n \to \infty$), $\|\Psi_n\|_{*,\sigma,d} = 1$, $\Phi_n = V\Psi_n$ and $\|h_n\|_{**,\sigma',d_n} \to 0$ such that $L(\Phi_n) = Vh_n$. Then, following the vortices (by a translation and up to a subsequence, using standards compactness arguments), $\Phi_n(.\mp d_n\vec{e}_1) \to \Phi$ with $L_{V_{\pm1}}(\Phi) = 0$, thus $\Phi = 0$ by the orthogonality condition and the symmetries. This implies that locally near the vortices, $\Phi_n(.\mp d\vec{e}_1) \to 0$ when $n \to \infty$. Then, to show that $\|\Psi_n\|_{*,\sigma,d_n} \to 0$ when $n \to \infty$ (and thus contradicting $\|\Psi_n\|_{*,\sigma,d_n} = 1$), this becomes an elliptic estimate problem. We want to use the equation $L(\Phi_n) = Vh_n$, the fact that h_n is small ($\|h_n\|_{**,\sigma',d_n} \to 0$ when $n \to \infty$), and that Φ_n is small locally near the vortices to show that Φ_n is small in the whole space. We use for that the Gross-Pitaevskii kernels, that have been studied in [19]. Writing $\Phi = V\Psi$, the equation $L(\Phi) = Vh$ becomes, at infinity in position and at first order,

$$-ic\partial_{x_2}\Psi - \Delta\Psi + 2\mathfrak{Re}(\Psi) = h.$$

The Gross-Pitaevskii kernels are used to invert this problem, writing Ψ as a convolution using h, and estimates on these convolution kernels (done in [19]) are enough to show the smallness of Φ (through Ψ) given the smallnes of h in the right norms.

Now, to show the existence of a solution Φ to the problem $L(\Phi) = Vh$, we use the Fredholm alternative. To add the required compactness to apply it, we look at the same equation in a bounded domain (large compared to the distance between the vortices) with a Dirichlet boundary condition. Here, the existence is thus a consequence of another a priori estimate, that will be a consequence of the previous one (see Lemma 2.1.19). Then, we let the size of the domain goes to infinity, and that provide the full inversion theorem for L (see Proposition 2.1.20), with one local orthogonality condition on $\partial_d V$.

The next difficulty for the existence of the solution are the nonlinear and source terms. We will use a fixed point theorem, by looking at the operator

$$(\eta L(.) + (1-\eta)VL'(./V))^{-1}(\Pi_d^\perp(-F(./V))) : \mathcal{E}_{*,\sigma,d} \to \mathcal{E}_{*,\sigma,d},$$

where F contains the source term and nonlinear terms, and $\Pi_d^{\perp}$ is a projector encoding the orthogonality condition, as we have yet to deal with this other problem. This operator is a contraction (for small perturbations, in order to kill the nonlinearity), and thus we can, at this point, construct a solution $\Phi_{c,d} = V\Psi_{c,d}$ to the problem

$$(\mathrm{TW}_c)(Q_c) = \lambda(c,d)Z_d,$$

where Z_d is a localized version of $\partial_d V$, and $\lambda(c,d) \in \mathbb{R}$ is a Lagrange multiplier (coming from the orthogonality condition). We thus look for a good choice of $d \in \mathbb{R}$ to cancel it (with $d = O_{c\to 0}\big(\frac{1}{c}\big)$). This is now only a one dimensional problem. For that, an estimate shows that

$$\lambda(c,d) = \pi\left(\frac{1}{d} - c\right) + O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

At this point, we do not know if the $O^{\sigma}_{c\to 0}(c^{2-\sigma})$ is continuous with respect to c and d. We want to apply the intermediate value theorem to cancel $\lambda(c,\, d)$. For that, we need to show that the functions $c, d \mapsto \Phi_{c,d}$, with $\Phi_{c,d}$ the perturbation constructed by the Lyapunov-Schmidt reduction described above, are continuous functions.

Since our goal is to show the differentiability of the branch, we will show a stronger result, that is that $\Phi_{c,d}$ is a C^1 function of c and d in the weighed L^∞ spaces. Leaving some technical details aside, the main ingredient is the implicit function theorem. We look at the functional

$$H(\Phi, c, d) := (\eta L(.) + (1-\eta)\, VL'(./V))^{-1}(-\Pi_d^{\perp}(F(\Phi/V))) + \Phi,$$

for which $H(\Phi_{c,d}, c, d) = 0$. We compute its differential with respect to its first variable

$$d_\Phi H(\Phi, c, d)(\varphi) = (\eta L(.) + (1-\eta)VL'(./V))^{-1}(\Pi_d^{\perp}(-d_\Psi F(\varphi/V))) + \varphi,$$

and thus, if $\Phi_{c,d}$ is small (which would make small the term $d_\Psi F(\varphi/V)$, since it is the differential of terms at least quadratic, and thus still containing $\Phi_{c,d}$), we can apply the implicit function theorem ($d_\Phi H(\Phi, c, d)$ is then a perturbation of the identity). This will follow from smallness and decays estimates on $\Phi_{c,d}$ and its derivatives in position (for general values of c and d, without requiring that $\lambda(c,d) = 0$).

We can now finally finish the construction of the travelling wave. We fix d_c a value such that $\lambda(c, d_c) = 0$ (several can exist at this point), and now $(\mathrm{TW}_c)(Q_c) = 0$. We check that Q_c has finite energy, and from the smallness of $\Psi_{c,d}$ in $\mathcal{E}_{*,\sigma,d}$, we give some estimates on Q_c and its derivatives in position (see Lemma 2.2.8).

The remaining difficulty to show that the branch $c \mapsto Q_c$ is C^1 with respect to the speed, is to show that $c \to d_c$ is a C^1 function. Unfortunately, this is quite convoluted. We will use the implicit function theorem on the equation $\lambda(c,d) = 0$ that defined d_c, and for that, we have to show that $\partial_d\lambda(c,\, d_c) \neq 0$. We recall that $\lambda(c,\, d) = \pi\big(\frac{1}{d} - c\big) + O^{\sigma}_{c\to 0}(c^{2-\sigma})$, and from the implicit function theorem on $\Psi_{c,d}$, we have that $\|\partial_d\Psi_{c,d}\|_{*,\sigma,d} = O^{\sigma,\sigma'}_{c\to 0}(c^{1-\sigma'})$ (since $\|\Psi_{c,d}\|_{*,\sigma,d} = O^{\sigma,\sigma'}_{c\to 0}(c^{1-\sigma'})$). In particular, we check that $\partial_d\lambda(c,d) = -\pi c^2 + O^{\sigma}_{c\to 0}(c^{2-\sigma})$ for any $\sigma > 0$, and thus we cannot conclude that $\partial_d\lambda \neq 0$ for $c > 0$ small enough a priori.

There is a moral reason for this. When c moves, the vortices move, and thus the error term Φ_{c,d_c} is, at least at first order and near the vortices, translated. Therefore, $\partial_d\Phi_{c,d_c}$ will be of the same size as Φ_{c,d_c} (that is $O^{\sigma'}_{c\to 0}(c^{1-\sigma'})$), but, if we remove the translation, by looking at $\partial_d\Phi_{c,d_c} - \partial_{x_1}\Phi_{c,d_c}$ near the vortex $+1$ for instance, we could expect a better estimate. For that term, we would only see the change of influence of the second vortex, which is already far away. We can check that the difficulty when computing $\partial_d\lambda$ is indeed local, near the vortices, is coming from $\partial_d\Phi_{c,d_c}$, and that the translation part cancels out exactly.

This is a simple idea, but it yields a fair amount of technical difficulties. We have to recast the way to choose d_c, change the norms and the spaces. In Proposition 2.3.5, we compute this gain, and we show that $\|\partial_d\Phi_{c,d} - \partial_{x_1}\Phi_{c,d}\| = O_{c\to 0}(c^{1+\varepsilon})$ in L^∞ near the vortex $+1$, for some $\varepsilon > 0$. We in fact have a better but more technical estimate. Using this estimate in the equation on $\partial_d\lambda$, we find that $\partial_d\lambda(c,d) = -\pi c^2 + O_{c\to 0}(c^{2+\varepsilon})$ for some $\varepsilon > 0$. This ends the sketch of the proof of Theorem 1.3.1.

1.4 Coercivity results on the branch and applications

The results presented in this section have been submitted for publication as a paper in collaboration with David Chiron.

1.4.1 Some particular values for the branch

With the solution Q_c constructed in Theorem 1.3.1, we can construct travelling waves of any small speed, i.e. solutions of

$$(\mathrm{TW}_{\vec{c}})(v) := i\,\vec{c}.\nabla v - \Delta v - (1-|v|^2)v$$

for any $\vec{c}\in\mathbb{R}^2$ of small modulus. For $\vec{c} = |\vec{c}|e^{i(\theta_{\vec{c}}-\pi/2)}\in\mathbb{R}^2$, $|\vec{c}|\leqslant c_0$, we have that

$$Q_{\vec{c}} := Q_{|\vec{c}|}\circ R_{-\theta_{\vec{c}}} \tag{1.4.1}$$

is a solution of $(\mathrm{TW}_{\vec{c}})$, with R_α being the rotation of angle α and $Q_{|\vec{c}|}$ defined in Theorem 1.3.1. Furthermore, the equation is invariant by translation and by changing the phase. Thus, we have a family of solutions of (GP) depending on five real parameters, $\vec{c}\in\mathbb{R}^2$, $|\vec{c}|\leqslant c_0$, $X\in\mathbb{R}^2$ and $\gamma\in\mathbb{R}$:

$$Q_{\vec{c}}(.-X-\vec{c}\,t)e^{i\gamma}.$$

We remark that, for a vortex of degree ±1, the family of solutions has three parameters (the two translations and the phase): $V_{\pm1}(.-X)e^{i\gamma}$ is solution of (GP) for $X\in\mathbb{R}^2$, $\gamma\in\mathbb{R}$. In particular, between a travelling wave and the two vortices that compose it, we lose a parameter (since the phase is global). This is one of the difficulty that will appear when we study the stability of this branch.

First, we give additional results on this branch of travelling waves: we will study the position of its zeros, its energy and momentum, as well as some particular values appearing in the linearization. The (additive) linearized operator around Q_c is

$$L_{Q_c}(\varphi) := -\Delta\varphi - ic\partial_{x_2}\varphi - (1-|Q_c|^2)\varphi + 2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c.$$

We want to define and use four particular directions for the linearized operator around Q_c, which are

$$\partial_{x_1}Q_c, \partial_{x_2}Q_c,$$

related to the translations (i.e. related to the parameter $X\in\mathbb{R}^2$ in the family of travelling waves), and

$$\partial_c Q_c, \partial_{c^\perp}Q_c,$$

related to the variation of speed (i.e. related to the parameter $\vec{c}\in\mathbb{R}^2$), if we change respectively its modulus or its direction. The functions $\partial_{x_1}Q_c$, $\partial_{x_2}Q_c$ and $\partial_c Q_c$ are defined in Theorem 1.3.1, and we will show that

$$\partial_{c^\perp}Q_c(x) := \partial_\alpha(Q_c\circ R_{-\alpha})_{|\alpha=0} = -x^\perp.\nabla Q_c(x),$$

with $x^\perp = (-x_2, x_1)$ (see Lemma 3.1.6). We infer the following properties.

Proposition 1.4.1. *There exists $c_0>0$ such that, for $0<c\leqslant c_0$, the momentum $\vec{P}(Q_c) = (P_1(Q_c), P_2(Q_c))$ of Q_c from Theorem 1.3.1, defined by*

$$P_1(Q_c) := \frac{1}{2}\langle i\partial_{x_1}Q_c, Q_c - 1\rangle,$$

$$P_2(Q_c) := \frac{1}{2}\langle i\partial_{x_2}Q_c, Q_c - 1\rangle,$$

verifies $c\mapsto\vec{P}(Q_c)\in C^1(]0,c_0[,\mathbb{R}^2)$,

$$P_1(Q_c) = \partial_c P_1(Q_c) = 0,$$

$$P_2(Q_c) = \frac{2\pi + o_{c\to0}(1)}{c}$$

and

$$\partial_c P_2(Q_c) = \frac{-2\pi + o_{c\to 0}(1)}{c^2}.$$

Furthermore, the energy satisfies $c \mapsto E(Q_c) \in C^1(]0, c_0[, \mathbb{R})$*, and*

$$E(Q_c) = (2\pi + o_{c\to 0}(1))\ln\left(\frac{1}{c}\right).$$

Additionally, $\mathfrak{Re}(L_{Q_c}(A)\bar{A}) \in L^1(\mathbb{R}^2, \mathbb{R})$ *for* $A \in \left\{\partial_{x_1} Q_c, \partial_{x_2} Q_c, \partial_c Q_c, \partial_{c^\perp} Q_c\right\}$*, and*

$$\langle L_{Q_c}(\partial_{x_1} Q_c), \partial_{x_1} Q_c\rangle = \langle L_{Q_c}(\partial_{x_2} Q_c), \partial_{x_2} Q_c\rangle = 0,$$

$$\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle = \partial_c P_2(Q_c) = \frac{-2\pi + o_{c\to 0}(1)}{c^2},$$

$$\langle L_{Q_c}(\partial_{c^\perp} Q_c), \partial_{c^\perp} Q_c\rangle = c P_2(Q_c) = 2\pi + o_{c\to 0}(1)$$

and

$$\partial_c E(Q_c) = c\partial_c P_2(Q_c) = \frac{-2\pi + o_{c\to 0}(1)}{c}.$$

Finally, the function Q_c *has exactly two zeros. Their positions are* $\pm\tilde{d}_c\vec{e}_1$*, with*

$$\left|d_c - \tilde{d}_c\right| = o_{c\to 0}(1),$$

where d_c *is defined in Theorem 1.3.1.*

The momentum has a generalized definition for finite energy functions (see [32] in $3d$ and [8]). For travelling waves going to 1 at infinity, it is equal to the quantity defined in Proposition 1.4.1.

The equality $\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle = \partial_c P_2(Q_c)$ is a general property for Hamiltonian system, see [23]. The equality $\partial_c E(Q_c) = c\partial_c P_2(Q_c)$ has been conjectured and formally shown in [26], provided we have a smooth branch $c \mapsto Q_c$, which is precisely shown in Theorem 1.3.1. We remark that the energy $E(Q_c)$ is of same order as the energy of the travelling waves constructed in [4], which also exhibit two vortices at distance of order $\frac{1}{c}$. We believe that both construction give the same branch, and that this branch minimises globally the energy at fixed momentum. However, we were not able to show even a local minimisation result of the energy for Q_c defined in Theorem 1.3.1.

In the limit $c \to 0$, the four directions $(\partial_{x_1} Q_c, \partial_{x_2} Q_c, c^2\partial_c Q_c, c\partial_{c^\perp} Q_c)$ are going to zeros of the quadratic form (while being of size of order one), and we see here the splitting of this kernel for small values of c. In particular, two directions give zero ($\partial_{x_1} Q_c$ and $\partial_{x_2} Q_c$), one becomes positive ($\partial_{c^\perp} Q_c$) and one negative ($\partial_c Q_c$).

1.4.2 Coercivity results

One of the main ideas is to reduce the problem of the coercivity of a travelling wave to the coercivity of vortices. We will first state such a result for vortices (Proposition 1.4.2) before the results on the travelling waves (see in particular Theorem 1.4.4).

1.4.2.1 Coercivity in the case of one vortex

A coercivity result for one vortex of degree ± 1 is already known, see [10], and in particular equation (2.42) there. We consider both vortices of degrees $+1$ and -1 here at the same time, since $V_1 = \overline{V_{-1}}$. Here, we present a slight variation of the results in [10] that will be useful for the coercivity of the travelling waves. We recall from [10] the quadratic form around V_1:

$$B_{V_1}(\varphi) = \int_{\mathbb{R}^2} |\nabla\varphi|^2 - (1 - |V_1|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\bar{V}_1\varphi),$$

for functions in the energy space

$$H_{V_1} = \left\{\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C}), \|\varphi\|^2_{H_{V_1}} := \int_{\mathbb{R}^2} |\nabla\varphi|^2 + (1 - |V_1|^2)|\varphi|^2 + \mathfrak{Re}^2(\bar{V}_1\varphi) < +\infty\right\}.$$

As the family of vortices has three parameters, we expect a coercivity result under three orthogonality conditions. The three associated directions are $\partial_{x_1}V_1, \partial_{x_2}V_1$ (for the translations) and iV_1 (for the phase).

Proposition 1.4.2. *There exist $K>0$, $R>5$, such that, if the following three orthogonality conditions are satisfied for $\varphi = V_1\psi \in C_c^\infty(\mathbb{R}^2\backslash\{0\},\mathbb{C})$,*

$$\int_{B(0,R)} \mathfrak{Re}(\partial_{x_1}V_1\overline{V_1\psi}) = \int_{B(0,R)} \mathfrak{Re}(\partial_{x_2}V_1\overline{V_1\psi}) = \int_{B(0,R)\backslash B(0,R/2)} \mathfrak{Im}(\psi) = 0,$$

then,

$$B_{V_1}(\varphi) \geqslant K\left(\int_{B(0,10)} |\nabla\varphi|^2 + |\varphi|^2 + \int_{\mathbb{R}^2\backslash B(0,5)} |\nabla\psi|^2|V_1|^2 + \mathfrak{Re}^2(\psi)|V_1|^4 + \frac{|\psi|^2}{r^2\ln^2(r)}\right).$$

The same result holds if we replace V_1 by V_{-1}. We remark that the coercivity norm is not $\|.\|_{H_{V_1}}$, but is weaker (the decay in position is stronger), and this is due to the fact that $iV_1 \notin H_{V_1}$. That is why this result is stated for compactly supported function. The fact that the support of φ avoids 0 is technical and can be removed by density (see Lemma 3.2.4).

Proposition 1.4.2 is shown in subsection 3.3.2. The proofs there are mostly slight variations or improvements of proofs given in [10].

1.4.2.2 Coercivity and kernel in the energy space

The main part of this section consists of coercivity results for the family of travelling waves constructed in Theorem 1.3.1. We will show it on Q_c defined in Theorem 1.3.1, and with (1.4.1), it extends to all speed values $\vec{c}$ of small norm. We recall the linearized operator around Q_c:

$$L_{Q_c}(\varphi) = -\Delta\varphi - ic\partial_{x_2}\varphi - (1-|Q_c|^2)\varphi + 2\,\mathfrak{Re}(\overline{Q_c}\varphi)Q_c.$$

The natural associated energy space is

$$H_{Q_c} := \left\{\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^2), \|\varphi\|_{H_{Q_c}} < +\infty\right\},$$

where

$$\|\varphi\|^2_{H_{Q_c}} := \int_{\mathbb{R}^2} |\nabla\varphi|^2 + |1-|Q_c|^2||\varphi|^2 + \mathfrak{Re}^2(\overline{Q_c}\varphi).$$

First, there are difficulties in the definition of the quadratic form for $\varphi \in H_{Q_c}$, because of the transport term. A natural definition for the associated quadratic form for $\varphi \in H_{Q_c}$ could be

$$\int_{\mathbb{R}^2} |\nabla\varphi|^2 - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi) - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}), \tag{1.4.2}$$

unfortunately the last term is not well defined for $\varphi \in H_{Q_c}$, because we lack a control on $\mathfrak{Im}(\overline{Q_c}\varphi)$ in $L^2(\mathbb{R}^2)$ in $\|.\|_{H_{Q_c}}$, see [32]. We can resolve this issue by decomposing this term and doing an integration by parts, but the proof of the integration by parts can not be done if we only suppose $\varphi \in H_{Q_c}$ (see section 3.2 for more details). We therefore define the quadratic form with the integration by parts already done. Take a smooth cutoff function η such that $\eta(x)=0$ on $B(\pm\tilde{d}_c\vec{e}_1, 1)$, $\eta(x)=1$ on $\mathbb{R}^2\backslash B(\pm\tilde{d}_c\vec{e}_1, 2)$, where $\pm\tilde{d}_c\vec{e}_1$ are the zeros of Q_c. We define, for $\varphi = Q_c\psi \in H_{Q_c}$,

$$\begin{aligned} B_{Q_c}(\varphi) \;:=\; & \int_{\mathbb{R}^2} |\nabla\varphi|^2 - (1-|Q_c|^2)|\varphi|^2 + 2\,\mathfrak{Re}^2(\overline{Q_c}\varphi) \\ - \;& c\int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}) - c\int_{\mathbb{R}^2} \eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2 \\ + \;& 2c\int_{\mathbb{R}^2} \eta\mathfrak{Re}\,\psi\,\mathfrak{Im}(\partial_{x_2}\psi)|Q_c|^2 + c\int_{\mathbb{R}^2} \partial_{x_2}\eta\mathfrak{Re}\,\psi\,\mathfrak{Im}\,\psi|Q_c|^2 \\ + \;& c\int_{\mathbb{R}^2} \eta\mathfrak{Re}\,\psi\,\mathfrak{Im}\,\psi\partial_{x_2}(|Q_c|^2). \end{aligned} \tag{1.4.3}$$

See subsection 3.2.3 for the details of the computation. For functions $\varphi \in H^1(\mathbb{R}^2)$ for instance, both quadratic forms (1.4.2) and (1.4.3) are well defined and are equal (see Lemma 3.4.7). We will show that B_{Q_c} is well defined for $\varphi \in H_{Q_c}$ (see Lemma 3.2.3), and that for $A \in \{\partial_{x_1}Q_c, \partial_{x_2}Q_c, \partial_c Q_c, \partial_{c^\perp}Q_c\}$, $B_{Q_c}(A) = \langle L_{Q_c}(A), A\rangle$.

From Proposition 1.4.1, we know that Q_c has only two zeros. We will write the quadratic form B_{Q_c} around the zeros of Q_c (for a function $\varphi = Q_c\psi \in H_{Q_c}$) as the quadratic form for one vortex (computed in Proposition 1.4.2), up to some small error. As we want to avoid to add an orthogonality on the phase, we change the coercivity norm to a weaker semi-norm, that avoids iQ_c, the direction connected to the shift of phase.

We will therefore infer a coercivity result under four orthogonality conditions near the zeros of Q_c (two for each zero). Then, we shall show that far from the zeros of Q_c, the coercivity holds, without any additional orthogonality conditions.

Proposition 1.4.3. *There exists $c_0, R > 0$ such that, for $0 < c \leqslant c_0$, if one defines $\tilde{V}_{\pm 1}$ to be the vortices centered around $\pm\tilde{d}_c\vec{e}_1$ ($\tilde{d}_c$ is defined in Proposition 1.4.1), there exist $K > 0$ such that for $\varphi = Q_c\psi \in H_{Q_c}$, $0 < c < c_0$, if the four orthogonality conditions*

$$\int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big) = \int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big) = 0,$$

$$\int_{B(-\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi}\Big) = \int_{B(-\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi}\Big) = 0$$

are satisfied, then, for

$$\|\varphi\|_{\mathcal{C}}^2 := \int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4,$$

the following coercivity result holds:

$$B_{Q_c}(\varphi) \geqslant K\|\varphi\|_{\mathcal{C}}^2.$$

We will check that $\|\varphi\|_{\mathcal{C}}$ is well defined for $\varphi \in H_{Q_c}$ (see section 3.2). Proposition 1.4.3 is proven in subsection 3.3.4.

We point out that $\varphi = Q_c\psi \mapsto \|\varphi\|_{\mathcal{C}}$ is not a norm but a seminorm since $\int_{\mathbb{R}^2}|\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4 = 0$ implies only that $\varphi = \lambda iQ_c$ for some $\lambda \in \mathbb{R}$, and iQ_c is the direction connected to the shift of phase.

Now, we want to change the orthogonality conditions in Proposition 1.4.3 to quantities linked to the parameters $\vec{c}$ and X of the travelling waves, that is $\partial_{x_1}Q_c, \partial_{x_2}Q_c, \partial_c Q_c$ and $\partial_{c^\perp}Q_c$. We can show that for $\varphi = Q_c\psi \in H_{Q_c}$, for instance

$$\left|\int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big)\right| \leqslant K\|\varphi\|_{\mathcal{C}},$$

but such an estimate might not hold for $\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)} \partial_{x_1}Q_c\overline{Q_c\psi}$ (because of the lack of control on $\mathfrak{Im}(\psi)$ in $L^2(\mathbb{R}^2)$ in the coercivity norm $\|.\|_{\mathcal{C}}$). It is therefore difficult to have a local orthogonality condition directly on $\partial_{x_1}Q_c$ for instance.

To solve this issue, we shall use the harmonic decomposition around $\pm\tilde{d}_c\vec{e}_1$. For the constructed travelling wave Q_c, two distances play a particular role, they are d_c (defined in Theorem 1.3.1) and $\tilde{d}_c$ (defined in Proposition 1.4.1 and is connected to the position of the zeros of Q_c). In particular, we define the following polar coordinates for $x \in \mathbb{R}^2$:

$$re^{i\theta} := x \in \mathbb{R}^2,$$

$$r_{\pm 1}e^{i\theta_{\pm 1}} := x - (\pm d_c)\vec{e}_1 \in \mathbb{R}^2,$$

$$\tilde{r}_{\pm 1}e^{i\tilde{\theta}_{\pm 1}} := x - (\pm\tilde{d}_c)\vec{e}_1 \in \mathbb{R}^2.$$

We will also use $\tilde{r} := \min(r_1, r_{-1})$ and $\check{r} := \min(\tilde{r}_1, \tilde{r}_{-1})$. For a function ψ such that $Q_c\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ and $j \in \mathbb{Z}$, we define its $j-$harmonic around $\pm\tilde{d}_c\vec{e}_1$ by the radial function around $\pm\tilde{d}_c\vec{e}_1$:

$$\psi^{j,\pm 1}(\tilde{r}_{\pm 1}) := \frac{1}{2\pi}\int_0^{2\pi} \psi\big(\tilde{r}_{\pm 1}e^{i\tilde{\theta}_{\pm 1}}\big)\, e^{-ij\tilde{\theta}_{\pm 1}} d\tilde{\theta}_{\pm 1}.$$

Summing over the Fourier modes leads to

$$\psi(x)=\sum_{j\in\mathbb{Z}}\psi^{j,\pm1}(\tilde{r}_{\pm1})e^{ij\tilde{\theta}_{\pm1}}.$$

and we define, to simplify the notations later on, the function $\psi^{\neq0}$, by

$$\psi^{\neq0}(x):=\psi(x)-\psi^{0,1}(\tilde{r}_1)$$

in the right half-plane, and

$$\psi^{\neq0}(x):=\psi(x)-\psi^{0,-1}(\tilde{r}_{-1})$$

in the left half-plane. This notation will only be used far from the line $\{x_1=0\}$. We now state the main coercivity result.

Theorem 1.4.4. *There exist $c_0,K,\beta_0>0$ such that, for $R>0$ defined in Proposition 1.4.3, for any $0<\beta<\beta_0$, there exists $c_0(\beta),K(\beta)>0$ such that, for $c<c_0(\beta)$, if $\varphi=Q_c\psi\in H_{Q_c}$ satisfies the following three orthogonality conditions:*

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_{x_1}Q_c\overline{Q_c\psi^{\neq0}}=\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_{x_2}Q_c\overline{Q_c\psi^{\neq0}}=0$$

and

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_{c}Q_c\overline{Q_c\psi^{\neq0}}=0,$$

then,

$$B_{Q_c}(\varphi)\geqslant K(\beta)c^{2+\beta}\|\varphi\|_{\mathcal{C}}^2,$$

with

$$\|\varphi\|_{\mathcal{C}}^2=\int_{\mathbb{R}^2}|\nabla\psi|^2|Q_c|^4+\mathfrak{Re}^2(\psi)|Q_c|^4.$$

If $\varphi=Q_c\psi$ also satisfies the fourth orthogonality condition (with $0<c<c_0$)

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_{c^\perp}Q_c\overline{Q_c\psi^{\neq0}}=0,$$

then

$$B_{Q_c}(\varphi)\geqslant K\|\varphi\|_{\mathcal{C}}^2.$$

Theorem 1.4.4 shows that under four orthogonality conditions, we have a coercivity result in a weaker norm $\|.\|_{\mathcal{C}}$, instead of $\|.\|_{H_{Q_c}}$ with a constant independent of c, and with only three orthogonality conditions, we have the coercivity but the constant is a $O^{\beta}_{c\to0}(c^{2+\beta})$. This is because, of the four particular directions of the linearized operator, $\partial_{x_1}Q_c,\partial_{x_2}Q_c$ are in its kernel, ∂_cQ_c is a small negative direction, and $\partial_{c^\perp}Q_c$ is a small positive direction (see Proposition 1.4.1). About the orthogonality conditions, we remark that, for $\varphi=Q_c\psi\in H_{Q_c}$,

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_{x_1}Q_c\overline{Q_c\psi^{\neq0}}$$

is close to

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_{x_1}Q_c\overline{Q_c\psi}$$

(we have $\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)}\partial_{x_1}Q_c\overline{Q_c\psi^{0,1}}=o_{c\to0}(1)\|\varphi\|_{H_{Q_c}}$ for instance), but the first quantity can be controlled by $\|\varphi\|_{\mathcal{C}}$, and the second can not be.

Theorem 1.4.4 is a consequence of Proposition 1.4.3, and is shown in section 3.4. From this result, we can also deduce the kernel of the linearized operator in H_{Q_c}.

Corollary 1.4.5. *There exists $c_0>0$ such that, for $0<c<c_0$, Q_c defined in Theorem 1.3.1, for $\varphi\in H_{Q_c}$, the following properties are equivalent:*

i. $L_{Q_c}(\varphi)=0$ in $H^{-1}(\mathbb{R}^2)$, that is, $\forall\varphi^*\in H^1(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2}\mathfrak{Re}(\nabla\varphi.\nabla\overline{\varphi^*})-(1-|Q_c|^2)\mathfrak{Re}(\varphi\overline{\varphi^*})+2\,\mathfrak{Re}(\overline{Q_c}\varphi)\mathfrak{Re}(\overline{Q_c}\varphi^*)-\mathfrak{Re}(ic\partial_{x_2}\varphi\overline{\varphi^*})=0.$$

ii. $\varphi \in \mathrm{Span}_{\mathbb{R}}(\partial_{x_1} Q_c, \partial_{x_2} Q_c)$.

This corollary is proven in subsection 3.4.5. This nondegeneracy result is, to our knowledge, the first one on this type of model. It is a building block in the analysis of the dynamical stability of the travelling wave and the construction of multi-travelling wave. Here, the travelling wave is not radial, nor has a simple profile, which means that we can not use classical technics for radial ground states for instance (see [43]).

1.4.2.3 Spectral stability in $H^1(\mathbb{R}^2)$

In this subsection, we give some result on the spectrum of $L_{Q_c}: H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$. In particular, we are interested in negative eigenvalues of the linearized operator. We can show that $H^1(\mathbb{R}^2) \subset H_{Q_c}$ and prove the following corollary of Theorem 1.4.4.

Corollary 1.4.6. *There exists $c_0 > 0$ such that, for $0 < c \leqslant c_0$, Q_c defined in Theorem 1.3.1, if $\varphi \in H^1(\mathbb{R}^2)$ satisfies*

$$\langle \varphi, i\partial_{x_2} Q_c \rangle = 0,$$

then

$$B_{Q_c}(\varphi) \geqslant 0.$$

We can show that $L_{Q_c}(\partial_c Q_c) = i\partial_{x_2} Q_c \in L^2(\mathbb{R}^2)$, and thus $\varphi i\overline{\partial_{x_2} Q_c} \in L^1(\mathbb{R}^2)$ for $\varphi \in H^1(\mathbb{R}^2)$. This result shows that we expect only one negative direction for the linearized operator, and it should also hold in H_{Q_c}. For $\varphi \in H^1(\mathbb{R}^2)$, we have that $B_{Q_c}(\varphi)$ is equal to the expression (1.4.2).

Now, we define $\mathfrak{G}$ to be the collection of subspaces $S \subset H^1(\mathbb{R}^2)$ such that $B_{Q_c}(\varphi) < 0$ for all $\varphi \neq 0, \varphi \in S$, and we define

$$n^-(L_{Q_c}) := \max \left\{ \dim S, S \in \mathfrak{G} \right\}.$$

Proposition 1.4.7. *There exists $c_0 > 0$ such that, for $0 < c < c_0$, for Q_c defined in Theorem 1.3.1,*

$$n^-(L_{Q_c}) = 1.$$

Furthermore, L_{Q_c}: $H^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ has exactly one negative eigenvalue with eigenvector in $L^2(\mathbb{R}^2)$.

With this result, Theorem 1.3.1 and Proposition 1.4.1, we have met all the conditions to show the spectral stability of the travelling wave:

Theorem 1.4.8. (Theorem 11.8 (*i*) of [30]) *For $0 < c_1 < c_2$ and $c \mapsto U_c$ a C^1 branch of solutions of $(\mathrm{TW}_c)(U_c) = 0$ on $]c_1, c_2[$ with finite energy, for $c_* \in]c_1, c_2[$, under the following conditions:*

i. for all $c \in]c_1, c_2[$, $\mathfrak{Re}(U_c - 1) \in H^1(\mathbb{R}^2)$, $\mathfrak{Im}(\nabla U_c) \in L^2(\mathbb{R}^2)$, $|U_c| \to 1$ at infinity and $\|U_c\|_{C^1(\mathbb{R}^2)} < +\infty$

ii. $n^-(L_{Q_{c_*}}) \leqslant 1$

iii. $\partial_c P_2(U_c)_{|c=c_*} < 0$,

then U_{c_*} is spectrally stable. That is, it is not an exponentially unstable solution of the linearized equation in $\dot{H}^1(\mathbb{R}^2, \mathbb{C})$.

Corollary 1.4.9. *There exists $c_0 > 0$ such that, for any $0 < c < c_0$, the function Q_c defined in Theorem 1.3.1 is spectrally stable in the sense of Theorem 1.4.8.*

The notion of spectral stability of [30] is the following: for any $u_0 \in H^1(\mathbb{R}^2, \mathbb{C})$, the solution to the problem

$$\begin{cases} i\partial_t u = L_{Q_c}(u) \\ u(t=0) = u_0 \end{cases}$$

satisfies that, for all $\lambda > 0$,

$$\left(\int_{\mathbb{R}^2} |\nabla u|^2(t)dx\right) e^{-\lambda t} \to 0$$

when $t \to \infty$. The result of [30] is a little stronger: the norm that does not grow exponentially in time is better than the one on $\dot{H}^1(\mathbb{R}^2, \mathbb{C})$, but weaker than the one on $H^1(\mathbb{R}^2, \mathbb{C})$, and is not explicit.

1.4.3 Generalisation to a larger energy space and use of the phase

There are two main difficulties with the phase. The first one, as previously stated, is that we lose a parameter when passing from two vortices to a travelling wave. The second one is that for the direction linked to the phase shift, namely iQ_c, we have $iQ_c \notin H_{Q_c}$ (and even for one vortex, $iV_1 \notin H_{V_1}$). This will be an obstacle when we modulate on the phase for the local uniqueness result. Therefore, we define here a space larger than H_{Q_c}.

1.4.3.1 Definition and properties of the space $H^{\exp}_{Q_c}$

We define the space $H^{\exp}_{Q_c}$, the expanded energy space, by

$$H^{\exp}_{Q_c} := \left\{\varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^2), \|\varphi\|_{H^{\exp}_{Q_c}} < +\infty\right\},$$

with the norm, for $\varphi = Q_c\psi \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$,

$$\|\varphi\|^2_{H^{\exp}_{Q_c}} := \|\varphi\|^2_{H^1(\{\tilde{r}\leqslant 10\})} + \int_{\{\tilde{r}\geqslant 5\}} |\nabla\psi|^2 + \mathfrak{Re}^2(\psi) + \frac{|\psi|^2}{\tilde{r}^2\ln^2(\tilde{r})},$$

where $\tilde{r} = \min(\tilde{r_1}, \tilde{r}_{-1})$, the minimum of the distance to the zeros of Q_c. It is easy to check that that there exists $K > 0$ independent of c such that, for $\varphi = Q_c\psi \in H^{\exp}_{Q_c}$,

$$\frac{1}{K}\|\varphi\|^2_{H^1(\{5\leqslant\tilde{r}\leqslant 10\})} \leqslant \int_{\{5\leqslant\tilde{r}\leqslant 10\}} |\nabla\psi|^2 + \mathfrak{Re}^2(\psi) + \frac{|\psi|^2}{\tilde{r}^2\ln(\tilde{r})^2} \leqslant K\|\varphi\|^2_{H^1(\{5\leqslant\tilde{r}\leqslant 10\})}.$$

We will show that $H_{Q_c} \subset H^{\exp}_{Q_c}$ and $iQ_c \in H^{\exp}_{Q_c}$, whereas $iQ_c \notin H_{Q_c}$. This space will appear in the proof of the local uniqueness (Theorem 1.4.13 below). The main difficulty is that $B_{Q_c}(\varphi)$ is not well defined for $\varphi \in H^{\exp}_{Q_c}$ because for instance of the term $(1-|Q_c|^2)|\varphi|^2$ integrated at infinity. If we write the linearized operator multiplicatively, for $\varphi = Q_c\psi$ (using $(\mathrm{TW}_c)(Q_c) = 0$),

$$Q_c L'_{Q_c}(\psi) := L_{Q_c}(\varphi) = Q_c\left(-ic\partial_{x_2}\psi - \Delta\psi - 2\frac{\nabla Q_c}{Q_c}.\nabla\psi + 2\mathfrak{Re}(\psi)|Q_c|^2\right),$$

then there will be no problem at infinity for $\varphi \in H^{\exp}_{Q_c}$ for the associated quadratic form (in ψ), but there are instead some integrability issues near the zeros of Q_c. We take as before a smooth cutoff function η such that $\eta(x) = 0$ on $B(\pm\tilde{d_c}\vec{e}_1, 1)$, $\eta(x) = 1$ on $\mathbb{R}^2\backslash B(\pm\tilde{d_c}\vec{e}_1, 2)$, where $\pm\tilde{d_c}\vec{e}_1$ are the zeros of Q_c. The natural linear operator for which we want to consider the quadratic form is then

$$L^{\exp}_{Q_c}(\varphi) := (1-\eta)L_{Q_c}(\varphi) + \eta Q_c L'_{Q_c}(\psi),$$

and we therefore define, for $\varphi = Q_c\psi \in H^{\exp}_{Q_c}$,

$$\begin{aligned}
B^{\exp}_{Q_c}(\varphi) \;:=\; & \int_{\mathbb{R}^2} (1-\eta)(|\nabla\varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) - (1-|Q_c|^2)|\varphi|^2 + 2\,\mathfrak{Re}^2(\overline{Q_c}\varphi)) \\
- & \int_{\mathbb{R}^2} \nabla\eta.(\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2 - 2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)) \\
+ & \int_{\mathbb{R}^2} c\partial_{x_2}\eta\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)|Q_c|^2 \\
+ & \int_{\mathbb{R}^2} \eta(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4) \\
+ & \int_{\mathbb{R}^2} \eta(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)). \qquad (1.4.4)
\end{aligned}$$

This quantity is independent of the choice of η.

We will show that $B^{\exp}_{Q_c}(\varphi)$ is well defined for $\varphi \in H^{\exp}_{Q_c}$ and that, if $\varphi \in H_{Q_c} \subset H^{\exp}_{Q_c}$, then $B^{\exp}_{Q_c}(\varphi) = B_{Q_c}(\varphi)$. Writing the quadratic form $B^{\exp}_{Q_c}$ is a way to enlarge the space of possible perturbations to add in particular the remaining zero of the linearized operator. We infer the following result.

Proposition 1.4.10. *There exist $c_0, K, R, \beta_0 > 0$ such that, for any $0 < \beta < \beta_0$, there exists $c_0(\beta)$, $K(\beta) > 0$ such that, for $0 < c < c_0(\beta)$, if $\varphi = Q_c\psi \in H^{\exp}_{Q_c}$ satisfies the following three orthogonality conditions:*

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{x_1} Q_c \overline{Q_c\psi^{\neq 0}} = \mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{x_2} Q_c \overline{Q_c\psi^{\neq 0}} = 0$$

and

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_c Q_c \overline{Q_c\psi^{\neq 0}} = 0,$$

then,

$$B^{\exp}_{Q_c}(\varphi) \geqslant K(\beta) c^{2+\beta} \|\varphi\|^2_{\mathcal{C}},$$

with

$$\|\varphi\|^2_{\mathcal{C}} = \int_{\mathbb{R}^2} |\nabla\psi|^2 |Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4.$$

If $\varphi = Q_c\psi$ also satisfies the fourth orthogonality condition (with $0 < c < c_0$)

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{c^\perp} Q_c \overline{Q_c\psi^{\neq 0}} = 0,$$

then

$$B^{\exp}_{Q_c}(\varphi) \geqslant K \|\varphi\|^2_{\mathcal{C}}.$$

Furthermore, for $\varphi \in H^{\exp}_{Q_c}$, the following properties are equivalent:

i. $L_{Q_c}(\varphi) = 0$ *in* $H^{-1}(\mathbb{R}^2)$, *that is,* $\forall \varphi^* \in H^1(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} \mathfrak{Re}(\nabla\varphi.\nabla\overline{\varphi^*}) - (1-|Q_c|^2)\mathfrak{Re}(\varphi\overline{\varphi^*}) + 2\,\mathfrak{Re}(\overline{Q_c}\varphi)\mathfrak{Re}(\overline{Q_c}\varphi^*) - \mathfrak{Re}(ic\partial_{x_2}\varphi\overline{\varphi^*}) = 0.$$

ii. $\varphi \in \mathrm{Span}_{\mathbb{R}}(iQ_c, \partial_{x_1}Q_c, \partial_{x_2}Q_c)$

Proposition 1.4.10 is proven in subsection 3.5.1. The additional direction in the kernel comes from the invariance of phase ($L_{Q_c}(iQ_c) = 0$). The main difficulties, compared to Theorem 1.4.4, is to show that the considered quantities are well defined with only $\varphi \in H^{\exp}_{Q_c}$, and that we can conclude by density in this bigger space.

1.4.3.2 Coercivity results with an orthogonality on the phase

The main problem with adding a local orthogonality condition on iQ_c is to choose where to put it. Indeed, we want this condition near both zeros of Q_c, or else the coercivity constant will depend on the distance between the vortices, which itself depends on c.

The first option is to let the coercivity constant depend on c. In that case, we can also remove the orthogonality condition on $\partial_{c^\perp}Q_c$, the small positive direction. We infer the following result.

Proposition 1.4.11. *There exist universal constants K_1, $c_0 > 0$ such that, with $R > 0$ defined in Proposition 1.4.3, for $0 < c < c_0$, for the function Q_c defined in Theorem 1.3.1, there exists $K_2(c) > 0$ depending on c such that, if $\varphi = Q_c\psi \in H^{\exp}_{Q_c}$ satisfies the following four orthogonality conditions:*

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{x_1} Q_c \overline{Q_c\psi^{\neq 0}} = \mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{x_2} Q_c \overline{Q_c\psi^{\neq 0}} = 0,$$

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_c Q_c \overline{Q_c\psi^{\neq 0}} = \mathfrak{Re}\int_{B(0,R)} i\psi = 0,$$

then

$$K_1\|\varphi\|^2_{H^{\exp}_{Q_c}} \geqslant B^{\exp}_{Q_c}(\varphi) \geqslant K_2(c)\|\varphi\|^2_{H^{\exp}_{Q_c}}.$$

Here, the orthogonality condition on iQ_c is around 0, between the two vortices, but it can be chosen near one of the vortices for instance, and the result still holds.

The second possibility is to work with symmetric perturbations, since the orthogonality condition can then be at both the zeros of Q_c. We then study the space

$$H^{\exp,s}_{Q_c} := \left\{\varphi \in H^{\exp}_{Q_c}, \forall x=(x_1,x_2)\in\mathbb{R}^2, \varphi(x_1,x_2)=\varphi(-x_1,x_2)\right\}.$$

We show that, under three orthogonality conditions, the quadratic form is equivalent to the norm on $H^{\exp}_{Q_c}$.

Theorem 1.4.12. *There exist $R, K, c_0 > 0$ such that, for $0 < c \leqslant c_0$, Q_c defined in Theorem 1.3.1, if a function $\varphi \in H^{\exp,s}_{Q_c}$ satisfies the three orthogonality conditions:*

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_c Q_c\bar{\varphi} = \mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} \partial_{x_2} Q_c\bar{\varphi} = 0,$$

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)} iQ_c\bar{\varphi} = 0,$$

then

$$\frac{1}{K}\|\varphi\|^2_{H^{\exp}_{Q_c}} \geqslant B^{\exp}_{Q_c}(\varphi) \geqslant K\|\varphi\|^2_{H^{\exp}_{Q_c}}.$$

We remark that here, the orthogonality condition to $\partial_{x_1}Q_c$ and $\partial_{c^\perp}Q_c$ are freely given by the symmetry. We also do not need to remove the 0-harmonic near the zeros of Q_c.

Propositions 1.4.11 and Theorem 1.4.12 hold if we replace $B^{\exp}_{Q_c}$ by B_{Q_c} for $\varphi = Q_c\psi \in H_{Q_c}$ with the symmetry, but the coercivity norm will still be $\|.\|_{H^{\exp}_{Q_c}}$.

1.4.4 Local uniqueness result

With Propositions 1.4.10 and 1.4.11, we can modulate on the five parameters $(\vec{c},\ X,\ \gamma)$ of the travelling wave, and these coercivity results will be enough to show the following theorem.

Theorem 1.4.13. *There exist constants $K,\ c_0,\ \varepsilon_0,\ \mu_0 > 0$ such that, for $0 < c < c_0$, Q_c defined in Theorem 1.3.1, there exists $R_c > 0$ depending on c such that, for any $\lambda > R_c$, if a function $Z \in C^2(\mathbb{R}^2,\mathbb{C})$ satisfies, for some small constant $\varepsilon(c,\lambda) > 0$, depending on c and λ,*

- $(\mathrm{TW}_c)(Z) = 0$
- $E(Z) < +\infty$
- $\|Z - Q_c\|_{C^1(\mathbb{R}^2\setminus B(0,\lambda))} \leqslant \mu_0$
- $\|Z - Q_c\|_{H^{\exp}_{Q_c}} \leqslant \varepsilon(c,\lambda)$,

then, there exists $X \in \mathbb{R}^2$ such that $|X| \leqslant K\|Z - Q_c\|_{H^{\exp}_{Q_c}}$, and

$$Z = Q_c(.\,- X).$$

The conditions $E(Z) < +\infty$ and $\|Z - Q_c\|_{H^{\exp}_{Q_c}} \leqslant \varepsilon(c,\lambda)$ imply that the travelling wave $Z \to 1$ at infinity, and therefore $Z = Q_c e^{i\gamma}$ with $\gamma \in \mathbb{R}, \gamma \neq 0$ is excluded. The fact that $\varepsilon(c,\lambda)$ depends on c comes in part from the constant of coercivity in Proposition 1.4.11, which depends itself on c. The condition that $\|Z - Q_c\|_{C^1(\mathbb{R}^2\setminus B(0,\lambda))} \leqslant \mu_0$ outside of $B(0,\lambda)$ is mainly technical. We believe that this condition is automatically satisfied with the other ones (with λ depending only on c), but we were not able to show it.

To the best of our knowledge, this is the first result of local uniqueness for travelling waves in (GP). It does not suppose any symmetries on Z, and therefore shows that we can not bifurcate from this branch, even to nonsymmetric travelling waves.

We believe that, at least in the symmetric case, Theorem 1.4.13 should hold for $\|Z-Q_c\|_{H^{\exp}_{Q_c}}\leqslant\varepsilon$ with $\varepsilon>0$ independent of c and λ. We also remark that the condition $\|Z-Q_c\|_{H^{\exp}_{Q_c}}\leqslant\varepsilon(c,\lambda)$ is weaker than $\|Z-Q_c\|_{H_{Q_c}}\leqslant\varepsilon(c,\lambda)$, and thus we can state a result in H_{Q_c}.

1.4.5 Sketch of the proofs

1.4.5.1 Sketch of the proof of Proposition 1.4.1

In Theorem 1.3.1, we have shown that $Q_c=V_1(x-d_c\vec{e}_1)V_{-1}(x+d_c\vec{e}_1)+\Gamma_c$ and

$$\partial_c Q_c=\frac{1}{c^2}\big(\partial_d(V_1(x-d\vec{e}_1)V_{-1}(x+d\vec{e}_1))_{|d=d_c}+\tilde{\Gamma}_c\big),$$

where Γ_c and $\tilde{\Gamma}_c$ are small perturbations when $c\to 0$ (in some weighed L^∞ spaces). For all the first order computations of this proposition when $c\to 0$, it boils down to showing that the error terms, Γ_c and $\tilde{\Gamma}_c$, contribute less than the main terms. For the main terms, the computations are (almost) explicit, and, for some of them, were done in formal computations in physical works. For the different equalities on the linearized operator, this is simply coming from straight forward computations, with estimates from Theorem 1.2.11 to show that all the quantities are well defined, and to justify some integrations by parts.

For the position of the zeros, this is a consequence of the fact that, for a vortex $V_{\pm1}$ centered at 0, $V_{\pm1}(0)=0$, and the Jacobian of $\nabla V_{\pm1}(0)$ is not 0. Thus, the zero of vortices are simple, and adding a small perturbation might change slightly its position, but not its order, nor its existence.

1.4.5.2 Sketch of the proofs of the coercivity results

We give here a sketch of the proofs of Propositions 1.4.2, 1.4.3, Theorem 1.4.4, Propositions 1.4.10, 1.4.11 and Theorem 1.4.12. We will not discuss here the proofs of the well posedness of the different terms. We take here functions smooth and compactly supported away from the zeros of Q_c, that will allow all the computations needed. We refer to Lemmas 3.2.4 for a density argument and Lemmas 3.2.1 to 3.2.3 for the well posedness of the quantities.

We consider V_1 a vortex centered at 0. We recall that

$$B_{V_1}(\varphi)=\int_{\mathbb{R}^2}|\nabla\varphi|^2-(1-|V_1|^2)|\varphi|^2+2\mathfrak{Re}^2(\bar{V_1}\varphi).$$

The result of Proposition 1.4.2 is a simple variation of results in [10]. The linearized operator around V_1 has two elements in its kernel in the energy space, $\partial_{x_1}V_1$ and $\partial_{x_2}V_1$. The third orthogonality is on the phase, iV_1, which is not in the energy space, but can be approximated by functions in it, and thus still require an orthogonality to avoid it. Once these three directions are removed, the coercivity follows.

Now, we infer that, with $\varphi=V_1\psi$ (compactly supported away from 0),

$$B_{V_1}(\varphi)=\tilde{B}_{V_1}(\psi):=\int_{\mathbb{R}^2}|\nabla\psi|^2|V_1|^2+2\mathfrak{Re}^2(\psi)|V_1|^4+4\mathfrak{Im}(\nabla V_1\bar{V_1})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi).$$

Remark that, with Lemma 1.2.1, $|V_1|\simeq 1$ and $|\nabla V_1|\simeq 0$ far from zero, thus, by Cauchy-Scwharz,

$$\tilde{B}_{V_1}(\psi)\geqslant K\int_{\mathbb{R}^2}|\nabla\psi|^2|V_1|^2+\mathfrak{Re}^2(\psi)|V_1|^4$$

for functions ψ supported outside of $B(0,\lambda)$ for some large (but independent of φ) $\lambda>0$. Thus, the coercivity hold without orthogonality conditions at infinity. We can therefore localized the coercivity result (see equation (3.3.6)). Writing

$$B^D_{V_1}(\varphi)=\int_{B(0,D)}|\nabla\varphi|^2-(1-|V_1|^2)|\varphi|^2+2\mathfrak{Re}^2(\bar{V_1}\varphi),$$

we infer that, under the three orthogonality conditions of Proposition 1.4.2, for $D>0$ a large but universal constant,

$$B^D_{V_1}(\varphi) \geqslant K(D)\|\varphi\|^2_{H^1(B(0,D))}.$$

Here, the coercivity norm has been replaced by $\|.\|_{H^1(B(0,D))}$ since they are equivalent (using $\int_{B(0,R)\setminus B(0,R/2)}\mathfrak{Im}(\psi)=0$). We use a cutoff function to write the quadratic form in the form $B_{V_1}(\varphi)$ near zero, and in the form $\tilde{B}_{V_1}(\psi)$ far from it.

We then compare it to the quadratic form around Q_c, written in the form (1.4.4). Locally, that is near the zeros of the two vortices that composes it, it is close to $B^D_{\tilde{V}_{\pm 1}}(\varphi)$, where $\tilde{V}_{\pm 1}$ is centered around $\pm\tilde{d}_c\vec{e}_1$. Indeed,

$$B_{Q_c}(\varphi)=\int_{\mathbb{R}^2}|\nabla\varphi|^2-(1-|Q_c|^2)|\varphi|^2+2\mathfrak{Re}^2(\overline{Q_c}\varphi)-c\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi})$$

and

$$B^D_{\tilde{V}_1}(\varphi)=\int_{B(\tilde{d}_c\vec{e}_1,D)}|\nabla\varphi|^2-\big(1-|\tilde{V}_1|^2\big)|\varphi|^2+2\mathfrak{Re}^2(\bar{V}_1\varphi),$$

with $Q_c=\tilde{V}_1+o_{c\to 0}(1)$ and $c>0$ is small.

Thus, taking $c>0$ small enough, the coercivity on B_{Q_c} localized in $B\big(\tilde{d}_c\vec{e}_1,D\big)$ holds. Now, we infer that the same result holds with the coercivity norm (for $\varphi=Q_c\psi$)

$$\int_{B(\tilde{d}_c\vec{e}_1,D)}|\nabla\psi|^2|Q_c|^4+\mathfrak{Re}^2(\psi)|Q_c|^4\leqslant K(D)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))}.$$

This norm does not see the phase (for $\psi=i$, hence $\varphi=iQ_c$, the direction connected to the shift of phase, it is zero), and we check that the quadratic form and the two orthogonality conditions on the translations does not see the phase either (their values for φ and $\varphi-i\lambda Q_c$ are identical for all $\lambda\in\mathbb{R}$). Thus, we can modulate on λ to remove the orthogonality condition on the phase around both vortices. We have a local coercivity result for B_{Q_c} near the vortices.

Now, at infinity in position, as for $\tilde{B}_{V_1}(\psi)$, the coercivity for $B_{Q_c}(\varphi)$ (that can also be written in term of ψ) is obtained without orthogonality conditions, with the same coercivity norm. Regrouping these two estimates, we conclude the proof of Proposition 1.4.3.

For the proof of Theorem 1.4.4, the idea is simply to change the orthogonality conditions to ones that are close to a linear combinaison of the previous ones. The main difference is that they are more adapted to the four particular directions computed in Proposition 1.4.1. This uses classical arguments when changing the orthogonality conditions in a coercivity result. The main point is, although the coercivity norm $\|.\|_{\mathcal{C}}$ is not H^1_{loc} and was reduce to a semi norm to remove the orthogonality condition on the phase, it still control the four previous orthogonality conditions, and the four new ones. In fact, the error between them is small in this coercivity semi norm.

Now, one of the direction is a positive one, on $\partial_{c^\perp}Q_c$, by Proposition 1.4.1. We can therefore remove it, but the coercivity constant will then depend on c (as this is a small positive direction when c is small). It uses the fact that the four orthgonality are orthogonal between themselves. This completes the proof of Theorem 1.4.4.

Now, we focus on the proof of the coercivity results with an additional orthogonality on the phase. For the symmetric case, we simply keep the coercivity norm $\|\varphi\|_{H^1(B(0,D))}$ locally, with the three orthogonality condition around each vortices. Then, by symmetry, the two orthogonality conditions on the phase (one for each vortex) are in fact the same. To complete the coercivity norm to have $\|.\|_{H^{\text{exp}}_{Q_c}}$, this is simply a Hardy type inequality.

In the non symmetric case, in the proof of the coercivity for one vortex, we move the orthogonality on the phase far from the vortex, so that it is the same orthogonality condition for both vortices. Since the distance we use depends on c, so will the coercivity norm. This gives Proposition 1.4.11.

1.4.5.3 On the Proofs of the corollaries of the coercivity and of the local uniqueness

With Theorem 1.4.4, and the fact that only one of the direction is negative, we can use classical methods to show results on the linearized operator, for instance the computation of its kernel. The only difficulty is that the coercivity norm is only a semi norm, but if $\|\varphi\|_{\mathcal{C}}=0$, it implies that $\varphi=i\lambda Q_c$ for some $\lambda\in\mathbb{R}$, but if we know that $\varphi\in H_{Q_c}$, since $iQ_c\notin H_{Q_c}$, then $\lambda=0$. The semi norm $\|.\|_{\mathcal{C}}$ is in fact a norm on the energy space H_{Q_c}. Since $H^1(\mathbb{R}^2,\mathbb{C})\subset H_{Q_c}$, with this same argument, we check that the operator has only one negative eigenvalue, thanks to Theorem 1.4.4. Then, the spectral stability follows from [30], a general work on Hamiltonian problems.

Now, we focus on the local uniqueness result, Theorem 1.4.13. By Proposition 1.4.11, using the fact that $iQ_c\in H^{\exp}_{Q_c}$, we now have all the tools to do a classical proof of local uniqueness, using a coercivity result in $H^{\exp}_{Q_c}$ by modulating on the parameters. There are two difficulties. First, the coercivity norm in that case depends on c, and thus, the error term has to be small with respect to c. Secondly, we need to show that, with the notations of Theorem 1.4.13, $\|Z-Q_c\|_{C^1(\mathbb{R}^2)}$ is small (see Lemma 3.6.1). This require a technical condition, and is used to write the perturbation exponentially far from the zeros of Q_c, and to estimate some nonlinear terms.

With this technical result, we can modulate on the five parameters of the travelling wave (two parameters for the translation, two for the speed, and one for the phase) so that the error term between Z and the travelling wave has the orthogonality conditions of Propositions 1.4.10 and 1.4.11 (both coercivity are required). A few computations are required to show that, when taking the scalar product of the equation with the perturbation, all the terms are well defined, and the quadratic form appears (see Lemma 3.6.3). Furthermore, since we modulate on the speed, a source term appears, but by taking the scalar product of the equation with the two small directions of the linearized operator ($\partial_c Q_c$ and $\partial_{c^\perp} Q_c$), we can estimate them with respect to the perturbation. We then conclude as in classical proofs of local uniqueness using a coercivity result.

1.5 Inversion of the linearized operator around Q_c

Our goal in this section is to improve some results on the branch $c\mapsto Q_c$ constructed in Theorem 1.3.1, by giving two new properties. The first one is about the inversion of the linearized operator around Q_c, and the second one is about the smoothness of the branch with respect to the speed. In the rest of this section, Q_c refers to the solution of (TW_c) from Theorem 1.3.1.

1.5.1 Inversion result for the linearized operator around Q_c

We want to invert the (additive) linearized operator around Q_c in some weighed L^∞ spaces:

$$L_{Q_c}(\varphi)=-\Delta\varphi-(1-|Q_c|^2)\varphi+2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c-ic\partial_{x_2}\varphi.$$

We have computed its kernel in Corollary 1.4.5. It is $\mathrm{Span}_{\mathbb{R}}(\partial_{x_1}Q_c,\partial_{x_2}Q_c)$ in the energy space

$$H_{Q_c}=\left\{\varphi\in H^1_{\mathrm{loc}}(\mathbb{R}^2,\mathbb{C}),\int|\nabla\varphi|^2+|1-|Q_c|^2||\varphi|^2+\mathfrak{Re}^2(\overline{Q_c}\varphi)<+\infty\right\}.$$

It has also a resonance, $L_{Q_c}(iQ_c)=0$, due to the invariance by shifting the phase, with $iQ_c\notin H_{Q_c}$. That poses an issue when trying to invert L_{Q_c}. In the proof of Theorem 1.3.1 (see Proposition 2.1.20), the operator

$$L_V(\varphi):=-\Delta\varphi-(1-|V|^2)\varphi+2\mathfrak{Re}(\bar{V}\varphi)V-ic\partial_{x_2}\varphi$$

with $V=V_1(.-d_c\vec{e_1})V_{-1}(.+d_c\vec{e_1})$ (which is close to Q_c) was inverted in a space with two symmetries, where the problem of the resonance disappears. Here, we invert L_{Q_c} in a space with only one symmetry (even in x_1), that do not avoid the resonance. By adding the second symmetry, the space is also orthogonal to the kernel of L_{Q_c}, and in that case we can invert the operator without any orthogonality condition.

With d_c defined in Theorem 1.3.1 ($\pm d_c\vec{e}_1$ are the center of the vortices from which Q_c is constructed as a perturbation of), we define

$$\tilde{r} := \min\left(|x - d_c\vec{e}_1|, |x + d_c\vec{e}_1|\right),$$

as well as the two norms, for $\sigma \in \mathbb{R}$, $\varphi = Q_c\psi \in C^2(\mathbb{R}^2, \mathbb{C})$, $\psi = \psi_1 + i\psi_2$, and $Q_c h \in C^1(\mathbb{R}^2, \mathbb{C})$, $h = h_1 + i\,h_2$,

$$\begin{aligned}\|\psi\|_{\otimes,\sigma} &:= \|Q_c\,\psi\|_{C^2(\{\tilde{r}\leqslant 3\})}\\ &+ \|\tilde{r}^{1+\sigma}\,\psi_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\nabla\psi_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\nabla^2\psi_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})}\\ &+ \|\tilde{r}^{\sigma}\,\psi_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{1+\sigma}\nabla\psi_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\nabla^2\psi_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})}\end{aligned} \tag{1.5.1}$$

and

$$\begin{aligned}\|h\|_{\otimes\otimes,\sigma} &:= \|Q_c h\|_{C^1(\{\tilde{r}\leqslant 3\})}\\ &+ \|\tilde{r}^{1+\sigma}\,h_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\,\nabla h_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})}\\ &+ \|\tilde{r}^{2+\sigma}\,h_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\,\nabla h_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})}.\end{aligned} \tag{1.5.2}$$

We define the spaces, for $\sigma \in \mathbb{R}$,

$$\mathcal{E}_{\otimes,\sigma} := \left\{\varphi = Q_c\psi \in C^2(\mathbb{R}^2, \mathbb{C}), \|\psi\|_{\otimes,\sigma} < +\infty, \forall (x_1, x_2) \in \mathbb{R}^2, \varphi(x_1, x_2) = \varphi(-x_1, x_2)\right\},$$

$$\mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma} := \left\{\varphi = Q_c\psi \in \mathcal{E}_{\otimes,\sigma}, \forall (x_1, x_2) \in \mathbb{R}^2, \varphi(x_1, x_2) = \overline{\varphi(x_1, -x_2)}\right\}$$

and

$$\mathcal{E}_{\otimes\otimes,\sigma} := \left\{Q_c h \in C^1(\mathbb{R}^2, \mathbb{C}), \|h\|_{\otimes\otimes,\sigma} < +\infty, \forall (x_1, x_2) \in \mathbb{R}^2, (Q_c h)(x_1, x_2) = (Q_c h)(-x_1, x_2)\right\},$$

$$\mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma} := \left\{Q_c h \in \mathcal{E}_{\otimes\otimes,\sigma}, \forall (x_1, x_2) \in \mathbb{R}^2, Q_c h(x_1, x_2) = \overline{Q_c h(x_1, -x_2)}\right\}.$$

These spaces are close to the spaces $\mathcal{E}_{*,\sigma}, \mathcal{E}_{**,\sigma}$ introduced the proof of Theorem 1.3.1 (see subsection 2.1.3). The decays in position are related, but we change the symmetries, added estimates on the second derivatives, and locally we look at $Q_c\psi$ instead of $V_1(. - d_c\vec{e}_1)V_{-1}(. + d_c\vec{e}_1)\psi$ (and similarly for h). Remark in particular that $\mathcal{E}_{\otimes,\sigma} \subset H_{Q_c}$ for $\sigma > 0$. Also, for $\varepsilon > 0$, $\varphi \in \mathcal{E}_{\otimes,-\varepsilon}$ is not necessarily bounded, and not a priori in the energy space (nor in the extended energy space, $H^{\exp}_{Q_c}$).

Furthermore, $\frac{1}{Q_c} \in C^\infty(\left\{\tilde{r} \geqslant 2\right\}, \mathbb{C})$, and is uniformly bounded in this space. It explains why the norm is different on $\left\{\tilde{r} \leqslant 2\right\}$ and outside of this domain (Q_c has zeros there, see Proposition 1.4.1). Finally, with the first symmetry (being even in x_1), functions in $\mathcal{E}_{\otimes,\sigma}$ are orthogonal to $\partial_{x_1}Q_c$, one of the elements of the kernel of L_{Q_c}.

To infer the inversion result, we need to deal with a difficulty coming from a resonance in L_{Q_c} by removing some harmonics around $\pm\tilde{d}_c\vec{e}_1$, the two zeros of Q_c (see Proposition 1.4.1). This is reminiscent of the requirement on the orthogonality condition in Theorem 1.4.4. For $R > 0$ and $h \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$, $\check{r} < R$ and $\check{\theta}$ the polar coordinates around $\pm\tilde{d}_c\vec{e}_1$, we define

$$h^0(\check{r}) := \frac{1}{2\pi}\int_0^{2\pi} h(\check{r}e^{i\,\check{\theta}})d\,\check{\theta},$$

and

$$h^{\neq 0}(x) := h - h^0(\check{r}). \tag{1.5.3}$$

Theorem 1.5.1. *There exists $R, c_0 > 0$ such that, for any $\sigma > 0$, $\varepsilon > 0$, there exists $K(\sigma, \varepsilon) > 0$ such that, for $Q_c h \in \mathcal{E}_{\otimes\otimes,\sigma}$, $0 < c < c_0$ with*

$$\mathfrak{Re}\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)} \partial_{x_2}Q_c\overline{Q_c h^{\neq 0}} = 0,$$

there exists $\varphi = Q_c\psi \in \mathcal{E}_{\otimes,-\varepsilon}$ such that

$$L_{Q_c}(\varphi) = Q_c h,$$

with $\|\psi\|_{\otimes,-\varepsilon} \leqslant \frac{K(\varepsilon,\sigma)}{c^2}\|h\|_{\otimes\otimes,\sigma}$.

Furthermore, for any $0<\sigma<\sigma'<1$ *and* $Q_c h \in \mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma'}$, *without any orthogonality condition, there exists a unique function* $\varphi \in \mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma}$ *such that*

$$L_{Q_c}(\varphi) = Q_c h,$$

and it satisfies $\|\psi\|_{\otimes,\sigma} \leqslant \frac{K(\sigma,\sigma')}{c^2}\|h\|_{\otimes\otimes,\sigma'}$. *There,*

$$(c, Q_c h) \to \varphi \in C^0(]0, c_0[, \mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma'}) \to \mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma}).$$

A few remarks on this result. First, if $\sigma<\sigma'$, $\mathcal{E}_{\otimes,\sigma}\subset\mathcal{E}_{\otimes,\sigma'}$ and $\mathcal{E}_{\otimes\otimes,\sigma}\subset\mathcal{E}_{\otimes\otimes,\sigma'}$. In particular, the solutions φ constructed does not depend on the choice of $\varepsilon>0$ in the non 2-symmetry case, and $\sigma'<\sigma$ in the 2-symmetry case. In the estimates of the norm, the constant being in $\frac{1}{c^2}$ comes from the fact that the smallest nonzero eigenvalue of L_{Q_c} is of order $-c^2$ when $c\to 0$ (see Proposition 1.4.1). This constant can be made independant of c, provided that we add a local orthogonality condition (on $\partial_c Q_c$, see the proof of Proposition 4.2.10).

In the case with one symmetry, we can be more precise on the part of the function φ that grows at infinity. There, the function φ will be the sum of two functions, one in $\mathcal{E}_{\otimes,\sigma'}$ for some $0<\sigma'<\sigma$, that decays well at infinity, and $\lambda(h,c)\varphi_\Upsilon$, where $\lambda(h,c)\in\mathbb{R}$ and $\varphi_\Upsilon\in\mathcal{E}_{\otimes,-\varepsilon}$ $(\forall\varepsilon>0)$ is a particular function, connected to the resonance. See Proposition 4.2.10 for more details. In the case with two symmetries, the orthogonality condition on $\partial_{x_2}Q_c$ is automatically satisfied.

Let us consider a model to understand the difficulties from the resonance $L_{Q_c}(iQ_c)=0$, $iQ_c\notin\mathcal{E}_{\otimes,\sigma}$, with $\sigma>0$. Consider the equation $\Delta u = f$ in $\mathbb{R}^2$, with $f\in C^\infty_c(\mathbb{R}^2,\mathbb{R})$. The Green function for the Laplacian in $\mathbb{R}^2$ is $\frac{\ln(r)}{2\pi}$, and thus the fundamental solution is $u_0=\frac{\ln(r)}{2\pi}*f$. We can check that this function is well defined, C^∞ and at infinity, $u_0\sim\frac{\ln(r)}{2\pi}\int_{\mathbb{R}^2}f$. If we want this solution to be bounded, we must impose that $\int_{\mathbb{R}^2}f=0$. In that case, we can check that $|u_0|\leqslant\frac{K(f)}{(1+r)}$. But if we instead looked at the equation $\Delta u - V(x)u = f$, where $V\geqslant 0, V\in C^\infty_c(\mathbb{R}^2,\mathbb{R})$ and $f\in C^\infty_c(\mathbb{R}^2,\mathbb{R})$, the condition $\int_{\mathbb{R}^2}f=0$ has no reason to be enough to show that a typical solution u_0 is bounded. In fact, we then must show that $\int_{\mathbb{R}^2}f+Vu=0$, which is more complicated to understand what it means on the source f. We remark that if we impose f to be odd, and the potential V to be even, this problem disappears.

The situation is very similar here, on the equation of the imaginary part of ψ, with $\varphi=Q_c\psi$. The element $\varphi=iQ_c$ is an element of the kernel, that can not be dealt with a local orthogonality condition, similarly as 1 for the operator Δ, if we want to stay in function spaces where functions are bounded.

In the case with one symmetry, we believe that the growth of φ at infinity is of order $\ln(r)$ instead of smaller than $r^{-\varepsilon}$ for all $\varepsilon>0$, as it is shown here.

This result could be interesting for the construction of a multi travelling wave solution of (GP). One of the step there is to construct an approximate solution, and to compute the error terms, it is necessary to invert the linearized operator around Q_c. We can also use it to improve the differentiability of the branch with respect to the speed.

1.5.2 Infinite differentiability of the branch of travelling waves

We will show that, for all $0<\sigma<1$, there exists $c_0(\sigma)>0$ such that

$$c\to\partial_c Q_c\in C^0(]0,c_0(\sigma)[,\mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma})$$

(see Lemma 4.1.1). Furthermore, from Lemma 3.1.6, we have

$$L_{Q_c}(\partial_c Q_c) = i\,\partial_{x_2}Q_c.$$

We want to differentiate this equation with respect to c. Formally, this would yield

$$L_{Q_c}(\partial_c^2 Q_c) = 2|\partial_c Q_c|^2 Q_c + 4\mathfrak{Re}(\overline{Q_c}\partial_c Q_c)\partial_c Q_c - 2i\partial_{x_2}\partial_c Q_c \in \mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma}$$

for all $0 < \sigma < 1$. This gives us a way to define $\partial_c^2 Q_c$ with Theorem 1.5.1. We use similar computations to show that the branch is infinitely differentiable.

Theorem 1.5.2. *For all $0<\sigma<1$, there exists $c_0, c_0(\sigma)>0$ such that*

$$c \to \partial_c Q_c \in C^\infty(]0, c_0(\sigma)[, \mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma}).$$

Furthermore,

$$c \to E(Q_c) \in C^\infty(]0, c_0[, \mathbb{R}),$$

and the momentum $P_2(Q_c) = \frac{1}{2}\langle i\partial_{x_2} Q_c, Q_c - 1\rangle$ satisfies

$$c \to P_2(Q_c) \in C^\infty(]0, c_0[, \mathbb{R}).$$

This result implies in particular that $c \to Q_c - 1 \in C^\infty(]0, c_0(p)[, X_p)$ for all $2 < p \leqslant +\infty$, with X_p defined in Theorem 1.3.1. The fact that the branch, as well as the energy and momentum, are C^2 with respect to the speed could be useful for the proof of the orbital stability of this branch of travelling wave. Remark that we do not give an equivalent of $\partial_c^2 Q_c$ when $c \to 0$. This seems to be a rather complicated computation, but we fully expect that $\partial_c^2 Q_c \sim \frac{1}{c^4}\partial_d^2 V$ when $c \to 0$.

1.5.3 Sketch of the proofs of Theorems 1.5.1 and 1.5.2

We want to invert the problem $L_{Q_c}(\varphi) = Q_c h$ in the weighed L^∞ spaces $\mathcal{E}_{\otimes,\sigma}$ and $\mathcal{E}_{\otimes\otimes,\sigma}$. Writing $\varphi = Q_c \psi$, at first order at infinity in position, the problem become $-ic\partial_{x_2}\psi - \Delta\psi + 2\mathfrak{Re}(\psi) = h$, as it was the case for the proof of Theorem 1.3.1. But here, we suppose only one symmetry, and we can invert this problem in the required space under the condition that $\int_{\mathbb{R}^2} \mathfrak{Im}(h) = 0$. That is why we show the inversion in a large settingn in Lemma 2.1.15. This condition was freely given when we inverted this problem with two symmetries, for the proof of Theorem 1.3.1.

Therefore, we look at the problem $L_{Q_c}(\varphi) = Q_c h - \mu(h, \psi) i\Upsilon$, where Υ is a particular smooth and compactly supported function, and $\mu(h, \psi)$ is linear in h, ψ, with values in $\mathbb{C}$, and they are choosen such that, when writing the problem at infinity in position on the form $-ic\partial_{x_2}\psi - \Delta\psi + 2\mathfrak{Re}(\psi) = \tilde{h}$, where $\tilde{h}$ depends on both h, ψ and $\mu(h, \psi)$, then $\int_{\mathbb{R}^2} \mathfrak{Im}(\tilde{h}) = 0$. This allows us to invert the problem at infinity. Since the kernel of L_{Q_c} is known (see Corollary 1.4.5), to invert locally the operator (with the additional term $-\mu(h, \psi) i\Upsilon$), we also require a local orthogonality condition (one of the two elements of the kernel is avoided by symmetry). Here, the constant of inversion depends on c, since L_{Q_c} has some small directions (see Proposition 1.4.1). The construction still requires a Fredholm alternative argument, that has to be modified because of the difficulties on the phase. This changes only slightly the proofs compared to Chapter 2, since the change is only on a one dimensional direction, thus the compactness arguments are identical.

To complete the inversion of L_{Q_c}, we need to invert the particular direction $i\Upsilon$. This is done explicitely for one vortex (see [10]), since there, the problem is an ODE. With an ansatz using this solution for each vortices, by constructing an inverse as a perturbation of it, we can invert this direction for L_{Q_c} (see Lemma 4.2.9). This particular solution does not decay as well at infinity compare to the function φ in $L_{Q_c}(\varphi) = Q_c h - \mu(h, \psi) i\Upsilon$, but this is expected, since it is also the case for a single vortex (where it grows like $\ln(r)$).

Now, in the case with two symmetries, $\mu(h, \psi) = 0$ and thus the solution has a better decay, and both elements of the kernel of L_{Q_c} are avoided. We can thus invert it, without any orthogonality condition.

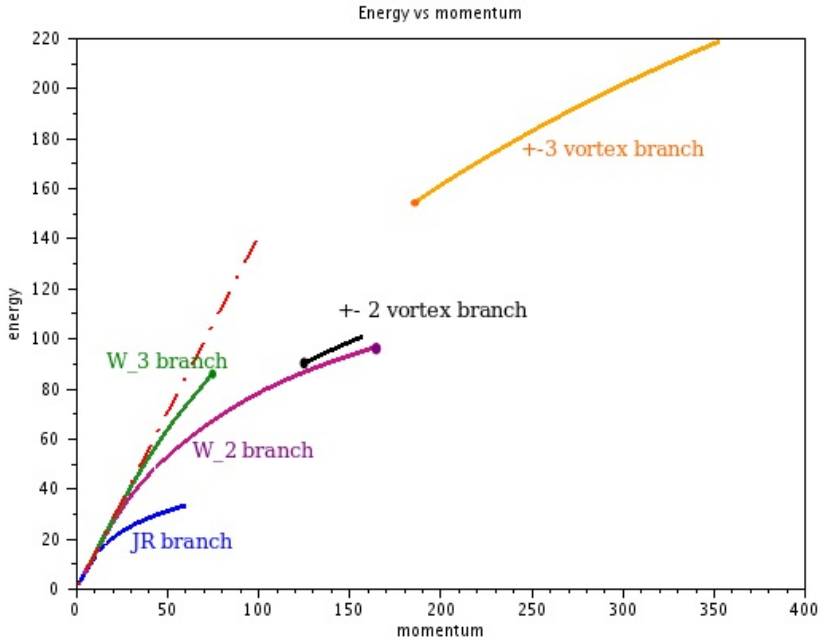
For the proof of Theorem 1.5.2, we use the inversion in the case with two symmetries, and the equality $L_{Q_c}(\partial_c Q_c) = i\partial_{x_2} Q_c$ (both $\partial_c Q_c$ and $i\partial_{x_2} Q_c$ have the two required symmetries). We thus write $\partial_c Q_c = L_{Q_c}^{-1}(i\partial_{x_2} Q_c)$, and we check that the operator $L_{Q_c}^{-1}$ is differentiable with respect to the speed. This shows that $\partial_c Q_c$ is C^1 with respect to the speed, thus $c \mapsto Q_c$ is C^2. By induction, we show that $c \mapsto Q_c$ is C^∞ with respect to the speed.

1.6 Some open problems and conjectures

We present here some questions that are only partially answered by the results of this thesis, or that are natural follow up problems.

- **Construction of a travelling wave behaving like two vortices of degree $\pm n$ for $n \geqslant 2$.**

From [25], there exists vortices of any degree $n \in \mathbb{Z}^*$. We constructed a travelling wave (of speed $c > 0$) behaving like two vortices of degree ± 1 at distance of order $2/c$ in Theorem 1.3.1. It used the fact that the kernel of the linearized operator around $V_{\pm 1}$ is known, and contains, in the associated energy space, only the translations (see [10]). Such a property ($\mathrm{Ker}_{H_{V_{\pm n}}} = \mathrm{Span}_{\mathbb{R}}(\partial_{x_1} V_{\pm n}, \partial_{x_2} V_{\pm n})$) is not known on vortices of degree $n \geqslant 2$. However, if it is shown that it holds for such vortices, then the construction done in Chapter 2 should work similarly (the distance between the vortices will be of order $2n/c$ in that case). Such branches have been seen numerically in [9]:



This graph represents different branches of travelling waves constructed numerically. We recall that large momentum yield speeds close to 0, and small momentum speeds close to $\sqrt{2}$. The JR (for Jones-Roberts) branch is the one constructed in Theorem 1.3.1. The ± 2 and ± 3 vortex branch are the ones descibed above. The W_2 and W_3 branches are constructed from the limit $c \to \sqrt{2}$.

Furthermore, if it is also possible to show a coercivity result on B_{V_n}, the quadratic form associated to the vortex V_n, then the coercivity results (such as Theorems 1.4.4 and 1.4.12) should also hold. It has been shown numerically that B_{V_n} can take negative values ([40]). But, if a coercivity result is shown, with several local orthogonality conditions, for B_{V_n} (to kill the finitely many negative directions), we should have a coercivity result for the branch $V_n V_{-n}$, with twice as many orthogonality conditions. This would show that this branch is likely unstable, but has no additional unstability directions than those of the vortices that compose it.

- **The constructions of Theorems 1.2.6, 1.2.7 and 1.3.1 yield the same branch**

The constructions of Theorems 1.2.7 and 1.2.6 respectively minimize locally the energy at fixed momentum, or is a critical point of a well chosen Lagrangian. This is not shown for the one from Theorem 1.3.1. However, in this last construction, the branch is C^∞ with respect to the speed, and the structure in term of vortices is well understood, and these properties are not shown in Theorems 1.2.6 and 1.2.7.

Showing that these branches are identical would combine these properties. Furthermore, the proof of such a result would most likely give the fact that the branch is isolated in the Energy/Momentum graph above (for large momentum), which would be a major step in the completion of the Jones-Roberts program.

To show such a result, it might be possible to improve the local uniqueness result of Theorem 1.3.1, to show for instance that any travelling wave behaving like two vortices (with a small error in L^∞ for instance) is an element of the branch of Theorem 1.3.1. With such a result, it would only require an improvement on the structure of the branches in Theorems 1.2.6 and 1.2.7 (namely, showing that they behave like two vortices in L^∞).

- **Extension of the branch of Theorem 1.3.1 for large speeds**

The construction of Theorem 1.3.1 is done for small speeds, less than some small $c_0 > 0$. It is conjectured that this branch extend in all speeds in $]0, \sqrt{2}[$. Adapting some proof in Chapter 4, using that $L_{Q_c}(\partial_c Q_c) = i\partial_{x_2} Q_c$, it is possible to show that, as long as $i\partial_{x_2} Q_c$ remains orthogonal to the kernel (which is true only shown for small speeds), then, for any for $0 < c < \sqrt{2}$, if the branch is still defined, $\partial_c Q_c$ is bounded in some weighed L^∞ space. This could give a way to continue the branch, even when the speed is no longer small, by integrating $\partial_c Q_c$ with respect to the speed.

- **Construction of smooth branches of solutions in other problems**

The method of the construction of travelling waves of Theorem 1.3.1 has been used in other cases, for instance [11] or [29]. In these other cases, only the construction was done, not the differentiability with respect to the parameter. By adapting elements of section 2.3, it might be possible to show the differentiability with respect to the parameter in these other cases.

- **Orbital stability and multi travelling wave solutions**

With a coercivity result such as Theorem 1.4.12, we could expect (at least in the symmetric case) to have an orbital stability result. However, there are some technical difficulties, connected to the weakness of the coercivity norm compared to the norm of the energy space. Furthermore, another difficulty is that we need to modulate on the speed, which makes the functional $E(u) - cP_2(u)$ not independent of time.

Theorem 1.5.1 is a first step in the construction of a multi travelling wave solutions, at least with a symmetry (for instance, one travelling wave moving in the direction $\vec{e}_2$ and the other in the direction $-\vec{e}_2$), using methods developed in [28], [33] or [34]. With the equivalents of Theorem 1.2.11, it is possible to compute the first order of the interaction between them. There are some technical difficulties left to complete such a construction, in particular with respect to the phase.

If such multi travelling wave solutions exists, their stability would be an interesting question, as for other multi soliton solutions, see for instance [35] or [36].

Chapter 2
Smooth branch of travelling waves

This chapter is devoted to the proof of Theorem 1.3.1. We start by reducing the problem to a one dimensional one in section 2.1. The construction of the travelling wave Q_c is completed in section 2.2. Furthermore, in subsection 2.2.2, we show that Q_c has finite energy and we compute some estimates particular to the branch of solutions. Finally section 2.3 is devoted to the proof of the differentiability of the branch.

2.1 Lyapunov-Schmidt reduction

The proof of Theorem 1.3.1 follows closely the construction done in [11] or [29]. The main idea is to use perturbation methods on an approximate solution.

In subsection 2.1.1 we define this approximate solution V which consists in two vortices at distance $2d$ from each other. We then look for a solution of (TW_c) as a perturbation of V, with an additive perturbation close to the vortices and a multiplicative one far from them. This is computed in subsection 2.1.2. We define suitable spaces in subsection 2.1.3 that we will use to invert the linear part and use a contraction argument. We ask for an orthogonality on the perturbation, and the norms are a little better but more technical than the ones in Theorem 1.3.1. In particular Γ_{c,d_c} in Theorem 1.3.1 verifies better estimates which are discussed for instance in Corollary 2.1.25 and in Lemma 2.2.8. We invert the linearized operator in Proposition 2.1.17 and show that the perturbation is a fixed point of a contracting functional in Proposition 2.1.21. The orthogonality condition create a Lagragian multiplier (see subsection 2.1.6), which left us with a problem in one dimension. This multiplier will be cancelled for a good choice of the parameter d in section 2.2.

2.1.1 Estimates on vortices

From [25], we can find stationary solution of (GP):

$$V_n(x) = \rho_n(r)e^{in\theta}$$

where $x = re^{\mathrm{i}\theta}, n \in \mathbb{Z}^*$, solving

$$\begin{cases} \Delta V_n - (|V_n|^2 - 1)V_n = 0 \\ |V_n| \to 1 \quad \text{as} \quad |x| \to \infty. \end{cases}$$

These solutions are well understood and, in particular, we have some estimates (see [25] for instance) that we will use. We also know the kernel of the linearized operator around $V_{\pm 1}$ ([10]), which we will need for inverting the linearized operator around the approximate solution V defined using these vortices

$$V(x) := V_1(x - d\vec{e}_1)V_{-1}(x + d\vec{e}_1)$$

where $d > 0$, $x = (x_1, x_2)$. The function V is the product of two vortices with opposite degrees at a distance $2d$ from each other. One vortex alone in $\mathbb{R}^2$ is a stationary solution, and it is expected that two vortices interact and translate at a constant speed of order $c \simeq \frac{1}{d}$, see [3]. Hence for the two parameters of this problem $c, d > 0$, we let them be free from each other, but with the condition c is of order $1/d$ by imposing that $\frac{1}{2c} < d < \frac{2}{c}$.

We will study in particular areas near the center of each vortices. We will use coordinates adapted to this problem:

$$\begin{aligned} x &= (x_1, x_2) = re^{i\theta}, \\ y &= (y_1, y_2) := x - d\vec{e_1} = r_1 e^{i\theta_1}, \\ z &= (z_1, z_2) := y + 2\, d\vec{e_1} = x + d\vec{e_1} = r_{-1} e^{i\theta_{-1}}, \\ \tilde{r} &:= \min(r_1, r_{-1}). \end{aligned} \tag{2.1.1}$$

Using y coordinate mean that we are centered around V_1, and z coordinate for around V_{-1}. Note that we have

$$V(x) = V_1(y)V_{-1}(z)$$

using these notations. If it is not precised, V will be taken in x, V_1 in y and V_{-1} in z. If we compute (TW_c) for V, i.e. $-ic\partial_{x_2}V - \Delta V - (1-|V|^2)V$, we get

$$(\mathrm{TW}_c)(V) = E - ic\partial_{x_2}V,$$

where we defined

$$E := -\Delta V - (1-|V|^2)V.$$

We have $V = V_1V_{-1}$ and, by using $-\Delta V_\varepsilon = (1-|V_\varepsilon|^2)V_\varepsilon$ for $\varepsilon = \pm 1$, we compute

$$E = -2\nabla V_1.\nabla V_{-1} + V_1V_{-1}(1-|V_1|^2 + 1 - |V_{-1}|^2 - 1 + |V_1V_{-1}|^2).$$

Hence

$$E = -2\nabla V_1.\nabla V_{-1} + (1-|V_1|^2)(1-|V_{-1}|^2)V_1V_{-1}. \tag{2.1.2}$$

The rest of this subsection is devoted to the computation of estimates on $V, E, \partial_d V$ and $ic\partial_{x_2}V$ using estimates on V_1 and V_{-1}. Let us start with the properties on $V_{\pm 1}$ we need.

Lemma 2.1.1. ([25]) *$V_1(x) = \rho_1(r)e^{i\theta}$ verifies $V_1(0) = 0$, and there exists a constant $\kappa > 0$ such that, for all $r > 0$, $0 < \rho_1(r) < 1$, $\rho_1'(r) > 0$, and*

$$\rho_1(r) \sim_{r\to 0} \kappa r,$$

$$\rho_1'(r) = O_{r\to\infty}\left(\frac{1}{r^3}\right),$$

$$\rho_1''(r) = o_{r\to\infty}\left(\frac{1}{r^3}\right),$$

$$1 - |V_1(x)| = \frac{1}{2r^2} + O_{r\to\infty}\left(\frac{1}{r^3}\right),$$

$$\nabla V_1(x) = iV_1(x)\frac{x^\perp}{r^2} + O_{r\to\infty}\left(\frac{1}{r^3}\right)$$

where $x^\perp = (-x_2, x_1)$, $x = re^{i\theta}$. Furthermore we have similar properties for V_{-1} since

$$V_{-1}(x) = \overline{V_1(x)}.$$

We will use the O notation for convergence independent of any other quantity. Now let us write all the derivatives of a vortex in polar coordinate, which will be useful all along the proof of the results.

Lemma 2.1.2. *We define* $u := \frac{\rho_1'(r_1)}{\rho_1(r_1)}$. *Then,*

$$\partial_{x_1} V_1(y) = \left(\cos(\theta_1)u - \frac{i}{r_1}\sin(\theta_1)\right)V_1,$$

$$\partial_{x_2} V_1(y) = \left(\sin(\theta_1)u + \frac{i}{r_1}\cos(\theta_1)\right)V_1,$$

$$\partial_{x_1x_1} V_1(y) = \left(\cos^2(\theta_1)(u^2+u') + \sin^2(\theta_1)\left(\frac{u}{r_1} - \frac{1}{r_1^2}\right) + 2i\sin(\theta_1)\cos(\theta_1)\left(\frac{1}{r_1^2} - \frac{u}{r_1}\right)\right)V_1,$$

$$\partial_{x_1x_2} V_1(y) = \left(\sin(\theta_1)\cos(\theta_1)\left(u^2 + u' + \frac{1}{r_1^2} - \frac{u}{r_1}\right) - i\cos(2\theta_1)\left(\frac{1}{r_1^2} - \frac{u}{r_1}\right)\right)V_1.$$

We obtain the derivatives of V_{-1} by changing $i \to -i, y \to z$, $\theta_1 \to \theta_{-1}$, $r_1 \to r_{-1}$ and $V_1 \to V_{-1}$. We remark in particular that the first derivatives are of first order $\frac{1}{r_1}$ and the second derivatives are of first order $\frac{1}{r_1^2}$ for large values of r_1. From [25], we can check that, more generally, we have

$$|D^{(n)}V_1(y)| \leqslant \frac{K(n)}{(1+r_1)^n}. \tag{2.1.3}$$

Proof. With the notation of (2.1.1) in radial coordinate around $d\vec{e}_1$, the center of V_1:

$$\partial_{x_1} = \cos(\theta_1)\partial_{r_1} - \frac{\sin(\theta_1)}{r_1}\partial_{\theta_1}$$

$$\partial_{x_2} = \sin(\theta_1)\partial_{r_1} + \frac{\cos(\theta_1)}{r_1}\partial_{\theta_1},$$

we compute directly the first two equalities of the lemma. Now, we compute

$$\partial_{x_1x_1} V_1 = \cos(\theta_1)\partial_{r_1}(\partial_{x_1}V_1) - \frac{\sin(\theta_1)}{r_1}\partial_{\theta_1}(\partial_{x_1}V_1)$$

with

$$\partial_{r_1}(\partial_{x_1}V_1) = \left(u\left(\cos(\theta_1)u - \frac{i}{r_1}\sin(\theta_1)\right) + \cos(\theta_1)u' + \frac{i}{r_1^2}\sin(\theta)\right)V_1$$

and

$$\partial_{\theta_1}(\partial_{x_1}V_1) = \left(i\cos(\theta_1)u + \frac{1}{r_1}\sin(\theta_1) - \sin(\theta_1)u - \frac{i}{r_1}\cos(\theta_1)\right)V_1$$

for the third inequality. We use them also in

$$\partial_{x_1x_2} V_1 = \sin(\theta_1)\partial_{r_1}(\partial_{x_1}V_1) + \frac{\cos(\theta_1)}{r_1}\partial_{\theta_1}(\partial_{x_1}V_1)$$

for the fourth relation, with $\cos^2(\theta_1) - \sin^2(\theta_1) = \cos(2\theta_1)$. □

Now, we compute some basic estimates on V.

Lemma 2.1.3. *There exists a universal constant* $K > 0$ *and a constant* $K(d) > 0$ *depending only on* $d > 1$ *such that*

$$|1 - V|^2 \leqslant \frac{K(d)}{(1+r)^2},$$

$$0 \leqslant 1 - |V|^2 \leqslant \frac{K}{(1+\tilde{r})^2},$$

$$|\nabla(|V|)| \leqslant \frac{K}{(1+\tilde{r})^3},$$

and we have

$$|\nabla V| \leqslant \frac{K}{(1+\tilde{r})},$$

as well as

$$|\nabla V| \leqslant \frac{Kd}{(1+\tilde{r})^2},$$

where $\tilde{r} = \min(r_1, r_{-1})$. *Furthermore,*

$$|\nabla^2 V| \leqslant \frac{K}{(1+\tilde{r})^2}$$

and

$$|\nabla^2 V| \leqslant \frac{Kd}{(1+\tilde{r})^3}.$$

Proof. For the first inequality, we are at fixed d. Since $V = |V_1 V_{-1}| e^{i(\theta_1 - \theta_{-1})}$ and θ_1, θ_{-1} are angles from points separated by $2d$, we infer

$$e^{i(\theta_1-\theta_{-1})} = 1 + O^d_{r\to\infty}\left(\frac{1}{r}\right),$$

and $|V_1 V_{-1}| = 1 + O^d_{r\to\infty}\left(\frac{1}{r^2}\right)$ from Lemma 2.1.1 where $O^d_{r\to\infty}\left(\frac{1}{r}\right)$ is a quantity that decay in $\frac{1}{r}$ is at fixed d. Therefore,

$$|1 - V|^2 = |1 - |V_1 V_{-1}| e^{i(\theta_1-\theta_{-1})}|^2 = \left| K(d) O_{r\to\infty}\left(\frac{1}{r}\right)\right|^2 \leqslant \frac{K(d)}{(1+r)^2}.$$

From Lemma 2.1.1, we compute

$$1 - |V|^2 = 1 - |V_1|^2 + |V_1|^2(1 - |V_{-1}|^2) \leqslant K\left(\frac{1}{(1+r_1)^2} + \frac{1}{(1+r_{-1})^2}\right) \leqslant \frac{K}{(1+\tilde{r})^2},$$

and

$$|\nabla(|V|)| \leqslant |\nabla(|V_1|)|V_{-1}|| + |\nabla(|V_{-1}|)|V_1|| \leqslant K\left(\frac{1}{(1+r_1)^3} + \frac{1}{(1+r_{-1})^3}\right) \leqslant \frac{K}{(1+\tilde{r})^3}.$$

We check that $\nabla V = \nabla V_1 V_{-1} + \nabla V_{-1} V_1$, and therefore, with Lemma 2.1.2, we have

$$|\nabla V| \leqslant \frac{K}{(1+r_1)} + \frac{K}{(1+r_{-1})} \leqslant \frac{K}{(1+\tilde{r})}.$$

Furthermore, by Lemma 2.1.1,

$$\nabla V_{\pm 1} = \frac{\pm i}{r_{\pm 1}} \vec{e}_{\theta_{\pm 1}} + O_{r_{\pm 1}\to\infty}\left(\frac{1}{r^3_{\pm 1}}\right).$$

For $\tilde{r} \geqslant 1$ (the last estimate on $|\nabla V|$ for $\tilde{r} \leqslant 1$ is a consequence of $|\nabla V| \leqslant \frac{K}{(1+\tilde{r})}$), since $r_{\pm 1} e^{i\theta_{\pm 1}} = x \mp d\vec{e}_1$,

$$\begin{aligned}
\frac{\cos(\theta_1)}{r_1} - \frac{\cos(\theta_{-1})}{r_{-1}} &= \frac{x_1 - d}{(x_1-d)^2 + x_2^2} - \frac{x_1 + d}{(x_1+d)^2 + x_2^2} \\
&= \frac{x_1}{r_1^2 r_{-1}^2}((x_1+d)^2 + x_2^2 - ((x_1-d)^2 + x_2^2)) - d\left(\frac{1}{r_1^2} + \frac{1}{r_{-1}^2}\right) \\
&= \frac{d}{r_1^2 r_{-1}^2}(2x_1^2 - r_1^2 - r_{-1}^2), \qquad (2.1.4)
\end{aligned}$$

therefore

$$\left|\frac{\cos(\theta_1)}{r_1} - \frac{\cos(\theta_{-1})}{r_{-1}}\right| \leqslant \frac{Kd}{(1+\tilde{r})^2}$$

since $\frac{x_1}{r_1 r_{-1}} \leqslant \frac{1}{\tilde{r}}$ if $\tilde{r} \geqslant 1$. With a similar estimation for $\frac{\sin(\theta_1)}{r_1} - \frac{\sin(\theta_{-1})}{r_{-1}}$, we infer

$$\begin{aligned}
|\nabla V| &\leqslant \left|\frac{\vec{e}_{\theta_1}}{r_1} - \frac{\vec{e}_{\theta_{-1}}}{r_{-1}}\right| + \frac{K}{(1+\tilde{r})^3} \\
&\leqslant \frac{Kd}{(1+\tilde{r})^2} + \frac{K}{(1+\tilde{r})^3} \\
&\leqslant \frac{Kd}{(1+\tilde{r})^2}.
\end{aligned}$$

Finally, for the second derivatives, we have for $j,k\in\{1,2\}$

$$\partial_{x_jx_k}V=\partial_{x_jx_k}V_1V_{-1}+\partial_{x_j}V_1\partial_{x_k}V_{-1}+\partial_{x_k}V_1\partial_{x_j}V_{-1}+\partial_{x_jx_k}V_{-1}V_1,$$

therefore, with (2.1.3),

$$|\nabla^2V|\leqslant\frac{K}{(1+r_1)^2}+\frac{K}{(1+r_{-1})(1+r_1)}+\frac{K}{(1+r_{-1})^2}\leqslant\frac{K}{(1+\tilde r)^2}.$$

We check with (2.1.4), $\frac{x_1}{r_1r_{-1}}\leqslant\frac{1}{\tilde r}$ if $\tilde r\geqslant1$ and $\left|\nabla\left(\frac{1}{r_{\pm1}}\right)\right|\leqslant\frac{K}{r_{\pm1}^2}$ that, for $\tilde r\geqslant1$,

$$\left|\nabla\left(\frac{\cos(\theta_1)}{r_1}-\frac{\cos(\theta_{-1})}{r_{-1}}\right)\right|\leqslant\frac{Kd}{(1+\tilde r)^3}.$$

With a similar estimation for $\nabla\left(\frac{\sin(\theta_1)}{r_1}-\frac{\sin(\theta_{-1})}{r_{-1}}\right)$ and Lemma 2.1.1, we conclude with

$$|\nabla^2V|\leqslant\left|\nabla\left(\frac{\vec e_{\theta_1}}{r_1}-\frac{\vec e_{\theta_{-1}}}{r_{-1}}\right)\right|+\frac{K}{(1+\tilde r)^3}\leqslant\frac{Kd}{(1+\tilde r)^3}.$$ □

Now we look at the convergence of some quantities when we are near the center of V_1 and $d\to\infty$. When we are close to the center of V_1 and d goes to infinity, we expect that the second vortex as no influence.

Lemma 2.1.4. *As $d\to\infty$, we have, locally uniformly in $\mathbb{R}^2$,*

$$V(.+d\vec e_1)=V_1(.)V_{-1}(.+2d\vec e_1)\to V_1(.),$$
$$E(.+d\vec e_1)\to0$$

and

$$\partial_dV(.+d\vec e_1)\to-\partial_{x_1}V_1(.).$$

Proof. In the limit $d\to\infty$, for $y\in\mathbb{R}^2$,

$$V(y+d\vec e_1)=V_1(y)e^{-i\theta_{-1}}\left(1+O\left(\frac{1}{r_{-1}^2}\right)\right)$$

by Lemma 2.1.1, hence

$$V(.)\to V_1(.)$$

locally uniformly since $\theta_{-1}\to0,r_{-1}\to+\infty$ when $d\to\infty$ locally uniformly. On the other hand, since $V(x)=V_1(y)V_{-1}(y+2\,d\vec e_1)$, we have

$$(\partial_dV)(y+d\vec e_1)=-\partial_{x_1}V_1(y)V_{-1}(y+2\,d\vec e_1)+V_1(y)\partial_{x_1}V_{-1}(y+2\,d\vec e_1).$$

Since $\partial_{x_1}V_{-1}(y+2\,d\vec e_1)=\nabla V_{-1}(y+2\,d\vec e_1).\vec e_1\to0$ locally uniformly as $d\to\infty$, we have

$$\partial_dV(.)\to-\partial_{x_1}V_1(.)$$

locally uniformly. Finally, from (2.1.2), we have that

$$E(x)=-2\nabla V_1(y).\nabla V_{-1}(z)+(1-|V_1(y)|^2)(1-|V_{-1}(z)|^2)V_1(y)V_{-1}(z)$$

with the notations from (2.1.1), therefore, locally uniformly,

$$E(.+d\vec e_1)\to0$$

as $\nabla V_{-1}\to0$ and $|V_{-1}|\to1$ locally uniformly when $d\to\infty$. □

We now do a precise computation on the term $ic\partial_{x_2}V$, which appears in $(\mathrm{TW}_c)(V)$.

Lemma 2.1.5. *There exists a universal constant $C>0$ (independent of d) such that if $r_1,r_{-1}\geqslant1$,*

$$\left|i\frac{\partial_{x_2}V}{V}-2d\frac{x_1^2-d^2-x_2^2}{r_1^2r_{-1}^2}\right|\leqslant C\left(\frac{1}{r_1^3}+\frac{1}{r_{-1}^3}\right).$$

Remark that this shows that the first order term of $i\frac{\partial_{x_2}V}{V}$ is real-valued and the dependence on d of this term is explicit.

Proof. Recall from Lemma 2.1.2 that for $\varepsilon = \pm 1$,

$$\partial_{x_2} V_\varepsilon = \frac{i\varepsilon}{r_\varepsilon}\cos(\theta_\varepsilon)V_\varepsilon + O_{r_1\to\infty}\left(\frac{1}{r_1^3}\right).$$

We have

$$\frac{\partial_{x_2}V}{V} = \frac{\partial_{x_2}V_1}{V_1} + \frac{\partial_{x_2}V_{-1}}{V_{-1}}$$

and

$$\cos(\theta_\varepsilon) = \frac{x_1 - \varepsilon d}{r_\varepsilon},$$

yielding

$$\begin{aligned}\frac{\partial_{x_2}V}{V} &= i\left(\frac{x_1-d}{r_1^2} - \frac{x_1+d}{r_{-1}^2}\right)\\ &= i\left(x_1\left(\frac{1}{r_1^2}-\frac{1}{r_{-1}^2}\right) - d\left(\frac{1}{r_1^2}+\frac{1}{r_{-1}^2}\right)\right) + O_{r_1\to\infty}\left(\frac{1}{r_1^3}\right) + O_{r_{-1}\to\infty}\left(\frac{1}{r_{-1}^3}\right).\end{aligned}$$

We compute with (2.1.1) that

$$\frac{1}{r_1^2} - \frac{1}{r_{-1}^2} = \frac{(x_1+d)^2 + x_2^2 - (x_1-d)^2 - x_2^2}{r_1^2 r_{-1}^2} = \frac{4dx_1}{r_1^2 r_{-1}^2}$$

and

$$\frac{1}{r_1^2} + \frac{1}{r_{-1}^2} = \frac{(x_1+d)^2 + x_2^2 + (x_1-d)^2 + x_2^2}{r_1^2 r_{-1}^2} = 2\frac{x_1^2 + d^2 + x_2^2}{r_1^2 r_{-1}^2},$$

yielding the estimate. □

Finally, we show an estimate on $\partial_d V = \partial_d(V_1(x - d\vec{e_1})V_{-1}(x + d\vec{e_1})) = -\partial_{x_1}V_1 V_{-1} + \partial_{x_1}V_{-1}V_1$.

Lemma 2.1.6. *There exists a constant $K > 0$ such that*

$$|\partial_d V| \leqslant \frac{K}{(1+\tilde{r})},$$

$$|\nabla\partial_d V| \leqslant \frac{K}{(1+\tilde{r})^2}$$

and

$$|\mathfrak{Re}(\bar{V}\partial_d V)| \leqslant \frac{K}{(1+\tilde{r})^3}.$$

Furthermore,

$$|\partial_d^2 V| \leqslant \frac{K}{(1+\tilde{r})^2}$$

and

$$|\partial_d^2 \nabla V| \leqslant \frac{K}{(1+\tilde{r})^3}.$$

Proof. We have that $\partial_d V = -\partial_{x_1}V_1V_{-1} + \partial_{x_1}V_{-1}V_1$ and from Lemma 2.1.2,

$$|\partial_{x_1}V_1| \leqslant \frac{K}{(1+r_1)} \leqslant \frac{K}{(1+\tilde{r})}.$$

Similarly, $|\partial_{x_1}V_{-1}| \leqslant \frac{K}{(1+\tilde{r})}$ and this proves the first inequality. Furthemore, for $\nabla\partial_d V$, every terms has two derivatives, each one bringing a $\frac{1}{(1+\tilde{r})}$ by (2.1.3), this shows the second inequality. Finally, we compute

$$\mathfrak{Re}(\bar{V}\partial_d V) = -|V_{-1}|^2\mathfrak{Re}(\bar{V}_1\partial_{x_1}V_1) + |V_1|^2\mathfrak{Re}(\overline{V_{-1}}\partial_{x_1}V_{-1}).$$

From Lemma 2.1.1, $|\mathfrak{Re}(\bar{V_1}\partial_{x_1}V_1)| \leqslant \frac{K}{(1+r_1)^3} \leqslant \frac{K}{(1+\tilde{r})^3}$ and $|V_{-1}|^2 \leqslant 1$. Similarly we have

$$||V_1|^2\mathfrak{Re}(\overline{V_{-1}}\partial_{x_1}V_{-1})| \leqslant \frac{K}{(1+\tilde{r})^3}.$$

Furthermore, since $\partial_d^2 V = \partial_{x_1}^2 V_1 V_{-1} - 2\partial_{x_1}V_1\partial_{x_1}V_{-1} + \partial_{x_1}^2 V_{-1}V_1$, with equation (2.1.3), we check easily the estimations on $\partial_d^2 V$ and $\partial_d^2\nabla V$. □

2.1.2 Setup of the proof

In the same way as in [11] (see also [29]), we will look at a solution of (TW_c) as a perturbation of V of the form

$$v := \eta V(1+\Psi) + (1-\eta)Ve^{\Psi}$$

where $\eta(x) = \tilde{\eta}(r_1) + \tilde{\eta}(r_{-1})$ and $\tilde{\eta}$ is a C^∞ positive cutoff with $\tilde{\eta}(r) = 1$ if $r \leqslant 1$ and 0 if $r \geqslant 2$. The perturbation is Ψ and we will also use

$$\Phi := V\Psi.$$

We use such a perturbation because we want it to be additive (in Φ) near the center of the vortices (where $v = V + \Phi$), and multiplicative (in Ψ) far from them (where $v = Ve^{\Psi}$). We shall require Φ to be bounded (and small) near the vortices. The problem becomes an equation on Ψ, with the following Lemma 2.1.7, we shall write

$$\eta L(\Phi) + (1-\eta)VL'(\Psi) + F(\Psi) = 0$$

where L and L' are linear. The main part of the proof of the construction consists of inverting the linearized operator $\eta L(\Phi) + (1-\eta)VL'(\Psi)$ in suitable spaces, and then use a contraction argument by showing that F is small and conclude on the existence of a solution Ψ by a fixed point theorem.

Lemma 2.1.7. *The function $v = \eta V(1+\Psi) + (1-\eta)Ve^{\Psi}$ is solution of (TW_c) if and only if*

$$\eta L(\Phi) + (1-\eta)VL'(\Psi) + F(\Psi) = 0,$$

where $\Phi = V\Psi$,

$$L'(\Psi) := -\Delta\Psi - 2\frac{\nabla V}{V}.\nabla\Psi + 2|V|^2\mathfrak{Re}(\Psi) - ic\partial_{x_2}\Psi,$$

$$L(\Phi) := -\Delta\Phi - (1-|V|^2)\Phi + 2\mathfrak{Re}(\bar{V}\Phi)V - ic\partial_{x_2}\Phi,$$

$$F(\Psi) := E - ic\partial_{x_2}V + V(1-\eta)(-\nabla\Psi.\nabla\Psi + |V|^2 S(\Psi)) + R(\Psi),$$

with

$$E = -\Delta V - (1-|V|^2)V,$$

$$S(\Psi) := e^{2\mathfrak{Re}(\Psi)} - 1 - 2\mathfrak{Re}(\Psi)$$

and $R(\Psi)$ is a sum of terms at least quadratic in Ψ or Φ localized in the area where $\eta \neq 0$. Furthermore, there exists $C, C_0 > 0$ such that the estimate

$$|R(\Psi)| + |\nabla R(\Psi)| \leqslant C\|\Phi\|^2_{C^2(\{\tilde{r}\leqslant 2\})}$$

holds if $\|\Phi\|_{C^2(\mathbb{R}^2)} \leqslant C_0$ (a constant independent of c), where $\tilde{r} = \min(|x - d\vec{e}_1|, |x + d\vec{e}_1|)$ for $x \in \mathbb{R}^2$. Additionally, $L(\Phi)$ and $L'(\Psi)$ are related by

$$L(\Phi) = (E - ic\partial_{x_2}V)\Psi + VL'(\Psi).$$

The main reason for such a perturbation ansatz is because $V(d\vec{e}_1) = V(-d\vec{e}_1) = 0$, so we can not divide by V as done in L' for instance when we look near the vortices, therefore an additive perturbation is more suitable. But far from the vortices, the perturbation is easier to compute when written multiplicatively with a factorisation by V. Remark also that this allows us to take Ψ to explode at $d\vec{e}_1$ and $-d\vec{e}_1$ as long as $\Phi = V\Psi$ does not. This is needed for the norm we use in subsection 2.1.3.

As we look for Φ small (it is a perturbation), the conditions $\|\Phi\|_{C^2(\mathbb{R}^2)} \leqslant C_0$ will always be true. We need them because some of the error terms have an exponential contribution in Ψ, and not only quadratic. We recall that, with our notations, $\nabla\Psi.\nabla\Psi$ is complex-valued.

Remark that the quantity F contains only nonlinear terms and the source term, which is $E - ic\partial_{x_2}V$. Furthermore, contrary to the work [29], the transport term is in the linearized operator, and not considered as an error term in F.

The rest of this subsection is devoted to the proof of Lemma 2.1.7.

Proof. First we show that $L(\Phi) = (E - ic\partial_{x_2}V)\Psi + L'(\Psi)V$. We use $\Phi = V\Psi$ in $L(\Phi)$ to compute

$$L(\Phi) = -\Delta V\,\Psi - \Delta\Psi\, V - 2\nabla\Psi.\nabla V - (1-|V|^2)\,V\Psi + 2|V|^2 V\mathfrak{Re}(\Psi) - ic V\partial_{x_2}\Psi - ic\partial_{x_2}V\Psi.$$

We have that $E = -\Delta V - (1-|V|^2)V$ hence $(E - ic\partial_{x_2}V)\Psi = -\Delta V\Psi - (1-|V|^2)V\Psi - ic\partial_{x_2}V\Psi$ and the remaining terms are exactly equal to $VL'(\Psi)$.

We denote $\zeta := 1 + \Psi - e^{\Psi}$. Remark that ζ is at least quadratic in Ψ. We compute the different terms in (TW_c):

$$-ic\partial_{x_2}v - \Delta v - (1-|v|^2)v = 0$$

with

$$v = \eta V(1+\Psi) + (1-\eta)Ve^{\Psi}.$$

We have $v = V + \Phi - (1-\eta)\zeta$. In general, our goal in this computation is to factorize any term when possible by $V(\eta + (1-\eta)e^{\Psi})$ and compute the other terms, which will be supported in the area $\eta(1-\eta) \neq 0$. First compute

$$\partial_{x_2}v =$$

$$\eta(\partial_{x_2}V\,(1+\Psi) + \partial_{x_2}\Psi V) + \partial_{x_2}\eta V(1+\Psi) + (1-\eta)\,e^{\Psi}(\partial_{x_2}V + \partial_{x_2}\Psi V) - \partial_{x_2}\eta V e^{\Psi},$$

therefore

$$-ic\partial_{x_2}v = V(\eta + (1-\eta)e^{\Psi})\left(-ic\frac{\partial_{x_2}V}{V} - ic\partial_{x_2}\Psi\right) - ic\eta\partial_{x_2}V\Psi - ic\partial_{x_2}\eta V\zeta. \tag{2.1.5}$$

For the second term, we compute

$$\begin{aligned}\Delta v &= \Delta\eta V(1+\Psi-e^{\Psi}) + 2\nabla\eta.\nabla(V(1+\Psi-e^{\Psi}))\\ &+ \eta(\Delta V(1+\Psi) + 2\nabla V.\nabla\Psi + V\Delta\Psi)\\ &+ (1-\eta)(\Delta V e^{\Psi} + 2\nabla V.\nabla\Psi e^{\Psi} + V(\Delta\Psi + \nabla\Psi.\nabla\Psi)e^{\Psi}),\end{aligned}$$

hence

$$\begin{aligned}-\Delta v &= V(\eta + (1-\eta)e^{\Psi})\left(-\frac{\Delta V}{V} - 2\frac{\nabla V}{V}.\nabla\Psi - \Delta\Psi\right)\\ &- \eta\Delta V\Psi - (1-\eta)\,V\nabla\Psi.\nabla\Psi\, e^{\Psi} - V\,\Delta\eta\zeta - 2\nabla\eta.\nabla(V\zeta).\end{aligned} \tag{2.1.6}$$

Finally, let us write $A := V(1+\Psi)$ and $B := Ve^{\Psi}$, so that $v = \eta A + (1-\eta)B$, and remark that $V\zeta = A - B$. We then have

$$(1-|v|^2)v = (1-\eta^2|A|^2 - (1-\eta)^2|B|^2 - 2\eta(1-\eta)\mathfrak{Re}(A\bar{B}))(\eta A + (1-\eta)B).$$

We want to bring out the terms not related to the interaction between A and B, namely $\eta(1-|A|^2)A + (1-\eta)(1-|B|^2)B$. We have

$$\begin{aligned}(1-|v|^2)v &= \eta(1-|A|^2)A + \eta A[(1-\eta^2)|A|^2 - (1-\eta)^2|B|^2 - 2\eta(1-\eta)\mathfrak{Re}(A\bar{B})]\\ &+ (1-\eta)(1-|B|^2)B + (1-\eta)B[(1-(1-\eta)^2)|B|^2 - \eta^2|A|^2 - 2\eta(1-\eta)\mathfrak{Re}(A\bar{B})].\end{aligned}$$

Now, factorizing $\eta(1-\eta)$ we get

$$\begin{aligned}(1-|v|^2)v &= \eta(1-|A|^2)A + (1-\eta)(1-|B|^2)B\\ &+ \eta(1-\eta)[(1+\eta)A|A|^2 - (1-\eta)A|B|^2 - 2\eta A\mathfrak{Re}(A\bar{B})]\\ &+ \eta(1-\eta)[(2-\eta)B|B|^2 - \eta B|A|^2 - 2(1-\eta)B\mathfrak{Re}(A\bar{B})].\end{aligned}$$

Remark that the last two lines yield 0 if we take $A=B$, since $V\zeta=A-B$, we can write

$$(1-|v|^2)v=\eta(1-|A|^2)A+(1-\eta)(1-|B|^2)B+\eta(1-\eta)(V\zeta G(\Psi)+\overline{V\zeta}H(\Psi))$$

where G,H are functions satisfying $|H(\Psi)|,|G(\Psi)|,|\nabla H(\Psi)|,|\nabla G(\Psi)|\leqslant C(1+|\Psi|+|\nabla\Psi|+|e^{\Psi}|+|\nabla\Psi e^{\Psi}|)$ for some universal constant $C>0$. We recall that $A=V(1+\Psi)$ hence

$$(1-|A|^2)A=(1-|V|^2|1+\Psi|^2)V(1+\Psi),$$

therefore we get a constant (in Φ), a linear and a nonlinear part in Ψ:

$$\begin{aligned}(1-|A|^2)A &= (1-|V|^2)V+(1-|V|^2)V\Psi-2|V|^2V\mathfrak{Re}(\Psi)\\ &\quad -2|V|^2V\mathfrak{Re}(\Psi)\Psi-|V\Psi|^2V(1+\Psi).\end{aligned}$$

We have $B=Ve^{\Psi}$, hence

$$(1-|B|^2)B=e^{\Psi}((1-|V|^2)V-2\,\mathfrak{Re}(\Psi)|V|^2V-|V|^2VS(\Psi)),$$

where $S(\Psi)=e^{2\mathfrak{Re}(\Psi)}-1-2\,\mathfrak{Re}(\Psi)$ is nonlinear in Ψ. We add these relations and obtain

$$\begin{aligned}\eta(1-|A|^2)A+(1-\eta)(1-|B|^2)B &= V(\eta+(1-\eta)e^{\Psi})((1-|V|^2)-2\,\mathfrak{Re}(\Psi)|V|^2)\\ &+ \eta(1-\eta)(V\zeta G(\Psi)+\overline{V\zeta}H(\Psi))\\ &+ \eta((1-|V|^2)V\Psi-2|V|^2V\mathfrak{Re}(\Psi)\Psi-|V\Psi|^2V(1+\Psi))\\ &- (1-\eta)e^{\Psi}|V|^2VS(\Psi). \qquad (2.1.7)\end{aligned}$$

Now adding the computations (2.1.5), (2.1.6) and (2.1.7) in $-ic\partial_{x_2}v-\Delta v-(1-|v|^2)v=0$ yields

$$\begin{aligned}&V(\eta+(1-\eta)\,e^{\Psi})\left(\frac{E-ic\partial_{x_2}V}{V}+L'(\Psi)\right)\\ &+\eta((E-ic\partial_{x_2}V)\Psi+2|V|^2V\mathfrak{Re}(\Psi)\Psi+|V\Psi|^2V(1+\Psi))\\ &+V(1-\eta)\,e^{\Psi}(|V|^2S(\Psi)-\nabla\Psi.\nabla\Psi)\\ &-ic\partial_{x_2}\eta V\zeta-V\Delta\eta\zeta-2\nabla\eta.\nabla(V\zeta)-\eta(1-\eta)(V\zeta G(Z)+\overline{V\zeta}H(\Psi)) = 0. \qquad (2.1.8)\end{aligned}$$

We divide by $\eta+(1-\eta)e^{\Psi}$, which is allowed since $\eta+(1-\eta)e^{\Psi}=1+(1-\eta)(e^{\Psi}-1)$ and in $\{\eta\neq 1\}$, $|\Psi|\leqslant\frac{|\Phi|}{|V|}\leqslant K\|\Phi\|_{L^\infty(\mathbb{R}^2)}\leqslant KC_0$ by our assumption $\|\Phi\|_{L^\infty(\mathbb{R}^2)}\leqslant C_0$, therefore, choosing C_0 small enough, in $\{\eta\neq 1\}$, we have $|e^{\Psi}-1|\leqslant 1/2$. We also remark that

$$\frac{(1-\eta)e^{\Psi}}{(\eta+(1-\eta)e^{\Psi})}=(1-\eta)+\eta(1-\eta)\left(\frac{e^{\Psi}-1}{\eta+(1-\eta)\,e^{\Psi}}\right),$$

therefore (2.1.8) become

$$\begin{aligned}&E-ic\partial_{x_2}V+VL'(\Psi)\\ &+V(1-\eta)(-\nabla\Psi.\nabla\Psi+|V|^2S(\Psi))\\ &+\frac{\eta}{(\eta+(1-\eta)\,e^{\Psi})}((E-ic\partial_{x_2}V)\Psi+2|V|^2V\mathfrak{Re}(\Psi)\Psi+|V\Psi|^2V(1+\Psi))\\ &+R_1(\Psi) = 0,\end{aligned}$$

where

$$\begin{aligned}R_1(\Psi) &:= \frac{1}{(\eta+(1-\eta)\,e^{\Psi})}(-ic\partial_{x_2}\eta V\zeta-V\Delta\eta\zeta-2\nabla\eta.\nabla(V\zeta)-\eta(1-\eta)(V\zeta G(\Psi)+\overline{V\zeta}H(\Psi)))\\ &+ V\eta(1-\eta)\left(\frac{e^{\Psi}-1}{\eta+(1-\eta)\,e^{\Psi}}\right)(-\nabla\Psi.\nabla\Psi+|V|^2S(\Psi)).\end{aligned}$$

Remark that $R_1(\Psi)$ is nonzero only in the rings where $\eta(1-\eta)\neq 0$, i.e. $1\leqslant\tilde{r}\leqslant 2$, since every term has either $\partial_{x_2}\eta,\Delta\eta$ or $\eta(1-\eta)$ as a factor. Furthermore they all have as an additional factor $\zeta,\nabla\zeta$, S or $\nabla\Psi.\nabla\Psi$. Hence, if we suppose that $|\Psi|,|\nabla\Psi|,|\nabla^2\Psi|\leqslant KC_0$ in the rings (which is a consequence of $\Phi=V\Psi$ and $\|\Phi\|_{C^2(\mathbb{R}^2)}\leqslant C_0$), then those terms can be bounded by $C\|\Psi\|^2_{C^1(\{1\leqslant\tilde{r}\leqslant 2\})}$. Therefore if $|\Psi|,|\nabla\Psi|,|\nabla^2\Psi|\leqslant KC_0$ in the rings, then

$$|R_1(\Psi)|+|\nabla R_1(\Psi)|\leqslant K\|\Psi\|^2_{C^2(\{1\leqslant\tilde{r}\leqslant 2\})}\leqslant K\|\Phi\|^2_{C^2(\{\tilde{r}\leqslant 2\})}$$

for some universal constant $K>0$, since in the rings, V is bounded from below by a nonzero constant. Now, we use

$$\frac{\eta}{(\eta+(1-\eta)\,e^{\Psi})}=\eta+\eta(1-\eta)\,\frac{1-e^{\Psi}}{\eta+(1-\eta)\,e^{\Psi}}$$

to compute

$$\frac{\eta}{(\eta+(1-\eta)\,e^{\Psi})}(E-ic\partial_{x_2}V)\,\Psi=\eta(E-ic\partial_{x_2}V)\Psi+R_2(\Psi),$$

where

$$R_2(\Psi):=\eta(1-\eta)\,\frac{(1-e^{\Psi})(E-ic\partial_{x_2}V)}{\eta+(1-\eta)\,e^{\Psi}}\Psi.$$

We show easily that $R_2(\Psi)$ satisfies the same estimates as $R_1(\Psi)$. Remark that, using $\Phi=V\Psi$,

$$\begin{aligned}\left|\frac{\eta}{(\eta+(1-\eta)\,e^{\Psi})}(2|V|^2V\mathfrak{Re}(\Psi)\Psi+|V\Psi|^2V(1+\Psi))\right| &= \\ \left|\frac{\eta}{(\eta+(1-\eta)\,e^{\Psi})}(2\mathfrak{Re}(\Phi\bar{V})\Phi+|\Phi|^2(V+\Phi))\right| &\leqslant K\|\Phi\|^2_{C^1(\{\tilde r\leqslant 2\})}\end{aligned}$$

and

$$\left|\nabla\left(\frac{\eta}{(\eta+(1-\eta)\,e^{\Psi})}(2\mathfrak{Re}(\Phi\bar{V})\Phi+|\Phi|^2(V+\Phi))\right)\right|\leqslant K\|\Phi\|^2_{C^1(\{\tilde r\leqslant 2\})}$$

if $\|\Phi\|_{L^\infty(\mathbb{R}^2)}\leqslant C_0$ (so that the term in e^{Ψ} is bounded) since $\eta\neq 0$ only if $\tilde r\leqslant 2$. We define

$$R(\Psi):=R_1(\Psi)+R_2(\Psi)+\frac{\eta}{(\eta+(1-\eta)\,e^{\Psi})}(2|V|^2V\mathfrak{Re}(\Psi)\Psi+|V\Psi|^2V(1+\Psi)),$$

which satisfies

$$|R(\Psi)|,|\nabla(R(\Psi))|\leqslant K\|\Phi\|^2_{C^2(\{\tilde r\leqslant 2\})}$$

for some universal constant $K>0$, provided that $\|\Phi\|_{C^2(\mathbb{R}^2)}\leqslant C_0$. The equation (2.1.8) then becomes

$$\begin{aligned}E-ic\partial_{x_2}V+VL'(\Psi)+V(1-\eta)(-\nabla\Psi.\nabla\Psi+|V|^2S(\Psi))& \\ +\eta(E-ic\partial_{x_2}V)\Psi+R(\Psi) &= 0.\end{aligned}$$

Now we finish by using $-icV\partial_{x_2}\Psi=-\eta icV\partial_{x_2}\Psi-(1-\eta)icV\partial_{x_2}\Psi$ and

$$\partial_{x_2}V\Psi+\partial_{x_2}\Psi V=\partial_{x_2}\Phi$$

to obtain

$$VL'(\Psi)+\eta\,(E-ic\partial_{x_2}V)\Psi-ic\eta\partial_{x_2}\Phi+V(1-\eta)(-\nabla\Psi.\nabla\Psi+|V|^2S(\Psi))+R(\Psi)=0.$$

Finally, since we have shown that $L(\Phi)=(E-ic\partial_{x_2}V)\Psi+L'(\Psi)V$, we infer

$$VL'(\Psi)+\eta(E-ic\partial_{x_2}V)\,\Psi=\eta L(\Phi)+(1-\eta)VL'(\Psi).$$

The proof is complete. □

2.1.3 Setup of the norms

For a given $\sigma\in\mathbb{R}$, we define, similarly as in [11] and [29], for $\Psi=\Psi_1+i\Psi_2$ and $h=h_1+ih_2$, the norms

$$\begin{aligned}\|\Psi\|_{*,\sigma,d} &:= \|V\Psi\|_{C^2(\{\tilde r\leqslant 3\})}\\ &+ \|\tilde r^{1+\sigma}\Psi_1\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\nabla\Psi_1\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\nabla^2\Psi_1\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ &+ \|\tilde r^{\sigma}\Psi_2\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{1+\sigma}\nabla\Psi_2\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\nabla^2\Psi_2\|_{L^\infty(\{\tilde r\geqslant 2\})},\end{aligned}$$

$$\begin{aligned}\|h\|_{**,\sigma,d} &:= \|Vh\|_{C^1(\{\tilde r\leqslant 3\})}\\ &+ \|\tilde r^{1+\sigma}h_1\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\nabla h_1\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ &+ \|\tilde r^{2+\sigma}h_2\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\nabla h_2\|_{L^\infty(\{\tilde r\geqslant 2\})},\end{aligned}$$

where $\tilde{r} = \min(r_1, r_{-1})$ (which depends on d). These are the spaces we shall use for the inversion of the linear operator for suitable values of σ.

This norm is not the "natural" energy norm that we could expect, for instance:

$$\|\Phi\|_{H_V}^2 := \int_{\mathbb{R}^2} |\nabla\Phi|^2 + (1-|V|^2)|\Phi|^2 + \mathfrak{Re}(\bar{V}\Phi)^2.$$

In particular, we require different conditions on the decay at infinity (with, in a way, less decay). As a consequence, the decay we have in Theorem 1.3.1 is not optimal (see [22]). This decay will be recovered later on by showing that the solution has finite energy. The main advantage of the norms $\|.\|_{*,\sigma,d}$ and $\|.\|_{**,\sigma,d}$ is that they will allow us to have uniform estimates on the error, without constants depending on c or d.

We are looking for a solution Ψ on a space of symmetric functions: we suppose that

$$\forall x = (x_1, x_2) \in \mathbb{R}^2, \Psi(x_1, x_2) = \overline{\Psi(x_1, -x_2)} = \Psi(-x_1, x_2)$$

because V and the equation has the same symmetries. With only those symmetries we will not be able to invert the linearized operator because it has a kernel, we also need an orthogonal condition. We define

$$Z_d(x) := \partial_d V(x)(\tilde{\eta}(4r_1) + \tilde{\eta}(4r_{-1})),$$

where $\tilde{\eta}$ is the same function as the one used for v: it is a C^∞ non negative smooth cutoff with $\tilde{\eta}(r) = 1$ if $r \leqslant 1$ and 0 if $r \geqslant 2$. In particular $Z_d(x) = 0$ if $\tilde{r} \geqslant 1/2$, which will make some computations easier. The other interest of the cutoff function is that without it

$$\partial_d V(x) = -\partial_{x_1} V_1 V_{-1} + \partial_{x_1} V_{-1} V_1$$

is not integrable in all $\mathbb{R}^2$. We define the Banach spaces we shall use for inverting the linear part:

$$\mathcal{E}_{*,\sigma,d} :=$$

$$\left\{\Phi = V\Psi \in C^2(\mathbb{R}^2, \mathbb{C}), \|\Psi\|_{*,\sigma,d} < +\infty; \langle \Phi, Z_d\rangle = 0; \forall x \in \mathbb{R}^2, \Psi(x_1, x_2) = \overline{\Psi(x_1, -x_2)} = \Psi(-x_1, x_2)\right\},$$

$$\mathcal{E}_{**,\sigma',d} := \left\{Vh \in C^1(\mathbb{R}^2, \mathbb{C}), \|h\|_{**,\sigma',d} < +\infty\right\}$$

for $\sigma, \sigma' \in \mathbb{R}$. We shall omit the subscript d in the construction and use only $\mathcal{E}_{*,\sigma}$, $\mathcal{E}_{**,\sigma'}$. Remark that $\mathcal{E}_{*,\sigma}$ contains an orthogonality condition as well as the symmetries.

Our first goal is to invert the linearized operator. This is a difficult part, which requires a lot of computations and critical elliptic estimates. The next subsection is devoted to the proof of the elliptic tools use in the proof of the inversion. In particular, our paper diverges here from [29] (see Remark 2.1.11 thereafter).

2.1.4 Some elliptic estimates

In this subsection, we provide some tools for elliptic estimate adapted to L^∞ norms.

2.1.4.1 Weighted L^∞ estimates on a Laplacian problem

Lemma 2.1.8. *For $d \geqslant 5$, $0 < \alpha < 1$, there exists a constant $K(\sigma) > 0$ such that, for $f \in C^0(\mathbb{R}^2, \mathbb{C})$ such that*

$$\forall (x_1, x_2) \in \mathbb{R}^2, \quad f(x_1, x_2) = -f(x_1, -x_2)$$

and with

$$\varepsilon_{f,\alpha} := \|f(x)(1+\tilde{r})^{2+\alpha}\|_{L^\infty(\mathbb{R}^2)} < +\infty,$$

there exists a unique $C^1(\mathbb{R}^2)$ function ζ such that

$$\Delta\zeta = f$$

in the distribution sense,

$$\forall (x_1, x_2) \in \mathbb{R}^2, \quad \zeta(x_1, x_2) = -\zeta(x_1, -x_2)$$

and ζ satisfies the following two estimates:

$$\forall x\in\mathbb{R}^2, |\zeta(x)|\leqslant \frac{K(\sigma)\varepsilon_{f,\alpha}}{(1+\tilde{r})^{\alpha}}$$

and

$$\forall x\in\mathbb{R}^2, |\nabla\zeta(x)|\leqslant \frac{K(\sigma)\varepsilon_{f,\alpha}}{(1+\tilde{r})^{1+\alpha}}.$$

Remark here that for a given function f, if it satisfies two inequalities with different values of $(\varepsilon_{f,\alpha},\ \alpha)$, then the associated function ζ satisfies the estimates with both sets of values by uniqueness. Furthermore, with only the hypothesis $f\in C^0(\mathbb{R}^2)$, we do not have $\zeta\in C^2_{\rm loc}(\mathbb{R}^2)$ a priori.

Proof. The uniqueness of such a function ζ is a consequence of the fact that ζ is bounded (by $\forall x\in\mathbb{R}^2, |\zeta(x)|\leqslant\frac{K\varepsilon_{f,\alpha}}{(1+\tilde{r})^{\alpha}}$), the linearity of the Laplacian, and that the only weak solution to $\Delta\zeta=0$ that tends to 0 at infinity is 0. We define

$$\zeta := G*f,$$

where G is the fundamental solution of the Laplacian in dimension 2, namely $G(x):=\frac{1}{2\pi}\ln(|x|)$. Since $\|f(x)(1+\tilde{r})^{2+\alpha}\|_{L^\infty(\mathbb{R}^2)}<+\infty$, we check that ζ is well defined. Let us show that $\zeta\in C^1(\mathbb{R}^2,\mathbb{C})$. If $f\in C^\infty_c(\mathbb{R}^2)$, then, for $j\in\{1,2\}$,

$$\begin{aligned}\frac{\zeta(x+h\vec{e}_j)-\zeta(x)}{|h|} &= \frac{1}{2\pi}\int_{\mathbb{R}^2}\ln(|x-Y|))\frac{f(Y+h\vec{e}_j)-f(Y)}{|h|}\,dY\\ &\to \frac{1}{2\pi}\int_{\mathbb{R}^2}\ln(|x-Y|))\partial_{Y_j}f(Y)\,dY\end{aligned}$$

when $|h|\to 0$. Then, for $\varepsilon>0$,

$$\left|\frac{1}{2\pi}\int_{B(x,\varepsilon)}\ln(|x-Y|))\partial_{Y_j}f(Y)\,dY\right|\leqslant K\varepsilon^2|\ln(\varepsilon)|\|\nabla f\|_{L^\infty(\mathbb{R}^2)}$$

and by integration by parts,

$$\begin{aligned}\frac{1}{2\pi}\int_{\mathbb{R}^2\setminus B(x,\varepsilon)}\ln(|x-Y|))\partial_{Y_j}f(Y)\,dY &= \frac{1}{2\pi}\int_{\mathbb{R}^2\setminus B(x,\varepsilon)}\frac{x_j-Y_j}{|x-Y|^2}f(Y)dY\\ &- \frac{1}{2\pi}\int_{\partial B(x,\varepsilon)}\ln(|x-Y|))\,f(Y)\vec{e}_j.\vec{\nu}\,d\sigma\end{aligned}$$

and since $\left|\frac{1}{2\pi}\int_{\partial B(x,\varepsilon)}\ln(|x-Y|))\,f(Y)\vec{e}_j.\vec{\nu}\,d\sigma\right|\leqslant K\|f\|_{L^\infty(\mathbb{R}^2)}\varepsilon|\ln(\varepsilon)|$, taking $\varepsilon\to 0$ we deduce that

$$\frac{\zeta(x+h\vec{e}_j)-\zeta(x)}{|h|}\to\frac{1}{2\pi}\int_{\mathbb{R}^2}\ln(|x-Y|))\partial_{Y_j}f(Y)\,dY=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x_j-Y_j}{|x-Y|^2}f(Y)dY$$

when $|h|\to 0$. This implies that, for $f\in C^\infty_c(\mathbb{R}^2)$,

$$\nabla\zeta(x)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-Y}{|x-Y|^2}f(Y)dY.$$

Now, for $f\in C^0(\mathbb{R}^2,\mathbb{C})$ such that $\|f(x)(1+\tilde{r})^{2+\alpha}\|_{L^\infty(\mathbb{R}^2)}<+\infty$, we take $f_n\in C^\infty_c(\mathbb{R}^2,\mathbb{C})$ such that $f_n\to f$ in $L^3(\mathbb{R}^2)$ and $(1+\tilde{r})^{\alpha/2}f_n\to(1+\tilde{r})^{\alpha/2}f$ in $L^1(\mathbb{R}^2)$ (we check easily that $f\in L^3(\mathbb{R}^2)$ and $(1+\tilde{r})^{\alpha/2}f\in L^1(\mathbb{R}^2)$). In particular, $f_n\to f$ in $L^1(\mathbb{R}^2)$. Then, for ζ_n such that $\Delta\zeta_n=f_n$, we check that, by Hölder inequality,

$$\left|\nabla\zeta_n(x)-\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-Y}{|x-Y|^2}f(Y)dY\right|\leqslant\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{|f_n(Y)-f(Y)|}{|x-Y|}dY,$$

$$\int_{\{|x-Y|\leqslant 1\}}\frac{|f_n(Y)-f(Y)|}{|x-Y|}dY\leqslant\|f_n-f\|_{L^3(\mathbb{R}^2)}\left(\int_{\{|x-Y|\leqslant 1\}}\frac{dY}{|x-Y|^{3/2}}\right)^{2/3}\leqslant K\|f_n-f\|_{L^3(\mathbb{R}^2)}$$

and

$$\int_{\{|x-Y|\geqslant 1\}} \frac{|f_n(Y)-f(Y)|}{|x-Y|}dY \leqslant \|f_n-f\|_{L^1(\mathbb{R}^2)},$$

therefore $\nabla\zeta_n \to \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-Y}{|x-Y|^2}f(Y)dY$ uniformly in $\mathbb{R}^2$.

Similarly, we estimate

$$\left|\zeta_n(x)-\frac{1}{2\pi}\int_{\mathbb{R}^2}\ln(|x-Y|)f(Y)dY\right| \leqslant \frac{1}{2\pi}\int_{\mathbb{R}^2}|f_n(Y)-f(Y)|\,|\ln(|x-Y|)|dY,$$

$$\begin{aligned}\int_{\{|x-Y|\leqslant 1\}}|f_n(Y)-f(Y)|\,|\ln(|x-Y|)|dY &\leqslant \|f_n-f\|_{L^3(\mathbb{R}^2)}\left(\int_{\{|x-Y|\leqslant 1\}}|\ln(|x-Y|)|^{3/2}dY\right)^{2/3}\\ &\leqslant K\|f_n-f\|_{L^3(\mathbb{R}^2)}\end{aligned}$$

and

$$\int_{\{|x-Y|\geqslant 1\}}|f_n(Y)-f(Y)||\ln(|x-Y|)|dY \leqslant K\|(1+\tilde{r})^{\alpha/2}f_n-(1+\tilde{r})^{\alpha/2}f\|_{L^1(\mathbb{R}^2)},$$

thus $\zeta_n \to G*f=\zeta$ uniformly in $\mathbb{R}^2$, which implies by differentiation of a sequence of functions, that $\zeta\in C^1(\mathbb{R}^2,\mathbb{C})$ and

$$\nabla\zeta(x)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-Y}{|x-Y|^2}f(Y)dY.$$

We check that ζ satisfies

$$\Delta\zeta=f$$

in the distribution sense. Indeed, for $\varphi\in C_c^\infty(\mathbb{R}^2)$, (see [12], chapter 2, Theorem 1)

$$\int_{\mathbb{R}^2}(G*f)\Delta\varphi=\int_{\mathbb{R}^2}f(G*\Delta\varphi)=\int_{\mathbb{R}^2}f\varphi.$$

It is also easy to check that

$$\forall(x_1,x_2)\in\mathbb{R}^2,\quad \zeta(x_1,x_2)=-\zeta(x_1,-x_2).$$

Now, if $|x-d\vec{e}_1|\leqslant 1$, we check that

$$|\nabla\zeta(x)|\leqslant K\int_{\mathbb{R}^2}\frac{1}{|Y|}|f(x-Y)|dY \leqslant K\varepsilon_{f,\alpha}\int_{\mathbb{R}^2}\frac{dY}{|Y|(1+\tilde{r}(Y-x))^{2+\alpha}}\leqslant K\varepsilon_{f,\alpha},$$

and, similarly,

$$|\zeta(x)|\leqslant K\varepsilon_{f,\alpha},$$

which is enough to show the required estimate of this lemma for these values of x. We can make the same estimate if $|x+d\vec{e}_1|\leqslant 1$, we therefore suppose from now on that $|x-d\vec{e}_1|,|x+d\vec{e}_1|\geqslant 1$.

First, let us show that

$$\int_{\{Y_1\geqslant 0\}}f(Y)dY=\int_{\{Y_1\leqslant 0\}}f(Y)dY=0. \tag{2.1.9}$$

The integrals are well defined because $|f(x)|\leqslant\frac{\varepsilon_{f,\alpha}}{(1+\tilde{r})^{2+\alpha}}$ and therefore f is integrable. Since f is odd with respect to x_2, (2.1.9) holds. We deduce that

$$\begin{aligned}|\nabla\zeta(x)| &\leqslant \frac{1}{2\pi}\left|\int_{\{Y_1\geqslant 0\}}\left(\frac{x-Y}{|x-Y|^2}-\frac{x-d\vec{e}_1}{|x-d\vec{e}_1|^2}\right)f(Y)dY\right|\\ &+ \frac{1}{2\pi}\left|\int_{\{Y_1\leqslant 0\}}\left(\frac{x-Y}{|x-Y|^2}-\frac{x+d\vec{e}_1}{|x+d\vec{e}_1|^2}\right)f(Y)dY\right|.\end{aligned}$$

Now, using $|f(x)| \leqslant \frac{\varepsilon_{f,\alpha}}{(1+\tilde{r})^{2+\alpha}}$, we estimate

$$\begin{aligned} 2\pi|\nabla\zeta(x)| &\leqslant \varepsilon_{f,\alpha}\int_{\{Y_1\geqslant 0\}}\left|\frac{x-Y}{|x-Y|^2}-\frac{x-d\vec{e_1}}{|x-d\vec{e_1}|^2}\right|\frac{dY}{(1+r_1(Y))^{2+\alpha}} \\ &+ \varepsilon_{f,\alpha}\int_{\{Y_1\leqslant 0\}}\left|\frac{x-Y}{|x-Y|^2}-\frac{x+d\vec{e_1}}{|x+d\vec{e_1}|^2}\right|\frac{dY}{(1+r_{-1}(Y))^{2+\alpha}}. \end{aligned}$$

By the change of variable $Y = Z + d\vec{e_1}$, we have

$$\begin{aligned} &\int_{\{Y_1\geqslant 0\}}\left|\frac{x-Y}{|x-Y|^2}-\frac{x-d\vec{e_1}}{|x-d\vec{e_1}|^2}\right|\frac{dY}{(1+r_1(Y))^{2+\alpha}} \\ =\ &\int_{\{Z_1\geqslant -d\}}\left|\frac{(x-d\vec{e_1})-Z}{|(x-d\vec{e_1})-Z|^2}-\frac{x-d\vec{e_1}}{|x-d\vec{e_1}|^2}\right|\frac{dZ}{(1+|Z|)^{2+\alpha}}, \\ \leqslant\ &\int_{\mathbb{R}^2}\left|\frac{(x-d\vec{e_1})-Z}{|(x-d\vec{e_1})-Z|^2}-\frac{x-d\vec{e_1}}{|x-d\vec{e_1}|^2}\right|\frac{dZ}{(1+|Z|)^{2+\alpha}}. \end{aligned}$$

Now, if $|Z| \geqslant 2|x-d\vec{e_1}|$, by triangular inequality, we check that

$$\left|\frac{(x-d\vec{e_1})-Z}{|(x-d\vec{e_1})-Z|^2}-\frac{x-d\vec{e_1}}{|x-d\vec{e_1}|^2}\right| \leqslant \frac{K}{|x-d\vec{e}_1|},$$

hence

$$\begin{aligned} &\int_{\{|Z|\geqslant 2|x-d\vec{e_1}|\}}\left|\frac{(x-d\vec{e_1})-Z}{|(x-d\vec{e_1})-Z|^2}-\frac{x-d\vec{e_1}}{|x-d\vec{e_1}|^2}\right|\frac{dZ}{(1+|Z|)^{2+\alpha}} \\ \leqslant\ &\frac{K}{|x-d\vec{e_1}|}\int_{\{|Z|\geqslant 2|x-d\vec{e_1}|\}}\frac{dZ}{(1+|Z|)^{2+\alpha}} \leqslant \frac{K(\alpha)}{|x-d\vec{e_1}|^{1+\alpha}}. \end{aligned} \tag{2.1.10}$$

We now work for $|Z| \leqslant 2|x-d\vec{e_1}|$. We remark that

$$\begin{aligned} &\left|\frac{(x-d\vec{e_1})-Z}{|(x-d\vec{e_1})-Z|^2}-\frac{x-d\vec{e_1}}{|x-d\vec{e_1}|^2}\right||x-d\vec{e_1}|^2|(x-d\vec{e_1})-Z|^2 \\ =\ &|(x-d\vec{e_1})(|x-d\vec{e_1}|^2-|(x-d\vec{e_1})-Z|^2)-Z|x-d\vec{e_1}|^2| \\ =\ &|(x-d\vec{e_1})(2(x-d\vec{e_1}).\bar{Z}-|Z|^2)-Z|x-d\vec{e_1}|^2| \\ =\ &|((x-d\vec{e_1})-Z)(2(x-d\vec{e_1}).\bar{Z}-|Z|^2)-Z|(x-d\vec{e_1})-Z|^2| \\ =\ &|(x-d\vec{e_1})-Z||Z|\left|\frac{(x-d\vec{e_1})-Z}{|(x-d\vec{e_1})-Z|}\left(2(x-d\vec{e_1}).\frac{\bar{Z}}{|Z|}-|Z|\right)-\frac{Z}{|Z|}|(x-d\vec{e_1})-Z|\right|, \end{aligned}$$

and we estimate

$$\begin{aligned} &\left|\frac{(x-d\vec{e_1})-Z}{|(x-d\vec{e_1})-Z|}\left(2(x-d\vec{e_1}).\frac{\bar{Z}}{|Z|}-|Z|\right)-\frac{Z}{|Z|}|(x-d\vec{e_1})-Z|\right| \\ \leqslant\ &2|x-d\vec{e_1}|+\left|\frac{(x-d\vec{e_1})-Z}{|(x-d\vec{e_1})-Z|}(-|Z|)-\frac{Z}{|Z|}|(x-d\vec{e_1})-Z|\right|. \end{aligned}$$

Furthermore,

$$\begin{aligned} &\left|\frac{(x-d\vec{e_1})-Z}{|(x-d\vec{e_1})-Z|}(-|Z|)-\frac{Z}{|Z|}|(x-d\vec{e_1})-Z|\right|^2|Z|^2|(x-d\vec{e_1})-Z|^2 \\ =\ &|\,((x-d\vec{e_1})-Z)|Z|^2+Z|(x-d\vec{e_1})-Z|^2\,|^2 \\ =\ &|(x-d\vec{e_1})-Z|^2|Z|^4+|Z|^2|(x-d\vec{e_1})-Z|^4+2(x-d\vec{e_1}-Z).\bar{Z}|Z|^2|(x-d\vec{e_1})-Z|^2 \\ =\ &|(x-d\vec{e_1})-Z|^2|Z|^2(-|Z|^2+|(x-d\vec{e_1})-Z|^2+2(x-d\vec{e_1}).\bar{Z}) \\ =\ &|(x-d\vec{e_1})-Z|^2|Z|^2|x-d\vec{e_1}|^2, \end{aligned}$$

therefore

$$\left|\frac{(x-d\vec{e_1})-Z}{|(x-d\vec{e_1})-Z|^2}-\frac{x-d\vec{e_1}}{|x-d\vec{e_1}|^2}\right| \leqslant \frac{3|Z|}{|x-d\vec{e_1}|\times|(x-d\vec{e_1})-Z|}.$$

We deduce that

$$\begin{aligned} &\int_{\{|Z|\leqslant 2|x-d\vec{e_1}|\}}\left|\frac{(x-d\vec{e_1})-Z}{|(x-d\vec{e_1})-Z|^2}-\frac{x-d\vec{e_1}}{|x-d\vec{e_1}|^2}\right|\frac{dZ}{(1+|Z|)^{2+\alpha}} \\ \leqslant\ &\frac{3}{|x-d\vec{e_1}|}\int_{\{|Z|\leqslant 2|x-d\vec{e_1}|\}}\frac{|Z|dZ}{|(x-d\vec{e_1})-Z|(1+|Z|)^{2+\alpha}}. \end{aligned}$$

We remark that, either $|(x-d\vec{e_1})-Z|\geqslant\frac{|x-d\vec{e_1}|}{2}$, and then

$$\begin{aligned}&\int_{\{|Z|\leqslant 2|x-d\vec{e_1}|\}\cap\left\{|(x-d\vec{e_1})-Z|\geqslant\frac{|x-d\vec{e_1}|}{2}\right\}}\frac{|Z|dZ}{|(x-d\vec{e_1})-Z|(1+|Z|)^{2+\alpha}}\\ \leqslant\ &\frac{2}{|x-d\vec{e_1}|}\int_{\{|Z|\leqslant 2|x-d\vec{e_1}|\}\cap\left\{|(x-d\vec{e_1})-Z|\geqslant\frac{|x-d\vec{e_1}|}{2}\right\}}\frac{|Z|dZ}{(1+|Z|)^{2+\alpha}}\\ \leqslant\ &\frac{K(\alpha)}{|x-d\vec{e_1}|^{\alpha}}\end{aligned}$$

since $\alpha<1$, or $|(x-d\vec{e_1})-Z|\leqslant\frac{|x-d\vec{e_1}|}{2}$, and then $|Z|\geqslant\frac{|x-d\vec{e_1}|}{2}$, therefore

$$\begin{aligned}&\int_{\{|Z|\leqslant 2|x-d\vec{e_1}|\}\cap\left\{|(x-d\vec{e_1})-Z|\leqslant\frac{|x-d\vec{e_1}|}{2}\right\}}\frac{|Z|dZ}{|(x-d\vec{e_1})-Z|(1+|Z|)^{2+\alpha}}\\ \leqslant\ &\int_{\left\{\frac{|x-d\vec{e_1}|}{2}\leqslant|Z|\leqslant 2|x-d\vec{e_1}|\right\}}\frac{|Z|dZ}{|(x-d\vec{e_1})-Z|(1+|Z|)^{2+\alpha}}\\ \leqslant\ &\frac{K}{|x-d\vec{e_1}|^{2+\alpha}}\int_{\{|Z-(x-d\vec{e_1})|\leqslant 3|x-d\vec{e_1}|\}}\frac{|Z|dZ}{|(x-d\vec{e_1})-Z|}\\ \leqslant\ &\frac{K}{|x-d\vec{e_1}|^{\alpha}}.\end{aligned}$$

We conclude that

$$\int_{\{|Z|\leqslant 2|x-d\vec{e_1}|\}}\left|\frac{(x-d\vec{e_1})-Z}{|(x-d\vec{e_1})-Z|^2}-\frac{x-d\vec{e_1}}{|x-d\vec{e_1}|^2}\right|\frac{dZ}{(1+|Z|)^{2+\alpha}}\leqslant\frac{K(\alpha)}{|x-d\vec{e_1}|^{1+\alpha}}.\tag{2.1.11}$$

Combining (2.1.9) and (2.1.11), and by symmetry, we deduce that

$$\begin{aligned}&\int_{\{Y_1\geqslant 0\}}\left|\frac{x-Y}{|x-Y|^2}-\frac{x-d\vec{e_1}}{|x-d\vec{e_1}|^2}\right|\frac{dY}{(1+r_1(Y))^{2+\alpha}}\\ +\ &\int_{\{Y_1\leqslant 0\}}\left|\frac{x-Y}{|x-Y|^2}-\frac{x+d\vec{e_1}}{|x+d\vec{e_1}|^2}\right|\frac{dY}{(1+r_{-1}(Y))^{2+\alpha}}\\ \leqslant\ &\frac{K(\alpha)}{|x-d\vec{e_1}|^{1+\alpha}}+\frac{K(\alpha)}{|x+d\vec{e_1}|^{1+\alpha}}\\ \leqslant\ &\frac{K(\alpha)}{\tilde{r}(x)^{1+\alpha}},\end{aligned}$$

and therefore (recall that $|x-d\vec{e_1}|,|x+d\vec{e_1}|\geqslant 1$),

$$|\nabla\zeta(x)|\leqslant\frac{K\varepsilon_{f,\alpha}}{(1+\tilde{r}(x))^{1+\alpha}}.$$

Now, let us show that $\zeta(x)\to 0$ when $|x|\to\infty$. We recall that

$$\zeta(x)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\ln(|x-Y|)f(Y)dY,$$

and since $\int_{\mathbb{R}^2}f(Y)dY=0$, for large values of x (in particular $|x|\gg d$),

$$\zeta(x)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\ln\left(\frac{|x-Y|}{|x|}\right)f(Y)dY.$$

If $|x-Y|\leqslant 1$, then $|f(Y)|\leqslant\frac{K\varepsilon_{f,\alpha}}{(1+|x|)^{2+\sigma}}$, hence

$$\begin{aligned}\int_{\{|x-Y|\leqslant 1\}}\left|\ln\left(\frac{|x-Y|}{|x|}\right)f(Y)\right|\ &\leqslant\ \frac{K\varepsilon_{f,\alpha}}{(1+|x|)^{2+\sigma}}\int_{\{|x-Y|\leqslant 1\}}|\ln(|x-Y|)-\ln(|x|)|\\ &\leqslant\ \frac{K\varepsilon_{f,\alpha}(1+\ln(|x|))}{(1+|x|)^{2+\sigma}}\to 0\end{aligned}$$

when $x\to\infty$. If $|x-Y|\geqslant 1$, then $\ln\Big(\frac{|x-Y|}{|x|}\Big)\to 0$ when $|x|\to\infty$ and we recall that f is bounded in L^∞. We have, for $|x|\geqslant 2$ that $|x-Y|\leqslant |x|(|y|+2)$ and therefore, for $|x-Y|\geqslant 1$, $|x|\geqslant 2$, $\Big|\ln\Big(\frac{|x-Y|}{|x|}\Big)\Big|\leqslant K\ln(|y|+2)$, hence

$$\left|1_{\{|x-Y|\geqslant 1\}}\ln\left(\frac{|x-Y|}{|x|}\right)f(Y)\right|\leqslant K\ln(|Y|+2)f(Y)\in L^1(\mathbb{R}^2,\mathbb{C}).$$

By dominated convergence theorem, we deduce that $\zeta(x)\to 0$ when $|x|\to\infty$. Now, to estimate ζ, we integrate from infinity. For instance, in the case $x_1\geqslant 0, x_2\geqslant 0$, we estimate

$$|\zeta(x)|\leqslant\left|\int_{x_2}^{+\infty}\partial_{x_2}\zeta(x_1,t)dt\right|\leqslant K\varepsilon_{f,\alpha}\int_{x_2}^{+\infty}\frac{dt}{(1+|x_1-d\vec{e_1}|+t)^{1+\alpha}}\leqslant\frac{K\varepsilon_{f,\alpha}}{\alpha(1+\tilde{r}(x))^\alpha}.$$ □

2.1.4.2 Fundamental solution for $-\Delta+2$

We will use the fundamental solution of $-\Delta+1$ and its following properties.

Lemma 2.1.9. ([1]) *The fundamental solution of $-\Delta+1$ in $\mathbb{R}^2$ is $\frac{1}{2\pi}K_0(|.|)$, where K_0 is the modified Bessel function of second kind. It satifies $K_0\in C^\infty(\mathbb{R}^{+*})$ and*

$$K_0(r)\sim_{r\to\infty}\sqrt{\frac{\pi}{2r}}e^{-r},$$

$$K_0(r)\sim_{r\to 0}-\ln(r),$$

$$K_0'(r)\sim_{r\to\infty}-\sqrt{\frac{\pi}{2r}}e^{-r},$$

$$K_0'(r)\sim_{r\to 0}\frac{-1}{r},$$

$$\forall r>0, K_0(r)>0, K_0'(r)<0\quad\text{and}\quad K_0''(r)>0.$$

Proof. The first three equivalents are respectively equations 9.7.2, 9.6.8 and 9.7.4 of [1]. The fourth one can be deduced from equations 9.6.27 and 9.6.9 of [1]. For $\nu\in\mathbb{N}$, K_ν is $C^\infty(\mathbb{R},\mathbb{R})$ since it solves 9.6.1 of [1] and from the end of 9.6 of [1], we have that K_ν has no zeros. In particular with the asymptotics of 9.6.8, this implies that $K_\nu(r)>0$. Furthermore, from 9.6.27 of [1], we have $K_0'=-K_1<0$ and $K_0''=-K_1'=\frac{K_0+K_2}{2}>0$. □

We end this subsection by the proof an elliptic estimate that will be used in the proof of Proposition 2.1.17.

Lemma 2.1.10. *For any $\alpha>0$, there exists a constant $C(\alpha)>0$ such that, for any $d>1$, if two real-valued functions $\Psi\in H^1(\mathbb{R}^2), h\in C^0(\mathbb{R}^2)$ satisfy in the distribution sense*

$$(-\Delta+2)\Psi=h,$$

and

$$\|(1+\tilde{r})^\alpha h\|_{L^\infty(\mathbb{R}^2)}<+\infty,$$

then $\Psi\in C^1(\mathbb{R}^2)$ with

$$|\Psi|+|\nabla\Psi|\leqslant\frac{C(\alpha)\|(1+\tilde{r})^\alpha h\|_{L^\infty(\mathbb{R}^2)}}{(1+\tilde{r})^\alpha}.$$

Proof. The fundamental solution of $-\Delta+2$ in $\mathbb{R}^2$ is $\frac{1}{2\pi}K_0\big(\sqrt{2}|.|\big)$ where K_0 is the modified Bessel function of the second kind with the properties described in Lemma 2.1.9. Since $\Psi\in H^1(\mathbb{R}^2)$ and the equation $-\Delta+2$ is strictly elliptic, we have

$$\Psi=\frac{1}{2\pi}K_0\big(\sqrt{2}|.|\big)*h,$$

therefore (using $K_0 \geqslant 0$), for $x \in \mathbb{R}^2$,

$$|\Psi(x)| \leqslant K\|(1+\tilde{r})^{\alpha}h\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2} K_0\big(\sqrt{2}|x-Y|\big)\frac{dY}{(1+\tilde{r}(Y))^{\alpha}}.$$

If $|x-d\vec{e}_1| \leqslant 1$ or $|x+d\vec{e}_1| \leqslant 1$, we have

$$\int_{\mathbb{R}^2} K_0\big(\sqrt{2}|x-Y|\big)\frac{1}{(1+\tilde{r}(Y))^{\alpha}}dY \leqslant \int_{\mathbb{R}^2} K_0\big(\sqrt{2}|x-Y|\big)dY \leqslant \int_{\mathbb{R}^2} K_0\big(\sqrt{2}|Y|\big)dY \leqslant K,$$

therefore the estimate holds. We now suppose that $|x-d\vec{e}_1|, |x+d\vec{e}_1| \geqslant 1$. We decompose

$$\begin{aligned}\int_{\mathbb{R}^2} K_0\big(\sqrt{2}|x-Y|\big)\frac{1}{(1+\tilde{r}(Y))^{\alpha}}dY &= \int_{\{Y_1\geqslant 0\}} K_0\big(\sqrt{2}|x-Y|\big)\frac{dY}{(1+|Y-d\vec{e}_1|)^{\alpha}} \\ &+ \int_{\{Y_1\leqslant 0\}} K_0\big(\sqrt{2}|x-Y|\big)\frac{dY}{(1+|Y+d_n\vec{e}_1|)^{\alpha}},\end{aligned}$$

and we estimate, by a change of variable,

$$\int_{\{Y_1\geqslant 0\}} K_0\big(\sqrt{2}|x-Y|\big)\frac{dY}{(1+|Y-d\vec{e}_1|)^{\alpha}} \leqslant \int_{\mathbb{R}^2} K_0\big(\sqrt{2}|Y|\big)\frac{dY}{(1+|x-d\vec{e}_1-Y|)^{\alpha}}.$$

Now, if $|Y| \leqslant \frac{|x-d\vec{e}_1|}{2}$, by Lemma 2.1.9 we have

$$\begin{aligned}&\int_{\left\{|Y|\leqslant\frac{|x-d\vec{e}_1|}{2}\right\}} K_0\big(\sqrt{2}|Y|\big)\frac{dY}{(1+|x-d\vec{e}_1-Y|)^{\alpha}} \\ \leqslant\ & \frac{K}{(1+|x-d\vec{e}_1|)^{\alpha}}\int_{\left\{|Y|\leqslant\frac{|x-d\vec{e}_1|}{2}\right\}} K_0\big(\sqrt{2}|Y|\big)dY \\ \leqslant\ & \frac{K}{(1+|x-d\vec{e}_1|)^{\alpha}}.\end{aligned}$$

If $|Y| \geqslant \frac{|x-d_n\vec{e}_1|}{2}$, by Lemma 2.1.9 we have

$$\begin{aligned}&\int_{\left\{|Y|\geqslant\frac{|x-d\vec{e}_1|}{2}\right\}} K_0\big(\sqrt{2}|Y|\big)\frac{dY}{(1+|x-d\vec{e}_1-Y|)^{\alpha}} \\ \leqslant\ & Ke^{-|x-d\vec{e}_1|/4}\int_{\left\{|Y|\geqslant\frac{|x-d\vec{e}_1|}{2}\right\}} e^{-|Y|/4}dY \\ \leqslant\ & \frac{K(\alpha)}{(1+|x-d\vec{e}_1|)^{\alpha}}.\end{aligned}$$

By symmetry, we have

$$\int_{\{Y_1\leqslant 0\}} K_0\big(\sqrt{2}|x-Y|\big)\frac{dY}{(1+|Y+d\vec{e}_1|)^{\alpha}} \leqslant \frac{K}{(1+|x+d\vec{e}_1|)^{\alpha}},$$

and this shows that

$$|\Psi(x)| \leqslant \frac{K(\alpha)\|(1+\tilde{r})^{\alpha}h\|_{L^\infty(\mathbb{R}^2)}}{(1+\tilde{r}(x))^{\alpha}}. \tag{2.1.12}$$

For $\nabla\Psi$, we have the similar integral form

$$\nabla\Psi = \frac{1}{2\pi}\nabla\big(K_0\big(\sqrt{2}|.|\big)\big) * h.$$

Once again, we can show the estimate if $|x-d\vec{e}_1| \leqslant 1$ or $|x+d\vec{e}_1| \leqslant 1$, and otherwise, we estimate as previously

$$\begin{aligned}|\nabla\Psi(x)| &\leqslant K\|(1+\tilde{r})^{\alpha}h\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2}\big|\nabla K_0\big(\sqrt{2}|x-Y|\big)\big|\frac{1}{(1+\tilde{r}(Y))^{\alpha}}dY \\ &\leqslant K\|(1+\tilde{r})^{\alpha}h\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2} -K_0'\big(\sqrt{2}|x-Y|\big)\big|\frac{1}{(1+\tilde{r}(Y))^{\alpha}}dY\end{aligned}$$

since $K_0' < 0$ (from Lemma 2.1.9). Now, we can do the same computation as for the estimation of $|\Psi|$, using the properties of K_0' instead of K_0 in Lemma 2.1.9. The same proof works, since the two main ingredients were the integrability near 0 and an exponential decay at infinity of K_0, and $-K_0'$ verifies this too. We deduce

$$|\nabla\Psi(x)| \leqslant \frac{C(\alpha)\|(1+\tilde{r})^{\alpha}h\|_{L^\infty(\mathbb{R}^2)}}{(1+\tilde{r}(x))^{\alpha}}. \tag{2.1.13}$$ □

Remark 2.1.11. Lemma 2.1.10 is different from the equivalent one of [29] for the gradient, which is equation (5.21) there. They claim that:

for any $0 < \sigma < 1$, there exists $C > 0$ such that, if two real-valued functions $\Psi \in C^1(\mathbb{R}^2)$, $h \in C^0(\mathbb{R}^2)$ satisfy

$$(-\Delta + 2)\Psi = h$$

in the distribution sense, and

$$\|\Psi(1+\tilde{r})^{1+\sigma}\|_{L^\infty(\mathbb{R}^2)} + \|\nabla\Psi(1+\tilde{r})^{2+\sigma}\|_{L^\infty(\mathbb{R}^2)} + \|(1+\tilde{r})^{1+\sigma}h\|_{L^\infty(\mathbb{R}^2)} < +\infty,$$

then

$$|\Psi| \leqslant \frac{C\|(1+\tilde{r})^{1+\sigma}h\|_{L^\infty(\mathbb{R}^2)}}{(1+\tilde{r})^{1+\sigma}}$$

and

$$|\nabla\Psi| \leqslant \frac{C\|(1+\tilde{r})^{1+\sigma}h\|_{L^\infty(\mathbb{R}^2)}}{(1+\tilde{r})^{2+\sigma}}.$$

The main difference they claim would be a stronger decay for the gradient. However, such a result can not hold, because of the following counterexample:

$$\Psi_\varepsilon(x) = \begin{cases} 0 & \text{if } |x| \leqslant 1/\varepsilon \\ \frac{\sin^2(r)}{(1+r)^{2+\sigma}} & \text{if } |x| \geqslant 1/\varepsilon. \end{cases}$$

For $\varepsilon > 0$ small enough (in particular such that $\frac{1}{\varepsilon} \gg \frac{1}{c}$, and such that $\frac{1}{\varepsilon}$ is an integer multiple of π, so that Ψ_ε is C^2), we have

$$\|(1+\tilde{r})^{1+\sigma}h(x)\|_{L^\infty(\mathbb{R}^2)} = \|(1+\tilde{r})^{1+\sigma}((-\Delta+2)\Psi)(x)\|_{L^\infty(\mathbb{R}^2)} \leqslant K\varepsilon$$

and

$$\|(1+\tilde{r})^{2+\sigma}|\nabla\Psi(x)|\|_{L^\infty(\mathbb{R}^2)} \geqslant 1/2.$$

Therefore, taking $\varepsilon \to 0$, we see that the estimate $|\nabla\Psi(x)| \leqslant \frac{C\|(1+\tilde{r})^{1+\sigma}h\|_{L^\infty(\mathbb{R}^2)}}{(1+\tilde{r})^{2+\sigma}}$ can not hold.

For our proof of the inversion of the linearized operator (Proposition 2.1.17 below), we did not choose the same norms $\|.\|_{*,\sigma,d}$ and $\|.\|_{**,\sigma',d}$ as in [29] (at the beginning of subsection 2.1.3). In particular, we require decays on the second derivatives for $\|.\|_{**,\sigma',d}$. Our proof of the inversion of the linearized operator (the equivalent of Lemma 5.1 of [29]) will be different, and will follow more closely the proof of [11].

2.1.4.3 Estimates for the Gross-Pitaevskii kernels

We are interested here in solving the following equation on ψ, given a source term h and $c \in \left]0, \sqrt{2}\right[$:

$$-ic\partial_{x_2}\psi - \Delta\psi + 2\mathfrak{Re}(\psi) = h.$$

It will appear in the inversion of the linearized operator around V. See Lemma 2.1.15 for the exact result. We give here a way to construct a solution formally. We will highlight all the important quantities, as well as all the difficulties that arise when trying to solve this equation rigorously.

In this subsection, we want to check that a solution of this equation, with $\psi = \psi_1 + i\psi_2$ and $h = h_1 + ih_2$ (where ψ_1, ψ_2, h_1, h_2 are real valued) can be written

$$\psi_1 = K_0 * h_1 + cH, \tag{2.1.14}$$

with H a function that satisfies

$$\partial_{x_j} H := K_j * h_2,$$

and

$$\partial_{x_j}\psi_2 = G_j - cK_j * h_1, \tag{2.1.15}$$

where similarly G_j satisfies

$$\partial_{x_k} G_j := (c^2 L_{j,k} - R_{j,k}) * h_2,$$

where, for $j,k \in \{1,2\}$, $\xi = (\xi_1,\xi_2) \in \mathbb{R}^2$,

$$\widehat{K_0}(\xi) := \frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_2^2},$$

$$\widehat{K_j}(\xi) := \frac{\xi_2\xi_j}{|\xi|^4 + 2|\xi|^2 - c^2\xi_2^2},$$

$$\widehat{L_{j,k}}(\xi) := \frac{\xi_2^2\xi_j\xi_k}{|\xi|^2(|\xi|^4 + 2|\xi|^2 - c^2\xi_2^2)},$$

and

$$\widehat{R_{j,k}}(\xi) := \frac{\xi_j\xi_k}{|\xi|^2}.$$

We will check later on that, for continuous and sufficiently decaying functions h, these quantities are well defined, and that H, G_j, ψ_2 can be defined from there derivatives. The Gross-Pitaevskii kernels, $K_0, K_j, L_{j,k}$, and the Riesz kernels $R_{j,k}$ have been studied in [19], and we will recall some of the results obtained there.

We write the system in real and imaginary part:

$$\begin{cases} c\partial_{x_2}\psi_2 - \Delta\psi_1 + 2\psi_1 = h_1 \\ -c\partial_{x_2}\psi_1 - \Delta\psi_2 = h_2. \end{cases}$$

Now, taking the Fourier transform of the system, we have

$$\begin{cases} i\xi_2 c\widehat{\psi_2} + (|\xi|^2+2)\widehat{\psi_1} = \hat{h_1} \\ -i\xi_2 c\widehat{\psi_1} + |\xi|^2\widehat{\psi_2} = \hat{h_2}, \end{cases}$$

and we write it

$$\begin{pmatrix} |\xi|^2+2 & ic\xi_2 \\ -ic\xi_2 & |\xi|^2 \end{pmatrix}\begin{pmatrix} \widehat{\psi_1} \\ \widehat{\psi_2} \end{pmatrix} = \begin{pmatrix} \hat{h_1} \\ \hat{h_2} \end{pmatrix}.$$

Here, we suppose that ψ is a tempered distributions and $h \in L^p(\mathbb{R}^2,\mathbb{C})$ for some $p>1$.

Now, we want to invert the matrix, and for that, we have to divide by its determinant, $|\xi|^4 + 2|\xi|^2 - c^2\xi_2^2$. For $0<c<\sqrt{2}$, this quantity is zero only for $\xi = 0$. Thus, for $\xi \neq 0$,

$$\begin{pmatrix} \widehat{\psi_1} \\ \widehat{\psi_2} \end{pmatrix} = \frac{1}{|\xi|^4 + 2|\xi|^2 - c^2\xi_2^2}\begin{pmatrix} |\xi|^2\hat{h_1} - ic\xi_2\hat{h_2} \\ (|\xi|^2+2)\hat{h_2} + ic\xi_2\hat{h_1} \end{pmatrix},$$

which implies that

$$\widehat{\psi_1} = \frac{|\xi|^2\hat{h_1}}{|\xi|^4 + 2|\xi|^2 - c^2\xi_2^2} + \frac{-ic\xi_2\hat{h_2}}{|\xi|^4 + 2|\xi|^2 - c^2\xi_2^2}.$$

With the definition of K_0, we have $\frac{|\xi|^2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_2^2}\hat{h_1} = \widehat{K_0}\hat{h_1}$ and, defining the distribution H by $\hat{H} = \frac{-i\xi_2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_2^2}\hat{h_2}$, we have, for $\xi \neq 0$,

$$\widehat{\partial_{x_j} H} = \frac{\xi_j\xi_2\hat{h_2}}{|\xi|^4 + 2|\xi|^2 - c^2\xi_2^2} = \widehat{K_j}\hat{h_2}.$$

Remark that $\frac{-i\xi_2}{|\xi|^4 + 2|\xi|^2 - c^2\xi_2^2} \in L^{3/2}(\mathbb{R}^2,\mathbb{C})$ and thus is a tempered distribution.

Now, we have

$$\widehat{\partial_{x_j}\psi_2} = \frac{i\xi_j(|\xi|^2+2)\hat{h_2}}{|\xi|^4+2|\xi|^2-c^2\xi_2^2} + \frac{-c\xi_j\xi_2\hat{h_1}}{|\xi|^4+2|\xi|^2-c^2\xi_2^2}.$$

We check that $\frac{-c\xi_j\xi_2}{|\xi|^4+2|\xi|^2-c^2\xi_2^2}\hat{h_1} = -c\,\widehat{K_j}\hat{h_1}$, and we compute

$$\frac{|\xi|^2+2}{|\xi|^4+2|\xi|^2-c^2\xi_2^2} = \frac{1}{|\xi|^2}\left(1-\frac{c^2\xi_2^2}{|\xi|^4+2|\xi|^2-c^2\xi_2^2}\right) = \frac{1}{|\xi|^2} - c^2\frac{\xi_2^2}{|\xi|^2(|\xi|^4+2|\xi|^2-c^2\xi_2^2)},$$

thus, denoting $\widehat{G_j} = \frac{i\xi_j(|\xi|^2+2)}{|\xi|^4+2|\xi|^2-c^2\xi_2^2}\hat{h_2}$, we have

$$\widehat{\partial_{x_k}G_j} := \left(c^2\widehat{L_{j,k}} - \widehat{R_{j,k}}\right)\hat{h_2}.$$

We therefore have that, at least formally, for $\xi \neq 0$, $\widehat{-ic\partial_{x_2}\psi - \Delta\psi + 2\mathfrak{Re}(\psi) - h}(\xi) = 0$. We deduce that there exists $P \in \mathbb{C}[X_1, X_2]$ such that $-ic\partial_{x_2}\psi - \Delta\psi + 2\mathfrak{Re}(\psi) - h = P$ Now, if the function ψ and h are such that the left hand side is bounded and goes to 0 at infinity, this implies that $P = 0$. This will hold under a condition on h (which will be $\int_{\mathbb{R}^2} h_2 = 0$ and some decay estimates, that ψ will inherit). Another remark is that ψ is here in part defined through its derivatives, and we need an argument to construct a primitive. See Lemma 2.1.15 for a rigorous proof of this construction. Remark that $-ic\partial_{x_2}\psi - \Delta\psi + 2\mathfrak{Re}(\psi) = 0$ has some nonzero or unbounded polynomial solutions, for instance $\psi = i$ or $\psi = ix_2 - \frac{c}{2}$.

The kernels K_0, K_j and $L_{j,k}$ have been studied in details in [19], [21] and [22]. In particular, we recall the following result.

Theorem 2.1.12. ([19], Theorems 5 and 6) *For $\mathcal{K} \in \{K_0, K_j, L_{j,k}\}$ and any $0 < c_0 < \sqrt{2}$, there exist a constant $K(c_0) > 0$ such that, for all $0 < c < c_0$,*

$$|\mathcal{K}(x)| \leqslant \frac{K(c_0)}{|x|^{1/2}(1+|x|)^{3/2}}$$

and

$$|\nabla\mathcal{K}(x)| \leqslant \frac{K(c_0)}{|x|^{3/2}(1+|x|)^{3/2}}.$$

Proof. This is the main result of Theorems 5 and 6 of [19]. We added the fact that the constant K is uniform in c, given that c is small. This can be easily shown by following the proof of Theorem 5 and 6 of [19], and verifying that the constants depends only on weighed L^∞ norms on $\hat{\mathcal{K}}$ and its first derivatives, which are uniforms in c if $c > 0$ is small. The condition $c < c_0$ is taken in ordrer to avoid $c \to \sqrt{2}$, where this does not hold (the singularity near $\xi = 0$ of $\hat{\mathcal{K}}$ changes of order at the limit). We will take often $c_0 = 1$. Furthermore, the factor $1/2$ for the growth near $x = 0$ is not at all optimal, but we will not require more here.

Remark that the speed in [19] is in the direction $\vec{e}_1$, whereas it is in the direction $\vec{e}_2$ in our case, which explains the swap between ξ_2 and ξ_1 in the two papers. □

We recall that $\tilde{r} = \min(r_1, r_{-1})$ with $r_{\pm 1} = |x \mp d\vec{e_1}|$, $\frac{1}{2c} < d < \frac{2}{c}$. We give some estimates of convolution with these kernels.

Lemma 2.1.13. *Take $K \in \{K_0, K_j, L_{j,k}\}$ and $h \in C^0(\mathbb{R}^2, \mathbb{R})$, and suppose that, for some $\alpha > 0$, $\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)} < +\infty$. Then, for any $0 < \alpha' < \alpha$, there exists $C(\alpha, \alpha') > 0$ such that, for $0 < c < 1$, if either*

- $\alpha < 2$
- $2 < \alpha < 3$, $\forall (x_1, x_2) \in \mathbb{R}^2, h(-x_1, x_2) = h(x_1, x_2)$ *and* $\int_{\mathbb{R}^2} h = 0$,

then

$$|K * h| \leqslant \frac{C(\alpha, \alpha')\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+\tilde{r})^{\alpha'}}.$$

Furthermore, if $\alpha<3$ (without any other conditions), then

$$|\nabla K*h|\leqslant\frac{C(\alpha,\alpha')\|h(1+\tilde r)^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+\tilde r)^{\alpha'}}.$$

The symmetry and $\int_{\mathbb{R}^2}h=0$ in the case $2<\alpha<3$ could be removed, if we suppose instead that $\int_{\{x_1\geqslant 0\}}h=\int_{\{x_1\leqslant 0\}}h=0$. In particular, if we suppose that $\forall(x_1,x_2)\in\mathbb{R}^2$, $h(x_1,x_2)=-h(x_1,-x_2)$, then the condition $\int_{\mathbb{R}^2}h=0$ is automatically satisfied.

Proof. First, since $\alpha>0$, $h\in L^p(\mathbb{R}^2,\mathbb{C})$ for some large $p>1$ (depdending on α), and ∇K, $K\in L^q(\mathbb{R}^2,\mathbb{C})$ for any $\frac{4}{3}>q>1$ by Theorem 2.1.12, thus $K*h$ and $\nabla K*h$ are well defined. We only look at the estimates for $x\in\mathbb{R}^2$ with $x_1\geqslant 0$. The case $x_1\leqslant 0$ can be done similarly. In this case, we have $\tilde r(x)=|x-d_c\vec{e}_1|$.

We first look at the case $0<\alpha<2$. By Theorem 2.1.12 and the change of variables $z=x-y$, we have

$$\begin{aligned}
&|K*h|(x)\\
\leqslant\ & C\|h(1+\tilde r)^\alpha\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2}\frac{dy}{|x-y|^{1/2}(1+|x-y|)^{3/2}(1+\tilde r(y))^\alpha}\\
\leqslant\ & C(\alpha)\|h(1+\tilde r)^\alpha\|_{L^\infty(\mathbb{R}^2)}\int_{\{y_1\geqslant 0\}}\frac{dy}{|x-y|^{1/2}(1+|x-y|)^{3/2}(1+|y-d\vec{e}_1|)^\alpha}\\
+\ & C(\alpha)\|h(1+\tilde r)^\alpha\|_{L^\infty(\mathbb{R}^2)}\int_{\{y_1\leqslant 0\}}\frac{dy}{|x-y|^{1/2}(1+|x-y|)^{3/2}(1+|y+d\vec{e}_1|)^\alpha}\\
\leqslant\ & C(\alpha)\|h(1+\tilde r)^\alpha\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2}\frac{dz}{|z|^{1/2}(1+|z|)^{3/2}(1+|z-(x-d\vec{e}_1)|)^\alpha}\\
+\ & C(\alpha)\|h(1+\tilde r)^\alpha\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2}\frac{dz}{|z|^{1/2}(1+|z|)^{3/2}(1+|z-(x+d\vec{e}_1)|)^\alpha}.
\end{aligned}\tag{2.1.16}$$

We focus on the estimation of $\int_{\mathbb{R}^2}\frac{dz}{|z|^{1/2}(1+|z|)^{3/2}(1+|z-(x-d\vec{e}_1)|)^\alpha}$. If $|x-d\vec{e}_1|\leqslant 1$, since $\alpha>0$,

$$\int_{\mathbb{R}^2}\frac{dz}{|z|^{1/2}(1+|z|)^{3/2}(1+|z-(x-d\vec{e}_1)|)^\alpha}\leqslant C(\alpha)\int_{\mathbb{R}^2}\frac{dz}{|z|^{1/2}(1+|z|)^{3/2}(1+|z|)^\alpha}\leqslant C(\alpha).$$

Now, for $|x-d\vec{e}_1|\geqslant 1$, we decompose

$$\begin{aligned}
&\int_{\mathbb{R}^2}\frac{dz}{|z|^{1/2}(1+|z|)^{3/2}(1+|z-(x-d\vec{e}_1)|)^\alpha}\\
=\ &\int_{\left\{|z|\leqslant\frac{|x-d\vec{e}_1|}{2}\right\}}\frac{dz}{|z|^{1/2}(1+|z|)^{3/2}(1+|z-(x-d\vec{e}_1)|)^\alpha}\\
+\ &\int_{\left\{|z|\geqslant\frac{|x-d\vec{e}_1|}{2}\right\}}\frac{dz}{|z|^{1/2}(1+|z|)^{3/2}(1+|z-(x-d\vec{e}_1)|)^\alpha}.
\end{aligned}$$

In $\left\{|z|\leqslant\frac{|x-d\vec{e}_1|}{2}\right\}$, we have $|z-(x-d\vec{e}_1)|\geqslant\frac{|x-d\vec{e}_1|}{2}$ and $|z-(x-d\vec{e}_1)|\geqslant|z|$, thus, since $\alpha-\alpha'>0$ and $|x-d\vec{e}_1|\geqslant 1$,

$$\begin{aligned}
&\int_{\left\{|z|\leqslant\frac{|x-d\vec{e}_1|}{2}\right\}}\frac{dz}{|z|^{1/2}(1+|z|)^{3/2}(1+|z-(x-d\vec{e}_1)|)^\alpha}\\
\leqslant\ &\frac{C}{|x-d\vec{e}_1|^{\alpha'}}\int_{\mathbb{R}^2}\frac{dz}{|z|^{1/2}(1+|z|)^{3/2}(1+|z|)^{\alpha-\alpha'}}\\
\leqslant\ &\frac{C(\alpha-\alpha')}{|x-d\vec{e}_1|^{\alpha'}}\\
\leqslant\ &\frac{C(\alpha-\alpha',\alpha')}{(1+|x-d\vec{e}_1|)^{\alpha'}}.
\end{aligned}$$

In $\left\{|z|\geqslant\frac{|x-d\vec{e}_1|}{2}\right\}$, we have $|z|\geqslant\frac{|z-(x-d\vec{e}_1)|}{3}$ since

$$|z-(x-d\vec{e}_1)|\leqslant|z|+|x-d\vec{e}_1|\leqslant|z|+2|z|\leqslant 3|z|,$$

and $|z| \geqslant K(1+|z|)$ since $|z| \geqslant \frac{|x-d\vec{e_1}|}{2} \geqslant \frac{1}{2}$. We then estimate, with $0<\alpha'<\alpha<2$,

$$\begin{aligned}
&\int_{\left\{|z|\geqslant\frac{|x-d\vec{e_1}|}{2}\right\}} \frac{dz}{|z|^{1/2}(1+|z|)^{3/2}(1+|z-(x-d\vec{e_1})|)^{\alpha}} \\
\leqslant\ & \frac{C}{(1+|x-d\vec{e_1}|)^{\alpha'}} \int_{\left\{|z|\geqslant\frac{|x-d\vec{e_1}|}{2}\right\}} \frac{dz}{(1+|z|)^{2-\alpha'}(1+|z-(x-d\vec{e_1})|)^{\alpha}} \\
\leqslant\ & \frac{C(\alpha,\alpha')}{(1+|x-d\vec{e_1}|)^{\alpha'}} \int_{\mathbb{R}^2} \frac{dz}{(1+|z-(x-d\vec{e_1})|)^{2+\alpha-\alpha'}} \\
\leqslant\ & \frac{C(\alpha,\alpha')}{(1+|x-d\vec{e_1}|)^{\alpha'}}.
\end{aligned}$$

With similar computations, we check that, since $x_1 \geqslant 0$,

$$\int_{\mathbb{R}^2} \frac{dz}{|z|^{1/2}(1+|z|)^{3/2}(1+|z-(x+d\vec{e_1})|)^{\alpha}} \leqslant \frac{C(\alpha-\alpha',\alpha')}{(1+|x+d\vec{e_1}|)^{\alpha'}} \leqslant \frac{C(\alpha-\alpha',\alpha')}{(1+|x-d\vec{e_1}|)^{\alpha'}}.$$

Therefore, for $0<\alpha<2$, we have

$$|K*h| \leqslant \frac{C(\alpha-\alpha',\alpha')\|h(1+\tilde{r})^{\alpha}\|_{L^\infty(\mathbb{R}^2)}}{(1+\tilde{r})^{\alpha'}}.$$

Now, if we consider ∇K instead of K and $\alpha<3$, a similar proof gives the result. The only change is that we now use $3-\alpha'>0$ since $\alpha'<\alpha<3$ in the estimate of the integral in $\left\{|z| \geqslant \frac{|x-d\vec{e_1}|}{2}\right\}$, with the extra decay coming from ∇K instead of K.

We now look at the case $2<\alpha<3$ and $\int_{\mathbb{R}^2} h = 0$. In particular, since $\alpha>2$, we indeed have $h \in L^1(\mathbb{R}^2)$. For $\tilde{r}(x) = |x-d\vec{e_1}| \leqslant 1$, the proof is the same as in the case $\alpha<2$.

We now suppose that $\tilde{r}(x) = |x-d\vec{e_1}| \geqslant 1$. Since $\int_{\mathbb{R}^2} h = 0$ and $\forall x \in \mathbb{R}^2, h(-x_1,x_2) = h(x_1,x_2)$, we have

$$\int_{\{y_1\leqslant 0\}} h(y)dy = \int_{\{y_1\geqslant 0\}} h(y)dy = 0,$$

hence

$$\int_{\{y_1\leqslant 0\}} K(x+d\vec{e_1})h(y)dy = \int_{\{y_1\geqslant 0\}} K(x-d\vec{e_1})h(y)dy = 0.$$

Therefore, we decompose

$$\begin{aligned}
&|(K*h)(x)| \\
=\ & \left|\int_{\mathbb{R}^2} K(x-y)h(y)dy\right| \\
=\ & \left|\int_{\{y_1\geqslant 0\}} (K(x-y)-K(x-d\vec{e_1}))h(y)dy\right| + \left|\int_{\{y_1\leqslant 0\}} (K(x-y)-K(x+d\vec{e_1}))h(y)dy\right| \\
\leqslant\ & \int_{\{y_1\geqslant 0\}\cap\{|y-d\vec{e_1}|\leqslant|x-d\vec{e_1}|/2\}} |K(x-y)-K(x-d\vec{e_1})||h(y)|dy \\
+\ & \int_{\{y_1\geqslant 0\}\cap\{|x-y|\leqslant|x-d\vec{e_1}|/2\}} |K(x-y)-K(x-d\vec{e_1})||h(y)|dy \\
+\ & \int_{\{y_1\geqslant 0\}\cap\{|x-y|\geqslant|x-d\vec{e_1}|/2\}\cap\{|y-d\vec{e_1}|\geqslant|x-d\vec{e_1}|/2\}} |K(x-y)-K(x-d\vec{e_1})||h(y)|dy. \\
+\ & \int_{\{y_1\leqslant 0\}} |K(x-y)-K(x+d\vec{e_1})||h(y)|dy.
\end{aligned}$$

In $\{y_1\geqslant 0\}\cap\{|y-d\vec{e_1}|\leqslant|x-d\vec{e_1}|/2\}$, by Theorem 2.1.12,

$$\begin{aligned}
&|K(x-y)-K(x-d\vec{e_1})| \\
\leqslant\ & |K((x-d\vec{e_1})-(y-d\vec{e_1}))-K(x-d\vec{e_1})| \\
\leqslant\ & |y-d\vec{e_1}|\left(\sup_{B(x-d\vec{e_1},|x-d\vec{e_1}|/2)} |\nabla K|\right) \\
\leqslant\ & \frac{C|y-d\vec{e_1}|}{(1+|x-d\vec{e_1}|)^3}.
\end{aligned}$$

With $|x-d\vec{e}_1|\geqslant 1$, $\alpha<3$ and the fact that in $\{y_1\geqslant 0\}\cap\{|y-d\vec{e}_1|\leqslant|x-d\vec{e}_1|/2\}$, $\tilde{r}(y)=|y-d\vec{e}_1|$, we estimate

$$\begin{aligned}
&\int_{\{y_1\geqslant 0\}\cap\{|y-d\vec{e}_1|\leqslant|x-d\vec{e}_1|/2\}}|K(x-y)-K(x-d\vec{e}_1)||h(y)|dy\\
\leqslant\ &\int_{\{|y-d\vec{e}_1|\leqslant|x-d\vec{e}_1|/2\}}\frac{C\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}|y-d\vec{e}_1|}{(1+|x-d\vec{e}_1|)^3(1+|y-d\vec{e}_1|)^\alpha}dy\\
\leqslant\ &\frac{C\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^3}\int_{\{|y-d\vec{e}_1|\leqslant|x-d\vec{e}_1|/2\}}\frac{|y-d\vec{e}_1|}{(1+|y-d\vec{e}_1|)^\alpha}dy\\
\leqslant\ &\frac{C\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^3}\int_{\{|z|\leqslant|x-d\vec{e}_1|/2\}}\frac{|z|}{(1+|z|)^\alpha}dz\\
\leqslant\ &\frac{C(\alpha)\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^3}\times\frac{1}{(1+|x-d\vec{e}_1|)^{\alpha-3}}\\
\leqslant\ &\frac{C(\alpha)\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^\alpha}.
\end{aligned}$$

Now, in $\{y_1\geqslant 0\}\cap\{|x-y|\leqslant|x-d\vec{e}_1|/2\}$, we have $|y-d\vec{e}_1|\geqslant|x-d\vec{e}_1|/2$, and thus

$$|h(y)|\leqslant\frac{C(\alpha)\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^\alpha}.$$

We deduce that

$$\begin{aligned}
&\int_{\{y_1\geqslant 0\}\cap\{|x-y|\leqslant|x-d\vec{e}_1|/2\}}|K(x-y)-K(x-d\vec{e}_1)||h(y)|dy\\
\leqslant\ &\frac{C\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^\alpha}\int_{\{y_1\geqslant 0\}\cap\{|x-y|\leqslant|x-d\vec{e}_1|/2\}}|K(x-y)-K(x-d\vec{e}_1)|dy\\
\leqslant\ &\frac{C\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^\alpha}\left(\int_{\{|x-y|\leqslant|x-d\vec{e}_1|/2\}}|K(x-y)|dy+|K(x-d\vec{e}_1)|\int_{\{|x-y|\leqslant|x-d\vec{e}_1|/2\}}dy\right)\\
\leqslant\ &\frac{C\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^\alpha}\left(\int_{\{|z|\leqslant|x-d\vec{e}_1|/2\}}|K(z)|dz+|K(x-d\vec{e}_1)||x-d\vec{e}_1|^2\right)\\
\leqslant\ &\frac{C\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^\alpha}(\ln(1+|x-d\vec{e}_1|)+1)\\
\leqslant\ &\frac{C(\alpha-\alpha')\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^{\alpha'}}
\end{aligned}$$

since $|x-d\vec{e}_1|\geqslant 1$.

Now, in $\{y_1\geqslant 0\}\cap\{|x-y|\geqslant|x-d\vec{e}_1|/2\}\cap\{|y-d\vec{e}_1|\geqslant|x-d\vec{e}_1|/2\}$, we have

$$|K(x-y)-K(x-d\vec{e}_1)|\leqslant|K(x-y)|+|K(x-d\vec{e}_1)|\leqslant\frac{C}{(1+|x-d\vec{e}_1|)^2}$$

and

$$|h(y)|\leqslant\frac{\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^\alpha},$$

as well as

$$|h(y)|\leqslant\frac{\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|y-d\vec{e}_1|)^\alpha}.$$

We deduce, since $\alpha-\alpha'>0$, that

$$\begin{aligned}
&\int_{\{y_1\geqslant 0\}\cap\{|x-y|\geqslant|x-d\vec{e}_1|/2\}\cap\{|y-d\vec{e}_1|\geqslant|x-d\vec{e}_1|/2\}}|K(x-y)-K(x-d\vec{e}_1)||h(y)|dy.\\
\leqslant\ &\frac{C\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^{2+(\alpha'-2)}}\int_{\mathbb{R}^2}\frac{dy}{(1+|y-d\vec{e}_1|)^{\alpha-\alpha'+2}}\\
\leqslant\ &\frac{C(\alpha-\alpha')\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^{\alpha'}}.
\end{aligned}$$

We are left with the estimation of $\int_{\{y_1\leqslant 0\}}|K(x-y)-K(x+d\vec{e_1})||h(y)|dy$. We decompose it,

$$\begin{aligned}
&\int_{\{y_1\leqslant 0\}}|K(x-y)-K(x+d\vec{e_1})||h(y)|dy\\
=&\int_{\{y_1\leqslant 0\}\cap\left\{|y+d\vec{e_1}|\leqslant\frac{|x+d\vec{e_1}|}{2}\right\}}|K(x-y)-K(x+d\vec{e_1})||h(y)|dy\\
+&\int_{\{y_1\leqslant 0\}\cap\left\{|y+d\vec{e_1}|\geqslant\frac{|x+d\vec{e_1}|}{2}\right\}}|K(x-y)-K(x+d\vec{e_1})||h(y)|dy.
\end{aligned}$$

In $\left\{y_1\leqslant 0\right\}\cap\left\{|y+d\vec{e_1}|\leqslant\frac{|x+d\vec{e_1}|}{2}\right\}$, we have

$$|h(y)|\leqslant\frac{\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|y+d\vec{e_1}|)^\alpha},$$

and

$$\begin{aligned}
&|K(x-y)-K(x+d\vec{e_1})|\\
=&|K((x+d\vec{e_1})-(y+d\vec{e_1}))-K(x+d\vec{e_1})|\\
\leqslant&|y+d\vec{e_1}|\sup_{B(x+d\vec{e_1},|x+d\vec{e_1}|/2)}|\nabla K|\\
\leqslant&\frac{C|y+d\vec{e_1}|}{(1+|x+d\vec{e_1}|)^3},
\end{aligned}$$

thus

$$\begin{aligned}
&\int_{\{y_1\leqslant 0\}\cap\left\{|y+d\vec{e_1}|\leqslant\frac{|x+d\vec{e_1}|}{2}\right\}}|K(x-y)-K(x+d\vec{e_1})||h(y)|dy\\
\leqslant&\frac{C\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x+d\vec{e_1}|)^3}\int_{\left\{|y+d\vec{e_1}|\leqslant\frac{|x+d\vec{e_1}|}{2}\right\}}\frac{|y+d\vec{e_1}|}{(1+|y+d\vec{e_1}|)^\alpha}dy\\
\leqslant&\frac{C\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x+d\vec{e_1}|)^3}\times\frac{C(\alpha)}{(1+|x+d\vec{e_1}|)^{\alpha-3}}\\
\leqslant&\frac{C(\alpha)\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x+d\vec{e_1}|)^\alpha}\\
\leqslant&\frac{C(\alpha)\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e_1}|)^\alpha}
\end{aligned}$$

since $x_1\geqslant 0$ (which implies that $|x+d\vec{e_1}|\geqslant|x-d\vec{e_1}|$).

Finally, in $\left\{y_1\leqslant 0\right\}\cap\left\{|y+d\vec{e_1}|\geqslant\frac{|x+d\vec{e_1}|}{2}\right\}$, we first suppose that $|x-y|\geqslant\frac{|x+d\vec{e_1}|}{2}$, thus

$$|K(x-y)-K(x+d\vec{e_1})|\leqslant|K(x-y)|+|K(x+d\vec{e_1})|\leqslant\frac{C}{(1+|x+d\vec{e_1}|)^2},$$

and we have

$$|h(y)|\leqslant\frac{K(\alpha)\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x+d\vec{e_1}|)^\alpha},$$

as well as

$$|h(y)|\leqslant\frac{K(\alpha)\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|y+d\vec{e_1}|)^\alpha}.$$

We therefore estimate, since $\alpha-\alpha'>0$, $|x+d\vec{e_1}|\geqslant|x-d\vec{e_1}|$,

$$\begin{aligned}
&\int_{\{y_1\leqslant 0\}\cap\left\{|y+d\vec{e_1}|\geqslant\frac{|x+d\vec{e_1}|}{2}\right\}\cap\left\{|x-y|\geqslant\frac{|x+d\vec{e_1}|}{2}\right\}}|K(x-y)-K(x+d\vec{e_1})||h(y)|dy\\
\leqslant&\frac{C\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x+d\vec{e_1}|)^{2+(\alpha'-2)}}\int_{\mathbb{R}^2}\frac{1}{(1+|y+d\vec{e_1}|)^{\alpha-\alpha'+2}}\\
\leqslant&\frac{C(\alpha-\alpha')\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x+d\vec{e_1}|)^{\alpha'}}\\
\leqslant&\frac{C(\alpha-\alpha')\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e_1}|)^{\alpha'}}.
\end{aligned}$$

The other case is when $|x-y| \leqslant \frac{|x+d\vec{e}_1|}{2}$, where we still have $|h(y)| \leqslant \frac{\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x+d\vec{e}_1|)^\alpha}$ and we estimate

$$\begin{aligned}
&\int_{\{y_1\leqslant 0\}\cap\{|x-y|\leqslant|x+d\vec{e}_1|/2\}} |K(x-y)-K(x+d\vec{e}_1)||h(y)|dy\\
\leqslant\ & \frac{C\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x+d\vec{e}_1|)^\alpha}\int_{\{y_1\leqslant 0\}\cap\{|x-y|\leqslant|x+d\vec{e}_1|/2\}} |K(x-y)-K(x+d\vec{e}_1)|dy\\
\leqslant\ & \frac{C\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x+d\vec{e}_1|)^\alpha}\left(\int_{\{|x-y|\leqslant|x+d\vec{e}_1|/2\}} |K(x-y)|dy + |K(x+d\vec{e}_1)|\int_{\{|x-y|\leqslant|x+d\vec{e}_1|/2\}} dy\right)\\
\leqslant\ & \frac{C\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x+d\vec{e}_1|)^\alpha}\left(\int_{\{|z|\leqslant|x+d\vec{e}_1|/2\}} |K(z)|dz + |K(x+d\vec{e}_1)||x+d\vec{e}_1|^2\right)\\
\leqslant\ & \frac{C\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x+d\vec{e}_1|)^\alpha}(\ln(1+|x+d\vec{e}_1|)+1)\\
\leqslant\ & \frac{C(\alpha-\alpha')\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x+d\vec{e}_1|)^{\alpha'}}\\
\leqslant\ & \frac{C(\alpha-\alpha')\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^{\alpha'}},
\end{aligned}$$

which concludes the estimates of this lemma. □

We complete these estimates with some for $R_{j,k}$.

Lemma 2.1.14. *Take $h\in C^1(\mathbb{R}^2,\mathbb{R})$ with $\forall x=(x_1,x_2)\in\mathbb{R}^2, h(-x_1,x_2)=h(x_1,x_2)$, and suppose that for some $\alpha>0$, $\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}+\|\nabla h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}<+\infty$. Then, for any $0<\alpha'<\alpha$, for $0<c<1$, if either*

- *$\alpha<2$*
- *$2<\alpha<3$ and $\int_{\mathbb{R}^2}h=0$,*

then, there exists $C(\alpha,\alpha')>0$ such that

$$|R_{j,k}*h|\leqslant\frac{C(\alpha,\alpha')(\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}+\|\nabla h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)})}{(1+\tilde{r})^{\alpha'}}.$$

Proof. We recall from [19] (or see equation (30) of [21]) that

$$\begin{aligned}
(R_{j,k}*h)(x) &= \frac{1}{2\pi}\int_{|x-y|\geqslant 1}\frac{\delta_{j,k}|x-y|^2-2(x-y)_j(x-y)_k}{|x-y|^4}h(y)dy\\
&+ \frac{1}{2\pi}\int_{|x-y|\leqslant 1}\frac{\delta_{j,k}|x-y|^2-2(x-y)_j(x-y)_k}{|x-y|^4}(h(y)-h(x))dy. \qquad (2.1.17)
\end{aligned}$$

As in the proof of Lemma 2.1.13, we suppose $x_1\geqslant 0$. It implies that $\tilde{r}(x)=|x-d\vec{e}_1|$. The proof can be done similarly if $x_1\leqslant 0$.

First, we look at the case $0<\alpha<2$. We check that

$$\begin{aligned}
&\left|\int_{|x-y|\geqslant 1}\frac{\delta_{j,k}|x-y|^2-2(x-y)_j(x-y)_k}{|x-y|^4}h(y)dy\right|\\
\leqslant\ & K\int_{|x-y|\geqslant 1}\frac{|h(y)|dy}{(1+|x-y|)^2}\\
\leqslant\ & K\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2}\frac{dy}{(1+|x-y|)^2(1+\tilde{r}(y))^\alpha}.
\end{aligned}$$

The estimate of $\int_{\mathbb{R}^2}\frac{dy}{(1+|x-y|)^2(1+\tilde{r}(y))^\alpha}$ can be done exactly as the estimate of

$$\int_{\mathbb{R}^2}\frac{dy}{|x-y|^{1/2}(1+|x-y|)^{3/2}(1+\tilde{r}(y))^\alpha}$$

in the proof of Lemma 2.1.13 (see equation (2.1.16) and the proof below). We deduce that

$$\left|\int_{|x-y|\geqslant 1}\frac{\delta_{j,k}|x-y|^2-2(x-y)_j(x-y)_k}{|x-y|^4}h(y)dy\right|\leqslant\frac{K(\alpha,\alpha')\|h(1+\tilde{r})^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^{\alpha'}}.$$

Now, if $|x-y|\leqslant 1$, for $0<\alpha<3$, we have

$$|h(y)-h(x)|\leqslant |y-x|\sup_{B(x,1)}|\nabla h|\leqslant |y-x|\frac{\|\nabla h(1+\tilde r)^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+\tilde r(x))^\alpha},$$

thus

$$\begin{aligned}&\left|\int_{|x-y|\leqslant 1}\frac{\delta_{j,k}|x-y|^2-2(x-y)_j(x-y)_k}{|x-y|^4}(h(y)-h(x))dy\right|\\ \leqslant\ & \frac{K\|\nabla h(1+\tilde r)^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+\tilde r(x))^\alpha}\int_{|x-y|\leqslant 1}\frac{1}{|x-y|^2}|y-x|\,dy\\ \leqslant\ & \frac{K\|\nabla h(1+\tilde r)^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+\tilde r(x))^\alpha}.\end{aligned}$$

This concludes the proof of the estimate in the case $\alpha<2$. We now suppose that $2<\alpha<3$ and $\int_{\mathbb{R}^2}h=0$. We already have estimate the second integral in (2.1.17) (since the computations were done for $0<\alpha<3$), and for the first integral, the case $|x-d\vec{e}_1|\leqslant 1$ is done as previously.

We now suppose that $|x-d\vec{e}_1|\geqslant 1$. We are left with the estimation of

$$\int_{|x-y|\geqslant 1}\frac{\delta_{j,k}|x-y|^2-2(x-y)_j(x-y)_k}{|x-y|^4}h(y)dy.$$

We define $F_{j,k}(z):=\frac{\delta_{j,k}|z|^2-2z_jz_k}{|z|^4}$ and we check easily that, for $|z|\geqslant 1$,

$$|F_{j,k}(z)|\leqslant\frac{K}{|z|^2}.$$

Since $\forall x\in\mathbb{R}^2, h(-x_1,x_2)=h(x_1,x_2)$ and $\int_{\mathbb{R}^2}h=0$, we have

$$\int_{\{y_1\geqslant 0\}}F_{j,k}(x-d\vec{e}_1)h(y)dy+\int_{\{y_1\leqslant 0\}}F_{j,k}(x+d\vec{e}_1)h(y)dy=0.$$

Furthermore, we estimate (since $|x-d\vec{e}_1|\geqslant 1$)

$$\begin{aligned}&\int_{\{y_1\geqslant 0\}\cap\{|x-y|\leqslant 1\}}|F_{j,k}(x-d\vec{e}_1)h(y)|dy\\ \leqslant\ & |F_{j,k}(x-d\vec{e}_1)|\int_{\{y_1\geqslant 0\}\cap\{|x-y|\leqslant 1\}}|h(y)|dy\\ \leqslant\ & \frac{K}{(1+|x-d\vec{e}_1|)^2}\int_{\{y_1\geqslant 0\}\cap\{|x-y|\leqslant 1\}}|h(y)|dy.\end{aligned}$$

Now, in $\{y_1\geqslant 0\}\cap\{|x-y|\leqslant 1\}$, we check that $|h(y)|\leqslant\frac{K(\alpha)\|h(1+\tilde r)^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^\alpha}$ and thus

$$\int_{\{y_1\geqslant 0\}\cap\{|x-y|\leqslant 1\}}|F_{j,k}(x-d\vec{e}_1)h(y)|dy\leqslant\frac{K(\alpha)\|h(1+\tilde r)^\alpha\|_{L^\infty(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^{2+\alpha}}.$$

Similarly, since $|x+d\vec{e}_1|\leqslant|x-d\vec{e}_1|$ since $x_1\geqslant 0$,

$$\int_{\{y_1\leqslant 0\}\cap\{|x-y|\leqslant 1\}}|F_{j,k}(x+d\vec{e}_1)h(y)|dy\leqslant\frac{K(\alpha)\|h(1+\tilde r)^\alpha\|_{C^0(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^{2+\alpha}}.$$

Therefore, we estimate

$$\begin{aligned}&\left|\int_{|x-y|\geqslant 1}\frac{\delta_{j,k}|x-y|^2-2(x-y)_j(x-y)_k}{|x-y|^4}h(y)dy\right|\\ \leqslant\ & \int_{\{y_1\geqslant 0\}\cap\{|x-y|\geqslant 1\}}|F_{j,k}(x-y)-F_{j,k}(x-d\vec{e}_1)||h(y)|dy\\ +\ & \int_{\{y_1\leqslant 0\}\cap\{|x-y|\geqslant 1\}}|F_{j,k}(x-y)-F_{j,k}(x+d\vec{e}_1)||h(y)|dy\\ +\ & \frac{K(\alpha)\|h(1+\tilde r)^\alpha\|_{C^0(\mathbb{R}^2)}}{(1+|x-d\vec{e}_1|)^{2+\alpha}}.\end{aligned}$$

Now, we conclude as in the proof of Lemma 2.1.13 for the estimation of the two remaining integrals, replacing the function K by $F_{j,k}$, and having the domain of all integrals restricted to $\{|x-y|\geqslant 1\}$. We check that, in $\{|z|\geqslant 1\}$,

$$|F_{j,k}(z)|\leqslant \frac{K}{|z|^2}\leqslant \frac{K}{(1+|z|)^2},$$

and, in $\{|x-y|\geqslant 1\}$,

$$|F_{j,k}(x-y)-F_{j,k}(x)|\leqslant \frac{K|y|}{(1+|x|)^3}.$$

With these estimates replacing Theorem 2.1.12, we can do the proof of the estimates as in Lemma 2.1.13, in the case $2<\alpha<3$ and $\int_{\mathbb{R}^2}h=0$. □

We can now solve the problem

$$-ic\partial_{x_2}\psi-\Delta\psi+2\mathfrak{Re}(\psi)=h,$$

$$\int_{\mathbb{R}^2}\mathfrak{Im}(h)=0$$

in some suitable spaces. We define the norms, for $\sigma,\sigma'\in\mathbb{R}$,

$$\begin{aligned}\|\psi\|_{\otimes,\sigma,\infty} &:= \|(1+\tilde r)^{1+\sigma}\psi_1\|_{L^\infty(\mathbb{R}^2)}+\|(1+\tilde r)^{2+\sigma}\nabla\psi_1\|_{L^\infty(\mathbb{R}^2)}\\ &+ \|(1+\tilde r)^{2+\sigma}\nabla^2\psi_1\|_{L^\infty(\mathbb{R}^2)}+\|(1+\tilde r)^{\sigma}\psi_2\|_{L^\infty(\mathbb{R}^2)}\\ &+ \|(1+\tilde r)^{1+\sigma}\nabla\psi_2\|_{L^\infty(\mathbb{R}^2)}+\|(1+\tilde r)^{2+\sigma}\nabla^2\psi_2\|_{L^\infty(\mathbb{R}^2)}\end{aligned}$$

and

$$\begin{aligned}\|h\|_{\otimes\otimes,\sigma',\infty} &:= \|(1+\tilde r)^{1+\sigma'}h_1\|_{L^\infty(\mathbb{R}^2)}+\|(1+\tilde r)^{2+\sigma'}\nabla h_1\|_{L^\infty(\mathbb{R}^2)}\\ &+ \|(1+\tilde r)^{2+\sigma'}h_2\|_{L^\infty(\mathbb{R}^2)}+\|(1+\tilde r)^{2+\sigma'}\nabla h_2\|_{L^\infty(\mathbb{R}^2)},\end{aligned}$$

as well as the spaces

$$\mathcal{E}^\infty_{\otimes,\sigma}:=\{\psi\in C^2(\mathbb{R}^2,\mathbb{C}),\|\psi\|_{\otimes,\sigma,\infty}<+\infty,\forall(x_1,x_2)\in\mathbb{R}^2,\psi(x_1,x_2)=\psi(-x_1,x_2)\},$$

and

$$\mathcal{E}^\infty_{\otimes\otimes,\sigma'}:=\{h\in C^1(\mathbb{R}^2,\mathbb{C}),\|h\|_{\otimes\otimes,\sigma',\infty}<+\infty,\forall(x_1,x_2)\in\mathbb{R}^2,h(x_1,x_2)=h(-x_1,x_2)\}.$$

The norms $\|.\|_{\otimes,\sigma,\infty}$ and $\|.\|_{*,\sigma}$ differ only on $\{\tilde r\leqslant 3\}$, and $\mathcal{E}^\infty_{\otimes,\sigma}$ has one less symmetry than $\mathcal{E}_{*,\sigma}$. Same remarks hold for $\|.\|_{\otimes\otimes,\sigma',\infty}$ and $\|.\|_{**,\sigma'}$ and their associated spaces. Remark that if $\chi\geqslant 0$ is a smooth cutoff function with value 0 on $\{\tilde r\leqslant R/2\}$ and 1 on $\{\tilde r\geqslant R\}$, then for any $\sigma\in\mathbb{R}$,

$$\|\psi\|_{*,\sigma}\leqslant K(R,\sigma)\|V\psi\|_{C^2(\{\tilde r\leqslant R\})}+K\|\chi\psi\|_{\otimes,\sigma,\infty}. \tag{2.1.18}$$

Lemma 2.1.15. *Given $1>\sigma'>\sigma>0$, there exists $K(\sigma,\sigma')>0$ such that, for any $h\in\mathcal{E}^\infty_{\otimes\otimes,\sigma'}$ with $\int_{\mathbb{R}^2}\mathfrak{Im}(h)=0$ and $0<c<1$, there exists a unique solution to the problem*

$$-ic\partial_{x_2}\psi-\Delta\psi+2\mathfrak{Re}(\psi)=h,$$

in $\mathcal{E}^\infty_{\otimes,\sigma}$. This solution $\psi\in\mathcal{E}^\infty_{\otimes,\sigma}$ satisfies

$$\|\psi\|_{\otimes,\sigma,\infty}\leqslant K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma',\infty}.$$

Furthermore, if instead $\sigma\in]-1,0[$ and $1>\sigma'>\sigma$, there exists then $K(\sigma,\sigma')>0$ such that, for any $h\in\mathcal{E}^\infty_{\otimes\otimes,\sigma'}$ with $\forall(x_1,x_2)\in\mathbb{R}^2,h(x_1,x_2)=\overline{h(x_1,-x_2)}$, there exists a unique solution to the problem

$$-ic\partial_{x_2}\psi-\Delta\psi+2\mathfrak{Re}(\psi)=h$$

in $\{\Psi\in\mathcal{E}^\infty_{\otimes,\sigma},\forall(x_1,x_2)\in\mathbb{R}^2,\Psi(x_1,x_2)=\overline{\Psi(x_1,-x_2)}\}$. This solution $\psi\in\mathcal{E}^\infty_{\otimes,\sigma}$ satisfies

$$\|\psi\|_{\otimes,\sigma,\infty}\leqslant K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma',\infty}.$$

The case $\sigma\in]-1,0[$ is particular and such a norm will be used only in the proof of Lemma 2.1.18 (if $\|\psi\|_{\otimes,\sigma,\infty}<+\infty$ for $\sigma<0$, the function ψ is not necessarily bounded for instance). Remark that the condition $\int_{\mathbb{R}^2}\mathfrak{Im}(h)=0$ is automatically satisfied if $\forall(x_1,x_2)\in\mathbb{R}^2,h(x_1,x_2)=\overline{h(x_1,-x_2)}$.

Proof. For $1>\sigma'>\sigma>-1$, we write in real and imaginary parts $h=h_1+ih_2$. We define, for $j\in\{1,2\}$,

$$\Psi_{1,j}:=K_0*\partial_{x_j}h_1+cK_j*h_2.$$

If $1>\sigma'>\sigma>0$, since $\partial_{x_j}h_1,h_2\in L^1(\mathbb{R}^2)$ (because $\sigma'>0$ and $h\in\mathcal{E}^\infty_{\otimes\otimes,\sigma'}$), and $\int_{\mathbb{R}^2}h_2=\int_{\mathbb{R}^2}\partial_{x_2}h_1=0$, by Lemma 2.1.13 (applied with $0<\alpha=2+\sigma'<3$, $0<\alpha'=2+\sigma<\alpha$), the function $\Psi_{1,2}$ is well defined and satisfies

$$|\nabla\Psi_{1,2}|+|\Psi_{1,2}|\leqslant\frac{K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde{r})^{2+\sigma}}.$$

This result still holds if $\sigma\in]-1,0[$ and $1>\sigma'>\sigma$, since $0<\alpha=2+\sigma'<3, 0<\alpha'=2+\sigma<\alpha$. We check, still with Lemma 2.1.13 (applied with $0<\alpha=2+\sigma'<3$, $0<\alpha'=2+\sigma<\alpha$), that

$$|\nabla\Psi_{1,1}|\leqslant\frac{K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde{r})^{2+\sigma}}.$$

If $\sigma\in]-1,0[$, we have $|\Psi_{1,1}|\leqslant\frac{K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde{r})^{2+\sigma}}$ by Lemma 2.1.13 ($2+\sigma<2$). But since $\partial_{x_1}h_1$ is not even in x_1, we can not apply Lemma 2.1.13 to estimate $\Psi_{1,1}$ with the same decay in the case $\sigma>0$. However, following the proof of Lemma 2.1.13, we check that the estimate holds if $|x+d\vec{e}_1|\leqslant1$ or $|x-d\vec{e}_1|\leqslant1$, and that otherwise

$$|\Psi_{1,1}|\leqslant\frac{K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde{r})^{2+\sigma}}+\left|K(x+d\vec{e}_1)\int_{\{y_1\leqslant0\}}\partial_{x_1}h(y)dy+K(x-d\vec{e}_1)\int_{\{y_1\geqslant0\}}\partial_{x_1}h(y)dy\right|.$$

Since

$$\int_{\{y_1\leqslant0\}}\partial_{x_1}h(y)dy=-\int_{\{y_1\geqslant0\}}\partial_{x_1}h(y)dy=\int_{\mathbb{R}}h(0,y_2)dy_2,$$

and

$$\left|\int_{\mathbb{R}}h(0,y_2)dy_2\right|\leqslant\int_{\mathbb{R}}\frac{\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde{r})^{1+\sigma'}}dy_2\leqslant c^\sigma\|h\|_{\otimes\otimes,\sigma',\infty}\int_{\mathbb{R}}\frac{dy_2}{(1+|y_2|)^{1+\sigma'-\sigma}}\leqslant K(\sigma,\sigma')c^\sigma\|h\|_{\otimes\otimes,\sigma',\infty},$$

we have

$$\begin{aligned}&\left|K(x+d\vec{e}_1)\int_{\{y_1\leqslant0\}}\partial_{x_1}h(y)dy+K(x-d\vec{e}_1)\int_{\{y_1\geqslant0\}}\partial_{x_1}h(y)dy\right|\\ \leqslant\;& K(\sigma,\sigma')|K(x+d\vec{e}_1)-K(x-d\vec{e}_1)|c^\sigma\|h\|_{\otimes\otimes,\sigma',\infty}.\end{aligned}$$

By Theorem 2.1.12, if $|x+d\vec{e}_1|,|x-d\vec{e}_1|\geqslant1$,

$$|K(x+d\vec{e}_1)-K(x-d\vec{e}_1)|\leqslant\frac{K}{(1+|x+d\vec{e}_1|)^2}+\frac{K}{(1+|x-d\vec{e}_1|)^2}\leqslant\frac{K}{(1+\tilde{r})^2},$$

and, if $\tilde{r}\leqslant3d$,

$$|K(x+d\vec{e}_1)-K(x-d\vec{e}_1)|\leqslant\frac{K}{(1+\tilde{r})^2}\leqslant\frac{Kd}{(1+\tilde{r})^3},$$

or if $\tilde{r}\geqslant3d$,

$$|K(x+d\vec{e}_1)-K(x-d\vec{e}_1)|\leqslant Kd\sup_{\nu\in[-d,d]}|\nabla K(x+\nu\vec{e}_1)|\leqslant\frac{Kd}{(1+\tilde{r})^3},$$

therefore, by interpolation,

$$|K(x+d\vec{e}_1)-K(x-d\vec{e}_1)|\leqslant\left(\frac{K}{(1+\tilde{r})^2}\right)^{1-\sigma}\times\left(\frac{Kd}{(1+\tilde{r})^3}\right)^{\sigma}\leqslant\frac{Kd^\sigma}{(1+\tilde{r})^{2+\sigma}}.$$

We deduce

$$\begin{aligned}&\left|K(x+d\vec{e}_1)\int_{\{y_1\leqslant0\}}\partial_{x_1}h(y)dy+K(x-d\vec{e}_1)\int_{\{y_1\geqslant0\}}\partial_{x_1}h(y)dy\right|\\ \leqslant\;& K(\sigma,\sigma')|K(x+d\vec{e}_1)-K(x-d\vec{e}_1)|c^\sigma\|h\|_{\otimes\otimes,\sigma',\infty}\\ \leqslant\;&\frac{K(\sigma,\sigma')\,(dc)^\sigma}{(1+\tilde{r})^{2+\sigma}}\|h\|_{\otimes\otimes,\sigma',\infty}\\ \leqslant\;&\frac{K(\sigma,\sigma')}{(1+\tilde{r})^{2+\sigma}}\|h\|_{\otimes\otimes,\sigma',\infty}.\end{aligned}$$

Combining the previous estimates, we conclude that, for $j \in \{1,2\}$,

$$|\nabla \Psi_{1,j}| + |\Psi_{1,j}| \leqslant \frac{K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde{r})^{2+\sigma}}. \tag{2.1.19}$$

Let us show that $\Psi_{1,j} \in C^1(\mathbb{R}^2,\mathbb{C})$ by dominated convergence theorem. For $x,\varepsilon \in \mathbb{R}^2$,

$$\begin{aligned}\nabla\Psi_{1,j}(x+\varepsilon) - \nabla\Psi_{1,j}(x) &= \int_{\mathbb{R}^2} \nabla K_0(y)(\partial_{x_j} h_1(x+\varepsilon-y) - \partial_{x_j} h_1(x-y))dy,\\ &+ c\int_{\mathbb{R}^2} \nabla K_j(y)(h_2(x+\varepsilon-y) - h_2(x-y))dy.\end{aligned}$$

We check that for any $y \in \mathbb{R}^2$, $\partial_{x_j} h_1(x+\varepsilon-y) - \partial_{x_j} h_1(x-y) \to 0$, $h_2(x+\varepsilon-y) - h_2(x-y) \to 0$ pointwise when $|\varepsilon| \to 0$ (by continuity of $\partial_{x_j} h_1$ and h_2), and

$$\begin{aligned}&|\nabla K_0(y)(\partial_{x_j} h_1(x+\varepsilon-y) - \partial_{x_j} h_1(x-y))|\\ +\ & c|\nabla K_j(y)(h_2(x+\varepsilon-y) - h_2(x-y))|\\ \leqslant\ & K(\sigma)\frac{|\nabla K_0(y)|}{(1+\tilde{r}(x-y))^{2+\sigma'}}\|\partial_{x_j} h_1(1+\tilde{r})^{2+\sigma'}\|_{L^\infty(\mathbb{R}^2)}\\ +\ & K(\sigma)\frac{c|\nabla K_j(y)|}{(1+\tilde{r}(x-y))^{2+\sigma'}}\|h_2(1+\tilde{r})^{2+\sigma'}\|_{L^\infty(\mathbb{R}^2)}\\ \leqslant\ & K(\sigma,x)\frac{|\nabla K_0(y)|}{(1+\tilde{r}(y))^{2+\sigma'}}\|\partial_{x_j} h_1(1+\tilde{r})^{2+\sigma'}\|_{L^\infty(\mathbb{R}^2)}\\ +\ & K(\sigma,x)\frac{c|\nabla K_j(y)|}{(1+\tilde{r}(y))^{2+\sigma'}}\|h_2(1+\tilde{r})^{2+\sigma'}\|_{L^\infty(\mathbb{R}^2)} \in L^1(\mathbb{R}^2)\end{aligned}$$

for $|\varepsilon| \leqslant 1$ and a constant $K(\sigma,x) > 0$, giving the domination.

Now, we check, by taking their Fourier transforms, that $\partial_{x_1}\Psi_{1,2} = \partial_{x_2}\Psi_{1,1} \in L^2(\mathbb{R}^2,\mathbb{C})$ (see the computations at the beginning of subsection 2.1.4.3), and thus the integral of the vector field $\begin{pmatrix}\Psi_{1,1}\\ \Psi_{1,2}\end{pmatrix}$ on any closed curve of $\mathbb{R}^2$ is 0. For a large constant $D > 0$, taking, for $x_1 \in \mathbb{R}$, the path

$$\big\{(x_1,y), y \in [-D,D]\big\} \cup \big\{Y = (y_1,y_2) \in \mathbb{R}^2, |(x_1,0) - Y| = D, y_1 \geqslant 0\big\},$$

since $|\Psi_{1,2}| \leqslant \frac{K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde{r})^{2+\sigma}}$ and

$$\int_{\{Y=(y_1,y_2)\in\mathbb{R}^2, |(x_1,0)-Y|=D, y_1\geqslant 0\}} |\Psi_{1,2}| \leqslant \frac{K(c,\sigma,\sigma',h)}{D^{1+\sigma}} \to 0$$

when $D \to \infty$ (since $1+\sigma > 0$), we deduce that

$$\int_{-\infty}^{+\infty} \Psi_{1,2}(x_1,y_2)dy_2 = 0. \tag{2.1.20}$$

We then define for $(x_1,x_2) \in \mathbb{R}^2$,

$$\psi_1(x_1,x_2) = \int_{+\infty}^{x_2} \Psi_{1,2}(x_1,y_2)dy_2,$$

and thus, if $x_2 < 0$,

$$\psi_1(x_1,x_2) = \int_{-\infty}^{x_2} \Psi_{1,2}(x_1,y_2)dy_2.$$

With (2.1.19), we check that $\psi_1 \in C^1(\mathbb{R}^2,\mathbb{C})$, and by simple integration from infinity using the equations above (with $\tilde{r} = \min(|x - d_c\vec{e}_1|, |x + d_c\vec{e}_1|)$, and since $1+\sigma > 0$), that

$$|\psi_1| \leqslant \frac{K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde{r})^{1+\sigma}}.$$

Furthermore, we check that

$$\partial_{x_2}\psi_1 = \Psi_{1,2} \in C^1(\mathbb{R}^2,\mathbb{C}),$$

and (by taking their Fourier transforms)

$$\partial_{x_1}\psi_1 = \Psi_{1,1} \in C^1(\mathbb{R}^2,\mathbb{C}),$$

therefore $\psi_1 \in C^2(\mathbb{R}^2, \mathbb{C})$, and by (2.1.19),

$$|\nabla \psi_1| \leqslant |\Psi_{1,1}| + |\Psi_{1,2}| \leqslant \frac{K(\sigma, \sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde{r})^{2+\sigma}}.$$

For $j, k \in \{1, 2\}$, we have $\partial^2_{x_j x_k} \psi_1 = \partial_{x_j} \Psi_{1,k}$, thus, by (2.1.19),

$$|\nabla^2 \psi_1| \leqslant \frac{K(\sigma, \sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde{r})^{2+\sigma}}.$$

Now, we define

$$\Psi_{2,j,k} := (c^2 L_{j,k} - R_{j,k}) * h_2 - c K_j * \partial_{x_k} h_1.$$

In the case $1 > \sigma' > \sigma > 0$, $\partial_{x_k} h_1, h_2 \in L^1(\mathbb{R}^2)$ and since $\int_{\mathbb{R}^2} h_2 = \int_{\mathbb{R}^2} \partial_{x_k} h_1 = 0$, by Lemmas 2.1.13 and 2.1.14 (for $\alpha = 2 + \sigma' < 3$, $\alpha' = 2 + \sigma < \alpha$, and the same variant for $K_j * \partial_{x_1} h_1$ as in the proof of (2.1.19)), this function is well defined in $L^\infty(\mathbb{R}^2, \mathbb{C})$, and satisfies,

$$|\Psi_{2,j,k}| \leqslant \frac{K(\sigma, \sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde{r})^{2+\sigma}}. \tag{2.1.21}$$

We check, as for the proof of (2.1.19), that this result holds if $\sigma \in]-1, 0[$ and $1 > \sigma' > \sigma$.

Remark here that we do not have $\Psi_{2,j,k} \in C^1(\mathbb{R}^2, \mathbb{C})$, since in Lemma 2.1.14, the estimate on $R_{j,k} * h_2$ uses ∇h_2 in the norm (showing that $\Psi_{2,j,k} \in C^1(\mathbb{R}^2, \mathbb{C})$ would require estimates on $\nabla^2 h_2$). However, we have that $\Psi_{2,j,k} \in C^0(\mathbb{R}^2, \mathbb{C})$ by dominated convergence and continuity of h_2 and $\partial_{x_k} h_1$ (as for $\nabla \Psi_{1,j}$). Furthermore, we check (by taking their Fourier transforms) that $\partial_{x_1} \Psi_{2,j,2} = \partial_{x_2} \Psi_{2,j,1}$ in the distribution sense. We infer that the integral of $\begin{pmatrix} \Psi_{2,j,1} \\ \Psi_{2,j,2} \end{pmatrix}$ on any bounded closed curve of $\mathbb{R}^2$ is 0. Indeed, taking χ_n a mollifier sequence, then $\chi_n * \Psi_{2,j,1}, \chi_n * \Psi_{2,j,2} \in C^1(\mathbb{R}^2, \mathbb{C})$,

$$\partial_{x_1}(\chi_n * \Psi_{2,j,2}) - \partial_{x_2}(\chi_n * \Psi_{2,j,1}) = \chi_n * (\partial_{x_1} \Psi_{2,j,2} - \partial_{x_2} \Psi_{2,j,1}) = 0,$$

therefore, for any closed curve $\mathcal{C}$, the integral of the field $\begin{pmatrix} \chi_n * \Psi_{2,j,1} \\ \chi_n * \Psi_{2,j,2} \end{pmatrix}$ is 0. Using $\chi_n * \Psi_{2,j,k} \to \Psi_{2,j,k}$ pointwise (by continuity of $\Psi_{2,j,k}$) and the domination

$$\|\chi_n * \Psi_{2,j,1}\|_{L^\infty(\mathbb{R}^2)} \leqslant \|\Psi_{2,j,1}\|_{L^\infty(\mathbb{R}^2)} < +\infty,$$

we infer that this result holds for $\begin{pmatrix} \Psi_{2,j,1} \\ \Psi_{2,j,2} \end{pmatrix}$. We deduce, as for the proof of (2.1.20), that

$$\int_{-\infty}^{+\infty} \Psi_{2,j,2}(x_1, y_2) dy_2 = 0. \tag{2.1.22}$$

We then define for $(x_1, x_2) \in \mathbb{R}^2$, $j \in \{1, 2\}$,

$$\Psi_{2,j}(x_1, x_2) = \int_{+\infty}^{x_2} \Psi_{2,j,2}(x_1, y_2) dy_2,$$

and if $x_2 < 0$, by (2.1.22),

$$\Psi_{2,j}(x_1, x_2) = \int_{-\infty}^{x_2} \Psi_{2,j,2}(x_1, y_2) dy_2.$$

With arguments similar to the proof for ψ_1, we check that $\Psi_{2,j} \in C^1(\mathbb{R}^2, \mathbb{C})$ with $\partial_{x_k} \Psi_{2,j} = \Psi_{2,j,k}$,

$$|\Psi_{2,j}| \leqslant \frac{K(\sigma, \sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde{r})^{1+\sigma}},$$

as well as

$$|\nabla \Psi_{2,j}| \leqslant \frac{K(\sigma, \sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde{r})^{2+\sigma}}.$$

Finally, since $\partial_{x_1} \Psi_{2,2} = \Psi_{2,2,1} = \Psi_{2,1,2} = \partial_{x_2} \Psi_{2,1} \in L^2(\mathbb{R}^2, \mathbb{C})$ (by taking their Fourier transforms, it follows from $R_{j,k} = R_{k,j}$, $L_{j,k} = L_{k,j}$ and $\hat{K}_j \xi_k = \hat{K}_k \xi_j$), we have, as before, that

$$\int_{-\infty}^{+\infty} \Psi_{2,2}(x_1, y_2) dy_2 = 0.$$

We define

$$\psi_2(x_1,x_2)=\int_{+\infty}^{x_2}\Psi_{2,2}(x_1,y_2)dy_2,$$

and thus, if $x_2<0$,

$$\psi_2(x_1,x_2)=\int_{-\infty}^{x_2}\Psi_{2,2}(x_1,y_2)dy_2.$$

We check, as previously, by integration from infinity, that $\psi_2\in C^2(\mathbb{R}^2,\mathbb{C})$, $\partial^2_{x_jx_k}\psi_2=\Psi_{2,j,k}$, and

$$|\nabla^2\psi_2|\leqslant\frac{K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde r)^{2+\sigma}},$$

$$|\nabla\psi_2|\leqslant\frac{K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde r)^{1+\sigma}},$$

as well as (if $\sigma>0$)

$$|\psi_2|\leqslant\frac{K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde r)^{\sigma}}.$$

Remark that if h satisfies $\forall(x_1,x_2)\in\mathbb{R}^2, h(x_1,x_2)=\overline{h(x_1,-x_2)}$, then by the definition of ψ_1 and ψ_2 above, for $\psi=\psi_1+i\psi_2$, we have that $\forall(x_1,x_2)\in\mathbb{R}^2, \psi(x_1,x_2)=\overline{\psi(x_1,-x_2)}$. Therefore, in the case $\sigma\in]-1,0[$, since $\forall(x_1,x_2)\in\mathbb{R}^2$, $\psi_2(x_1,x_2)=-\psi_2(x_1,-x_2)$, we have $\psi_2(x_1,0)=0$, and we integrate $\nabla\psi_2$ from the line $\{x_2=0\}$ instead of infinity to show that $|\psi_2|\leqslant\frac{K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma',\infty}}{(1+\tilde r)^{\sigma}}$.

We deduce that, in either cases, $\psi=\psi_1+i\psi_2\in\mathcal{E}^\infty_{\otimes,\sigma}$, and it satisfies

$$\|\psi\|_{\otimes,\sigma,\infty}\leqslant K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma',\infty}.$$

Now, let us show that $-ic\partial_{x_2}\psi-\Delta\psi+2\mathfrak{Re}(\psi)=h$. From the computations at the beginning of subsection 2.1.4.3, we check that the Fourier transform (in the distribution sense) of both side of the equation are equals on $\{\xi\in\mathbb{R}^2,\xi\neq0\}$ (remark that they are both in $L^p(\mathbb{R}^2,\mathbb{C})$ for some $p>2$ large enough). This implies that

$$\mathrm{Supp}\left(\widehat{-ic\partial_{x_2}\psi-\Delta\psi+2\mathfrak{Re}(\psi)-h}\right)\subset\{0\},$$

and thus $-ic\partial_{x_2}\psi-\Delta\psi+2\mathfrak{Re}(\psi)-h=P\in\mathbb{C}[X_1,X_2]$. With the decay estimates on ψ and h, we check that P is bounded and goes to 0 at infinity (since $\sigma,\sigma'>-1$), thus $P=0$.

Finally, if $\tilde\psi\in\mathcal{E}^\infty_{\otimes,\sigma}$ satisfies $-ic\partial_{x_2}\tilde\psi-\Delta\tilde\psi+2\mathfrak{Re}(\tilde\psi)=h$, then $\psi-\tilde\psi\in C^2(\mathbb{R}^2,\mathbb{C})$ and

$$(-ic\partial_{x_2}-\Delta+2\mathfrak{Re})(\psi-\tilde\psi)=0.$$

With the computations at the beginning of subsection 2.1.4.3, since $\psi-\tilde\psi$ is a tempered distribution, we check that $\mathrm{Supp}\,\widehat{\psi-\tilde\psi}\subset\{0\}$, therefore $\psi-\tilde\psi=P\in\mathbb{C}[X_1,X_2]$. If $\sigma>0$, since $\psi-\tilde\psi$ goes to 0 at infinity, $P=0$. If $\sigma\in]-1,0[$, then $P=i\lambda$ for some $\lambda\in\mathbb{R}$ ($\mathfrak{Re}(\psi-\tilde\psi)\to0$ at infinity and $r^{-\sigma}\mathfrak{Im}(\psi-\tilde\psi)$ is bounded), and by the symmetry on $\psi,\tilde\psi$ we have in that case, $\lambda=0$. This shows the uniqueness of a solution in $\mathcal{E}^\infty_{\otimes,\sigma}$ (with the symmetry if $\sigma\in]-1,0[$), and thus concludes the proof of this lemma. □

2.1.5 Reduction of the problem

2.1.5.1 Inversion of the linearized operator

One of the key element in the inversion of the linearized operator is the computation of the kernel for only one vortex. The kernel of the linearized operator around one vortex has been studied in [10], with the following result.

Theorem 2.1.16. (Theorem 1.2 of [10]) *Consider the linearized operator around one vortex of degree $\varepsilon=\pm1$,*

$$L_{V_\varepsilon}(\Phi):=-\Delta\Phi-(1-|V_\varepsilon|^2)\Phi+2\,\mathfrak{Re}(\bar V_\varepsilon\Phi)\bar V_\varepsilon.$$

Suppose that

$$\|\Phi\|^2_{H_{V_\varepsilon}}:=\int_{\mathbb{R}^2}|\nabla\Phi|^2+(1-|V_\varepsilon|^2)|\Phi|^2+\mathfrak{Re}^2(\bar V_\varepsilon\Phi)<+\infty$$

and

$$L_{V_\varepsilon}(\Phi)=0.$$

Then, there exist two constants $c_1,c_2\in\mathbb{R}$ *such that*

$$\Phi=c_1\partial_{x_1}V_\varepsilon+c_2\partial_{x_2}V_\varepsilon.$$

This result describes the kernel of L_{V_ε} that will appear in the proof of Proposition 2.1.17. It shows that the kernel in $H_{V_\varepsilon}:=\left\{\Phi\in H^1_{\rm loc}(\mathbb{R}^2),\|\Phi\|_{H_{V_\varepsilon}}<+\infty\right\}$ contains only the two elements we expect: $\partial_{x_1}V_\varepsilon,\partial_{x_2}V_\varepsilon$, which are due to the invariance by translation of (GP). One of the directions will be killed by the symmetry and the other one by the orthogonality.

Now, we shall invert the linear part $\eta L(\Phi)+(1-\eta)VL'(\Psi)$. We recall that $\Phi=V\Psi$. We first state an a priori estimate result. We recall the definitions, for $\sigma,\sigma'\in]0,1[$,

$$\mathcal{E}_{*,\sigma,d}=$$

$$\left\{\Phi=V\Psi\in C^2(\mathbb{R}^2,\mathbb{C}),\|\Psi\|_{*,\sigma,d}<+\infty;\langle\Phi,Z_d\rangle=0;\forall x\in\mathbb{R}^2,\Psi(x_1,x_2)=\overline{\Psi(x_1,-x_2)}=\Psi(-x_1,x_2)\right\}$$

and

$$\mathcal{E}_{**,\sigma',d}=\left\{Vh\in C^1(\mathbb{R}^2,\mathbb{C}),\|h\|_{**,\sigma',d}<+\infty\right\},$$

with

$$\begin{aligned}\|\Psi\|_{*,\sigma,d} &= \|V\Psi\|_{C^2(\{\tilde r\leqslant 3\})}\\ &+ \|\tilde r^{1+\sigma}\Psi_1\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\nabla\Psi_1\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\nabla^2\Psi_1\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ &+ \|\tilde r^{\sigma}\Psi_2\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{1+\sigma}\nabla\Psi_2\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\nabla^2\Psi_2\|_{L^\infty(\{\tilde r\geqslant 2\})},\end{aligned}$$

$$\begin{aligned}\|h\|_{**,\sigma',d} &= \|Vh\|_{C^1(\{\tilde r\leqslant 3\})}\\ &+ \|\tilde r^{1+\sigma'}h_1\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma'}\nabla h_1\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ &+ \|\tilde r^{2+\sigma'}h_2\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma'}\nabla h_2\|_{L^\infty(\{\tilde r\geqslant 2\})}.\end{aligned}$$

Proposition 2.1.17. *For* $1>\sigma'>\sigma>0$*, consider the problem, in the distribution sense*

$$\begin{cases}\eta L(\Phi)+(1-\eta)\,VL'(\Psi)=Vh\\ \Phi=V\Psi\in\mathcal{E}_{*,\sigma},Vh\in\mathcal{E}_{**,\sigma'}.\end{cases}$$

Then, there exist constants $c_0(\sigma,\sigma')>0$ *small and* $C(\sigma,\sigma')>0$ *depending only on* σ *and* σ'*, such that, for any solution of this problem with* $0<c\leqslant c_0(\sigma,\sigma')$*,* $\frac12<cd<2$*, it holds*

$$\|\Psi\|_{*,\sigma,d}\leqslant C(\sigma,\sigma')\|h\|_{**,\sigma',d}.$$

Proof. This proof is similar to the ones done in [11] for the inversion of their linearized operator. The main difference is that we have a transport term. Fix $1>\sigma'>\sigma>0$. We argue by contradiction. Suppose that for given $1>\sigma'>\sigma>0$, there is no threshold $c_0(\sigma,\sigma')>0$ such that, if $0<c\leqslant c_0(\sigma,\sigma')$ we have $\|\Psi\|_{*,\sigma,d}\leqslant C(\sigma,\sigma')\|h\|_{**,\sigma',d}$. We can then find a sequence of $c_n\to 0$ (and so $d_n\to\infty$), functions $\Phi_n=V\Psi_n\in\mathcal{E}_{*,\sigma}$ and $Vh_n\in\mathcal{E}_{**,\sigma'}$ solutions of the problem and such that

$$\|\Psi_n\|_{*,\sigma,d_n}=1$$

and

$$\|h_n\|_{**,\sigma',d_n}\to 0.$$

We look in the region $\Sigma:=\left\{x_1\geqslant 0\right\}$ thanks to the symetry $\Psi(x_1,\,x_2)=\Psi(-x_1,\,x_2)$. The orthogonality condition of $\mathcal{E}_{*,\sigma}$ becomes $2\mathfrak{Re}\int_\Sigma\overline{\Phi_n}\,Z_{d_n}=0$.

Step 1. Inner estimates.

The problem can be written (using $VL'(\Psi_n)=-(E-ic_n\,\partial_{x_2}V)\Psi_n+L(\Phi_n)$ from Lemma 2.1.7) as

$$Vh_n=L(\Phi_n)-(1-\eta)(E-ic_n\,\partial_{x_2}V)\Psi_n.$$

First, we recall that V and E are depending on n. The sequence $(\Phi_n(.+d_n\vec{e_1}))_{n\in\mathbb{N}}$ is equicontinuous and bounded ($1=\|\Psi_n\|_{*,\sigma,d}$ controls Φ_n and its derivatives in $L^\infty(\mathbb{R}^2)$ uniformly in n).

Such a function Φ_n, as a solution of

$$\Delta\Phi_n=-(1-|V|^2)\Phi_n+2\mathfrak{Re}(\bar{V}\Phi_n)V-ic\partial_{x_2}\Phi_n-(1-\eta)(E-ic_n\partial_{x_2}V)\Psi_n-Vh_n \tag{2.1.23}$$

in the distribution sense, by Theorem 8.8 of [15] is $H^2_{\mathrm{loc}}(\mathbb{R}^2)$ (since the right hand side is $C^0(\mathbb{R}^2)$). Furthermore, still by Theorem 8.8 of [15], we have, for $x\in\mathbb{R}^2$,

$$\|\Phi_n\|_{H^2(B(x,1))}\leqslant K(\|\Phi_n\|_{H^1(B(x,2))}+\|\Delta\Phi_n\|_{L^2(B(x,2))}).$$

By $\|\Psi_n\|_{*,\sigma,d}=1$, the quantities $\|\Phi_n\|_{L^\infty(B(x,2))}$, $\|\nabla\Phi_n\|_{L^\infty(B(x,2))}$ and $\|\Delta\Phi_n\|_{L^\infty(B(x,2))}$ are bounded by a constant independent of n. Therefore, $(\Phi_n)_{n\in\mathbb{N}}$ is bounded in $H^2_{\mathrm{loc}}(\mathbb{R}^2)$.

We deduce, by compact embedding, that there exists a function Φ such that $\Phi_n(.+d_n\vec{e_1})\to\Phi$ in $H^1_{\mathrm{loc}}(\mathbb{R}^2)$ (up to a subsequence).

Now, since $L(\Phi_n)=-\Delta\Phi_n-(1-|V|^2)\Phi_n+2\,\mathfrak{Re}(\bar{V}\Phi_n)V-ic\partial_{x_2}\Phi_n$, we have, in the weak sense,

$$\Delta\Phi_n+Vh_n=-(1-|V|^2)\Phi_n+2\mathfrak{Re}(\bar{V}\Phi_n)V-ic_n\,\partial_{x_2}\Phi_n-(1-\eta)(E-ic_n\,\partial_{x_2}V)\,\Psi_n,$$

therefore $\Delta\Phi_n(.+d_n\vec{e_1})+Vh_n(.+d_n\vec{e_1})$ is equicontinuous and bounded uniformly and then, by Ascoli's Theorem, up to a subsequence converges to a limit l in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^2)$. Since $Vh_n(.+d_n\vec{e_1})\to 0$ in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^2)$ by $\|h_n\|_{**,\sigma',d}\to 0$ and $\Delta\Phi_n(.+d_n\vec{e_1})\to\Delta\Phi$ in the distribution sense, this limit must be $\Delta\Phi$ (in the $H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)$ sense).

We have locally uniformly that $Vh_n(.+d_n\vec{e_1})\to 0$ because $\|h_n\|_{**,\sigma',d}\to 0$ and $|V|\leqslant 1$, and we have, from Lemma 2.1.4, that $E(y+d_n\vec{e_1})\to 0$ and $V(y+d_n\vec{e_1})\to V_1(y)$ when $n\to\infty$ locally uniformly. Lastly, $\partial_{x_2}\Phi_n$ and $(1-\eta)\partial_{x_2}V\Psi_n$ are uniformly bounded in $\mathbb{R}^2$ independently of n. Therefore when we take the locally uniform limit when $d_n\to\infty$ in

$$(Vh_n)(y+d_n\vec{e_1})=(L(\Phi_n))(y+d_n\vec{e_1})-((1-\eta)(E-ic_n\partial_{x_2}V)\Psi_n)(y+d_n\vec{e_1}),$$

we have (in the distribution sense)

$$L_{V_1}(\Phi)=0.$$

Using $\partial_d V(.+d\vec{e_1})\to-\partial_{x_1}V_1(.)$ locally uniformly from Lemma 2.1.4, we show that

$$0=2\mathfrak{Re}\int_\Sigma\overline{\Phi_n}\,Z_d\to 2\langle\Phi\,|\,\tilde{\eta}(./4)\partial_{x_1}V_1\rangle$$

since Z_d is compactly supported around 0 when we take the equation in $y+d_n\vec{e_1}$. The problem at the limit $n\to\infty$ becomes (in the $H^{-1}_{\mathrm{loc}}(\mathbb{R}^2)$ sense)

$$\begin{cases} L_{V_1}(\Phi)=0\\ \langle\Phi|\,\tilde{\eta}(\frac{\cdot}{4})\partial_{x_1}V_1\rangle=0,\end{cases}$$

with $\Phi=V_1\Psi$ (since $V(y+d\vec{e_1})\to V_1(y)$ from Lemma 2.1.4).

Let us show that $\|\Phi\|_{H_{V_1}}<+\infty$. For that, we will show that

$$\int_{B(d_n\vec{e_1},d_n^{1/2})}|\nabla\Phi_n|^2+\frac{|\Phi_n|^2}{(1+r_1)^2}+\mathfrak{Re}^2(\bar{V_1}(.-d_n\vec{e_1})\Phi_n)\leqslant K(\sigma),$$

where $K(\sigma)>0$ is independent of n, which shall imply (by Lemma 2.1.3)

$$\|\Phi\|^2_{H_{V_1}}\leqslant\limsup_{n\to\infty}\int_{B(d_n\vec{e_1},d_n^{1/2})}|\nabla\Phi_n|^2+\frac{|\Phi_n|^2}{(1+r_1)^2}+\mathfrak{Re}(\bar{V_1}(.-d_n\vec{e_1})\Phi_n)^2\leqslant K(\sigma)<+\infty.$$

First, $\Phi_n\in C^2(\mathbb{R}^2)$ hence $\Phi_n\in H^1_{\mathrm{loc}}(\mathbb{R}^2)$. We have

$$|\nabla\Phi_n|^2\leqslant 2|\nabla V_1|^2|\Psi_n|^2+2|\nabla\Psi_n|^2|V_1|^2,$$

with $|\nabla V_1|^2 = O_{r_1\to\infty}\Big(\frac{1}{r_1^2}\Big)$ by Lemma 2.1.2, and, in $B\big(d_n\vec{e_1},\, d_n^{1/2}\big)$, $|\Psi_n|^2 \leqslant \frac{C}{(1+r_1)^{2\sigma}}\|\Psi_n\|^2_{*,\sigma,d}$, $|\nabla\Psi_n|^2 \leqslant \frac{C}{(1+r_1)^{2+2\sigma}}\|\Psi_n\|^2_{*,\sigma,d}$. Therefore since $\|\Psi_n\|_{*,\sigma,d}\leqslant 1$,

$$\int_{B(d_n\vec{e}_1,d_n^{1/2})}|\nabla\Phi_n|^2 \leqslant \int_{B(d_n\vec{e}_1,d_n^{1/2})}\frac{K}{(1+r_1)^{2+2\sigma}} \leqslant K(\sigma).$$

In addition, in $B\big(d_n\vec{e_1}, d_n^{1/2}\big)$, $|\Phi_n|^2 = |V_1|^2|\Psi_n|^2 \leqslant \frac{K}{(1+r_1)^{2\sigma}}\|\Psi_n\|^2_{*,\sigma,d_n}$ hence $\frac{|\Phi_n|^2}{(1+r_1)^2} \leqslant \frac{K}{(1+r_1)^{2+2\sigma}}$ and

$$\int_{B(d_n\vec{e}_1,d_n^{1/2})}\frac{|\Phi_n|^2}{(1+r_1)^2} \leqslant \int_{B(d_n\vec{e}_1,d_n^{1/2})}\frac{K}{(1+r_1)^{2+2\sigma}} \leqslant K(\sigma).$$

Lastly, still in $B\big(d_n\vec{e_1}, d_n^{1/2}\big)$, by Lemma 2.1.3,

$$\mathfrak{Re}(\bar{V_1}\Phi_n)^2 = |V_1|^4\mathfrak{Re}(V_{-1}\Psi_n)^2 \leqslant |V_1|^4(\mathfrak{Re}(\Psi_n)^2 + (1-|V_{-1}|^2)|\Psi_n|^2) \leqslant \frac{K}{(1+r_1)^{2+2\sigma}},$$

giving the same result. We then have $\|\Phi\|_{H_{V_1}} < +\infty$, therefore, we can apply Theorem 2.1.16. We deduce that

$$\Phi = c_1\partial_{x_1}V_1 + c_2\partial_{x_2}V_1$$

for some constants $c_1, c_2 \in \mathbb{R}$.

Since $\forall x \in \mathbb{R}^2, \Psi_n(x_1, x_2) = \overline{\Psi_n(x_1, -x_2)}$, we have $\forall y \in \mathbb{R}^2, \Phi(y_1, y_2) = \overline{\Phi(y_1, -y_2)}$. The function $\partial_{x_1}V_1$ enjoys also this symmetry, therefore so does $c_2\partial_{x_2}V_1$. It is possible only if $c_2 = 0$. The orthogonality condition then imposes

$$c_1\int_\Sigma |\partial_{x_1}V_1(y)|^2\tilde{\eta}\Big(\frac{y}{4}\Big)\,dy = 0,$$

implying that $c_1 = 0$. Hence

$$\Phi_n(.\,+d_n\vec{e_1}) \to 0$$

in $\mathcal{C}^1_{\mathrm{loc}}(\mathbb{R}^2)$. By equation (2.1.23) and standard elliptic estimates, this convergence also hold in $\mathcal{C}^2_{\mathrm{loc}}(\mathbb{R}^2)$. The same proof works for the z coordinate (around the center of the -1 vortex). As a consequence, for any $R > 0$, we have

$$\|\Phi_n\|_{L^\infty(\{\tilde{r}\leqslant R\})} + \|\nabla\Phi_n\|_{L^\infty(\{\tilde{r}\leqslant R\})} + \|\nabla^2\Phi_n\|_{L^\infty(\{\tilde{r}\leqslant R\})} \to 0 \tag{2.1.24}$$

as $n \to \infty$. With this result, to obtain a contradiction (which will be $\|\Psi_n\|_{*,\sigma,d} \to 0$) we still need to have estimates near the infinity in space.

Step 2. Outer computations.

Thanks to the previous step, we can take a cutoff to look only at the infinity in space. For $R \geqslant 4$, we define χ_R a smooth cutoff function with value $\chi_R(x) = 1$ if $\tilde{r} \geqslant R$ and $\chi_R(x) = 0$ if $\tilde{r} \leqslant \frac{R}{2}$, with $|\nabla\chi_R| \leqslant \frac{4}{R}$. We then define

$$\tilde{\Psi}_n := \chi_R\Psi_n,$$

$$\tilde{h}_n := \chi_R h_n$$

and we choose χ_R such that $\tilde{\Psi}_n$ and $\tilde{h}_n$ enjoy the same symmetries than Ψ_n and h_n respectively. We compute on $\mathbb{R}^2\backslash(B(d_n\vec{e_1}, R)\cup B(-d_n\vec{e_1}, R))$:

$$\nabla\tilde{\Psi}_n = \nabla\chi_R\Psi_n + \chi_R\nabla\Psi_n = \nabla\Psi_n,$$

$$\Delta\tilde{\Psi}_n = \Delta\chi_R\Psi_n + 2\nabla\chi_R\nabla\Psi_n + \chi_R\Delta\Psi_n = \Delta\Psi_n.$$

We deduce that $\tilde{\Psi}_n \in \mathcal{E}^\infty_{\otimes,\sigma}$ and $\tilde{h}_n \in \mathcal{E}^\infty_{\otimes\otimes,\sigma'}$ by (2.1.18), since $\tilde{\Psi}_n \in C^2(B(d_n\vec{e_1}, R)\cup B(-d_n\vec{e_1}, R), \mathbb{C})$, $\tilde{h}_n \in C^1(B(d_n\vec{e_1}, R)\cup B(-d_n\vec{e_1}, R), \mathbb{C})$ and, outside of $B(d_n\vec{e_1}, R)\cup B(-d_n\vec{e_1}, R)$, $\tilde{\Psi}_n = \Psi_n$ with $\|\Psi_n\|_{*,\sigma,d_n} = 1$, as well as $\tilde{h}_n = h_n$, with $\|h_n\|_{**,\sigma',d_n} \to 0$ when $n \to \infty$. In particular,

$$\big\|\tilde{h}_n\big\|_{\otimes\otimes,\sigma',\infty} = o^R_{n\to\infty}(1),$$

where $o^R_{n\to\infty}(1)$ is a sequence that, for fixed $R\geqslant 4$, goes to 0 when $n\to\infty$ (it also depends on σ and σ').

Since $\chi_R=1$ on $\mathbb{R}^2\backslash(B(d_n\vec{e}_1,R)\cup B(-d_n\vec{e}_1,R))$, we have there $L'\big(\tilde{\Psi}_n\big)=\tilde{h}_n$. Therefore, we can write that in $\mathbb{R}^2$ that $L'\big(\tilde{\Psi}_n\big)=\tilde{h}_n+\mathrm{Loc}(\Psi_n)$, with

$$\mathrm{Loc}(\Psi_n):=-\frac{\chi_R\eta}{V}L(V\Psi_n)+(1-\eta)(L'(\chi_R\Psi_n)-\chi_R L'(\Psi_n)),$$

a term that is supported in $\mathbb{R}^2\backslash(B(d_n\vec{e}_1,R)\cup B(-d_n\vec{e}_1,R))$. By (2.1.24) and $\|h_n\|_{**,\sigma',d_n}\to 0$ when $n\to\infty$, it satisfies

$$\begin{aligned}
&\|\mathrm{Loc}(\Psi_n)\|_{\otimes\otimes,\sigma',\infty}\\
\leqslant\ & K(R)\|\mathrm{Loc}(\Psi_n)\|_{C^1(\mathbb{R}^2\backslash(B(d_n\vec{e}_1,R)\cup B(-d_n\vec{e}_1,R)))}\\
\leqslant\ & K(R)\|\Phi_n\|_{C^2(\mathbb{R}^2\backslash(B(d_n\vec{e}_1,R)\cup B(-d_n\vec{e}_1,R)))}\\
=\ & o^R_{n\to\infty}(1).
\end{aligned}$$

We recall that $L'(\Psi)=-\Delta\Psi-2\frac{\nabla V}{V}.\nabla\Psi+2|V|^2\mathfrak{Re}(\Psi)-ic\partial_{x_2}\Psi$, therefore

$$-\Delta\tilde{\Psi}_n-ic\partial_{x_2}\tilde{\Psi}_n+2\mathfrak{Re}\big(\tilde{\Psi}_n\big)=\tilde{h}_n+\mathrm{Loc}(\Psi_n)+2\frac{\nabla V}{V}.\nabla\tilde{\Psi}_n+2(1-|V|^2)\mathfrak{Re}\big(\tilde{\Psi}_n\big). \tag{2.1.25}$$

We define

$$\tilde{h}'_n:=\tilde{h}_n+\mathrm{Loc}(\Psi_n)+2\frac{\nabla V}{V}.\nabla\tilde{\Psi}_n+2(1-|V|^2)\mathfrak{Re}\big(\tilde{\Psi}_n\big).$$

Let us show that $\tilde{h}'_n\in\mathcal{E}^\infty_{\otimes\otimes,\sigma'}$ with

$$\big\|\tilde{h}'_n\big\|_{\otimes\otimes,\sigma',\infty}\leqslant o^R_{n\to\infty}(1)+o_{R\to\infty}(1),$$

where $o_{R\to\infty}(1)$ is a quantity that goes to 0 when $R\to\infty$ (in particular, independently of n). By Lemma 2.1.15, (the condition $\int_{\mathbb{R}^2}\mathfrak{Im}\big(\tilde{h}'_n\big)=0$ is a consequence of the symmetries on $\tilde{h}_n$ and $\tilde{\Psi}_n$), this would imply, with equation (2.1.25) (and since $\tilde{\Psi}_n\in\mathcal{E}^\infty_{\otimes,\sigma}$), that

$$\|\tilde{\Psi}\|_{\otimes,\sigma,\infty}\leqslant o^R_{n\to\infty}(1)+o_{R\to\infty}(1). \tag{2.1.26}$$

This estimate has already been done for the terms $\mathrm{Loc}(\Psi_n)$ and $\tilde{h}_n$. Therefore, we only have to check that

$$\left\|2\frac{\nabla V}{V}.\nabla\tilde{\Psi}_n+2(1-|V|^2)\mathfrak{Re}\big(\tilde{\Psi}_n\big)\right\|_{\otimes\otimes,\sigma',\infty}\leqslant o^R_{n\to\infty}(1)+o_{R\to\infty}(1).$$

First, remark that the term $(1-|V|^2)\mathfrak{Re}\big(\tilde{\Psi}_n\big)$ is real-valued. By Lemma 2.1.3,

$$|1-|V|^2|+\nabla(|V|^2)\leqslant\frac{K}{(1+\tilde{r})^2},$$

and with (2.1.24), $\tilde{\Psi}_n=\Psi_n$ in $\big\{\tilde{r}\geqslant R\big\}$, $\|\Psi_n\|_{*,\sigma}=1$, $0<\sigma<\sigma'<1$,

$$\begin{aligned}
&\big\|(1+\tilde{r})^{1+\sigma'}(1-|V|^2)\mathfrak{Re}\big(\tilde{\Psi}_n\big)\big\|_{L^\infty(\mathbb{R}^2)}\\
\leqslant\ & o^R_{n\to\infty}(1)+K\left\|\frac{(1+\tilde{r})^{1+\sigma'}}{(1+\tilde{r})^{3+\sigma}}\right\|_{L^\infty(\{\tilde{r}\geqslant R\})}\\
\leqslant\ & o^R_{n\to\infty}(1)+o_{R\to\infty}(1)
\end{aligned}$$

and

$$\begin{aligned}
&\|(1+\tilde{r})^{2+\sigma'}\nabla((1-|V|^2)\mathfrak{Re}(\tilde{\Psi}))\|_{L^\infty(\mathbb{R}^2)}\\
\leqslant\ & \|(1+\tilde{r})^{2+\sigma'}\nabla(|V|^2)\mathfrak{Re}(\tilde{\Psi})\|_{L^\infty(\mathbb{R}^2)}+\|(1+\tilde{r})^{2+\sigma'}(1-|V|^2)\mathfrak{Re}(\nabla\tilde{\Psi})\|_{L^\infty(\mathbb{R}^2)}\\
\leqslant\ & o^R_{n\to\infty}(1)+K\left(\left\|\frac{(1+\tilde{r})^{2+\sigma'}}{(1+\tilde{r})^{3+\sigma}}\right\|_{L^\infty(\{\tilde{r}\geqslant R\})}\right)\\
\leqslant\ & o^R_{n\to\infty}(1)+o_{R\to\infty}(1).
\end{aligned}$$

This concludes the proof of

$$\left\|2(1-|V|^2)\mathfrak{Re}\big(\tilde{\Psi}_n\big)\right\|_{\otimes\otimes,\sigma',\infty}\leqslant o^R_{n\to\infty}(1)+o_{R\to\infty}(1).$$

Now, we compute

$$\frac{\nabla V}{V}(x)=\frac{\nabla V_1}{V_1}(y)+\frac{\nabla V_{-1}}{V_{-1}}(z),$$

and recall, by Lemma 2.1.1, that $\nabla V_\varepsilon(x)=i\varepsilon V_\varepsilon(x)\frac{x^\perp}{|x|^2}+O\big(\frac{1}{r^3}\big)$ for $\varepsilon=\pm1$. We deduce that, far from the vortices (for instance on $\mathbb{R}^2\backslash(B(d\vec{e}_1,4)\cup B(-d\vec{e}_1,4))$), we have

$$\frac{\nabla V}{V}(x)=i\left(\frac{y^\perp}{r_1^2}-\frac{z^\perp}{r_{-1}^2}\right)+O_{r_1\to\infty}\left(\frac{1}{r_1^3}\right)+O_{r_{-1}\to\infty}\left(\frac{1}{r_{-1}^3}\right).\tag{2.1.27}$$

In particular, the first order of $\frac{\nabla V}{V}$ is purely imaginary, and the next term is of order $\frac{1}{r_1^3}+\frac{1}{r_{-1}^3}$. We check in particular, using (2.1.27) and Lemma 2.1.3, that on $\mathbb{R}^2\backslash(B(d\vec{e}_1,4)\cup B(-d\vec{e}_1,4))$,

$$\begin{aligned}&\left|(1+\tilde{r})\mathfrak{Im}\left(\frac{\nabla V}{V}\right)\right|+\left|(1+\tilde{r})^3\mathfrak{Re}\left(\frac{\nabla V}{V}\right)\right|\\ +\ &\left|(1+\tilde{r})^2\nabla\mathfrak{Im}\left(\frac{\nabla V}{V}\right)\right|+\left|(1+\tilde{r})^3\nabla\mathfrak{Re}\left(\frac{\nabla V}{V}\right)\right|\\ \leqslant\ & K.\end{aligned}\tag{2.1.28}$$

Therefore, with $R\geqslant4$, equation (2.1.24), $\tilde{\Psi}_n=\Psi_n$ in $\big\{\tilde{r}\geqslant R\big\}$, $\|\Psi_n\|_{*,\sigma}=1$ and $0<\sigma<\sigma'<1$,

$$\begin{aligned}&\left\|(1+\tilde{r})^{1+\sigma'}\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\right)\right\|_{L^\infty(\mathbb{R}^2)}\\ \leqslant\ & o^R_{n\to\infty}(1)+K\left\|\frac{(1+\tilde{r})^{1+\sigma'}}{(1+\tilde{r})^{2+\sigma}}\right\|_{L^\infty(\{\tilde{r}\geqslant R\})}\\ \leqslant\ & o^R_{n\to\infty}(1)+o_{R\to\infty}(1),\\ &\left\|(1+\tilde{r})^{2+\sigma'}\nabla\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\right)\right\|_{L^\infty(\mathbb{R}^2)}\\ \leqslant\ & o^R_{n\to\infty}(1)+K\left\|\frac{(1+\tilde{r})^{2+\sigma'}}{(1+\tilde{r})^{3+\sigma}}\right\|_{L^\infty(\{\tilde{r}\geqslant R\})}\\ \leqslant\ & o^R_{n\to\infty}(1)+o_{R\to\infty}(1),\end{aligned}$$

$$\begin{aligned}&\left\|(1+\tilde{r})^{2+\sigma'}\mathfrak{Im}\left(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\right)\right\|_{L^\infty(\mathbb{R}^2)}\\ \leqslant\ & \left\|(1+\tilde{r})^{2+\sigma'}\mathfrak{Im}\left(\frac{\nabla V}{V}\right).\mathfrak{Re}(\nabla\tilde{\Psi})\right\|_{L^\infty(\mathbb{R}^2)}+\left\|(1+\tilde{r})^{2+\sigma'}\mathfrak{Re}\left(\frac{\nabla V}{V}\right).\mathfrak{Im}(\nabla\tilde{\Psi})\right\|_{L^\infty(\mathbb{R}^2)}\\ \leqslant\ & o^R_{n\to\infty}(1)+K\left\|\frac{(1+\tilde{r})^{2+\sigma'}}{(1+\tilde{r})^{3+\sigma}}\right\|_{L^\infty(\{\tilde{r}\geqslant R\})}+K\left\|\frac{(1+\tilde{r})^{2+\sigma'}}{(1+\tilde{r})^{3+\sigma}}\right\|_{L^\infty(\{\tilde{r}\geqslant R\})}\\ \leqslant\ & o^R_{n\to\infty}(1)+o_{R\to\infty}(1),\end{aligned}$$

and, with a similar decomposition,

$$\left\|(1+\tilde{r})^{2+\sigma'}\nabla\mathfrak{Im}\left(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\right)\right\|_{L^\infty(\mathbb{R}^2)}\leqslant o^R_{n\to\infty}(1)+o_{R\to\infty}(1).$$

This conclude the proof of $\left\|2\frac{\nabla V}{V}.\nabla\tilde{\Psi}\right\|_{\otimes\otimes,\sigma',\infty}\leqslant o^R_{n\to\infty}(1)+o_{R\to\infty}(1)$, and thus of (2.1.26).

Step 3. Conclusion.

We have $\|\Psi_n\|_{*,\sigma,d_n}\leqslant K(R)\|\Phi_n\|_{C^2(\{\tilde{r}\leqslant R\})}+K\big\|\tilde{\Psi}_n\big\|_{*,\sigma,d_n}$ by (2.1.18), therefore, with equations (2.1.24) and (2.1.26),

$$\|\Psi_n\|_{*,\sigma,d_n}\leqslant o^R_{n\to\infty}(1)+o_{R\to\infty}(1).$$

If we take R large enough (depending on σ, σ') so that $o_{R\to\infty}(1) \leqslant 1/10$ and then n large enough (depending on R, σ and σ') so that $o^R_{n\to\infty}(1) \leqslant 1/10$, we have, for n large, $\|\Psi_n\|_{*,\sigma,d_n} \leqslant 1/5$, which is in contradiction with

$$\|\Psi_n\|_{*,\sigma,d_n} = 1. \qquad \square$$

2.1.5.2 Existence of a solution

At this point, we do not have existence of a solution to the linear problem

$$\begin{cases} \eta L(\Phi) + (1-\eta)\, VL'(\Psi) = Vh \\ \Phi \in \mathcal{E}_{*,\sigma}, Vh \in \mathcal{E}_{**,\sigma'}, \end{cases}$$

only an a priori estimate. The existence of a solution is done in Proposition 2.1.20, its proof being the purpose of this subsection. In [11], the existence proof is done using mainly the fact that the domain is bounded. We provide here a proof of existence by approximation on balls of large radii for a particular Hilbertian norm. Given $c > 0$ and $a > 10/c^2$, we define

$$H_a := \left\{ \Phi = Q_c\Psi \in H^1_{\mathrm{loc}}(B(0,a)), \|\Phi\|^2_{H_a} := \|\Phi\|^2_{H^1(\{\tilde r \leqslant 3\})} + \int_{\{\tilde r \geqslant 2\}\cap\{r\leqslant a\}} |\nabla\Psi|^2 + \mathfrak{Re}^2(\Psi) + \frac{\mathfrak{Im}^2(\Psi)}{(1+r)^{5/2}} \right\},$$

and we also allow $a = +\infty$. We first state a result on functions in H_∞.

Lemma 2.1.18. *There exists $c_0 > 0$ such that, for $0 < c < c_0$, $0 < \sigma < \sigma' < 1$, $Vh \in \mathcal{E}_{**,\sigma'}$, if a function $\Phi \in H_\infty \cap C^1(\mathbb{R}^2)$ satisfies, in the weak sense,*

$$\eta L(\Phi) + (1-\eta)\, VL'(\Psi) = Vh,$$

and $\Phi = V\Psi$, $\langle V\Psi, Z_d\rangle = 0; \forall x \in \mathbb{R}^2, \Psi(x_1,x_2) = \overline{\Psi(x_1,-x_2)} = \Psi(-x_1,x_2)$, then

$$\Phi \in \mathcal{E}_{*,\sigma}.$$

Proof. First, we check that, as a solution of $\eta L(\Phi) + (1-\eta)\, VL'(\Psi) = Vh$, $\Phi \in C^2(\mathbb{R}^2, \mathbb{C})$ and

$$\|\Phi\|_{L^\infty(\{r<10/c^2\})} + \|\nabla\Phi\|_{L^\infty(\{r<10/c^2\})} + \|\nabla^2\Phi\|_{L^\infty(\{r<10/c^2\})} \leqslant K(c, \|\Phi\|_{H_\infty}, \|h\|_{**,\sigma'}) < +\infty.$$

Since $\Phi \in C^2(\mathbb{R}^2, \mathbb{C})$ and it satisfies the symmetries and the orthogonality condition, to show that $\Phi = V\Psi \in \mathcal{E}_{*,\sigma}$, we only have to show that $\|\Psi\|_{*,\sigma,d} < +\infty$. Now, similarly as in the proof of Proposition 2.1.17, we add a cutoff function χ_R, writing $\tilde\Psi = \tilde\Psi_1 + i\tilde\Psi_2 = \chi_R\Psi$, $\tilde h = \tilde h_1 + i\tilde h_2 = \chi_R h$ but this time its value is 1 if $r \geqslant 10/c^2$ and 0 if $r \leqslant 5/c^2$. In particular, its support is far from both vortices. We check similarly that, with the same notations, we obtain the equation (2.1.25) that we write in real and imaginary parts:

$$\begin{cases} \Delta\tilde\Psi_1 - 2|V|^2\tilde\Psi_1 = -\tilde h_1 - 2\mathfrak{Re}\Big(\frac{\nabla V}{V}.\nabla\tilde\Psi\Big) + c\partial_{x_2}\tilde\Psi_2 + \mathrm{Loc}_1(\Psi) \\ \Delta\tilde\Psi_2 + c\partial_{x_2}\tilde\Psi_1 = -\tilde h_2 - 2\mathfrak{Im}\Big(\frac{\nabla V}{V}.\nabla\tilde\Psi\Big) + \mathrm{Loc}_2(\Psi), \end{cases} \tag{2.1.29}$$

where $\mathrm{Loc}(\Psi) = \mathrm{Loc}_1(\Psi) + i\,\mathrm{Loc}_2(\Psi)$, and this time the local terms is in $\{5/c^2 \leqslant r \leqslant 10/c^2\}$. Recall that $\tilde\Psi = 0$ on $\{r \leqslant 5/c^2\}$. In particular, we look only at values of x such that $|x| \geqslant 5/c^2$. Now, we define a function ζ, solution of $\Delta\zeta = -\tilde h_2 - 2\mathfrak{Im}\Big(\frac{\nabla V}{V}.\nabla\tilde\Psi\Big) + \mathrm{Loc}_2(\Psi)$ as in Lemma 2.1.8. With Lemma 2.1.3 and $\nabla\tilde\Psi \in L^2(\mathbb{R}^2)$ (since $\Phi \in H_\infty$), we have $Y \mapsto \frac{(\ln|x-Y|)}{(1+\tilde r)^{1/10}}\mathfrak{Im}\Big(\frac{\nabla V}{V}.\nabla\tilde\Psi\Big)(Y) \in L^1(\mathbb{R}^2)$ and thus ζ is well defined. By Hölder inequality, we can check that $\mathfrak{Im}\Big(\frac{\nabla V}{V}.\nabla\tilde\Psi\Big) \in L^3(\mathbb{R}^2)$. We check, with the same computations as in the proof of Lemma 2.1.8 (with $\alpha = 1/10$ in the computations), that $\zeta \in C^1(\mathbb{R}^2)$ and that we have

$$|\nabla\zeta(x)| \leqslant \frac{1}{2\pi}\int_{\mathbb{R}^2} \frac{1}{|x-Y|}\left| -\tilde h_2 - 2\mathfrak{Im}\Big(\frac{\nabla V}{V}.\nabla\tilde\Psi\Big) + \mathrm{Loc}_2(\Psi)\right|(Y)dY,$$

under the condition that $\nabla\tilde{\Psi}\in L^2(\mathbb{R}^2)\cap L^3(\mathbb{R}^2)$. With the upcoming estimates, we will check in particular that this condition is satisfied. From the proof of Lemma 2.1.8, we check that, since $Vh\in\mathcal{E}_{**,\sigma'}$ and $\frac{1+\sigma}{2}<1$,

$$\sup_{x\in\mathbb{R}^2}(1+|x|)^{\frac{1+\sigma}{2}}\int_{\mathbb{R}^2}\frac{1}{|x-Y|}\Big|-\tilde{h}_2+\mathrm{Loc}_2(\Psi)\Big|(Y)dY\Big|<+\infty$$

(here, its size may depend on $\sigma,\sigma',c,R,\|\Phi\|_{H_\infty}$ and $\|h\|_{**,\sigma'}$). Now, from Lemma 2.1.3, we have, outside of $\{\chi_R=0\}$ that $|\nabla V|\leqslant\frac{K(c)}{(1+r)^2}$. We deduce

$$\int_{\mathbb{R}^2}\frac{1}{|x-Y|}\left|\mathfrak{Im}\left(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\right)\right|(Y)dY\leqslant K(c,R)\int_{\mathbb{R}^2}\frac{|\nabla\tilde{\Psi}|(Y)}{|x-Y|(1+|Y|)^2}dY.$$

We focus now on the estimation of $\int_{\mathbb{R}^2}\frac{|\nabla\tilde{\Psi}|(Y)}{|x-Y|(1+|Y|)^2}dY$. From [15], Theorem 8.8, we check that $\|\nabla\tilde{\Psi}\|_{H^1(\mathbb{R}^2)}\leqslant K(c,R,\|\Phi\|_{H_\infty},\|h\|_{**,\sigma'})$. In particular, by Sobolev embedding, $\|\nabla\tilde{\Psi}\|_{L^3(\mathbb{R}^2)}\leqslant K(c,R,\|\Phi\|_{H_\infty},\|h\|_{**,\sigma'})$. In the area $\{|x-Y|\leqslant 1\}$, we have $(1+|Y|)^2\geqslant K(1+|x|)^2$ and therefore, by Hölder inequality,

$$\begin{aligned}\int_{\{|x-Y|\leqslant 1\}}\frac{|\nabla\tilde{\Psi}|(Y)}{|x-Y|(1+|Y|)^2}dY &\leqslant \frac{K}{(1+|x|)^2}\int_{\{|x-y|\leqslant 1\}}\frac{|\nabla\tilde{\Psi}|(Y)}{|x-Y|}dY\\ &\leqslant \frac{K\,\|\nabla\tilde{\Psi}\|_{L^3(\mathbb{R}^2)}}{(1+|x|)^2}\left(\int_{\{|x-Y|\leqslant 1\}}\frac{dY}{|x-Y|^{3/2}}\right)^{2/3}\\ &\leqslant \frac{K(c,R,\|\Phi\|_{H_\infty},\|h\|_{**,\sigma})}{(1+|x|)^2}.\end{aligned}$$

In the area $\{1\leqslant|x-Y|\leqslant|x|/2\}$, we have $|Y|\geqslant\frac{|x-Y|}{2}$ and $|Y|\geqslant\frac{|x|}{2}$, therefore, by Cauchy-Schwarz (since $\frac{1+\sigma}{2}<1$),

$$\begin{aligned}&\int_{\{1\leqslant|x-Y|\leqslant|x|/2\}}\frac{|\nabla\tilde{\Psi}|(Y)\,dY}{|x-Y|(1+|Y|)^2}\\ &\leqslant \frac{K(\sigma,c,R)}{(1+|x|)^{\frac{1+\sigma}{2}}}\int_{\{1\leqslant|x-Y|\leqslant|x|/2\}}\frac{|\nabla\tilde{\Psi}|(Y)\,dY}{|x-Y|(1+|x-Y|)^{2-\left(\frac{1+\sigma}{2}\right)}}\\ &\leqslant \frac{K(\sigma,c,R)}{(1+|x|)^{\frac{1+\sigma}{2}}}\sqrt{\int_{\{1\leqslant|x-Y|\leqslant|x|/2\}}|\nabla\tilde{\Psi}|^2(Y)\,dY\int_{\{1\leqslant|x-Y|\leqslant|x|/2\}}\frac{dY}{|x-Y|^{3-\left(\frac{1+\sigma}{2}\right)}}}\\ &\leqslant \frac{K(c,R,\sigma,\|\Phi\|_{H_\infty})}{(1+|x|)^{\frac{1+\sigma}{2}}}.\end{aligned}$$

Finally, in the area $\{|x-Y|\geqslant|x|/2\}$, we estimate by Cauchy-Schwarz that

$$\begin{aligned}&\int_{\{|x-Y|\geqslant|x|/2\}}\frac{|\nabla\tilde{\Psi}|(Y)}{|x-Y|(1+|Y|)^2}dY\\ &\leqslant \frac{K}{1+|x|}\sqrt{\int_{\{|x-Y|\geqslant|x|/2\}}|\nabla\tilde{\Psi}|^2\int_{\{|x-Y|\geqslant|x|/2\}}\frac{dY}{(1+|Y|)^4}}\\ &\leqslant \frac{K(\|\Phi\|_{H_\infty})}{1+|x|}.\end{aligned}$$

Combining these estimates, we conclude that

$$|\nabla\zeta|(x)\leqslant\frac{K(c,R,\sigma,\sigma',\|\Phi\|_{H_\infty},\|h\|_{**,\sigma})}{(1+|x|)^{\frac{1+\sigma}{2}}}.$$

Now, we write $\tilde{\Psi}_2'=\tilde{\Psi}_2-\zeta$, and the system becomes

$$\begin{cases}\Delta\tilde{\Psi}_1-2\tilde{\Psi}_1-c\partial_{x_2}\tilde{\Psi}_2'=-\tilde{h}_1-2\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\right)+\mathrm{Loc}_1(\Psi)-c\partial_{x_2}\zeta-2(1-|V|^2)\tilde{\Psi}_1\\ \Delta\tilde{\Psi}_2'+c\partial_{x_2}\tilde{\Psi}_1=0.\end{cases}$$

We deduce, as for equation (2.1.15), that for $j \in \{1,2\}$,

$$\partial_{x_j}\tilde{\Psi}_2' = cK_j * \Big(-\tilde{h}_1 - 2\mathfrak{Re}\Big(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\Big) + \mathrm{Loc}_1(\Psi) - c\partial_{x_2}\zeta - 2(1-|V|^2)\tilde{\Psi}_1\Big).$$

We check that, with Lemma 2.1.13 (for $1 > \alpha = \frac{1+\sigma}{2} > 0$, $\alpha' = \sigma < \alpha$),

$$|K_j * \big(-\tilde{h}_1 + \mathrm{Loc}_1(\Psi) - c\partial_{x_2}\zeta\big)| \leqslant \frac{K(c,R,\sigma,\|\Phi\|_{H_\infty},\|h\|_{**,\sigma'})}{(1+|x|)^\sigma},$$

since

$$\big|-\tilde{h}_1 + \mathrm{Loc}_1(\Psi) - c\partial_{x_2}\zeta\big| \leqslant \frac{K(c,R,\sigma,\|\Phi\|_{H_\infty},\|h\|_{**,\sigma'})}{(1+|x|)^{\frac{1+\sigma}{2}}}.$$

Furthermore, from Lemma 2.1.3, outside of $\{\chi_R = 0\}$, $|\nabla V| \leqslant \frac{K(c)}{(1+r)^2}$. We check, with Theorem 2.1.12, that on $\Big\{|x-Y| \leqslant \frac{|x|}{2}\Big\}$, we have $|Y| \geqslant \frac{|x|}{2}$ and

$$\begin{aligned}
&\int_{\{|x-Y|\leqslant|x|/2\}} \Big| K_j(x-Y)\mathfrak{Re}\Big(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\Big)(Y)\Big| dY \\
\leqslant\ & \frac{K(c,R)}{(1+|x|)^2}\int_{\{|x-Y|\leqslant|x|/2\}} \frac{|\nabla\tilde{\Psi}|(Y)\,dY}{|x-Y|^{1/2}(1+|x-Y|)^{3/2}}.
\end{aligned}$$

By Cauchy-Schwarz, we estimate

$$\begin{aligned}
&\int_{\{|x-Y|\leqslant|x|/2\}} \frac{|\nabla\tilde{\Psi}|(Y)\,dY}{|x-Y|^{1/2}(1+|x-Y|)^{3/2}} \\
\leqslant\ & \|\nabla\tilde{\Psi}\|_{L^2(\mathbb{R}^2)}\sqrt{\int_{\{|x-Y|\leqslant|x|/2\}} \frac{dY}{|x-Y|(1+|x-Y|)^3}} \\
<\ & +\infty,
\end{aligned}$$

and in $\Big\{|x-Y| \geqslant \frac{|x|}{2}\Big\}$, we estimate

$$\int_{\{|x-Y|\geqslant|x|/2\}} \Big| K_j(x-Y)\mathfrak{Re}\Big(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\Big)(Y)\Big| \leqslant \frac{K(c,R)}{(1+|x|)^2}\int_{\{|x-Y|\leqslant|x|/2\}} \frac{|\nabla\tilde{\Psi}|(Y)\,dY}{(1+|Y|)^2},$$

and we conclude by Cauchy-Schwarz that

$$\int_{\{|x-Y|\geqslant|x|/2\}} \Big| K_j(x-Y)\mathfrak{Re}\Big(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\Big)(Y)\Big| dY \leqslant \frac{K(c,R,\|\Phi\|_{H_\infty})}{(1+|x|)^2}.$$

Since $\big\|\tilde{\Psi}_1\big\|_{L^2(\mathbb{R}^2)} \leqslant K(c,R,\|\Phi\|_{H_\infty})$, we estimate similarly

$$\int_{\mathbb{R}^2} \big|K_j(x-Y)(1-|V|^2)\tilde{\Psi}_1(Y)\big| dY \leqslant \frac{K(c,R,\|\Phi\|_{H_\infty})}{(1+|x|)^2},$$

and we conclude that $\big|\partial_{x_j}\tilde{\Psi}_2'\big| \leqslant \frac{K(c,R,\|\Phi\|_{H_\infty})}{(1+|x|)^2}$. Therefore, since $\tilde{\Psi}_2 = \zeta + \tilde{\Psi}_2'$,

$$\big|\nabla\tilde{\Psi}_2\big| \leqslant \frac{K(c,R,\sigma,\sigma',\|\Phi\|_{H_\infty},\|h\|_{**,\sigma'})}{(1+|x|)^\sigma}.$$

By integration from the origin (using $\big\|\tilde{\Psi}_2\big\|_{L^\infty(\{r<10/c^2\})} \leqslant K(c,\|\Phi\|_{H_\infty},\|h\|_{**})$), we deduce also that

$$\big|\tilde{\Psi}_2\big| \leqslant \frac{K(c,R,\sigma,\sigma',\|\Phi\|_{H_\infty},\|h\|_{**,\sigma'})}{(1+|x|)^{-1+\sigma}}. \tag{2.1.30}$$

With these estimates and the equation

$$\Delta\tilde{\Psi}_1 - 2\tilde{\Psi}_1 = -\tilde{h}_1 + c\partial_{x_2}\tilde{\Psi}_2 + \mathrm{Loc}_1(\Psi) - 2\mathfrak{Re}\Big(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\Big) - 2(1-|V|^2)\tilde{\Psi}_1,$$

we check that $\left|-\tilde{h}_1+c\partial_{x_2}\tilde{\Psi}_2+\mathrm{Loc}_1(\Psi)\right|\leqslant\frac{K(c,R,\sigma,\sigma',\|\Phi\|_{H_\infty},\|h\|_{**,\sigma'})}{(1+|x|)^\sigma}$, and by Lemma 2.1.10 (for $\alpha=\sigma>0$),

$$|\tilde{\Psi}_1|+|\nabla\tilde{\Psi}_1|\leqslant\frac{K(c,R,\sigma,\sigma',\|\Phi\|_{H_\infty},\|h\|_{**,\sigma'})}{(1+|x|)^\sigma}$$

(where the estimation for the terms $\mathfrak{Re}\Big(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\Big)$ and $2(1-|V|^2)\Psi_1$ are similar to what has already been done since we only have $\nabla\tilde{\Psi},\Psi_1\in L^2(\mathbb{R}^2)$ at this point).

With this first set of estimates, looking at equation (2.1.29), we have enough to show that

$$\left|\Delta\tilde{\Psi}_1-2\tilde{\Psi}_1-c\partial_{x_2}\tilde{\Psi}_2\right|\leqslant\frac{K(c,R,\sigma,\|\Phi\|_{H_\infty},\|h\|_{**,\sigma'})}{(1+|x|)^{1+\sigma}}$$

and

$$\left|\Delta\tilde{\Psi}_2+c\partial_{x_2}\tilde{\Psi}_1\right|\leqslant\frac{K(c,R,\sigma,\|\Phi\|_{H_\infty},\|h\|_{**,\sigma'})}{(1+|x|)^{2+\sigma}}.$$

From the computations at the beginning of subsection 2.1.4.3, we have that, for $j\in\{1,2\}$,

$$\partial_{x_j}\tilde{\Psi}_1=\partial_{x_j}K_0*\left(\Delta\tilde{\Psi}_1-2\tilde{\Psi}_1-c\partial_{x_2}\tilde{\Psi}_2\right)+cK_j*\left(\Delta\tilde{\Psi}_2+c\partial_{x_2}\tilde{\Psi}_1\right),$$

therefore, by Lemma 2.1.13, taking $\alpha=1+\sigma<2$ and $\alpha'=1+\sigma'<\alpha$,

$$\left|\nabla\tilde{\Psi}_1\right|\leqslant\frac{K(c,R,\sigma,\sigma',\|\Phi\|_{H_\infty},\|h\|_{**,\sigma'})}{(1+|x|)^{1+\sigma'}}.$$

Furthermore, by Lemma 2.1.13, $\left|K_j*\left(\Delta\tilde{\Psi}_2+c\partial_{x_2}\tilde{\Psi}_1\right)\right|\leqslant\frac{K(c,R,\sigma,\sigma',\|\Phi\|_{H_\infty})}{(1+|x|)^{2+\sigma/2}}$, hence, since for $x_j>0$,

$$\tilde{\Psi}_1=K_0*\left(\Delta\tilde{\Psi}_1-2\tilde{\Psi}_1-c\partial_{x_2}\tilde{\Psi}_2\right)+c\int_{x_j}^{+\infty}K_j*\left(\Delta\tilde{\Psi}_2+c\partial_{x_2}\tilde{\Psi}_1\right)dy_j$$

by integration from infinity, we also have (with a similar computation if $x_j<0$)

$$\left|\tilde{\Psi}_1\right|\leqslant\frac{K(c,R,\sigma,\sigma',\|\Phi\|_{H_\infty},\|h\|_{**,\sigma'})}{(1+|x|)^{1+\sigma/2}}.$$

Now, using Theorem 8.10 from [15], we have for any $x\in\mathbb{R}^2$ that

$$|\nabla^2\tilde{\Psi}|(x)\leqslant K(\|\Delta\tilde{\Psi}\|_{L^\infty(B(x,1))}+\|\tilde{\Psi}\|_{L^\infty(B(x,1))}+\|\nabla\tilde{\Psi}\|_{L^\infty(B(x,1))}),$$

therefore (the limiting decay coming from (2.1.30))

$$|\nabla^2\tilde{\Psi}|\leqslant\frac{K(c,R,\sigma,\sigma',\|\Phi\|_{H_\infty},\|h\|_{**,\sigma'})}{(1+|x|)^{-1+\sigma}}.$$

With these estimates, we have that $\tilde{\Psi}\in\mathcal{E}_{\otimes,-3+\sigma,\infty}$. Now, we define

$$\check{h}:=\tilde{h}+2\frac{\nabla V}{V}.\nabla\tilde{\Psi}+2(1-|V|^2)\mathfrak{Re}(\tilde{\Psi})+\mathrm{Loc}(\Psi),$$

and we infer that, for any $\alpha\leqslant\sigma'$

$$\|\check{h}\|_{\otimes\otimes,\alpha,\infty}\leqslant K(\alpha,c,R,\sigma,\sigma',\delta,\|\Phi\|_{H_\infty},\|h\|_{**,\sigma'})(1+\|\tilde{\Psi}\|_{\otimes,\delta,\infty})\tag{2.1.31}$$

given that $\delta\geqslant-2+\alpha$. Indeed, we have that, for $\alpha\leqslant\sigma'$, $\|\tilde{h}\|_{\otimes\otimes,\alpha,\infty}\leqslant K(\alpha,\sigma')\|h\|_{**,\sigma'}$, and

$$\|\mathrm{Loc}(\Psi)\|_{\otimes\otimes,\alpha,\infty}\leqslant K(c,\alpha)\|\Phi\|_{C^2(\{r\leqslant10/c^2\})}\leqslant K(c,\alpha,\|\Phi\|_{H_\infty},\|h\|_{**,\sigma'}).$$

We recall that $(1-|V|^2)\mathfrak{Re}(\tilde{\Psi})$ is a real-valued term, and with Lemma 2.1.3, $0<\sigma<\sigma'<1$, we estimate

$$\|(1+\tilde{r})^{1+\alpha}(1-|V|^2)\mathfrak{Re}(\tilde{\Psi})\|_{L^\infty(\mathbb{R}^2)}\leqslant K\left\|\frac{(1+\tilde{r})^{1+\alpha}}{(1+\tilde{r})^{3+\delta}}\right\|_{L^\infty(\mathbb{R}^2)}\|\tilde{\Psi}\|_{\otimes,\delta,\infty}\leqslant K(\alpha,\delta)\|\tilde{\Psi}\|_{\otimes,\delta,\infty}$$

if $1+\alpha \geqslant 3+\delta$ (which is a consequence of $\delta \geqslant -2+\alpha$), and

$$\|(1+\tilde{r})^{2+\alpha}\nabla((1-|V|^2)\mathfrak{Re}(\tilde{\Psi}))\|_{L^\infty(\mathbb{R}^2)} \leqslant K\left\|\frac{(1+\tilde{r})^{2+\alpha}}{(1+\tilde{r})^{4+\delta}}\right\|_{L^\infty(\mathbb{R}^2)}\|\tilde{\Psi}\|_{\otimes,\delta,\infty} \leqslant K(\alpha,\delta)\|\tilde{\Psi}\|_{\otimes,\delta,\infty}.$$

Now, we estimate similarly (still using Lemma 2.1.3)

$$\left\|(1+\tilde{r})^{1+\alpha}\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\right)\right\|_{L^\infty(\mathbb{R}^2)} \leqslant K(c)\left\|\frac{(1+\tilde{r})^{1+\alpha}}{(1+\tilde{r})^{3+\delta}}\right\|_{L^\infty(\mathbb{R}^2)}\|\tilde{\Psi}\|_{\otimes,\delta,\infty} \leqslant K(c,\alpha,\delta)\|\tilde{\Psi}\|_{\otimes,\delta,\infty},$$

$$\left\|(1+\tilde{r})^{2+\alpha}\nabla\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\right)\right\|_{L^\infty(\mathbb{R}^2)} \leqslant K(c)\left\|\frac{(1+\tilde{r})^{2+\alpha}}{(1+\tilde{r})^{4+\delta}}\right\|_{L^\infty(\mathbb{R}^2)}\|\tilde{\Psi}\|_{\otimes,\delta,\infty} \leqslant K(c,\alpha,\delta)\|\tilde{\Psi}\|_{\otimes,\delta,\infty},$$

and since

$$\mathfrak{Im}\left(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\right) = \mathfrak{Im}\left(\frac{\nabla V}{V}\right).\mathfrak{Re}(\nabla\tilde{\Psi}) + \mathfrak{Re}\left(\frac{\nabla V}{V}\right).\mathfrak{Im}(\nabla\tilde{\Psi}),$$

with Lemma 2.1.3 and estimate (2.1.28), we infer that

$$\begin{aligned}
&\left\|(1+\tilde{r})^{2+\alpha}\mathfrak{Im}\left(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\right)\right\|_{L^\infty(\mathbb{R}^2)}\\
\leqslant\ &\left\|(1+\tilde{r})^{2+\alpha}\mathfrak{Im}\left(\frac{\nabla V}{V}\right).\mathfrak{Re}(\nabla\tilde{\Psi})\right\|_{L^\infty(\mathbb{R}^2)} + \left\|(1+\tilde{r})^{2+\alpha}\mathfrak{Re}\left(\frac{\nabla V}{V}\right).\mathfrak{Im}(\nabla\tilde{\Psi})\right\|_{L^\infty(\mathbb{R}^2)}\\
\leqslant\ &K(c)\left\|\frac{(1+\tilde{r})^{2+\alpha}}{(1+\tilde{r})^{4+\delta}}\right\|_{L^\infty(\mathbb{R}^2)}\|\tilde{\Psi}\|_{\otimes,\delta,\infty} + K\left\|\frac{(1+\tilde{r})^{2+\alpha}}{(1+\tilde{r})^{4+\delta}}\right\|_{L^\infty(\mathbb{R}^2)}\|\tilde{\Psi}\|_{\otimes,\delta,\infty}\\
\leqslant\ &K(c,\alpha,\delta)\|\tilde{\Psi}\|_{\otimes,\delta,\infty},
\end{aligned}$$

and with similar estimates,

$$\left\|(1+\tilde{r})^{2+\alpha}\nabla\mathfrak{Im}\left(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\right)\right\|_{L^\infty(\mathbb{R}^2)} \leqslant K(c,\alpha,\delta)\|\tilde{\Psi}\|_{\otimes,\delta,\infty}.$$

This concludes the proof of (2.1.31). With $\tilde{\Psi} \in \mathcal{E}_{\otimes,-3+\sigma,\infty}$, we therefore deduce that for $\varepsilon > 0$ a small constant, $\|\check{h}\|_{\otimes\otimes,-1+\sigma-\varepsilon,\infty} < +\infty$, hence $\check{h} \in \mathcal{E}_{\otimes\otimes,-1+\sigma-\varepsilon}$. With estimate (2.1.31), Lemma 2.1.15 and

$$-\Delta\tilde{\Psi} - ic\partial_{x_2}\tilde{\Psi} + 2\mathfrak{Re}(\tilde{\Psi}) = \check{h},$$

and with the symmetries on $\tilde{\Psi}$ and $\check{h}$, we can bootstrap our estimates on $\tilde{\Psi}$ and then on $\check{h}$, and we conclude that $\tilde{\Psi} \in \mathcal{E}_{\otimes,\sigma}$ (since $\sigma < \sigma'$). □

The next step is to construct a solution on a large ball in the space H_a.

Lemma 2.1.19. *For $0 < \sigma' < 1$, there exists $c_0(\sigma') > 0$ such that, for $0 < c < c_0(\sigma')$, there exists $a_0(c,\sigma') > \frac{10}{c^2}$ such that, for $Vh \in \mathcal{E}_{**,\sigma'}$, $a > a_0(c,\sigma')$, the problem*

$$\begin{cases}
\eta L(\Phi) + (1-\eta)\,VL'(\Psi) = Vh \quad \text{in } B(0,a)\\
\Phi \in H_a, \Phi = V\Psi, \langle V\Psi, Z_d\rangle = 0; \forall x \in B(0,a), \Psi(x_1,x_2) = \overline{\Psi(x_1,-x_2)} = \Psi(-x_1,x_2)\\
\Phi = 0 \quad \text{on } \partial B(0,a)\\
\langle Vh, Z_d\rangle = 0
\end{cases}$$

admits a unique solution, and furthermore, there exists $K(\sigma',c) > 0$ independent of a such that

$$\|\Phi\|_{H_a} \leqslant K(\sigma',c)\|h\|_{**,\sigma'}.$$

Here, $a > 10/c^2$ is not necessary, the condition $a > 10/c$ should be enough. However, this simplifies some estimates in the proof, and it will be enough for us here. Here, we require $\langle Vh, Z_d\rangle = 0$ in order to apply the Fredholm alternative in $\left\{\varphi \in H_0^1(B(0,a)), \langle\varphi, Z_d\rangle = 0\right\}$ to show the existence of a solution.

Proof. We argue by contradiction on the estimation. Assuming the existence, take any $0<\sigma'<1$, and choose $c_0(\sigma')>0$ smaller than the one from Proposition 2.1.17, and $0<c<c_0(\sigma')$. Suppose that there exists a sequence $a_n>\frac{10}{c^2}$, $a_n\to\infty$, functions $\Phi_n\in H_{a_n}$, $\Phi_n=0$ on $\partial B(0,a_n)$ and $Vh_n\in\mathcal{E}_{**,\sigma'}$ such that $\|\Phi_n\|_{H_{a_n}}=1$, $\|h_n\|_{**,\sigma'}\to 0$ and $\eta L(\Phi_n)+(1-\eta)VL'(\Psi_n)=Vh_n$ on $B(0,a_n)$. In particular, remark here that c is independent of n, only the size of the ball grows. Our goal is to show that $\|\Phi_n\|_{H_{a_n}}=o^c_{n\to\infty}(1)$, where $o^c_{n\to\infty}(1)$ is a quantity going to 0 when $n\to\infty$ at fixed c, which leads to the contradiction.

Following the same arguments as in step 1 of the proof of Proposition 2.1.17, we check that $\Phi_n\to\Phi$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ and $\eta L(\Phi)+(1-\eta)\,VL'(\Psi)=0$ in $\mathbb{R}^2$. Furthermore, it is easy to check that, since $\|\Phi_n\|_{H_{a_n}}=1$, we have $\|\Phi\|_{H_\infty}\leqslant 1$. Then, by Lemma 2.1.18, since the orthogonality and the symmetries pass at the limit, this implies that $\Phi\in\mathcal{E}_{*,\sigma}$ for any $0<\sigma<\sigma'$, and therefore, by Proposition 2.1.17, $\Phi=0$.

We deduce that $\|\Phi_n\|_{C^2(B(0,10/c^2))}=o^c_{n\to\infty}(1)$. Now, we use the same cutoff as in the proof of Lemma 2.1.18, and we have the system on $\tilde\Psi_n=\tilde\Psi_1+i\tilde\Psi_2$ (see equation (2.1.29)):

$$\begin{cases}\Delta\tilde\Psi_1-2\tilde\Psi_1-c\partial_{x_2}\tilde\Psi_2=-\tilde h_1-2\mathfrak{Re}\Big(\frac{\nabla V}{V}.\nabla\tilde\Psi_n\Big)+\mathrm{Loc}_1(\Psi_n)-2(1-|V|^2)\tilde\Psi_1\\ \Delta\tilde\Psi_2+c\partial_{x_2}\tilde\Psi_1=-\tilde h_2-2\mathfrak{Im}\Big(\frac{\nabla V}{V}.\nabla\tilde\Psi_n\Big)+\mathrm{Loc}_2(\Psi_n).\end{cases}$$

Now, multiplying the first equation by $\tilde\Psi_1$ and integrating on $\Omega=B(0,a)\backslash B(0,5/c^2)$, we have

$$\int_\Omega\big(\Delta\tilde\Psi_1-2\tilde\Psi_1\big)\tilde\Psi_1=$$

$$\int_\Omega\Big(c\partial_{x_2}\tilde\Psi_2-\tilde h_1-2\mathfrak{Re}\Big(\frac{\nabla V}{V}.\nabla\tilde\Psi_n\Big)+\mathrm{Loc}_1(\Psi_n)-2(1-|V|^2)\tilde\Psi_1\Big)\tilde\Psi_1. \tag{2.1.32}$$

We integrate by parts. Recall that $\|\Phi_n\|_{C^2(B(0,10/c^2))}=o^c_{n\to\infty}(1)$ and $\Phi_n=V\Psi_n=0$ on $\partial B(0,a_n)$, thus

$$\int_\Omega\Delta\tilde\Psi_1\tilde\Psi_1=-\int_\Omega|\nabla\tilde\Psi_1|^2+o^c_{n\to\infty}(1).$$

Furthermore, since $Vh_n\in\mathcal{E}_{**,\sigma'}$, we check easily that $\big\|\tilde h_1\big\|_{L^2(\Omega)}\leqslant o^{\sigma'}_{c\to 0}(1)$, and we compute with Lemma 2.1.3 and $\|\Phi_n\|_{C^2(B(0,10/c^2))}=o^c_{n\to\infty}(1)$ that, since for $x\in\Omega, r\geqslant 5/c^2$,

$$\Big\|\frac{\nabla V}{V}\Big\|_{L^\infty(\Omega)}+\|\mathrm{Loc}_1(\Psi_n)\|_{L^\infty(\Omega)}+\|\mathrm{Loc}_2(\Psi_n)\|_{L^\infty(\Omega)}+\|(1-|V|^2)\|_{L^\infty(\Omega)}\leqslant o_{c\to 0}(1)+o^c_{n\to\infty}(1).$$

This allows us to estimate the right hand side of (2.1.32): by Cauchy-Schwarz,

$$\big\|\nabla\tilde\Psi_1\big\|^2_{L^2(\Omega)}+2\big\|\tilde\Psi_1\big\|^2_{L^2(\Omega)}\leqslant$$

$$c\big\|\nabla\tilde\Psi_2\big\|_{L^2(\Omega)}\big\|\tilde\Psi_1\big\|_{L^2(\Omega)}+(o_{c\to 0}(1)+o^c_{n\to\infty}(1))\big(\big\|\nabla\tilde\Psi_n\big\|_{L^2(\Omega)}+\big\|\tilde\Psi_1\big\|_{L^2(\Omega)}\big)+o^c_{n\to\infty}(1).$$

Now, we multiply the second equation by $\tilde\Psi_2$, and we integrate on Ω. By integration by parts, we check

$$\big\|\nabla\tilde\Psi_2\big\|^2_{L^2(\Omega)}\leqslant$$

$$c\Big|\int_\Omega\partial_{x_2}\tilde\Psi_1\tilde\Psi_2\Big|+\Big|\int_\Omega\tilde h_2\tilde\Psi_2\Big|+2\int_\Omega\Big|\mathfrak{Im}\Big(\frac{\nabla V}{V}.\nabla\tilde\Psi_n\Big)\tilde\Psi_2\Big|+\int_\Omega\big|\mathrm{Loc}_2(\Psi_n)\tilde\Psi_2\big|+o^c_{n\to\infty}(1).$$

By integration by parts, since $\|\Phi_n\|_{C^2(B(0,10/c^2))}=o^c_{n\to\infty}(1)$ and $\Phi_n=0$ on $\partial B(0,a_n)$, we have

$$c\Big|\int_\Omega\partial_{x_2}\tilde\Psi_1\tilde\Psi_2\Big|\leqslant o^c_{n\to\infty}(1)+c\Big|\int_\Omega\tilde\Psi_1\partial_{x_2}\tilde\Psi_2\Big|\leqslant o^c_{n\to\infty}(1)+c\,\big\|\tilde\Psi_1\big\|_{L^2(\Omega)}\big\|\nabla\tilde\Psi_2\big\|_{L^2(\Omega)}.$$

We recall that $\big|\tilde\Psi_2\big| = o^c_{n\to\infty}(1)$ on $\partial B(0,5/c^2)$, therefore

$$\begin{aligned}\int_{r=5/c^2}^a \frac{|\tilde\Psi_2|^2}{r^{2+\sigma'}} r\,dr &= \frac{-1}{\sigma'}\int_{r=5/c^2}^a \partial_r\Big(\frac{1}{r^{\sigma'}}\Big)\big|\tilde\Psi_2\big|^2 dr\\ &\leqslant \frac{K(c)}{\sigma'}\big|\tilde\Psi_2\big|^2(5/c^2) + \frac{2}{\sigma'}\int_{r=5/c^2}^a \frac{1}{r^{\sigma'}}\big|\nabla\tilde\Psi_2\big|\big|\tilde\Psi_2\big| dr\\ &\leqslant o^{c,\sigma'}_{n\to\infty}(1) + \frac{2}{\sigma'}\sqrt{\int_{r=5/c^2}^a \big|\nabla\tilde\Psi_2\big|^2 r\,dr \int_{r=5/c^2}^a \frac{|\tilde\Psi_2|^2}{r^{2+\sigma'}} r\,dr}.\end{aligned}$$

We deduce that

$$\int_{r=5/c^2}^a \frac{|\tilde\Psi_2|^2}{r^{2+\sigma'}} r\,dr \leqslant o^{c,\sigma'}_{n\to\infty}(1) + \frac{K}{\sigma'}\int_{r=5/c^2}^a \big|\nabla\tilde\Psi_2\big|^2 r\,dr,$$

and therefore

$$\left|\int_\Omega \frac{|\tilde\Psi_2|^2}{(1+|x|)^{2+\sigma'}}\right| \leqslant o^{c,\sigma'}_{n\to\infty}(1) + \frac{K}{\sigma'}\big\|\nabla\tilde\Psi_2\big\|^2_{L^2(\Omega)}.$$

Since $Vh_n \in \mathcal{E}_{**,\sigma'}$, we estimate, by Cauchy-Schwarz, that

$$\left|\int_\Omega \tilde h_2\tilde\Psi_2\right| \leqslant o_{c\to 0}(1)\sqrt{\int_\Omega \frac{|\tilde\Psi_2|^2}{(1+|x|)^{2+\sigma'}}} \leqslant o^{\sigma'}_{c\to 0}(1)\big\|\nabla\tilde\Psi_2\big\|_{L^2(\Omega)} + o^{c,\sigma}_{n\to\infty}(1).$$

Furthermore, since $\mathrm{Loc}_2(\Psi_n)$ is supported in $B(0,\,10\,/\,c^2)$ and $\|\Phi_n\|_{C^1(B(0,10/c^2))} = o^c_{n\to\infty}(1)$, we check that

$$\int_\Omega \big|\mathrm{Loc}_2(\Psi_n)\tilde\Psi_2\big| \leqslant o^c_{n\to\infty}(1).$$

Finally, from Lemma 2.1.2, we check that, in $\mathbb{R}^2$,

$$\left|\frac{\nabla V}{V}\right| \leqslant K\left|i\Big(\frac{y^\perp}{|y|^2} - \frac{z^\perp}{|z|^2}\Big)\right| + \frac{K}{c(1+|x|)^2} \leqslant \frac{K}{c(1+|x|)^2},$$

and thus, by Cauchy-Schwarz,

$$\begin{aligned}\int_\Omega \left|\mathfrak{Im}\Big(\frac{\nabla V}{V}.\nabla\tilde\Psi_n\Big)\tilde\Psi_2\right| &\leqslant \big\|\nabla\tilde\Psi_n\big\|_{L^2(\Omega)}\sqrt{\int_\Omega \left|\frac{\nabla V}{V}\right|^2\big|\tilde\Psi_2\big|^2}\\ &\leqslant \frac{K\,\|\nabla\tilde\Psi_n\|_{L^2(\Omega)}}{c}\sqrt{\int_\Omega \frac{|\tilde\Psi_2|^2}{(1+|x|)^4}}.\end{aligned}$$

In Ω, $|x| \geqslant 5/c^2$, thus

$$\int_\Omega \frac{|\tilde\Psi_2|^2}{(1+|x|)^4} \leqslant c^{2(2-\sigma')}\int_\Omega \frac{|\tilde\Psi_2|^2}{(1+|x|)^{2+\sigma'}} \leqslant c^{2(2-\sigma')}K(\sigma')\big\|\nabla\tilde\Psi_2\big\|^2_{L^2(\Omega)} + o^c_{n\to\infty}(1),$$

hence

$$\int_\Omega \left|\mathfrak{Im}\Big(\frac{\nabla V}{V}.\nabla\tilde\Psi_n\Big)\tilde\Psi_2\right| \leqslant o^{\sigma'}_{c\to 0}(1)\big\|\nabla\tilde\Psi_2\big\|_{L^2(\Omega)} + o^c_{n\to\infty}(1).$$

We conclude that

$$\begin{aligned}&\big\|\nabla\tilde\Psi_1\big\|^2_{L^2(\Omega)} + 2\big\|\tilde\Psi_1\big\|^2_{L^2(\Omega)}\\ \leqslant\ & c\big\|\nabla\tilde\Psi_2\big\|_{L^2(\Omega)}\big\|\tilde\Psi_1\big\|_{L^2(\Omega)} + (o_{c\to 0}(1) + o^c_{n\to\infty}(1))\big(\big\|\nabla\tilde\Psi_n\big\|_{L^2(\Omega)} + \big\|\tilde\Psi_1\big\|_{L^2(\Omega)}\big) + o^c_{n\to\infty}(1),\end{aligned}$$

and

$$\big\|\nabla\tilde\Psi_2\big\|^2_{L^2(\Omega)} \leqslant o^c_{n\to\infty}(1) + c\big\|\tilde\Psi_1\big\|_{L^2(\Omega)}\big\|\nabla\tilde\Psi_2\big\|_{L^2(\Omega)} + o^{\sigma'}_{c\to 0}(1)\big\|\nabla\tilde\Psi_2\big\|_{L^2(\Omega)},$$

therefore

$$\big\|\nabla\tilde\Psi_1\big\|_{L^2(\Omega)} + \big\|\tilde\Psi_1\big\|_{L^2(\Omega)} + \big\|\nabla\tilde\Psi_2\big\|_{L^2(\Omega)} \leqslant o^c_{n\to\infty}(1) + o^{\sigma'}_{c\to 0}(1).$$

We have shown that for any $\sigma' > 0$,

$$\left|\int_\Omega \frac{|\tilde\Psi_2|^2}{(1+|x|)^{2+\sigma}}\right| \leqslant \Big(o^{c,\sigma'}_{n\to\infty}(1) + \frac{K}{\sigma'}\big\|\nabla\tilde\Psi_2\big\|^2_{L^2(\Omega)}\Big),$$

thus

$$\left|\int_\Omega \frac{|\tilde{\Psi}_2|^2}{(1+|x|)^{5/2}}\right| \leqslant o^c_{n\to\infty}(1) + o_{c\to 0}(1).$$

Together with $\|\Phi_n\|_{C^2(B(0,10/c^2))} = o^c_{n\to\infty}(1)$, this is in contradiction with $\|\Phi_n\|_{H_{a_n}} = 1$.

This concludes the proof of the estimation. Now, for the existence, we argue by Fredholm's alternative in $\{\varphi \in H^1_0(B(0,\, a)),\, \langle \varphi,\, Z_d\rangle = 0\}$, and we remark that the norms $\|.\|_{H_a}$ and $\|.\|_{H^1}$ are equivalent on $B(0,\, a)$. By Riesz's representation theorem, the elliptic equation $\eta L(\Phi) + (1-\eta)\, VL'(\Psi) = Vh$ can be rewritten in the operational form $\Phi + \mathcal{K}(\Phi) = \mathcal{S}(h)$ where $\mathcal{K}$ is a compact operator in $H^1_0(B(0,a))$, and it has no kernel in H_a (i.e. in $\{\varphi \in H^1_0(B(0,a)), \langle \varphi, Z_d\rangle = 0\}$) by the estimation we just showed. Therefore, there exists a unique solution $\Phi \in H_a$, and it then satisfies

$$\|\Phi\|_{H_a} \leqslant K(\sigma', c)\|h\|_{**,\sigma'}. \qquad \square$$

Proposition 2.1.20. *Consider the problem, for $0 < \sigma < \sigma' < 1$,*

$$\begin{cases} \eta L(\Phi) + (1-\eta)\, VL'(\Psi) = Vh \\ Vh \in \mathcal{E}_{**,\sigma'}, \langle Vh, Z_d\rangle = 0. \end{cases}$$

Then, there exist constants $c_0(\sigma,\sigma') > 0$ small and $C(\sigma,\sigma') > 0$ depending only on σ, σ', such that, for $0 < c \leqslant c_0(\sigma,\sigma')$ and $Vh \in \mathcal{E}_{**,\sigma'}$ with $\langle Vh, Z_d\rangle = 0$, there exists $\Phi \in \mathcal{E}_{*,\sigma}$, $\Phi = V\Psi$ solution of this problem, with

$$\|\Psi\|_{*,\sigma,d} \leqslant C(\sigma,\sigma')\|h\|_{**,\sigma',d}.$$

Proof. By Lemma 2.1.19, For $a > a_0(c,\sigma')$, there exists a solution to the problem

$$\begin{cases} \eta L(\Phi_a) + (1-\eta)\, VL'(\Psi_a) = Vh \quad \text{on } B(0,a) \\ \Phi_a \in H_a, \Phi_a = V\Psi_a, \langle V\Psi_a, Z_d\rangle = 0; \forall x \in B(0,a), \Psi_a(x_1,x_2) = \overline{\Psi_a(x_1,-x_2)} = \Psi_a(-x_1,x_2) \\ \Phi_a = 0 \quad \text{on } \partial B(0,a) \\ \langle h, Z_d\rangle = 0 \end{cases}$$

with $\|\Phi_a\|_{H_a} \leqslant K(\sigma',\, c)\|h\|_{**,\sigma'}$. Taking a sequence of values $a_n > a_0$ going to infinity, we can construct by a diagonal argument a function $\Phi \in H^1_{\text{loc}}(\mathbb{R}^2)$ which satisfies in the distribution sense

$$\eta L(\Phi) + (1-\eta)\, VL'(\Phi) = Vh$$

(hence $\Phi \in C^2(\mathbb{R}^2)$ by standard elliptic arguments), such that

$$\|\Phi\|_{H_\infty} \leqslant \limsup_{n\to\infty} \|\Phi_n\|_{H_{a_n}} \leqslant K(\sigma',c)\|h\|_{**,\sigma'},$$

thus $\Phi \in H_\infty$, and $\Phi = V\Psi$, $\langle V\Psi, Z_d\rangle = 0; \forall x \in \mathbb{R}^2, \Psi(x_1,x_2) = \overline{\Psi(x_1,-x_2)} = \Psi(-x_1,x_2)$. From Lemma 2.1.18, we deduce that $\Phi \in \mathcal{E}_{*,\sigma}$, and is thus a solution to the problem. Furthermore, by Proposition 2.1.17, $\|\Psi\|_{*,\sigma,d} \leqslant C(\sigma,\sigma')\|h\|_{**,\sigma',d}$. Still by Proposition 2.1.17, this solution is unique in $\mathcal{E}_{*,\sigma'}$. $\square$

2.1.5.3 Estimates for the contraction in the orthogonal space

We showed in Proposition 2.1.20 that the operator $\eta L(.) + (1-\eta)VL'(./V)$ is invertible from $\mathcal{E}_{**,\sigma',d} \cap \{\langle ., Z_d\rangle = 0\}$ to $\mathcal{E}_{*,\sigma,d}$. The operator $(\eta L(.) + (1-\eta)VL'(./V))^{-1}$ is the one that, for a given $Vh \in \mathcal{E}_{**,\sigma',d}$ such that $\langle Vh, Z_d\rangle = 0$, returns the unique function $\Phi = V\Psi \in \mathcal{E}_{*,\sigma,d}$ such that $\eta L(\Phi) + (1-\eta)\, VL'(\Psi) = Vh$ in the distribution sense, and this function satisfies the estimate $\|\Psi\|_{*,\sigma,d} \leqslant C(\sigma,\sigma')\|h\|_{**,\sigma',d}$.

Now, we define (for $\Phi \in C^0(\mathbb{R}^2, \mathbb{C})$)

$$\Pi^\perp_d(\Phi) := \Phi - \langle \Phi, Z_d\rangle \frac{Z_d}{\|Z_d\|^2_{L^2(\mathbb{R}^2)}},$$

the projection on the orthogonal of Z_d. We want to apply a fixed-point theorem on the functional

$$(\eta L(.) + (1-\eta)\, VL'(./V))^{-1}(\Pi^\perp_d(-F(./V))) : \mathcal{E}_{*,\sigma} \to \mathcal{E}_{*,\sigma},$$

and for that we need some estimates on the function $\Pi_d^{\perp} oF(./V): \mathcal{E}_{*,\sigma} \to \{Vh \in \mathcal{E}_{**,\sigma'}, \langle Vh, Z_d\rangle = 0\}$. The function F contains the source term $E - ic\partial_{x_2}V$ and nonlinear terms. The source term requires a precise computation (see Lemma 2.1.22) to show its smallness in the spaces of invertibility. The nonlinear terms will be small if we do the contraction in an area with small Ψ (which is the case since we will do it in the space of function $\Phi = V\Psi \in \mathcal{E}_{*,\sigma}$ such that $\|\Psi\|_{*,\sigma,d} \leqslant K_0(\sigma,\sigma')c^{1-\sigma'}$ for a well chosen constant $K_0(\sigma,\sigma') > 0$). This subsection is devoted to the proof of the following result.

Proposition 2.1.21. *For $0 < \sigma < \sigma' < 1$, there exist constants $K_0(\sigma,\sigma'), c_0(\sigma,\sigma') > 0$ depending only on σ, σ' such that for $0 < c < c_0(\sigma,\sigma')$, the function (from $\mathcal{E}_{*,\sigma,d}$ to $\mathcal{E}_{*,\sigma,d}$)*

$$\Phi \mapsto (\eta L(.) + (1-\eta)\, VL'(./V))^{-1}(\Pi_d^{\perp}(-F(\Phi/V)))$$

is a contraction in the space of functions $\Phi = V\Psi \in \mathcal{E}_{*,\sigma,d}$ such that $\|\Psi\|_{*,\sigma,d} \leqslant K_0(\sigma,\sigma')c^{1-\sigma'}$. As such, by the contraction mapping theorem, it admits a unique fixed point $\Phi \in \mathcal{E}_{*,\sigma,d}$ in $\{\Phi \in \mathcal{E}_{*,\sigma,d}, \|\Psi\|_{*,\sigma,d} \leqslant K_0(\sigma,\sigma')c^{1-\sigma'}\}$, and there exists $\lambda(c,d) \in \mathbb{R}$ such that

$$\eta L(\Phi) + (1-\eta)\, VL'(\Psi) + F(\Psi) = \lambda(c,d) Z_d$$

in the distribution sense.

We recall that, from the definition of $\mathcal{E}_{*,\sigma,d}$ in subsection 2.1.3, $\Phi \in \mathcal{E}_{*,\sigma,d}$ implies that $\langle \Phi, Z_d\rangle = 0$, which is the origin of the fact that $\eta L(\Phi) + (1-\eta)\, VL'(\Psi) + F(\Psi)$ is not zero, but only proportional to Z_d.

We start with some estimates on the terms contained in $F(\Psi)$. These are done in the following three lemmas.

Lemma 2.1.22. *For any $0 < \sigma' < 1$, there exists a constant $C_1(\sigma') > 0$ depending only on σ' such that*

$$\left\| \frac{ic\partial_{x_2}V}{V} \right\|_{**,\sigma',d} + \left\| \frac{E}{V} \right\|_{**,\sigma',d} \leqslant C_1(\sigma')c^{1-\sigma'}.$$

Proof. We have defined the norm

$$\|h\|_{**,\sigma',d} = \|Vh\|_{C^1(\{\tilde{r}\leqslant 3\})} + \|\tilde{r}^{1+\sigma'}h_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma'}h_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma'}\nabla h\|_{L^\infty(\{\tilde{r}\geqslant 2\})},$$

thus we separate two areas for the computation: the first one is where $\tilde{r} \leqslant 3$ which will be easy and then far from the vortices, i.e. in $\{\tilde{r} \geqslant 2\}$, where the division by V is not a problem.

Step 1. Estimates for E.

In (2.1.2), we showed that

$$E = (1-|V_1|^2)(1-|V_{-1}|^2)V_1V_{-1} - 2\nabla V_1.\nabla V_{-1}.$$

Near V_1, i.e. in $B(d\vec{e_1}, 3)$, we have from Lemma 2.1.1,

$$\|(1-|V_{-1}|^2)\|_{C^1(\{r_1\leqslant 3\})} \leqslant Kc^2 \quad \text{and} \quad \|\nabla V_{-1}\|_{C^1(\{r_1\leqslant 3\})} \leqslant Kc,$$

hence

$$\left\| \frac{E}{V} V \right\|_{C^1(\{r_1\leqslant 3\})} \leqslant Kc \leqslant o^{\sigma'}_{c\to 0}(1)c^{1-\sigma'}, \tag{2.1.33}$$

where $o^{\sigma'}_{c\to 0}(1)$ is a quantity that for a fixed $\sigma' > 0$, goes to 0 when $c \to 0$. By symmetry, the result holds in the area where $\tilde{r} \leqslant 3$.

We now turn to the estimates for $\tilde{r} \geqslant 2$. The first term $(1-|V_1|^2)(1-|V_{-1}|^2)$ of $\frac{E}{V}$ is real valued. Using the definition of r_1 and r_{-1} from (2.1.1), in the right half-plane, where $r_1 \leqslant r_{-1}$ and $r_{-1} \geqslant d \geqslant \frac{K}{c}$, we have from Lemma 2.1.1

$$\|r_1^{1+\sigma'}(1-|V_1|^2)(1-|V_{-1}|^2)\|_{L^\infty(\{2\leqslant r_1\leqslant r_{-1}\})} \leqslant K \left\| \frac{1}{r_1^{1-\sigma'} r_{-1}^2} \right\|_{L^\infty(\{2\leqslant r_1\leqslant r_{-1}\})}$$

and

$$\|r_1^2(1-|V_1|^2)(1-|V_{-1}|^2)\|_{L^\infty(\{2\leqslant r_1\leqslant r_{-1}\})}\leqslant K.$$

In this area, $\frac{1}{r_{-1}^2}\leqslant Kc^2$ and $\frac{1}{r_1^{1-\sigma'}}\leqslant\frac{1}{2^{1-\sigma'}}$, thus

$$\|r_1^{1+\sigma'}(1-|V_1|^2)(1-|V_{-1}|^2)\|_{L^\infty(\{2\leqslant r_1\leqslant r_{-1}\})}\leqslant K(\sigma')c^2\leqslant o^{\sigma'}_{c\to 0}(1)\,c^{1-\sigma'}.$$

By symmetry, the same result holds for the other half-plane, hence

$$\|\tilde r^{1+\sigma'}(1-|V_1|^2)(1-|V_{-1}|^2)\|_{L^\infty(\{\tilde r\geqslant 2\})}\leqslant o^{\sigma'}_{c\to 0}(1)\,c^{1-\sigma'}. \tag{2.1.34}$$

From Lemma 2.1.1, we have

$$\nabla V_\varepsilon(x)=i\varepsilon V_\varepsilon(x)\frac{x^\perp}{r^2}+O\Big(\frac{1}{r^3}\Big),$$

hence

$$\frac{\nabla V_1.\nabla V_{-1}}{V_1\,V_{-1}}=\frac{y^\perp.\,z^\perp}{r_1^2\,r_{-1}^2}+O\Big(\frac{1}{r_1^3\,r_{-1}}\Big)+O\Big(\frac{1}{r_{-1}^3\,r_1}\Big).$$

Remark that the first term is real-valued. We compute first in the right half-plane, where $r_1\leqslant r_{-1}$ and $r_{-1}\geqslant d\geqslant\frac{K}{c}$,

$$\left\|r_1^{1+\sigma'}\frac{y^\perp.\,z^\perp}{r_1^2\,r_{-1}^2}\right\|_{L^\infty(\{2\leqslant r_1\leqslant r_{-1}\})}\leqslant\left\|\frac{r_1^{1+\sigma'}}{r_1\,r_{-1}}\right\|_{L^\infty(\{2\leqslant r_1\leqslant r_{-1}\})}.$$

Since

$$\frac{r_1^{1+\sigma'}}{r_1\,r_{-1}}=\Big(\frac{r_1}{r_{-1}}\Big)^{\sigma'}\frac{1}{r_{-1}^{1-\sigma'}}\leqslant K(\sigma')\,c^{1-\sigma'},$$

we deduce

$$\left\|r_1^{1+\sigma'}\frac{y^\perp.\,z^\perp}{r_1^2\,r_{-1}^2}\right\|_{L^\infty(\{2\leqslant r_1\leqslant r_{-1}\})}\leqslant K(\sigma')c^{1-\sigma'}$$

and by symmetry,

$$\left\|\tilde r^{1+\sigma'}\frac{y^\perp.\,z^\perp}{r_1^2\,r_{-1}^2}\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\leqslant K(\sigma')c^{1-\sigma'}. \tag{2.1.35}$$

For the last two terms $O\Big(\frac{1}{r_1^3\,r_{-1}}\Big)+O\Big(\frac{1}{r_{-1}^3\,r_1}\Big)$,we will show that in the right half-plane

$$\left\|r_1^{2+\sigma'}\frac{1}{r_1^3\,r_{-1}}\right\|_{L^\infty(\{2\leqslant r_1\leqslant r_{-1}\})}+\left\|r_1^{2+\sigma'}\frac{1}{r_{-1}^3\,r_1}\right\|_{L^\infty(\{2\leqslant r_1\leqslant r_{-1}\})}\leqslant o^{\sigma'}_{c\to 0}(1)c^{1-\sigma'}. \tag{2.1.36}$$

This immediately implies

$$\left\|r_1^{1+\sigma'}\frac{1}{r_1^3\,r_{-1}}\right\|_{L^\infty(\{2\leqslant r_1\leqslant r_{-1}\})}+\left\|r_1^{1+\sigma'}\frac{1}{r_{-1}^3\,r_1}\right\|_{L^\infty(\{2\leqslant r_1\leqslant r_{-1}\})}\leqslant o^{\sigma'}_{c\to 0}(1)\,c^{1-\sigma'}. \tag{2.1.37}$$

We compute in the right half-plane where $r_1\leqslant r_{-1}$ and $r_{-1}\geqslant d\geqslant\frac{K}{c}$, $\frac{1}{r_{-1}}\leqslant Kc$ and $\frac{1}{r_1^{1-\sigma'}}\leqslant K(\sigma')$, thus

$$r_1^{2+\sigma'}\frac{1}{r_1^3\,r_{-1}}=\frac{1}{r_1^{1-\sigma'}\,r_{-1}}\leqslant Kc\leqslant o^{\sigma'}_{c\to 0}(1)\,c^{1-\sigma'}.$$

Furthermore, still in the right half-plane,

$$r_1^{2+\sigma'}\frac{1}{r_{-1}^3\,r_1}=\Big(\frac{r_1}{r_{-1}}\Big)^{1+\sigma'}\frac{1}{r_{-1}^{2-\sigma'}}\leqslant K(\sigma')\,c^{2-\sigma'}\leqslant o^{\sigma'}_{c\to 0}(1)\,c^{1-\sigma'}.$$

Gathering (2.1.36) to (2.1.37) and using the symmetry for the left half-plane, we deduce with the previous esimates (2.1.33), (2.1.34), (2.1.35) that

$$\left\|V\Big(\frac{E}{V}\Big)\right\|_{C^1(\{\tilde r\leqslant 3\})}+\left\|\tilde r^{1+\sigma'}\mathfrak{Re}\Big(\frac{E}{V}\Big)\right\|_{L^\infty(\{\tilde r\geqslant 2\})}+\left\|\tilde r^{2+\sigma'}\mathfrak{Im}\Big(\frac{E}{V}\Big)\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\leqslant K(\sigma')c^{1-\sigma'}.$$

Now, for the estimate on $\nabla\Big(\frac{E}{V}\Big)$, we have from Lemma 2.1.1, for $\tilde r\geqslant 2$,

$$|\nabla((1-|V_1|^2)(1-|V_{-1}|^2))|\leqslant|\nabla|V_1|^2(1-|V_{-1}|^2)|+|(1-|V_1|^2)\nabla|V_{-1}|^2|\leqslant\frac{K}{r_1^3r_{-1}^2}+\frac{K}{r_1^2r_{-1}^3},$$

and

$$\left|\nabla\Big(\frac{\nabla V_1.\nabla V_{-1}}{V_1\,V_{-1}}\Big)\right|\leqslant\left|\nabla\Big(\frac{\nabla V_1}{V_1}\Big).\frac{\nabla V_{-1}}{V_{-1}}\right|+\left|\frac{\nabla V_1}{V_1}.\nabla\Big(\frac{\nabla V_{-1}}{V_{-1}}\Big)\right|\leqslant\frac{K}{r_1^2r_{-1}}+\frac{K}{r_1r_{-1}^2},$$

thus, with similar estimates as previously, we deduce

$$\left\|\tilde r^{2+\sigma'}\nabla\Big(\frac{E}{V}\Big)\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\leqslant K(\sigma')c^{1-\sigma'}. \tag{2.1.38}$$

This concludes the proof of

$$\left\|\frac{E}{V}\right\|_{**,\sigma',d}\leqslant C_1'(\sigma')c^{1-\sigma'}$$

for some constant $C_1'(\sigma')>0$ depending only on σ'.

Step 2. Estimates for $ic\frac{\partial_{x_2}V}{V}$.

First, near the vortices, we have $|\partial_{x_2}V|+|\nabla\partial_{x_2}V|\leqslant K$ a universal constant, therefore

$$\left\|ic\frac{\partial_{x_2}V}{V}V\right\|_{C^1(\{\tilde r\leqslant 3\})}\leqslant Kc\leqslant o_{c\to 0}^{\sigma'}(1)c^{1-\sigma'}.$$

We now turn to the estimate for $\tilde r\geqslant 2$. Recall Lemma 2.1.5, stating that for a universal constant $C>0$, since $r_1,r_{-1}\geqslant 2$,

$$\left|ic\frac{\partial_{x_2}V}{V}-2\,cd\frac{x_1^2-d^2-x_2^2}{r_1^2r_{-1}^2}\right|\leqslant C\Big(\frac{c}{r_1^3}+\frac{c}{r_{-1}^3}\Big).$$

Remark that $2\,cd\frac{x_1^2-d^2-x_2^2}{r_1^2r_{-1}^2}$ is real-valued. Using that $cd\leqslant 2$, that

$$|x_1^2-d^2|=|(x_1-d)(x_1+d)|\leqslant r_1\,r_{-1}$$

and also that $x_2^2\leqslant r_1\,r_{-1}$, we deduce that in the right half-plane, where $r_1\leqslant r_{-1}$ and $r_{-1}\geqslant d\geqslant\frac{K}{c}$,

$$\left\|r_1^{1+\sigma'}\,2\,cd\frac{x_1^2-d^2-x_2^2}{r_1^2r_{-1}^2}\right\|_{L^\infty(\{2\leqslant r_1\leqslant r_{-1}\})}\leqslant K\left\|\frac{r_1^{1+\sigma'}}{r_1\,r_{-1}}\right\|_{L^\infty(\{2\leqslant r_1\leqslant r_{-1}\})},$$

and since we have

$$\frac{r_1^{1+\sigma'}}{r_1\,r_{-1}}=\Big(\frac{r_1}{r_{-1}}\Big)^{\sigma'}\frac{1}{r_{-1}^{1-\sigma'}}\leqslant K(\sigma')c^{1-\sigma'},$$

we infer

$$\left\|2r_1^{1+\sigma'}cd\frac{x_1^2-d^2-x_2^2}{r_1^2r_{-1}^2}\right\|_{L^\infty(\{2\leqslant r_1\leqslant r_{-1}\})}\leqslant K(\sigma')c^{1-\sigma'}.$$

It is easy to check that in the right half-plane

$$r_1^{2+\sigma'}\Big(\frac{c}{r_1^3}+\frac{c}{r_{-1}^3}\Big)\leqslant Kc\leqslant o_{c\to 0}^{\sigma'}(1)c^{1-\sigma},$$

and therefore by symmetry for the left half-plane,

$$\begin{aligned}&\left\|V\Big(ic\frac{\partial_{x_2}V}{V}\Big)\right\|_{C^1(\{\tilde r\leqslant 3\})}+\left\|\tilde r^{1+\sigma'}\mathfrak{Re}\Big(ic\frac{\partial_{x_2}V}{V}\Big)\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\\+&\left\|\tilde r^{2+\sigma'}\mathfrak{Im}\Big(ic\frac{\partial_{x_2}V}{V}\Big)\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\\\leqslant&\ K(\sigma')c^{1-\sigma'}.\end{aligned}$$

From the proof of Lemma 2.1.5, we check (using Lemma 2.1.3) that, if $\tilde{r} \geqslant 1$

$$\left|\nabla\left(ic\frac{\partial_{x_2}V}{V} - 2cd\frac{x_1^2-d^2-x_2^2}{r_1^2 r_{-1}^2}\right)\right| \leqslant K\left(\frac{c}{r_1^3}+\frac{c}{r_{-1}^3}\right).$$

With $\left|\nabla\left(\frac{1}{r_{\pm 1}}\right)\right| \leqslant \frac{K}{r_{\pm 1}^2}$ if $\tilde{r} \geqslant 1$ and similar computations as previously, we check that

$$\left|\nabla\left(2cd\frac{x_1^2-d^2-x_2^2}{r_1^2 r_{-1}^2}\right)\right| \leqslant K(\sigma')c^{1-\sigma'}.$$

Therefore, there exists $C_1''(\sigma') > 0$ such that

$$\left\|ic\frac{\partial_{x_2}V}{V}\right\|_{**,\sigma',d} \leqslant C_1'(\sigma')c^{1-\sigma'}.$$

We conclude by taking $C_1(\sigma') = \max\left(C_1'(\sigma'), C_1''(\sigma')\right)$. □

Lemma 2.1.23. *For $0 < \sigma < \sigma' < 1$, for $\Phi = V\Psi$, $\Phi' = V\Psi' \in \mathcal{E}_{*,\sigma,d}$ such that $\|\Psi\|_{*,\sigma,d}$, $\|\Psi'\|_{*,\sigma,d} \leqslant C_0$ with C_0 defined in Lemma 2.1.7, if there exists $K(\sigma,\sigma') > 0$ such that $\|\Psi\|_{*,\sigma,d}$, $\|\Psi'\|_{*,\sigma,d} \leqslant K(\sigma,\sigma')c^{1-\sigma'}$, then*

$$\left\|\frac{R(\Psi)}{V}\right\|_{**,\sigma',d} \leqslant o_{c\to 0}^{\sigma'}(1)c^{1-\sigma'}$$

and

$$\left\|\frac{R(\Psi')-R(\Psi)}{V}\right\|_{**,\sigma',d} \leqslant o_{c\to 0}^{\sigma'}(1)\|\Psi'-\Psi\|_{*,\sigma,d},$$

where the $o_{c\to 0}^{\sigma;\sigma'}(1)$ is a quantity that, for fixed σ and σ', goes to 0 when $c \to 0$.

Proof. Since $\eta \neq 0$ only in the domain where $\|.\|_{**,\sigma',d} = \|V.\|_{C^1(\{\tilde{r}\leqslant 3\})}$ and $\|.\|_{*,\sigma,d} = \|V.\|_{C^2(\{\tilde{r}\leqslant 3\})}$, we will work only with these two norms. Recall from Lemma 2.1.7 that $R(\Psi)$ is supported in $\{\eta \neq 0\}$ and

$$|R(\Psi)| + |\nabla R(\Psi)| \leqslant C\|\Phi\|^2_{C^2(\{\tilde{r}\leqslant 2\})}$$

since $\|\Psi\|_{*,\sigma,d} \leqslant C_0$. We deduce

$$\left\|\frac{R(\Psi)}{V}\right\|_{**,\sigma',d} = \|R(\Psi)\|_{C^1(\{\tilde{r}\leqslant 3\})} \leqslant K(\sigma')c^{2-2\sigma'} \leqslant o_{c\to 0}^{\sigma'}(1)c^{1-\sigma'}.$$

Furthermore, using the definition of $R(\Psi)$ in the proof of Lemma 2.1.7 we check that every term is at least quadratic in Ψ (or its real or imaginary part), therefore, with $\|\Psi\|_{*,\sigma,d}, \|\Psi'\|_{*,\sigma,d} \leqslant C_0$, $R(\Psi') - R(\Psi)$ can be estimated by

$$\begin{aligned}\left\|\frac{R(\Psi')-R(\Psi)}{V}\right\|_{**,\sigma',d} &= \|R(\Psi')-R(\Psi)\|_{C^1(\{\tilde{r}\leqslant 3\})}\\ &\leqslant K(\|\Psi\|_{*,\sigma,d}+\|\Psi'\|_{*,\sigma,d})\|\Psi'-\Psi\|_{*,\sigma,d}\\ &\leqslant o_{c\to 0}(1)\|\Psi'-\Psi\|_{*,\sigma,d}.\end{aligned}$$

□

Lemma 2.1.24. *For $0 < \sigma < \sigma' < 1$, for $\Phi = V\Psi$, $\Phi' = V\Psi' \in \mathcal{E}_{*,\sigma,d}$ such that $\|\Psi\|_{*,\sigma,d}$, $\|\Psi'\|_{*,\sigma,d} \leqslant C_0$ with C_0 defined in Lemma 2.1.7, if there exists $K(\sigma,\sigma') > 0$ such that $\|\Psi\|_{*,\sigma,d}$, $\|\Psi'\|_{*,\sigma,d} \leqslant K(\sigma,\sigma')c^{1-\sigma'}$, then*

$$\|(1-\eta)(-\nabla\Psi.\nabla\Psi + |V|^2 S(\Psi))\|_{**,\sigma',d} \leqslant o_{c\to 0}^{\sigma;\sigma'}(1)c^{1-\sigma'},$$

$$\|(1-\eta)(-\nabla\Psi'.\nabla\Psi' + \nabla\Psi.\nabla\Psi + |V|^2(S(\Psi')-S(\Psi)))\|_{**,\sigma',d} \leqslant o_{c\to 0}^{\sigma;\sigma'}(1)\|\Psi'-\Psi\|_{*,\sigma,d}.$$

Proof. As done in Lemma 2.1.23, we check easily that

$$\|(1-\eta)(\nabla\Psi.\nabla\Psi + |V|^2 S(\Psi))V\|_{C^1(\{\tilde{r}\leqslant 3\})} \leqslant K(\sigma,\sigma')c^{1-\sigma'}\|\Phi\|_{C^2(\{\tilde{r}\leqslant 3\})},$$

since in the area where $(1-\eta)\neq 0$, $C_1\leqslant|V|\leqslant 1$ for a universal constant $C_1>0$, $\Phi=V\Psi$ and using $\|V\Psi\|_{C^1(\{\tilde r\leqslant 3\})}\leqslant K(\sigma,\sigma')c^{1-\sigma'}$.

We then estimate (with $\eta=0$ in $\{\tilde r\geqslant 2\}$)

$$\begin{aligned}&\|\tilde r^{1+\sigma'}\mathfrak{Re}(\nabla\Psi.\nabla\Psi)\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ \leqslant\ & K\|\Psi\|^2_{*,\sigma,d}\left\|\frac{\tilde r^{1+\sigma'}}{\tilde r^{2+2\sigma}}\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ \leqslant\ & K(\sigma,\sigma')c^{2-2\sigma'}\\ \leqslant\ & o^{\sigma,\sigma'}_{c\to 0}(1)c^{1-\sigma'},\end{aligned}$$

and

$$\begin{aligned}&\|\tilde r^{2+\sigma'}\mathfrak{Im}(\nabla\Psi.\nabla\Psi)\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ \leqslant\ & 2\|\tilde r^{2+\sigma'}\mathfrak{Im}(\nabla\Psi).\mathfrak{Re}(\nabla\Psi)\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ \leqslant\ & K\|\Psi\|^2_{*,\sigma,d}\left\|\frac{\tilde r^{2+\sigma'}}{\tilde r^{3+2\sigma}}\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ \leqslant\ & o^{\sigma,\sigma'}_{c\to 0}(1)c^{1-\sigma'},\end{aligned}$$

and we check that with similar computations, that

$$\|\tilde r^{2+\sigma'}\nabla(\nabla\Psi.\nabla\Psi)\|_{L^\infty(\{\tilde r\geqslant 2\})}\leqslant o^{\sigma,\sigma'}_{c\to 0}(1)c^{1-\sigma'},$$

thus

$$\|(1-\eta)(-\nabla\Psi.\nabla\Psi)\|_{**,\sigma',d}\leqslant o^{\sigma,\sigma'}_{c\to 0}(1)c^{1-\sigma'}.$$

Now, since $(1-\eta)(-\nabla\Psi'.\nabla\Psi'+\nabla\Psi.\nabla\Psi)=-(1-\eta)(\nabla(\Psi'-\Psi).\nabla(\Psi'+\Psi))$, with similar computations (and $\|\Psi'+\Psi\|_{*,\sigma,d}\leqslant 2K(\sigma,\sigma')c^{1-\sigma'}$), we have

$$\|(1-\eta)(-\nabla\Psi'.\nabla\Psi'+\nabla\Psi.\nabla\Psi)\|_{**,\sigma',d}\leqslant o^{\sigma,\sigma'}_{c\to 0}(1)\|\Psi'-\Psi\|_{*,\sigma,d}.$$

Finally, recall that

$$S(\Psi)=e^{2\mathfrak{Re}(\Psi)}-1-2\,\mathfrak{Re}(\Psi).$$

Moreover, $e^{2\mathfrak{Re}(\Psi)}-1-2\,\mathfrak{Re}(\Psi)$ is real-valued and for $\tilde r\geqslant 2$, if $\|\Psi\|_{*,\sigma,d}\leqslant C_0$,

$$|\tilde r^{1+\sigma'}|V|^2(e^{2\mathfrak{Re}(\Psi)}-1-2\mathfrak{Re}(\Psi))|\leqslant K|\tilde r^{1+\sigma'}\mathfrak{Re}^2(\Psi)|\leqslant K(\sigma,\sigma')\|\Psi\|^2_{*,\sigma,d}\leqslant o^{\sigma,\sigma'}_{c\to 0}(1)c^{1-\sigma'},$$

and with Lemma 2.1.3,

$$\begin{aligned}&|\tilde r^{2+\sigma'}\nabla(|V|^2(e^{2\mathfrak{Re}(\Psi)}-1-2\mathfrak{Re}(\Psi)))|\\ \leqslant\ & 2|\tilde r^{2+\sigma'}\nabla\mathfrak{Re}(\Psi)(e^{2\mathfrak{Re}(\Psi)}-1)|+2|\tilde r^{2+\sigma'}\nabla(|V|^2)(e^{2\mathfrak{Re}(\Psi)}-1-2\mathfrak{Re}(\Psi))|\\ \leqslant\ & K\left(|\tilde r^{2+\sigma'}\nabla\mathfrak{Re}(\Psi)\mathfrak{Re}(\Psi)|+\left|\frac{\tilde r^{2+\sigma'}}{\tilde r^3}\mathfrak{Re}^2(\Psi)\right|\right)\\ \leqslant\ & K(\sigma,\sigma')\|\Psi\|^2_{*,\sigma,d}\left\|\frac{\tilde r^{2+\sigma'}}{\tilde r^{3+2\sigma}}\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ \leqslant\ & o^{\sigma,\sigma'}_{c\to 0}(1)c^{1-\sigma'},\end{aligned}$$

hence

$$\|(1-\eta)|V|^2S(\Psi)\|_{**,\sigma',d}\leqslant o^{\sigma,\sigma'}_{c\to 0}(1)c^{1-\sigma'}.$$

With similar comutations on

$$|V|^2(S(\Psi')-S(\Psi))=2|V|^2(\mathfrak{Re}(\Psi')-\mathfrak{Re}(\Psi))\sum_{n=2}^{+\infty}2^{n-1}\sum_{k=0}^{n-1}\frac{\mathfrak{Re}(\Psi)^{n-1-k}\mathfrak{Re}(\Psi')^k}{n!},$$

we conclude with

$$\|(1-\eta)(|V|^2(S(\Psi')-S(\Psi)))\|_{**,\sigma',d}\leqslant o^{\sigma,\sigma'}_{c\to 0}(1)\|\Psi'-\Psi\|_{*,\sigma,d}.\qquad\square$$

Now, we end the proof of Proposition 2.1.21

Proof. (of Proposition 2.1.21) We take the constants $C(\sigma,\sigma')$ defined in Proposition 2.1.17 and $C_1(\sigma')$ from Lemma 2.1.22. We then define $K_0(\sigma,\sigma') := C(\sigma,\sigma')(C_1(\sigma')+1)$.

To apply the contraction mapping theorem, we need to show that for $\Phi = V\Psi, \Phi' = V\Psi' \in \mathcal{E}_{*,\sigma,d}$ with

$$\|\Psi\|_{*,\sigma,d}, \|\Psi'\|_{*,\sigma,d} \leqslant K_0(\sigma,\sigma')c^{1-\sigma'},$$

we have for small $c>0$,

$$\left\|\frac{F(\Psi)}{V}\right\|_{**,\sigma',d} \leqslant \frac{K_0(\sigma,\sigma')}{C(\sigma,\sigma')}c^{1-\sigma'} \tag{2.1.39}$$

and

$$\left\|\frac{F(\Psi')-F(\Psi)}{V}\right\|_{**,\sigma',d} \leqslant o^{\sigma,\sigma'}_{c\to 0}(1)\|\Psi'-\Psi\|_{*,\sigma,d}. \tag{2.1.40}$$

If these estimates hold, using Proposition 2.1.17, we have that the closed ball $B_{\|\cdot\|_{*,\sigma,d}}(0, K_0(\sigma,\sigma')c^{1-\sigma'})$ is stable by $\Phi \mapsto V(\eta L(V.) + (1-\eta)\, VL'(.))^{-1}(\Pi_d^{\perp}(-F(\Phi/V)))$ and this operator is a contraction in the ball (for c small enough, depending on σ,σ'), hence we can apply the contraction mapping theorem.

From Lemma 2.1.7, we have

$$F(\Psi) = E - ic\partial_{x_2}V + V(1-\eta)(-\nabla\Psi.\nabla\Psi + |V|^2 S(\Psi)) + R(\Psi).$$

By Lemmas 2.1.22 to 2.1.24, we have, given that c is small enough (depending only on σ,σ'), that both (2.1.39) and (2.1.40) hold. Therefore, defining $c_0(\sigma,\sigma')>0$ small enough such that all the required conditions on c are satisfied if $c<c_0(\sigma,\sigma')$, we end the proof of Proposition 2.1.21.

We have therefore constructed a function $\Phi = V\Psi \in \mathcal{E}_{*,\sigma,d}$ such that

$$\Phi = (\eta L(.) + (1-\eta)\, VL'(./V))^{-1}(\Pi_d^{\perp}(-F(\Phi/V))).$$

Therefore, by definition of the operator $(\eta L(.) + (1-\eta)\, VL'(./V))^{-1}$, we have, in the distribution sense,

$$\eta L(\Phi) + (1-\eta)\, VL'(\Psi) = \Pi_d^{\perp}(-F(\Phi/V)),$$

and thus, there exists $\lambda(c,d)\in\mathbb{R}$ such that

$$\eta L(\Phi) + (1-\eta)VL'(\Psi) + F(\Psi) = \lambda(c,d)Z_d. \qquad \square$$

At this point, we have the existence of a function $\Phi = V\Psi \in \mathcal{E}_{*,\sigma,d}$ depending on c,d and a priori σ,σ', such that $\|\Psi\|_{*,\sigma,d} \leqslant K(\sigma,\sigma')c^{1-\sigma'}$ and

$$\eta L(\Phi) + (1-\eta)VL'(\Psi) + F(\Psi) = \lambda(c,d)Z_d \tag{2.1.41}$$

in the distribution sense for some $\lambda(c,d)\in\mathbb{R}$. By using elliptic regularity, we show easily that $\Phi \in C^{\infty}(\mathbb{R}^2,\mathbb{C})$ and that (2.1.41) is verified in the strong sense. The goal is now to show that we can take $\lambda(c,d)=0$ for a good choice of d, but first we need a better estimate on Φ using the parameters σ and σ'. We denote by $\Phi_{\sigma,\sigma'} = V\Psi_{\sigma,\sigma'}$ the solution obtained by Proposition 2.1.21 for the values $\sigma<\sigma'$.

Corollary 2.1.25. *For $0<\sigma_1<\sigma_1'<1$, $0<\sigma_2<\sigma_2'<1$, there exists $c_0(\sigma_1,\sigma_1',\sigma_2,\sigma_2')>0$ such that for $0<c<c_0(\sigma_1,\sigma_1',\sigma_2,\sigma_2')$, $\Phi_{\sigma_1,\sigma_1'} = V\Psi_{\sigma_1,\sigma_1'} = V\Psi_{\sigma_2,\sigma_2'} = \Phi_{\sigma_2,\sigma_2'}$. We can thus take any values of σ,σ' with $\sigma<\sigma'$ and the estimate*

$$\|\Psi\|_{*,\sigma,d} \leqslant K(\sigma,\sigma')c^{1-\sigma'}$$

holds for $0<c<c_0(\sigma,\sigma')$. In particular, for c small enough,

$$\|\Phi\|_{C^2(\{\tilde{r}\leqslant 3\})} \leqslant Kc^{3/4}.$$

Proof. This is because for $\sigma_1<\sigma_2$, $\mathcal{E}_{*,\sigma_2}\subset\mathcal{E}_{*,\sigma_1}$ hence the fixed point for σ_2 (for any $\sigma_2'>\sigma_2$) yields the same value of Ψ as the fixed point for σ_1 for c small enough (for any $\sigma_1'>\sigma_1$). In particular, this implies also that $\lambda(c,d)$ is independent of σ,σ' (for c small enough). $\square$

2.1.6 Estimation on the Lagrange multiplier $\lambda(c,d)$

To finish the construction of a solution of (TW_c), we need to find a link between d and c such that $\lambda(c,d)=0$ in (2.1.41). Here, we give an estimate of $\lambda(c,d)$ for small values of c.

Proposition 2.1.26. *For $\lambda(c,d), \Phi = V\Psi$ defined in the equation of Proposition 2.1.21, namely*

$$\eta L(\Phi) + (1-\eta)\, VL'(\Psi) + F(\Psi) = \lambda(c,d) Z_d,$$

we have, for any $0<\sigma<1$,

$$\lambda(c,d)\int_{\mathbb{R}^2} |\partial_d V|^2 \eta = \pi\left(\frac{1}{d} - c\right) + O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

We will take the scalar product of $\eta L(\Phi) + (1-\eta)\, VL'(\Psi) + F(\Psi) - \lambda(c,d)Z_d$ with $\partial_d V$. We will show in the proof that in the term $\langle \eta L(\Phi) + (1-\eta) VL'(\Psi) + F(\Psi), \partial_d V\rangle$, the largest contribution come from the source term $E - ic\partial_{x_2} V$ in $F(\Psi)$. We will show that $\langle E, \partial_d V\rangle \simeq \frac{\pi}{d}$ and $\langle -ic\partial_{x_2} V, \partial_d V\rangle \simeq -\pi c$, so that, at the leading order, $\lambda(c,d) \sim K\big(\frac{1}{d} - c\big)$. In the proof, steps 1, 2 and 7 show that the terms other than $E - ic\partial_{x_2} V$ are of lower order, and steps 3-6 compute exactly the contribution of these leading order terms.

Proof. Recall from Lemma 2.1.7 that $L(\Phi) = (E - ic\partial_{x_2} V)\Psi + VL'(\Psi)$, hence we write the equation under the form

$$L(\Phi) - (1-\eta)\,(E - ic\partial_{x_2} V)\Psi + F(\Psi) = \lambda(c,d) Z_d.$$

We want to take the scalar product with $\partial_d V$. We will compute the terms $(1-\eta)E\Psi$ (step 1), $F(\Psi)$ (steps 2 to 6) and in step 7 we will show that we can do an integration by parts for $\langle L(\Phi), Z_d\rangle$ and compute its contribution.

We have by definition $Z_d = \eta \partial_d V$, hence

$$\langle Z_d, \partial_d V\rangle = \int_{\mathbb{R}^2} |\partial_d V|^2 \eta$$

which is finite and independent of d since $\eta = 0$ outside $\big\{\tilde r \leqslant 2\big\}$. Recall that $\|\Psi\|_{*,\sigma} \leqslant K(\sigma,\sigma') c^{1-\sigma'}$ where

$$\begin{aligned} \|\Psi\|_{*,\sigma} &= \|V\Psi\|_{C^2(\{\tilde r\leqslant 3\})} + \|\tilde r^{1+\sigma}\Psi_1\|_{L^\infty(\{\tilde r\geqslant 2\})} + \|\tilde r^{2+\sigma}\nabla\Psi_1\|_{L^\infty(\{\tilde r\geqslant 2\})} \\ &+ \|\tilde r^{\sigma}\Psi_2\|_{L^\infty(\{\tilde r\geqslant 2\})} + \|\tilde r^{1+\sigma}\nabla\Psi_2\|_{L^\infty(\{\tilde r\geqslant 2\})} + \|\tilde r^{2+\sigma}\nabla^2\Psi\|_{L^\infty(\{\tilde r\geqslant 2\})}, \end{aligned}$$

which we will heavily use with several values of σ, σ' in the following computations, in particular for $\sigma \in]0,1[$, the estimate

$$\|\Psi\|_{*,\sigma/2,d} \leqslant K(\sigma) c^{1-\sigma}.$$

Step 1. We have $\langle (1-\eta)(E - ic\partial_{x_2} V)\Psi, \partial_d V\rangle = O^{\sigma}_{c\to 0}(c^{2-\sigma})$.

From Lemma 2.1.6, we have

$$|\partial_d V| \leqslant \frac{K}{1+\tilde r}. \tag{2.1.42}$$

In (2.1.2), we showed that

$$E = -2\nabla V_1.\nabla V_{-1} + (1-|V_1|^2)(1-|V_{-1}|^2) V_1 V_{-1},$$

hence, with Lemmas 2.1.1 and 2.1.5 (estimating $ic\partial_{x_2} V$ as in step 2 of the proof of Lemma 2.1.22), we have

$$|E - ic\partial_{x_2} V| \leqslant \frac{Kc}{1+\tilde r}$$

by using $|\nabla V_1| \leqslant \frac{K}{1+\tilde r}$, $|\nabla V_{-1}| \leqslant \frac{K}{d} \leqslant Kc$ and $|1-|V_{-1}|^2| \leqslant Kc^2$ in the right half-plane and the symmetric estimate in the other one. We also have, in $\big\{1-\eta \neq 0\big\}$,

$$|\Psi| \leqslant K\frac{\|\Psi\|_{*,\sigma/2,d}}{(1+\tilde r)^{\sigma/2}} \leqslant \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde r)^{\sigma/2}},$$

hence

$$\left|\langle (1-\eta)(E-ic\partial_{x_2}V)\Psi, \partial_d V\rangle\right| \leqslant K(\sigma)\int_{\mathbb{R}^2}\frac{c^{2-\sigma}}{(1+\tilde{r})^{2+\sigma/2}} = O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

Step 2. We have $\langle F(\Psi), \partial_d V\rangle = \langle E - ic\partial_{x_2}V, \partial_d V\rangle + O^{\sigma}_{c\to 0}(c^{2-\sigma})$.

In this step, we want to show that the nonlinear terms in $F(\Psi)$ are negligible. Recall that

$$F(\Psi) = E - ic\partial_{x_2}V + R(\Psi) + V(1-\eta)(-\nabla\Psi.\nabla\Psi + |V|^2 S(\Psi)).$$

We first show that

$$\langle R(\Psi), \partial_d V\rangle = O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

Indeed, $R(\Psi)$ is localized in $\{\tilde{r}\leqslant 2\}$ and $|R(\Psi)|\leqslant C\|\Phi\|^2_{C^1(\{\tilde{r}\leqslant 3\})}$ (since $\|\Psi\|_{*,\sigma,d}\leqslant C_0$, see Lemma 2.1.7), and using that in $\{\tilde{r}\leqslant 3\}$, $|\Phi|+|\nabla\Phi|\leqslant K(\sigma)c^{1-\sigma/2}$ yields

$$|R(\Psi)|\leqslant c\|\partial_{x_2}\Phi\|_{C^0(\{\tilde{r}\leqslant 3\})} + C\|\Phi\|^2_{C^1(\{\tilde{r}\leqslant 3\})} = O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

Now, we use $\|\Psi\|_{*,\sigma/2,d}\leqslant K(\sigma)c^{1-\sigma}$ to estimate, in $\{1-\eta\neq 0\}$,

$$|\nabla\Psi.\nabla\Psi|\leqslant \frac{K\|\Psi\|^2_{*,\sigma,d}}{(1+\tilde{r})^{2+\sigma}}\leqslant \frac{K(\sigma)c^{2-\sigma}}{(1+\tilde{r})^{2+\sigma}},$$

therefore

$$\left|\langle -\nabla\Psi.\nabla\Psi V(1-\eta), \partial_d V\rangle\right| \leqslant Kc^{2-\sigma}\int_{\mathbb{R}^2}\frac{1}{(1+\tilde{r})^{3+\sigma}} = O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

The same argument can be made for

$$\left|\langle -|V|^2 S(\Psi)V(1-\eta), \partial_d V\rangle\right| = O^{\sigma}_{c\to 0}(c^{2-\sigma})$$

by using $S(\Psi) = e^{2\mathfrak{Re}(\Psi)} - 1 - 2\mathfrak{Re}(\Psi)$ and the fact that it is real-valued.

Step 3. We have $\langle E - ic\partial_{x_2}V, \partial_d V\rangle = -2\int_{\{x_1\geqslant 0\}}\mathfrak{Re}((E-ic\partial_{x_2}V)\overline{\partial_{x_1}V_1 V_{-1}}) + O^{\sigma}_{c\to 0}(c^{2-\sigma})$.

The goal of this step is to simplify the computation by using the symmetry. By symmetry, we can only look in the right half-plane:

$$\langle E - ic\partial_{x_2}V, \partial_d V\rangle = 2\int_{\{x_1\geqslant 0\}}\mathfrak{Re}((E-ic\partial_{x_2}V)\overline{\partial_d V}).$$

Recall that $\partial_d V = -\partial_{x_1}V_1 V_{-1} + \partial_{x_1}V_{-1}V_1$, hence we need to show that

$$\int_{\{x_1\geqslant 0\}}\mathfrak{Re}((E-ic\partial_{x_2}V)\overline{\partial_{x_1}V_{-1}V_1}) = O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

We compute

$$\begin{aligned}\int_{\{x_1\geqslant 0\}}\mathfrak{Re}((E-ic\partial_{x_2}V)\overline{\partial_{x_1}V_{-1}V_1}) &= \int_{\{x_1\geqslant 0\}}\mathfrak{Re}\left(\left(\frac{E-ic\partial_{x_2}V}{V}|V|^2\right)\overline{\frac{\partial_{x_1}V_{-1}}{V_{-1}}}\right)\\ &= \int_{\{x_1\geqslant 0\}}\mathfrak{Re}\left(\frac{E-ic\partial_{x_2}V}{V}|V|^2\right)\mathfrak{Re}\left(\frac{\partial_{x_1}V_{-1}}{V_{-1}}\right)\\ &+ \int_{\{x_1\geqslant 0\}}\mathfrak{Im}\left(\frac{E-ic\partial_{x_2}V}{V}|V|^2\right)\mathfrak{Im}\left(\frac{\partial_{x_1}V_{-1}}{V_{-1}}\right).\end{aligned}$$

In the right half-plane, we have $d\leqslant r_{-1}$ and $\tilde{r}\leqslant r_1$, hence

$$\left|\mathfrak{Re}\left(\frac{\partial_{x_1}V_{-1}}{V_{-1}}\right)\right|\leqslant \frac{K}{r^3_{-1}}\leqslant \frac{Kc^{1-\sigma/2}}{(1+\tilde{r})^{2+\sigma/2}},$$

$$\left|\mathfrak{Im}\left(\frac{\partial_{x_1}V_{-1}}{V_{-1}}\right)\right|\leqslant \frac{K}{r_{-1}}\leqslant \frac{Kc^{1-\sigma/2}}{(1+\tilde{r})^{\sigma/2}},$$

from Lemma 2.1.1. Moreover,

$$\left|\mathfrak{Re}\left(\frac{E-ic\partial_{x_2}V}{V}|V|^2\right)\right|\leqslant\frac{Kc^{1-\sigma/2}}{(1+\tilde{r})^{1+\sigma/2}},$$

$$\left|\mathfrak{Im}\left(\frac{E-ic\partial_{x_2}V}{V}|V|^2\right)\right|\leqslant\frac{Kc^{1-\sigma/2}}{(1+\tilde{r})^{2+\sigma/2}},$$

from Lemma 2.1.22. We thus deduce that

$$\left|\int_{\{x_1\geqslant 0\}}\mathfrak{Re}((E-ic\partial_{x_2}V)\overline{\partial_{x_1}V_{-1}V_1})\right|\leqslant Kc^{1-\sigma/2}\int_{\mathbb{R}^2}\frac{c^{1-\sigma/2}}{(1+\tilde{r})^{2+\sigma}}=O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

Step 4. We have

$$\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(E\overline{\partial_{x_1}V_1\,V_{-1}})=-2\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(\partial_{x_2}V_1\overline{\partial_{x_1}V_1}\partial_{x_2}V_{-1}\overline{V_{-1}})+O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

The goal of this step is to compute the part of E that produces the higher order term. Recall from (2.1.2) that

$$E=-2\nabla V_1.\nabla V_{-1}+(1-|V_1|^2)(1-|V_{-1}|^2)V_1V_{-1}$$

and since

$$|(1-|V_1|^2)(1-|V_{-1}|^2)|\leqslant\frac{Kc^2}{(1+\tilde{r})^2}$$

by Lemma 2.1.1, we deduce

$$\int_{\{x_1\geqslant 0\}}\mathfrak{Re}((1-|V_1|^2)(1-|V_{-1}|^2)V_1V_{-1}\overline{\partial_{x_1}V_1\,V_{-1}})=O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

Now we compute the first contribution from $-2\nabla V_1.\nabla V_{-1}=-2\partial_{x_1}V_1\partial_{x_1}V_{-1}-2\partial_{x_2}V_1\partial_{x_2}V_{-1}$,

$$\int_{\{x_1\geqslant 0\}}\mathfrak{Re}((-2\partial_{x_1}V_1\partial_{x_1}V_{-1})\overline{\partial_{x_1}V_1\,V_{-1}})=-2\int_{\{x_1\geqslant 0\}}|\partial_{x_1}V_1|^2\,\mathfrak{Re}(\partial_{x_1}V_{-1}\overline{V_{-1}}).$$

From Lemma 2.1.1 we have

$$\mathfrak{Re}(\partial_{x_1}V_{-1}\overline{V_{-1}})=O\left(\frac{1}{r_{-1}^3}\right)$$

since the main part in $\partial_{x_1}V_{-1}\overline{V_{-1}}$ is purely imaginary. Using $r_1\leqslant r_{-1}$ and $r_{-1}\geqslant d\geqslant\frac{K}{c}$ in the right half-plane, we have $\frac{1}{r_{-1}^3}\leqslant\frac{Kc^{2-\sigma}}{(1+\tilde{r})^{1+\sigma}}$ and, noting that $|\partial_{x_1}V_1|^2\leqslant\frac{K}{(1+\tilde{r})^2}$, we obtain

$$\int_{\{x_1\geqslant 0\}}|\partial_{x_1}V_1|^2\,|\mathfrak{Re}(\partial_{x_1}V_{-1}\overline{V_{-1}})|\leqslant Kc^{2-\sigma}\int_{\{x_1\geqslant 0\}}\frac{1}{(1+\tilde{r})^{3+\sigma}}=O_{c\to 0}(c^{5/4}).$$

Finally, the second contribution from $-2\nabla V_1.\nabla V_{-1}$ is

$$\int_{\{x_1\geqslant 0\}}\mathfrak{Re}((-2\partial_{x_2}V_1\partial_{x_2}V_{-1})\overline{\partial_{x_1}V_1\,V_{-1}})=-2\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(\partial_{x_2}V_1\overline{\partial_{x_1}V_1}\partial_{x_2}V_{-1}\overline{V_{-1}})$$

which concludes the proof of this step.

Step 5. We have $\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(E\overline{\partial_{x_1}V_1\,V_{-1}})=\frac{\pi}{d}+O^{\sigma}_{c\to 0}(c^{2-\sigma})$.

By Lemma 2.1.1, we have

$$\partial_{x_2}V_{-1}\overline{V_{-1}}=-i|V_{-1}|^2\frac{y_1+2d}{r_{-1}^2}+O\left(\frac{1}{r_{-1}^3}\right).$$

The $O\Big(\frac{1}{r_{-1}^3}\Big)$ yielding a term which is a $O^\sigma_{c\to 0}(c^{2-\sigma})$ as in step 4, therefore

$$\int_{\{x_1\geqslant 0\}}\mathfrak{Re}((-2\partial_{x_2}V_1\partial_{x_2}V_{-1})\overline{\partial_{x_1}V_1\,V_{-1}})=2\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(i\partial_{x_2}V_1\overline{\partial_{x_1}V_1})|V_{-1}|^2\frac{y_1+2d}{r_{-1}^2}+O^\sigma_{c\to 0}(c^{2-\sigma}).$$

Now we compute in polar coordinate around $d\vec{e}_1$, writing $V_1=\rho_1(r_1)e^{i\theta_1}$. From Lemma 2.1.2, we have

$$\partial_{x_1}V_1=\left(\cos(\theta_1)\frac{\rho_1'(r_1)}{\rho_1(r_1)}-\frac{i}{r_1}\sin(\theta_1)\right)V_1,$$

$$\partial_{x_2}V_1=\left(\sin(\theta_1)\frac{\rho_1'(r_1)}{\rho_1(r_1)}+\frac{i}{r_1}\cos(\theta_1)\right)V_1.$$

We then compute

$$\mathfrak{Re}(i\partial_{x_2}V_1\overline{\partial_{x_1}V_1})=-|V_1|^2\left(\cos^2(\theta_1)\frac{\rho_1'}{r_1\rho_1}+\sin^2(\theta_1)\frac{\rho_1'}{r_1\rho_1}\right)=-|V_1|^2\frac{\rho_1'}{r_1\rho_1}.$$

From Lemma 2.1.1, we have $\rho_1'(r_1)=O_{r_1\to\infty}\Big(\frac{1}{r_1^3}\Big)$. As a consequence

$$\begin{aligned}&\left|\int_{\{x_1\geqslant 0\}}|V_1|^2\frac{\rho_1'}{r_1\rho_1}|V_{-1}|^2\frac{y_1+2d}{r_{-1}^2}-\int_{\{r_1\leqslant d^{1/2}\}}|V_1|^2\frac{\rho_1'}{r_1\rho_1}|V_{-1}|^2\frac{y_1+2d}{r_{-1}^2}\right|\\ \leqslant\;& Kc^{2-\sigma}\int_{\{r_1\geqslant d^{1/2}\}}\frac{1}{(1+\tilde{r})^{2+2\sigma}}\end{aligned}$$

because when $x_1\geqslant 0$ and $r_1\geqslant d^{1/2}$, we have $\left||V_1|^2\frac{\rho_1'}{r_1\rho_1}|V_{-1}|^2\frac{y_1+2d}{r_{-1}^2}\right|\leqslant\frac{Kc^{2-\sigma}}{(1+\tilde{r})^{2+2\sigma}}$. We deduce that

$$\int_{\{x_1\geqslant 0\}}|V_1|^2\frac{\rho_1'}{r_1\rho_1}|V_{-1}|^2\frac{y_1+2d}{r_{-1}^2}=\int_{\{r_1\leqslant d^{1/2}\}}|V_1|^2\frac{\rho_1'}{r_1\rho_1}|V_{-1}|^2\frac{y_1+2d}{r_{-1}^2}+O^\sigma_{c\to 0}(c^{2-\sigma}).$$

In the ball $\big\{r_1\leqslant d^{1/2}\big\}$, we have

$$r_{-1}^2=4\,d^2\left(1+O_{d\to\infty}\left(\frac{1}{d}\right)\right)\quad\text{and}\quad|V_{-1}|^2=1+O\left(\frac{1}{d^2}\right)$$

therefore

$$\int_{\{x_1\geqslant 0\}}|V_1|^2\frac{\rho_1'}{r_1\rho_1}|V_{-1}|^2\frac{y_1+2d}{r_{-1}^2}=\frac{1}{4\,d^2}\int_{\{r_1\leqslant d^{1/2}\}}|V_1|^2\frac{\rho_1'}{r_1\,\rho_1}(y_1+2d)+O^\sigma_{c\to 0}(c^{2-\sigma}).$$

Since $y_1=r_1\cos(\theta_1)$, by integration in polar coordinates we have

$$\int_{\{r_1\leqslant d^{1/2}\}}|V_1|^2\frac{\rho_1'}{r_1\rho_1}\,y_1=0$$

hence

$$\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(E\overline{\partial_{x_1}V_1\,V_{-1}})=\frac{1}{d}\int_{\{r_1\leqslant d^{1/2}\}}|V_1|^2\frac{\rho_1'}{r_1\rho_1}+O^\sigma_{c\to 0}(c^{2-\sigma}).$$

Remark that $|V_1|^2=\rho_1^2$ hence

$$\int_{\{r_1\leqslant d^{1/2}\}}|V_1|^2\frac{\rho_1'}{r_1\rho_1}=2\pi\int_0^{d^{1/2}}\rho_1\rho_1'\,dr_1=\pi[\rho_1^2]_0^{d^{1/2}}=\pi+O_{d\to\infty}\left(\frac{1}{d}\right)$$

Since $\rho_1=1+O\Big(\frac{1}{r_1^2}\Big)$ when $r_1\to\infty$ and $\rho_1(0)=0$ by Lemma 2.1.1. Therefore, as claimed,

$$\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(E\overline{\partial_{x_1}V_1\,V_{-1}})=\frac{\pi}{d}+O^\sigma_{c\to 0}(c^{2-\sigma}).$$

Notice that we have shown in particular that

$$\int_{\mathbb{R}^2}\mathfrak{Re}(i\partial_{x_2}V_1\overline{\partial_{x_1}V_1})|V_{-1}|^2=-\pi+O^\sigma_{c\to 0}(c^{1-\sigma}).\tag{2.1.43}$$

Step 6. We have $\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(-ic\partial_{x_2}V\overline{\partial_{x_1}V_1\,V_{-1}})=-\pi c+O^{\sigma}_{c\to 0}(c^{2-\sigma})$.

We are left with the computation of

$$\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(-ic\partial_{x_2}V\overline{\partial_{x_1}V_1\,V_{-1}})=$$
$$\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(-ic\partial_{x_2}V_1\overline{\partial_{x_1}V_1})|V_{-1}|^2+\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(-ic\partial_{x_2}V_{-1}V_1\overline{\partial_{x_1}V_1\,V_{-1}}) \tag{2.1.44}$$

since $\partial_{x_2}V=\partial_{x_2}V_1V_{-1}+\partial_{x_2}V_{-1}V_1$. For the second term in (2.1.44), we compute

$$-c\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(i\partial_{x_2}V_{-1}\,V_1\overline{\partial_{x_1}V_1\,V_{-1}})=c\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(\overline{\partial_{x_1}V_1}V_1)|V_{-1}|^2\frac{y_1+2d}{r_{-1}^2}+O^{\sigma}_{c\to 0}(c^{2-s})$$

in view of the relation

$$i\partial_{x_2}V_{-1}\overline{V_{-1}}=-|V_{-1}|^2\frac{y_1+2d}{r_{-1}^2}+O\left(\frac{1}{r_{-1}^3}\right)$$

from Lemma 2.1.1 and the fact that $\int_{\{x_1\geqslant 0\}}c\,O\Big(\frac{1}{r_{-1}^3}\Big)=O^{\sigma}_{c\to 0}(c^{2-\sigma})$ (as in step 4). Now recall from Lemma 2.1.2 that

$$\partial_{x_1}V_1=\left(\cos(\theta_1)\frac{\rho_1'(r_1)}{\rho_1(r_1)}-\frac{i}{r_1}\sin(\theta_1)\right)V_1$$

therefore

$$\mathfrak{Re}(\overline{\partial_{x_1}V_1}V_1)=\cos(\theta_1)\frac{\rho_1'}{\rho_1}|V_1|^2.$$

In particular, $|\mathfrak{Re}(\overline{\partial_{x_1}V_1}V_1)|\leqslant\frac{K}{1+r_1^3}$ is integrable. Furthermore, $\Big||V_{-1}|^2\frac{y_1+2d}{r_{-1}^2}\Big|=O_{c\to 0}(c)$ in the right half-plane, therefore

$$-c\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(i\partial_{x_2}V_{-1}V_1\overline{\partial_{x_1}V_1\,V_{-1}})=O_{c\to 0}(c^2)=O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

The first contribution in (2.1.44) is

$$c\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(i\partial_{x_2}V_1\overline{\partial_{x_1}V_1})|V_{-1}|^2=c\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(i\partial_{x_2}V_1\overline{\partial_{x_1}V_1})+O^{\sigma}_{c\to 0}(c^{2-\sigma})$$

using that $|V_{-1}|^2=1+O\Big(\frac{1}{r_{-1}^2}\Big)$. From (2.1.43), we have

$$\int_{\{x_1\geqslant 0\}}\mathfrak{Re}(i\partial_{x_2}V_1\overline{\partial_{x_1}V_1})=-\pi+O^{\sigma}_{c\to 0}(c^{1-\sigma}).$$

This conclude the proof of step 6, and combining step 4, 5 and 6 we deduce

$$\int_{\{x_1\geqslant 0\}}\mathfrak{Re}((E-ic\partial_{x_2}V)\overline{\partial_{x_1}V_1\,V_{-1}})=\pi\left(\frac{1}{d}-c\right)+O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

Step 7. We have $\langle L(\Phi),\partial_d V\rangle=O^{\sigma}_{c\to 0}(c^{2-\sigma})$.

We want to compute, by integration by parts, that

$$\langle L(\Phi),\partial_d V\rangle=\langle\Phi,L(\partial_d V)\rangle.$$

First, we recall that the left hand side is well defined, because we showed in the previous steps that all the other terms are bounded, therefore this one is also bounded. We have

$$\int_{B(0,R)}\mathfrak{Re}(\Delta\Phi\overline{\partial_d V})=\int_{\partial B(0,R)}\mathfrak{Re}(\nabla\Phi\overline{\partial_d V}).\vec{n}-\mathfrak{Re}(\Phi\overline{\nabla\partial_d V}).\vec{n}+\int_{B(0,R)}\mathfrak{Re}(\Phi\overline{\Delta\partial_d V}),$$

and

$$|\mathfrak{Re}(\nabla\Phi\overline{\partial_d V})|+|\mathfrak{Re}(\Phi\overline{\nabla\partial_d V})|\leqslant\frac{K}{(1+\tilde{r})^{2+1/2}},$$

therefore

$$\int_{\partial B(0,R)} \mathfrak{Re}(\nabla\Phi\overline{\partial_d V}).\vec{n} - \mathfrak{Re}(\Phi\overline{\nabla\partial_d V}).\vec{n} = o_{R\to\infty}(1)$$

and the integration by parts holds.

Recall that

$$L(h) = -\Delta h - (1-|V|^2)h + 2\,\mathfrak{Re}(\bar{V}h)V - ic\partial_{x_2}h$$

and

$$L_{V_1}(h) = -\Delta h - (1-|V_1|^2)h + 2\,\mathfrak{Re}(\bar{V}_1 h)V_1.$$

From Lemma 2.1.6 and $\|\Psi\|_{*,\sigma/2} \leqslant K(\sigma)c^{1-\sigma}$, we check easily that

$$|\langle\Phi, -ic\partial_{x_2}\partial_d V\rangle| \leqslant \int_{\mathbb{R}^2} \frac{K(\sigma)c^{2-\sigma}}{(1+\tilde{r})^{2+\sigma/2}} = O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

We therefore focus on the remaining part, with the operator

$$\tilde{L}(h) := -\Delta h - (1-|V|^2)h + 2\,\mathfrak{Re}(\bar{V}h)V - ic\partial_{x_2}h.$$

We remark that we have $L_{V_1}(\partial_{x_1}V_1) = 0$, since $\partial_{x_1}(-\Delta V_1 - (1-|V_1|^2)V_1) = 0$. Recall that $\partial_d V = -\partial_{x_1}V_1 V_{-1} + \partial_{x_1}V_{-1}V_1$ and let us compute

$$\tilde{L}(V_{-1}h) = L_{V_1}(h)V_{-1} - \Delta(V_{-1}h) + \Delta h V_{-1} + (|V|^2 - |V_1|^2)hV_{-1} + 2\mathfrak{Re}(\bar{V}_1 h)(1-|V_{-1}|^2)V,$$

therefore, using the equation or V_{-1},

$$\tilde{L}(V_{-1}\,h) = L_{V_1}(h)V_{-1} - 2\nabla V_{-1}.\nabla h + (1-|V_{-1}|^2)(1-|V_1|^2)\,V_{-1}h + 2\mathfrak{Re}(\bar{V}_1 h)(1-|V_{-1}|^2)V.$$

Taking $h = \partial_{x_1}V_1$ then yields

$$\tilde{L}(V_{-1}\partial_{x_1}V_1) = -2\nabla V_{-1}.\nabla\partial_{x_1}V_1 + (1-|V_{-1}|^2)(1-|V_1|^2)V_{-1}\partial_{x_1}V_1 + 2\mathfrak{Re}(\bar{V}_1\partial_{x_1}V_1)(1-|V_{-1}|^2)V.$$

Remark that $|\nabla V_{-1}.\nabla\partial_{x_1}V_1| \leqslant \frac{K}{(1+r_1)(1+r_{-1})^2}$, $|(1-|V_{-1}|^2)(1-|V_1|^2)V_{-1}\partial_{x_1}V_1| \leqslant \frac{K}{(1+r_1)^3(1+r_{-1})^2}$ and $|2\mathfrak{Re}(\bar{V}_1\partial_{x_1}V_1)(1-|V_{-1}|^2)V| \leqslant \frac{K}{(1+r_1)^3(1+r_{-1})^2}$ for a universal constant $K>0$ by Lemma 2.1.1, therefore

$$\langle\Phi, \tilde{L}(\partial_{x_1}V_1\,V_{-1})\rangle = O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

Exchanging the roles of V_1 and V_{-1}, we have similarly

$$\tilde{L}(V_1\partial_{x_1}V_{-1}) = -2\nabla V_1.\nabla\partial_{x_1}V_{-1} + (1-|V_{-1}|^2)(1-|V_1|^2)V_1\partial_{x_1}V_{-1}.$$

We then conclude that

$$\langle\tilde{L}(\Phi), \partial_d V\rangle = O^{\sigma}_{c\to 0}(c^{2-\sigma}),$$

which end the proof of this step. Notice that we have shown

$$|L(\partial_d V)| \leqslant \frac{Kc}{(1+\tilde{r})^2} \tag{2.1.45}$$

because $\frac{1}{(1+r_1)(1+r_{-1})} \leqslant \frac{Kc}{(1+\tilde{r})}$ in the whole space.

Step 8. Conclusion.

Adding all the results obtained in steps 1 to 7, we deduce

$$\lambda(c,d)\int_{\mathbb{R}^2}|\partial_d V|^2\eta = \pi\left(\frac{1}{d} - c\right) + O^{\sigma}_{c\to 0}(c^{2-\sigma}). \qquad \square$$

At this point, we cannot conclude that there exists d such that $\lambda(c,d) = 0$. For that, we need to show that the $O^{\sigma}_{c\to 0}(c^{2-\sigma})$ is continuous with respect to c and d. This will be shown in section 2.2.

2.2 Construction and properties of the travelling wave

Given $0<\sigma<\sigma'<1$, $d,c>0$ satisfying $\frac{1}{2c}<d<\frac{2}{c}$ and $c<c_0(\sigma,\sigma')$ defined in Proposition 2.1.21, we define $\Phi_{c,d}=V\Psi_{c,d}\in\mathcal{E}_{*,\sigma,d}$ the fonction constructed by the contraction mapping theorem in Proposition 2.1.21. From Corollary 2.1.25, for any $0<\sigma_1<\sigma_1'<1$, this function satisfies, for $c<c_0(\sigma_1,\sigma_1')$, that

$$\|\Psi_{c,d}\|_{*,\sigma_1,d}\leqslant K(\sigma_1,\sigma_1')c^{1-\sigma_1'}.$$

With equation (2.1.41) and Proposition 2.1.26, if we show that $\Phi_{c,d}$ is a continuous function of c and d, then there exists $c_0>0$ such that, for any $0<c<c_0$, by the intermediate value theorem, there exists $d_c>0$ such that $\lambda(c,d_c)=0$. This would conclude the construction of the travelling wave, and is done in subsection 2.2.1. In subsection 2.2.2, we compute some estimates on Q_c which will be usefull for understanding the linearized operator around Q_c. We also show there that Q_c is a travelling wave solution with finite energy.

2.2.1 Proof that $\Phi_{c,d}$ is a C^1 function of c and d

To end the construction of the travelling wave, we only need the continuity of $\Phi_{c,d}$ with respect to c and d. But for the construction of the C^1 branch of travelling wave in section 2.3, we need its differentiability.

2.2.1.1 Setup of the problem

From Proposition 2.1.21, the function $\Phi_{c,d}$ is defined by the implicit equation on $\mathcal{E}_{*,\sigma,d}$

$$(\eta L(.)+(1-\eta)\,VL'(./V))^{-1}(\Pi_d^{\perp}(-F(\Phi_{c,d}/V)))+\Phi_{c,d}=0,$$

where $(\eta L(.)+(1-\eta)\,VL'(./V))^{-1}$ is the linear operator from $\mathcal{E}_{**,\sigma',d}\cap\{\langle .,Z_d\rangle=0\}$ to $\mathcal{E}_{*,\sigma,d}$, that, for a function $Vh\in\mathcal{E}_{**,\sigma',d}$ with $\langle Vh,Z_d\rangle=0$, yields the unique function $\Phi=V\Psi\in\mathcal{E}_{*,\sigma,d}$ such that

$$\eta L(\Phi)+(1-\eta)VL'(\Psi)=Vh$$

in the distribution sense. We recall the quantity $Z_d(x)=\partial_d V(x)(\tilde\eta(4r_1)+\tilde\eta(4r_{-1}))$ defined in subsection 2.1.3 and we have defined the projection

$$\Pi_d^{\perp}(\Phi)=\Phi-\langle\Phi,Z_d\rangle\frac{Z_d}{\|Z_d\|^2_{L^2(\mathbb{R}^2)}}.$$

We want to show that $(c,d)\mapsto\Phi_{c,d}$ is of class C^1 from values of c,d such that $0<c<c_0(\sigma)$ and $\frac{1}{2d}<c<\frac{2}{d}$ to $\mathcal{E}_{*,\sigma,d}$. The first obstacle is that $\mathcal{E}_{*,\sigma,d}$ depends on d (through $\tilde r$), both in the weights in $\|.\|_{*,\sigma,d}$ and in the orthogonality required: $\langle\Phi,Z_d\rangle=0$. To be able to use the implicit function theorem, we first need to write an equation on Φ in a space that does not depend on d. The norm $\|.\|_{*,\sigma,d}$ depends on d (through $\tilde r$):

$$\begin{aligned}\|\Psi\|_{*,\sigma,d} &= \|V\Psi\|_{C^2(\{\tilde r\leqslant 3\})}+\|\tilde r^{1+\sigma}\Psi_1\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\nabla\Psi_1\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ &+ \|\tilde r^{\sigma}\Psi_2\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{1+\sigma}\nabla\Psi_2\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\nabla^2\Psi\|_{L^\infty(\{\tilde r\geqslant 2\})}.\end{aligned}$$

For $d_\circledast\in\mathbb{R}$, $d_\circledast\geqslant 10$ and $d\in\mathbb{R}$ such that $|d-d_\circledast|<\delta$ for some small $\delta>0$ (that we will fix later on), we define

$$\begin{aligned}\|\Phi\|_{\circledast,\sigma,d_\circledast} &:= \|\Phi\|_{C^2(\{\tilde r_\circledast\leqslant 3\})}+\left\|\tilde r_\circledast^{1+\sigma}\,\mathfrak{Re}\left(\frac{\Phi}{V_\circledast}\right)\right\|_{L^\infty(\{\tilde r_\circledast\geqslant 2\})}+\left\|\tilde r_\circledast^{2+\sigma}\nabla\mathfrak{Re}\left(\frac{\Phi}{V_\circledast}\right)\right\|_{L^\infty(\{\tilde r_\circledast\geqslant 2\})}\\ &+ \left\|\tilde r_\circledast^{\sigma}\,\mathfrak{Im}\left(\frac{\Phi}{V_\circledast}\right)\right\|_{L^\infty(\{\tilde r_\circledast\geqslant 2\})}+\left\|\tilde r_\circledast^{1+\sigma}\nabla\mathfrak{Im}\left(\frac{\Phi}{V_\circledast}\right)\right\|_{L^\infty(\{\tilde r_\circledast\geqslant 2\})}\\ &+ \left\|\tilde r_\circledast^{2+\sigma}\nabla^2\left(\frac{\Phi}{V_\circledast}\right)\right\|_{L^\infty(\{\tilde r_\circledast\geqslant 2\})},\end{aligned}$$

where $V_\circledast=V_1(x-d_\circledast\vec{e_1})V_{-1}(x+d_\circledast\vec{e_1})$ and $\tilde r_\circledast=\min(r_{1,\circledast},r_{-1,\circledast})$ with $r_{1,\circledast}=|x-d_\circledast\vec{e_1}|$, $r_{-1,\circledast}=|x+d_\circledast\vec{e_1}|$. Then, for $\Phi=V\Psi\in\mathcal{E}_{*,\sigma,d}$ (V taken in d),

$$K_1\|\Psi\|_{*,\sigma,d}\leqslant\|\Phi\|_{\circledast,\sigma,d_\circledast}\leqslant K_2\|\Psi\|_{*,\sigma,d}\tag{2.2.1}$$

where $K_{1,2}>0$ are absolute when $|d-d_\circledast|<\delta$. Indeed, we check with simple geometric arguments that if $\tilde r_\circledast \geqslant 1$, V taken in d, then $\tilde r \geqslant 1/2$ and we have

$$\left|\frac{V}{V_\circledast}-1\right| \leqslant \frac{K}{(1+\tilde r)} \quad \text{and} \quad \left|\nabla\left(\frac{V}{V_\circledast}\right)\right| \leqslant \frac{K}{(1+\tilde r)^2} \tag{2.2.2}$$

for a universal constant $K>0$. Moreover, we have, for instance, if $\tilde r_\circledast \geqslant 2$ (hence $\tilde r_\circledast \leqslant 2\tilde r$),

$$\begin{aligned}\left|\tilde r_\circledast^{1+\sigma}\,\mathfrak{Re}\left(\frac{\Phi}{V_\circledast}\right)\right| &\leqslant \left|\tilde r_\circledast^{1+\sigma}\,\mathfrak{Re}\left(\frac{\Phi}{V}\right)\right| + \left|\tilde r_\circledast^{1+\sigma}\,\mathfrak{Re}\left(\frac{\Phi}{V}\left(\frac{V}{V_\circledast}-1\right)\right)\right| \\ &\leqslant K\|\Psi\|_{*,\sigma,d} + K\left|r_\circledast^{\sigma}\frac{\Phi}{V}\right| \leqslant K\|\Psi\|_{*,\sigma,d}.\end{aligned}$$

Using (2.2.2), we can estimate similarly all the terms in (2.2.1).

We define similarly, for $g=V_\circledast(g_1+ig_2)\in C^1(\mathbb{R}^2)$, $\sigma'>0$

$$\|g\|_{\circledast\circledast,\sigma',d_\circledast} := \|g\|_{C^1(\{\tilde r_\circledast\leqslant 3\})} + \|\tilde r_\circledast^{1+\sigma'} g_1\|_{L^\infty(\{\tilde r_\circledast\geqslant 2\})} + \|\tilde r_\circledast^{2+\sigma'} g_2\|_{L^\infty(\{\tilde r_\circledast\geqslant 2\})} + \|\tilde r_\circledast^{2+\sigma'}\nabla g\|_{L^\infty(\{\tilde r_\circledast\geqslant 2\})}.$$

We have that there exist $C_1,C_2>0$ universal constants such that, for $0<\sigma'<1$ and any $d,d_\circledast\geqslant 10$ with $|d-d_\circledast|<\delta$, for any $Vh\in\mathcal{E}_{**,\sigma',d}$, $g=Vh$,

$$C_1\|h\|_{**,\sigma',d} \leqslant \|g\|_{\circledast\circledast,\sigma',d_\circledast} \leqslant C_2\|h\|_{**,\sigma',d}.$$

We define the spaces, for $\sigma,\sigma'>0$,

$$\mathcal{E}_{\circledast,\sigma,d_\circledast} :=$$

$$\left\{\Phi\in C^2(\mathbb{R}^2,\mathbb{C}), \|\Phi\|_{\circledast,\sigma,d_\circledast}<+\infty, \langle\Phi, Z_{d_\circledast}\rangle=0, \forall x\in\mathbb{R}^2, \Phi(x_1,x_2)=\overline{\Phi(x_1,-x_2)}=\Phi(-x_1,x_2)\right\}$$

and

$$\mathcal{E}_{\circledast\circledast,\sigma',d_\circledast} := \left\{g\in C^1(\mathbb{R}^2,\mathbb{C}), \|g\|_{\circledast\circledast,\sigma',d_\circledast}<+\infty\right\}.$$

We infer that, from Proposition 2.1.17, that the operator

$$(\eta L(.)+(1-\eta)\,VL'(./V))^{-1} o\,\Pi_d^\perp$$

goes from $\mathcal{E}_{\circledast\circledast,\sigma',d_\circledast}$ to $\mathcal{E}_{\circledast,\sigma,d_\circledast}$, and that (for $0<\sigma<\sigma'<1$)

$$\left\|\left|(\eta L(.)+(1-\eta)\,VL'(./V))^{-1} o\,\Pi_d^\perp\right|\right\|_{\mathcal{E}_{\circledast\circledast,\sigma',d_\circledast}\to\mathcal{E}_{\circledast,\sigma,d_\circledast}}$$

is bounded independently of c,d and $d_\circledast$ if $|d-d_\circledast|<\delta$. Indeed, the norms $\|.\|_{*,\sigma,d}$ and $\|.\|_{\circledast,\sigma,d_\circledast}$ are equivalent, as well as the norms $\|.\|_{**,\sigma',d}$ and $\|.\|_{\circledast\circledast,\sigma',d_\circledast}$ for any σ, $\sigma'>0$. About the orthogonality, we replaced $\langle\Phi,\,Z_d\rangle=0$ by $\langle\Phi,\,Z_{d_\circledast}\rangle=0$. This does not change the proof of Proposition 2.1.17, since when we argue by contradiction, if for a universal constant $|\lambda|\leqslant\delta$ we took the orthogonality $\langle\Phi,Z_{d+\lambda}\rangle=0$ instead of $\langle\Phi,Z_d\rangle=0$, the proof does not change, given that δ is small enough (independently of d). To be specific, we have to take δ small enough such that $\langle\partial_{x_1}V_1,\partial_{x_1}V_1(.+\lambda)\rangle>0$ for all $\lambda\in\,]-\delta,\delta[$.

Therefore, we take a sequence $\mathcal{D}^{(n)}>0$ going to infinity such that $|\mathcal{D}^{(n+1)}-\mathcal{D}^{(n)}|<\delta/2$, and for any given d large enough, there exists $k(d)$ such that $d\in\,]D^{(k(d))}-\delta/2,\,D^{(k(d))}+\delta/2[$, and the proof of Proposition 2.1.17 holds with the orthogonality $\langle\Phi,Z_{D^{(k(d))}}\rangle=0$ for any value of d in $]D^{(k(d))}-\delta/2,D^{(k(d))}+\delta/2[$. We denote $D^{(k(d))}=d_\circledast$. The inversion of the linearized operator then holds for $d\in\,]D^{(n)}-\delta/2,D^{(n)}+\delta/2[$ with $D^{(n)}=d_\circledast$, for all $n\in\mathbb{N}$ large enough.

Furthermore, the contraction arguments given in the proof of Proposition 2.1.21 still hold (because the norms are equivalent), hence we can define $\Phi_{c,d}$ by a fixed point argument if $\frac{1}{2d}\leqslant c\leqslant\frac{2}{d}$ and $|d-d_\circledast|<\delta$ in the space $\mathcal{E}_{\circledast,\sigma,d_\circledast}$ that does not depend on d.

We want to emphasize the fact that we change a little the definition of the spaces compared to section 2.1. In particular, for $\Phi=V\Psi$, the norm $\|.\|_{\circledast,\sigma,d_\circledast}$ is on the function Φ, and before, for $\|.\|_{*,\sigma,d}$, it was on Ψ. This is because V depends on d, and we want to remove any dependence on d. The same remark holds for $\|.\|_{\circledast\circledast,\sigma',d_\circledast}$ and $\|.\|_{\circledast\circledast,\sigma',d}$ (with $g=Vh$).

We continue, and we define

$$H(\Phi,c,d):=(\eta L(.)+(1-\eta)\,VL'(./V))^{-1}(-\Pi_d^{\perp}(F(\Phi/V)))+\Phi.$$

The function $\Phi_{c,d}\in\mathcal{E}_{\circledast,\sigma,d_\circledast}$ is defined, for $\frac{1}{2d}<c<\frac{2}{d}$ and $|d-d_\circledast|<\delta$, by being the only solution in a ball of $\mathcal{E}_{\circledast,\sigma,d_\circledast}$ (with a radius depending on σ,σ' and c but not d) to the implicit equation on Φ: $H(\Phi,c,d)=0$. This means that we shall be able to use the implicit function theorem in the space $\mathcal{E}_{\circledast,\sigma,d_\circledast}$ on the equation $H(\Phi,c,d)=0$ to show that $\Phi_{c,d}$ is a C^1 function of d in $\mathcal{E}_{\circledast,\sigma,d_\circledast}$ (for values of d such that $\frac{1}{2d}<c<\frac{2}{d}$ and $|d-d_\circledast|<\delta$). We want to differentiate this equation with respect to Φ at a fixed c and d, and show that we can invert the operator obtained when we take Φ close to $\Phi_{c,d}$. Since $(\eta L(.)+(1-\eta)VL'(./V))^{-1}$ and $\Pi_d^{\perp}$ are linear operators that do not depend on Φ, it is easy to check that $H(\Phi,c,d)$ is differentiable with respect to Φ, and we compute

$$d_\Phi H(\Phi,c,d)(\varphi)=(\eta L(.)+(1-\eta)VL'(./V))^{-1}(\Pi_d^{\perp}(-d_\Psi F(\varphi/V)))+\varphi.$$

To show that $d_\Phi H(\Phi,c,d)\colon\mathcal{E}_{\circledast,\sigma,d_\circledast}\to\mathcal{E}_{\circledast,\sigma,d_\circledast}$ and that it is invertible, it is enough to check that

$$\|\!|(\eta L(.)+(1-\eta)\,VL'(./V))^{-1}(\Pi_d^{\perp}(d_\Psi F(./V)))|\!\|_{\mathcal{E}_{\circledast,\sigma,d_\circledast}\to\mathcal{E}_{\circledast,\sigma,d_\circledast}}=o^{\sigma}_{c\to 0}(1),\tag{2.2.3}$$

which implies that $d_\Phi H(\Phi,c,d)$ is a small perturbation of Id for small values of c (at fixed σ). From Proposition 2.1.17, we have that $\|\!|(\eta L(.)+(1-\eta)VL'(./V))^{-1}o\Pi_d^{\perp}|\!\|_{\mathcal{E}_{\circledast\circledast,\sigma',d_\circledast}\to\mathcal{E}_{\circledast,\sigma,d_\circledast}}$ is bounded independently of d and $d_\circledast$ if $|d-d_\circledast|<\delta$, thus it is enough to check that, for some $\sigma'>\sigma$ (we will take $\sigma'=\frac{1+\sigma}{2}>\sigma$),

$$\|\!|d_\Psi F(.)|\!\|_{\mathcal{E}_{\circledast,\sigma,d_\circledast}\to\mathcal{E}_{\circledast\circledast,\sigma',d_\circledast}}=o^{\sigma,\sigma'}_{c\to 0}(1).$$

This is a consequence of the following lemma (for functions $\Phi=V\Psi$ such that $\|\Psi\|_{*,\sigma,d}=o^{\sigma}_{c\to 0}(1)$, which is the case if Φ is near $\Phi_{c,d}$ since $\|\Psi_{c,d}\|_{*,\sigma,d}\leqslant K(\sigma,\sigma')c^{1-\sigma'}$), where we do the computations with the $*-$norms since they are equivalent, with uniform constants, to the $\circledast$-norms. We define

$$\gamma(\sigma):=\frac{1+\sigma}{2}>\sigma.$$

Lemma 2.2.1. *There exists $C>0$ such that, for $0<\sigma<1$ and functions $\Phi=V\Psi,\varphi=V\psi\in\mathcal{E}_{*,\sigma,d}$, if $\frac{1}{2d}<c<\frac{2}{d}$ and $\|\Psi\|_{*,\sigma,d}\leqslant 1$, then*

$$\|d_\Psi F(\psi)\|_{**,\gamma(\sigma),d}\leqslant C\|\Psi\|_{*,\sigma,d}\|\psi\|_{*,\sigma,d}.$$

Proof. Recall from Lemma 2.1.7 that

$$F(\Psi)=E-ic\partial_{x_2}V+V(1-\eta)(-\nabla\Psi.\nabla\Psi+|V|^2S(\Psi))+R(\Psi)$$

with $S(\Psi)=e^{2\mathfrak{Re}(\Psi)}-1-2\mathfrak{Re}(\Psi)$ and $R(\Psi)$ at least quadratic in Φ and supported in $\{\tilde r\leqslant 2\}$. We compute

$$d_\Psi F(\psi)=V(1-\eta)(-2\,\nabla\Psi.\nabla\psi+|V|^2\,dS(\psi))+d_\Psi R(\psi).$$

We recall the condition $\frac{1}{2d}<c<\frac{2}{d}$. For the term $d_\Psi R(\psi)$, since R is a sum of terms at least quadratic in Φ and is supported in $\{\tilde r\leqslant 2\}$ (see the proof of Lemma 2.1.7), when we differentiate, every term has Ψ or $\nabla\Psi$ as a factor. Therefore,

$$\begin{aligned}\|d_\Psi R(\psi)\|_{**,\gamma(\sigma),d}&\leqslant K\|\Phi\|_{C^2(\{\tilde r\leqslant 2\})}\|V\psi\|_{C^2(\{\tilde r\leqslant 2\})}\\&\leqslant K\|\Psi\|_{*,\sigma,d}\|\psi\|_{*,\sigma,d}.\end{aligned}$$

Now, for $\mathfrak{Re}(\nabla\Psi.\nabla\psi)$, since $\sigma>0,\gamma(\sigma)<1$, we estimate

$$\begin{aligned}\|\tilde r^{1+\gamma(\sigma)}\mathfrak{Re}(\nabla\Psi.\nabla\psi)\|_{L^\infty(\{\tilde r\geqslant 2\})}&\leqslant\|\tilde r^{1+\gamma(\sigma)}|\nabla\Psi|\times|\nabla\psi|\|_{L^\infty(\{\tilde r\geqslant 2\})}\\&\leqslant K\|\Psi\|_{*,\sigma,d}\|\psi\|_{*,\sigma,d}\left\|\frac{\tilde r^{1+\gamma(\sigma)}}{\tilde r^{2+2\sigma}}\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\\&\leqslant K\|\Psi\|_{*,\sigma,d}\|\psi\|_{*,\sigma,d}.\end{aligned}$$

Similarly,

$$\begin{aligned}\|\tilde r^{2+\gamma(\sigma)}\mathfrak{Im}(\nabla\Psi.\nabla\psi)\|_{L^\infty(\{\tilde r\geqslant 2\})} &\leqslant \|\tilde r^{2+\gamma(\sigma)}\nabla\mathfrak{Re}\,\Psi.\nabla\mathfrak{Im}\,\psi\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ &+ \|\tilde r^{2+\gamma(\sigma)}\nabla\mathfrak{Im}\,\Psi.\nabla\mathfrak{Re}\,\psi\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ &\leqslant K\|\Psi\|_{*,\sigma,d}\|\psi\|_{*,\sigma,d}\left\|\frac{\tilde r^{2+\gamma(\sigma)}}{\tilde r^{3+2\sigma}}\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ &\leqslant K\|\Psi\|_{*,\sigma,d}\|\psi\|_{*,\sigma,d}.\end{aligned}$$

With similar computation, we check that

$$\|\tilde r^{2+\gamma(\sigma)}\nabla(\nabla\Psi.\nabla\psi)\|_{L^\infty(\{\tilde r\geqslant 2\})}\leqslant K\|\Psi\|_{*,\sigma,d}\|\psi\|_{*,\sigma,d}.$$

Finally, we have

$$d_\Psi S(\psi)=2\mathfrak{Re}(\psi)(e^{2\mathfrak{Re}(\Psi)}-1),$$

a real-valued term, and since $\|\Psi\|_{*,\sigma,d}\leqslant 1$, we estimate

$$\begin{aligned}\|\tilde r^{1+\gamma(\sigma)}\mathfrak{Re}(\psi)(e^{2\mathfrak{Re}(\Psi)}-1)\|_{L^\infty(\{\tilde r\geqslant 2\})} &\leqslant K\|\tilde r^{1+\gamma(\sigma)}\mathfrak{Re}(\psi)\mathfrak{Re}(\Psi)\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ &\leqslant K\|\psi\|_{*,\sigma,d}\|\Psi\|_{*,\sigma,d}\left\|\frac{\tilde r^{1+\gamma(\sigma)}}{\tilde r^{2+2\sigma}}\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ &\leqslant K\|\Psi_{c,d}\|_{*,\sigma,d}\|\psi\|_{*,\sigma,d},\end{aligned}$$

as well as

$$\begin{aligned}\|\tilde r^{2+\gamma(\sigma)}\nabla(\mathfrak{Re}(\psi)(e^{2\mathfrak{Re}(\Psi)}-1))\|_{L^\infty(\{\tilde r\geqslant 2\})} &\leqslant K\|\tilde r^{2+\gamma(\sigma)}\mathfrak{Re}(\nabla\psi)\mathfrak{Re}(\Psi)\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ &+ K\|\tilde r^{2+\gamma(\sigma)}\mathfrak{Re}(\psi)\mathfrak{Re}(\nabla\Psi)\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ &\leqslant K\|\Psi_{c,d}\|_{*,\sigma,d}\|\psi\|_{*,\sigma,d}\left\|\frac{\tilde r^{2+\gamma(\sigma)}}{\tilde r^{3+2\sigma}}\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ &\leqslant K\|\Psi_{c,d}\|_{*,\sigma,d}\|\psi\|_{*,\sigma,d}.\end{aligned}$$

These estimates imply

$$\|d_\Psi F(\psi)\|_{**,\gamma(\sigma),d}\leqslant C\|\Psi_{c,d}\|_{*,\sigma,d}\|\psi\|_{*,\sigma,d}.$$ □

2.2.1.2 Proof of the differentiabilities of $\Phi_{c,d}$ with respect of c and d

We shall now show that $c\mapsto\Phi_{c,d}$ is C^1 and compute estimates on $\partial_c\Psi_{c,d}$ at fixed d, and then show that $d\mapsto\Phi_{c,d}$ is C^1 at fixed c and estimate $\partial_d\Phi_{c,d}$. These estimates will be usefull in subsection 2.3.6. For $d\mapsto\Phi_{c,d}$, we will use the implicit function theorem (see Lemma 2.2.3), but we start here with the derivation with respect to c.

Lemma 2.2.2. *For $0<\sigma<1$, there exists $c_0(\sigma)>0$ such that, at fixed $d>\frac{1}{2c_0(\sigma)}$,*

$$c\mapsto\Phi_{c,d}\in C^1\left(\left]\frac{1}{2d},\frac{2}{d}\right[\cap]0,c_0(\sigma)[,\mathcal{E}_{*,\sigma,d}\right).$$

Remark that, at fixed d, $\partial_c\Phi_{c,d}=V\partial_c\Psi_{c,d}$.

Proof. In this proof, we consider a fixed $d>\frac{1}{2c_0(\sigma)}$. We define, for $c\in\mathbb{R}$ such that $\frac{1}{2d}<c<\frac{2}{d}$ and $0<c<c_0(\sigma)$, the operator

$$\mathbb{H}_c\colon\Phi\mapsto(\eta L(.)+(1-\eta)\,VL'(./V))^{-1}(\Pi_d^\perp(F(\Phi/V)))$$

from $\mathcal{E}_{\circledast,\sigma,d_\circledast}$ to $\mathcal{E}_{\circledast,\sigma,d_\circledast}$. The dependency on c is coming from both F and $(\eta L(.)+(1-\eta)\,VL'(./V))^{-1}$, and in this proof, we will add a subscript on these functions giving the value of c where it is taken. Take $c'\in\mathbb{R}$ such that $\frac{1}{2d}<c'<\frac{2}{d}$ and $0<c'<c_0(\sigma)$, and let us show that

$$\|\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})-\mathbb{H}_c(\Phi_{c',d})\|_{\circledast,\sigma,d_\circledast}=o^{\sigma,c}_{\varepsilon\to 0}(1).$$

In particular, remark that we look for a convergence uniform in c'. By definition of the operator $(\eta L(.)+(1-\eta)\,VL'(./V))^{-1}$, the function $\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})$ (in $\mathcal{E}_{\circledast,\sigma,d_\circledast}$) is such that, in the distribution sense,

$$\Big(\eta L(.)+(1-\eta)\,VL'\Big(\frac{\cdot}{V}\Big)\Big)_{c+\varepsilon}(\mathbb{H}_{c+\varepsilon}(\Phi_{c',d}))=\Pi_d^{\perp}(F_{c+\varepsilon}(\Phi_{c',d}/V)).$$

Since $\Phi_{c',d}\in C^\infty(\mathbb{R}^2)$, we have that $\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})\in C^\infty(\mathbb{R}^2)$ and the equation is satisfied in the strong sense. Furtheremore, since $\Pi_d^{\perp}(F_{c+\varepsilon}(\Phi_{c',d}/V))\in\mathcal{E}_{\circledast\circledast,\frac{2+\sigma}{3},d_\circledast}$ by Lemmas 2.1.22 to 2.1.24 with $\|\Pi_d^{\perp}(F_{c+\varepsilon}(\Phi_{c',d}/V))\|_{\circledast\circledast,\frac{2+\sigma}{3},d_\circledast}\leqslant K(\sigma)$ (since $\Phi_{c',d}\in\mathcal{E}_{\circledast,\frac{2+\sigma}{3},d_\circledast}$ with $\|\Phi_{c',d}\|_{\circledast,\frac{2+\sigma}{3},d_\circledast}\leqslant K(\sigma)$), we have, by Lemma 2.1.18, that $\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})\in\mathcal{E}_{\circledast,\gamma(\sigma),d_\circledast}$ (since $\gamma(\sigma)<\frac{2+\sigma}{3}$) with, fom Proposition 2.1.17, $\|\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})\|_{\circledast,\gamma(\sigma),d_\circledast}\leqslant K(\sigma)$. We check similarly that

$$\Big(\eta L(.)+(1-\eta)\,VL'\Big(\frac{\cdot}{V}\Big)\Big)_{c}(\mathbb{H}_{c}(\Phi_{c',d}))=\Pi_d^{\perp}(F_{c}(\Phi_{c',d}/V)).$$

Now, from the definitions of L and L' from Lemma 2.1.7, we have

$$\begin{aligned}\Big(\eta L(.)+(1-\eta)\,VL'\Big(\frac{\cdot}{V}\Big)\Big)_{c+\varepsilon}(\mathbb{H}_{c+\varepsilon}(\Phi_{c',d}))\;&=\;\Big(\eta L(.)+(1-\eta)\,VL'\Big(\frac{\cdot}{V}\Big)\Big)_{c}(\mathbb{H}_{c+\varepsilon}(\Phi_{c',d}))\\&-\;i\varepsilon\eta\partial_{x_2}\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})\\&-\;i\varepsilon(1-\eta)V\partial_{x_2}\Big(\frac{\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})}{V}\Big),\end{aligned}$$

and therefore

$$\begin{aligned}&\Big(\eta L(.)+(1-\eta)\,VL'\Big(\frac{\cdot}{V}\Big)\Big)_{c}(\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})-\mathbb{H}_{c}(\Phi_{c',d}))\\=\;&-(\Pi_d^{\perp}(F_{c+\varepsilon}(\Phi_{c',d}/V)-F_c(\Phi_{c',d}/V)))\\-\;&i\varepsilon\Big(\eta\partial_{x_2}\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})+(1-\eta)V\partial_{x_2}\Big(\frac{\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})}{V}\Big)\Big).\end{aligned}$$

We check, using $\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})\in\mathcal{E}_{\circledast,\gamma(\sigma),d_\circledast}$, $\|\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})\|_{\circledast,\gamma(\sigma),d_\circledast}\leqslant K(\sigma)$ that

$$i\varepsilon\Big(\eta\partial_{x_2}\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})+(1-\eta)V\partial_{x_2}\Big(\frac{\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})}{V}\Big)\Big)\in\mathcal{E}_{\circledast\circledast,\gamma(\sigma),d_\circledast},$$

with

$$\Big\|i\varepsilon\Big(\eta\partial_{x_2}\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})+(1-\eta)V\partial_{x_2}\Big(\frac{\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})}{V}\Big)\Big)\Big\|_{\circledast\circledast,\gamma(\sigma),d_\circledast}\leqslant K(\sigma)\varepsilon.$$

In particular, by Proposition 2.1.17 (from $\mathcal{E}_{\circledast\circledast,\gamma(\sigma),d_\circledast}$ to $\mathcal{E}_{\circledast,\sigma,d_\circledast}$), we have

$$\begin{aligned}&\|\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})-\mathbb{H}_{c}(\Phi_{c',d})\|_{\circledast,\sigma,d_\circledast}\\\leqslant\;&K(\sigma)\|\Pi_d^{\perp}(F_{c+\varepsilon}(\Phi_{c',d}/V)-F_c(\Phi_{c',d}/V))\|_{\circledast\circledast,\gamma(\sigma),d_\circledast}\\+\;&K(\sigma)\varepsilon.\end{aligned}$$

We recall that

$$F_c(\Psi)=E-ic\partial_{x_2}V+V(1-\eta)(-\nabla\Psi.\nabla\Psi+|V|^2S(\Psi))+R_c(\Psi),$$

therefore

$$F_{c+\varepsilon}(\Phi_{c',d}/V)-F_c(\Phi_{c',d}/V)=-i\varepsilon\partial_{x_2}V+R_{c+\varepsilon}(\Phi_{c',d}/V)-R_c(\Phi_{c',d}/V).$$

By Lemma 2.1.5 (for $i\partial_{x_2}V$) and the definition of R_c (in the proof of Lemma 2.1.7), we check that, for any $0<\sigma<1$, since $\|\Psi_{c',d}\|_{*,\sigma,d}\leqslant K(\sigma)c_0(\sigma)^{1-\gamma(\sigma)}\leqslant K(\sigma)$,

$$\|\Pi_d^{\perp}(F_{c+\varepsilon}(\Phi_{c',d}/V)-F_c(\Phi_{c',d}/V))\|_{\circledast\circledast,\sigma,d_\circledast}\leqslant K(\sigma)\frac{\varepsilon}{c}.$$

We conclude that

$$\|\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})-\mathbb{H}_{c}(\Phi_{c',d})\|_{\circledast,\sigma,d_\circledast}=o^{\sigma,c}_{\varepsilon\to0}(1),$$

thus $\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})\to\mathbb{H}_c(\Phi_{c',d})$ when $\varepsilon\to0$ in $\mathcal{E}_{\circledast,\sigma,d_\circledast}$ uniformly in c'. We remark that it is also uniform in d in any compact set of $]0,c_0(\sigma)[$.

The next step is to show that $c\mapsto\Phi_{c,d}$ is a continuous function in $\mathcal{E}_{*,\sigma,d}$. Take ε_n a sequence such that $\varepsilon_n\to 0$ when $n\to\infty$, then $\|\Phi_{c+\varepsilon_n,d}\|_{*,\sigma,d}\leqslant K_0(\sigma,\sigma')(c+\varepsilon_n)^{1-\sigma'}$ (for $K_0(\sigma,\sigma')$ the constant in Proposition 2.1.21), and (in the strong sense)

$$\Big(\eta L(.)+(1-\eta)\,VL'\Big(\frac{\cdot}{V}\Big)\Big)_{c+\varepsilon_n}(\Phi_{c+\varepsilon_n,d})+\Pi_d^{\perp}(F_{c+\varepsilon_n}(\Phi_{c+\varepsilon_n,d}/V))=0.$$

With the same arguments as in step 1 of the proof of Proposition 2.1.17, we check that, up to a subsequence, $\Phi_{c+\varepsilon_n}\to\Phi$ locally uniformly in $\mathbb{R}^2$ for some function $\Phi\in\mathcal{E}_{*,\sigma,d}$ such that $\|\Phi\|_{*,\sigma,d}\leqslant K_0(\sigma,\sigma')c^{1-\sigma'}$. Then, since

$$\mathbb{H}_{c+\varepsilon_n}(\Phi_{c+\varepsilon_n,d})+\Phi_{c+\varepsilon_n,d}=0,$$

by taking the limit when $n\to\infty$, up to a subsequence, since $\mathbb{H}_{c+\varepsilon}(\Phi_{c',d})\to\mathbb{H}_c(\Phi_{c',d})$ when $\varepsilon\to 0$ in $\mathcal{E}_{*,\sigma,d}$ (the norm is equivalent to the one of $\mathcal{E}_{\circledast,\sigma,d_\circledast}$) uniformly in c', we have

$$\mathbb{H}_c(\Phi)+\Phi=0.$$

But then, $\Phi\in\mathcal{E}_{*,\sigma,d}$, $\|\Phi\|_{*,\sigma,d}\leqslant K_0(\sigma,\sigma')c^{1-\sigma'}$ and $\mathbb{H}_c(\Phi)+\Phi=H(\Phi,c,d)=0$. By Proposition 2.1.21, this implies that $\Phi=\Phi_{c,d}$, therefore $\Phi_{c,d}$ is an accumulation point of $\Phi_{c+\varepsilon_n,d}$. It is the only accumulation point, since any other will also satisfy $\Phi\in\mathcal{E}_{*,\sigma,d}$, $\|\Phi\|_{*,\sigma,d}\leqslant K_0(\sigma,\sigma')c^{1-\sigma'}$ and $H(\Phi,c,d)=0$. Therefore, $\Phi_{c+\varepsilon_n,d}\to\Phi_{c,d}$ in $\mathcal{E}_{*,\sigma,d}$, hence $c\mapsto\Phi_{c,d}$ is a continuous function in $\mathcal{E}_{*,\sigma,d}$.

Now, let us show that it is a C^1 function in $\mathcal{E}_{*,\sigma,d}$. Since $\mathbb{H}_c(\Phi_{c,d})+\Phi_{c,d}=0$, we have

$$\begin{aligned}&\Big(\eta L(.)+(1-\eta)\,VL'\Big(\frac{\cdot}{V}\Big)\Big)_c(\Phi_{c+\varepsilon,d}-\Phi_{c,d})\\ =\;&-(\Pi_d^{\perp}(F_{c+\varepsilon}(\Phi_{c+\varepsilon,d}/V)-F_c(\Phi_{c,d}/V)))\\ -\;&i\varepsilon\Big(\eta\partial_{x_2}\Phi_{c+\varepsilon,d}+(1-\eta)V\partial_{x_2}\Big(\frac{\Phi_{c+\varepsilon,d}}{V}\Big)\Big).\end{aligned}$$

Furthermore, from $\|\Pi_d^{\perp}(F_{c+\varepsilon}(\Phi_{c',d}/V)-F_c(\Phi_{c',d}/V))\|_{\circledast\circledast,\sigma,d_\circledast}\leqslant K(\sigma,c)\varepsilon$ and

$$\Big\|i\varepsilon\Big(\eta\partial_{x_2}\Phi_{c+\varepsilon,d}+(1-\eta)V\partial_{x_2}\Big(\frac{\Phi_{c+\varepsilon,d}}{V}\Big)\Big)\Big\|_{\circledast\circledast,\sigma,d_\circledast}\leqslant K(\sigma,c)\varepsilon,$$

we deduce that $\|\Phi_{c+\varepsilon,d}-\Phi_{c,d}\|_{\circledast,\sigma,d_\circledast}\leqslant K(\sigma,c)\varepsilon$.

From the definition of F, we infer that

$$\begin{aligned}F_{c+\varepsilon}(\Phi_{c+\varepsilon,d}/V)-F_c(\Phi_{c,d}/V)\;=\;&-i\varepsilon\partial_{x_2}V\\ +\;&V(1-\eta)(-\nabla\Psi_{c+\varepsilon,d}.\nabla\Psi_{c+\varepsilon,d}+\nabla\Psi_{c,d}.\nabla\Psi_{c,d})\\ +\;&V(1-\eta)|V|^2(S(\Psi_{c+\varepsilon,d})-S(\Psi_{c,d}))\\ +\;&R_{c+\varepsilon}(\Psi_{c+\varepsilon,d})-R_c(\Psi_{c,d}).\end{aligned}$$

Now, regrouping the terms of $\Pi_d^{\perp}(d_\Psi F_c((\Phi_{c+\varepsilon,d}-\Phi_{c,d})/V))$ and using $\|\Phi_{c+\varepsilon,d}-\Phi_{c,d}\|_{\circledast,\sigma,d_\circledast}\leqslant K(\sigma,c)\varepsilon$ for the remaining nonlinear terms (which will be at least quadratic in $\Phi_{c+\varepsilon,d}-\Phi_{c,d}$, since F is C^∞ with respect to Ψ), as well as the fact that $c\mapsto R_c\in C^\infty(]0,c_0(\sigma)[,C^1(\mathbb{R}^2))$, for any $0<\sigma<1$,

$$\begin{aligned}\Pi_d^{\perp}(F_{c+\varepsilon}(\Phi_{c+\varepsilon,d}/V)-F_c(\Phi_{c,d}/V))\;=\;&\Pi_d^{\perp}(d_\Psi F_c((\Phi_{c+\varepsilon,d}-\Phi_{c,d})/V))\\ +\;&\varepsilon\Pi_d^{\perp}(-i\partial_{x_2}V)\\ +\;&O^{\sigma,c}_{\|.\|_{**,\sigma,d}}(\varepsilon^2),\end{aligned}$$

where $O^{\sigma,c}_{\|.\|_{**,\sigma,d}}(\varepsilon^2)$ is a quantity going to 0 as ε^2 when $\varepsilon\to 0$ in the norm $\|.\|_{**,\sigma,d}$ at fixed σ,c. We deduce that

$$\begin{aligned}&\Big(\mathrm{Id}+\Big(\eta L(.)+(1-\eta)\,VL'\Big(\frac{\cdot}{V}\Big)\Big)_c^{-1}(\Pi_d^{\perp}(d_\Psi F_c(./V)))\Big)((\Phi_{c+\varepsilon,d}-\Phi_{c,d}))\\ =\;&\Big(\eta L(.)+(1-\eta)\,VL'\Big(\frac{\cdot}{V}\Big)\Big)_c^{-1}\Big(-\varepsilon\Pi_d^{\perp}(-i\partial_{x_2}V)-i\varepsilon\Big(\eta\partial_{x_2}\Phi_{c+\varepsilon,d}+(1-\eta)V\partial_{x_2}\Big(\frac{\Phi_{c+\varepsilon,d}}{V}\Big)\Big)\Big)\\ +\;&\Big(\eta L(.)+(1-\eta)\,VL'\Big(\frac{\cdot}{V}\Big)\Big)_c^{-1}(O^{\sigma,c}_{\|.\|_{**,\sigma,d}}(\varepsilon^2)),\end{aligned}$$

and we have shown that $\big(\mathrm{Id}+\big(\eta L(.)+(1-\eta)\,VL'\big(\frac{\cdot}{V}\big)\big)_c^{-1}\big(\Pi_d^{\perp}\big(\frac{1}{V}d_\Phi F_c(./V)\big)\big)\big)$ is invertible from $\mathcal{E}_{\circledast,\sigma,d_\circledast}$ to $\mathcal{E}_{\circledast,\sigma,d_\circledast}$ (with an operator norm equal to $1+o^{\sigma}_{c\to 0}(1)$ if taken in $\Phi=\Phi_{c,d}$, see Lemma 2.2.1). Furthermore, $\Phi_{c,d}$ is continuous with respect to c in $\mathcal{E}_{\circledast,\gamma(\sigma),d_\circledast}$ (with the same computations as previously, replacing σ by $\gamma(\sigma)$), therefore

$$\eta\partial_{x_2}\Phi_{c+\varepsilon,d}+(1-\eta)V\partial_{x_2}\left(\frac{\Phi_{c+\varepsilon,d}}{V}\right)\to\eta\partial_{x_2}\Phi_{c,d}+(1-\eta)V\partial_{x_2}\left(\frac{\Phi_{c,d}}{V}\right)$$

in $\mathcal{E}_{\circledast\circledast,\gamma(\sigma),d_\circledast}$ when $\varepsilon\to 0$. We deduce that $c\mapsto\Phi_{c,d}$ is C^1 in $\mathcal{E}_{\circledast,\sigma,d_\circledast}$ (and therefore in $\mathcal{E}_{*,\sigma,d}$). □

Now, we show the differentiablity of $\Phi_{c,d}$ with respect to d.

Lemma 2.2.3. *For $0<\sigma<1$, there exists $c_0(\sigma)>0$ such that, for $0<c<c_0(\sigma)$,*

$$d\mapsto\Phi_{c,d}\in C^1\left(\left]\frac{1}{2c},\frac{2}{c}\right[\cap\left]d_\circledast-\frac{\delta}{2},d_\circledast+\frac{\delta}{2}\right[,\mathcal{E}_{\circledast,\sigma,d_\circledast}\right).$$

We recall that $\delta>0$ is defined at the beginning of this subsection.

Proof. We fix $0<c<c_0(\sigma)$. We define, for $d\in\left]\frac{1}{2c},\frac{2}{c}\right[\cap\left]d_\circledast-\frac{\delta}{2},d_\circledast+\frac{\delta}{2}\right[$, the function

$$\mathbb{H}_d\colon\Phi\mapsto(\eta L(.)+(1-\eta)\,VL'(./V))_d^{-1}(\Pi_d^{\perp}(F_d(\Phi/V)))$$

from $\mathcal{E}_{\circledast,\sigma,d_\circledast}$ to $\mathcal{E}_{\circledast,\sigma,d_\circledast}$, so that

$$H(\Phi,c,d)=\mathbb{H}_d(\Phi)+\Phi.$$

We took the same convention as in the proof of Lemma 2.2.2: we added a subscript in d in the operators to describe at which values of d this operator is taken.

Step 1. Differentiability of $\mathbb{H}_d$ with respect to d.

To apply the implicit function theorem, we have to check that $H(\Phi,\,c,\,d)$ (or, equivalently $\mathbb{H}_d(\Phi)$) is differentiable with respect to d, and that $\partial_d H(\Phi,\,c,\,d)\in\mathcal{E}_{\circledast,\sigma,d_\circledast}$. By definition of the operator $(\eta L(.)+(1-\eta)\,VL'(./V))^{-1}$, we have, in the distribution sense,

$$\left(\eta L(\mathbb{H}_{d+\varepsilon}(\Phi))+(1-\eta)\,VL'\left(\frac{\mathbb{H}_{d+\varepsilon}(\Phi)}{V}\right)\right)_{d+\varepsilon}+\Pi_{d+\varepsilon}^{\perp}(F_{d+\varepsilon}(\Phi/V_{d+\varepsilon}))=0$$

and

$$\left(\eta L(\mathbb{H}_d(\Phi))+(1-\eta)\,VL'\left(\frac{\mathbb{H}_d(\Phi)}{V}\right)\right)_{d}+\Pi_{d}^{\perp}(F_{d}(\Phi/V_{d}))=0.$$

From Lemma 2.1.7, we have, for any $\Phi=V_d\Psi\in\mathcal{E}_{\circledast,\sigma,d_\circledast}$ that

$$\left(\eta L(.)+(1-\eta)\,VL'\left(\frac{\cdot}{V}\right)\right)_d(\Phi)=L_d(\Phi)-(1-\eta_d)(E-ic\partial_{x_2}V)_d\Psi,$$

and with the definition of L_d (in Lemma 2.1.7), we check that, for any $\Phi\in\mathcal{E}_{\circledast,\sigma,d_\circledast}$, in the distribution sense,

$$\begin{aligned}&\left(\left(\eta L(.)+(1-\eta)\,VL'\left(\frac{\cdot}{V}\right)\right)_{d+\varepsilon}-\left(\eta L(.)+(1-\eta)\,VL'\left(\frac{\cdot}{V}\right)\right)_{d}\right)(\Phi)\\ =\;&(|V_{d+\varepsilon}|^2-|V_d|^2)\Phi+2\mathfrak{Re}(\overline{V_{d+\varepsilon}}\Phi)V_{d+\varepsilon}-2\mathfrak{Re}(\bar V_d\Phi)V_d\\ -\;&(1-\eta_{d+\varepsilon})(E-ic\partial_{x_2}V)_{d+\varepsilon}+(1-\eta_d)(E-ic\partial_{x_2}V)_d.\end{aligned}$$

We therefore compute that, in the distribution sense,

$$\begin{aligned}&\left(\eta L(.)+(1-\eta)\,VL'\left(\frac{\cdot}{V}\right)\right)_d(\mathbb{H}_{d+\varepsilon}(\Phi)-\mathbb{H}_d(\Phi))\\ =\;&-((|V_{d+\varepsilon}|^2-|V_d|^2)\mathbb{H}_{d+\varepsilon}(\Phi)+2\mathfrak{Re}(\overline{V_{d+\varepsilon}}\mathbb{H}_{d+\varepsilon}(\Phi))V_{d+\varepsilon}-2\mathfrak{Re}(\bar V_d\mathbb{H}_{d+\varepsilon}(\Phi))V_d)\\ +\;&((1-\eta_{d+\varepsilon})(E-ic\partial_{x_2}V)_{d+\varepsilon}-(1-\eta_d)(E-ic\partial_{x_2}V)_d)\mathbb{H}_{d+\varepsilon}(\Phi)\\ -\;&(\Pi_{d+\varepsilon}^{\perp}(F_{d+\varepsilon}(\Phi/V_{d+\varepsilon}))-\Pi_d^{\perp}(F_d(\Phi/V_d))).\end{aligned}$$

Since

$$\partial_d^2 V = \partial_{x_1}^2 V_1 V_{-1} + \partial_{x_1}^2 V_{-1} V_1 - 2\partial_{x_1} V_1 \partial_{x_1} V_{-1},$$

with Lemmas 2.1.1, 2.1.6 and equation (2.1.3), we check easily that

$$|V_{d+\varepsilon}|^2 - |V_d|^2 = \varepsilon\,\partial_d(|V|^2) + \frac{O^{c,d}_{\varepsilon\to 0}(\varepsilon^2)}{(1+\tilde{r})^3}$$

and

$$\nabla(|V_{d+\varepsilon}|^2) - \nabla(|V_d|^2) = \varepsilon\,\partial_d(\nabla|V|^2) + \frac{O^{c,d}_{\varepsilon\to 0}(\varepsilon^2)}{(1+\tilde{r})^3}.$$

It implies in particular that $(|V_{d+\varepsilon}|^2 - |V_d|^2)\mathbb{H}_{d+\varepsilon}(\Phi) \in \mathcal{E}_{\circledast\circledast,\gamma(\sigma),d_\circledast}$, with

$$\|(|V_{d+\varepsilon}|^2 - |V_d|^2)\mathbb{H}_{d+\varepsilon}(\Phi)\|_{\circledast\circledast,\gamma(\sigma),d_\circledast} \to 0$$

when $\varepsilon \to 0$. We check similarly

$$\begin{aligned} & 2\mathfrak{Re}(\overline{V_{d+\varepsilon}}\mathbb{H}_{d+\varepsilon}(\Phi))V_{d+\varepsilon} - 2\mathfrak{Re}(\bar{V}_d\mathbb{H}_{d+\varepsilon}(\Phi))V_d \\ = \;& \varepsilon(2\mathfrak{Re}(\overline{\partial_d V}\mathbb{H}_{d+\varepsilon}(\Phi))V_d + 2\mathfrak{Re}(\bar{V}_d\mathbb{H}_{d+\varepsilon}(\Phi))\partial_d V_d) + O^{c,d}_{\|.\|_{\circledast\circledast,\gamma(\sigma),d_\circledast}}(\varepsilon^2), \end{aligned}$$

and that $2\mathfrak{Re}(\overline{\partial_d V}\mathbb{H}_{d+\varepsilon}(\Phi))V_d + 2\mathfrak{Re}(\bar{V}_d\mathbb{H}_{d+\varepsilon}(\Phi))\partial V_d \in \mathcal{E}_{\circledast\circledast,\gamma(\sigma),d_\circledast}$. We continue, still with Lemmas 2.1.1, 2.1.6 and equation (2.1.3), we infer

$$\begin{aligned} & ((1-\eta_{d+\varepsilon})(E - ic\partial_{x_2}V)_{d+\varepsilon} - (1-\eta_d)(E - ic\partial_{x_2}V)_d)\mathbb{H}_{d+\varepsilon}(\Phi) \\ = \;& \varepsilon\partial_d((1-\eta_d)(E - ic\partial_{x_2}V)_d)\mathbb{H}_{d+\varepsilon}(\Phi) + O^{c,d}_{\|.\|_{\circledast\circledast,\gamma(\sigma),d_\circledast}}(\varepsilon^2) \end{aligned}$$

and $\partial_d((1-\eta_d)(E - ic\partial_{x_2}V)_d)\mathbb{H}_{d+\varepsilon}(\Phi) \in \mathcal{E}_{\circledast\circledast,\gamma(\sigma),d_\circledast}$. Finally, we recall that

$$F_d(\Psi) = (E - ic\partial_{x_2}V)_d + V_d(1-\eta)(-\nabla\Psi.\nabla\Psi + |V|^2 S(\Psi)) + R_d(\Psi),$$

and we check similarly that

$$\Pi^\perp_{d+\varepsilon}(F_{d+\varepsilon}(\Phi/V_{d+\varepsilon})) - \Pi^\perp_d(F_d(\Phi/V_d)) = \varepsilon\partial_d(\Pi^\perp_d(F_d(\Phi/V_d))) + O^{c,d}_{\|.\|_{\circledast\circledast,\gamma(\sigma),d_\circledast}}(\varepsilon^2).$$

We have

$$\partial_d(\Pi^\perp_d(F_d(\Phi/V_d))) = (\partial_d\Pi^\perp_d)(F_d(\Phi/V_d)) + \Pi^\perp_d(\partial_d(F_d(\Phi/V_d))),$$

and since $(\partial_d\Pi^\perp_d)(F_d(\Phi/V))$ is compactly supported, $(\partial_d\Pi^\perp_d)(F_d(\Phi/V)) \in \mathcal{E}_{\circledast\circledast,\gamma(\sigma),d_\circledast}$. We will check in the next step that $\partial_d(F_d(\Phi/V_d)) \in \mathcal{E}_{\circledast\circledast,\gamma(\sigma),d_\circledast}$. Let us suppose this result for now and finish the proof of the differentiability.

Combining the different estimates, we have in particular that

$$\left(\eta L(.) + (1-\eta)VL'\left(\frac{\cdot}{V}\right)\right)_d(\mathbb{H}_{d+\varepsilon}(\Phi) - \mathbb{H}_d(\Phi)) \to 0$$

in $\mathcal{E}_{\circledast\circledast,\gamma(\sigma),d_\circledast}$ when $\varepsilon \to 0$. By Proposition 2.1.17 (from $\mathcal{E}_{\circledast\circledast,\gamma(\sigma),d_\circledast}$ to $\mathcal{E}_{\circledast,\sigma,d_\circledast}$), this implies that

$$\mathbb{H}_{d+\varepsilon}(\Phi) \to \mathbb{H}_d(\Phi)$$

in $\mathcal{E}_{\circledast,\sigma,d_\circledast}$ when $\varepsilon \to 0$. Now, taking the equation

$$\begin{aligned} & \left(\eta L(.) + (1-\eta)VL'\left(\frac{\cdot}{V}\right)\right)_d(\mathbb{H}_{d+\varepsilon}(\Phi) - \mathbb{H}_d(\Phi)) \\ = \;& -((|V_{d+\varepsilon}|^2 - |V_d|^2)\mathbb{H}_{d+\varepsilon}(\Phi) + 2\mathfrak{Re}(\overline{V_{d+\varepsilon}}\mathbb{H}_{d+\varepsilon}(\Phi))V_{d+\varepsilon} - 2\mathfrak{Re}(\bar{V}_d\mathbb{H}_{d+\varepsilon}(\Phi))V_d) \\ + \;& ((1-\eta_{d+\varepsilon})(E - ic\partial_{x_2}V)_{d+\varepsilon} - (1-\eta_d)(E - ic\partial_{x_2}V)_d)\mathbb{H}_{d+\varepsilon}(\Phi) \\ - \;& (\Pi^\perp_{d+\varepsilon}(F_{d+\varepsilon}(\Phi/V_{d+\varepsilon})) - \Pi^\perp_d(F_d(\Phi/V_d))) \end{aligned}$$

and dividing it by ε, and then taking $\varepsilon \to 0$, we check that $d \mapsto \mathbb{H}_d(\Phi)$ is a C^1 function in $\mathcal{E}_{\circledast,\sigma,d_\circledast}$, with

$$\partial_d H(\Phi, c, d) = \partial_d\mathbb{H}_d(\Phi) = \left(\eta L(.) + (1-\eta)VL'\left(\frac{\cdot}{V}\right)\right)^{-1}(G(d,\Phi)),$$

with

$$\begin{aligned} G(d,\Phi) \;:=\; & \partial_d(|V|^2)\mathbb{H}_d(\Phi) + 2\mathfrak{Re}(\overline{\partial_d V}\mathbb{H}_d(\Phi))V_d + 2\mathfrak{Re}(\bar{V}_d\mathbb{H}_d(\Phi))\partial_d V_d \\ + \;& \partial_d((1-\eta_d)(E - ic\partial_{x_2}V)_d)\mathbb{H}_d(\Phi) - \partial_d(\Pi^\perp_d(F_d(\Phi/V_d))). \end{aligned}$$

By the implicit function theorem, with Lemma 2.2.1, since $\|\Psi_{c,d}\|_{*,\sigma,d}\leqslant K(\sigma,\sigma')c^{1-\sigma'}$ this implies that, for c small enough, $d\mapsto\Phi_{c,d}$ is a C^1 function, and

$$\partial_d\Phi_{c,d}=-d_\Phi H^{-1}(\partial_d H(\Phi_{c,d},d,c)).$$

Now, let us check that indeed $\partial_d(F_d(\Phi/V_d))\in\mathcal{E}_{\circledast\circledast,\gamma(\sigma),d_\circledast}$ for $\Phi\in\mathcal{E}_{\circledast,\sigma,d_\circledast}$.

Step 2. Proof of $\left\|\frac{\partial_d(F_d(\Phi/V_d))}{V}\right\|_{**,\gamma(\sigma),d}\leqslant K(\sigma)c^{1-\gamma(\sigma)}+K\|\Psi\|_{*,\sigma,d}$.

By the equivalence of the $*$ and $\circledast$ norms, these estimates imply that $\partial_d(F_d(\Phi\ /\ V_d))\ \in\ \mathcal{E}_{\circledast\circledast,\gamma(\sigma),d_\circledast}$. We suppose from now on that $\|\Psi\|_{*,\sigma,d}\leqslant 1$. From Lemma 2.1.7, we have

$$F_d\Big(\frac{\Phi}{V_d}\Big)=(E-ic\partial_{x_2}V)_d+R_d\Big(\frac{\Phi}{V_d}\Big)+V_d(1-\eta_d)\Big(-\nabla\Big(\frac{\Phi}{V_d}\Big).\nabla\Big(\frac{\Phi}{V_d}\Big)+|V_d|^2S\Big(\frac{\Phi}{V_d}\Big)\Big).$$

It is easy to check that at fixed Φ,c,

$$\left\|\frac{\partial_d\Big(R_d\Big(\frac{\Phi}{V_d}\Big)\Big)}{V}\right\|_{**,\gamma(\sigma),d}\leqslant K(\sigma)c^{1-\gamma(\sigma)}+K\|\Psi\|_{*,\sigma,d},$$

since it is localized near the vortices. For the nonlinear part, we have

$$\begin{aligned}\frac{\partial_d\Big(V(1-\eta)\Big(-\nabla\Big(\frac{\Phi}{V}\Big).\nabla\Big(\frac{\Phi}{V}\Big)+|V|^2S\Big(\frac{\Phi}{V}\Big)\Big)\Big)}{V}\ &=\ \frac{\partial_dV}{V}(1-\eta)(-\nabla\Psi.\nabla\Psi+|V|^2S(\Psi))\\ &-\ \partial_d\eta(-\nabla\Psi.\nabla\Psi+|V|^2S(\Psi))\\ &+\ (1-\eta)\Big(-2\nabla\Psi.\partial_d\Big(\nabla\Big(\frac{\Phi}{V_d}\Big)\Big)\Big)\\ &+\ (1-\eta)2\mathfrak{Re}(\bar{V}\partial_dV)S(\Psi)\\ &+\ (1-\eta)|V|^2\partial_d\Big(S\Big(\frac{\Phi}{V_d}\Big)\Big).\end{aligned}$$

For the first line, from Lemma 2.1.6, $\|\Psi\|_{*,\sigma,d}\leqslant 1$ and the definition of $\|.\|_{*,\sigma,d}$, we have

$$\left|\frac{\partial_dV}{V}(1-\eta)(-\nabla\Psi.\nabla\Psi+|V|^2S(\Psi))\right|\ \leqslant\ \frac{K\|\Psi\|^2_{*,\sigma,d}}{(1+\tilde{r})^3}\leqslant\frac{K\|\Psi\|_{*,\sigma,d}}{(1+\tilde{r})^3}$$

and

$$\left|\nabla\Big(\frac{\partial_dV}{V}(1-\eta)(-\nabla\Psi.\nabla\Psi+|V|^2S(\Psi))\Big)\right|\ \leqslant\ \frac{K\|\Psi\|^2_{*,\sigma,d}}{(1+\tilde{r})^3}\leqslant\frac{K\|\Psi\|_{*,\sigma,d}}{(1+\tilde{r})^3},$$

which is enough the estimate. Similarly, since $\partial_d\eta$ is compactly supported, we have

$$|\partial_d\eta(-\nabla\Psi.\nabla\Psi+|V|^2S(\Psi))|+|\nabla(\partial_d\eta(-\nabla\Psi.\nabla\Psi+|V|^2S(\Psi)))|\leqslant\frac{K\|\Psi\|^2_{*,\sigma,d}}{(1+\tilde{r})^3}\leqslant\frac{K\|\Psi\|_{*,\sigma,d}}{(1+\tilde{r})^3}.$$

Now, we develop

$$\partial_d\Big(\nabla\Big(\frac{\Phi}{V}\Big)\Big)=-\frac{\partial_dV\nabla\Phi}{V^2}-\frac{\nabla\partial_dV\Phi}{V^2}+\frac{\partial_dV\Phi\nabla V}{V^3},$$

and we check, with Lemma 2.1.6, that

$$\left|(1-\eta)\Big(-2\nabla\Psi.\partial_d\Big(\nabla\Big(\frac{\Phi}{V_d}\Big)\Big)\Big)\right|\leqslant\frac{K\|\Psi\|^2_{*,\sigma,d}}{(1+\tilde{r})^3}\leqslant\frac{K\|\Psi\|_{*,\sigma,d}}{(1+\tilde{r})^3},$$

as well as

$$\left|\nabla\Big((1-\eta)\Big(-2\nabla\Psi.\partial_d\Big(\nabla\Big(\frac{\Phi}{V_d}\Big)\Big)\Big)\Big)\right|\leqslant\frac{K\|\Psi\|^2_{*,\sigma,d}}{(1+\tilde{r})^3}\leqslant\frac{K\|\Psi\|_{*,\sigma,d}}{(1+\tilde{r})^3}.$$

Since $|\mathfrak{Re}(\bar{V}\partial_dV)|\leqslant\frac{K}{(1+\tilde{r})^3}$ from Lemma 2.1.6 and $|S(\Psi)|\leqslant K|\mathfrak{Re}(\Psi)|$ (since $\|\Psi\|_{*,\sigma,d}\leqslant 1$), we have similarly

$$|(1-\eta)2\mathfrak{Re}(\bar{V}\partial_dV)S(\Psi)|\leqslant\frac{K\|\Psi\|_{*,\sigma,d}}{(1+\tilde{r})^3},$$

and finally, since

$$\partial_d\Big(S\Big(\frac{\Phi}{V_d}\Big)\Big)=-2\mathfrak{Re}\Big(\frac{\Phi\partial_d V}{V^2}\Big)(e^{2\mathfrak{Re}(\Psi)}-1)$$

is real-valued, we check that

$$\Big|\partial_d\Big(S\Big(\frac{\Phi}{V_d}\Big)\Big)\Big|\leqslant\frac{K\|\Psi\|^2_{*,\sigma,d}}{(1+\tilde r)^{2+2\sigma}}\leqslant\frac{K\|\Psi\|_{*,\sigma,d}}{(1+\tilde r)^{1+\gamma(\sigma)}}$$

and

$$\Big|\nabla\partial_d\Big(S\Big(\frac{\Phi}{V_d}\Big)\Big)\Big|\leqslant\frac{K\|\Psi\|^2_{*,\sigma,d}}{(1+\tilde r)^{3+2\sigma}}\leqslant\frac{K\|\Psi\|_{*,\sigma,d}}{(1+\tilde r)^{2+\gamma(\sigma)}}.$$

and this is enough for the estimate. Finally, we will show that for any $0<\sigma<1$,

$$\Big\|\frac{\partial_d(E-ic\partial_{x_2}V)}{V}\Big\|_{**,\sigma,d}\leqslant K(\sigma)c^{1-\sigma},$$

which would conclude the proof of this step (taking $\gamma(\sigma)$ instead of σ).

Let us show first that

$$|\partial_d E|\leqslant\frac{Kc^{1-\sigma}}{(1+\tilde r)^{2+\sigma}}. \tag{2.2.4}$$

We have from (2.1.2) that

$$E=-2\nabla V_1.\nabla V_{-1}+(1-|V_1|^2)(1-|V_{-1}|^2)\,V_1\,V_{-1},$$

hence

$$\partial_d E=2\nabla\partial_{x_1}V_1.\nabla V_{-1}-2\nabla V_1.\nabla\partial_{x_1}V_{-1}+\partial_d((1-|V_1|^2)(1-|V_{-1}|^2)V_1\,V_{-1}).$$

With Lemmas 2.1.1 and 2.1.2, we easily check that

$$|\nabla\partial_{x_1}V_1.\nabla V_{-1}|\leqslant\frac{K}{(1+r_1)^2(1+r_{-1})},$$

$$|\nabla V_1.\nabla\partial_{x_1}V_{-1}|\leqslant\frac{K}{(1+r_1)(1+r_{-1})^2}$$

and

$$|\partial_d((1-|V_1|^2)(1-|V_{-1}|^2)V_1V_{-1})|\leqslant\frac{K}{(1+r_1)^3(1+r_{-1})^2}+\frac{K}{(1+r_1)^2(1+r_{-1})^3}.$$

In the right half-plane, where $r_1\leqslant r_{-1}$ and $r_{-1}\geqslant d$, we use

$$\frac{1}{(1+r_{-1})^{1-\sigma}}\leqslant Kc^{1-\sigma}$$

and

$$\frac{1}{(1+r_1)^\alpha}+\frac{1}{(1+r_{-1})^\alpha}\leqslant\frac{2}{(1+\tilde r)^\alpha}$$

for $\alpha>0$ on the three previous estimates to show that

$$|\partial_d E|\leqslant\frac{Kc^{1-\sigma}}{(1+\tilde r)^{2+\sigma}}$$

in the right half-plane. Similarly, the result holds in the left half-plane, and this proves (2.2.4). With similar computations, we can estimate $\nabla\big(\frac{\partial_d E}{V}\big)$ and show that

$$\Big\|\frac{\partial_d E}{V}\Big\|_{**,\sigma,d}\leqslant K(\sigma)c^{1-\sigma}.$$

Let us now prove that

$$\Big\|\frac{\partial_d(ic\partial_{x_2}V)}{V}\Big\|_{**,\sigma,d}\leqslant K(\sigma)c^{1-\sigma}. \tag{2.2.5}$$

We show easily that

$$\|ic\partial_{x_2}\partial_d V\|_{C^1(\{\tilde r\leqslant 3\})}\leqslant Kc\leqslant Kc^{1-\sigma},$$

and since $\partial_{x_2}\partial_d V = -\partial_{x_1x_2}V_1V_{-1} + \partial_{x_1x_2}V_{-1}V_1 - \partial_{x_1}V_1\partial_{x_2}V_{-1} + \partial_{x_1}V_{-1}\partial_{x_2}V_1$, by Lemma 2.1.2 we have

$$|\partial_{x_2}\partial_d V| \leqslant \frac{K}{(1+\tilde r)^2}, |\nabla\partial_{x_2}\partial_d V| \leqslant \frac{K}{(1+\tilde r)^3}$$

therefore

$$\left\| \tilde r^{1+\sigma}\mathfrak{Re}\left(\frac{ic\partial_{x_2}\partial_d V}{V}\right)\right\|_{L^\infty(\{\tilde r\geqslant 2\})} + \left\| \tilde r^{2+\sigma}\nabla\left(\frac{ic\partial_{x_2}\partial_d V}{V}\right)\right\|_{L^\infty(\{\tilde r\geqslant 2\})} \leqslant Kc \leqslant Kc^{1-\sigma}.$$

This proves that (2.2.5) is true for the real part contribution. We are left with the proof of

$$\left\| c\,\tilde r^{2+\sigma}\mathfrak{Im}\left(\frac{i\,\partial_{x_2}\partial_d V}{V}\right)\right\|_{L^\infty(\{\tilde r\geqslant 2\})} \leqslant K(\sigma)c^{1-\sigma},$$

which is more delicate and relies on some cancelations. We compute

$$\mathfrak{Im}\left(\frac{i\,\partial_{x_2}\partial_d V}{V}\right) = -\mathfrak{Re}\left(-\frac{\partial_{x_1x_2}V_1}{V_1} + \frac{\partial_{x_1x_2}V_{-1}}{V_{-1}}\right) - \mathfrak{Re}\left(-\frac{\partial_{x_1}V_1}{V_1}\frac{\partial_{x_2}V_{-1}}{V_{-1}} + \frac{\partial_{x_1}V_{-1}}{V_{-1}}\frac{\partial_{x_2}V_1}{V_{-1}}\right).$$

From Lemma 2.1.2, we have

$$\frac{\partial_{x_1}V_1}{V_1} = -\frac{i}{r_1}\sin(\theta_1) + O_{r_1\to\infty}\left(\frac{1}{r_1^3}\right)$$

and the part in $O_{r_1\to\infty}\left(\frac{1}{r_1^3}\right)$ can be estimated as in the proof of Lemma 2.1.22 for $\left\|\frac{ic\partial_{x_2}V}{V}\right\|_{**,\sigma,d}$. In particular, we will just compute the terms of order less than $\frac{1}{r_1^3}$ or $\frac{1}{r_{-1}^3}$. From Lemma 2.1.2, we have also

$$\frac{\partial_{x_2}V_1}{V_1} = -\frac{i}{r_1}\cos(\theta_1) + O_{r_1\to\infty}\left(\frac{1}{r_1^3}\right)$$

and

$$\mathfrak{Re}\left(\frac{\partial_{x_1x_2}V_1}{V_1}\right) = \frac{\cos(\theta_1)\sin(\theta_1)}{r_1^2} + O_{r_1\to\infty}\left(\frac{1}{r_1^3}\right).$$

These two estimates hold by changing $i\to -i$, $\theta_1\to\theta_{-1}$, $r_1\to r_{-1}$ and $V_1\to V_{-1}$. We then deduce that

$$\begin{aligned}\mathfrak{Im}\left(\frac{i\,\partial_{x_2}\partial_d V}{V}\right) &= -\left(-\frac{\cos(\theta_1)\sin(\theta_1)}{r_1^2} + \frac{\cos(\theta_{-1})\sin(\theta_{-1})}{r_{-1}^2}\right)\\ &- \left(-\frac{\sin(\theta_1)}{r_1}\frac{\cos(\theta_{-1})}{r_{-1}} + \frac{\sin(\theta_{-1})}{r_{-1}}\frac{\cos(\theta_1)}{r_1}\right)\\ &+ O_{r_1\to\infty}\left(\frac{1}{r_1^3}\right) + O_{r_{-1}\to\infty}\left(\frac{1}{r_{-1}^3}\right).\end{aligned} \tag{2.2.6}$$

We start with the second term of (2.2.6) which is the easiest one. We use for $\epsilon = \pm 1$ that

$$\cos(\theta_\epsilon) = \frac{x_1 - d\epsilon}{r_\epsilon} \quad\text{and}\quad \sin(\theta_\epsilon) = \frac{x_2}{r_\epsilon}$$

to compute

$$\sin(\theta_1)\cos(\theta_{-1}) = \frac{(x_1+d)x_2}{r_1r_{-1}}$$

and

$$\sin(\theta_{-1})\cos(\theta_1) = \frac{(x_1-d)x_2}{r_1r_{-1}},$$

therefore

$$-\frac{\sin(\theta_1)}{r_1}\frac{\cos(\theta_{-1})}{r_{-1}} + \frac{\sin(\theta_{-1})}{r_{-1}}\frac{\cos(\theta_1)}{r_1} = \frac{2dx_2}{(r_1r_{-1})^2}.$$

We have, in the right half-plane, where $r_1\leqslant r_{-1}$ and $r_{-1}\geqslant d\geqslant \frac{K}{c}$,

$$\left| c\,\tilde r^{2+\sigma}\frac{2dx_2}{(r_1r_{-1})^2}\right| = 2\left| cd\,\frac{\tilde r^{2+\sigma}}{r_1^2r_{-1}^\sigma}\frac{x_2}{r_{-1}}\frac{1}{r_{-1}^{1-\sigma}}\right| \leqslant Kc^{1-\sigma}$$

since $\frac{\tilde r^{2+\sigma}}{r_1^2 r_{-1}^\sigma}\leqslant 1$, $\frac{|x_2|}{r_{-1}}\leqslant 1$ and $cd\leqslant K$. Similarly, we have the same estimate in the left half-plane.

Now for the first term of (2.2.6), we have, for $\epsilon=\pm 1$,

$$\sin(\theta_\epsilon)\cos(\theta_\epsilon)=\frac{(x_1-\epsilon d)x_2}{r_\epsilon^2}.$$

Therefore,

$$-\frac{\cos(\theta_1)\sin(\theta_1)}{r_1^2}+\frac{\cos(\theta_{-1})\sin(\theta_{-1})}{r_{-1}^2}=\frac{x_2}{(r_1r_{-1})^4}(r_1^4(x_1+d)-r_{-1}^4(x_1-d)).$$

We compute, for $\epsilon=\pm 1$,

$$r_\epsilon^4=((x_1-\epsilon d)^2+x_2^2)^2=(x_1-\epsilon d)^4+2(x_1-\epsilon d)^2x_2^2+x_2^4,$$

hence

$$\begin{aligned}&-\frac{\cos(\theta_1)\sin(\theta_1)}{r_1^2}+\frac{\cos(\theta_{-1})\sin(\theta_{-1})}{r_{-1}^2}\\ =\;&\frac{x_2}{(r_1r_{-1})^4}(x_1-d)(x_1+d)((x_1-d)^3-(x_1+d)^3+2x_2^2((x_1-d)-(x_1+d)))\\ +\;&\frac{x_2}{(r_1r_{-1})^4}x_2^4(x_1+d-(x_1-d)).\end{aligned}$$

We simplify this equation to

$$-\frac{\cos(\theta_1)\sin(\theta_1)}{r_1^2}+\frac{\cos(\theta_{-1})\sin(\theta_{-1})}{r_{-1}^2}=\frac{-x_2(x_1-d)(x_1+d)}{(r_1r_{-1})^4}(2d^3+6x_1^2d-4x_2^2d)+\frac{2x_2^5d}{(r_1r_{-1})^4}.\qquad(2.2.7)$$

We now estimate separately each contribution of (2.2.7). We have, in the right half-plane, where $r_1\leqslant r_{-1}$ and $r_{-1}\geqslant d\geqslant\frac{K}{c}$,

$$\left|c\,\tilde r^{2+\sigma}\frac{2x_2^5d}{(r_1r_{-1})^4}\right|=2\left|cd\frac{x_2^5}{r_1^2r_{-1}^3}\frac{\tilde r^{2+\sigma}}{r_1^2r_{-1}^\sigma}\frac{1}{r_{-1}^{1-\sigma}}\right|\leqslant Kc^{1-\sigma}$$

since $|x_2|\leqslant r_1, |x_2|\leqslant r_{-1}$ and $\frac{\tilde r^{2+\sigma}}{r_1^2r_{-1}^\sigma}\leqslant 1$. Still in the right half-plane,

$$\left|c\,\tilde r^{2+\sigma}\frac{x_2(x_1-d)(x_1+d)}{(r_1r_{-1})^4}2d^3\right|=2\left|cd\frac{d^2}{r_{-1}^2}\frac{(x_1-d)}{r_1}\frac{(x_1+d)}{r_{-1}}\frac{x_2}{r_1}\frac{\tilde r^{2+\sigma}}{r_1^2r_{-1}^\sigma}\frac{1}{r_{-1}^{1-\sigma}}\right|\leqslant Kc^{1-\sigma}$$

since $d\leqslant Kr_{-1}$, $|x_1-d|\leqslant r_1$ and $|x_1+d|\leqslant r_{-1}$. For the next term, we write $x_1^2=x_1^2-d^2+d^2$ in

$$\frac{x_2(x_1-d)(x_1+d)}{(r_1r_{-1})^4}6x_1^2d=\frac{x_2(x_1-d)(x_1+d)}{(r_1r_{-1})^4}6(x_1^2-d^2)d+\frac{x_2(x_1-d)(x_1+d)}{(r_1r_{-1})^4}6d^3.$$

In the right half-plane, using $x_1^2-d^2=(x_1-d)(x_1+d)$,

$$\left|c\,\tilde r^{2+\sigma}\frac{x_2(x_1-d)(x_1+d)}{(r_1r_{-1})^4}6(x_1^2-d^2)d\right|=6\left|cd\frac{(x_1-d)^2}{r_1^2}\frac{(x_1+d)^2}{r_{-1}^2}\frac{x_2}{r_{-1}}\frac{\tilde r^{2+\sigma}}{r_1^2r_{-1}^\sigma}\frac{1}{r_{-1}^{1-\sigma}}\right|\leqslant Kc^{1-\sigma}$$

using previous estimates. We continue in the right half-plane with

$$\left|c\,\tilde r^{2+\sigma}\frac{x_2(x_1-d)(x_1+d)}{(r_1r_{-1})^4}6d^3\right|=6\left|cd\frac{(x_1-d)}{r_1}\frac{(x_1+d)}{r_{-1}}\frac{d^2}{r_{-1}^2}\frac{x_2}{r_1}\frac{\tilde r^{2+\sigma}}{r_1^2r_{-1}^\sigma}\frac{1}{r_{-1}^{1-\sigma}}\right|\leqslant Kc^{1-\sigma}$$

and

$$\left|c\,\tilde r^{2+\sigma}\frac{x_2(x_1-d)(x_1+d)}{(r_1r_{-1})^4}4x_2^2d\right|=4\left|cd\frac{(x_1-d)}{r_1}\frac{(x_1+d)}{r_{-1}}\frac{x_2^3}{r_1r_{-1}^2}\frac{\tilde r^{2+\sigma}}{r_1^2r_{-1}^\sigma}\frac{1}{r_{-1}^{1-\sigma}}\right|\leqslant Kc^{1-\sigma}$$

using previous estimates. Similarly, all these estimates hold in the left half-plane, which ends the proof of

$$\left\|\frac{\partial_d(ic\partial_{x_2}V)}{V}\right\|_{**,\sigma,d_c}\leqslant Kc^{1-\sigma}.\qquad\square$$

We check easily by standard elliptic regularity arguments that $\partial_c\Phi_{c,d}\in C^\infty(\mathbb{R}^2,\mathbb{C})$. Furthermore, $c\mapsto\Phi_{c,d}$ is C^1 with values in $\mathcal{E}_{*,\sigma,d}$, therefore $\partial_c\nabla\Phi_{c,d}$ is well defined (in $C^0(\mathbb{R}^2,\mathbb{C})$). Let us show that it is equal to $\nabla\partial_c\Phi_{c,d}$. For $\varphi\in C_c^\infty(\mathbb{R}^2,\mathbb{C})$, we have, by derivation under an integral, that

$$\begin{aligned}\int_{\mathbb{R}^2}\partial_c\nabla\Phi_{c,d}\varphi &= \partial_c\int_{\mathbb{R}^2}\nabla\Phi_{c,d}\varphi\\ &= -\partial_c\int_{\mathbb{R}^2}\Phi_{c,d}\nabla\varphi\\ &= -\int_{\mathbb{R}^2}\partial_c\Phi_{c,d}\nabla\varphi\\ &= \int_{\mathbb{R}^2}\nabla\partial_c\Phi_{c,d}\varphi.\end{aligned}$$

Therefore $\partial_c\nabla\Phi_{c,d}=\nabla\partial_c\Phi_{c,d}$ in the distribution sense, and thus in the strong sense. Furthermore, thanks to the equation $\eta L(\Phi_{c,d})+(1-\eta)\,VL'(\Psi_{c,d})+F(\Psi_{c,d})=\lambda(c,d)Z_d$, we can isolate $\Delta\Phi_{c,d}$ as in (2.1.23), and show in particular that it is a C^1 function of c. By similar arguments as for the gradient, we can show that $\partial_c\Delta\Phi_{c,d}=\Delta\partial_c\Phi_{c,d}$. Furthermore, the same proof holds if we differentiate $\Phi_{c,d}$ with respect to d. We can therefore inverse derivatives in position and derivatives with respect to c or d on $\Phi_{c,d}$.

Let us also show that $(c,d)\mapsto\partial_c\Phi_{c,d}$ is a continuous function from $\Omega:=\big\{(c,d)\in\mathbb{R}^2,0<c<c_0(\sigma),\ \frac{1}{2c}<d<\frac{2}{c}\big\}$ to $\mathcal{E}_{*,\sigma,d}$. With the same compactness argument used in the proof of the continuity of $c\mapsto\Phi_{c,d}$, we can show that $(c,d)\mapsto\Phi_{c,d}$ is continuous from Ω to $\mathcal{E}_{*,\sigma,d}$. From the proof of Lemma 2.2.2, we have that

$$\begin{aligned}&\Big(\mathrm{Id}+\Big(\eta L(.)+(1-\eta)\,VL'\Big(\frac{\cdot}{V}\Big)\Big)^{-1}(\Pi_d^\perp(d_\Psi F(./V)))\Big)(\partial_c\Phi_c)\\ =\ &\Pi_d^\perp(\partial_c F(\Phi_{c,d}/V))-i\eta\partial_{x_2}\Phi_{c,d}+(1-\eta)V\partial_{x_2}\Big(\frac{\Phi_{c,d}}{V}\Big).\end{aligned}$$

Since $(c,\,d)\mapsto\Phi_{c,d}$ is continuous from Ω to $\mathcal{E}_{*,\sigma,d}$, and that the dependence on $(c,\,d)$ of the other terms of the right-hand side is explicit, we check that $\Pi_d^\perp(\partial_c F(\Phi_{c,d}/V))-i\eta\partial_{x_2}\Phi_{c,d}+(1-\eta)V\partial_{x_2}\Big(\frac{\Phi_{c,d}}{V}\Big)$ is continuous from Ω to $\mathcal{E}_{**,\gamma(\sigma),d}$. We check also that $(c,d)\mapsto\big(\mathrm{Id}+\big(\eta L(.)+(1-\eta)\,VL'\big(\frac{\cdot}{V}\big)\big)^{-1}(\Pi_d^\perp(d_\Psi F(./V)))\big)$ is continuous from Ω to $\mathcal{E}_{**,\gamma(\sigma),d}\to\mathcal{E}_{*,\sigma,d}$, and thus $(c,d)\mapsto\partial_c\Phi_{c,d}$ is a continuous function from Ω to $\mathcal{E}_{*,\sigma,d}$. The same proof holds for $(c,d)\mapsto\partial_d\Phi_{c,d}$.

We end this subsection with the symmetries of $\partial_d\Phi_{c,d}$.

Lemma 2.2.4. *The function $\partial_d\Phi_{c,d}$ satisfies the symmetries: for $x=(x_1,x_2)\in\mathbb{R}^2$,*

$$\partial_d\Phi_{c,d}(x_1,x_2)=\partial_d\Phi_{c,d_c}(-x_1,x_2)=\overline{\partial_d\Phi_{c,d}(x_1,-x_2)}.$$

Proof. From subsection 2.1.3,

$$\forall x=(x_1,x_2)\in\mathbb{R}^2,\ \Psi_{c,d}(x_1,x_2)=\overline{\Psi_{c,d}(x_1,-x_2)}=\Psi_{c,d}(-x_1,x_2)$$

and V enjoys the same symmetries, therefore for all $d\in\mathbb{R}$ such that $\frac{1}{2c}<d<\frac{2}{c}$,

$$\Phi_{c,d}(x_1,x_2)=\Phi_{c,d}(-x_1,x_2)=\overline{\Phi_{c,d}(x_1,-x_2)}.$$

Since

$$\partial_d\Phi_{c,d}=\lim_{\varepsilon\to0}\frac{\Phi_{c,d+\varepsilon}-\Phi_{c,d}}{\varepsilon},$$

these symmetries also hold for $\partial_d\Phi_{c,d}$. □

2.2.2 End of the construction and properties of Q_c

A consequence of equation (2.1.41) and Proposition 2.1.26 is that, for $0<\sigma<1$, there exists $c_0(\sigma)>0$ such that, for $0<c<c_0(\sigma)$,

$$\eta L(\Phi_{c,d})+(1-\eta)VL'(\Psi_{c,d})+F(\Psi_{c,d})=\lambda(c,d)Z_d$$

with

$$\lambda(c,d)\int_{\mathbb{R}^2}|\partial_d V|^2\eta = \pi\left(\frac{1}{d}-c\right)+O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

Following the proof of Proposition 2.1.26, with Lemmas 2.2.2 and 2.2.3, we can check that the $O^{\sigma}_{c\to 0}(c^{2-\sigma})$ is continuous with respect of c and d. Therefore, by the intermediate value theorem, there exists $d_c>0$ such that $\lambda(c,d_c)=0$, with

$$d_c = \frac{1}{c}+O^{\sigma}_{c\to 0}(c^{-\sigma}),$$

for $c>0$ small enough. Then, for the function $\Phi_{c,d_c}=V\Psi_{c,d_c}$ with $\|\Psi_{c,d_c}\|_{*,\sigma,d_c}\leqslant K(\sigma,\sigma')c^{1-\sigma'}$, we have

$$\eta L(\Phi_{c,d_c})+(1-\eta)VL'(\Psi_{c,d_c})+F(\Psi_{c,d_c})=0,$$

meaning that if we define

$$Q_c := \eta V(1+\Psi_{c,d_c})+(1-\eta)Ve^{\Psi_{c,d_c}},$$

then Q_c solves (TW_c).

2.2.2.1 Behaviour at infinity and energy estimation

Lemma 2.2.5. *The function Q_c satisfies $Q_c(x)\to 1$ when $|x|\to\infty$.*

Proof. From $\|\Psi_{c,d_c}\|_{*,\sigma,d_c}\leqslant K(\sigma,\sigma')c^{1-\sigma'}$ we have $\Psi_{c,d_c}(x)\to 0$ when $|x|\to\infty$. Furthermore $|1-V|^2\leqslant\frac{C(d_c)}{1+r^2}$ by Lemma 2.1.3 and $Q_c=Ve^{\Psi_{c,d_c}}$ for large values of $|x|$, hence $Q_c(x)\to 1$ when $|x|\to\infty$. □

In the statement of Theorem 1.3.1, we have set $Q_c=V+\Gamma_{c,d_c}$, we therefore define

$$\Gamma_{c,d_c} := \eta V\Psi_{c,d_c}+(1-\eta)V(e^{\Psi_{c,d_c}}-1). \tag{2.2.8}$$

We compute that

$$\left\|\frac{\Gamma_{c,d_c}}{V}\right\|_{*,\sigma,d_c}\leqslant K\|\Psi_{c,d_c}\|_{*,\sigma,d_c}+\|(1-\eta)(e^{\Psi_{c,d_c}}-1-\Psi_{c,d_c})\|_{*,\sigma,d_c},$$

and since $\|\Psi_{c,d_c}\|_{*,\sigma,d_c}\leqslant 1$ for c small enough (depending on σ), we have

$$\|(1-\eta)(e^{\Psi_{c,d_c}}-1-\Psi_{c,d_c})\|_{*,\sigma,d_c}\leqslant K\left\|(1-\eta)\Psi^2_{c,d_c}\sum_{n=2}^{+\infty}\frac{\Psi^{n-2}_{c,d_c}}{n!}\right\|_{*,\sigma,d_c}.$$

Now, for $0<\sigma<\sigma'<1$, we have $\frac{1+\sigma'}{2}>\frac{1+\sigma}{2}$, hence

$$|\Psi_{c,d_c}|\leqslant K(\sigma,\sigma')\frac{c^{1-\frac{1+\sigma'}{2}}}{(1+\tilde{r})^{\frac{1+\sigma}{2}}}\quad\text{and}\quad|\nabla\Psi_{c,d_c}|\leqslant K(\sigma,\sigma')\frac{c^{1-\frac{1+\sigma'}{2}}}{(1+\tilde{r})^{1+\frac{1+\sigma}{2}}},$$

therefore

$$|\Psi_{c,d_c}|^2\leqslant K(\sigma,\sigma')\frac{c^{1-\sigma'}}{(1+\tilde{r})^{1+\sigma}}\quad\text{and}\quad|\nabla\Psi_{c,d_c}|^2\leqslant K(\sigma,\sigma')\frac{c^{1-\sigma'}}{(1+\tilde{r})^{2+\sigma}}.$$

Thus, with $|\nabla^2\Psi_{c,d_c}|\leqslant K(\sigma,\sigma')\frac{c^{1-\sigma'}}{(1+\tilde{r})^{1+\sigma}}$, we check that, for any $0<\sigma<\sigma'<1$,

$$\left\|(1-\eta)\Psi^2_{c,d_c}\sum_{n=2}^{+\infty}\frac{\Psi^{n-2}_{c,d_c}}{n!}\right\|_{*,\sigma,d_c}\leqslant K(\sigma,\sigma')c^{1-\sigma'}.$$

Combining this result with $\|\Psi_{c,d_c}\|_{*,\sigma,d_c}\leqslant K(\sigma,\sigma')c^{1-\sigma'}$, we deduce that

$$\left\|\frac{\Gamma_{c,d_c}}{V}\right\|_{*,\sigma,d_c}\leqslant K(\sigma,\sigma')c^{1-\sigma'}. \tag{2.2.9}$$

In particular, we have, for any $0<\sigma<\sigma'<1$, $0<c<c_0(\sigma,\sigma')$, that

$$|\Gamma_{c,d_c}|\leqslant\frac{K(\sigma,\sigma')c^{1-\sigma'}}{(1+\tilde r)^{\sigma}},\tag{2.2.10}$$

$$\left|\mathfrak{Re}\left(\frac{\Gamma_{c,d_c}}{V}\right)\right|\leqslant\frac{K(\sigma,\sigma')c^{1-\sigma'}}{(1+\tilde r)^{1+\sigma}},\tag{2.2.11}$$

and, if $\tilde r\geqslant 2$,

$$|\nabla\Gamma_{c,d_c}|\leqslant\left|\nabla\left(\frac{\Gamma_{c,d_c}}{V}\right)\right|+\left|\frac{\nabla V}{V}\right|\times\left|\frac{\Gamma_{c,d_c}}{V}\right|,$$

therefore, using $|\nabla V|\leqslant\frac{K}{(1+\tilde r)}$ from Lemma 2.1.1, we have

$$|\nabla\Gamma_{c,d_c}|\leqslant\frac{K(\sigma,\sigma')c^{1-\sigma'}}{(1+\tilde r)^{1+\sigma}}.\tag{2.2.12}$$

Estimate (2.2.12) remains true in $\{\tilde r\leqslant 2\}$ since $\|\Gamma_{c,d_c}\|_{C^1(\{\tilde r\leqslant 2\})}\leqslant\left\|\frac{\Gamma}{V}\right\|_{*,\sigma,d_c}\leqslant K(\sigma,\sigma')c^{1-\sigma'}$. We now show the estimates on Γ_{c,d_c} of Theorem 1.3.1.

Lemma 2.2.6. *For $+\infty\geqslant p>2$, there exists $c_0(p)>0$ such that if $0<c<c_0(p)$, we have $\Gamma_{c,d_c}\in L^p(\mathbb{R}^2)$, $\nabla\Gamma_{c,d_c}\in L^{p-1}(\mathbb{R}^2)$ and*

$$\|\Gamma_{c,d_c}\|_{L^p(\mathbb{R}^2)}+\|\nabla\Gamma_{c,d_c}\|_{L^{p-1}(\mathbb{R}^2)}=o_{c\to 0}(1).$$

Proof. If $p=+\infty$, using (2.2.10) and (2.2.12), we infer

$$\|\Gamma_{c,d_c}\|_{L^\infty(\mathbb{R}^2)}\leqslant K(\sigma)c^{1-\sigma},$$
$$\|\nabla\Gamma_{c,d_c}\|_{L^\infty(\mathbb{R}^2)}\leqslant K(\sigma)c^{1-\sigma},$$

hence the result holds. If $2<p<+\infty$ then, by (2.2.10),

$$\int_{\mathbb{R}^2}|\Gamma_{c,d_c}|^p\leqslant\int_{\mathbb{R}^2}\frac{\|\Gamma_{c,d_c}\|^p_{*,\sigma,d_c}}{(1+\tilde r)^{p\sigma}}dx\leqslant\int_{\mathbb{R}^2}\frac{K(\sigma,\sigma')\,c^{(1-\sigma')p}}{(1+\tilde r)^{p\sigma}}dx.$$

Taking $0<\sigma<\sigma'<1$ such that $p\sigma>2$ then gives the result. Furthermore, by (2.2.12),

$$\int_{\mathbb{R}^2}|\nabla\Gamma_{c,d_c}|^p\leqslant\int_{\mathbb{R}^2}\frac{K(\sigma,\sigma')c^{(1-\sigma')p}}{(1+\tilde r)^{p(\sigma+1)}}dx,$$

so for $p>1$ we can take $0<\sigma<\sigma'<1$ such that $p(\sigma+1)>2$ and we have the result. □

Remark that we can have better estimates on Γ_{c,d_c}, in particular if we look at real and imaginary parts of $\frac{\Gamma_{c,d_c}}{V}$. For instance it is possible to show that

$$\left\|\mathfrak{Re}\left(\frac{\Gamma_{c,d_c}}{V}\right)\right\|_{L^p(\{\tilde r\geqslant 1\})}=o_{c\to 0}(1)$$

for $p>1$ instead of $p>2$. This estimate does not hold for $\tilde r$ small since it is not clear that Ψ_{c,d_c} is bounded there (but Φ_{c,d_c} is). This is due to the fact that the zeros of Q_c are not exactly those of V.

Lemma 2.2.7. *The travelling wave Q_c has finite energy, that is:*

$$E(Q_c)=\frac12\int_{\mathbb{R}^2}|\nabla Q_c|^2+\frac14\int_{\mathbb{R}^2}(1-|Q_c|^2)^2<+\infty.$$

Proof. Far from the vortices, $\nabla Q_c=\nabla(V_1\,V_{-1})\,e^{\Psi_{c,d_c}}+\nabla\Psi_{c,d_c}V_1V_{-1}\,e^{\Psi_{c,d_c}}$. We know that, for $\tilde r\geqslant 1$,

$$|\nabla\Psi_{c,d_c}|\leqslant\frac{K(\sigma)}{\tilde r^{1+\sigma}}$$

and (by Lemma 2.1.3)

$$|\nabla(V_1V_{-1})|\leqslant\frac{K(c)}{\tilde r^2},$$

hence

$$|\nabla Q_c|^2 \leqslant \frac{K(c,\sigma)}{\tilde{r}^{2+2\sigma}}$$

and is therefore integrable. On the other hand,

$$|1-|Q_c|^2| = |1-|V_1V_{-1}|^2 e^{2\mathfrak{Re}(\Psi_{c,d_c})}| \leqslant K(1-|V_1V_{-1}|^2+|V_1V_{-1}|^2|\mathfrak{Re}(\Psi_{c,d_c})|),$$

and we have

$$1-|V_1V_{-1}|^2 = O\Big(\frac{1}{\tilde{r}^2}\Big) \quad \text{and} \quad \mathfrak{Re}(\Psi_{c,d_c}) = O^\sigma\Big(\frac{1}{\tilde{r}^{1+\sigma}}\Big),$$

therefore

$$(1-|Q_c|^2)^2 = O\Big(\frac{1}{\tilde{r}^{2+2\sigma}}\Big)$$

and is integrable. □

At this point, we have finish the proof of the construction of Q_c. In the next two subsection, we add some estimates on Q_c that will be usefull for the differentiability of the branch, and others that are interesting in themselves.

2.2.2.2 A set of estimations on Q_c

The next Lemma gives additional estimates on Q_c which are more precise but more technical than the ones in Theorem 1.3.1.

Lemma 2.2.8. *For any $0<\sigma<\sigma'<1$, there exists $c_0(\sigma,\sigma')$, $K(\sigma,\sigma')>0$ such that for $0<c<c_0(\sigma,\sigma')$ we have*

$$\|\Psi_{c,d_c}\|_{*,\sigma,d_c} \leqslant K(\sigma,\sigma')c^{1-\sigma'}. \tag{2.2.13}$$

Furthermore, for any $0<\sigma<1$, there exist $c_0(\sigma), K(\sigma)>0$ such that for $0<c<c_0(\sigma)$ we have

$$\begin{aligned} & \|V\Psi_{c,d_c}\|_{C^1(\tilde{r}\leqslant 3)} + \|\tilde{r}^\sigma \mathfrak{Im}(\Psi_{c,d_c})\|_{L^\infty(\tilde{r}\geqslant 2)} + \|\tilde{r}^{1+\sigma}\mathfrak{Re}(\Psi_{c,d_c})\|_{L^\infty(\tilde{r}\geqslant 2)} \\ + \;& \|\tilde{r}^{1+\sigma}\mathfrak{Im}(\nabla\Psi_{c,d_c})\|_{L^\infty(\tilde{r}\geqslant 2)} + \|\tilde{r}^{2+\sigma}\mathfrak{Re}(\nabla\Psi_{c,d_c})\|_{L^\infty(\tilde{r}\geqslant 2)} \\ \leqslant \;& K(\sigma)c^{1-\sigma}, \end{aligned} \tag{2.2.14}$$

$$|1-|Q_c|| \leqslant \frac{K(\sigma)}{(1+\tilde{r})^{1+\sigma}}, \tag{2.2.15}$$

$$|Q_c - V| \leqslant \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^\sigma}, \tag{2.2.16}$$

$$||Q_c|^2-|V|^2| \leqslant \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^{1+\sigma}}, \tag{2.2.17}$$

$$|\mathfrak{Re}(\nabla Q_c\overline{Q_c})| \leqslant \frac{K(\sigma)}{(1+\tilde{r})^{1+\sigma}}, \tag{2.2.18}$$

$$|\mathfrak{Im}(\nabla Q_c\overline{Q_c})| \leqslant \frac{K}{1+\tilde{r}} \tag{2.2.19}$$

Equation (2.2.14) is a slight improvements of (2.2.13). It is, except for the second derivatives, the estimate in the case $\sigma'=\sigma$.

Proof. The first estimate comes from the construction of the solution.

We now take χ a cutoff function with value 1 in $\{\tilde{r}\geqslant 2\}$ and 0 in $\{\tilde{r}\leqslant 1\}$, we write $\tilde{\Psi}=\chi\Psi_{c,d_c}$ and $\tilde{h}=\chi h$, where h contains the nonlinear and source terms. We recall from (2.1.29) that $\tilde{\Psi}=\tilde{\Psi}_1+i\tilde{\Psi}_2$ and $\tilde{h}=\tilde{h}_1+i\tilde{h}_2$ satisfy the system

$$\begin{cases} \Delta\tilde{\Psi}_1 - 2\tilde{\Psi}_1 = -\tilde{h}_1 - 2\mathfrak{Re}\Big(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\Big) - 2(1-|V|^2)\tilde{\Psi}_1 + c\partial_{x_2}\tilde{\Psi}_2 + \mathrm{Loc}_1(\Psi) \\ \Delta\tilde{\Psi}_2 = -\tilde{h}_2 - 2\mathfrak{Im}\Big(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\Big) + \mathrm{Loc}_2(\Psi) - c\partial_{x_2}\tilde{\Psi}_1, \end{cases}$$

where $\mathrm{Loc}_1(\Psi), \mathrm{Loc}_2(\Psi)$ are localized terms. From Lemmas 2.1.22 to 2.1.24, we check that for any $0<\sigma<1$,

$$\|\tilde{h}\|_{**,\sigma,d} \leqslant K(\sigma)c^{1-\sigma}.$$

Furthermore, as in the proof of Proposition 2.1.17, we check that (using $\|\tilde{\Psi}\|_{*,\sigma/2,d} \leqslant K(\sigma)c^{1-\sigma}$)

$$\left\| \frac{\nabla V}{V}.\nabla\tilde{\Psi} - 2(1-|V|^2)\mathfrak{Re}(\tilde{\Psi}) + \mathrm{Loc}(\Psi) \right\|_{**,\sigma,d} \leqslant K(\sigma)c^{1-\sigma}.$$

Finally, with (2.2.13), for $\sigma' = \frac{1+\sigma}{2} > \sigma$,

$$\|c\partial_{x_2}\tilde{\Psi}\|_{**,\sigma,d} \leqslant K(\sigma)\, c\, \|\tilde{\Psi}\|_{*,\sigma,d} \leqslant K(\sigma)c^{1+1-\frac{1+\sigma}{2}} \leqslant K(\sigma)c^{1-\sigma}.$$

With Lemma 2.1.10 for $\alpha = 1+\sigma > 0$, we deduce from the first equation of the system that

$$\begin{aligned} &\left\|(1+\tilde{r})^{1+\sigma}\tilde{\Psi}_1\right\|_{L^\infty(\mathbb{R}^2)} \\ \leqslant\ & K(\sigma)\left\|(1+\tilde{r})^{1+\sigma}\left(-\tilde{h}_1 - 2\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\right) - 2(1-|V|^2)\tilde{\Psi}_1 + c\partial_{x_2}\tilde{\Psi}_2 + \mathrm{Loc}_1(\Psi)\right)\right\|_{L^\infty(\mathbb{R}^2)} \\ \leqslant\ & K(\sigma)c^{1-\sigma}, \end{aligned}$$

and, by differentiating the equation, by Lemma 2.1.10 for $\alpha = 2+\sigma > 0$

$$\begin{aligned} &\left\|(1+\tilde{r})^{2+\sigma}\nabla\tilde{\Psi}_1\right\|_{L^\infty(\mathbb{R}^2)} \\ \leqslant\ & K(\sigma)\left\|(1+\tilde{r})^{2+\sigma}\nabla\left(-\tilde{h}_1 - 2\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\right) - 2(1-|V|^2)\tilde{\Psi}_1 + c\partial_{x_2}\tilde{\Psi}_2 + \mathrm{Loc}_1(\Psi)\right)\right\|_{L^\infty(\mathbb{R}^2)} \\ \leqslant\ & K(\sigma)c^{1-\sigma}. \end{aligned}$$

Now, using Lemma 2.1.8 and $\left\|(1+\tilde{r})^{2+\sigma}\nabla\tilde{\Psi}_1\right\|_{L^\infty(\mathbb{R}^2)} \leqslant K(\sigma)c^{1-\sigma}$, we infer that

$$\begin{aligned} &\left\|(1+\tilde{r})^{\sigma}\tilde{\Psi}_2\right\|_{L^\infty(\mathbb{R}^2)} + \left\|(1+\tilde{r})^{1+\sigma}\nabla\tilde{\Psi}_2\right\|_{L^\infty(\mathbb{R}^2)} \\ \leqslant\ & K(\sigma)\left\|(1+\tilde{r})^{2+\sigma}\left(-\tilde{h}_2 - 2\mathfrak{Im}\left(\frac{\nabla V}{V}.\nabla\tilde{\Psi}\right) + \mathrm{Loc}_2(\Psi) - c\partial_{x_2}\tilde{\Psi}_1\right)\right\|_{L^\infty(\mathbb{R}^2)} \\ \leqslant\ & K(\sigma)c^{1-\sigma}, \end{aligned}$$

which concludes the proof of (2.2.14).

The estimate (2.2.15) is clear if $\tilde{r} \leqslant 3$. If $\tilde{r} \geqslant 3$, then $Q_c = Ve^{\Psi_{c,d_c}}$ and, for c small enough (depending on σ), $|\mathfrak{Re}(\Psi_{c,d_c})| \leqslant 1$, thus

$$\begin{aligned} |1-|Q_c|| &= |1-|V|-|V|(e^{\mathfrak{Re}(\Psi_{c,d_c})}-1)| \\ &\leqslant |1-|V|| + K|\mathfrak{Re}(\Psi_{c,d_c})| \\ &\leqslant \frac{K}{(1+\tilde{r})^2} + \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^{1+\sigma}} \\ &\leqslant \frac{K(\sigma)}{(1+\tilde{r})^{1+\sigma}} \end{aligned}$$

by Lemma 2.1.3 and (2.2.14). For (2.2.16), if $\tilde{r} \geqslant 3$, we compute

$$|Q_c - V| = |V| \times |e^{\Psi_{c,d_c}} - 1| \leqslant C|\Psi_{c,d_c}| \leqslant \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^{\sigma}}$$

and if $\tilde{r} \leqslant 3$, $|Q_c - V| \leqslant C\|\Psi_{c,d_c}\|_{*,\sigma,d_c}$ and the estimate (2.2.16) holds. Similarly, for $\tilde{r} \geqslant 3$,

$$||Q_c|^2 - |V|^2| \leqslant |V|^2|e^{2\mathfrak{Re}(\Psi_{c,d_c})} - 1| \leqslant \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^{1+\sigma}}$$

and for the same reason if $\tilde{r} \leqslant 3$ the estimate (2.2.17) holds. Inequalities (2.2.18) and (2.2.19) are clear if $\tilde{r} \leqslant 3$ and we compute, for $\tilde{r} \geqslant 3$,

$$\nabla Q_c\overline{Q_c} = \nabla(Ve^{\Psi_{c,d_c}})\bar{V}e^{\bar{\Psi}_{c,d_c}} = \nabla V\bar{V}e^{2\mathfrak{Re}(\Psi c,d_c)} + |V|^2\nabla\Psi_{c,d_c}e^{2\mathfrak{Re}(\Psi_{c,d_c})}.$$

We have $|e^{2\mathfrak{Re}(\Psi_{c,d_c})}|\leqslant 1$ for c small enough and by Lemma 2.1.1 we have $|\mathfrak{Im}(\nabla V\bar{V})|\leqslant\frac{K}{1+\tilde{r}}$ and $|\mathfrak{Re}(\nabla V\bar{V})|\leqslant\frac{K}{(1+\tilde{r})^3}$. Combining it with $|\nabla\Psi_{c,d_c}|\leqslant\frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^{1+\sigma}}$ from (2.2.14), estimates (2.2.18) and (2.2.19) hold. □

2.2.2.3 Estimations on derivatives of $\Phi_{c,d}$ with respect to c and d at $d=d_c$.

We cannot easily compute $\partial_d\Psi_{c,d\,|\,d=d_c}$ because of issues locally around the vortices (due to the fact that $\Psi_{c,d}$ is unbounded near $\tilde{r}=0$, and changing d change the position of the vortices). We shall prove here instead an estimate on $\partial_d\Phi_{c,d\,|\,d=d_c}$, as well as an estimate on $\partial_c\Psi_{c,d\,|\,d=d_c}$.

Lemma 2.2.9. *For any $0<\sigma<\sigma'<1, c\in\mathbb{R}$ such that $\frac{1}{2d}<c<\frac{2}{d}$ and $0<c<c_0(\sigma,\sigma')$, we have*

$$\|\partial_c\Psi_{c,d\,|\,d=d_c}\|_{*,\sigma,d}\leqslant K(\sigma,\sigma')c^{-\sigma'}$$

and

$$\left\|\frac{\partial_d\Phi_{c,d}}{V}_{\,|\,d=d_c}\right\|_{*,\sigma,d_c}\leqslant K(\sigma,\sigma')c^{1-\sigma'},$$

with $K(\sigma,\sigma')>0$ depending only on σ,σ'.

Proof. From the proof (and with the notations) of Lemma 2.2.2,

$$\begin{aligned}
&\left(\mathrm{Id}+\left(\eta L(.)+(1-\eta)\,VL'\left(\frac{\cdot}{V}\right)\right)_c^{-1}(\Pi_d^{\perp}(d_\Psi F_c(./V)))\right)((\Phi_{c+\varepsilon,d}-\Phi_{c,d}))\\
=\;&\left(\eta L(.)+(1-\eta)\,VL'\left(\frac{\cdot}{V}\right)\right)_c^{-1}\left(-\varepsilon\Pi_d^{\perp}(-i\partial_{x_2}V)-i\varepsilon\left(\eta\partial_{x_2}\Phi_{c+\varepsilon,d}+(1-\eta)V\partial_{x_2}\left(\frac{\Phi_{c+\varepsilon,d}}{V}\right)\right)\right)\\
+\;&\left(\eta L(.)+(1-\eta)\,VL'\left(\frac{\cdot}{V}\right)\right)_c^{-1}(O^{\sigma,c}_{\|.\|_{**,\sigma,d}}(\varepsilon^2)),
\end{aligned}$$

thus, taking $\varepsilon\to 0$, we deduce that (with Lemma 2.2.2)

$$\begin{aligned}
&\left(\mathrm{Id}+\left(\eta L(.)+(1-\eta)\,VL'\left(\frac{\cdot}{V}\right)\right)^{-1}(\Pi_d^{\perp}(d_\Psi F_c(./V)))\right)(\partial_c\Phi_{c,d})\\
=\;&\left(\eta L(.)+(1-\eta)\,VL'\left(\frac{\cdot}{V}\right)\right)^{-1}\left(\Pi_d^{\perp}(\partial_c F(\Phi_{c,d}/V))-i\eta\partial_{x_2}\Phi_{c,d}+(1-\eta)V\partial_{x_2}\left(\frac{\Phi_{c,d}}{V}\right)\right).
\end{aligned}$$

Since at $d=d_c$, $\lambda(c,d_c)=0$, we have

$$\Pi_d^{\perp}(\partial_c F(\Phi_{c,d}/V))-i\eta\partial_{x_2}\Phi_{c,d}+(1-\eta)V\partial_{x_2}\left(\frac{\Phi_{c,d}}{V}\right)_{\,|\,d=d_c}=-i\,\partial_{x_2}Q_c,$$

hence, with Proposition 2.1.17,

$$\begin{aligned}
\|\partial_c\Psi_{c,d\,|\,d=d_c}\|_{*,\sigma,d}\;&\leqslant\;K\|\partial_c\Psi_{c,d\,|\,d=d_c}\|_{\circledast,\sigma,d_\circledast}\\
&\leqslant\;K(\sigma,\sigma')\left\|\frac{i\partial_{x_2}Q_c}{V}\right\|_{\circledast\circledast,\sigma',d_\circledast}.
\end{aligned}$$

We will conclude by showing that for any $0<\sigma<\sigma'<1$,

$$\left\|\frac{i\partial_{x_2}Q_c}{V}\right\|_{\circledast\circledast,\sigma,d_\circledast}\leqslant K(\sigma,\sigma')c^{-\sigma'}.$$

By Lemma 2.1.22, we have

$$\left\|\frac{i\partial_{x_2}V}{V}\right\|_{\circledast\circledast,\sigma,d_\circledast}\leqslant K(\sigma)c^{-\sigma},$$

and using $\|\Psi_{c,d_c}\|_{*,\sigma,d_c}\leqslant K(\sigma,\sigma')c^{1-\sigma'}$ with Lemma 2.1.3, we check easily that, for c small enough,

$$\left\|\frac{i\partial_{x_2}Q_c}{V}\right\|_{\circledast\circledast,\sigma,d_\circledast}\leqslant K(\sigma,\sigma')c^{-\sigma'}.$$

We now focus on the estimation of $\partial_d\Phi_{c,d\,|\,d=d_c}$. At the end of step 1 of the proof of Lemma 2.2.3, we have shown that

$$\partial_d\Phi_{c,d\,|\,d=d_c}=-d_\Phi H^{-1}(\partial_d H(\Phi_{c,d_c},c,d_c)).$$

From Lemma 2.2.1, we have that, at $d=d_c, \Phi=\Phi_{c,d_c}$, the operator $d_\Phi H^{-1}$ is invertible from $\mathcal{E}_{\circledast,\sigma,d_\circledast}$ to $\mathcal{E}_{\circledast,\sigma,d_\circledast}$, with an operator norm with size $1+o^{\sigma}_{c\to 0}(1)$. We therefore only have to check that

$$\|\partial_d H(\Phi_{c,d_c},c,d_c)\|_{*,\sigma,d_c}\leqslant K(\sigma,\sigma')c^{1-\sigma'}.$$

Since $\partial_d H(\Phi_{c,d_c},c,d_c)=\big(\eta L(.)+(1-\eta)\,VL'\big(\frac{\cdot}{V}\big)\big)^{-1}(G(d_c,\Phi_{c,d_c}))$, By Proposition 2.1.17 (from $\mathcal{E}_{\circledast\circledast,\sigma',d_\circledast}$ to $\mathcal{E}_{\circledast,\sigma,d_\circledast}$), it will be a consequence of

$$\left\|\frac{G(d_c,\Phi_{c,d_c})}{V}\right\|_{**,\sigma',d}\leqslant K(\sigma,\sigma')c^{1-\sigma'}$$

for any $0<\sigma<\sigma'<1$.

We have, since $\mathbb{H}_{d_c}(\Phi_{c,d_c})=\Phi_{c,d_c}$, that

$$\begin{aligned}\frac{G(d_c,\Phi_{c,d_c})}{V} &= \partial_d(|V|^2)\frac{\Phi_{c,d_c}}{V}+2\mathfrak{Re}(\overline{\partial_d V}\Phi_{c,d_c})+2\mathfrak{Re}(\bar V\Phi_{c,d_c})\frac{\partial_d V}{V}\\ &+ \partial_d((1-\eta)(E-ic\partial_{x_2}V))_{|d=d_c}\frac{\Phi_{c,d_c}}{V}-\frac{1}{V}\partial_d(\Pi_d^{\perp}(F_d(\Phi/V)))_{|d=d_c}.\end{aligned}$$

Since $\partial_d(|V|^2)=2\mathfrak{Re}(\partial_d V\bar V)$, we check, with Lemma 2.1.6 that

$$\left|\partial_d(|V|^2)\frac{\Phi_{c,d_c}}{V}\right|+\left|2\mathfrak{Re}(\bar V\Phi_{c,d_c})\frac{\partial_d V}{V}\right|\leqslant\frac{K(\sigma,\sigma')c^{1-\sigma'}}{(1+\tilde r)^{2+\sigma}},$$

and

$$|\mathfrak{Re}(\overline{\partial_d V}\Phi_{c,d_c})|\leqslant\frac{K(\sigma,\sigma')c^{1-\sigma'}}{(1+\tilde r)^{1+\sigma}},$$

as well as

$$\left|\nabla\left(\partial_d(|V|^2)\frac{\Phi_{c,d_c}}{V}+2\mathfrak{Re}(\bar V\Phi_{c,d_c})\frac{\partial_d V}{V}+\mathfrak{Re}(\overline{\partial_d V}\Phi_{c,d_c})\right)\right|\leqslant\frac{K(\sigma,\sigma')c^{1-\sigma'}}{(1+\tilde r)^{2+\sigma}},$$

and this estimate a real valued quantity. From step 2 of the proof of Lemma 2.2.3, we have

$$\left\|\frac{1}{V}\partial_d((1-\eta)(E-ic\partial_{x_2}V))\right\|_{**,\sigma,d}\leqslant K(\sigma)c^{1-\sigma},$$

which is enough to show that

$$\left\|\partial_d((1-\eta)(E-ic\partial_{x_2}V))_{|d=d_c}\frac{\Phi_{c,d_c}}{V}\right\|_{**,\sigma,d}\leqslant K(\sigma,\sigma')c^{1-\sigma'}.$$

Finally, in step 2 of the proof of Lemma 2.2.3, we have shown that (taking the estimate for $\Phi=\Phi_{c,d_c}$)

$$\left\|\frac{1}{V}\partial_d(\Pi_d^{\perp}(F_d(\Phi/V)))_{|d=d_c}\right\|_{**,\sigma,d}\leqslant K(\sigma,\sigma')c^{1-\sigma'},$$

which conclude the proof of this lemma. □

2.3 Differentiability of the branch $c\mapsto Q_c$

The goal of this section is to prove that the constructed branch is C^1, and to give the leading order term of $\partial_c Q_c$ as $c\to 0$. The result is the following one.

Proposition 2.3.1. *For any $+\infty\geqslant p>2$, there exists $c_0(p)>0$ such that*

$$c\mapsto Q_c-1\in C^1(]0,c_0(p)[,X_p),$$

with the estimate

$$\left\|\partial_c Q_c+\left(\frac{1+o_{c\to 0}(1)}{c^2}\right)\partial_d(V_1(.-d\vec{e_1})V_{-1}(.+d\vec{e_1}))_{|d=d_c}\right\|_{X_p}=o_{c\to 0}\left(\frac{1}{c^2}\right).$$

Proposition 2.3.1, together with subsection 2.2.2, ends the proof of Theorem 1.3.1. Subsections 2.3.1 to 2.3.7 are devoted to the proof of Proposition 2.3.1.

In this section, to make the dependances on c and d clear, we use the following notations. We denote $\Phi_{c,d}, \Psi_{c,d}$ and $\Gamma_{c,d}$ in order to emphasize the dependence of Φ, Ψ and Γ in Proposition 2.1.21 on c and d. A value of d that makes $\lambda(c,d)=0$ in Proposition 2.1.26 is written d_c. We will show later on that there exist one and only one value of d_c satisfying this in $\left]\frac{c}{2}, 2c\right[$. With these notations, $Q_c = V_1(.-d_c\vec{e_1})V_{-1}(.+d_c\vec{e_1}) + \Gamma_{c,d_c}$ is the solution of (TW_c) we constructed in section 2.2.

In subsection 2.2.1 we showed that $\Phi_{c,d}$ is a C^1 function of both c and d. We also have computed estimates for the derivatives of $\Phi_{c,d}$ with respect to c and d in Lemma 2.2.9, that will be usefull here.

The goal is to show that d_c is a C^1 function of c. We will do this by the implicit function theorem, but this requires a lot of computations. In particular, in Proposition 2.1.26, d_c was choosen so that

$$\langle L(\Phi_{c,d}) - (1-\eta)(E - ic\partial_{x_2}V)\Psi_{c,d} + F(\Psi_{c,d}), \partial_d V\rangle = 0,$$

but we may equivalently define it by the implicit equation

$$\int_{B(d\vec{e_1}, d^{\varepsilon'})\cup B(-d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}((L(\Phi_{c,d}) - (1-\eta)E\Psi_{c,d} + F(\Psi_{c,d}))\overline{\partial_d V}) = 0.$$

This is the same equation but the scalar product is not taken on the whole space but only on $B(d\vec{e_1}, d^{\varepsilon'}) \cup B(-d\vec{e_1}, d^{\varepsilon'})$ for some $0 < \varepsilon' < 1$ (we will take $\varepsilon' = 13/24$ but this value is purely technical, other values are possible). The only reason why we take it in the whole space in Lemma 2.1.26 was because of the boundary terms that will appear in the integration by parts when we write

$$\langle L(\Phi), \partial_d V\rangle = \langle \Phi, L(\partial_d V)\rangle.$$

With the boundary terms on the boundary of $B(\pm d\vec{e_1}, d^{\varepsilon'})$, $\varepsilon' > 0$, we are far enough from the vortices to make them small enough for our estimations. Thanks to this we can separate what happens near the vortex V_1 from what happens near the vortex V_{-1} because now the integrals are in two well separated domain, one around each vortex. We use this in subsection 2.3.1. We need to differentiate the equation with respect to d. If we write $Q_{c,d} = V + \Gamma_{c,d}$, then $\partial_d Q_c = \partial_d V + \partial_d(\Gamma_{c,d})$. The term $\partial_d V$ is easy to compute and to understand: we just move both vortices in opposite direction. But $\partial_d \Gamma_{c,d}$ is very difficult to understand, and our estimations on Γ_{c,d_c} are not enough to compute easily what happens with sufficient precision to control its contribution. We would rather write $Q_{c,d}$ under the form

$$Q_{c,d}(x) = \big(V_1(x - d\vec{e_1}) + \tilde{\Gamma}_1(x - d\vec{e_1})\big) + \big(V_{-1}(x + d\vec{e_1}) + \tilde{\Gamma}_{-1}(x + d\vec{e_1})\big) + \mathrm{Err}$$

where $\tilde{\Gamma}_1(x - d\vec{e_1})$ is centered near V_1, is small and is here because of the existence of V_{-1} far away. Then the term we understand is

$$\partial_{x_1+d}\big(V_1(x - d\vec{e_1}) + \tilde{\Gamma}_1(x - d\vec{e_1})\big)$$

which is what changes near the center of V_1 when we move only the other vortex. This can be computed more easily and that is what we do in subsection 2.3.3. This term is easy to compute only near the vortex V_1, and that is one of the reasons we work only on $B(d\vec{e_1}, d^{\varepsilon'})$. The main contribution to the variation of the position of V_{-1} is as expected from the source term $E - ic\partial_{x_2}V$. This is the computation of subsection 2.3.4.

Furthermore, most estimations boils down to what happen near each vortex, see for instance the contribution of E in step 5 of the proof of Proposition 2.1.26, where we separate the contribution far from both vortices and close to them. By integrating only on $B(d\vec{e_1}, d^{\varepsilon'})$ we reduce the number of estimations we need to do. Moreover, in such a ball the contribution of the vortex V_{-1} and its derivatives are easy to compute, see subsection 2.3.2.

Subsection 2.3.5 gathers all the estimations needed to show that only the contribution from the source term is of leading order. Subsection 2.3.6 and 2.3.7 are easy computations using previous subsections to compute the first order term of $\partial_c Q_c$.

The main and most difficult part is subsection 2.3.3. We want to show that $\partial_{x_1+d}\big(\tilde{\Gamma}_1(x - d\vec{e_1})\big)$ is much smaller than $\tilde{\Gamma}_1(x - d\vec{e_1})$, i.e. that the derivative with respect to $x_1 + d$ gives us additional smallness in c. For this we do a proof by contradiction which follows closely what was done in the proof of Proposition 2.1.17.

We define the following differential operators:

$$\partial_{y_1} := \partial_{x_1} - \partial_d,$$
$$\partial_{z_1} := \partial_{x_1} + \partial_d.$$

These notations follow the definitions of $y_1 = x_1 - d$ and $z_1 = x_1 + d$ from (2.1.1). The derivative in d is taken at fixed c. The function $\partial_d \Phi_{c,d}$ is the derivative of Φ with respect to d at fixed c and we shall use the notation

$$\partial_d \Phi_{c,d_c} := \partial_d \Phi_{c,d\,|\,d=d_c},$$

and similarly for $\partial_d \Gamma_{c,d_c}$ and $\partial_d \Psi_{c,d_c}$. The derivatives ∂_{y_1} and ∂_{z_1} behave naturally on function depending on x and d only through y or z, as shown in the following lemma.

Lemma 2.3.2. *For any $\mathfrak{F} \in C^1(\mathbb{R}^2, \mathbb{C})$, we have*

$$\partial_{y_1}(\mathfrak{F}(z)) = \partial_{z_1}(\mathfrak{F}(y)) = 0$$

and

$$\partial_{y_1}(\mathfrak{F}(y)) = 2\partial_{x_1}\mathfrak{F}(y),$$
$$\partial_{z_1}(\mathfrak{F}(z)) = 2\partial_{x_1}\mathfrak{F}(z).$$

Proof. We compute

$$\partial_{y_1}(\mathfrak{F}(z)) = \partial_{x_1}(\mathfrak{F}(x_1+d, x_2)) - \partial_d(\mathfrak{F}(x_1+d, x_2)) = \partial_{x_1}\mathfrak{F}(z) - \partial_{x_1}\mathfrak{F}(z) = 0.$$

Similarly we have $\partial_{z_1}(\mathfrak{F}(y)) = 0$. Moreover,

$$\partial_{y_1}(\mathfrak{F}(y)) = \partial_{x_1}(\mathfrak{F}(x_1-d, x_2)) - \partial_d(\mathfrak{F}(x_1-d, x_2)) = \partial_{x_1}\mathfrak{F}(y) + \partial_{x_1}\mathfrak{F}(y) = 2\partial_{x_1}\mathfrak{F}(y)$$

and similarly, $\partial_{z_1}(\mathfrak{F}(z)) = 2\partial_{x_1}\mathfrak{F}(z)$. □

We have an estimate on $\partial_d \Phi_{c,d\,|\,d=d_c}$, but it is not enough to show that d_c is a C^1 function of c. The main idea of the proof is to compute an estimate on $\partial_{z_1}\Phi_{c,d_c} = \partial_{x_1}\Phi_{c,d_c} + \partial_d \Phi_{c,d_c}$ near the vortex V_1 which is better than the ones on $\partial_{x_1}\Phi_{c,d_c}$ and $\partial_d \Phi_{c,d_c}$. In particular we will have $\partial_{z_1}\Phi_{c,d_c} = o_{c\to 0}(c^{1+\lambda})$ for some $\lambda > 0$ instead of $o_{c\to 0}(c^{1-\sigma})$ for $\sigma > 0$. This estimate is done in Proposition 2.3.5. First, we compute a first rough estimate on $\partial_{z_1}\Psi_{c,d}$ which is a corollary of Lemma 2.2.3.

Corollary 2.3.3. *For χ a smooth cutoff function with value 1 in $\{r_{-1} \geqslant 3\}$ and 0 in $\{r_{-1} \leqslant 2\}$, for $0 < \sigma < \sigma' < 1$, there exist $c_0(\sigma, \sigma') > 0$ such that, for $0 < c < c_0(\sigma, \sigma')$, we have*

$$\begin{aligned} & \left\| V\chi\, \partial_{z_1}\Psi_{c,d\,|\,d=d_c} \right\|_{C^1(\{\tilde r \leqslant 3\})} \\ + \ & \left\| \tilde r^{1+\sigma} \mathfrak{Re}\big(\partial_{z_1}\Psi_{c,d\,|\,d=d_c}\big) \right\|_{L^\infty(\{\tilde r \geqslant 2\})} + \left\| \tilde r^{2+\sigma} \nabla \mathfrak{Re}\big(\partial_{z_1}\Psi_{c,d\,|\,d=d_c}\big) \right\|_{L^\infty(\{\tilde r \geqslant 2\})} \\ + \ & \left\| \tilde r^{\sigma} \mathfrak{Im}\big(\partial_{z_1}\Psi_{c,d\,|\,d=d_c}\big) \right\|_{L^\infty(\{\tilde r \geqslant 2\})} + \left\| \tilde r^{1+\sigma} \nabla \mathfrak{Im}\big(\partial_{z_1}\Psi_{c,d\,|\,d=d_c}\big) \right\|_{L^\infty(\{\tilde r \geqslant 2\})} \\ \leqslant \ & K(\sigma, \sigma') c^{1-\sigma'}. \end{aligned}$$

Proof. Remark that $V_1 \partial_d \Psi_{c,d}$ might not be bounded near $d\vec{e}_1$, but $V_1 \partial_{z_1}\Psi_{c,d}$ is, since, by Lemma 2.3.2, $\partial_{z_1} V_1 = 0$ hence

$$V_1 \partial_{z_1}\Psi_{c,d} = \partial_{z_1}\Phi_{c,d} = \partial_d \Phi_{c,d} + \partial_{x_1}\Phi_{c,d},$$

with $\partial_d \Phi_{c,d}$ bounded by Lemma 2.2.3. We take a cutoff χ to avoid the fact that $V_{-1}\partial_{z_1}\Psi_{c,d}$ is not necessary bounded near $-d\vec{e}_1$. In particular, with these remarks, we easily check, with Lemma 2.2.3, that

$$\left\| V\chi \partial_{z_1}\Psi_{c,d\,|\,d=d_c} \right\|_{C^1(\{\tilde r \leqslant 3\})} \leqslant K(\sigma, \sigma') c^{1-\sigma'}.$$

We now focus on the region $\{\tilde r \geqslant 2\}$. From the definition of ∂_{z_1}, we have that

$$\partial_{z_1}\Psi_{c,d_{|\,d=d_c}} = \partial_d \Psi_{c,d_c} + \partial_{x_1}\Psi_{c,d_c}.$$

We compute

$$\partial_d \Psi_{c,d_c} = \frac{\partial_d \Phi_{c,d_c}}{V} + \frac{\partial_d V}{V}\Psi_{c,d_c},$$

and from Lemma 2.2.3, we have

$$\left\| \frac{\partial_d \Phi_{c,d_c}}{V} \right\|_{*,\sigma,d_c} \leqslant K(\sigma,\sigma')c^{1-\sigma'}.$$

From Lemma 2.1.6, we have

$$|\partial_d V| \leqslant \frac{K}{(1+\tilde{r})}$$

and

$$|\nabla \partial_d V| \leqslant \frac{K}{(1+\tilde{r})^2},$$

and together with $\|\Psi_{c,d_c}\|_{*,\sigma,d_c} \leqslant K(\sigma,\sigma')c^{1-\sigma'}$, we check that

$$\begin{aligned} & \left\| \tilde{r}^{1+\sigma} \mathfrak{Re}\left(\frac{\partial_d V}{V}\Psi_{c,d_c} \right) \right\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \left\| \tilde{r}^{2+\sigma} \nabla \mathfrak{Re}\left(\frac{\partial_d V}{V}\Psi_{c,d_c} \right) \right\|_{L^\infty(\{\tilde{r}\geqslant 2\})} \\ + & \left\| \tilde{r}^{\sigma} \mathfrak{Im}\left(\frac{\partial_d V}{V}\Psi_{c,d_c} \right) \right\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \left\| \tilde{r}^{1+\sigma} \nabla \mathfrak{Im}\left(\frac{\partial_d V}{V}\Psi_{c,d_c} \right) \right\|_{L^\infty(\{\tilde{r}\geqslant 2\})} \\ \leqslant & \; K(\sigma,\sigma')c^{1-\sigma'}. \end{aligned}$$

Finally, for the contribution of $\partial_{x_1}\Psi_{c,d_c}$, using $\|\Psi_{c,d_c}\|_{*,\sigma,d_c} \leqslant K(\sigma,\sigma')c^{1-\sigma'}$, we show that, with some margin,

$$\begin{aligned} & \|\tilde{r}^{1+\sigma}\mathfrak{Re}(\partial_{x_1}\Psi_{c,d_c})\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\nabla\mathfrak{Re}(\partial_{x_1}\Psi_{c,d_c})\|_{L^\infty(\{\tilde{r}\geqslant 2\})} \\ + & \|\tilde{r}^{\sigma}\,\mathfrak{Im}(\partial_{x_1}\Psi_{c,d_c})\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{1+\sigma}\nabla\mathfrak{Im}(\partial_{x_1}\Psi_{c,d_c})\|_{L^\infty(\{\tilde{r}\geqslant 2\})} \\ \leqslant & \; K(\sigma,\sigma')c^{1-\sigma'}, \end{aligned}$$

which ends the proof of this corollary. □

2.3.1 Recasting the implicit equation defining d_c

At this point, we do not know if d_c is uniquely defined for $c>0$. We denote by d_c a value defined by the implicit equation on d:

$$\langle \mathrm{TW}_c(Q_{c,d}), \partial_d V \rangle = 0,$$

where

$$Q_{c,d} := V + \Gamma_{c,d},$$

with $\Gamma_{c,d} = \eta V \Psi_{c,d} + (1-\eta)\, V(e^{\Psi_{c,d}} - 1)$, which is a C^1 function of d and c in $\mathcal{E}_{*,\sigma,d}$ thanks to subsection 2.2.1. Remark that d_c is also defined by the implicit equation for $0<\varepsilon'<1$:

$$\int_{B(d\vec{e}_1,d^{\varepsilon'})\cup B(-d\vec{e}_1,d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\mathrm{TW}_c(Q_{c,d})) = 0,$$

that we will use instead because of the reasons explained at the begining of section 2.3. We can check easily that $\partial_d Q_{c,d}$, $\partial_c Q_{c,d} \in C^\infty(\mathbb{R}^2)$ (by looking at the equations they satisfy in the distribution sense and using standard elliptic regularity arguments), and furthermore, that $d \mapsto \partial_d Q_{c,d}$ and $c \mapsto \partial_c Q_c$ are continuous functions (on their domain of definition in $C^\infty_{\mathrm{loc}}(\mathbb{R}^2)$ for instance). From now on, we take any $0<\varepsilon'<1$, but we will fix its value later on. We want to differentiate this quantity with respect to d and take the result at a value d_c such that $\mathrm{TW}_c(Q_{c,d_c}) = 0$ in $\mathbb{R}^2$. In particular, we have

$$\partial_d \int_{B(d\vec{e}_1,d^{\varepsilon'})\cup B(-d\vec{e}_1,d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\mathrm{TW}_c(Q_{c,d}))_{|\,d=d_c} =$$

$$\int_{B(d\vec{e}_1,d^{\varepsilon'})\cup B(-d\vec{e}_1,d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\partial_d(\mathrm{TW}_c(Q_{c,d})))_{|\,d=d_c}.$$

Now, by symmetry, we remark that

$$\int_{B(d\vec{e_1},d^{\varepsilon'})\cup B(-d\vec{e_1},d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\partial_d(\mathrm{TW}_c(Q_{c,d}))) = 2\int_{B(d\vec{e_1},d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\partial_d(\mathrm{TW}_c(Q_{c,d}))).$$

We will use the two operators we have already defined:

$$\partial_{y_1} = \partial_{x_1} - \partial_d \quad \text{and} \quad \partial_{z_1} = \partial_{x_1} + \partial_d.$$

Since $\mathrm{TW}_c(Q_{c,d_c}) = 0$ everywhere in $\mathbb{R}^2$, we therefore have $\partial_{x_1}(\mathrm{TW}_c(Q_{c,d_c})) = 0$, hence, at $d = d_c$,

$$\partial_d(\mathrm{TW}_c(Q_{c,d})) = \partial_{z_1}(\mathrm{TW}_c(Q_{c,d})).$$

We write

$$\mathrm{TW}_c(Q_{c,d}) = \mathrm{TW}_c(V) + L(\Gamma_{c,d}) + \mathrm{NL}_V(\Gamma_{c,d}),$$

with

$$L(\Gamma_{c,d}) = -\Delta\Gamma_{c,d} - ic\partial_{x_2}\Gamma_{c,d} - (1-|V|^2)\,\Gamma_{c,d} + 2\,\mathfrak{Re}(\bar{V}\,\Gamma_{c,d})V$$

and

$$\mathrm{NL}_V(\Gamma_{c,d}) := 2\,\mathfrak{Re}(\bar{V}\,\Gamma_{c,d})\,\Gamma_{c,d} + |\Gamma_{c,d}|^2(V+\Gamma_{c,d}).$$

We compute

$$\partial_{z_1}(\mathrm{TW}_c(Q_{c,d})) = \partial_{z_1}(\mathrm{TW}_c(V)) + L(\partial_{z_1}\Gamma_{c,d}) + (\partial_{z_1}L)(\Gamma_{c,d}) + \partial_{z_1}(\mathrm{NL}_V(\Gamma_{c,d})),$$

therefore, at $d = d_c$,

$$\begin{aligned}\partial_d \int_{B(d\vec{e_1},d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\,\mathrm{TW}_c(Q_{c,d})) &= \int_{B(d\vec{e_1},d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\partial_{z_1}(\mathrm{TW}_c(V)))\\ &+ \int_{B(d\vec{e_1},d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}L(\partial_{z_1}\Gamma_{c,d})) + \int_{B(d\vec{e_1},d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}(\partial_{z_1}L)(\Gamma_{c,d}))\\ &+ \int_{B(d\vec{e_1},d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\partial_{z_1}(\mathrm{NL}_V(\Gamma_{c,d}))) \qquad (2.3.1)\end{aligned}$$

since the boundary term is 0 (when the differentiation is on the d in $B(d\vec{e_1},\ d^{\varepsilon'})$) because $\mathrm{TW}_c(Q_{c,d_c}) = 0$. We need to estimate those four terms at $d = d_c$, and that is the goal of the next subsections. Subsections 2.3.2 and 2.3.3 yield estimates on the derivatives of V_{-1} and $\partial_{z_1}\Psi_{c,d}$ respectively in $B'_d := B(d\vec{e_1}, d^{\varepsilon'})$. Subsection 2.3.4 is about the estimation of

$$\int_{B(d\vec{e_1},d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\partial_{z_1}(\mathrm{TW}_c(V)))$$

which will be the leading order term, and subsection 2.3.5 shows that all the other terms are smaller for d_c large enough.

2.3.2 Estimates on the derivatives of V_{-1} in $B(d\vec{e_1}, d^{\varepsilon})$

Lemma 2.3.4. *For $0 < \varepsilon < 1$, in $B(d\vec{e_1}, d^{\varepsilon})$, with the $O(.)$ being always real valued, we have*

$$\partial_{x_1}V_{-1} = \left(O_{d\to\infty}\left(\frac{1}{d^3}\right) + iO_{d\to\infty}\left(\frac{1}{d^{2-\varepsilon}}\right)\right)V_{-1},$$

$$\partial_{x_2}V_{-1} = \left(O_{d\to\infty}\left(\frac{1}{d^{4-\varepsilon}}\right) + iO_{d\to\infty}\left(\frac{1}{d}\right)\right)V_{-1},$$

$$\partial_{x_1x_1}V_{-1} = \left(O_{d\to\infty}\left(\frac{1}{d^{4-2\varepsilon}}\right) + iO_{d\to\infty}\left(\frac{1}{d^{3-\varepsilon}}\right)\right)V_{-1},$$

$$\partial_{x_1x_2}V_{-1} = \left(O_{d\to\infty}\left(\frac{1}{d^{3-\varepsilon}}\right) + \frac{i}{4d^2}\left(1 + O_{d\to\infty}\left(\frac{1}{d^{1-\varepsilon}}\right)\right)\right)V_{-1}.$$

Proof. Recall from Lemma 2.1.2 that, with $u=\frac{\rho'_{-1}(r_{-1})}{\rho_{-1}(r_{-1})}$,

$$\partial_{x_1}V_{-1}=\left(\cos(\theta_{-1})u+\frac{i}{r_{-1}}\sin(\theta_{-1})\right)V_{-1},$$

$$\partial_{x_2}V_{-1}=\left(\sin(\theta_{-1})u-\frac{i}{r_{-1}}\cos(\theta_{-1})\right)V_{-1},$$

$$\partial_{x_1x_1}V_{-1}=\left(\cos^2(\theta_{-1})(u^2+u')+\sin^2(\theta_{-1})\left(\frac{u}{r_{-1}}-\frac{1}{r_{-1}^2}\right)-2\,i\sin(\theta_{-1})\cos(\theta_{-1})\left(\frac{1}{r_{-1}^2}-\frac{u}{r_{-1}}\right)\right)V_{-1}$$

and

$$\partial_{x_1x_2}V_{-1}=\left(\sin(\theta_{-1})\cos(\theta_{-1})\left(u^2+u'+\frac{1}{r_{-1}^2}-\frac{u}{r_{-1}}\right)+i\cos(2\theta_{-1})\left(\frac{1}{r_{-1}^2}-\frac{u}{r_{-1}}\right)\right)V_{-1}.$$

In the ball $B(d\vec{e_1},d^\varepsilon)$, we have, by Lemma 2.1.1, that $\frac{1}{r_{-1}}\leqslant\frac{K}{d}$,

$$u=O_{d\to\infty}\left(\frac{1}{d^3}\right)\quad\text{and}\quad\sin(\theta_{-1})=O_{d\to\infty}\left(\frac{1}{d^{1-\varepsilon}}\right),$$

the last one is because for $(y_1,y_2)\in B(d\vec{e_1},d^\varepsilon)$, we have $|\,y_2\,|\leqslant d^\varepsilon$ hence

$$|\sin(\theta_{-1})|=\frac{|y_2|}{r_{-1}}\leqslant\frac{K}{d^{1-\varepsilon}}.$$

We also compute in the same way that

$$\cos(\theta_{-1})=\sqrt{1-\sin^2(\theta_{-1})}=1+O_{d\to\infty}\left(\frac{1}{d^{2-2\varepsilon}}\right).$$

With the equation on ρ_{-1} coming fom $-\Delta V_{-1}-(1-|V_{-1}|^2)V_{-1}=0$, we check easily that

$$u'=O_{d\to\infty}\left(\frac{1}{d^4}\right)$$

as well (or see [25]). Finally, we estimate

$$\cos(2\theta_{-1})=1-2\sin^2(\theta_{-1})=1+O_{d\to\infty}\left(\frac{1}{d^{2-2\varepsilon}}\right)$$

and

$$\frac{1}{r_{-1}^2}=(2d+O_{d\to\infty}(d^\varepsilon))^{-2}=\frac{1}{4d^2}+O_{d\to\infty}\left(\frac{1}{d^{3-\varepsilon}}\right).$$

With this estimations, we end the proof of the lemma. □

2.3.3 Estimate on $\partial_{z_1}\Psi_{c,d}$ in $B(d\vec{e_1},d^{\varepsilon'})$

We define the following norms for $\Psi=\Psi_1+i\Psi_2$ and $h=h_1+ih_2$, $0<\alpha<1, 0<\varepsilon'<\varepsilon<1$:

$$\begin{aligned}\|\Psi\|_{*,B'_d} &:= \|V\Psi\|_{C^1(\{r_1\leqslant 2\})}\\ &+ \|r_1^{1-\alpha}\,\Psi_1\|_{L^\infty(\{d^{\varepsilon'}\geqslant r_1\geqslant 2\})}+\|r_1^{1-\alpha}\nabla\Psi_1\|_{L^\infty(\{d^{\varepsilon'}\geqslant r_1\geqslant 2\})}\\ &+ \|r_1^{-\alpha}\,\Psi_2\|_{L^\infty(\{d^{\varepsilon'}\geqslant r_1\geqslant 2\})}+\|r_1^{1-\alpha}\nabla\Psi_2\|_{L^\infty(\{d^{\varepsilon'}\geqslant r_1\geqslant 2\})}\end{aligned}$$

and

$$\begin{aligned}\|h\|_{**,B_d} &:= \|Vh\|_{C^0(\{r_1\leqslant 3\})}\\ &+ \|r_1^{1-\alpha}\,h_1\|_{L^\infty(\{d^{\varepsilon}\geqslant r_1\geqslant 2\})}+\|r_1^{2-\alpha}\,h_2\|_{L^\infty(\{d^{\varepsilon}\geqslant r_1\geqslant 2\})}.\end{aligned}$$

They are the norms $\|.\|_{*,-\alpha,d}$ and $\|.\|_{**,-\alpha,d}$ of subsection 2.1.3, but without the second derivatives, less decay on the gradient of the real part for $\|.\|_{*,B'_d}$, and only on $B'_d = B(d\vec{e_1}, d^{\varepsilon'})$ for $\|.\|_{*,B'_d}$ and on $B_d := B(d\vec{e_1}, d^{\varepsilon})$ for $\|.\|_{**,B_d}$. The other main difference with the previous norms is that we require less decay (we take $-\alpha < 0$ instead of $\sigma > 0$ in the decay) in space, which here, since the norms are only in $\{r_1 \leqslant d^{\varepsilon}\}$, can be compensated by some smallness in c.

From Corollary 2.3.3, we have that $\|\partial_{z_1}\Psi_{c,d_c}\|_{*,B'_{d_c}} < +\infty$. We want to show the following proposition.

Proposition 2.3.5. *For $0<\alpha<1, 0<\varepsilon'<\varepsilon<1, 0<\lambda<1$, if*

$$\lambda < (1+\alpha)\varepsilon',$$

$$\lambda + (1-\alpha)\varepsilon' < 2\varepsilon - \varepsilon'$$

and

$$\lambda < 2 - \varepsilon(2-\alpha),$$

we have

$$\left\|\partial_{z_1}\Psi_{c,d_{|d=d_c}}\right\|_{*,B'_{d_c}} = o_{c\to 0}(c^{1+\lambda}).$$

Such a choice of parameters $(\lambda, \alpha, \varepsilon, \varepsilon')$ exists, we can take for instance $\alpha = 1/2$, $\lambda = 3/4$, $\varepsilon = 19/24$ and $\varepsilon' = 13/24$. Furthermore, with this particular choice of parameters, we also have

$$\lambda + (1-\alpha)\varepsilon' > 1, \tag{2.3.2}$$

which will be usefull later on. These conditions are bounds on how much additional smallness we can have on $\partial_{z_1}\Psi_{c,d}$ near $d_c\vec{e_1}$.

The main goal of this proposition is to have a decay in c better than $O_{c\to 0}(c)$, which is not obvious from the estimates we have done until now. The estimate on $\partial_{z_1}\Psi_{c,d_{|d=d_c}}$ from Corollary 2.3.3 will not be enough in the computation of $\partial_c d_c$ for the nonlinear terms. The proof of Proposition 2.3.5 follows closely the proof of the inversibility of the linearized operator in Proposition 2.1.17. We want to invert the same linearized operator, but with a different norm, which is better locally around the vortex V_1.

The reason why we take B_d a little bigger than B'_d is to make the elliptic estimates of step 2 in Proposition 2.1.17 work here too. The main idea of this proposition is to show that if we move V_{-1} a little, then locally around V_1 the change is very small. We now start the proof of Proposition 2.3.5.

Proof. First, we remark that in B_d, since $\varepsilon < 1$, $\tilde{r} = r_1$.

Step 1. Computation of the equation on $\partial_{z_1}\Psi_{c,d}$.

Recall that $\Phi_{c,d}$ solves the equation (with $\Phi_{c,d} = V\Psi_{c,d}$)

$$\eta L(\Phi_{c,d}) + (1-\eta)\, VL'(\Psi_{c,d}) + F(\Psi_{c,d}) = \lambda(c,d)Z_d,$$

and we recall that $\lambda(c,d) = \frac{\langle F(\Psi_{c,d}), Z_d\rangle}{\|Z_d\|^2_{L^2(\mathbb{R}^2)}}$, and we check easily, with Lemma 2.2.3, that it is a C^1 function of d. The equation on $\Phi_{c,d}$ holds for any $x \in \mathbb{R}^2$ and any $d \in \mathbb{R}$, $\frac{1}{2d} < c < \frac{2}{d}$, hence

$$\partial_{z_1}(\eta L(\Phi_{c,d}) + (1-\eta)\, VL'(\Psi_{c,d}) + \Pi_d^{\perp}(F(\Psi_{c,d})) - \lambda(c,d)Z_d) = 0.$$

We compute

$$\begin{aligned}\partial_{z_1}(\lambda(c,d)Z_d) &= (\partial_{x_1} + \partial_d)(\lambda(c,d)Z_d)\\ &= \partial_d\lambda(c,d)Z_d + \lambda(c,d)\partial_{z_1}Z_d,\end{aligned}$$

and we recall, from the proof of Proposition 2.1.26 that

$$\lambda(c,d)\int_{\mathbb{R}^2}|\partial_d V|^2\eta^2 = \pi\left(\frac{1}{d} - c\right) + O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

With Lemma 2.2.3 and Corollary 2.3.3, as well as Lemma 2.1.6, we infer that the terms contributing to the $O^{\sigma}_{c\to 0}(c^{2-\sigma})$ are such that, when differentiated with respect to d, their contributions are still a $O^{\sigma}_{c\to 0}(c^{2-\sigma})$. Indeed, if the derivative with respect to d fall on a $\Psi_{c,d}$, then by Lemma 2.2.3 and Corollary 2.3.3, the same estimates used in the proof of Proposition 2.1.26 still hold. If the derivative fall on a term depending on V, by Lemma 2.1.6, we gain some decay in the integrals. We deduce that, since $\lambda(c,d_c)=0$,

$$\partial_d\lambda(c,d)_{|d=d_c}=\frac{-\pi}{d_c^2}+O^{\sigma}_{c\to 0}(c^{2-\sigma})=O^{\sigma}_{c\to 0}(c^{2-\sigma}).$$

Here, we see why the fact that d is differentiable with respect to c is not obvious. The main contribution is at this point not enough to beat the error terms. Therefore, showing that $\partial_d\lambda(c,d)\neq 0$ is not simple here. This is why we need improved estimations on $\partial_{z_1}\Psi_{c,d_c}$, that will give us the fact that the error terms are a $O^{\varepsilon}_{c\to 0}(c^{2+\varepsilon})$ for some $\varepsilon>0$.

Now, writing

$$\mathrm{TW}_c(Q_{c,d})=\eta L(\Phi_{c,d})+(1-\eta)VL'(\Psi_{c,d})+F(\Psi_{c,d}),$$

(with the notations of Lemma 2.1.7), we have (since $\lambda(c,d_c)=0$)

$$(\partial_{z_1}(\mathrm{TW}_c(Q_{c,d}))-\partial_d\lambda(c,d)Z_d)_{|d=d_c}=0.$$

We recall that

$$F(\Psi_{c,d})=E-ic\partial_{x_2}V+V(1-\eta)(-\nabla\Psi_{c,d}.\nabla\Psi_{c,d}+|V|^2S(\Psi_{c,d}))+R(\Psi_{c,d}),$$

where $R(\Psi_{c,d})$ is a sum of terms at least quadratic in $\Psi_{c,d}$ or $\Phi_{c,d}$ localized in the area where $\eta\neq 0$.

We compute

$$\begin{aligned}
\partial_{z_1}(\mathrm{TW}_c(Q_{c,d})) &= \eta L(V\partial_{z_1}\Psi_{c,d})+(1-\eta)VL'(\partial_{z_1}\Psi_{c,d})\\
&+ \eta\partial_{z_1}L(\Phi_{c,d})+(1-\eta)V\partial_{z_1}L'(\Psi_{c,d})+\partial_{z_1}(E-ic\partial_{x_2}V)\\
&+ \eta L(\partial_{z_1}V\Psi_{c,d})+(1-\eta)\partial_{z_1}VL'(\Psi_{c,d})\\
&+ \partial_{z_1}\eta(L(\Phi_{c,d})-VL'(\Psi_{c,d})-ic\partial_{x_2}\Phi_{c,d})\\
&- \partial_{z_1}\eta V(-ic\partial_{x_2}\Psi_{c,d}-\nabla\Psi_{c,d}.\nabla\Psi_{c,d}+|V|^2S(\Psi_{c,d}))\\
&+ \partial_{z_1}(R(\Psi_{c,d}))\\
&+ \partial_{z_1}V(1-\eta)(-ic\partial_{x_2}\Psi_{c,d}-\nabla\Psi_{c,d}.\nabla\Psi_{c,d}+|V|^2S(\Psi_{c,d}))\\
&+ V(1-\eta)\partial_{z_1}(-ic\partial_{x_2}\Psi_{c,d}-\nabla\Psi_{c,d}.\nabla\Psi_{c,d}+|V|^2S(\Psi_{c,d})).
\end{aligned}$$

We regroup the terms in the following way. We define

$$\mathcal{L}(\partial_{z_1}\Psi_{c,d}):=\eta L(V\partial_{z_1}\Psi_{c,d})+(1-\eta)VL'(\partial_{z_1}\Psi_{c,d}),$$

which is the same linearized operator we have inverted in Proposition 2.1.17 (taken in $\partial_{z_1}\Psi_{c,d}$), and we define the operator

$$\mathcal{L}_{\partial_{z_1}}(\Psi_{c,d}):=\eta\partial_{z_1}L(\Phi_{c,d})+(1-\eta)V\partial_{z_1}L'(\Psi_{c,d})+\eta L(\partial_{z_1}V\Psi_{c,d})+(1-\eta)\partial_{z_1}VL'(\Psi_{c,d}).$$

We already have $\mathrm{TW}_c(V)=E-ic\partial_{x_2}V$, therefore

$$\partial_{z_1}(\mathrm{TW}_c(V))=\partial_{z_1}(E-ic\partial_{x_2}V).$$

We define the local error

$$\mathrm{Err}_{\mathrm{loc}}:=\partial_{z_1}(R(\Psi_{c,d}))-\partial_d\lambda(c,d)Z_d,$$

the far away error

$$\mathrm{Err}_{\mathrm{far}}:=\partial_{z_1}V(1-\eta)(-\nabla\Psi_{c,d}.\nabla\Psi_{c,d}+|V|^2S(\Psi))$$

and the nonlinear terms

$$\mathrm{NL}_{\partial_{z_1}}(\Psi_{c,d}):=V(1-\eta)\partial_{z_1}(-\nabla\Psi_{c,d}.\nabla\Psi_{c,d}+|V|^2S(\Psi_{c,d})).$$

Finally, we write the cutoff error

$$\mathrm{Err}_{\mathrm{cut}}:=\partial_{z_1}\eta(L(\Phi_{c,d})-VL'(\Psi_{c,d})+ic\partial_{x_2}\Psi_{c,d}+\nabla\Psi_{c,d}.\nabla\Psi_{c,d}-|V|^2S(\Psi_{c,d}))$$

which is supported in the area $\{2 \leqslant r_{-1} \leqslant 3\}$, and in particular is zero in $B(d_c\vec{e}_1, d_c^{\varepsilon})$. With these definitions, we have, at $d = d_c$,

$$\begin{aligned} & (\partial_{z_1}(\eta L(\Phi_{c,d}) + (1-\eta)\, VL'(\Psi_{c,d}) + F(\Psi_{c,d})) - \partial_d \lambda(c,d) Z_d)_{|\, d=d_c} \\ = \;& \mathcal{L}(\partial_{z_1}\Psi_{c,d})_{|\, d=d_c} \\ + \;& (\partial_{z_1}(\mathrm{TW}_c(V)) + \mathcal{L}_{\partial_{z_1}}(\Psi_{c,d}) + \mathrm{NL}_{\partial_{z_1}}(\Psi_{c,d}))_{|\, d=d_c} \\ + \;& (\mathrm{Err}_{\mathrm{loc}} + \mathrm{Err}_{\mathrm{far}} + \mathrm{Err}_{\mathrm{cut}})_{|\, d=d_c}. \end{aligned}$$

The equation satisfied by $\partial_{z_1}\Psi_{c,d}$ at $d = d_c$ is therefore

$$(\mathcal{L}(\partial_{z_1}\Psi_{c,d}) + \partial_{z_1}(\mathrm{TW}_c(V)) + \mathcal{L}_{\partial z_1}(\Psi_{c,d}) + \mathrm{NL}_{\partial_{z_1}}(\Psi_{c,d}) + \mathrm{Err}_{\mathrm{loc}} + \mathrm{Err}_{\mathrm{far}} + \mathrm{Err}_{\mathrm{cut}})_{|\, d=d_c} = 0.$$

Step 2. Beginning of the contradiction argument.

Now, suppose that the result of Proposition 2.3.5 is false. The scheme of this proof is the same as in Proposition 2.1.17. Then, there exist an absolute constant $\delta > 0$ and sequences $\partial_{z_1}\Psi_n$, $c_n \to 0$, $d_n \to \infty$ such that

$$d_n^{1+\lambda} \| \partial_{z_1}\Psi_{n\, |\, d=d_n} \|_{*, B'_{d_n}} \geqslant \delta,$$

where we write $d_n = d_{c_n}$ (a value such that $\lambda(c_n, d_n) = 0$ in Proposition 2.1.26). We have just shown that Ψ_n (where we omit the subscripts in d_n, c_n) satisfies

$$\mathcal{L}(\partial_{z_1}\Psi_n) + \partial_{z_1}(\mathrm{TW}_{c_n}(V)) + \mathcal{L}_{\partial z_1}(\Psi_n) + \mathrm{NL}_{\partial_{z_1}}(\Psi_n) + \mathrm{Err}_{\mathrm{loc}} + \mathrm{Err}_{\mathrm{far}} + \mathrm{Err}_{\mathrm{cut}} \;=\; 0.$$

The function

$$\frac{(V\partial_{z_1}\Psi_n)(. - d_n\vec{e}_1)}{\|\partial_{z_1}\Psi_n\|_{*, B'_{d_n}}}$$

converges locally uniformly up to a subsequence to a limit $\mathfrak{G}$, since it is bounded in $\|.\|_{*, B'_\lambda}$ for any $\lambda > 0$ (for the same reasons that $\Psi_n \to \Psi$ locally uniformly in the beginning of the proof of Proposition 2.1.17).

The equation on $\partial_{z_1}\Psi_n$ is

$$\mathcal{L}(\partial_{z_1}\Psi_n) + Vh_n = 0, \tag{2.3.3}$$

with

$$Vh_n \;:=\; \partial_{z_1}(\mathrm{TW}_{c_n}(V)) + \mathcal{L}_{\partial z_1}(\Psi_n) + \mathrm{NL}_{\partial_{z_1}}(\Psi_n) + \mathrm{Err}_{\mathrm{loc}} + \mathrm{Err}_{\mathrm{far}} + \mathrm{Err}_{\mathrm{cut}}.$$

The goal of Proposition 2.1.17 was to estimate $\|\Psi\|_{*,\sigma,d}$ with $\|h\|_{**,\sigma',d}$ for the equation $\mathcal{L}(\Psi) = h$ if d is large enough (given an orthogonality condition on Ψ). Here we do the same thing, but localized in space, and with a very particular h_n that we will estimate. To continue as in the proof of Proposition 2.1.17, we want to show that

$$\frac{h_n(. - d_n\vec{e}_1)}{\|\partial_{z_1}\Psi_n\|_{*, B'_{d_n}}} \to 0$$

in $\mathcal{C}^0_{\mathrm{loc}}$ so that we get at the limit (following the $+1$ vortex) in (2.3.3)

$$L_{V_1}(\mathfrak{G}) = 0,$$

using the sames techniques as in the proof of Proposition 2.1.17. It will be enough for that to show that

$$\left\| \frac{h_n}{\|\partial_{z_1}\Psi_n\|_{*, B'_{d_n}}} \right\|_{**, B_{d_n}} \to 0 \tag{2.3.4}$$

and we will also use this estimate later on. Remark that here, the problem is no longer symmetric in x_1, in particular, we cannot use the same argument near the -1 vortex, but it is not needed.

Step 3. Proof of (2.3.4).

Recall the definition of $\|.\|_{**,B_{d_n}}$:

$$\begin{aligned}\|h\|_{**,B_{d_n}} &= \|Vh\|_{C^0(\{r_1\leqslant 3\})}\\ &+ \|r_1^{1-\alpha}h_1\|_{L^\infty(\{d_n^\varepsilon\geqslant r_1\geqslant 2\})}+\|r_1^{2-\alpha}h_2\|_{L^\infty(\{d_n^\varepsilon\geqslant r_1\geqslant 2\})}.\end{aligned}$$

Since

$$d_n^{1+\lambda}\|\partial_{z_1}\Psi_{n\,|\,d=d_n}\|_{*,B'_{d_n}}\geqslant\delta,$$

we have

$$\frac{1}{\|\partial_{z_1}\Psi_n\|_{*,B'_{d_n}}}\leqslant\frac{1}{\delta c_n^{1+\lambda}},$$

therefore it is enough to show that

$$\|h_n\|_{**,B_{d_n}}=o_{c_n\to 0}(c_n^{1+\lambda}) \tag{2.3.5}$$

to have (2.3.4). We recall that

$$Vh_n = \partial_{z_1}(\mathrm{TW}_{c_n}(V))+\mathcal{L}_{\partial z_1}(\Psi_n)+\mathrm{NL}_{\partial_{z_1}}(\Psi_n)+\mathrm{Err}_{\mathrm{loc}}+\mathrm{Err}_{\mathrm{far}}+\mathrm{Err}_{\mathrm{cut}}.$$

The contribution of $\partial_{z_1}(\mathrm{TW}_{c_n}(V))$ will be established in step 3.1, $\mathcal{L}_{\partial_{z_1}}(\Psi_n)$ in step 3.2, $\mathrm{NL}_{\partial_{z_1}}(\Psi_n)$ in step 3.3, and finally, $\mathrm{Err}_{\mathrm{loc}}+\mathrm{Err}_{\mathrm{far}}+\mathrm{Err}_{\mathrm{cut}}$ in step 3.4.

Step 3.1. Proof of $\left\|\frac{\partial_{z_1}\mathrm{TW}_{c_n}(V)}{V}\right\|_{**,B_{d_n}}=o_{c_n\to 0}(c_n^{1+\lambda})$.

Recall from (2.1.2) that

$$\mathrm{TW}_c(V)=E-ic\partial_{x_2}V=-2\nabla V_1.\nabla V_{-1}+(1-|V_1|^2)(1-|V_{-1}|^2)V_1V_{-1}-ic\partial_{x_2}V,$$

therefore, with Lemma 2.3.2, we have

$$\partial_{z_1}(\mathrm{TW}_c(V))=-4\nabla V_1.\nabla\partial_{x_1}V_{-1}+2(1-|V_1|^2)\,V_1\partial_{x_1}((1-|V_{-1}|^2)\,V_{-1})-2\,ic\partial_{x_2}(V_1\partial_{x_1}V_{-1}).$$

We now estimate this quantity at $d=d_n$. We have

$$|(1-|V_1|^2)\,V_1\partial_{x_1}((1-|V_{-1}|^2)\,V_{-1})|\leqslant\frac{K}{1+r_1^2}\times\frac{1}{d_n^3},$$

and using $\lambda<1$, $\alpha>0$, we deduce

$$\left\|\frac{(1-|V_1|^2)\,V_1\partial_{x_1}((1-|V_{-1}|^2)\,V_{-1})}{V}\right\|_{**,B_{d_n}}=o_{c_n\to 0}(c_n^{1+\lambda}).$$

We compute with Lemmas 2.1.2 and 2.3.4 that

$$\mathfrak{Re}\left(\frac{4\nabla V_1.\nabla\partial_{x_1}V_{-1}}{V}\right)=4\mathfrak{Re}\left(\frac{\nabla V_1}{V_1}\right).\mathfrak{Re}\left(\frac{\nabla\partial_{x_1}V_{-1}}{V_{-1}}\right)-4\mathfrak{Im}\left(\frac{\nabla V_1}{V_1}\right).\mathfrak{Im}\left(\frac{\nabla\partial_{x_1}V_{-1}}{V_{-1}}\right),$$

leading to

$$\left|\mathfrak{Re}\left(\frac{4\nabla V_1.\nabla\partial_{x_1}V_{-1}}{V}\right)\right|\leqslant\frac{K}{(1+r_1^3)d_n^{3-\varepsilon}}+\frac{K}{(1+r_1)d_n^2}$$

for a universal constant K. Since $\lambda<1$ and $\alpha>0$, we have

$$\left\|\mathfrak{Re}\left(\frac{4\nabla V_1.\nabla\partial_{x_1}V_{-1}}{V}\right)\right\|_{**,B_{d_n}}=o_{c_n\to 0}(c_n^{1+\lambda}).$$

Similarly, we have

$$\left|\mathfrak{Im}\left(\frac{4\nabla V_1.\nabla\partial_{x_1}V_{-1}}{V}\right)\right|\leqslant\frac{K}{(1+r_1^3)d_n^2}+\frac{K}{(1+r_1)d_n^{3-\varepsilon}}.$$

Therefore, using

$$\frac{1}{d_n}\leqslant\frac{K}{(1+r_1)^{1/\varepsilon}},$$

since we are in $B_{d_n} = B(d_n \vec{e_1}, d_n^\varepsilon)$, and

$$\lambda < 2 - \varepsilon(2-\alpha),$$

which is one of the hypothesis of the lemma, we have

$$\left\| i \mathfrak{Im}\left(\frac{4 \nabla V_1 . \nabla \partial_{x_1} V_{-1}}{V} \right) \right\|_{**, B_{d_n}} = o_{c_n \to 0}(c_n^{1+\lambda}).$$

Now, for $2i\,c_{\ n}\partial_{x_2}(V_1 \partial_{x_1} V_{-1}) = 2i\,c_n \partial_{x_2} V_1 \partial_{x_1} V_{-1} + 2i\,c_n \partial_{x_1 x_2} V_{-1} V_1$, we estimate (still using Lemma 2.1.2 and 2.3.4)

$$\left| \mathfrak{Re}\left(\frac{i c_n\, \partial_{x_2} V_1 \partial_{x_1} V_{-1}}{V} \right) \right| \leqslant \frac{K}{(1+r_1^3) d_n^{3-\varepsilon}} + \frac{K}{(1+r_1) d_n^4},$$

$$\left| \mathfrak{Im}\left(\frac{i c_n\, \partial_{x_2} V_1 \partial_{x_1} V_{-1}}{V} \right) \right| \leqslant \frac{K}{(1+r_1^3) d_n^4} + \frac{K}{(1+r_1) d_n^{3-\varepsilon}},$$

therefore, using $\frac{1}{d_n} \leqslant \frac{K}{(1+r_1)^{1/\varepsilon}}$, we have, under the condition

$$\lambda < 2 - \varepsilon(2-\alpha)$$

for the imaginary part (as for the previous term) and with no condition for the real part (since $\alpha > 0, \lambda < 1$), that

$$\left\| \frac{2\, i\, c_n\, \partial_{x_2} V_1 \partial_{x_1} V_{-1}}{V} \right\|_{**, B_{d_n}} = o_{c_n \to 0}(c_n^{1+\lambda}).$$

We then compute (still using Lemma 2.1.2 and 2.3.4)

$$\left| \mathfrak{Re}\left(\frac{i\, c_{\ n} \partial_{x_1 x_2} V_{-1}\, V_1}{V} \right) \right| \leqslant \frac{K}{d_n^3},$$

$$\left| \mathfrak{Im}\left(\frac{i\, c_{\ n} \partial_{x_1 x_2} V_{-1}\, V_1}{V} \right) \right| \leqslant \frac{K}{d_n^{4-\varepsilon}},$$

therefore, using $\frac{1}{d_n} \leqslant \frac{K}{(1+r_1)^{1/\varepsilon}}$, we have, under the conditions

$$\lambda < 2 - \varepsilon(1-\alpha) \quad \text{and} \quad \lambda < 3 - \varepsilon(3-\alpha),$$

which are met since

$$\lambda < 2 - \varepsilon(2-\alpha) = 2 - \varepsilon(1-\alpha) - \varepsilon < 2 - \varepsilon(1-\alpha),$$

and $\lambda < 2 - \varepsilon(2-\alpha) = 3 - \varepsilon(3-\alpha) - 1 + \varepsilon < 3 - \varepsilon(3-\alpha)$, that

$$\left\| \frac{i\, c \partial_{x_2}(V_1 \partial_{x_1 x_2} V_{-1})}{V} \right\|_{**, B_{d_n}} = o_{c_n \to 0}(c_n^{1+\lambda}).$$

This concludes the proof of step 3.1.

Step 3.2. Proof of $\left\| \frac{\mathcal{L}_{\partial_{z_1}}(\Psi_n)}{V} \right\|_{**, B_{d_n}} = o_{c_n \to 0}(c_n^{1+\lambda})$.

We have defined

$$\mathcal{L}_{\partial_{z_1}}(\Psi_n) = \eta\, (\partial_{z_1} L)(\Phi_n) + (1-\eta) V\, (\partial_{z_1} L')(\Psi_n) + \eta L((\partial_{z_1} V)\Psi_n) + (1-\eta)\partial_{z_1} V L'(\Psi_n).$$

We recall from Lemma 2.1.7 that

$$L'(\Psi_n) = -\Delta \Psi_n - 2\frac{\nabla V}{V} . \nabla \Psi_n + 2|V|^2 \mathfrak{Re}(\Psi_n) - i\, c_n\, \partial_{x_2} \Psi_n,$$

$$L(\Phi_n) = -\Delta \Phi_n - (1 - |V|^2)\Phi_n + 2\mathfrak{Re}(\bar{V}\Phi_n) V - i\, c_n\, \partial_{x_2} \Phi_n,$$

hence

$$(\partial_{z_1} L)(\Phi_n) = 4\, \mathfrak{Re}(\overline{V_{-1}} \partial_{x_1} V_{-1})\Phi_n + 4\, \mathfrak{Re}(\overline{\partial_{x_1} V_{-1}\, V_1}\Phi_n) V + 4\, \mathfrak{Re}(\bar{V}\Phi_n) V_1 \partial_{x_1} V_{-1}.$$

We shall now estimate all these terms one by one.

Since $\eta\partial_{z_1}L(\Phi_n)$ is compactly supported in $\{\tilde{r}\leqslant 2\}$ and $\|.\|_{**,B_{d_n}}$ looks at the function only on $\{r_1\leqslant d^{\varepsilon}\}$, using Lemma 2.3.4 ($\nabla V_{-1}=O_{c\to 0}(c)$) and $\|\Psi_n\|_{*,\frac{1-\lambda}{4},d_n}\leqslant K(\lambda)\,c^{\frac{1+\lambda}{2}}$, we check that

$$\left\|\frac{\eta\partial_{z_1}L(\Phi_n)}{V}\right\|_{**,B_{d_n}}=o_{c_n\to 0}(c_n^{1+\lambda}).$$

With the same arguments, we also check that

$$\left\|\frac{\eta L(\partial_{z_1}V\Psi_n)}{V}\right\|_{**,B_{d_n}}=o_{c_n\to 0}(c_n^{1+\lambda}).$$

Now, with $\|\Psi_n\|_{*,\sigma,d_n}\leqslant K(\sigma,\sigma')c_n^{1-\sigma'}$, we check that for any $0<\sigma<\sigma'<1$,

$$|L'(\Psi_n)|\leqslant\frac{K(\sigma,\sigma')}{(1+r_1)^{1+\sigma}d_n^{1-\sigma'}},$$

therefore, with Lemma 2.3.4, we have

$$|(1-\eta)\partial_{z_1}VL'(\Psi_n)|\leqslant\frac{K(\sigma,\sigma')}{(1+r_1)^{1+\sigma}d_n^{3-\varepsilon-\sigma'}}.$$

In particular, we check that if

$$\lambda<2-\varepsilon(2-\alpha),$$

we can take σ,σ' such that $0<\sigma<\sigma'<\frac{2-\varepsilon(2-\alpha)-\lambda}{1-\varepsilon}$, hence

$$\left\|\frac{(1-\eta)\partial_{z_1}VL'(\Psi_n)}{V}\right\|_{**,B_{d_n}}=o_{c_n\to 0}(c_n^{1+\lambda}).$$

Finally, we estimate

$$|\partial_{z_1}L'(\Psi_n)|\leqslant K\left|\partial_{x_1}\frac{\nabla V_{-1}}{V_{-1}}.\nabla\Psi_n\right|+K\,|\mathfrak{Re}(\partial_{x_1}V_{-1}\overline{V_{-1}})\mathfrak{Re}(\Psi_n)|.$$

With Lemma 2.3.4 and $\|\Psi_n\|_{*,\sigma,d_n}\leqslant K(\sigma,\sigma')\,c_n^{1-\sigma'}$ (from (2.2.13)), we check that

$$|(1-\eta)V\partial_{z_1}L'(\Psi_n)|\leqslant\frac{K(\sigma,\sigma')(1-\eta)}{r_1^{1+\sigma}d_n^{4-\varepsilon-\sigma'}},$$

therefore, with the same condition as for the previous term, namely

$$\lambda<2-(2-\alpha)\varepsilon,$$

we infer, taking $\sigma<\sigma'$ small enough,

$$\left\|\frac{(1-\eta)V\partial_{z_1}L'(\Psi_n)}{V}\right\|_{**,B_{d_n}}=o_{c_n\to 0}(c_n^{1+\lambda}).$$

This concludes the proof of step 3.2.

Step 3.3. Proof of $\left\|\frac{\mathrm{NL}_{\partial_{z_1}}(\Psi_n)}{V}\right\|_{**,B_{d_n}}=o_{c_n\to 0}(c_n^{1+\lambda})$.

We recall

$$\mathrm{NL}_{\partial_{z_1}}(\Psi_n)=V(1-\eta)\partial_{z_1}(-\nabla\Psi_n.\nabla\Psi_n+|V|^2S(\Psi_n)),$$

with $S(\Psi_n)=e^{2\mathfrak{Re}(\Psi_n)}-1-2\mathfrak{Re}(\Psi_n)$. We compute

$$\begin{aligned}\partial_{z_1}(-\nabla\Psi_n.\nabla\Psi_n+|V|^2\,S(\Psi_n)) &= -2\nabla\partial_{z_1}\Psi_n.\nabla\Psi_n\\ &+ 4\mathfrak{Re}(\partial_{x_1}V_{-1}\overline{V_{-1}})S(\Psi_n)\\ &+ |V|^2\partial_{z_1}S(\Psi_n).\end{aligned}$$

Now, with Corollary 2.3.3 and (2.2.13), we check that, for any $0<\sigma<\sigma'<1$, $r_1\geqslant 2$,

$$|\nabla\partial_{z_1}\Psi_n.\nabla\Psi_n|\leqslant\frac{K(\sigma,\sigma')}{r_1^{2+2\sigma}d_n^{2-2\sigma'}},$$

$$|4\mathfrak{Re}(\partial_{x_1}V_{-1}\overline{V_{-1}})S(\Psi_n)+|V|^2\partial_{z_1}S(\Psi_n)|\leqslant\frac{K(\sigma,\sigma')}{r_1^{2+2\sigma}d_n^{2-2\sigma'}},$$

therefore, taking $\sigma<\sigma'<\frac{1-\lambda}{2}$, we check that

$$\|(1-\eta)(-2\nabla\partial_{z_1}\Psi_n.\nabla\Psi_n+4\mathfrak{Re}(\partial_{x_1}V_{-1}\overline{V_{-1}})S(\Psi_n)+|V|^2\partial_{z_1}S(\Psi_n))\|_{**,B_{d_n}}=o_{c_n\to 0}(c_n^{1+\lambda}).$$

The proof of step 3.3 is complete.

Step 3.4. Proof of $\left\|\frac{\mathrm{Err}_{\mathrm{loc}}+\mathrm{Err}_{\mathrm{far}}+\mathrm{Err}_{\mathrm{cut}}}{V}\right\|_{**,B_{d_n}}=o_{c_n\to 0}(c_n^{1+\lambda})$.

We recall

$$\mathrm{Err}_{\mathrm{cut}}=\partial_{z_1}\eta(L(\Phi_n)-VL'(\Psi_n)+ic\partial_{x_2}\Psi_n+\nabla\Psi_n.\nabla\Psi_n-|V|^2S(\Psi_n)),$$

$$\mathrm{Err}_{\mathrm{loc}}=\partial_{z_1}(R(\Psi_n))-\partial_d\lambda(c_n,d_n)Z_{d_n},$$

$$\mathrm{Err}_{\mathrm{far}}=\partial_{z_1}V(1-\eta)(-\nabla\Psi_n.\nabla\Psi_n+|V|^2S(\Psi_n)).$$

$\mathrm{Err}_{\mathrm{cut}}$ is compactly supported in $\{r_{-1}\leqslant 2\}$, therefore $\mathrm{Err}_{\mathrm{cut}}=0$ in B_{d_n}, hence

$$\left\|\frac{\mathrm{Err}_{\mathrm{cut}}}{V}\right\|_{**,B_{d_n}}=0.$$

Now, $\mathrm{Err}_{\mathrm{loc}}$ is supported in $\{r_1\leqslant 2\}$, and from Lemma 2.1.7, we know that $R(\Psi_n)$ is a sum of terms at least quadratic in Ψ_n or Φ_n localized in the area where $\eta\neq 0$. Therefore, from Corollary 2.3.3 and (2.2.14), we check that

$$|\partial_{z_1}(R(\Psi_n))|\leqslant\frac{K(\sigma)}{d_n^{2-2\sigma}},$$

and we have check in step 1 that $|\partial_d\lambda(c_n,d_n)|=O^{\sigma}_{c_n\to 0}(c_n^{2-\sigma})$. Thus, taking $\sigma<\frac{1-\lambda}{2}$,

$$\left\|\frac{\mathrm{Err}_{\mathrm{loc}}}{V}\right\|_{**,B_{d_n}}=o_{c_n\to 0}(c_n^{1+\lambda}).$$

From (2.2.13), we check that, for any $1>\sigma'>\sigma>0$,

$$|-\nabla\Psi_n.\nabla\Psi_n+|V|^2S(\Psi_n)|\leqslant\frac{K(\sigma,\sigma')}{(1+r_1)^{2+2\sigma}d_n^{2-2\sigma'}},$$

and from Lemma 2.3.4, we have

$$|\partial_{z_1}V|\leqslant\frac{K}{d_n^{2-\varepsilon}},$$

therefore, choosing $\sigma<\sigma'$ small enough, we have

$$\left\|\frac{\partial_{z_1}V}{V}(1-\eta)(-\nabla\Psi_n.\nabla\Psi_n+|V|^2S(\Psi_n))\right\|_{**,B_{d_n}}=o_{c_n\to 0}(c_n^{1+\lambda}).$$

This ends the proof of step 3.4 and hence of (2.3.4).

Step 4. Three additional estimates on h_n.

This step is devoted to the proof of the following three estimates:

$$\|Vh_n\|_{L^\infty(\{\tilde r\leqslant 3\})}+\|\tilde r^{1+\sigma}\mathfrak{Re}(h_n)\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\mathfrak{Im}(h_n)\|_{L^\infty(\{\tilde r\geqslant 2\})}\leqslant K(\sigma,\sigma')c_n^{1-\sigma'}. \tag{2.3.6}$$

In the right half-plane, we want to show that

$$|h_n| \leqslant \frac{K(\sigma)c_n^{1+\sigma}}{(1+r_1)}, \tag{2.3.7}$$

and, in the left half-plane,

$$|h_n| \leqslant \frac{K(\sigma)c_n^{1-\sigma}}{(1+r_{-1})^2}. \tag{2.3.8}$$

Observe that h_n is not symmetrical with respect to x_1 because of the cutoff. Recall that

$$Vh_n = \partial_{z_1}(\mathrm{TW}_{c_n}(V)) + \mathcal{L}_{\partial z_1}(\Psi_n) + \mathrm{NL}_{\partial_{z_1}}(\Psi_n) + \mathrm{Err}_{\mathrm{loc}} + \mathrm{Err}_{\mathrm{far}} + \mathrm{Err}_{\mathrm{cut}}.$$

We complete estimates done in the previous step to show that (2.3.6), (2.3.7) and (2.3.8) hold.

Step 4.1. Estimates for $\partial_{z_1}(\mathrm{TW}_{c_n}(V))$.

From Step 3.1, we have

$$\partial_{z_1}(\mathrm{TW}_c(V)) = -4\nabla V_1.\nabla\partial_{x_1}V_{-1} + 2(1-|V_1|^2)\,V_1\partial_{x_1}((1-|V_{-1}|^2)\,V_{-1}) - 2ic\partial_{x_2}(V_1\partial_{x_1}V_{-1}).$$

In view of Lemma 2.1.1, equation (2.1.3) and the estimate $(1+r_1)(1+r_{-1}) \geqslant d_n(1+\tilde{r})$, we have

$$\|\partial_{z_1}(\mathrm{TW}_c(V))/V\|_{**,\sigma,d_n} \leqslant K(\sigma)c_n^{1-\sigma}.$$

Furthermore, in the left half-plane, with Lemma 2.1.1 and equation (2.1.3), we check easily that

$$|\partial_{z_1}(\mathrm{TW}_c(V))| \leqslant \frac{Kc_n}{(1+r_1)^2}.$$

Furthermore, in the right half-plane, we have $\frac{1}{(1+r_{-1})} \leqslant Kc_n$, therefore, still using Lemma 2.1.1 and equation (2.1.3), we check that

$$|\partial_{z_1}(\mathrm{TW}_c(V))| \leqslant \frac{Kc_n^2}{(1+r_1)}.$$

Step 4.2. Estimates for $\mathcal{L}_{\partial_{z_1}}(\Psi_n)$.

We have, from Step 3.2, that

$$\mathcal{L}_{\partial_{z_1}}(\Psi_n) = \eta\partial_{z_1}L(\Phi_n) + (1-\eta)V\partial_{z_1}L'(\Psi_n) + \eta L(\partial_{z_1}V\Psi_n) + (1-\eta)\partial_{z_1}VL'(\Psi_n),$$

with

$$(\partial_{z_1}L)(\Phi_n) = 4\,\mathfrak{Re}(\overline{V_{-1}}\partial_{x_1}V_{-1})\Phi_n + 4\,\mathfrak{Re}(\overline{\partial_{x_1}V_{-1}\,V_1}\Phi_n)V + 4\,\mathfrak{Re}(\bar{V}\Phi_n)V_1\partial_{x_1}V_{-1},$$

$$L'(\Psi_n) = -\Delta\Psi_n - 2\frac{\nabla V}{V}.\nabla\Psi_n + 2|V|^2\mathfrak{Re}(\Psi_n) - ic_n\,\partial_{x_2}\Psi_n$$

and

$$|\partial_{z_1}L'(\Psi_n)| \leqslant K\left|\partial_{x_1}\frac{\nabla V_{-1}}{V_{-1}}.\nabla\Psi_n\right| + K|\mathfrak{Re}(\partial_{x_1}V_{-1}\overline{V_{-1}})\mathfrak{Re}(\Psi_n)|. \tag{2.3.9}$$

Similarly as in Step 4.1, every local term (in the area $\{\eta \neq 0\}$) satisfies the two estimates, using $\|\Psi_n\|_{*,\frac{1-\sigma}{2},d_n} \leqslant K(\sigma)c_n^{\sigma}$. The two nonlocal terms are $(1-\eta)V\partial_{z_1}L'(\Psi_n)$ and $(1-\eta)\partial_{z_1}VL'(\Psi_n)$. For the first term, in view of Lemma 2.1.1, equations (2.1.3), (2.2.14) and (2.3.9), we check that

$$\begin{aligned} & \|V\,(1-\eta)\partial_{z_1}L'(\Psi_n)\|_{L^\infty(\{\tilde{r}\leqslant 3\})} \\ + \;& \|\tilde{r}^{1+\sigma}\mathfrak{Re}((1-\eta)\partial_{z_1}L'(\Psi_n))\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\mathfrak{Im}((1-\eta)\partial_{z_1}L'(\Psi_n))\|_{L^\infty(\{\tilde{r}\geqslant 2\})} \\ \leqslant \;& K(\sigma,\sigma')c_n^{1-\sigma'} \end{aligned}$$

and, in the left-half plane,

$$|(1-\eta)V\partial_{z_1}L'(\Psi_n)| \leqslant \frac{K(\sigma)c_n^{1-\sigma}}{(1+r_{-1})^2}$$

Furthermore, using now $\|\Psi_n\|_{*,\frac{1-\sigma}{2},d_n} \leqslant K(\sigma)c_n^\sigma$, we check that, in the right half-plane,

$$|(1-\eta)V\partial_{z_1}L'(\Psi_n)| \leqslant \frac{K(\sigma)c_n^{1+\sigma}}{(1+r_1)}.$$

Finally, for the term $(1-\eta)\partial_{z_1}VL'(\Psi_n)$, we use $\|\Psi_n\|_{*,\sigma,d_n} \leqslant K(\sigma,\sigma')c_n^{1-\sigma'}$ and (2.2.14) to check that

$$|L'(\Psi_n)| \leqslant \frac{K(\sigma)c_n^{1-\sigma'}}{(1+\tilde{r})^{1+\sigma}}.$$

Combining this estimate with $|\partial_{z_1}V| \leqslant \frac{K}{(1+\tilde{r})}$, we show that

$$\left\| (1+\tilde{r})^{2+\sigma}\left((1-\eta)\frac{\partial_{z_1}V}{V}L'(\Psi_n)\right)\right\|_{L^\infty(\mathbb{R}^2)} \leqslant K(\sigma,\sigma')c_n^{1-\sigma'},$$

and, in the left half-plane,

$$|(1-\eta)\partial_{z_1}VL'(\Psi_n)| \leqslant \frac{K(\sigma)c_n^{1-\sigma}}{(1+r_{-1})^2}.$$

Furtherore, using $\|\Psi_n\|_{*,\frac{1-\sigma}{2},d_n} \leqslant K(\sigma)c_n^\sigma$ and (2.2.14), we also have the estimate

$$|L'(\Psi_n)| \leqslant \frac{K(\sigma)c_n^{\sigma}}{(1+\tilde{r})},$$

and using $|\partial_{z_1}V| \leqslant Kc_n$ in the right half-plane, we estimate in this same area that

$$|(1-\eta)\partial_{z_1}VL'(\Psi_n)| \leqslant \frac{K(\sigma)c_n^{1+\sigma}}{(1+\tilde{r})}.$$

Step 4.3. Estimates for $\mathrm{NL}_{\partial_{z_1}}(\Psi_n)$.

From Step 3.3,

$$\mathrm{NL}_{\partial_{z_1}}(\Psi_n) = V(1-\eta)\partial_{z_1}(-\nabla\Psi_n.\nabla\Psi_n + |V|^2S(\Psi_n)).$$

Using equation (2.2.14) for $\frac{1+\sigma}{2}$ and Corollary 2.3.3 (also for $\frac{1+\sigma}{2}$), we check without difficulties that

$$\begin{aligned} & \|\mathrm{NL}_{\partial_{z_1}}(\Psi_n)\|_{L^\infty(\{\tilde{r}\leqslant 3\})} \\ + \ & \|\tilde{r}^{1+\sigma}\mathfrak{Re}(\mathrm{NL}_{\partial_{z_1}}(\Psi_n)/V)\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\mathfrak{Im}(\mathrm{NL}_{\partial_{z_1}}(\Psi_n)/V)\|_{L^\infty(\{\tilde{r}\geqslant 2\})} \\ \leqslant \ & K(\sigma)c_n^{1-\sigma}, \end{aligned}$$

and, with, some margin, that in the left half-plane,

$$|\mathrm{NL}_{\partial_{z_1}}(\Psi_n)| \leqslant \frac{K(\sigma)c_n^{1-\sigma}}{(1+r_{-1})^2}.$$

Now, using $\|\Psi_n\|_{*,\frac{1-\sigma}{4},d_n} \leqslant K(\sigma)c_n^{\frac{1+\sigma}{2}}$ and Corollary 2.3.3 (for $\frac{1-\sigma}{2}$), we have, in the right half-plane,

$$|\mathrm{NL}_{\partial_{z_1}}(\Psi_n)| \leqslant \frac{K(\sigma)c_n^{1+\sigma}}{(1+\tilde{r})}.$$

Step 4.4. Estimates for $\mathrm{Err}_{\mathrm{loc}} + \mathrm{Err}_{\mathrm{far}} + \mathrm{Err}_{\mathrm{cut}}$.

For $\mathrm{Err}_{\mathrm{loc}} = \partial_{z_1}(R(\Psi_n)) - \partial_d\lambda(c_n, d_n)Z_{d_n}$, the same computations as in Step 4.3 yield the estimates (because this term is compactly supported in the area $\{\eta \neq 0\}$) needed for (2.3.6) to (2.3.8).

For $\mathrm{Err}_{\mathrm{cut}} = \partial_{z_1}\eta(L(\Phi_n) - VL'(\Psi_n) + ic\partial_{x_2}\Psi_n + \nabla\Psi_n.\nabla\Psi_n - |V|^2S(\Psi_n))$, this term is compactly supported near the vortex -1, hence is 0 in the right half-plane. Furthermore, using $\|\Psi_n\|_{*,\sigma/2,d_n} \leqslant K(\sigma)c_n^{1-\sigma}$, we check easily that

$$\|\mathrm{Err}_{\mathrm{cut}}/V\|_{**,\sigma,d_n} \leqslant K(\sigma)c_n^{1-\sigma}$$

and, since it is compactly supported, in the left half-plane,

$$|\mathrm{Err}_{\mathrm{cut}}| \leqslant \frac{K(\sigma)c_n^{1-\sigma}}{(1+r_{-1})^2}$$

Finally, for $\mathrm{Err}_{\mathrm{far}} = \partial_{z_1} V(1-\eta)(-\nabla\Psi_n.\nabla\Psi_n + |V|^2 S(\Psi_n))$, from (2.2.14) we have

$$|(1-\eta)(-\nabla\Psi_n.\nabla\Psi_n + |V|^2 S(\Psi_n))| \leqslant \frac{K(\sigma)c_n^{1-\sigma}}{(1+\tilde{r})^{2+\sigma}},$$

and we conclude as in Step 4.2.

This concludes the proof of estimates (2.3.6), (2.3.7) and (2.3.8).

Step 5. Inner estimates.

By the estimation we have just proved, we have in particular

$$\frac{h_n(.-d_n\vec{e_1})}{\|\partial_{z_1}\Psi_n\|_{*,B'_{d_n}}} \to 0$$

in $\mathcal{C}^0_{\mathrm{loc}}$ (which corresponds to follow the $+1$ vortex). Therefore, at the limit, in the distribution sense,

$$L_{V_1}(\mathfrak{G}) = 0$$

in all $\mathbb{R}^2$. If we show that $\langle \mathfrak{G}, \chi\partial_{x_1} V_1\rangle = 0$ for χ a cutoff near 0, we can then use Theorem 2.1.16 to show, similarly as in the proof of Proposition 2.1.17, that $\mathfrak{G} = 0$ since

$$\left\| \frac{(V\partial_{z_1}\Psi_n)(.-d_n\vec{e_1})}{\|\partial_{z_1}\Psi_n\|_{*,B'_{d_n}}} \right\|_{*,B_{d_n}} = 1,$$

hence $\|\mathfrak{G}\|_{H_{V_1}} < +\infty$. We recall that, by construction, we have $\langle \Phi_{c,d}, Z_d\rangle = 0$. By symmetry, this implies that $\langle \Phi_{c,d}, \eta(y)\partial_d V\rangle = 0$. Both $\Phi_{c,d}$ and $\eta(y)\partial_d V$ are C^1 with respect to d, and therefore

$$0 = \partial_d\langle \Phi_{c,d}, \eta(y)\partial_d V\rangle = \langle \partial_d\Phi_{c,d}, \eta(y)\partial_d V\rangle + \langle \Phi_{c,d}, \partial_d(\eta(y)\partial_d V)\rangle.$$

Furthermore, $\langle \partial_{x_1}\Phi_{c,d}, \eta(y)\partial_d V\rangle = -\langle \Phi_{c,d}, \partial_{x_1}\eta(y)\partial_d V\rangle$, thus

$$\langle \partial_{z_1}\Phi_{c,d}, \eta(y)\partial_d V\rangle = -\langle \Phi_{c,d}, \eta(y)\partial_{z_1}\partial_d V\rangle,$$

and we check easily that $|\eta(y)\partial_{z_1}\partial_d V| \leqslant Kc\eta(y)$, therefore, since $\|\Psi_{c,d}\|_{*,\sigma,d} \leqslant K(\sigma,\sigma')c^{1-\sigma'}$, we have $|\langle \partial_{z_1}\Phi_{c,d}, \eta(y)\partial_d V\rangle| \leqslant K(\sigma,\sigma')c^{2-\sigma'}$, and thus, taking $0 < \sigma' < 1-\lambda$, for c_n and d_n, $n \to \infty$, we infer that $\langle \mathfrak{G}, \eta\partial_{x_1} V_1\rangle = 0$.

We continue as in the proof of Proposition 2.1.17. The fact that $\mathfrak{G} = 0$ gives us that for any $R > 0$, we have

$$\frac{\|V\partial_{z_1}\Psi_n\|_{L^\infty(\{r_1\leqslant R\})} + \|\nabla(V\partial_{z_1}\Psi_n)\|_{L^\infty(\{r_1\leqslant R\})}}{\|\partial_{z_1}\Psi_n\|_{*,B'_{d_n}}} \to 0.$$

Step 6. Outer computations.

We have the same outer computations as in step 2 of the proof of Proposition 2.1.17, but with $\mathcal{Y}_n = \frac{\partial_{z_1}\Psi_n}{\|\partial_{z_1}\Psi_n\|_{*,B_{d_n}}}$ playing the role of Ψ_n and $\mathcal{H}_n = \frac{h_n}{\|\partial_{z_1}\Psi_n\|_{*,B_{d_n}}}$ playing the role of h_n, since they satisfy the same equation. We showed in (2.3.4) that

$$\|\mathcal{H}_n\|_{**,B_{d_n}} = o_{n\to\infty}(1),$$

and the system of equation is, with $\mathcal{Y}_n = \mathcal{Y}_1 + i\mathcal{Y}_2$ and $\mathcal{H}_n = \mathcal{H}_1 + i\mathcal{H}_2$,

$$\begin{cases} \Delta\mathcal{Y}_1 - 2|V|^2\mathcal{Y}_1 = -\mathcal{H}_1 - 2\mathfrak{Re}\Big(\frac{\nabla V}{V}.\nabla\mathcal{Y}_n\Big) + c\partial_{x_2}\mathcal{Y}_2 \\ \Delta\mathcal{Y}_2 + c\partial_{x_2}\mathcal{Y}_1 = -\mathcal{H}_2 - 2\mathfrak{Im}\Big(\frac{\nabla V}{V}.\nabla\mathcal{Y}_n\Big). \end{cases}$$

Recall the two balls $B_{d_n} = B(d_n\vec{e}_1,\, d_n^{\varepsilon})$ and $B'_{d_n} = B(d_n\vec{e}_1,\, d_n^{\varepsilon'})$. We have, as in the proof of Proposition 2.1.17, outside $\{r_1 \leqslant R\}$ but in B'_{d_n}, that $\|\mathcal{Y}_n\|_{*,B'_{d_n}} = 1$ and $\|\mathcal{H}_n\|_{**,B_{d_n}} = o_{n\to\infty}(1)$, therefore

$$|\Delta\mathcal{Y}_1 - 2\mathcal{Y}_1| \leqslant \frac{o_{R\to\infty}(1) + o^R_{n\to\infty}(1)}{(1+r_1)^{1-\alpha}} \tag{2.3.10}$$

and

$$|\Delta\mathcal{Y}_2 + c\partial_{x_2}\mathcal{Y}_1| \leqslant \frac{o_{R\to\infty}(1) + o^R_{n\to\infty}(1)}{(1+r_1)^{2-\alpha}}. \tag{2.3.11}$$

We want to extend these estimates in $B_{d_n} = B(d_n\vec{e}_1, d_n^{\varepsilon})$ and not only on $B'_{d_n} = (d_n\vec{e}_1, d_n^{\varepsilon'})$. Since $\|\mathcal{H}_n\|_{**,B_{d_n}} = o_{n\to\infty}(1)$ from (2.3.4), the estimates on $\mathcal{H}_1$ and $\mathcal{H}_2$ are already on B_{d_n}, leaving $c\partial_{x_2}\mathcal{Y}_2$ and the real and imaginary parts of $\frac{\nabla V}{V}.\nabla\mathcal{Y}_n$ to estimate.

First, we check that, in $B_{d_n}\backslash B'_{d_n}$,

$$|c_n\partial_{x_2}\mathcal{Y}_2| \leqslant \frac{d_n^{1+\lambda}c_n^{2-\sigma}}{(1+r_1)^{1+\sigma}} = \frac{o_{n\to\infty}(1)}{(1+r_1)^{1-\sigma}}$$

taking $\sigma > 0$ small enough. We use $\mathcal{Y}_n = \frac{\partial_{z_1}\Psi_n}{\|\partial_{z_1}\Psi_n\|_{*,B'_{d_n}}}$, $\frac{1}{\|\partial_{z_1}\Psi_n\|_{*,B'_{d_n}}} \leqslant K d_n^{1+\lambda}$ and Corollary 2.3.3 to compute, for any $1 > \sigma > 0$,

$$\left|\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\mathcal{Y}_n\right)\right| \leqslant \left|\frac{\nabla V}{V}\right| \times |\nabla\mathcal{Y}_n| \leqslant \frac{K(\sigma)d_n^{1+\lambda}}{r_1^{2+\sigma}d_n^{1-\sigma}} \leqslant \frac{K(\sigma)}{r_1^{2+\sigma}d_n^{-\sigma-\lambda}}.$$

In $B_{d_n}\backslash B'_{d_n}$, we have $r_1 \geqslant d_n^{\varepsilon'}$, therefore

$$\left|\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\mathcal{Y}_n\right)\right| \leqslant \frac{K(\sigma)}{r_1^{1-\alpha}d_n^{-\sigma-\lambda+(1+\alpha+\sigma)\varepsilon'}}.$$

Since we assume

$$\lambda < (1+\alpha)\varepsilon',$$

then we can choose $\sigma > 0$ small such that $-\sigma - \lambda + (1+\alpha+\sigma)\varepsilon' > 0$ and deduce, in $B_{d_n}\backslash B'_{d_n}$, that

$$\left|\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\mathcal{Y}_n\right)\right| \leqslant \frac{o_{n\to\infty}(1)}{r_1^{1-\alpha}}.$$

This result shows that (2.3.10) holds on B_{d_n}. Now, we compute

$$\left|\mathfrak{Im}\left(\frac{\nabla V}{V}.\nabla\mathcal{Y}_n\right)\right| \leqslant \left|\mathfrak{Re}\left(\frac{\nabla V}{V}\right).\mathfrak{Im}(\nabla\mathcal{Y}_n)\right| + \left|\mathfrak{Re}(\nabla\mathcal{Y}_n).\mathfrak{Im}\left(\frac{\nabla V}{V}\right)\right|,$$

and with Corollary 2.3.3, Lemma 2.1.2 and 2.3.4, we estimate

$$\left|\mathfrak{Re}\left(\frac{\nabla V}{V}\right).\mathfrak{Im}(\nabla\mathcal{Y}_n)\right| \leqslant K(\sigma)\left(\frac{1}{d_n^3} + \frac{1}{r_1^3}\right)\frac{d_n^{1+\lambda}}{r_1^{1+\sigma}d_n^{1-\sigma}}$$

and

$$\left|\mathfrak{Re}(\nabla\mathcal{Y}_n).\mathfrak{Im}\left(\frac{\nabla V}{V}\right)\right| \leqslant K(\sigma)\frac{d_n^{1+\lambda}}{r_1^{2+\sigma}d_n^{1-\sigma}}\left(\frac{1}{d_n^{2-\varepsilon}} + \frac{1}{r_1}\right).$$

In $B_{d_n}\backslash B'_{d_n}$, we have $d_n^{\varepsilon} \geqslant r_1 \geqslant d_n^{\varepsilon'}$, and with similar estimates as for the previous term, we check that, since $\lambda < (1+\alpha)\varepsilon'$, we have

$$\lambda < (2+\alpha)\varepsilon',$$

for the first term, and

$$\lambda < (1+\alpha)\varepsilon'$$

for the second one. We can find $\sigma > 0$ such that

$$\left|\mathfrak{Im}\left(\frac{\nabla V}{V}.\nabla\mathcal{Y}_n\right)\right| \leqslant \frac{o_{n\to\infty}(1)}{(1+r_1)^{2-\alpha}}$$

in $B_{d_n}\backslash B'_{d_n}$. We deduce that (2.3.11) holds on B_{d_n}. Additionally, we will use (from Lemma 2.2.3) for $0<\sigma<\sigma'<1$,

$$\begin{aligned} & \|V\chi\mathcal{Y}_n\|_{C^1(\{\tilde r\leqslant 3\})} \\ + \ & \|\tilde r^{1+\sigma}\mathfrak{Re}(\mathcal{Y}_n)\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{1+\sigma}\nabla\mathfrak{Re}(\mathcal{Y}_n)\|_{L^\infty(\{\tilde r\geqslant 2\})} \\ + \ & \|\tilde r^{\sigma}\,\mathfrak{Im}(\mathcal{Y}_n)\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{1+\sigma}\nabla\mathfrak{Im}(\mathcal{Y}_n)\|_{L^\infty(\{\tilde r\geqslant 2\})} \\ \leqslant \ & K(\sigma,\sigma')c_n^{1-\sigma'}d_n^{1+\lambda} \\ \leqslant \ & K(\sigma,\sigma')d_n^{\lambda+\sigma''} \end{aligned} \tag{2.3.12}$$

and from (2.3.6),

$$\|V\mathcal{H}_n\|_{L^\infty(\{\tilde r\leqslant 3\})}+\|\tilde r^{1+\sigma}\mathfrak{Re}(\mathcal{H}_n)\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\mathfrak{Im}(\mathcal{H}_n)\|_{L^\infty(\{\tilde r\geqslant 2\})}\leqslant K(\sigma,\sigma')d_n^{\lambda+\sigma''} \tag{2.3.13}$$

to do estimates outside of B_{d_n}. These estimates are not optimal (in particular in the smallness in c_n) but we will only use them on parts far away from the center of V_1. Thanks to (2.3.7), we have a slightly better estimate in the right half-plane, that is, for $0<\sigma<1$,

$$|\mathcal{H}_n|\leqslant K|h_n|d_n^{1+\lambda}\leqslant\frac{K(\sigma)d_n^{\lambda-\sigma}}{(1+r_1)}. \tag{2.3.14}$$

Step 7. Elliptic estimates.

We follow the proof of Proposition 2.1.17. At this point, we have on $\mathcal{Y}_n$ that $\|\mathcal{Y}_n\|_{*,B'_{d_n}}=1$, $\|V\mathcal{Y}_n\|_{L^\infty(\{r_1\leqslant R\})}+\|\nabla(V\mathcal{Y}_n)\|_{L^\infty(\{r_1\leqslant R\})}\to 0$ as $n\to\infty$ for any $R>1$, and with $\mathcal{Y}_n=\mathcal{Y}_1+i\mathcal{Y}_2$,

$$|\Delta\mathcal{Y}_2+c\partial_{x_2}\mathcal{Y}_1|\leqslant\frac{o_{R\to\infty}(1)+o^R_{n\to\infty}(1)}{(1+r_1)^{2-\alpha}},$$

$$|\Delta\mathcal{Y}_1-2|V|^2\mathcal{Y}_1|\leqslant\frac{o_{R\to\infty}(1)+o^R_{n\to\infty}(1)}{(1+r_1)^{1-\alpha}}.$$

We want to show that $\|\mathcal{Y}_n\|_{*,B'_{d_n}}=o_{R\to\infty}(1)+o^R_{n\to\infty}(1)$. We want to use similar elliptic estimates as in the proof of Proposition 2.1.17, but we have to show that they still work if we only have the estimate in $B_{d_n}=B(d_n\vec{e}_1,d_n^{\varepsilon})$ and we want the final estimates in $B'_{d_n}=B(d_n\vec{e}_1,d_n^{\varepsilon'})$, with $\varepsilon'<\varepsilon$.

Step 7.1. Elliptic estimate for $\mathcal{Y}_2$.

We start by solving the following problem in $\mathbb{R}^2$:

$$\Delta\zeta=f,$$

with

$$f:=-\mathcal{H}_2-2\,\mathfrak{Im}\Big(\frac{\nabla V}{V}.\nabla\mathcal{Y}_n\Big),$$

which is odd in x_2 (the derivation with respect to z_1 breaks the symmetry on x_1, but not on x_2) and satisfies

$$|f|\leqslant\frac{o_{R\to\infty}(1)+o^R_{n\to\infty}(1)}{(1+r_1)^{2-\alpha}}$$

in $B_{d_n}=B(d_n\vec{e}_1,d_n^{\varepsilon})$, and, from (2.3.11) and (2.3.13),

$$|f|\leqslant\frac{K(\sigma,\sigma')d_n^{\lambda+\sigma'}}{(1+\tilde r)^{2+\sigma}} \tag{2.3.15}$$

in $\mathbb{R}^2$, for any $1>\sigma'>\sigma>0$. Similarly as in the proof of Lemma 2.1.8, we write, for $x\in B(d_n\vec{e}_1,d_n^{\varepsilon'})$,

$$\nabla\zeta(x) \ = \ \frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{x-Y}{|x-Y|^2}f(Y)dY. \tag{2.3.16}$$

Furthermore, we have the same local result, that is, if $|x - d\vec{e}_1| \leqslant 1$,

$$|\nabla \zeta(x)| \leqslant o_{R\to\infty}(1) + o^R_{n\to\infty}(1).$$

By symmetry (see in particular Lemma 2.2.4), we have

$$\int_{B(d_n\vec{e}_1, 2|x-d_n\vec{e}_1|)} f(Y)dY = 0,$$

hence

$$\begin{aligned}
&\int_{B(d_n\vec{e}_1,d_n^\varepsilon)} \frac{x-Y}{|x-Y|^2} f(Y)dY \\
&= \int_{B(d_n\vec{e}_1,d_n^\varepsilon)} f(Y)\left(\frac{x-Y}{|x-Y|^2} - \mathbf{1}_{\{|Y-d_n\vec{e}_1|\leqslant 2|x-d_n\vec{e}_1|\}}\frac{x-d_n\vec{e}_1}{|x-d_n\vec{e}_1|^2}\right)dY,
\end{aligned}$$

and then, we infer

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_{B(d_n\vec{e}_1,d_n^\varepsilon)} f(Y)\left(\frac{x-Y}{|x-Y|^2} - \mathbf{1}_{\{|Y-d_n\vec{e}_1|\leqslant 2|x-d_n\vec{e}_1|\}}\frac{x-d_n\vec{e}_1}{|x-d_n\vec{e}_1|^2}\right)dY\right| \\
\leqslant\ & \int_{B(d_n\vec{e}_1,d_n^\varepsilon)} \frac{(o_{R\to\infty}(1)+o^R_{n\to\infty}(1))}{(1+|Y|)^{2-\alpha}}\left|\frac{x-Y}{|x-Y|^2} - \mathbf{1}_{\{|Y-d_n\vec{e}_1|\leqslant 2|x-d_n\vec{e}_1|\}}\frac{x-d_n\vec{e}_1}{|x-d_n\vec{e}_1|^2}\right|dY.
\end{aligned}$$

We do the same change of variable $Z = Y - d_n\vec{e}_1$ as in the proof of lemma 2.1.8, and we are now at

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_{B(d_n\vec{e}_1,d_n^\varepsilon)} f(Y)\left(\frac{x-Y}{|x-Y|^2} - \mathbf{1}_{\{|Y-d_n\vec{e}_1|\leqslant 2|x-d_n\vec{e}_1|\}}\frac{x-d_n\vec{e}_1}{|x-d_n\vec{e}_1|^2}\right)dY\right| \\
\leqslant\ & \int_{B(0,d_n^\varepsilon)} \frac{(o_{R\to\infty}(1)+o^R_{n\to\infty}(1))}{(1+|Z|)^{2-\alpha}}\left|\frac{x-d_n\vec{e}_1-Z}{|x-d_n\vec{e}_1-Z|^2} - \mathbf{1}_{\{|Z|\leqslant 2|x-d_n\vec{e}_1|\}}\frac{x-d_n\vec{e}_1}{|x-d_n\vec{e}_1|^2}\right|dZ.
\end{aligned}$$

We want to follow the same computations as in the proof of Lemma 2.1.8, but now $\frac{1}{(1+|Z|)^{2-\alpha}}$ is no longer integrable, and this is why we added the function $\mathbf{1}_{\{|Z|\leqslant 2|x-d_n\vec{e}_1|\}}$. If $|Z| \geqslant 2|x-d_n\vec{e}_1|$, then $|x-d_n\vec{e}_1-Z| \geqslant |Z|/2$ and

$$\begin{aligned}
&\int_{B(0,d_n^\varepsilon)\cap\{|Z|\geqslant 2|x-d_n\vec{e}_1|\}} \frac{(o_{R\to\infty}(1)+o^R_{n\to\infty}(1))}{(1+|Z|)^{2-\alpha}}\left|\frac{x-d_n\vec{e}_1-Z}{|x-d_n\vec{e}_1-Z|^2}\right|dZ \\
\leqslant\ & \int_{B(0,d_n^\varepsilon)\cap\{|Z|\geqslant 2|x-d_n\vec{e}_1|\}} \frac{(o_{R\to\infty}(1)+o^R_{n\to\infty}(1))}{(1+|Z|)^{2-\alpha}|Z|}dZ \\
\leqslant\ & \frac{o_{R\to\infty}(1)+o^R_{n\to\infty}(1)}{(1+|x-d_n\vec{e}_1|)^{1-\alpha}}.
\end{aligned}$$

Then, in $\{|Z| \leqslant 2|x-d_n\vec{e}_1|\}$, we follow exactly the same computation as in the proof of the proof of Lemma 2.1.8 for the remaining part of the integral, and we conclude that

$$\begin{aligned}
&\left|\frac{1}{2\pi}\int_{B(d_n\vec{e}_1,d_n^\varepsilon)} f(Y)\left(\frac{x-Y}{|x-Y|^2} - \mathbf{1}_{\{|Y-d_n\vec{e}_1|\leqslant 2|x-d_n\vec{e}_1|\}}\frac{x-d_n\vec{e}_1}{|x-d_n\vec{e}_1|^2}\right)dY\right| \\
\leqslant\ & \frac{o_{R\to\infty}(1)+o^R_{n\to\infty}(1)}{(1+|x-d_n\vec{e}_1|)^{1-\alpha}}.
\end{aligned}$$

We are left with the estimation of (after a translation)

$$\int_{\mathbb{R}^2\setminus B(0,d_n^\varepsilon)} \frac{|f(Z+d_n\vec{e}_1)|}{|Z-(x-d_n\vec{e}_1)|}dZ.$$

By symmetry (see Lemma 2.2.4), we have

$$\int_{\mathbb{R}^2\setminus B(0,d_n^\varepsilon)} \frac{f(Z+d_n\vec{e}_1)}{|Z|}dZ = 0,$$

therefore

$$\left|\int_{\mathbb{R}^2\setminus B(0,d_n^\varepsilon)} \frac{f(Z+d_n\vec{e}_1)}{|Z-(x-d_n\vec{e}_1)|}dZ\right| = \left|\int_{\mathbb{R}^2\setminus B(0,d_n^\varepsilon)} f(Z+d_n\vec{e}_1)\left(\frac{1}{|Z-(x-d_n\vec{e}_1)|} - \frac{1}{|Z|}\right)dZ\right|.$$

Since $|x-d_n\vec{e}_1|\leqslant d_n^{\varepsilon'}\ll d_n^{\varepsilon}\leqslant|Z|$, we have, for $Z\in\mathbb{R}^2\backslash B(0,d_n^{\varepsilon})$,

$$\left|\frac{1}{|Z-(x-d_n\vec{e}_1)|}-\frac{1}{|Z|}\right|\leqslant\frac{K|x-d_n\vec{e}_1|}{|Z|^2}\leqslant\frac{Kd_n^{\varepsilon'}}{d_n^{2\varepsilon}},$$

thus, with (2.3.15),

$$\begin{aligned}&\left|\int_{\mathbb{R}^2\backslash B(0,d_n^{\varepsilon})}f(Z+d_n\vec{e}_1)\left(\frac{1}{|Z-(x-d_n\vec{e}_1)|}-\frac{1}{|Z|}\right)dZ\right|\\ \leqslant\;&\frac{K(\sigma,\sigma')d_n^{\varepsilon'+\lambda+\sigma'}}{d_n^{2\varepsilon}}\left|\int_{\mathbb{R}^2\backslash B(d_n\vec{e}_1,d_n^{\varepsilon})}\frac{1}{(1+\tilde{r})^{2+\sigma}}\right|\\ \leqslant\;&K(\sigma,\sigma')d_n^{\varepsilon'+\lambda-2\varepsilon+\sigma'}.\end{aligned}$$

In particular, we have

$$\left|\int_{\mathbb{R}^2\backslash B(0,d_n^{\varepsilon})}f(Y+d_n\vec{e}_1)\left(\frac{1}{|Y-(x-d_n\vec{e}_1)|}-\frac{1}{|Y|}\right)dY\right|\leqslant\frac{o_{n\to\infty}(1)}{(1+|x-d_n\vec{e}_1|)^{1-\alpha}}$$

if, since $|x-d_n\vec{e}_1|\leqslant d_n^{\varepsilon'}$,

$$K(\sigma,\sigma')d_n^{\varepsilon'+\lambda-2\varepsilon+\sigma'}\leqslant\frac{o_{n\to\infty}(1)}{d_n^{\varepsilon'(1-\alpha)}},$$

hence, since we make the assumption

$$\lambda+\varepsilon'(1-\alpha)<2\varepsilon-\varepsilon',$$

we can find $\sigma'>0$ such that, for $x\in B(d_n\vec{e}_1,d_n^{\varepsilon})$,

$$|\nabla\zeta(x)|\leqslant\frac{o_{R\to\infty}(1)+o^R_{n\to\infty}(1)}{(1+|x-d_n\vec{e}_1|)^{1-\alpha}}.\tag{2.3.17}$$

Using Lemma 2.1.8 and (2.3.15), we also have, in all $\mathbb{R}^2$ this time, that

$$|\nabla\zeta(x)|\leqslant\frac{K(\sigma,\sigma')d_n^{\lambda+\sigma'}}{(1+\tilde{r})^{1+\sigma}}.\tag{2.3.18}$$

Here, we cannot integrate from infinity (since the estimate is only on a ball) to get an estimation on ζ, but this will be dealt with later on.

Now, following the proof of Proposition 2.1.17, we define $\mathcal{Y}_2':=\mathcal{Y}_2-\zeta$, and we have, for $j\in\{1,2\}$,

$$\partial_{x_j}\mathcal{Y}_2'=K_j*f',$$

where

$$f':=-\mathcal{H}_1-2\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\mathcal{Y}_n\right)-(1-|V|^2)\mathcal{Y}_1-c\partial_{x_2}\zeta.$$

We first estimate the convolution in $B(d_n\vec{e}_1,d_n^{\varepsilon})$. With $\|\mathcal{Y}_n\|_{*,B'_{d_n}}=1$, we check that, with some margin in $B(d_n\vec{e}_1,d_n^{\varepsilon'})$,

$$\left|2\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\mathcal{Y}_n\right)-(1-|V|^2)\mathcal{Y}_1\right|\leqslant\frac{o_{R\to\infty}(1)}{(1+r_1)^{3/2-\alpha}}.$$

Now, we have shown in step 6 that

$$\left|\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\mathcal{Y}_n\right)\right|\leqslant\frac{o_{n\to\infty}(1)}{r_1^{1-\alpha+\sigma''}}$$

for some $\sigma''>0$. In $B(d_n\vec{e}_1,d_n^{\varepsilon})\backslash B(d_n\vec{e}_1,d_n^{\varepsilon'})$, we have

$$|(1-|V|^2)\mathcal{Y}_1|\leqslant\frac{d_n^{\lambda+\sigma'}}{r_1^{3+\sigma}}\leqslant\frac{d_n^{\lambda+\sigma'-(2+\alpha-\sigma'')\varepsilon'}}{r_1^{1-\alpha+\sigma''}}=\frac{o_{n\to\infty}(1)}{r_1^{1-\alpha+\sigma''}}$$

given that σ' and σ'' are small enough since $\lambda-(2+\alpha)\varepsilon'<0$. Therefore, following the proof of Lemma 2.1.13 (only changing the integral from $\mathbb{R}^2$ to $B(d_n\vec{e}_1, d_n^{\varepsilon})$), we check with the same computations (since we have some margin $\sigma''>0$ on the decay) that

$$\left|\int_{B(d_n\vec{e}_1,d_n^{\varepsilon})} K_j(x-Y)\left(2\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\mathcal{Y}_n\right)-(1-|V|^2)\mathcal{Y}_1\right)(Y)dY\right|\leqslant\frac{o_{R\to\infty}(1)}{(1+r_1)^{1-\alpha}}.$$

Now, using (2.3.17), we check that, following the proof of Lemma 2.1.13 (using Hölder inequality instead of Cauchy-Schwarz in the last estimate to make sur that the two integrals are well defined, this does not change the final estimate),

$$\left|\int_{B(d_n\vec{e}_1,d_n^{\varepsilon})} K_j(x-Y)(c\partial_{x_2}\zeta)(Y)dY\right|\leqslant\frac{c(o_{R\to\infty}(1)+o^R_{n\to\infty}(1))}{(1+r_1)^{1-\alpha-1/10}}.$$

And, since $x\in B(d_n\vec{e}_1,d_n^{\varepsilon'})$, $c(1+r_1)^{1/10}\leqslant K$, therefore

$$\left|\int_{B(d_n\vec{e}_1,d_n^{\varepsilon})} K_j(x-Y)(c\partial_{x_2}\zeta)(Y)dY\right|\leqslant\frac{o_{R\to\infty}(1)+o^R_{n\to\infty}(1)}{(1+r_1)^{1-\alpha}}.$$

For the last remaining term, we use (2.3.7) with $\sigma=\frac{\lambda+1}{2}$ to estimate

$$|\mathcal{H}_1|\leqslant\frac{o^R_{n\to 0}(1)}{(1+r_1)},$$

and then, from Lemma 2.1.13 (only changing the integral from $\mathbb{R}^2$ to $B(d_n\vec{e}_1,d_n^{\varepsilon})$ in the proof), we infer

$$\left|\int_{B(d_n\vec{e}_1,d_n^{\varepsilon})} K_j(x-Y)\mathcal{H}_1(Y)dY\right|\leqslant\frac{o_{R\to\infty}(1)+o^R_{n\to\infty}(1)}{(1+r_1)^{1-\alpha}}.$$

Combining these estimates, we have shown that

$$\left|\int_{B(d_n\vec{e}_1,d_n^{\varepsilon})} K_j(x-Y)f'(Y)dY\right|\leqslant\frac{o_{R\to\infty}(1)+o^R_{n\to\infty}(1)}{(1+r_1)^{1-\alpha}}.$$

Now, we focus on the left half-plane. From (2.3.8), we have

$$|\mathcal{H}_1|\leqslant\frac{K(\sigma)c_n^{1-\sigma}d_n^{1+\lambda}}{(1+r_{-1})^2}.$$

Furthermore, we check, using (2.3.11) and (2.3.18) that, in the left half-plane,

$$\left|-2\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\mathcal{Y}_n\right)-(1-|V|^2)\mathcal{Y}_1\right|\leqslant\frac{K(\sigma,\sigma')d_n^{1+\lambda}c_n^{1-\sigma'}}{(1+r_{-1})^{2+\sigma}}$$

and

$$|c_n\partial_{x_2}\zeta|\leqslant\frac{K(\sigma,\sigma')d_n^{\lambda+\sigma'}c_n}{(1+r_{-1})^{1+\sigma}}.$$

We have by Theorem 2.1.12 (since $x\in B(d_n\vec{e}_1,d_n^{\varepsilon'})$) that $|K_j(x-Y)|\leqslant\frac{K}{d_n^{\beta}(1+\tilde{r}(Y))^{2-\beta}}$ for Y in the left half-plane, for any $0\leqslant\beta\leqslant 2$. Therefore, taking $\beta=2-\sigma$, we have

$$\left|\int_{\{y_1\leqslant 0\}} K_j(x-Y)\mathcal{H}_1(Y)dY\right|\leqslant\int_{\mathbb{R}^2}\frac{K(\sigma,\sigma')d_n^{\lambda+\sigma+\sigma'-2}}{(1+\tilde{r})^{2+\sigma}}\leqslant\frac{K(\sigma,\sigma')d_n^{\lambda+\sigma+\sigma'-2+(1-\alpha)\varepsilon'}}{(1+|x-d_n\vec{e}_1|)^{1-\alpha}}.$$

Taking $\beta=2$, we have

$$\begin{aligned}&\left|\int_{\{Y_1\leqslant 0\}} K_j(x-Y)\left(-2\mathfrak{Re}\left(\frac{\nabla V}{V}.\nabla\mathcal{Y}_n\right)-(1-|V|^2)\mathcal{Y}_1\right)(Y)dY\right|\\ \leqslant\ &\int_{\mathbb{R}^2}\frac{K(\sigma,\sigma')d_n^{\lambda+\sigma'-2}}{(1+\tilde{r})^{2+\sigma}}\\ \leqslant\ &\frac{K(\sigma,\sigma')d_n^{\lambda+\sigma'-2+(1-\alpha)\varepsilon'}}{(1+|x-d_n\vec{e}_1|)^{1-\alpha}},\end{aligned}$$

and finally, taking $\beta=1$, we estimate

$$\left|\int_{\{Y_1\leqslant 0\}} K_j(x-Y)c_n\,\partial_{x_2}\zeta(Y)dY\right| \leqslant \int_{\mathbb{R}^2}\frac{K(\sigma,\sigma')d_n^{\lambda+\sigma'-2}}{(1+\tilde r)^{2+\sigma}} \leqslant \frac{K(\sigma,\sigma')d_n^{\lambda+\sigma'-2+(1-\alpha)\varepsilon'}}{(1+|x-d_n\vec e_1|)^{1-\alpha}}.$$

Thus, taking $\sigma'>\sigma>0$ small enough, since $\lambda-2+(1-\alpha)\varepsilon'<0$, we have

$$\left|\int_{\{Y_1\leqslant 0\}} K_j(x-Y)f'(Y)dY\right| \leqslant \frac{o_{n\to 0}(1)}{(1+|x-d_n\vec e_1|)^{1-\alpha}}.$$

We are left with the estimation in $\Omega:=\{Y_1\geqslant 0\}\setminus B(d_n\vec e_1,d_n^{\varepsilon})$. We infer that, in Ω, we have, for $0<\sigma<\sigma'<1$

$$|f'|\leqslant \frac{K(\sigma')d_n^{\lambda-\sigma'}}{(1+r_1)}+\frac{K(\sigma)d_n^{\lambda+\sigma}}{(1+r_1)^{2+\sigma}}.$$

Indeed, from equation (2.3.14) and (2.3.18), we have $|\mathcal{H}_1-c\partial_{x_2}\zeta|\leqslant\frac{K(\sigma)d_n^{\lambda+\sigma}}{(1+r_1)}$, and using (2.3.11), we check that

$$\left|2\mathfrak{Re}\Big(\frac{\nabla V}{V}.\nabla\mathcal{Y}_n\Big)-(1-|V|^2)\mathcal{Y}_1\right|\leqslant\frac{K(\sigma)d_n^{\lambda+\sigma}}{(1+\tilde r)^{2+\sigma}}.$$

Now, for $y\in\Omega$, $x\in B(d_n\vec e_1,d_n^{\varepsilon'})$, we have from Theorem 2.1.12 that

$$|K_j(x-Y)|\leqslant\frac{K}{d_n^{2\varepsilon}}$$

and

$$|K_j(x-Y)|\leqslant\frac{K}{(1+\tilde r(Y))^{3/2}d_n^{\varepsilon/2}}.$$

We deduce that, for $x\in B(d_n\vec e_1,d_n^{\varepsilon'})$,

$$\begin{aligned}\int_\Omega|K_j(x-Y)|\frac{K(\sigma')d_n^{\lambda-\sigma'}}{(1+r_1(Y))}dY &\leqslant K(\sigma')d_n^{\lambda-\sigma'-\varepsilon/2}\int_{\mathbb{R}^2}\frac{K}{(1+\tilde r(Y))^{5/2}}dY\\ &\leqslant \frac{K(\sigma')d_n^{\lambda-\sigma'+(1-\alpha)\varepsilon'-\varepsilon/2}}{(1+|x-d_n\vec e_1|)^{1-\alpha}}=\frac{o_{n\to 0}(1)}{(1+|x-d_n\vec e_1|)^{1-\alpha}}\end{aligned}$$

taking $\sigma'<1$ large enough (since $\lambda+(1-\alpha)\varepsilon'-1-\varepsilon/2<0$), and

$$\begin{aligned}\int_\Omega|K_j(x-Y)|\frac{K(\sigma)d_n^{\lambda+\sigma}}{(1+\tilde r(Y))^{2+\sigma}}dY &\leqslant K(\sigma)d_n^{\lambda+\sigma-2\varepsilon}\int_{\mathbb{R}^2}\frac{1}{(1+\tilde r(Y))^{2+\sigma}}dY\\ &\leqslant \frac{K(\sigma)d_n^{\lambda+\sigma+(1-\alpha)\varepsilon'-2\varepsilon}}{(1+|x-d_n\vec e_1|)^{1-\alpha}}=\frac{o_{n\to 0}(1)}{(1+|x-d_n\vec e_1|)^{1-\alpha}}\end{aligned}$$

taking $\sigma>0$ small enough (since $\lambda+(1-\alpha)\varepsilon'-2\varepsilon<0$). We deduce that, for $x\in B(d_n\vec e_1,d_n^{\varepsilon'})$,

$$|\partial_{x_j}\mathcal{Y}_2'|=|K_j*f'|\leqslant\frac{o_{n\to 0}(1)+o_{R\to\infty}(1)}{(1+|x-d_n\vec e_1|)^{1-\alpha}}.$$

With (2.3.17), we have shown that

$$|\partial_{x_j}\mathcal{Y}_2|\leqslant\frac{o_{n\to 0}(1)+o_{R\to\infty}(1)}{(1+|x-d_n\vec e_1|)^{1-\alpha}}.$$

Now, since $|\mathcal{Y}_2|+|\nabla\mathcal{Y}_2|=o_{R\to\infty}(1)$ in $B(d_n\vec e_1,10)$, by integration from $d_n\vec e_1$, we check that, since $\alpha>0$,

$$|\mathcal{Y}_2|\leqslant\frac{o_{n\to 0}(1)+o_{R\to\infty}(1)}{(1+|x-d_n\vec e_1|)^{-\alpha}}.$$

Step 7.2. Elliptic estimate for $\mathcal{Y}_1$.

For $\mathcal{Y}_1$ we also use the function K_0 and we have

$$\mathcal{Y}_1=\frac{1}{2\pi}K_0\big(\sqrt{2}|.|\big)*(-\Delta\mathcal{Y}_1+2\mathcal{Y}_1),$$

therefore

$$\begin{aligned}|\mathcal{Y}_1|(x) \;\leqslant\; & \int_{\tilde{B}_{d_n}(x)} \frac{1}{2\pi} K_0\big(\sqrt{2}|x-Y|\big)|(-\Delta\mathcal{Y}_1+2\mathcal{Y}_1)(Y)|dY \\ + \; & \int_{\mathbb{R}\setminus\tilde{B}_{d_n}(x)} \frac{1}{2\pi} K_0\big(\sqrt{2}|x-Y|\big)|(-\Delta\mathcal{Y}_1+2\mathcal{Y}_1)(Y)|dY,\end{aligned}$$

where $B_{d_n}(x) = B(x - d_n\vec{e}_1, d_n^{\varepsilon})$. The first term can be computed as in the proof of Lemma 2.1.10, and for the second term, in $\mathbb{R}\setminus B_{d_n}$, we have

$$K_0\big(\sqrt{2}|x|\big) \leqslant K e^{-d_n^{\varepsilon/2}} e^{-|x|^{1/2}}$$

from Lemma 2.1.9, which, with (2.3.11) and (2.3.13), make the term integrable and a $o_{d_n\to\infty}\big(e^{-d_n^{\varepsilon/4}}\big)$, which is enough to show that

$$|\nabla\mathcal{Y}_1| + |\mathcal{Y}_1| \leqslant \frac{o_{n\to\infty}(1) + o_{R\to\infty}(1)}{(1+r_1)^{1-\alpha}}.$$

Step 8. Conclusion.

We conclude that there is a contradiction, as in the end of the proof of Proposition 2.1.17. This ends the proof of Proposition 2.3.5. □

In the rest of this chapter, we take $\alpha, \varepsilon, \varepsilon', \lambda$ such that they satisfy the conditions of Proposition 2.3.5, and

$$\lambda + (1-\alpha)\varepsilon' > 1.$$

2.3.4 Proof of $\int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\partial_{z_1}(\mathrm{TW}_c(V)))_{|d=d_c} = \frac{-\pi}{d_c^2} + o_{d_c\to\infty}\Big(\frac{1}{d_c^2}\Big)$

From (2.1.2), the equation on V is

$$\mathrm{TW}_c(V) = E - ic\partial_{x_2}V = -2\nabla V_1.\nabla V_{-1} + (1-|V_1|^2)(1-|V_{-1}|^2)\,V_1 V_{-1} - ic\partial_{x_2}(V_1 V_{-1}).$$

We use Lemma 2.3.2 to compute

$$\partial_{z_1}V = \partial_{x_1}V_1V_{-1} + \partial_{x_1}V_{-1}V_1 - (-\partial_{x_1}V_1V_{-1} + \partial_{x_1}V_{-1}V_1) = 2\partial_{x_1}V_1V_{-1}.$$

Therefore

$$\partial_{z_1}\mathrm{TW}_c(V) = -4\nabla V_1.\nabla\partial_{x_1}V_{-1} + 2(1-|V_1|^2)V_1\partial_{x_1}((1-|V_{-1}|^2)V_{-1}) - 2\,ic\partial_{x_2}(V_1\partial_{x_1}V_{-1}),$$

and then

$$\begin{aligned}\int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\partial_{z_1}(\mathrm{TW}_c(V))) \;=\; & -4\int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\nabla V_1.\nabla\partial_{x_1}V_{-1}) \\ & + 2\int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}(1-|V_1|^2)\,V_1\partial_{x_1}((1-|V_{-1}|^2)\,V_{-1})) \\ & - 2\int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V} i\,c\partial_{x_2}(V_1\partial_{x_1}V_{-1})).\end{aligned}$$

We want to compute this quantity at $d = d_c$. We omit the subscript and use only d in this proof. In fact, it works for any d such that $\frac{1}{2d} \leqslant c \leqslant \frac{2}{d}$.

Step 1. Proof of $\int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}(1-|V_1|^2)\,V_1\partial_{x_1}((1-|V_{-1}|^2)\,V_{-1})) = o_{d\to\infty}\big(\frac{1}{d^2}\big)$.

First remark that $\partial_{x_1}((1-|V_{-1}|^2)V_{-1}) = O_{d\to\infty}\big(\frac{1}{d^3}\big)$ in $B(d\vec{e_1}, d^{\varepsilon'})$ by Lemma 2.3.4 and

$$(1-|V_1|^2)V_1\overline{\partial_d V} = O_{r_1\to\infty}\Big(\frac{1}{r_1^3}\Big)$$

therefore

$$\int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}(1-|V_1|^2)\,V_1\partial_{x_1}((1-|V_{-1}|^2)V_{-1})) = o_{d\to\infty}\Big(\frac{1}{d^2}\Big).$$

Step 2. Proof of $\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_d V i}\, c\partial_{x_2}(V_1\partial_{x_1}V_{-1}))=o_{d\to\infty}\big(\frac{1}{d^2}\big)$.

Now we compute

$$ic\partial_{x_2}(V_1\partial_{x_1}V_{-1})=ic\partial_{x_2}V_1\partial_{x_1}V_{-1}+ic\partial_{x_1x_2}V_{-1}V_1,$$

hence

$$\begin{aligned}\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_d V i}\, c\partial_{x_2}(V_1\partial_{x_1}V_{-1})) &= -c\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_{x_1}V_1\,V_{-1}}i\partial_{x_2}V_1\partial_{x_1}V_{-1})\\ &- c\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_{x_1}V_1\,V_{-1}}i\partial_{x_1x_2}V_{-1}V_1)\\ &+ c\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_{x_1}V_{-1}\,V_1}i\partial_{x_2}V_1\partial_{x_1}V_{-1})\\ &+ c\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_{x_1}V_{-1}\,V_1}i\partial_{x_1x_2}V_{-1}V_1).\end{aligned}\tag{2.3.19}$$

Now, using Lemma 2.3.4, we estimate the first term of (2.3.19),

$$c\left|\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_{x_1}V_1\,V_{-1}}i\partial_{x_2}V_1\partial_{x_1}V_{-1})\right|\leqslant$$

$$c\int_{B(d\vec{e}_1,d^{\varepsilon'})}|\overline{\partial_{x_1}V_1}\partial_{x_2}V_1|\times|\overline{V_{-1}}\partial_{x_1}V_{-1}|\leqslant K\int_{B(d\vec{e}_1,d^{\varepsilon'})}\frac{1}{(1+r_1^2)}\,\frac{1}{d^{3-\varepsilon'}}\leqslant\frac{K\ln(d^{\varepsilon'})}{d^{3-\varepsilon'}}.$$

Since $\varepsilon'>0$, we have

$$c\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_{x_1}V_1\,V_{-1}}i\partial_{x_2}V_1\partial_{x_1}V_{-1})=o_{d\to\infty}\left(\frac{1}{d^2}\right).\tag{2.3.20}$$

Using Lemma 2.3.4, for the second term of (2.3.19), we have

$$\begin{aligned}\left|c\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_{x_1}V_1\,V_{-1}}i\partial_{x_1x_2}V_{-1}V_1)\right| &\leqslant \left|c\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Im}(\overline{\partial_{x_1}V_1}V_1)\,\mathfrak{Re}(\partial_{x_1x_2}V_{-1}\overline{V_{-1}})\right|\\ &+ \left|c\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_{x_1}V_1}V_1)\mathfrak{Im}(\partial_{x_1x_2}V_{-1}\overline{V_{-1}})\right|\\ &\leqslant \int_{B(d\vec{e}_1,d^{\varepsilon'})}\frac{K}{(1+r_1)d^{4-\varepsilon'}}\leqslant\frac{K}{d^{4-2\varepsilon'}}=o_{d\to\infty}\left(\frac{1}{d^2}\right)\end{aligned}$$

since $c\leqslant\frac{2}{d}$ and $\varepsilon'<1$. For the third term of (2.3.19), we obtain similarly

$$\begin{aligned}\left|c\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_{x_1}V_{-1}\,V_1}i\partial_{x_2}V_1\partial_{x_1}V_{-1})\right| &\leqslant \left|c\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Im}(\bar{V}_1\partial_{x_2}V_1)\,\mathfrak{Re}(\partial_{x_1}V_{-1}\overline{\partial_{x_1}V_{-1}})\right|\\ &+ \left|c\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\bar{V}_1\partial_{x_2}V_1)\mathfrak{Im}(\partial_{x_1}V_{-1}\overline{\partial_{x_1}V_{-1}})\right|\\ &\leqslant \int_{B(d\vec{e}_1,d^{\varepsilon'})}\frac{K}{(1+r_1)d^{5-2\varepsilon'}}=o_{d\to\infty}\left(\frac{1}{d^2}\right).\end{aligned}$$

Finally, for the last term of (2.3.19),

$$\begin{aligned}\left|c\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_{x_1}V_{-1}\,V_1}i\partial_{x_1x_2}V_{-1}V_1)\right| &\leqslant \left|c\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Im}(\bar{V}_1V_1)\,\mathfrak{Re}(\partial_{x_1x_2}V_{-1}\overline{\partial_{x_1}V_{-1}})\right|\\ &+ \left|c\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\bar{V}_1V_1)\mathfrak{Im}(\partial_{x_1x_2}V_{-1}\overline{\partial_{x_1}V_{-1}})\right|\\ &\leqslant \int_{B(d\vec{e}_1,d^{\varepsilon'})}\frac{K}{d^{5-\varepsilon'}}\leqslant\frac{K}{d^{5-3\varepsilon'}}=o_{d\to\infty}\left(\frac{1}{d^2}\right).\end{aligned}$$

This conclude the proof of step 2.

Step 3. Proof of $\int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}(-4\nabla V_1.\nabla \partial_{x_1} V_{-1})) = -\frac{\pi}{d^2} + o_{d\to\infty}\big(\frac{1}{d^2}\big)$.

We have

$$-4\nabla V_1.\nabla \partial_{x_1} V_{-1} = -4\partial_{x_1} V_1 \partial_{x_1 x_1} V_{-1} - 4\partial_{x_2} V_1 \partial_{x_1 x_2} V_{-1}.$$

Remark that using $|\partial_d V| \leqslant \frac{K}{(1+r_1)}$ and Lemma 2.3.4 once again,

$$\left| \int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V} \partial_{x_1} V_1 \partial_{x_1 x_1} V_{-1}) \right| \leqslant \int_{B(d\vec{e}_1, d^{\varepsilon'})} \frac{K}{(1+r_1^2) d^{3-\varepsilon'}} = o_{d\to\infty}\left(\frac{1}{d^2}\right)$$

as for (2.3.20). Moreover,

$$-4 \int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V} \partial_{x_2} V_1 \partial_{x_1 x_2} V_{-1}) =$$

$$4 \int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_{x_1} V_1\, V_{-1}} \partial_{x_2} V_1 \partial_{x_1 x_2} V_{-1}) - 4 \int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}\big(\overline{\partial_{x_1} V_{-1}\, V_1} \partial_{x_2} V_1 \partial_{x_1 x_2} V_{-1}\big). \tag{2.3.21}$$

For the first integral in (2.3.21), we write

$$4 \int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_{x_1} V_1\, V_{-1}} \partial_{x_2} V_1 \partial_{x_1 x_2} V_{-1}) =$$

$$4 \int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_{x_1} V_1} \partial_{x_2} V_1)\, \mathfrak{Re}(\overline{V_{-1}} \partial_{x_1 x_2} V_{-1}) - \mathfrak{Im}(\overline{\partial_{x_1} V_1} \partial_{x_2} V_1) \mathfrak{Im}(\overline{V_{-1}} \partial_{x_1 x_2} V_{-1}).$$

For the first contribution, we have

$$\left| \int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_{x_1} V_1} \partial_{x_2} V_1)\, \mathfrak{Re}(\overline{V_{-1}} \partial_{x_1 x_2} V_{-1}) \right| \leqslant \int_{B(d\vec{e}_1, d^{\varepsilon'})} \frac{K}{(1+r_1^2) d^{3-\varepsilon'}} = o_{d\to\infty}\left(\frac{1}{d^2}\right)$$

by the same computations as (2.3.20). For the second contribution, recall from Lemma 2.1.2 that

$$\partial_{x_1} V_1 = \left(\cos(\theta_1)\, u - \frac{i}{r_1} \sin(\theta_1) \right) V_1 \quad \text{and} \quad \partial_{x_2} V_1 = \left(\sin(\theta_1)\, u + \frac{i}{r_1} \cos(\theta_1) \right) V_1,$$

therefore

$$\mathfrak{Im}(\overline{\partial_{x_1} V_1} \partial_{x_2} V_1) = \frac{u}{r_1} |V_1|^2,$$

and then, by Lemma 2.3.4,

$$-4 \int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Im}(\overline{\partial_{x_1} V_1} \partial_{x_2} V_1) \mathfrak{Im}(\overline{V_{-1}} \partial_{x_1 x_2} V_{-1}) = -4 \int_{B(d\vec{e}_1, d^{\varepsilon'})} \frac{u}{r_1} \frac{1}{4\, d^2} |V_1|^2\, dr_1 + o_{d\to\infty}\left(\frac{1}{d^2}\right)$$

since

$$\int_{B(d\vec{e}_1, d^{\varepsilon'})} \frac{u}{r_1} \frac{1}{4\, d^{2+1/4}} |V_1|^2\, dr_1 = o_{d\to\infty}\left(\frac{1}{d^2}\right).$$

We compute, using $|V_1|^2 = \rho_1^2$, $u = \frac{\rho_1'}{\rho_1}$ and Lemma 2.1.1,

$$-4 \int_{B(d\vec{e}_1, d^{\varepsilon'})} \frac{u}{r_1} \frac{|V_1|^2}{4\, d^2}\, dr_1 = \frac{-2\pi}{d^2} \int_0^{d^{\varepsilon'}} \rho_1'(r_1) \rho(r_1)\, dr_1 = \frac{-\pi}{d^2} [\rho_1^2]_0^{d^{\varepsilon'}} = \frac{-\pi}{d^2} + o_{d\to\infty}\left(\frac{1}{d^2}\right).$$

We obtain the estimate for the first integral in (2.3.21):

$$4 \int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_{x_1} V_1\, V_{-1}} \partial_{x_2} V_1 \partial_{x_1 x_2} V_{-1}) = \frac{-\pi}{d^2} + o_{d\to\infty}\left(\frac{1}{d^2}\right).$$

For the second integral in (2.3.21), we estimate

$$\left| \int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}\big(\overline{\partial_{x_1} V_{-1}\, V_1} \partial_{x_2} V_1 \partial_{x_1 x_2} V_{-1}\big) \right| \leqslant \int_{B(d\vec{e}_1, d^{\varepsilon'})} \frac{K}{(1+r_1) d^{4-\varepsilon'}} = o_{d\to\infty}\left(\frac{1}{d^2}\right).$$

This ends the proof of this subsection.

2.3.5 Proof of $\partial_d \int_{B(d\vec{e}_1,d^{\varepsilon'})\cup B(-d\vec{e}_1,d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\mathrm{TW}_c(Q_{c,d}))_{|d=d_c} = \frac{-2\pi}{d_c^2} + o_{d_c\to\infty}\left(\frac{1}{d_c^2}\right)$

In order to prove the result of this subsection, by using (2.3.1) and the result of subsection 2.3.4 we just have to show that at $d=d_c$,

$$\int_{B(d\vec{e}_1,\varepsilon')} \mathfrak{Re}(\overline{\partial_d V}L(\partial_{z_1}\Gamma_{c,d_c})) + \int_{B(d\vec{e}_1,d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}(\partial_{z_1}L)(\Gamma_{c,d_c}))$$
$$+\int_{B(d\vec{e}_1,d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\partial_{z_1}(\mathrm{NL}_V(\Gamma_{c,d_c}))) = o_{d_c\to\infty}\left(\frac{1}{d_c^2}\right).$$

Similarly to subsection 2.3.4, we omit the subscript on d_c in the proof.

Step 1. Proof of $\int_{B(d\vec{e}_1,d^{3/4})} \mathfrak{Re}(\overline{\partial_d V}L(\partial_{z_1}\Gamma_{c,d})) = o_{d\to\infty}(\frac{1}{d^2})$.

For this term, we want to do integration by parts and use that $L(\partial_d V)$ is very small, but since the integral is not on the whole space, there are the two boundary terms:

$$\left|\int_{B(d\vec{e}_1,d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}L(\partial_{z_1}\Gamma_{c,d}))\right| \leqslant \left|\int_{B(d\vec{e}_1,d^{\varepsilon'})} \mathfrak{Re}(L(\partial_d V)\overline{\partial_{z_1}\Gamma_{c,d}})\right|$$
$$+\left|\int_{\partial B(d\vec{e}_1,d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\nabla\partial_{z_1}\Gamma_{c,d})\right| + \left|\int_{\partial B(d\vec{e}_1,d^{\varepsilon'})} \mathfrak{Re}(\overline{\nabla\partial_d V}\partial_{z_1}\Gamma_{c,d})\right|,$$

where $\partial B(d\vec{e}_1,d^{\varepsilon'})$ is the boundary of $B(d\vec{e}_1,d^{\varepsilon'})$. On $\partial B(d\vec{e}_1,d^{\varepsilon'})$, we have

$$\Gamma_{c,d} = V(e^{\Psi_{c,d}}-1),$$

hence

$$\partial_{z_1}\Gamma_{c,d} = 2V_1\partial_{x_1}V_{-1}(e^{\Psi_{c,d}}-1) + V\partial_{z_1}\Psi_{c,d}e^{\Psi_{c,d}} \tag{2.3.22}$$

and

$$\begin{aligned}\nabla\partial_{z_1}\Gamma_{c,d} &= 2\nabla V_1\partial_{x_1}V_{-1}(e^{\Psi_{c,d}}-1) + 2V_1\nabla\partial_{x_1}V_{-1}(e^{\Psi_{c,d}}-1) + 2V_1\partial_{x_1}V_{-1}\nabla\Psi_{c,d}e^{\Psi_{c,d}}\\ &+ \nabla V\partial_{z_1}\Psi_{c,d}e^{\Psi_{c,d}} + V\nabla\partial_{z_1}\Psi_{c,d}e^{\Psi_{c,d}} + V\partial_{z_1}\Psi_{c,d}\nabla\Psi_{c,d}e^{\Psi_{c,d}}.\end{aligned} \tag{2.3.23}$$

By Lemmas 2.1.2 and 2.3.4, Proposition 2.3.5 and (2.2.14), we infer on $\partial B(d\vec{e}_1,d^{\varepsilon'})$ that, for any $1>\sigma>0$,

$$|\partial_{z_1}\Gamma_{c,d}| \leqslant \frac{K(\sigma)}{d^{2-\varepsilon'}d^{1-\sigma}} + \frac{K}{d^{1+\lambda-\alpha\varepsilon'}}. \tag{2.3.24}$$

Thus, still on $\partial B(d\vec{e}_1,d^{\varepsilon'})$, from Lemma 2.1.6 we compute

$$\left|\int_{\partial B(d\vec{e}_1,d^{\varepsilon'})} \mathfrak{Re}(\overline{\nabla\partial_d V}\partial_{z_1}\Gamma_{c,d})\right| \leqslant \frac{K}{d^{\varepsilon'}}\left(\frac{K(\sigma)}{d^{2-\varepsilon'}d^{1-\sigma}} + \frac{K}{d^{1+\lambda-\alpha\varepsilon'}}\right) \leqslant \frac{K(\sigma)}{d^{3-\sigma}} + \frac{K}{d^{1+\lambda+(1-\alpha)\varepsilon'}}.$$

Since $3-\sigma>2$ and $\lambda+(1-\alpha)\varepsilon'>1$ by (2.3.2), we have

$$\left|\int_{\partial B(d\vec{e}_1,d^{\varepsilon'})} \mathfrak{Re}(\overline{\nabla\partial_d V}\partial_{z_1}\Gamma_{c,d})\right| = o_{d\to\infty}\left(\frac{1}{d^2}\right).$$

For (2.3.23), we estimate on $\partial B(d\vec{e}_1,d^{\varepsilon'})$, still using Lemmas 2.1.2 and 2.3.4, Proposition 2.3.5 and (2.2.14), for any $1>\sigma>0$,

$$|2\nabla V_1\partial_{x_1}V_{-1}(e^{\Psi_{c,d}}-1) + 2V_1\nabla\partial_{x_1}V_{-1}(e^{\Psi_{c,d}}-1) + 2V_1\partial_{x_1}V_{-1}\nabla\Psi_{c,d}e^{\Psi_{c,d}}| \leqslant \frac{K(\sigma)}{d^{3-\sigma}},$$

and

$$|\nabla V\partial_{z_1}\Psi_{c,d}e^{\Psi_{c,d}} + V\nabla\partial_{z_1}\Psi_{c,d}e^{\Psi_{c,d}} + V\partial_{z_1}\Psi_{c,d}\nabla\Psi_{c,d}e^{\Psi_{c,d}}| \leqslant \frac{K}{d^{1+\lambda+(1-\alpha)\varepsilon'}} + \frac{K(\sigma)}{e^{2+\lambda+(1-\alpha)\varepsilon'-\sigma}}.$$

In particular, from (2.3.23), we can find $1 > \sigma > 0$ such that, on $\partial B(d\vec{e}_1, d^{\varepsilon'})$,

$$|\nabla \partial_{z_1} \Gamma_{c,d}| = o_{d\to\infty}\left(\frac{1}{d^2}\right),$$

thus

$$\left| \int_{\partial B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V} \nabla \partial_{z_1} \Gamma_{c,d}) \right| = o_{d\to\infty}\left(\frac{1}{d^2}\right).$$

From (2.1.45), we know that

$$|L(\partial_d V)| \leqslant \frac{K}{(1+\tilde{r}^2)d}.$$

Moreover, by Proposition 2.3.5, we have $|\partial_{z_1}\Gamma_{c,d}| \leqslant \frac{K}{d^{1+\lambda-\alpha\varepsilon'}}$ in $B(d\vec{e}_1, d^{\varepsilon'})$, which is enough to show that

$$\int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(L(\overline{\partial_d V})\partial_{z_1}\Gamma_{c,d}) = o_{d\to\infty}\left(\frac{1}{d^2}\right).$$

Step 2. Proof of $\int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}(\partial_{z_1} L)(\Gamma_{c,d})) = o_{d\to\infty}(\frac{1}{d^2})$.

We have

$$(\partial_{z_1} L)(\Gamma_{c,d}) = 4\,\mathfrak{Re}(\overline{V_{-1}}\partial_{x_1} V_{-1})\,\Gamma_{c,d} + 4\,\mathfrak{Re}(\overline{\partial_{x_1} V_{-1}\, V_1}\Gamma_{c,d})V + 4\,\mathfrak{Re}(\bar{V}\,\Gamma_{c,d})\,V_1 \partial_{x_1} V_{-1},$$

thus

$$\begin{aligned}\int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}(\partial_{z_1} L)(\Gamma_{c,d})) &= 4\int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\Gamma_{c,d})\mathfrak{Re}(\overline{V_{-1}}\partial_{x_1} V_{-1}) \\ &+ 4\int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_{x_1} V_{-1}\, V_1}\Gamma_{c,d})\mathfrak{Re}(\overline{\partial_d V} V) \\ &+ 4\int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V} V_1 \partial_{x_1} V_{-1})\mathfrak{Re}(\bar{V}\Gamma_{c,d}). \qquad (2.3.25)\end{aligned}$$

Using $|\partial_d V| \leqslant \frac{K}{1+r_1}$,

$$\mathfrak{Re}(\overline{V_{-1}}\partial_{x_1} V_{-1}) = O_{d\to\infty}\left(\frac{1}{d^3}\right) \quad \text{and} \quad |\Gamma_{c,d}| \leqslant \frac{K}{(1+r_1)^{1/2} d^{1/2}}$$

from Lemma 2.1.6, Lemma 2.1.2 and (2.2.9) respectively, we may bound

$$\left| \int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\,\Gamma_{c,d})\mathfrak{Re}(\overline{V_{-1}}\partial_{x_1} V_{-1}) \right| \leqslant \int_{B(d\vec{e}_1, d^{\varepsilon'})} \frac{K}{(1+r_1)^{1+1/2} d^{3+1/2}} = o_{d\to\infty}\left(\frac{1}{d^2}\right).$$

The second term of (2.3.25) is

$$4\int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_{x_1} V_{-1}\, V_1}\Gamma_{c,d})\mathfrak{Re}(\overline{\partial_d V} V).$$

We compute that

$$|\mathfrak{Re}(\overline{\partial_{x_1} V_{-1}\, V_1}\Gamma_{c,d})| \leqslant \frac{K}{(1+r_1)^{1/8} d^{17/8}} \quad \text{and} \quad |\mathfrak{Re}(\overline{\partial_d V} V)| \leqslant \frac{K}{(1+r_1)^3}$$

in $B(d\vec{e}_1, d^{\varepsilon'})$ using

$$|\Gamma_{c,d}| \leqslant \frac{K}{(1+r_1)^{1/8} d^{7/8}}$$

by (2.2.14) and the definition of $\Gamma_{c,d}$. Therefore, since $17/8 > 2$,

$$\int_{B(d\vec{e}_1, d^{\varepsilon'})} 4\,\mathfrak{Re}(\overline{\partial_{x_1} V_{-1}\, V_1}\Gamma_{c,d})\,\mathfrak{Re}(\overline{\partial_d V} V) = o_{d\to\infty}\left(\frac{1}{d^2}\right).$$

The last term of (2.3.25) is

$$\int_{B(d\vec{e}_1, d^{\varepsilon'})} 4\,\mathfrak{Re}(\bar{V}\Gamma_{c,d})\,\mathfrak{Re}(V_1 \partial_{x_1} V_{-1}\overline{\partial_d V}).$$

Recalling that

$$|\mathfrak{Re}(\bar{V}\Gamma_{c,d})| \leqslant K\,|\mathfrak{Re}(\Psi)| \leqslant \frac{K}{(1+r_1)^{1+1/8}\,d^{7/8}}$$

and

$$|\mathfrak{Re}(V_1\partial_{x_1}V_{-1}\overline{\partial_d V})| \leqslant \frac{K}{d^{5/4}(1+r_1)},$$

we deduce

$$\int_{B(d\vec{e}_1,d^{\varepsilon'})} 4\,\mathfrak{Re}(\bar{V}\Gamma_{c,d})\mathfrak{Re}(V_1\partial_{x_1}V_{-1}\overline{\partial_d V}) = o_{d\to\infty}\Big(\frac{1}{d^2}\Big).$$

Step 3. Proof of $\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_d V}\bar{\partial}_{z_1}(\mathrm{NL}_V(\Gamma_{c,d}))) = o_{d\to\infty}\big(\frac{1}{d^2}\big)$.

Recall that

$$\begin{aligned}\partial_{z_1}\mathrm{NL}_V\big(\Gamma_{c,d}\big) \;=\; & 4\,\mathfrak{Re}(\overline{\partial_{x_1}V_{-1}\,V_1}\Gamma_{c,d})\Gamma_{c,d} + 2\,\mathfrak{Re}(\bar{V}\bar{\partial}_{z_1}\Gamma_{c,d})\Gamma_{c,d} + 2\,\mathfrak{Re}(\bar{V}\Gamma_{c,d})\partial_{z_1}\Gamma_{c,d}\\ & +2\,\mathfrak{Re}\big(\overline{\Gamma_{c,d}}\partial_{z_1}\Gamma_{c,d}\big)(V+\Gamma_{c,d}) + |\Gamma_{c,d}|^2(2\partial_{x_1}V_{-1}\,V_1 + \partial_{z_1}\Gamma_{c,d}).\end{aligned}$$

We write

$$\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_d V}\bar{\partial}_{z_1}(\mathrm{NL}_V(\Gamma_{c,d}))) = I_1+I_2+I_3+I_4+I_5,$$

with

$$I_1 = \int_{B(d\vec{e}_1,d^{\varepsilon'})} 4\mathfrak{Re}(\overline{\partial_d V}\Gamma_{c,d})\mathfrak{Re}(\overline{\partial_{x_1}V_{-1}\,V_1}\Gamma_{c,d}),$$

$$I_2 = \int_{B(d\vec{e}_1,d^{\varepsilon'})} 2\mathfrak{Re}(\overline{\partial_d V}\Gamma_{c,d})\mathfrak{Re}(\bar{V}\bar{\partial}_{z_1}\Gamma_{c,d}),$$

$$I_3 = \int_{B(d\vec{e}_1,d^{\varepsilon'})} 2\mathfrak{Re}(\bar{V}\Gamma_{c,d})\mathfrak{Re}(\overline{\partial_d V}\partial_{z_1}\Gamma_{c,d}),$$

$$I_4 = \int_{B(d\vec{e}_1,d^{\varepsilon'})} 2\mathfrak{Re}\big(\overline{\Gamma_{c,d}}\partial_{z_1}\Gamma_{c,d}\big)\mathfrak{Re}(\overline{\partial_d V}V) + 2\mathfrak{Re}\big(\overline{\Gamma_{c,d}}\partial_{z_1}\Gamma_{c,d}\big)\mathfrak{Re}(\overline{\partial_d V}\Gamma_{c,d}),$$

$$I_5 = \int_{B(d\vec{e}_1,d^{\varepsilon'})} |\Gamma_{c,d}|^2\mathfrak{Re}(\overline{\partial_d V}(2\partial_{x_1}V_{-1}V_1 + \partial_{z_1}\Gamma_{c,d})).$$

Estimate for I_1.

We estimate, by using $|\Gamma_{c,d}| \leqslant \frac{K}{(1+r_1)^{9/16}\,d^{7/16}}$ that

$$|\mathfrak{Re}(\overline{\partial_d V}\Gamma_{c,d})\,\mathfrak{Re}(\overline{\partial_{x_1}V_{-1}\,V_1}\Gamma_{c,d})| \leqslant |\Gamma_{c,d}|^2\frac{K}{(1+r_1)\,d^{5/4}} \leqslant \frac{K}{(1+r_1)^{2+1/8}\,d^{17/8}}$$

Then, since $17/8 > 2$,

$$\int_{B(d\vec{e}_1,d^{\varepsilon'})} 4\,\mathfrak{Re}(\overline{\partial_d V}\Gamma_{c,d})\,\mathfrak{Re}(\overline{\partial_{x_1}V_{-1}\,V_1}\Gamma_{c,d}) = o_{d\to\infty}\Big(\frac{1}{d^2}\Big).$$

Estimate for I_2.

From (2.3.22), we have

$$\partial_{z_1}\Gamma_{c,d} = 2V_1\partial_{x_1}V_{-1}(e^{\Psi_{c,d}}-1) + V\partial_{z_1}\Psi_{c,d}e^{\Psi_{c,d}},$$

therefore, on $B(d\vec{e}_1,d^{\varepsilon'})$, by Lemma 2.3.4, Proposition 2.3.5 and (2.2.13), for any $1>\sigma>0$,

$$|\mathfrak{Re}(\bar{V}\bar{\partial}_{z_1}\Gamma_{c,d})| \leqslant \frac{K(\sigma)}{d^{3-\varepsilon'-\sigma}} + \frac{K}{(1+r_1)^{1-\alpha}d^{1+\lambda}} + \frac{K(\sigma)}{d^{2+\lambda-\sigma}(1+r_1)^{-\alpha}}.$$

Combining this with

$$|\mathfrak{Re}(\overline{\partial_d V}\Gamma_{c,d})| \leqslant \frac{K(\sigma)}{(1+r_1)\,d^{1-\sigma}}$$

since $|\Gamma_{c,d}|\leqslant\frac{K(\sigma)}{d^{1-\sigma}}$, we infer

$$\begin{aligned}\left|\int_{B(d\vec{e}_1,d^{\varepsilon'})}2\,\mathfrak{Re}(\overline{\partial_d V}\Gamma_{c,d})\,\mathfrak{Re}(\bar{V}\partial_{z_1}\Gamma_{c,d})\right| &\leqslant \left|\int_{B(d\vec{e}_1,d^{\varepsilon'})}\frac{K(\sigma)}{(1+r_1)\,d^{4-\varepsilon'-2\sigma}}\right|\\ &+\left|\int_{B(d\vec{e}_1,d^{\varepsilon'})}\frac{K(\sigma)}{(1+r_1)^{2-\alpha}d^{2+\lambda-\sigma}}\right|\\ &+\left|\int_{B(d\vec{e}_1,d^{\varepsilon'})}\frac{K(\sigma)}{(1+r_1)^{1-\alpha}d^{3+\lambda-2\sigma}}\right|,\end{aligned}$$

and since $\lambda+(1-\alpha)\varepsilon'>1$, we conclude, taking $\sigma>0$ small enough,

$$\int_{B(d\vec{e}_1,d^{\varepsilon'})}2\,\mathfrak{Re}(\overline{\partial_d V}\Gamma_{c,d})\,\mathfrak{Re}(\bar{V}\partial_{z_1}\Gamma_{c,d})=o_{d\to\infty}\Big(\frac{1}{d^2}\Big).$$

Estimate for I_3.

We have from (2.3.22) that

$$\partial_{z_1}\Gamma_{c,d}=2V_1\partial_{x_1}V_{-1}(e^{\Psi_{c,d}}-1)+V\partial_{z_1}\Psi_{c,d}e^{\Psi_{c,d}},$$

therefore

$$|\mathfrak{Re}(\overline{\partial_d V}\partial_{z_1}\Gamma_{c,d})|\leqslant K\Big(\frac{1}{(1+r_1)d^{3-2\varepsilon'}}+\frac{1}{(1+r_1)^{1-\alpha}d^{1+\lambda}}\Big),$$

and $|\mathfrak{Re}(\bar{V}\Gamma_{c,d})|\leqslant K\,|\mathfrak{Re}(\Psi_{c,d})|$, hence

$$|\mathfrak{Re}(\bar{V}\Gamma_{c,d})|\leqslant\frac{K(\sigma)}{(1+r_1)^{1+\sigma}\,d^{1-\sigma}},$$

then

$$\begin{aligned}\left|\int_{B(d\vec{e}_1,d^{\varepsilon'})}2\,\mathfrak{Re}(\overline{\partial_d V}\partial_{z_1}\Gamma_{c,d})\,\mathfrak{Re}(\bar{V}\Gamma_{c,d})\right| &\leqslant \int_{B(d\vec{e}_1,d^{\varepsilon'})}\frac{K(\sigma)}{(1+r_1)^{2+\sigma}\,d^{4-2\varepsilon'-\sigma}}\\ &+\int_{B(d\vec{e}_1,d^{\varepsilon'})}\frac{K(\sigma)}{(1+r_1)^{2+\sigma-\alpha}\,d^{2+\lambda-\sigma}}\\ &=o_{d\to\infty}\Big(\frac{1}{d^2}\Big)\end{aligned}$$

by taking $\sigma>0$ small enough and using $\lambda+(1-\alpha)\varepsilon'>1$.

Estimate for I_4.

Recall that

$$|\mathfrak{Re}(\overline{\partial_d V}V)|\leqslant\frac{K}{(1+r_1)^3},$$

and we have

$$|\mathfrak{Re}(\overline{\partial_d V}\Gamma_{c,d})|\leqslant\frac{K}{(1+r_1)^{1+6/8}\,d^{2/8}}$$

since $|\Gamma_{c,d}|\leqslant\frac{K}{(1+r_1)^{1+6/8}\,d^{2/8}}$. Therefore, with $\frac{1}{d}\leqslant\frac{K}{(1+r_1)}$,

$$|\mathfrak{Re}(\overline{\partial_d V}V)+\mathfrak{Re}(\overline{\partial_d V}\Gamma_{c,d})|\leqslant\frac{K}{(1+r_1)^2}$$

Now, we use $|\Gamma_{c,d}|\leqslant\frac{K(\sigma)}{(1+r_1)^{\sigma}\,d^{1-\sigma}}$ and Proposition 2.3.5 to get

$$|\mathfrak{Re}\big(\overline{\Gamma_{c,d}}\partial_{z_1}\Gamma_{c,d}\big)|\leqslant\frac{K}{(1+r_1)^{\sigma-\alpha}\,d^{2+\lambda-\sigma'}}.$$

We conclude as for the previous estimates,

$$\int_{B(d\vec{e}_1,d^{\varepsilon'})}2(\mathfrak{Re}(\overline{\partial_d V}V)+\mathfrak{Re}(\overline{\partial_d V}\Gamma_{c,d}))\,\mathfrak{Re}\big(\overline{\Gamma_{c,d}}\partial_{z_1}\Gamma_{c,d}\big)=o_{d\to\infty}\Big(\frac{1}{d^2}\Big).$$

Estimate for I_5.

We have, by Proposition 2.3.5,

$$|\mathfrak{Re}(\overline{\partial_d V}(\partial_{x_1} V_{-1}\, V_1 + \partial_{z_1}\Gamma_{c,d}))| \leqslant \frac{K}{(1+r_1)}\left(\frac{1}{d^{2-\varepsilon'}} + \frac{1}{(1+r_1)^{1-\alpha} d^{2+\lambda-\sigma}}\right)$$

and using $|\Gamma_{c,d}| \leqslant \frac{K(\sigma)}{d^{1-\sigma}}$, we have

$$|\Gamma_{c,d}|^2 \leqslant \frac{K(\sigma)}{d^{2-2\sigma}}.$$

Therefore, for $\sigma > 0$ small enough, since $\lambda + (1-\alpha)\varepsilon' > 1$,

$$\int_{B(d\vec{e}_1, d^{\varepsilon'})} |\Gamma_{c,d}|^2 \mathfrak{Re}(\overline{\partial_d V}(2\partial_{x_1} V_{-1}\, V_1 + \partial_{z_1}\Gamma_{c,d})) = o_{d\to\infty}\left(\frac{1}{d^2}\right)$$

which concludes the estimates.

2.3.6 Proof of $\partial_c d_c = -\frac{1}{c^2} + o_{c\to 0}\left(\frac{1}{c^2}\right)$

Recall that d_c is defined by the implicit equation

$$\int_{B(d\vec{e}_1, d^{\varepsilon'}) \cup B(-d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\, \mathrm{TW}_c(Q_{c,d})) = 0.$$

We showed in subsection 2.3.5 that

$$\partial_d \int_{B(d\vec{e}_1, d^{\varepsilon'}) \cup B(-d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\, \mathrm{TW}_c(Q_{c,d}))_{|\, d=d_c} = \frac{-2\,\pi}{d_c^2} + o_{d_c\to\infty}\left(\frac{1}{d_c^2}\right).$$

Therefore, by the implicit function theorem,

$$\partial_c d_c = \frac{\partial_c \int_{B(d\vec{e}_1, d^{\varepsilon'}) \cup B(-d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\, \mathrm{TW}_c(Q_{c,d}))_{|\, d=d_c}}{\frac{-2\,\pi}{d_c^2} + o_{d_c\to\infty}\left(\frac{1}{d_c^2}\right)}.$$

We compute for

$$\mathrm{TW}_c(Q_{c,d}) = -ic\partial_{x_2} Q_{c,d} - \Delta\, Q_{c,d} - (1 - |Q_{c,d}|^2)\, Q_{c,d}$$

that, with $\partial_c Q_{c,d} = \partial_c(V + \Gamma_{c,d}) = \partial_c \Gamma_{c,d}$ at fixed d, we have (still at fixed d)

$$\partial_c(\mathrm{TW}_c(Q_{c,d})) = -i\,\partial_{x_2} Q_{c,d} - L_{Q_{c,d}}(\partial_c \Gamma_{c,d}),$$

where

$$L_{Q_{c,d}}(h) := -\Delta h - ic\partial_{x_2} h - (1 - |Q_{c,d}|^2)h + 2\mathfrak{Re}(\overline{Q_{c,d}} h) Q_{c,d}.$$

We are left with the computation of

$$\partial_c \int_{B(d\vec{e}_1, d^{\varepsilon'}) \cup B(-d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\, \mathrm{TW}_c(Q_{c,d}))_{|\, d=d_c} =$$

$$-\int_{B(d\vec{e}_1, d^{\varepsilon'}) \cup B(-d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}(i\,\partial_{x_2} Q_{c,d}))_{|\, d=d_c}$$

$$-\int_{B(d\vec{e}_1, d^{\varepsilon'}) \cup B(-d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V} L_{Q_c}(\partial_c \Gamma_{c,d}))_{|\, d=d_c}.$$

As above, we omit the subscript in d_c for the computations.

Step 1. Proof of $\int_{B(d\vec{e}_1, d^{\varepsilon'}) \cup B(-d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}(-i\,\partial_{x_2} Q_c))_{|\, d=d_c} = 2\pi + o_{c\to 0}(1)$.

We have $\partial_{x_2}Q_c=\partial_{x_2}V+\partial_{x_2}\Gamma_{c,d}$, hence

$$-\int_{B(d\vec{e}_1,d^{\varepsilon'})\cup B(-d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_d V}(i\partial_{x_2}Q_c))=$$

$$-\int_{B(d\vec{e}_1,d^{\varepsilon'})\cup B(-d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(i\overline{\partial_d V}\partial_{x_2}V))-\int_{B(d\vec{e}_1,d^{\varepsilon'})\cup B(-d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(i\,\overline{\partial_d V}\partial_{x_2}\Gamma_{c,d}).$$

Since

$$|\partial_d V|\leqslant\frac{K}{(1+r_1)}$$

and

$$|\partial_{x_2}\Gamma_{c,d}|\leqslant\frac{K}{(1+r_1)^{1+1/2}\,d^{1/2}},$$

we have

$$\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(i\overline{\partial_d V}\partial_{x_2}\Gamma_{c,d})=o_{c\to 0}(1).$$

Furthermore,

$$-\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(i\overline{\partial_d V}\partial_{x_2}V)=\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}\big(i\,\overline{\partial_{x_1}V_1}\partial_{x_2}V_1\big)+o_{c\to 0}(1),$$

and we already computed in (2.1.43) that

$$\int_{\mathbb{R}^2}\mathfrak{Re}(i\,\partial_{x_2}V_1\overline{\partial_{x_1}V_1})=-\pi+o_{c\to 0}(c^{1/4})$$

hence

$$\int_{B(d\vec{e}_1,d^{\varepsilon'})\cup B(-d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_d V}(-i\partial_{x_2}Q_c))_{|\,d=d_c}=2\pi+o_{c\to 0}(1).$$

Step 2. Proof of $\int_{B(d\vec{e}_1,d^{\varepsilon'})\cup B(-d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_d V}L_{Q_c}(\partial_c\Gamma_{c,d}))_{|\,d=d_c}=o_{c\to 0}(1)$.

From the definition of $\Gamma_{c,d}$, at fixed d, we have

$$\partial_c\Gamma_{c,d}=\eta V\partial_c\Psi_{c,d}+(1-\eta)V\partial_c\Psi_{c,d}e^{\Psi_{c,d}}. \tag{2.3.26}$$

We have, by definition,

$$L_{Q_c}(\partial_c\Gamma_{c,d})=-ic\partial_{x_2}\partial_c\Gamma_{c,d}-\Delta\partial_c\Gamma_{c,d}-(1-|Q_c|^2)\partial_c\Gamma_{c,d}+2\mathfrak{Re}(\overline{Q_c}\partial_c\Gamma_{c,d})Q_c,$$

and using $|\partial_d V|\leqslant\frac{K}{(1+r_1)}$ with $|\partial_{x_2}\partial_c\Gamma_{c,d}|\leqslant\frac{Kc^{-1/2}}{(1+r_1)^{1+1/2}}$ since $\left\|\frac{\partial_c\Gamma_{c,d}}{V}\right\|_{*,1/2,d}\leqslant Kc^{-3/4}$ from Lemma 2.2.9 and (2.3.26), we have

$$\left|\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_d V}(-ic\partial_{x_2}\partial_c\Gamma_{c,d}))\right|\leqslant K\int_{B(d\vec{e}_1,d^{\varepsilon'})}\frac{c^{1/4}}{(1+r_1)^{2+1/2}}=o_{c\to 0}(1).$$

The estimate on $B(-d\vec{e}_1,d^{\varepsilon'})$ is similar.

We define

$$\tilde{L}_{Q_c}(h):=-\Delta h-(1-|Q_c|^2)h+2\mathfrak{Re}(\overline{Q_c}h)Q_c$$

and we are then left with the computation of

$$\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}\big(\overline{\partial_d V}\tilde{L}_{Q_c}(\partial_c\Gamma_{c,d})\big),$$

the part on $B(-d\vec{e}_1,d^{\varepsilon'})$ being symmetrical. We want to put the linear operator onto $\partial_d V$ since $\tilde{L}_{Q_c}(\partial_d V)$ is close to $L_V(\partial_d V)$ which is itself small. We then integrate by parts:

$$\left|\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}\big(\overline{\partial_d V}\tilde{L}_{Q_c}(\partial_c\Gamma_{c,d})\big)\right|\leqslant\left|\int_{B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}\big(\tilde{L}_{Q_c}(\partial_d V)\,\overline{\partial_c\Gamma_{c,d}}\big)\right|$$

$$+\left|\int_{\partial B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\overline{\partial_d V}\nabla\partial_c\Gamma_{c,d})\right|+\left|\int_{\partial B(d\vec{e}_1,d^{\varepsilon'})}\mathfrak{Re}(\nabla\overline{\partial_d V}\partial_c\Gamma_{c,d})\right|.$$

We have on $\partial B(d\vec{e_1}, d^{\varepsilon'})$, that $|\partial_d V| \leqslant \frac{K}{d^{3/4}}$, $|\nabla\partial_d V| \leqslant \frac{K}{d^{3/2}}$ from Lemma 2.1.6. Moreover, by $\left\|\frac{\partial_c\Gamma_{c,d}}{V}\right\|_{*,1/2,d} \leqslant K(\sigma)\, c^{-1/2-\sigma}$ from Lemma 2.2.9 and (2.3.26), we deduce $|\nabla\partial_c\Gamma_{c,d}| \leqslant \frac{K(\sigma)\, d^{1/2+\sigma}}{d^{(3/4)(3/2)}} \leqslant \frac{K(\sigma)}{d^{5/8-\sigma}}$ and $|\partial_c\Gamma_{c,d}| \leqslant \frac{K(\sigma) d^{1/2-\sigma}}{d^{(3/4)(1/2)}} \leqslant K(\sigma) d^{1/8-\sigma}$. We then obtain, for $\sigma > 0$ small enough,

$$\left|\int_{\partial B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V}\nabla\partial_c\Gamma_{c,d})\right| \leqslant \int_{\partial B(d\vec{e_1}, d^{\varepsilon'})} |\partial_d V||\nabla\partial_c\Gamma_{c,d}| \leqslant d^{3/4}\frac{K(\sigma) d^{2\sigma}}{d^{3/4} d^{5/8}} = o_{c\to 0}(1),$$

$$\left|\int_{\partial B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}(\nabla\overline{\partial_d V}\partial_c\Gamma_{c,d})\right| \leqslant \int_{\partial B(d\vec{e_1}, d^{\varepsilon'})} |\nabla\partial_d V||\partial_c\Gamma_{c,d}| \leqslant d^{3/4}\frac{K(\sigma)\, d^{1/8+\sigma}}{d^{3/2}} = o_{c\to 0}(1).$$

Therefore,

$$\int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}\big(\overline{\partial_d V}\tilde{L}_{Q_c}(\partial_c\Gamma_{c,d})\big) = \int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}\big(\tilde{L}_{Q_c}(\partial_d V)\,\overline{\partial_c\Gamma_{c,d}}\big) + o_{c\to 0}(1).$$

Now, from (2.1.45), we have that that

$$|L_V(\partial_d V)| \leqslant \frac{K}{(1+\tilde{r}^2)d}$$

and by Lemma 2.2.9 and (2.3.26), we have $|\partial_c\Gamma_{c,d}| \leqslant \frac{K d^{1/4}}{(1+r_1)^{1/2}}$, hence

$$\left|\int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}(L_V(\partial_d V)\,\overline{\partial_c\Gamma_{c,d}})\right| \leqslant K\int_{B(d\vec{e_1}, d^{\varepsilon'})} \frac{1}{(1+r_1)^{2+1/2} d^{1/4}} = o_{c\to 0}(1).$$

We deduce from this that

$$\int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}\big(\overline{\partial_d V}\tilde{L}_{Q_c}(\partial_c\Gamma_{c,d})\big) = \int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}\big(\big(\tilde{L}_{Q_c} - L_V\big)(\partial_d V)\,\overline{\partial_c\Gamma_{c,d}}\big) + o_{c\to 0}(1).$$

We have $\tilde{L}_{Q_c}(h) = -\Delta h - (1-|Q_c|^2)h + 2\mathfrak{Re}(\overline{Q_c}h)Q_c$ and $L_V(h) = -\Delta h - (1-|V|^2)h + 2\mathfrak{Re}(\bar{V}h)V$, therefore

$$\big(\tilde{L}_{Q_c} - L_V\big)(\partial_d V) = (|Q_c|^2 - |V|^2)\partial_d V + 2\mathfrak{Re}(\bar{V}\partial_d V)(Q_c - V) + 2\mathfrak{Re}(\overline{Q_c - V}\partial_d V)Q_c.$$

We have by (2.2.17) that $||Q_c|^2 - |V|^2| \leqslant \frac{K c^{3/4}}{(1+\tilde{r})^{1+1/4}}$, hence

$$\left|\int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}(\,(|Q_c|^2 - |V|^2)\partial_d V\overline{\partial_c\Gamma_{c,d}})\right| \leqslant K\int_{B(d\vec{e_1}, d^{\varepsilon'})} \frac{c^{1/4}}{(1+r_1)^{2+3/4}} = o_{c\to 0}(1).$$

We have from (2.2.16) that $|Q_c - V| \leqslant \frac{c^{3/4}}{(1+\tilde{r})^{1/4}}$, and, in $B(d\vec{e_1}, d^{\varepsilon'})$, we have (by Lemmas 2.1.1 and 2.1.2) that $|\mathfrak{Re}(\bar{V}\partial_d V)| \leqslant \frac{K}{(1+r_1)^3}$, therefore

$$\left|\int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}(\,2\mathfrak{Re}(\bar{V}\partial_d V)(Q_c - V)\overline{\partial_c\Gamma_{c,d}})\right| \leqslant K\int_{B(d\vec{e_1}, d^{\varepsilon'})} \frac{c^{1/4}}{(1+r_1)^{3+3/4}} = o_{c\to 0}(1).$$

Finally, by using the same estimates, we have

$$\left|\int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}(\,2\mathfrak{Re}(\overline{Q_c - V}\partial_d V)Q_c\overline{\partial_c\Gamma_{c,d}})\right| \leqslant K\int_{B(d\vec{e_1}, d^{\varepsilon'})} \frac{c^{3/4}}{(1+r_1)^{1+1/4}}|\mathfrak{Re}(Q_c\overline{\partial_c\Gamma_{c,d}})|.$$

We compute

$$\mathfrak{Re}(Q_c\overline{\partial_c\Gamma_{c,d}}) = \mathfrak{Re}(V\overline{\partial_c\Gamma_{c,d}}) + \mathfrak{Re}(\Gamma_{c,d}\overline{\partial_c\Gamma_{c,d}}).$$

By using $\left\|\frac{\partial_c\Gamma_{c,d}}{V}\right\|_{*,1/2,d} \leqslant K(\sigma)c^{-1/2-\sigma}$ from Lemma 2.2.2 and (2.3.26), we have $|\mathfrak{Re}(V\overline{\partial_c\Gamma_{c,d_c}})| \leqslant \frac{K(\sigma)\, c^{-1/2-\sigma}}{(1+r_1)^{3/2}}$. Furthermore, with $|\Gamma_{c,d}| \leqslant \frac{K c^{1/2}}{(1+r_1)^{1/2}}$, we have $|\mathfrak{Re}(\Gamma_{c,d}\overline{\partial_c\Gamma_{c,d}})| \leqslant \frac{K(\sigma)c^{-\sigma}}{(1+r_1)}$. With these estimates, we infer, taking $\sigma > 0$ small enough,

$$\left|\int_{B(d\vec{e_1}, d^{\varepsilon'})} \mathfrak{Re}(\,2\mathfrak{Re}(\overline{Q_c - V}\partial_d V)Q_c\overline{\partial_c\Gamma_{c,d}})\right| = o_{c\to 0}(1)$$

which ends the proof of

$$\int_{B(d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}\big(\big(\tilde{L}_{Q_c} - L_V \big) (\partial_d V)\, \overline{\partial_c \Gamma_{c,d}} \big) = o_{c\to 0}(1).$$

Step 3. Conclusion.

We showed that

$$\partial_c d_c = \frac{2\pi + o_{c\to 0}(1)}{\frac{-2\,\pi}{d_c^2} + o_{d_c\to\infty}\Big(\frac{1}{d_c^2}\Big)},$$

therefore, with $d_c = \frac{1+o_{c\to 0}(1)}{c}$ from Proposition 2.1.26 we have

$$\partial_c d_c = -\frac{1+o_{c\to 0}(1)}{c^2}.$$

As a result of subsection 2.3.5, at fixed c,

$$\partial_d \int_{B(d\vec{e}_1, d^{\varepsilon'}) \cup B(-d\vec{e}_1, d^{\varepsilon'})} \mathfrak{Re}(\overline{\partial_d V} \mathrm{TW}_c(Q_{c,d}))_{|\,d=d_c} \neq 0$$

for c small enough. By the implicit function theorem, taking some $0 < c_* < c_0(\sigma)$, we can construct a C^1 branch $c \mapsto d_c$ in a vicinity of c_*. We define $\mathbf{C}$ as the set of $c_* > c_\circledast \geqslant 0$ such that there exists a C^1 branch $c \mapsto d_c$ on $]c_\circledast, c_*[$. We have just shown that $\mathbf{C}$ is not empty. Let us suppose that $c_\circledast := \inf \mathbf{C} \neq 0$. Then, $c \mapsto d_c$ is uniformly bounded on $]c_\circledast, c_*[$ in C^1 by subsection 2.3.6, and can therefore be extended by continuity to $c_\circledast$, and we denote $d_\circledast$ its value there. We can construct the perturbation $\Phi_{c_\circledast, d_\circledast}$ by continuity since c, $d \mapsto \Phi_{c,d}$ are C^1 functions in the Banach space $\big\{\Phi \in C^1(\mathbb{R}^2, \mathbb{C}),\, \|\Phi\|_{*,\sigma,d_\circledast} < +\infty\big\}$ for its canonical norm (which is equivalent to $\|.\|_{*,\sigma,d}$ for any $d \in [d_\circledast, d_{c_*}]$). By passing to the limit, we have $\|\Phi_{c_\circledast,d_\circledast}\|_{*,\sigma,d_\circledast} \leqslant K_0(\sigma,\sigma') c_\circledast^{1-\sigma'}$ for $K_0(\sigma,\sigma')$ defined in Proposition 2.1.21. By continuity of λ, we check that we have $\lambda(c_\circledast, d_\circledast) = 0$ (for the perturbation $\Phi_{c_\circledast,d_\circledast}$). Therefore, by the implicit function theorem, there exists a unique branch $c \mapsto d_c$ in a vicinity of $(c_\circledast, d_\circledast)$ such that $\lambda(c, d_c) = 0$. This branch, by uniqueness, corresponds to the branch we had on $]c_\circledast, c_*[$, and is also C^1 by the implicit function theorem. Therefore $\inf \mathbf{C} < c_\circledast$, which is in contradiction with $c_\circledast = \inf \mathbf{C}$, and thus $\inf \mathbf{C} = 0$.

In particular, the travelling wave Q_c on this branch is uniquely defined by this construction and is a C^1 function of c. Indeed, we shall now show that there is only one choice of d_c such that $\lambda(c, d_c) = 0$ in $\big]\frac{1}{2c}, \frac{2}{c}\big[$. If there exist $d_1 \neq d_2$ in $\big]\frac{1}{2c}, \frac{2}{c}\big[$ such that $\lambda(c, d_1) = \lambda(c, d_2) = 0$, by Subsection 2.3.5, we have

$$\partial_d(\lambda(c,d))_{|\,d=d_1} < 0 \quad \text{and} \quad \partial_d(\lambda(c,d))_{|\,d=d_2} < 0,$$

therefore, there exists d' such that $\lambda(c, d') = 0$ and $\partial_d(\lambda(c,d'))_{|\,d=d'} \geqslant 0$, but then, since $\lambda(c,d') = 0$, we have $\partial_d(\lambda(c,d))_{|\,d=d'} < 0$, which is in contradiction with $\partial_d(\lambda(c,d'))_{|\,d=d'} \geqslant 0$. Now that we have uniqueness in the choice of d_c (in $\big]\frac{1}{2c}, \frac{2}{c}\big[$), we have uniqueness of $\Phi_{c,d}$ in the set

$$\big\{\Phi \in C^1(\mathbb{R}^2, \mathbb{C}),\, \|\Phi\|_{*,\sigma,d_c} \leqslant K_0(\sigma,\sigma') c^{1-\sigma'}\big\}$$

for $K_0(\sigma,\sigma') > 0$ defined in Proposition 2.1.21.

2.3.7 Proof of the estimate on $\partial_c Q_c$

We conclude the proof of Theorem 1.3.1 with the following lemma.

Lemma 2.3.6. *For any $0 < \sigma < 1$, there exist $c_0(\sigma) > 0$ such that for any $c < c_0(\sigma)$,*

$$\left\| \frac{\partial_c Q_c}{V} + \left(\frac{1+o_{c\to 0}(1)}{c^2} \right) \frac{\partial_d V_{|\,d=d_c}}{V} \right\|_{*,\sigma,d_c} = o_{c\to 0}\left(\frac{1}{c^2}\right).$$

With this estimate and by using the same computations as in the proof of Lemma 2.2.6, we show that

$$\left\| \partial_c Q_c + \left(\frac{1+o_{c\to 0}(1)}{c^2} \right) \partial_d (V_1(.-d\vec{e}_1) V_{-1}(.+d\vec{e}_1))_{|\,d=d_c} \right\|_p = o_{c\to 0}\left(\frac{1}{c^2}\right).$$

for all $+\infty \geqslant p > 2$ if c is small enough, which ends the proof of Theorem 1.3.1.

Proof. From subsection 2.3.5, we know that Q_c is a C^1 function of c. We have $Q_c = V + \Gamma_{c,d_c}$, hence

$$\partial_c Q_c = \partial_c V + \partial_c(\Gamma_{c,d_c}) = \frac{-1 + o_{c\to 0}(1)}{c^2}\partial_d V + \partial_c(\Gamma_{c,d_c}),$$

where we used $\partial_c V = \left(-\frac{1}{c^2} + o_{c\to 0}\left(\frac{1}{c^2}\right)\right)\partial_d V$ thanks to subsection 2.3.6. Γ_{c,d_c} depends on c directly and through d_c. We will write $\partial_c \Gamma_{c,d_c}$ for the derivatives with respect to c but at a fixed d_c, and $\partial_d \Gamma_{c,d_c}$ for the derivate with respect to d_c but at fixed c. In particular,

$$\partial_c(\Gamma_{c,d_c}) = \partial_c \Gamma_{c,d_c} + \partial_c d_c\, \partial_d \Gamma_{c,d_c}.$$

From Lemma 2.2.9 and (2.3.26), we showed that

$$\left\| \frac{\partial_c \Gamma_{c,d_c}}{V} \right\|_{*,\sigma,d_c} \leqslant K(\sigma,\sigma') c^{-\sigma'},$$

and from Lemma 2.2.3 with the definition of $\Gamma_{c,d}$, we show easily that

$$\left\| \frac{\partial_d \Gamma_{c,d_c}}{V} \right\|_{*,\sigma,d_c} \leqslant K(\sigma,\sigma') c^{1-\sigma'}.$$

Finally, from subsection 2.3.6, we have $\partial_c d_c = \frac{1 + o_{c\to 0}(1)}{c^2}$, therefore

$$\left\| \frac{\partial_c(\Gamma_{c,d_c})}{V} \right\|_{*,\sigma,d_c} \leqslant K(\sigma,\sigma')(c^{-\sigma'} + c^{-2}(1 + o_{c\to 0}(1))c^{1-\sigma'}) = o_{c\to 0}\left(\frac{1}{c^2}\right)$$

since $0 < \sigma < \sigma' < 1$. □

This concludes the proof of Lemma 2.3.6, which itself concludes the proof of Theorem 1.3.1.

Chapter 3
Coercivity and applications

This chapter is devoted to the proofs of the results in section 1.4. Section 3.1 is devoted to the proof of Proposition 1.4.1. We start by giving some estimates on the branch of travelling waves in subsection 3.1.1, we then show the equivalents when $c \to 0$ for the energy and momentum, as well as the relations between them and some specific values of the quadratic form in subsection 3.1.2. Finally, in subsection 3.1.3, we study the travelling wave near its zeros.

In section 3.2, we infer some properties of the space H_{Q_c}. First, we explain why we can not have a coercivity result in the energy norm in subsection 3.2.1, and we show the well posedness of several quantities in subsections 3.2.2 and 3.2.3. A density argument is given in subsection 3.2.4, that will be needed for the proof of Proposition 1.4.3.

Section 3.3 is devoted to the proofs of Propositions 1.4.2 and 1.4.3. We start by writing the quadratic form for test functions in a particular form (subsection 3.3.1), and we then show Proposition 1.4.2 and 1.4.3 respectively in subsections 3.3.2 and 3.3.4. To show Proposition 1.4.3, we use Proposition 1.4.2 and the fact that we know well the travelling wave near its zeros from subsection 3.1.3.

The next part, section 3.4, is devoted to the proof of Theorem 1.4.4 and its corollaries. We show the coercivity under four orthogonality conditions by showing that we can modify the initial function by a small amount to have the four orthogonality conditions of Proposition 1.4.3, and that the error commited is small in the coercivity norm. We then focus on the corollaries of Theorem 1.4.4 in subsection 3.4.5. We show there composition of the kernel of L_{Q_c} (Corollary 1.4.5), and the results in $H^1(\mathbb{R}^2)$: Corollary 1.4.6, Proposition 1.4.7 and Corollary 1.4.9.

The penultimate section (3.5) is devoted to the proofs of Proposition 1.4.10, 1.4.11 and Theorem 1.4.12. In subsection 3.5.1, we study the space $H^{\exp}_{Q_c}$, in particular we give a density argument, that allows us to finish the proof of Proposition 1.4.10. Then, in subsection 3.5.2, we compute how the additional orthogonality condition improves the coercivity norm, both in the symmetric and non symmetric case, and we can then show Proposition 1.4.11 and Theorem 1.4.12.

Section 3.6 is devoted to the proof of Theorem 1.4.13. We use here classical methods for the proof of local uniqueness, by modulating on the five parameters of the family, and using a coercivity result. One of the main point is to write the problem additively near the zeros of Q_c and multiplicatively far from them. The reason for that is that we do not know the link between the speed and the position of the zeros of a travelling wave in general, and we therefore cannot write a perturbation multiplicatively in the whole space. Because of that, we require here an orthogonality on the phase, and we cannot avoid it, as we did for instance the proof of Proposition 1.4.3 by choosing correctly the position of the vortices.

We will use many cutoffs in the proofs. As a rule of thumb, a function written as η, χ or $\tilde{\chi}$ will be smooth and have value 1 at infinity and 0 in some compact domain. The function η itself is reserved for B_{Q_c} and $B^{\exp}_{Q_c}$ (see equations (1.4.3) and (1.4.4)).

3.1 Properties of the branch of travelling waves

This section is devoted to the proof of Proposition 1.4.1. In subsection 3.1.1, we recall some estimates on Q_c defined in Theorem 1.3.1 from previous works ([7], [19], [25] and Chapter 2). In subsection 3.1.2, we compute some equalities and equivalents when $c \to 0$ on the energy, momentum and the four particular directions ($\partial_{x_1} Q_c$, $\partial_{x_2} Q_c$, $\partial_c Q_c$ and $\partial_{c^\perp} Q_c$). Finally, the properties of the zeros of Q_c are studied in subsection 3.1.3.

3.1.1 Decay estimates

3.1.1.1 Estimates on vortices

We recall that vortices are stationary solutions of (GP) of degrees $n \in \mathbb{Z}^*$ (see [7]):

$$V_n(x) = \rho_n(r)e^{in\theta},$$

where $x = re^{i\theta}$, solving

$$\begin{cases} \Delta V_n - (|V_n|^2 - 1)V_n = 0 \\ |V_n| \to 1 \text{ as } |x| \to \infty. \end{cases}$$

We regroup here estimates on quantities involving vortices. We also define, as in Chapter 2

$$V(.) = V_1(. - d_c\vec{e_1})V_{-1}(. + d_c\vec{e_1})$$

and

$$\partial_d V(.) = \partial_d(V_1(. - d\vec{e_1})V_{-1}(. + d\vec{e_1}))_{|d=d_c}.$$

We will also estimate

$$\partial_d^2 V := \partial_d^2(V_1(. - d\vec{e_1})V_{-1}(. + d\vec{e_1}))_{|d=d_c}.$$

The function $V(x) = V_1(x - d_c\vec{e_1})V_{-1}(x + d_c\vec{e_1})$ is close to $V_1(x - d_c\vec{e_1})$ in $B\big(d_c\vec{e_1}, 2d_c^{1/2}\big)$, since, from Lemma 1.2.1 and [7], we have, uniformly in $B\big(d_c\vec{e_1}, 2d_c^{1/2}\big)$,

$$V_{-1}(. + d_c\vec{e_1}) = 1 + O_{c\to 0}(c^{1/2}) \tag{3.1.1}$$

and

$$|\nabla V_{-1}(. + d_c\vec{e_1})| \leqslant \frac{o_{c\to 0}(c^{1/2})}{(1 + \tilde{r_1})}. \tag{3.1.2}$$

We recall that $B\big(d_c\vec{e_1}, 2d_c^{1/2}\big)$ is near the vortex of degree $+1$ of Q_c and that $\tilde{r} = \min(r_1, r_{-1})$, with $r_{\pm 1} = |x \mp d_c\vec{e_1}|$.

3.1.1.2 Estimates on Q_c from Chapter 2

We recall, for the function Q_c defined in Theorem 1.3.1, that

$$\forall (x_1, x_2) \in \mathbb{R}^2, Q_c(x_1, x_2) = \overline{Q_c(x_1, -x_2)} = Q_c(-x_1, x_2). \tag{3.1.3}$$

In particular, $\partial_c Q_c$ enjoys the same symmetries, since (3.1.3) holds for any $c > 0$ small enough. We recall that $Q_c \in C^\infty(\mathbb{R}^2, \mathbb{C})$ by standard elliptic regularity arguments.

Finally, we recall some estimates on Q_c and its derivatives, coming from Lemma 2.2.8 and equations (2.2.10), (2.2.12). We denote $\tilde{r} = \min(r_1, r_{-1})$, the minimum of the distances to $d_c\vec{e}_1$ and $-d_c\vec{e}_1$, and we recall that $V(x) = V_1(x - d_c\vec{e_1})V_{-1}(x + d_c\vec{e_1})$.

We write $Q_c = V + \Gamma_c$ or $Q_c = (1 - \eta)V\Psi_c + \eta V e^{\Psi_c}$, where $\Gamma_c = (1 - \eta)V\Psi_c + \eta V\,(e^{\Psi_c} - 1)$ (see equation (2.2.8)). There exists $K > 0$ and, for any $0 < \sigma < 1$, there exists $K(\sigma) > 0$ such that

$$|\Gamma_c| \leqslant \frac{K(\sigma)c^{1-\sigma}}{(1 + \tilde{r})^\sigma} \tag{3.1.4}$$

$$|\nabla\Gamma_c| \leqslant \frac{K(\sigma)c^{1-\sigma}}{(1 + \tilde{r})^{1+\sigma}}. \tag{3.1.5}$$

$$|1 - |Q_c|| \leqslant \frac{K(\sigma)}{(1 + \tilde{r})^{1+\sigma}}, \tag{3.1.6}$$

$$|Q_c - V| \leqslant \frac{K(\sigma)c^{1-\sigma}}{(1 + \tilde{r})^\sigma}, \tag{3.1.7}$$

$$||Q_c|^2 - |V|^2| \leqslant \frac{K(\sigma)c^{1-\sigma}}{(1 + \tilde{r})^{1+\sigma}}, \tag{3.1.8}$$

$$|\mathfrak{Re}(\nabla Q_c\overline{Q_c})| \leqslant \frac{K(\sigma)}{(1 + \tilde{r})^{2+\sigma}}, \tag{3.1.9}$$

$$|\mathfrak{Im}(\nabla Q_c\overline{Q_c})| \leqslant \frac{K}{1 + \tilde{r}} \tag{3.1.10}$$

and for $0<\sigma<\sigma'<1$, there exists $K(\sigma,\sigma')>0$ such that

$$|D^2\mathfrak{Im}(\Psi_c)|+|\nabla\mathfrak{Re}(\Psi_c)|+|\nabla^2\mathfrak{Re}(\Psi_c)|\leqslant\frac{K(\sigma,\sigma')c^{1-\sigma'}}{(1+\tilde{r})^{2+\sigma}}. \tag{3.1.11}$$

From Lemmas 1.2.1, with Theorem 1.3.1, we deduce in particular that for c small enough, there exist universal constants $K_1,K_2>0$ such that on $\mathbb{R}^2\backslash B(\pm d_c\vec{e}_1,1)$ we have

$$K_1\leqslant|Q_c|\leqslant K_2. \tag{3.1.12}$$

To these estimates, we add two additional lemmas. We write

$$\begin{array}{rcl}\|\psi\|_{\sigma,d_c} & := & \|V\psi\|_{C^1(\{\tilde{r}\leqslant 3\})}+\|\tilde{r}^{1+\sigma}\mathfrak{Re}(\psi)\|_{L^\infty(\{\tilde{r}\geqslant 2\})}+\|\tilde{r}^{2+\sigma}\nabla\mathfrak{Re}(\psi)\|_{L^\infty(\{\tilde{r}\geqslant 2\})}\\ & + & \|\tilde{r}^{\sigma}\mathfrak{Im}(\psi)\|_{L^\infty(\{\tilde{r}\geqslant 2\})}+\|\tilde{r}^{1+\sigma}\nabla\mathfrak{Im}(\psi)\|_{L^\infty(\{\tilde{r}\geqslant 2\})},\end{array}$$

where $\tilde{r}=\min(r_1,r_{-1})$, with

$$r_{\pm 1}=|x\mp d_c\vec{e}_1|, \tag{3.1.13}$$

and with d_c defined in Theorem 1.3.1. The first lemma is about Q_c and the second one about $\partial_c Q_c$.

Lemma 3.1.1. *For any $0<\sigma<1$, there exist $c_0(\sigma),K(\sigma)>0$ such that, for $0<c<c_0(\sigma)$ and Q_c defined in Theorem 1.3.1, if*

$$\Gamma_c=Q_c-V,$$

then

$$\left\|\frac{\Gamma_c}{V}\right\|_{\sigma,d_c}\leqslant K(\sigma)c^{1-\sigma}.$$

Proof. This estimate is a consequence of

$$\Gamma_c=(1-\eta)V\Psi_c+\eta V\left(e^{\Psi_c}-1\right)$$

and equation (2.2.14). □

Lemma 3.1.2. (Lemma 2.3.6) *There exists $1>\beta_0>0$ such that, for all $0<\sigma<\beta_0<\sigma'<1$, There exists $c_0(\sigma,\sigma')>0$ such that for any $0<c<c_0(\sigma,\sigma')$, Q_c defined in Theorem 1.3.1, $c\mapsto Q_c$ is a C^1 function from $]0,c_0(\sigma,\sigma')[$ to $C^1(\mathbb{R}^2,\mathbb{C})$, and*

$$\left\|\frac{\partial_c Q_c}{V}+\left(\frac{1+o_{c\to 0}(c^{1-\sigma'})}{c^2}\right)\frac{\partial_d V_{|d=d_c}}{V}\right\|_{\sigma,d_c}=o^{\sigma,\sigma'}_{c\to 0}\left(\frac{c^{1-\sigma'}}{c^2}\right).$$

These results are technical, but quite precise. They give both a decay in position and the size in c of the error term. The statement of Lemma 2.3.6 has $o_{c\to 0}(1)$ and $o_{c\to 0}\left(\frac{1}{c^2}\right)$ instead of respectively $o_{c\to 0}(c^{1-\sigma'})$ and $o_{c\to 0}\left(\frac{c^{1-\sigma'}}{c^2}\right)$, but its proof gives this better estimate (given that σ' is close enough to 1). We recall that $o^{\sigma,\sigma'}_{c\to 0}(1)$ is a quantity going to 0 when $c\to 0$ at fixed σ,σ'. We recall that $\partial_c\nabla Q_c=\nabla\partial_c Q_c$. We conclude this subsection with a link between the $\|.\|_\sigma$ norms and $\|.\|_{H_{Q_c}}$. We recall

$$\|\varphi\|^2_{H_{Q_c}}=\int_{\mathbb{R}^2}|\nabla\varphi|^2+|1-|Q_c|^2||\varphi|^2+\mathfrak{Re}^2(\overline{Q_c}\varphi).$$

Lemma 3.1.3. *There exists a universal constant $K>0$ (independent of c) such that, for Q_c defined in Theorem 1.3.1,*

$$\|h\|_{H_{Q_c}}\leqslant K\left\|\frac{h}{V}\right\|_{3/4,d_c}.$$

The value $\sigma=3/4$ is arbitrary here, this estimate holds for other values of σ.

Proof. We compute, using Lemma 1.2.1, that

$$\int_{\mathbb{R}^2}|\nabla h|^2\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}+\int_{\{\tilde{r}\geqslant 1\}}\left|\nabla\left(\frac{h}{V}V\right)\right|^2\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}+2\int_{\{\tilde{r}\geqslant 1\}}\left|\nabla\left(\frac{h}{V}\right)\right|^2+|\nabla V|^2\frac{|h|^2}{|V|^2}.$$

From Lemma 1.2.1 and the definition of $\|.\|_{3/4,d_c}$, we check that

$$2\int_{\{\tilde{r}\geqslant 1\}}\left|\nabla\left(\frac{h}{V}\right)\right|^2+|\nabla V|^2\frac{|h|^2}{|V|^2}\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}\int_{\{\tilde{r}\geqslant 1\}}\frac{1}{(1+\tilde{r})^{3+1/2}}\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}.$$

Furthermore, from equation (3.1.6) with $\sigma=1/2$, we have the estimate

$$\int_{\mathbb{R}^2}|1-|Q_c|^2||h|^2\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}\int_{\mathbb{R}^2}\frac{1}{(1+\tilde{r})^{9/4}}\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}.$$

Finally, we compute

$$\int_{\mathbb{R}^2}\mathfrak{Re}^2(\overline{Q_c}h)\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}+\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Re}^2(\overline{Q_c}h),$$

and

$$\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Re}^2(\overline{Q_c}h)=\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Re}^2\left(V\overline{Q_c}\frac{h}{V}\right)\leqslant 2\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Re}^2\left(\frac{h}{V}\right)\mathfrak{Re}^2(V\overline{Q_c})+\mathfrak{Im}^2\left(\frac{h}{V}\right)\mathfrak{Im}^2(V\overline{Q_c}).$$

With the definition of $\|.\|_{3/4,d_c}$, Lemmas 1.2.1 and 3.1.1, we check that

$$\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Re}^2\left(\frac{h}{V}\right)\mathfrak{Re}^2(V\overline{Q_c})\leqslant K\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Re}^2\left(\frac{h}{V}\right)\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}\int_{\{\tilde{r}\geqslant 1\}}\frac{1}{(1+\tilde{r})^{3+1/2}}\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}.$$

From Lemma 3.1.1 with $\sigma=1/2$, we check that, since $\mathfrak{Im}^2(V\overline{Q_c})=\mathfrak{Im}^2(V\overline{V+\Gamma_c})=\mathfrak{Im}^2(V\bar{\Gamma}_c)$, we have

$$\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Im}^2\left(\frac{h}{V}\right)\mathfrak{Im}^2(V\overline{Q_c})\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}\int_{\{\tilde{r}\geqslant 1\}}\frac{1}{(1+\tilde{r})^{2+1/2}}\leqslant K\left\|\frac{h}{V}\right\|^2_{3/4,d_c}.$$

Combining, these estimates, we end the proof of this lemma. □

3.1.1.3 Faraway estimates on Q_c

Since $E(Q_c)<+\infty$ thanks to Theorem 1.3.1, from Theorem 7 of [19], we have the following result.

Theorem 3.1.4. ([19], Theorem 7) *There exists a constant $C(c)>0$ (depending on c) such that, for Q_c defined in Theorem 1.3.1,*

$$|1-|Q_c|^2|\leqslant\frac{C(c)}{(1+r)^2},$$

$$|1-Q_c|\leqslant\frac{C(c)}{1+r},$$

$$|\nabla Q_c|\leqslant\frac{C(c)}{(1+r)^2}$$

and

$$|\nabla|Q_c||\leqslant\frac{C(c)}{(1+r)^3}.$$

Furthermore, such estimates hold for any travelling waves with finite energy (but then the constant $C(c)$ also depends on the travelling wave, and not only on its speed).

This result is crucial to show that some terms are well defined, since it gives better decay estimates in position than the estimates in subsection 3.1.1.2 (but with no smallness in c). Remark that $1-|Q_c|^2$ is not necessarily positive. In fact it is not at infinity (see [21]). In particular, the estimate

$$|1-|Q_c|^2|\geqslant\frac{C(c)}{1+r^2}$$

does not hold because of the possibility of $|Q_c|=1$. This happens, but only for few directions and it can be catched up. We show the following sufficient result, which is needed to show that some quantities we will use are well defined. Furthermore, in these estimates, the constant depends on c, and thus can not be used in error estimates (since the smallness of the errors there will depend on c).

Lemma 3.1.5. *There exists $c_0>0$ such that, for $0<c<c_0$, there exists $C(c)>0$ such that for $\varphi\in H_{Q_c}$ and the function Q_c defined in Theorem 1.3.1,*

$$\int_{\mathbb{R}^2}\frac{|\varphi|^2}{(1+|x|)^2}dx\leqslant C(c)\bigg(\int_{\mathbb{R}^2}|\nabla\varphi|^2+|1-|Q_c|^2||\varphi|^2\bigg).$$

Proof. From Propositions 5 and 7 of [21] (where $\eta=1-|Q_c|^2$), we have in our case, for $x=r\sigma\in\mathbb{R}^2$ with $r\in\mathbb{R}^+$, $|\sigma|=1$, $\sigma=(\sigma_1,\sigma_2)\in\mathbb{R}^2$, that

$$r^2(1-|Q_c|^2)(r\sigma)\to c\alpha(c)\left(\frac{1}{1-\frac{c^2}{2}+\frac{c^2\sigma_2^2}{2}}-\frac{2\sigma_2^2}{\left(1-\frac{c^2}{2}+\frac{c^2\sigma_2^2}{2}\right)^2}\right)$$

uniformly in $\sigma\in S^1$ when $r\to+\infty$, where $\alpha(c)>0$ depends on c and Q_c. Remark that our travelling wave is axisymmetric around axis x_2 (and not x_1 for which the results of [21] are given), hence the swap between σ_1 and σ_2 between the two papers. We have

$$\frac{1}{1-\frac{c^2}{2}+\frac{c^2\sigma_2^2}{2}}-\frac{2\sigma_2^2}{\left(1-\frac{c^2}{2}+\frac{c^2\sigma_2^2}{2}\right)^2}=\frac{1-\frac{c^2}{2}-\left(2-\frac{c^2}{2}\right)\sigma_2^2}{\left(1-\frac{c^2}{2}+\frac{c^2\sigma_2^2}{2}\right)^2},$$

this shows in particular that $|Q_c|=1$ when $r\gg\frac{1}{c}$ is possible only in cones around $\sin(\theta)=\sigma_2=\pm\sqrt{\frac{1-c^2/2}{2-c^2/2}}$. Therefore, for c small enough, for some $\gamma>0$ small and $R>0$ large (that may depend on c), we have

$$\int_{\mathbb{R}^2}|1-|Q_c|^2||\varphi|^2\geqslant K(c,\beta,R)\int_{\mathbb{R}^2\setminus(B(0,R)\cup D(\gamma))}\frac{|\varphi|^2}{(1+r)^2},$$

where $D(\gamma)=\left\{re^{i\theta}\in\mathbb{R}^2,\left|\sin(\theta)\pm\sqrt{\frac{1-c^2/2}{2-c^2/2}}\right|\leqslant\gamma\right\}$. We want to show that for $\varphi\in H_{Q_c}$,

$$\int_{D(\gamma)\cup(\mathbb{R}^2\setminus B(0,R))}\frac{|\varphi|^2}{(1+r)^2}\leqslant C(c,\gamma,R)\bigg(\int_{\mathbb{R}^2}|\nabla\varphi|^2+\int_{\mathbb{R}^2\setminus(B(0,R)\cup D(\gamma))}\frac{|\varphi|^2}{(1+r)^2}\bigg).$$

For θ_0 any of the four angles such that $\sin(\theta)\pm\sqrt{\frac{1-c^2/2}{2-c^2/2}}=0$, we fix $r>0$ and look at $\varphi(\theta)$ as a function of the angle only. We compute, for $\theta\in[\theta_0-2\beta,\theta_0+2\beta]$ ($\beta>0$ being a small constant depending on γ such that $\{x=re^{i\theta}\in\mathbb{R}^2,\theta\in[\theta_0+3\beta,\theta_0+\beta]\}\cap D(\gamma)=\emptyset$, and such that $D(\gamma)$ is included in the union of the $[\theta_0-\beta,\theta_0+\beta]$ for the four possible values of θ_0),

$$\varphi(\theta)=\varphi(2\beta+\theta)-\int_\theta^{2\beta+\theta}\partial_\theta\varphi(\Theta)d\Theta,$$

hence,

$$|\varphi(\theta)|\leqslant|\varphi(2\beta+\theta)|+\int_{\theta_0-\beta}^{\theta_0+3\beta}|\partial_\theta\varphi(\Theta)|d\Theta.$$

This implies that

$$|\varphi(\theta)|^2\leqslant 2|\varphi(2\beta+\theta)|^2+K\int_0^{2\pi}|\partial_\theta\varphi(\Theta)|^2d\Theta$$

by Cauchy-Schwarz, and, integrating between $\theta_0-\beta$ and $\theta_0+\beta$ yields

$$\int_{\theta_0-\beta}^{\theta_0+\beta}|\varphi(\theta)|^2d\theta\leqslant 2\int_{\theta_0+\beta}^{\theta_0+3\beta}|\varphi(\theta)|^2d\theta+K\int_0^{2\pi}|\partial_\theta\varphi(\theta)|^2d\theta.$$

Now multiplying by $\frac{r}{(1+r)^2}$ and integrating in r on $[R,+\infty[$, we infer

$$\begin{aligned}\int_{\theta-\theta_0\in[-\beta,\beta]}\int_{r\in[R,+\infty[}\frac{|\varphi|^2}{(1+r)^2}rdrd\theta &\leqslant 2\int_{\theta-\theta_0\in[\beta,3\beta]}\int_{r\in[R,+\infty[}\frac{|\varphi|^2}{(1+r)^2}rdrd\theta\\ &+ K(c,\beta,R)\int_{\mathbb{R}^2}|\nabla\varphi|^2dx\\ &\leqslant 2\int_{\mathbb{R}^2\setminus(B(0,R)\cup D(\gamma))}\frac{|\varphi|^2dx}{(1+|x|)^2}+K(c,\beta,R)\int_{\mathbb{R}^2}|\nabla\varphi|^2dx,\end{aligned}$$

using

$$\frac{|\partial_\theta\varphi|^2}{(1+r)^2}\leqslant\frac{|\partial_\theta\varphi|^2}{r^2}\leqslant|\nabla\varphi|^2.$$

Therefore,

$$\int_{D(\gamma)\cup(\mathbb{R}^2\setminus B(0,R))}\frac{|\varphi|^2}{(1+r)^2}\leqslant K\int_{\mathbb{R}^2\setminus(B(0,R)\cup D(\gamma))}\frac{|\varphi|^2}{(1+r)^2}dx+K(c,\beta,\gamma,R)\int_{\mathbb{R}^2}|\nabla\varphi|^2dx,$$

and thus

$$\int_{\mathbb{R}^2\setminus B(0,R)}\frac{|\varphi|^2}{(1+r)^2}\leqslant K(c,\beta,\gamma,R)\int_{\mathbb{R}^2}|\nabla\varphi|^2+|1-|Q_c|^2||\varphi|^2.$$

We are left with the proof of

$$\int_{B(0,R)}\frac{|\varphi|^2}{(1+r)^2}\leqslant K(c,\beta,R)\bigg(\int_{\mathbb{R}^2}|\nabla\varphi|^2+\int_{\mathbb{R}^2\setminus B(0,R)}\frac{|\varphi|^2}{(1+r)^2}\bigg).\tag{3.1.14}$$

We argue by contradiction. We suppose that there exists a sequence $\varphi_n\in H_{Q_c}$ such that $\int_{B(0,R)}\frac{|\varphi_n|^2}{(1+r)^2}=1$ and $\int_{\mathbb{R}^2}|\nabla\varphi_n|^2+\int_{\mathbb{R}^2\setminus B(0,R)}\frac{|\varphi_n|^2}{(1+r)^2}\to 0$. Since φ_n is bounded in $H^1(B(0,R+1))$, by Rellich's Theorem, up to a subsequence, we have the convergences $\varphi_n\to\varphi$ strongly in L^2 and weakly in H^1 to some function φ in $B(0,R+1)$. In particular $\int_{B(0,R+1)}|\nabla\varphi|^2=0$, hence φ is constant on $B(0,R+1)$, and with $\int_{B(0,R+1)\setminus B(0,R)}\frac{|\varphi|^2}{(1+r)^2}=0$ we have $\varphi=0$, which is in contradiction with $1=\int_{B(0,R)}\frac{|\varphi_n|^2}{(1+r)^2}\to\int_{B(0,R)}\frac{|\varphi|^2}{(1+r)^2}$ by $L^2(B(0,R+1))$ strong convergence. This concludes the proof of this lemma. □

3.1.2 Construction and properties of the four particular directions

3.1.2.1 Definitions

The four directions we want to study here are $\partial_{x_1}Q_c$, $\partial_{x_2}Q_c$, ∂_cQ_c and $\partial_{c^\perp}Q_c$. The first two are derivatives of Q_c with respect to the position, the third one is the derivative of Q_c with respect of the speed, and we have its first order term in Theorem 1.3.1. The fourth direction is defined in Lemma 3.1.6 below. The directions $\partial_{x_1}Q_c$ and $\partial_{x_2}Q_c$ correspond to the translations of the travelling wave, ∂_cQ_c and $\partial_{c^\perp}Q_c$ to changes respectively in the modulus and direction of its speed. These directions will also appear in the orthogonality conditions for some of the coercivity results.

Lemma 3.1.6. *Take $\vec{c}\in\mathbb{R}^2$ such that $|\vec{c}|<c_0$ for c_0 defined in Theorem 1.3.1. Define α such that $\vec{c}=|\vec{c}|R_\alpha(-\vec{e}_2)$, where $R_\theta\colon\mathbb{R}^2\to\mathbb{R}^2$ is the rotation of angle θ. Then, $Q_{\vec{c}}:=Q_{|\vec{c}|}\circ R_{-\alpha}$ solves*

$$\begin{cases}(\mathrm{TW}_{\vec{c}})(v)=i\,\vec{c}.\nabla v-\Delta v-(1-|v|^2)v=0\\ |v|\to 1\text{ as }|x|\to+\infty,\end{cases}$$

where $Q_{|\vec{c}|}$ is the solution of $(\mathrm{TW}_{|\vec{c}|})$ in Theorem 1.3.1. In particular, $Q_{\vec{c}}$ is a C^1 function of α and

$$\partial_\alpha Q_{\vec{c}}(x)=-R_{-\alpha}(x^\perp).\nabla Q_{|\vec{c}|}(R_{-\alpha}(x)).$$

Furthermore, at $\alpha=0$, the quantity

$$\partial_{c^\perp}Q_c:=(\partial_\alpha Q_{\vec{c}})_{|\,\alpha=0}$$

satisfies

$$\partial_{c^\perp}Q_c(x)=-x^\perp.\nabla Q_c(x),$$

is in $C^\infty(\mathbb{R}^2,\mathbb{C})$ and

$$L_{Q_c}(\partial_{c^\perp}Q_c)=-ic\partial_{x_1}Q_c.$$

Proof. Since the Laplacian operator is invariant by rotation, it is easy to check that $Q_{|\vec{c}|}\circ R_{-\alpha}$ solves $(\mathrm{TW}_{\vec{c}})(Q_{|\vec{c}|}\circ R_{-\alpha})=0$. The function $\theta\mapsto R_\theta$ is C^1, hence $(\alpha,x)\mapsto Q_{\vec{c}}(x)$ is a C^1 function, and we compute

$$(\partial_\alpha Q_{\vec{c}})(x)=\partial_\alpha(Q_{|\vec{c}|}\circ R_{-\alpha})(x)=\partial_\alpha(R_{-\alpha}(x)).\nabla Q_{|\vec{c}|}(R_{-\alpha}(x)).$$

We remark that

$$\partial_\alpha(R_{-\alpha}(x)) = -R_{-\alpha}(x^\perp),$$

where $x^\perp = (-x_2, x_1)$, hence

$$\partial_\alpha Q_{\vec{c}}(x) = -R_{-\alpha}(x^\perp).\nabla Q_{|\vec{c}|}(R_{-\alpha}(x)).$$

In particular, for $\alpha = 0$,

$$\partial_\alpha Q_{\vec{c}}(x)_{\,|\,\alpha=0} = -x^\perp.\nabla Q_c(x).$$

We recall that $Q_{\vec{c}}$ solves

$$i\,\vec{c}.\nabla Q_{\vec{c}} - \Delta Q_{\vec{c}} - (1-|Q_{\vec{c}}|^2)Q_{\vec{c}} = 0,$$

and when we differentiate this equation with respect to α (with $|\vec{c}| = c$), we have

$$-i\partial_\alpha\vec{c}.(\nabla Q_{\vec{c}}) + L_{Q_{\vec{c}}}(\partial_\alpha Q_{\vec{c}}) = 0.$$

At $\alpha = 0$, $Q_{\vec{c}} = Q_c$, $\partial_\alpha\vec{c} = -c\,\vec{e}_1$ and $\partial_\alpha Q_{\vec{c}\,|\,\alpha=0} = \partial_{c^\perp} Q_c$, therefore

$$L_{Q_c}(\partial_{c^\perp} Q_c) = -i\,c\partial_{x_1} Q_c. \qquad \square$$

3.1.2.2 Estimates on the four directions

We shall now show that the functions $\partial_{x_1}Q_c, \partial_{x_2}Q_c, \partial_c Q_c$ and $\partial_{c^\perp}Q_c$ are in the energy space and we will also compute their values through the linearized operator around Q_c, namely

$$L_{Q_c}(\varphi) = -\Delta\varphi - i\,c\partial_{x_2}\varphi - (1-|Q_c|^2)\varphi + 2\,\mathfrak{Re}(\overline{Q_c}\varphi)Q_c.$$

Lemma 3.1.7. *There exists $c_0 > 0$ such that, for $0 < c < c_0$, Q_c defined in Theorem 1.3.1, we have*

$$\partial_{x_1}Q_c, \partial_{x_2}Q_c, \partial_c Q_c, \partial_{c^\perp}Q_c \in H_{Q_c},$$

and

$$L_{Q_c}(\partial_{x_1}Q_c) = L_{Q_c}(\partial_{x_2}Q_c) = 0,$$
$$L_{Q_c}(\partial_c Q_c) = i\,\partial_{x_2}Q_c,$$
$$L_{Q_c}(\partial_{c^\perp}Q_c) = -i\,c\partial_{x_1}Q_c.$$

We could check that we also have $\partial_{x_1}Q_c, \partial_{x_2}Q_c \in H^1(\mathbb{R}^2)$ (see [21]), but we expect that $\partial_c Q_c$, $\partial_{c^\perp}Q_c \notin L^2(\mathbb{R}^2)$. For $\partial_{c^\perp}Q_c$, this can be shown with Lemma 3.1.6 and [21].

Proof. We have defined

$$\|\varphi\|^2_{H_{Q_c}} = \int_{\mathbb{R}^2} |\nabla\varphi|^2 + |1-|Q_c|^2||\varphi|^2 + \mathfrak{Re}^2(\overline{Q_c}\varphi).$$

For any of the four functions, since they are in $C^\infty(\mathbb{R}^2, \mathbb{C})$, the only possible problem for the integrability is at infinity.

Step 1. We have $\partial_{x_1}Q_c, \partial_{x_2}Q_c \in H_{Q_c}$.

From Lemma 1.2.1 and equation (3.1.11) (for $1 > \sigma' > \sigma = 3/4$), we have

$$\int_{\mathbb{R}^2} |\nabla\partial_{x_1}Q_c|^2 + \int_{\mathbb{R}^2} |\nabla\partial_{x_2}Q_c|^2 \leqslant \int_{\mathbb{R}^2} \frac{K(c,\sigma')}{(1+r)^{7/2}} < +\infty.$$

From Theorem 3.1.4, we have

$$\int_{\mathbb{R}^2} |1-|Q_c|^2||\nabla Q_c|^2 + \mathfrak{Re}^2(\overline{Q_c}\nabla Q_c) \leqslant \int_{\mathbb{R}^2} \frac{K(c)}{(1+r)^4} < +\infty,$$

hence $\partial_{x_1}Q_c, \partial_{x_2}Q_c \in H_{Q_c}$.

Step 2. We have $\partial_c Q_c \in H_{Q_c}$.

From Lemmas 3.1.2 and 3.1.3, we have that for $\sigma > 0$ small enough

$$\partial_c Q_c + \frac{1 + o^{\sigma}_{c\to 0}(c^{\sigma})}{c^2}\partial_d V_{|d=d_c} \in H_{Q_c},$$

therefore we just have to check that $\partial_d V_{|d=d_c} \in H_{Q_c}$, which is a direct consequence of Lemma 2.1.6.

Step 3. We have $\partial_{c^\perp} Q_c \in H_{Q_c}$.

From Lemma 3.1.6, we have $\partial_{c^\perp} Q_c = -x^\perp.\nabla Q_c$. With Theorem 3.1.4, Lemma 1.2.1 and equation (3.1.11), we check that

$$\int_{\mathbb{R}^2} |\nabla \partial_{c^\perp} Q_c|^2 + |(1-|Q_c|^2)||\partial_{c^\perp} Q_c|^2 < +\infty.$$

Now, from Lemma 1.2.1 and equation (3.1.6) (with $\sigma = 1/2$), we have

$$\int_{\mathbb{R}^2} \mathfrak{Re}^2(\overline{Q_c}\partial_{c^\perp} Q_c) \leqslant K\int_{\mathbb{R}^2}(1+r^2)\mathfrak{Re}^2(\nabla Q_c \overline{Q_c}) \leqslant K(c)\int_{\mathbb{R}^2}\frac{1}{(1+r)^3} < +\infty,$$

thus $\partial_{c^\perp} Q_c \in H_{Q_c}$.

Step 4. Computation of the linearized operator on $\partial_{x_1} Q_c, \partial_{x_2} Q_c, \partial_c Q_c, \partial_{c^\perp} Q_c$.

For the values in the linearized operator, since

$$-ic\partial_{x_2} Q_c - \Delta Q_c - (1-|Q_c|^2)Q_c = (\mathrm{TW}_c)(Q_c) = 0,$$

by differentiating it with respect to x_1 and x_2, we have

$$L_{Q_c}(\partial_{x_1} Q_c) = L_{Q_c}(\partial_{x_2} Q_c) = 0.$$

By differentiating it with respect to c, we have (we recall that $\partial_c Q_c \in C^\infty(\mathbb{R}^2, \mathbb{C})$)

$$-i\partial_{x_2} Q_c + L_{Q_c}(\partial_c Q_c) = 0.$$

Finally, the quantity $L_{Q_c}(\partial_{c^\perp} Q_c)$ is given by Lemma 3.1.6. □

The next two lemmas are additional estimates on the four directions that will be useful later on. They estimate in particular the dependence on c of $\|.\|_{\mathcal{C}}$ on these four directions.

Lemma 3.1.8. *There exists $K > 0$ a universal constant, independent of c, such that, for Q_c defined in Theorem 1.3.1,*

$$\|\partial_{x_1} Q_c\|_{\mathcal{C}} + \|\partial_{x_2} Q_c\|_{\mathcal{C}} + \|c^2 \partial_c Q_c\|_{\mathcal{C}} \leqslant K.$$

Furthermore, for any $1 > \beta > 0$,

$$\|c\partial_{c^\perp} Q_c\|_{\mathcal{C}} = o^{\beta}_{c\to 0}(c^{-\beta}).$$

Proof. We defined, for $\varphi = Q_c\psi \in H_{Q_c}$,

$$\|\varphi\|^2_{\mathcal{C}} = \int_{\mathbb{R}^2} |\nabla\psi|^2 |Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4.$$

We recall that, since $\varphi = Q_c\psi$,

$$\int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 = \int_{\mathbb{R}^2} |\nabla\varphi - \nabla Q_c\psi|^2|Q_c|^2 \leqslant K\int_{\mathbb{R}^2} |\nabla\varphi|^2|Q_c|^2 + |\nabla Q_c|^2|\varphi|^2 \tag{3.1.15}$$

Step 1. We have $\|\partial_{x_1} Q_c\|_{\mathcal{C}} + \|\partial_{x_2} Q_c\|_{\mathcal{C}} \leqslant K$.

From Lemmas 1.2.1 and 3.1.1 and equations (3.1.9) to (3.1.11), we have that, for $\tilde{r} = \min(r_1, r_{-1})$,

$$|\nabla Q_c| \leqslant \frac{K}{(1+\tilde{r})} \quad \text{and} \quad |\nabla^2 Q_c| \leqslant \frac{K}{(1+\tilde{r})^2}.$$

Therefore,

$$\int_{\mathbb{R}^2}|\nabla(\partial_{x_1}Q_c)|^2|Q_c|^2+|\nabla(\partial_{x_2}Q_c)|^2|Q_c|^2\leqslant K,$$

and we also have

$$\int_{\mathbb{R}^2}|\nabla Q_c|^2|\nabla Q_c|^2\leqslant K,$$

thus, with equation (3.1.15),

$$\int_{\mathbb{R}^2}\Big|\nabla\Big(\frac{\partial_{x_1}Q_c}{Q_c}\Big)\Big|^2|Q_c|^4+\int_{\mathbb{R}^2}\Big|\nabla\Big(\frac{\partial_{x_2}Q_c}{Q_c}\Big)\Big|^2|Q_c|^4\leqslant K.$$

By equation (3.1.9) (for $\sigma=1/4$), we have

$$\int_{\mathbb{R}^2}\mathfrak{Re}^2\Big(\frac{\nabla Q_c}{Q_c}\Big)|Q_c|^4\leqslant K\int_{\mathbb{R}^2}\mathfrak{Re}^2(\nabla Q_c\overline{Q_c})\leqslant K\int_{\mathbb{R}^2}\frac{1}{(1+\tilde r)^{5/2}}\leqslant K.$$

We conclude that $\|\partial_{x_1}Q_c\|_{\mathcal{C}}+\|\partial_{x_2}Q_c\|_{\mathcal{C}}\leqslant K$.

Step 2. We have $\|c^2\partial_cQ_c\|_{\mathcal{C}}\leqslant K$.

From Lemma 3.1.2, we have, writing $c^2\partial_cQ_c=(1+o_{c\to0}(1))\partial_dV_{|d=d_c}+h$, that $\Big\|\frac{h}{V}\Big\|_{\sigma,d_c}=o_{c\to0}(1)$. In particular if we show that $\|\partial_dV_{|d=d_c}\|_{\mathcal{C}}\leqslant K$ and $\|h\|_{\mathcal{C}}\leqslant K$, then $\|c^2\partial_cQ_c\|_{\mathcal{C}}\leqslant K$. From Lemma 2.1.6, we check directly that

$$\int_{\mathbb{R}^2}|\nabla\partial_dV_{|d=d_c}|^2+\frac{|\partial_dV_{|d=d_c}|^2}{(1+\tilde r)^{3/2}}+\mathfrak{Re}^2(V\partial_dV_{|d=d_c})\leqslant K.$$

In particular, with (3.1.15), it implies that

$$\int_{\mathbb{R}^2}\Big|\nabla\Big(\frac{\partial_dV_{|d=d_c}}{Q_c}\Big)\Big|^2|Q_c|^4\leqslant K$$

and we estimate

$$\int_{\mathbb{R}^2}\mathfrak{Re}^2\Big(\frac{\partial_dV_{|d=d_c}}{Q_c}\Big)|Q_c|^4\leqslant K\int_{\mathbb{R}^2}\mathfrak{Re}^2(\bar V\partial_dV_{|d=d_c})+|V-Q_c|^2|\partial_dV_{|d=d_c}|^2\leqslant K$$

with the same arguments and equation (3.1.7). Similarly,

$$\int_{\mathbb{R}^2}\Big|\nabla\frac{\partial_dV_{|d=d_c}}{Q_c}\Big|^2|Q_c|^4\leqslant 2\int_{\mathbb{R}^2}|\nabla\partial_dV_{|d=d_c}|^2|Q_c|^2+|\nabla Q_c\partial_dV_{|d=d_c}|^2\leqslant K,$$

therefore $\|\partial_dV_{|d=d_c}\|_{\mathcal{C}}\leqslant K$. We now have to estimate $\|h\|_{\mathcal{C}}$. The computations are similar, since we check easily that

$$\int_{\mathbb{R}^2}|\nabla h|^2+|\nabla Q_c|^2|h|^2\leqslant K\Big\|\frac{h}{V}\Big\|^2_{3/4,d_c}$$

and

$$\int_{\mathbb{R}^2}\mathfrak{Re}^2(\bar Q_ch)\leqslant K\int_{\mathbb{R}^2}\mathfrak{Re}^2(\bar Vh)+|V-Q_c|^2|h|^2\leqslant K\Big\|\frac{h}{V}\Big\|^2_{3/4,d_c}.$$

Step 3. We have $\|c\partial_{c^\perp}Q_c\|_{\mathcal{C}}=o^{\beta}_{c\to0}(c^{-\beta})$.

By definition, $c\partial_{c^\perp}Q_c=-cx^\perp.\nabla Q_c(x)$, and we check by triangular inequality that $c|x^\perp|\leqslant K(1+\tilde r)$ since $\tilde r=\min(|x-\tilde d_c\vec e_1|,|x+\tilde d_c\vec e_1|)$ and $c\,\tilde d_c\to1$. Therefore,

$$\int_{\mathbb{R}^2}|\nabla(c\partial_{c^\perp}Q_c)|^2\leqslant c^2\int_{\mathbb{R}^2}|\nabla Q_c|^2+\int_{\mathbb{R}^2}(c|x^\perp|)^2|\nabla^2Q_c|^2\leqslant K\Big(1+\int_{\mathbb{R}^2}|\nabla^2Q_c|^2(1+\tilde r)^2\Big).$$

We have $|\nabla^2Q_c|\leqslant|\nabla^2V|+|\nabla^2\Gamma_c|$, and with equation (3.1.11), we check that $\int_{\mathbb{R}^2}|\nabla^2\Gamma_c|^2(1+\tilde r)^2\leqslant K$. With computations similar to the ones of Lemmas 2.1.3 and 1.2.1, we can show that

$$|\nabla^2V|\leqslant\frac{K}{(1+\tilde r)^2}\quad\text{and}\quad|\nabla^2V|\leqslant\frac{K}{c\,(1+\tilde r)^3},$$

therefore, for any $1>\beta>0$,

$$|\nabla^2 V| \leqslant \frac{Kc^{-\beta}}{(1+\tilde{r})^{2+\beta}},$$

and thus, by (3.1.15),

$$\int_{\mathbb{R}^2}\left|\nabla\left(\frac{c\partial_{c^\perp}Q_c}{Q_c}\right)\right|^2|Q_c|^4 \leqslant K\int_{\mathbb{R}^2}|\nabla c\partial_{c^\perp}Q_c|^2|Q_c|^2+|\nabla Q_c|^2|c\partial_{c^\perp}Q_c|^2 \leqslant K(\beta)c^{-2\beta}.$$

Furthermore, by equations (3.1.9) (for $\sigma=1/2$) and (3.1.12), we have

$$\int_{\mathbb{R}^2}\mathfrak{Re}^2\left(\frac{cx^\perp.\nabla Q_c(x)}{Q_c}\right)|Q_c|^4 \leqslant K\int_{\mathbb{R}^2}(1+\tilde{r})^2\mathfrak{Re}^2(\nabla Q_c\overline{Q_c}) \leqslant K\int_{\mathbb{R}^2}\frac{1}{(1+\tilde{r})^3} \leqslant K.$$

We conclude that $\|c\partial_{c^\perp}Q_c\|_{\mathcal{C}} = o^{\beta}_{c\to 0}(c^{-\beta})$. □

3.1.2.3 Link with the energy and momentum and computations of equivalents

In this subsection, we compute the value of the four previous particular direction $\partial_{x_1}Q_c$, $\partial_{x_2}Q_c$, $\partial_c Q_c, \partial_{c^\perp}Q_c$ on the quadratic form. In particular, we shall show that one of them is negative.

Lemma 3.1.9. *There exists $c_0>0$ such that for $0<c<c_0$, and for Q_c defined in Theorem 1.3.1, for $A\in\{\partial_{x_1}Q_c,\partial_{x_2}Q_c,\partial_c Q_c,\partial_{c^\perp}Q_c\}$, $\mathfrak{Re}(L_{Q_c}(A)\bar{A})\in L^1(\mathbb{R}^2)$ and*

$$\langle L_{Q_c}(\partial_{x_1}Q_c),\partial_{x_1}Q_c\rangle = \langle L_{Q_c}(\partial_{x_2}Q_c),\partial_{x_2}Q_c\rangle = 0,$$

$$\langle L_{Q_c}(\partial_c Q_c),\partial_c Q_c\rangle = \frac{-2\pi+o_{c\to 0}(1)}{c^2},$$

$$\langle L_{Q_c}(\partial_{c^\perp}Q_c),\partial_{c^\perp}Q_c\rangle = 2\pi+o_{c\to 0}(1).$$

Proof. For $A\in\{\partial_{x_1}Q_c,\partial_{x_2}Q_c,\partial_c Q_c,\partial_{c^\perp}Q_c\}$, we recall from Lemma 3.1.7 that $A\in H_{Q_c}$. To show that $\mathfrak{Re}(L_{Q_c}(A)\bar{A})\in L^1(\mathbb{R}^2)$, we need to show that

$$-\mathfrak{Re}(\Delta A\bar{A})-\mathfrak{Re}(ic\partial_{x_2}A\bar{A})-(1-|Q_c|^2)|A|^2+2\mathfrak{Re}^2(\overline{Q_c}A)\in L^1(\mathbb{R}^2).$$

For that, we check that, for some $\sigma>1/2$,

$$\begin{array}{rl} & \|(1+r)^\sigma A\|_{L^\infty(\mathbb{R}^2)}+\|(1+r)^{1+\sigma}(|\nabla A|+|\mathfrak{Re}(A)|)\|_{L^\infty(\mathbb{R}^2)} \\ + & \|(1+r)^{2+\sigma}\mathfrak{Im}(\Delta A)\|_{L^\infty(\mathbb{R}^2)}+\|(1+r)^{1+\sigma}\mathfrak{Re}(\Delta A)\|_{L^\infty(\mathbb{R}^2)} \\ < & +\infty. \end{array} \tag{3.1.16}$$

For $\partial_{x_1}Q_c$ and $\partial_{x_2}Q_c$, this follows from Theorem 3.1.4, and, since $L_{Q_c}(\partial_{x_{1,2}}Q_c)=0$, from

$$\Delta(\partial_{x_{1,2}}Q_c) = -ic\partial^2_{x_2x_{1,2}}Q_c-(1-|Q_c|^2)\partial_{x_{1,2}}Q_c+2\mathfrak{Re}(\overline{Q_c}\partial_{x_{1,2}}Q_c)Q_c,$$

which allows to estimate $\Delta(\partial_{x_{1,2}}Q_c)$ with Theorem 3.1.4, Lemma 1.2.1 and equation (3.1.11) for any $\sigma>1/2$.

Now, for $\partial_c Q_c$, the estimates not on its Laplacian are a consequence of Lemma 3.1.2, Theorem 3.1.4 and Lemma 2.1.6. Then, from Lemma 3.1.7, we have $L_{Q_c}(\partial_c Q_c)=i\partial_{x_2}Q_c$, thus

$$\Delta(\partial_c Q_c) = -i\partial_{x_2}Q_c-ic\partial_{x_2}\partial_c Q_c-(1-|Q_c|^2)\partial_c Q_c+2\mathfrak{Re}(\overline{Q_c}\partial_c Q_c)Q_c.$$

By Theorem 3.1.4 and Lemma 3.1.2, we have, for any $\sigma>1/2$,

$$|(1-|Q_c|^2)\partial_c Q_c|+|2\mathfrak{Re}(\overline{Q_c}\partial_c Q_c)Q_c| \leqslant \frac{K(c,\sigma)}{(1+r)^{2+\sigma}},$$

$$|\partial_{x_2}Q_c|+|\partial_{x_2}\partial_c Q_c| \leqslant \frac{K(c,\sigma)}{(1+r)^{1+\sigma}}$$

and

$$|\mathfrak{Re}(\partial_{x_2}Q_c)|+|\mathfrak{Re}(\partial_{x_2}\partial_c Q_c)| \leqslant \frac{K(c,\sigma)}{(1+r)^{2+\sigma}},$$

which is enough to show the estimates for $\partial_c Q_c$.

Finally, from Lemma 3.1.6 we recall that

$$\partial_{c^\perp} Q_c = -x^\perp.\nabla Q_c(x)$$

and

$$L_{Q_c}(\partial_{c^\perp} Q_c) = -ic\partial_{x_1} Q_c.$$

Similarly, the estimates not on its Laplacian follow from Theorem 3.1.4, Lemmas 1.2.1 and 3.1.1 and equation (3.1.11). We also have

$$\Delta(\partial_{c^\perp} Q_c) = ic\partial_{x_1} Q_c - ic\partial_{x_2}\partial_{c^\perp} Q_c - (1-|Q_c|^2)\partial_{c^\perp} Q_c + 2\mathfrak{Re}(\overline{Q_c}\partial_{c^\perp} Q_c)Q_c,$$

and with the same previous estimates, we conclude that $\partial_{c^\perp} Q_c$ satisfies the required estimates. With the definition $\|.\|_{H_{Q_c}}$, we check that the last two terms are in $L^1(\mathbb{R}^2)$, and for the first two, the integrands are in $L^1(\mathbb{R}^2,\mathbb{R})$ by estimates in subsections 3.1.1.1 and (3.1.16).

Step 1. We have $\langle L_{Q_c}(\partial_{x_1} Q_c), \partial_{x_1} Q_c\rangle = \langle L_{Q_c}(\partial_{x_2} Q_c), \partial_{x_2} Q_c\rangle = 0$.

From Lemma 3.1.7, we have $L_{Q_c}(\partial_{x_1} Q_c) = L_{Q_c}(\partial_{x_2} Q_c) = 0$, hence

$$\langle L_{Q_c}(\partial_{x_1} Q_c), \partial_{x_1} Q_c\rangle = \langle L_{Q_c}(\partial_{x_2} Q_c), \partial_{x_2} Q_c\rangle = 0.$$

Step 2. We have $\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle = \frac{-2\pi + o_{c\to 0}(1)}{c^2}$.

From Lemma 3.1.7, we have

$$L_{Q_c}(\partial_c Q_c) = i\partial_{x_2} Q_c,$$

therefore

$$\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle = \langle i\partial_{x_2} Q_c, \partial_c Q_c\rangle. \tag{3.1.17}$$

From Lemma 3.1.2, we can write $\partial_c Q_c = -\Big(\frac{1+o_{c\to 0}(1)}{c^2}\Big)\partial_d V_{|d=d_c} + h$ with $\Big\|\frac{h}{V}\Big\|_{\sigma,d_c} = o_{c\to 0}\big(\frac{1}{c^2}\big)$. Similarly, from Lemma 3.1.1, we write $Q_c = V + \Gamma_c$ with $\Big\|\frac{\Gamma_c}{V}\Big\|_{\sigma,d_c} = o_{c\to 0}(1)$, and we compute

$$\begin{aligned}\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle &= \left\langle i\,\partial_{x_2} V, -\left(\frac{1+o_{c\to 0}(1)}{c^2}\right)\partial_d V_{|d=d_c}\right\rangle + \langle i\,\partial_{x_2} V, h\rangle \\ &+ \left\langle i\,\partial_{x_2}\Gamma_c, -\left(\frac{1+o_{c\to 0}(1)}{c^2}\right)\partial_d V_{|d=d_c}\right\rangle + \langle i\,\partial_{x_2}\Gamma_c, h\rangle.\end{aligned} \tag{3.1.18}$$

By symmetry in x_1 of V, we compute

$$\langle i\partial_{x_2} V, \partial_d V_{|d=d_c}\rangle = -2\langle i\partial_{x_2} V_1 V_{-1}, \partial_{x_1} V_1 V_{-1}\rangle + 2\langle i\partial_{x_2} V_1 V_{-1}, \partial_{x_1} V_{-1} V_1\rangle.$$

In equation (2.1.43), we computed

$$\langle i\partial_{x_2} V_1 V_{-1}, \partial_{x_1} V_1 V_{-1}\rangle = -\pi + o_{c\to 0}(1).$$

Furthermore,

$$|\langle i\partial_{x_2} V_1 V_{-1}, \partial_{x_1} V_{-1} V_1\rangle| = \left|\int_{\mathbb{R}^2} \mathfrak{Re}(i\partial_{x_2} V_1 \bar{V}_1\, \overline{\partial_{x_1} V_{-1}} V_{-1})\right| \leqslant$$

$$\left|\int_{\mathbb{R}^2} \mathfrak{Re}(\partial_{x_2} V_1 \bar{V}_1)\mathfrak{Im}(\overline{\partial_{x_1} V_{-1}} V_{-1})\right| + \left|\int_{\mathbb{R}^2} \mathfrak{Im}(\partial_{x_2} V_1 \bar{V}_1)\mathfrak{Re}(\overline{\partial_{x_1} V_{-1}} V_{-1})\right|.$$

From Lemma 1.2.1, we have the estimates

$$|\mathfrak{Re}(\partial_{x_2} V_{-1}\overline{V_{-1}})| \leqslant \frac{K}{(1+r_{-1})^3} \quad \text{and} \quad |\mathfrak{Re}(\overline{\partial_{x_1} V_1} V_1)| \leqslant \frac{K}{(1+r_1)^3},$$

as well as

$$|\mathfrak{Im}(\partial_{x_2} V_{-1}\overline{V_{-1}})| \leqslant \frac{K}{1+r_{-1}} \quad \text{and} \quad |\mathfrak{Im}(\overline{\partial_{x_1} V_1} V_1)| \leqslant \frac{K}{1+r_1}.$$

We deduce, in the right half-plane, where $r_{-1} \geqslant d_c$, that $|\mathfrak{Im}(\nabla V_{-1}\overline{V_{-1}})| = o_{c\to 0}(1)$ and thus

$$\left|\int_{\{x_1\geqslant 0\}} \mathfrak{Re}(\partial_{x_2}V_1\bar{V}_1)\mathfrak{Im}(\overline{\partial_{x_1}V_{-1}}V_{-1})\right| \leqslant o_{c\to 0}(1)\int_{\{x_1\geqslant 0\}}\frac{1}{(1+r_1)^3} = o_{c\to 0}(1).$$

In the left half-plane, we have $\frac{1}{1+r_1}\leqslant \frac{K}{1+r_{-1}}$ and $\frac{1}{1+r_1} = o_{c\to 0}(1)$, therefore

$$\left|\int_{\{x_1\leqslant 0\}} \mathfrak{Re}(\partial_{x_2}V_1\bar{V}_1)\mathfrak{Im}(\overline{\partial_{x_1}V_{-1}}V_{-1})\right| \leqslant o_{c\to 0}(1)\int_{\{x_1\leqslant 0\}}\frac{1}{(1+r_{-1})^3} = o_{c\to 0}(1).$$

We therefore have

$$\left|\int_{\mathbb{R}^2} \mathfrak{Re}(\partial_{x_2}V_1\bar{V}_1)\mathfrak{Im}(\overline{\partial_{x_1}V_{-1}}V_{-1})\right| = o_{c\to 0}(1),$$

and by similar estimates,

$$\left|\int_{\mathbb{R}^2} \mathfrak{Im}(\partial_{x_2}V_1\bar{V}_1)\mathfrak{Re}(\overline{\partial_{x_1}V_{-1}}V_{-1})\right| = o_{c\to 0}(1).$$

We can thus conclude that $\langle i\partial_{x_2}V_1V_{-1}, \partial_{x_1}V_{-1}V_1\rangle = o_{c\to 0}(1)$. Therefore,

$$\left(\frac{1+o_{c\to 0}(1)}{c^2}\right)\langle i\,\partial_{x_2}V, -\partial_d V_{|\,d=d_c}\rangle = \frac{-2\pi}{c^2} + o\left(\frac{1}{c^2}\right). \tag{3.1.19}$$

Now, we estimate

$$\begin{aligned}
|\langle i\,\partial_{x_2}V, h\rangle| &= \left|\int_{\mathbb{R}^2}\mathfrak{Re}(i\partial_{x_2}V\bar{h})\right| \\
&\leqslant o_{c\to 0}(1) + \left|\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Re}(i\partial_{x_2}V\bar{h})\right| \\
&\leqslant o_{c\to 0}(1) + \left|\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Re}\left(i\partial_{x_2}V\overline{\left(\frac{h}{V}\right)}\right)\right|
\end{aligned}$$

because $\|h\|_{L^\infty} = o_{c\to 0}(1)$ and $|\partial_{x_2}V|$ is bounded near $\tilde{d}_c$ by a universal constant. Furthermore,

$$\left|\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Re}\left(i\partial_{x_2}V\overline{\left(\frac{h}{V}\right)}\right)\right| \leqslant \left|\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Re}(\partial_{x_2}V\bar{V})\mathfrak{Im}\left(\frac{h}{V}\right)\right| + \left|\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Im}(\partial_{x_2}V\bar{V})\mathfrak{Re}\left(\frac{h}{V}\right)\right|.$$

From Lemmas 1.2.1 and 3.1.2 (taking $\sigma = 1/2$), we have

$$\left|\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Re}(\partial_{x_2}V\bar{V})\mathfrak{Im}\left(\frac{h}{V}\right)\right| \leqslant K\left\|\frac{h}{V}\right\|_{1/2,d_c}\int_{\{\tilde{r}\geqslant 1\}}\frac{1}{(1+\tilde{r})^{3+1/2}} = o_{c\to 0}\left(\frac{1}{c^2}\right)$$

and

$$\left|\int_{\{\tilde{r}\geqslant 1\}}\mathfrak{Im}(\partial_{x_2}V\bar{V})\mathfrak{Re}\left(\frac{h}{V}\right)\right| \leqslant K\left\|\frac{h}{V}\right\|_{1/2,d_c}\int_{\{\tilde{r}\geqslant 1\}}\frac{1}{(1+\tilde{r})^{2+1/2}} = o_{c\to 0}\left(\frac{1}{c^2}\right),$$

therefore

$$|\langle i\,\partial_{x_2}V, h\rangle| = o_{c\to 0}\left(\frac{1}{c^2}\right). \tag{3.1.20}$$

Now, by Lemmas 1.2.1 and 3.1.1 (taking $\sigma = 1/2$), we have

$$\left(\frac{1+o_{c\to 0}(1)}{c^2}\right)|\langle i\,\partial_{x_2}\Gamma_c, \partial_d V_{|\,d=d_c}\rangle| \leqslant \frac{K}{c^2}\left\|\frac{\Gamma_c}{V}\right\|_{1/2,d_c}\int_{\mathbb{R}^2}\frac{1}{(1+\tilde{r})^{2+1/2}} = o_{c\to 0}\left(\frac{1}{c^2}\right). \tag{3.1.21}$$

Finally, by Lemmas 3.1.1 and 3.1.2, we check easily that

$$|\langle i\,\partial_{x_2}\Gamma_c, h\rangle| \leqslant K\left\|\frac{\Gamma_c}{V}\right\|_{3/4,d_c}\left\|\frac{h}{V}\right\|_{1/2,d_c}\int_{\mathbb{R}^2}\frac{1}{(1+\tilde{r})^{2+1/4}} = o_{c\to 0}\left(\frac{1}{c^2}\right). \tag{3.1.22}$$

Combining (3.1.19) to (3.1.22) in (3.1.18), we conclude that

$$\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle = \frac{-2\pi + o_{c\to 0}(1)}{c^2}.$$

Step 3. We have $\langle L_{Q_c}(\partial_{c^\perp} Q_c), \partial_{c^\perp} Q_c\rangle = 2\pi + o_{c\to 0}(1)$.

From Lemma 3.1.7, we have $L_{Q_c}(\partial_{c^\perp} Q_c) = -ic\partial_{x_1} Q_c$ and from Lemma 3.1.6, we have $\partial_{c^\perp} Q_c = -x^\perp.\nabla Q_c$. Therefore,

$$\langle L_{Q_c}(\partial_{c^\perp} Q_c), \partial_{c^\perp} Q_c\rangle = c\langle i\partial_{x_1} Q_c, x^\perp.\nabla Q_c\rangle.$$

We have

$$\langle i\partial_{x_1} Q_c, -x_2\,\partial_{x_1} Q_c\rangle = -\int_{\mathbb{R}^2} \mathfrak{Re}(ix_2|\partial_{x_1} Q_c|^2) = 0,$$

hence

$$\langle L_{Q_c}(\partial_{c^\perp} Q_c), \partial_{c^\perp} Q_c\rangle = c\langle i\partial_{x_1} Q_c, x_1\partial_{x_2} Q_c\rangle. \tag{3.1.23}$$

From Lemma 3.1.1, we write $Q_c = V + \Gamma_c$ with $\left\|\frac{\Gamma_c}{V}\right\|_{\sigma,d_c} \leqslant K(\sigma)c^{1-\sigma}$ for any $0<\sigma<1$, and we compute

$$\langle i\partial_{x_1} Q_c, x_1\partial_{x_2} Q_c\rangle = \langle i\partial_{x_1} V, x_1\partial_{x_2} V\rangle + \langle i\partial_{x_1} V, x_1\partial_{x_2}\Gamma_c\rangle + \langle i\partial_{x_1}\Gamma_c, x_1\partial_{x_2} V\rangle + \langle i\partial_{x_1}\Gamma_c, x_1\partial_{x_2}\Gamma_c\rangle.$$

We write $x_1 = d_c + y_1$, therefore

$$\langle i\partial_{x_1} V, x_1\partial_{x_2} V\rangle = d_c\langle i\partial_{x_1} V, \partial_{x_2} V\rangle + \langle i\partial_{x_1} V, y_1\partial_{x_2} V\rangle.$$

We have

$$\begin{aligned}\langle i\partial_{x_1} V, \partial_{x_2} V\rangle &= \langle i\partial_{x_1} V_1V_{-1}, \partial_{x_2} V_1V_{-1}\rangle + \langle i\partial_{x_1} V_{-1}V_1, \partial_{x_2} V_{-1}V_1\rangle\\ &+ \langle i\partial_{x_1} V_1V_{-1}, \partial_{x_2} V_{-1}V_1\rangle + \langle i\partial_{x_1} V_{-1}V_1, \partial_{x_2} V_1V_{-1}\rangle,\end{aligned}$$

and, from the previous step and by symmetry, we have

$$\langle i\partial_{x_1} V_1V_{-1}, \partial_{x_2} V_1V_{-1}\rangle = \langle i\partial_{x_1} V_{-1}V_1, \partial_{x_2} V_{-1}V_1\rangle = \pi + o_{c\to 0}(1)$$

and

$$|\langle i\partial_{x_1} V_1V_{-1}, \partial_{x_2} V_{-1}V_1\rangle| + |\langle i\partial_{x_1} V_{-1}V_1, \partial_{x_2} V_1V_{-1}\rangle| = o_{c\to 0}(1),$$

thus

$$\langle i\partial_{x_1} V, \partial_{x_2} V\rangle = 2\pi + o_{c\to 0}(1).$$

With $V_{\pm 1}$ centered around $\pm d_c\vec{e_1}$, we write $V = V_1V_{-1}$ and we compute

$$\begin{aligned}\langle i\partial_{x_1} V, y_1\partial_{x_2} V\rangle &= \int_{\mathbb{R}^2} \mathfrak{Re}(iy_1\partial_{x_1} V_1\overline{\partial_{x_2} V_1}|V_{-1}|^2 + iy_1\partial_{x_1} V_{-1}\overline{\partial_{x_2} V_{-1}}|V_1|^2)\\ &+ \int_{\mathbb{R}^2} \mathfrak{Re}\big(iy_1\partial_{x_1} V_1\bar{V}_1\overline{V_{-1}}\partial_{x_2} V_{-1} + iy_1\partial_{x_1} V_{-1}\overline{V_{-1}}\bar{V}_1\partial_{x_2} V_1\big).\end{aligned}$$

By decomposition in polar coordinates, with the notation of (3.1.13) and Lemma 1.2.1, we compute

$$\int_{\mathbb{R}^2} \mathfrak{Re}(iy_1\partial_{x_1} V_1\overline{\partial_{x_2} V_1}|V_{-1}|^2) = \int_0^{+\infty}\int_0^{2\pi} |V_{-1}|^2\rho_1(r_1)\rho_1'(r_1)\cos(\theta_1)\, r_1\, dr_1 d\theta_1.$$

By integration in polar coordinates, we check that

$$\int_0^{+\infty}\int_0^{2\pi} \rho_1(r_1)\rho_1'(r_1)\cos(\theta_1) = 0,$$

hence

$$\int_{\mathbb{R}^2} \mathfrak{Re}(iy_1\partial_{x_1} V_1\overline{\partial_{x_2} V_1}|V_{-1}|^2) = \int_{\mathbb{R}^2} (1-|V_{-1}|^2)\mathfrak{Re}(iy_1\partial_{x_1} V_1\overline{\partial_{x_2} V_1}).$$

In particular, since, from Lemma 1.2.1, we have

$$(1-|V_{-1}|^2) \leqslant \frac{K}{(1+r_{-1})^2}$$

and

$$|\rho_1'(r_1)| \leqslant \frac{K}{(1+r_1)^3},$$

we can deduce that

$$\int_{\mathbb{R}^2} \mathfrak{Re}(iy_1\partial_{x_1}V_1\overline{\partial_{x_2}V_1}|V_{-1}|^2) = o_{c\to 0}(1)$$

and, similarly,

$$\int_{\mathbb{R}^2} \mathfrak{Re}(iy_1\partial_{x_1}V_{-1}\overline{\partial_{x_2}V_{-1}}|V_1|^2) = o_{c\to 0}(1).$$

Therefore, we conclude that

$$\langle i\partial_{x_1}V, x_1\partial_{x_2}V\rangle = (2\pi + o_{c\to 0}(1))\tilde{d}_c = \frac{2\pi + o_{c\to 0}(1)}{c}.$$

Now, we want to show that

$$|\langle i\partial_{x_1}V, x_1\partial_{x_2}\Gamma_c\rangle| + |\langle i\partial_{x_1}\Gamma_c, x_1\partial_{x_2}V\rangle| + |\langle i\partial_{x_1}\Gamma_c, x_1\partial_{x_2}\Gamma_c\rangle| = o_{c\to 0}\left(\frac{1}{c}\right),$$

which is enough to end the proof of this step.

By triangular inequality, we have $|x_1| \leqslant \frac{K(1+\tilde{r})}{c}$, and with Lemmas 1.2.1 and 3.1.1 (for $\sigma = 1/2$), we estimate

$$\begin{aligned}
|\langle i\partial_{x_1}V, x_1\partial_{x_2}\Gamma_c\rangle| &= \left|\int_{\mathbb{R}^2} x_1\mathfrak{Re}(\partial_{x_1}V\bar{V})\mathfrak{Im}\big(\overline{\partial_{x_2}\Gamma_c\bar{V}}\big)\right| + \left|\int_{\mathbb{R}^2} x_1\mathfrak{Im}(\partial_{x_1}V\bar{V})\mathfrak{Re}\big(\overline{\partial_{x_2}\Gamma_c\bar{V}}\big)\right| \\
&\leqslant \frac{K}{c}\left(\int_{\mathbb{R}^2}\frac{(1+\tilde{r})}{(1+\tilde{r})^3}\times\frac{c^{1/2}}{(1+\tilde{r})^{3/2}} + \frac{(1+\tilde{r})}{(1+\tilde{r})}\times\frac{c^{1/2}}{(1+\tilde{r})^{5/2}}\right) \\
&= o_{c\to 0}\left(\frac{1}{c}\right).
\end{aligned}$$

Similarly, we check with the same computations that $|\langle i\partial_{x_1}\Gamma_c, x_1\partial_{x_2}V\rangle| = o_{c\to 0}\big(\frac{1}{c}\big)$.

Finally, using Lemma 3.1.1 (for $\sigma = 1/4$), we estimate

$$|\langle i\partial_{x_1}\Gamma_c, x_1\partial_{x_2}\Gamma_c\rangle| \leqslant Kc^{3/2}\|x_1\|_{L^\infty(\{\tilde{r}\leqslant 1\})} + K\left|\int_{\{\tilde{r}\geqslant 1\}} \mathfrak{Re}\left(ix_1\frac{\partial_{x_1}\Gamma_c}{V}\overline{\frac{\partial_{x_2}\Gamma_c}{V}}\right)\right|.$$

We have $\|x_1\|_{L^\infty(\{\tilde{r}\leqslant 1\})} \leqslant \frac{K}{c}$. Moreover, we infer

$$\begin{aligned}
\left|\int_{\{\tilde{r}\geqslant 1\}} \mathfrak{Re}\left(ix_1\frac{\partial_{x_1}\Gamma_c}{V}\overline{\frac{\partial_{x_2}\Gamma_c}{V}}\right)\right| &\leqslant \int_{\{\tilde{r}\geqslant 1\}} |x_1|\left|\mathfrak{Re}\left(\frac{\partial_{x_1}\Gamma_c}{V}\right)\mathfrak{Im}\left(\frac{\partial_{x_2}\Gamma_c}{V}\right)\right| \\
&\quad + \int_{\{\tilde{r}\geqslant 1\}} |x_1|\left|\mathfrak{Im}\left(\frac{\partial_{x_1}\Gamma_c}{V}\right)\mathfrak{Re}\left(\frac{\partial_{x_2}\Gamma_c}{V}\right)\right|,
\end{aligned}$$

and, with Lemma 3.1.1 (for $\sigma = 1/4$), we have

$$\left|\int_{\{\tilde{r}\geqslant 1\}} \mathfrak{Re}\left(ix_1\frac{\partial_{x_1}\Gamma_c}{V}\overline{\frac{\partial_{x_2}\Gamma_c}{V}}\right)\right| \leqslant K\int_{\{\tilde{r}\geqslant 1\}} |x_1|\frac{c^{3/2}}{(1+\tilde{r})^{3+1/2}} = o_{c\to 0}(1),$$

since $\frac{|x_1|c}{(1+\tilde{r})} \leqslant K$ by triangular inequality. We conclude that

$$\langle i\partial_{x_1}\Gamma_c, x_1\partial_{x_2}\Gamma_c\rangle = o_{c\to 0}(1),$$

which, together with the previous estimates, shows that

$$\langle L_{Q_c}(\partial_{c^\perp}Q_c), \partial_{c^\perp}Q_c\rangle = 2\pi + o_{c\to 0}(1). \qquad \square$$

These quantities are connected to the energy and momentum. This is shown in this next lemma.

Lemma 3.1.10. *There exists $c_0 > 0$ such that for $0 < c < c_0$, Q_c defined in Theorem 1.3.1, we have*

$$P_1(Q_c) = \partial_c P_1(Q_c) = 0,$$

$$P_2(Q_c) = \frac{1}{c}B_{Q_c}(\partial_{c^\perp}Q_c) = \frac{2\pi + o_{c\to 0}(1)}{c}$$

and

$$\partial_c P_2(Q_c) = B_{Q_c}(\partial_c Q_c) = \frac{-2\pi + o_{c\to 0}(1)}{c^2}.$$

Furthermore,

$$\partial_c E(Q_c) = c\partial_c P_2(Q_c),$$

and

$$E(Q_c) = (2\pi + o_{c\to 0}(1))\ln\left(\frac{1}{c}\right).$$

Proof. We have

$$P_1(Q_c) = \frac{1}{2}\langle i\partial_{x_1} Q_c, Q_c - 1\rangle,$$

by the symmetries (3.1.3), $\partial_{x_1} Q_c$ is odd in x_1 and $Q_c - 1$ is even. Therefore,

$$P_1(Q_c) = \partial_c P_1(Q_c) = 0.$$

We have

$$P_2(Q_c) = \frac{1}{2}\langle i\partial_{x_2} Q_c, Q_c - 1\rangle,$$

and from Lemma 3.1.9 and (3.1.23), we have

$$2\pi + o_{c\to 0}(1) = B_{Q_c}(\partial_{c^\perp} Q_c) = c\langle i\,\partial_{x_1} Q_c, x_1\partial_{x_2} Q_c\rangle.$$

By integration by parts (which can be done thanks to Theorem 3.1.4, Lemma 1.2.1 and equation (3.1.11)), we compute

$$\langle i\,\partial_{x_1} Q_c, x_1\partial_{x_2} Q_c\rangle = -\langle i(Q_c - 1), \partial_{x_2} Q_c\rangle - \langle i(Q_c - 1), x_1\partial_{x_1 x_2} Q_c\rangle,$$

and

$$\langle i(Q_c - 1), x_1\partial_{x_1 x_2} Q_c\rangle = -\langle i\partial_{x_2} Q_c, x_1\partial_{x_1} Q_c\rangle = \langle i\partial_{x_1} Q_c, x_1\partial_{x_2} Q_c\rangle.$$

Therefore,

$$P_2(Q_c) = \frac{1}{2}\langle i\,\partial_{x_1} Q_c, x_1\partial_{x_2} Q_c\rangle = \frac{1}{c} B_{Q_c}(\partial_{c^\perp} Q_c) = \frac{2\pi + o_{c\to 0}(1)}{c}.$$

We have $P_2(Q_c) = \frac{1}{2}\int_{\mathbb{R}^2} \mathfrak{Re}(i\partial_{x_2} Q_c(\overline{Q_c} - 1))$, and we check that, with Lemmas 3.1.1 and 3.1.2 that

$$|\partial_c\partial_{x_2} Q_c(\overline{Q_c} - 1)| + |\partial_{x_2} Q_c\partial_c\overline{Q_c}| \leqslant \frac{K}{(1+\tilde{r})^{5/2}},$$

and is therefore dominated by an integrable function independent of $c \in]c_1, c_2[$ given that $c_1, c_2 > 0$ are small enough. We deduce that $c \mapsto P_2(Q_c) \in C^1(]0, c_0[, \mathbb{R})$ for some small $c_0 > 0$ and that, by integration by parts,

$$2\partial_c P_2(Q_c) = \langle i\partial_{x_2}\partial_c Q_c, Q_c - 1\rangle + \langle i\partial_{x_2} Q_c, \partial_c Q_c\rangle = 2\langle i\partial_{x_2} Q_c, \partial_c Q_c\rangle,$$

and, from Lemma 3.1.9 and equation (3.1.17), we have

$$\langle i\partial_{x_2} Q_c, \partial_c Q_c\rangle = B_{Q_c}(\partial_c Q_c) = \frac{-2\pi + o_{c\to 0}(1)}{c^2},$$

therefore

$$\partial_c P_2(Q_c) = \frac{-2\pi + o_{c\to 0}(1)}{c^2}.$$

We recall that

$$E(Q_c) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla Q_c|^2 + \frac{1}{4}\int_{\mathbb{R}^2} (1 - |Q_c|^2)^2.$$

We check with Lemmas 3.1.1, 3.1.2 that

$$|\partial_c \nabla Q_c.\overline{\nabla Q_c}| + |\partial_c(|Q_c|^2)(1 - |Q_c|^2)| \leqslant \frac{K}{(1+\tilde{r})^{5/2}}$$

and is therefore dominated by an integrable function independent of $c \in]c_1, c_2[$ given that $c_1, c_2 > 0$ are small enough. We deduce that $c \mapsto E(Q_c) \in C^1(]0, c_0[, \mathbb{R})$ for some small $c_0 > 0$ and that,

$$\partial_c\left(\frac{1}{2}\int_{\mathbb{R}^2} |\nabla Q_c|^2\right) = \frac{1}{2}\int_{\mathbb{R}^2} \mathfrak{Re}(\nabla Q_c\overline{\nabla\partial_c Q_c}) + \mathfrak{Re}(\nabla\partial_c Q_c\overline{\nabla Q_c}).$$

We check, with Theorem 3.1.4 and $(\mathrm{TW}_c)(Q_c)=0$, that we can do the integration by parts, which yields

$$\partial_c\left(\frac{1}{2}\int_{\mathbb{R}^2}|\nabla Q_c|^2\right)=\langle -\Delta Q_c,\partial_c Q_c\rangle.$$

We check similarly that

$$\partial_c\left(\frac{1}{4}\int_{\mathbb{R}^2}(1-|Q_c|^2)^2\right)=-\int_{\mathbb{R}^2}(1-|Q_c|^2)\mathfrak{Re}(\partial_c Q_c\overline{Q_c}),$$

hence

$$\partial_c\left(\frac{1}{4}\int_{\mathbb{R}^2}(1-|Q_c|^2)^2\right)=\langle -(1-|Q_c|^2)Q_c,\partial_c Q_c\rangle.$$

Now, since $-ic\partial_{x_2}Q_c-\Delta Q_c-(1-|Q_c|^2)Q_c=0$, we have

$$\partial_c E(Q_c)=\langle -\Delta Q_c-(1-|Q_c|^2)Q_c,\partial_c Q_c\rangle=c\langle -i\partial_{x_2}Q_c,\partial_c Q_c\rangle.$$

Now, since $P_2(Q_c)=\frac{1}{2}\langle i\partial_{x_2}Q_c,Q_c-1\rangle$, we have

$$\partial_c P_2(Q_c)=\frac{1}{2}(\langle i\partial_{x_2}\partial_c Q_c,Q_c-1\rangle+\langle i\partial_{x_2}Q_c,\partial_c Q_c\rangle).$$

By integrations by parts, we compute

$$\partial_c P_2(Q_c)=\langle -i\partial_{x_2}Q_c,\partial_c Q_c\rangle.$$

We deduce that $\partial_c E(Q_c)=c\partial_c P_2(Q_c)$, and in particular, we deduce that

$$\partial_c E(Q_c)=\frac{-2\pi+o_{c\to 0}(1)}{c}.$$

By integration (from some fixed $c_0>c>0$), we check that $E(Q_c)=(2\pi+o_{c\to 0}(1))\ln\left(\frac{1}{c}\right)$. □

We conclude this subsection with an estimate on Q_c connected to the energy that will be useful later on.

Lemma 3.1.11. *There exists $K>0$, a universal constant independent of c, such that, if c is small enough, for Q_c defined in Theorem 1.3.1,*

$$\int_{\mathbb{R}^2}\frac{|\mathfrak{Im}(\nabla Q_c\overline{Q_c})|^2}{|Q_c|^2}\leqslant K\ln\left(\frac{1}{c}\right).$$

Proof. We recall that $r_{\pm 1}=|x\mp d_c\vec{e_1}|$. Since ∇Q_c is bounded near the zeros of Q_c (by Lemmas 1.2.1 and 3.1.1), and $|Q_c|\geqslant K$ on $\mathbb{R}^2\backslash B(\pm\tilde{d_c}\vec{e_1},1)$ by (3.1.12), we have

$$\int_{\mathbb{R}^2}\frac{|\mathfrak{Im}(\nabla Q_c\overline{Q_c})|^2}{|Q_c|^2}\leqslant K\left(1+\int_{\{\tilde{r}\geqslant 1\}}|\mathfrak{Im}(\nabla Q_c\overline{Q_c})|^2\right).$$

Now, by (3.1.12), Lemma 3.1.10 and the definition of the energy,

$$\int_{\{\tilde{r}\geqslant 1\}}|\mathfrak{Im}(\nabla Q_c\overline{Q_c})|^2\leqslant\int_{\{\tilde{r}\geqslant 1\}}|\nabla Q_c|^2|Q_c|^2\leqslant K\int_{\mathbb{R}^2}|\nabla Q_c|^2\leqslant KE(Q_c)\leqslant K\ln\left(\frac{1}{c}\right).$$ □

We could check that this estimate is optimal with respect to its growth in c when $c\to 0$.

3.1.3 Zeros of Q_c

In this subsection, we show that Q_c has only two zeros and we compute estimates on Q_c around them. In a bounded domain, a general result about the zeros of solutions to the Ginzburg-Landau problem is already known, see [42].

Lemma 3.1.12. *For $c>0$ small enough, the function Q_c defined in Theorem 1.3.1 has exactly two zeros. Their positions are $\pm\tilde{d_c}\vec{e_1}$, and, for any $0<\sigma<1$,*

$$|d_c-\tilde{d_c}|=o^{\sigma}_{c\to 0}(c^{1-\sigma}),$$

where d_c is defined in Theorem 1.3.1.

The notation $o^{\sigma}_{c\to 0}(1)$ denotes a quantity going to 0 when $c\to 0$ at fixed σ. Combining Lemmas 3.1.9, 3.1.10 and 3.1.12, we end the proof of Proposition 1.4.1.

Proof. From (3.1.3), we know that Q_c enjoys the symmetry $Q_c(x_1, x_2) = Q_c(-x_1, x_2)$ for $(x_1, x_2) \in \mathbb{R}^2$, hence we look at zeros only in the right half-plane. From Theorem 1.3.1, we have $Q_c = V_1\big(.-d_c\vec{e}_1\big)V_{-1}(.+d_c\vec{e}_1)+\Gamma_c$ with $\|\Gamma_c\|_{L^\infty(\mathbb{R}^2)}+\|\nabla\Gamma_c\|_{L^\infty(\mathbb{R}^2)} = o_{c\to 0}(1)$. In the right half-plane and outside of $B(d_c\vec{e}_1,\Lambda)$ for any $\Lambda>0$, by Lemma 1.2.1, we estimate

$$|Q_c| \geqslant |V_1\big(.-d_c\vec{e}_1\big)\,V_{-1}(.+d_c\vec{e}_1)| - o_{c\to 0}(1) \geqslant K(\Lambda) > 0$$

if c is small enough (depending on Λ). Now, we consider the smooth function $F\colon \mathbb{R}\times\mathbb{R}^2\to\mathbb{C}$ defined by

$$F(\mu, z) := \big(V_1\big(.-d_c\vec{e}_1\big)V_{-1}(.+d_c\vec{e}_1)+\mu\Gamma_c(.)\big)(z+d_c\vec{e}_1).$$

We have $F(0,0) = V_1(0)V_{-1}(2d_c\vec{e}_1) = 0$ by Lemma 1.2.1 and $F(1,z) = Q_c(z+d_c\vec{e}_1)$. For $|\mu|\leqslant 1$ and $|z|\leqslant 1$, since $\|\nabla\Gamma_c\|_{L^\infty(\mathbb{R}^2)} = o^{\sigma}_{c\to 0}(c^{1-\sigma})$ by equation (3.1.5), with Lemma 1.2.1 and equation (3.1.1), we check that

$$|d_zF_{(\mu,z)}(\xi) - \nabla V_1(z).\xi| = o_{c\to 0}(1)|\xi| \tag{3.1.24}$$

uniformly in $\mu\in[0,1]$.

Now, from Lemma 1.2.1, we estimate (for $x = re^{i\theta}\neq 0\in\mathbb{R}^2$)

$$\begin{aligned}\partial_{x_1}V_1(x) &= \left(\cos(\theta)\rho'(r) - \frac{i}{r}\sin(\theta)\rho(r)\right)e^{i\theta}\\ &= \kappa(\cos(\theta) - i\sin(\theta))e^{i\theta} + o_{r\to 0}(1)\\ &= \kappa + o_{r\to 0}(1),\end{aligned}$$

and thus, by continuity, $\partial_{x_1}V_1(0) = \kappa > 0$. Similarly, we check that $\partial_{x_2}V_1(0) = -i\kappa$, and therefore,

$$\nabla V_1(z) = \kappa\begin{pmatrix}1\\ -i\end{pmatrix} + o_{|z|\to 0}(1).$$

Identifying $\mathbb{C}$ with $\mathbb{R}^2$ canonically, we deduce that the Jacobian determinant of F in z, $J(F)$, satisfies

$$J(F)(\mu,z) = J(V_1)(z) + o_{c\to 0}(1) = -\kappa^2 + o_{c\to 0}(1) + o_{|z|\to 0}(1) \neq 0,$$

given that c and $|z|$ are small enough. By the implicit function theorem, there exists $\mu_0>0$ such that, for $|\mu|\leqslant\mu_0$, there exists a unique value $z(\mu)$ in a vicinity of 0 such that $F(\mu, z(\mu)) = 0$, and since $\partial_\mu F(\mu, z) = \Gamma_c(d_c\vec{e}_1 + z) = o^{\sigma}_{c\to 0}(c^{1-\sigma})$ uniformly in z (by (3.1.4)), it satisfies additionally $z(\mu) = o^{\sigma}_{c\to 0}(c^{1-\sigma})$.

Now, let us show that we can take $\mu_0 = 1$. Indeed, if we define $\mu_0 = \sup\big\{\nu>0, \mu\to z(\mu)\in C^1([0,\nu],\mathbb{R}^2)\big\} > 0$ and we have $\mu_0 < 1$, since $\mu\to z(\mu)\in C^1([0,\mu_0],\mathbb{R}^2)$ with $|d_\mu z|(\mu) = o^{\sigma}_{c\to 0}(c^{1-\sigma})$ uniformly in $[0,\mu_0]$, it can be continuously extended to μ_0 with $F(\mu_0, z(\mu_0)) = 0$ and $z(\mu_0) = o^{\sigma}_{c\to 0}(c^{1-\sigma})$. Then, by the implicit function theorem at $(\mu_0, z(\mu_0))$ (since $\mu_0<1$ with equation (3.1.24)), it can be extended above μ_0, which is in contradiction with the definition of μ_0.

Since $F(1,.) = Q_c(.+d_c\vec{e}_1)$, we have shown that there exists $z\in\mathbb{R}^2$ with $|z| = o^{\sigma}_{c\to 0}(c^{1-\sigma})$ such that $Q_c(z+d_c\vec{e}_1) = 0$. Now, for c small enough and $|\xi|\leqslant 1$, we have

$$\nabla(Q_c(\xi+z+d_c\vec{e}_1)) = \nabla V_1(z) + o_{c\to 0}(1) + o_{|\xi|\to 0}(1) = \kappa\begin{pmatrix}1\\ -i\end{pmatrix} + o_{c\to 0}(1) + o_{|\xi|\to 0}(1).$$

We deduce, with $Q_c(\zeta+z+d_c\vec{e}_1) = \int_0^{|\zeta|}\nabla Q_c\Big(\xi\frac{\zeta}{|\zeta|}+z+d_c\vec{e}_1\Big).\frac{\zeta}{|\zeta|}d\xi$, that

$$\left|Q_c(\zeta+z+d_c\vec{e}_1) - \zeta.\begin{pmatrix}1\\ -i\end{pmatrix}\kappa\right| = o_{|\zeta|\to 0}(|\zeta|) + o_{c\to 0}(1)|\zeta|.$$

Therefore, Q_c has no other zeros in $B(z+d_c\vec{e}_1,\Lambda)$ for some $\Lambda>0$ independent of c. Therefore, since for c small enough, $|Q_c| > K(\Lambda) > 0$ outside of $B(z+d_c\vec{e}_1,\Lambda)$ in the right half-plane, Q_c has only one zero in the right half-plane.

By the symmetry $Q_c(x_1,x_2)=\overline{Q_c(x_1,-x_2)}$ (see (3.1.3)), z must be colinear to $\vec{e_1}$, therefore we define $\tilde{d_c}\in\mathbb{R}$ by $\tilde{d_c}\vec{e_1}:=z+d_c\vec{e_1}$, and we conclude that, since $|z|=o^{\sigma}_{c\to 0}(c^{1-\sigma})$,

$$|d_c-\tilde{d_c}|=o^{\sigma}_{c\to 0}(c^{1-\sigma}). \qquad \square$$

We define the vortices around the zeros of Q_c by

$$\tilde{V_{\pm 1}}(x):=V_{\pm 1}(x\mp\tilde{d_c}\vec{e_1}),$$

and we will use the already defined polar coordinates around $\pm\tilde{d_c}\vec{e_1}$ of $x\in\mathbb{R}^2$, namely

$$\tilde{r}_{\pm 1}=|x\mp\tilde{d_c}\vec{e_1}|,\tilde{\theta}_{\pm 1}=\arg(x\mp\tilde{d_c}\vec{e_1}).$$

One of the idea of the proof is to understand how Q_c is close, multiplicatively, to vortices $\tilde{V_{\pm 1}}$ centered at its zeros, since by construction it is close to a vortex centered around $\pm d_c\vec{e_1}$, which is itself close to $\pm\tilde{d_c}\vec{e_1}$. In particular, Lemma 3.1.14 below will show that the ratio $\left|\frac{Q_c}{\tilde{V_1}}\right|$ is bounded and close to 1 near $\tilde{d_c}\vec{e_1}$.

In Lemma 3.1.13 to follow, we compute the additive perturbation between derivatives of Q_c and a vortex $\tilde{V_{\pm 1}}$ centered around one of its zeros. In Lemma 3.1.14, we compute the multiplicative perturbation. All along, we work in $B\big(\tilde{d_c}\vec{e_1},\tilde{d_c}^{1/2}\big)$, the size of the ball $\tilde{d_c}^{1/2}$ being arbitrary (any quantity that both goes to infinity when $c\to 0$ and is a $o_{c\to 0}(\tilde{d_c})$ should work). We recall that $\tilde{r}_{\pm 1}=|x\mp\tilde{d_c}\vec{e_1}|$.

Lemma 3.1.13. *Uniformly in $B\big(\tilde{d_c}\vec{e_1},\tilde{d_c}^{1/2}\big)$, for Q_c defined in Theorem 1.3.1, one has*

$$|Q_c-\tilde{V_1}|=o_{c\to 0}(1),$$

$$|\nabla Q_c-\nabla\tilde{V_1}|\leqslant\frac{o_{c\to 0}(1)}{1+\tilde{r}_1}$$

and

$$|\nabla^2 Q_c-\nabla^2\tilde{V_1}|\leqslant\frac{o_{c\to 0}(1)}{1+\tilde{r}_1}.$$

Proof. From equations (3.1.7) and (3.1.1), as well as Lemmas 2.1.6, 3.1.12 and the mean value theorem, in $B\big(\tilde{d_c}\vec{e_1},\tilde{d_c}^{1/2}\big)$,

$$\begin{aligned}|Q_c-\tilde{V_1}| &\leqslant |Q_c-V|+|V-\tilde{V_1}|\\ &\leqslant o_{c\to 0}(1)+|V_1(.-\tilde{d_c}\vec{e_1})-\tilde{V_1}|\\ &\leqslant o_{c\to 0}(1)+|d_c-\tilde{d_c}|\,\|\partial_{x_1}V\|_{L^\infty(\mathbb{R}^2)}\\ &\leqslant o_{c\to 0}(1),\end{aligned}\tag{3.1.25}$$

which is the first statement.

For the second statement, we write $Q_c=V_1(.-d_c\vec{e_1})V_{-1}(.-d_c\vec{e_1})+\Gamma_c$, and from equation (3.1.5) (with some margin), we have

$$|\nabla\Gamma_c|\leqslant\frac{o_{c\to 0}(1)}{1+\tilde{r}_1}.$$

Furthermore, since $\tilde{V_1}=V_1(.-\tilde{d_c}\vec{e_1})$,

$$\nabla(V_1(.-d_c\vec{e_1})V_{-1}(.+d_c\vec{e_1}))-\nabla\tilde{V_1}=$$

$$\nabla V_1(.-d_c\vec{e_1})V_{-1}(.+d_c\vec{e_1})-\nabla\tilde{V_1}+V_1(.-d_c\vec{e_1})\nabla V_{-1}(.+d_c\vec{e_1}),$$

and from (3.1.2), in $B\big(\tilde{d_c}\vec{e_1},\tilde{d_c}^{1/2}\big)$, we have

$$|\nabla V_{-1}(.+d_c\vec{e_1})|\leqslant\frac{o_{c\to 0}(1)}{1+\tilde{r}_1}.$$

We compute

$$\nabla V_1(.-d_c\vec{e_1})V_{-1}(.+d_c\vec{e_1})-\nabla\tilde{V_1}=\nabla V_1(.-d_c\vec{e_1})(V_{-1}(.+d_c\vec{e_1})-1)-\nabla\tilde{V_1}+\nabla V_1(.-d_c\vec{e_1})$$

and, from (3.1.1), in $B\big(\tilde{d}_c\vec{e}_1, \tilde{d}_c^{1/2}\big)$, we have $|V_{-1}(.+d_c\vec{e}_1)-1| = o_{c\to 0}(1)$. Finally, from Lemmas 1.2.1 and 3.1.12, we estimate (with the mean value theorem)

$$|\nabla V_1(.-d_c\vec{e}_1)-\nabla\tilde{V}_1| \leqslant |d_c-\tilde{d}_c| \sup_{d\in[d_c,\tilde{d}_c]\cup[\tilde{d}_c,d_c]} |\nabla^2 V_1(x-d)| \leqslant K\frac{|d_c-\tilde{d}_c|}{(1+\tilde{r}_1)^2} = \frac{o_{c\to 0}(1)}{(1+\tilde{r}_1)^2},$$

hence

$$|\nabla Q_c - \nabla\tilde{V}_1| \leqslant \frac{o_{c\to 0}(1)}{1+\tilde{r}_1}. \tag{3.1.26}$$

Now, writing $w = Q_c - \tilde{V}_1$, in $B\big(\tilde{d}_c\vec{e}_1,\, 2\tilde{d}_c^{1/2}\big)$, we estimate (since $\mathrm{TW}_c(Q_c)=0$ and $\Delta\tilde{V}_1 - (|\tilde{V}_1|^2-1)\tilde{V}_1 = 0$)

$$|\Delta w| = |-ic\partial_{x_2}Q_c - (1-|Q_c|^2)Q_c + (1-|\tilde{V}_1|^2)\tilde{V}_1| \leqslant \frac{o_{c\to 0}(1)}{1+\tilde{r}_1}$$

by equations (3.1.6) to (3.1.10) and (3.1.1). Furthermore, by equations (3.1.6) to (3.1.2), we have

$$|\nabla(\Delta w)| \leqslant \frac{o_{c\to 0}(1)}{(1+\tilde{r}_1)}.$$

We check, as the proof of (3.1.25), that, in $B\big(\tilde{d}_c\vec{e}_1, 2\tilde{d}_c^{1/2}\big)$,

$$|w| = o_{c\to 0}(1),$$

and, similarly, with equations (3.1.2) and (3.1.26), that

$$|\nabla w| = o_{c\to 0}(1)$$

in $B\big(\tilde{d}_c\vec{e}_1, 2\tilde{d}_c^{1/2}\big)$. By Theorem 6.2 of [15] (taking a domain $\Omega = B\Big(x-\tilde{d}_c\vec{e}_1, \frac{|x-\tilde{d}_c\vec{e}_1|}{2}\Big)$, and $\alpha = 1/2$, but it also holds for any $0<\alpha<1$), we have, for $x\in B\big(\tilde{d}_c\vec{e}_1, 2\tilde{d}_c^{1/2}\big)$,

$$(1+\tilde{r}_1)^2|\nabla^2 w(x-\tilde{d}_c\vec{e}_1)| \leqslant K(\|w\|_{C^1(\Omega)} + (1+\tilde{r}_1)^2\|\Delta w\|_{C^1(\Omega)}),$$

and from the previous estimates, we have $\|w\|_{C^1(\Omega)} = o_{c\to 0}(1)$ and $\|\Delta w\|_{C^1(\Omega)} \leqslant \frac{o_{c\to 0}(1)}{(1+\tilde{r}_1)}$, therefore

$$|\nabla^2(Q_c-\tilde{V}_1)| = |\nabla^2 w| \leqslant \frac{o_{c\to 0}(1)}{(1+\tilde{r}_1)}. \qquad \square$$

Lemma 3.1.14. *In $B\big(\tilde{d}_c\vec{e}_1, \tilde{d}_c^{1/2}\big)$, for Q_c defined in Theorem 1.3.1, we have*

$$\left|\frac{Q_c}{\tilde{V}_1} - 1\right| = o_{c\to 0}(c^{1/10}).$$

In particular,

$$\left|\frac{Q_c}{\tilde{V}_1}\right| = 1 + o_{c\to 0}(c^{1/10}).$$

The power $1/10$ is arbitrary, but enough here for the upcoming estimations.

Proof. We recall that both Q_c and $\tilde{V}_1$ are C^∞ since they are solutions of elliptic equations. We have that $Q_c(\tilde{d}_c\vec{e}_1) = 0$ by Lemma 3.1.12, thus, for $x\in\mathbb{R}^2$, by Taylor expansion, for $|x|\leqslant 1$,

$$Q_c(x+\tilde{d}_c\vec{e}_1) = x.\nabla Q_c(\tilde{d}_c\vec{e}_1) + O_{|x|\to 0}(|x|^2).$$

From Theorem 1.3.1, we have $Q_c = V_1\big(.-d_c\vec{e}_1\big)V_{-1}(.+d_c\vec{e}_1) + \Gamma_c$, therefore, with $V_{\pm 1}$ being centered around $\pm d_c\vec{e}_1$ for the rest of the proof,

$$\nabla Q_c(\tilde{d}_c\vec{e}_1) = \nabla V_1(\tilde{d}_c\vec{e}_1)V_{-1}(\tilde{d}_c\vec{e}_1) + V_1(\tilde{d}_c\vec{e}_1)\nabla V_{-1}(\tilde{d}_c\vec{e}_1) + \nabla\Gamma_c(\tilde{d}_c\vec{e}_1).$$

We have $V_1(\tilde{d}_c\vec{e}_1)\nabla V_{-1}(\tilde{d}_c\vec{e}_1) + \nabla\Gamma_c(\tilde{d}_c\vec{e}_1) = o_{c\to 0}(c^{1/2})$ by Theorem 1.3.1, Lemma 1.2.1 and (3.1.2). Furthermore, by (3.1.1), Lemmas 1.2.1 and 3.1.12,

$$\nabla V_1(\tilde{d}_c\vec{e}_1)V_{-1}(\tilde{d}_c\vec{e}_1) \;=\; \nabla V_1(\tilde{d}_c\vec{e}_1) + o_{c\to 0}(c^{1/4})$$

We deduce that

$$Q_c(x+\tilde{d_c}\vec{e_1}) = x.(\nabla V_1(d_c\vec{e_1}) + o_{c\to 0}(c^{1/4})) + O_{x\to 0}(|x|^2). \tag{3.1.27}$$

We also have $\tilde{V_1}(x+\tilde{d_c}\vec{e_1}) = x.\nabla\tilde{V_1}(\tilde{d_c}\vec{e_1}) + O_{x\to 0}(|x|^2)$ (since $\tilde{V_1}(\tilde{d_c}\vec{e_1}) = 0$) and $\nabla V_1(d_c\vec{e_1}) = \nabla\tilde{V_1}(\tilde{d_c}\vec{e_1})$, hence

$$Q_c(x+\tilde{d_c}\vec{e_1}) = \tilde{V_1}(x+\tilde{d_c}\vec{e_1}) + x.o_{c\to 0}(c^{1/4}) + O_{|x|\to 0}(|x|^2).$$

Now, by Lemma 1.2.1, there exists $K>0$ such that, in $B(\tilde{d_c}\vec{e_1}, c^{1/4})$ for c small enough, $|\tilde{V_1}(x+\tilde{d_c}\vec{e_1})| \geqslant K|x|$. We deduce that

$$\begin{aligned}\left|\frac{Q_c}{\tilde{V_1}} - 1\right| &\leqslant \frac{|x|o_{c\to 0}(c^{1/4})}{|\tilde{V_1}(x+\tilde{d_c}\vec{e_1})|} + \frac{O_{|x|\to 0}(|x|^2)}{|\tilde{V_1}(x+\tilde{d_c}\vec{e_1})|}\\ &\leqslant o_{c\to 0}(c^{1/4}) + O_{|x|\to 0}(|x|)\\ &\leqslant o_{c\to 0}(c^{1/5}).\end{aligned}$$

Outside of $B(\tilde{d_c}\vec{e_1}, c^{1/4})$ and in $B\big(\tilde{d_c}\vec{e_1}, \tilde{d_c}^{1/2}\big)$, we have $|\tilde{V_1}| \geqslant Kc^{1/4}$ by Lemma 1.2.1, and

$$Q_c = V_1 + O_{c\to 0}(c^{1/2})$$

by Theorem 1.3.1, equations (3.1.7) and (3.1.1). We deduce

$$\left|\frac{Q_c}{\tilde{V_1}} - 1\right|(x) = \left|\frac{V_1 + O_{c\to 0}(c^{1/2})}{\tilde{V_1}} - 1\right|(x) = \left|\frac{V_1(x)}{\tilde{V_1}(x)} - 1\right| + o_{c\to 0}(c^{1/10}).$$

Furthermore, by Lemma 3.1.12 (for $\sigma = 1/2$), we have

$$\left|\frac{V_1(x)}{\tilde{V_1}(x)} - 1\right| = \left|\frac{\tilde{V_1}(x) + O_{|d_c-\tilde{d}_c|\to 0}(|d_c-\tilde{d_c}|)}{\tilde{V_1}(x)} - 1\right| = \frac{O_{|d_c-\tilde{d}_c|\to 0}(|d_c-\tilde{d_c}|)}{c^{1/4}} = o_{c\to 0}(c^{1/10}).$$

We conclude that $\left|\frac{Q_c}{\tilde{V_1}} - 1\right| = o_{c\to 0}(c^{1/10})$ in $B\big(\tilde{d_c}\vec{e_1}, \tilde{d_c}^{1/2}\big)$. □

By the symmetries of Q_c (see (3.1.3)), the result of Lemma 3.1.14 holds if we change $\vec{e_1}$ by $-\vec{e_1}$ and $\tilde{V_1}$ by $\tilde{V}_{-1}$.

We conclude this section with the proof that in $B\big(\pm\tilde{d_c}\vec{e_1}, \tilde{d_c}^{1/2}\big)$, we have, for $\psi \in C_c^\infty\big(\mathbb{R}^2\backslash\{\pm\tilde{d_c}\vec{e_1}\}, \mathbb{C}\big)$,

$$\int_0^{2\pi} |\psi^{\neq 0}|^2 d\tilde{\theta}_{\pm 1} \leqslant \tilde{r}_{\pm 1}^2 \int_0^{2\pi} |\nabla\psi|^2 d\tilde{\theta}_{\pm 1}. \tag{3.1.28}$$

We recall that the function $\psi^{\neq 0}$ is defined by

$$\psi^{\neq 0}(x) = \psi(x) - \psi^{0,1}(\tilde{r}_1)$$

in the right half-plane, and

$$\psi^{\neq 0}(x) = \psi(x) - \psi^{0,-1}(\tilde{r}_{-1})$$

in the left half-plane.

To show (3.1.28), it is enough to show that, for $\psi \in C_c^\infty(\mathbb{R}^2\backslash\{0\}, \mathbb{C})$, we have, with $x = re^{i\theta}$,

$$\int_0^{2\pi} \left|\psi - \int_0^{2\pi} \psi d\gamma\right|^2 d\theta \leqslant r^2 \int_0^{2\pi} |\nabla\psi|^2 d\theta.$$

This is a Poincaré inequality. By decomposition in harmonics and Parseval's equality, we have

$$\begin{aligned}\int_0^{2\pi} \left|\psi - \int_0^{2\pi} \psi(\gamma) d\gamma\right|^2 d\theta &= \int_0^{2\pi} \left|\sum_{n\in\mathbb{Z}^*} \psi_n(r) e^{in\theta}\right|^2 d\theta\\ &= \int_0^{2\pi} \sum_{n\in\mathbb{Z}^*} |\psi_n(r)|^2 d\theta,\end{aligned}$$

and

$$\begin{aligned}\int_0^{2\pi}|\nabla\psi|^2d\theta &\geqslant \int_0^{2\pi}\frac{1}{r^2}|\partial_\theta\psi|^2d\theta\\ &\geqslant \int_0^{2\pi}\Big|\sum_{n\in\mathbb{Z}^*} i\frac{n\psi_n(r)}{r}e^{in\theta}\Big|^2d\theta\\ &\geqslant \frac{1}{r^2}\int_0^{2\pi}\sum_{n\in\mathbb{Z}^*} n^2|\psi_n(r)|^2d\theta\\ &\geqslant \frac{1}{r^2}\int_0^{2\pi}\sum_{n\in\mathbb{Z}^*} |\psi_n(r)|^2d\theta.\end{aligned}$$

This concludes the proof of (3.1.28). With $|Q_c\big(x\pm\tilde{d}_c\vec{e_1}\big)|=O_{\tilde{r}_{\pm1}\to0}(\tilde{r}_{\pm1})$ and (3.1.28), we have, for $\tilde{r}_{\pm1}\leqslant R$,

$$\begin{aligned}\int_0^{2\pi}|Q_c|^2|\psi^{\neq0}|^2d\tilde{\theta}_{\pm1} &\leqslant K\int_0^{2\pi}\tilde{r}_{\pm1}^2|\psi^{\neq0}|^2d\tilde{\theta}_{\pm1}\\ &\leqslant K\int_0^{2\pi}\tilde{r}_{\pm1}^4|\nabla\psi|^2d\tilde{\theta}_{\pm1}\\ &\leqslant K(R)\int_0^{2\pi}|Q_c|^4|\nabla\psi|^2d\tilde{\theta}_{\pm1}.\end{aligned}\tag{3.1.29}$$

This result will be usefull to estimate the quantities in the orthogonality conditions.

3.2 Estimations in H_{Q_c}

We give several estimates for functions in H_{Q_c}. They will in particular allow us to use a density argument to show Proposition 1.4.3 once it is shown for test function in section 3.3. We will also explain why a coercivity result with the energy norm $\|.\|_{H_{Q_c}}$ is impossible with any number of local orthogonality conditions, and show that the quadratic form and the coercivity norm are well defined for functions in H_{Q_c}.

3.2.1 Comparaison of the energy and coercivity norms

In the introduction, we have defined the quadratic form by

$$\begin{aligned}B_{Q_c}(\varphi) &= \int_{\mathbb{R}^2}|\nabla\varphi|^2-(1-|Q_c|^2)|\varphi|^2+2\,\mathfrak{Re}^2(\overline{Q_c}\varphi)\\ &- c\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi})-c\int_{\mathbb{R}^2}\eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2\\ &+ 2c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\,\psi\,\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2+c\int_{\mathbb{R}^2}\partial_{x_2}\eta\mathfrak{Re}\,\psi\,\mathfrak{Im}\,\psi|Q_c|^2\\ &+ c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\,\psi\,\mathfrak{Im}\,\psi\partial_{x_2}(|Q_c|^2)\end{aligned}$$

(see (1.4.3)). We will show in Lemma 3.2.3 below that this quantity is well defined for $\varphi\in H_{Q_c}$. As we have seen, the natural energy space H_{Q_c} is given by the norm

$$\|\varphi\|^2_{H_{Q_c}}=\int_{\mathbb{R}^2}|\nabla\varphi|^2+|1-|Q_c|^2||\varphi|^2+\mathfrak{Re}^2(\overline{Q_c}\varphi).$$

We could expect to remplace Theorem 1.4.4 by a result of the form: up to some local orthogonality conditions, for $\varphi\in H_{Q_c}$ we have

$$B_{Q_c}(\varphi)\geqslant K(c)\|\varphi\|^2_{H_{Q_c}}.$$

However such a result can not hold. This is because of a formal zero of L_{Q_c} which is not in the space H_{Q_c}: iQ_c (which comes from the phase invariance of the equation). We have $L_{Q_c}(iQ_c)=0$ and $iQ_c\notin H_{Q_c}$ because

$$(1-|Q_c|^2)|iQ_c|^2$$

is not integrable at infinity (see [21], where it is shown that this quantity decays like $1/r^2$). We can then create functions in H_{Q_c} getting close to iQ_c, for instance

$$f_R = \eta_R i Q_c,$$

where η_R is a C^∞ real function with value 1 if $R_0 < |x| < R$ and value 0 if $|x| < R_0 - 1$ or $|x| > 2R$. In that case, when $R \to +\infty$, $\|f_R\|_{H_{Q_c}} \to +\infty$ and $B_{Q_c}(f_R) \to C$ a constant independent of R, making the inequality $B_{Q_c}(\varphi) \geqslant K\|\varphi\|^2_{H_{Q_c}}$ impossible (and the local orthogonality conditions are verified for R_0 large enough since $f_R = 0$ on $B(0, R_0 - 1)$). That is why we get the result in a weaker norm in Proposition 1.4.11: we will only get for $\varphi \in H_{Q_c}$, up to some local orthogonality conditions,

$$B_{Q_c}(\varphi) \geqslant K(c)\|\varphi\|^2_{H^{\exp}_{Q_c}},$$

where $\|.\|_{H^{\exp}_{Q_c}}$ is defined in subsection 1.4.3.1. In particular, $\|.\|_{H^{\exp}_{Q_c}}$ is not equivalent to $\|.\|_{H_{Q_c}}$.

3.2.2 The coercivity norm and other quantities are well defined in H_{Q_c}

We have defined the energy space H_{Q_c} by the norm

$$\|\varphi\|^2_{H_{Q_c}} = \int_{\mathbb{R}^2} |\nabla\varphi|^2 + |1 - |Q_c|^2||\varphi|^2 + \mathfrak{Re}^2(\overline{Q_c}\varphi).$$

By Lemma 3.1.5, we have that, for $\varphi \in H_{Q_c}$,

$$\int_{\mathbb{R}^2} \frac{|\varphi|^2}{(1+|x|)^2} dx \leqslant C(c)\|\varphi\|^2_{H_{Q_c}}. \tag{3.2.1}$$

The goal of this subsection is to show that for $\varphi \in H_{Q_c}$, $\|\varphi\|_{\mathcal{C}}$ and $B_{Q_c}(\varphi)$, as well as the quantities in the orthogonality conditions of Proposition 1.4.3 and Theorem 1.4.4, are well defined. This is done in Lemmas 3.2.1 to 3.2.3.

Lemma 3.2.1. *There exists $c_0 > 0$ such that for $0 < c \leqslant c_0$, there exists $C(c) > 0$ such that, for Q_c defined in Theorem 1.3.1 and for any $\varphi = Q_c\psi \in H_{Q_c}$,*

$$\|\varphi\|^2_{\mathcal{C}} = \int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4 \leqslant C(c)\|\varphi\|^2_{H_{Q_c}}.$$

Proof. We estimate for $\varphi = Q_c\psi \in H_{Q_c}$, using equations (3.1.12), (3.2.1) and $|\nabla Q_c| \leqslant \frac{C(c)}{(1+r)^2}$ from Theorem 3.1.4, that

$$\begin{aligned}
\int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 &= \int_{\mathbb{R}^2} |\nabla\varphi - \nabla Q_c\psi|^2|Q_c|^2 \\
&\leqslant K\int_{\mathbb{R}^2} |\nabla\varphi|^2|Q_c|^2 + |\nabla Q_c|^2|Q_c\psi|^2 \\
&\leqslant K(c)\int_{\mathbb{R}^2} |\nabla\varphi|^2 + \frac{|\varphi|^2}{(1+r)^4} \\
&\leqslant K(c)\|\varphi\|^2_{H_{Q_c}}.
\end{aligned}$$

Similarly, for $\varphi = Q_c\psi$,

$$\int_{\mathbb{R}^2} \mathfrak{Re}^2(\psi)|Q_c|^4 = \int_{\mathbb{R}^2} \mathfrak{Re}^2(\overline{Q_c}\varphi) \leqslant \|\varphi\|^2_{H_{Q_c}}.$$

We conclude that

$$\int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4 \leqslant C(c)\|\varphi\|^2_{H_{Q_c}}. \tag{3.2.2}$$ □

We conclude this subsection with the proof that the quantities in the orthogonality conditions are well defined for $\varphi \in H_{Q_c}$.

Lemma 3.2.2. *There exists $K>0$ and, for c small enough, there exists $K(c)>0$ such that, for Q_c defined in Theorem 1.3.1 and $\varphi=Q_c\psi\in H_{Q_c}$, $0<R<\tilde{d}_c^{1/2}$, we have*

$$\int_{B(\pm\tilde{d}_c\vec{e}_1,R)}\left|\mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_{\pm1}\overline{\tilde{V}_{\pm1}\psi}\Big)\right|+\int_{B(\pm\tilde{d}_c\vec{e}_1,R)}\left|\mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_{\pm1}\overline{\tilde{V}_{\pm1}\psi}\Big)\right|\leqslant K(c)\|\varphi\|_{H_{Q_c}},$$

$$\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)}\left|\mathfrak{Re}\Big(\partial_{x_{1,2}}Q_c\overline{Q_c\psi^{\neq0}}\Big)\right|\leqslant K(c)\|\varphi\|_{H_{Q_c}},$$

$$\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)}\left|\mathfrak{Re}\Big(\partial_cQ_c\overline{Q_c\psi^{\neq0}}\Big)\right|\leqslant K(c)\|\varphi\|_{H_{Q_c}}$$

and

$$\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)}\left|\mathfrak{Re}\Big(-x^\perp.\nabla Q_c\overline{Q_c\psi^{\neq0}}\Big)\right|\leqslant K(c)\|\varphi\|_{H_{Q_c}}.$$

We recall that $\psi^{\neq0}(x)=\psi(x)-\psi^{0,1}(\tilde{r}_1)$ in the right half-plane and $\psi^{\neq0}(x)=\psi(x)-\psi^{0,-1}(\tilde{r}_{-1})$ in the left half-plane, with $\tilde{r}_{\pm1}=|x\mp\tilde{d}_c\vec{e}_1|$ and $\psi^{0,\pm1}(\tilde{r}_{\pm1})$ the 0-harmonic of ψ around $\pm\tilde{d}_c\vec{e}_1$.

Proof. From Lemma 3.1.14, we have, for $\varphi=Q_c\psi\in H_{Q_c}$,

$$|\tilde{V}_{\pm1}\psi|=|\varphi|\times\left|\frac{\tilde{V}_{\pm1}}{Q_c}\right|\leqslant2|\varphi|$$

given that c is small enough. We deduce by Cauchy-Schwarz, Lemmas 1.2.1 and 3.1.5 that

$$\begin{aligned}\int_{B(\pm\tilde{d}_c\vec{e}_1,R)}\left|\mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_{\pm1}\overline{\tilde{V}_{\pm1}\psi}\Big)\right| &\leqslant 2\int_{B(\pm\tilde{d}_c\vec{e}_1,R)}|\partial_{x_1}\tilde{V}_{\pm1}|\times|\varphi|\leqslant K(c)\|\varphi\|_{H^1(B(\pm\tilde{d}_c\vec{e}_1,R))}\\ &\leqslant K(c)\|\varphi\|_{H_{Q_c}},\end{aligned}$$

and similarly $\int_{B(\pm\tilde{d}_c\vec{e}_1,R)}\left|\mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_{\pm1}\overline{\tilde{V}_{\pm1}\psi}\Big)\right|\leqslant K(c)\|\varphi\|_{H_{Q_c}}$.

By Cauchy-Schwarz, equation (3.2.2) and Theorem 1.3.1 (for $p=+\infty$), we conclude that

$$\begin{aligned}\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)}\left|\mathfrak{Re}\Big(\partial_cQ_c\overline{Q_c\psi^{\neq0}}\Big)\right| &\leqslant K(c)\sqrt{\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)}|\nabla\psi|^2|Q_c|^4}\\ &\leqslant K(c)\|\varphi\|_{H_{Q_c}}.\end{aligned}$$

We can estimate the other terms similarly. □

3.2.3 On the definition of B_{Q_c}

We start by explaining how to get $B_{Q_c}(\varphi)$ from the "natural" quadratic form

$$\int_{\mathbb{R}^2}|\nabla\varphi|^2-(1-|Q_c|^2)|\varphi|^2+2\mathfrak{Re}^2(\overline{Q_c}\varphi)-\mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}).$$

For the first three terms of this quantity, it is obvious that they are well defined for $\varphi\in H_{Q_c}$, but the term $-\mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi})$ is not clearly integrable.

Take a smooth cutoff function η such that $\eta(x)=0$ on $B(\pm\tilde{d}_c\vec{e}_1,1)$, $\eta(x)=1$ on $\mathbb{R}^2\backslash B(\pm\tilde{d}_c\vec{e}_1,2)$. Then, taking for now $\varphi=Q_c\psi\in C_c^\infty(\mathbb{R}^2)$,

$$\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi})=\eta\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi})+(1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}),$$

and writing $\varphi=Q_c\psi$,

$$\begin{aligned}\eta\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}) &= \eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2+\eta\mathfrak{Re}(i\partial_{x_2}\psi\bar{\psi})|Q_c|^2\\ &= \eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2-\eta\mathfrak{Re}\psi\,\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2\\ &+ \eta\mathfrak{Re}\partial_{x_2}\psi\,\mathfrak{Im}\psi|Q_c|^2.\end{aligned}$$

Furthermore,

$$\begin{aligned}\eta\mathfrak{Re}\partial_{x_2}\psi\mathfrak{Im}\,\psi|Q_c|^2 &= \partial_{x_2}(\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2)\\ &- \partial_{x_2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2-\eta\mathfrak{Re}\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2\\ &- \eta\mathfrak{Re}\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2),\end{aligned}$$

thus we can write

$$\begin{aligned}\int_{\mathbb{R}^2}\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi}) &= \int_{\mathbb{R}^2}\partial_{x_2}(\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2)\\ &+ \int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi})+\int_{\mathbb{R}^2}\eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2\\ &- 2\int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2-\int_{\mathbb{R}^2}\partial_{x_2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2\\ &- \int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2).\end{aligned}$$

The only difficulty here is that the first integral is not well defined for $\varphi\in H_{Q_c}$, but it is the integral of a derivative. Therefore, this is why we defined instead the quadratic form

$$\begin{aligned}B_{Q_c}(\varphi) &= \int_{\mathbb{R}^2}|\nabla\varphi|^2-(1-|Q_c|^2)|\varphi|^2+2\mathfrak{Re}^2(\overline{Q_c}\varphi)\\ &- c\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi})-c\int_{\mathbb{R}^2}\eta\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2\\ &+ 2c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2+c\int_{\mathbb{R}^2}\partial_{x_2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2\\ &+ c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2).\end{aligned}$$

It is easy to check that this quantity is independent of the choice of η. We will show in Lemma 3.2.3 that this quantity is well defined for $\varphi\in H_{Q_c}$. By adding some conditions on φ, for instance if $\varphi\in H^1(\mathbb{R}^2)$, we can show that $\int_{\mathbb{R}^2}\partial_{x_2}(\eta\mathfrak{Re}\psi\mathfrak{Im}\,\psi|Q_c|^2)$ is well defined and is 0. In these cases, we therefore have

$$B_{Q_c}(\varphi)=\int_{\mathbb{R}^2}|\nabla\varphi|^2-(1-|Q_c|^2)|\varphi|^2+2\,\mathfrak{Re}^2(\overline{Q_c}\varphi)-\mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}).$$

This is a classical situation for Schrödinger equations with nonzero limit at infinity (see [8] or [32]): the quadratic form is defined up to a term which is a derivative of some function in some L^p space.

Lemma 3.2.3. *There exists $c_0>0$ such that, for $0<c\leqslant c_0$, Q_c defined in Theorem 1.3.1, there exists a constant $C(c)>0$ such that, for $\varphi=Q_c\psi\in H_{Q_c}$ and η a smooth cutoff function such that $\eta(x)=0$ on $B(\pm\tilde{d}_c\vec{e}_1,1)$, $\eta(x)=1$ on $\mathbb{R}^2\backslash B(\pm\tilde{d}_c\vec{e}_1,2)$, we have*

$$\begin{aligned}&\int_{\mathbb{R}^2}|(1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar{\varphi})|+\int_{\mathbb{R}^2}|\eta\mathfrak{Re}\,(i\partial_{x_2}Q_c\overline{Q_c})|\psi|^2|\\ +\;&\int_{\mathbb{R}^2}|\eta\mathfrak{Re}\,\psi\mathfrak{Im}(\partial_{x_2}\psi)|Q_c|^2|+\int_{\mathbb{R}^2}|\partial_{x_2}\eta\mathfrak{Re}\psi\mathfrak{Im}\psi|Q_c|^2|\\ +\;&\int_{\mathbb{R}^2}|\eta\mathfrak{Re}\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2)|\\ \leqslant\;& C(c)\|\varphi\|^2_{H_{Q_c}}.\end{aligned}$$

Proof. Since $|1-|Q_c|^2|\geqslant K>0$ on $B(\pm\tilde{d}_c\vec{e}_1,2)$ for c small enough by Lemma 1.2.1 and Theorem 1.3.1, we estimate

$$\int_{\mathbb{R}^2}|(1-\eta)\mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi})|\leqslant C(c)\int_{B(\tilde{d}_c\vec{e}_1,2)\cup B(-\tilde{d}_c\vec{e}_1,2)}|1-|Q_c|^2||\varphi||\partial_{x_2}\varphi|\leqslant C(c)\|\varphi\|^2_{H_{Q_c}}.$$

Furthermore, by (3.1.12) and Lemma 3.1.5,

$$\int_{\mathbb{R}^2}|\eta\mathfrak{Re}(ic\partial_{x_2}Q_c\overline{Q_c})|\psi|^2|\leqslant C(c)\int_{\mathbb{R}^2}\eta|\nabla Q_c||\psi|^2\leqslant C(c)\int_{\mathbb{R}^2}\eta|\nabla Q_c||\varphi|^2\leqslant C(c)\|\varphi\|^2_{H_{Q_c}}$$

since $|\nabla Q_c| \leqslant \frac{C(c)}{(1+r)^2}$ from Theorem 3.1.4. By Cauchy-Schwarz, equations (3.1.12) and Lemma 3.2.1,

$$\int_{\mathbb{R}^2} |\eta \mathfrak{Re}\psi \mathfrak{Im}\partial_{x_2}\psi |Q_c|^2| \leqslant K\sqrt{\int_{\mathbb{R}^2} \eta \mathfrak{Re}^2(\psi) \int_{\mathbb{R}^2} \eta |\nabla\psi|^2} \leqslant C(c)\|\varphi\|^2_{H_{Q_c}}. \tag{3.2.3}$$

Now, still by equations (3.1.12) and Lemma 3.2.1, since $\partial_{x_2}\eta$ is supported in $B(\pm\tilde{d}_c\vec{e}_1, 2)\backslash B(\pm\tilde{d}_c\vec{e}_1, 1)$,

$$\int_{\mathbb{R}^2} |\partial_{x_2}\eta \mathfrak{Re}\,\psi \mathfrak{Im}\,\psi |Q_c|^2| \leqslant K\|\varphi\|^2_{H_{Q_c}}.$$

Finally, since $|\nabla Q_c| \leqslant \frac{C(c)}{(1+r)^2}$ by Theorem 3.1.4, by Cauchy-Schwarz and Lemma 3.1.5,

$$\int_{\mathbb{R}^2} |\eta \mathfrak{Re}\,\psi \mathfrak{Im}\,\psi \partial_{x_2}(|Q_c|^2)| \leqslant C(c)\sqrt{\int_{\mathbb{R}^2} \eta \mathfrak{Re}^2(\psi) \int_{\mathbb{R}^2} \eta \frac{\mathfrak{Im}^2\psi}{(1+r)^4}} \leqslant C(c)\|\varphi\|^2_{H_{Q_c}}. \qquad \square$$

3.2.4 Density of test functions in H_{Q_c}

We shall prove the coercivity with test functions, that are 0 in a vicinity of the zeros of Q_c. This will allow us to divide by Q_c in several computations. We give here a density result to show that it is not a problem to remove a vicinity of the zeros of Q_c for test functions.

Lemma 3.2.4. *$C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\}, \mathbb{C})$ is dense in H_{Q_c} for the norm $\|.\|_{H_{Q_c}}$.*

This result uses similar arguments as [10] for the density in H_{V_1}. For the sake of completeness, we give a proof of it.

Proof. We recall that

$$\|\varphi\|^2_{H_{Q_c}} = \int_{\mathbb{R}^2} |\nabla\varphi|^2 + |1-|Q_c|^2||\varphi|^2 + \mathfrak{Re}^2(\overline{Q_c}\varphi),$$

and since, for all $\lambda > 0$,

$$K_1(\lambda)\int_{B(0,\lambda)} |\nabla\varphi|^2 + |\varphi|^2 \leqslant \int_{B(0,\lambda)} |\nabla\varphi|^2 + |1-|Q_c|^2||\varphi|^2 + \mathfrak{Re}^2(\overline{Q_c}\varphi) \leqslant K_2(\lambda)\int_{B(0,\lambda)} |\nabla\varphi|^2 + |\varphi|^2,$$

by standard density argument, we have that $C_c^\infty(\mathbb{R}^2, \mathbb{C})$ is dense in H_{Q_c} for the norm $\|.\|_{H_{Q_c}}$.

We are therefore left with the proof that $C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\}, \mathbb{C})$ is dense in $C_c^\infty(\mathbb{R}^2, \mathbb{C})$ for the norm $\|.\|_{H_{Q_c}}$. For that, it is enough to check that $C_c^\infty(B(0,2)\backslash\{0\}, \mathbb{C})$ is dense in $C_c^\infty(B(0,2), \mathbb{C})$ for the norm $\|.\|_{H^1(B(0,2))}$. This result is a consequence of the fact that the capacity of a point in a ball in dimension 2 is 0. For the sake of completeness, we give here a proof of this result.

We define $\eta_\varepsilon \in C^0(B(0,2), \mathbb{R})$ the radial function with $\eta_\varepsilon(x) = 0$ if $|x| \leqslant \varepsilon$, $\eta_\varepsilon(x) = -\frac{\ln(|x|)}{\ln(\varepsilon)} + 1$ if $|x| \in [\varepsilon, 1]$ and $\eta_\varepsilon(x) = 1$ if $2 \geqslant |x| \geqslant 1$. Then, we define $\eta_{\varepsilon,\lambda} \in C^\infty(B(0,2), \mathbb{R})$ a radial regularisation of η_ε with $\eta_{\varepsilon,\lambda}(x) = 0$ if $|x| \leqslant \varepsilon/2$ such that $\eta_{\varepsilon,\lambda} \to \eta_\varepsilon$ in $H^1(B(0,2))$ when $\lambda \to 0$. Finally, we define $\eta_{\varepsilon,\lambda,\delta} = \eta_{\varepsilon,\lambda}\left(\frac{x}{\delta}\right)$ for a small $\delta > 0$.

Now, given $\varphi \in C_c^\infty(B(0,2), \mathbb{C})$, $\eta_{\varepsilon,\lambda,\delta}\varphi \in C_c^\infty(B(0,2)\backslash\{0\}, \mathbb{C})$ for all $\varepsilon > 0, \lambda > 0, \delta > 0$, and by dominated convergence, we check that

$$\int_{B(0,2)} |\eta_{\varepsilon,\lambda,\delta}\varphi|^2 \to \int_{B(0,2)} |\varphi|^2$$

when $\delta \to 0$. Furthermore, we compute by integration by parts

$$\begin{aligned}
\int_{B(0,2)} |\nabla(\eta_{\varepsilon,\lambda,\delta}\varphi)|^2 &= \int_{B(0,2)} \eta^2_{\varepsilon,\lambda,\delta}|\nabla\varphi|^2 + 2\int_{B(0,2)} \nabla\eta_{\varepsilon,\lambda,\delta}\eta_{\varepsilon,\lambda,\delta}\mathfrak{Re}(\nabla\varphi\bar{\varphi}) \\
&+ \int_{B(0,2)} |\nabla\eta_{\varepsilon,\lambda,\delta}|^2|\varphi|^2 \\
&= \int_{B(0,2)} \eta^2_{\varepsilon,\lambda,\delta}|\nabla\varphi|^2 - \int_{B(0,2)} |\varphi|^2\Delta\eta_{\varepsilon,\lambda,\delta}\eta_{\varepsilon,\lambda,\delta}.
\end{aligned}$$

Now, extending φ to $\mathbb{R}^2$ by $\varphi = 0$ outside of $B(0,2)$, we have by change of variables

$$\int_{B(0,2)} |\varphi|^2 \Delta\eta_{\varepsilon,\lambda,\delta}\eta_{\varepsilon,\lambda,\delta} = \int_{\mathbb{R}^2} |\varphi|^2 \Delta\eta_{\varepsilon,\lambda,\delta}\eta_{\varepsilon,\lambda,\delta} = \int_{\mathbb{R}^2} |\varphi|^2(x\delta)\Delta\eta_{\varepsilon,\lambda}\eta_{\varepsilon,\lambda}.$$

When $\delta \to 0$, we have by dominated convergence that $\int_{B(0,2)} \eta^2_{\varepsilon,\lambda,\delta}|\nabla\varphi|^2 \to \int_{B(0,2)} |\nabla\varphi|^2$ and

$$\int_{\mathbb{R}^2} |\varphi|^2(x\delta)\Delta\eta_{\varepsilon,\lambda}\eta_{\varepsilon,\lambda} \to |\varphi|^2(0)\int_{\mathbb{R}^2} \Delta\eta_{\varepsilon,\lambda}\eta_{\varepsilon,\lambda} = -|\varphi|^2(0)\int_{\mathbb{R}^2} |\nabla\eta_{\varepsilon,\lambda}|^2.$$

Now, taking $\lambda \to 0$, we deduce that

$$\lim_{\lambda\to 0}\lim_{\delta\to 0} \int_{B(0,2)} |\nabla(\eta_{\varepsilon,\lambda,\delta}\varphi)|^2 = \int_{B(0,2)} |\nabla\varphi|^2 - |\varphi|^2(0)\int_{\mathbb{R}^2} |\nabla\eta_\varepsilon|^2.$$

From the definition of η_ε, we compute

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla\eta_\varepsilon|^2 &= \int_\varepsilon^1 \frac{1}{\ln(\varepsilon)^2 r^2} r\,dr \\ &= \frac{1}{\ln(\varepsilon)^2}\int_\varepsilon^1 \frac{1}{r} dr \\ &= \frac{-1}{\ln(\varepsilon)} \to 0\end{aligned}$$

when $\varepsilon \to 0$. We deduce that

$$\lim_{\varepsilon\to 0}\lim_{\lambda\to 0}\lim_{\delta\to 0} \int_{B(0,2)} |\nabla(\eta_{\varepsilon,\lambda,\delta}\varphi)|^2 = \int_{B(0,2)} |\nabla\varphi|^2.$$

This concludes the proof of this lemma. □

3.3 Coercivity results in H_{Q_c}

This section is devoted to the proofs of Propositions 1.4.2 and 1.4.3. Here, we will do most of the computations with test functions, that is functions in $C_c^\infty\big(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\}, \mathbb{C}\big)$. This will allow to do many computations, including dividing by Q_c in some quantities.

3.3.1 Expression of the quadratic forms

We recall that η if a smooth cutoff function such that $\eta(x) = 0$ on $B(\pm\tilde{d}_c\vec{e}_1, 1)$, $\eta(x) = 1$ on $\mathbb{R}^2\backslash(B(\tilde{d}_c\vec{e}_1, 2)\cup B(-\tilde{d}_c\vec{e}_1, 2))$, where $\pm\tilde{d}_c\vec{e}_1$ are the zeros of Q_c. Furthermore, from [10], we recall the quadratic form around a vortex V_1:

$$B_{V_1}(\varphi) = \int_{\mathbb{R}^2} |\nabla\varphi|^2 - (1-|V_1|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\bar{V}_1\varphi).$$

We want to write the quadratic form around V_1 and Q_c in a special form. For the one around Q_c, it will be of the form $B^{\exp}_{Q_c}$, defined in (1.4.4).

Lemma 3.3.1. *For $\varphi = Q_c\psi \in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\}, \mathbb{C})$, we have*

$$\langle L_{Q_c}(\varphi), \varphi\rangle = B^{\exp}_{Q_c}(\varphi),$$

where $B^{\exp}_{Q_c}(\varphi)$ is defined in (1.4.4). Furthermore, for $\varphi = V_1\psi \in C_c^\infty(\mathbb{R}^2\backslash\{0\}, \mathbb{C})$, where V_1 is centered at 0, and $\tilde{\eta}$ a smooth radial cutoff function with value 0 in $B(0,1)$, and value 1 outside of $B(0,2)$,

$$\begin{aligned}B_{V_1}(\varphi) &= \int_{\mathbb{R}^2} (1-\tilde{\eta})(|\nabla\varphi|^2 - (1-|V_1|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\bar{V}_1\varphi)) \\ &- \int_{\mathbb{R}^2} \nabla\tilde{\eta}.(\mathfrak{Re}(\nabla V_1\bar{V}_1)|\psi|^2 - 2\mathfrak{Im}(\nabla V_1\bar{V}_1)\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)) \\ &+ \int_{\mathbb{R}^2} \tilde{\eta}(|\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 + 4\mathfrak{Im}(\nabla V_1\bar{V}_1)\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)).\end{aligned}$$

Proof. We recall that $L_{Q_c}(\varphi) = -ic\partial_{x_2}\varphi - \Delta\varphi - (1-|Q_c|^2)\varphi + 2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c$. Writing $\varphi = Q_c\psi \in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d_c}\vec{e_1}, -\tilde{d_c}\vec{e_1}\}, \mathbb{C})$, we decompose

$$L_{Q_c}(\varphi) = -ic\partial_{x_2}\psi Q_c - \Delta\psi Q_c - 2\nabla Q_c.\nabla\psi + 2\mathfrak{Re}(\psi)|Q_c|^2Q_c + \mathrm{TW}_c(Q_c)\psi.$$

Since $\mathrm{TW}_c(Q_c) = 0$,

$$\begin{aligned}
&\langle L_{Q_c}(\varphi), \varphi\rangle \\
=\ &\langle (1-\eta)L_{Q_c}(\varphi), \varphi\rangle + \langle \eta L_{Q_c}(\varphi), Q_c\psi\rangle \\
=\ &\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}((-ic\partial_{x_2}\varphi - \Delta\varphi - (1-|Q_c|^2)\varphi + 2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c)\bar{\varphi}) \\
+\ &\int_{\mathbb{R}^2}\eta\mathfrak{Re}((-ic\partial_{x_2}\psi Q_c - \Delta\psi Q_c - 2\nabla Q_c.\nabla\psi + 2\mathfrak{Re}(\psi)|Q_c|^2Q_c)\overline{Q_c\psi}).
\end{aligned}$$

By integration by parts,

$$\begin{aligned}
&\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}((-ic\partial_{x_2}\varphi - \Delta\varphi - (1-|Q_c|^2)\varphi + 2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c)\bar{\varphi}) \\
=\ &\int_{\mathbb{R}^2}(1-\eta)(|\nabla\varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) - (1-|Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi)) \\
-\ &\int_{\mathbb{R}^2}\nabla\eta.\mathfrak{Re}(\nabla\varphi\bar{\varphi}).
\end{aligned}$$

Similarly, we compute

$$\begin{aligned}
&\int_{\mathbb{R}^2}\eta\mathfrak{Re}((-ic\partial_{x_2}\psi Q_c - \Delta\psi Q_c - 2\nabla Q_c.\nabla\psi + 2\mathfrak{Re}(\psi)|Q_c|^2Q_c)\overline{Q_c\psi}) \\
=\ &\int_{\mathbb{R}^2}\eta(\mathfrak{Re}(-ic\partial_{x_2}\psi\bar{\psi}|Q_c|^2) - \mathfrak{Re}(\Delta\psi\bar{\psi})|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4 - 2\mathfrak{Re}(\nabla Q_c.\nabla\psi\overline{Q_c\psi})) \\
=\ &\int_{\mathbb{R}^2}\eta(c|Q_c|^2(\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi) - \mathfrak{Re}(\partial_{x_2}\psi)\mathfrak{Im}(\psi)) + 2\mathfrak{Re}^2(\psi)|Q_c|^4 - 2\mathfrak{Re}(\nabla Q_c.\nabla\psi\overline{Q_c\psi})) \\
+\ &\int_{\mathbb{R}^2}\eta|\nabla\psi|^2|Q_c|^2 + 2\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla Q_c\overline{Q_c}).\mathfrak{Re}(\nabla\psi\bar{\psi}) + \int_{\mathbb{R}^2}\nabla\eta.\mathfrak{Re}(\nabla\psi\bar{\psi})|Q_c|^2.
\end{aligned}$$

We continue, we have

$$\begin{aligned}
&-\int_{\mathbb{R}^2}\eta|Q_c|^2\mathfrak{Re}(\partial_{x_2}\psi)\mathfrak{Im}(\psi) \\
=\ &\int_{\mathbb{R}^2}\eta|Q_c|^2\mathfrak{Re}(\psi)\mathfrak{Im}(\partial_{x_2}\psi) + \int_{\mathbb{R}^2}\partial_{x_2}\eta|Q_c|^2\mathfrak{Re}(\psi)\mathfrak{Im}(\psi) + 2\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\partial_{x_2}Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi),
\end{aligned}$$

as well as

$$\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla Q_c.\nabla\psi\overline{Q_c\psi}) = \int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla Q_c\overline{Q_c}).\mathfrak{Re}(\nabla\psi\bar{\psi}) + \int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi\bar{\psi}),$$

therefore

$$\begin{aligned}
&\int_{\mathbb{R}^2}\eta\mathfrak{Re}((-ic\partial_{x_2}\psi Q_c - \Delta\psi Q_c - 2\nabla Q_c.\nabla\psi + 2\mathfrak{Re}(\psi)|Q_c|^2Q_c)\overline{Q_c\psi}) \\
=\ &\int_{\mathbb{R}^2}\eta(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4 + 2c\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)) \\
+\ &\int_{\mathbb{R}^2}\eta(2c\mathfrak{Re}(\partial_{x_2}Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi) - 2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi\bar{\psi})) \\
+\ &c\int_{\mathbb{R}^2}\partial_{x_2}\eta\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)|Q_c|^2 + \int_{\mathbb{R}^2}\nabla\eta.\mathfrak{Re}(\nabla\psi\bar{\psi})|Q_c|^2.
\end{aligned}$$

Since $ic\partial_{x_2}Q_c = \Delta Q_c + (1-|Q_c|^2)Q_c$, we have $c\mathfrak{Re}(\partial_{x_2}Q_c\overline{Q_c}) = \mathfrak{Re}(i\Delta Q_c\overline{Q_c})$. By integration by parts,

$$\begin{aligned}
&2\int_{\mathbb{R}^2}\eta\mathfrak{Re}(i\Delta Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi) \\
=\ &2\int_{\mathbb{R}^2}\nabla\eta.\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi) \\
-\ &2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q_c\overline{Q_c}).\mathfrak{Re}(\nabla\psi)\mathfrak{Im}(\psi) - 2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q_c\overline{Q_c}).\mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi),
\end{aligned}$$

and

$$\begin{aligned} & -2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi\bar{\psi}) \\ = \; & -2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q_c\overline{Q_c})(\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)-\mathfrak{Im}(\psi)\mathfrak{Re}(\nabla\psi)). \end{aligned}$$

Combining these estimates, with

$$\int_{\mathbb{R}^2}\nabla\eta.\mathfrak{Re}(\nabla\varphi\bar{\varphi})=\int_{\mathbb{R}^2}\nabla\eta.(\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2+\mathfrak{Re}(\nabla\psi\bar{\psi})|Q_c|^2),$$

we conclude the proof of

$$\langle L_{Q_c}(\varphi),\varphi\rangle=B^{\exp}_{Q_c}(\varphi).$$

Now, for the proof for $B_{V_1}(\varphi)$, the computations are identical, simply replacing c by 0, η by $\tilde{\eta}$, and Q_c by V_1. □

3.3.2 A coercivity result for the quadratic form around one vortex

This subsection is devoted to the proof of Proposition 1.4.2, and a localized version of it (see Lemma 3.3.2).

3.3.2.1 Coercivity for test functions

Proof. (of Proposition 1.4.2) We recall the result from [10], see Lemma 3.1 and equation (2.42) there. If $\varphi=V_1\psi\in C_c^\infty(\mathbb{R}^2\backslash\{0\},\mathbb{C})$ with the two orthogonality conditions

$$\int_{B(0,R)}\mathfrak{Re}(\partial_{x_1}V_1\bar{\varphi})=\int_{B(0,R)}\mathfrak{Re}(\partial_{x_1}V_1\bar{\varphi})=0,$$

then, writing $\psi^0(x)=\frac{1}{2\pi}\int_0^{2\pi}\psi(|x|\cos(\theta),|x|\sin(\theta)d\theta)$, the 0-harmonic around 0 of ψ, and $\psi^{\neq 0}=\psi-\psi^0$, then

$$B_{V_1}(\varphi)\geqslant K\int_{\mathbb{R}^2}|\nabla(V_1\psi^{\neq 0})|^2+|\nabla\psi^0|^2|V_1|^2+\frac{|V_1\psi^{\neq 0}|^2}{(1+r)^2}+\mathfrak{Re}^2(\psi)|V_1|^4.$$

We recall from Lemma 1.2.1 that there exists $K_1>0$ such that $K_1\leqslant\frac{|V_1|}{r}\leqslant\frac{1}{K_1}$, and that $|V_1|$ is a radial function around 0. Therefore, by Hardy inequality in dimension 4,

$$\int_{B(0,1)}|\psi^0|^2\leqslant K\bigg(\int_{B(0,2)}|\nabla\psi^0|^2|V_1|^2+\int_{B(0,2)\backslash B(0,1)}|\psi^0|^2\bigg).$$

By Poincaré inéquality, using $\int_{B(0,R)\backslash B(0,R/2)}\mathfrak{Im}(\psi)=0$ and $|V_1|^2\geqslant K$ outside of $B(0,1)$, we have

$$\int_{B(0,10)\backslash B(0,1)}|\psi^0|^2\leqslant K\bigg(\int_{B(0,R)}|\nabla\psi^0|^2|V_1|^2+\mathfrak{Re}^2(\psi^0)|V_1|^4\bigg).$$

Here, the constant $K>0$ depends on $R>0$, but we consider R as a universal constant. We deduce that

$$\begin{aligned} \int_{B(0,10)}|\varphi|^2 &\leqslant \int_{B(0,10)}|V_1\psi|^2 \\ &\leqslant K\bigg(\int_{B(0,10)}|V_1\psi^0|^2+\int_{B(0,10)}|V_1\psi^{\neq 0}|^2\bigg) \\ &\leqslant K\bigg(\int_{\mathbb{R}^2}|\nabla(V_1\psi^{\neq 0})|^2+|\nabla\psi^0|^2|V_1|^2+\frac{|V_1\psi^{\neq 0}|^2}{(1+r)^2}+\mathfrak{Re}^2(\psi)|V_1|^4\bigg). \end{aligned}$$

Similarly,

$$\begin{aligned}\int_{B(0,10)}|\nabla\varphi|^2 &\leqslant \int_{B(0,10)}|\nabla(V_1(\psi^0+\psi^{\neq 0}))|^2\\ &\leqslant K\Big(\int_{B(0,10)}|\nabla(V_1\psi^0)|^2+\int_{B(0,10)}|\nabla(V_1\psi^{\neq 0})|^2\Big)\\ &\leqslant K\Big(\int_{B(0,10)}|\nabla\psi^0|^2|V_1|^2+|\psi^0|^2|\nabla V_1|^2+\int_{B(0,10)}|\nabla(V_1\psi^{\neq 0})|^2\Big)\\ &\leqslant K\Big(\int_{\mathbb{R}^2}|\nabla(V_1\psi^{\neq 0})|^2+|\nabla\psi^0|^2|V_1|^2+\frac{|V_1\psi^{\neq 0}|^2}{(1+r)^2}+\mathfrak{Re}^2(\psi)|V_1|^4\Big).\end{aligned}$$

Finally, outside of $B(0,5)$, we have, by Lemma 1.2.1, that

$$\int_{\mathbb{R}^2\setminus B(0,5)}|\nabla\psi|^2\leqslant K\int_{\mathbb{R}^2\setminus B(0,5)}|\nabla\psi|^2|V_1|^2.$$

Let us show that

$$\int_{\mathbb{R}^2\setminus B(0,5)}\frac{|\psi|^2}{r^2\ln^2(r)}\leqslant K\Big(\int_{\mathbb{R}^2\setminus B(0,5)}|\nabla\psi|^2+\int_{B(0,10)\setminus B(0,5)}|\psi|^2\Big).$$

This is a Hardy type inequality, and it would conclude the proof of this proposition. Remark that for the harmonics other than zeros, this is a direct consequence of

$$\int_{\mathbb{R}^2\setminus B(0,5)}\frac{|\psi^{\neq 0}|^2}{r^2}\leqslant\int_{\mathbb{R}^2\setminus B(0,5)}|\nabla\psi|^2.$$

We therefore suppose that ψ is a radial compactly supported function. We define χ a smooth radial cutoff function with $\chi(r)=0$ if $r\leqslant 4$ and $\chi(r)=1$ if $r\geqslant 5$. Then, by Cauchy-Schwarz,

$$\begin{aligned}\Big|\int_{\mathbb{R}^2\setminus B(0,5)}\frac{\chi(r)|\psi|^2}{r^2\ln^2(r)}\Big| &= \Big|-\int_5^{+\infty}\chi(r)|\psi|^2(r)\partial_r\Big(\frac{1}{\ln(r)}\Big)dr\Big|\\ &= \Big|\int_5^{+\infty}\partial_r(\chi|\psi|^2)(r)\frac{dr}{\ln(r)}\Big|\\ &\leqslant K\Big(\int_{B(0,10)\setminus B(0,5)}|\psi|^2+\int_5^{+\infty}\chi(r)|\psi|(r)\partial_r|\psi|(r)\frac{dr}{\ln(r)}\Big)\\ &\leqslant K\Big(\int_{B(0,10)\setminus B(0,5)}|\psi|^2+\sqrt{\int_{\mathbb{R}^2\setminus B(0,5)}\frac{\chi(r)|\psi|^2}{r^2\ln^2(r)}\int_{\mathbb{R}^2\setminus B(0,5)}|\nabla\psi|^2}\Big).\end{aligned}$$

The proof is complete. □

3.3.2.2 Localisation of the coercivity for one vortex

Now, we want to localize the coercivity result. We define, for $D>10$, $\varphi=V_1\psi\in H_{V_1}$,

$$\begin{aligned}B_{V_1}^{\mathrm{loc}_D}(\varphi) &:= \int_{B(0,D)}(1-\tilde\eta)(|\nabla\varphi|^2-(1-|V_1|^2)|\varphi|^2+2\mathfrak{Re}^2(\bar V_1\varphi))\\ &- \int_{B(0,D)}\nabla\tilde\eta.(\mathfrak{Re}(\nabla V_1\bar V_1)|\psi|^2-2\mathfrak{Im}(\nabla V_1\bar V_1)\mathfrak{Re}(\psi)\mathfrak{Im}(\psi))\\ &+ \int_{B(0,D)}\tilde\eta(|\nabla\psi|^2|V_1|^2+2\mathfrak{Re}^2(\psi)|V_1|^4+4\mathfrak{Im}(\nabla V_1\bar V_1)\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)),\end{aligned}$$

where $\tilde\eta$ is a smooth radial cutoff function such that $\tilde\eta(x)=0$ on $B(0,1)$, $\tilde\eta(x)=1$ on $\mathbb{R}^2\setminus B(0,2)$.

Lemma 3.3.2. *There exist K, R, $D_0>0$ with $D_0>R$, such that, for $D>D_0$ and $\varphi=V_1\psi\in C_c^\infty(\mathbb{R}^2\setminus\{0\},\mathbb{C})$, if the following three orthogonality conditions*

$$\int_{B(0,R)}\mathfrak{Re}(\partial_{x_1}V_1\bar\varphi)=\int_{B(0,R)}\mathfrak{Re}(\partial_{x_2}V_1\bar\varphi)=\int_{B(0,R)\setminus B(0,R/2)}\mathfrak{Im}(\psi)=0$$

are satisfied, then

$$B_{V_1}^{\mathrm{loc}_D}(\varphi) \geqslant K\bigg(\int_{B(0,10)} |\nabla\varphi|^2 + |\varphi|^2 + \int_{B(0,D)\setminus B(0,5)} |\nabla\psi|^2|V_1|^2 + \mathfrak{Re}^2(\psi)|V_1|^4 + \frac{|\psi|^2}{r^2\ln^2(r)}\bigg).$$

Proof. We decompose ψ in harmonics $j\in\mathbb{N}, l\in\{1,2\}$, with the same decomposition as (2.5) of [10]. This decomposition is adapted to the quadratic form $B_{V_1}^{\mathrm{loc}_D}$, see equation (2.4) of [10], that also holds if the integral is only on $B(0,D)$.

For $j=0$, the proof is identical. For $j\geqslant 2$, $l\in\{1,2\}$ from equation (2.38) of [10] (that holds on $B(0,D)$ as the inequality is pointwise), the proof holds if it does for $j=1,\ \ l\in\{1,2\}$.

We therefore focus on the case $j=l=1$. We write $\psi=\psi_1(r)\cos(\theta)+i\psi_2(r)\sin(\theta)$, with ψ_1, $\psi_2\in C_c^\infty(\mathbb{R}^{+*},\mathbb{R})$. The other possibility ($l=2$) is $\psi=\psi_1(r)i\cos(\theta)+\psi_2(r)\sin(\theta)$, which is done similarly. We will show a more general result, that is, for any $\varphi=V_1\psi\in C_c^\infty(\mathbb{R}^2\setminus\{0\},\mathbb{C})$ satisfying the orthogonality conditions,

$$\begin{aligned}&B_{V_1}^{\mathrm{loc}_D}(V_1\psi^{\neq 0})\\ \geqslant\ & K\bigg(\int_{B(0,10)} |\nabla(V_1\psi^{\neq 0})|^2 + |V_1\psi^{\neq 0}|^2 + \int_{B(0,D)\setminus B(0,5)} |\nabla\psi^{\neq 0}|^2|V_1|^2 + \mathfrak{Re}^2(\psi^{\neq 0})|V_1|^4 + \frac{|\psi^{\neq 0}|^2}{r^2}\bigg).\end{aligned}$$

With the previous remark, it is enough to conlcude the proof of this lemma. In the rest of the proof, to simplify the notation, we write ψ instead of $\psi^{\neq 0}$, but it still has no 0-harmonic.

We remark that, for $D>R_0>2$,

$$\begin{aligned}&\int_{B(0,D)\setminus B(0,R_0)} |\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 + 4\mathfrak{Im}(\nabla V_1\bar{V_1}).\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)\\ \geqslant\ & \int_{B(0,D)\setminus B(0,R_0)} |\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 - \frac{K|V_1|^2}{R_0}|\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)|\\ \geqslant\ & \frac{1}{2}\int_{B(0,D)\setminus B(0,R_0)} |\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 \qquad (3.3.1)\end{aligned}$$

if R_0 is large enough. We therefore take $R_0>R$ large enough such that (3.3.1) holds. For $\frac{D}{2}>\lambda>R_0$, we define χ_λ a smooth cutoff function such that $\chi_\lambda(r)=1$ if $r\leqslant\lambda$, $\chi_\lambda=0$ if $r\geqslant 2\lambda$, and $|\chi_\lambda'|\leqslant\frac{K}{\lambda}$. In particular, since $R_0>2$, we have $\mathrm{Supp}(\chi_\lambda')\subset\mathrm{Supp}(\tilde{\eta})$ and $\mathrm{Supp}(1-\tilde{\eta})\subset\mathrm{Supp}(\chi_\lambda)$. This implies that

$$\begin{aligned}&\int_{B(0,D)} (1-\tilde{\eta})(|\nabla\varphi|^2 - (1-|V_1|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\bar{V_1}\varphi))\\ =\ & \int_{B(0,D)} (1-\tilde{\eta})(|\nabla(\chi_\lambda\varphi)|^2 - (1-|V_1|^2)|\chi_\lambda\varphi|^2 + 2\mathfrak{Re}^2(\bar{V_1}\chi_\lambda\varphi))\end{aligned}$$

and

$$\begin{aligned}&\int_{B(0,D)} \nabla\tilde{\eta}.(\mathfrak{Re}(\nabla V_1\bar{V_1})|\psi|^2 - 2\mathfrak{Im}(\nabla V_1\bar{V_1})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi))\\ =\ & \int_{B(0,D)} \nabla\tilde{\eta}.(\mathfrak{Re}(\nabla V_1\bar{V_1})|\chi_\lambda\psi|^2 - 2\mathfrak{Im}(\nabla V_1\bar{V_1})\mathfrak{Re}(\chi_\lambda\psi)\mathfrak{Im}(\chi_\lambda\psi)).\end{aligned}$$

Now, we decompose

$$\begin{aligned}&\int_{B(0,D)} \tilde{\eta}(|\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 + 4\mathfrak{Im}(\nabla V_1\bar{V_1})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi))\\ =\ & \int_{B(0,D)} (1-\chi_\lambda^2)\tilde{\eta}(|\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 + 4\mathfrak{Im}(\nabla V_1\bar{V_1})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi))\\ +\ & \int_{B(0,D)} \chi_\lambda^2\tilde{\eta}(|\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 + 4\mathfrak{Im}(\nabla V_1\bar{V_1})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)),\end{aligned}$$

and by equation (3.3.1),

$$\begin{aligned}&\int_{B(0,D)} (1-\chi_\lambda^2)\tilde{\eta}(|\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4 + 4\mathfrak{Im}(\nabla V_1\bar{V_1})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi))\\ \geqslant\ & K\int_{B(0,D)} (1-\chi_\lambda^2)|\nabla\psi|^2|V_1|^2 + 2\mathfrak{Re}^2(\psi)|V_1|^4.\end{aligned}$$

Furthermore,

$$\begin{aligned}
&\int_{B(0,D)} \chi_\lambda^2\tilde{\eta}(|\nabla\psi|^2|V_1|^2+2\mathfrak{Re}^2(\psi)|V_1|^4+4\mathfrak{Im}(\nabla V_1\bar{V_1})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi))\\
=&\int_{B(0,D)} \tilde{\eta}(|\nabla(\chi_\lambda\psi)|^2|V_1|^2+2\mathfrak{Re}^2(\chi_\lambda\psi)|V_1|^4+4\mathfrak{Im}(\nabla V_1\bar{V_1})\mathfrak{Im}(\nabla(\chi_\lambda\psi))\mathfrak{Re}(\chi_\lambda\psi))\\
-&\int_{B(0,D)} \tilde{\eta}((|\nabla(\chi_\lambda\psi)-\nabla\chi_\lambda\psi|^2-|\nabla(\chi_\lambda\psi)|^2)|V_1|^2-4\mathfrak{Im}(\nabla V_1\bar{V_1}).\nabla\chi_\lambda\mathfrak{Im}(\psi)\mathfrak{Re}(\chi_\lambda\psi)),
\end{aligned}$$

and thus

$$\begin{aligned}
&B_{V_1}^{\mathrm{loc}_D}(V_1\psi)\\
\geqslant\ & B_{V_1}^{\mathrm{loc}_D}(V_1\chi_\lambda\psi)+K\int_{B(0,D)}(1-\chi_\lambda^2)|\nabla\psi|^2|V_1|^2+2\mathfrak{Re}^2(\psi)|V_1|^4\\
-&\int_{B(0,D)} \tilde{\eta}((|\nabla(\chi_\lambda\psi)-\nabla\chi_\lambda\psi|^2-|\nabla(\chi_\lambda\psi)|^2)|V_1|^2-4\mathfrak{Im}(\nabla V_1\bar{V_1}).\nabla\chi_\lambda\mathfrak{Im}(\psi)\mathfrak{Re}(\chi_\lambda\psi)).
\end{aligned}$$

Since $V_1\chi_\lambda\psi\in C_c^\infty(B(0,D))$, we have $B_{V_1}^{\mathrm{loc}_D}(V_1\chi_\lambda\psi)=B_{V_1}(V_1\chi_\lambda\psi)$, and since $\chi_\lambda=1$ in $B(0,R)$ and $V_1\psi$ satisfied the orthogonality conditions, so does $V_1\chi_\lambda\psi$. By Proposition 1.4.2, we deduce that

$$\begin{aligned}
&B_{V_1}^{\mathrm{loc}_D}(V_1\chi_\lambda\psi)\\
\geqslant\ & K\int_{B(0,10)}|\nabla(V_1\chi_\lambda\psi)|^2+|V_1\chi_\lambda\psi|^2\\
+\ & K\int_{B(0,D)\setminus B(0,5)}|\nabla(\chi_\lambda\psi)|^2|V_1|^2+\mathfrak{Re}^2(\chi_\lambda\psi)|V_1|^4+\frac{|\chi_\lambda\psi|^2}{r^2\ln^2(r)}.
\end{aligned}$$

Now, remarking that

$$|\nabla(\chi_\lambda\psi)|^2|V_1|^2\geqslant K_1|\nabla\psi|^2\chi_\lambda^2|V_1|^2-K_2|\nabla\chi_\lambda|^2|\psi|^2|V_1|^2,$$

and since $\chi_\lambda=1$ in $B(0,10)$, we deduce that

$$\begin{aligned}
&B_{V_1}^{\mathrm{loc}_D}(V_1\psi)\\
\geqslant\ & K\left(\int_{B(0,10)}|\nabla\varphi|^2+|\varphi|^2+\int_{B(0,D)\setminus B(0,5)}|\nabla\psi|^2|V_1|^2+\mathfrak{Re}^2(\psi)|V_1|^4\right)\\
-\ & K\int_{B(0,D)}\tilde{\eta}(|(|\nabla(\chi_\lambda\psi)-\nabla\chi_\lambda\psi|^2-|\nabla(\chi_\lambda\psi)|^2)||V_1|^2+|\mathfrak{Im}(\nabla V_1\bar{V_1}).\nabla\chi_\lambda\mathfrak{Im}(\psi)\mathfrak{Re}(\chi_\lambda\psi)|)\\
-\ & K\int_{B(0,D)\setminus B(0,5)}|\nabla\chi_\lambda|^2|\psi|^2|V_1|^2. \qquad (3.3.2)
\end{aligned}$$

Since $\nabla\chi_\lambda$ is supported in $B(0,2\lambda)\setminus B(0,\lambda)$ with $|\nabla\chi_\lambda|\leqslant\frac{K}{\lambda}$, we have

$$\int_{B(0,D)\setminus B(0,5)}|\nabla\chi_\lambda|^2|\psi|^2|V_1|^2\leqslant K\int_{B(0,2\lambda)\setminus B(0,\lambda)}\frac{|\psi|^2}{(1+r)^2}, \qquad (3.3.3)$$

and by Cauchy-Schwarz, we have that

$$\int_{B(0,D)}\tilde{\eta}|\mathfrak{Im}(\nabla V_1\bar{V_1}).\nabla\chi_\lambda\mathfrak{Im}(\psi)\mathfrak{Re}(\chi_\lambda\psi)|\leqslant K\sqrt{\int_{B(0,2\lambda)\setminus B(0,\lambda)}\frac{|\psi|^2}{(1+r)^2}\int_{B(0,D)\setminus B(0,5)}\mathfrak{Re}^2(\psi)}$$

and

$$\begin{aligned}
&\int_{B(0,D)}\tilde{\eta}(|(|\nabla(\chi_\lambda\psi)-\nabla\chi_\lambda\psi|^2-|\nabla(\chi_\lambda\psi)|^2)||V_1|^2)\\
\leqslant\ & K\left(\sqrt{\int_{B(0,2\lambda)\setminus B(0,\lambda)}\frac{|\psi|^2}{(1+r)^2}\int_{B(0,D)\setminus B(0,5)}|\nabla\psi|^2|V_1|^2}+\int_{B(0,2\lambda)\setminus B(0,\lambda)}\frac{|\psi|^2}{(1+r)^2}\right). \qquad (3.3.4)
\end{aligned}$$

Since ψ has no 0 harmonics, we have that

$$\int_{B(0,D)\setminus B(0,5)}\frac{|\psi|^2}{(1+r)^2}\leqslant K\int_{B(0,D)\setminus B(0,5)}|\nabla\psi|^2|V_1|^2.$$

We infer that there exists $D_0 > R_0$ a large constant such that, for $D > D_0$, for all $\varphi = V_1\psi \in C_c^\infty(\mathbb{R}^2\backslash\{0\},\mathbb{C})$, there exists $\lambda \in \left[R_0, \frac{D_0}{2}\right]$ such that

$$\int_{B(0,2\lambda)\backslash B(0,\lambda)} \frac{|\psi|^2}{(1+r)^2} \leqslant \varepsilon \int_{B(0,D)\backslash B(0,5)} |\nabla\psi|^2 |V_1|^2 \tag{3.3.5}$$

for some small fixed constant $\varepsilon > 0$. Indeed, if this does not hold, then $\int_{B(0,D)\backslash B(0,5)} |\nabla\psi|^2|V_1|^2 \neq 0$ and

$$\begin{aligned}
&\int_{B(0,D)\backslash B(0,5)} \frac{|\psi|^2}{(1+r)^2} \\
\geqslant\ & \int_{R_0}^{D_0} \frac{|\psi|^2}{(1+r)^2} r\,dr \\
\geqslant\ & \sum_{n=0}^{\left\lfloor \log_2\left(\frac{D_0}{2R_0}\right)\right\rfloor - 2} \int_{2^n R_0}^{2^{n+1}R_0} \frac{|\psi|^2}{(1+r)^2} r\,dr \\
\geqslant\ & \sum_{n=0}^{\left\lfloor \log_2\left(\frac{D_0}{2R_0}\right)\right\rfloor - 2} \varepsilon \int_{B(0,D)\backslash B(0,5)} |\nabla\psi|^2|V_1|^2 \\
\geqslant\ & \varepsilon\left(\left\lfloor \log_2\left(\frac{D_0}{2R_0}\right)\right\rfloor - 1\right) \int_{B(0,D)\backslash B(0,5)} |\nabla\psi|^2|V_1|^2 \\
>\ & \frac{1}{K} \int_{B(0,D)\backslash B(0,5)} |\nabla\psi|^2|V_1|^2
\end{aligned}$$

for D_0 large enough. Taking $\varepsilon > 0$ small enough, with equation (3.3.2) to (3.3.5), we conclude the proof of this lemma. □

A consequence of Lemma 3.3.2 is that, for a function $\varphi = V_1\psi \in C_c^\infty(\mathbb{R}^2\backslash\{0\},\mathbb{C})$ satisfying the three orthogonality conditions in Lemma 3.3.2 and $D > D_0$, then

$$B_{V_1}^{\mathrm{loc}_D}(\varphi) \geqslant K(D)\|\varphi\|^2_{H^1(B(0,D))}. \tag{3.3.6}$$

3.3.3 Coercivity for a travelling wave near its zeros

We recall from Lemma 3.3.1 that, for $\varphi \in C_c^\infty\big(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\},\mathbb{C}\big)$, we have

$$\begin{aligned}
\langle L_{Q_c}(\varphi), \varphi\rangle \ =\ & \int_{\mathbb{R}^2} (1-\eta)(|\nabla\varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) - (1-|Q_c|^2)|\varphi|^2 + 2\,\mathfrak{Re}^2(\overline{Q_c}\varphi)) \\
&- \int_{\mathbb{R}^2} \nabla\eta.(\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2 - 2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)) \\
&+ \int_{\mathbb{R}^2} c\partial_{x_2}\eta\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)|Q_c|^2 \\
&+ \int_{\mathbb{R}^2} \eta(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4) \\
&+ \int_{\mathbb{R}^2} \eta(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)).
\end{aligned}$$

For $D > D_0$ ($D_0 > 0$ being defined in Lemma 3.3.2), we define, with $\varphi = Q_c\psi$,

$$\begin{aligned}
B_{Q_c}^{\mathrm{loc}_{\pm 1,D}}(\varphi) \ :=\ & \int_{B(\pm\tilde{d}_c\vec{e}_1,D)} (1-\eta)(|\nabla\varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) - (1-|Q_c|^2)|\varphi|^2 + 2\,\mathfrak{Re}^2(\overline{Q_c}\varphi)) \\
&- \int_{B(\pm\tilde{d}_c\vec{e}_1,D)} \nabla\eta.(\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2 - 2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)) \\
&+ \int_{B(\pm\tilde{d}_c\vec{e}_1,D)} c\partial_{x_2}\eta\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)|Q_c|^2 \\
&+ \int_{B(\pm\tilde{d}_c\vec{e}_1,D)} \eta(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4) \\
&+ \int_{B(\pm\tilde{d}_c\vec{e}_1,D)} \eta(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)).
\end{aligned}$$

We infer that this quantity is close enough to $B^{\mathrm{loc}_D}_{\tilde{V}_{\pm 1}}(\varphi)$ for the coercivity to hold, with $\tilde{V}_{\pm 1}$ being centered at $\pm\tilde{d}_c\vec{e}_1$, the zero of Q_c in the right half plane.

Lemma 3.3.3. *There exist R, $D_0 > 0$ with $D_0 > R$, such that, for $D > D_0$, $0 < c < c_0(D)$ and $\varphi = Q_c\psi \in C_c^\infty\big(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1\},\mathbb{C}\big)$, if the following three orthogonality conditions*

$$\int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\big(\partial_{x_1}\tilde{V}_1\bar{\varphi}\big) = \int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\big(\partial_{x_2}\tilde{V}_1\bar{\varphi}\big) = \int_{B(\tilde{d}_c\vec{e}_1,R)\backslash B(\tilde{d}_c\vec{e}_1,R/2)} \mathfrak{Im}(\psi) = 0$$

are satisfied, then

$$B^{\mathrm{loc}_{1,D}}_{Q_c}(\varphi) \geqslant K(D)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))}.$$

Proof. First, remark that we write $\varphi = Q_c\psi$ and not $\varphi = \tilde{V}_1\psi$, as we did in the proof of Proposition 1.4.2. Hence, to apply Lemma 3.3.2, the third orthogonality condition becomes

$$\int_{B(\tilde{d}_c\vec{e}_1,R)\backslash B(\tilde{d}_c\vec{e}_1,R/2)} \mathfrak{Im}\left(\psi\frac{Q_c}{\tilde{V}_1}\right) = 0.$$

With Lemma 3.1.14, we check that

$$\begin{aligned}
&\left|\int_{B(\tilde{d}_c\vec{e}_1,R)\backslash B(\tilde{d}_c\vec{e}_1,R/2)} \mathfrak{Im}\left(\psi\frac{Q_c}{\tilde{V}_1}\right)\right| \\
\leqslant\ &\left|\int_{B(\tilde{d}_c\vec{e}_1,R)\backslash B(\tilde{d}_c\vec{e}_1,R/2)} \mathfrak{Im}(\psi)\right| + o_{c\to 0}(1)\|\psi\|_{L^2(B(\tilde{d}_c\vec{e}_1,R)\backslash B(\tilde{d}_c\vec{e}_1,R/2))} \\
\leqslant\ &\left|\int_{B(\tilde{d}_c\vec{e}_1,R)\backslash B(\tilde{d}_c\vec{e}_1,R/2)} \mathfrak{Im}(\psi)\right| + o^D_{c\to 0}(1)\|\varphi\|_{H^1(B(\tilde{d}_c\vec{e}_1,D))},
\end{aligned}$$

therefore, by standard coercivity argument, we can change this orthogonality condition, given that c is small enough (depending on D). With equation (3.3.6), it is therefore enough to show that

$$|B^{\mathrm{loc}_D}_{Q_c}(\varphi) - B^{\mathrm{loc}_D}_{\tilde{V}_1}(\varphi)| \leqslant o^D_{c\to 0}(1)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))}$$

to complete the proof of this lemma. Thus, for $\varphi = Q_c\psi \in C_c^\infty\big(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1\},\mathbb{C}\big)$, writing $\varphi = V_1\Big(\frac{Q_c}{V_1}\psi\Big)$ in $B^{\mathrm{loc}_D}_{\tilde{V}_1}(\varphi)$, we have

$$\begin{aligned}
&B^{\mathrm{loc}_{1,D}}_{Q_c}(\varphi) - B^{\mathrm{loc}_D}_{\tilde{V}_1}(\varphi) \\
=\ &\int_{B(\tilde{d}_c\vec{e}_1,D)} -\mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) + \big(|Q_c|^2 - |\tilde{V}_1|^2\big)|\varphi|^2 + 2\big(\mathfrak{Re}^2(\overline{Q_c}\varphi) - \mathfrak{Re}^2\big(\bar{\tilde{V}}_1\varphi\big)\big) \\
-\ &\int_{B(\tilde{d}_c\vec{e}_1,D)} \nabla\eta.\big(\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2 - 2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)\big) \\
+\ &\int_{B(\tilde{d}_c\vec{e}_1,D)} \nabla\eta.\left(\mathfrak{Re}\big(\nabla\tilde{V}_1\bar{\tilde{V}}_1\big)\Big|\frac{Q_c}{\tilde{V}_1}\psi\Big|^2 - 2\mathfrak{Im}\big(\nabla\tilde{V}_1\bar{\tilde{V}}_1\big)\mathfrak{Re}\left(\frac{Q_c}{\tilde{V}_1}\psi\right)\mathfrak{Im}\left(\frac{Q_c}{\tilde{V}_1}\psi\right)\right) \\
+\ &\int_{B(\tilde{d}_c\vec{e}_1,D)} c\partial_{x_2}\eta\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)|Q_c|^2 \\
+\ &\int_{B(\tilde{d}_c\vec{e}_1,D)} \eta(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4) \\
-\ &\int_{B(\tilde{d}_c\vec{e}_1,D)} \eta\left(\left|\nabla\left(\frac{Q_c}{\tilde{V}_1}\psi\right)\right|^2|Q_c|^2 + 2\mathfrak{Re}^2\left(\frac{Q_c}{\tilde{V}_1}\psi\right)|Q_c|^4\right) \\
+\ &\int_{B(\tilde{d}_c\vec{e}_1,D)} \eta(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)) \\
-\ &\int_{B(\tilde{d}_c\vec{e}_1,D)} \eta\left(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}\left(\nabla\left(\frac{Q_c}{\tilde{V}_1}\psi\right)\right)\mathfrak{Re}\left(\frac{Q_c}{\tilde{V}_1}\psi\right)\right).
\end{aligned}$$

With Theorem 1.3.1 (for $p=+\infty$) and Cauchy-Schwarz, we check easily that

$$\begin{aligned}&\int_{B(\tilde{d}_c\vec{e}_1,D)}|\mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi})|+||Q_c|^2-|\tilde{V}_1|^2||\varphi|^2+2\big|\mathfrak{Re}^2(\overline{Q_c}\varphi)-\mathfrak{Re}^2\big(\bar{\tilde{V}}_1\varphi\big)\big|\\ \leqslant\ & o^D_{c\to 0}(1)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))}.\end{aligned}$$

Since $\nabla\eta$ is supported in $B\big(\tilde{d}_c\vec{e}_1,2\big)\backslash B\big(\tilde{d}_c\vec{e}_1,1\big)$, still with Theorem 1.3.1 (for $p=+\infty$), we check that

$$\begin{aligned}&\int_{B(\tilde{d}_c\vec{e}_1,D)}\Big|\nabla\eta.\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2-\nabla\eta\mathfrak{Re}\big(\nabla\tilde{V}_1\bar{\tilde{V}}_1\big)\Big|\frac{Q_c}{\tilde{V}_1}\psi\Big|^2\Big|\\ \leqslant\ & K\int_{B(\tilde{d}_c\vec{e}_1,D)}\Big|\nabla\eta.\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\varphi|^2-\nabla\eta\mathfrak{Re}\big(\nabla\tilde{V}_1\bar{\tilde{V}}_1\big)\Big|\frac{Q_c}{\tilde{V}_1}\varphi\Big|^2\Big|\\ \leqslant\ & \Big\|\nabla\eta.\mathfrak{Re}(\nabla Q_c\overline{Q_c})-\nabla\eta\mathfrak{Re}\big(\nabla\tilde{V}_1\bar{\tilde{V}}_1\big)\Big|\frac{Q_c}{\tilde{V}_1}\Big|^2\Big\|_{L^\infty((\tilde{d}_c\vec{e}_1,D))}\|\varphi\|_{H^1(B(\tilde{d}_c\vec{e}_1,D))}\\ \leqslant\ & o^D_{c\to 0}(1)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))}.\end{aligned}$$

We check similarly that the same estimate hold for all the remaining error terms, using the fact that η is supported in $\mathbb{R}^2\backslash B\big(\tilde{d}_c\vec{e}_1,1\big)$. □

Remark that, by density argument (see the proof of Lemma 3.2.4), Lemma 3.3.3 holds for any $\varphi\in H^1(B(0,D))$. Now, we want to remove the orthogonality condition on the phase. For that, we have to change the coercivity norm

Lemma 3.3.4. *There exist R, $D_0>0$ with $D_0>R$, such that, for $D>D_0$, $0<c<c_0(D)$ and $\varphi=Q_c\psi\in C^\infty_c\big(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1\},\mathbb{C}\big)$, if the following two orthogonality conditions*

$$\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big)=\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big)=0$$

are satisfied, then

$$B^{\mathrm{loc}_{1,D}}_{Q_c}(\varphi)\geqslant K(D)\int_{B(\tilde{d}_c\vec{e}_1,D)}|\nabla\psi|^2|Q_c|^4+\mathfrak{Re}^2(\psi)|Q_c|^4.$$

Proof. Take a function $\varphi\in H^1(B(0,D))$ that satisfies the orthogonality conditions

$$\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big)=\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big)=\int_{B(\tilde{d}_c\vec{e}_1,R)\backslash B(\tilde{d}_c\vec{e}_1,R/2)}\mathfrak{Im}(\psi)=0,$$

and let us show that $B^{\mathrm{loc}_{1,D}}_{Q_c}(\varphi)\geqslant K\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))}$. Take $\varepsilon_1,\varepsilon_2,\varepsilon_3\in\mathbb{R}$ and we define

$$\tilde{\varphi}=\varphi-\varepsilon_1\partial_{x_1}Q_c-\varepsilon_2\partial_{x_2}Q_c-\varepsilon_3 iQ_c.$$

We have, for $\varphi=Q_c\psi$, by Theorem 1.3.1 (for $p=+\infty$) and Lemma 3.1.14,

$$\begin{aligned}&\left|\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big)-\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}(\partial_{x_1}Q_c\overline{Q_c\psi})\right|\\ \leqslant\ & \left|\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\left(\partial_{x_1}\tilde{V}_1\frac{\tilde{V}_1}{Q_c}\bar{\varphi}-\partial_{x_1}Q_c\bar{\varphi}\right)\right|\\ \leqslant\ & K\left\|\partial_{x_1}\tilde{V}_1\frac{\tilde{V}_1}{Q_c}-\partial_{x_1}Q_c\right\|_{L^\infty(B(\tilde{d}_c\vec{e}_1,R))}\|\varphi\|_{H^1(B(\tilde{d}_c\vec{e}_1,D))}\\ \leqslant\ & o^D_{c\to 0}(1)\|\varphi\|_{H^1(B(\tilde{d}_c\vec{e}_1,D))}.\end{aligned}$$

Similar estimates hold for $\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big)$. By standard arguments, we check that there exists $\varepsilon_1,\varepsilon_2,\varepsilon_3\in\mathbb{R}$ with $|\varepsilon_1|+|\varepsilon_2|+|\varepsilon_2|\leqslant o_{c\to 0}(1)\|\varphi\|_{H^1(B(\tilde{d}_c\vec{e}_1,D))}$ such that $\tilde{\varphi}$ satisfies the three orthogonality conditions of Lemma 3.3.3. We deduce that, since (by Theorem 1.3.1 for $p=+\infty$)

$$\|\partial_{x_1}Q_c\|_{H^1(B(\tilde{d}_c\vec{e}_1,D))}+\|\partial_{x_2}Q_c\|_{H^1(B(\tilde{d}_c\vec{e}_1,D))}+\|iQ_c\|_{H^1(B(\tilde{d}_c\vec{e}_1,D))}\leqslant K(D),$$

$$\begin{aligned}B^{\mathrm{loc}_1,D}_{Q_c}(\varphi) &\geqslant B^{\mathrm{loc}_1,D}_{Q_c}(\tilde{\varphi})-o^D_{c\to 0}(1)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))}\\ &\geqslant K(D)\|\tilde{\varphi}\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))}-o^D_{c\to 0}(1)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))}\\ &\geqslant K(D)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))}-o^D_{c\to 0}(1)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))}\\ &\geqslant K(D)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))},\end{aligned}$$

given that c is small enough (depending on D). For $\varphi=Q_c\psi$, we infer that

$$\int_{B(\tilde{d}_c\vec{e}_1,D)}|\nabla\psi|^2|Q_c|^4+\mathfrak{Re}^2(\psi)|Q_c|^4\leqslant K(D)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))}.$$

Indeed, we have

$$\int_{B(\tilde{d}_c\vec{e}_1,D)}\mathfrak{Re}^2(\psi)|Q_c|^4\leqslant K\int_{B(\tilde{d}_c\vec{e}_1,D)}\mathfrak{Re}^2(\varphi)\leqslant K\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))},$$

and

$$\begin{aligned}\int_{B(\tilde{d}_c\vec{e}_1,D)}|\nabla\psi|^2|Q_c|^4 &= \int_{B(\tilde{d}_c\vec{e}_1,D)}|\nabla\varphi-\nabla Q_c\psi|^2|Q_c|^2\\ &\leqslant K\bigg(\int_{B(\tilde{d}_c\vec{e}_1,D)}|\nabla\varphi|^2+\int_{B(\tilde{d}_c\vec{e}_1,D)}|\nabla Q_c\psi|^2|Q_c|^2\bigg)\\ &\leqslant K\bigg(\int_{B(\tilde{d}_c\vec{e}_1,D)}|\nabla\varphi|^2+\int_{B(\tilde{d}_c\vec{e}_1,D)}|\varphi|^2\bigg).\end{aligned}$$

We deduce that, under the three orthogonality conditions, for $\varphi=Q_c\psi$,

$$\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big)=\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big)=\int_{B(\tilde{d}_c\vec{e}_1,R)\setminus B(\tilde{d}_c\vec{e}_1,R/2)}\mathfrak{Im}(\psi)=0,$$

then

$$B^{\mathrm{loc}_1,D}_{Q_c}(\varphi)\geqslant K(D)\int_{B(\tilde{d}_c\vec{e}_1,D)}|\nabla\psi|^2|Q_c|^4+\mathfrak{Re}^2(\psi)|Q_c|^4.$$

Now, let us show that for any $\lambda\in\mathbb{R}$, $\varphi\in H^1\big(B\big(\tilde{d}_c\vec{e}_1,D\big)\big)$,

$$B^{\mathrm{loc}_1,D}_{Q_c}(\varphi-i\lambda Q_c)=B^{\mathrm{loc}_1,D}_{Q_c}(\varphi).$$

For $\varphi\in C^\infty_c(\mathbb{R}^2,\mathbb{C})$, we have $L_{Q_c}(\varphi-i\lambda Q_c)=L_{Q_c}(\varphi)\in C^\infty_c(\mathbb{R}^2,\mathbb{C})$, thus $\langle L_{Q_c}(\varphi-i\lambda Q_c),\varphi-i\lambda Q_c\rangle$ is well defined, and

$$\langle L_{Q_c}(\varphi-i\lambda Q_c),\varphi-i\lambda Q_c\rangle=\langle L_{Q_c}(\varphi),\varphi-i\lambda Q_c\rangle=\langle\varphi,L_{Q_c}(\varphi-i\lambda Q_c)\rangle=\langle L_{Q_c}(\varphi),\varphi\rangle.$$

With computations similar to the one of the proof of Lemma 3.3.1 and by density, using $\nabla(\psi-i\lambda)=\nabla\psi$ and $\mathfrak{Re}(\psi-i\lambda)=\mathfrak{Re}(\psi)$, we deduce that $B^{\mathrm{loc}_1,D}_{Q_c}(\varphi-i\lambda Q_c)=B^{\mathrm{loc}_1,D}_{Q_c}(\varphi)$.

Now, for $\lambda\in\mathbb{R}$, $\tilde{\varphi}=\varphi-i\lambda Q_c$, $\tilde{\psi}=\psi-i\lambda$, $\tilde{\varphi}=Q_c\tilde{\psi}$, we have $B^{\mathrm{loc}_1,D}_{Q_c}(\varphi)=B^{\mathrm{loc}_1,D}_{Q_c}(\tilde{\varphi})$,

$$\int_{B(\tilde{d}_c\vec{e}_1,D)}|\nabla\psi|^2|Q_c|^4+\mathfrak{Re}^2(\psi)|Q_c|^4=\int_{B(\tilde{d}_c\vec{e}_1,D)}|\nabla\tilde{\psi}|^2|Q_c|^4+\mathfrak{Re}^2(\tilde{\psi})|Q_c|^4$$

and

$$\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\nabla\tilde{V}_1\overline{\tilde{V}_1\psi}\Big)=\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\nabla\tilde{V}_1\overline{\tilde{V}_1\tilde{\psi}}\Big).$$

For this last equality, it comes from the fact that $\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\big(i\nabla\tilde{V}_1\bar{\tilde{V}}_1\big)=0$, since $\mathfrak{Re}\big(i\nabla\tilde{V}_1\bar{\tilde{V}}_1\big)$ has no zero harmonic (see Lemma 1.2.1). We also check that

$$\int_{B(\tilde{d}_c\vec{e}_1,R)\setminus B(\tilde{d}_c\vec{e}_1,R/2)}\mathfrak{Im}(\psi)=\int_{B(\tilde{d}_c\vec{e}_1,R)\setminus B(\tilde{d}_c\vec{e}_1,R/2)}\mathfrak{Im}(\tilde{\psi})+K\lambda$$

for a universal constant $K>0$. Therefore, choosing $\lambda\in\mathbb{R}$ such that $\int_{B(\tilde{d}_c\vec{e}_1,R)\setminus B(\tilde{d}_c\vec{e}_1,R/2)}\mathfrak{Im}(\tilde{\psi})=0$, we have, for a function $\varphi=Q_c\psi$ that satisfies

$$\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big)=\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big)=0,$$

that

$$\begin{aligned}B^{\mathrm{loc}_1,D}_{Q_c}(\varphi) &= B^{\mathrm{loc}_1,D}_{Q_c}(\tilde{\varphi})\\ &\geqslant \int_{B(\tilde{d}_c\vec{e}_1,D)}|\nabla\tilde{\psi}|^2|Q_c|^4+\mathfrak{Re}^2(\tilde{\psi})|Q_c|^4\\ &= \int_{B(\tilde{d}_c\vec{e}_1,D)}|\nabla\psi|^2|Q_c|^4+\mathfrak{Re}^2(\psi)|Q_c|^4.\end{aligned}$$

This concludes the proof of this lemma. □

3.3.4 Proof of Proposition 1.4.3

Proof. (of Proposition 1.4.3) From Lemma 3.3.1, we have, for $\varphi=Q_c\psi\in C^\infty_c\big(\mathbb{R}^2\setminus\{\tilde{d}_c\vec{e}_1,-\tilde{d}_c\vec{e}_1\},\mathbb{C}\big)$ that

$$\begin{aligned}B_{Q_c}(\varphi) &= \int_{\mathbb{R}^2}(1-\eta)(|\nabla\varphi|^2-\mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi})-(1-|Q_c|^2)|\varphi|^2+2\,\mathfrak{Re}^2(\overline{Q_c}\varphi))\\ &- \int_{\mathbb{R}^2}\nabla\eta.(\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2-2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi))\\ &+ \int_{\mathbb{R}^2}c\partial_{x_2}\eta\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)|Q_c|^2\\ &+ \int_{\mathbb{R}^2}\eta(|\nabla\psi|^2|Q_c|^2+2\mathfrak{Re}^2(\psi)|Q_c|^4)\\ &+ \int_{\mathbb{R}^2}\eta(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)+2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)).\end{aligned}$$

We decompose the integral in three domains, $B\big(\pm\tilde{d}_c\vec{e}_1,\ D\big)$ (which yield $B^{\mathrm{loc}_{\pm1},D}_{Q_c}(\varphi)$) and $\mathbb{R}^2\setminus\big(B\big(\tilde{d}_c\vec{e}_1,D\big)\cup B\big(-\tilde{d}_c\vec{e}_1,D\big)\big)$ for some $D>D_0>0$, where D_0 is defined in Lemma 3.3.3.

Then, with the four orthogonality conditions and Lemma 3.3.3, we check that

$$B^{\mathrm{loc}_1,D}_{Q_c}(\varphi)\geqslant K(D)\int_{B(\tilde{d}_c\vec{e}_1,D)}|\nabla\psi|^2|Q_c|^4+\mathfrak{Re}^2(\psi)|Q_c|^4,$$

and, by symmetry of the problem around $B\big(\pm\tilde{d}_c\vec{e}_1,D\big)$, since $Q_c=-V_{-1}\big(.+\tilde{d}_c\vec{e}_1\big)+o_{c\to0}(1)$ in $L^\infty\big(B\big(-\tilde{d}_c\vec{e}_1,D\big)\big)$, and checking that multiplying the vortex by -1 does not change the result, that

$$B^{\mathrm{loc}_{-1},D}_{Q_c}(\varphi)\geqslant K(D)\int_{B(-\tilde{d}_c\vec{e}_1,D)}|\nabla\psi|^2|Q_c|^4+\mathfrak{Re}^2(\psi)|Q_c|^4.$$

Furthermore, there exist $K_1,K_2>0$, universal constants, such that, outside of $B(\tilde{d}_c\vec{e}_1,1)\cup B(-\tilde{d}_c\vec{e}_1,1)$ for c small enough, we have

$$K_1\geqslant|Q_c|^2\geqslant K_2$$

by (3.1.12). We also have

$$|\mathfrak{Im}(\nabla Q_c\overline{Q_c})|\leqslant K\bigg(\frac{1}{(1+\tilde{r}_1)}+\frac{1}{(1+\tilde{r}_{-1})}\bigg)$$

by (3.1.10). With these estimates and by Cauchy-Schwarz, for $D > D_0$,

$$\begin{aligned}&\int_{\mathbb{R}^2\setminus(B(\tilde{d}_c\vec{e_1},D)\cup B(-\tilde{d}_c\vec{e_1},D))} 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)\\ \geqslant\ & -Kc\int_{\mathbb{R}^2\setminus(B(\tilde{d}_c\vec{e_1},D)\cup B(-\tilde{d}_c\vec{e_1},D))} |\nabla\psi|^2|Q_c|^4+\mathfrak{Re}^2(\psi)|Q_c|^4,\end{aligned}$$

and

$$\begin{aligned}&\int_{\mathbb{R}^2\setminus(B(\tilde{d}_c\vec{e_1},D)\cup B(-\tilde{d}_c\vec{e_1},D))} 4\mathfrak{Im}(\nabla Q_c\overline{Q_c}).\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)\\ \geqslant\ & \frac{-K}{(1+D)}\int_{\mathbb{R}^2\setminus(B(\tilde{d}_c\vec{e_1},D)\cup B(-\tilde{d}_c\vec{e_1},D))} |\nabla\psi|^2|Q_c|^4+\mathfrak{Re}^2(\psi)|Q_c|^4.\end{aligned}$$

Therefore, taking $D > D_0$ large enough (independently of c or c_0, $D \geqslant 10K+1$) and c small enough ($c \leqslant \frac{10}{K}$), we have

$$\begin{aligned}&\int_{\mathbb{R}^2\setminus(B(\tilde{d}_c\vec{e_1},D)\cup B(-\tilde{d}_c\vec{e_1},D))} |\nabla\psi|^2|Q_c|^2+2\mathfrak{Re}^2(\psi)|Q_c|^4\\ +\ & \int_{\mathbb{R}^2\setminus(B(\tilde{d}_c\vec{e_1},D)\cup B(-\tilde{d}_c\vec{e_1},D))} 4\mathfrak{Im}(\nabla Q_c\overline{Q_c}).\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)+2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)\\ \geqslant\ & K\int_{\mathbb{R}^2\setminus(B(\tilde{d}_c\vec{e_1},D)\cup B(-\tilde{d}_c\vec{e_1},D))} |\nabla\psi|^2|Q_c|^4+\mathfrak{Re}^2(\psi)|Q_c|^4.\end{aligned}$$

We deduce that, for $\varphi = Q_c\psi \in C_c^\infty(\mathbb{R}^2\setminus\{\tilde{d_c}\vec{e_1}, -\tilde{d_c}\vec{e_1}\},\mathbb{C})$,

$$B_{Q_c}(\varphi) \geqslant K\|\varphi\|_{\mathcal{C}}^2$$

if

$$\int_{B(\tilde{d}_c\vec{e_1},R)}\mathfrak{Re}\Big(\partial_{x_1}\tilde{V_1}\overline{\tilde{V_1}\psi}\Big)=\int_{B(\tilde{d}_c\vec{e_1},R)}\mathfrak{Re}\Big(\partial_{x_2}\tilde{V_1}\overline{\tilde{V_1}\psi}\Big)=0,$$

$$\int_{B(-\tilde{d}_c\vec{e_1},R)}\mathfrak{Re}\Big(\partial_{x_1}\tilde{V_{-1}}\overline{\tilde{V_{-1}}\psi}\Big)=\int_{B(-\tilde{d}_c\vec{e_1},R)}\mathfrak{Re}\Big(\partial_{x_2}\tilde{V_{-1}}\overline{\tilde{V_{-1}}\psi}\Big)=0.$$

We argue by density to show this result in H_{Q_c}. From Lemma 3.2.1, we know that $\|.\|_{\mathcal{C}}$ is continuous with respect to $\|.\|_{H_{Q_c}}$. Furthermore, we recall from Lemma 3.2.2, that

$$\int_{B(\tilde{d}_c\vec{e_1},R)}\Big|\mathfrak{Re}\Big(\partial_{x_1}\tilde{V_1}\overline{\tilde{V_1}\psi}\Big)\Big| \leqslant K(c)\|\varphi\|_{H_{Q_c}},$$

and similar estimates hold for

$$\int_{B(\tilde{d}_c\vec{e_1},R)}\mathfrak{Re}\Big(\partial_{x_2}\tilde{V_1}\overline{\tilde{V_1}\psi}\Big), \int_{B(-\tilde{d_c}\vec{e_1},R)}\mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi}\Big)$$

and

$$\int_{B(-\tilde{d}_c\vec{e_1},R)}\mathfrak{Re}\Big(\partial_{x_2}\tilde{V_{-1}}\overline{\tilde{V_{-1}}\psi}\Big). \tag{3.3.7}$$

In particular, we check that these quantities are continuous for the norm $\|.\|_{H_{Q_c}}$, and that we can pass to the limit by density in these quantities by Lemma 3.2.4.

We are left with the passage to the limit for the quadratic form. For $\varphi \in H_{Q_c}$, we recall from (1.4.3) that

$$\begin{aligned}B_{Q_c}(\varphi) \ =\ & \int_{\mathbb{R}^2}|\nabla\varphi|^2-(1-|Q_c|^2)|\varphi|^2+2\,\mathfrak{Re}^2(\overline{Q_c}\varphi)\\ +\ & c\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(i\,\partial_{x_2}\varphi\bar{\varphi})+c\int_{\mathbb{R}^2}\eta\mathfrak{Re}(i\,\partial_{x_2}Q_c\overline{Q_c})|\psi|^2\\ -\ & 2c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\,\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2-c\int_{\mathbb{R}^2}\partial_{x_2}\eta\mathfrak{Re}\,\psi\mathfrak{Im}\,\psi|Q_c|^2\\ -\ & c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\,\psi\mathfrak{Im}\,\psi\partial_{x_2}(|Q_c|^2).\end{aligned}$$

Following the proof of Lemma 3.2.3, we check easily that, for $\varphi_1 = Q_c\psi_1$, $\varphi_2 = Q_c\psi_2 \in H_{Q_c}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} |\nabla\varphi_1\overline{\nabla\varphi_2}| + |(1-|Q_c|^2)\varphi_1\overline{\varphi_2}| + |\mathfrak{Re}(\overline{Q_c}\varphi_1)\mathfrak{Re}(\overline{Q_c}\varphi_2)| \\
+ &\int_{\mathbb{R}^2} (1-\eta)|\mathfrak{Re}(i\partial_{x_2}\varphi_1\overline{\varphi_2})| + \int_{\mathbb{R}^2} \eta\,|\mathfrak{Re}(i\partial_{x_2}Q_c\overline{Q_c})||\psi_1\psi_2| \\
+ &\int_{\mathbb{R}^2} \eta\,|\mathfrak{Re}\,\psi_1\mathfrak{Im}\partial_{x_2}\psi_2||Q_c|^2 + \int_{\mathbb{R}^2} |\partial_{x_2}\eta\mathfrak{Re}\,\psi_1\mathfrak{Im}\,\psi_2||Q_c|^2 \\
+ &\int_{\mathbb{R}^2} \eta\,|\mathfrak{Re}\,\psi_1\mathfrak{Im}\,\psi_2\partial_{x_2}(|Q_c|^2)| \\
\leqslant\ & K(c)\|\varphi_1\|_{H_{Q_c}}\|\varphi_2\|_{H_{Q_c}},
\end{aligned}$$

and thus we can pass at the limit in B_{Q_c} by Lemma 3.2.4. This concludes the proof of Proposition 1.4.3. □

3.4 Proof of Theorem 1.4.4 and its corollaries

3.4.1 Link between the sets of orthogononality conditions

The first goal of this subsection is to show that the four particular directions $(\partial_{x_1}Q_c, \partial_{x_2}Q_c, c^2\partial_c Q_c, c\partial_{c^\perp}Q_c)$ are almost orthogonal between them near the zeros of Q_c, and that they can replace the four orthogonality conditions of Proposition 1.4.3. This is computed in the following lemma.

Lemma 3.4.1. *For $R > 0$ given by Proposition 1.4.3, there exist K_1, $K_2 > 0$, two constants independent of c, such that, for Q_c defined in Theorem 1.3.1,*

$$K_1 \leqslant \int_{B(\pm\tilde{d}_c\vec{e}_1,R)} |\partial_{x_1}Q_c|^2 + \int_{B(\pm\tilde{d}_c\vec{e}_1,R)} |\partial_{x_2}Q_c|^2 + \int_{B(\pm\tilde{d}_c\vec{e}_1,R)} |c^2\partial_c Q_c|^2 + \int_{B(\pm\tilde{d}_c\vec{e}_1,R)} |c\partial_{c^\perp}Q_c|^2 \leqslant K_2.$$

Furthermore, for $A, B \in \{\partial_{x_1}Q_c, \partial_{x_2}Q_c, c^2\partial_c Q_c, c\partial_{c^\perp}Q_c\}$, $A \neq B$, we have that, for $1 > \beta_0 > 0$ a small constant,

$$\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}(A\bar{B}) = o_{c\to 0}(c^{\beta_0}).$$

Proof. From Lemma 3.1.1, we have, in $B(\pm\tilde{d}_c\vec{e}_1, R)$, that (for $0 < \sigma = 1-\beta_0 < 1$)

$$Q_c(x) = V_1(x - d_c\vec{e}_1)V_{-1}(x + d_c\vec{e}_1) + o_{c\to 0}(c^{\beta_0})$$

and

$$\nabla Q_c(x) = \nabla(V_1(x - d_c\vec{e}_1)V_{-1}(x + d_c\vec{e}_1)) + o_{c\to 0}(c^{\beta_0}).$$

In this proof a $o_{c\to 0}(c^{\beta_0})$ may depend on R, but we consider R as a universal constant. From Lemmas 1.2.1 and 3.1.12 and equation (3.1.7), we show that, by the mean value theorem, in $B(\pm\tilde{d}_c\vec{e}_1, R)$,

$$Q_c = V_1V_{-1} + o_{c\to 0}(c^{\beta_0}) = V_{\pm 1} + o_{c\to 0}(c^{\beta_0}) = \tilde{V}_{\pm 1} + o_{c\to 0}(c^{\beta_0}) \tag{3.4.1}$$

and, similarly,

$$\nabla Q_c = \nabla\tilde{V}_{\pm 1} + o_{c\to 0}(c^{\beta_0}). \tag{3.4.2}$$

Thus, in $B(\pm\tilde{d}_c\vec{e}_1, R)$, we have

$$\partial_{x_1}Q_c = \partial_{x_1}\tilde{V}_{\pm 1} + o_{c\to 0}(c^{\beta_0}) \tag{3.4.3}$$

and

$$\partial_{x_2}Q_c = \partial_{x_2}\tilde{V}_{\pm 1} + o_{c\to 0}(c^{\beta_0}). \tag{3.4.4}$$

Furthermore, by Lemma 3.1.2, we have in particular that in $B(\pm\tilde{d}_c\vec{e}_1, R)$,

$$c^2\partial_c Q_c = (1 + o_{c\to 0}(c^{\beta_0}))\partial_d(V_1(x - d\vec{e}_1)V_{-1}(x + d\vec{e}_1))_{|d=d_c} + o_{c\to 0}(c^{\beta_0}).$$

Thus, in $B(\pm\tilde{d}_c\vec{e}_1,R)$, with Lemmas 1.2.1 and 3.1.12, we estimate

$$c^2\partial_c Q_c = \mp\partial_{x_1}\tilde{V}_{\pm1} + o_{c\to0}(c^{\beta_0}). \tag{3.4.5}$$

Finally, from Lemma 3.1.6, we have

$$c\partial_{c^\perp}Q_c = -cx^\perp.\nabla Q_c$$

with $x^\perp=(-x_2,x_1)$. In $B(\pm\tilde{d}_c\vec{e}_1,R)$, we have, since $c\tilde{d}_c=1+o_{c\to0}(c^{\beta_0})$ and Lemma 3.1.12,

$$cx^\perp = \mp\vec{e}_2 + o_{c\to0}(c^{\beta_0}).$$

Therefore, in $B(\pm\tilde{d}_c\vec{e}_1,R)$, we have

$$c\partial_{c^\perp}Q_c = \pm\partial_{x_2}\tilde{V}_{\pm1} + o_{c\to0}(c^{\beta_0}). \tag{3.4.6}$$

Now, from Lemma 1.2.1, we have

$$K_1 \leqslant \int_{B(\pm\tilde{d}_c\vec{e}_1,R)} \big|\partial_{x_1}\tilde{V}_{\pm1}\big|^2 + \int_{B(\pm\tilde{d}_c\vec{e}_1,R)} \big|\partial_{x_2}\tilde{V}_{\pm1}\big|^2 \leqslant K_2 \tag{3.4.7}$$

for universal constant $K_1,K_2>0$ (depending only on R). By a change of variable, we have, writing $\tilde{V}_{\pm1}=\rho(\tilde{r}_{\pm1})e^{i\tilde{\theta}_{\pm1}}$ (with the notations of Lemma 1.2.1),

$$\partial_{x_1}\tilde{V}_{\pm1} = \left(\cos\big(\tilde{\theta}_{\pm1}\big)\frac{\rho'(\tilde{r}_{\pm1})}{\rho(\tilde{r}_{\pm1})} - \frac{\pm i}{\tilde{r}_{\pm1}}\sin\big(\tilde{\theta}_{\pm1}\big)\right)\tilde{V}_{\pm1} \tag{3.4.8}$$

and

$$\partial_{x_2}\tilde{V}_{\pm1} = \left(\sin\big(\tilde{\theta}_{\pm1}\big)\frac{\rho'(\tilde{r}_{\pm1})}{\rho(\tilde{r}_{\pm1})} + \frac{\pm i}{\tilde{r}_{\pm1}}\cos\big(\tilde{\theta}_{\pm1}\big)\right)\tilde{V}_{\pm1}. \tag{3.4.9}$$

Since

$$\mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_{\pm1}\overline{\partial_{x_2}\tilde{V}_{\pm1}}\Big) = 2\cos\big(\tilde{\theta}_{\pm1}\big)\sin\big(\tilde{\theta}_{\pm1}\big)\frac{\rho'(\tilde{r}_{\pm1})}{\tilde{r}_{\pm1}\rho(\tilde{r}_{\pm1})}\big|\tilde{V}_{\pm1}\big|^2,$$

by integration in polar coordinates, we have

$$\int_{B(\pm\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_{\pm1}\overline{\partial_{x_2}\tilde{V}_{\pm1}}\Big) = 0. \tag{3.4.10}$$

Combining (3.4.3) to (3.4.6) with (3.4.7) and (3.4.10), we can do every estimate stated in the lemma. □

With (3.4.3) to (3.4.6), we check that these four directions are close to the ones in the orthogonality conditions of Proposition 1.4.3. This will appear in the proof of Lemma 3.4.5. Now, we give a way to develop the quadratic form for some particular functions.

Lemma 3.4.2. *For $\varphi\in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1,-\tilde{d}_c\vec{e}_1\},\mathbb{C})$ and $A\in\mathrm{Span}\{\partial_{x_1}Q_c,\partial_{x_2}Q_c,\partial_cQ_c,\partial_{c^\perp}Q_c\}$, we have*

$$\langle L_{Q_c}(\varphi+A),\varphi+A\rangle = \langle L_{Q_c}(\varphi),\varphi\rangle + \langle 2L_{Q_c}(A),\varphi\rangle + \langle L_{Q_c}(A),A\rangle.$$

Furthermore, $\langle L_{Q_c}(\varphi+A),\varphi+A\rangle = B_{Q_c}(\varphi+A)$ and $\langle L_{Q_c}(A),A\rangle = B_{Q_c}(A)$.

Proof. Since $\varphi\in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1,-\tilde{d}_c\vec{e}_1\},\mathbb{C})$, it is enough to check that $\mathfrak{Re}(L_{Q_c}(A)\bar{A})\in L^1(\mathbb{R}^2,\mathbb{R})$ for $A\in\mathrm{Span}\{\partial_{x_1}Q_c,\partial_{x_2}Q_c,\partial_cQ_c,\partial_{c^\perp}Q_c\}$ to show that

$$\langle L_{Q_c}(\varphi+A),\varphi+A\rangle = \langle L_{Q_c}(\varphi),\varphi\rangle + \langle 2L_{Q_c}(A),\varphi\rangle + \langle L_{Q_c}(A),A\rangle.$$

From Lemma 3.1.7, we have, for $A=\mu_1\partial_{x_1}Q_c+\mu_2\partial_{x_2}Q_c+\mu_3\partial_cQ_c+\mu_4\partial_{c^\perp}Q_c$, that

$$L_{Q_c}(A) = \mu_3 i\partial_{x_2}Q_c - \mu_4 i\partial_{x_1}Q_c.$$

Now, with (3.1.16) (that holds also for A by linearity) and (3.1.9), (3.1.10), we check easily that $\mathfrak{Re}(L_{Q_c}(A)\bar{A})\in L^1(\mathbb{R}^2,\mathbb{R})$.

Now, from subsection 3.2.3, to show that for $\Phi=Q_c\Psi\in H_{Q_c}\cap C^2(\mathbb{R}^2,\mathbb{C})$, we have $\langle L_{Q_c}(\Phi),\Phi\rangle = B_{Q_c}(\Phi)$, it is enough to show that $\int_{\mathbb{R}^2}\partial_{x_2}(\eta\mathfrak{Re}\Psi\mathfrak{Im}\Psi|Q_c|^2)$ is well defined and is 0. For $\Phi=A$ or $\Phi=\varphi+A$, this is a consequence of (3.1.16), Lemma 3.1.16 and $\varphi\in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1,-\tilde{d}_c\vec{e}_1\},\mathbb{C})$. □

3.4.2 Some useful elliptic estimates

We want to improve slightly the coercivity norm near the zeros of Q_c. This is done in the following lemma. The improvement is in the exponent of the weight in front of f^2.

Lemma 3.4.3. *There exists a universal constant $K>0$ such that, for any $D>2$, for V_1 centered at 0 and any function $f\in C_c^\infty\big(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1,-\tilde{d}_c\vec{e}_1\},\mathbb{R}\big)$, we have*

$$\int_{B(\pm\tilde{d}_c\vec{e}_1,D)} f^2|V_1|^3\,dx\leqslant K\int_{B(\pm\tilde{d}_c\vec{e}_1,D)}|\nabla f|^2|V_1|^4+f^2|V_1|^4\,dx.$$

In particular, this implies that, for $\psi\in C_c^\infty(\mathbb{R}^2\backslash\{0\},\mathbb{C})$,

$$\int_{B(0,D)}\mathfrak{Re}^2(\psi)|V_1|^3\,dx\leqslant K\int_{B(0,D)}|\nabla\psi|^2|V_1|^4+\mathfrak{Re}^2(\psi)|V_1|^4\,dx.$$

This lemma, with Lemmas 3.1.14 and 3.2.4, implies that, for $\varphi=Q_c\psi\in H_{Q_c}$,

$$\int_{\mathbb{R}^2}\mathfrak{Re}^2(\psi)|Q_c|^3\leqslant K\|\varphi\|_{\mathcal{C}}^2. \tag{3.4.11}$$

Proof. Since $|V_1|\geqslant K>0$ outside of $B(0,1)$, we take χ a radial smooth non negative cutoff with value 0 in $B(0,1)$ and value 1 outside $B(0,3/2)$. We have

$$\int_{B(0,D)}\chi f^2|V_1|^3\,dx\leqslant K\int_{B(0,D)}\chi f^2|V_1|^4\,dx\leqslant K\int_{B(0,D)}f^2|V_1|^4\,dx.$$

In $B(0,2)$, from Lemma 1.2.1, there exists $K_1,K_2>0$ such that $K_1\geqslant\frac{|V_1|}{r}\geqslant K_2$, and thus

$$\int_{B(0,D)}(1-\chi)f^2|V_1|^3\,dx\leqslant K\bigg(\int_0^{2\pi}\int_0^2(1-\chi(r))f^2(x)r^4dr\bigg)d\theta.$$

For $g\in C_c^\infty(\mathbb{R}\backslash\{0\},\mathbb{R})$, we have

$$\begin{aligned}\int_0^2(1-\chi(r))g^2(r)r^4dr &= \frac{-1}{5}\int_0^2\partial_r((1-\chi)g^2)r^5dr\\ &= \frac{-2}{5}\int_0^2(1-\chi(r))\partial_r g(r)g(r)r^5dr+\frac{1}{4}\int_0^2\chi'(r)g^2(r)r^5dr,\end{aligned}$$

and since $\chi'(r)\neq0$ only for $r\in[1,2]$, we have

$$\int_0^2|\chi'(r)|g^2(r)r^5dr\leqslant K\int_0^2g^2(r)r^4dr,$$

and, by Cauchy-Schwarz,

$$\int_0^2(1-\chi(r))|\partial_r g(r)g(r)|r^5dr\leqslant\sqrt{\int_0^2(\partial_r g)^2r^5dr\int_0^2g^2(r)r^5dr}.$$

We deduce that

$$\int_0^2(1-\chi(r))g^2(r)r^4dr\leqslant K\bigg(\int_0^2(\partial_r g)^2r^5dr+\int_0^2g^2(r)r^5dr\bigg),$$

and taking, for any $\theta\in[0,2\pi]$, $g(r)=f(r\cos(\theta),r\sin(\theta))$, and since $r\leqslant K|V_1|$ in $B(0,2)$ (by Lemma 1.2.1), by integration with respect to θ, we conclude that

$$\int_{B(0,D)}(1-\chi)f^2|V_1|^3\,dx\leqslant K\int_{B(0,D)}|\nabla f|^2|V_1|^4+f^2|V_1|^4\,dx,$$

which ends the proof of this lemma. □

We estimate here some quantities with the coercivity norm. These computations will be useful later on.

Lemma 3.4.4. *There exists $K>0$, a universal constant independent of c, such that, if c is small enough, for Q_c defined in Theorem 1.3.1, for $\varphi = Q_c\psi \in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\}, \mathbb{C})$, we have*

$$\left|\int_{\mathbb{R}^2} \mathfrak{Re}(\psi)\mathfrak{Im}(\nabla Q_c\overline{Q_c})\right| \leqslant K\ln\left(\frac{1}{c}\right)\|\varphi\|_{\mathcal{C}}$$

and

$$\left|\int_{\mathbb{R}^2} \mathfrak{Im}(\psi)\mathfrak{Re}(\nabla Q_c\overline{Q_c})\right| \leqslant K\|\varphi\|_{\mathcal{C}}.$$

Proof. By Cauchy-Schwarz, Lemmas 3.1.11 (with a slight modification near the zeros of Q_c that does not change the result) and 3.4.3,

$$\begin{aligned}\left|\int_{\mathbb{R}^2} \mathfrak{Re}(\psi)\mathfrak{Im}(\nabla Q_c\overline{Q_c})\right| &\leqslant \sqrt{\int_{\mathbb{R}^2}\mathfrak{Re}^2(\psi)|Q_c|^3\int_{\mathbb{R}^2}\frac{|\mathfrak{Im}(\nabla Q_c\overline{Q_c})|^2}{|Q_c|^3}}\\ &\leqslant K\ln\left(\frac{1}{c}\right)\sqrt{\int_{\mathbb{R}^2}\mathfrak{Re}^2(\psi)|Q_c|^3}\\ &\leqslant K\ln\left(\frac{1}{c}\right)\|\varphi\|_{\mathcal{C}}.\end{aligned}$$

We now focus on the second estimate. We take χ a smooth function with value 1 outside of $\{\tilde{r}\geqslant 2\}$ and 0 inside $\{\tilde{r}\leqslant 1\}$, and that is radial around $\pm\tilde{d}_c\vec{e}_1$ in $B(\pm\tilde{d}_c\vec{e}_1, 2)$. We remark that

$$\mathfrak{Re}(\nabla Q_c\overline{Q_c}) = \frac{1}{2}\nabla(|Q_c|^2) = \frac{1}{2}\nabla(\chi(|Q_c|^2-1)+(1-\chi)|Q_c|^2)+\frac{1}{2}\nabla\chi,$$

thus, by integration by parts, we have

$$\begin{aligned}\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\mathfrak{Re}(\nabla Q_c\overline{Q_c}) &= \frac{1}{2}\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\nabla(\chi(|Q_c|^2-1)+(1-\chi)|Q_c|^2)+\frac{1}{2}\int_{\mathbb{R}^2}\nabla\chi\mathfrak{Im}(\psi)\\ &= \frac{-1}{2}\int_{\mathbb{R}^2}\mathfrak{Im}(\nabla\psi)\chi(|Q_c|^2-1)-\frac{1}{2}\int_{\mathbb{R}^2}\mathfrak{Im}(\nabla\psi)(1-\chi)|Q_c|^2\\ &+ \frac{1}{2}\int_{\mathbb{R}^2}\nabla\chi\mathfrak{Im}(\psi).\end{aligned}$$

and, since χ is radial around $\pm\tilde{d}_c\vec{e}_1$ in $B(\pm\tilde{d}_c\vec{e}_1, 2)$,

$$\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\nabla\chi = \int_{B(\tilde{d}_c\vec{e}_1,2)\cup B(-\tilde{d}_c\vec{e}_1,2)}\mathfrak{Im}(\psi^{\neq 0})\nabla\chi.$$

Since $\nabla\chi$ is supported in $\big(B(\tilde{d}_c\vec{e}_1, 2)\cup B(-\tilde{d}_c\vec{e}_1, 2)\big)\backslash\big(B(\tilde{d}_c\vec{e}_1, 1)\cup B(-\tilde{d}_c\vec{e}_1, 1)\big)$, by equations (3.1.12), (3.1.28) and Cauchy-Schwarz,

$$\left|\int_{B(\tilde{d}_c\vec{e}_1,2)\cup B(-\tilde{d}_c\vec{e}_1,2)}\mathfrak{Im}(\psi^{\neq 0})\nabla\chi\right| \leqslant K\sqrt{\int_{\mathbb{R}^2}|\nabla\psi|^2|Q_c|^4}.$$

Now, by Cauchy-Schwarz, we check that

$$\left|\int_{\mathbb{R}^2}\mathfrak{Im}(\nabla\psi)(1-\chi)|Q_c|^2\right| \leqslant K\sqrt{\int_{\mathbb{R}^2}|\nabla\psi|^2|Q_c|^4\int_{\mathbb{R}^2}(1-\chi)^2} \leqslant K\sqrt{\int_{\mathbb{R}^2}|\nabla\psi|^2|Q_c|^4}.$$

Furthermore, we check that (χ being supported in $\{\tilde{r}\geqslant 1\}$)

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\mathfrak{Im}(\nabla\psi)\chi(|Q_c|^2-1)\right| &\leqslant \sqrt{\int_{\mathbb{R}^2}|\nabla\psi|^2\chi\int_{\mathbb{R}^2}(|Q_c|^2-1)^2}\\ &\leqslant K\sqrt{\int_{\mathbb{R}^2}|\nabla\psi|^2|Q_c|^4}.\end{aligned}$$

Indeed, we have, from equation (3.1.6) (for $\sigma = 1/2$), that

$$||Q_c|^2-1| \leqslant \frac{K}{(1+\tilde{r})^{3/2}},$$

which is enough to show that

$$\int_{\mathbb{R}^2}(|Q_c|^2-1)^2\leqslant K.$$

Combining these estimates, we conclude the proof of

$$\left|\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\mathfrak{Re}(\nabla Q_c\overline{Q_c})\right|\leqslant K\sqrt{\int_{\mathbb{R}^2}|\nabla\psi|^2|Q_c|^4}\leqslant K\|\varphi\|_{\mathcal{C}}. \qquad \square$$

3.4.3 Coercivity result under four othogonality conditions

The next result is the first part of Theorem 1.4.4, the second part (for the coercivity under three orthogonalities) is done in Lemma 3.4.6 below. We recall that, in $B\big(\pm\tilde{d}_c\vec{e}_1,\,R\big)$, we have $\psi^{\neq 0}(x)=\psi(x)-\psi^{0,\pm1}(\tilde{r}_{\pm1})$ with $\psi^{0,\pm1}(\tilde{r}_{\pm1})$ the 0-harmonic centered around $\pm\tilde{d}_c\vec{e}_1$ of ψ.

Lemma 3.4.5. *There exist $R,K,c_0>0$ such that, for $0<c\leqslant c_0$ and $\varphi=Q_c\psi\in H_{Q_c}$, Q_c defined in Theorem 1.3.1, if*

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)}\partial_{x_1}Q_c\overline{Q_c\psi^{\neq0}}=\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)}\partial_{x_2}Q_c\overline{Q_c\psi^{\neq0}}=0,$$

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)}\partial_{c}Q_c\overline{Q_c\psi^{\neq0}}=\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)}\partial_{c^\perp}Q_c\overline{Q_c\psi^{\neq0}}=0,$$

then

$$B_{Q_c}(\varphi)\geqslant K\|\varphi\|_{\mathcal{C}}^2.$$

Proof. For $\varphi=Q_c\psi\in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1,-\tilde{d}_c\vec{e}_1\},\mathbb{C})$, we take $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$ four real parameters and we define

$$\psi^*:=\psi+\varepsilon_1\frac{\partial_{x_1}Q_c}{Q_c}+\varepsilon_2\frac{c^2\partial_cQ_c}{Q_c}+\varepsilon_3\frac{\partial_{x_2}Q_c}{Q_c}+\varepsilon_4\frac{c\partial_{c^\perp}Q_c}{Q_c}.$$

Since, by Lemma 3.1.7, $\partial_{x_1}Q_c,\partial_{x_2}Q_c,\partial_cQ_c,\partial_{c^\perp}Q_c\in H_{Q_c}$, we deduce that $Q_c\psi^*\in H_{Q_c}$. Furthermore, we have

$$\begin{aligned}\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi^*}\Big) &= \int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big)\\ &+ \varepsilon_1\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\left(\partial_{x_1}\tilde{V}_1\overline{\partial_{x_1}Q_c\frac{\tilde{V}_1}{Q_c}}\right)\\ &+ \varepsilon_2\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\left(\partial_{x_1}\tilde{V}_1c^2\overline{\partial_cQ_c\frac{\tilde{V}_1}{Q_c}}\right)\\ &+ \varepsilon_3\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\left(\partial_{x_1}\tilde{V}_1\overline{\partial_{x_2}Q_c\frac{\tilde{V}_1}{Q_c}}\right)\\ &+ \varepsilon_4\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\left(\partial_{x_1}\tilde{V}_1c\overline{\partial_{c^\perp}Q_c\frac{\tilde{V}_1}{Q_c}}\right).\end{aligned}$$

From (3.4.8), we compute

$$\partial_{x_1}\tilde{V}_1\bar{\tilde{V}}_1=\left(\cos\big(\tilde{\theta}_1\big)\frac{\rho'(\tilde{r}_1)}{\rho(\tilde{r}_1)}-\frac{i}{\tilde{r}_1}\sin\big(\tilde{\theta}_1\big)\right)|\tilde{V}_1|^2,$$

and in particular, it has no 0-harmonic (since $|\tilde{V}_1|^2$ is radial). Therefore,

$$\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big)=\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi^{\neq0}}\Big)=$$

$$\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\Big(\partial_{x_1}Q_c\overline{Q_c\psi^{\neq0}}\Big)+\int_{B(\tilde{d}_c\vec{e}_1,R)}\mathfrak{Re}\big(\big(\partial_{x_1}\tilde{V}_1\bar{\tilde{V}}_1-\partial_{x_1}Q_c\overline{Q_c}\big)\psi^{\neq0}\big).$$

By Cauchy-Schwarz and equation (3.1.28),

$$\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}|Q_c\psi^{\neq 0}|^2\leqslant K\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}|Q_c|^4|\nabla\psi|^2\leqslant K\|\varphi\|_{\mathcal{C}}^2. \tag{3.4.12}$$

Here, K depends on R, but we consider R as a universal constant. We remark, by equations (3.4.3), (3.4.5) and (3.4.12) that

$$\begin{aligned}&\frac{1}{2}\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}(\partial_{x_1}Q_c-c^2\partial_cQ_c)\overline{Q_c\psi^{\neq 0}}\\ =\;&\int_{B(\tilde{d}_c\vec{e_1},R)}\mathfrak{Re}\Big(\partial_{x_1}Q_c\overline{Q_c\psi^{\neq 0}}\Big)+o_{c\to 0}(c^{\beta_0})K\|\varphi\|_{\mathcal{C}}^2,\end{aligned}$$

where $\beta_0>0$ is a small constant. We supposed that

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_{x_1}Q_c\overline{Q_c\psi^{\neq 0}}=\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_cQ_c\overline{Q_c\psi^{\neq 0}}=0,$$

therefore

$$\int_{B(\tilde{d}_c\vec{e_1},R)}\mathfrak{Re}\Big(\partial_{x_1}Q_c\overline{Q_c\psi^{\neq 0}}\Big)=o_{c\to 0}(c^{\beta_0})K\|\varphi\|_{\mathcal{C}}^2.$$

Furthermore, by equations (3.1.7), (3.1.28), (3.4.3), Lemma 3.1.14 and Cauchy-Schwarz,

$$\begin{aligned}\left|\int_{B(\tilde{d}_c\vec{e_1},R)}\mathfrak{Re}\big(\big(\partial_{x_1}\tilde{V_1}\bar{\tilde{V}}_1-\partial_{x_1}Q_c\overline{Q_c}\big)\psi^{\neq 0}\big)\right|&\leqslant o_{c\to 0}(c^{\beta_0})\sqrt{\int_{B(\tilde{d}_c\vec{e_1},R)}|\psi^{\neq 0}|^2|Q_c|^2}\\&\leqslant o_{c\to 0}(c^{\beta_0})K\|\varphi\|_{\mathcal{C}}\end{aligned}$$

Now, from Lemma 3.1.14 and equation (3.4.3), we estimate

$$\int_{B(\tilde{d}_c\vec{e_1},R)}\mathfrak{Re}\left(\partial_{x_1}\tilde{V_1}\overline{\partial_{x_1}Q_c\frac{\tilde{V_1}}{Q_c}}\right)=\int_{B(\tilde{d}_c\vec{e_1},R)}|\partial_{x_1}\tilde{V_1}|^2+o_{c\to 0}(1).$$

With (3.4.4), we check

$$\int_{B(\tilde{d}_c\vec{e_1},R)}\mathfrak{Re}\left(\partial_{x_1}\tilde{V_1}\overline{\partial_{x_2}Q_c\frac{\tilde{V_1}}{Q_c}}\right)=o_{c\to 0}(1).$$

Similarly, by (3.4.5) and Lemma 3.1.14, we have

$$\int_{B(\tilde{d}_c\vec{e_1},R)}\mathfrak{Re}\left(\partial_{x_1}\tilde{V_1}\overline{c^2\partial_cQ_c\frac{\tilde{V_1}}{Q_c}}\right)=-\int_{B(\tilde{d}_c\vec{e_1},R)}|\partial_{x_1}\tilde{V_1}|^2+o_{c\to 0}(1)$$

and by (3.4.6), we have

$$\int_{B(\tilde{d}_c\vec{e_1},R)}\mathfrak{Re}\left(\partial_{x_1}\tilde{V_1}\overline{c\partial_{c^\perp}Q_c\frac{\tilde{V_1}}{Q_c}}\right)=o_{c\to 0}(1).$$

Thus, with (3.4.7) we deduce that, writing

$$K(R)=\int_{B(0,R)}|\partial_{x_1}V_1(x)|^2dx,$$

since

$$K(R)=\int_{B(\tilde{d}_c\vec{e_1},R)}|\partial_{x_1}\tilde{V_1}|^2=\int_{B(-\tilde{d}_c\vec{e_1},R)}|\partial_{x_1}\tilde{V}_{-1}|^2=\int_{B(\tilde{d}_c\vec{e_1},R)}|\partial_{x_2}\tilde{V_1}|^2=\int_{B(-\tilde{d}_c\vec{e_1},R)}|\partial_{x_2}\tilde{V}_{-1}|^2,$$

we have

$$\begin{aligned}&\int_{B(\tilde{d}_c\vec{e_1},R)}\mathfrak{Re}\Big(\partial_{x_1}\tilde{V_1}\overline{\tilde{V_1}\psi^*}\Big)\\=\;&(\varepsilon_1-\varepsilon_2)K(R)+o_{c\to 0}(1)(\varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4)+o_{c\to 0}(c^{\beta_0})K\|\varphi\|_{\mathcal{C}}\end{aligned}$$

Similarly we can do the same computation for every orthogonalities, and we have the system

$$\begin{pmatrix} \int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi^*}\Big) \\ \int_{B(-\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi^*}\Big) \\ \int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_1\overline{\tilde{V}_1\psi^*}\Big) \\ \int_{B(-\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi^*}\Big) \end{pmatrix} = \left(K(R)\begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix} + o_{c\to 0}(1)\right)\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix}$$
$$+ \; o_{c\to 0}(c^{\beta_0})K\|\varphi\|_{\mathcal{C}}.$$

Therefore, since the matrix is invertible and $K(R)>0$, for c small enough, we can find $\varepsilon_1,\varepsilon_2,\varepsilon_3$, $\varepsilon_4\in\mathbb{R}$ such that

$$|\varepsilon_1|+|\varepsilon_2|+|\varepsilon_3|+|\varepsilon_4|\leqslant o_{c\to 0}(c^{\beta_0})K\|\varphi\|_{\mathcal{C}} \tag{3.4.13}$$

and

$$\int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi^*}\Big) = \int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_1\overline{\tilde{V}_1\psi^*}\Big) = 0,$$

$$\int_{B(-\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi^*}\Big) = \int_{B(-\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi^*}\Big) = 0.$$

Therefore, by Proposition 1.4.3, since $Q_c\psi^*\in H_{Q_c}$, we have

$$B_{Q_c}(Q_c\psi^*)\geqslant K\|Q_c\psi^*\|_{\mathcal{C}}^2.$$

From Lemma 3.1.8, we have,

$$\|\partial_{x_1}Q_c\|_{\mathcal{C}}+\|\partial_{x_2}Q_c\|_{\mathcal{C}}+\|c^2\partial_cQ_c\|_{\mathcal{C}}+c^{\beta_0/2}\|c\partial_{c^\perp}Q_c\|_{\mathcal{C}}\leqslant K(\beta_0)$$

hence, since $Q_c(\psi^*-\psi)=\varepsilon_1\partial_{x_1}Q_c+\varepsilon_2c^2\partial_cQ_c+\varepsilon_3\partial_{x_2}Q_c+\varepsilon_4c\partial_{c^\perp}Q_c$,

$$\begin{aligned} & \|Q_c\psi\|_{\mathcal{C}}^2 \\ \leqslant\; & \|Q_c\psi^*\|_{\mathcal{C}}^2+\|Q_c(\psi-\psi^*)\|_{\mathcal{C}}^2 \\ \leqslant\; & \|Q_c\psi^*\|_{\mathcal{C}}^2+K(\beta_0)\left(|\varepsilon_1|+|\varepsilon_2|+|\varepsilon_3|+c^{-\beta_0/2}|\varepsilon_4|\right)^2, \end{aligned}$$

therefore, for c small enough, by (3.4.13), we have

$$\|Q_c\psi^*\|_{\mathcal{C}}^2\geqslant K\|Q_c\psi\|_{\mathcal{C}}^2$$

and

$$B_{Q_c}(Q_c\psi^*)\geqslant K\|Q_c\psi\|_{\mathcal{C}}^2$$

Finally, we compute, since $Q_c(\psi-\psi^*)=\varepsilon_1\partial_{x_1}Q_c+\varepsilon_2c^2\partial_cQ_c+\varepsilon_3\partial_{x_2}Q_c+\varepsilon_4c\partial_{c^\perp}Q_c$, by Lemma 3.4.2, that

$$B_{Q_c}(\varphi)=B_{Q_c}(Q_c\psi^*)+B_{Q_c}(Q_c(\psi-\psi^*))+2\langle Q_c\psi^*,L_{Q_c}(Q_c(\psi-\psi^*))\rangle.$$

Furthermore, we compute, still by Lemma 3.4.2,

$$\langle Q_c\psi^*,L_{Q_c}(Q_c(\psi-\psi^*))\rangle=-B_{Q_c}(Q_c(\psi-\psi^*))+\langle Q_c\psi,L_{Q_c}(Q_c(\psi-\psi^*))\rangle,$$

therefore

$$\begin{aligned} B_{Q_c}(\varphi) &= B_{Q_c}(Q_c\psi^*)-B_{Q_c}(Q_c(\psi-\psi^*))+2\langle Q_c\psi,L_{Q_c}(Q_c(\psi-\psi^*))\rangle \\ &\geqslant K\|Q_c\psi\|_{\mathcal{C}}^2-B_{Q_c}(Q_c(\psi-\psi^*))+2\langle Q_c\psi,L_{Q_c}(Q_c(\psi-\psi^*))\rangle. \end{aligned}$$

We have

$$Q_c(\psi-\psi^*)=-(\varepsilon_1\partial_{x_1}Q_c+\varepsilon_2c^2\partial_cQ_c+\varepsilon_3\partial_{x_2}Q_c+\varepsilon_4c\partial_{c^\perp}Q_c),$$

and from Lemma 3.1.7, we have

$$L_{Q_c}(Q_c(\psi-\psi^*))=-c^2\varepsilon_2i\,\partial_{x_2}Q_c+c^2\varepsilon_4i\partial_{x_1}Q_c.$$

We compute

$$\begin{aligned} & B_{Q_c}(Q_c(\psi-\psi^*)) \\ = \ & \langle -(\varepsilon_1\partial_{x_1}Q_c+\varepsilon_2c^2\partial_cQ_c+\varepsilon_3\partial_{x_2}Q_c+\varepsilon_4c\partial_{c^\perp}Q_c), -c^2\varepsilon_2i\partial_{x_2}Q_c+c^2\varepsilon_4i\partial_{x_1}Q_c\rangle, \end{aligned}$$

and with (3.1.3), we check that

$$B_{Q_c}(Q_c(\psi-\psi^*))=\varepsilon_2^2c^4\langle L_{Q_c}(\partial_cQ_c),\partial_cQ_c\rangle-\varepsilon_4^2c^2\langle L_{Q_c}(\partial_{c^\perp}Q_c),\partial_{c^\perp}Q_c\rangle.$$

With Lemma 3.1.9 and equation (3.4.13), we estimate

$$|B_{Q_c}(Q_c(\psi-\psi^*))|\leqslant Kc^2(\varepsilon_2^2+\varepsilon_4^2)\leqslant o_{c\to0}(1)\|Q_c\psi\|_{\mathcal{C}}^2.$$

Finally, we have

$$\langle Q_c\psi,L_{Q_c}(Q_c(\psi-\psi^*))\rangle=\langle Q_c\psi,-c^2\varepsilon_2i\partial_{x_2}Q_c+c^2\varepsilon_4i\partial_{x_1}Q_c\rangle.$$

We compute

$$c^2\langle Q_c\psi,i\nabla Q_c\rangle=c^2\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\mathfrak{Re}(\nabla Q_c\overline{Q_c})-c^2\int_{\mathbb{R}^2}\mathfrak{Re}(\psi)\mathfrak{Im}(\nabla Q_c\overline{Q_c}),$$

and to finish the proof, we use

$$|c\langle Q_c\psi,i\nabla Q_c\rangle|\leqslant Kc\ln\left(\frac{1}{c}\right)\|Q_c\psi\|_{\mathcal{C}} \tag{3.4.14}$$

for a constant $K>0$ independent of c by Lemma 3.4.4, which is enough to show that

$$\begin{aligned} & |\langle Q_c\psi,L_{Q_c}(Q_c(\psi-\psi^*))\rangle| \\ \leqslant \ & o_{c\to0}(1)(|\varepsilon_2|+|\varepsilon_4|)\|Q_c\psi\|_{\mathcal{C}} \\ \leqslant \ & o_{c\to0}(1)\|Q_c\psi\|_{\mathcal{C}}^2, \end{aligned}$$

since $c\ln\left(\frac{1}{c}\right)=o_{c\to0}(1)$. We have shown that, for $\varphi\in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d_c}\vec{e_1},-\tilde{d_c}\vec{e_1}\},\mathbb{C})$

$$\begin{aligned} B_{Q_c}(\varphi) \ & \geqslant \ K\|Q_c\psi\|_{\mathcal{C}}^2-B_{Q_c}(Q_c(\psi-\psi^*))+2\langle Q_c\psi,L_{Q_c}(Q_c(\psi-\psi^*))\rangle \\ & \geqslant \ (K-o_{c\to0}(1))\|Q_c\psi\|_{\mathcal{C}}^2 \\ & \geqslant \ \frac{K}{2}\|Q_c\psi\|_{\mathcal{C}}^2 \end{aligned}$$

for c small enough. Now, by Lemma 3.2.4, we conclude by density as in the proof of Proposition 1.4.3. □

3.4.4 Coercivity under three orthogonality conditions

Lemma 3.4.6. *There exists $R,K>0$ such that, for $0<\beta<\beta_0$, β_0 a small constant, there exists $c_0(\beta),K(\beta)>0$ with, for $0<c<c_0(\beta)$, Q_c defined in Theorem 1.3.1, $\varphi=Q_c\psi\in H_{Q_c}$, if*

$$\mathfrak{Re}\int_{B(\tilde{d_c}\vec{e_1},R)\cup B(-\tilde{d_c}\vec{e_1},R)}\partial_{x_1}Q_c\overline{Q_c\psi^{\neq0}}=\mathfrak{Re}\int_{B(\tilde{d_c}\vec{e_1},R)\cup B(-\tilde{d_c}\vec{e_1},R)}\partial_{x_2}Q_c\overline{Q_c\psi^{\neq0}}=0,$$

$$\mathfrak{Re}\int_{B(\tilde{d_c}\vec{e_1},R)\cup B(-\tilde{d_c}\vec{e_1},R)}\partial_cQ_c\overline{Q_c\psi^{\neq0}}=0,$$

then

$$B_{Q_c}(\varphi)\geqslant K(\beta)c^{2+\beta}\|\varphi\|_{\mathcal{C}}^2.$$

Proof. As for the proof of Lemma 3.4.5, we show the result for $\varphi=Q_c\psi\in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d_c}\vec{e_1},-\tilde{d_c}\vec{e_1}\},\mathbb{C})$, and we conclude by density for $\varphi\in H_{Q_c}$.

For $\varphi=Q_c\psi\in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d_c}\vec{e_1},-\tilde{d_c}\vec{e_1}\},\mathbb{C})$, we take $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4$ four real parameters and we define

$$\psi^*:=\psi+\varepsilon_1\frac{\partial_{x_1}Q_c}{Q_c}+\varepsilon_2\frac{c^2\partial_cQ_c}{Q_c}+\varepsilon_3\frac{\partial_{x_2}Q_c}{Q_c}+\varepsilon_4\frac{c\partial_{c^\perp}Q_c}{Q_c}.$$

With the same computation as in the proof of Lemma 3.4.5, we check that $Q_c\psi^* \in H_{Q_c}$, and using similarly the estimates of Lemma 3.4.1, we can take $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4 \in \mathbb{R}$ such that

$$|\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| = o_{c\to 0}(c^{\beta_0})\|\varphi\|_{\mathcal{C}},$$

$|\varepsilon_4| \leqslant K\|\varphi\|_{\mathcal{C}}$ and such that ψ^* satisfies the four orthogonality conditions of Lemma 3.4.5. Therefore,

$$B_{Q_c}(Q_c\psi^*) \geqslant K\|Q_c\psi^*\|_{\mathcal{C}}^2. \tag{3.4.15}$$

We write

$$T = \varepsilon_1\partial_{x_1}Q_c + \varepsilon_2 c^2\partial_c Q_c + \varepsilon_3\partial_{x_2}Q_c,$$

and we develop, by Lemma 3.4.2,

$$\begin{aligned} & B_{Q_c}(Q_c\psi) \\ = \;& B_{Q_c}(Q_c\psi^*) + c^2\varepsilon_4^2 B_{Q_c}(\partial_{c^\perp}Q_c) + B_{Q_c}(T) \\ - \;& 2\langle Q_c\psi^*, c\varepsilon_4 L_{Q_c}(\partial_{c^\perp}Q_c)\rangle - 2\langle Q_c\psi^*, L_{Q_c}(T)\rangle + 2c\varepsilon_4\langle L_{Q_c}(\partial_{c^\perp}Q_c), T\rangle. \end{aligned}$$

Using Lemmas 3.1.7 and 3.1.9, we compute

$$\begin{aligned} |B_{Q_c}(T)| &= |\langle L_{Q_c}(T), T\rangle| = |\langle L_{Q_c}(\varepsilon_2 c^2\partial_c Q_c), \varepsilon_2 c^2\partial_c Q_c\rangle| \\ &= \varepsilon_2^2 c^4 |\langle L_{Q_c}(\partial_c Q_c), \partial_c Q_c\rangle| \\ &\leqslant K\varepsilon_2^2 c^2 = o_{c\to 0}(c^{2+2\beta_0})\|\varphi\|_{\mathcal{C}}^2 \end{aligned} \tag{3.4.16}$$

Now, we compute, by Lemma 3.1.7, that

$$\langle Q_c\psi^*, c\varepsilon_4 L_{Q_c}(\partial_{c^\perp}Q_c)\rangle = \varepsilon_4 c^2\langle Q_c\psi^*, i\partial_{x_1}Q_c\rangle.$$

From Lemma 3.4.4, we have

$$|c\langle Q_c\psi^*, i\partial_{x_1}Q_c\rangle| \leqslant o_{c\to 0}(c^{1-\beta_0/2})\|\varphi^*\|_{\mathcal{C}},$$

therefore

$$|\langle Q_c\psi^*, c\varepsilon_4 L_{Q_c}(\partial_{c^\perp}Q_c)\rangle| \leqslant o_{c\to 0}(c^{1+\beta_0/2})\|\varphi^*\|_{\mathcal{C}}\|\varphi\|_{\mathcal{C}}. \tag{3.4.17}$$

Similarly, we compute

$$\langle Q_c\psi^*, L_{Q_c}(T)\rangle = \langle Q_c\psi^*, \varepsilon_2 c^2 L_{Q_c}(\partial_c Q_c)\rangle = \varepsilon_2 c^2\langle Q_c\psi^*, i\partial_{x_2}Q_c\rangle.$$

Still from Lemma 3.4.4, we have

$$|c\langle Q_c\psi^*, i\partial_{x_2}Q_c\rangle| \leqslant Kc\ln\left(\frac{1}{c}\right)\|\varphi^*\|_{\mathcal{C}},$$

therefore

$$|\langle Q_c\psi^*, L_{Q_c}(T)\rangle| \leqslant K|\varepsilon_2|c^2\ln\left(\frac{1}{c}\right)\|\varphi^*\|_{\mathcal{C}} \leqslant o_{c\to 0}(c^{1+\beta_0})\|\varphi^*\|_{\mathcal{C}}\|\varphi\|_{\mathcal{C}}. \tag{3.4.18}$$

Finally, we compute similarly that

$$c|\varepsilon_4\langle L_{Q_c}(\partial_{c^\perp}Q_c), T\rangle| = c|\varepsilon_4\langle ic\partial_{x_1}Q_c, T\rangle| = c^2|\varepsilon_4\langle i\,\partial_{x_1}Q_c, \varepsilon_2 c^2\partial_c Q_c + \varepsilon_3\partial_{x_2}Q_c\rangle|.$$

Using Lemma 3.4.4 for $\varphi = c^2\partial_c Q_c$ (with Lemma 3.2.4), we infer

$$|\langle i\,\partial_{x_1}Q_c, c^2\partial_c Q_c\rangle| \leqslant K\|c^2\partial_c Q_c\|_{\mathcal{C}},$$

and $\|c^2\partial_c Q_c\|_{\mathcal{C}} \leqslant K$ by Lemma 3.1.8. Furthermore, since $Q_c(-x_1, x_2) = Q_c(x_1, x_2)$, we have

$$\langle i\,\partial_{x_1}Q_c, \partial_{x_2}Q_c\rangle = 0.$$

We conclude that

$$|c\varepsilon_4\langle L_{Q_c}(\partial_{c^\perp}Q_c), T\rangle| \leqslant Kc^2\,|\varepsilon_4|(|\varepsilon_2| + |\varepsilon_3|) = o_{c\to 0}(c^{2+\beta_0/2})\|\varphi\|_{\mathcal{C}}^2. \tag{3.4.19}$$

Now, combining (3.4.15) to (3.4.19), and with $B_{Q_c}(\partial_{c^\perp}Q_c) = 2\pi + o_{c\to 0}(1)$ from Lemma 3.1.9, we have

$$B_{Q_c}(\varphi) \geqslant K\|\varphi^*\|_{\mathcal{C}}^2 + K\varepsilon_4^2 c^2 - o_{c\to 0}(c^{2+\beta_0/2})\|\varphi\|_{\mathcal{C}}^2 - o_{c\to 0}(c^{1+\beta_0/2})\|\varphi^*\|_{\mathcal{C}}\|\varphi\|_{\mathcal{C}}.$$

Similarly as in the proof of Lemma 3.4.5, we have from Lemma 3.1.8 that, for any $\beta_0/2>\beta>0$,

$$\|\varphi\|_{\mathcal{C}}^2\leqslant K\|\varphi^*\|_{\mathcal{C}}^2+K(\beta)\varepsilon_4^2c^{-\beta},$$

hence

$$\varepsilon_4^2c^2\geqslant K(\beta)c^{2+\beta}(\|\varphi\|_{\mathcal{C}}^2-\|\varphi^*\|_{\mathcal{C}}^2),$$

therefore

$$\begin{aligned}B_{Q_c}(\varphi)&\geqslant K_1(\beta)(\|\varphi^*\|_{\mathcal{C}}^2+c^{2+\beta}\|\varphi\|_{\mathcal{C}}^2)-K_2(\beta)c^{2+\beta}\|\varphi^*\|_{\mathcal{C}}^2-o_{c\to0}(c^{2+\beta_0/2})\|\varphi\|_{\mathcal{C}}^2\\&-o_{c\to0}(c^{1+\beta_0})\|\varphi^*\|_{\mathcal{C}}\|\varphi\|_{\mathcal{C}}\\&\geqslant K(\beta)c^{2+\beta}\|\varphi\|_{\mathcal{C}}^2\end{aligned}$$

for c small enough (depending on β). □

Lemmas 3.1.12, 3.4.5 and 3.4.6 together end the proof of Theorem 1.4.4. Remark that in both Lemmas 3.4.5 and 3.4.6, we could replace the orthogonality condition $\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)}\partial_cQ_cQ_c\psi^{\neq0}=0$ by

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)}\partial_d(V_1(x-d\vec{e}_1)V_{-1}(x+d\vec{e}_1))_{|d=d_c}\overline{Q_c\psi^{\neq0}}(x)dx=0,\tag{3.4.20}$$

since, by Theorem 1.3.1 (for $p=+\infty$),

$$\|c^2\partial_cQ_c-\partial_d(V_1(x-d\vec{e}_1)V_{-1}(x+d\vec{e}_1))_{|d=d_c}\|_{C^1(B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R))}=o_{c\to0}(1),$$

and thus this replacement creates an error term that can be estimate as the other ones in the proof of Lemma 3.4.5.

3.4.5 Proof of the corollaries of Theorem 1.4.4

3.4.5.1 Proof of Corollary 1.4.5

Proof. We start with the proof that (i) implies (ii). We start by showing that, for $\varphi_0\in C_c^\infty(\mathbb{R}^2,\mathbb{C})$,

$$B_{Q_c}(\varphi+\varphi_0)=B_{Q_c}(\varphi_0).$$

We take $\varphi_0=Q_c\psi_0\in C_c^\infty(\mathbb{R}^2,\mathbb{C})$ and, by integration by parts, from (i), we check that

$$\langle L_{Q_c}(\varphi_0),\varphi\rangle=0.$$

Furthermore, we check (for $\varphi\in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1,-\tilde{d}_c\vec{e}_1\},\mathbb{C})$ and then by density for $\varphi\in H_{Q_c}$) that for $\varphi_0\in C_c^\infty(\mathbb{R}^2,\mathbb{C})$,

$$B_{Q_c}(\varphi+\varphi_0)=B_{Q_c}(\varphi)+B_{Q_c}(\varphi_0)+2\langle\varphi,L_{Q_c}(\varphi_0)\rangle,$$

hence

$$B_{Q_c}(\varphi+\varphi_0)=B_{Q_c}(\varphi)+B_{Q_c}(\varphi_0).\tag{3.4.21}$$

Similarly as in the proof of Proposition 1.4.3, we argue by density that this result holds for $\varphi_0\in H_{Q_c}$. Now, taking $\varphi_0=-\varphi$, we infer from (3.4.21) that $B_{Q_c}(\varphi)=0$, thus, for $\varphi\in H_{Q_c}$,

$$B_{Q_c}(\varphi+\varphi_0)=B_{Q_c}(\varphi_0).\tag{3.4.22}$$

Now, similarly as the proof of Lemma 3.4.5, we decompose $\varphi=Q_c\psi\in H_{Q_c}$ in

$$\varphi=\varphi^*+\varepsilon_1\partial_{x_1}Q_c+\varepsilon_2\partial_{x_2}Q_c+\varepsilon_3c^2\partial_cQ_c$$

with

$$|\varepsilon_1|+|\varepsilon_2|+|\varepsilon_3|\leqslant K\|\varphi\|_{\mathcal{C}},$$

such that φ^* verifies the three orthogonality conditions of Lemma 3.4.6. We write

$$A=\varepsilon_1\partial_{x_1}Q_c+\varepsilon_2\partial_{x_2}Q_c+\varepsilon_3c^2\partial_cQ_c\in H_{Q_c}$$

by Lemma 3.1.7, and using (3.4.22), we have

$$B_{Q_c}(\varphi^*) = B_{Q_c}(\varphi - A) = B_{Q_c}(A).$$

From Lemma 3.4.6, we have $B_{Q_c}(\varphi^*) \geqslant K c^{2+\beta_0/2} \|\varphi^*\|_{\mathcal{C}}^2$. Furthermore, from Lemmas 3.1.7 and 3.1.9,

$$B_{Q_c}(A) = \varepsilon_3^2 c^2 B_{Q_c}(\partial_c Q_c) = (-2\pi + o_{c\to 0}(1))\varepsilon_3^2 \leqslant 0.$$

We deduce that $\varepsilon_3 = 0$ and $\|\varphi^*\|_{\mathcal{C}} = 0$, hence $\varphi^* = i\mu Q_c$ for some $\mu \in \mathbb{R}$. Since $\varphi^* = \varphi - R \in H_{Q_c}$, we deduce that $\mu = 0$ (or else $\|\varphi^*\|_{H_{Q_c}}^2 \geqslant \int_{\mathbb{R}^2} \frac{|\varphi^*|^2}{(1+\tilde{r})^2} = +\infty$). Therefore,

$$\varphi = \varepsilon_1 \partial_{x_1} Q_c + \varepsilon_2 \partial_{x_2} Q_c \in \mathrm{Span}_{\mathbb{R}}(\partial_{x_1} Q_c, \partial_{x_2} Q_c).$$

Finally, the fact that (ii) implies (i) is a consequence of Lemma 3.1.7. This concludes the proof of this lemma. □

3.4.5.2 Spectral stability

We have $H^1(\mathbb{R}^2) \subset H_{Q_c}$, therefore $B_{Q_c}(\varphi)$ is well defined for $\varphi \in H^1(\mathbb{R}^2)$. Furthermore, the fact that $i\partial_{x_2} Q_c \in L^2(\mathbb{R}^2)$ is a consequence of Theorem 3.1.4, and in particular this justifies that $\langle \varphi, i\partial_{x_2} Q_c \rangle$ is well defined for $\varphi \in H^1(\mathbb{R}^2)$. For $\varphi \in H^1(\mathbb{R}^2)$, there are no issue in the definition of the quadratic form, as shown in the following lemma.

Lemma 3.4.7. *There exists $c_0 > 0$ such that, for $0 < c < c_0$, Q_c defined in Theorem 1.3.1, if $\varphi \in H^1(\mathbb{R}^2)$, then*

$$B_{Q_c}(\varphi) = \int_{\mathbb{R}^2} |\nabla \varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) - (1 - |Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi).$$

Proof. We recall that $H^1(\mathbb{R}^2) \subset H_{Q_c}$ and, for $\varphi = Q_c\psi \in H^1(\mathbb{R}^2)$,

$$\begin{aligned} B_{Q_c}(\varphi) &= \int_{\mathbb{R}^2} |\nabla\varphi|^2 - (1-|Q_c|^2)|\varphi|^2 + 2\,\mathfrak{Re}^2(\overline{Q_c}\varphi) \\ &- c\int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}(i\,\partial_{x_2}\varphi\bar{\varphi}) - c\int_{\mathbb{R}^2} \eta\mathfrak{Re}\, i\,\partial_{x_2} Q_c\overline{Q_c}|\psi|^2 \\ &+ 2c\int_{\mathbb{R}^2} \eta\mathfrak{Re}\,\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2 + c\int_{\mathbb{R}^2} \partial_{x_2}\eta\mathfrak{Re}\,\psi\mathfrak{Im}\,\psi|Q_c|^2 \\ &+ c\int_{\mathbb{R}^2} \eta\mathfrak{Re}\,\psi\mathfrak{Im}\,\psi\partial_{x_2}(|Q_c|^2). \end{aligned}$$

Since $\varphi \in H^1(\mathbb{R}^2)$, the integral $\int_{\mathbb{R}^2} \mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi})$ is well defined as the scalar product of two $L^2(\mathbb{R}^2)$ functions. Now, still because $\varphi = Q_c\psi \in H^1(\mathbb{R}^2)$, we can integrate by parts, and we check that

$$\begin{aligned} \int_{\mathbb{R}^2} \eta\mathfrak{Re}\,\psi\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2 &= -\int_{\mathbb{R}^2} \eta\mathfrak{Re}\,\partial_{x_2}\psi\mathfrak{Im}\psi|Q_c|^2 \\ &- \int_{\mathbb{R}^2} \partial_{x_2}\eta\mathfrak{Re}\,\psi\mathfrak{Im}\psi|Q_c|^2 - \int_{\mathbb{R}^2} \eta\mathfrak{Re}\,\psi\mathfrak{Im}\psi\partial_{x_2}(|Q_c|^2). \end{aligned}$$

We conclude by expanding

$$\begin{aligned} \int_{\mathbb{R}^2} \eta\mathfrak{Re}(i\,\partial_{x_2}\varphi\bar{\varphi}) &= \int_{\mathbb{R}^2} \eta\mathfrak{Re}(i\,\partial_{x_2} Q_c\overline{Q_c})|\psi|^2 + \int_{\mathbb{R}^2} \eta\mathfrak{Re}(i\,\partial_{x_2}\psi\bar{\psi})|Q_c|^2 \\ &= \int_{\mathbb{R}^2} \eta\mathfrak{Re}(i\,\partial_{x_2} Q_c\overline{Q_c})|\psi|^2 + \int_{\mathbb{R}^2} \eta\mathfrak{Re}(\partial_{x_2}\psi)\mathfrak{Im}\psi|Q_c|^2 \\ &+ \int_{\mathbb{R}^2} \eta\mathfrak{Re}(\psi)\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2. \end{aligned}$$

□

The rest of this subsection is devoted to the proofs of Corollary 1.4.6, Proposition 1.4.7 and Corollary 1.4.9.

Proof. (of Corollary 1.4.6) For $\varphi \in H^1(\mathbb{R}^2)$ such that $\langle \varphi, i\partial_{x_2} Q_c\rangle = 0$, we decompose it in

$$\varphi = \varphi^* + \varepsilon_1 \partial_{x_1} Q_c + \varepsilon_2 \partial_{x_2} Q_c + c^2 \varepsilon_3 \partial_c Q_c.$$

Similarly as in the proof of Lemma 3.4.5, we can find $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}$ such that φ^* satisfies the three orthogonality conditions of Lemma 3.4.6, and thus (since $\varphi \in H^1(\mathbb{R}^2) \subset H_{Q_c}$, for $\beta = \beta_0/2$)

$$B_{Q_c}(\varphi^*) \geqslant K c^{2+\beta_0/2} \|\varphi^*\|_{\mathcal{C}}^2.$$

Now, we compute, by Lemma 3.4.2 and with a density argument, that

$$B_{Q_c}(\varphi) = B_{Q_c}(\varphi^*) + 2\langle \varphi^*, L_{Q_c}(\varepsilon_1 \partial_{x_1} Q_c + \varepsilon_2 \partial_{x_2} Q_c + c^2 \varepsilon_3 \partial_c Q_c)\rangle + \varepsilon_3^2 c^4 B_{Q_c}(\partial_c Q_c).$$

We have from Lemma 3.1.7 that $L_{Q_c}(\varepsilon_1 \partial_{x_1} Q_c + \varepsilon_2 \partial_{x_2} Q_c + c^2 \varepsilon_3 \partial_c Q_c) = c^2 \varepsilon_3 i \partial_{x_2} Q_c$, therefore

$$B_{Q_c}(\varphi) \geqslant K c^{2+\beta_0/2} \|\varphi^*\|_{\mathcal{C}}^2 + 2c^2 \varepsilon_3 \langle \varphi^*, i\partial_{x_2} Q_c\rangle + \varepsilon_3^2 c^4 B_{Q_c}(\partial_c Q_c).$$

Since $\langle \varphi, i\partial_{x_2} Q_c\rangle = 0$ and $\varphi = \varphi^* + \varepsilon_1 \partial_{x_1} Q_c + \varepsilon_2 \partial_{x_2} Q_c + c^2 \varepsilon_3 \partial_c Q_c$, we have

$$\langle \varphi^*, i\partial_{x_2} Q_c\rangle = -\langle \varepsilon_1 \partial_{x_1} Q_c + \varepsilon_2 \partial_{x_2} Q_c + c^2 \varepsilon_3 \partial_c Q_c, i\partial_{x_2} Q_c\rangle.$$

Since $\partial_{x_1} Q_c$ is odd in x_1 and $i\partial_{x_2} Q_c$ is even in x_1, we have $\langle \varepsilon_1 \partial_{x_1} Q_c, i\partial_{x_2} Q_c\rangle = 0$. Furthermore,

$$\langle \varepsilon_2 \partial_{x_2} Q_c, i\partial_{x_2} Q_c\rangle = \varepsilon_2 \int_{\mathbb{R}^2} \mathfrak{Re}(i|\partial_{x_2} Q_c|^2) = 0,$$

and, from Lemma 3.1.9, we have

$$B_{Q_c}(\partial_c Q_c) = \langle \partial_c Q_c, i\partial_{x_2} Q_c\rangle = \frac{-2\pi + o_{c\to 0}(1)}{c^2},$$

thus

$$\langle \varphi^*, L_{Q_c}(\varepsilon_1 \partial_{x_1} Q_c + \varepsilon_2 \partial_{x_2} Q_c + c^2 \varepsilon_3 \partial_c Q_c)\rangle = (2\pi + o_{c\to 0}(1))\varepsilon_3 B_{Q_c}(\partial_c Q_c),$$

and

$$B_{Q_c}(\varphi) \geqslant K c^{2+\beta_0/2} \|\varphi^*\|_{\mathcal{C}}^2 - \varepsilon_3^2 c^4 B_{Q_c}(\partial_c Q_c) \geqslant K c^{2+\beta_0/2} \|\varphi^*\|_{\mathcal{C}}^2 + 2\pi \varepsilon_3^2 c^2 (1 + o_{c\to 0}(1)) \geqslant 0$$

for c small enough. This also shows that if $\varphi \in H^1(\mathbb{R}^2)$, $B_{Q_c}(\varphi) = 0$ and $\langle \varphi, i\partial_{x_2} Q_c\rangle = 0$, then $\varphi \in \mathrm{Span}_{\mathbb{R}}\{\partial_{x_1} Q_c, \partial_{x_2} Q_c\}$. □

We can now finish the proof of Proposition 1.4.7.

Proof. (of Proposition 1.4.7) First, we have from Theorem 3.1.4 that $i\partial_{x_2} Q_c \in L^2(\mathbb{R}^2)$. Now, with Corollary 1.4.6, it is easy to check that $n^-(L_{Q_c}) \leqslant 1$. Indeed, if it is false, we can find u, $v \in H^1(\mathbb{R}^2)$ such that for all $\lambda, \mu \in \mathbb{R}$ with $(\lambda, \mu) \neq (0,0)$, $\lambda u + \mu v \neq 0$ and $B_{Q_c}(\lambda u + \mu v) < 0$. Then, we can take $(\lambda, \mu) \neq (0,0)$ such that

$$\langle \lambda u + \mu v, i\partial_{x_2} Q_c\rangle = 0,$$

which implies $B_{Q_c}(\lambda u + \mu v) \geqslant 0$ and therefore a contradiction.

Let us show that L_{Q_c} has at least one negative eigenvalue (with eigenvector in $H^1(\mathbb{R}^2)$), which implies that $n^-(L_{Q_c}) = 1$ and that it is the only negative eigenvalue. We consider

$$\alpha_c := \inf_{\varphi \in H^1(\mathbb{R}^2), \|\varphi\|_{L^2(\mathbb{R}^2)} = 1} B_{Q_c}(\varphi).$$

We recall, from Lemma 3.4.7, that (since $\varphi \in H^1(\mathbb{R}^2)$)

$$B_{Q_c}(\varphi) = \int_{\mathbb{R}^2} |\nabla \varphi|^2 - \mathfrak{Re}(ic\partial_{x_2} \varphi \bar{\varphi}) - (1 - |Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi),$$

and if $\varphi \in H^1(\mathbb{R}^2)$ with $\|\varphi\|_{L^2(\mathbb{R}^2)} = 1$, we have, by Cauchy-Schwarz,

$$B_{Q_c}(\varphi) \geqslant \int_{\mathbb{R}^2} |\nabla \varphi|^2 - Kc \|\partial_{x_2} \varphi\|_{L^2(\mathbb{R}^2)} - K \geqslant -K(c).$$

In particular, this implies that $\alpha_c \neq -\infty$.

Now, assume that there exists no $\varphi \in C_c^\infty(\mathbb{R}^2, \mathbb{C})$ such that $B_{Q_c}(\varphi) < 0$. Then, for any $\varphi \in C_c^\infty(\mathbb{R}^2, \mathbb{C})$, we have $B_{Q_c}(\varphi) \geqslant 0$. Following the density argument at the end of the proof of Proposition 1.4.3, we have $B_{Q_c}(\varphi) \geqslant 0$ for all $\varphi \in H_{Q_c}$, and in particular $B_{Q_c}(\partial_c Q_c) \geqslant 0$ (we recall that $\partial_c Q_c \in H_{Q_c}$ but is not a priori in $H^1(\mathbb{R}^2)$), which is in contradiction with Lemma 3.1.9. Therefore, there exists $\varphi \in C_c^\infty(\mathbb{R}^2, \mathbb{C}) \subset H^1(\mathbb{R}^2)$ such that $B_{Q_c}(\varphi) < 0$, and in particular $B_{Q_c}\Big(\frac{\varphi}{\|\varphi\|_{L^2(\mathbb{R}^2)}}\Big) < 0$ and $\Big\|\frac{\varphi}{\|\varphi\|_{L^2(\mathbb{R}^2)}}\Big\|_{L^2(\mathbb{R}^2)} = 1$, hence $\alpha_c < 0$.

Remark that we did not show that $\partial_c Q_c \in L^2(\mathbb{R}^2)$, and we believe this to be false. This estimation on α_c is the only time we need to work specifically with Q_c from Theorem 1.3.1. From now on, we can suppose that Q_c is a travelling wave with finite energy such that $\alpha_c < 0$.

To show that there exists at least one negative eigenvalue, it is enough to show that α_c is achieved for a function $\varphi \in H^1(\mathbb{R}^2)$. Let us take a minimizing sequence $\varphi_n \in H^1(\mathbb{R}^2)$ such that $\|\varphi_n\|_{L^2(\mathbb{R}^2)} = 1$ and $B_{Q_c}(\varphi_n) \to \alpha_c$. We have

$$\int_{\mathbb{R}^2} |\nabla \varphi_n|^2 = B_{Q_c}(\varphi_n) + \int_{\mathbb{R}^2} \mathfrak{Re}(ic\partial_{x_2}\varphi_n \overline{\varphi_n}) + (1 - |Q_c|^2)|\varphi_n|^2 - 2\mathfrak{Re}^2(\overline{Q_c}\varphi_n),$$

therefore, by Cauchy-Schwarz,

$$\int_{\mathbb{R}^2} |\nabla \varphi_n|^2 \leqslant |\alpha_c| + Kc\|\nabla \varphi_n\|_{L^2(\mathbb{R}^2)} + K.$$

We deduce that, for c small enough,

$$\|\nabla \varphi_n\|^2_{L^2(\mathbb{R}^2)} - Kc\|\nabla \varphi_n\|_{L^2(\mathbb{R}^2)} \leqslant K(c),$$

hence $\|\nabla \varphi_n\|^2_{L^2(\mathbb{R}^2)}$ is bounded uniformly in n given that $c < c_0$ for some constant c_0 small enough. We deduce that φ_n is bounded in $H^1(\mathbb{R}^2)$, therefore, up to a subsequence, $\varphi_n \to \varphi$ weakly in $H^1(\mathbb{R}^2)$.

Now, we remark that for any $\varphi \in H^1(\mathbb{R}^2)$, by integration by parts (see Lemma 3.4.7),

$$\begin{aligned}\int_{\mathbb{R}^2} -\mathfrak{Re}(ic\partial_{x_2}\varphi\bar{\varphi}) &= -c\int_{\mathbb{R}^2} \mathfrak{Re}(\partial_{x_2}\varphi)\mathfrak{Im}(\varphi) + c\int_{\mathbb{R}^2} \mathfrak{Re}(\varphi)\mathfrak{Im}(\partial_{x_2}\varphi) \\ &= 2c\int_{\mathbb{R}^2} \mathfrak{Re}(\varphi)\mathfrak{Im}(\partial_{x_2}\varphi).\end{aligned}$$

For $R > 0$, since $\varphi_n \to \varphi$ weakly in $H^1(\mathbb{R}^2)$, this implies that $\varphi_n \to \varphi$ strongly in $L^2(B(0,R))$ by Rellich-Kondrakov theorem. In particular, we have

$$\int_{B(0,R)} \mathfrak{Re}(\varphi_n)\mathfrak{Im}(\partial_{x_2}\varphi_n) \to \int_{B(0,R)} \mathfrak{Re}(\varphi)\mathfrak{Im}(\partial_{x_2}\varphi).$$

since $\varphi_n \to \varphi$ strongly in $L^2(B(0,R))$ and $\partial_{x_2}\varphi_n \to \partial_{x_2}\varphi$ weakly in $L^2(B(0,R))$. We deduce that, up to a subsequence,

$$\begin{aligned}&\int_{B(0,R)} |\nabla\varphi|^2 + 2c\mathfrak{Re}(\varphi)\mathfrak{Im}(\partial_{x_2}\varphi) - (1 - |Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi) \\ \leqslant\ &\liminf_{n\to\infty} \int_{B(0,R)} |\nabla\varphi_n|^2 + 2c\mathfrak{Re}(\varphi_n)\mathfrak{Im}(\partial_{x_2}\varphi_n) - (1 - |Q_c|^2)|\varphi_n|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi_n) + o^R_{n\to\infty}(1).\end{aligned}$$

Furthermore, we have, by weak convergence

$$\|\varphi\|_{H^1(\mathbb{R}^2)} \leqslant \liminf_{n\to\infty} \|\varphi_n\|_{H^1(\mathbb{R}^2)} \leqslant K(c)$$

therefore, we estimate

$$\begin{aligned}&\int_{\mathbb{R}^2\setminus B(0,R)} |\nabla\varphi|^2 + 2c\mathfrak{Re}(\varphi)\mathfrak{Im}(\partial_{x_2}\varphi) - (1 - |Q_c|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi) \\ \leqslant\ &K\|\varphi\|^2_{H^1(\mathbb{R}^2\setminus B(0,R))} = o_{R\to\infty}(1).\end{aligned}$$

We deduce that

$$\begin{aligned}B_{Q_c}(\varphi) \leqslant\ &\liminf_{n\to\infty} \int_{B(0,R)} |\nabla\varphi_n|^2 + 2c\mathfrak{Re}(\varphi_n)\mathfrak{Im}(\partial_{x_2}\varphi_n) - (1 - |Q_c|^2)|\varphi_n|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi_n) \\ +\ &o^R_{n\to\infty}(1) + o_{R\to\infty}(1).\end{aligned}$$

Now, we have

$$\begin{aligned} & \liminf_{n\to\infty} \int_{B(0,R)} |\nabla\varphi_n|^2 + 2c\mathfrak{Re}(\varphi_n)\mathfrak{Im}(\partial_{x_2}\varphi_n) - (1-|Q_c|^2)|\varphi_n|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi_n) \\ = & \liminf_{n\to\infty} B_{Q_c}(\varphi_n), \\ - & \liminf_{n\to\infty} \int_{\mathbb{R}^2\setminus B(0,R)} |\nabla\varphi_n|^2 + 2c\mathfrak{Re}(\varphi_n)\mathfrak{Im}(\partial_{x_2}\varphi_n) - (1-|Q_c|^2)|\varphi_n|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi_n) \end{aligned}$$

and $B_{Q_c}(\varphi_n) \to \alpha_c$, therefore

$$\begin{aligned} B_{Q_c}(\varphi) \leqslant & \ \alpha_c + o^R_{n\to\infty}(1) + o_{R\to\infty}(1) \\ & - \liminf_{n\to\infty} \int_{\mathbb{R}^2\setminus B(0,R)} |\nabla\varphi_n|^2 + 2c\mathfrak{Re}(\varphi_n)\mathfrak{Im}(\partial_{x_2}\varphi_n) - (1-|Q_c|^2)|\varphi_n|^2 + 2\mathfrak{Re}^2(\overline{Q_c}\varphi_n). \end{aligned}$$

From Theorem 3.1.4, we have $(1-|Q_c|^2)(x) \to 0$ when $|x| \to \infty$, therefore, since $\|\varphi_n\|_{L^2(\mathbb{R}^2)} = 1$, we have by dominated convergence that

$$\int_{\mathbb{R}^2\setminus B(0,R)} (1-|Q_c|^2)|\varphi_n|^2 \leqslant \sqrt{\int_{\mathbb{R}^2\setminus B(0,R)} (1-|Q_c|^2)^2 \int_{\mathbb{R}^2} |\varphi_n|^2} \leqslant o_{R\to\infty}(1).$$

Furthermore, we check easily that (since $(A+B)^2 \geqslant \frac{1}{2}A^2 - B^2$)

$$\int_{\mathbb{R}^2\setminus B(0,R)} \mathfrak{Re}^2(\overline{Q_c}\varphi_n) \geqslant \frac{1}{2}\int_{\mathbb{R}^2\setminus B(0,R)} \mathfrak{Re}^2(Q_c)\mathfrak{Re}^2(\varphi_n) - \int_{\mathbb{R}^2\setminus B(0,R)} \mathfrak{Im}^2(Q_c)\mathfrak{Im}^2(\varphi_n),$$

and from Theorem 3.1.4, $\mathfrak{Im}(Q_c)(x) \to 0$ and $\mathfrak{Re}(Q_c)(x) \to 1$ when $|x| \to \infty$, therefore, since $\|\varphi_n\|_{L^2(\mathbb{R}^2)} = 1$, by dominated convergence,

$$\int_{\mathbb{R}^2\setminus B(0,R)} 2\mathfrak{Re}^2(\overline{Q_c}\varphi_n) \geqslant \int_{\mathbb{R}^2\setminus B(0,R)} \mathfrak{Re}^2(\varphi_n) - o_{R\to\infty}(1).$$

We deduce that, since $c < \sqrt{2}$,

$$\begin{aligned} B_{Q_c}(\varphi) \leqslant & \ \alpha_c + o^R_{n\to\infty}(1) + o_{R\to\infty}(1) \\ & - \liminf_{n\to\infty} \left(\int_{\mathbb{R}^2\setminus B(0,R)} |\nabla\varphi_n|^2 + 2c\mathfrak{Re}(\varphi_n)\mathfrak{Im}(\partial_{x_2}\varphi_n) + \mathfrak{Re}^2(\varphi_n) \right) \\ \leqslant & \ \alpha_c + o^R_{n\to\infty}(1) + o_{R\to\infty}(1) \\ & - \liminf_{n\to\infty} \left(\int_{\mathbb{R}^2\setminus B(0,R)} (|\nabla\varphi_n| + c\mathfrak{Re}(\varphi_n))^2 + (2-c^2)\mathfrak{Re}^2(\varphi_n) \right) \\ \leqslant & \ \alpha_c + o^R_{n\to\infty}(1) + o_{R\to\infty}(1). \end{aligned}$$

Thus, by letting $n \to \infty$ and then $R \to \infty$,

$$B_{Q_c}(\varphi) \leqslant \alpha_c.$$

In particular, this implies that $\|\varphi\|_{L^2(\mathbb{R}^2)} \neq 0$, or else $B_{Q_c}(\varphi) = 0 \leqslant \alpha_c$ and we know that $\alpha_c < 0$. Furthermore, by weak convergence, we have $\|\varphi\|_{L^2(\mathbb{R}^2)} \leqslant 1$, and if it is not 1, then, since $\alpha_c < 0$,

$$B_{Q_c}\left(\frac{\varphi}{\|\varphi\|_{L^2(\mathbb{R}^2)}} \right) \leqslant \frac{\alpha_c}{\|\varphi\|^2_{L^2(\mathbb{R}^2)}} < \alpha_c$$

which is in contradiction with the definition of α_c. Therefore $\|\varphi\|_{L^2(\mathbb{R}^2)} = 1$ and $B_{Q_c}(\varphi) = \alpha_c$. This concludes the proof of Proposition 1.4.7. □

Proof. (of Corollary 1.4.9) The hypothesis to have the spectral stability from Theorem 11.8 of [30] are:

- The curve of travelling waves is C^1 from $]0, c_0[$ to $C^1(\mathbb{R}^2, \mathbb{C})$ with respect to the speed. This is a consequence of Theorem 1.3.1. This is enough to legitimate the computations done in the proof of Theorem 11.8 of [30].

- $\mathfrak{Re}(Q_c) - 1 \in H^1(\mathbb{R}^2)$, $\nabla Q_c \in L^2(\mathbb{R}^2)$, $|Q_c| \to 1$ at infinity and $\|Q_c\|_{C^1(\mathbb{R}^2)} \leqslant K$. These are consequences of Theorem 7 of [22].

- $n^-(L_{Q_c}) \leqslant 1$. This is a consequence of Proposition 1.4.7.
- $\partial_c P_2(Q_c) < 0$. This is a consequence of Proposition 1.4.1. □

3.5 Coercivity results with an orthogonality on the phase

This section is devoted to the proofs of Proposition 1.4.10, 1.4.11 and Theorem 1.4.12.

3.5.1 Properties of the space $H^{\exp}_{Q_c}$

In this subsection, we look at the space $H^{\exp}_{Q_c}$. We recall the norm

$$\|\varphi\|^2_{H^{\exp}_{Q_c}} = \|\varphi\|^2_{H^1(\{\tilde r\leqslant 10\})} + \int_{\{\tilde r\geqslant 5\}} |\nabla\psi|^2 + \mathfrak{Re}^2(\psi) + \frac{|\psi|^2}{\tilde r^2 \ln(\tilde r)^2}.$$

The quadratic form we look at is

$$\begin{aligned}
B^{\exp}_{Q_c}(\varphi) &= \int_{\mathbb{R}^2} \eta(|\nabla\varphi|^2 - \mathfrak{Re}(ic\partial_{x_2}\varphi\bar\varphi) - (1-|Q_c|^2)|\varphi|^2 + 2\,\mathfrak{Re}^2(\overline{Q_c}\varphi))\\
&- \int_{\mathbb{R}^2} \nabla\eta.(\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2 - 2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi))\\
&+ \int_{\mathbb{R}^2} c\partial_{x_2}\eta|Q_c|^2\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)\\
&+ \int_{\mathbb{R}^2} (1-\eta)(|\nabla\psi|^2|Q_c|^2 + 2\mathfrak{Re}^2(\psi)|Q_c|^4)\\
&+ \int_{\mathbb{R}^2} (1-\eta)(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi) + 2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi))
\end{aligned}$$

We will show in Lemma 3.5.1 that $B^{\exp}_{Q_c}(\varphi)$ is well defined for $\varphi \in H^{\exp}_{Q_c}$. The main difference between B_{Q_c} and $B^{\exp}_{Q_c}$ is the space on which they are defined. In particular, we can check easily for instance that, for $\varphi \in C^\infty_c(\mathbb{R}^2)$ with support far from the zeros of Q_c, we have $B^{\exp}_{Q_c}(\varphi) = B_{Q_c}(\varphi)$, as the terms with the gradient of the cutoff are exactly the ones coming from the integrations by parts. We start with a lemma about the space $H^{\exp}_{Q_c}$.

Lemma 3.5.1. *The following properties of $H^{\exp}_{Q_c}$ hold:*

$$H_{Q_c} \subset H^{\exp}_{Q_c},$$

$$iQ_c \in H^{\exp}_{Q_c}.$$

Furthermore, there exists $K(c) > 0$ such that, for $\varphi \in H^{\exp}_{Q_c}$,

$$\|\varphi\|_{\mathcal{C}} \leqslant K\|\varphi\|_{H^{\exp}_{Q_c}}, \tag{3.5.1}$$

$$\|\varphi\|_{H^{\exp}_{Q_c}} \leqslant K(c)\|\varphi\|_{H_{Q_c}}. \tag{3.5.2}$$

and the integrands of $B^{\exp}_{Q_c}(\varphi)$, defined in (1.4.4), are in $L^1(\mathbb{R}^2)$ for $\varphi \in H^{\exp}_{Q_c}$, and $B^{\exp}_{Q_c}$ does not depend on the choice of η. Finally, if $\varphi \in H_{Q_c} \subset H^{\exp}_{Q_c}$,

$$B_{Q_c}(\varphi) = B^{\exp}_{Q_c}(\varphi).$$

Proof. First, let us show (3.5.2). We have

$$\|\varphi\|_{H^1(\{\tilde r\leqslant 10\})} \leqslant K\|\varphi\|_{H_{Q_c}},$$

and, by equation (3.1.12) and Lemma 3.1.5, we check that

$$\int_{\{\tilde r\geqslant 5\}} \mathfrak{Re}^2(\psi) \leqslant K\|\varphi\|^2_{H_{Q_c}},$$

and also that

$$\int_{\{\tilde r\geqslant 5\}} \frac{|\psi|^2}{\tilde r^2\ln(\tilde r)^2} \leqslant K\int_{\{\tilde r\geqslant 5\}} \frac{|\varphi|^2}{(1+\tilde r)^2} \leqslant K(c)\|\varphi\|^2_{H_{Q_c}}.$$

Furthermore, we compute, by equations (3.1.12), (3.2.1) and Proposition 3.1.4,

$$\int_{\{\tilde r\geqslant 5\}}|\nabla\psi|^2\leqslant K\int_{\{\tilde r\geqslant 5\}}|\nabla\psi|^2|Q_c|^4\leqslant K\bigg(\int_{\{\tilde r\geqslant 5\}}|\nabla\varphi|^2+\int_{\{\tilde r\geqslant 5\}}|\nabla Q_c|^2|\varphi|^2\bigg)\leqslant K(c)\|\varphi\|^2_{H_{Q_c}}.$$

We deduce that (3.5.2) holds, and therefore $H_{Q_c}\subset H^{\exp}_{Q_c}$. Now, we check that

$$\|iQ_c\|^2_{H^{\exp}_{Q_c}}\leqslant\|iQ_c\|^2_{H^1(\{\tilde r\leqslant 10\})}+K\int_{\{\tilde r\geqslant 5\}}\frac{|i|^2}{\tilde r^2\ln(\tilde r)^2}+\int_{\{\tilde r\geqslant 5\}}|\nabla i|^2<+\infty. \tag{3.5.3}$$

With regards to the definition of $\|.\|_{\mathcal{C}}$, we check easily that

$$\|\varphi\|_{\mathcal{C}}\leqslant\|\varphi\|_{H^{\exp}_{Q_c}}.$$

Finally, we recall the definition of $B^{\exp}_{Q_c}(\varphi)$ from equation (1.4.4),

$$\begin{aligned}
B^{\exp}_{Q_c}(\varphi) &= \int_{\mathbb{R}^2}(1-\eta)(|\nabla\varphi|^2-\mathfrak{Re}(ic\partial_{x_2}\varphi\bar\varphi)-(1-|Q_c|^2)|\varphi|^2+2\,\mathfrak{Re}^2(\overline{Q_c}\varphi))\\
&- \int_{\mathbb{R}^2}\nabla\eta.(\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2-2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi))\\
&+ \int_{\mathbb{R}^2}c\partial_{x_2}\eta|Q_c|^2\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)\\
&+ \int_{\mathbb{R}^2}\eta(|\nabla\psi|^2|Q_c|^2+2\mathfrak{Re}^2(\psi)|Q_c|^4)\\
&+ \int_{\mathbb{R}^2}\eta(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)+2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)).
\end{aligned}$$

For $\lambda>0$, we have $\|\varphi\|_{H^1(B(0,\lambda))}\leqslant K(c,\lambda)\|\varphi\|_{H^{\exp}_{Q_c}}$, therefore (since $1-\eta$ is compactly supported) we only have to check that the integrands in the last two lines are in $L^1(\mathbb{R}^2)$, and this is a consequence of Cauchy-Schwarz, since

$$\begin{aligned}
&\int_{\mathbb{R}^2}\eta(|\nabla\psi|^2|Q_c|^2+2\mathfrak{Re}^2(\psi)|Q_c|^4+4|\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)|+2c|Q_c|^2|\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi)|)\\
&\qquad\leqslant K\int_{\mathbb{R}^2}\eta(|\nabla\psi|^2+\mathfrak{Re}^2(\psi))\leqslant K\|\varphi\|^2_{H^{\exp}_{Q_c}}.
\end{aligned}$$

Furthermore, for two cutoffs η,η' such that they are both 0 near the zeros of Q_c and 1 at infinity, we have

$$\begin{aligned}
&B^{\exp}_{Q_c,\eta}(\varphi)-B^{\exp}_{Q_c,\eta'}(\varphi)\\
=&\int_{\mathbb{R}^2}(\eta'-\eta)(|\nabla\varphi|^2-\mathfrak{Re}(ic\partial_{x_2}\varphi\bar\varphi)-(1-|Q_c|^2)|\varphi|^2+2\mathfrak{Re}^2(\overline{Q_c}\varphi))\\
+&\int_{\mathbb{R}^2}\nabla(\eta-\eta').(\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\psi|^2-2\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi))-c\partial_{x_2}(\eta-\eta')|Q_c|^2\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)\\
+&\int_{\mathbb{R}^2}(\eta'-\eta)(|\nabla\psi|^2|Q_c|^2+2\mathfrak{Re}^2(\psi)|Q_c|^4)\\
+&\int_{\mathbb{R}^2}(\eta'-\eta)(4\mathfrak{Im}(\nabla Q_c\overline{Q_c})\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi)+2c|Q_c|^2\mathfrak{Im}(\partial_{x_2}\psi)\mathfrak{Re}(\psi))
\end{aligned}$$

and, developping $\varphi=Q_c\psi$ (see the proof of Lemma 3.3.1) and by integration by parts using that $\eta-\eta'\neq 0$ only in a compact domain far from the zeros of Q_c, we check that it is 0.

Finally, for $\varphi\in H_{Q_c}$, $B_{Q_c}(\varphi)$ and $B^{\exp}_{Q_c}(\varphi)$ are both well defined. We recall

$$\begin{aligned}
B_{Q_c}(\varphi) &= \int_{\mathbb{R}^2}|\nabla\varphi|^2-(1-|Q_c|^2)|\varphi|^2+2\,\mathfrak{Re}^2(\overline{Q_c}\varphi)\\
&- c\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(i\partial_{x_2}\varphi\bar\varphi)-c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\,i\,\partial_{x_2}Q_c\overline{Q_c}|\psi|^2\\
&+ 2c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\,\psi\,\mathfrak{Im}\partial_{x_2}\psi|Q_c|^2+c\int_{\mathbb{R}^2}\partial_{x_2}\eta\mathfrak{Re}\,\psi\,\mathfrak{Im}\,\psi|Q_c|^2\\
&+ c\int_{\mathbb{R}^2}\eta\mathfrak{Re}\,\psi\,\mathfrak{Im}\,\psi\partial_{x_2}(|Q_c|^2).
\end{aligned}$$

With the same computation as in the proof of Lemma 3.3.1, we check that for $\varphi \in C_c^\infty\big(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\},\mathbb{C}\big)$, we have

$$B_{Q_c}(\varphi) = B_{Q_c}^{\exp}(\varphi).$$

With the same arguments as in the density proof at the end of the proof of Proposition 1.4.3, we check that this equality holds for $\varphi \in H_{Q_c}$. □

Now, we state some lemmas that where shown previously in H_{Q_c}, that we have to extend to $H_{Q_c}^{\exp}$ to replace some arguments that were used in the proof of Propositions 1.4.3 for the proofs of Propositions 1.4.10, 1.4.11 and Theorem 1.4.12. We start with the density argument.

Lemma 3.5.2. *$C_c^\infty\big(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\},\mathbb{C}\big)$ is dense in $H_{Q_c}^{\exp}$ for $\|.\|_{H_{Q_c}^{\exp}}$.*

Proof. The proof is identical to the one of Lemma 3.2.4, as we check easily that, for $\lambda > \frac{10}{c}$ large enough,

$$\|\varphi\|^2_{H^1(\{\tilde{r}\leqslant 10\})} + \int_{\{\tilde{r}\geqslant 5\}\cap B(0,\lambda)} |\nabla\psi|^2 + \mathfrak{Re}^2(\psi) + \frac{|\psi|^2}{\tilde{r}^2\ln(\tilde{r})^2} \leqslant K_1(\lambda,c)\|\varphi\|^2_{H^1(B(0,\lambda))}$$

and

$$\|\varphi\|^2_{H^1(\{\tilde{r}\leqslant 10\})} + \int_{\{\tilde{r}\geqslant 5\}\cap B(0,\lambda)} |\nabla\psi|^2 + \mathfrak{Re}^2(\psi) + \frac{|\psi|^2}{\tilde{r}^2\ln(\tilde{r})^2} \geqslant K_2(\lambda,c)\|\varphi\|^2_{H^1(B(0,\lambda))}$$ □

We also want to decompose the quadratic form, but with a fifth possible direction: iQ_c.

Lemma 3.5.3. *For $\varphi \in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\},\mathbb{C})$ and $A \in \mathrm{Span}\{\partial_{x_1}Q_c, \partial_{x_2}Q_c, \partial_c Q_c, \partial_{c^\perp}Q_c, iQ_c\}$, we have*

$$\langle L_{Q_c}(\varphi + A), \varphi + A\rangle = \langle L_{Q_c}(\varphi), \varphi\rangle + \langle 2L_{Q_c}(A), \varphi\rangle + \langle L_{Q_c}(A), A\rangle.$$

Furthermore, $\langle L_{Q_c}(\varphi + A), \varphi + A\rangle = B_{Q_c}^{\exp}(\varphi + A)$, $L_{Q_c}(iQ_c) = 0$ and

$$\|\partial_{x_1}Q_c\|_{H_{Q_c}^{\exp}} + \|\partial_{x_2}Q_c\|_{H_{Q_c}^{\exp}} + \|c^2\partial_c Q_c\|_{H_{Q_c}^{\exp}} + c^{\beta_0/2}\|c\partial_{c^\perp}Q_c\|_{H_{Q_c}^{\exp}} + \|iQ_c\|_{H_{Q_c}^{\exp}} \leqslant K(\beta_0).$$

Proof. As for the proof of Lemma 3.4.2, we only have to show that $\mathfrak{Re}(L_{Q_c}(A)\bar{A}) \in L^1(\mathbb{R}^2)$ to show the first equality.

By simple computation (or by invariance of the phase), we check that $L_{Q_c}(iQ_c) = 0$. Writing $A = T + \varepsilon iQ_c$ for $\varepsilon \in \mathbb{R}, T \in \mathrm{Span}\{\partial_{x_1}Q_c, \partial_{x_2}Q_c, \partial_c Q_c, \partial_{c^\perp}Q_c\}$, we compute from Lemma 3.1.7 that

$$L_{Q_c}(A) = L_{Q_c}(T) \in \mathrm{Span}_{\mathbb{R}}(i\partial_{x_1}Q_c, i\partial_{x_2}Q_c),$$

thus

$$\mathfrak{Re}(L_{Q_c}(A)\bar{A}) = \mathfrak{Re}(L_{Q_c}(T)\overline{T + \varepsilon iQ_c}) = \mathfrak{Re}(L_{Q_c}(T)\bar{T}) + \varepsilon\mathfrak{Re}(L_{Q_c}(T)\overline{iQ_c}).$$

From the proof of Lemma 3.4.2, we have $\mathfrak{Re}(L_{Q_c}(T)\bar{T}) \in L^1(\mathbb{R}^2)$, and since $L_{Q_c}(T) \in \mathrm{Span}_{\mathbb{R}}(i\partial_{x_1}Q_c, i\partial_{x_2}Q_c)$, with Theorem 3.1.4, we have

$$|\mathfrak{Re}(L_{Q_c}(T)\overline{iQ_c})| \leqslant \frac{K(c)}{(1+r)^3} \in L^1(\mathbb{R}^2).$$

Let us check that, for $\varphi \in H_{Q_c}^{\exp}$, $B_{Q_c}^{\exp}(\varphi + \varepsilon iQ_c) = B_{Q_c}^{\exp}(\varphi)$ for $\varepsilon \in \mathbb{R}$.

We check, from (1.4.4), that, for $\varphi \in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\},\mathbb{C})$, this equality holds by integration by parts and because $\mathfrak{Re}(\psi + i) = \mathfrak{Re}(\psi)$, $\mathfrak{Im}(\nabla(\psi + i)) = \mathfrak{Im}(\nabla\psi)$. We then argue by density, as in the proof of Proposition 1.4.3.

We deduce, from Lemmas 3.1.7 and 3.4.2, that for $\varphi \in C_c^\infty(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\},\mathbb{C})$,

$$\begin{aligned} B_{Q_c}^{\exp}(\varphi + A) &= B_{Q_c}^{\exp}(\varphi + T) = B_{Q_c}(\varphi + T) \\ &= \langle L_{Q_c}(\varphi + T), \varphi + T\rangle = \langle L_{Q_c}(\varphi + A), \varphi + T\rangle \\ &= \langle L_{Q_c}(\varphi + A), \varphi + A\rangle - \langle L_{Q_c}(\varphi + A), \varepsilon iQ_c\rangle, \end{aligned}$$

and we check, with Lemma 3.1.7, that for some $v\in\mathbb{R}^2$ depending on A,

$$\begin{aligned}\langle L_{Q_c}(\varphi+A),\varepsilon iQ_c\rangle &= \langle L_{Q_c}(\varphi),\varepsilon iQ_c\rangle+\langle L_{Q_c}(P),\varepsilon iQ_c\rangle\\ &= \varepsilon\langle\varphi,L_{Q_c}(iQ_c)\rangle+\varepsilon v.\int_{\mathbb{R}^2}\mathfrak{Re}(\nabla Q_c\overline{Q_c})\\ &= 0.\end{aligned}$$

From Lemma 3.1.8, we have,

$$\|\partial_{x_1}Q_c\|_{\mathcal{C}}+\|\partial_{x_2}Q_c\|_{\mathcal{C}}+\|c^2\partial_cQ_c\|_{\mathcal{C}}+c^{\beta_0/2}\|c\partial_{c^\perp}Q_c\|_{\mathcal{C}}\leqslant K(\beta_0),$$

and with Lemmas 1.2.1, 3.1.2 and equations (3.1.9), (3.1.10), (3.1.11), we check with the definition of $\|.\|_{H^{\exp}_{Q_c}}$ and $\|.\|_{\mathcal{C}}$ that, for $A\in\{\partial_{x_1}Q_c,\partial_{x_2}Q_c,c^2\partial_cQ_c,c^{1+\beta_0/2}\partial_{c^\perp}Q_c\}$,

$$\|A\|^2_{H^{\exp}_{Q_c}}\leqslant K\|A\|^2_{H^1(\{\tilde r\leqslant 10\})}+\|A\|^2_{\mathcal{C}}\leqslant K(\beta_0).$$

Finally, we check that

$$\|iQ_c\|^2_{H^{\exp}_{Q_c}}=\|iQ_c\|^2_{H^1(\{\tilde r\leqslant 10\})}+\int_{\{\tilde r\geqslant 5\}}|\nabla i|^2+\mathfrak{Re}^2(i)+\frac{|i|^2}{\tilde r^2\ln(\tilde r)^2}\leqslant K.$$ □

We can now end the proof of Proposition 1.4.10.

Proof. (of Proposition 1.4.10) From Theorem 1.4.4, for $\varphi\in C^\infty_c(\mathbb{R}^2\backslash\{\tilde d_c\vec{e_1},-\tilde d_c\vec{e_1}\},\mathbb{C})$, under the four orthogonality conditions of Proposition 1.4.10, we have, by lemma 3.5.1,

$$B^{\exp}_{Q_c}(\varphi)=B_{Q_c}(\varphi)=\langle L_{Q_c}(\varphi),\varphi\rangle\geqslant K\|\varphi\|^2_{\mathcal{C}}.$$

We then conclude by density, as in the proof of Proposition 1.4.3, using Lemma 3.5.2. The proof for the density in $B^{\exp}_{Q_c}$ is similar to the one for B_{Q_c} in the proof of Proposition 1.4.3. The coercivity under three orthogonality conditions can be shown similarly.

Then, for the computation of the kernel, the proof is identical to the one of Corollary 1.4.5. With Lemma 3.5.1, we check easily that we can do the same computation simply by replacing $B_{Q_c}(\varphi)$ by $B^{\exp}_{Q_c}(\varphi)$. The only difference is at the end, when we have $\|\varphi^*\|_{\mathcal{C}}=0$, it implies that $\varphi^*=\lambda iQ_c$ for some $\lambda\in\mathbb{R}$, and we can not conclude that $\lambda=0$, since we only have $\varphi^*\in H^{\exp}_{Q_c}$ instead of $\varphi^*\in H_{Q_c}$. This implies that

$$\varphi\in\mathrm{Span}_{\mathbb{R}}(\partial_{x_1}Q_c,\partial_{x_2}Q_c,iQ_c).$$

Using Lemma 3.1.7 and 3.5.3, we check easily the implication from (ii) to (i). □

3.5.2 Change of the coercivity norm with an orthogonality on the phase

We now focus on the proofs of Proposition 1.4.11 and Theorem 1.4.12. In these results, we add an orthogonality condition on the phase. We start with a lemma giving the coercivity result but with the original orthogonality conditions on the vortices, adding the one on the phase.

Lemma 3.5.4. *For $\varphi=Q_c\psi\in H^{\exp}_{Q_c}$, if the following four orthogonality conditions are satisfied:*

$$\int_{B(\tilde d_c\vec{e_1},R)}\mathfrak{Re}\Big(\partial_{x_1}\tilde V_1\overline{\tilde V_1\psi}\Big)=\int_{B(\tilde d_c\vec{e_1},R)}\mathfrak{Re}\Big(\partial_{x_2}\tilde V_1\overline{\tilde V_1\psi}\Big)=0,$$

$$\int_{B(-\tilde d_c\vec{e_1},R)}\mathfrak{Re}\Big(\partial_{x_1}\tilde V_{-1}\overline{\tilde V_{-1}\psi}\Big)=\int_{B(-\tilde d_c\vec{e_1},R)}\mathfrak{Re}\Big(\partial_{x_2}\tilde V_{-1}\overline{\tilde V_{-1}\psi}\Big)=0,$$

then, if $\mathfrak{Re}\int_{B(0,R)}i\psi=0$, we have (with $K(c)\leqslant 1$)

$$B^{\exp}_{Q_c}(\varphi)\geqslant K(c)\|\varphi\|^2_{H^{\exp}_{Q_c}}+K\|\varphi\|^2_{\mathcal{C}},$$

or if $\forall x \in \mathbb{R}, \varphi(x_1,x_2) = \varphi(-x_1,x_2)$ *and* $\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)} iQ_c\bar{\varphi} = 0$, *then*

$$B^{\exp}_{Q_c}(\varphi) \geqslant K\|\varphi\|^2_{H^{\exp}_{Q_c}}.$$

Proof. Let us show these results for $\varphi = Q_c\psi \in C_c^\infty\big(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\}, \mathbb{C}\big)$. We then conclude by density. We start with the nonsymmetric case.

By Lemma 3.3.4, for $\varphi = Q_c\psi \in C_c^\infty\big(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\}, \mathbb{C}\big)$ such that

$$\int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big) = \int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big) = 0,$$

we have

$$B^{\mathrm{loc}_{1,D}}_{Q_c}(\varphi) \geqslant K(D)\int_{B(\tilde{d}_c\vec{e}_1,D)} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4.$$

By Lemma 3.3.3, we infer, by a standard proof by contradiction (with the first two orthogonality conditions),

$$B^{\mathrm{loc}_{1,D}}_{Q_c}(\varphi) \geqslant K_1(D)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))} - K_2(D)\left(\int_{B(\tilde{d}_c\vec{e}_1,R)\backslash B(\tilde{d}_c\vec{e}_1,R/2)} \mathfrak{Im}(\psi)\right)^2.$$

We deduce, with Lemma 3.3.3, that for any small $\varepsilon > 0$

$$\begin{aligned} B^{\mathrm{loc}_{1,D}}_{Q_c}(\varphi) \;&\geqslant\; K(D)(1-\varepsilon)\int_{B(\tilde{d}_c\vec{e}_1,D)} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4 \\ &+\; K_1(D)\varepsilon\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))} - K_2(D)\varepsilon\left(\int_{B(\tilde{d}_c\vec{e}_1,R)\backslash B(\tilde{d}_c\vec{e}_1,R/2)} \mathfrak{Im}(\psi)\right)^2. \end{aligned}$$

By Poincaré inéquality, if $\mathfrak{Re}\int_{B(0,R)} i\psi = 0$, then

$$\begin{aligned} \int_{B(\tilde{d}_c\vec{e}_1,R)\backslash B(\tilde{d}_c\vec{e}_1,R/2)} \mathfrak{Im}(\psi) \;&\leqslant\; K(c)\sqrt{\int_{\mathbb{R}^2\backslash(B(\tilde{d}_c\vec{e}_1,R/2)\cup B(-\tilde{d}_c\vec{e}_1,R/2))} |\nabla\psi|^2} \\ &\leqslant\; K(c)\sqrt{\int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4}. \end{aligned}$$

Therefore, for any small $\mu > 0$, taking $\varepsilon > 0$ small enough (depending on c, D and μ),

$$\begin{aligned} B^{\mathrm{loc}_{1,D}}_{Q_c}(\varphi) \;&\geqslant\; K(D)\int_{B(\tilde{d}_c\vec{e}_1,D)} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4 \\ &+\; K_1(D,c,\mu)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,D))} - \mu\int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4. \end{aligned}$$

With similar arguments, we have a similar result for $B^{\mathrm{loc}_{-1,D}}_{Q_c}(\varphi)$. Now, as in the proof of Proposition 1.4.3, we have, taking $\mu > 0$ small enough and $D > 0$ large enough,

$$\begin{aligned} B_{Q_c}(\varphi) \;&\geqslant\; B^{\mathrm{loc}_{1,D}}_{Q_c}(\varphi) + B^{\mathrm{loc}_{-1,D}}_{Q_c}(\varphi) \\ &+\; K\left(\int_{\mathbb{R}^2\backslash(B(\tilde{d}_c\vec{e}_1,D)\cup B(-\tilde{d}_c\vec{e}_1,D))} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4\right) \\ &\geqslant\; K\int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 + \mathfrak{Re}^2(\psi)|Q_c|^4 + K_1(c,\mu)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,10))} \\ &-\; \mu\int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4 \\ &\geqslant\; K\|\varphi\|^2_{\mathcal{C}} + K(c)\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,10))}. \end{aligned}$$

Then, by the same Hardy type inequality as in the proof of Proposition 1.4.3, we show that

$$\int_{\mathbb{R}^2} \frac{|\varphi|^2}{(1+\tilde{r})^2\ln^2(2+\tilde{r})} \leqslant K\left(\|\varphi\|^2_{H^1(B(\tilde{d}_c\vec{e}_1,10))} + \int_{\mathbb{R}^2} |\nabla\psi|^2|Q_c|^4\right),$$

therefore

$$B_{Q_c}(\varphi) \geqslant K\|\varphi\|_{\mathcal{C}}^2 + K(c)\|\varphi\|_{H^{\exp}_{Q_c}}^2.$$

In the symmetric case, the proof is identical, exept that, by symmetry,

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e}_1,R)} iQ_c\bar{\varphi} = 0,$$

and we check by Poincaré inequality that for a function φ satisfying this orthogonality condition, $\varphi = Q_c\psi$,

$$\left|\int_{B(\tilde{d}_c\vec{e}_1,R)\setminus B(\tilde{d}_c\vec{e}_1,R/2)} \mathfrak{Im}(\psi)\right| \leqslant K\|\varphi\|_{H^1(B(\tilde{d}_c\vec{e}_1,R))},$$

for a universal constant $K>0$. By a similar computation as previously, we conclude the proof of this lemma. □

We now have all the elements necessary to conclude the proof of Proposition 1.4.11.

Proof. (of Proposition 1.4.11) This proof follows the proof of Lemma 3.4.5. For $\varphi \in C_c^\infty\big(\mathbb{R}^2\setminus\{\tilde{d}_c\vec{e}_1, -\tilde{d}_c\vec{e}_1\}, \mathbb{C}\big)$ and five real-valued parameters $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5$ we define $\varphi^* = Q_c\psi^*$ by

$$\psi^* = \psi + \varepsilon_1\frac{\partial_{x_1}Q_c}{Q_c} + \varepsilon_2\frac{c^2\partial_c Q_c}{Q_c} + \varepsilon_3\frac{\partial_{x_2}Q_c}{Q_c} + \varepsilon_4\frac{c\partial_{c^\perp}Q_c}{Q_c} + \varepsilon_5 i.$$

From Lemma 3.5.3, we check that $\varphi^* \in H^{\exp}_{Q_c}$. Now, similarly as the proof of Lemma 3.4.5, we check that

$$\begin{aligned}
\int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi^*}\Big) &= \int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi}\Big) \\
&+ \varepsilon_1\int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\left(\partial_{x_1}\tilde{V}_1\overline{\partial_{x_1}Q_c\frac{\tilde{V}_1}{Q_c}}\right) \\
&+ \varepsilon_2\int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\left(\partial_{x_1}\tilde{V}_1\overline{c^2\partial_c Q_c\frac{\tilde{V}_1}{Q_c}}\right) \\
&+ \varepsilon_3\int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\left(\partial_{x_1}\tilde{V}_1\overline{\partial_{x_2}Q_c\frac{\tilde{V}_1}{Q_c}}\right) \\
&+ \varepsilon_4\int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\left(\partial_{x_1}\tilde{V}_1\overline{c\partial_{c^\perp}Q_c\frac{\tilde{V}_1}{Q_c}}\right) \\
&+ \varepsilon_5\int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{i\tilde{V}_1}\Big).
\end{aligned}$$

Furthermore, with Lemma 1.2.1, we check that

$$\int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_1}V_1\overline{i\tilde{V}_1}\Big) = 0,$$

and the other terms are estimated as in the proof of Lemma 3.4.5. Similarly,

$$\int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_2}V_1\overline{i\tilde{V}_1}\Big) = \int_{B(-\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\big(\partial_{x_1}V_{-1}\overline{i\widetilde{V_{-1}}}\big) = \int_{B(-\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\big(\partial_{x_2}V_{-1}\overline{i\widetilde{V_{-1}}}\big) = 0.$$

We also check that, from (3.1.9), (3.1.10), Lemmas 3.1.2 and 3.1.6 that

$$\begin{aligned}
&\left|\int_{B(0,R)} \mathfrak{Re}\left(i\frac{\partial_{x_1}Q_c}{Q_c}\right)\right| + \left|\int_{B(0,R)} \mathfrak{Re}\left(i\frac{\partial_{x_2}Q_c}{Q_c}\right)\right| \\
+ &\left|\int_{B(0,R)} \mathfrak{Re}\left(ic^2\frac{\partial_c Q_c}{Q_c}\right)\right| + \left|\int_{B(0,R)} \mathfrak{Re}\left(ci\frac{\partial_{c^\perp}Q_c}{Q_c}\right)\right| \\
= &\ o_{c\to 0}(1),
\end{aligned}$$

and

$$\int_{B(0,R)} \mathfrak{Re}(i\times i) = -\pi R^2 < 0.$$

We deduce, as in the proof of Lemma 3.4.5, that

$$\begin{pmatrix} \int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_1\overline{\tilde{V}_1\psi^*}\Big) \\ \int_{B(-\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_1}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi^*}\Big) \\ \int_{B(\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_1\overline{\tilde{V}_1\psi^*}\Big) \\ \int_{B(-\tilde{d}_c\vec{e}_1,R)} \mathfrak{Re}\Big(\partial_{x_2}\tilde{V}_{-1}\overline{\tilde{V}_{-1}\psi^*}\Big) \\ \mathfrak{Re}\int_{B(0,R)} i\psi = 0 \end{pmatrix}$$

$$= \left(\begin{pmatrix} K(R) & -K(R) & 0 & 0 & 0 \\ K(R) & K(R) & 0 & 0 & 0 \\ 0 & 0 & K(R) & -K(R) & 0 \\ 0 & 0 & K(R) & K(R) & 0 \\ 0 & 0 & 0 & 0 & -\pi R^2 \end{pmatrix} + o_{c\to 0}(1)\right)\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \end{pmatrix}$$

$$+ \quad o_{c\to 0}(c^{\beta_0})K\|\varphi\|_{\mathcal{C}}.$$

Therefore, we can find $\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5\in\mathbb{R}$ such that

$$|\varepsilon_1|+|\varepsilon_2|+|\varepsilon_3|+|\varepsilon_4|+|\varepsilon_5| \leqslant o_{c\to 0}(c^{\beta_0})\|\varphi\|_{\mathcal{C}}$$

and φ^* satisfies the five orthogonality conditions of Lemma 3.5.4. Therefore,

$$B^{\exp}_{Q_c}(\varphi^*) \geqslant K(c)\|\varphi^*\|^2_{H^{\exp}_{Q_c}} + K\|\varphi^*\|^2_{\mathcal{C}}.$$

We continue as in the proof of Lemma 3.4.5, and with the same arguments, we have

$$B^{\exp}_{Q_c}(\varphi) \geqslant K(c)\|\varphi^*\|^2_{H^{\exp}_{Q_c}} + K\|\varphi\|^2_{\mathcal{C}}.$$

Now, by Lemma 3.5.3, we have

$$\begin{aligned} \|\varphi^*\|_{H^{\exp}_{Q_c}} &\geqslant \|\varphi\|_{H^{\exp}_{Q_c}} - \|\varepsilon_1\partial_{x_1}Q_c + \varepsilon_2 c^2\partial_c Q_c + \varepsilon_3\partial_{x_2}Q_c + \varepsilon_4 c\partial_{c^\perp}Q_c + \varepsilon_5 i\|_{H^{\exp}_{Q_c}} \\ &\geqslant \|\varphi\|_{H^{\exp}_{Q_c}} - o_{c\to 0}(c^{\beta_0/2})\|\varphi\|_{\mathcal{C}}, \end{aligned}$$

thus, since we can take $K(c)\leqslant 1$, we have

$$B^{\exp}_{Q_c}(\varphi) \geqslant K(c)\|\varphi\|^2_{H^{\exp}_{Q_c}}.$$

We conclude by density as in the proof of Proposition 1.4.3, thanks to Lemma 3.5.2. We are left with the proof of $B^{\exp}_{Q_c}(\varphi)\leqslant K\|\varphi\|^2_{H^{\exp}_{Q_c}}$. With regards to (1.4.4), the local terms can be estimated by $K\|\varphi\|^2_{H^1(\{\tilde{r}\leqslant 10\})} \leqslant K\|\varphi\|^2_{H^{\exp}_{Q_c}}$ and the terms at infinity, by Cauchy Schwarz, can be estimated by $K\int_{\{\tilde{r}\geqslant 5\}} |\nabla\psi|^2 + \mathfrak{Re}^2(\psi) + \frac{|\psi|^2}{\tilde{r}^2\ln^2(\tilde{r})} \leqslant K\|\varphi\|^2_{H^{\exp}_{Q_c}}$. □

As for the remark above equation (3.4.20), we can replace the orthogonality condition $\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)} \partial_c Q_c\overline{Q_c\psi^{\neq 0}} = 0$ by

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e}_1,R)\cup B(-\tilde{d}_c\vec{e}_1,R)} \partial_d(V_1(x-d\vec{e}_1)V_{-1}(x+d\vec{e}_1))_{|\,d=d_c}\overline{Q_c\psi^{\neq 0}}(x)dx = 0 \tag{3.5.4}$$

in Propositions 1.4.10 and 1.4.11.

Proof. (of Theorem 1.4.12) This proof follows closely the proof of Proposition 1.4.11,

First, From Lemma 3.1.3 and the definition of $\partial_{c^\perp}Q_c$ in Lemma 3.1.6, we check that $\partial_{x_1}Q_c$ and $\partial_{c^\perp}Q_c$ are odd in x_1, and for $\varphi = Q_c\psi \in C^\infty_c\big(\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1,\,-\tilde{d}_c\vec{e}_1\},\,\mathbb{C}\big)$ with $\forall(x_1,\,x_2)\in\mathbb{R}^2$, $\varphi(x_1,x_2)=\varphi(-x_1,x_2)$, we check that in $B\big(\tilde{d}_c\vec{e}_1,R\big)\cup B\big(-\tilde{d}_c\vec{e}_1,R\big)$, $Q_c\psi^{\neq 0}$ is even in x_1. Therefore, these two orthogonality conditions are freely given.

We decompose as previously for, $\varepsilon_1, \varepsilon_2, \varepsilon_3$ three real-valued parameters,

$$\varphi = \varphi^* + \varepsilon_1 i Q_c + \varepsilon_2 \partial_{x_2} Q_c + \varepsilon_3 c^2 \partial_c Q_c.$$

We suppose that

$$\mathfrak{Re} \int_{B(\tilde{d}_c \vec{e_1}, R) \cup B(-\tilde{d}_c \vec{e_1}, R)} \partial_c Q_c \bar{\varphi} = \mathfrak{Re} \int_{B(\tilde{d}_c \vec{e_1}, R) \cup B(-\tilde{d}_c \vec{e_1}, R)} \partial_{x_2} Q_c \bar{\varphi} = 0,$$

$$\mathfrak{Re} \int_{B(\tilde{d}_c \vec{e_1}, R) \cup B(-\tilde{d}_c \vec{e_1}, R)} i Q_c \bar{\varphi} = 0,$$

and we show, as in the proof of Lemma 3.4.5, that we can find $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \mathbb{R}$ such that

$$|\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| \leqslant o_{c \to 0}(c^{\beta_0}) \|\varphi\|_{H^{\exp}_{Q_c}},$$

and φ^* satisfies the five orthogononality conditions of Lemma 3.5.4 (we recall that two of them are given by symmetry). Here, since we did not remove the 0-harmonics, the error is only controlled by $\|\varphi\|_{H^{\exp}_{Q_c}}$ instead of $\|\varphi\|_{\mathcal{C}}$. For instance, we have

$$\int_{B(\tilde{d}_c \vec{e_1}, R)} \left| \mathfrak{Re}\left(\left(\partial_{x_2} \tilde{V_1} \bar{\tilde{V}}_1 - \partial_{x_2} Q_c \overline{Q_c} \right) \psi \right) \right| \leqslant o_{c \to 0}(1) \|Q_c \psi\|_{L^2(B(\tilde{d}_c \vec{e_1}, R))} = o_{c \to 0}(1) \|\varphi\|_{H^{\exp}_{Q_c}}.$$

Now, from Lemma 3.5.4, since $\varphi^* \in H^{\exp}_{Q_c}$, we have

$$B^{\exp}_{Q_c}(\varphi^*) \geqslant K \|\varphi^*\|^2_{H^{\exp}_{Q_c}}.$$

We continue, as in the proof of Lemma 3.4.5, with $|\varepsilon_1| + |\varepsilon_2| + |\varepsilon_3| = o_{c \to 0}(1) \|\varphi\|_{H^{\exp}_{Q_c}}$ and Lemma 3.5.3. We show that

$$B^{\exp}_{Q_c}(\varphi) \geqslant K \|\varphi\|^2_{H^{\exp}_{Q_c}}.$$

We conclude the proof of Theorem 1.4.12 by density. □

3.6 Local uniqueness result

This section is devoted to the proof of Theorem 1.4.13. This proof will follow classical schemes for local uniqueness using the coercivity. Here, we will use Propositions 1.4.10 and 1.4.11, with the remark (3.5.4).

3.6.1 Construction of a perturbation

For a given $\vec{c}' \in \mathbb{R}^2$, $0 < |\vec{c}'| \leqslant c_0$ (c_0 defined in Theorem 1.3.1), $X \in \mathbb{R}^2$ and $\gamma \in \mathbb{R}$, we define, thanks to (1.4.1), the travelling wave

$$Q := Q_{\vec{c}'}(. - X) e^{i\gamma}. \tag{3.6.1}$$

We define a smooth cutoff function η, with value 0 in $B\left(\pm \tilde{d}_c \vec{e_1}, R+1\right)$ ($R > 10$ is defined in Theorem 1.4.4), and 1 outside of $B\left(\tilde{d}_c \vec{e_1}, R+2\right) \cup B\left(-\tilde{d}_c \vec{e_1}, R+2\right)$. The first step is to define a function ψ such that

$$(1-\eta) Q\psi + \eta Q(e^{\psi} - 1) = Z - Q, \tag{3.6.2}$$

with $Q\psi$ satisfying the orthogonality conditions of Propositions 1.4.10 and 1.4.11. We start by showing that there exists a function ψ solution of (3.6.2). We denote $\delta^{|.|}(c\vec{e_2}, \vec{c}') := |c\vec{e_2}.\frac{\vec{c}'}{|\vec{c}'|} - \vec{c}'|$ and $\delta^{\perp}(c\vec{e_2}, \vec{c}') := |c\vec{e_2}.\frac{\vec{c}'^{\perp}}{|\vec{c}'|} - \vec{c}'|$. At fixed c, these two quantities characterize $\vec{c}'$. We will use them as variables instead of $\vec{c}'$, this decomposition being well adapted to the problem.

Since both Z and $|Q|$ go to 1 at infinity, we have that such a function ψ is bounded at infinity.

The perturbation here is chosen additively close to the zeros of the travelling wave, and multiplicatively at infinity. This seems to be a fit form for the perturbation, and we have already used it in the construction of Q_c.

Lemma 3.6.1. *There exits $c_0 > 0$ such that, for $0 < c < c_0$ and any $\Lambda > \frac{10}{c}$, with Z a function satisfying the hypothesis of Theorem 1.4.13 and Q defined by (3.6.1) with $\frac{c}{2} \leqslant |\vec{c}'| \leqslant 2c$, there exist $K, K(\Lambda) > 0$ such that*

$$\|Z - Q\|_{C^1(B(0,\Lambda))} \leqslant K(\Lambda)\|Z - Q_c\|_{H^{\exp}_{Q_c}} + K\left(|X| + \frac{\delta^{|\cdot|}(c\vec{e}_2, \vec{c}')}{c^2} + \frac{\delta^{\perp}(c\vec{e}_2, \vec{c}')}{c} + |\gamma| \right).$$

We will mainly use this result for $\Lambda = \lambda + 1$, $\lambda > 0$ defined in Theorem 1.4.13.

Proof. We recall that such a function Z is in $C^\infty(\mathbb{R}^2, \mathbb{C})$ by elliptic regularity.

We start with the estimation of $w := Q_c - Z$ in $B(0, \Lambda)$. Since both Z and Q_c solve (TW_c), we have

$$-\Delta w = (1 - |Q_c|^2)Q_c - (1 - |Z|^2)Z + ic\partial_{x_2} w.$$

From Theorem 8.8 of [15], we have that for $x \in \mathbb{R}^2$, $\Omega := B(0, \Lambda)$, $2\Omega = B(0, 2\Lambda)$,

$$\|w\|_{W^{2,2}(\Omega)} \leqslant K(\Lambda)(\|w\|_{H^1(\Omega)} + \|ic\partial_{x_2} w + (1 - |Q_c|^2)Q_c - (1 - |Z|^2)Z\|_{L^2(2\Omega)}).$$

We compute that

$$(1 - |Q_c|^2)Q_c - (1 - |Z|^2)Z = (Q_c - Z)(1 - |Q_c|^2) + Z(|Q_c| - |Z|)(|Q_c| + |Z|).$$

From [13], we have that any travelling wave of finite energy is bounded in $L^\infty(\mathbb{R}^2)$ by a universal constant, i.e.

$$|Q_c| + |Z| \leqslant K, \tag{3.6.3}$$

therefore

$$|1 - |Q_c|^2| + |Z|\,(|Q_c| + |Z|) \leqslant K$$

for a universal constant K. Thus,

$$\|(1 - |Q_c|^2)Q_c - (1 - |Z|^2)Z\|_{L^2(2\Omega)} \leqslant K\|w\|_{L^2(2\Omega)},$$

and we deduce, from Lemma 3.1.5, that

$$\|w\|_{W^{2,2}(\Omega)} \leqslant K(\Lambda)(\|w\|_{H^1(2\Omega)} + \|ic\partial_{x_2} w\|_{L^2(2\Omega)} + \|w\|_{L^2(2\Omega)}) \leqslant K(\Lambda)\|w\|_{H^{\exp}_{Q_c}}.$$

By standard elliptic arguments, we have that for every $k \geqslant 2$,

$$\|w\|_{W^{k,2}(\Omega)} \leqslant K(\Lambda, k)\|w\|_{H^{\exp}_{Q_c}}.$$

By Sobolev embeddings, we estimate

$$\|w\|_{C^1(\Omega)} \leqslant K(\Lambda)\|w\|_{W^{4,2}(\Omega)} \leqslant K(\Lambda)\|w\|_{H^{\exp}_{Q_c}}. \tag{3.6.4}$$

From (3.6.4), we have

$$\|Z - Q\|_{L^\infty(\Omega)} \leqslant \|Q - Q_c\|_{L^\infty(\Omega)} + \|w\|_{L^\infty(\Omega)} \leqslant \|Q - Q_c\|_{L^\infty(\mathbb{R}^2)} + K(\Lambda)\|w\|_{H^{\exp}_{Q_c}}.$$

We estimate

$$\begin{aligned} \|Q - Q_c\|_{L^\infty(\mathbb{R}^2)} &= \|Q_{\vec{c}'}(. - X)e^{i\gamma} - Q_c\|_{L^\infty(\mathbb{R}^2)} \\ &\leqslant \|Q_{\vec{c}'}(. - X)e^{i\gamma} - Q_{\vec{c}'}(. - X)\|_{L^\infty(\mathbb{R}^2)} + \|Q_{\vec{c}'}(. - X) - Q_{\vec{c}'}\|_{L^\infty(\mathbb{R}^2)} \\ &+ \|Q_{\vec{c}'} - Q_{|\vec{c}'|\vec{e}_2}\|_{L^\infty(\mathbb{R}^2)} + \|Q_{|\vec{c}'|\vec{e}_2} - Q_c\|_{L^\infty(\mathbb{R}^2)}. \end{aligned}$$

We check, with Theorem 1.3.1 and Lemma 3.1.6 that $\|\nabla Q\|_{L^\infty(\mathbb{R}^2)} + c^2\|\partial_c Q\|_{L^\infty(\mathbb{R}^2)} + c\|\partial_{c^\perp} Q\|_{L^\infty(\mathbb{R}^2)} + \|iQ\|_{L^\infty(\mathbb{R}^2)} \leqslant K$, and that it also holds for any travelling wave of the form $Q_{\vec{\zeta}}(. - Y)e^{i\beta}$ if $2c \geqslant |\vec{\zeta}| \geqslant c/2, Y \in \mathbb{R}^2$ and $\beta \in \mathbb{R}$.

We check that $\|Q_{\vec{c}'}(. - X)e^{i\gamma} - Q_{\vec{c}'}(. - X)\|_{L^\infty(\mathbb{R}^2)} \leqslant |e^{i\gamma} - 1|\|Q_{\vec{c}'}(. - X)\|_{L^\infty(\mathbb{R}^2)} \leqslant K|\gamma|$, and we estimate (by the mean value theorem)

$$\|Q_{\vec{c}'}(. - X) - Q_{\vec{c}'}\|_{L^\infty(\mathbb{R}^2)} \leqslant K|X|\|\nabla Q_{\vec{c}'}\|_{L^\infty(\mathbb{R}^2)} \leqslant K|X|.$$

Similarly, we have

$$\|Q_{\vec{c}'}-Q_{|\vec{c}'|\vec{e}_2}\|_{L^\infty(\mathbb{R}^2)}\leqslant K\frac{\delta^\perp(c\vec{e}_2,\vec{c}')+\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c}$$

and $\|Q_{|\vec{c}'|\vec{e}_2}-Q_c\|_{L^\infty(\mathbb{R}^2)}\leqslant K\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}$. We deduce that (since $c\leqslant 1$)

$$\|Q-Q_c\|_{L^\infty(\mathbb{R}^2)}\leqslant K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^\perp(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\right), \tag{3.6.5}$$

and thus

$$\|Z-Q\|_{L^\infty(B(0,\Lambda))}\leqslant K(\Lambda)\|Z-Q_c\|_{H^{\exp}_{Q_c}}+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^\perp(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\right). \tag{3.6.6}$$

Finally, from Lemmas 1.2.1, 3.1.1 and 3.1.2, $\partial_{c^\perp}Q_c=-x^\perp.\nabla Q_c$ and equation (3.1.11), we have

$$\|\nabla\partial_{x_2}Q\|_{L^\infty(\mathbb{R}^2)}+c^2\|\nabla\partial_cQ\|_{L^\infty(\mathbb{R}^2)}+c\|\nabla\partial_{c^\perp}Q\|_{L^\infty(\mathbb{R}^2)}+\|i\nabla Q_c\|_{L^\infty(\mathbb{R}^2)}\leqslant K.$$

We deduce that

$$\|\nabla(Q-Q_c)\|_{L^\infty(\mathbb{R}^2)}\leqslant K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^\perp(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\right),$$

and, by (3.6.4),

$$\|\nabla(Z-Q)\|_{L^\infty(B(0,\Lambda))}\leqslant K(\Lambda)\|Z-Q_c\|_{H^{\exp}_{Q_c}}+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^\perp(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\right).$$

□

Lemma 3.6.2. *There exists $\varepsilon_0(c)>0$ small such that, for Z a function satisfying the hyptothesis of Theorem 1.4.13 with*

$$|X|+\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^\perp(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\leqslant\varepsilon_0(c),$$

there exists a function $Q\psi\in C^1(\mathbb{R}^2,\mathbb{C})$ such that (3.6.2) holds. Furthermore, for any $\Lambda>\frac{10}{c}$, there exists $K,K(\Lambda)>0$ such that

$$\|Q\psi\|_{C^1(B(0,\Lambda))}\leqslant K(\Lambda)\|Z-Q_c\|_{H^{\exp}_{Q_c}}+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^\perp(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\right).$$

Proof. First, taking $\varepsilon_0(c)$ small enough (depending on c), we check that $\frac{c}{2}\leqslant|\vec{c}'|\leqslant 2c$.

We recall equation (3.6.2):

$$(1-\eta)Q\psi+\eta Q(e^\psi-1)=Z-Q.$$

We write it in the form

$$\psi+\eta(e^\psi-1-\psi)=\frac{Z-Q}{Q},$$

and in $\{\eta=0\}$, we therefore define

$$\psi=\frac{Z-Q}{Q}. \tag{3.6.7}$$

Now, we define the set $\Omega:=B(0,\lambda+1)\backslash(B(d_c\vec{e}_1,R-1)\cup B(-d_c\vec{e}_1,R-1))$. In this set, we have that

$$\left\|\frac{Z-Q}{Q}\right\|_{C^1(\Omega)}\leqslant K\varepsilon_0(c)+K(\lambda)\|Z-Q_c\|_{H^{\exp}_{Q_c}}$$

by Lemma 3.6.1 and (3.1.12). Therefore, since $e^\psi-1-\psi$ is at least quadratic in $\psi\in C^1(\Omega,\mathbb{C})$, by a fixed point argument (on $H(\psi):=\frac{Z-Q}{Q}-\eta(e^\psi-1-\psi)$, which is a contraction on $\|\psi\|_{L^\infty(\{\eta\neq 0\})}<\mu$ for $\mu>0$ small enough), we deduce that on Ω, given that ε_0 and $\|Z-Q_c\|_{H^{\exp}_{Q_c}}$ are small enough (depending on λ for $\|Z-Q_c\|_{H^{\exp}_{Q_c}}$), there exists a unique function $\psi\in C^1(\Omega,\mathbb{C})$ such that $\psi+\eta(e^\psi-1-\psi)=\frac{Z-Q}{Q}$ in Ω. By unicity, since we have a solution of the same problem on $\{\eta=0\}$ which intersect Ω, we can construct $Q\psi\in C^1(B(0,\lambda+1),\mathbb{C})$ such that $\eta Q\psi+(1-\eta)Q(e^\psi-1)=Z-Q$ in $B(0,\lambda+1)$.

Furthermore, we use here the hypothesis that, outside of $B(0,\lambda)$, $|Z-Q_c|\leqslant\mu_0$. We deduce that (taking $\mu_0<\frac{1}{4}$) there exists $\delta>0$ such that $|Z|>\delta$ outside of $B(0,\lambda)$. In particular, since λ can be taken large, we have that outside of $B(0,\lambda)$, $\eta=1$. The equation on ψ then becomes

$$e^{\psi}=\frac{Z}{Q},$$

and by equation (3.1.12) and $|Z|>\delta$, we deduce that there exists a unique solution to this problem in $C^1(\mathbb{R}^2\backslash B(0,\lambda),\mathbb{C})$ that is equal on $B(0,\lambda+1)\backslash B(0,\lambda)$ to the previously constructed function ψ.

Therefore, there exists $Q\psi\in C^1(\mathbb{R}^2,\mathbb{C})$ such that $(1-\eta)Q\psi+\eta Q(e^{\psi}-1)=Z-Q$ in $\mathbb{R}^2$. Furthermore, we check that (by the fixed point argument), since $\{\eta\neq 1\}\subset B(0,\lambda)$,

$$\begin{aligned}\|\psi\|_{C^1(\{\eta\neq 1\})} &\leqslant K\left\|\frac{Z-Q}{Q}\right\|_{C^1(\{\eta\neq 1\})}\\ &\leqslant K(\lambda)\|Z-Q_c\|_{H^{\exp}_{Q_c}}+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^{\perp}(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\right).\end{aligned}$$

From equation (3.1.12) and Lemma 3.6.1, we have

$$\begin{aligned}\|Q\psi\|_{C^1(B(0,\Lambda))} &\leqslant \|Z-Q\|_{C^1(B(0,\Lambda))}+K\|\psi\|_{C^1(\{\eta\neq 1\})}+K(\Lambda)\|Z-Q_c\|_{H^{\exp}_{Q_c}}\\ &\leqslant K(\Lambda)\|Z-Q_c\|_{H^{\exp}_{Q_c}}+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^{\perp}(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\right).\end{aligned}$$

This concludes the proof of the lemma. □

Lemma 3.6.3. *The functions Q and ψ, defined respectively in (3.6.1) and Lemma 3.6.2, satisfy*

$$\varphi:=Q\psi\in H^{\exp}_{Q}.$$

Furthermore, $\varphi\in C^2(\mathbb{R}^2,\mathbb{C})$ and there exists $K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)>0$ such that, in $\{\eta=1\}$ (i.e. far from the vortices),

$$|\nabla\psi|+|\mathfrak{Re}(\psi)|+|\Delta\psi|\leqslant\frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{(1+r)^2},$$

$$|\nabla\mathfrak{Re}(\psi)|\leqslant\frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{(1+r)^3}$$

and

$$|\mathfrak{Im}(\psi+i\gamma)|\leqslant\frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{(1+r)}.$$

Proof. From Lemma 3.6.2, for any $\Lambda>\frac{10}{c}$,

$$\|Q\psi\|_{C^1(B(0,\Lambda))}\leqslant K(\Lambda)\|Z-Q_c\|_{H^{\exp}_{Q_c}}+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^{\perp}(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\right),\tag{3.6.8}$$

therefore, we only have to check the integrability at infinity of $Q\psi$ to show that $\varphi=Q\psi\in H^{\exp}_{Q}$. In $\{\eta=1\}$, we have

$$e^{\psi}=\frac{Z}{Q}.$$

We have shown in the proof of Lemma 3.6.2 that $K>\left|\frac{Z}{Q}\right|>\delta/2$ outside of $B(0,\lambda)$ for some $\delta>0$, and together with (3.6.8), we check that

$$\|\psi\|_{C^0(\{\eta=1\})}\leqslant K(\lambda,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0).\tag{3.6.9}$$

This implies that

$$\int_{\{\eta=1\}}\frac{|Q\psi|^2}{\tilde{r}^2\ln(\tilde{r})^2}<+\infty.$$

Similarly, we check that, in $\{\eta=1\}$, since $e^{\psi}=\frac{Z}{Q}$,

$$\nabla\psi=\frac{e^{-\psi}}{Q}\nabla(Z-Q)-\frac{\nabla Q}{Q}(1-e^{-\psi}),$$

therefore

$$|\nabla\psi|\leqslant K(\lambda,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0)(|\nabla(Z-Q)|+|\nabla Q|). \tag{3.6.10}$$

From Theorem 3.1.4, we have

$$|\nabla Z|+|\nabla Q|\leqslant\frac{K(c,Z)}{(1+r)^2},$$

therefore,

$$\int_{\{\eta=1\}}|\nabla Q|^2|\psi|^2<+\infty$$

and

$$\int_{\{\eta=1\}}|\nabla(Z-Q)|^2\ \leqslant\ \int_{\{\eta=1\}}\frac{K(c,Z)}{(1+r)^4}<+\infty.$$

We deduce that $\int_{\{\eta=1\}}|\nabla\psi|^2<+\infty$, and, furthermore, equation (3.6.10) shows that

$$|\nabla\psi|\leqslant\frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{(1+r)^2}$$

in $\{\eta=1\}$.

Now, still in $\{\eta=1\}$, we have

$$Qe^{\psi}=Z,$$

we deduce that $Qe^{-i\gamma}(e^{\psi+i\gamma}-1)=Z-Qe^{-i\gamma}$. Now, we recall that $\|\psi\|_{C^0(\{\eta=1\})}\leqslant K(\lambda,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0)$, thus $|\mathfrak{Re}(e^{\psi+i\gamma}-1-(\psi+i\gamma))|\leqslant K(\lambda,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0)|\mathfrak{Re}(e^{\psi+i\gamma}-1)|$. We deduce from this, with (3.6.8) that, in $\{\eta=1\}$, with $\frac{1}{4}\|\psi+i\gamma\|_{L^\infty(\mathbb{R}^2)}\leqslant|\mathfrak{Re}(e^{\psi+i\gamma}-1)|\leqslant K\|\psi+i\gamma\|_{L^\infty(\mathbb{R}^2)}$,

$$\begin{aligned}
|\mathfrak{Re}(\psi)| &= |\mathfrak{Re}(\psi+i\gamma)|\\
&\leqslant |\mathfrak{Re}(e^{\psi+i\gamma}-1)|+|\mathfrak{Re}(e^{\psi+i\gamma}-1-(\psi+i\gamma))|\\
&\leqslant K(\lambda,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0)|\mathfrak{Re}(e^{\psi+i\gamma}-1)|\\
&\leqslant K(\lambda,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0)\left|\mathfrak{Re}\left(\frac{(Z-Qe^{-i\gamma})\overline{Qe^{i\gamma}}}{|Q|^2}\right)\right|\\
&\leqslant K(\lambda,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0)(|\mathfrak{Re}(Z-Qe^{-i\gamma})|+|\mathfrak{Im}(Z-Qe^{-i\gamma})\mathfrak{Im}(Qe^{i\gamma}-1)|).
\end{aligned}$$

From Theorem 3.1.4,

$$|\mathfrak{Re}(Z-Qe^{-i\gamma})|\leqslant|\mathfrak{Re}(Z-1)|+|\mathfrak{Re}(1-Qe^{-i\gamma})|\leqslant\frac{K(c,Z)}{(1+r)^2}$$

and

$$|\mathfrak{Im}(Z-Qe^{-i\gamma})\mathfrak{Im}(Qe^{i\gamma}-1)|\leqslant\frac{K(c,Z)}{(1+r)^2}.$$

We conclude that, in $\{\eta=1\}$, we have $|\mathfrak{Re}(\psi)|\leqslant\frac{K\left(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z\right)}{(1+r)^2}$ hence

$$\int_{\{\eta=1\}}\mathfrak{Re}^2(\psi)<+\infty.$$

This conclude the proof of $\varphi=Q\psi\in H^{\exp}_{Q_c}$. We are left with the proof of the following estimates, $|\Delta\psi|\leqslant\frac{K\left(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z\right)}{(1+r)^2}$, $|\mathfrak{Im}(\psi+i\gamma)|\leqslant\frac{K\left(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z\right)}{(1+r)}$ and $|\mathfrak{Re}(\nabla\psi)|\leqslant\frac{K\left(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z\right)}{(1+r)^3}$ in $\{\eta=1\}$.

We recall that, in $\{\eta=1\}$, $\nabla\psi=\frac{e^{-\psi}}{Q}\nabla(Z-Q)-\frac{\nabla Q}{Q}(1-e^{-\psi})$, from which we compute, by differentiating a second time,

$$\begin{aligned}\Delta\psi &= -\frac{\nabla\psi.\nabla(Z-Q)}{Q}e^{-\psi}-\frac{\nabla Q}{Q}e^{-\psi}.\nabla(Z-Q)+\frac{e^{-\psi}}{Q}\Delta(Z-Q)\\ &- \frac{\Delta Q}{Q}(1-e^{-\psi})+\frac{\nabla Q.\nabla Q}{Q^2}(1-e^{-\psi})-\frac{\nabla Q}{Q}.\nabla\psi e^{-\psi}.\end{aligned}$$

Using Theorem 3.1.4, $\Delta Q=-i\vec{c}'.\nabla Q-(1-|Q|^2)Q$, $\Delta Z=-ic\partial_{x_2}Z-(1-|Z|^2)Z$ and previous estimates on ψ, we check that, in $\{\eta=1\}$,

$$|\Delta\psi|\leqslant\frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{(1+r)^2}.$$

We have $Qe^{-i\gamma}(e^{\psi+i\gamma}-1)=Z-Qe^{-i\gamma}$ in $\{\eta=1\}$, therefore

$$e^{\psi+i\gamma}-1=\frac{Z}{Qe^{-i\gamma}}-1$$

We check, since $\|\psi\|_{C^0(\{\eta=1\})}\leqslant K(\lambda,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0)$, that we have by Theorem 3.1.4

$$\begin{aligned}|\mathfrak{Im}(\psi+i\gamma)| &\leqslant K(\lambda,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0)|\mathfrak{Im}(e^{\psi+i\gamma}-1)|\\ &\leqslant K(\lambda,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0)\left|\frac{Z}{Qe^{-i\gamma}}-1\right|\\ &\leqslant \frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{(1+r)}.\end{aligned}$$

Finally, since $\nabla\psi=\frac{e^{-\psi}}{Q}\nabla(Z-Q)-\frac{\nabla Q}{Q}(1-e^{-\psi})=\frac{\nabla Z}{Q}e^{-\psi}-\frac{\nabla Q}{Q}$, we check with Theorem 3.1.4 that, in $\{\eta=1\}$,

$$\begin{aligned}|\nabla\mathfrak{Re}(\psi)| &\leqslant \left|\mathfrak{Re}\left(\frac{\nabla Z}{Q}e^{-\psi}\right)\right|+\left|\mathfrak{Re}\left(\frac{\nabla Q}{Q}\right)\right|\\ &\leqslant \left|\mathfrak{Re}\left(\nabla Z\bar{Z}\frac{e^{-\psi}}{Q\bar{Z}}\right)\right|+\frac{|\mathfrak{Re}(\nabla Q\bar{Q})|}{|Q|^2}\\ &\leqslant \left|\mathfrak{Im}(\nabla Z\bar{Z})\mathfrak{Im}\left(\frac{e^{-\psi}}{Q\bar{Z}}\right)\right|+|\mathfrak{Re}(\nabla Z\bar{Z})|\left|\mathfrak{Re}\left(\frac{e^{-\psi}}{Q\bar{Z}}\right)\right|+\frac{|\nabla(|Q|^2)|}{2|Q|^2}\\ &\leqslant \frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{(1+r)^2}\left|\mathfrak{Im}\left(\frac{e^{-\psi}}{Q\bar{Z}}\right)\right|+\frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{(1+r)^3}.\end{aligned}$$

We compute in $\{\eta=1\}$, still using Theorem 3.1.4,

$$\begin{aligned}\left|\mathfrak{Im}\left(\frac{e^{-\psi}}{Q\bar{Z}}\right)\right| &= \frac{1}{|QZ|^2}|\mathfrak{Im}(e^{-\psi-i\gamma}\bar{Q}Ze^{i\gamma})|\\ &\leqslant K(|\mathfrak{Im}(e^{-\psi-i\gamma}-1)\mathfrak{Re}(\bar{Q}Ze^{i\gamma})|+|\mathfrak{Re}(e^{-\psi-i\gamma})\mathfrak{Im}(\bar{Q}Ze^{i\gamma})|)\\ &\leqslant \frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{1+r}+K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)|\mathfrak{Im}(\bar{Q}Ze^{i\gamma})|\\ &\leqslant \frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{(1+r)}+K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)(|Qe^{-i\gamma}-1|+|Z-1|)\\ &\leqslant \frac{K(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z)}{(1+r)}.\end{aligned}$$

This concludes the proof of this lemma. □

We remark that here, since $\psi\nrightarrow 0$ at infinity (if $\gamma\neq 0$), we do not have $Q\psi\in H_Q$. This is one of the main reasons to introduce the space $H^{\exp}_Q$.

Lemma 3.6.4. *The functions Q and ψ, defined respectively in (3.6.1) and Lemma 3.6.2, satisfy, with $\varphi=Q\psi$,*

$$\langle L^{\exp}_Q(\varphi),(\varphi+i\gamma Q)\rangle=B^{\exp}_Q(\varphi),$$

where $L_Q^{\exp}(\varphi) = (1-\eta)L_Q(\varphi) + \eta Q L'_Q(\psi)$, *with*

$$L'_Q(\psi) = -\Delta\psi - 2\frac{\nabla Q}{Q}.\nabla\psi + i\,\vec{c}.\nabla\psi + 2\mathfrak{Re}(\psi)|Q|^2.$$

Furthermore,

$$L_Q(\varphi) = Q L'_Q(\psi).$$

The equality $\langle L_Q^{\exp}(\varphi), (\varphi + i\gamma Q)\rangle = B_Q^{\exp}(\varphi)$ is not obvious for functions $\varphi \in C^2(\mathbb{R}^2,\mathbb{C}) \cap H_{Q_c}^{\exp}$ (because of some integration by parts to justify) and we need to check that, for the particular function φ we have contructed, this result holds. We will use mainly the decay estimates of Lemma 3.6.3.

Morally, we are showing that, since $L_Q(i\gamma Q) = 0$, that we can do the following computation: $\langle L_Q(\varphi), \varphi + i\gamma Q\rangle = \langle \varphi, L_Q(\varphi + i\gamma Q)\rangle = \langle \varphi, L_Q(\varphi)\rangle = B_Q(\varphi)$. The goal of this lemma is simply to check that, with the estimates of Lemma 3.6.3, the integrands are integrable and the integration by parts can be done to have $\langle L_Q^{\exp}(\varphi), (\varphi + i\gamma Q)\rangle = B_Q^{\exp}(\varphi)$.

Proof. First, let us show that $L_{Q_c}(\Phi) = Q_c L'_{Q_c}(\Psi)$ if $\Phi = Q_c\Psi \in C^2(\mathbb{R}^2,\mathbb{C})$. With equation (3.6.1), it implies that $L_Q(\varphi) = Q L'_Q(\psi)$. We recall that

$$L_{Q_c}(\Phi) = -\Delta\Phi - i\,c\partial_{x_2}\Phi - (1-|Q_c|^2)\Phi + 2\mathfrak{Re}(\overline{Q_c}\Phi)Q_c,$$

and we develop with $\Phi = Q_c\Psi$,

$$L_{Q_c}(\Phi) = \mathrm{TW}_c(Q_c)\Psi - Q_c\Delta\Psi - 2\nabla Q_c.\nabla\Psi - i\,cQ_c\partial_{x_2}\Psi + 2\mathfrak{Re}(\Psi)|Q_c|^2Q_c,$$

thus, since $(\mathrm{TW}_c)(Q_c) = 0$, we have $L_{Q_c}(\Phi) = Q_c L'_{Q_c}(\Psi)$.

Now, for $\varphi = Q\psi$, we have

$$\begin{aligned}
&\langle (1-\eta)L_Q(\varphi) + \eta Q L'_Q(\psi), (\varphi + i\gamma Q)\rangle \\
= &\int_{\mathbb{R}^2} \mathfrak{Re}((1-\eta)L_Q(\varphi)\overline{(\varphi + i\gamma Q)}) \\
+ &\int_{\mathbb{R}^2} \eta|Q|^2\mathfrak{Re}\left(\left(-\Delta\psi - 2\frac{\nabla Q}{Q}.\nabla\psi + i\,\vec{c}.\nabla\psi\right)(\overline{\psi + i\gamma})\right) + \eta|Q|^4\mathfrak{Re}^2(\psi).
\end{aligned}$$

With Lemma 3.6.3, we check that all the terms are integrable independently (in particular since $\varphi + i\gamma Q = Q(\psi + i\gamma)$ and $\|(\psi + i\gamma)(1+r)\|_{L^\infty(\{\eta=1\})} < +\infty$ by Lemma 3.6.3). We recall that $L_Q(\varphi) = -\Delta\varphi + i\,\vec{c}.\nabla\varphi - (1-|Q|^2)\varphi + 2\mathfrak{Re}(\bar{Q}\varphi)Q$, and thus

$$\begin{aligned}
\int_{\mathbb{R}^2} \mathfrak{Re}((1-\eta)L_Q(\varphi)\overline{(\varphi + i\gamma Q)}) &= \int_{\mathbb{R}^2} (1-\eta)(\mathfrak{Re}(i\,\vec{c}.\nabla\varphi\bar{\varphi}) - (1-|Q|^2)|\varphi|^2 + 2\mathfrak{Re}^2(\bar{Q}\varphi)) \\
&+ \int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}(-\Delta\varphi\bar{\varphi}) + \gamma\int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}(L_Q(\varphi)\overline{iQ}).
\end{aligned}$$

We recall that $1-\eta$ is compactly supported and that $\varphi \in C^2(\mathbb{R}^2,\mathbb{C})$. By integration by parts,

$$\int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}(-\Delta\varphi\bar{\varphi}) = \int_{\mathbb{R}^2} (1-\eta)|\nabla\varphi|^2 - \int_{\mathbb{R}^2} \nabla\eta.\mathfrak{Re}(\nabla\varphi\bar{\varphi}),$$

and we decompose

$$\begin{aligned}
\int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}(\eta L_Q(\varphi)\overline{iQ}) &= \int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}(-\Delta\varphi\overline{iQ} + \vec{c}.\nabla\varphi\,\bar{Q}) \\
&- \int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}((1-|Q|^2)\varphi\overline{iQ}).
\end{aligned}$$

By integration by parts, that we have

$$\int_{\mathbb{R}^2} (1-\eta)\mathfrak{Re}(\vec{c}.\nabla\varphi\,\bar{Q}) = -\vec{c}.\int_{\mathbb{R}^2} -\nabla\eta\mathfrak{Re}(\varphi\bar{Q}) + (1-\eta)\mathfrak{Re}(\varphi\nabla\bar{Q})$$

and

$$\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(-\Delta\varphi\overline{iQ}) \;=\; \int_{\mathbb{R}^2}-\nabla\eta.(\mathfrak{Re}(i\varphi\nabla\bar{Q})-\mathfrak{Re}(i\nabla\varphi\bar{Q}))+\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(i\varphi\Delta\bar{Q}).$$

Combining this computations, we infer

$$\begin{aligned}
&\int_{\mathbb{R}^2}\mathfrak{Re}((1-\eta)L_Q(\varphi)\overline{(\varphi+i\gamma Q)})\\
=\;&\int_{\mathbb{R}^2}(1-\eta)(|\nabla\varphi|^2+\mathfrak{Re}(i\,\vec{c}.\nabla\varphi\bar{\varphi})-(1-|Q|^2)|\varphi|^2+2\mathfrak{Re}^2(\bar{Q}\varphi))\\
-\;&\int_{\mathbb{R}^2}\nabla\eta.\mathfrak{Re}(\nabla\varphi\bar{\varphi})\gamma\vec{c}.\int_{\mathbb{R}^2}\nabla\eta\mathfrak{Re}(\varphi\bar{Q})\\
-\;&\gamma\Big(\int_{\mathbb{R}^2}\nabla\eta.(\mathfrak{Re}(i\varphi\nabla\bar{Q})-\mathfrak{Re}(i\nabla\varphi\bar{Q}))\Big)\\
+\;&\gamma\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Re}(\varphi(-\vec{c}.\nabla\bar{Q}+i(1-|Q|^2)\bar{Q}+i\Delta\bar{Q})).
\end{aligned}$$

Since $-\Delta Q+i\,\vec{c}.\nabla Q-(1-|Q|^2)Q=0$, we have $-\vec{c}.\nabla\bar{Q}+i(1-|Q|^2)\bar{Q}+i\Delta\bar{Q}=0$, therefore

$$\begin{aligned}
&\int_{\mathbb{R}^2}\mathfrak{Re}((1-\eta)L_Q(\varphi)\overline{(\varphi+i\gamma Q)})\\
=\;&\int_{\mathbb{R}^2}(1-\eta)(|\nabla\varphi|^2+\mathfrak{Re}(i\vec{c}.\nabla\varphi\bar{\varphi})-(1-|Q|^2)|\varphi|^2+2\mathfrak{Re}^2(\bar{Q}\varphi))\\
-\;&\int_{\mathbb{R}^2}\nabla\eta.\mathfrak{Re}(\nabla\varphi\bar{\varphi})\\
-\;&\gamma\Big(-\vec{c}.\int_{\mathbb{R}^2}\nabla\eta\mathfrak{Re}(\varphi\bar{Q})+\int_{\mathbb{R}^2}\nabla\eta.(\mathfrak{Re}(i\varphi\nabla\bar{Q})-\mathfrak{Re}(i\nabla\varphi\bar{Q}))\Big).
\end{aligned}$$

Until now, all the integrals were on bounded domain (since $1-\eta$ is compactly supported).

Now, by integration by parts, (that can be done thanks to Lemma 3.6.3 and Theorem 3.1.4)

$$\begin{aligned}
\int_{\mathbb{R}^2}\eta|Q|^2\mathfrak{Re}(-\Delta\psi(\overline{\psi+i\gamma})) \;=\;&\int_{\mathbb{R}^2}\nabla\eta.|Q|^2\mathfrak{Re}(\nabla\psi(\overline{\psi+i\gamma}))\\
+\;&\int_{\mathbb{R}^2}\eta\nabla(|Q|^2).\mathfrak{Re}(\nabla\psi(\overline{\psi+i\gamma}))\\
+\;&\int_{\mathbb{R}^2}\eta|Q|^2|\nabla\psi|^2.
\end{aligned}$$

Now, we decompose (and we check that each term is well defined at each step with Lemma 3.6.3 and Theorem 3.1.4)

$$\begin{aligned}
&\int_{\mathbb{R}^2}\eta|Q|^2\mathfrak{Re}\Big(-2\frac{\nabla Q}{Q}.\nabla\psi\Big)(\overline{\psi+i\gamma})\Big)\\
=\;&-2\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla Q\bar{Q}.\nabla\psi\bar{\psi})-2\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla Q\bar{Q}.\nabla\psi\overline{(i\gamma)}),
\end{aligned}$$

with

$$\begin{aligned}
-2\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla Q\bar{Q}.\nabla\psi\bar{\psi}) \;=\;&-2\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla Q\bar{Q}).\mathfrak{Re}(\nabla\psi\bar{\psi})\\
+\;&2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q\bar{Q}).\mathfrak{Im}(\nabla\psi\bar{\psi}),
\end{aligned}$$

and since $\nabla(|Q|^2)=2\mathfrak{Re}(\nabla Q\bar{Q})$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2}\eta|Q|^2\mathfrak{Re}\Big(\Big(-\Delta\psi-2\frac{\nabla Q}{Q}.\nabla\psi\Big)(\overline{\psi+i\gamma})\Big)\\
=\;&\int_{\mathbb{R}^2}\eta|Q|^2|\nabla\psi|^2+2\int_{\mathbb{R}^2}(1-\eta)\mathfrak{Im}(\nabla Q\bar{Q}).\mathfrak{Im}(\nabla\psi\bar{\psi})\\
+\;&\int_{\mathbb{R}^2}\nabla\eta.|Q|^2\mathfrak{Re}(\nabla\psi(\overline{\psi+i\gamma}))+2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q\bar{Q}).\mathfrak{Im}(\nabla\psi\overline{(i\gamma)})).
\end{aligned}$$

We continue. We have

$$\begin{aligned}2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q\bar{Q}).\mathfrak{Im}(\nabla\psi\bar{\psi}) &= 2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q\bar{Q}).\mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi)\\ &- 2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q\bar{Q}).\mathfrak{Re}(\nabla\psi)\mathfrak{Im}(\psi),\end{aligned}$$

and by integration by parts (still using Lemma 3.6.3 and Theorem 3.1.4),

$$\begin{aligned}&-2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q\bar{Q}).\mathfrak{Re}(\nabla\psi)\mathfrak{Im}(\psi)\\ =\ & 2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q\bar{Q}).\mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi)\\ +\ & 2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\Delta Q\bar{Q})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)\\ +\ & 2\int_{\mathbb{R}^2}\nabla\eta.\mathfrak{Im}(\nabla Q\bar{Q})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi).\end{aligned}$$

We have $\mathfrak{Im}(\Delta Q\bar{Q}) = \mathfrak{Im}(i\,\vec{c}.\nabla Q - (1-|Q|^2Q)\bar{Q}) = \mathfrak{Re}(\vec{c}.\nabla Q\bar{Q})$, therefore

$$\begin{aligned}&\int_{\mathbb{R}^2}\eta|Q|^2\mathfrak{Re}\Big(\Big(-\Delta\psi - 2\frac{\nabla Q}{Q}.\nabla\psi\Big)(\overline{\psi+i\gamma})\Big)\\ =\ & \int_{\mathbb{R}^2}\eta|Q|^2|\nabla\psi|^2 + 4\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q\bar{Q}).\mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi)\\ +\ & 2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q\bar{Q}).\mathfrak{Im}(\nabla\psi\overline{(i\gamma)}))\\ +\ & 2\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\vec{c}.\nabla Q\bar{Q})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)\\ +\ & \int_{\mathbb{R}^2}\nabla\eta(|Q|^2\mathfrak{Re}(\nabla\psi(\overline{\psi+i\gamma})) + 2\mathfrak{Im}(\nabla Q\bar{Q})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi)).\end{aligned}$$

Now, we compute

$$\begin{aligned}\vec{c}.\int_{\mathbb{R}^2}\eta|Q|^2\mathfrak{Re}(i\nabla\psi(\overline{\psi+i\gamma})) &= \vec{c}.\int_{\mathbb{R}^2}\eta|Q|^2\mathfrak{Re}(\nabla\psi)\mathfrak{Im}(\psi+i\gamma)\\ &- \vec{c}.\int_{\mathbb{R}^2}\eta|Q|^2\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi),\end{aligned}$$

and by integration by parts (still using Lemma 3.6.3 and Theorem 3.1.4),

$$\begin{aligned}\vec{c}.\int_{\mathbb{R}^2}\eta|Q|^2\mathfrak{Re}(\nabla\psi)\mathfrak{Im}(\psi+i\gamma) &= -\vec{c}.\int_{\mathbb{R}^2}\nabla\eta|Q|^2\mathfrak{Re}(\psi)\mathfrak{Im}(\psi+i\gamma)\\ &- \vec{c}.\int_{\mathbb{R}^2}\eta\nabla(|Q|^2)\mathfrak{Re}(\psi)\mathfrak{Im}(\psi+i\gamma)\\ &- \vec{c}.\int_{\mathbb{R}^2}\eta|Q|^2\mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi).\end{aligned}$$

Since $\nabla(|Q|^2) = 2\mathfrak{Re}(\nabla Q\bar{Q})$, we infer

$$\begin{aligned}&\int_{\mathbb{R}^2}\eta|Q|^2\mathfrak{Re}\Big(\Big(-\Delta\psi - 2\frac{\nabla Q}{Q}.\nabla\psi - i\vec{c}.\nabla\psi\Big)(\overline{\psi+i\gamma})\Big)\\ =\ & \int_{\mathbb{R}^2}\eta(|Q|^2|\nabla\psi|^2 + 4\mathfrak{Im}(\nabla Q\bar{Q}).\mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi) - 2\vec{c}.\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(\psi))\\ +\ & 2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q\bar{Q}).\mathfrak{Im}(\nabla\psi\overline{(i\gamma)}))\\ -\ & 2\gamma\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\vec{c}.\nabla Q\bar{Q})\mathfrak{Re}(\psi)\\ +\ & \int_{\mathbb{R}^2}\nabla\eta.(|Q|^2\mathfrak{Re}(\nabla\psi(\overline{\psi+i\gamma})) + 2\mathfrak{Im}(\nabla Q\bar{Q})\mathfrak{Re}(\psi)\mathfrak{Im}(\psi))\\ +\ & \vec{c}.\int_{\mathbb{R}^2}\nabla\eta|Q|^2\mathfrak{Re}(\psi)\mathfrak{Im}(\psi+i\gamma).\end{aligned}$$

Combining these computation yields

$$\begin{aligned}\int_{\mathbb{R}^2}\mathfrak{Re}(L_Q^{\exp}(\varphi)\overline{(\varphi+i\gamma Q)}) &= B_Q^{\exp}(\varphi)\\ &- \gamma\Big(-\vec{c}.\int_{\mathbb{R}^2}\nabla\eta\mathfrak{Re}(\varphi\bar{Q})+\int_{\mathbb{R}^2}\nabla\eta.(\mathfrak{Re}(i\varphi\nabla\bar{Q})-\mathfrak{Re}(i\nabla\varphi\bar{Q}))\Big)\\ &+ 2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q\bar{Q}).\mathfrak{Im}(\nabla\psi\overline{(i\gamma)}))\\ &- 2\gamma\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\vec{c}.\nabla Q\bar{Q})\mathfrak{Re}(\psi)\\ &+ \int_{\mathbb{R}^2}\nabla\eta.|Q|^2\mathfrak{Re}(\nabla\psi\overline{(i\gamma)})\\ &- \vec{c}.\gamma\int_{\mathbb{R}^2}\nabla\eta|Q|^2\mathfrak{Re}(\psi).\end{aligned}$$

We compute, by integration by parts (still using Lemma 3.6.3 and Theorem 3.1.4), that

$$\begin{aligned}2\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q\bar{Q}).\mathfrak{Im}(\nabla\psi\overline{(i\gamma)})) &= -2\gamma\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla Q\bar{Q}).\mathfrak{Re}(\nabla\psi)\\ &= 2\gamma\int_{\mathbb{R}^2}\nabla\eta.\mathfrak{Im}(\nabla Q\bar{Q})\mathfrak{Re}(\psi)\\ &+ 2\gamma\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\Delta Q\bar{Q})\mathfrak{Re}(\psi),\end{aligned}$$

and since $\mathfrak{Im}(\Delta Q\bar{Q})=\mathfrak{Re}(\vec{c}.\nabla Q\bar{Q})$ and $\mathfrak{Re}(\nabla\psi\overline{(i\gamma)})=\gamma\mathfrak{Im}(\nabla\psi)$, we have

$$\begin{aligned}\int_{\mathbb{R}^2}\mathfrak{Re}(L_Q^{\exp}(\varphi)\overline{(\varphi+i\gamma Q)}) &= B_Q^{\exp}(\varphi)\\ &- \gamma\Big(-\vec{c}.\int_{\mathbb{R}^2}\nabla\eta\mathfrak{Re}(\varphi\bar{Q})+\int_{\mathbb{R}^2}\nabla\eta.(\mathfrak{Re}(i\varphi\nabla\bar{Q})-\mathfrak{Re}(i\nabla\varphi\bar{Q}))\Big)\\ &+ 2\gamma\int_{\mathbb{R}^2}\nabla\eta.\mathfrak{Im}(\nabla Q\bar{Q})\mathfrak{Re}(\psi)\\ &+ \gamma\int_{\mathbb{R}^2}\nabla\eta.|Q|^2\mathfrak{Im}(\nabla\psi)\\ &- \vec{c}.\gamma\int_{\mathbb{R}^2}\nabla\eta|Q|^2\mathfrak{Re}(\psi).\end{aligned}$$

we check that $\mathfrak{Re}(\varphi\bar{Q})=|Q|^2\mathfrak{Re}(\psi)$, $\mathfrak{Re}(i\varphi\nabla\bar{Q})=-\mathfrak{Re}(\nabla Q\bar{Q})\mathfrak{Im}(\psi)+\mathfrak{Im}(\nabla Q\bar{Q})\mathfrak{Re}(\psi)$ and that

$$\begin{aligned}-\mathfrak{Re}(i\nabla\varphi\bar{Q}) &= -\mathfrak{Re}(i\nabla Q_c\bar{Q}\psi)-\mathfrak{Re}(i\nabla\psi)|Q|^2\\ &= \mathfrak{Im}(\nabla Q\bar{Q})\mathfrak{Re}(\psi)+\mathfrak{Re}(\nabla Q\bar{Q})\mathfrak{Im}(\psi)+\mathfrak{Im}(\nabla\psi)|Q|^2,\end{aligned}$$

thus concluding the proof of

$$\int_{\mathbb{R}^2}\mathfrak{Re}(L_Q^{\exp}(\varphi)\overline{(\varphi+i\gamma Q)})=B_Q^{\exp}(\varphi).$$ □

3.6.2 Properties of the perturbation

We look for the equation satisfied by $\varphi=Q\psi$ in the next lemma.

Lemma 3.6.5. *The functions Q and ψ, defined respectively in (3.6.1) and Lemma 3.6.2, with $\varphi=Q\psi$, satisfy the equation*

$$L_Q(Q\psi)-i(c\vec{e_2}-\vec{c}').H(\psi)+\mathrm{NL}_{\mathrm{loc}}(\psi)+F(\psi)=0,$$

with L_Q the linearized operator around Q: $L_Q(\varphi):=-\Delta\varphi-i\,\vec{c}.\nabla\varphi-(1-|Q|^2)\varphi+2\mathfrak{Re}(\bar{Q}\varphi)Q$,

$$S(\psi):=e^{2\mathfrak{Re}(\psi)}-1-2\mathfrak{Re}(\psi),$$

$$F(\psi):=Q\eta(-\nabla\psi.\nabla\psi+|Q|^2S(\psi)),$$

$$H(\psi):=\nabla Q+\frac{\nabla(Q\psi)(1-\eta)+Q\nabla\psi\eta e^{\psi}}{(1-\eta)+\eta e^{\psi}}$$

and $\mathrm{NL}_{\mathrm{loc}}(\psi)$ *is a sum of terms at least quadratic in* ψ*, localized in the area where* $\eta \neq 1$*. Furthermore,*

$$|\langle \mathrm{NL}_{\mathrm{loc}}(\psi), Q(\psi + i\gamma)\rangle| \leqslant K(\|Q\psi\|_{C^1(\{\eta\neq 1\})} + |\gamma|)\|Q\psi\|^2_{H^1(\{\eta\neq 1\})}.$$

Remark that here, the equation satisfied by φ has a "source" term, $i(c\vec{e_2} - \vec{c}').H(\psi)$, coming from the fact that Z and Q_c might not have the same speed at this point. We will estimate it later on.

Proof. The function Z solves (TW_c), hence,

$$i(c\vec{e_2} - \vec{c}').\nabla Z = -i\,\vec{c}'.\nabla Z - \Delta Z - (1 - |Z|^2)Z.$$

From (3.6.2), we have

$$Z = Q + (1-\eta)Q\psi + \eta Q(e^{\psi} - 1).$$

We define

$$\zeta := 1 + \psi - e^{\psi}.$$

We replace $Z = Q + (1-\eta)Q\psi + \eta Q(e^{\psi}-1)$ in $-i\,\vec{c}'.\nabla Z - \Delta Z - (1-|Z|^2)Z$ exactly as in the proof of Lemma 2.1.7, by simply changing $V, \Psi, c\vec{e}_2, \eta$ respectively to $Q, \psi, \vec{c}', 1-\eta$. In particular, $E - ic\partial_{x_2}V$ becomes 0 (since $\mathrm{TW}_{\vec{c}'}(Q) = 0$). This computation yields

$$i(c\vec{e_2} - \vec{c}').\nabla Z = ((1-\eta) + \eta e^{\psi})(L_Q(Q\psi) + \widetilde{\mathrm{NL}}_{\mathrm{loc}}(\psi) + F(\psi)).$$

Furthermore, we have that $((1-\eta)+\eta e^{\psi}) \neq 0$ by Lemma 3.6.2 and equation (3.6.9) (for the same reason as in the proof of Lemma 2.1.7), and we compute (as in Lemma 2.1.7) that

$$\frac{\eta e^{\psi}}{(1-\eta)+\eta e^{\psi}} = \eta + \eta(1-\eta)\left(\frac{e^{\psi}-1}{(1-\eta)+\eta e^{\psi}}\right). \tag{3.6.11}$$

Furthermore, we have

$$\begin{aligned}\nabla Z &= \nabla Q - Q\nabla\eta\zeta + \nabla Q((1-\eta)\psi + \eta(e^{\psi}-1)) + Q\nabla\psi((1-\eta)+\eta e^{\psi}) \\ &= \nabla Q(1-\eta+\eta e^{\psi}) - Q\nabla\eta\zeta + \nabla(Q\psi)(1-\eta) + Q\nabla\psi\eta e^{\psi},\end{aligned}$$

hence

$$\frac{\nabla Z}{(1-\eta)+\eta e^{\psi}} = \nabla Q - \frac{Q\nabla\eta\zeta}{(1-\eta)+\eta e^{\psi}} + \frac{\nabla(Q\psi)(1-\eta) + Q\nabla\psi\eta e^{\psi}}{(1-\eta)+\eta e^{\psi}},$$

therefore, with $\mathrm{NL}_{\mathrm{loc}}(\psi) = \widetilde{\mathrm{NL}}_{\mathrm{loc}}(\psi) + i(c\vec{e_2} - \vec{c}').\frac{-Q\nabla\eta\zeta}{(1-\eta)+\eta e^{\psi}}$, we have

$$L_Q(Q\psi) - i(c\vec{e_2} - \vec{c}').H(\psi) + \mathrm{NL}_{\mathrm{loc}}(\psi) + F(\psi) = 0.$$

Finally, we check, similarly as in the proof of Lemma 2.1.7, that

$$|\langle \mathrm{NL}_{\mathrm{loc}}(\psi), Q(\psi+i\gamma)\rangle| \leqslant K(\|Q\psi\|_{C^1(\{\eta\neq 1\})} + |\gamma|)\int_{\mathbb{R}^2} |\mathrm{NL}_{\mathrm{loc}}(\psi)|,$$

hence

$$|\langle \mathrm{NL}_{\mathrm{loc}}(\psi), Q(\psi+i\gamma)\rangle| \leqslant K(\|Q\psi\|_{C^1(\{\eta\neq 1\})} + |\gamma|)\|Q\psi\|^2_{H^1(\{\eta\neq 1\})}. \qquad \square$$

Now, we want to choose the right parameters γ, $\vec{c}'$, X so that φ satisfies the orthogonality conditions of Proposition 1.4.10 and 1.4.11 (with remark 3.5.4).

Lemma 3.6.6. *For the functions* Q *and* ψ*, defined respectively in (3.6.1) and Lemma 3.6.2, there exist* $X, \vec{c}' \in \mathbb{R}^2, \gamma \in \mathbb{R}$ *such that*

$$|X| + \frac{\delta^{|.|}(c\vec{e_2}, \vec{c}')}{c^2} + \frac{\delta^{\perp}(c\vec{e_2}, \vec{c}')}{c} + |\gamma| \leqslant o^{\lambda,c}_{\|Z - Q_c\|_{H^{\exp}_{Q_c}} \to 0}(1),$$

and

$$\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c}',1},R)\cup B(\boldsymbol{d}_{\vec{c}',2},R)}\partial_{x_1}Q\overline{Q\psi^{\neq 0}}=\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c}',1},R)\cup B(\boldsymbol{d}_{\vec{c}',2},R)}\partial_{x_2}Q\overline{Q\psi^{\neq 0}}=0,$$

$$\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c}',1},R)\cup B(\boldsymbol{d}_{\vec{c}',2},R)}\partial_{c^\perp}Q\overline{Q\psi^{\neq 0}}=0,$$

$$\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c}',1},R)\cup B(\boldsymbol{d}_{\vec{c}',2},R)}\partial_{\boldsymbol{d}}\boldsymbol{V}\overline{Q\psi^{\neq 0}}=0$$

$$\mathfrak{Re}\int_{B((\boldsymbol{d}_{\vec{c}',1}+\boldsymbol{d}_{\vec{c}',2})/2,R)}i\psi=0,$$

where $\boldsymbol{d}_{\vec{c}',1}$ *and* $\boldsymbol{d}_{\vec{c}',2}$ *are the zeros of* Q, $\boldsymbol{d}_{\vec{c}',1}$ *being the closest one of* $\tilde{d}_c\vec{e_1}$, *and* $\partial_{\boldsymbol{d}}\boldsymbol{V}$ *is the first order of* Q *by Theorem 1.3.1 and (1.4.1).*

Here, the notations for the harmonics are done for Q, and are therefore centered around $\boldsymbol{d}_{\vec{c}',1}$ or $\boldsymbol{d}_{\vec{c}',2}$. This means that $\psi^{\neq 0}(x)=\psi(x)-\psi^{\boldsymbol{0},1}(\boldsymbol{r}_1)$ with $\boldsymbol{r}_1:=|x-\boldsymbol{d}_{\vec{c}',1}|$, $x-\boldsymbol{d}_{\vec{c}',1}=\boldsymbol{r}_1e^{i\boldsymbol{\theta}_1}\in\mathbb{R}^2$ and $\psi^{\boldsymbol{0},1}$ being the 0-harmonic of ψ around $\boldsymbol{d}_{\vec{c}',1}$ in $B(\boldsymbol{d}_{\vec{c}',1},R)$, and $\psi^{\neq 0}(x)=\psi(x)-\psi^{\boldsymbol{0},2}(\boldsymbol{r}_2)$ with $\boldsymbol{r}_2:=|x-\boldsymbol{d}_{\vec{c}',2}|$ in $B(\boldsymbol{d}_{\vec{c}',2},R)$ and $\psi^{\boldsymbol{0},1}$ being the 0-harmonic of ψ around $\boldsymbol{d}_{\vec{c}',2}$. We will denote $\psi^{\boldsymbol{0}}(x)$ the quantity equal to $\psi^{\boldsymbol{0},1}(\boldsymbol{r}_1)$ in the right half-plane and to $\psi^{\boldsymbol{0},2}(\boldsymbol{r}_2)$ in the left half-plane. Remark that $\boldsymbol{d}_{\vec{c}',1}\in\mathbb{R}^2$, whereas $\tilde{d}_c\in\mathbb{R}$. We recall that, taking $\|Z-Q_c\|_{H^{\exp}_{Q_c}}$ small enough, we have $\frac{\delta^{|\cdot|}(c\vec{e_2},\vec{c}')}{c^2}\leqslant 1$, and in particular, for c small enough, it implies that $\frac{c}{2}\leqslant|\vec{c}'|\leqslant 2c$. We recall that $o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)$ is a quantity going to 0 when $\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0$ at fixed λ and c.

Proof. For $X=(X_1,X_2),\vec{c}'\in\mathbb{R}^2$, we define, as previously, the function

$$Q=Q_{\vec{c}'}(.-X)e^{i\gamma}.$$

We define, to simplify the notations,

$$\Omega:=B(\boldsymbol{d}_{\vec{c}',1},R)\cup B(\boldsymbol{d}_{\vec{c}',2},R)$$

and

$$\Omega':=B\left(\frac{(\boldsymbol{d}_{\vec{c}',1}+\boldsymbol{d}_{\vec{c}',2})}{2},R\right),$$

which is between the two vortices. We define

$$G\begin{pmatrix}X_1\\X_2\\\delta_1\\\delta_2\\\gamma\end{pmatrix}:=\begin{pmatrix}\mathfrak{Re}\int_\Omega\partial_{x_1}Q\overline{Q\psi^{\neq 0}}\\\mathfrak{Re}\int_\Omega\partial_{x_2}Q\overline{Q\psi^{\neq 0}}\\c^2\mathfrak{Re}\int_\Omega\partial_{\boldsymbol{d}}\boldsymbol{V}\overline{Q\psi^{\neq 0}}\\c\mathfrak{Re}\int_\Omega\partial_{c^\perp}Q\overline{Q\psi^{\neq 0}}\\\mathfrak{Re}\int_{\Omega'}i\psi\end{pmatrix},$$

where $\vec{c}'$ (used to defined $Q=Q_{\vec{c}'}(.-X)e^{i\gamma}$) is given by $\delta_1=\delta^{|\cdot|}(c\vec{e_2},\vec{c}')$ and $\delta_2=\delta^{\perp}(c\vec{e_2},\vec{c}')$.

Here, we use the notation ∂_cQ for $\partial_cQ_{c\,|\,c=c'}$. We remark from (3.6.7) and the definition of η, that in Ω, we have

$$Q\psi=Z-Q.$$

First, we have

$$\|Q\psi\|_{C^1(\Omega)}\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e_2},\vec{c}')}{c^2}+\frac{\delta^{\perp}(c\vec{e_2},\vec{c}')}{c}+|\gamma|\right),\tag{3.6.12}$$

which is a consequence of Lemma 3.6.1. By Lemma 3.4.1, we compute that

$$\left| G\begin{pmatrix} 0\\0\\0\\0\\0 \end{pmatrix}\right| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1).$$

Let us compute $\partial_{X_2} G$. We recall that $Q\psi \in C^1(\mathbb{R}^2,\mathbb{C})$. Since Ω depends on X, we have

$$\begin{aligned}\partial_{X_2}\mathfrak{Re}\int_\Omega \partial_{x_2} Q\overline{Q\psi^{\neq 0}} &= \int_{\partial\Omega}\mathfrak{Re}\Big(\partial_{x_2} Q\overline{Q\psi^{\neq 0}}\Big)\\ &- \int_\Omega \mathfrak{Re}\Big(\partial^2_{x_2x_2} Q\overline{Q\psi^{\neq 0}}\Big)\\ &+ \int_\Omega \mathfrak{Re}\Big(\partial_{x_2} Q\overline{\partial_{X_2}(Q\psi^{\neq 0})}\Big).\end{aligned}$$

By estimate (3.6.12), we have

$$\left|\int_{\partial\Omega}\mathfrak{Re}\Big(\partial_{x_2} Q\overline{Q\psi^{\neq 0}}\Big)\right| + \left|\int_\Omega \mathfrak{Re}\Big(\partial^2_{x_2x_2} Q\overline{Q\psi^{\neq 0}}\Big)\right| \leqslant$$

$$o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1) + K\left(|X| + \frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}\,')}{c^2} + \frac{\delta^{\perp}(c\vec{e}_2,\vec{c}\,')}{c}\right),$$

and since $Q\psi = Z - Q$ and $\psi^{\neq 0} = \psi - \psi^{\mathbf{0}}$ in Ω, we check that,

$$\int_\Omega \mathfrak{Re}\Big(\partial_{x_2} Q\overline{\partial_{X_2}(Q\psi^{\neq 0})}\Big) = -\int_\Omega |\partial_{x_2} Q|^2 + \int_\Omega \mathfrak{Re}\big(\partial_{x_2} Q\overline{\partial_X(Q\psi^{\mathbf{0}})}\big).$$

Now, using $Q\psi = Z - Q$, we check that, in $B(\boldsymbol{d}_{\vec{c}\,',1}, R)$, where $x = \boldsymbol{r}_1 e^{i\boldsymbol{\theta}_1}$,

$$\begin{aligned}2\pi\partial_{X_2}(Q\psi^0) &= \partial_{X_2}\Big(Q\int_0^{2\pi}\frac{Z-Q}{Q}d\boldsymbol{\theta}_1\Big)\\ &= \partial_{x_2}Q\int_0^{2\pi}\frac{Z-Q}{Q}d\boldsymbol{\theta}_1\\ &+ Q\int_0^{2\pi}\frac{-\partial_{x_2}Q}{Q}d\boldsymbol{\theta}_1 + Q\int_0^{2\pi}\frac{-(Z-Q)\partial_{x_2}Q}{Q^2}d\boldsymbol{\theta}_1\\ &+ Q\int_0^{2\pi}\partial_{x_2}\Big(\frac{Z-Q}{Q}\Big)d\boldsymbol{\theta}_1\end{aligned}$$

Therefore, we estimate (since R is a universal constant)

$$\left|\int_{B(\boldsymbol{d}_{\vec{c}\,',1},R)}\mathfrak{Re}\big(\partial_{x_2}Q\overline{\partial_X(Q\psi^0)}\big)\right| \leqslant$$

$$\left|\int_{B(\boldsymbol{d}_{\vec{c}\,',1},R)}\mathfrak{Re}\Big(\overline{\partial_{x_2}Q}Q\int_0^{2\pi}\frac{-\partial_{x_2}Q}{Q}d\boldsymbol{\theta}_1\Big)\right| + K\|Z-Q\|_{C^1(\Omega)}.$$

Let us show that, in $B(\boldsymbol{d}_{\vec{c}\,',1}, R)$,

$$Q\int_0^{2\pi}\frac{-\partial_{x_2}Q}{Q}d\boldsymbol{\theta}_1 = o_{c\to 0}(1). \tag{3.6.13}$$

We have in this domain that $\frac{Q}{\boldsymbol{V}_1} = 1 + o_{c\to 0}(1)$ and $\big|\nabla Q_c - \nabla\tilde{V}_1\big| = o_{c\to 0}(1)$ by Lemmas 3.1.13 and 3.1.14, where $\mathbf{V}_1$ is the vortex centered at $\boldsymbol{d}_{\vec{c}\,',1}$. We deduce that, in $B(\boldsymbol{d}_{\vec{c}\,',1}, R)$,

$$Q\int_0^{2\pi}\frac{-\partial_{x_2}Q}{Q}d\boldsymbol{\theta}_1 = \mathbf{V}_1\int_0^{2\pi}\frac{-\partial_{x_2}\mathbf{V}_1}{\mathbf{V}_1}d\boldsymbol{\theta}_1 + o_{c\to 0}(1).$$

Finally, by Lemma 1.2.1, we check that $\frac{\partial_{x_2}\mathbf{V}_1}{\mathbf{V}_1}$ has no 0-harmonic around $\boldsymbol{d}_{\vec{c}',1}$, therefore

$$\mathbf{V}_1\int_0^{2\pi}\frac{-\partial_{x_2}\mathbf{V}_1}{\mathbf{V}_1}d\boldsymbol{\theta}_1=0. \tag{3.6.14}$$

By symmetry, the same proof holds in $B(\boldsymbol{d}_{\vec{c}',2},R)$.

Adding up these estimates, we get

$$\left|\partial_{X_2}\mathfrak{Re}\int_\Omega\partial_{x_2}Q\overline{Q\psi^{\neq0}}+\int_\Omega|\partial_{x_2}Q|^2\right|\leqslant$$

$$o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to0}(1)+o_{c\to0}(1)+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^{\perp}(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\right).$$

By a similar computation, we have

$$\left|\partial_{X_2}\mathfrak{Re}\int_\Omega\partial_{\boldsymbol{d}}\boldsymbol{V}\overline{Q\psi^{\neq0}}-\int_\Omega\mathfrak{Re}(\partial_{\boldsymbol{d}}\boldsymbol{V}\overline{\partial_{x_2}Q})\right|\leqslant$$

$$o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to0}(1)+o_{c\to0}(1)+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^{\perp}(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\right).$$

By Lemma 3.4.1 and Theorem 1.3.1 (for $p=+\infty$), we have

$$\left|\int_\Omega\mathfrak{Re}(\partial_{\boldsymbol{d}}\boldsymbol{V}\overline{\partial_{x_2}Q})\right|\leqslant\left|\int_\Omega\mathfrak{Re}(c^2\partial_cQ\overline{\partial_{x_2}Q})\right|+\left|\int_\Omega\mathfrak{Re}((\partial_{\boldsymbol{d}}\boldsymbol{V}-c^2\partial_cQ)\overline{\partial_{x_2}Q})\right|=o_{c\to0}(1).$$

Similarly, we check

$$\left|\partial_{X_2}\int_\Omega\partial_{x_1}Q\overline{Q\psi^{\neq0}}\right|-\left|\int_\Omega\mathfrak{Re}(\partial_{x_1}Q\overline{\partial_{x_2}Q})\right|\leqslant$$

$$o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to0}(1)+o_{c\to0}(1)+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^{\perp}(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\right).$$

Still by Lemma 3.4.1, we have

$$\left|\int_\Omega\mathfrak{Re}(\partial_{x_1}Q\overline{\partial_{x_2}Q})\right|=o_{c\to0}(1).$$

With the same arguments, we check that

$$\left|\partial_{X_2}\int_\Omega c\partial_{c^\perp}Q\overline{Q\psi^{\neq0}}\right|\leqslant$$

$$o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to0}(1)+o_{c\to0}(1)+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^{\perp}(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\right).$$

Finally, with equations (3.1.6) to (3.1.10) and (3.6.12), we check easily that

$$\partial_{X_2}\left(\mathfrak{Re}\int_{\Omega'}i\psi\right)\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to0}(1)+o_{c\to0}(1)+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^{\perp}(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\right).$$

We deduce that

$$\left|\partial_{X_2}G\begin{pmatrix}X_1\\X_2\\\delta_1\\\delta_2\\\gamma\end{pmatrix}+\begin{pmatrix}0\\\int_\Omega|\partial_{x_2}Q|^2\\0\\0\\0\end{pmatrix}\right|\leqslant$$

$$o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to0}(1)+o_{c\to0}(1)+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^{\perp}(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\right).$$

We can also check, with similar computations, that

$$\left|\partial_{X_1} G\begin{pmatrix} X_1\\ X_2\\ \delta_1\\ \delta_2\\ \gamma\end{pmatrix}+\begin{pmatrix} \int_\Omega|\partial_{x_1}Q|^2\\ 0\\ 0\\ 0\\ 0\end{pmatrix}\right|\leqslant$$

$$o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)+o_{c\to 0}(1)+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e_2},\vec{c}')}{c^2}+\frac{\delta^{\perp}(c\vec{e_2},\vec{c}')}{c}+|\gamma|\right).$$

We infer that this also holds with a similar proof for the last two directions, namely

$$\left|c^2\partial_{\delta_1} G\begin{pmatrix} X_1\\ X_2\\ \delta_1\\ \delta_2\\ \gamma\end{pmatrix}+\begin{pmatrix} 0\\ 0\\ \int_\Omega|c^2\partial_c Q|^2\\ 0\\ 0\end{pmatrix}\right|\leqslant$$

$$o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)+o_{c\to 0}(1)+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e_2},\vec{c}')}{c^2}+\frac{\delta^{\perp}(c\vec{e_2},\vec{c}')}{c}+|\gamma|\right)$$

(using the fact that $\partial_{\boldsymbol{d}}\boldsymbol{V}$ is differentiable with respect to δ_1, which is not obvious for $c^2\partial_c Q$ and is the reason we have to use this orthogonality) and

$$\left|c\partial_{\delta_2} G\begin{pmatrix} X_1\\ X_2\\ \delta_1\\ \delta_2\\ \gamma\end{pmatrix}+\begin{pmatrix} 0\\ 0\\ 0\\ \int_\Omega|c\partial_{c^\perp} Q|^2\\ 0\end{pmatrix}\right|\leqslant$$

$$o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)+o_{c\to 0}(1)+K\left(|X|+\frac{\delta^{|\cdot|}(c\vec{e_2},\vec{c}')}{c^2}+\frac{\delta^{\perp}(c\vec{e_2},\vec{c}')}{c}+|\gamma|\right).$$

We will only show for these directions that, in $B(\boldsymbol{d}_{\vec{c}',1},R)$,

$$\left|Q\int_0^{2\pi}\frac{c^2\partial_c Q}{Q}d\boldsymbol{\theta}_1\right|+\left|Q\int_0^{2\pi}\frac{c\partial_{c^\perp} Q}{Q}d\boldsymbol{\theta}_1\right|=o_{c\to 0}(1),$$

the other computations are similar to the ones done for $\partial_{X_2}F$ (using Lemma 3.4.1).

We recall from Lemma 3.1.2 that, in $B(\boldsymbol{d}_{\vec{c}',1},R)$,

$$\|c^2\partial_c Q-\partial_{\boldsymbol{d}}\boldsymbol{V}\|_{C^1(B(\boldsymbol{d}_{\vec{c}',1},R))}=o_{c\to 0}(1),$$

where $\|\partial_{\boldsymbol{d}}\boldsymbol{V}+\partial_{x_1}V_1\|_{C^1(B(\boldsymbol{d}_{\vec{c}',1},R))}=o_{c\to 0}(1)$, V_1 being centered around a point $d_{\vec{c}'}\in\mathbb{R}^2$ such that

$$|d_{\vec{c}'}-\boldsymbol{d}_{\vec{c}',1}|=o_{c\to 0}(1).$$

Therefore, we check that

$$\begin{aligned}\left|Q\int_0^{2\pi}\frac{c^2\partial_c Q}{Q}d\boldsymbol{\theta}_1\right| &\leqslant \left|\mathbf{V}_1\int_0^{2\pi}\frac{\partial_{x_1}\mathbf{V}_1}{\mathbf{V}_1}d\boldsymbol{\theta}_1\right|+o_{c\to 0}(1)\\ &= o_{c\to 0}(1)\end{aligned}$$

from (3.6.14). Finally, we have, from Lemma 3.1.6 that $\partial_{c^\perp}Q=-x^{\perp,\delta^\perp(c\vec{e_2},\vec{c}')}.\nabla Q$, where $x^{\perp,\delta^\perp(c\vec{e_2},\vec{c}')}$ is $x^\perp$ rotated by an angle $\delta^\perp(c\vec{e_2},\vec{c}')$. We remark that, in $B(\boldsymbol{d}_{\vec{c}',1},R)$,

$$\left|Q\int_0^{2\pi}\frac{c\boldsymbol{d}_{\vec{c}',1}.\nabla Q}{Q}d\boldsymbol{\theta}_1\right|\leqslant\left|\mathbf{V}_1\int_0^{2\pi}\frac{c\boldsymbol{d}_{\vec{c}',1}.\nabla\mathbf{V}_1}{\mathbf{V}_1}d\boldsymbol{\theta}_1\right|+o_{c\to 0}(1)$$

and

$$\left|\mathbf{V}_1\int_0^{2\pi}\frac{c\boldsymbol{d}_{\vec{c}',1}.\nabla\mathbf{V}_1}{\mathbf{V}_1}d\boldsymbol{\theta}_1\right|=0$$

by (3.6.14) and the same result for ∂_{x_1} instead of ∂_{x_2}. Therefore, since $|x^{\perp,\delta^\perp(c\vec{e}_2,\vec{c}')}-\boldsymbol{d}_{\vec{c}',1}|\leqslant K$ in $B(\boldsymbol{d}_{\vec{c}',1},R)$,

$$\begin{aligned}\left|Q\int_0^{2\pi}\frac{c\partial_{c^\perp}Q}{Q}d\boldsymbol{\theta}_1\right| &\leqslant \left|Q\int_0^{2\pi}\frac{c(x^{\perp,\delta^\perp(c\vec{e}_2,\vec{c}')}-\boldsymbol{d}_{\vec{c}',1}).\nabla Q}{Q}d\boldsymbol{\theta}_1\right|+o_{c\to0}(1)\\ &\leqslant Kc+o_{c\to0}(1)\\ &= o_{c\to0}(1).\end{aligned}$$

Finally, we infer that

$$\left|\partial_\gamma G\begin{pmatrix}X_1\\X_2\\\delta_1\\\delta_2\\\gamma\end{pmatrix}+\begin{pmatrix}0\\0\\0\\0\\\mathfrak{Re}\int_{\Omega'}Q\end{pmatrix}\right|\leqslant$$

$$o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to0}(1)+o_{c\to0}(1)+K\left(|X|+\frac{\delta^{|.|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^\perp(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\right).$$

The proof is similar of the previous computations, and we will only show that, in Ω,

$$|\partial_\gamma(Q\psi^{\neq0})|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to0}(1).$$

We have

$$\begin{aligned}|\partial_\gamma(Q\psi^{\neq0})| &= |\partial_\gamma(Q\psi)-\partial_\gamma(Q\psi^{\mathbf{0}})|\\ &\leqslant \left|-iQ-\frac{Q}{2\pi}\int_0^{2\pi}\frac{-iQ}{Q}d\theta\right|+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to0}(1)\\ &\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to0}(1).\end{aligned}$$

From Theorem 1.3.1, $\mathfrak{Re}\int_{\Omega'}Q=\mathfrak{Re}\int_{\Omega'}-1+o_{c\to0}(1)\leqslant-K<0$. We conclude, by Lemma 3.4.1, that, for c and $\|Z-Q_c\|_{H^{\exp}_{Q_c}}$ small enough, dG is invertible in a vicinity of $(0,0,0,0,0)$ of size independent of $\|Z-Q_c\|_{H^{\exp}_{Q_c}}$. Therefore, by the implicit function theorem, taking c small enough and $\varepsilon(c,\lambda)$ small enough, we can find $X,\vec{c}'\in\mathbb{R}^2,\gamma\in\mathbb{R}$ such that

$$|X|+\frac{\delta^{|.|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^\perp(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to0}(1),$$

and satisfying

$$\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c}',1},R)\cup B(\boldsymbol{d}_{\vec{c}',2},R)}\partial_{x_1}Q\overline{Q\psi^{\neq0}}=\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c}',1},R)\cup B(\boldsymbol{d}_{\vec{c}',2},R)}\partial_{x_2}Q\overline{Q\psi^{\neq0}}=0,$$

$$\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c}',1},R)\cup B(\boldsymbol{d}_{\vec{c}',2},R)}\partial_{\boldsymbol{d}}\boldsymbol{V}\overline{Q\psi^{\neq0}}=\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c}',1},R)\cup B(\boldsymbol{d}_{\vec{c}',2},R)}\partial_{c^\perp}Q\overline{Q\psi^{\neq0}}=0,$$

$$\mathfrak{Re}\int_{B((\boldsymbol{d}_{\vec{c}',1}+\boldsymbol{d}_{\vec{c}',2})/2,R)}i\psi=0.$$

□

3.6.3 End of the proof of Theorem 1.4.13

From Lemmas 3.6.3 and 3.6.6, we can find $\varphi=Q\psi\in H^{\exp}_Q$ such that

$$|X|+\frac{\delta^{|.|}(c\vec{e}_2,\vec{c}')}{c^2}+\frac{\delta^\perp(c\vec{e}_2,\vec{c}')}{c}+|\gamma|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to0}(1),\tag{3.6.15}$$

and

$$\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c}',1},R)\cup B(\boldsymbol{d}_{\vec{c}',2},R)}\partial_{x_1}Q\overline{Q\psi^{\neq 0}}=\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c}',1},R)\cup B(\boldsymbol{d}_{\vec{c}',2},R)}\partial_{x_2}Q\overline{Q\psi^{\neq 0}}=0,$$

$$\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c}',1},R)\cup B(\boldsymbol{d}_{\vec{c}',2},R)}\partial_{c}Q\overline{Q\psi^{\neq 0}}=\mathfrak{Re}\int_{B(\boldsymbol{d}_{\vec{c}',1},R)\cup B(\boldsymbol{d}_{\vec{c}',2},R)}\partial_{c^\perp}Q\overline{Q\psi^{\neq 0}}=0,$$

$$\mathfrak{Re}\int_{B((\boldsymbol{d}_{\vec{c}',1}+\boldsymbol{d}_{\vec{c}',2})/2,R)}i\psi=0.$$

Now, from Lemma 3.6.5, ψ satisfies the equation

$$L_Q(Q\psi)-i(\vec{c}'-c\vec{e_2}).H(\psi)+\mathrm{NL}_{\mathrm{loc}}(\psi)+F(\psi)=0. \tag{3.6.16}$$

We remark that

$$L_Q(Q\psi)=(1-\eta)L_Q(Q\psi)+\eta Q\,L'_Q(\psi),$$

and by Lemmas 3.6.3 and 3.6.4,

$$\langle(1-\eta)L_Q(Q\psi)+\eta Q\,L'_Q(\psi),Q(\psi+i\gamma)\rangle=B_Q^{\exp}(\varphi).$$

We deduce that

$$B_Q^{\exp}(\varphi)-\langle i(\vec{c}'-c\vec{e_2}).H(\psi),Q(\psi+i\gamma)\rangle+\langle\mathrm{NL}_{\mathrm{loc}}(\psi),Q(\psi+i\gamma)\rangle+\langle F(\psi),Q(\psi+i\gamma)\rangle=0. \tag{3.6.17}$$

Since $Q\psi\in H_Q^{\exp}$ by Lemma 3.6.3, with the orthogonality conditions satisfied (see Lemma 3.6.6), we can apply Propositions 1.4.10 and 1.4.11 with remark (3.5.4). We have

$$B_Q^{\exp}(\varphi)\geqslant K\|\varphi\|_{\mathcal{C}}^2+K(c)\|\varphi\|_{H_Q^{\exp}}^2. \tag{3.6.18}$$

3.6.3.1 Better estimates on $\vec{c}'-c\vec{e_2}$

The term $i(\vec{c}'-c\vec{e_2}).H(\psi)$ contains a "source" term, because Z and Q do not satisfy the same equation (since the travelling waves Z and Q may not have the same speed at this point). We want to show the following estimates:

$$\delta^{|\cdot|}(c\vec{e_2},\vec{c}')\leqslant\left(Kc^2\ln\left(\frac{1}{c}\right)+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\right)\|\varphi\|_{\mathcal{C}}+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_{Q_c}} \tag{3.6.19}$$

and

$$\delta^{\perp}(c\vec{e_2},\vec{c}')\leqslant\left(Kc^2\ln\left(\frac{1}{c}\right)+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\right)\|\varphi\|_{\mathcal{C}}+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_{Q_c}}, \tag{3.6.20}$$

where $\delta^{|\cdot|}(c\vec{e_2},\vec{c}')=|c\vec{e_2}.\frac{\vec{c}'}{|\vec{c}'|}-\vec{c}'|$ and $\delta^{\perp}(c\vec{e_2},\vec{c}')=|c\vec{e_2}.\frac{\vec{c}'^{\perp}}{|\vec{c}'|}-\vec{c}'|$.

This subsection is devoted to the proof of (3.6.19) and (3.6.20).

Step 1. We have the estimate (3.6.19).

We take the scalar product of (3.6.16) with $c^2\partial_cQ$, which yields

$$\langle i(\vec{c}'-c\vec{e_2}).H(\psi),c^2\partial_cQ\rangle=\langle Q\psi,c^2L_Q(\partial_cQ)\rangle+\langle\mathrm{NL}_{\mathrm{loc}}(\psi)+F(\psi),c^2\partial_cQ\rangle.$$

We check here, with the L^∞ estimates on ψ and its derivatives, as well as on ∂_cQ (see Lemma 3.1.2 and 3.6.3), that $\langle L_Q(Q\psi),c^2\partial_cQ\rangle$ is well defined and that all the integrations by parts can be done.

We recall that $H(\psi)=\nabla Q+\frac{\nabla(Q\psi)(1-\eta)+Q\nabla\psi\eta e^{\psi}}{(1-\eta)+\eta e^{\psi}}$, and we check that, since $1-\eta$ is compactly supported (in a domain with size independent of $c,\vec{c}'$), with equation (3.6.11)

$$\begin{aligned}\left|\left\langle i(\vec{c}'-c\vec{e_2}).\frac{\nabla(Q\psi)(1-\eta)+Q\nabla\psi\eta e^{\psi}}{(1-\eta)+\eta e^{\psi}},c^2\partial_cQ\right\rangle\right| &\leqslant K|(\vec{c}'-c\vec{e_2}).\langle\eta iQ\nabla\psi,c^2\partial_cQ\rangle|\\ &+ K|\vec{c}'-c\vec{e_2}|\|\varphi\|_{H^{\exp}_{Q_c}}.\end{aligned}$$

We compute with Lemma 3.1.2 that

$$\begin{aligned}|\langle \eta i Q\nabla\psi, c^2\partial_c Q\rangle| &= \left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla\psi i Q c^2\overline{\partial_c Q})\right|\\ &\leqslant \left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla\psi)\mathfrak{Im}(Qc^2\overline{\partial_c Q})\right| + \left|\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(Qc^2\overline{\partial_c Q})\right|\\ &\leqslant \left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\psi)\nabla(\mathfrak{Im}(Qc^2\overline{\partial_c Q}))\right| + K\|\varphi\|_{H^{\exp}_{Q_c}}\\ &+ \|\varphi\|_{\mathcal{C}}\sqrt{\int_{\mathbb{R}^2}\eta\mathfrak{Re}^2(Qc^2\overline{\partial_c Q})}.\end{aligned}$$

From Lemmas 3.1.1 and 3.1.2, we check that $\int_{\mathbb{R}^2}\eta\mathfrak{Re}^2(Qc^2\overline{\partial_c Q})\leqslant K$, and furthermore,

$$|\nabla(\mathfrak{Im}(Qc^2\overline{\partial_c Q}))|\leqslant c^2|\partial_c Q||\nabla Q| + Kc^2|\nabla\partial_c Q|$$

and with Lemma 3.1.2 (with $\sigma = 1/2$), we check that

$$|\nabla(\mathfrak{Im}(Qc^2\overline{\partial_c Q}))|\leqslant \frac{K}{(1+\tilde{r})^{3/2}},$$

thus, by Cauchy-Schwarz,

$$\left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\psi)\nabla(\mathfrak{Im}(Qc^2\overline{\partial_c Q}))\right|\leqslant K\|\varphi\|_{\mathcal{C}}.$$

Using $|\vec{c}' - c\vec{e_2}|\leqslant K(c)(\delta^{|.|}(c\vec{e_2},\vec{c}') + \delta^{\perp}(c\vec{e_2},\vec{c}'))\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)$ and $\|\varphi\|_{\mathcal{C}}\leqslant K\|\varphi\|_{H^{\exp}_{Q_c}}$, we deduce that

$$\left|\left\langle i(\vec{c}' - c\vec{e_2}).\frac{(1-\eta)\nabla(Q\psi) + \eta e^{\psi}Q\nabla\psi}{(1-\eta)+\eta e^{\psi}}, c^2\partial_c Q\right\rangle\right|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1).$$

Furthermore, we check that, by symmetry (see (3.1.3)),

$$\langle i(\vec{c}' - c\vec{e_2}).\nabla_x Q, c^2\partial_c Q\rangle = \delta^{|.|}(c\vec{e_2},\vec{c}')\left\langle i\frac{\vec{c}'}{|\vec{c}'|}.\nabla Q, c^2\partial_c Q\right\rangle.$$

Furthermore, from Lemma 3.1.7, we have $L_Q(\partial_c Q) = i\nabla_{\vec{c}'}Q$, therefore, from Proposition 1.4.1,

$$\left\langle i\frac{\vec{c}'}{|\vec{c}'|}.\nabla Q, c^2\partial_c Q\right\rangle = c^2 B_Q(\partial_c Q) = -2\pi + o_{c\to 0}(1).$$

We deduce that

$$\delta^{|.|}(c\vec{e_2},\vec{c}')\leqslant K|\langle Q\psi, c^2 L_Q(\partial_c Q)\rangle + \langle \mathrm{NL}_{\mathrm{loc}}(\psi) + F(\psi), c^2\partial_c Q\rangle| + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_{Q}}.$$

Now, since $L_Q(\partial_c Q) = i\frac{\vec{c}'}{|\vec{c}'|}.\nabla Q$, we check that

$$\langle Q\psi, c^2 L_Q(\partial_c Q)\rangle = c^2\left\langle Q\psi, i\frac{\vec{c}'}{|\vec{c}'|}.\nabla Q\right\rangle,$$

and

$$\left|\left\langle Q\psi, i\frac{\vec{c}'}{|\vec{c}'|}.\nabla Q\right\rangle\right|\leqslant\left|\int_{\mathbb{R}^2}\mathfrak{Re}(\psi)\mathfrak{Im}\left(\frac{\vec{c}'}{|\vec{c}'|}.\nabla Q\bar{Q}\right)\right| + \left|\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\mathfrak{Re}\left(\frac{\vec{c}'}{|\vec{c}'|}.\nabla Q\bar{Q}\right)\right|.$$

From Lemma 3.4.4, we deduce that

$$|\langle Q\psi, c^2 L_Q(\partial_c Q)\rangle|\leqslant Kc^2\ln\left(\frac{1}{c}\right)\|\varphi\|_{\mathcal{C}}.$$

Now, we check easily that, with Lemmas 3.6.1 and 3.6.5,

$$|\langle \mathrm{NL}_{\mathrm{loc}}(\psi), c^2\partial_c Q\rangle|\leqslant K(c)\|\varphi\|_{H^{\exp}_{Q}}\|\varphi\|_{C^1(B(0,\lambda))}\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_{Q}}.$$

To conclude the proof of estimate (3.6.19), we shall estimate

$$|\langle F(\psi), c^2\partial_c Q\rangle|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_{Q}} + \left(K\lambda_0 + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\right)\|\varphi\|_{C},$$

with $F(\psi)=Q\eta(-\nabla\psi.\nabla\psi+|Q|^2S(\psi))$. First, we estimate, for $\Lambda>\lambda>\frac{10}{c}$, with Lemma 3.6.2,

$$\begin{aligned}|\langle -Q\eta\nabla\psi.\nabla\psi, c^2\partial_c Q\rangle| &= \left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla\psi.\nabla\psi c^2\bar{Q}\partial_c Q)\right|\\ &\leqslant \int_{\mathbb{R}^2}\eta\,|\nabla\psi|^2|c^2\bar{Q}\partial_c Q|\\ &\leqslant K\|\nabla\psi\|_{L^\infty(B(0,\lambda)\cap\{\eta\neq 0\})}\sqrt{\int_{B(0,\lambda)}\eta|\nabla\psi|^2}\sqrt{\int_{B(0,\lambda)}\eta|c^2\bar{Q}\partial_c Q|^2}\\ &+ \|c^2\bar{Q}\partial_c Q\|_{L^\infty(\mathbb{R}^2\setminus B(0,\Lambda))}\int_{\mathbb{R}^2\setminus B(0,\lambda)}\eta\,|\nabla\psi|^2\\ &\leqslant o^{\Lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{\mathcal{C}}+o_{\Lambda\to\infty}(1)\|\varphi\|_{\mathcal{C}},\end{aligned}$$

since, by Lemma 3.1.2, $|c^2\bar{Q}\partial_c Q|\leqslant\frac{K}{(1+\tilde{r})^{1/2}}$. We deduce that

$$|\langle -Q\eta\nabla\psi.\nabla\psi, c^2\partial_c Q\rangle|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{\mathcal{C}}.$$

Now, in $\{\eta=1\}$, since $e^\psi=\frac{Z}{Q}$ and $1-K\lambda_0\leqslant\frac{|Z|}{|Q|}\leqslant 1+K\mu_0$ (by our assumptions on Z), we have $|\mathfrak{Re}(\psi)|\leqslant K\mu_0$. We deduce, with Lemma 3.6.1, that in $\{\eta\neq 0\}$,

$$|\mathfrak{Re}(\psi)|\leqslant K\mu_0+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1).$$

With $S(\psi)=e^{2\mathfrak{Re}(\psi)}-1-2\mathfrak{Re}(\psi)$, we check that, in $\eta\neq 0$, $|S(\psi)|\leqslant K|\mathfrak{Re}(\psi)|^2$ (given that μ_0 and $\|Z-Q_c\|_{H^{\exp}_{Q_c}}$ are small enough), and with similar computations as for $|\langle -Q\eta\nabla\psi.\nabla\psi, c^2\partial_c Q\rangle|$, we conclude that

$$|\langle F(\psi), c^2\partial_c Q\rangle|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{\mathcal{C}}.$$

This concludes the proof of

$$\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_Q}+\left(Kc^2\ln\left(\frac{1}{c}\right)+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\right)\|\varphi\|_{\mathcal{C}}.$$

Step 2. We have the estimate (3.6.20).

Now, we take the scalar product of (3.6.16) with $c\partial_{c^\perp}Q$:

$$\langle i(\vec{c}'-c\vec{e}_2).H(\psi), c\partial_{c^\perp}Q\rangle=\langle Q\psi, cL_Q(\partial_{c^\perp}Q)\rangle+\langle \mathrm{NL}_{\mathrm{loc}}(\psi)+F(\psi), c\partial_{c^\perp}Q\rangle.$$

We check that, since

$$\begin{aligned}\left\langle i(\vec{c}'-c\vec{e}_2).\frac{\nabla(Q\psi)(1-\eta)+Q\nabla\psi\eta e^\psi}{(1-\eta)+\eta e^\psi}, c\partial_{c^\perp}Q\right\rangle &\leqslant K|(\vec{c}'-c\vec{e}_2).\langle(1-\eta)iQ\nabla\psi, c\partial_{c^\perp}Q\rangle|\\ &+ K|\vec{c}'-c\vec{e}_2|\|\varphi\|_{H^{\exp}_{Q_c}},\end{aligned}$$

and

$$\begin{aligned}|\langle\eta iQ\nabla\psi, c\partial_{c^\perp}Q\rangle| &= \left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla\psi iQc\overline{\partial_{c^\perp}Q})\right|\\ &\leqslant \left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\nabla\psi)\mathfrak{Im}(Qc\overline{\partial_{c^\perp}Q})\right|+\left|\int_{\mathbb{R}^2}\eta\mathfrak{Im}(\nabla\psi)\mathfrak{Re}(Qc\overline{\partial_{c^\perp}Q})\right|\\ &\leqslant \left|\int_{\mathbb{R}^2}\eta\mathfrak{Re}(\psi)\nabla(\mathfrak{Im}(Qc\overline{\partial_{c^\perp}Q}))\right|+K\|\varphi\|_{H^{\exp}_{Q_c}}\\ &+ \|\varphi\|_{\mathcal{C}}\int_{\mathbb{R}^2}\eta\mathfrak{Re}^2(Qc\overline{\partial_{c^\perp}Q}).\end{aligned}$$

We check, with Lemmas 3.1.1 and 3.1.2, that

$$\int_{\mathbb{R}^2}\eta\mathfrak{Re}^2(Qc\overline{\partial_{c^\perp}Q})\leqslant K$$

and

$$|\nabla(\mathfrak{Im}(Q\overline{\partial_{c^\perp}Q}))| \leqslant |\nabla Q||\partial_{c^\perp}Q| + |\nabla\partial_{c^\perp}Q| \leqslant \frac{K(c)}{(1+r)^2},$$

therefore, as for the previous estimation,

$$\left|\left\langle i(\vec{c}\,' - c\vec{e_2}).\frac{(1-\eta)\nabla(Q\psi) + \eta e^{\psi}Q\nabla\psi}{(1-\eta)+\eta e^{\psi}}, c\partial_{c^\perp}Q\right\rangle\right| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_Q}.$$

We check that, by symmetry (see equation (3.1.3))

$$\langle i(\vec{c}\,' - c\vec{e_2}).\nabla Q, c\partial_{c^\perp}Q\rangle = \delta^\perp(c\vec{e_2},\vec{c}\,')\left\langle i\frac{\vec{c}\,'}{|\vec{c}\,'|}.\nabla Q, c\partial_{c^\perp}Q\right\rangle$$

Furthermore, from Lemma 3.1.7, we have $L_Q(\partial_{c^\perp}Q) = -ic\frac{\vec{c}\,'^{\perp}}{|\vec{c}\,'|}.\nabla Q$, therefore, from Proposition 1.4.1,

$$c\left\langle i\frac{\vec{c}\,'^{\perp}}{|\vec{c}\,'|}.\nabla Q, \partial_{c^\perp}Q\right\rangle = -B_Q(\partial_{c^\perp}Q) = -2\pi + o_{c\to 0}(1).$$

We deduce that

$$\delta^\perp(c\vec{e_2},\vec{c}\,') \leqslant$$
$$K|\langle Q\psi, cL_Q(\partial_{c^\perp}Q)\rangle + \langle \mathrm{NL}_{\mathrm{loc}}(\psi) + F(\psi), c\partial_{c^\perp}Q\rangle| + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_Q}.$$

As previously, we check that

$$|\langle \mathrm{NL}_{\mathrm{loc}}(\psi) + F(\psi), c\partial_{c^\perp}Q\rangle| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_Q} + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{\mathcal{C}}$$

and from Lemma 3.1.7, we have

$$\begin{aligned}|\langle Q\psi, L_Q(\partial_{c^\perp}Q)\rangle| &= \left|\left\langle Q\psi, i\frac{\vec{c}\,'^{\perp}}{|\vec{c}\,'|}.\nabla Q\right\rangle\right| \\ &\leqslant \left|\int_{\mathbb{R}^2}\mathfrak{Re}(\psi)\mathfrak{Im}\left(\frac{\vec{c}\,'^{\perp}}{|\vec{c}\,'|}.\nabla Q\bar{Q}\right)\right| + \left|\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\mathfrak{Re}\left(\frac{\vec{c}\,'^{\perp}}{|\vec{c}\,'|}.\nabla Q\bar{Q}\right)\right|,\end{aligned}$$

and with Lemma 3.4.4, we deduce that

$$c|\langle Q\psi, L_Q(\partial_{c^\perp}Q)\rangle| \leqslant Kc\ln\left(\frac{1}{c}\right)\|\varphi\|_{\mathcal{C}}.$$

We conclude that

$$\delta^\perp(c\vec{e_2},\vec{c}\,') \leqslant \left(Kc^2\ln\left(\frac{1}{c}\right) + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\right)\|\varphi\|_{\mathcal{C}} + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_{Q_c}}.$$

3.6.3.2 Estimations on the remaining terms

Let us show in this subsection that

$$\begin{aligned}&|\langle i(\vec{c}\,' - c\vec{e_2}).H(\psi), Q(\psi+i\gamma)\rangle| + |\langle \mathrm{NL}_{\mathrm{loc}}(\psi), Q(\psi+i\gamma)\rangle| + |\langle F(\psi), Q(\psi+i\gamma)\rangle| \\ \leqslant\ &\left(o_{c\to 0}(1) + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1) + K\lambda_0\right)\|\varphi\|^2_{\mathcal{C}} + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}. \qquad (3.6.21)\end{aligned}$$

Step 1. Proof of $|\langle \mathrm{NL}_{\mathrm{loc}}(\psi), Q(\psi+i\gamma)\rangle| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}$.

From Lemma 3.6.5, we have

$$|\langle \mathrm{NL}_{\mathrm{loc}}(\psi), Q(\psi+i\gamma)\rangle| \leqslant K(\|Q\psi\|_{C^1(\{\eta\neq 1\})} + |\gamma|)\|\varphi\|^2_{H^1(\{\eta\neq 1\})},$$

therefore, from Lemmas 3.6.2, 3.6.6 and equation (3.6.15), we deduce

$$|\langle \mathrm{NL}_{\mathrm{loc}}(\psi), Q\psi\rangle| \leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}.$$

Step 2. Proof of

$$|\langle i(\vec{c}'-c\vec{e}_2).H(\psi),Q(\psi+i\gamma)\rangle|$$
$$\leqslant\Big(o_{c\to 0}(1)+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\Big)\|\varphi\|^2_{\vec{\mathcal{C}}}+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}.$$

We separate the estimation in two parts. First, we look at $\langle i(\vec{c}'-c\vec{e}_2).H(\psi),Q\psi\rangle$. We recall that $H(\psi)=\nabla Q+\frac{(1-\eta)\nabla(Q\psi)+\eta e^{\psi}Q\nabla\psi}{(1-\eta)+\eta e^{\psi}}$, and, since $|\vec{c}'-c\vec{e}_2|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)$ and $1-\eta$ is compactly supported, we check easily that

$$\left|\left\langle i(\vec{c}'-c\vec{e}_2).\frac{(1-\eta)\nabla(Q\psi)+\eta e^{\psi}Q\nabla\psi}{(1-\eta)+\eta e^{\psi}},Q\psi\right\rangle\right|\leqslant$$
$$o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\big(|\langle\eta i Q\nabla\psi,Q\psi\rangle|+K(c)\|\varphi\|^2_{H^{\exp}_Q}\big).$$

Furthermore, we check that

$$|\langle\eta i Q\nabla\psi,Q\psi\rangle|\leqslant\left|\int_{\mathbb{R}^2}\mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi)|Q|^2\eta\right|+\left|\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\mathfrak{Re}(\nabla\psi)|Q|^2\eta\right|,$$

and by Cauchy-Scwharz, $|\int_{\mathbb{R}^2}\mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi)|Q|^2\eta|\leqslant K\|\varphi\|^2_{\vec{\mathcal{C}}}$. Now, by integration by parts (using Lemma 3.6.3), we have

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\mathfrak{Re}(\nabla\psi)|Q|^2\eta\right| &\leqslant \left|\int_{\mathbb{R}^2}\mathfrak{Re}(\psi)\mathfrak{Im}(\nabla\psi)|Q|^2\eta\right|\\ &+ \left|\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\mathfrak{Re}(\psi)\nabla(|Q|^2)\eta\right|\\ &+ \left|\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\mathfrak{Re}(\psi)|Q|^2\nabla\eta\right|,\end{aligned}$$

and by Cauchy-Schwarz, we check that

$$\left|\int_{\mathbb{R}^2}\mathfrak{Im}(\psi)\mathfrak{Re}(\nabla\psi)|Q|^2\eta\right|\leqslant K\|\varphi\|^2_{H^{\exp}_Q}.$$

We deduce that

$$\left|\left\langle i(\vec{c}'-c\vec{e}_2).\frac{(1-\eta)\nabla(Q\psi)+\eta e^{\psi}Q\nabla\psi}{(1-\eta)+\eta e^{\psi}},Q\psi\right\rangle\right|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}.$$

Finally, we write

$$|\langle i(\vec{c}'-c\vec{e}_2).\nabla Q,Q\psi\rangle|\leqslant\delta^{|\cdot|}(c\vec{e}_2,\vec{c}')\left|\left\langle i\frac{\vec{c}'}{|\vec{c}'|}.\nabla Q,Q\psi\right\rangle\right|+\delta^{\perp}(c\vec{e}_2,\vec{c}')\left|\left\langle i\frac{\vec{c}'^{\perp}}{|\vec{c}'|}.\nabla Q,Q\psi\right\rangle\right|.$$

With Lemma 3.4.4, we check that

$$\left|\left\langle i\frac{\vec{c}'}{|\vec{c}'|}.\nabla Q,Q\psi\right\rangle\right|+\left|\left\langle i\frac{\vec{c}'^{\perp}}{|\vec{c}'|}.\nabla Q,Q\psi\right\rangle\right|\leqslant K\ln\left(\frac{1}{c}\right)\|\varphi\|_{\mathcal{C}}.$$

With (3.6.19) and (3.6.20), we deduce that

$$\begin{aligned}|\langle i(\vec{c}'-c\vec{e}_2).\nabla Q,Q\psi\rangle| &\leqslant \left(Kc\ln^2\left(\frac{1}{c}\right)+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\right)\|\varphi\|^2_{\vec{\mathcal{C}}}+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}\\ &\leqslant \Big(o_{c\to 0}(1)+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\Big)\|\varphi\|^2_{\vec{\mathcal{C}}}+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}.\end{aligned}$$

Now, we look at $\langle i(\vec{c}'-c\vec{e}_2).H(\psi),Qi\gamma\rangle$. We check that

$$\langle i\nabla Q,Qi\gamma\rangle=\gamma\int_{\mathbb{R}^2}\mathfrak{Re}(\nabla Q\bar{Q})=\frac{\gamma}{2}\int_{\mathbb{R}^2}\nabla(|Q|^2-1)=0,$$

thus

$$\langle i(\vec{c}'-c\vec{e_2}).H(\psi),Qi\gamma\rangle=\left\langle i(\vec{c}'-c\vec{e_2}).\frac{(1-\eta)\nabla(Q\psi)+\eta e^{\psi}Q\nabla\psi}{(1-\eta)+\eta e^{\psi}},Qi\gamma\right\rangle.$$

In the area $\{\eta\neq 0\}$, since $|\gamma|=o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)$ by Lemma 3.6.6, since

$$|\vec{c}'-c\vec{e_2}|\leqslant K\left(c\ln\left(\frac{1}{c}\right)+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\right)\|\varphi\|c+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_Q}$$

by estimates (3.6.19) and (3.6.20), we check that

$$\int_{\{\eta\neq 0\}}\mathfrak{Re}\left(i(\vec{c}'-c\vec{e_2}).\frac{(1-\eta)\nabla(Q\psi)+\eta e^{\psi}Q\nabla\psi}{(1-\eta)+\eta e^{\psi}}\overline{Qi\gamma}\right)\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q},$$

and therefore (with Lemma 3.6.3 that justifies the integrability)

$$|\langle i(\vec{c}'-c\vec{e_2}).H(\psi),Qi\gamma\rangle|\leqslant\left|\gamma(\vec{c}'-c\vec{e_2}).\int_{\mathbb{R}^2}\eta|Q|^2\mathfrak{Re}(\nabla\psi)\right|+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}.$$

By integration by parts (since $|\mathfrak{Re}(\psi)|\leqslant\frac{K\left(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z\right)}{(1+r)^2}$ and $|\mathfrak{Re}(\nabla\psi)|\leqslant\frac{K\left(\lambda,c,\|Z-Q_c\|_{H^{\exp}_{Q_c}},\varepsilon_0,Z\right)}{(1+r)^3}$ by Lemma 3.6.3) and Cauchy-Schwarz,

$$\begin{aligned}\left|\int_{\mathbb{R}^2}\eta|Q|^2\mathfrak{Re}(\nabla\psi)\right|&\leqslant\left|\int_{\mathbb{R}^2}\nabla\eta|Q|^2\mathfrak{Re}(\psi)\right|+\left|\int_{\mathbb{R}^2}\eta\nabla(|Q|^2)\mathfrak{Re}(\psi)\right|\\&\leqslant K(c)\|\varphi\|_{H^{\exp}_Q}.\end{aligned}$$

Since $|\gamma|=o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)$ by Lemma 3.6.6 and $|\vec{c}'-c\vec{e_2}|\leqslant\left(K(c)+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\right)\|\varphi\|_{H^{\exp}_Q}$ by (3.6.19), (3.6.20) and Lemma 3.5.1, we conclude that

$$|\langle i(\vec{c}'-c\vec{e_2}).H(\psi),Qi\gamma\rangle|\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{H^{\exp}_Q}.$$

Step 3. Proof of $|\langle F(\psi),Q(\psi+i\gamma)\rangle|\leqslant\left(o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)+K\lambda_0\right)\|\varphi\|^2_{\mathcal{C}}$.

We recall

$$F(\psi)=Q\eta(-\nabla\psi.\nabla\psi+|Q|^2S(\psi)),$$
$$S(\psi)=e^{2\mathfrak{Re}(\psi)}-1-2\,\mathfrak{Re}(\psi).$$

First, we look at $\langle F(\psi),Q\psi\rangle$. We have

$$|\langle F(\psi),Q\psi\rangle|\leqslant|\langle Q(1-\eta)\nabla\psi.\nabla\psi,Q\psi\rangle|+|\langle Q(1-\eta)|Q|^2S(\psi),Q\psi\rangle|.$$

We check that $\|\varphi\|_{L^\infty(\mathbb{R}^2)}\leqslant K\|\psi\|_{L^\infty(\mathbb{R}^2\setminus B(0,\lambda))}+K\|\varphi\|_{L^\infty(B(0,\lambda))}\leqslant K\lambda_0+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)$

$$|\langle Q\eta\nabla\psi.\nabla\psi,Q\psi\rangle|\leqslant\|\varphi\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2}\eta|\nabla\psi|^2\leqslant\left(K\lambda_0+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\right)\|\varphi\|^2_{\mathcal{C}}.$$

Finally, since $\|\varphi\|_{L^\infty(\mathbb{R}^2)}\leqslant K$ a uniform constant for c and $\|Z-Q_c\|_{H^{\exp}_{Q_c}}$ small enough,

$$|\langle Q\eta|Q|^2S(\psi),Q\psi\rangle|\leqslant\|\varphi\|_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^2}\eta\mathfrak{Re}^2(\psi)\leqslant\left(K\lambda_0+o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\right)\|\varphi\|^2_{\mathcal{C}}.$$

Now, we compute

$$|\langle F(\psi),Qi\gamma\rangle|\leqslant|\gamma|\left|\int_{\mathbb{R}^2}-\mathfrak{Re}(\eta i\nabla\psi.\nabla\psi)|Q|^2+\eta|Q|^4\mathfrak{Re}(S(\psi)i)\right|,$$

and since $S(\psi)$ is real-valued, we check that, since $|\gamma|=o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)$ by Lemma 3.6.6,

$$|\langle F(\psi),Qi\gamma\rangle|\leqslant|\gamma|\int_{\mathbb{R}^2}\eta|\nabla\psi|^2|Q|^2\leqslant o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|^2_{\mathcal{C}}.$$

3.6.3.3 Conclusion

Combining the steps 1 to 3 and (3.6.18) in (3.6.17), we deduce that, taking c small enough, and then $\|Z-Q_c\|_{H^{\exp}_{Q_c}}$ small enough (depending on c and λ), we have

$$\begin{aligned} 0 \;&\geqslant\; K\|\varphi\|_{\mathcal{C}}^2 + K(c)\|\varphi\|_{H^{\exp}_{Q}}^2 \\ &-\; \Big(o_{c\to 0}(1) + K\mu_0 + o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\Big)\|\varphi\|_{\mathcal{C}}^2 - o^{\lambda,c}_{\|Z-Q_c\|_{H^{\exp}_{Q_c}}\to 0}(1)\|\varphi\|_{H^{\exp}_{Q}}^2, \end{aligned}$$

hence, if μ_0 is taken small enough (independently of any other parameters) then c small enough and $\|Z-Q_c\|_{H^{\exp}_{Q_c}}$ small enough (depending on λ and c),

$$K(c)\|\varphi\|_{H^{\exp}_{Q_c}}^2 + K\|\varphi\|_{\mathcal{C}}^2 \leqslant 0.$$

We deduce that $\varphi = 0$, thus $Z = Q$. Furthermore, from (3.6.19) and (3.6.20) we deduce that $\vec{c}\,' = c\vec{e}_2$, and since $Z \to 1$ at infinity, we also have $\gamma = 0$ (or else $\|Z-Q_c\|_{H^{\exp}_{Q_c}} = +\infty$). This concludes the proof of Theorem 1.4.13.

Chapter 4

Inversion of the linearized operator and applications

In section 4.1, we recall previous results (mainly from the previous chapters) on the branch of travelling wave of Theorem 1.3.1, and show some corollaries that will be useful here. Section 4.2 is devoted to the proof of Theorem 1.5.1, and section 4.3 to the proof of Theorem 1.5.2, which relies mainly on Theorem 1.5.1.

We recall from subsection 1.5 that, with d_c defined in Theorem 1.3.1 ($\pm d_c\vec{e}_1$ are the center of the vortices from which Q_c is constructed as a perturbation of), we defined

$$\tilde{r} = \min\left(|x - d_c\vec{e}_1|, |x + d_c\vec{e}_1|\right),$$

as well as the two norms, for $\sigma, \sigma' \in \mathbb{R}$, $\varphi = Q_c\psi \in C^2(\mathbb{R}^2, \mathbb{C})$, $\psi = \psi_1 + i\psi_2$, and $Q_c h \in C^1(\mathbb{R}^2, \mathbb{C})$, $h = h_1 + i\,h_2$,

$$\begin{aligned}\|\psi\|_{\otimes,\sigma} &= \|Q_c\,\psi\|_{C^2(\{\tilde{r}\leqslant 3\})}\\ &+ \|\tilde{r}^{1+\sigma}\,\psi_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\nabla\psi_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\nabla^2\psi_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})}\\ &+ \|\tilde{r}^{\sigma}\,\psi_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{1+\sigma}\nabla\psi_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma}\nabla^2\psi_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})}\end{aligned}$$

and

$$\begin{aligned}\|h\|_{\otimes\otimes,\sigma'} &= \|Q_c h\|_{C^1(\{\tilde{r}\leqslant 3\})}\\ &+ \|\tilde{r}^{1+\sigma'}\,h_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma'}\,\nabla h_1\|_{L^\infty(\{\tilde{r}\geqslant 2\})}\\ &+ \|\tilde{r}^{2+\sigma'}\,h_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})} + \|\tilde{r}^{2+\sigma'}\,\nabla h_2\|_{L^\infty(\{\tilde{r}\geqslant 2\})}.\end{aligned}$$

We defined the spaces, for $\sigma, \sigma' \in \mathbb{R}$,

$$\mathcal{E}_{\otimes,\sigma} = \left\{\varphi = Q_c\psi \in C^2(\mathbb{R}^2, \mathbb{C}), \|\psi\|_{\otimes,\sigma} < +\infty, \forall (x_1, x_2) \in \mathbb{R}^2, \varphi(x_1, x_2) = \varphi(-x_1, x_2)\right\},$$

$$\mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma} = \left\{\varphi = Q_c\psi \in \mathcal{E}_{\otimes,\sigma}, \forall (x_1, x_2) \in \mathbb{R}^2, \varphi(x_1, x_2) = \overline{\varphi(x_1, -x_2)}\right\}$$

and

$$\mathcal{E}_{\otimes\otimes,\sigma'} = \left\{Q_c h \in C^1(\mathbb{R}^2, \mathbb{C}), \|h\|_{\otimes\otimes,\sigma'} < +\infty, \forall (x_1, x_2) \in \mathbb{R}^2, Q_c h(x_1, x_2) = Q_c h(-x_1, x_2)\right\},$$

$$\mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma'} = \left\{Q_c h \in \mathcal{E}_{\otimes\otimes,\sigma'}, \forall (x_1, x_2) \in \mathbb{R}^2, Q_c h(x_1, x_2) = \overline{Q_c h(x_1, -x_2)}\right\}.$$

4.1 Previous results on the branch

This section contains mostly results previously known on the branch of Theorem 1.3.1. This allows us to present them in a way adapted to the proofs to come, and to regroup the properties from other chapters and articles.

From Proposition 1.4.1, Q_c has exactly two zeros, and they are near $\pm d_c\vec{e}_1$. In particular, with $\tilde{r} = \min\left(|x - d_c\vec{e}_1|, |x + d_c\vec{e}_1|\right)$, we have shown (see equation (3.1.12)) that outside of $\left\{\tilde{r} \leqslant 2\right\}$,

$$\frac{1}{K} \leqslant |Q_c| \leqslant K$$

for a universal constant $K > 0$. This is why the norms $\|.\|_{\otimes,\sigma}$ and $\|.\|_{\otimes\otimes,\sigma}$ are separated in an estimate on $\left\{\tilde{r} \leqslant 3\right\}$ and another outside this domain, to allow the division by Q_c.

4.1.1 Symmetries of the travelling wave

We recall that for all $x=(x_1,x_2)\in\mathbb{R}^2$,

$$Q_c(x_1,x_2)=Q_c(-x_1,x_2)=\overline{Q_c(x_1,-x_2)}.$$

This implies that

$$\partial_c Q_c(x_1,x_2)=\partial_c Q_c(-x_1,x_2)=\overline{\partial_c Q_c(x_1,-x_2)},$$
$$\partial_{x_1} Q_c(x_1,x_2)=-\partial_{x_1} Q_c(-x_1,x_2)=\overline{\partial_{x_1} Q_c(x_1,-x_2)},$$

and

$$\partial_{x_2} Q_c(x_1,x_2)=\partial_{x_2} Q_c(-x_1,x_2)=-\overline{\partial_{x_2} Q_c(x_1,-x_2)}.$$

Remark that these three quantities all have different symmetries. We also check that

$$\partial_{c^\perp} Q_c(x_1,x_2)=-\partial_{c^\perp} Q_c(-x_1,x_2)=-\overline{\partial_{c^\perp} Q_c(x_1,-x_2)},$$

see Lemma 3.1.6. We will not need it, since functions even in x_1 satisfied the orthogonality on this direction.

4.1.2 Decay estimates for the travelling wave

In this subsection, we recall some decay in position satisfied by the travelling wave. First, from subsection 3.1.1.2, we recall that for all $0<\sigma<1$,

$$|1-|Q_c|^2|\leqslant\frac{K(\sigma)}{(1+\tilde{r})^{1+\sigma}},\tag{4.1.1}$$

$$|\mathfrak{Re}(\nabla Q_c\overline{Q_c})|\leqslant\frac{K(\sigma)}{(1+\tilde{r})^{1+\sigma}},\tag{4.1.2}$$

$$|\mathfrak{Im}(\nabla Q_c\overline{Q_c})|\leqslant\frac{K}{1+\tilde{r}}\tag{4.1.3}$$

and from equations (2.2.13) and (2.2.15), with the fact that $(\mathrm{TW}_c)(Q_c)=0$ (or see [6]),

$$\|Q_c\|_{C^2(\mathbb{R}^2,\mathbb{C})}\leqslant K.\tag{4.1.4}$$

We now give an estimate of Q_c using the norm $\|.\|_{\otimes\otimes,\sigma}$.

Lemma 4.1.1. *For all $0<\sigma<1$, there exists $c_0(\sigma),K(\sigma)>0$ such that, for all $0<c<c_0(\sigma)$,*

$$|Q_c-1|\leqslant\frac{K(\sigma)c^{-\sigma}}{(1+\tilde{r})^{\sigma}},$$

and

$$\left\|\frac{i\nabla Q_c}{Q_c}\right\|_{\otimes\otimes,\sigma}\leqslant K(\sigma)c^{-\sigma}.$$

Proof. From equation (4.1.4), we check that

$$\|Q_c\|_{C^2(\{\tilde{r}\leqslant 3\})}\leqslant\|Q_c\|_{C^2(\mathbb{R}^2,\mathbb{C})}\leqslant K.$$

Now, from section 2.2.2, outside of $\{\tilde{r}\leqslant 3\}$, we can write $Q_c=Ve^{\Psi_{c,d_c}}$ with $\|\Psi_{c,d_c}\|_{L^\infty(\mathbb{R}^2)}\leqslant 1$ for $c>0$ small enough, and $V=V_1(.-d_c\vec{e_1})V_{-1}(.+d_c\vec{e_1})$. Thus

$$\begin{aligned}|Q_c-1| &\leqslant |V-1|+|V||e^{\Psi_{c,d_c}}-1|\\ &\leqslant |V-1|+K|\Psi_{c,d_c}|.\end{aligned}$$

From equation (2.2.13), we have $|\Psi_{c,d_c}|\leqslant\frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^{\sigma}}$, and from Lemma 2.1.3, $|\nabla V|\leqslant\frac{Kc^{-1}}{(1+\tilde{r})^2}$, therefore, integrating from infinity (on axes where x_1 is constant), we check that

$$|V-1|\leqslant\frac{Kc^{-1}}{(1+\tilde{r})}.$$

Now, since $|V|\leqslant 1$ by Lemma 1.2.1, $|V-1|\leqslant 2$, and by interpolation,

$$|V-1|\leqslant \frac{Kc^{-\sigma}}{(1+\tilde r)^{\sigma}},$$

thus, for c small enough,

$$|Q_c-1|\leqslant \frac{K(\sigma)c^{-\sigma}}{(1+\tilde r)^{\sigma}}.$$

We recall that, with $\|.\|_{\otimes\otimes,\sigma}$ defined in (1.5.2),

$$\begin{aligned}
\left\|\frac{i\nabla Q_c}{Q_c}\right\|_{\otimes\otimes,\sigma} &= \|i\nabla Q_c\|_{C^1(\{\tilde r\leqslant 3\})}\\
&+ \left\|\tilde r^{1+\sigma}\mathfrak{Re}\left(\frac{i\nabla Q_c}{Q_c}\right)\right\|_{L^\infty(\{\tilde r\geqslant 2\})} + \left\|\tilde r^{2+\sigma}\nabla\mathfrak{Re}\left(\frac{i\nabla Q_c}{Q_c}\right)\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\\
&+ \left\|\tilde r^{2+\sigma}\mathfrak{Im}\left(\frac{i\nabla Q_c}{Q_c}\right)\right\|_{L^\infty(\{\tilde r\geqslant 2\})} + \left\|\tilde r^{2+\sigma}\nabla\mathfrak{Im}\left(\frac{i\nabla Q_c}{Q_c}\right)\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\\
&= \|\nabla Q_c\|_{C^1(\{\tilde r\leqslant 3\})}\\
&+ \left\|\tilde r^{1+\sigma}\mathfrak{Im}\left(\frac{\nabla Q_c}{Q_c}\right)\right\|_{L^\infty(\{\tilde r\geqslant 2\})} + \left\|\tilde r^{2+\sigma}\nabla\mathfrak{Im}\left(\frac{\nabla Q_c}{Q_c}\right)\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\\
&+ \left\|\tilde r^{2+\sigma}\mathfrak{Re}\left(\frac{\nabla Q_c}{Q_c}\right)\right\|_{L^\infty(\{\tilde r\geqslant 2\})} + \left\|\tilde r^{2+\sigma}\nabla\mathfrak{Re}\left(\frac{\nabla Q_c}{Q_c}\right)\right\|_{L^\infty(\{\tilde r\geqslant 2\})}.
\end{aligned}$$

We have $\|i\nabla Q_c\|_{C^1(\{\tilde r\leqslant 3\})}\leqslant \|Q_c\|_{C^2(\mathbb{R}^2,\mathbb{C})}\leqslant K$, and

$$\frac{\nabla Q_c}{Q_c}=\frac{\nabla Q_c\overline{Q_c}}{|Q_c|^2}.$$

Outside of $\{\tilde r\leqslant 3\}$, $Q_c=Ve^{\Psi_{c,d_c}}$, and thus

$$\frac{\nabla Q_c}{Q_c}=\frac{\nabla W\bar V+\nabla\Psi_{c,d_c}|V|^2}{|Q_c|^2}e^{2\mathfrak{Re}(\Psi_{c,d_c})}.$$

From equation (2.2.14), we have

$$|(1+\tilde r)^{2+\sigma}\mathfrak{Re}(\nabla\Psi_{c,d_c})|+|(1+\tilde r)^{1+\sigma}\mathfrak{Im}(\nabla\Psi_{c,d_c})|\leqslant K(\sigma)c^{1-\sigma},$$

and since, in $\{\tilde r\geqslant 3\}$,

$$\left|\frac{|V|^2}{|Q_c|^2}e^{2\mathfrak{Re}(\Psi_{c,d_c})}\right|\leqslant K,$$

we have

$$\left\|\tilde r^{1+\sigma}\mathfrak{Im}\left(\frac{\nabla\Psi_{c,d_c}|V|^2}{|Q_c|^2}e^{2\mathfrak{Re}(\Psi_{c,d_c})}\right)\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\leqslant K(\sigma)c^{1-\sigma}$$

and

$$\left\|\tilde r^{2+\sigma}\mathfrak{Re}\left(\frac{\nabla\Psi_{c,d_c}|V|^2}{|Q_c|^2}e^{2\mathfrak{Re}(\Psi_{c,d_c})}\right)\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\leqslant K(\sigma)c^{1-\sigma}.$$

Now, from Lemma 1.2.1, we have, for a vortex $V_{\pm 1}$ centered at 0, $|2\mathfrak{Re}(\nabla V_{\pm 1}\overline{V_{\pm 1}})|=|\nabla|V_{\pm 1}|^2|\leqslant \frac{K}{(1+r)^3}$, thus, by translation,

$$|\mathfrak{Re}(\nabla W\bar V)|\leqslant\frac{K}{(1+\tilde r)^3}.$$

Still from Lemma 1.2.1, we check that $|\nabla V|\leqslant\frac{K}{(1+\tilde r)}$, and from Lemma 2.1.3, we have $|\nabla V|\leqslant\frac{K}{(1+\tilde r)^2c}$, therefore, by interpolation,

$$|\mathfrak{Im}(\nabla W\bar V)|\leqslant K|\nabla V|\leqslant\frac{Kc^{-\sigma}}{(1+\tilde r)^{1+\sigma}}.$$

We deduce

$$\left\|\tilde r^{1+\sigma}\mathfrak{Im}\left(\frac{\nabla Q_c}{Q_c}\right)\right\|_{L^\infty(\{\tilde r\geqslant 2\})} + \left\|\tilde r^{2+\sigma}\mathfrak{Re}\left(\frac{\nabla Q_c}{Q_c}\right)\right\|_{L^\infty(\{\tilde r\geqslant 2\})}\leqslant K(\sigma)c^{-\sigma}.$$

We now focus on the derivatives of $\frac{\nabla Q_c}{Q_c}$. We compute

$$\begin{aligned}\nabla\left(\frac{\nabla Q_c}{Q_c}\right) &= \nabla\left(\frac{e^{2\mathfrak{Re}(\Psi_{c,d_c})}}{|Q_c|^2}\right)(\nabla W\bar{V}+\nabla\Psi_{c,d_c}|V|^2)\\ &+ \frac{\nabla(\nabla W\bar{V})+|V|^2\nabla^2\Psi_{c,d_c}}{|Q_c|^2}e^{2\mathfrak{Re}(\Psi_{c,d_c})}\\ &+ \frac{\nabla\Psi_{c,d_c}\nabla|V|^2}{|Q_c|^2}e^{2\mathfrak{Re}(\Psi_{c,d_c})}. \qquad (4.1.5)\end{aligned}$$

Remark that $\nabla\left(\frac{e^{2\mathfrak{Re}(\Psi_{c,d_c})}}{|Q_c|^2}\right)$ is real valued, and

$$\nabla\left(\frac{e^{2\mathfrak{Re}(\Psi_{c,d_c})}}{|Q_c|^2}\right)=\frac{2\nabla\mathfrak{Re}(\Psi_{c,d_c})e^{2\mathfrak{Re}(\Psi_{c,d_c})}}{|Q_c|^2}-\frac{\nabla(|Q_c|^2)}{|Q_c|^4}e^{2\mathfrak{Re}(\Psi_{c,d_c})},$$

and by equation (2.2.13), (4.1.2) and (4.1.3), we check that

$$\left|\nabla\left(\frac{e^{2\mathfrak{Re}(\Psi_{c,d_c})}}{|Q_c|^2}\right)\right|\leqslant\frac{K}{(1+\tilde{r})}.$$

This is enough to show the estimates for the first term of (4.1.5). For the second term, from equation (2.2.14) (with $\sigma'=\frac{1+\sigma}{2}>\sigma$),

$$|(1+\tilde{r})^{2+\sigma}\nabla^2\Psi_{c,d_c}|\leqslant K(\sigma)c^{1-\frac{1+\sigma}{2}}\leqslant K(\sigma)c^{-\sigma}.$$

Now, with Lemma 1.2.1,

$$|\nabla\mathfrak{Re}(\nabla W\bar{V})|=\left|\frac{1}{2}\nabla^2(|V|^2)\right|\leqslant\frac{K}{(1+\tilde{r})^3},$$

and with Lemmas 1.2.1 and 2.1.2, we check easily that $|\nabla(\nabla W\bar{V})|\leqslant\frac{K}{(1+\tilde{r})^2}$. To conclude the estimation of this term, we are left with the proof of

$$|\nabla(\nabla W\bar{V})|\leqslant\frac{Kc^{-1}}{(1+\tilde{r})^3}.$$

We compute

$$\begin{aligned}\nabla W\bar{V} &= (\nabla V_1\bar{V}_1)(.-d_c\vec{e}_1)|V_{-1}(.+d_c\vec{e}_1)|^2+(\nabla V_{-1}\overline{V_{-1}})(.+d_c\vec{e}_1)|V_1(.-d_c\vec{e}_1)|^2\\ &= (\nabla V_1\bar{V}_1)(.-d_c\vec{e}_1)+(\nabla V_{-1}\overline{V_{-1}})(.+d_c\vec{e}_1)\\ &+ (\nabla V_1\bar{V}_1)(.-d_c\vec{e}_1)(|V_{-1}(.+d_c\vec{e}_1)|^2-1)+(\nabla V_{-1}\overline{V_{-1}})(.+d_c\vec{e}_1)(|V_1(.-d_c\vec{e}_1)|^2-1).\end{aligned}$$

We check easily, with Lemma 1.2.1, that

$$|\nabla((\nabla V_1\bar{V}_1)(.-d_c\vec{e}_1)(|V_{-1}(.+d_c\vec{e}_1)|^2-1)+(\nabla V_{-1}\overline{V_{-1}})(.+d_c\vec{e}_1)(|V_1(.-d_c\vec{e}_1)|^2-1))|\leqslant\frac{K}{(1+\tilde{r})^3}.$$

Furthermore, with Lemma 1.2.1, we have that $\mathfrak{Im}(\nabla(\nabla V_1\bar{V}_1))(x)=-\mathfrak{Im}(\nabla(\nabla V_{-1}\overline{V_{-1}}))(x)$, therefore, with Theorem 1.3.1 (stating that $d_c\leqslant Kc^{-1}$),

$$\begin{aligned}&|\mathfrak{Im}(\nabla((\nabla V_1\bar{V}_1)(.-d_c\vec{e}_1)+(\nabla V_{-1}\overline{V_{-1}})(.+d_c\vec{e}_1)))|\\ =\ &|\mathfrak{Im}(\nabla(\nabla V_1V_1)(.-d_c\vec{e}_1)+\nabla(\nabla V_{-1}V_{-1}))(.+d_c\vec{e}_1)|\\ =\ &\left|\int_{-2d_c}^{0}\partial_d(\nabla(\nabla V_1V_1))(.-d_c\vec{e}_1)\right|\\ \leqslant\ &\frac{Kd_c}{(1+\tilde{r})^3}\leqslant\frac{Kc^{-1}}{(1+\tilde{r})^3}\end{aligned}$$

by Lemmas 2.1.2 and 2.1.6. Finally, for the third term, we recall that $|\nabla|V|^2|\leqslant\frac{K}{(1+\tilde{r})^3}$, which is more than enough to do the required estimates. □

We recall the definition of the energy space:

$$H_{Q_c}=\left\{\varphi\in H^1_{\mathrm{loc}}(\mathbb{R}^2),\ \int_{\mathbb{R}^2}|\nabla\varphi|^2+|1-|Q_c|^2||\varphi|^2+\mathfrak{Re}^2(\overline{Q_c}\varphi)<+\infty\right\},$$

Remark in particular that for all $0<\sigma<1$, with (4.1.1) to (4.1.3) and the definition of $\|.\|_{\otimes,\sigma}$, that

$$\mathcal{E}_{\otimes,\sigma}\subset H_{Q_c}.$$

4.2 Inversion of the linearized operator around Q_c

This section is devoted to the proof of Theorem 1.5.1.

4.2.1 Inversion of the linearized operator at infinity

The (additive) linearized operator around Q_c is defined by

$$L_{Q_c}(\varphi)=-\Delta\varphi-ic\partial_{x_2}\varphi-(1-|Q_c|^2)\varphi+2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c,$$

and, for $\varphi=Q_c\psi$, we define the multiplicative linearized operator

$$L'_{Q_c}(\psi):=-\Delta\psi-ic\partial_{x_2}\psi-2\frac{\nabla Q_c}{Q_c}.\nabla\psi+2\mathfrak{Re}(\psi)|Q_c|^2.$$

The first step of the proof of Theorem 1.5.1 is to invert these operators in suitable spaces.

Since $(\mathrm{TW}_c)(Q_c)=-\Delta Q_c-ic\partial_{x_2}Q_c-(1-|Q_c|^2)Q_c=0$, we can check that, on $\mathbb{R}^2\backslash\{\tilde{d}_c\vec{e}_1,-\tilde{d}_c\vec{e}_1\}$,with $\varphi=Q_c\psi$, $L_{Q_c}(\varphi)=Q_cL'_{Q_c}(\psi)$ (we recall that $\pm\tilde{d}_c\vec{e}_1$ are the zeros of Q_c, see Proposition 1.4.1). Formally, at infinity in position, the equation $L_{Q_c}(\varphi)=Q_cL'_{Q_c}(\psi)=Q_ch$ becomes

$$-ic\partial_{x_2}\psi-\Delta\psi+2\mathfrak{Re}(\psi)=h.$$

We have already inverted this operator, see Lemma 2.1.15 for the result. We will use this result extensively in this subsection. A consequence of this lemma is an estimate on ∂_cQ_c.

Corollary 4.2.1. *For $0<\sigma<1$, there exists $c_0(\sigma)>0$ such that, for all $0<c<c_0(\sigma)$,*

$$\left\|\frac{\partial_cQ_c}{Q_c}\right\|_{\otimes,\sigma}\leqslant K(\sigma)c^{-2}.$$

Proof. We recall from Lemma 3.1.7 that

$$L_{Q_c}(\partial_cQ_c)=i\partial_{x_2}Q_c,$$

and by elliptic estimates on this equation, as well as Lemmas 2.1.6 and 2.3.6, we check that

$$\|\partial_cQ_c\|_{C^3(\{\tilde{r}\leqslant 3\})}\leqslant Kc^{-2}. \tag{4.2.1}$$

Now, take χ a smooth cutoff function with value 1 outside of $\{\tilde{r}\leqslant 3\}$ and 0 in $\{\tilde{r}\leqslant 2\}$. We have

$$L_{Q_c}(\chi\partial_cQ_c)=Q_cL'_{Q_c}\left(\frac{\chi\partial_cQ_c}{Q_c}\right)=i\partial_{x_2}Q_c+E_\chi,$$

where E_χ are error term supported in $\{2\leqslant\tilde{r}\leqslant 3\}$, and since

$$L'_{Q_c}(\psi)=-ic\partial_{x_2}\psi-\Delta\psi-2\frac{\nabla Q_c}{Q_c}.\nabla\psi+2\mathfrak{Re}(\psi)|Q_c|^2,$$

we have

$$(-ic\partial_{x_2}-\Delta+2\mathfrak{Re})\left(\frac{\chi\partial_cQ_c}{Q_c}\right)=\frac{i\partial_{x_2}Q_c+E_\chi}{Q_c}+2\frac{\nabla Q_c}{Q_c}.\nabla\left(\frac{\chi\partial_cQ_c}{Q_c}\right)-2\mathfrak{Re}\left(\frac{\chi\partial_cQ_c}{Q_c}\right)(1-|Q_c|^2).$$

From Lemma 4.1.1, for $1>\sigma'>\sigma$,

$$\left\|\frac{i\partial_{x_2}Q_c}{Q_c}\right\|_{\otimes\otimes,\sigma'}\leqslant\frac{K(\sigma')}{c^{\sigma'}},$$

and with (4.2.1), we check easily that $\left\|\frac{E_\chi}{Q_c}\right\|_{\otimes\otimes,\sigma'}\leqslant\frac{K}{c^2}$. Finally, we check with Lemma 4.1.1, as well as equation (2.1.3), (2.2.14), Lemmas 2.1.6 and 2.3.6, that

$$\left\|2\frac{\nabla Q_c}{Q_c}.\nabla\left(\frac{\chi\partial_c Q_c}{Q_c}\right)-2\mathfrak{Re}\left(\frac{\chi\partial_c Q_c}{Q_c}\right)(1-|Q_c|^2)\right\|_{\otimes\otimes,\sigma'}\leqslant\frac{K}{c^2}.$$

We deduce, with Lemma 2.1.15 (taking $\sigma'=\frac{1+\sigma}{2}>\sigma$), that

$$\left\|\frac{\chi\partial_c Q_c}{Q_c}\right\|_{\otimes,\sigma}\leqslant\frac{K(\sigma)}{c^2}.$$

□

It will be useful to invert the equation $-ic\partial_{x_2}\psi-\Delta\psi+2\mathfrak{Re}(\psi)=h$ in the case $h\in\mathcal{E}^\infty_{\otimes\otimes,0}$. There, the function ψ will be in $\mathcal{E}^\infty_{\otimes,-\varepsilon}$ for all $\varepsilon>0$.

Lemma 4.2.2. *For $h\in\mathcal{E}^\infty_{\otimes\otimes,0}$ and $1>\varepsilon>0$, there exists a function $\psi\in\mathcal{E}^\infty_{\otimes,-\varepsilon}$, such that*

$$-ic\partial_{x_2}\psi-\Delta\psi+2\mathfrak{Re}(\psi)=h,$$

and this function satisfies

$$\|\psi\|_{\otimes,-\varepsilon,\infty}\leqslant K(\varepsilon)\|h\|_{\otimes\otimes,0,\infty}.$$

Furthermore, all solutions of this problem in $\mathcal{E}^\infty_{\otimes,-\varepsilon}$ differ by an element of $\mathrm{Span}_{\mathbb{R}}(i)$.

In particular, remark that such a solution does not necessarily go to 0 at infinity on its imaginary part, but it does on its real part and for its derivatives. We believe that we could show that

$$\|(1+\ln(1+\tilde{r}))\psi_2\|_{L^\infty(\mathbb{R}^2)}\leqslant K(c)\|h\|_{\otimes\otimes,0,\infty},$$

but it is not necessary for the computations to come. Remark also that, for $0<\varepsilon'<\varepsilon$, $\mathcal{E}_{\otimes,-\varepsilon'}\subset\mathcal{E}_{\otimes,-\varepsilon}$, and thus the function ψ does not depend on $\varepsilon>0$. Also, we do not require that $\int_{\mathbb{R}^2}\mathfrak{Im}(h)=0$ here. This proof is similar to the proof of Lemma 2.1.15, with some slightly different technical points.

Proof. For $j\in\{1,2\}$, we define the function

$$\Psi_{1,j}:=K_0*\partial_{x_j}h_1+cK_j*h_2.$$

From Lemma 2.1.13, we have (for $\alpha'=2-\varepsilon<2$ and $\alpha=\frac{2+\alpha'}{2}$) that $\Psi_{1,j}\in C^1(\mathbb{R}^2,\mathbb{C})$, with

$$|\Psi_{1,j}|+|\nabla\Psi_{1,j}|\leqslant\frac{K(\varepsilon)\|h\|_{\otimes\otimes,0,\infty}}{(1+\tilde{r})^{2-\varepsilon}}.$$

As in the proof of Lemma 2.1.15, we define, if $x_2\geqslant 0$,

$$\psi_1(x_1,x_2)=\int_{+\infty}^{x_2}\Psi_{1,2}(x_1,y_2)dy_2,$$

and if $x_2<0$,

$$\psi_1(x_1,x_2)=\int_{-\infty}^{x_2}\Psi_{1,2}(x_1,y_2)dy_2.$$

We have $\partial_{x_1}\Psi_{1,2}=\partial_{x_2}\Psi_{2,1}$ and thus $\int_{-\infty}^{+\infty}\Psi_{1,2}(x_1,y_2)dy_2=0$ as previously. We then check similarly that

$$\|(1+\tilde{r})^{1-\varepsilon}\psi_1\|_{L^\infty(\mathbb{R}^2)}+\|(1+\tilde{r})^{2-\varepsilon}\nabla\psi_1\|_{L^\infty(\mathbb{R}^2)}+\|(1+\tilde{r})^{2-\varepsilon}\nabla^2\psi_1\|_{L^\infty(\mathbb{R}^2)}\leqslant K(\varepsilon)\|h\|_{\otimes\otimes,0,\infty}.$$

Now, we define

$$\Psi_{2,j,k}:=(c^2L_{j,k}-R_{j,k})*h_2-cK_j*\partial_{x_k}h_1,$$

and from Lemma 2.1.14, we have, for $\alpha' = 2-\varepsilon < 2$, $\alpha = \frac{2+\alpha'}{2}$, $\Psi_{2,j,k} \in C^0(\mathbb{R}^2, \mathbb{C})$, with

$$|\Psi_{2,j,k}| \leqslant \frac{K(\varepsilon)\|h\|_{\otimes\otimes,0,\infty}}{(1+\tilde{r})^{2-\varepsilon}}.$$

For $0 < \varepsilon < 1$, the decay is still enough to construct $\Psi_{2,j} \in C^1(\mathbb{R}^2, \mathbb{C})$ as in the proof of Lemma 2.1.15. We now diverge from the proof of Lemma 2.1.15, and we define ψ_2 using $\psi_2(x_1, x_2) = \psi_2(-x_1, x_2)$, by $\psi_2(\pm d_c \vec{e_1}) = 0$, and

$$\nabla \psi_2 = \begin{pmatrix} \Psi_{2,1} \\ \Psi_{2,2} \end{pmatrix}.$$

We then check that $\psi_2 \in C^0(\mathbb{R}^2, \mathbb{C})$, and by integration from infinity for $\nabla\psi_2$ and integration from $\pm d_c\vec{e_1}$ for ψ_2, that

$$\|(1+\tilde{r})^{-\varepsilon}\psi_2\|_{L^\infty(\mathbb{R}^2)} + \|(1+\tilde{r})^{1-\varepsilon}\nabla\psi_2\|_{L^\infty(\mathbb{R}^2)} + \|(1+\tilde{r})^{2-\varepsilon}\nabla^2\psi_2\|_{L^\infty(\mathbb{R}^2)} \leqslant K(\varepsilon)\|h\|_{\otimes\otimes,0,\infty}.$$

Finally, as in the proof of Lemma 2.1.15, we check that

$$-ic\partial_{x_2}\psi - \Delta\psi + 2\mathfrak{Re}(\psi) = h,$$

since both sides of the equations are still temperated distribution, are bounded and goes to 0 at infinity in position. Furthermore, if $\tilde{\psi} \in \mathcal{E}^\infty_{\otimes,-\varepsilon}$ is another solution of this problem, then

$$(-ic\partial_{x_2} - \Delta + 2\mathfrak{Re})(\psi - \tilde{\psi}) = 0,$$

thus $\psi - \tilde{\psi} \in \mathbb{C}[X_1, X_2]$, and using the decays of $\psi - \tilde{\psi} \in \mathcal{E}^\infty_{\otimes,-\varepsilon}$, we check that (with $\varepsilon < 1$)

$$|\mathfrak{Re}(\psi - \tilde{\psi})| + \frac{|\mathfrak{Im}(\psi - \tilde{\psi})|}{(1+\tilde{r})^\varepsilon} \to 0$$

at infinity, hence it is $i\lambda$ for some $\lambda \in \mathbb{R}$. □

4.2.2 Inversion of the linearized operator around Q_c

We recall

$$L_{Q_c}(\varphi) = -\Delta\varphi - ic\partial_{x_2}\varphi - (1-|Q_c|^2)\varphi + 2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c$$

and, for $\varphi = Q_c\psi$,

$$L'_{Q_c}(\psi) = -ic\partial_{x_2}\psi - \Delta\psi - 2\frac{\nabla Q_c}{Q_c}.\nabla\psi + 2\,\mathfrak{Re}(\psi)|Q_c|^2.$$

We also recall that, since $(\mathrm{TW}_c)(Q_c) = 0$, we have $L_{Q_c}(\varphi) = Q_c L'_{Q_c}(\psi)$ (where $Q_c \neq 0$).

4.2.2.1 Inversion of the linearized operator around a vortex

This subsection uses mainly arguments from [10]. We recall the linearized operator around a vortex:

$$L_{V_1}(\varphi) = -\Delta\varphi - (1-|V_1|^2)\varphi + 2\mathfrak{Re}(\bar{V}_1\varphi)V_1,$$

and with $\varphi = V_1\psi$,

$$L'_{V_1}(\psi) := -\Delta\psi - 2\frac{\nabla V_1}{V_1}.\nabla\psi + 2\mathfrak{Re}(\psi)|V_1|^2,$$

we have (where $V_1 \neq 0$)

$$L_{V_1}(\varphi) = V_1 L'_{V_1}(\psi).$$

This operator also has a resonance: $L_{V_1}(iV_1) = 0$. We give here a way to invert L_{V_1} on this direction. For $R > 0$ a large constant, we define $\chi \in C^\infty_c(\mathbb{R}\backslash[0,R], \mathbb{R}^+)$ such that (with $\rho(r) = |V_1(x)|$ where $|x| = r$)

$$\int_0^{+\infty} s\rho^2(s)\chi(s)ds = 1. \tag{4.2.2}$$

We recall

$$H_{V_1} = \left\{ \varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{C}), \|\varphi\|^2_{H_{V_1}} = \int_{\mathbb{R}^2} |\nabla\varphi|^2 + (1-|V_1|^2)|\varphi|^2 + \mathfrak{Re}^2(\bar{V}_1\varphi) < +\infty \right\}.$$

Lemma 4.2.3. *The problem*

$$L_{V_1}(\varphi) = i\chi(|.|)V_1$$

in the distribution sense admits the solution

$$\varphi_1 := iV_1\psi_1(|.|)$$

with

$$\psi_1(r) := -\int_0^r \left(\frac{1}{u\rho^2(u)}\int_0^u s\rho^2(s)\chi(s)ds\right)du,$$

which satisfies $\varphi_1 \in C^\infty(\mathbb{R}^2, \mathbb{C})$ *and*

$$\psi_1(r) \sim -\ln(r),\ \psi_1'(r) \sim \frac{-1}{r},\ \psi_1''(r) \sim \frac{1}{r^2}$$

when $r \to \infty$.

Remark that $\varphi_1 = 0$ in $B(0,R)$ since $\chi = 0$ in $B(0,R)$. See [10] for more general results on the inversion of the linearized operator around $V_{\pm 1}$.

Proof. We look for an ansatz of the form $\varphi_1 = iV_1\psi_1(r)$ with $\psi_1 \in C^\infty(\mathbb{R}^{+*}, \mathbb{R})$. The equation $L_{V_1}(\varphi_1) = i\chi(|.|)V_1$ then becomes

$$-\Delta_r\psi_1 - 2\frac{\nabla V_1}{V_1}.\vec{e_r}\psi_1'(r) = \chi(r).$$

From Lemma 2.1.2, we have $\frac{\nabla V_1}{V_1}.\vec{e_r} = \frac{\rho'(r)}{\rho(r)}$, and therefore

$$-\psi_1''(r) - \left(\frac{1}{r} + \frac{2\rho'(r)}{\rho(r)}\right)\psi_1'(r) = \chi(r).$$

This equation can be factorized in

$$(r\rho^2(r)\psi_1'(r))' = -r\rho^2(r)\chi(r),$$

and therefore a solution is

$$\psi_1(r) = -\int_0^r \left(\frac{1}{u\rho^2(u)}\int_0^u s\rho^2(s)\chi(s)ds\right)du.$$

Furthermore, since

$$\int_0^{+\infty} s\rho^2(s)\chi(s)ds = 1,$$

we have, by Lemma 1.2.1,

$$\psi_1(r) \sim -\int_1^r \frac{du}{u\rho^2(u)} \sim -\int_1^r \frac{du}{u} \sim -\ln(r)$$

when $r \to \infty$. Now, we compute

$$\psi_1'(r) = \frac{-1}{r\rho^2(r)}\int_0^r s\rho^2(s)\chi(s)ds$$

and

$$\psi_1''(r) = \frac{1}{r^2\,\rho^2(r)}\int_0^r s\rho^2(s)\chi(s)ds - \frac{1}{r}\partial_r\left(\frac{1}{\rho^2(r)}\int_0^r s\rho^2(s)\chi(s)ds\right),$$

and with Lemma 1.2.1, we infer the equivalents of $\psi_1'(r)$ and $\psi_1''(r)$ when $r \to \infty$. □

We deduce the following small improvement of Theorem 1.2 of [10], since we removed an orthogonality condition. It is also a good first step to understand some of the ideas of the proof of Theorem 1.5.1.

Lemma 4.2.4. *For* $h \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{C})$ *such that* $\int_{\mathbb{R}^2} |h|^2(1+r)^{2+\sigma} < +\infty$ *for some* $\sigma > 0$, *and with* $\langle h, \partial_{x_1}V_1\rangle = \langle h, \partial_{x_2}V_1\rangle = 0$, *there exists* $\varphi \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{C})$ *such that*

$$L_{V_1}(\varphi) = h.$$

Furthermore, $\varphi = \varphi_0 + \frac{\langle h, iV_1\rangle}{2\pi}\varphi_1$*, where* φ_1 *is defined in Lemma 4.2.3, and* $\varphi_0 \in H_{V_1}$*, with*

$$\|\varphi_0\|^2_{H_{V_1}} \leqslant K(\chi)\int_{\mathbb{R}^2} |h - \langle h, iV_1\rangle i\chi(|.|)V_1|^2(1+r)^{2+\sigma}.$$

Proof. we consider $h' := h - i\chi(|.|)V_1\frac{\langle h, iV_1\rangle}{2\pi}$. We have $\int_{\mathbb{R}^2}|h'|^2(1+r)^{2+\sigma} < +\infty$,

$$\langle h', \partial_{x_1}V_1\rangle = \langle h, \partial_{x_1}V_1\rangle - \frac{\langle h, iV_1\rangle}{2\pi}\langle i\chi(|.|)V_1, \partial_{x_1}V_1\rangle = 0$$

since, from Lemma 2.1.2,

$$\langle i\chi(|.|)V_1, \partial_{x_1}V_1\rangle = -\int_0^{+\infty}\left(\int_0^{2\pi}\sin(\theta)d\theta\right)\chi(r)\rho^2(r)\,dr = 0.$$

Similarly, $\langle h', \partial_{x_2}V_1\rangle = 0$, and

$$\langle h', iV_1\rangle = \langle h, iV_1\rangle - \frac{\langle h, iV_1\rangle}{2\pi}\langle i\chi(|.|)V_1, iV_1\rangle = 0,$$

since

$$\langle i\chi(|.|)V_1, iV_1\rangle = 2\pi\int_0^{+\infty}\chi(r)\rho^2(r)r\,dr = 2\pi$$

by (4.2.2). From Theorem 1.2 of [10], we deduce that there exists $\varphi_0 \in H_{V_1}$ such that

$$L_{V_1}(\varphi_0) = h' = h - i\chi(|.|)V_1\frac{\langle h, iV_1\rangle}{2\pi}.$$

Now, from Lemma 4.2.3, since $L_{V_1}\left(\frac{\langle h, iV_1\rangle}{2\pi}\varphi_1\right) = \frac{\langle h, iV_1\rangle}{2\pi}i\chi(|.|)V_1$, we have

$$L_{V_1}\left(\varphi_0 + \frac{\langle h, iV_1\rangle}{2\pi}\varphi_1\right) = h. \qquad \square$$

We also infer the following result, that will be useful in the proof of Lemma 4.2.6.

Lemma 4.2.5. *The problem*

$$L_{V_1}(\varphi) = i\chi(|.|)V_1$$

has no solution in H_{V_1}*.*

Proof. By standard elliptic estimates, we have that if such a function $\varphi \in H_{V_1}$ exists, then $\varphi \in C^\infty(\mathbb{R}^2, \mathbb{C})$. Following the proof of Lemma 4.2.3, writing $\varphi = V_1\psi$ and decomposing ψ in harmonics, we check that $(r\rho^2(r)\psi_1'(r))' = -r\rho^2(r)\chi(r)$, with $\psi_1 \in C^\infty(\mathbb{R}^{+*}, \mathbb{R})$ being the 0 harmonic of ψ.

We deduce that $r\rho^2(r)\psi_1'(r) = -\int_0^r u\rho^2(u)\chi(u)du + K_1$ for some $K_1 \in \mathbb{R}$, and since $\varphi \in H_{V_1}$, $K_1 = 0$, or else $\psi_1'(r) \sim \frac{K}{\alpha^2 r^3}$ near $r = 0$. Therefore,

$$\psi_1(r) = -\int_0^r\left(\frac{1}{u\rho^2(u)}\int_0^u s\rho^2(s)\chi(s)ds\right)du + K_2$$

for some $K_2 \in \mathbb{R}$. By Lemma 4.2.3, this implies that $\psi_1(r) \sim -\ln(r)$ when $r \to \infty$, which leads to the contradiction

$$+\infty > \|\varphi\|^2_{H_{V_1}} \geqslant \int_{\mathbb{R}^2}(1 - |V_1|^2)|\varphi|^2 \geqslant K\int_{\{r>1\}}\frac{\ln^2(r)}{(1+r)^2} = +\infty. \qquad \square$$

4.2.2.2 Inversion of the linearized operator around Q_c

We recall that

$$\tilde{r} = \min\left(|x - d_c\vec{e_1}|, |x + d_c\vec{e_1}|\right)$$

is the minimum of the distances to $d_c\vec{e}_1$ and $-d_c\vec{e}_1$, and that we have defined the norms, for $\sigma\in\mathbb{R}$, $\varphi=Q_c\psi\in C^2(\mathbb{R}^2,\mathbb{C})$, $\psi=\psi_1+i\psi_2$, and $Q_ch\in C^1(\mathbb{R}^2,\mathbb{C})$, $h=h_1+ih_2$,

$$\begin{aligned}\|\psi\|_{\otimes,\sigma} &= \|Q_c\,\psi\|_{C^2(\{\tilde r\leqslant 3\})}\\ &+ \|\tilde r^{1+\sigma}\,\psi_1\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\nabla\psi_1\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\nabla^2\psi_1\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ &+ \|\tilde r^{\sigma}\,\psi_2\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{1+\sigma}\nabla\psi_2\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\nabla^2\psi_2\|_{L^\infty(\{\tilde r\geqslant 2\})}\end{aligned}$$

and

$$\begin{aligned}\|h\|_{\otimes\otimes,\sigma} &= \|Q_ch\|_{C^1(\{\tilde r\leqslant 3\})}\\ &+ \|\tilde r^{1+\sigma}\,h_1\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\,\nabla h_1\|_{L^\infty(\{\tilde r\geqslant 2\})}\\ &+ \|\tilde r^{2+\sigma}\,h_2\|_{L^\infty(\{\tilde r\geqslant 2\})}+\|\tilde r^{2+\sigma}\,\nabla h_2\|_{L^\infty(\{\tilde r\geqslant 2\})}.\end{aligned}$$

We want to invert the linearized operator around Q_c from $\mathcal{E}_{\otimes\otimes,\sigma}$ to $\mathcal{E}_{\otimes,-\varepsilon}$ for $1>\sigma>0$, $\varepsilon>0$.

These spaces are close to $\mathcal{E}_{**,\sigma}$ and $\mathcal{E}_{*,-\varepsilon}$ in Chapter 2. In fact, in Proposition 2.1.20, we inverted the linearized operator around $V=V_1V_{-1}$ in the $*$-spaces, and here we want to invert it around $Q_c=V+o_{c\to 0}(1)$ for the $\otimes$-spaces. Furthermore, in Chapter 2, we supposed two symmetries in the space $\mathcal{E}_{*,\sigma}$, and here we only have one. Therefore, we will need to add an orthogonality condition on $\partial_{x_2}Q_c$, but we will also have to deal with the phase. For that, we define

$$\Upsilon:=(\chi(|x-d_c|)+\chi(|x+d_c|))Q_c,$$

where χ is the cutoff function from Lemma 4.2.3. As we have done for one vortex in Lemma 4.2.3, we will look for a solution of $L_{Q_c}(\varphi)=i\Upsilon$ to deal with the phase. This solution will also grow at infinity.

We define η a cutoff function, whith $\eta(x)=\tilde\eta(r_1)+\tilde\eta(r_{-1})$ and $\tilde\eta$ is a C^∞ positive cutoff with $\tilde\eta(r)=0$ if $r\leqslant R+1$ and 1 if $r\geqslant R+2$ (R is considered as a universal constant). We then define, for $\varphi=Q_c\psi\in\mathcal{E}_{\otimes,\sigma}$, $Q_ch\in\mathcal{E}_{\otimes\otimes,\sigma'}$, $1>\sigma'>\sigma>0$,

$$\mu(\psi,h):=\frac{\int_{\mathbb{R}^2}\mathfrak{Im}\Big(h\eta-ic\partial_{x_2}\eta\psi-\Delta\eta\psi-2\nabla\eta.\nabla\psi+2\eta\frac{\nabla Q_c}{Q_c}.\nabla\psi\Big)}{\int_{\mathbb{R}^2}\mathfrak{Im}\Big(\frac{i\Upsilon\eta}{Q_c}\Big)}.$$

Let us show that

$$|\mu(\psi,h)|\leqslant K(\sigma,\sigma')(\|\psi\|_{\otimes,\sigma}+\|h\|_{\otimes\otimes,\sigma'}). \tag{4.2.3}$$

First, since $\eta=1$ on $\mathbb{R}^2\backslash(B(d_c\vec{e_1},\,R+2)\cup B(-d_c\vec{e_1},\,R+2))$, $\Delta\eta$, $\nabla\eta$ and $\partial_{x_2}\eta$ are compactly supported in $B(d_c\vec{e}_1,R+2)\cup B(-d_c\vec{e}_1,R+2)$. We deduce that

$$\int_{\mathbb{R}^2}|\mathfrak{Im}(-ic\partial_{x_2}\eta\psi-\Delta\eta\psi-2\nabla\eta.\nabla\psi)|\leqslant K(\sigma)\|\psi\|_{\otimes,\sigma}.$$

With regards to the definition of $\|h\|_{\otimes\otimes,\sigma'}$, we check easily that

$$\int_{\mathbb{R}^2}|\eta\mathfrak{Im}(h)|\leqslant K\int_{\mathbb{R}^2}\frac{\|h\|_{\otimes\otimes,\sigma'}}{(1+\tilde r)^{2+\sigma'}}\leqslant K(\sigma')\|h\|_{\otimes\otimes,\sigma'}.$$

Now, with (4.1.2) and (4.1.3), we check easily that

$$\int_{\mathbb{R}^2}\left|\mathfrak{Im}\Big(2\eta\frac{\nabla Q_c}{Q_c}.\nabla\psi\Big)\right|\leqslant\int_{\mathbb{R}^2}\frac{K\|\psi\|_{\otimes,\sigma}}{(1+\tilde r)^{2+\sigma}}\leqslant K(\sigma)\|\psi\|_{\otimes,\sigma}.$$

Finally,

$$\int_{\mathbb{R}^2}\mathfrak{Im}\Big(\frac{i\Upsilon\eta}{Q_c}\Big)=\int_{\mathbb{R}^2}\mathfrak{Im}(i(\chi(x-d_c)+\chi(x+d_c))\eta)=\int_{\mathbb{R}^2}\eta(\chi(x-d_c)+\chi(x+d_c)),$$

and since $\operatorname{Supp}\chi(|\,.\,|)\subset\{\eta=1\}$, $\int_{\mathbb{R}^2}\eta(\chi(x-d_c)+\chi(x+d_c))=\int_{\mathbb{R}^2}(\chi(x-d_c)+\chi(x+d_c))>K>0$. This concludes the proof of (4.2.3).

The proof of the inversion will be done in Lemmas 4.2.6 to 4.2.8. They follow closely the proofs of Proposition 2.1.17, Lemma 2.1.19 and Proposition 2.1.20. To show the existence of a solution, we start with an a priori estimate, then we solve the equation on a large bounded domain (to have compactness), then we extend it to the whole space. The next lemma is the a priori estimate. We will use the notation $\psi^{\neq 0}$, defined in (1.5.3).

Lemma 4.2.6. *For $0<\sigma'<\sigma<1$, there exists $K(\sigma,\sigma')$, $c_0(\sigma,\sigma')$, $R>0$ such that, if for $0<c<c_0(\sigma,\sigma')$ and some $Q_c h\in\mathcal{E}_{\otimes\otimes,\sigma'}$, $\varphi=Q_c\psi\in\mathcal{E}_{\otimes,\sigma}$,*

$$L_{Q_c}(\varphi)=Q_c h-\mu(\psi,h)i\Upsilon$$

and

$$\mathfrak{Re}\int_{B(d_c\vec{e_1},R)\cup B(-d_c\vec{e_1},R)}\partial_c Q_c\bar{\varphi}=\mathfrak{Re}\int_{B(d_c\vec{e_1},R)\cup B(-d_c\vec{e_1},R)}\partial_{x_2}Q_c\overline{Q_c\psi^{\neq 0}}=0,$$

then

$$\|\psi\|_{\otimes,\sigma}\leqslant K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma'}.$$

Proof. This proof follows closely the proof of Proposition 2.1.17. We argue by contradiction. Suppose that it does not hold. Then there exists $c_n\to 0$, $\varphi_n=Q_{c_n}\psi_n\in\mathcal{E}_{\otimes,\sigma}$, $Q_{c_n}h_n\in\mathcal{E}_{\otimes\otimes,\sigma'}$ such that $L_{Q_{c_n}}(\varphi_n)=Q_{c_n}h_n-\mu(\psi_n,h_n)i\Upsilon$ (remark that Υ and η depend on n through d_{c_n}),

$$\mathfrak{Re}\int_{B(d_{c_n}\vec{e_1},R)\cup B(-d_{c_n}\vec{e_1},R)}\partial_c Q_{c_n}\overline{\varphi_n}=\mathfrak{Re}\int_{B(d_{c_n}\vec{e_1},R)\cup B(-d_{c_n}\vec{e_1},R)}\partial_{x_2}Q_{c_n}\overline{\varphi_n}=0$$

and

$$\|\psi_n\|_{\otimes,\sigma}=1,\|h_n\|_{\otimes\otimes,\sigma'}\to 0.$$

with (4.2.3), we have $|\mu(\psi_n,h_n)|\leqslant K(\sigma,\sigma')$, thus $\mu(\psi_n,h_n)\to\mu\in\mathbb{R}$ up to a subsequence.

We argue as in step 1 of the proof of Proposition 2.1.17. The functions $\varphi_n(.-d_{c_n}\vec{e_1})$ are equicontiuous and bounded, as we check with the same arguments as in Chapter 2, that, up to a subsequence, it converges to some limit $\Phi\in H_{V_1}$.

We check similarly that $Q_{c_n}(.-d_{c_n}\vec{e_1})\to V_1$ and $\Upsilon(.-d_{c_n}\vec{e_1})\to\chi V_1$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ by Theorem 1.3.1 (for $p=+\infty$), and therefore Φ satisfies the equation

$$L_{V_1}(\Phi)=\mu i\chi V_1.$$

By Lemma 4.2.5, this implies that $\mu=0$ (since $\Phi\in H_{V_1}$). Furthermore, we have

$$\mathfrak{Re}\int_{B(d_{c_n}\vec{e_1},R)\cup B(-d_{c_n}\vec{e_1},R)}\partial_c Q_{c_n}\overline{\varphi_n}=0,$$

and since $\varphi_n(-x_1,x_2)=\varphi_n(x_1,x_2)$, we deduce that

$$\mathfrak{Re}\int_{B(d_{c_n}\vec{e_1},R)}\partial_c Q_{c_n}\overline{\varphi_n}=0.$$

When $n\to\infty$, from Theorem 1.3.1 (for $p=+\infty$), we have $c_n^2\partial_c Q_{c_n}(.-d_{c_n}\vec{e_1})\to\partial_{x_1}V_1$, thus

$$\mathfrak{Re}\int_{B(0,R)}\partial_{x_1}V_1\bar{\Phi}=0.$$

Similarly, since $\partial_{x_2}Q_{c_n}(.-d_{c_n}\vec{e_1})\to\partial_{x_2}V_1$, and decomposing V_1 in harmonics, we check

$$\mathfrak{Re}\int_{B(0,R)}\partial_{x_2}V_1\overline{V_1\left(\frac{\Phi}{V_1}\right)^{\neq 0}}=\mathfrak{Re}\int_{B(0,R)}\partial_{x_2}V_1\bar{\Phi}=0.$$

Since $\Phi\in H_{V_1}$ and $L_{V_1}(\Phi)=0$, by Theorem 1.1 of [10], we have $\Phi=\varepsilon_1\partial_{x_1}V_1+\varepsilon_2\partial_{x_2}V_1$ for $\varepsilon_1,\varepsilon_2\in\mathbb{R}$. With the two previous orthogonality conditions on Φ, we deduce that $\varepsilon_1=\varepsilon_2=0$, and thus $\Phi=0$. By symmetry, the same result holds if we shift by $+d_{c_n}\vec{e_1}$ instead of $-d_{c_n}\vec{e_1}$.

Now, since

$$\mu(\psi_n,h_n)\to 0$$

when $n\to\infty$ (since $\mu=0$), we check, as in the proof of Proposition 2.1.17, that this implies

$$\|\varphi_n\|_{C^2(|\vec{r}\leqslant\lambda|)}=o^{\lambda}_{n\to\infty}(1)$$

for any $\lambda>R$.

We now define $\tilde{\psi}_n := \eta\psi_n$. Since $L_{Q_{c_n}}(\varphi_n) = Q_{c_n}h_n - \mu(\psi_n, h_n)i\Upsilon$, multiplying this equation by η, we have

$$\eta L'(\psi_n) = h_n\eta - \mu(\psi_n, h_n)\frac{i\Upsilon\eta}{Q_c}.$$

Now, we compute

$$\begin{aligned}\eta L'(\psi_n) &= \eta\Big(-ic\partial_{x_2}\psi_n - \Delta\psi_n - 2\frac{\nabla Q_{c_n}}{Q_{c_n}}.\nabla\psi_n + 2\mathfrak{Re}(\psi_n)|Q_{c_n}|^2\Big)\\ &= -ic\partial_{x_2}\tilde{\psi}_n - \Delta\tilde{\psi}_n + 2\mathfrak{Re}\big(\tilde{\psi}_n\big)\\ &+ ic\partial_{x_2}\eta\psi_n + \Delta\eta\psi_n + 2\nabla\eta.\nabla\psi_n + 2\eta\mathfrak{Re}(\psi_n)(|Q_{c_n}|^2 - 1)\\ &- 2\eta\frac{\nabla Q_{c_n}}{Q_{c_n}}.\nabla\psi_n.\end{aligned}$$

We deduce that

$$\begin{aligned}-ic\partial_{x_2}\tilde{\psi}_n - \Delta\tilde{\psi}_n + 2\mathfrak{Re}\big(\tilde{\psi}_n\big) &= h_n\eta - \mu(\psi_n, h_n)\frac{i\Upsilon\eta}{Q_c}\\ &- ic\partial_{x_2}\eta\psi_n - \Delta\eta\psi_n - 2\nabla\eta.\nabla\psi_n - 2\eta\mathfrak{Re}(\psi_n)(|Q_{c_n}|^2 - 1)\\ &+ 2\eta\frac{\nabla Q_{c_n}}{Q_{c_n}}.\nabla\psi_n.\end{aligned}$$

We denote

$$\tilde{h}_n = h_n\eta - ic\partial_{x_2}\eta\psi_n - \Delta\eta\psi_n - 2\nabla\eta.\nabla\psi_n - 2\eta\mathfrak{Re}(\psi_n)(|Q_{c_n}|^2 - 1) + 2\eta\frac{\nabla Q_{c_n}}{Q_{c_n}}.\nabla\psi_n - \mu(\psi_n, h_n)\frac{i\Upsilon\eta}{Q_{c_n}}.$$

We check, as in the proof of Proposition 2.1.17, that

$$\big\|\tilde{h}_n\big\|_{\otimes\otimes,\sigma',\infty} = o_{\lambda\to\infty}(1) + o^{\lambda}_{n\to\infty}(1).$$

The only additional term we have to check is $\mu(\psi_n, h_n)\frac{i\Upsilon\eta}{Q_c}$, and since $\mu(\psi_n, h_n) = o^{\lambda}_{n\to\infty}(1)$, and $\frac{i\Upsilon\eta}{Q_{c_n}}$ is compactly supported in $B(d_{c_n}\vec{e}_1, R+2)\cup B(-d_{c_n}\vec{e}_1, R+2)$, we check easily that

$$\Big\|\mu(\psi_n, h_n)\frac{i\Upsilon\eta}{Q_{c_n}}\Big\|_{\otimes\otimes,\sigma',\infty} = o^{\lambda}_{n\to\infty}(1).$$

Furthermore, we have that

$$\int_{\mathbb{R}^2}\mathfrak{Im}\big(\tilde{h}_n\big) = 0,$$

since

$$\mu(\psi_n, h_n) = \frac{\int_{\mathbb{R}^2}\mathfrak{Im}\Big(h_n\eta - ic\partial_{x_2}\eta\psi_n - \Delta\eta\psi_n - 2\nabla\eta.\nabla\psi_n + 2\eta\frac{\nabla Q_{c_n}}{Q_{c_n}}.\nabla\psi_n\Big)}{\int_{\mathbb{R}^2}\mathfrak{Im}\Big(\frac{i\Upsilon\eta}{Q_{c_n}}\Big)}.$$

Now, from Lemma 2.1.15, since $-ic\partial_{x_2}\tilde{\psi}_n - \Delta\tilde{\psi}_n + 2\mathfrak{Re}\big(\tilde{\psi}_n\big) = \tilde{h}_n$, $\int_{\mathbb{R}^2}\mathfrak{Im}\big(\tilde{h}_n\big) = 0$, $\tilde{h}_n \in \mathcal{E}^{\infty}_{\otimes\otimes,\sigma}$ and $\tilde{\psi}_n \in \mathcal{E}^{\infty}_{\otimes,\sigma}$, we deduce

$$\big\|\tilde{\psi}_n\big\|_{\otimes,\sigma,\infty} \leqslant K(\sigma,\sigma')\big\|\tilde{h}_n\big\|_{\otimes\otimes,\sigma',\infty} \leqslant o^{\lambda}_{n\to\infty}(1) + o_{\lambda\to\infty}(1).$$

It implies, with $\|\varphi_n\|_{C^2(\{\tilde{r}\leqslant\lambda\})} = o^{\lambda}_{n\to\infty}(1)$ and (2.1.18), that

$$\|\psi_n\|_{\otimes,\sigma} \leqslant K\big(\|\varphi_n\|_{C^2(\{\tilde{r}\leqslant\lambda\})} + \big\|\tilde{\psi}_n\big\|_{\otimes,\sigma,\infty}\big) = o^{\lambda}_{n\to\infty}(1) + o_{\lambda\to\infty}(1),$$

hence, taking λ large enough and then n large enough, this is in contradiction with $\|\psi_n\|_{\otimes,\sigma} = 1$. □

We continue as in Chapter 2. We want to show existence of a solution by constructing one on a large ball $B(0,a)$ by Fredholm alternative, then pass at the limit $a\to\infty$ to have a solution in $\mathbb{R}^2$.

We define, for $a > 10/c^2$, $R > 0$ a large constant and $\varphi = Q_c\psi \in H^1_{\mathrm{loc}}(B(0,a))$ the norm

$$\|\varphi\|^2_{H_a} := \|\varphi\|^2_{H^1(\{\tilde{r}\leqslant 2R\})} + \int_{\mathbb{R}^2\setminus\{\tilde{r}\leqslant R\}}|\nabla\psi|^2 + \mathfrak{Re}^2(\psi) + \frac{|\psi|^2}{(1+\tilde{r})^{5/2}},$$

as well as the space

$$H_a := \{\varphi \in H^1_{\rm loc}(B(0,a)), \|\varphi\|_{H_a} < +\infty\}.$$

We define

$$\mu_a(\psi,h) := \frac{\int_{B(0,a)} \mathfrak{Im}\Big(h\eta - ic\partial_{x_2}\eta\psi - \Delta\eta\psi - 2\nabla\eta.\nabla\psi + 2\eta\frac{\nabla Q_c}{Q_c}.\nabla\psi\Big)}{\int_{B(0,a)}\mathfrak{Im}\Big(\frac{i\Upsilon\eta}{Q_c}\Big)}.$$

Let us check that, for $\sigma' > 0$,

$$|\mu_a(\psi,h)| \leqslant K(c,\sigma')(\|\psi\|_{H_a} + \|h\|_{\otimes\otimes,\sigma'}). \tag{4.2.4}$$

As for the proof of (4.2.3), for $a > 10/c^2$, we have $\int_{B(0,a)}\mathfrak{Im}\Big(\frac{i\Upsilon\eta}{Q_c}\Big) > K > 0$. Since $\eta = 1$ on $\mathbb{R}^2\backslash(B(d_c\vec{e_1}, R+2)\cup B(-d_c\vec{e_1}, R+2))$, we check easily by Cauchy-Schwarz that

$$\int_{B(0,a)} |\mathfrak{Im}(h\eta - ic\partial_{x_2}\eta\psi - \Delta\eta\psi - 2\nabla\eta.\nabla\psi)| \leqslant K(\sigma')(\|\psi\|_{H_a} + \|h\|_{\otimes\otimes,\sigma'}).$$

Now, we estimate by Cauchy-Schwarz that

$$\begin{aligned}
&\int_{B(0,a)}\Big|\mathfrak{Im}\Big(2\eta\frac{\nabla Q_c}{Q_c}.\nabla\psi\Big)\Big| \\
\leqslant\ & \sqrt{\int_{B(0,a)}(|Q_c|^2-1)^2\int_{\mathbb{R}^2\backslash\{\tilde{r}\leqslant R\}}\mathfrak{Re}^2(\psi)} + K\sqrt{\int_{B(0,a)}|\nabla Q_c|^2\int_{\mathbb{R}^2\backslash\{\tilde{r}\leqslant R\}}|\nabla\psi|^2} \\
\leqslant\ & K(E(Q_c))\|\psi\|_{H_a}.
\end{aligned}$$

We recall the notation, around $\pm\tilde{d}_c\vec{e_1}$, $h^{\neq 0} = h - h^0$,

$$h^0(x) = \frac{1}{2\pi}\int_0^{2\pi} h\big(|x \mp \tilde{d}_c\vec{e_1}|e^{i\theta}\big)d\theta.$$

Finally, we define

$$\lambda(\psi) := \frac{1}{V(B(0,10/c^2)\backslash B(0,5/c^2))}\int_{B(0,10/c^2)\backslash B(0,5/c^2)}\mathfrak{Im}(\psi),$$

where $V(\Omega)$ is the volume of Ω, the average of the imaginary part of ψ in $B(0,10/c^2)\backslash B(0,5/c^2)$.

Lemma 4.2.7. *For $0 < \sigma' < 1$ there exists $c_0(\sigma') > 0$ such that, for $0 < c < c_0(\sigma')$, there exists $K(\sigma',c), R > 0$ such that there exists $a_0(c,\sigma) > 10/c^2$ such that, for any $Q_c h \in \mathcal{E}_{\otimes\otimes,\sigma'}$, $a > a_0(c,\sigma')$, the problem*

$$\begin{cases}
L_{Q_c}(\varphi) = Q_c h - \mu_a(\psi,h) i\Upsilon \text{ in } B(0,a) \\
\varphi = 0 \text{ on } \partial B(0,a) \\
\varphi = Q_c\psi \in H_a, \mathfrak{Re}\int_{B(d_c\vec{e_1},R)\cup B(-d_c\vec{e_1},R)}\partial_c Q_c\bar{\varphi} = \mathfrak{Re}\int_{B(d_c\vec{e_1},R)\cup B(-d_c\vec{e_1},R)}\partial_{x_2}Q_c\overline{Q_c\psi^{\neq 0}} = 0 \\
\mathfrak{Re}\int_{B(d_c\vec{e_1},R)\cup B(-d_c\vec{e_1},R)}\partial_c Q_c\overline{Q_c h} = \mathfrak{Re}\int_{B(d_c\vec{e_1},R)\cup B(-d_c\vec{e_1},R)}\partial_{x_2}Q_c\overline{Q_c h^{\neq 0}} = 0
\end{cases}$$

admits a unique solution with

$$\|\varphi - i\lambda(\psi)Q_c\|_{H_a} \leqslant K(\sigma',c)\|h\|_{\otimes\otimes,\sigma'}.$$

This proof follows closely the proof of Lemma 2.1.19. The orthogonality conditions on h are required to apply the Fredholm alternative.

Proof. We argue by contradiction on the estimation, assuming the existence. Suppose that there exists a sequence $a_n > \frac{10}{c^2}$, $a_n \to \infty$, functions $\varphi_n = Q_c\psi_n \in H_{a_n}$, $\varphi_n = 0$ on $\partial B(0, a_n)$ and $Q_{c_n}h_n \in \mathcal{E}_{\otimes\otimes,\sigma'}$ such that $\|\varphi_n - i\lambda(\psi_n)Q_c\|_{H_{a_n}} = 1$, $\|h_n\|_{\otimes\otimes,\sigma'} \to 0$ and $L_{Q_c}(\varphi_n) = Q_c h_n - \mu_{a_n}(\psi_n, h_n)i\Upsilon$ on $B(0,a_n)$. In particular, remark here that c is independent of n, only the size of the ball grows. Our goal is to show that $\|\varphi_n - i\lambda(\psi_n)Q_c\|_{H_{a_n}} = o_{n\to\infty}(1)$, which leads to the contradiction.

As in the proof of Lemma 2.1.19, we pass at the limit when $n\to\infty$, and up to a subsequence, in $C^1_{\rm loc}(\mathbb{R}^2)$, $\varphi_n - i\lambda(\psi_n)Q_c \to \varphi = Q_c\psi \in H_\infty$ with $L_{Q_c}(\varphi) = -\mu(\psi,0)i\Upsilon$ in $\mathbb{R}^2$ (the convergence $\mu_{a_n}(\psi_n - i\lambda(\psi_n), h_n) \to \mu(\psi,0)$ up to a subsequence comes from the bilinearity of μ and (4.2.4)), and, since they are invariant by adding $i\lambda Q_c$ to φ,

$$\mathfrak{Re}\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)} \partial_c Q_c\bar{\varphi} = \mathfrak{Re}\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)} \partial_{x_2} Q_c\overline{Q_c\psi^{\neq 0}} = 0.$$

Let us check that this implies that $\varphi = i\lambda Q_c$ for some $\lambda\in\mathbb{R}$.

As in the proof of Lemma 4.2.6, with the same cutoff function χ, we define $\tilde{\psi} = \chi\psi$, that satisfies

$$\begin{aligned}&(-ic\partial_{x_2} - \Delta + 2\mathfrak{Re})(\tilde{\psi})\\ =\ & -ic\partial_{x_2}\eta\psi - \Delta\eta\psi - 2\nabla\eta.\nabla\psi + 2\eta\frac{\nabla Q_c}{Q_c}.\nabla\psi - 2\eta\mathfrak{Re}(\psi)(|Q_c|^2-1) - \mu(\psi,0)i\Upsilon.\end{aligned}$$

Let us check that the right hand side is in $\mathcal{E}_{\otimes\otimes,\sigma'}$ with $0<\sigma'<1$. Thanks to the equation $L_{Q_c}(\varphi) = -\mu(\psi,0)i\Upsilon$, we have that $\varphi\in C^\infty_{\rm loc}(\mathbb{R}^2,\mathbb{C})$, thus

$$-ic\partial_{x_2}\eta\psi - \Delta\eta\psi - 2\nabla\eta.\nabla\psi - \mu(\psi,0)i\Upsilon \in \mathcal{E}_{\otimes\otimes,\sigma'}$$

for any $0<\sigma'<1$, as these terms are compactly supported. For the two remainings term, the proof is identical as the proof of Lemma 2.1.18.

From Lemma 2.1.15, we deduce that there exists $\zeta\in\mathcal{E}_{\otimes,\sigma}$ for some $0<\sigma<\sigma'$ such that

$$\begin{aligned}&(-ic\partial_{x_2} - \Delta + 2\mathfrak{Re})(\zeta)\\ =\ & -ic\partial_{x_2}\eta\psi - \Delta\eta\psi - 2\nabla\eta.\nabla\psi + 2\eta\frac{\nabla Q_c}{Q_c}.\nabla\psi - 2\eta\mathfrak{Re}(\psi_n)(|Q_{c_n}|^2-1) - \mu(\psi,0)i\Upsilon.\end{aligned}$$

Now, we have that

$$(-ic\partial_{x_2} - \Delta + 2\mathfrak{Re})(\tilde{\psi} - \zeta) = 0,$$

and we check that, for $\sigma>0$, $\mathcal{E}_{\otimes,\sigma}\subset H_\infty$, thus $\tilde{\psi}-\zeta\in H_\infty$. From the proof of Lemma 2.1.15, we deduce that $\tilde{\psi}-\zeta\in\mathbb{C}[X_1,X_2]$, and we check easily that $H_\infty\cap\mathbb{C}[X_1,X_2] = \mathrm{Span}_{\mathbb{R}}(i)$, thus there exists $\lambda\in\mathbb{R}$ such that $\tilde{\psi} - i\lambda = \zeta\in\mathcal{E}_{\otimes,\sigma}$. In particular, if we define $\check{\varphi} = \varphi - i\lambda Q_c$, $\check{\psi} = \psi - i\lambda$, then $\check{\varphi}\in\mathcal{E}_{\otimes,\sigma}$ (since $\check{\psi} = \psi - i\lambda = \zeta + \psi - \tilde{\psi}$, where $\psi-\tilde{\psi}$ is compactly supported) with

$$L_{Q_c}(\check{\varphi}) = -\mu(\check{\psi},0)i\Upsilon,$$

since $L_{Q_c}(iQ_c) = \mu(i,0) = 0$. By Lemma 4.2.6 (for $h=0\in\mathcal{E}_{\otimes\otimes,\sigma'}$), since

$$\mathfrak{Re}\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)} \partial_c Q_c\bar{\check{\varphi}} = \mathfrak{Re}\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)} \partial_c Q_c\bar{\varphi} = 0$$

by symmetry, and

$$\mathfrak{Re}\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)} \partial_{x_2} Q_c\overline{Q_c\check{\psi}^{\neq 0}} = \mathfrak{Re}\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)} \partial_{x_2} Q_c\overline{Q_c\psi^{\neq 0}} = 0,$$

we have $\check{\varphi}=0$, and thus $\varphi_n - i\lambda(\psi_n)Q_c \to i\lambda Q_c$ in $C^1(B(0,\Lambda))$ for all $\Lambda>0$. Furthermore, since $\lambda(\psi_n - i\lambda(\psi_n)) = 0$, taking $\Lambda > 10/c^2$, we deduce that $\lambda = 0$.

This implies that, for all $\Lambda>0$, $\|\varphi_n - i\lambda(\psi_n)\ Q_c\|_{C^1(B(0,\Lambda))} = o^{\Lambda}_{n\to\infty}(1)$. Furthermore, since $\mu(\lambda i,0) = 0$ for any $\lambda\in\mathbb{R}$, we have $\mu_{a_n}(\psi_n - i\lambda(\psi_n), h_n)\to 0$ when $n\to\infty$.

Now, as in the proof of Lemma 4.2.6, multiplying the equation by χ, we write it, with $\tilde{\psi}_n = \eta\psi_n$, on the form

$$-ic\partial_{x_2}\tilde{\psi}_n - \Delta\tilde{\psi}_n + 2\mathfrak{Re}\big(\tilde{\psi}_n\big) = \tilde{h}_n,$$

where

$$\begin{aligned}\tilde{h}_n \ &=\ h_n\eta - ic\partial_{x_2}\eta\psi_n - \Delta\eta\psi_n - 2\nabla\eta.\nabla\psi_n - 2\eta\mathfrak{Re}(\psi_n)(|Q_{c_n}|^2-1)\\ &+\ 2\eta\frac{\nabla Q_{c_n}}{Q_{c_n}}.\nabla\psi_n - \mu_{a_n}(\psi_n,h_n)\frac{i\Upsilon\eta}{Q_c}.\end{aligned}$$

Now, the proof varies a little from the one of Lemma 2.1.19. Taking the scalar product of the imaginary part of the equation with $\mathfrak{Im}(\tilde{\psi}_n)$, we infer

$$\int_{B(0,a_n)} c\partial_{x_2}\mathfrak{Re}(\tilde{\psi}_n)\mathfrak{Im}(\tilde{\psi}_n)+\int_{B(0,a_n)} -\Delta\mathfrak{Im}(\tilde{\psi}_n)\mathfrak{Im}(\tilde{\psi}_n)=\int_{B(0,a_n)}\mathfrak{Im}(\tilde{h}_n)\mathfrak{Im}(\tilde{\psi}_n).$$

By integration by parts, since $\tilde{\psi}_n=0$ on $\partial B(0,a_n)$, we have

$$\int_{B(0,a_n)} -\Delta\mathfrak{Im}(\tilde{\psi}_n)\mathfrak{Im}(\tilde{\psi}_n)=\int_{B(0,a_n)}|\nabla\mathfrak{Im}(\tilde{\psi}_n)|^2$$

and

$$\begin{aligned}\left|\int_{B(0,a_n)} c\partial_{x_2}\mathfrak{Re}(\tilde{\psi}_n)\mathfrak{Im}(\tilde{\psi}_n)\right| &\leqslant \left|c\int_{B(0,a_n)}\mathfrak{Re}(\tilde{\psi}_n)\partial_{x_2}\mathfrak{Im}(\tilde{\psi}_n)\right|\\ &\leqslant c\sqrt{\int_{B(0,a_n)}\mathfrak{Re}^2(\tilde{\psi}_n)\int_{B(0,a_n)}|\nabla\mathfrak{Im}(\tilde{\psi}_n)|^2}.\end{aligned}$$

Furthermore, since $\int_{B(0,a_n)}\mathfrak{Im}(\tilde{h}_n)=0$, we have

$$\int_{B(0,a_n)}\mathfrak{Im}(\tilde{h}_n)\mathfrak{Im}(\tilde{\psi}_n)=\int_{B(0,a_n)}\mathfrak{Im}(\tilde{h}_n)\mathfrak{Im}(\tilde{\psi}_n-i\lambda(\psi_n)),$$

and we estimate, since $\|\varphi_n-i\lambda(\psi_n)\,Q_c\|_{C^1(B(0,\Lambda))}=o^{\Lambda}_{n\to\infty}(1)$, that

$$\int_{B(0,\Lambda)}\left|\mathfrak{Im}(\tilde{h}_n)\mathfrak{Im}(\tilde{\psi}_n-i\lambda(\psi_n))\right|=o^{\Lambda,\lambda}_{n\to\infty}(1),$$

and

$$\begin{aligned}&\int_{B(0,a_n)\setminus B(0,\Lambda)}\left|\mathfrak{Im}(\tilde{h}_n)\mathfrak{Im}(\tilde{\psi}_n-i\lambda(\psi_n))\right|\\ =\;&\int_{B(0,a_n)\setminus B(0,\Lambda)}\left|\mathfrak{Im}\Big(2\eta\frac{\nabla Q_{c_n}}{Q_{c_n}}.\nabla\tilde{\psi}_n\Big)\mathfrak{Im}(\tilde{\psi}_n-i\lambda(\psi_n))\right|\\ \leqslant\;& o^c_{\Lambda\to\infty}(1)\sqrt{\int_{B(0,a_n)}|\nabla\tilde{\psi}_n|^2\int_{B(0,a_n)\setminus B(0,\Lambda)}\left|\frac{\nabla Q_{c_n}}{Q_{c_n}}\right|^2(1+r)^{1/8}\mathfrak{Im}^2(\tilde{\psi}_n-i\lambda(\psi_n))}\\ \leqslant\;& o^c_{\Lambda\to\infty}(1)\sqrt{\int_{B(0,a_n)}|\nabla\tilde{\psi}_n|^2\int_{B(0,a_n)\setminus B(0,\Lambda)}\frac{\mathfrak{Im}^2(\tilde{\psi}_n-i\lambda(\psi_n))}{(1+r)^{5/2}}}.\end{aligned}$$

We deduce that

$$\int_{B(0,a_n)}|\nabla\mathfrak{Im}(\tilde{\psi}_n)|^2\leqslant c\sqrt{\int_{B(0,a_n)}\mathfrak{Re}^2(\tilde{\psi}_n)\int_{B(0,a_n)}|\nabla\mathfrak{Im}(\tilde{\psi}_n)|^2}+o^c_{\Lambda\to\infty}(1)\|\tilde{\varphi}_n-i\lambda(\psi_n)Q_c\|^2_{H_{a_n}}.$$

Now, taking the scalar product of the real part of the equation with $\mathfrak{Re}(\tilde{\psi}_n)$, the computation is identical to the one in Lemma 2.1.19, and we have

$$\begin{aligned}&\int_{B(0,a_n)}|\nabla\mathfrak{Re}(\tilde{\psi}_n)|^2+\int_{B(0,a_n)}\mathfrak{Re}^2(\tilde{\psi}_n)\\ \leqslant\;& c\sqrt{\int_{B(0,a_n)}\mathfrak{Re}^2(\tilde{\psi}_n)\int_{B(0,a_n)}|\nabla\mathfrak{Im}(\tilde{\psi}_n)|^2}+o^c_{\Lambda\to\infty}(1)\|\tilde{\varphi}_n-i\lambda(\psi_n)Q_c\|_{H_{a_n}}.\end{aligned}$$

Now, since $\nabla\tilde{\psi}_n=\nabla(\tilde{\psi}_n-i\lambda(\psi_n))$, $\mathfrak{Re}(\tilde{\psi}_n)=\mathfrak{Re}(\tilde{\psi}_n-i\lambda(\psi_n))$, and $\|\varphi_n-i\lambda(\psi_n)\,Q_c\|_{C^1(B(0,\Lambda))}=o^{\Lambda}_{n\to\infty}(1)$, we compute, with the same Hardy type inequality as in the proof of Lemma 2.1.19, that

$$\|\varphi_n-i\lambda(\psi_n)\,Q_c\|^2_{H_{a_n}}\leqslant o^c_{n\to\infty}(1)+K\left(\int_{B(0,a_n)}|\nabla\tilde{\psi}_n|^2+\int_{B(0,a_n)}\mathfrak{Re}^2(\tilde{\psi}_n)\right).$$

Combining these estimates, we deduce that

$$\|\varphi_n-i\lambda(\psi_n)\,Q_c\|^2_{H_{a_n}}=o^c_{n\to\infty}(1)+o^c_{\Lambda\to\infty}(1).$$

We concluded the proof of the estimation, taking Λ and n large enough. Now, for the existence, we argue by Fredholm's alternative in

$$\left\{\varphi\in H_0^1(B(0,a)),\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_c Q_c\bar{\varphi}=\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_{x_2}Q_c\overline{Q_c\psi^{\neq 0}}=0\right\},$$

and we remark that the norms $\left\|.-i\lambda\left(\frac{.}{Q_c}\right)Q_c\right\|_{H_a}$ and $\|.\|_{H^1}$ are equivalent on $H_0^1(B(0,\,a))$. By Riesz's representation theorem, the elliptic equation $L_{Q_c}(\varphi)=Q_ch-\mu_a(\psi,\,h)i\Upsilon$ can be rewritten in the operational form $\Phi+\mathcal{K}(\Phi)=\mathcal{S}(h)$ where $\mathcal{K}$ is a compact operator in $H_0^1(B(0,\,a))$, and it has no kernel in H_a (i.e. in $\left\{\varphi\in H_0^1(B(0,\,a)),\ \mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_c Q_c\bar{\varphi}=\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_{x_2}Q_c\overline{Q_c\psi^{\neq 0}}=0\right\}$) by the estimation we just showed and the boundary condition. Therefore, there exists a unique solution $\varphi=Q_c\psi\in H_a$, and it then satisfies

$$\|\varphi-i\lambda(\psi)Q_c\|_{H_a}\leqslant K(\sigma',c)\|h\|_{\otimes\otimes,\sigma'}.$$ □

Lemma 4.2.8. *There exists $R>0$ such that, for $0<\sigma<\sigma'<1$, $Q_ch\in\mathcal{E}_{\otimes\otimes,\sigma'}$ with*

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_c Q_c\overline{Q_ch}=\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_{x_2}Q_c\overline{Q_ch^{\neq 0}}=0,$$

the problem

$$L_{Q_c}(\varphi)=Q_ch-\mu(\psi,h)i\Upsilon$$

admits a unique solution $\varphi=Q_c\psi\in\mathcal{E}_{\otimes,\sigma}$ such that

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_c Q_c\bar{\varphi}=\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_{x_2}Q_c\overline{Q_c\psi^{\neq 0}}=0.$$

Furthermore, this solution satisfies

$$\|\psi\|_{\otimes,\sigma}\leqslant K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma'}.$$

Proof. The proof is identical to the one of Proposition 2.1.20, using Lemma 4.2.7 instead of Lemma 2.1.19. The other difference is that, when we have a solution in the whole space which is in H_∞, we have $\varphi-i\lambda Q_c\in\mathcal{E}_{\otimes,\sigma}$ for some $\lambda\in\mathbb{R}$ (as in the proof of Lemma 4.2.7). The consider solution is $\varphi-i\lambda Q_c$, as we check that

$$L_{Q_c}(\varphi)=L_{Q_c}(\varphi-i\lambda Q_c),$$

$$\mu(\psi,h)=\mu(\psi-i\lambda,h),$$

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_c Q_c\overline{(\varphi-i\lambda Q_c)}=\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_c Q_c\bar{\varphi}$$

and

$$\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_{x_2}Q_c\overline{Q_c(\psi-i\lambda)^{\neq 0}}=\mathfrak{Re}\int_{B(\tilde{d}_c\vec{e_1},R)\cup B(-\tilde{d}_c\vec{e_1},R)}\partial_{x_2}Q_c\overline{Q_c\psi^{\neq 0}}.$$ □

To complete the inversion of L_{Q_c}, we need to inverse the problem $L_{Q_c}(\varphi)=i\Upsilon$.

Lemma 4.2.9. *For $c>0$ small enough, there exists a function $\varphi_\Upsilon\in C^2(\mathbb{R}^2,\mathbb{C})$ such that*

$$L_{Q_c}(\varphi_\Upsilon)=i\Upsilon.$$

For all $\varepsilon>0$, this function $\varphi_\Upsilon=Q_c\psi_\Upsilon$ is in $\mathcal{E}_{\otimes,-\varepsilon}$ and, for $c>0$ small enough (depending on ε),

$$\left\|\frac{\Psi_\Upsilon-i\psi_1(x-d_c)-i\psi_{-1}(x+d_c)}{Q_c}\right\|_{\mathcal{E}_{\otimes,-\varepsilon}}\leqslant K(\varepsilon)c^{1-\varepsilon}.$$

Proof. The equation, on $\mathbb{R}^2\backslash\{d_c\vec{e_1}, -d_c\vec{e_1}\}$, is $L'_{Q_c}(\psi) = i\frac{\Upsilon}{Q_c}$. We look for an ansatz of the form

$$\psi = i\psi_1(x-d_c) + i\psi_{-1}(x+d_c) + \chi\Xi + \Lambda,$$

where $\psi_{\pm 1}$ are defined in Lemma 4.2.3, $R>0$ is given by Lemma 4.2.8, χ is a smooth cutoff function with value 0 in $B(d_c\vec{e_1}, 2R)\cup B(-d_c\vec{e_1}, 2R)$ and 1 in $B(d_c\vec{e_1}, 2R+1)\cup B(-d_c\vec{e_1}, 2R+1)$, Ξ will be the solution of (using Lemma 4.2.2)

$$-ic\partial_{x_2}\Xi - \Delta\Xi + 2\mathfrak{Re}(\Xi) = L'_{Q_c}(i\psi_1 + i\psi_{-1}) - i\Upsilon,$$

and Λ is a remainder, that will solve

$$\begin{aligned} L'_{Q_c}(\Lambda) &= -ic\partial_{x_2}(\chi\Xi) - \Delta(\chi\Xi) + 2\mathfrak{Re}(\chi\Xi) - L'_{Q_c}(\chi\Xi) \\ &- \mu(\Lambda, -ic\partial_{x_2}(\chi\Xi) - \Delta(\chi\Xi) + 2\mathfrak{Re}(\chi\Xi) - L'_{Q_c}(\chi\Xi)). \end{aligned}$$

The idea of this ansatz is to compare Q_c with two vortices, where $i\psi_{\pm 1}$ are a solution of this problem. The error terms are then small when $c\to 0$, but still does not decay enough to use Lemma 4.2.8. This is why we introduce Ξ, that solves this problem at infinity. The reminder is then small when $c\to 0$, and has now enough decay, i.e. is in $\mathcal{E}_{\otimes\otimes,\sigma}$ for some $\sigma>0$, and Λ ties the reminders up, and will be constructed using Lemma 4.2.8.

First, let us estimate

$$L'_{Q_c}(i\psi_1 + i\psi_{-1}) - i\Upsilon.$$

By Lemma 4.2.3, $i\Upsilon = L'_{V_1}(i\psi_1) + L'_{V_{-1}}(i\psi_{-1})$, where $V_{\pm 1}$ are centered at $\pm d_c\vec{e_1}$, and

$$L'_{V_1}(\psi) = -\Delta\psi - 2\frac{\nabla V_1}{V_1}.\nabla\psi + 2\mathfrak{Re}(\psi)|V_1|^2.$$

We have

$$L'_{Q_c}(i\psi_1 + i\psi_{-1}) - i\Upsilon = L'_{Q_c}(i\psi_1) - L'_{V_1}(i\psi_1) + L'_{Q_c}(i\psi_{-1}) - L'_{V_{-1}}(i\psi_{-1}).$$

We recall

$$L'_{Q_c}(\psi) = -\Delta\psi - ic\partial_{x_2}\psi + 2\mathfrak{Re}(\psi)|Q_c|^2 - 2\frac{\nabla Q_c}{Q_c}.\nabla\psi,$$

thus, since ψ_1 is real-valued,

$$L'_{Q_c}(i\psi_1) - L'_{V_1}(i\psi_1) = c\partial_{x_2}\psi_1 + 2i\left(\frac{\nabla V_1}{V_1} - \frac{\nabla Q_c}{Q_c}\right).\nabla\psi_1.$$

We write $Q_c = V_1V_{-1} + \tilde{\Gamma}_c$, where $V_{\pm 1}$ is centered at $\pm d_c\vec{e_1}$. We compute

$$L'_{Q_c}(i\psi_1) - L'_{V_1}(i\psi_1) = c\partial_{x_2}\psi_1 + 2i\left(-\frac{\nabla V_{-1}}{V_{-1}} - \frac{\nabla\tilde{\Gamma}_c}{\tilde{\Gamma}_c}\right).\nabla\psi_1.$$

We estimate, for all $0<\sigma<1$, with Lemmas 2.2.8, 4.2.3 and $\frac{1}{(1+r_1)(1+r_{-1})} \leqslant \frac{Kc}{(1+\tilde{r})}$, that

$$\begin{aligned} |\mathfrak{Re}(L'_{Q_c}(i\psi_1) - L'_{V_1}(i\psi_1))| &\leqslant \frac{Kc}{(1+r_1)} + \frac{K}{(1+r_{-1})(1+r_1)} + \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^{1+\sigma}(1+r_1)}. \\ &\leqslant \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})}, \end{aligned}$$

$$\begin{aligned} |\nabla(\mathfrak{Re}(L'_{Q_c}(i\psi_1) - L'_{V_1}(i\psi_1)))| &\leqslant \frac{Kc}{(1+r_1)^2} + \frac{K}{(1+r_{-1})^2(1+r_1)} + \frac{K}{(1+r_{-1})(1+r_1)^2} \\ &+ \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^{2+\sigma/2}(1+r_1)} \\ &\leqslant \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^2}, \end{aligned}$$

as well as

$$\begin{aligned}|\mathfrak{Im}(L'_{Q_c}(i\psi_1)-L'_{V_1}(i\psi_1))| &\leqslant \frac{K}{(1+r_1)(1+r_{-1})^3}+\frac{K(\sigma)c^{1-\sigma}}{(1+r_1)(1+\tilde{r})^{2+\sigma/2}}\\ &\leqslant \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^{2+\sigma}}\end{aligned}$$

and

$$\begin{aligned}|\nabla(\mathfrak{Im}(L'_{Q_c}(i\psi_1)-L'_{V_1}(i\psi_1)))| &\leqslant \frac{K}{(1+r_1)(1+r_{-1})^3}+\frac{K(\sigma)c^{1-\sigma}}{(1+r_1)(1+\tilde{r})^{2+\sigma}}\\ &\leqslant \frac{K(\sigma)c^{1-\sigma}}{(1+\tilde{r})^{2+\sigma}}.\end{aligned}$$

We deduce that $-L'_{Q_c}(i\psi_1+i\psi_{-1})+i\Upsilon\in\mathcal{E}_{\otimes\otimes,0}$ with, for all $\varepsilon>0$,

$$\|L'_{Q_c}(i\psi_1+i\psi_{-1})+i\Upsilon\|_{\otimes\otimes,0}\leqslant K(\varepsilon)c^{1-\varepsilon}.$$

From Lemma 4.2.2, there exists $\Xi=\Xi_1+i\Xi_2\in C^2(\mathbb{R}^2,\mathbb{C})$ such that

$$-ic\partial_{x_2}\Xi-\Delta\Xi+2\mathfrak{Re}(\Xi)=-L'_{Q_c}(i\psi_1+i\psi_{-1})+i\Upsilon,$$

and

$$\|\Xi\|_{\otimes,-\varepsilon,\infty}\leqslant K(\varepsilon)\|L'_{Q_c}(i\psi_1+i\psi_{-1})+i\Upsilon\|_{\otimes\otimes,0}\leqslant K(\varepsilon)c^{1-\varepsilon}. \tag{4.2.5}$$

Now, since $L'_{Q_c}(i\psi_1+i\psi_{-1})-i\Upsilon=0$ in $B(d_c\vec{e_1},2R+1)\cup B(-d_c\vec{e_1},2R+1)$, we have

$$\chi(L'_{Q_c}(i\psi_1+i\psi_{-1})-i\Upsilon)=L'_{Q_c}(i\psi_1+i\psi_{-1})-i\Upsilon,$$

and therefore

$$-ic\partial_{x_2}(\chi\Xi)-\Delta(\chi\Xi)+2\mathfrak{Re}(\chi\Xi)=-L'_{Q_c}(i\psi_1+i\psi_{-1})+i\Upsilon-ic\partial_{x_2}\chi\Xi-\Delta\chi\Xi-2\nabla\chi.\nabla\Xi.$$

We deduce that, writing $\psi=i\psi_1+i\psi_{-1}+\chi\Xi+\Lambda$ for some function Λ,

$$\begin{aligned}&L'_{Q_c}(\psi)-i\Upsilon\\ =\;&L'_{Q_c}(i\psi_1(x-d_c)+i\psi_{-1}(x+d_c)+\chi\Xi+\Lambda)-i\Upsilon\\ =\;&L'_{Q_c}(i\psi_1(x-d_c)+i\psi_{-1}(x+d_c))-i\Upsilon+L'_{Q_c}(\chi\Xi)+L'_{Q_c}(\Lambda)\\ =\;&ic\partial_{x_2}(\chi\Xi)+\Delta(\chi\Xi)-2\mathfrak{Re}(\chi\Xi)+L'_{Q_c}(\chi\Xi)\\ +\;&ic\partial_{x_2}\chi\Xi+\Delta\chi\Xi+2\nabla\chi.\nabla\Xi+L'_{Q_c}(\Lambda).\end{aligned}$$

We therefore look at

$$h=-ic\partial_{x_2}(\chi\Xi)-\Delta(\chi\Xi)+2\mathfrak{Re}(\chi\Xi)-L'_{Q_c}(\chi\Xi)-ic\partial_{x_2}\chi\Xi-\Delta\chi\Xi-2\nabla\chi.\nabla\Xi.$$

Since $L'_{Q_c}(\psi)=-\Delta\psi-ic\partial_{x_2}\psi+2\mathfrak{Re}(\psi)|Q_c|^2-2\frac{\nabla Q_c}{Q_c}.\nabla\psi$, we deduce

$$h=2\chi\mathfrak{Re}(\Xi)(1-|Q_c|^2)-2\frac{\nabla Q_c}{Q_c}.\nabla(\chi\Xi).$$

We check, with Lemma 2.2.8 and (4.2.5), that $2\mathfrak{Re}(\chi\Xi)(1-|Q_c|^2)-2\frac{\nabla Q_c}{Q_c}.\nabla(\chi\Xi)\in\mathcal{E}_{\otimes\otimes,\sigma}$ for some $\sigma>0$, and

$$\left\|2\mathfrak{Re}(\chi\Xi)(1-|Q_c|^2)-2\frac{\nabla Q_c}{Q_c}.\nabla(\chi\Xi)\right\|_{\otimes\otimes,\sigma}\leqslant K(\varepsilon)c^{1-\varepsilon}.$$

Similarly, as it is compatcly supported, with the estimates on Ξ we check easily that

$$\|ic\partial_{x_2}\chi\Xi-\Delta\chi\Xi-2\nabla\chi.\nabla\Xi\|_{\otimes\otimes,\sigma}\leqslant K(\varepsilon)c^{1-\varepsilon}.$$

We deduce, from Lemma 4.2.8, the orthogonality conditions being satisfied since the source is 0 in $B\big(\tilde{d}_c\vec{e_1},R\big)\cup B\big(-\tilde{d}_c\vec{e_1},R\big)$ (because of the cutoff χ), that there exists $\Lambda\in\mathcal{E}_{\otimes,\sigma'}$, $0<\sigma'<\sigma$ such that

$$L'_{Q_c}(\Lambda)=2\mathfrak{Re}(\Xi)(1-|Q_c|^2)-2\frac{\nabla Q_c}{Q_c}.\nabla\Xi+(1-\eta)L'_{Q_c}(i\psi_1+i\psi_{-1})-\mu(\Lambda,h)Q_ci\Upsilon,$$

with

$$\left\| \frac{\Lambda}{Q_c} \right\|_{\otimes,\sigma} + |\mu(\Lambda,h)| \leqslant K(\varepsilon)c^{1-\varepsilon}.$$

We deduce that

$$L'_{Q_c}(i\psi_1(x-d_c)+i\psi_{-1}(x+d_c)+\Xi+\Lambda) = (1-\mu(\Lambda,h))Q_c i\Upsilon,$$

thus

$$L'_{Q_c}\left(\frac{i\psi_1(x-d_c)+i\psi_{-1}(x+d_c)+\Xi+\Lambda}{1-\mu(\Lambda,h)}\right) = i\Upsilon.$$

For $c>0$ small enough (depending on ε), $|\mu(\Lambda,h)|<1/2$. We therefore define

$$\varphi_\Upsilon := Q_c\left(\frac{i\psi_1(x-d_c)+i\psi_{-1}(x+d_c)+\Xi+\Lambda}{1-\mu(\Lambda,h)}\right).$$

Now, with

$$1-\mu(\Lambda,h) = 1+o^{\varepsilon}_{c\to 0}(1),$$

the estimates on Ξ and $\left\|\frac{\Lambda}{Q_c}\right\|_{\otimes,-\varepsilon} \leqslant \left\|\frac{\Lambda}{Q_c}\right\|_{\otimes,\sigma} \leqslant K(\varepsilon)c^{1-\varepsilon}$, we check the estimates on

$$\Psi_\Upsilon - i\psi_1(x-d_c) - i\psi_{-1}(x+d_c). \qquad \square$$

Now, we can invert the problem $L_{Q_c}(Q_c\psi) = Q_c h$ without requiring that $\mu(\psi,h)=0$.

Proposition 4.2.10. *Given $0<\sigma<\sigma'<1$ and $Q_c h \in \mathcal{E}_{\otimes\otimes,\sigma'}$ with*

$$\mathfrak{Re}\int_{B(d_c\vec{e_1},R)\cup B(-d_c\vec{e_1},R)} \partial_{x_2} Q_c \overline{Q_c h^{\neq 0}} = 0,$$

there exists $\mu\in\mathbb{R}$ and $\varphi = Q_c\psi \in \mathcal{E}_{\otimes,\sigma}$ such that

$$L_{Q_c}(\varphi+\mu\varphi_\Upsilon) = Q_c h.$$

Furthermore,

$$\|\psi\|_{\otimes,\sigma} \leqslant \frac{K(\sigma,\sigma')}{c^2}\|h\|_{\otimes\otimes,\sigma'}$$

and

$$|\mu| \leqslant \frac{K(\sigma,\sigma')}{c^{\sigma'}}\|h\|_{\otimes\otimes,\sigma'}.$$

Proof. First, we suppose additionally that

$$\mathfrak{Re}\int_{B(d_c\vec{e_1},R)\cup B(-d_c\vec{e_1},R)} \partial_c Q_c \overline{Q_c h} = 0.$$

Then, from Lemma 4.2.6, there exists $\mu\in\mathbb{R}$, $\varphi = Q_c\psi\in\mathcal{E}_{\otimes,\sigma}$ such that

$$L_{Q_c}(\varphi) = h - \mu i\Upsilon,$$

with $\|\psi\|_{\otimes,\sigma}\leqslant K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma'}$ and $|\mu|\leqslant K(\sigma,\sigma')(\|h\|_{\otimes\otimes,\sigma'}+\|\psi\|_{\otimes,\sigma})\leqslant K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma'}$. From Lemma 4.2.9, $L_{Q_c}(\varphi_\Upsilon) = i\Upsilon$, therefore

$$L_{Q_c}(\varphi+\mu\varphi_\Upsilon) = Q_c h.$$

In that case, $\|\psi\|_{\otimes,\sigma}\leqslant K(\sigma,\sigma')\|h\|_{\otimes\otimes,\sigma'}$, where $K(\sigma,\sigma')$ does not depend on $c>0$.

Now, in the general case, we decompose for some $\alpha\in\mathbb{R}$, $Q_c h = Q_c h' + \alpha i\partial_{x_2}Q_c$ with

$$Q_c h' := Q_c h - \alpha i\partial_{x_2}Q_c.$$

We have, by symmetry,

$$\mathfrak{Re}\int_{B(d_c\vec{e_1},R)\cup B(-d_c\vec{e_1},R)} \partial_{x_2} Q_c \overline{Q_c h'^{\neq 0}} = 0,$$

and

$$\begin{aligned}&\mathfrak{Re}\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)}\partial_c Q_c\overline{Q_c h'}\\ =\;&\mathfrak{Re}\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)}\partial_c Q_c\overline{Q_c h}-\alpha\mathfrak{Re}\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)}\partial_c Q_c\overline{i\partial_{x_2}Q_c}.\end{aligned}$$

From and Theorem 1.3.1 (for $p=+\infty$), we have

$$c^2\mathfrak{Re}\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)}\partial_c Q_c\overline{i\partial_{x_2}Q_c}\leqslant K.$$

For $R>0$ large enough, using Lemma 4.1.1,

$$\begin{aligned}c^2\mathfrak{Re}\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)}\partial_c Q_c\overline{i\partial_{x_2}Q_c}&=2\mathfrak{Re}\int_{B(0,R)}\partial_{x_1}V_1(x)\overline{i\partial_{x_2}V_1}(x)+o_{c\to 0}(1)\\&=2\mathfrak{Re}\int_{B(0,R)}\partial_{x_1}V_1(x)\overline{i\partial_{x_2}V_1}(x)+o_{c\to 0}(1)\\&=4\pi\int_0^R\rho'(r)\rho(r)dr+o_{c\to 0}(1)\\&=4\pi(1-o_{R\to\infty}(1))+o_{c\to 0}(1),\end{aligned}$$

and thus

$$c^2\mathfrak{Re}\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)}\partial_c Q_c\overline{i\partial_{x_2}Q_c}\geqslant\frac{1}{K}.$$

We choose $\alpha\in\mathbb{R}$ such that

$$\mathfrak{Re}\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)}\partial_c Q_c\overline{Q_c h'}=0.$$

We check with Theorem 1.3.1 (for $p=+\infty$) that

$$\begin{aligned}|\alpha|&\leqslant Kc^2\,\mathfrak{Re}\int_{B(d_c\vec{e}_1,R)\cup B(-d_c\vec{e}_1,R)}|\partial_c Q_c||Q_c h|\\&\leqslant K\|h\|_{\otimes\otimes,\sigma},\end{aligned}$$

where $K>0$ does not depend on c. We deduce that there exists $\varphi'=Q_c\psi'\in\mathcal{E}_{\otimes,\sigma'}$, $\mu\in\mathbb{R}$ such that

$$L_{Q_c}(\varphi'+\mu\varphi_\Upsilon)=Q_c h',$$

with (using Lemma 4.1.1)

$$\begin{aligned}|\mu|+\|\varphi'\|_{\otimes,\sigma'}&\leqslant K(\sigma,\sigma')\|h'\|_{\otimes\otimes,\sigma}\\&\leqslant K(\sigma,\sigma')(\|h\|_{\otimes\otimes,\sigma}+|\alpha|\,\|i\partial_{x_2}Q_c\|_{\otimes\otimes,\sigma})\\&\leqslant\frac{K(\sigma,\sigma')}{c^\sigma}\|h\|_{\otimes\otimes,\sigma}.\end{aligned}$$

Now, we recall from Lemma 3.1.7 that $L_{Q_c}(\partial_c Q_c)=i\partial_{x_2}Q_c$, thus

$$L_{Q_c}(\varphi'+\mu\varphi_\Upsilon+\alpha\partial_c Q_c)=Q_c h.$$

Therefore, defining $\varphi=Q_c\psi:=\varphi'+\alpha\partial_c Q_c$, we check that $L_{Q_c}(\varphi+\mu\varphi_\Upsilon)=Q_c h$ and

$$\begin{aligned}\|\psi\|_{\otimes,\sigma}&\leqslant K(\sigma,\sigma')(\|h\|_{\otimes\otimes,\sigma'}+|\alpha|\,\|\partial_c Q_c\|_{\otimes,\sigma})\\&\leqslant\frac{K(\sigma,\sigma')}{c^2}\|h\|_{\otimes\otimes,\sigma'},\end{aligned}$$

which concludes the proof of this proposition. □

4.2.3 Inversion with two symmetries

We recall the spaces

$$\mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma}=\left\{\varphi\in\mathcal{E}_{\otimes,\sigma},\forall(x_1,x_2)\in\mathbb{R}^2,\varphi(x_1,x_2)=\overline{\varphi(x_1,-x_2)}\right\}$$

and

$$\mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma'} = \left\{ Q_c h \in \mathcal{E}_{\otimes\otimes,\sigma'}, \forall (x_1,x_2)\in\mathbb{R}^2, Q_c h(x_1,x_2) = \overline{Q_c h(x_1,-x_2)} \right\}.$$

Proposition 4.2.11. *Given $0<\sigma<\sigma'<1$ and $Q_c h \in \mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma'}$, there exists a unique function $\varphi = Q_c\psi \in \mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma'}$ such that*

$$L_{Q_c}(\varphi) = Q_c h.$$

Furthermore,

$$\|\psi\|_{\otimes,\sigma} \leqslant \frac{K(\sigma,\sigma')}{c^2}\|h\|_{\otimes\otimes,\sigma'}$$

and

$$(c, Q_c h) \to \varphi$$

is a continuous function from $(]0,c_0(\sigma,\sigma')[, \mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma'})$ to $\mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma}$ for some small constant $c_0(\sigma,\sigma')>0$.

Proof. With this second symmetry, we can check that $\mu(\psi, h) = 0$ and that the orthogonality condition on $\partial_{x_2} Q_c$ is automatically satisfied in Proposition 4.2.10. This implies the existence of a solution with the require estimate.

To show uniqueness, suppose that $\varphi' \in \mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma}$ also satisfies this equation. Then $\varphi - \varphi' \in \mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma} \subset H_{Q_c}$, and $L_{Q_c}(\varphi-\varphi') = 0$. From Corollary 1.4.5, this implies that $\varphi - \varphi' = \alpha\partial_{x_1} Q_c + \beta\partial_{x_2} Q_c$ for some $\alpha,\beta\in\mathbb{R}$, and by the symmetries (see subsection 4.1.1), $\alpha = \beta = 0$.

We now focus on the continuity. One difficulty is that the spaces $\mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma}$, $\mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma'}$ and their associated norms depends on c. Similarly as in subsection 2.2.1, we recall that for $c, c' > 0$ small and close enough, the norms between the associated spaces are equivalent, with a constant independent of c, c'. Here, to show the continuity, we take $c_n \to c$ and $Q_{c_n} h_n \to Q_c h$ in $\mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma'}$, using for all the norms the limit value c for the speed. Given that n is large enough, this choice does not change the spaces.

Now, there exists $\varphi_n = Q_{c_n}\psi_n \in \mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma''}$ for all $\sigma' > \sigma'' > 0$ such that

$$L_{Q_{c_n}}(\varphi_n) = Q_{c_n} h_n.$$

We also define $\varphi = Q_c\psi \in \mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma''}$ such that $L_{Q_c}(\varphi) = Q_c h$. To show the continuity, it is enough to show that $\varphi_n \to \varphi$ in $\mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma''}$. First, we remark that

$$\|\psi_n\|_{\otimes,\sigma''} \leqslant K(\sigma',\sigma'')\|h_n\|_{\otimes\otimes,\sigma'},$$

and since $\|h_n\|_{\otimes\otimes,\sigma'} \to \|h\|_{\otimes\otimes,\sigma'}$, is bounded uniformly in n. We compute

$$L_{Q_c}(\varphi_n - \varphi) = Q_{c_n} h_n - Q_c h + (L_{Q_c} - L_{Q_{c_n}})(\varphi_n),$$

and therefore we simply have to show that

$$\left\| \frac{Q_{c_n} h_n - Q_c h}{Q_c} \right\|_{\otimes\otimes,\sigma'} + \left\| \frac{(L_{Q_c} - L_{Q_{c_n}})(\varphi_n)}{Q_c} \right\|_{\otimes\otimes,\frac{\sigma+\sigma'}{2}} \to 0$$

when $n\to\infty$. The fact that $\left\| \frac{Q_{c_n} h_n - Q_c h}{Q_c} \right\|_{\otimes\otimes,\sigma'} \to 0$ comes from the hypothesis, and we are left with the proof of $\left\| \frac{(L_{Q_c} - L_{Q_{c_n}})(\varphi_n)}{Q_c} \right\|_{\otimes\otimes,\frac{\sigma+\sigma'}{2}} \to 0$. Since $c \to Q_c - 1 \in C^1(]0,c_0(\varsigma)[, \mathcal{E}^{2\mathrm{sym}}_{\otimes,\varsigma})$ for all $0<\varsigma<1$, we have that

$$\left\| \frac{(L_{Q_c} - L_{Q_{c_n}})(\varphi_n)}{Q_c} \right\|_{\otimes\otimes,\frac{\sigma+\sigma'}{2}} \leqslant |c_n - c| \left\| \frac{\partial_c L_{Q_{c^*}}(\varphi_n)}{Q_c} \right\|_{\otimes\otimes,\frac{\sigma+\sigma'}{2}}$$

for some $c^* \in [\min(c,c_n), \max(c,c_n)]$. Let us show that, more generally, for any $1 > \sigma'' > \sigma' > 0$, $\varphi = Q_c\psi \in \mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma''}$, we have

$$\left\| \frac{\partial_c L_{Q_c}(\varphi)}{Q_c} \right\|_{\otimes\otimes,\sigma'} \leqslant K(\sigma',\sigma'')\|\psi\|_{\otimes,\sigma''}. \tag{4.2.6}$$

Equality (4.2.6) is enough to conclude the proof of this proposition.

We recall that

$$L_{Q_c}(\varphi) = -\Delta\varphi - (1-|Q_c|^2)\varphi + 2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c - ic\partial_{x_2}\varphi,$$

and thus

$$\partial_c L_{Q_c}(\varphi) = 2\mathfrak{Re}(\overline{Q_c}\partial_c Q_c)\varphi + 2\mathfrak{Re}(\overline{\partial_c Q_c}\varphi)Q_c + 2\mathfrak{Re}(\overline{Q_c}\varphi)\partial_c Q_c - i\partial_{x_2}\varphi.$$

We check, with regards to the definition of the norms $\|.\|_{\otimes\otimes,\sigma'}$, $\|.\|_{\otimes,\sigma''}$ that, for $1>\sigma''>\sigma'>0$, (with computations similar to the proof of Lemma 4.1.1)

$$\|i\partial_{x_2}\varphi\|_{\otimes\otimes,\sigma'} \leqslant \|\varphi\|_{\otimes,\sigma''}.$$

For the other terms, the estimates are clear in the area $\{\tilde{r}\leqslant 3\}$. With $\varphi = Q_c\psi$, since $c\to Q_c - 1 \in C^1(]0,c_0(\varsigma)[,\mathcal{E}^{2\mathrm{sym}}_{\otimes,\varsigma})$ for all $0<\varsigma<1$, taking $\varsigma = 1+\sigma'-\sigma''<1$, outside of $\{\tilde{r}\leqslant 3\}$,

$$|\mathfrak{Re}(\overline{Q_c}\partial_c Q_c)\psi| \leqslant \frac{K(\sigma',\sigma'')}{(1+\tilde{r})^{1+\varsigma+\sigma''}} \leqslant \frac{K(\sigma',\sigma'')}{(1+\tilde{r})^{2+\sigma'}}.$$

We check similarly that

$$|\nabla(\mathfrak{Re}(\overline{Q_c}\partial_c Q_c)\psi)| \leqslant \frac{K(\sigma',\sigma'')}{(1+\tilde{r})^{2+\sigma'}}.$$

Now, $2\mathfrak{Re}(\overline{\partial_c Q_c}\varphi)$ is real valued, and still with $\varsigma = 1+\sigma'-\sigma''<1$, outside of $\{\tilde{r}\leqslant 3\}$,

$$|\mathfrak{Re}(\overline{\partial_c Q_c}\varphi)| \leqslant |\partial_c Q_c Q_c\psi| \leqslant \frac{K(\sigma',\sigma'')}{(1+\tilde{r})^{\varsigma+\sigma''}} \leqslant \frac{K(\sigma',\sigma'')}{(1+\tilde{r})^{1+\sigma''}},$$

and, with Lemma 4.1.1,

$$\begin{aligned}
|\nabla\mathfrak{Re}(\overline{\partial_c Q_c}\varphi)| &\leqslant K(|\nabla\partial_c Q_c||\psi| + |\nabla Q_c||\psi||\partial_c Q_c| + |\nabla\psi||\partial_c Q_c|) \\
&\leqslant K(\sigma',\sigma'')\left(\frac{1}{(1+\tilde{r})^{1+\varsigma+\sigma''}} + \frac{1}{(1+\tilde{r})^{1+\sigma''+\varsigma}} + \frac{1}{(1+\tilde{r})^{1+\sigma''+\varsigma}}\right) \\
&\leqslant \frac{K(\sigma',\sigma'')}{(1+\tilde{r})^{2+\sigma'}}.
\end{aligned}$$

Finally, still with Lemma 4.1.1, we check that

$$\begin{aligned}
|\mathfrak{Re}(\overline{Q_c}\varphi)\partial_c Q_c| &= |Q_c|^2|\mathfrak{Re}(\psi)\partial_c Q_c| \\
&\leqslant \frac{K(\sigma',\sigma'')}{(1+\tilde{r})^{1+\sigma''+\varsigma}} \\
&\leqslant \frac{K(\sigma',\sigma'')}{(1+\tilde{r})^{2+\sigma'}},
\end{aligned}$$

and we check similarly that $|\nabla(\mathfrak{Re}(\overline{Q_c}\varphi)\partial_c Q_c)| \leqslant \frac{K(\sigma',\sigma'')}{(1+\tilde{r})^{2+\sigma'}}$. This concludes the proof of (4.2.6). □

4.3 Smoothness of the branch of travelling wave

4.3.1 Second derivative with respect to the speed

4.3.1.1 Proof of the differentiability

We recall that

$$L_{Q_c}(\varphi) = -ic\partial_{x_2}\varphi - \Delta\varphi - (1-|Q_c|^2)\varphi + 2\mathfrak{Re}(\overline{Q_c}\varphi)Q_c$$

and that

$$L_{Q_c}(\partial_c Q_c) = i\partial_{x_2}Q_c.$$

We define the operator

$$\partial_c L_{Q_c}(\varphi) := 2\mathfrak{Re}(\partial_c Q_c\overline{Q_c})\varphi + 2\mathfrak{Re}(\overline{\partial_c Q_c}\varphi)Q_c + 2\mathfrak{Re}(\overline{Q_c}\varphi)\partial_c Q_c - i\partial_{x_2}\varphi.$$

Take $\varepsilon>0$ a small constant, and remark that $L_{Q_{c+\varepsilon}}(\partial_c Q_{c+\varepsilon})=i\partial_{x_2}Q_{c+\varepsilon}$. We compute

$$L_{Q_c}(\partial_c Q_{c+\varepsilon}-\partial_c Q_c)=(L_{Q_c}-L_{Q_{c+\varepsilon}})(\partial_c Q_{c+\varepsilon})-i\partial_{x_2}Q_{c+\varepsilon}+i\partial_{x_2}Q_c.$$

Let us show that for all $\varepsilon>0$, $(L_{Q_c}-L_{Q_{c+\varepsilon}})(\partial_c Q_{c+\varepsilon})-i\partial_{x_2}Q_{c+\varepsilon}+i\partial_{x_2}Q_c\in\mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma}$ and that

$$\frac{(L_{Q_c}-L_{Q_{c+\varepsilon}})(\partial_c Q_{c+\varepsilon})-i\partial_{x_2}Q_{c+\varepsilon}+i\partial_{x_2}Q_c}{\varepsilon}\to\partial_c L_{Q_c}(\partial_c Q_c)+i\partial_{x_2}\partial_c Q_c$$

when $\varepsilon\to 0$ for the norm $\left\|\frac{1}{Q_c}.\right\|_{\otimes\otimes,\sigma}$. Fort that, is is enough to show that $\partial_c L_{Q_c}(\partial_c Q_c)+i\partial_{x_2}\partial_c Q_c\in\mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma}$. From Corollary 4.2.1, we have $\partial_c Q_c\in\mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma}$, thus we check that $i\partial_{x_2}\partial_c Q_c\in\mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma}$ (as in the proof of Lemma 4.1.1). Now, still using Corollary 4.2.1, we check that

$$\partial_c L_{Q_c}(\partial_c Q_c):=4\mathfrak{Re}(\partial_c Q_c\overline{Q_c})\partial_c Q_c+2|\partial_c Q_c|^2Q_c-i\partial_{x_2}\partial_c Q_c\in\mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma},$$

using in particular that $|\partial_c Q_c|^2$ is real valued. We deduce that, with Proposition 4.2.11,

$$\frac{\partial_c Q_{c+\varepsilon}-\partial_c Q_c}{\varepsilon}\to L_{Q_c}^{-1}(\partial_c L_{Q_c}(\partial_c Q_c)+i\partial_{x_2}\partial_c Q_c)$$

when $\varepsilon\to 0$. In particular $c\to L_{Q_c}^{-1}(\partial_c L_{Q_c}(\partial_c Q_c)+i\partial_{x_2}\partial_c Q_c)$ is a continuous function (for the norm $\|.\|_{\otimes,\sigma}$) and thus $c\to Q_c-1\in C^2(]0,c_0(\sigma)[,\mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma})$ for $c_0(\sigma)>0$ small enough, depending only on σ.

4.3.1.2 Differentiation of the energy and momentum

First, we check that, if $A\in\mathcal{E}_{\otimes,\sigma}$ and $B\in\mathcal{E}_{\otimes\otimes,\sigma}$, then $i\partial_{x_2}A\in\mathcal{E}_{\otimes\otimes,\sigma}$ and $\mathfrak{Re}(A\bar{B})\in L^1(\mathbb{R}^2,\mathbb{R})$ since, outside of $\{\tilde{r}\leqslant 1\}$,

$$|\mathfrak{Re}(A\bar{B})|\leqslant K\left(\left|\mathfrak{Re}\left(\frac{A}{Q_c}\right)\right|\left|\mathfrak{Re}\left(\frac{B}{Q_c}\right)\right|+\left|\mathfrak{Im}\left(\frac{A}{Q_c}\right)\right|\left|\mathfrak{Im}\left(\frac{B}{Q_c}\right)\right|\right)\leqslant\frac{K(\sigma)}{(1+\tilde{r})^{2+\sigma}}\|A\|_{\otimes,\sigma}\|B\|_{\otimes\otimes,\sigma}$$

for some $\sigma>0$. From Proposition 1.4.1, we have

$$\partial_c P(Q_c)=\langle L_{Q_c}(\partial_c Q_c),\partial_c Q_c\rangle=\langle i\partial_{x_2}Q_c,\partial_c Q_c\rangle.$$

Now, we recall that $\partial_c Q_c,\partial_c^2 Q_c\in\mathcal{E}_{\otimes,\sigma}$, $i\partial_{x_2}Q_c\in\mathcal{E}_{\otimes\otimes,\sigma}$, and $i\partial_{x_2}\partial_c Q_c\in\mathcal{E}_{\otimes\otimes,\sigma}$. We deduce that

$$\mathfrak{Re}(i\partial_{x_2}\partial_c Q_c\overline{\partial_c Q_c}),\mathfrak{Re}\big(i\partial_{x_2}Q_c\overline{\partial_c^2 Q_c}\big)\in L^1(\mathbb{R}^2,\mathbb{R}),$$

and therefore $\langle i\partial_{x_2}Q_c,\partial_c Q_c\rangle\in C^1(]0,c_0[,\mathbb{R})$ (for $c_0=c_0(\sigma),\sigma=1/2$ for instance) with

$$\partial_c(\langle i\partial_{x_2}Q_c,\partial_c Q_c\rangle)=\langle i\partial_{x_2}\partial_c Q_c,\partial_c Q_c\rangle+\langle i\partial_{x_2}Q_c,\partial_c^2 Q_c\rangle.$$

We deduce that $P(Q_c)\in C^2(]0,c_0[,\mathbb{R})$ and

$$\partial_c^2 P(Q_c)=\langle i\partial_{x_2}\partial_c Q_c,\partial_c Q_c\rangle+\langle i\partial_{x_2}Q_c,\partial_c^2 Q_c\rangle.$$

Now, we recall from Proposition 1.4.1 that $\partial_c E(Q_c)=c\partial_c P(Q_c)$. We deduce that $E(Q_c)\in C^2(]0,c_0[,\mathbb{R})$ and

$$\partial_c^2 E(Q_c)=\partial_c P(Q_c)+c\partial_c^2 P(Q_c).\tag{4.3.1}$$

4.3.2 Generalisation to higher order derivatives

We argue by induction on $n\geqslant 1$. We define the set of functions

$$\mathcal{A}_n:=\mathrm{Span}_{0\leqslant k<n}(i\partial_{x_2}\partial_c^k Q_c)+\mathrm{Span}_{k,l,m\geqslant 0;0\leqslant k+l+m<n}\big(\mathfrak{Re}\big(\partial_c^k Q_c\overline{\partial_c^l Q_c}\big)\partial_c^m Q_c\big).$$

We suppose the following results for $n\geqslant 1$: for all $0<\sigma<1$, there exists $c_0(\sigma)>0$ such that

- $c\to Q_c-1\in C^n(]0,c_0(\sigma)[,\mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma})$
- $L_{Q_c}(\partial_c^n Q_c)=A_n(c)\in\mathcal{A}_n$.

In subsection 4.3.1.1, we have shown this results for $n=2$. Let us show that these results then holds for $n+1$ if they do for a given $n \geqslant 1$.

As in the previous subsection, we show that $A_n(c), \partial_c A_n(c) \in \mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma}$ and $\partial_c L_{Q_c}(\partial_c^n Q_c) \in \mathcal{E}^{2\mathrm{sym}}_{\otimes\otimes,\sigma}$, using $\partial_c^n Q_c \in \mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma}$ instead of $\partial_c Q_c \in \mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma}$. Now, as in the proof in subsection 4.3.1.1, we can show similarly that

$$\frac{\partial_c^n Q_{c+\varepsilon} - \partial_c^n Q_c}{\varepsilon} \to L_{Q_c}^{-1}(\partial_c L_{Q_c}(\partial_c^n Q_c) + \partial_c A_n(c))$$

when $\varepsilon \to 0$ for the norm $\left\| \frac{1}{Q_c} . \right\|_{\otimes,\sigma}$. We deduce that $c \to Q_c - 1 \in C^{n+1}(]0, c_0(\sigma)[, \mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma})$ and that, defining $A_{n+1}(c) := \partial_c L_{Q_c}(\partial_c^n Q_c) + \partial_c A_n(c)$, we have $L_{Q_c}(\partial_c^{n+1} Q_c) = A_{n+1}(c) \in \mathcal{A}_{n+1}$.

Finally, we check that for all $n, m \geqslant 0$, $\mathfrak{Re}(i\partial_{x_2}\partial_c^n Q_c \overline{\partial_c^m Q_c}) \in L^1(\mathbb{R}^2, \mathbb{R})$ since $\partial_c^n Q_c, \partial_c^m Q_c \in \mathcal{E}^{2\mathrm{sym}}_{\otimes,\sigma}$. Therefore, we check by induction that $P_2(Q_c) \in C^\infty(]0, c_0[, \mathbb{R})$, with

$$\partial_c^l P_2(Q_c) = \sum_{n+m=l} a_{n,m} \langle i\partial_{x_2}\partial_c^n Q_c, \partial_c^m Q_c \rangle$$

for some $(a_{n,m}) \in \mathbb{R}$. Using $\partial_c E(Q_c) = c\partial_c P_2(Q_c)$, we deduce that $E(Q_c) \in C^\infty(]0, c_0[, \mathbb{R})$. This concludes the proof of Theorem 1.5.2.

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